CONFORMALLY INVARIANT RELATIVISTIC SOLUTIONS

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Abstract

The study of exact solutions to the Einstein and Einstein–Maxwell field equations, by imposing a symmetry requirement on the manifold, has been the subject of much recent research. In this thesis we consider specifically conformal symmetries in static and nonstatic spherically symmetric spacetimes. We find conformally invariant solutions, for spherically symmetric vectors, to the Einstein–Maxwell field equations for static spacetimes. These solutions generalise results found previously and have the advantage of being regular in the interior of the sphere. The general solution to the conformal Killing vector equation for static spherically symmetric spacetimes is found. This solution is subject to integrability conditions that place restrictions on the metric functions. From the general solution we regain the special cases of Killing vectors, homothetic vectors and spherically symmetric vectors with a static conformal factor. Inheriting conformal vectors in static spacetimes are also identified. We find a new class of accelerating, expanding and shearing cosmological solutions in nonstatic spherically symmetric spacetimes. These solutions satisfy an equation of state which is a generalisation of the stiff equation of state. We also show that this solution admits a conformal Killing vector which is explicitly obtained.

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

M S Maharaj
December 1993

Dedication

To Rajashree, my family and friends

Acknowledgments

Without the excellent supervision of my friend Professor S D Maharaj, this work would not have been possible, thank you. I also thank my co–supervisor Dr R Maartens (of the University of the Witwatersrand) for assistance provided. My greatest appreciation is due to my friend Kesh Govinder for the excellent work he has done using LaTeX to prepare this work. Also to my family, especially my Dad, my brother and my Mum, for taking on the burdens that were really mine, thank you. To my fianceé, Rajashree, thanks for the understanding and support. I am grateful for financial assistance to the Foundation for Research and Development (University Development Programme) and to the Hanno Rund Fund of the University of Natal.

Contents

Introduction

Computer generated numerical solutions to the Einstein field equations are a powerful means of studying stellar structure. This is an area of rapidly expanding research in general relativity and cosmology. Although some good numerical results are available (MacCallum 1991) these numerical techniques obviously have their drawbacks. Primarily the problem is that computer generated solutions do not provide for a global understanding of the gravitational field. Such solutions are too closely tied to the parameters and initial conditions, and are often unstable in relation to some of these parameters.

Analytic solutions, on the other hand, do not suffer from these problems. They are not specifically tied to a choice of parameters and initial conditions. If a solution is found to be very sensitive to a variation in any parameter, then this may be confidently viewed as a property of the spacetime and not as an artificial property. A global analysis of exact solutions is possible, and furthermore the applicability of exact solutions to astrophysical and cosmological problems is better understood than the numerical case. Finding exact solutions to the field equations is however not necessarily very easy. We seek to model relativistic stars in astrophysics for which solutions of the Einstein field equations are in closed form (Shapiro and Teukolsky 1983).

The techniques used thus far to generate analytic solutions to the Einstein field equations for spherically symmetric spacetimes have been mainly *ad hoc* in nature (Kramer *et al* 1980). Normally we specify one or more of the geometric and matter variables, and use the field equations to determine the remaining variables. It is quite clear from previous research that due to the complexity of the Einstein field equations, simplifying assumptions need to be made in order to generate exact solutions. Such assumptions should incorporate symmetry properties of the manifold on geometrical grounds. The aim of our study is to systematise the search for exact solutions by introducing Lie symmetry requirements on the spacetime. Earlier works utilising this approach were concerned with self-similar spacetimes (Cahill and Taub 1971, Eardly 1974). More recently Herrera *et al* (1984) and Herrera and Ponce de Leon (1985) have assumed that the spherically symmetric spacetime manifold admits a static conformal Killing vector.

Our intention in this thesis is to find solutions to the field equations by imposing a conformal Killing symmetry requirement on spacetime. This assumption is not very restrictive and may be motivated on physical and geometrical grounds (Dyer *et al* 1987, Coley 1991). While this condition is essentially geometric it has two physical points of support. Firstly, it is a generalisation of self-similarity in hydrodynamics. Secondly, it is a generalisation of the property of the incompressible Schwarzchild interior solution, which has eleven independent conformal symmetries in addition to the four Killing symmetries, since it is conformally flat. The Schwarzchild interior solution is perhaps the most realistic known exact static spacetime. The conformal symmetry assumption therefore offers the possibility of finding exact solutions that are generalisations of the Schwarzchild interior solution.

Our specific task here is to investigate spherically symmetric spacetimes that admit a conformal Killing vector field. Spherically symmetric models have been widely applied to describe realistic gravitational fields. We consider both the static and nonstatic cases. Particular forms for the conformal Killing vector are investigated and we show that they lead to new solutions of the field equations. A significant result is that we generate the general conformal symmetry in static spacetimes. Our solutions have astrophysical (see chapter 2) and cosmological (see chapter 4) applications.

This thesis will follow the following broad outline:

- In chapter 1 we briefly consider the Einstein–Maxwell field equations and symmetries on manifolds. We present the field equations for a charged nonconducting perfect fluid. The various symmetries on the spacetime manifold, in terms of the Lie derivative, are defined. Their interrelationships are presented diagrammatically. In particular we consider conformal Killing symmetries as this concept underlies this thesis.

- In chapter 2 we analyse conformally symmetric charged static fluid spheres. The appropriate field equations are generated and matching conditions at the surface of the sphere are found. A necessary condition for a nonsingular stellar centre is established. For a nonstatic spherically symmetric conformal Killing vector with static conformal factor a number of new solutions to the Einstein–Maxwell system of equations are found. These solutions generalise the results of Herrera *et al* (1984) and Herrera and Ponce de Leon (1985). The physical properties of these solutions are investigated.

- Chapter 3 is concerned with the general conformal geometry of static spherically symmetric spacetimes. The conformal Killing vector equations are obtained and integrated. An outline of the integration procedure is provided. The general conformal Killing vector is obtained subject to integrability conditions. An analysis of the integrability conditions simplifies the form of the conformal Killing vector. We consider the special cases of Killing vectors, homothetic vectors, inheriting vectors and spherically symmetric conformal Killing vectors with a static conformal factor.

- In chapter 4 we consider nonstatic shearing solutions in spherically symmetric gravitational fields. The field equations are generated using an ansatz of Hajj–Boutros (1985). A class of new shearing solutions is presented and previously published results are related to special cases of our solution. We briefly investigate the physical properties of the solution. The solution is then characterised geometrically with a conformal symmetry.

- In the conclusion we highlight the results established in this thesis. We point out a number of generalisations that arise from our results for future research. In addition we motivate the application of symmetries other than conformal Killing vectors in the analysis of exact solutions to the field equations.

We should point out that some of the results contained in this thesis have been published. Many of the results in chapter 2, pertaining to conformally invariant charged static spheres, may be found in Maartens and Maharaj (1990). The general conformal solution in static spherically symmetric spacetimes has been submitted for publication (Maharaj *et al* 1993b). Aspects of the class of shearing solutions to the Einstein field equations in chapter 4 appear in Maharaj *et al* (1993a).

Chapter 1

Field Equations and Symmetries

1.1 Introduction

In this chapter we present a brief outline of the results required and the techniques used in this thesis. We do not give a detailed overview of general relativity. For the relevant background the reader is referred to Hawking and Ellis (1973), Misner *et al* (1973), Stephani (1990) and Wald (1984) among many other excellent works. Our intention is to briefly discuss the Einstein–Maxwell system of field equations and the role of symmetries in general relativity. These are necessary to find solutions modelling conformally invariant charged fluid spheres and perfect fluid cosmological models which are considered later. In §1.2 we introduce the Einstein–Maxwell system of equations. The energy–momentum tensor describes a charged nonconducting imperfect fluid without heat flow. In §1.3 we consider symmetries in general relativity. For the necessary differential geometry for an analysis of symmetries on manifolds the reader is referred to Dubrovin *et al* (1984), Kobayashi and Nomizu (1968) and Yano and Ishihira (1973). The Lie bracket is defined and a Lie algebra is introduced. A variety of symmetries are considered and the relationship between these symmetries is illustrated. The special case of conformal Killing symmetries is investigated in particular as they are of interest in subsequent chapters.

1.2 Field Equations

We take spacetime to be a four–dimensional differentiable manifold endowed with a symmetric, nondegenerate metric tensor field $\boldsymbol{g}$ of signature $(- + ++)$. Manifolds with an indefinite metric tensor field, as is the case in general relativity, are called pseudo–Riemannian manifolds. Points in the manifold are labelled by the real coordinates $x^i = (x^0, x^1, x^2, x^3)$ where x^0 is timelike and x^1, x^2, x^3 are spacelike. The metric tensor field is associated with the metric connection $\boldsymbol{\Gamma}$ by the fundamental theorem of Riemannian geometry. This theorem guarantees the existence of a unique symmetric connection preserving inner products under parallel transport. In terms of components we have

$$\Gamma^i{}_{jk} = \tfrac{1}{2} g^{il}(g_{kl,j} + g_{lj,k} - g_{jk,l})$$

where we use the notation that commas denote partial differentiation.

We can show that the noncommutativity of the covariant derivative generates the Riemann curvature tensor, whose components are given by

$$R^i{}_{jkl} = \Gamma^i{}_{jl,k} - \Gamma^i{}_{jk,l} + \Gamma^m{}_{jl}\Gamma^i{}_{mk} - \Gamma^m{}_{jk}\Gamma^i{}_{ml}$$

The Riemann tensor provides a measure of the curvature of spacetime, ie, it provides a measure of deviation from flatness. Contraction of the curvature tensor $R^i{}_{jkl}$ yields the Ricci tensor defined as

$$R_{ij} = R^k{}_{ikj}$$

$$= \Gamma^k{}_{ij,k} - \Gamma^k{}_{ik,j} + \Gamma^l{}_{ij}\Gamma^k{}_{lk} - \Gamma^l{}_{ik}\Gamma^k{}_{lj}$$

A further contraction yields the Ricci scalar

$$R = R^i{}_i$$

The symmetric Einstein tensor $\boldsymbol{G}$ is defined in terms of the Ricci tensor and Ricci scalar by

$$G_{ij} = R_{ij} - \tfrac{1}{2} R g_{ij}$$

It follows that the Einstein tensor has zero divergence

$$G^{ij}{}_{;j} = 0$$

called the Bianchi identity where we use the notation that semicolons denote covariant differentiation.

In this thesis we consider a charged nonconducting imperfect fluid without heat flow. The total energy–momentum tensor $\boldsymbol{T}$ is given by the sum

$$T_{ij} = M_{ij} + E_{ij} \tag{1.2.1}$$

The matter contribution is

$$M_{ij} = (\mu + p)u_i u_j + p g_{ij} + \pi_{ij} \tag{1.2.2}$$

where the energy density μ, the pressure p and the anisotropic stress tensor π_{ij} are measured relative to the fluid four–velocity vector $\boldsymbol{u}$. The electromagnetic contribution is

$$E_{ij} = F_{ik}F_j{}^k - \tfrac{1}{4} g_{ij} F_{kl} F^{kl} \tag{1.2.3}$$

where $\boldsymbol{F}$ is the skew–symmetric electromagnetic Maxwell tensor. Note that in our model we assume that there is no heat flow. The physical reason for this requirement is that our conformally invariant fluid spheres must match to the exterior Reissner–Nordstrom spacetime. The Einstein–Maxwell field equations are given by

$$R_{ij} - \tfrac{1}{2}Rg_{ij} = T_{ij} \tag{1.2.4a}$$

$$F_{[ij;k]} = 0 \tag{1.2.4b}$$

$$F^{ij}{}_{;j} = \epsilon u^i \tag{1.2.4c}$$

where ϵ is the charge density. This system describes a charged gravitating imperfect fluid. We seek solutions to the Einstein field equations (1.2.4a) and the Einstein–Maxwell system (1.2.4) in subsequent chapters by imposing a symmetry requirement on spacetime.

1.3 Symmetries in General Relativity

In this section we briefly discuss some concepts of symmetries in general relativity. The type of symmetries dealt with are those which arise from the existence of a Lie algebra of vector fields on the spacetime manifold which are invariant vector fields of geometrical objects on the manifold.

The Lie derivative plays an important role in describing symmetries of physical fields in general and the gravitational field in particular. It provides a coordinate independent description of a symmetry property in the manifold M and is defined naturally by the manifold structure. Unlike the covariant derivative the Lie derivative is an operation defined on a differentiable manifold without imposing additional structure. Let $\boldsymbol{X}$ be a vector field such that $\boldsymbol{X}$ operates on differentiable scalar fields f producing scalar fields $\boldsymbol{X}f$. The Lie derivative with respect to $\boldsymbol{X}$ is an extension of this operation to the action of an operator $\mathcal{L}_{\boldsymbol{X}}$ on all differentiable tensor fields (Hawking and Ellis 1973, Kramer *et al* 1980, Stephani 1990). The Lie derivative obeys all the usual rules for a derivative operator and always gives a tensor field of the same type as the tensor field differentiated.

The Lie bracket or commutator of two vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ is defined by

$$[\boldsymbol{X}, \boldsymbol{Y}] = \boldsymbol{X}\boldsymbol{Y} - \boldsymbol{Y}\boldsymbol{X} \tag{1.3.1}$$

The commutator is also a vector as $[\boldsymbol{X}, \boldsymbol{Y}]$ inherits the linearity properties of $\boldsymbol{X}$ and $\boldsymbol{Y}$. A Lie algebra is a finite dimensional vector space on which the bracket operation (1.3.1) has been defined. The Lie bracket (1.3.1) is skew–symmetric and bilinear. This operation is not associative but instead satisfies the Jacobi identity

$$[\boldsymbol{X}, [\boldsymbol{Y}, \boldsymbol{Z}]] + [\boldsymbol{Y}, [\boldsymbol{Z}, \boldsymbol{X}]] + [\boldsymbol{Z}, [\boldsymbol{X}, \boldsymbol{Y}]] = 0 \tag{1.3.2}$$

for arbitrary vector fields $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$. The Lie derivative and the Lie bracket are related by the identities

$$\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Y} = [\boldsymbol{X}, \boldsymbol{Y}]$$

$$\mathcal{L}_{[\boldsymbol{X}, \boldsymbol{Y}]} = \left[\mathcal{L}_{\boldsymbol{X}}, \mathcal{L}_{\boldsymbol{Y}}\right]$$

for vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$.

There is a close relationship between Lie algebras and Lie groups: every Lie algebra defines a unique, simply connected Lie group. An r–dimensional Lie group G_r is a group which is also a smooth r–dimensional differentiable manifold whose structure is such that the group composition $G_r \times G_r \longrightarrow G_r$ and the group inverse $G_r \longrightarrow G_r$ are smooth maps. The subject of Lie groups will not be pursued further as it is not relevant to this thesis. For further information on Lie groups and their application to physics see Choquet–Bruhat *et al* (1977), Dubrovin *et al* (1984), Kramer *et al* (1980), Sattinger and Weaver (1986) and Schutz (1980). Kramer *et al* in particular relate Lie groups and Lie algebras to various classes of solutions of the Einstein field equations.

A variety of symmetries may be defined on a manifold by the action of the operator $\mathcal{L}_{\boldsymbol{X}}$ on the metric tensor, the connection coefficients, the Ricci tensor, the Riemann tensor and the Weyl tensor. These symmetries are defined by

1. WPC – Weyl Projective Collineation

$$\mathcal{L}_{\boldsymbol{X}} W^i{}_{jkl} = 0$$

2. PC – Projective Collineation

$$\mathcal{L}_{\boldsymbol{X}} \Gamma^i{}_{jk} = \delta^i{}_j \psi_{;k} + \delta^i{}_k \psi_{;j}$$

3. SPC – Special Projective Collineation

$$\mathcal{L}_{\boldsymbol{X}} \Gamma^i{}_{jk} = \delta^i{}_j \psi_{;k} + \delta^i{}_k \psi_{;j} \qquad \psi_{;jk} = 0$$

4. RC – Ricci Collineation

$$\mathcal{L}_{\boldsymbol{X}} R_{ij} = 0$$

5. CC – Curvature Collineation

$$\mathcal{L}_{\boldsymbol{X}} R^i{}_{jkl} = 0$$

6. SCC – Special Curvature Collineation

$$(\mathcal{L}_{\boldsymbol{X}} \Gamma^i{}_{jk})_{;l} = 0$$

7. AC – Affine Collineation

$$\mathcal{L}_{\boldsymbol{X}}\Gamma^{i}{}_{jk} = 0$$

8. HM – Homothetic Motion

$$\mathcal{L}_{\boldsymbol{X}} g_{ij} = 2\psi g_{ij} \qquad \psi = \text{constant}$$

9. M – Motions

$$\mathcal{L}_{\boldsymbol{X}} g_{ij} = 0$$

10. S Conf C – Special Conformal Collineation

$$\mathcal{L}_{\boldsymbol{X}}\Gamma^{i}{}_{jk} = \delta^{i}{}_{j}\psi_{;k} + \delta^{i}{}_{k}\psi_{;j} - g_{jk}g^{il}\psi_{;l} \qquad \psi_{;jk} = 0$$

11. S Conf M – Special Conformal Motions

$$\mathcal{L}_{\boldsymbol{X}} g_{ij} = 2\psi g_{ij} \qquad \psi_{;jk} = 0$$

12. W Conf C – Weyl Conformal Collineation

$$\mathcal{L}_{\boldsymbol{X}} C^{i}{}_{jkl} = 0$$

13. Conf C – Conformal Collineation

$$\mathcal{L}_{\boldsymbol{X}}\Gamma^{i}{}_{jk} = \delta^{i}{}_{j}\psi_{;k} + \delta^{i}{}_{k}\psi_{;j} - g_{jk}g^{il}\psi_{;l}$$

14. Conf M – Conformal Motions

$$\mathcal{L}_{\boldsymbol{X}} g_{ij} = 2\psi g_{ij}$$

15. NC – Null Geodesic Collineation

$$\mathcal{L}_{\boldsymbol{X}}\Gamma^{i}{}_{jk} = g_{jk}g^{im}\psi_{;m}$$

16. SNC – Special Null Geodesic Collineation

$$\mathcal{L}_{X}\Gamma^{i}{}_{jk} = g_{jk}g^{im}\psi_{;m} \qquad \psi_{;jk} = 0$$

A detailed discussion of the geometric interpretations and interrelationships of these symmetries is given by Katzin and Levine (1972). Not contained in their list is the curvature inheritance symmetry of Duggal (1992) which generalises curvature collineations.

The relationship between the various symmetries is summarised in Figure 1. This table is to be understood as follows: The symmetry described in any block, when it exists, is automatically a subcase of the symmetries described in adjacent blocks indicated by arrows leading from the given block. For example if an HM is admitted then the transformation which defines the HM also satisfies the requirements for being a Conf M and AC and so on through the diagram. Note that the arrow

$$\dashrightarrow$$

should only be considered when the Ricci tensor vanishes. The arrow

$$\dashrightarrow$$

applies when the spacetime is Ricci flat. We have adapted Figure 1 from Katzin *et al* (1969) and Katzin and Levine (1972).

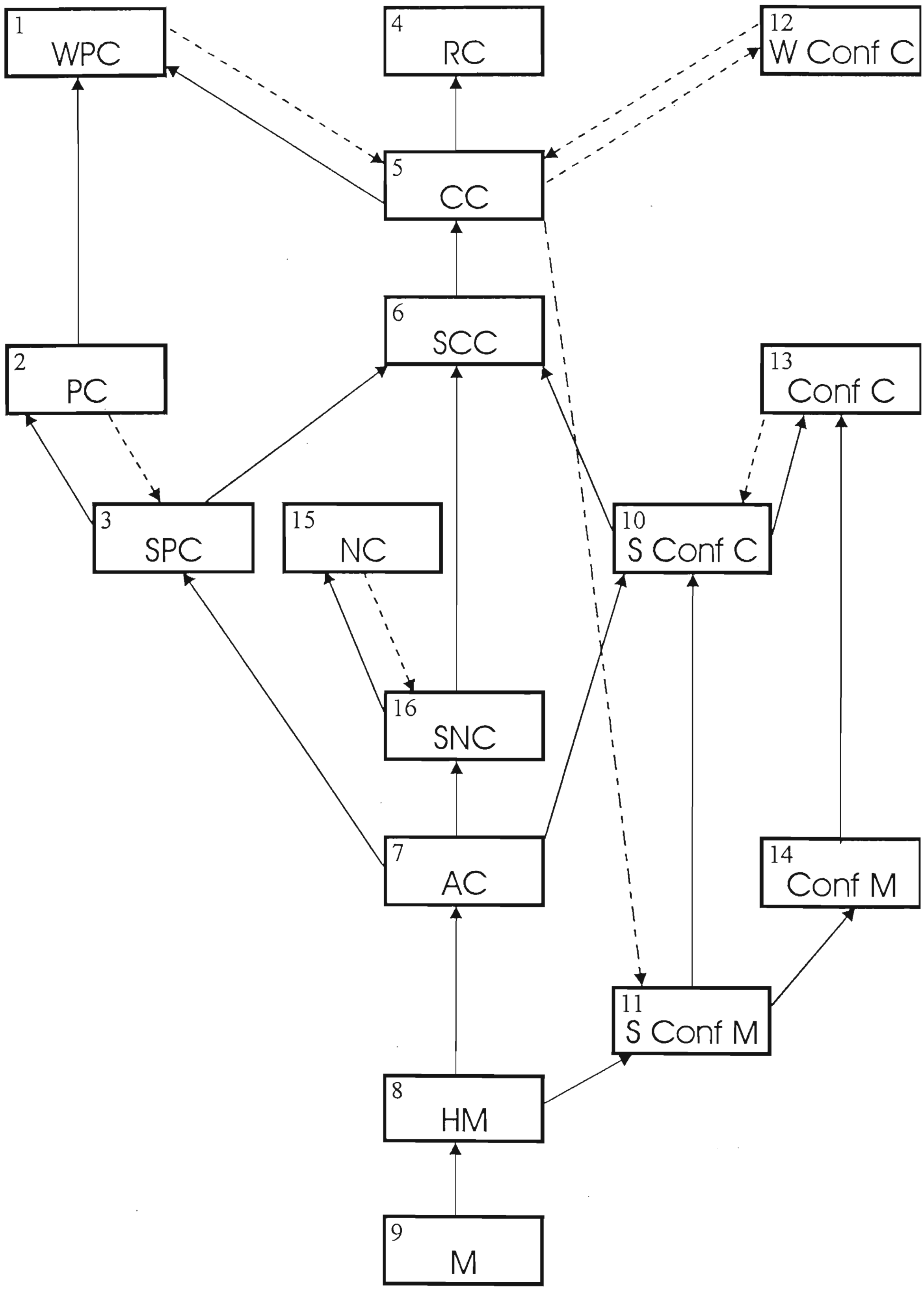

Figure 1. Relations between symmetries

Of the various symmetries given in Figure 1 we are primarily concerned with conformal motions. A conformal Killing vector $\boldsymbol{\xi}$ is defined by the action of $\mathcal{L}_{\boldsymbol{\xi}}$ on the metric tensor field $\boldsymbol{g}$:

$$\mathcal{L}_{\boldsymbol{\xi}} g_{ij} = 2\psi g_{ij} \tag{1.3.3}$$

where $\psi = \psi(x^i)$ is the conformal factor and $\boldsymbol{g}$ is the metric tensor field. If $\boldsymbol{g}$ is specified, then we solve (1.3.3) to obtain the conformal Killing vector $\boldsymbol{\xi}$. There are four cases associated with the equation (1.3.3):

(i) $\psi = 0$: $\boldsymbol{\xi}$ is a Killing vector,

(ii)$\psi_{,i} = 0 \neq \psi$: $\boldsymbol{\xi}$ is a homothetic Killing vector,

(iii) $\psi_{;ij} = 0 \neq \psi_{,i}$: $\boldsymbol{\xi}$ is a special conformal Killing vector and

(iv) $\psi_{;ij} \neq 0$: $\boldsymbol{\xi}$ is a nonspecial conformal Killing vector.

which are represented in their appropriate blocks in Figure 1. Killing vectors generate constants or first integrals of the motion along geodesics. The Killing vectors span a group of isometries which may be used to invariantly characterise solutions of the Einstein field equations (Kramer *et al* 1980). A homothetic Killing vector scales distances by the same constant factor and preserves the null geodesic affine parameters. Conformal Killing vectors generate constants of the motion along null geodesics for massless particles. A detailed account of the relationship between Lie symmetries, including conformal motions, and first integrals is given by Katzin and Levine (1972).

Suppose that G_r is a group of conformal motions with generators $\{\boldsymbol{\xi}_I\} =$

$\{\xi_1, \xi_2, \ldots, \xi_r\}$. The elements of the basis $\{\xi_I\}$ are related by

$$[\xi_I, \xi_J] = C^K{}_{IJ}\xi_K \tag{1.3.4}$$

where the $C^K{}_{IJ}$ are the structure constants of the group and satisfy

$$C^K{}_{IJ} = -C^K{}_{JI}$$

The Lie identity

$$C^K{}_{LM}C^M{}_{IJ} + C^K{}_{IM}C^M{}_{JL} + C^K{}_{JM}C^M{}_{LI} = 0$$

is obtained by substituting the relation (1.3.4) into the Jacobi identity (1.3.2). In the spacetime manifold of classical general relativity we have a maximal G_{15} Lie algebra of conformal motions. This is always possible if the spacetime is conformally flat (Choquet–Bruhat *et al* 1977).

In this thesis we are concerned with finding solutions to the conformal Killing equation (1.3.3) in spherically symmetric spacetimes. The following definitions are necessary for the spacetimes studied in this thesis (Demainski 1985):

- A spherically symmetric gravitational field is a spacetime in which the three parameter group of rotations acts as a group of isometries on spacelike two surfaces. This means that there exists three linearly independent Killing vector fields $\{\xi_\alpha\}$ such that

  $$[\xi_\alpha, \xi_\beta] = \epsilon^\gamma{}_{\alpha\beta}\xi_\gamma$$

  where $\epsilon^\gamma{}_{\alpha\beta}$ is the permutation index (for the explicit form of the Killing vectors see §3.4).

- If a spacetime admits a timelike Killing vector which is not orthogonal to any family of spacelike hypersurfaces, that is, $\xi_{[i;j}\xi_{k]} \neq 0$ then the spacetime is said to be stationary. If there exists a family of hypersurfaces to which the timelike Killing vector $\{\xi_0\}$ is orthogonal then the spacetime is said to be static.

The static spacetimes are considered in chapters 2 and 3. Nonstatic spherically symmetric gravitational fields are considered in chapter 4.

Chapter 2

Conformally Symmetric Static Fluid Spheres

2.1 Introduction

Solutions of the Einstein–Maxwell system of field equations are important in relativisitic astrophysics. In this chapter we model static spheres of charged imperfect fluids which match smoothly to the Reissner–Nordstrom exterior. The spacetime geometry is assumed to admit a nonstatic spherically symmetric conformal Killing vector with a static conformal factor. Our results generalise those of Herrera *et al* (1984) and Herrera and Ponce de Leon (1985) who investigated the special case of a static conformal Killing vector. In §2.2 we generate the Einstein field equations for the general case of a charged imperfect fluid energy–momentum tensor. We review particular results for a static conformal vector established previously and show why these are necessarily singular at the centre of the sphere. The case of the nonstatic conformal Killing vector, with a static conformal factor, is investigated in §2.3. These solutions may be regular at the centre of the sphere. A necessary condition on the conformal factor for regularity is established. In §2.4 we generate new solutions to the field equations for the nonstatic conformal vector. We establish an upper limit on the mass–radius ratio, discuss the regularity of the metric functions at the centre of the sphere and relate our results to those of Herrera and Ponce de Leon (1985). We also find regular perfect fluid solutions with uniform charge and regular charged imperfect fluid solutions by specifying a form for the conformal factor satisfying the regularity condition. The general conformal Killing vector in static spacetimes is obtained in chapter 3. Many of the results established in this chapter have been published (Maartens and Maharaj 1990).

2.2 Field Equations

We generate the Einstein field equations for a charged nonconducting imperfect fluid without heat flow. Our intention is to find solutions applicable to conformally symmetric static fluid spheres. The total energy–momentum tensor, including the electromagnetic contribution, is given by (1.2.1)–(1.2.3). The field equations governing this physical model are the Einstein–Maxwell system (1.2.4). The spacetime geometry is described by the line element

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.2.1}$$

where we have chosen coordinates $x^i = (t, r, \theta, \phi)$.

By symmetry the fluid 4–velocity $\boldsymbol{u}$, energy density μ, isotropic pressure p, and stress tensor $\boldsymbol{\pi}$ are restricted as established by Maartens *et al* (1986), Maharaj and Maartens (1986,1989). They have the form

$$u^i = e^{-\nu}\delta^i_0 \tag{2.2.2a}$$

$$\mu = \mu(r) \tag{2.2.2b}$$

$$p = \tfrac{1}{3}[p_R(r) + 2p_T(r)] \tag{2.2.2c}$$

$$\pi_{ij} = (p_R - p_T)(n_i n_j - \tfrac{1}{3}h_{ij}) \tag{2.2.2d}$$

$$n^i = e^{-\lambda}\delta^i_1 \tag{2.2.2e}$$

where $\boldsymbol{n}$ is a unit radial vector, p_R is the radial pressure, and p_T is the tangential pressure. Maharaj and Maartens (1986,1989) demonstrate that symmetry also implies that we can express the Maxwell field in the form

$$\boldsymbol{F} = f(r)dt \wedge dr \tag{2.2.3}$$

where $f(r)$ is an arbitrary function. From (2.2.3) we conclude that $d\boldsymbol{F} = 0$ and the first Maxwell equation (1.2.4b) is satisfied. The remaining Maxwell equation (1.2.4c) determines the charge density $\epsilon(r)$ in terms of the function f and the metric tensor $\boldsymbol{g}$. From the form (2.2.3) we conclude that the magnetic field vanishes in the fluid rest frame. Furthermore the electric field $E_i = F_{ij}u^j$ has the form

$$E_i = E(r)n_i \tag{2.2.4}$$

where $E(r)$ is an arbitrary function.

The Einstein field equations (1.2.4a), for the energy–momentum tensor (1.2.1), with the help of equations (2.2.1)–(2.2.4) may be expressed as the system

$$\mu + \tfrac{1}{2}E^2 = r^{-2} + r^{-2}e^{-2\lambda}(2r\lambda' - 1) \tag{2.2.5a}$$

$$p_R - \tfrac{1}{2}E^2 = -r^{-2} + r^{-2}e^{-2\lambda}(2r\nu' + 1) \tag{2.2.5b}$$

$$p_T + \tfrac{1}{2}E^2 = e^{-2\lambda}[\nu'' + r^{-1}(\nu' - \lambda')(r\nu' + 1)] \tag{2.2.5c}$$

where primes denote differentiation with respect to r. With $p_R = p_T$ and $E = 0$ we regain the field equations for an uncharged perfect fluid (Kramer *et al* 1980). The Maxwell equations (1.2.4b)–(1.2.4c) become

$$E^{ij}{}_{;j} = -F^{ij}F_j{}^k{}_{;k} = -\epsilon E n^i \tag{2.2.6a}$$

$$\epsilon = -u_i F^{ij}{}_{;j} = e^{-\lambda}r^{-2}(r^2E)' \tag{2.2.6b}$$

The contracted Bianchi identity is of the form

$$\epsilon E^i = M^{ij}{}_{;j}$$

in the case of a charged imperfect fluid. This gives the result

$$r^{-2}E(r^2E)' = (\mu + p_R)\nu' + p_R' + 2r^{-1}(p_R - p_T) \tag{2.2.7}$$

by (1.2.1)–(1.2.4), (2.2.1), (2.2.2) and (2.2.6). The field equations (2.2.5) imply (2.2.7). For some purposes it may be more convenient to replace one of the field equations with (2.2.7).

We briefly consider the boundary conditions at the radius $r = R$ of the fluid sphere. The metric functions and energy–momentum tensor interior to the sphere must match the Reissner–Nordstrom exterior spacetime

$$ds^2 = -\left(1 - 2\frac{M}{R} + \frac{Q^2}{2R^2}\right)dt^2 + \left(1 - 2\frac{M}{R} + \frac{Q^2}{2R^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

at the boundary. For the charged imperfect fluid sphere to match smoothly with the Reissner–Nordstrom exterior solution we require

$$e^{2\nu(R)} = e^{-2\lambda(R)} = 1 - \frac{2M}{R} + \frac{Q^2}{2R^2} \tag{2.2.8a}$$

$$p_R(R) = 0 \tag{2.2.8b}$$

$$E(R) = \frac{Q}{R^2} \tag{2.2.8c}$$

as junction conditions at the surface of the sphere (Herrera and Ponce de Leon 1985). In equations (2.2.8) the quantities M and Q represent the total mass and charge of the sphere respectively.

2.3 The Conformal Symmetry

The field equations (2.2.5) are underdetermined. One or more (depending on whether $E \neq 0$ and $p_R - p_T \neq 0$) functional relations need to be satisfied in order to generate a solution. One method is to assume that the spacetime is mapped conformally onto itself along the direction of the vector field $\boldsymbol{\xi}$, so that (1.3.3) is satisfied. This method avoids an *ad hoc* specification of one of the variables. We consider spherically symmetric conformal Killing vectors $\boldsymbol{\xi}$ of the form

$$\boldsymbol{\xi} = \xi^0(t,r)\frac{\partial}{\partial t} + \xi^1(t,r)\frac{\partial}{\partial r} \tag{2.3.1}$$

in this section. The nonstatic form of $\boldsymbol{\xi}$ generalises static conformal vectors considered previously by Herrera and Ponce de Leon (1985) and does generate regular static spheres. The conformal Killing equation (1.3.3) becomes

$$\nu'\xi^1 + {\xi^0}_t = \psi \tag{2.3.2a}$$

$$-e^{2\nu}\xi^0{}_{,r} + e^{2\lambda}\xi^1{}_{,t} = 0 \tag{2.3.2b}$$

$$\lambda'\xi^1 + \xi^1{}_{,r} = \psi \tag{2.3.2c}$$

$$\xi^1 = r\psi \tag{2.3.2d}$$

for the static line element (2.2.1). Note that the subscripts t, r, θ, ϕ denote partial differentiation. The general solution of the conformal equation (1.3.3) is investigated in the next chapter.

Herrera *et al* (1984) assume that

$$\boldsymbol{\xi} = \xi^0(r)\frac{\partial}{\partial t} + \xi^1(r)\frac{\partial}{\partial r} \tag{2.3.3}$$

Thus their form of the conformal Killing vector $\boldsymbol{\xi}$ is spherically symmetric and static. Using the line element (2.2.1) and the conformal vector (2.3.3) in the equations (2.3.2) we obtain

$$\xi^0 = A \tag{2.3.4a}$$

$$\xi^1 = r\psi \tag{2.3.4b}$$

$$\psi = Be^{-\lambda} \tag{2.3.4c}$$

$$e^{2\nu} = C^2r^2 \tag{2.3.4d}$$

where A, B and C are constants. This case has been analysed by Herrera *et al* (1984) and Herrera and Ponce de Leon (1985) in their analysis of static spheres in general relativity. We may set $A = 0$ since $A\partial/\partial t$ is a Killing vector. Also we may set $B = 1$ by the rescaling

$$\xi \longrightarrow \frac{\xi}{B} \qquad\qquad \psi \longrightarrow \frac{\psi}{B}$$

which leaves the conformal Killing vector equations (2.3.2) invariant. Thus the assumptions (2.3.2) and (2.3.3) for the static spacetime geometry (2.2.1) determine the metric component $e^{2\nu}$ explicitly, and fix the components ξ^0, ξ^1 and the conformal factor ψ.

The form (2.3.3) is the most general vector $\boldsymbol{\xi}$ invariant under the Killing symmetries of the metric tensor $\boldsymbol{g}$. Thus we have the relationships

$$\left[\frac{\partial}{\partial t}, \boldsymbol{\xi}\right] = 0$$

$$[\boldsymbol{\xi}_\alpha, \boldsymbol{\xi}] = 0$$

where $\{\boldsymbol{\xi}_\alpha\} = \{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$ generates SO(3). (For the explicit form of the Killing vectors in static spacetimes (2.2.1) see §3.4.) A nonisometric conformal Killing vector that is static and spherically symmetric is necessarily orthogonal to $\partial/\partial t$. As there are no such vectors in Minkowski spacetime the form of the static vector (2.3.3) rules out the limiting case of a regular vacuum solution. This suggests that the form (2.3.3) may lead to solutions of the field equations that are singular at the centre of the sphere. By spherical symmetry the world line $\{r = 0\}$ is a timelike geodesic. The regularity of spacetime along a geodesic imposes stringent conditions on the limiting behaviour of the metric tensor $\boldsymbol{g}$, obtained by expanding about the central geodesic.

If t measures proper time along $\{r = 0\}$, then this gives (Ellis *et al* 1978)

$$e^{2\nu} = 1 + O(l^2) \qquad r = l + O(l^3)$$

near $l = 0$, where l is the proper radial distance orthogonal to $\{r = 0\}$ $(dl = e^{\lambda}dr)$. By the rescaling

$$t \longrightarrow e^{-a}t \qquad \nu \longrightarrow \nu + a$$

which leaves the metric invariant, we obtain the more general necessary condition for a nonsingular stellar centre:

$$e^{2\nu} = e^{2a} + O(r^2) \tag{2.3.5a}$$

$$e^{2\lambda} = 1 + O(r^2) \tag{2.3.5b}$$

near $r = 0$. Then (2.3.4) shows that all the solutions obtained via (2.3.3) are necessarily singular.

The singularity of these relationships is clearly an undesirable feature. The requirement of a static conformal vector is too restrictive. For regular solutions we must weaken the static symmetry requirement of the vector (2.3.3). That is we assume the nonstatic spherically symmetric form (2.3.1) for the conformal Killing vector $\boldsymbol{\xi}$. Note that the form (2.3.1) generalizes the isotropic conformal vector

$$\boldsymbol{\xi} = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}$$

of Minkowski spacetime. This isotropic conformal symmetry generates nonstatic shear–free solutions for spherically symmetric gravitational fields (Dyer *et al* 1987, Maharaj *et al* 1991, Havas 1992). In addition we assume that the conformal factor is static so that

$$\psi = \psi(r) \tag{2.3.6}$$

Note that (2.3.6) follows from the static vector (2.3.3), but not necessarily from the nonstatic vector (2.3.1). The general case $\psi = \psi(r,t)$ will be the subject of later investigation. This should allow for a wider range of behaviour.

Using (2.3.1) and (2.3.6) we find that (2.3.2) generates the solution

$$\xi^0 = A + kt$$

$$\xi^1 = r\psi$$

$$\psi = Be^{-\lambda}$$

$$e^{2\nu} = C^2 r^2 \exp\left(-2kB^{-1}\int r^{-1}e^{\lambda}dr\right)$$

where A, B, C and k are constants. As for the static conformal vector we may set $A = 0$ and $B = 1$ without loss of generality. Thus we have

$$\boldsymbol{\xi} = kt\frac{\partial}{\partial t} + r\psi\frac{\partial}{\partial r} \tag{2.3.7a}$$

$$e^{2\lambda} = \psi^{-2} \tag{2.3.7b}$$

$$e^{2\nu} = C^2 r^2 \exp\left(-2k\int \frac{dr}{r\psi}\right) \tag{2.3.7c}$$

The solutions of Herrera *et al* (1984) and Herrera and Ponce de Leon (1985) discussed earlier belong to the class $k = 0$. The vacuum solution is given by $k = 1 = \psi$. The self–similar Tolman solutions (Wainwright 1985)

$$ds^2 = -r^{4(\gamma-1)/\gamma}dt^2 + b^2dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

are given by (2.3.7) with

$$k = \frac{(2-\gamma)}{b\gamma} \qquad \psi = \frac{1}{b} \qquad C = 1$$

There are many well–known spacetimes which are not contained in the class (2.3.7). An example is the constant density Schwarzschild interior solution (Kramer *et al* 1980)

$$e^{-2\lambda} = 1 - Ar^2 \tag{2.3.8a}$$

$$2e^{\nu} = 3(1 - AR^2)^{\frac{1}{2}} - (1 - Ar^2)^{\frac{1}{2}} \tag{2.3.8b}$$

If we take $\psi = \pm(1 - Ar^2)^{1/2}$, then

$$e^{-2\lambda} = 1 - Ar^2 \tag{2.3.9a}$$

$$e^{\nu} = Cr[2r^{-1}(1 + (1 - Ar^2)^{\frac{1}{2}})]^{\pm k} \tag{2.3.9b}$$

which cannot regain (2.3.8) for any C, k. This suggests that the Schwarzschild interior solution (2.3.8) requires a more general conformal factor $\psi = \psi(r, t)$.

For the conformal Killing vector (2.3.7) the field equations (2.2.5) become

$$\mu = \tfrac{1}{2}r^{-2}(1 - k^2) - 3r^{-1}\psi\psi' + \Delta \tag{2.3.10a}$$

$$p_R = \tfrac{1}{2}r^{-2}(k^2 - 1 + 4\psi^2 - 4k\psi) + r^{-1}\psi\psi' - \Delta \tag{2.3.10b}$$

$$E^2 = r^{-2}(k^2 + 1 - 2\psi^2) + 2r^{-1}\psi\psi' - 2\Delta \tag{2.3.10c}$$

In the above we have defined

$$\Delta = \tfrac{1}{2}(p_T - p_R) \tag{2.3.11}$$

as the measure of the pressure anisotropy. An exact solution of the Einstein–Maxwell equations (2.3.10) in the general case of an imperfect charged sphere ($E\Delta \neq 0$) requires a choice of $\psi(r)$ and of an equation of state $f(\mu, p_R, \Delta) = 0$ (or some other equivalent choice). We present a number of new solutions to the field equations (2.3.10) in §2.4.

On comparing equations (2.3.5) and (2.3.7) we obtain a necessary condition for regularity at the stellar centre. This may be expressed in terms of the conformal factor

$$\psi = 1 + O(r^2) \quad \text{near} \quad r = 0, \quad k = 1 \tag{2.3.12}$$

If the condition (2.3.12) is satisfied then the field equations (2.3.10) show that the dynamical variables μ, p_R, Δ and E are all bounded at $r = 0$. This is valid provided any one of the variables is bounded. This occurs naturally in the special case

$$E\Delta = 0$$

of an uncharged or perfect fluid sphere.

We should point out that each dynamical tensor is mapped conformally onto itself by the nonstatic vector (2.3.1). This property implies that

$$[\boldsymbol{\xi}, \boldsymbol{u}] = -\psi \boldsymbol{u}$$

$$[\boldsymbol{\xi}, \boldsymbol{n}] = -\psi \boldsymbol{n}$$

$$\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{\pi} = \left(2 + \frac{r\Delta'}{\Delta}\right) \psi \boldsymbol{\pi}$$

$$\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{F} = \left(\frac{f'}{f} r\psi + r\psi' + \psi + k\right) \boldsymbol{F}$$

$$[\xi, \boldsymbol{E}] = \psi\left(\frac{E'}{E}r - 1\right)\boldsymbol{E}$$

by (2.2.2)–(2.2.4) and (2.3.7). These relationships follow from our choice of nonstatic conformal vector ξ. Maartens *et al* (1986) prove that none of the above holds in general for conformal motions. The conformal Killing vector ξ forms a five–dimensional Lie algebra with the Killing vectors $\{\xi_0 = \partial/\partial t, \xi_\alpha\}$. The Lie brackets of the Lie algebra are given by

$$[\xi_0, \xi] = k\xi_0$$

$$[\xi_\alpha, \xi] = 0$$

$$[\xi_0, \xi_\alpha] = 0$$

$$[\xi_\alpha, \xi_\beta] = \epsilon^\gamma{}_{\alpha\beta}\xi_\gamma$$

where $\epsilon^\gamma{}_{\alpha\beta}$ is the standard permutation symbol.

2.4 New Solutions

We generate new solutions to the field equations (2.3.10) with the nonstatic spherically symmetric vector (2.3.7) and the static conformal factor (2.3.6). We begin by deriving the mass–charge–radius relations for solutions with the vector (2.3.7). By (2.2.8), (2.3.7), (2.3.10b), and (2.3.10c) we have

$$\frac{Q^2}{2R^2} = 1 - 3\psi^2(R) + 2k\psi(R) \tag{2.4.1a}$$

$$\frac{M}{R} = 1 - 2\psi^2(R) + k\psi(R) \tag{2.4.1b}$$

Since $Q^2 \geq 0$, $M \geq 0$, the relationships (2.4.1) imply

$$\max\{a_-, b_-\} \leq \psi(R) \leq \min\{a_+, b_+\} \tag{2.4.2}$$

where $3a_\pm = k \pm (k^2+3)^{\frac{1}{2}}$, $4b_\pm = k \pm (k^2+8)^{\frac{1}{2}}$. On eliminating $\psi^2(R)$ from (2.4.1) we obtain

$$3M = R + \frac{Q^2}{R} - kR\psi(R) \tag{2.4.3}$$

In the case of $k = 0$ we obtain the results of Herrera and Ponce de Leon (1985). In their solutions an increase in charge Q increases the mass M. For $k \neq 0$ it is no longer necessarily true that charge increases with mass. Furthermore, for uncharged spheres with conformal symmetry the value

$$\frac{M}{R} = \frac{1}{3}$$

is not an upper limit as stated by Herrera *et al* (1984) and Ponce de Leon (1988). We can show by (2.4.1) and (2.4.3) that the upper limit is

$$\frac{M}{R} = \tfrac{1}{3} - \tfrac{1}{9}k^2 + \tfrac{1}{9}|k|\left(k^2+3\right)^{\frac{1}{2}}$$

for uncharged spheres. Thus we have established that the limit on M/R is independent of the pressure anisotropy Δ. The limit depends only on the conformal symmetry parameter k. The quantity M/R may exceed $1/3$ and approach $1/2$ arbitrarily closely for large $|k|$. For regular uncharged spheres ($k = 1$) the maximum value is

$$\frac{M}{R} = \frac{4}{9}$$

which is the same as the perfect fluid limit. This was proved by Ponce de Leon (1988).

The condition for the existence of a horizon ($Q^2/2 \leq M^2$) follows from condition (2.4.1):

$$\psi(R) \leq \tfrac{1}{2}(k-1) \qquad \text{or} \qquad \psi(R) \geq \tfrac{1}{2}(k+1) \tag{2.4.4}$$

This condition yields the horizon

$$R_* = M + \left(M^2 - \tfrac{1}{2}Q^2\right)^{\frac{1}{2}} \tag{2.4.5}$$

From equation (2.4.1) it follows that $R_* \leq R$ for all k and all $\psi(R)$ satisfying the conditions (2.4.2) and (2.4.4) (which ensures the existence of R_*).

Solutions found earlier may be generalised with the assistance of our conformal Killing vector $\boldsymbol{\xi}$. We now describe briefly the generalisation of some of the singular solutions of Herrera and Ponce de Leon (1985). The generalised self-similar solutions are all singular, except for the vacuum limit. They are given by $\Delta = 0 = \psi'$ in the field equations (2.3.10):

$$\mu = \tfrac{1}{2}(1-k^2)r^{-2}$$

$$p = \tfrac{1}{2}(k^2 - 1 + 4\psi^2 - 4k\psi)r^{-2}$$

$$E^2 = (k^2 + 1 - 2\psi^2)r^{-2}$$

$$e^{2\lambda} = \psi^{-2}$$

$$e^{2\nu} = C^2 r^{2(1-k/\psi)}$$

for the nonstatic conformal vector (2.3.7). This is a charged generalisation of the Tolman solution (Wainwright 1985).

The generalisation of the charged dust solution follows by setting $\Delta = 0 = p$ in (2.3.10). This solution is also singular for all k:

$$\mu = \tfrac{1}{2}(1 - k^2 - 6r\psi\psi')r^{-2}$$

$$E^2 = (1 - 3\psi^2 + 2k\psi)r^{-2}$$

The conformal factor ψ satisfies

$$2r\psi\psi' + 4\psi^2 - 4k\psi + k^2 - 1 = 0$$

which is an Abel equation (Zwillinger 1989). The exceptional cases occur when $k = 0, 1$. When $k = 0$ we have

$$\psi^2 = 1/4 + Ar^{-4}$$

considered by Herrera and Ponce de Leon (1985). In the case $k = 1$ we generate the result

$$\psi = 1 + Ar^2 = e^{-\lambda}$$

$$e^{2\nu} = C^2r^2(A + r^2)^{-1}$$

by (2.3.7).

Another solution may also be generated. The generalised solution for perfect fluids with uniform charge density is obtained by solving (2.3.10c) for ψ with $\Delta = 0$, $E = Br$. Then substituting into (2.3.10a) and (2.3.10b) yields

$$\psi^2 = \tfrac{1}{2}(1 + k^2) - Ar^2 + \tfrac{1}{2}B^2r^4 = e^{-2\lambda}$$

$$\mu = \tfrac{1}{2}(1-k^2)r^{-2} + 3A - 3B^2r^2$$

$$p = \tfrac{1}{2}(1+3k^2)r^{-2} - 3A + 2B^2r^2 - 2k\psi r^{-2}$$

The metric function $e^{2\nu}$ is given by (2.3.7c). When $k=1$ we have that the regularity condition (2.3.12) is satisfied if $\psi \geq 0$, and further that μ, p, and E are bounded at $r=0$. In this case the solution takes the form

$$\psi = (1 - Ar^2 + \tfrac{1}{2}B^2r^4)^{\frac{1}{2}} = e^{-\lambda} \tag{2.4.6a}$$

$$e^{2\nu} = C^2r^2 \exp\left(-2\int \frac{dr}{r\psi}\right) \tag{2.4.6b}$$

$$\mu = 3A - 3B^2r^2 \tag{2.4.6c}$$

$$E = Br \tag{2.4.6d}$$

$$p = 2[1 - (1 - Ar^2 + \tfrac{1}{2}B^2r^4)^{\frac{1}{2}}]r^{-2} - 3A + 2B^2r^2 \tag{2.4.6e}$$

At the centre of the sphere we have the pressure $p(0) = -3A$ and the radius of the sphere is given by

$$4B^4R^6 - 12AB^2R^4 + (9A^2 + 6B^2)R^2 - 8A = 0$$

where $A > 0$ by (2.4.6) since $\mu > 0$. This implies that the cubic polynomial in R^2 always has a positive root. Thus it is possible to choose A and B in such a manner that μ remains positive throughout the sphere. A limiting case arises when μ and p vanish simultaneously at $r = R$. This is given by $B = A/\sqrt{2}$ and $R = (2/A)^{1/2}$.

Then the conformal factor vanishes at the boundary, $\psi(R) = 0$, so that by (2.4.1) and (2.4.5), the stellar surface is at the horizon R_*, with $M = R$ and $Q = \sqrt{2}R$. This limiting case is unstable. We should point out that all solutions in (2.4.6) have unstable features. Both the pressure p and the derivative $dp/d\mu$ are negative, at least near the centre. Physically we have that electric repulsion is holding the matter apart but this is not stable. Even though this regular solution is not stable it does generalise the singular solution of Herrera of Ponce de Leon (1985) and has the advantage of being regular.

The solution (2.4.6) suggests a general property of the class of solutions that obey the regularity conditions (2.3.12). All regular fluid spheres are characterised with nonpositive pressure at the centre. To establish this we utilise the necessary conditions for regularity: the quantities μ, p_R, Δ, E must be bounded at $r = 0$, and $\psi = 1 - Ar^2 + O(r^3)$ near $r = 0$ and $k = 1$. Then the field equations (2.3.10) give

$$\mu(0) = 6A + \Delta(0) \tag{2.4.7a}$$

$$p_R(0) = -4A - \Delta(0) \tag{2.4.7b}$$

$$E^2(0) = -2\Delta(0) \tag{2.4.7c}$$

Since $\mu, E^2 \geq 0$ we have $A \geq 0$ and $\Delta(0) \leq 0$. Furthermore, the isotropic pressure p given by (2.2.2c), satisfies

$$p(0) = \tfrac{1}{3}(\Delta(0) - 12A)$$

by (2.3.11) and (2.4.7). Thus, except for the special case $p(0) = 0$, the pressure is negative near the centre of the sphere. Note that neither of these cases is ruled out as negative pressures could occur in nonequilibrium metastable states (Maartens *et* *al* 1986), and zero central pressure may be possible for some astrophysical situations. However it is unlikely that $p(0) \leq 0$ is a realistic constraint for the interior of a stellar model.

We have exhibited a number of regular solutions to the field equations, with a central–pressure feature, for the spherically symmetric conformal symmetry (2.3.1) and static conormal factor (2.3.6). There are further general results that follow from (2.3.10)–(2.3.12):

(a) *Uncharged spheres:* There are no regular uncharged perfect fluid spheres. On setting $\Delta = 0 = E$ into (2.3.10), we find that $\psi = +(1 - Ar^2)^{\frac{1}{2}}$. This gives the solution (2.3.9) with $+k = +1$. Furthermore, we obtain the energy density $\mu = 3A > 0$ and the pressure

$$p = -A - 2r^{-2}[(1 - Ar^2)^{\frac{1}{2}} - (1 - Ar^2)],$$

which is negative for all r. Therefore there is no zero–pressure surface and we cannot have a regular uncharged perfect fluid sphere.

(b) *Incompressible spheres:* There are no regular incompressible perfect fluid spheres. This follows since $\Delta = 0 = \mu'$ implies $E = 0$ by (2.3.10a) and (2.3.10c). Then the argument in (a) applies and the result follows.

(c) *Equation of state:* There are no regular perfect fluid spheres with the linear barotropic equation of state

$$p = (\gamma - 1)\mu$$

where $1 \leq \gamma \leq 2$. If we put $\Delta = 0$ for this equation of state into (2.3.10), with $k = 1$, we obtain

$$\psi = 1 - Ar^{2/(2-3\gamma)}$$

which is singular at $r = 0$.

We also generate a solution for a charged and imperfect fluid sphere. It is convenient to choose the following simple regular polynomial form for the conformal factor

$$\psi = 1 - Ar^2$$

Then we choose a linear pressure anisotropy $\Delta(r)$. By (2.4.7) we have that $-6A \leq \Delta(0) \leq -4A$ ensures that $\mu(0) \geq 0$ and $p_R(0) \geq 0$. Also by (2.3.10b) the quantity $\Delta(R) = 4A(AR^2 - 1)$ ensures that $p_R(R) = 0$. Then (2.3.10a) yields $AR^2 \leq 1$ as the condition for $\mu(R) \geq 0$. We make the choice

$$A = \tfrac{1}{10}R^{-2}$$

$$50R^2\Delta = 7x - 25$$

where $x = r/R$. Then the field equations (2.3.10) generate the following simple polynomial solution

$$50R^2\mu = 5 + 7x - 3x^2 \tag{2.4.8a}$$

$$50R^2 p_R = 5 - 7x + 2x^2 \tag{2.4.8b}$$

$$50R^2 p_T = -45 + 7x + 2x^2 \tag{2.4.8c}$$

$$25R^2E^2 = 25 - 35x + 5x^2 \tag{2.4.8d}$$

The charge and mass are given by

$$Q = \tfrac{3}{10}\sqrt{26}R$$

$$M = \tfrac{27}{25}R$$

respectively where we have used (2.4.1). Thus the sphere is charge dominated and has no horizon, by (2.4.4). The radial pressure p_R is positive in the interior and decreases monotonically to zero at the boundary. However the tangential pressure p_T is negative throughout the interior. The metric components, from (2.3.7), inside the sphere are given by

$$e^{2\lambda} = \left(1 - \tfrac{1}{10}x^2\right)^{-2}$$

$$e^{2\nu} = \tfrac{9}{10}\left(1 - \tfrac{1}{10}x^2\right)$$

The conformal Killing vector

$$\boldsymbol{\xi} = \left[t\frac{\partial}{\partial t} + \left(1 - \tfrac{1}{10}x^2\right)x\frac{\partial}{\partial x}\right]$$

characterises this class of solutions.

The form of the conformal vector $\boldsymbol{\xi}$ chosen is crucial. The generalisations of the solutions of Herrera and Ponce de Leon (1985) and the new regular solutions (2.4.6) and (2.4.8) demonstrate the importance of the choice of conformal symmetry vector $\boldsymbol{\xi}$. The solutions exhibited by Herrera and de Leon (1985) are all infinite at $r = 0$ because $\boldsymbol{\xi}$ is static. We are able to overcome this singularity via the nonstatic generalisation (2.3.1). However our solutions suffer some drawbacks, particularly the problem of negative pressures. This is illustrated by the fact that the Schwarzschild interior solution is not contained in our class of solutions. The static nature of the conformal factor (2.3.6) is the essential cause of these limitations. When ψ is allowed to be nonstatic

$$\psi = \psi(t, r)$$

a new range of solutions is possible. In other words we require a nonstatic spherically symmetric conformal Killing vector with a nonstatic conformal factor in our search for realistic conformally symmetric static fluid spheres.

Chapter 3

Static Conformal Geometry

3.1 Introduction

In chapter 2 spherically symmetric conformal Killing vectors, with a static conformal factor, were considered. Solutions of the Einstein–Maxwell equations for static spheres of charged imperfect fluids were generated. Even though these solutions are regular they exhibit unstable features. Our intention in this chapter is to obtain the general conformal symmetry in static, spherically symmetric spacetimes without *ab initio* specifying a form for the conformal vectors or the conformal factor. This should allow for more general behaviour and may admit new classes of solutions to the Einstein field equations invariant under a conformal symmetry. Furthermore our general solution may help eliminate the instability inherent in the solutions in the previous chapter. The conformal Killing vector equations are presented and integrated in §3.2. As the integration procedure is complicated we present some details of the solution process. The general solution is subject to integrability conditions which are investigated in §3.3. A simplified form of the conformal symmetry is obtained. In §3.4 we investigate a number of special cases that arise from the general conformal Killing vector: Killing vectors, homothetic vectors, inheriting vectors and spherically symmetric conformal vectors with a static conformal factor. We should point out that aspects of this chapter have been submitted for publication (Maharaj *et al* 1993b).

3.2 Solution of the conformal equation

In this section we analyse the full conformal geometry for static, spherically symmetric spacetimes without making any assumptions about the form of the conformal symmetry. In standard coordinates $x^i = (t, r, \theta, \phi)$ the line element is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.2.1}$$

The conformal Killing vector equation (1.3.3) for the metric (3.2.1) reduces to the following system of equations

$$\nu'\xi^1 + \xi^0{}_t = \psi \tag{3.2.2a}$$

$$-e^{2\nu}\xi^0{}_r + e^{2\lambda}\xi^1{}_t = 0 \tag{3.2.2b}$$

$$-e^{2\nu}\xi^0{}_\theta + r^2\xi^2{}_t = 0 \tag{3.2.2c}$$

$$-e^{2\nu}\xi^0{}_\phi + r^2\sin^2\theta\xi^3{}_t = 0 \tag{3.2.2d}$$

$$\lambda'\xi^1 + \xi^1{}_r = \psi \tag{3.2.2e}$$

$$e^{2\lambda}\xi^1{}_\theta + r^2\xi^2{}_r = 0 \tag{3.2.2f}$$

$$e^{2\lambda}\xi^1{}_\phi + r^2\sin^2\theta\xi^3{}_r = 0 \tag{3.2.2g}$$

$$\xi^1 + r\xi^2{}_\theta = r\psi \tag{3.2.2h}$$

$$\xi^2{}_\phi + \sin^2\theta\xi^3{}_\theta = 0 \tag{3.2.2i}$$

$$\xi^1 + r\cot\theta\xi^2 + r\xi^3{}_\phi = r\psi \tag{3.2.2j}$$

The equations (3.2.2a–3.2.2j) are a coupled system of ten first order, linear partial differential equations. We seek to integrate this system in general to obtain the general conformal vector $\boldsymbol{\xi} = (\xi^0, \xi^1, \xi^2, \xi^3)$ and the conformal factor ψ.

From equations (3.2.2c, 3.2.2d, 3.2.2i) we establish the result

$$(\sin\theta\xi^3)_{\theta t} = 0$$

Also from (3.2.2f, 3.2.2g, 3.2.2i) we obtain

$$(\sin\theta\xi^3)_{\theta r} = 0$$

Thus we must have that $(\sin\theta\xi^3)_\theta$ is independent of t and r. Thus ξ^3 must be of the form

$$\xi^3 = \mathcal{A} + \csc\theta\mathcal{B} \tag{3.2.3}$$

where $\mathcal{A} = \mathcal{A}(\theta, \phi)$ and $\mathcal{B} = \mathcal{B}(t, r, \phi)$ are arbitrary functions.

On differentiating the difference (3.2.2h)–(3.2.2j) with respect to ϕ we obtain

$$\xi^2{}_{\phi\theta} - \cot\theta\xi^2{}_{\phi} - \xi^3{}_{\phi\phi} = 0$$

We may eliminate ξ^2 from the above with the help of (3.2.2i) to obtain

$$\sin\theta\cos\theta\xi^3{}_{\theta} + \sin^2\theta\xi^3{}_{\theta\theta} + \xi^3{}_{\phi\phi} = 0$$

where the dependent variable is ξ^3 only. Substituting (3.2.3) in this partial differential equation generates the integrability condition

$$\mathcal{B} + \mathcal{B}_{\phi\phi} + \sin^2\theta\cos\theta\mathcal{A}_{\theta} + \sin^3\theta\mathcal{A}_{\theta\theta} + \sin\theta\mathcal{A}_{\phi\phi} = 0 \tag{3.2.4}$$

Equation (3.2.4) implies that the function $\mathcal{B}$ must be of the form

$$\mathcal{B} = \cos\phi\mathcal{C} + \sin\phi\mathcal{D} + \mathcal{E} \tag{3.2.5}$$

where $\mathcal{C} = \mathcal{C}(t,r)$, $\mathcal{D} = \mathcal{D}(t,r)$ and $\mathcal{E} = \mathcal{E}(\phi)$ are arbitrary functions. Substituting (3.2.5) and (3.2.3) in (3.2.2i) and integrating yields

$$\xi^2 = -\sin^2\theta\mathcal{A}_{\theta}{}^{\phi} + \cos\theta[\sin\phi\mathcal{C} - \cos\phi\mathcal{D}] + \cos\theta\mathcal{E}^{\phi} + \mathcal{F} \tag{3.2.6}$$

where for convenience we have utilised the notation

$$\mathcal{A}_{\theta}{}^{\phi} = \int \mathcal{A}_{\theta} d\phi$$

$$\mathcal{E}^{\phi} = \int \mathcal{E} d\phi$$

and $\mathcal{F} = \mathcal{F}(t,r,\theta)$ is a function of the integration process.

With the help of (3.2.6) we integrate (3.2.2d) to obtain

$$\xi^0 = r^2 e^{-2\nu}\sin\theta[\sin\phi\mathcal{C}_t - \cos\phi\mathcal{D}_t] + \mathcal{G} \tag{3.2.7}$$

where $\mathcal{G} = \mathcal{G}(t, r, \theta)$ is a function of integration. Similarly (3.2.2g) implies

$$\xi^1 = r^2 e^{-2\lambda} \sin\theta[-\sin\phi \mathcal{C}_r + \cos\phi \mathcal{D}_r] + \mathcal{H} \tag{3.2.8}$$

where $\mathcal{H} = \mathcal{H}(t, r, \theta)$ is a function of integration. Substituting (3.2.7) and (3.2.8) into (3.2.2b) gives the condition

$$-e^{2\nu} \sin\theta[(r^2 e^{-2\nu}\mathcal{C}_t)_r \sin\phi - (r^2 e^{-2\nu}\mathcal{D}_t)_r \cos\phi] - e^{2\nu}\mathcal{G}_r$$

$$+r^2 \sin\theta[-\sin\phi \mathcal{C}_{tr} + \cos\phi \mathcal{D}_{tr}] + e^{2\lambda}\mathcal{H}_t = 0$$

This equation will be satisfied only if the following conditions hold

$$e^{2\nu}(r^2 e^{-2\nu}\mathcal{C}_t)_r + r^2 \mathcal{C}_{tr} = 0$$

$$e^{2\nu}(r^2 e^{-2\nu}\mathcal{D}_t)_r + r^2 \mathcal{D}_{tr} = 0$$

$$-e^{2\nu}\mathcal{G}_r + e^{2\lambda}\mathcal{H}_t = 0$$

Similarly substituting (3.2.4) and (3.2.7) in (3.2.2c) yields the condition

$$-e^{2\nu}\mathcal{G}_\theta + r^2 \mathcal{F}_t = 0$$

Now (3.2.6) and (3.2.8) in (3.2.2f) implies the restriction

$$e^{2\lambda}\mathcal{H}_\theta + r^2 \mathcal{F}_r = 0$$

for $\mathcal{H}$ and $\mathcal{F}$.

At this stage (3.2.2b), (3.2.2c), (3.2.2d), (3.2.2f), (3.2.2g), (3.2.2i) are satisfied. The components $\xi^0, \xi^1, \xi^2, \xi^3$ of the conformal vector have been obtained. It remains to integrate (3.2.2a), (3.2.2e), (3.2.2h), (3.2.2j) and obtain the conformal factor ψ. We now substitute (3.2.7) and (3.2.8) into (3.2.2a) to generate the conformal factor:

$$\psi = r^2 \sin\theta(-\nu' e^{-2\lambda}\mathcal{C}_r + e^{-2\nu}\mathcal{C}_{tt}) \sin\phi$$
$$+ r^2 \sin\theta(\nu' e^{-2\lambda}\mathcal{D}_r - e^{-2\nu}\mathcal{D}_{tt}) \cos\phi + \nu'\mathcal{H} + \mathcal{G}_t \tag{3.2.9}$$

Now substituting (3.2.8) and (3.2.9) in (3.2.2e) yields

$$\sin\theta \sin\phi[-\nu' r^2 e^{-2\lambda}\mathcal{C}_r + r^2 e^{-2\nu}\mathcal{C}_{tt} + \lambda' r^2 e^{-2\lambda} + (r^2 e^{-2\lambda}\mathcal{C}_r)_r]$$
$$+ \sin\theta \cos\phi[r^2 \nu' e^{-2\lambda}\mathcal{D}_r - r^2 e^{-2\nu}\mathcal{D}_{tt} - \lambda' r^2 e^{-2\lambda}\mathcal{D}_r + (r^2 e^{-2\lambda}\mathcal{D}_r)_r]$$
$$+ \nu'\mathcal{H} + \mathcal{G}_t + \lambda'\mathcal{H} + \mathcal{H}_r = 0$$

We obtain the three equations

$$r^2 \nu' e^{-2\lambda}\mathcal{C}_r - r^2 e^{-2\nu}\mathcal{C}_{tt} = \lambda' r^2 e^{-2\lambda}\mathcal{C}_r + (r^2 e^{-2\lambda}\mathcal{C}_r)_r$$

$$r^2 \nu' e^{-2\lambda}\mathcal{D}_r - r^2 e^{-2\nu}\mathcal{D}_{tt} = \lambda' r^2 e^{-2\lambda}\mathcal{D}_r + (r^2 e^{-2\lambda}\mathcal{D}_r)_r$$

$$\nu'\mathcal{H} + \mathcal{G}_t = \lambda'\mathcal{H} + \mathcal{H}_r$$

from the above condition.

Equation (3.2.2h) becomes

$$re^{-2\lambda}(-\sin\phi\mathcal{C}_r + \cos\phi\mathcal{D}_r) + \frac{\mathcal{H}}{r\sin\theta} - 2\cos\theta\mathcal{A}^{\phi}{}_{\theta} - \sin\theta\mathcal{A}^{\phi}{}_{\theta\theta}$$
$$- (\sin\phi\mathcal{C} - \cos\phi\mathcal{D}) - \mathcal{E}^{\phi} + \frac{\mathcal{F}_{\theta}}{\sin\theta} = r^2(-\nu' e^{-2\lambda}\mathcal{C}_r + e^{-2\nu}\mathcal{C}_{tt})\sin\phi$$
$$+ r^2(\nu' e^{-2\lambda}\mathcal{D}_r - e^{-2\nu}\mathcal{D}_{tt})\cos\phi + \frac{\nu'\mathcal{H}}{\sin\theta} + \frac{\mathcal{G}_t}{\sin\theta}$$

where we have used (3.2.8), (3.2.6) and (3.2.9). Similarly (3.2.3), (3.2.6), (3.2.8) (3.2.9) in (3.2.2j) yield

$$re^{-2\lambda}(-\sin\phi\mathcal{C}_r + \cos\phi\mathcal{D}_r) + \frac{\mathcal{H}}{r\sin\theta} - \cos\theta\mathcal{A}^{\phi}{}_{\theta}$$

$$-(\sin\phi\mathcal{C}-\cos\phi\mathcal{D})-\cot^2\theta\mathcal{E}^\phi+\frac{\cos\theta}{\sin^2\theta}\mathcal{F}+\csc\theta\mathcal{A}_\phi+\csc^2\theta\mathcal{E}_\phi=$$

$$r^2(-\nu'e^{-2\lambda}\mathcal{C}_r+e^{-2\nu}\mathcal{C}_{tt})\sin\phi+r^2(\nu'e^{-2\lambda}\mathcal{D}_r-e^{-2\nu}\mathcal{D}_{tt})\cos\phi+\frac{\nu'\mathcal{H}}{\sin\theta}+\frac{\mathcal{G}_t}{\sin\theta}$$

as the final condition on the solution. At this stage we have exhausted all ten equations in (3.2.2).

Collecting our results we obtain the conformal Killing vector $\boldsymbol{\xi}$ with components

$$\xi^0=r^2e^{-2\nu}\sin\theta[\sin\phi\mathcal{C}_t-\cos\phi\mathcal{D}_t]+\mathcal{G} \tag{3.2.10a}$$

$$\xi^1=r^2e^{-2\lambda}\sin\theta[-\sin\phi\mathcal{C}_r+\cos\phi\mathcal{D}_r]+\mathcal{H} \tag{3.2.10b}$$

$$\xi^2=-\sin^2\theta\mathcal{A}^\phi{}_\theta+\cos\theta[\sin\phi\mathcal{C}-\cos\phi\mathcal{D}]+\cos\theta\mathcal{E}^\phi+\mathcal{F} \tag{3.2.10c}$$

$$\xi^3=\mathcal{A}+\csc\theta[\cos\phi\mathcal{C}+\sin\phi\mathcal{D}+\mathcal{E}] \tag{3.2.10d}$$

and the conformal factor

$$\begin{aligned}\psi=&\,r^2\sin\theta(-\nu'e^{-2\lambda}\mathcal{C}_r+e^{-2\nu}\mathcal{C}_{tt})\sin\phi\\&+r^2\sin\theta(\nu'e^{-2\lambda}\mathcal{D}_r-e^{-2\nu}\mathcal{D}_{tt})\cos\phi+\nu'\mathcal{H}+\mathcal{G}_t\end{aligned} \tag{3.2.11}$$

This solution is subject to the conditions

$$e^{2\nu}(r^2e^{-2\nu}\mathcal{C}_t)_r+r^2\mathcal{C}_{tr}=0 \tag{3.2.12a}$$

$$e^{2\nu}(r^2e^{-2\nu}\mathcal{D}_t)_r+r^2\mathcal{D}_{tr}=0 \tag{3.2.12b}$$

$$-e^{2\nu}\mathcal{G}_r + e^{2\lambda}\mathcal{H}_t = 0 \tag{3.2.12c}$$

$$-e^{2\nu}\mathcal{G}_\theta + r^2\mathcal{F}_t = 0 \tag{3.2.12d}$$

$$e^{2\lambda}\mathcal{H}_\theta + r^2\mathcal{F}_r = 0 \tag{3.2.12e}$$

$$r^2\nu' e^{-2\lambda}\mathcal{C}_r - r^2 e^{-2\nu}\mathcal{C}_{tt} = \lambda' r^2 e^{-2\lambda}\mathcal{C}_r + (r^2 e^{-2\lambda}\mathcal{C}_r)_r \tag{3.2.12f}$$

$$r^2\nu' e^{-2\lambda}\mathcal{D}_r - r^2 e^{-2\nu}\mathcal{D}_{tt} = \lambda' r^2 e^{-2\lambda}\mathcal{D}_r + (r^2 e^{-2\lambda}\mathcal{D}_r)_r \tag{3.2.12g}$$

$$\nu'\mathcal{H} + \mathcal{G}_t = \lambda'\mathcal{H} + \mathcal{H}_r \tag{3.2.12h}$$

$$\mathcal{E} + \mathcal{E}_{\phi\phi} = -\sin^2\theta\cos\theta\mathcal{A}_\theta - \sin^3\theta\mathcal{A}_{\theta\theta} - \sin\theta\mathcal{A}_{\phi\phi} \tag{3.2.12i}$$

$$\begin{aligned}
re^{-2\lambda}(-\sin\phi\mathcal{C}_r + \cos\phi\mathcal{D}_r) &+ \frac{\mathcal{H}}{r\sin\theta} - 2\cos\theta\mathcal{A}^\phi{}_\theta - \sin\theta\mathcal{A}^\phi{}_{\theta\theta} \\
&- (\sin\phi\mathcal{C} - \cos\phi\mathcal{D}) - \mathcal{E}^\phi + \frac{\mathcal{F}_\theta}{\sin\theta} = r^2(-\nu' e^{-2\lambda}\mathcal{C}_r + e^{-2\nu}\mathcal{C}_{tt})\sin\phi \\
&+ r^2(\nu' e^{-2\lambda}\mathcal{D}_r - e^{-2\nu}\mathcal{D}_{tt})\cos\phi + \frac{\nu'\mathcal{H}}{\sin\theta} + \frac{\mathcal{G}_t}{\sin\theta}
\end{aligned} \tag{3.2.12j}$$

$$\begin{aligned}
re^{-2\lambda}(-\sin\phi\mathcal{C}_r + \cos\phi\mathcal{D}_r) &+ \frac{\mathcal{H}}{r\sin\theta} - \cos\theta\mathcal{A}^\phi{}_\theta - (\sin\phi\mathcal{C} - \cos\phi\mathcal{D}) \\
&+ \cot^2\theta\mathcal{E}^\phi + \frac{\cos\theta}{\sin^2\theta}\mathcal{F} + \csc\theta\mathcal{A}_\phi + \csc^2\theta\mathcal{E}_\phi = r^2(-\nu' e^{-2\lambda}\mathcal{C}_r + e^{-2\nu}\mathcal{C}_{tt})\sin\phi
\end{aligned}$$

$$+ r^2(\nu' e^{-2\lambda}\mathcal{D}_r - e^{-2\nu}\mathcal{D}_{tt})\cos\phi + \frac{\nu'\mathcal{H}}{\sin\theta} + \frac{\mathcal{G}_t}{\sin\theta} \tag{3.2.12k}$$

The quantities $\xi^0, \xi^1, \xi^2, \xi^3$ are the components of $\boldsymbol{\xi}$ with conformal factor ψ. The functions $\mathcal{A}(\theta,\phi), \mathcal{C}(t,r), \mathcal{D}(t,r), \mathcal{E}(\phi), \mathcal{F}(t,r,\theta), \mathcal{G}(t,r,\theta), \mathcal{H}(t,r,\theta)$ arise from the integration process.

Thus we have generated the general conformal Killing vector $\boldsymbol{\xi}$, given by (3.2.10), with conformal factor ψ, given by (3.2.11). This general solution of the conformal equations (3.2.2) is subject to the integrability conditions (3.2.12). The existence of a conformal Killing vector $\boldsymbol{\xi}$ for the spacetime (3.2.1) restricts the metric functions $\nu(r)$ and $\lambda(r)$ by the integrability conditions. These metric functions are further restricted by the Einstein field equations. This may help generate new solutions to the field equations with a conformal symmetry. We should point out that the full conformal geometry is known completely only in few spacetimes of cosmological interest. The Lie algebra of conformal motions in Minkowski space is given by Choquet–Bruhat *et al* (1977). Maartens and Maharaj (1986) have found the fifteen conformal vectors in the Robertson–Walker spacetimes for all three cases of the spatial geometry. The conformal geometry of some anisotropic locally rotationally symmetric spacetimes has been analysed by Moodley (1991). Lortan (1992) investigated the conformal symmetries of the Bianchi I spacetime which is homogeneous but anisotropic. Maartens and Maharaj (1991) found the conformal Killing vectors in the *pp*–wave spacetimes, the plane fronted gravitational waves with parallel rays, and have related their results to the Einstein–Maxwell and the Einstein–Klein–Gordon field equations.

3.3 Integrability Conditions

The conformal Killing vector ξ found in §3.2 is the most general conformal symmetry for static, spherically symmetric spacetimes (3.2.1). However the form of the solution is complicated and is difficult to utilise in applications. In fact the functions $\mathcal{A}(\theta,\phi), \mathcal{E}(\phi), \mathcal{F}(t,r,\theta)$ and $\mathcal{H}(t,r,\theta)$ may be reduced further. In this section we analyse the integrability conditions more closely and attempt to express the solution in a simpler form.

On differentiating the integrability condition (3.2.12j) with respect to θ and ϕ we obtain the third order equation

$$2(\cos\theta \mathcal{A}_\theta)_\theta + (\sin\theta \mathcal{A}_{\theta\theta})_\theta = 0$$

This has the general solution

$$\mathcal{A} = \csc\theta \mathcal{A}_1 - \cot\theta \mathcal{A}_2 + \mathcal{A}_3 \tag{3.3.1}$$

where $\mathcal{A}_1(\phi), \mathcal{A}_2(\phi)$ and $\mathcal{A}_3(\phi)$ are functions that arise from integration. With the help of (3.3.1), (3.2.12i) may be written as

$$\mathcal{E} + \mathcal{A}_1 + (\mathcal{E} + \mathcal{A}_1)_{\phi\phi} - \cos\theta(\mathcal{A}_{2\phi\phi} + \mathcal{A}_2) + \sin\theta \mathcal{A}_{3\phi\phi} = 0$$

This equation implies that

$$\mathcal{E} + \mathcal{A}_1 = a_1\cos\phi + a_2\sin\phi \tag{3.3.2a}$$

$$\mathcal{A}_2 = a_3\cos\phi + a_4\sin\phi \tag{3.3.2b}$$

$$\mathcal{A}_3 = a_5\phi + a_6 \tag{3.3.2c}$$

where a_1 – a_6 are strictly constants.

Utilising (3.3.1), (3.3.2) we may write (3.2.12j) as

$$re^{-2\lambda}(-\sin\phi\mathcal{C}_r + \cos\phi\mathcal{D}_r) + \frac{\mathcal{H}}{r\sin\theta} - (\sin\phi\mathcal{C} - \cos\phi\mathcal{D}) + \frac{\mathcal{F}_\theta}{\sin\theta}$$
$$-(a_1\sin\phi - a_2\cos\phi) = r^2(-\nu' e^{-2\lambda}\mathcal{C}_r + e^{-2\nu}\mathcal{C}_{tt})\sin\phi$$
$$+r^2(\nu' e^{-2\lambda}\mathcal{D}_r - e^{-2\nu}\mathcal{D}_{tt})\cos\phi + \frac{\nu'\mathcal{H}}{\sin\theta} + \frac{\mathcal{G}_t}{\sin\theta}$$

This condition implies the following three equations:

$$-re^{-2\lambda}\mathcal{C}_r - \mathcal{C} - a_1 = r^2(-\nu' e^{-2\lambda}\mathcal{C}_r + e^{-2\nu}\mathcal{C}_{tt}) \tag{3.3.3a}$$

$$re^{-2\lambda}\mathcal{D}_r + \mathcal{D} + a_2 = r^2(\nu' e^{-2\lambda}\mathcal{D}_r - e^{-2\nu}\mathcal{D}_{tt}) \tag{3.3.3b}$$

$$\frac{\mathcal{H}}{r\sin\theta} + \frac{\mathcal{F}_\theta}{\sin\theta} = \frac{\nu'\mathcal{H}}{\sin\theta} + \frac{\mathcal{G}_t}{\sin\theta} \tag{3.3.3c}$$

Similarly (3.3.1), (3.3.2) and (3.2.12k) yield

$$re^{-2\lambda}(-\sin\phi\mathcal{C}_r + \cos\phi\mathcal{D}_r) + \frac{\mathcal{H}}{r\sin\theta} - (\sin\phi\mathcal{C} - \cos\phi\mathcal{D}) + \frac{\cos\theta}{\sin^2\theta}\mathcal{F} - \sin\phi a_1$$
$$+a_2\cos\phi + a_5\csc\theta = r^2(-\nu' e^{-2\lambda}\mathcal{C}_r + e^{-2\nu}\mathcal{C}_{tt})\sin\phi$$
$$+r^2(\nu' e^{-2\lambda}\mathcal{D}_r - e^{-2\nu}\mathcal{D}_{tt})\cos\phi + \frac{\nu'\mathcal{H}}{\sin\theta} + \frac{\mathcal{G}_t}{\sin\theta}$$

This equation will be satisfied if

$$-re^{-2\lambda}\mathcal{C}_r - \mathcal{C} - a_1 = r^2(-\nu' e^{-2\lambda}\mathcal{C}_r + e^{-2\nu}\mathcal{C}_{tt}) \tag{3.3.4a}$$

$$re^{-2\lambda}\mathcal{D}_r + \mathcal{D} + a_2 = r^2(\nu' e^{-2\lambda}\mathcal{D}_r - e^{-2\nu}\mathcal{D}_{tt}) \tag{3.3.4b}$$

$$\frac{\mathcal{H}}{r\sin\theta}+\frac{\cos\theta}{\sin^2\theta}\mathcal{F}+a_5\csc\theta=\frac{\nu'\mathcal{H}}{\sin\theta}+\frac{\mathcal{G}_t}{\sin\theta} \tag{3.3.4c}$$

From the above we observe that (3.3.4a,3.3.4b) are equivalent to (3.3.3a,3.3.3b). Thus only four equations in (3.3.3) and (3.3.4) are independent.

A difference of (3.3.3c) and (3.3.4c) gives, after some simplification,

$$\left(\frac{\mathcal{F}}{\sin\theta}\right)_\theta=\csc\theta a_5$$

This differential equation is integrated to generate the function

$$\mathcal{F}=a_5\sin\theta\ln|\tan(\tfrac{1}{2}\theta)|+\sin\theta\mathcal{I} \tag{3.3.5}$$

where $\mathcal{I}(t,r)$ is a function of integration. With the functions $\mathcal{F}$ given by (3.3.5) we are in a position to integrate (3.2.12d), (3.2.12e) to obtain $\mathcal{G}$ and $\mathcal{H}$:

$$\mathcal{G}=-r^2e^{-2\nu}\cos\theta\mathcal{I}_t+\mathcal{J} \tag{3.3.6}$$

$$\mathcal{H}=r^2e^{-2\lambda}\cos\theta\mathcal{I}_t+\mathcal{K} \tag{3.3.7}$$

where $\mathcal{J}(t,r)$ and $\mathcal{K}(t,r)$ arise from the integration process. Substituting (3.3.5), (3.3.6), (3.3.7) in (3.3.3c) we obtain

$$\left(\frac{1}{r}-\nu'\right)(r^2e^{-2\lambda}\cos\theta\mathcal{I}_t+\mathcal{K})+a_5(\cos\theta\ln|\tan(\tfrac{1}{2}\theta)|+1)+\cos\theta\mathcal{I}$$
$$=-r^2e^{-2\nu}\cos\theta\mathcal{I}_{tt}+\mathcal{J}_t$$

which implies that

$$\left(\frac{1}{r}-\nu'\right)r^2e^{-2\lambda}\mathcal{I}_t+\mathcal{I}=-r^2e^{-2\nu}\mathcal{I}_{tt} \tag{3.3.8a}$$

$$\left(\frac{1}{r}-\nu'\right)\mathcal{K}+a_5=\mathcal{J}_t \tag{3.3.8b}$$

$$a_5 = 0 \tag{3.3.8c}$$

Substituting (3.3.6), (3.3.7) in (3.2.12c) yields

$$e^{2\lambda}(r^2e^{-2\nu})'\cos\theta\mathcal{I}_t + r^2\cos\theta\mathcal{I}_{tr} - e^{2\nu}\mathcal{J}_r + r^2\cos\theta\mathcal{I}_{tt} + e^{2\lambda}\mathcal{K}_t = 0$$

This equation is satisfied if

$$e^{2\nu}(r^2e^{-2\nu})'\mathcal{I}_t + r^2\mathcal{I}_{tr} + r^2\mathcal{I}_{tt} = 0 \tag{3.3.9a}$$

$$e^{2\nu}\mathcal{J}_r - e^{2\lambda}\mathcal{K}_t = 0 \tag{3.3.9b}$$

Similarly, substituting (3.3.6), (3.3.7) into (3.2.12h) gives

$$\begin{aligned}(\nu' - \lambda')r^2e^{-2\lambda}\cos\theta\mathcal{I}_t + \nu'\mathcal{K} - r^2e^{-2\nu}\cos\theta\mathcal{I}_{tt} + \mathcal{J}_t = \lambda'\mathcal{K} + (r^2e^{-2\lambda})'\cos\theta\mathcal{I}_t\\ + r^2e^{-2\lambda}\cos\theta\mathcal{I}_{tr} + \mathcal{K}_r\end{aligned}$$

This implies the two restrictions

$$(\nu' - \lambda')r^2e^{-2\lambda}\mathcal{I}_t - r^2e^{-2\nu}\mathcal{I}_{tt} = (r^2e^{-2\lambda})'\mathcal{I}_t + r^2e^{-2\lambda}\mathcal{I}_{tr} \tag{3.3.10a}$$

$$\nu'\mathcal{K} + \mathcal{J}_t = \lambda'\mathcal{K} + \mathcal{K}_r \tag{3.3.10b}$$

on the functions $\mathcal{I}, \mathcal{J}, \mathcal{K}$.

The above analysis of the integrability conditions has simplified the general conformal geometry obtained in §3.2. The conformal Killing vector $\boldsymbol{\xi}$ may be written explicitly as

$$\xi^0 = r^2e^{-2\nu}\sin\theta[\sin\phi\mathcal{C}_t - \cos\phi\mathcal{D}_t] - r^2e^{-2\nu}\cos\theta\mathcal{I}_t + \mathcal{J} \tag{3.3.11a}$$

$$\xi^1 = r^2 e^{-2\lambda} \sin\theta[-\sin\phi \mathcal{C}_r + \cos\phi \mathcal{D}_r] + r^2 e^{-2\lambda} \cos\theta \mathcal{I}_t + \mathcal{K} \tag{3.3.11b}$$

$$\xi^2 = \cos\theta[\sin\phi \mathcal{C} - \cos\phi \mathcal{D}] - a_3 \sin\phi + a_4 \cos\phi + \sin\theta \mathcal{I} \tag{3.3.11c}$$

$$\xi^3 = \csc\theta[\cos\phi \mathcal{C} + \sin\phi \mathcal{D}] - \cot\theta(a_3 \cos\phi + a_4 \sin\phi) + a_6 \tag{3.3.11d}$$

with the conformal factor

$$\psi = r^2 \sin\theta(-\nu' e^{-2\lambda} \mathcal{C}_r + e^{-2\nu} \mathcal{C}_{tt}) \sin\phi + r^2 \sin\theta(\nu' e^{-2\lambda} \mathcal{D}_r - e^{-2\nu} \mathcal{D}_{tt}) \cos\phi$$
$$+ r^2 \cos\theta(\nu' e^{-2\lambda} \mathcal{I}_t - e^{-2\nu} \mathcal{I}_{tt}) + \nu' \mathcal{K} + \mathcal{J}_t \tag{3.3.12}$$

This solution is subject to the twelve integrability conditions (3.2.12a), (3.2.12b), (3.2.12f), (3.2.12g), (3.3.3a), (3.3.3b), (3.3.8a), (3.3.8b), (3.3.9) and (3.3.10). Collecting the various conditions for easy reference we have

$$e^{2\nu}(r^2 e^{-2\nu} \mathcal{C}_t)_r + r^2 \mathcal{C}_{tr} = 0 \tag{3.3.13a}$$

$$r^2 \nu' e^{-2\lambda} \mathcal{C}_r - r^2 e^{-2\nu} \mathcal{C}_{tt} = \lambda' r^2 e^{-2\lambda} \mathcal{C}_r + (r^2 e^{-2\lambda} \mathcal{C}_r)_r \tag{3.3.13b}$$

$$-r e^{-2\lambda} \mathcal{C}_r - \mathcal{C} = r^2(-\nu' e^{-2\lambda} \mathcal{C}_r + e^{-2\nu} \mathcal{C}_{tt}) \tag{3.3.13c}$$

$$e^{2\nu}(r^2 e^{-2\nu} \mathcal{D}_t)_r + r^2 \mathcal{D}_{tr} = 0 \tag{3.3.13d}$$

$$r^2 \nu' e^{-2\lambda} \mathcal{D}_r - r^2 e^{-2\nu} \mathcal{D}_{tt} = \lambda' r^2 e^{-2\lambda} \mathcal{D}_r + (r^2 e^{-2\lambda} \mathcal{D}_r)_r \tag{3.3.13e}$$

$$r e^{-2\lambda} \mathcal{D}_r + \mathcal{D} = r^2(\nu' e^{-2\lambda} \mathcal{D}_r - e^{-2\nu} \mathcal{D}_{tt}) \tag{3.3.13f}$$

$$\left(\frac{1}{r}-\nu'\right)r^2e^{-2\lambda}\mathcal{I}_t+\mathcal{I}=-r^2e^{-2\nu}\mathcal{I}_{tt} \tag{3.3.13g}$$

$$e^{2\nu}(r^2e^{-2\nu})'\mathcal{I}_t+r^2\mathcal{I}_{tr}+r^2\mathcal{I}_{tt}=0 \tag{3.3.13h}$$

$$(\nu'-\lambda')r^2e^{-2\lambda}\mathcal{I}_t-r^2e^{-2\nu}\mathcal{I}_{tt}=(r^2e^{-2\lambda})'\mathcal{I}_t+r^2e^{-2\lambda}\mathcal{I}_{tr} \tag{3.3.13i}$$

$$\left(\frac{1}{r}-\nu'\right)\mathcal{K}=\mathcal{J}_t \tag{3.3.13j}$$

$$e^{2\nu}\mathcal{J}_r-e^{2\lambda}\mathcal{K}_t=0 \tag{3.3.13k}$$

$$\nu'\mathcal{K}+\mathcal{J}_t=\lambda'\mathcal{K}+\mathcal{K}_r \tag{3.3.13l}$$

In the above we have utilised the transformation

$$\mathcal{C}+a_1\longrightarrow\mathcal{C}$$

$$\mathcal{D}+a_2\longrightarrow\mathcal{D}$$

to eliminate the constants a_1, a_2 without any loss of generality. We believe that the general conformal solution given in (3.3.11) is new and has not been published previously (special cases are considered in the next section). The functions $\mathcal{C}(t,r), \mathcal{D}(t,r), \mathcal{I}(t,r), \mathcal{J}(t,r), \mathcal{K}(t,r)$ that appear in the conformal vector (3.3.11), the conformal factor (3.3.12) and the integrability conditions (3.3.13) arise from the integration process. Note that in the above conformal vector $\boldsymbol{\xi}$ the dependence on the spacetime coordinates θ and ϕ has been completely specified. The freedom in the coordinates t and r are restricted by the integrability conditions. It is remarkable that even though the spacetime is static, the general conformal symmetry permits a t–dependence. It should be emphasised that the requirement of a conformal symmetry is a geometrical condition. To study dynamical effects we have to consider, in addition, the Einstein field equations in conjunction with the conformal symmetry. This is the subject of ongoing research.

3.4 Special Cases

In this section we consider some special cases that arise from the general conformal symmetry presented in §3.3.

Case I: To obtain the Killing vectors of the spacetime (3.2.1) from the general conformal Killing vector (3.3.11) we need to set $\psi = 0$ in (3.3.12). This gives the conditions

$$-\nu' e^{-2\lambda}\mathcal{C}_r + e^{-2\nu}\mathcal{C}_{tt} = 0$$

$$\nu' e^{-2\lambda}\mathcal{D}_r - e^{-2\nu}\mathcal{D}_{tt} = 0$$

$$\nu' e^{-2\lambda}\mathcal{I}_t - e^{-2\nu}\mathcal{I}_{tt} = 0$$

$$\nu'\mathcal{K} + \mathcal{J}_t = 0$$

We can show that this system together with the integrability conditions (3.3.13) imply

$$\mathcal{C} = 0$$

$$\mathcal{D} = 0$$

$$\mathcal{I} = 0$$

$$\mathcal{K} = 0$$

$$\mathcal{J} = \text{constant}$$

Thus the components of the general Killing vector $\boldsymbol{\xi}$ are

$$\xi^0 = \mathcal{J}$$

$$\xi^1 = 0$$

$$\xi^2 = -a_3 \sin\phi + a_4 \cos\phi$$

$$\xi^3 = -\cot\theta(a_3 \cos\phi + a_4 \sin\phi) + a_6$$

so that

$$\boldsymbol{\xi} = \mathcal{J}\frac{\partial}{\partial t} + (-a_3 \sin\phi + a_4 \cos\phi)\frac{\partial}{\partial\theta} + (-\cot\theta(a_3 \cos\phi + a_4 \sin\phi) + a_6)\frac{\partial}{\partial\phi}$$

We generate the four–dimensional Lie algebra of the Killing vectors spanned by the vectors

$$\xi_0 = \frac{\partial}{\partial t}$$

$$\xi_1 = \frac{\partial}{\partial \phi}$$

$$\xi_2 = \cos\phi \frac{\partial}{\partial \theta} - \sin\phi \cot\theta \frac{\partial}{\partial \phi}$$

$$\xi_3 = \sin\phi \frac{\partial}{\partial \theta} + \cos\phi \cot\theta \frac{\partial}{\partial \phi}$$

by appropriate choices of the constants a_3, a_4, a_6 and $\mathcal{J}$. The set of Killing vectors $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ are consistent with those presented by Kramer *et al* (1980).

Case II: The homothetic Killing vector of the spacetime (3.2.1) is obtained as a special case of (3.3.11) by taking ψ to be a nonzero constant. We obtain from (3.3.12) the following conditions

$$-\nu' e^{-2\lambda} \mathcal{C}_r + e^{-2\nu} \mathcal{C}_{tt} = 0$$

$$\nu' e^{-2\lambda} \mathcal{D}_r - e^{-2\nu} \mathcal{D}_{tt} = 0$$

$$\nu' e^{-2\lambda} \mathcal{I}_t - e^{-2\nu} \mathcal{I}_{tt} = 0$$

$$\nu' \mathcal{K} + \mathcal{J}_t = \psi$$

These equations together with the integrability conditions (3.3.13) imply

$$\mathcal{C} = 0$$

$$\mathcal{D} = 0$$

$$\mathcal{I} = 0$$

$$\mathcal{K} = r\psi$$

$$\mathcal{J} = (1 - b_1)\psi t + b_2$$

$$\nu = \ln b_3 r^{b_1}$$

$$\lambda = \text{constant}$$

where b_1, b_2, b_3 are constants. Therefore the existence of a homothetic vector places restrictions on the gravitational potentials ν and λ. The components of the homothetic Killing vector $\boldsymbol{\xi}$ are given by

$$\xi^0 = (1 - b_1)\psi t + b_2$$

$$\xi^1 = r\psi$$

$$\xi^2 = -a_3 \sin\phi + a_4 \cos\phi$$

$$\xi^3 = -\cot\theta(a_3 \cos\theta + a_4 \sin\phi) + a_6$$

so that

$$\xi = ((1-b_1)\psi t + b_2)\frac{\partial}{\partial t} + r\psi\frac{\partial}{\partial r} + (-a_3\sin\phi + a_4\cos\phi)\frac{\partial}{\partial\theta}$$
$$+ (-\cot\theta(a_3\cos\phi + a_4\sin\phi) + a_6)\frac{\partial}{\partial\phi}$$

is the general homothetic vector.

We obtain the line element

$$ds^2 = -r^{4(\gamma-1)/\gamma}dt^2 + b^2dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

if we set

$$b_1 = 2(\gamma-1)/\gamma$$

$$b_3 = 1$$

$$e^\lambda = b$$

from the above. This spacetime is the familiar self-similar Tolman model (Wainwright 1985). Thus we have generated the general homothetic vector $\boldsymbol{\xi}$ corresponding to the self-similar Tolman model.

Case III: The concept of an inheriting conformal Killing vector was introduced by Coley and Tupper (1989). An inheriting conformal Killing vector satisfies the additional condition

$$\mathcal{L}_{\boldsymbol{\xi}} u_a = \psi u_a \tag{3.4.3}$$

where $\boldsymbol{u}$ is the relativistic fluid four-velocity. Inheriting conformal symmetries are important physically as fluid flow lines are mapped conformally by the vector $\boldsymbol{\xi}$ in

(3.4.3). However, note that in a general conformal symmetry fluid flow lines are not mapped conformally (Maartens *et al* 1986). Herrera *et al* (1984) and Mason and Maartens (1987) have studied the effects on the conformal geometry by the assumption that (3.4.4) is satisfied. We should point out that inheriting conformal vectors have a close connection with the relativistic thermodynamics of fluids (Israel 1972). A number of particular results in various spacetimes have been established by Coley and Tupper (1990a,b,c) and Coley (1991) by specifying the form of the energy–momentum tensor. However there are very few proper inheriting vectors ($\psi_{;ij} \neq 0$) known and Coley and Tupper (1990b) have conjectured that there exist no perfect fluid spacetimes admitting a proper inheriting conformal Killing vector apart from the three exceptional cases: spacetimes conformal to flat spacetime, spacetimes with the property that $\boldsymbol{\xi}$ is parallel to $\boldsymbol{u}$, spacetimes with $\boldsymbol{\xi}$ orthogonal to $\boldsymbol{u}$. Here we intend to find all general inheriting conformal vectors in static, spherically symmetric spacetimes.

The inheriting conformal equation (3.4.3) for the line element (3.2.1) becomes

$$\nu'\xi^1 + \xi^0{}_t = \psi \tag{3.4.4a}$$

$$\xi^0{}_r = 0 \tag{3.4.4b}$$

$$\xi^0{}_\theta = 0 \tag{3.4.4c}$$

$$\xi^0{}_\phi = 0 \tag{3.4.4d}$$

Equation (3.4.4a) is equivalent to the conformal equation (3.2.2a) and will consequently provide no further restriction. However equations (3.4.4b), (3.4.4c), (3.4.4d) imply

$$\xi^0 = \xi^0(t)$$

Then the component (3.3.11a) generates the conditions

$$\mathcal{C}_t = 0$$

$$\mathcal{D}_t = 0$$

$$\mathcal{I}_t = 0$$

$$\mathcal{J} = \mathcal{J}(t)$$

Now (3.3.13g) forces the function $\mathcal{I}$ to vanish:

$$\mathcal{I} = 0$$

and from (3.3.13k) we obtain

$$\mathcal{K}_t = 0$$

This means that

$$\xi^1 = \xi^1(r)$$

Thus the components of the vector (3.3.11) reduce to

$$\xi^0 = \mathcal{J}$$

$$\xi^1 = \mathcal{K}$$

$$\xi^2 = \cos\theta[\sin\phi\mathcal{C} - \cos\phi\mathcal{D}] + a_4\cos\phi$$

$$\xi^3 = \csc\theta[\cos\phi\mathcal{C} + \sin\phi\mathcal{D}] - \cot\theta(a_3\cos\phi + a_4\sin\phi) + a_6$$

The conformal factor is of the form

$$\psi = -\nu' r^2 e^{-2\lambda}\mathcal{C}_r \sin\theta\sin\phi + \nu' r^2 e^{-2\lambda}\mathcal{D}_r + \nu'\mathcal{K} + \mathcal{J}_t$$

The integrability conditions (3.3.13) simplify to

$$(\nu' - \lambda')\mathcal{C}_r = 0$$

$$(1 - r\nu')\mathcal{C}_r = 0$$

$$(\nu' - \lambda')\mathcal{D}_r = 0$$

$$(1 - r\nu')\mathcal{D}_r = 0$$

$$\left(\frac{1}{r} - \nu'\right)\mathcal{K} = \mathcal{J}_t$$

$$\nu'\mathcal{K} + \mathcal{J}_t = \lambda'\mathcal{K} + \mathcal{K}_r$$

In the above $\mathcal{J}$ has become a function of t and $\mathcal{C}, \mathcal{D}, \mathcal{K}$ are dependent only on r. Thus we have generated the general inheriting conformal Killing vector in static, spherically symmetric spacetimes. This vector will be proper if $\psi_{;ij} \neq 0$.

Two canonical cases arise from the integrability conditions. From the equations $(1/r - \nu')\mathcal{K} = \mathcal{J}_t$ and $\nu'\mathcal{K} + \mathcal{J}_t = \lambda'\mathcal{K} + \mathcal{K}_r$ we observe that if $\mathcal{J}_t = 0$ then

$1 - r\nu' = 0$ (otherwise $\mathcal{K} = 0$ in which case ξ^0 is a constant). If $\mathcal{J}_t \neq 0$ then clearly $1 - r\nu' \neq 0$ and $\mathcal{K} \neq 0$. If $\mathcal{J}_t = 0$ then we obtain

$$\mathcal{J} = \mathcal{J}_2$$

$$\mathcal{K} = \mathcal{K}_1 e^{\nu - \lambda}$$

$$\nu' = \frac{1}{r}$$

as the first class of inheriting vectors. For the case $\mathcal{J}_t \neq 0$ we have

$$\mathcal{J} = \mathcal{J}_1 t + \mathcal{J}_2$$

$$\mathcal{K} = \frac{\mathcal{J}_1 r}{1 - r\nu'}$$

$$\left[e^{\lambda}(1 - r\nu')\right]' = 0$$

as the second class of inheriting vectors. In the above $\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}_1$ are constants. We have established that an inheriting conformal vector places severe restrictions on the metric functions ν and λ.

Case IV: From our general conformal symmetry we can regain the conformal Killing vector

$$\boldsymbol{\xi} = \xi^0(t,r)\frac{\partial}{\partial t} + \xi^1(t,r)\frac{\partial}{\partial r}$$

with the static conformal factor $\psi = \psi(r)$ which has been comprehensively analysed in chapter 2. For consistency with this form of the conformal symmetry the functions $\mathcal{C}, \mathcal{D}, \mathcal{I}$ and the constants a_3, a_4, a_6 in (3.3.11)–(3.3.13) vanish. The conformal factor

(3.3.12) and the integrability conditions (3.3.13) may be expressed as

$$\psi = \frac{1}{r}\mathcal{K}$$

$$\left(\frac{1}{r} - \nu'\right)\mathcal{K} = \mathcal{J}_t$$

$$e^{2\nu}\mathcal{J}_r - e^{2\lambda}\mathcal{K}_t = 0$$

$$\nu'\mathcal{K} + \mathcal{J}_t = \lambda'\mathcal{K} + \mathcal{K}_r$$

We can integrate this system to obtain the vector

$$\boldsymbol{\xi} = (\mathcal{J}_1 t + \mathcal{J}_2)\frac{\partial}{\partial t} + r\psi\frac{\partial}{\partial r}$$

and the gravitational potentials are given by

$$e^{2\nu} = C^2 r^2 \exp\left(-2\mathcal{J}_1 B^{-1}\int r^{-1}e^{-\lambda}dr\right)$$

$$e^{2\lambda} = \frac{B^2}{\psi^2}$$

where $\mathcal{J}_1, \mathcal{J}_2, B, C$ are constants.

This form of the conformal Killing vector has been used by Maartens and Maharaj (1990) to construct static conformally invariant solutions. In fact if we set

$$B = 1 \qquad \mathcal{J}_1 = k \qquad \mathcal{J}_2 = A$$

then we find that $\boldsymbol{\xi}$ is identical to the nonstatic spherically symmetric conformal Killing vector (2.3.7) utilised in §2.3 to generate static spheres of charged imperfect fluids. Herrera *et al* (1984) and Herrera and Ponce de Leon (1985) considered the simpler case of a static vector $\boldsymbol{\xi} = (\xi^0(r), \xi^1(r), 0, 0)$. Their case is regained if we set $\mathcal{J}_1 = k = 0$.

Chapter 4

Nonstatic Shearing Solutions

4.1 Introduction

In chapters 2 and 3 we considered static, spherically symmetric gravitational fields. Here we consider the more general case of nonstatic spacetimes with spherical symmetry. In the literature (Kramer *et al* 1980) most of the exact solutions analysed have vanishing shear since this condition substantially simplifies the field equations. Only a few exact solutions with nonzero shear have been found. As we expect to find a large variety of perfect fluid spherically symmetric solutions in the general class of nonzero shear we place that requirement in this section. Our intention is to find a class of new shearing solutions to the field equations and characterise them geometrically with a conformal symmetry. In §4.2 we present the field equations for a spherically symmetric metric in a comoving frame of reference. The field equations are rewritten using the coordinate freedom in the line element for our class of solutions. The general solution to the field equations, for a class of accelerating, expanding and shearing gravitational fields, is presented in §4.3 in terms of elementary functions. In §4.4 we explicitly relate some of our solutions to those previously published. Some properties of the solutions are investigated in §4.5 and we establish the fact that our solutions are characterised by a generalisation of the stiff equation of state. This class of solutions is of cosmological interest. Finally, in §4.6 we exhibit a conformal Killing vector for our solutions which may be interpreted as a geometrical characterisation of our solutions. We should point out that some of the results in this chapter have been published (Maharaj *et al* 1993a).

4.2 Spherically Symmetric Equations

Here we introduce coordinates $x^i = (t, r, \theta, \phi)$ and a comoving frame of reference $u^i = (e^{-\nu}, 0, 0, 0)$ so that the spherically symmetric spacetime geometry is described by the line element (Kramer *et al* 1980)

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + Y^2(t,r)d\Omega^2 \tag{4.2.1}$$

where the quantity

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

is the metric of the unit two–sphere. The three potentials ν, λ and Y are functions of t and r. Unlike the previous chapters the gravitational field is nonstatic. By spherical symmetry the vorticity has to vanish. However the remaining kinematical quantities, namely the acceleration, the expansion and the shear need not vanish necessarily. The kinematical quantities are given respectively by

$$\omega = 0$$

$$\dot{u}_i = (0, \nu', 0, 0)$$

$$\Theta = e^{-\nu}\left(\dot{\lambda} + 2\frac{\dot{Y}}{Y}\right)$$

$$\sigma^r_r = \sigma^\theta_\theta = -\tfrac{1}{2}\sigma^\phi_\phi = \tfrac{1}{3}e^{-\nu}\left(\frac{\dot{Y}}{Y} - \dot{\lambda}\right)$$

where primes and dots denote differentiation with respect to r and t respectively. For nonzero shear we must have $\dot{Y}/Y \neq \dot{\lambda}$. In the case of vanishing shear σ we can introduce a coordinate system that is simultaneously comoving and isotropic, which is not possible when $\sigma \neq 0$. It is this property of vanishing shear that greatly simplifies the Einstein field equations.

We consider the Einstein field equations (1.2.4a) for an uncharged perfect fluid matter distribution (ie, we set $E_{ij} = 0 = \pi_{ij}$ in (1.2.1)–(1.2.2)). In the case of the spherically symmetric metric (4.2.1) the field equations are given by (Kramer *et al* 1980)

$$\mu = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - Y'\lambda' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}e^{-2\nu}\left(\dot{Y}\dot{\lambda} + \frac{\dot{Y}^2}{2Y}\right) \tag{4.2.2a}$$

$$p = -\frac{1}{Y^2} + \frac{2}{Y}e^{-2\lambda}\left(Y'\nu' + \frac{Y'^2}{2Y}\right) - \frac{2}{Y}e^{-2\nu}\left(\ddot{Y} - \dot{Y}\dot{\nu} + \frac{\dot{Y}^2}{2Y}\right) \tag{4.2.2b}$$

$$\begin{aligned} pY = {} & e^{-2\lambda}[(\nu'' + \nu'^2 - \nu'\lambda')Y + Y'' + Y'\nu' - Y'\lambda'] \\ & -e^{-2\nu}[(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu})Y + \ddot{Y} + \dot{Y}\dot{\lambda} - \dot{Y}\dot{\nu}] \end{aligned} \tag{4.2.2c}$$

$$0 = \dot{Y}' - \dot{Y}\nu' - Y'\dot{\lambda} \tag{4.2.2d}$$

Furthermore the conservation equations

$$T^{ij}{}_{;j} = 0$$

become

$$p' = -(\mu + p)\nu' \tag{4.2.3a}$$

$$\dot{\mu} = -(\mu + p)\left(\dot{\lambda} + 2\frac{\dot{Y}}{Y}\right) \tag{4.2.3b}$$

Of course the conservation equations (4.2.3) may be obtained directly from the field equations (4.2.2).

One cannot expect to make much progress with the field equations (4.2.2) in full generality without simplifying assumptions. Hajj–Boutros (1985) assumed that

$$\dot{\nu} = 0 = \dot{\lambda}$$

and imposed separability on the remaining metric function $Y(t, r)$ so that (4.2.1) can be written as

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + R^2(r)T^2(t)d\Omega^2 \tag{4.2.4}$$

This ansatz is a generalisation of the metric found by Gutman and Bespal'ko (1967) and assures the nonvanishing of the kinematical quantities $\dot{u}_i, \Theta$ and σ, provided $\nu' \neq 0 \neq \dot{T}$. The line element (4.2.4) may be simplified further; apparently Hajj–Boutros did not observe that the function $R(r)$ could be transformed away by the coordinate freedom in the metric.

We introduce a new coordinate $\tilde{r}$ and redefine the metric functions ν, λ. The appropriate transformation is given by

$$\tilde{r} = R(r)$$

$$\tilde{\nu}(\tilde{r}) = \nu(r)$$

$$\tilde{\lambda}(\tilde{r}) = \lambda(r) - \ln \frac{dR}{dr}$$

Then after dropping the tildes (4.2.4) becomes

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2T^2(t)d\Omega^2 \tag{4.2.5}$$

Therefore, we have established that we may take $R = r$ in (4.2.4) without loss of generality. With the simplified form of the line element (4.2.5) the field equations (4.2.2) become

$$\mu = \frac{1}{r^2T^2} + \frac{2}{r}e^{-2\lambda}\left[\lambda' - \frac{1}{2r}\right] + e^{-2\nu}\left(\frac{\dot{T}^2}{T^2}\right) \tag{4.2.6a}$$

$$p = \frac{1}{rT}\left\{-\frac{1}{rT} + T\left(2\nu' + \frac{1}{r}\right)e^{-2\lambda} - 2r\left(\ddot{T} + \frac{\dot{T}^2}{2T}\right)e^{-2\nu}\right\} \tag{4.2.6b}$$

$$p = \left\{e^{-2\lambda}\left[(\nu'' + \nu'^2 - \nu'\lambda') + \frac{1}{r}(\nu' - \lambda')\right]\right\} - \frac{\ddot{T}}{T}e^{-2\nu} \tag{4.2.6c}$$

$$0 = 1 - r\nu' \tag{4.2.6d}$$

We seek the general solution to (4.2.6) in the next section.

4.3 Solution of the Field Equations

Equation (4.2.6d) is immediately integrated to give the potential

$$e^{2\nu} = a^2r^2 \tag{4.3.1}$$

where a is a constant. Equations (4.2.6b) and (4.2.6c) generate the differential equation

$$\frac{1}{T^2} + \frac{r^2}{e^{2\nu}}\left(\frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2}\right) = \frac{r}{e^{2\lambda}}\left[\frac{1}{r} + (\nu' + \lambda') + r(\nu'\lambda' - \nu'^2 - \nu'')\right] \tag{4.3.2}$$

From relations (4.3.1) and (4.3.2) we obtain the partial differential equation

$$\frac{1}{T^2} + \frac{1}{a^2}\left(\frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2}\right) = \frac{2r}{e^{2\lambda}}\left(\frac{1}{r} + \lambda'\right) \tag{4.3.3}$$

in which the variables t and r have separated. Note that equation (4.3.3) is equivalent to equation (3.5) of Hajj–Boutros (1985). However our expression is simpler because we have used the coordinate freedom $\tilde{r} = R(r)$. Equation (4.3.3) is equivalent to

$$\frac{1}{T^2} + \frac{1}{a^2}\left(\frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2}\right) = 2k \tag{4.3.4}$$

$$\frac{r}{e^{2\lambda}}\left(\frac{1}{r} + \lambda'\right) = k \tag{4.3.5}$$

where k is an arbitrary constant. The change of variable $T \to T^2$ reduces (4.3.4) to the form

$$(T^2)^{\cdot\cdot} - 4a^2kT^2 + 2a^2 = 0 \tag{4.3.6}$$

which is obviously integrable. Equation (4.3.5) has solution

$$e^{2\lambda} = \frac{1}{k + br^2} \tag{4.3.7}$$

where b is a constant.

We are now in a position to present the line elements for our class of solutions. It is convenient to distinguish between the three cases

$$k = 0 \qquad k < 0 \qquad k > 0$$

that arise in the solution of (4.3.6). Then we find that the metric (4.2.5) assumes the following forms:

$k = 0:$

$$ds^2 = -a^2r^2dt^2 + \left(\frac{1}{br^2}\right) dr^2 + r^2(-a^2t^2 + ct + d)d\Omega^2 \tag{4.3.8a}$$

$k = -n^2 < 0:$

$$ds^2 = -a^2r^2dt^2 + \left(\frac{1}{-n^2 + br^2}\right) dr^2 + r^2 \left(c\sin(2ant) + d\cos(2ant) - \frac{1}{2n^2}\right) d\Omega^2 \tag{4.3.8b}$$

$k = n^2 > 0:$

$$ds^2 = -a^2r^2dt^2 + \left(\frac{1}{n^2 + br^2}\right) dr^2 + r^2 \left(ce^{2ant} + de^{-2ant} + \frac{1}{2n^2}\right) d\Omega^2 \tag{4.3.8c}$$

where c and d are constants of integration. Note that for $k \leq 0$ we must have $b > 0$; for $k < 0$ we have $r > \sqrt{-k/b}$. The line elements (4.3.8) comprise all possible solutions for the field equations (4.2.6). The acceleration $\dot{u}_i$, the expansion Θ, and the shear σ are nonvanishing for this class of solutions.

4.4 Special cases

Some of the solutions (4.3.8) have been found as special cases previously. In this section we obtain these cases from our general solutions (4.3.8).

Case I: If we set $a = 1/2$, $b = 0$, $k = 1$ then (4.3.8c) gives the particular case

$$ds^2 = -\left(\frac{r^2}{4}\right) dt^2 + dr^2 + r^2 \left(ce^t + de^{-t} + \frac{1}{2}\right) d\Omega^2 \tag{4.4.1}$$

which was first found by Gutman and Bespal'ko (1967). This solution satisfies the stiff equation of state $p = \mu$.

Case II: Another shearing solution with the stiff equation of state $p = \mu$ is given by

$$ds^2 = -\left(\frac{r^2}{t^2}\right) dt^2 + dr^2 + r^2 T^2(t) d\Omega^2 \tag{4.4.2}$$

which was reported by Wesson (1978). The function $T(t)$ satisfies

$$\left(T^4 - T^2 + \eta_0^2\right)^{1/2} + T^2 - \frac{1}{2} = \left(\frac{t}{T_0}\right)^{\pm 2} \tag{4.4.3}$$

where η_0 and T_0 are constants. In the above form it seems that the Wesson solution is a new solution to the Einstein field equations. However this solution is not new and is equivalent to (4.4.1) (and is therefore a special case of (4.3.8c)). To establish this we redefine the time variable by

$$\tilde{t} = \ln t^2$$

After squaring (4.4.3) we obtain

$$T^2 = ce^{\tilde{t}} + de^{-\tilde{t}} + \frac{1}{2} \tag{4.4.4}$$

where we have set

$$c = \frac{1}{T_0} \qquad\qquad d = \frac{1}{8} - \frac{\eta_0^2}{2}$$

for the positive exponent, and we must interchange c and d for the negative exponent in (4.4.4). With the new variable $\tilde{t}$, (4.4.2) becomes

$$ds^2 = -\left(\frac{r^2}{4}\right) d\tilde{t}^2 + dr^2 + r^2 \left(ce^{\tilde{t}} + de^{-\tilde{t}} + \frac{1}{2}\right) d\Omega^2 \tag{4.4.5}$$

which is of the same form as (4.4.1). Thus we have verified that the solution of Wesson and Gutman and Bespal'ko are equivalent and are contained in our general class (4.3.8c). In the literature the Wesson solution has been taken to be distinct from (4.4.1) (see p.173 of Kramer *et al* (1980)).

Case III: Lake (1983) also found the solutions (4.3.8a) and (4.3.8c), using a different approach, by imposing separability on the metric functions $\nu(t,r)$, $\lambda(t,r)$ and $Y(t,r)$. He did not obtain (4.3.8b) because he demanded $k \geq 0$ on physical grounds. However the case $k < 0$ given by (4.3.8b) is also a solution to the Einstein field equations for the restricted interval $r > \sqrt{-k/b}$. Lake also mentioned the equivalence between the metrics of Gutman and Bespal'ko and Wesson but did not explicitly provide the argument leading to our relationship (4.4.4).

Case IV: Hajj–Boutros (1985) did not use the coordinate freedom $\tilde{r} = R(r)$ that we utilised to write (4.2.6). Consequently his equation (3.7) that corresponds to our (4.3.5) also contains $R(r)$:

$$\frac{R}{e^{2\lambda}}\left(R'' + \frac{R'^2}{R} + R'\lambda'\right) = k$$

Solutions of this equation for specific choices of λ were obtained by Hajj–Boutros. Thus all the particular solutions of Hajj–Boutros are special cases of our general metrics (4.3.8). Furthermore we have obtained the general solution of the field equations – a possibility obscured by the failure to set $\tilde{r} = R(r)$ by Hajj–Boutros (1985). For the special choice $\lambda = \ln(a/r)$, solutions were presented by Hajj–Boutros in terms of the third Painlevé transcendent. However those forms of the metrics are redundant since they must be expressible in terms of elementary functions by the general solutions (4.3.8). It appears that the Painlevé transcendents arise in the Hajj–Boutros solutions because the coordinate freedom in the line element was not used to remove the function $R(r)$.

Case V: Shaver and Lake (1988) analysed separable metrics with spherical, plane and hyperbolic symmetries which obey the weak and strong energy conditions and do not contain scalar polynomial singularities. (Note that because of the energy conditions placed on their solutions the only physical metrics allowed in their class of solutions are Minkowski, de Sitter and anti–de Sitter.) Their results contain those of Lake (1983) in the special case of spherical symmetry. Shaver and Lake point out that the results given by Hajj–Boutros (1985) are reducible by coordinate transformations to those of Lake.

In addition to finding the general line elements for the field equations (4.2.6) we have collected and unified associated results in the literature.

4.5 Some properties of the solutions

The energy density and pressure are given by

$$\mu = -3b + \frac{k}{r^2} - \frac{1}{a^2 r^2}\frac{\ddot{T}}{T}, \tag{4.5.1}$$

$$p = 3b + \frac{k}{r^2} - \frac{1}{a^2 r^2}\frac{\ddot{T}}{T} \tag{4.5.2}$$

for the cases considered in this chapter. For our class of solutions the metric functions ν, λ, T are given in (4.3.8) and the dynamical quantities μ, p by (4.5.1) – (4.5.2). We observe, as a check, that the conservation equations (4.2.3) are identically satisfied. From (4.5.1) and (4.5.2) we obtain

$$p = \mu + 6b \tag{4.5.3}$$

relating the pressure and the energy density. The relationship (4.5.3) is a generalisation of the stiff equation of state $p = \mu$. Note that the apparent problem of nonzero pressure in the vacuum case does not in fact arise. When $\mu = 0$, (4.5.1) shows that $\ddot{T}/T = a^2(k - 3br^2)$, which forces $b = 0$, otherwise $a^2 = 0$. However this is inconsistent with (4.3.4). Thus there is no vacuum case.

It is clear from the kinematical quantities $\dot{u}_i, \Theta, \sigma$ that our solutions (4.3.8) are accelerating, expanding and shearing. In their study of anisotropic solutions, generated by neutral viscous fluids, Goenner and Kowalewski (1989) use the parameter

$$A = 3\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{\Theta^2}$$

as a measure of the anisotropy. With the help of our forms for σ and Θ we obtain

$$A = \tfrac{1}{2}$$

so that in our solutions the anisotropy is constant. This result follows essentially because $\lambda(r)$ is independent of t.

The constancy of anisotropy, together with the singularity in μ and p at $r = 0$ (see (4.5.1), (4.5.2)), mean that these solutions are unlikely to be suitable as cosmological models. For the case $k < 0$, $r > \sqrt{-k/b} > 0$, so that there is no singularity at $r = 0$. Furthermore (4.5.1) and (4.5.2) show that the singularity at $r = 0$ when $k > 0$ may be avoided for the special case $\mu = -p = -3b$ which occurs for specific values of the constants c and d. This spacetime has constant curvature (Lake 1983). Note that there may also be singularities in μ and p for $r > 0$ at certain times t. This follows since (4.3.4) and (4.5.1) imply

$$\mu = -3b - \frac{k}{r^2} + \frac{1}{a^2r^2}\left(\frac{a^2 + \dot{T}^2}{T^2}\right)$$

with $p = \mu + 6b$. Thus if there exists t_0 such that $T^2(t_0) = 0$ (which depends on the constants c, d in (4.3.8)), then μ and p become unbounded at $t = t_0$.

Finally we note that for $kb < 0$, there is a singularity in the metric function $e^{2\lambda}$ at $r = \sqrt{-k/b}$ (see (4.3.8b), (4.3.8c)). However μ, p and the kinematic quantities are regular there. The singularity in the metric tensor field g_{ij} may be removed by the coordinate change

$$r \rightarrow \int e^{\lambda} dr$$

for the function λ.

4.6 A Conformal Symmetry

In §4.3 we obtained a class of accelerating, expanding and shearing solutions to the Einstein field equations. These solutions satisfy the equation of state (4.5.3), which generalises the stiff equation of state. This may be viewed as a thermodynamical characterisation of the solution generated. In this section we seek a geometrical characterisation by investigating the existence of a conformal vector in the general solution (4.3.8).

For static spacetimes in chapter 2 we found a spherically symmetric conformal Killing vector ($\xi^2 = \xi^3 = 0$). Similarly we seek a conformal vector of the form

$$\xi = \xi^0 \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial r}$$

for the nonstatic solutions of this chapter. The conformal Killing vector equation (1.3.3) becomes

$$\nu' \xi^1 + \xi^0{}_t = \psi \qquad (4.6.1a)$$

$$e^{2\lambda}\xi^1{}_t - e^{2\nu}\xi^0{}_r = 0 \tag{4.6.1b}$$

$$\xi^0{}_\theta = 0 \tag{4.6.1c}$$

$$\xi^0{}_\phi = 0 \tag{4.6.1d}$$

$$\lambda'\xi^1 + \xi^1{}_r = \psi \tag{4.6.1e}$$

$$\xi^1{}_\theta = 0 \tag{4.6.1f}$$

$$\xi^1{}_\phi = 0 \tag{4.6.1g}$$

$$\frac{\dot{T}}{T}\xi^0 + \frac{1}{r}\xi^1 = \psi \tag{4.6.1h}$$

This system reduces to the partial differential equations

$$\frac{\dot{T}}{T}\xi^0 + \frac{1}{r}\xi^1 = \psi \tag{4.6.2a}$$

$$\lambda'\xi^1 + \xi^1{}_r = \psi \tag{4.6.2b}$$

$$\nu'\xi^1 + \xi^0{}_t = \psi \tag{4.6.2c}$$

$$e^{2\lambda}\xi^1{}_t - e^{2\nu}\xi^0{}_r = 0 \tag{4.6.2d}$$

where $\xi^0 = \xi(t,r)$ and $\xi^1 = \xi^1(t,r)$. This is a system of four equations in the three unknowns ξ^0, ξ^1, ψ.

We can rewrite (4.6.2d) as

$$\frac{\partial}{\partial t}\left(e^{(2\lambda-2\nu)}\xi^1\right) - \frac{\partial}{\partial r}\left(\xi^0\right) = 0$$

which implies that there exists a function $\Psi(t,r)$ such that

$$\xi^1 = e^{(2\nu-2\lambda)}\Psi_r$$

$$\xi^0 = \Psi_t$$

The conformal factor is given by

$$\psi = \frac{\dot{T}}{T}\Psi_t + \frac{1}{r}e^{(2\nu-2\lambda)}\Psi_r$$

from (4.6.2a). With the form for ψ the equations (4.6.2c), (4.6.2d) become

$$\lambda' e^{(2\nu-2\lambda)}\Psi_r + \left(e^{(2\nu-2\lambda)}\Psi_r\right)_r = \frac{\dot{T}}{T}\Psi_t + \frac{1}{r}e^{(2\nu-2\lambda)}\Psi_r$$

$$\nu' e^{(2\nu-2\lambda)}\Psi_r + \Psi_{tt} = \frac{\dot{T}}{T}\Psi_t + \frac{1}{r}e^{(2\nu-2\lambda)}\Psi_r$$

These may be expressed as

$$[a^2br^3 + a^2r(k+br^2)]\Psi_r + a^2r^2(k+br^2)\Psi_{rr} - \frac{\dot{T}}{T}\Psi_t = 0 \qquad (4.6.3)$$

$$\Psi_{tt} = \frac{\dot{T}}{T}\Psi_t \qquad (4.6.4)$$

where we have utilised the metric functions (4.3.1) and (4.3.7). Equation (4.6.4) implies that

$$\Psi = \alpha(r)\int T(t)dt + \beta(r)$$

where α and β are arbitrary functions of r. With this form of Ψ we see that (4.6.3) is satisfied only if

$$\alpha(r) = 0$$

as $T = T(t)$, otherwise the spacetime is static. This implies that the function $\Psi = \Psi(r)$ is independent of t.

Thus we require that

$$\xi^0 = 0$$

$$\xi^1 = \xi^1(r)$$

$$\psi = \psi(r)$$

in which case the conformal Killing vector equations (4.6.2) reduce to the two equations

$$\frac{1}{r}\xi^1 = \psi$$

$$\frac{-br}{k + br^2}\xi^1 + \xi^1{}_{,r} = \psi$$

This system of differential equations may be immediately integrated. The component ξ^1 is given by

$$\xi^1 = \begin{cases} C_0 & \text{if } k = 0 \\ C_+ r(n^2 + br^2)^{-\frac{1}{2}} & \text{if } k = n^2 \\ C_- r(-n^2 + br^2)^{-\frac{1}{2}} & \text{if } k = -n^2 \end{cases}$$

and the conformal factor has the form

$$\psi = \begin{cases} \frac{1}{r}C_0 & \text{if } k = 0 \\ \\ C_+(n^2 + br^2)^{-\frac{1}{2}} & \text{if } k = n^2 \\ \\ C_-(-n^2 + br^2)^{-\frac{1}{2}} & \text{if } k = -n^2 \end{cases}$$

where C_0, C_+ and C_- are constants. Thus our class of expanding, accelerating and shearing solutions admit a conformal Killing vector in the radial direction. This vector is very specialised: the nonvanishing component ξ^1 and conformal factor ψ are static. It is interesting to note that this conformal Killing vector is proper and does not contain the Killing vectors of spherically symmetric spacetimes.

Thus we have demonstrated that the class of solutions presented in this chapter admit a conformal Killing vector. This vector has only a radial component and a static conformal factor ψ. It would be interesting to find other conformal Killing vectors, if any, admitted by our general solution (4.3.8) other than the conformal vector

$$\boldsymbol{\xi} = \xi^1 \frac{\partial}{\partial r}$$

found above. This will be the object of future investigation. Conformal Killing vectors have been exhibited in other solutions in nonstatic spherically symmetric spacetimes, eg, Maharaj *et al* (1991). However those solutions have vanishing shear; we believe that our conformal symmetry is the first exhibited for nonvanishing shear in the case of the spherically symmetric line element (4.2.1). Our analysis in this section suggests the possibility of obtaining solutions to the Einstein field equations, with nonzero shear, by imposing a conformal symmetry on the spacetime. There are few solutions known to the field equations with nonzero shear (Kramer *et* al 1980) and our approach may in fact help to isolate other classes of solution. Clearly this is a nontrivial task as the analysis involves complicated systems of equations whose difficulty is directly related to the fact that we require the shear to be nonvanishing.

Conclusion

Our objective in this thesis was to find new solutions to the Einstein field equations for spherically symmetric gravitational fields. We considered both static and non-static gravitational fields. Such solutions were obtained by imposing the requirement of a conformal Killing vector symmetry on the spacetime manifold. These solutions have applications in relativistic astrophysics and may be used to model conformally invariant charged spheres. Shapiro and Teukolsky (1983) describe a number of likely scenerios in which such models are applicable for the description of realistic stars. Of course solutions with a conformal symmetry may also be applied in a cosmological setting (Dyer *al al* 1987, Maharaj *et al* 1991, Havas 1992).

We now provide an outline of the work carried out in this thesis with special attention given to the new results established in our investigation. These are given by the following items

- We presented the Einstein–Maxwell system of field equations for a charged nonconducting imperfect fluid in chapter 1. A variety of Lie symmetries on the spacetime manifold were reviewed, and presented diagramatically which highlighted their interrelationships. In particular some properties of conformal motions were studied in greater detail.

- Conformally symmetric static fluid spheres were studied under the assumption that they admitted a nonstatic spherically symmetric conformal Killing vector, with a static conformal factor. We found a necessary condition on the conformal factor for regularity. A number of new solutions to the Einstein–Maxwell system were found. These solutions generalise those of Herrera *et al* (1984) and Herrera and Ponce de Leon (1985).

- A particular feature of the solutions presented in chapter 2 is that they were regular in the interior of the sphere. This is an improvement on solutions obtained previously. In particular we exhibited a regular perfect fluid solution in which the charge is uniform. We also found a regular imperfect fluid solution which is charged by specifying a polynomial form for the conformal factor that satisfies the regularity condition. However we should point out that our solutions may be unstable. It is possible to overcome this difficulty by considering a nonstatic conformal factor.

- We found the general solution of the conformal Killing equation in static spherically symmetric spacetimes. This solution is subject to twelve integrability conditions which restrict the metric functions. We should point out that even though the spacetime is static the general conformal symmetry contains a t–dependence. In the past the form of the conformal symmetry used was *ad hoc*. Our general solution provides consistency requirements on the form of the conformal vector chosen. We expect that the general conformal vector will generate new solutions to the field equations.

- The special case of a Killing vector was regained from the general solution. The general homothetic vector was isolated and we established that this corresponds to the self–similar Tolman model (Wainwright 1985). All inheriting conformal vectors in static spherically symmetric spacetimes were found. This is a significant result as inheriting vectors are rare (Coley and Tupper 1990b). We confirmed that the existence of inheriting conformal vectors places restrictions on the metric functions, and we obtained these restrictions explicitly.

- We found a class of new shearing solutions to the Einstein field equations for nonstatic spherically symmetric spacetimes. These expanding, accelerating cosmological solutions satisfy an equation of state that is a generalisation of the stiff equation of state. We showed that earlier solutions in the literature are contained as special cases in our solution. We geometrically characterise our class of solutions by a conformal Killing vector.

In the above we have highlighted only those items of principal interest.

In this thesis we have analysed conformal symmetries in spherically symmetric spacetimes. The results of chapter 2 may be extended by considering a nonstatic conformal factor. This may generate conformally invariant charged spheres which are both regular and stable. The general conformal symmetry of chapter 3 may be utilised to find new solutions of astrophysical and cosmological significance. Furthermore the general solution may help in the analysis of the geometry of particular spacetimes, eg, the Schwarzschild interior solution. There are very few known shearing solutions to the field equations (Kramer *et al* 1980). Our analysis in chapter 4 suggests that we may find other solutions, with nonzero shear, by imposing a conformal symmetry requirement. In future work we may apply symmetries other than conformal to generate further exact solutions. For example curvature collineations (Collinson 1970a, Katzin and Levine 1970a,b,c) and Ricci collineations (Collinson 1970b, Melfo *et al* 1992, Oliver and Davis 1977,1979, Tsamparlis and Mason 1990) may generate new solutions. Further, in this regard, Duggal (1992) has suggested the use of curvature inheritance, which generalises the concept of curvature collineations.

We hope that we have demonstrated that the use of conformal symmetries in the search for exact solutions to the field equations is a fertile area of research. Clearly further investigation is indicated.

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