ON PURITY RELATIVE TO AN HEREDITARY TORSION THEORY

by

Derek Johanathan Gray

Submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics University of Natal 1992

DURBAN 1992

ABSTRACT

The thesis is mainly concerned with properties of the concept "σ-purity" introduced by J. Lambek in "Torsion Theories, Additive Semantics and Rings of Quotients", (Springer-Verlag, 1971).

In particular we are interested in modules M for which every exact sequence of the form $0 \to M \to K \to L \to 0$ (or $0 \to K \to M \to L \to 0$ or $0 \to K \to L \to M \to 0$) is σ-pure exact. Modules of the first type turn out to be precisely the σ-injective modules of O. Goldman (J. Algebra 13, (1969), 10-47). This characterization allows us to study σ-injectivity from the perspective of purity.

Similarly the demand that every short exact sequence of modules of the form $0 \to K \to M \to L \to 0$ or $0 \to K \to L \to M \to 0$ be σ-pure exact leads to concepts which generalize regularity and flatness respectively. The questions of which properties of regularity and flatness extend to these more general concepts of σ-regularity and σ-flatness are investigated.

For various classes of rings R and torsion radicals σ on R-mod, certain conditions equivalent to the σ-regularity and the σ-injectivity of R are found.

We also introduce some new dimensions and study semi-σ-flat and semi-σ-injective modules (defined by suitably restricting conditions on σ-flat and σ-injective modules). We further characterize those rings R for which every R-module is semi-σ-flat.

The related concepts of a projective cover and a perfect ring (introduced by H. Bass in Trans. Amer. Math. Soc. 95, (1960), 466-488) are extended in a natural way and, *inter alia*, we obtain a generalization of a famous theorem of Bass.

Lastly, we develop a relativized version of the Jacobson Radical which is shown to have properties analogous to both the classical Jacobson Radical and a radical due to J.S. Golan.

PREFACE

The work on this thesis was carried out at the University of Zululand, KwaDlangezwa under the supervision of Professor A.R. Meijer.

Originality can be claimed for all results where no acknowledgement is made. This work has not been submitted in any form to another University.

ACKNOWLEDGEMENTS

I would like to thank Professor Meijer for introducing me to σ-purity and for supervising my progress. His attention to detail taught me a lot about discipline in Mathematics and his meticulous reading of my work saved me from many a subtle error. I also would like to thank him for suggestions regarding style.

Finally, a word of thanks to Mrs G. E. MacLachlan who did a fine job of typing.

CONTENTS

Page

CONVENTIONS, NOTATION

Z,Q denote the ring of integers and the field of rational numbers respectively.

Rings R will be associative with identity but not necessarily commutative and, unless otherwise indicated, "R-modules" will be unital left R-modules, "ideals" will be left ideals.

R-mod will denote the category of all (left) R-modules and R-module homomorphisms.

"Homomorphism" will mean "R-module homomorphism" unless otherwise specified. $Hom_R(A,B)$ will denote the group of R-homomorphisms $\alpha:A\to B$. If R is understood we use Hom(A,B).

$\alpha\cdot\beta$ will mean the "composition of α with β" defined by $(\alpha\cdot\beta)(x) = \alpha(\beta(x))$.

If A is a subset of B and $g:B\to C$ is a homomorphism, $f:A\to C$ defined by $f(a) = g(a)$ for all $a\in A$ is called the *restriction* of g to A and we write $f = g/_A$. 1_A denotes the map from A to A defined by $1_A(x) = x$ for all $x\in A$.

We will usually use the terms "monic" and "epic" for "1-1" and "onto" respectively.

If there is a monomorphism $\alpha:N\to M$, N is said to be *embedded* in M.

A sequence $\ldots P_{n-1}\xrightarrow{\alpha_{n-1}}P_n\xrightarrow{\alpha_n}P_{n+1}\ldots$ of R-homomorphisms will be called *exact* iff $\ker\alpha_n = \operatorname{im}\alpha_{n-1}$ for all n.

An exact sequence of the form $0\to N\xrightarrow{i}M\xrightarrow{\pi}M/N\to 0$ is called a *short exact sequence*.

f.g. and f.p. will mean "finitely generated" and "finitely presented" respectively.

By an *integral domain* we mean a commutative ring with identity which has no divisors of zero. A ring is called (left) *Noetherian* iff every (left) ideal is finitely generated. A ring is called (left) *Artinian* iff every nonempty collection of (left) ideals of R has a minimal element. A ring will be called (left) *hereditary* iff every (left) ideal is projective and (left) *semi-hereditary* iff every finitely generated (left) ideal is projective. A ring R is called *local* iff it has a unique maximal ideal. An Artinian ring in which every ideal is principal (i.e. has a single generator) is called a *uniserial* ring.

An ideal P of a commutative ring R will be called *prime* (*semiprime*) iff whenever $A.B \leq P$ for ideals A,B of R ($A^2 \leq P$ for an ideal A of R) we must have $A \leq P$ or $B \leq P$ ($A \leq P$). A ring R will be called a *semiprime ring* iff it has no nonzero nilpotent ideals.

$N \trianglelefteq M$ will mean "N is a *large* submodule of M". i.e. $N \cap S \neq 0$ for all nonzero submodules S of M.

A class S of R-modules is said to be *closed under module extensions* iff whenever $N \leq M$ and both N and M/N belong to S, then M belongs to S.

A *chain* $\{N_i\}_i$ of R-modules is a collection such that for any i and j, $N_i \leq N_j$ or $N_j \leq N_i$.

A module M is said to satisfy the *ascending chain condition* on submodules iff for every countably infinite (ascending) chain of submodules $M_1 \leq M_2 \leq \ldots$ of M there is an n such that $M_n = M_{n+1} = \ldots$

If U is a class of modules, a subclass V of U is said to be *cofinal in* U iff every element of U contains an element of V.

For an element x, $\mathrm{Ann}\,x = \{r \in R \mid r.x = 0\}$ will denote the left annihilator of x. An R-module M is called *faithful* iff whenever $r.M = 0$ for $r \in R$, we must have $r = 0$.

If $\{M_i\}$ is a collection of R-modules, we denote their *direct product* by $\prod_i M_i = \{(m_i)_i \mid m_i \in M_i\}$ under componentwise operations and the *direct sum* by $\bigoplus_i M_i = \{(m_i)_i \in \prod_i M_i \mid$ only a finite number of the m_i are nonzero$\}$. (Each M_i is called a *direct summand* of $\bigoplus_i M_i$). The canonical projections: $(m_i)_i \mapsto m_i$ and injections: $m_i \mapsto (m_j)_j$ where $m_j = \begin{cases} 0 & \text{if } i \neq j \\ m_i & \text{if } i = j \end{cases}$ are often denoted by π_i and in_i respectively.

If A is a set we denote the cardinal number of A by $|A|$ and if M is an R-module, the direct product of $|A|$ copies of M is denoted by $M^{|A|}$ and $M^{(A)}$ their direct sum.

A module M is said to be a *subdirect sum* of modules, M_i $(i\in I)$ iff there is a monomorphism $\alpha: M \rightarrowtail \bigoplus_{i\in I} M_i$ such that $\pi_i \cdot \alpha$ is an epimorphism for each projection map $\pi_i : \bigoplus_{i\in I} M_i \to M_i$, $i \in I$.

A module M will be said to be *cogenerated by a class* U of modules iff M can be embedded in a direct product of copies of elements of U. A module M is called *finitely cogenerated* iff whenever M can be embedded in a direct product $\prod_{i\in I} U_i$ of modules U_i, then there is a finite subset J of I such that M can be embedded in $\prod_{j\in J} U_j$.

The (short) exact sequence $0 \to K \xrightarrow{\alpha} L \xrightarrow{\beta} M \to 0$ is said to *split* iff α has a left inverse or, equivalently, β has a right inverse. [When this happens $L \simeq K \oplus M$ ([45], Theorem 2.3)].

A nonzero R-module M is called *simple* iff M has only the trivial submodules 0 and M. A module M will be called *semisimple* iff it is a (direct) sum of simple submodules.

A ring R is called *semisimple* iff every ideal of R is a direct summand. In this case every R-module will be semisimple ([45], Theorem 5.1).

A ring R is called *Quasi-Frobenius* iff projectivity and injectivity are equivalent for R-modules.

If N is a right R-module and M is a left R-module, $N \otimes M$ denotes the *tensor product* of N and M.

$\lim_{\to I}$ denotes the *direct limit* of the directed system of modules indexed by I.

σ will always denote a torsion radical and L_σ its associated filter. T_σ and F_σ will denote the classes of σ-torsion and σ-torsion free modules respectively (see p7 ff.).

E(M) and E_σ(M) will denote respectively the injective hull ([45]) and the σ-injective hull ([37]) of the module M.

References used are not necessarily the only or even the original reference for the topic concerned.

There is an index of definitions at the end of the thesis.

GENERAL INTRODUCTION

For many of our results we find it is necessary to assume that R-mod be σ-pure-inductive, i.e. that the union of an ascending chain of σ-pure submodules of a module M be σ-pure in M. For this it is sufficient that every ideal in the associated filter L_σ is f.g. This condition is consequently assumed in many of our proofs, (but always explicitly stated). If R is Noetherian then this condition holds and our results of this type therefore fall within the scope of B. Stenström's Chapter 7 of [67] ("Hereditary Torsion Theories for Noetherian rings") and J.S. Golan's Chapter 42 of [35] ("Torsion Theories of Finite Type"). In §1.2.5.3 we show that there exist non-Noetherian rings which satisfy the condition that every $I \in L_\sigma$ be f.g. for an appropriate torsion radical σ. (The condition "every $I \in L_\sigma$ is f.g." has been studied in some detail in [36]).

CHAPTER ONE

PRELIMINARY MATERIAL

§1.1 INTRODUCTION

This chapter serves to provide the background needed for the rest of the thesis. The concept of σ-injectivity, which will be studied in some detail in Chapter Two, is introduced and a few of its well known properties are mentioned. The condition that every $I\in L_\sigma$ be f.g., which is fundamental to many later results, is studied here.

Our main field of study: σ-purity, is introduced and compared with Cohn purity (from which σ-purity was developed). For ease of reference we list some known results on σ-purity, in particular the construction of the σ-pure injective hull.

Lastly, we introduce two new (dual) dimensions and show that the corresponding global dimensions coincide under certain conditions.

§1.2 BACKGROUND IN R-MODULES

1.2.1 Fundamental definitions ([45], [61])

1.2.1.1 An exact sequence $0\to A\xrightarrow{\alpha}B\xrightarrow{\beta}C\to 0$ of R-modules induces exact sequences: $0\to \mathrm{Hom}(M,A)\xrightarrow{\alpha_*}\mathrm{Hom}(M,B)\xrightarrow{\beta_*}\mathrm{Hom}(M,C)$ and $0\to \mathrm{Hom}(C,M)\xrightarrow{\beta^*}\mathrm{Hom}(B,M)\xrightarrow{\alpha^*}\mathrm{Hom}(A,M)$ for any module M, where e.g. $\alpha_*(\phi) = \alpha\cdot\phi$ and $\alpha^*(\phi) = \phi\cdot\alpha$ ([45], Theorem 6.2).

In case $\mathrm{Hom}(M,B)\xrightarrow{\beta_*}\mathrm{Hom}(M,C)\to 0$ is exact, for each exact sequence $B\xrightarrow{\beta}C\to 0$, then M is called *projective*. If $\mathrm{Hom}(B,M)\xrightarrow{\alpha^*}\mathrm{Hom}(A,M)\to 0$ is exact, for each exact sequence $0\to A\xrightarrow{\alpha}B$, then M is called *injective*.

Every module M may be embedded in an injective module, called the *injective hull*, E(M), of M ([45]).

1.2.1.2 A module F is called *free* iff it is isomorphic to $R^{(I)}$ for some nonempty set I.

Free modules are projective and every R-module is an epimorphic image of a free module ([45]).

1.2.1.3 If in the expression $M \simeq F/H$, where F is free and H is a submodule of F, both F and H are finitely generated, M is called *finitely presented* (f.p.).

The following are well known:

(a) If $M \simeq F/H$ is f.p. and F is f.g., then H is f.g. ([61], Corollary 3.42).

(b) If $0 \to K \to L \to M \to 0$ is a short exact sequence with K and M f.p. modules, then L is f.p. ([65], Lemma 1(ii)).

(c) From (b) it follows easily that finite direct sums of f.p. modules are f.p.

(d) If M is f.p. and L is a f.g. submodule of M, then M/L is f.p. ([65], Lemma 1(i)).

1.2.1.4 For a module M, a *projective resolution* of M is an exact sequence $\dots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ where each P_i is projective.

The smallest n for which $\mathrm{im}\, d_n$ is projective (if it exists) is called the *length of the projective resolution*. All projective resolutions of a module M have the same length and this is called the *projective dimension*, pr M, of M (see [49] for a fuller discussion).

Let X be any module and consider the induced sequence: $\mathrm{Hom}(P_0,X) \xrightarrow{d_1'} \mathrm{Hom}(P_1,X) \xrightarrow{d_2'} \dots$ (where $d_n'(\phi) = \phi \cdot d_n$ for all n, $\phi \in \mathrm{Hom}(P_{n-1},X)$). This is not necessarily an exact sequence, although $\mathrm{im}\, d_n' \leq \ker d_{n+1}'$ for all n.

For each n, $(\ker d'_{n+1})/(\mathrm{im}\, d'_n)$ is denoted by $Ext^n(M,X)$. $\mathrm{Ext}^n(M,X)$ is independent (up to isomorphism) of the particular projective resolution used to define it ([49]).

Suppose $0\to N\to M\to M/N\to 0$ is an exact sequence. The following induced sequences of homology for Ext are exact for any module X (and for $n\geq 1$):

$$\ldots\ldots\to \mathrm{Ext}^{n-1}(N,X)\to \mathrm{Ext}^n(M/N,X)\to \mathrm{Ext}^n(M,X)\to \mathrm{Ext}^n(N,X)\to \mathrm{Ext}^{n+1}(M/N,X)\to\ldots (1)$$

and $\ldots\to \mathrm{Ext}^{n-1}(X,M/N)\to \mathrm{Ext}^n(X,N)\to \mathrm{Ext}^n(X,M)\to \mathrm{Ext}^n(X,M/N)\to \mathrm{Ext}^{n+1}(X,N)\to\ldots (2)$

We will also, on occasion, use the following properties of Ext:

(a) $\mathrm{Ext}^{n+1}(M,X) = 0$ for all X iff $\mathrm{Ext}^{m+1}(M,X) = 0$ for all X and for all $m\geq n$.

(b) Q is injective iff $\mathrm{Ext}^n(M,Q) = 0$ for all M and for all $n\geq 1$.

(c) $\mathrm{Ext}^0(A,B)\simeq \mathrm{Hom}(A,B)$ for all A and B ([49]).

1.2.1.5 Let A be a right R-module and B a left R-module. Let F be the free Abelian group on the set $\{(a_i,b_i)\mid a_i\in A, b_i\in B\}$. Let K be the subgroup of F generated by all elements of the form $(a+a',b) - (a,b) - (a',b)$, $(a,b+b') - (a,b) - (a,b')$ and $(ar,b) - (a,rb)$ where $r\in R, a,a'\in A, b,b'\in B$.

The Abelian group F/K is called the *tensor product* of A,B, written $A\otimes_R B$ (or just $A\otimes B$ if R is understood).

If $a\in A$, $b\in B$, then we denote $(a,b)+K$ by $a\otimes b$. If $f:N_1\to N_2$ is a homomorphism of left R-modules, and $g:M_1\to M_2$ a homomorphism of right R-modules then there is an induced homomorphism $g\otimes f$: $M_1\otimes N_1\to M_2\otimes N_2$ of Abelian groups defined by $g\otimes f(m\otimes n)=g(m)\otimes f(n)$.

If $0\to N\xrightarrow{\alpha} M\xrightarrow{\beta} P\to 0$ is an exact sequence of left R-modules and L is a right R-module, then $L\otimes N\xrightarrow{1_L\otimes\alpha} L\otimes M\xrightarrow{1_L\otimes\beta} L\otimes P\to 0$ is exact ([67], Proposition 8.6).

1.2.1.6 If $\ldots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ is a projective resolution of M, and X is any left R-module, we get an induced sequence $\ldots P_1 \otimes X \xrightarrow{d_1 \otimes 1_X} P_0 \otimes X \xrightarrow{d_0 \otimes 1_X} M \otimes X$ and if we define $Tor_n(M,X) = [\ker(d_n \otimes 1_X)]/[\mathrm{im}(d_{n+1} \otimes 1_X)]$, the exact sequence of right R-modules $0 \to N \to M \to M/N \to 0$ induces an exact sequence of homology:

$$\ldots Tor_2(M/N,X) \to Tor_1(N,X) \to Tor_1(M,X) \to Tor_1(M/N,X) \to N \otimes X \to M \otimes X \qquad (1)$$

for any left module X.

Similarly an exact sequence of left R-modules $0 \to N \to M \to M/N \to 0$ gives rise to an induced exact sequence:

$$\ldots Tor_2(X,M/N) \to Tor_1(X,N) \to Tor_1(X,M) \to Tor_1(X,M/N) \to X \otimes N \to X \otimes M \qquad (2)$$

for any right R-module X (see [61], §8).

1.2.1.7 If X is a left R-module, the *character module* $X^* = \mathrm{Hom}_Z(X,Q/Z)$ is a right R-module under $(\phi r)(x) = \phi(rx)$ $(r \in R,\ x \in X,\ \phi \in X^*)$. It follows from [11], Proposition 5.1 (page 120) that $\mathrm{Ext}^n(M,X^*) \simeq (Tor_n(M,X))^*$ for all right R-module M, $n \geq 1$.

1.2.1.8 Let $\{M_i\}_{i \in I}$ be a class of R-modules, where I is a directed partially ordered set (i.e. for i, $j \in I$ there is a $k \in I$ with $k \geq i$, $k \geq j$).

Suppose that for all i,j with $i \leq j$ there is homomorphism $\delta_i^j : M_i \to M_j$ satisfying:

(1) $\delta_i^i = 1_{M_i}$ for all i.

(2) if $i \leq j \leq k$ then $\delta_j^k \cdot \delta_i^j = \delta_i^k$.

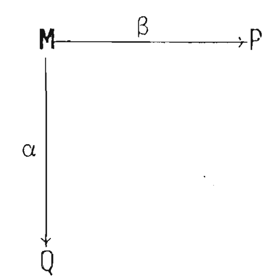
Then $\{M_i (i \in I),\ \delta_i^j\}$ is called a *directed system* and if S is the submodule of $\bigoplus_i M_i$ generated by all elements of the form $in_i(a_i) - in_j(\delta_i^j(a_i))$, $a_i \in M_i$, $i \leq j$, then $(\bigoplus_i M_i)/S$ is called the *direct limit* of the directed system, written $\varinjlim_I M_i$ (see [45], Chapter 4 or [63], §13 for properties of this concept). For each i, the homomorphism $\delta_i : M_i \to \varinjlim_I M_i$ defined by $\delta_i(m_i) = in_i(m_i) + S$ is called the *canonical map*.

1.2.1.9 A direct limit of a directed system of submodules of a module M, where the δ_i^j are inclusion maps, is called a *directed union.*

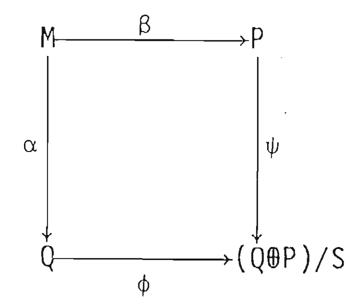
It is shown in [63] that every R-module is a directed union of its f.g. submodules and that both a direct sum and the union of an ascending chain of modules are special cases of a directed union.

1.2.2 Completion of Diagrams

(i) A pushout diagram for modules is constructed as follows: Given the diagram

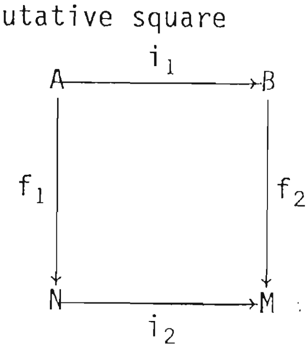


of left R-modules and homomorphisms, we may complete the square



commutatively by taking $S = \{(\alpha(m), -\beta(m)) \mid m \in M\}$, $\phi(q) = (q,o)+S$, for $q \in Q$ and $\psi(p) = (o,p)+S$, for $p \in P$.

(ii) Given a commutative square



where i_1, i_2 are the inclusion maps, we complete the rectangle

$$\begin{array}{ccccc} A & \xrightarrow{i_1} & B & \xrightarrow{\pi_1} & B/A \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ N & \xrightarrow{i_2} & M & \xrightarrow{\pi_2} & M/N \end{array}$$

commutatively, by defining $f_3(b+A) = f_2(b)+N$ for $b \in B$.

(iii) Dually, given the diagram

$$\begin{array}{ccc} & & Q \\ & & \downarrow{\scriptstyle \alpha} \\ P & \xrightarrow{\beta} & M \end{array}$$

we complete the *pullback* diagram

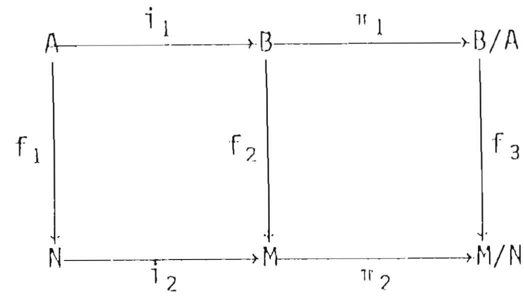
$$\begin{array}{ccc} X & \xrightarrow{\psi} & Q \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \alpha} \\ P & \xrightarrow{\beta} & M \end{array}$$

commutatively, by taking $X = \{(p,q) \in P \oplus Q \mid \beta(p) = \alpha(q)\}$ and $\phi:(p,q) \mapsto p$, $\psi:(p,q) \mapsto q$ for $(p,q) \in X$.

(iv) Given a commutative square

$$\begin{array}{ccc} B & \xrightarrow{\pi_1} & B/A \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ M & \xrightarrow{\pi_2} & M/N \end{array}$$

where A and N are submodules of B and M respectively, we complete the rectangle



commutatively by letting f_1 be the restriction of f_2 to A.

1.2.3 Torsion Theories

The following results on torsion theories are mainly taken from [37], [66] and [67]:

1.2.3.1 Definition ([66])

Let R be a ring. A *torsion radical* is a functor σ:R-mod→R-mod assigning to each R-module M a submodule $\sigma(M)$, and to every homomorphism $f:M\to P$ the homomorphism $f/_{\sigma(M)}$, such that.

(i) If N is a submodule of M then $\sigma(N) = N \cap \sigma(M)$.

(ii) If $f:M_1\to M_2$ is a homomorphism then $f[\sigma(M_1)] \leq \sigma(M_2)$.

(iii) $\sigma(M/\sigma(M)) = 0$ for all modules M.

Throughout this thesis torsion radicals will be denoted by σ,τ etc.

1.2.3.2 A module M is called *σ-torsion* iff $\sigma(M) = M$ and *σ-torsion free* if $\sigma(M) = 0$.

The class of σ-torsion modules will be denoted by $\mathcal{T}_\sigma$ and the class of σ-torsion free modules by $\mathcal{F}_\sigma$. $(\mathcal{T}_\sigma,\mathcal{F}_\sigma)$ is then called a (*hereditary*) *torsion theory*.

1.2.3.3 Note

It follows easily from 1.2.3.1(i) that $\sigma(M)$ is σ-torsion and contains every σ-torsion submodule N of M.

1.2.3.4 Theorem ([66])

Let R be a ring and σ a torsion radical on R-mod. $(\mathcal{T}_\sigma, \mathcal{F}_\sigma)$ satisfies the following:

(i) $\mathcal{T}_\sigma$ is closed under submodules, factor modules, direct sums and module extensions.

(ii) $\mathcal{F}_\sigma$ is closed under submodules, direct products, module extensions and essential extensions.

(iii) $F \in \mathcal{F}_\sigma$ iff $\mathrm{Hom}(T,F) = 0$ for all $T \in \mathcal{T}_\sigma$ and

(iv) $T \in \mathcal{T}_\sigma$ iff $\mathrm{Hom}(T,F) = 0$ for all $F \in \mathcal{F}_\sigma$.

1.2.3.5 Note

Given an hereditary torsion theory we define the associated filter $\mathcal{L}_\sigma = \{I \mid I$ is a left ideal of R and $R/I \in \mathcal{T}_\sigma\}$. It then follows easily that $\sigma(M) = \{m \in M \mid \mathrm{Ann}\ m \in \mathcal{L}_\sigma\}$.

1.2.3.6 Theorem ([37] and [67])

Let σ be a torsion radical and let $\mathcal{L}_\sigma$ be as defined above. Then $\mathcal{L}_\sigma$ has the following properties:

(i) If $I_1 \in \mathcal{L}_\sigma$ and I_2 is a left ideal of R such that $I_1 \leq I_2$, then $I_2 \in \mathcal{L}_\sigma$.

(ii) If $I_1, I_2 \in \mathcal{L}_\sigma$, then $I_1 \cap I_2 \in \mathcal{L}_\sigma$.

(iii) If $r \in R$ and $I \in \mathcal{L}_\sigma$, then $\mathrm{Ann}(r+I) \in \mathcal{L}_\sigma$.

(iv) If $I \in \mathcal{L}_\sigma$ and K is a left ideal of R such that $\mathrm{Ann}(a+K) \in \mathcal{L}_\sigma$ for all $a \in I$, then $K \in \mathcal{L}_\sigma$.

Given (i), (ii), (iii) and (iv), it is further true that

(v) If $I, J \in \mathcal{L}_\sigma$ then $I.J \in \mathcal{L}_\sigma$ ([37]).

1.2.3.7 Remark

A set L_σ of left ideals satisfying 1.2.3.6 is called a *Gabriel Topology* and is also uniquely determined by (T_σ, F_σ) ([67], Theorem 5.1, page 146).

Given a Gabriel Topology L_σ, we may in turn define σ by $\sigma(M) = \{x \in M \mid \text{Ann}\, x \in L_\sigma\}$ for any module M.

σ thus defined is a torsion radical and is again uniquely determined by L_σ ([67]). Thus a torsion theory may be defined by specifying either σ or L_σ.

1.2.3.8 Example

Let R be a commutative integral domain. Define for an R-module M, $\sigma(M) = \{m \in M \mid r.m = 0 \text{ for some } r \in R,\ r \neq 0\}$. It is routine to verify that σ is a torsion radical on R-mod and that L_σ is the set of nonzero ideals of R.

In particular if R = $\mathbb{Z}$, $\sigma(G)$ is the *torsion subgroup* of an Abelian group G (see [33]). We shall refer to this as the *usual torsion theory on the category of Abelian groups.*

1.2.4 σ-Injectivity

1.2.4.1 Definition ([37])

A module E is called *σ-injective* iff for each submodule N of any module M such that M/N is σ-torsion, any homomorphism $f:N \to E$ can be extended to a homomorphism $g:M \to E$.

1.2.4.2 Lemma ([37], Proposition 3.2)

The following are equivalent for a module E:

(i) E is σ-injective.

(ii) If $I \in L_\sigma$ and $f: I \to E$ is a homomorphism, then f can be extended to a homomorphism $g: R \to E$.

1.2.4.3 <u>Definition</u> ([44])

Let M be an R-module and let E(M) be its injective hull. $E_\sigma(M) = \{x \in E(M) | \mathrm{Ann}(x+M) \in L_\sigma\}$ is called the *σ-injective hull* of M.

1.2.4.4 <u>Theorem</u> ([53], Proposition 0.7)

$E_\sigma(M)$ is a σ-injective, essential extension of M and $E_\sigma(M)/M \in T_\sigma$. Moreover, any other σ-injective, essential extension E of M such that E/M is σ-torsion is isomorphic to $E_\sigma(M)$.

1.2.4.5 <u>Definition</u> ([44])

Let R be a ring and σ a torsion radical on R-mod.

Let N be a submodule of the module M, $I, J \in L_\sigma$ such that $I \subsetneq J$ and let $i_1: I \to J$, $i_2: N \to M$ be the inclusion maps. N is called a *σ-neat submodule* iff whenever homomorphisms $f: I \to N$, $g: J \to M$ exist such that $g \cdot i_1 = i_2 \cdot f$, there is a left ideal K such that $I \subsetneq K \leq J$ and a homomorphism $g: K \to N$ such that $g/_I = f$.

(This is an equivalent form of the definition given in [44] as noted on page 1139 of that article).

1.2.4.6 <u>Remarks</u>

(i) A σ-neat submodule of a σ-injective module is σ-injective ([44], Proposition 2).

(ii) Homomorphic images of σ-injective modules are σ-injective iff every $I \in \mathcal{L}_\sigma$ is projective ([34], Proposition 4.6).

(iii) Direct summands, finite direct sums and direct products of σ-injective modules are σ-injective ([35], Proposition 8.4).

(iv) Arbitrary direct sums of σ-injective modules need not be σ-injective:

Let R be any non-Noetherian ring and take $\mathcal{L}_\sigma$ to be the set of all left ideals in R. For this σ, σ-injectivity is equivalent to injectivity.

Since R is not Noetherian, it follows from Theorem 17.2 of [45] that there is a direct sum of injective R-modules which is not injective. Thus we have a family of σ-injective modules whose direct sum is not σ-injective.

(v) If $0 \to F \xrightarrow{i} E \to L \to 0$ is exact, F is σ-injective and L is σ-torsion, then the sequence splits. If, further, $\sigma(E) = 0$, then $F = E$ ([37], Proposition 3.4).

1.2.5: <u>**The Condition "Every $I \in \mathcal{L}_\sigma$ is f.g."**</u>

1.2.5.1 <u>**Theorem**</u>

Let R be a ring and σ a torsion radical on R-mod. The following are equivalent:

(i) Every direct sum of σ-injective modules is σ-injective and $\mathcal{L}_\sigma$ contains a cofinal subset of left ideals of the form $\bigoplus_\alpha J_\alpha$ where each J_α is countably generated.

(ii) If $I \in \mathcal{L}_\sigma$ then I is f.g.

(iii) Every essential (left) ideal in $\mathcal{L}_\sigma$ is f.g.

Proof

(L_σ is called *σ-Noetherian* iff whenever $I_1 \leq I_2 \leq \ldots$ is a countably infinite ascending chain of left ideals of R such that $\bigcup_k I_k \in L_\sigma$, then $I_n \in L_\sigma$ for some n([36])).

That (ii) implies (i) is in Theorem 2 of [36]. Suppose (i) holds. The condition "direct sums of σ-injective modules are σ-injective" is shown, in Theorem 1 of [36], to be equivalent to: "L_σ is σ-Noetherian and has the ascending chain condition".

This condition, together with "L_σ contains a cofinal subset of left ideals of the form $\bigoplus_\alpha J_\alpha$ where each J_α is countably generated" implies (ii) by Theorem 2 of [36].

To complete the proof we only need to show that (iii) implies (ii). Suppose, therefore, that every essential (left) ideal in L_σ is f.g. and let I be an arbitrary ideal in L_σ. Suppose I is not f.g. and let $S = \{J \in L_\sigma \mid I \leq J \text{ and } J \text{ is not f.g.}\}$. $S \neq \phi$; since $I \in S$, and is inductive, hence it has a maximal element M (say).

Let N be any nonzero left ideal of R. If $N \cap M = 0$ then, by maximality of M in S, M+N (=M⊕N) is f.g. Hence M is f.g., which is a contradiction. Hence, $M \trianglelefteq R$ and the fact that M is not f.g. is in conflict with our assumption. Thus every $I \in L_\sigma$ is f.g. and (ii) holds.

1.2.5.2 **Theorem**

Suppose R is a commutative ring and σ a torsion radical on R-mod. Then the following are equivalent:

(i) Every $I \in L_\sigma$ is f.g.

(ii) Every semiprime ideal $I \in L_\sigma$ is f.g.

(iii) Every prime ideal $I \in L_\sigma$ is f.g.

Proof

Suppose (iii) holds and let $S = \{I\epsilon L_\sigma | I \text{ is not f.g.}\}$. If $S \neq \phi$, it is inductive and has a maximal element M (say), by Zorn's Lemma.

M is not prime since it is not f.g. Thus there exist ideals A,B such that A.B≤M but A≰M and B≰M. Let $a\epsilon A-M$, $b\epsilon B-M$. By maximality of M, M+Ra is f.g. Suppose M+Ra can be generated by the set $\{m_1+r_1a, m_2+r_2a, \ldots, m_n+r_na\}$ for some $m_i\epsilon M$, $r_i\epsilon R$. Let J = Ann (a+M). Since $a.b\epsilon M$ and $b\notin M$, it is clear that $M \lneq M+Rb \leq J$. By maximality of M, J is f.g., say by the set $\{j_1, j_2, \ldots j_k\}$.

Let $x\epsilon M$, then $x\epsilon M+Ra$ and there exist $s_i\epsilon R$ such that $x = \sum_{i=1}^{n} s_i(m_i+r_ia)$. Thus $(\sum_{i=1}^{n} s_ir_i)a = x - \sum_{i=1}^{n} s_im_i\epsilon M$, whence $\sum_{i=1}^{n} s_ir_i\epsilon J$ and there exist $t_i\epsilon R$ such that $\sum_{i=1}^{n} s_ir_i = \sum_{i=1}^{k} t_ij_i$.

Hence $x = \sum_{i=1}^{n} s_im_i + \sum_{i=1}^{k} t_ij_ia$ and M can be generated by the set $\{m_1, m_2, \ldots m_n, j_1a, j_2a, \ldots j_ka\}$, which is a contradiction. That is, $S = \phi$ as required.

1.2.5.3 Example

If R is Noetherian then every $I\epsilon L_\sigma$ is f.g. for all torsion radicals σ on R-mod. There are non-Noetherian rings which admit a torsion radical σ satisfying this condition.

The following example is due to A.R. Meijer (personal communication):

Let S be a non-Noetherian, commutative ring with identity and F a field. Put $R = S\oplus F = \{(s,f)|s\epsilon S, f\epsilon F\}$ with componentwise operations.

$M = \{(s,0)|s\epsilon S\}$ is a maximal ideal of R (for $R/M\simeq F$ is simple). If we take $L_\sigma = \{M;R\}$, then L_σ contains only f.g. ideals. We show L_σ is indeed the filter of a torsion radical:

1.2.3.6(i) follows by maximality of M and 1.2.3.6(ii) is clearly valid in this case.

(iii): Suppose $I \in \mathcal{L}_\sigma$ and $r \in R$. Thus I = M or R. Let J = Ann(r+I). Clearly $M \leq J$, so J = M or R. Hence $J \in \mathcal{L}_\sigma$.

(iv) Suppose $I \in \mathcal{L}_\sigma$ and K is an ideal of R such that $\mathrm{Ann}(a+K) \in \mathcal{L}_\sigma$ for all $a \in I$. Thus Ann $((1,0)+K) \in \mathcal{L}_\sigma$ and therefore Ann((1,0)+K) = M or R. In either case $M \leq \mathrm{Ann}((1,0)+K)$ and $M(1,0) = M \leq K$. Hence K = M or R and $K \in \mathcal{L}_\sigma$ as required.

§1.3 PURITY:

1.3.1 Pure Theories

1.3.1.1 D.J. Fieldhouse in [29] describes a generalization of the concept of purity to arbitrary categories. We will adapt his definition of a Pure Theory to the category R-mod.

For this purpose we need the following result:

1.3.1.2 Lemma ([26], Theorem 3)

Suppose

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\beta_1} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow[\alpha_2]{} & B_2 & \xrightarrow[\beta_2]{} & B_3 & \longrightarrow & 0 \end{array}$$

is a commutative diagram of left R-modules with exact rows. Then the following are equivalent:

(i) There is a homomorphism $\mu: A_2 \to B_1$ such that $f_1 = \mu \cdot \alpha_1$.

(ii) There is a homomorphism $\tau: A_3 \to B_2$ such that $f_3 = \beta_2 \cdot \tau$.

1.3.1.3 Definitions ([29])

(i) Given short exact sequences:

$A:0\to A_1\xrightarrow{\alpha_1}A_2\xrightarrow{\beta_1}A_3\to 0$ and $B:0\to B_1\xrightarrow{\alpha_2}B_2\xrightarrow{\beta_2}B_3\to 0$ of (left) R-modules we write $A\Theta B$ iff for every commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\beta_1} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow[\alpha_2]{} & B_2 & \xrightarrow[\beta_2]{} & B_3 & \longrightarrow & 0 \end{array}$$

one of the equivalent conditions of Lemma 1.3.1.2 holds. (Note that Θ is not a symmetric relation).

(ii) The collection E of all short exact sequences of R-modules forms a category with morphisms $F = (f_1,f_2,f_3):A\to B$, where the $f_i:A_i\to B_i$ are R-homomorphisms making the above diagram commute.

If G is any collection of short exact sequences in E then we define $G^\star = \{B\in E \mid G\Theta B \text{ for all } G\in G\}$ and $G^+ = \{A\in E\mid A\Theta G \text{ for all } G\in G\}$. A *pure theory* is then an ordered pair (P,Q) of classes of short exact sequences such that $Q = P^+$ and $P = Q^\star$.

(iii) We say that P *"left generates"* and Q *"right generates"* the pure theory (P,Q) and note that $(P,Q) = ((P^+)^\star,P^+) = (Q^\star, (Q^\star)^+)$.

(iv) (P,Q) is called *projectively generated* iff the middle term of each short exact sequence in Q is projective.

(v) The elements of P are called the *pure exact sequences* and those of Q the *copure exact sequences* of the pure theory.

(vi) If $0\to P_1\xrightarrow{\alpha}P_2\xrightarrow{\beta}P_3\to 0$ is a pure exact sequence, we call α a *pure monomorphism* and β a *pure epimorphism*.

(vii) We will say that a submodule N of a module M is pure in M iff the inclusion map $i:N\to M$ is a pure monomorphism.

(viii) A module P is called *pure projective* iff $\mathrm{Hom}(P, P_2) \to \mathrm{Hom}(P, P_3) \to 0$ is exact for each pure exact sequence $0 \to P_1 \to P_2 \to P_3 \to 0$.

1.3.1.4 **<u>Lemma</u> ([29], Theorem 4.3)**

Let

$$\begin{array}{lccccccccc}
A: & 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\beta_1} & A_3 & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
B: & 0 & \longrightarrow & B_1 & \xrightarrow[\alpha_2]{} & B_2 & \xrightarrow[\beta_2]{} & B_3 & \longrightarrow & 0
\end{array}$$

be a commutative diagram with exact rows in R-mod and let $(\mathcal{P},\mathcal{Q})$ be a projectively generated pure theory in R-mod.

(i) If f_1 is monic, $f_1(A_1)$ is pure in B_1 and B is pure exact, then A is pure exact.

(ii) If f_3 is epic, the sequence $0 \to \ker f_3 \xrightarrow{i} A_3 \xrightarrow{f_3} B_3 \to 0$ is pure exact (where i is the inclusion map) and A is pure exact, then B is pure exact.

1.3.1.5 **<u>Theorem</u> ([31], Theorem 5.1)**

Let $E \leq F \leq G$ be R-modules and let $(\mathcal{P},\mathcal{Q})$ be a projectively generated pure theory in R-mod.

(i) If E is pure in F and F is pure in G, then E is pure in G.

(ii) If E is pure in G then E is pure in F.

(iii) If F is pure in G then F/E is pure in G/E.

(iv) If F/E is pure in G/E and E is pure in G, then F is pure in G.

1.3.2 <u>Cohn Purity</u>

1.3.2.1 **<u>Definition</u> ([12])**

We call a submodule N of a module M a *Cohn pure submodule* iff the solvability of the system of equations $\sum_{i=1}^{n} r_{ij}x_i = a_j$ (where $a_j \in N$, $r_{ij} \in R$, $j = 1,2,\ldots,m$) in M implies its solvability in N.

1.3.2.2 P.M. Cohn, in [12], shows that N is a Cohn pure submodule of the module M iff the induced mapping $K\otimes N \xrightarrow{1_K\otimes i} K\otimes M$ is monic for all right R-modules K (where $i:N\to M$ the inclusion map).

1.3.2.3 D.J. Fieldhouse in [26] and Döman in [13] discuss some of the properties of this concept. Many of these have counterparts for σ-purity. We list a few of the properties, which are important for our purposes, below:

(i) A module N is Cohn pure in M iff every f.p. module is projective with respect to the sequence $0\to N\xrightarrow{i} M\xrightarrow{\pi} M/N\to 0$ ([45], Theorem 16.5).

(ii) Cohn purity forms a pure theory right generated by the family $\mathcal{Q} = \{0\to G_1\xrightarrow{\alpha} G_2\xrightarrow{\beta} G_3\to 0 \mid G_1, G_2$ are f.g. and G_2 is free$\}$ ([29], Theorem 7.1).

(iii) For every module M there is a direct sum $\bigoplus_i A_i$ of f.p. modules A_i and a Cohn pure exact sequence $0\to \ker\alpha\to \bigoplus_i A_i \xrightarrow{\alpha} M\to 0$ ([45], Theorem 16.6).

(iv) A module M is f.g. and Cohn pure projective iff it is f.p.

This result is mentioned by Fieldhouse on page 15 of [29]. (He attributes it (without reference) to Zimmerman, and for completeness sake we would like to include a proof).

Proof

If M is f.p. then it is f.g. and Cohn pure projective by (i) above.

Conversely suppose M is f.g., Cohn pure projective. By (iii) above, there is a Cohn pure exact sequence $0\to \ker\alpha \xrightarrow{i} \bigoplus_{i\in I} A_i \xrightarrow{\alpha} M\to 0$ where the A_i are f.p. Since M is Cohn pure projective there is a homomorphism $\tau: M\to \bigoplus_{i\in I} A_i$ such that $\alpha\cdot\tau = 1_M$ and hence M is a direct summand of $\bigoplus_{i\in I} A_i$. Since M is f.g., $M \leq \bigoplus_{i=1}^{n} A_i$ for some integer n. It follows from 1.2.1.3(c) and (d) that $\bigoplus_{i=1}^{n} A_i$ is f.p. and, since M is f.g., $(\bigoplus_{i=1}^{n} A_i)/M$ is f.p. and hence Cohn pure projective.

$M \leq \bigoplus_{i=1}^{n} A_i \leq \bigoplus_{i\in I} A_i$ and M is Cohn pure in $\bigoplus_{i\in I} A_i$ (being a direct summand) hence, by 1.3.1.5(ii), M is Cohn pure in $\bigoplus_{i=1}^{n} A_i$. Thus the sequence $0\to M \xrightarrow{i} \bigoplus_{i=1}^{n} A_i \xrightarrow{\pi} (\bigoplus_{i=1}^{n} A_i)/M \to 0$ splits and M is therefore the quotient of a f.p. module (isomorphic to $\bigoplus_{i=1}^{n} A_i$) by a f.g. module (isomorphic to $(\bigoplus_{i=1}^{n} A_i)/M$), i.e. M is f.p. (1.2.1.3(d)).

(v) A module M is Cohn pure projective iff it is a direct summand of a direct sum of f.p. modules ([45], Theorem 16.7).

1.3.2.4 Definition

A left R-module F is called *flat* ([45]) iff for any exact sequence $0\to N\to M$ of right R-modules the induced sequence $0\to N\otimes F\to M\otimes F$ is exact.

1.3.2.5 There are a number of equivalent characterizations of flatness:

(i) F is flat iff $0\to J\otimes F\to R\otimes F$ is exact for all right ideals J of R ([45], Theorem 14.6).

(ii) Because of the exactness of the sequence $Tor_1(R/J,F)\to J\otimes F\to R\otimes F$, F is flat iff $Tor_1(R/J,F) = 0$ for all right ideals J of R ([61]).

(iii) F is flat iff its character module $F^* = Hom_Z(F,Q/Z)$ is injective. This is a famous characterization by Lambek ([54]).

(iv) F is flat iff every exact sequence of the form $0\to N\to M\to F\to 0$ is Cohn pure exact ([30]).

1.3.2.6 Definition

A ring is called *regular* iff $a\in aRa$ for all $a\in R$. This concept was introduced by von Neumann in [58].

The following interesting result will be used later on:

1.3.2.7 **Theorem** ([32], Theorem 2)

A ring R is regular iff every left ideal is Cohn pure in R.

1.3.3 σ- Purity : Some Fundamental Properties

1.3.3.1 **Remark**

The following are equivalent for an Abelian group G, subgroup H of G and the usual torsion theory on Z-mod:

(i) $nG \cap H = nH$ for all $n \in Z$ (i.e. H is pure in G in the sense usual for Abelian groups, see, for example, Fuchs, [33]).

(ii) Every cyclic, torsion group is projective with respect to the exact sequence $0 \to H \xrightarrow{i} G \xrightarrow{\pi} G/H \to 0$.

(iii) Every f.p. Abelian group is projective with respect to this sequence (i.e. H is Cohn pure in G).

Proof

Since Z is Noetherian, the concepts of finitely generated Abelian groups and finitely presented Abelian groups coincide ([45], Theorem 3.6). Furthermore, since any finite cyclic group is torsion (being isomorphic to $Z/(n)$ for some natural number $n>1$) and every infinite cyclic group is projective (being isomorphic to Z) the result follows from Theorem 6.18 of [6] and Theorem 29.3 of [33].

The previous result suggests the following definitions:

1.3.3.2 **Definition** ([44])

Let R be a ring and σ a torsion radical on R-mod.

A short exact sequence $0 \to P_1 \xrightarrow{\alpha} P_2 \xrightarrow{\beta} P_3 \to 0$ of R-modules is called *σ-pure exact* iff every cyclic, σ-torsion module P is projective with respect to this sequence. A submodule N of M is called a *σ-pure* submodule iff the sequence $0 \to N \xrightarrow{i} M \xrightarrow{\pi} M/N \to 0$ is σ-pure exact.

1.3.3.3 **Definition** ([44])

Let R be a ring and σ a torsion radical on R-mod.

A submodule N of an R-module M is called *strongly σ-pure* iff every σ-torsion module is projective with respect to the sequence $0 \to N \xrightarrow{i} M \xrightarrow{\pi} M/N \to 0$.

1.3.3.4 **Lemma**

Let R be a ring, N a submodule of an R-module M and let σ be a torsion radical on R-mod. Then the following are equivalent:

(i) N is σ-pure in M.

(ii) For each $x \in M$ and $I \in L_\sigma$ such that $Ix \leq N$, there is a $y \in N$ with $I(x-y) = 0$. (Thus σ-purity coincides with Lambek's purity introduced in [53]).

Proof

(i) implies (ii): Suppose that N is σ-pure in M and that $Ix \leq N$ for some $x \in M$, and some $I \in L_\sigma$. R/I is then cyclic, σ-torsion and $\alpha: R/I \to M/N$ defined by $\alpha(r+I) = rx+N$, for $r \in R$, is an R-homomorphism. Let $\pi: M \to M/N$ be the canonical epimorphism. Since N is σ-pure in M, there is a homomorphism $\beta: R/I \to M$ such that $\pi \cdot \beta = \alpha$. Let $\beta(1+I) = z$ and put $y = x-z$, then $\pi(y) = \pi(x)-\pi(z) = (x+N)-\pi \cdot \beta(1+I) = (x+N)-\alpha(1+I) = 0+N$, hence $y \in N$. Since $\beta(1+I) = z$ it follows that $Iz = 0$, and since $z = x-y$ with $y \in N$, this proves (ii).

(ii) implies (i): Suppose that $T \simeq R/I$ is cyclic, σ-torsion and let $\alpha: R/I \to M/N$ be a homomorphism, with $\alpha(1+I) = x+N$ for some $x \in M$. Then $Ix \leq N$ and, by (ii), there is a $y \in N$ with $I(x-y) = 0$. The map $\beta: R/I \to M$ defined by $\beta(1+I) = x-y$ is therefore a well defined R-homomorphism and $\pi \cdot \beta = \alpha$, whence N is σ-pure in M.

1.3.3.5 Lemma

Let R be a ring and σ a torsion radical on R-mod. If every $I \in \mathcal{L}_\sigma$ is f.g. then the union of an ascending chain $\{M_i\}_i$ of σ-pure submodules of a module M is σ-pure in M ([64], Proposition 7.3).

1.3.3.6 Theorem

Let R be a ring and σ a torsion radical on R-mod. Let I be some index set and, for each $i \in I$, let P_i be a submodule of a module M_i. Then

(i) $\bigoplus_{i\in I} P_i$ is σ-pure in $\bigoplus_{i\in I} M_i$ iff P_i is σ-pure in M_i for each i.

(ii) $\prod_i P_i$ is σ-pure in $\prod_i M_i$ iff each P_i is σ-pure in each M_i.

(iii) $\bigoplus_i P_i$ is σ-pure in $\prod_i P_i$.

(iv) $\bigoplus_i P_i$ is σ-pure in $\prod_i M_i$ iff P_i is σ-pure in M_i for each i.

Proof

(i) Suppose that $I \in \mathcal{L}_\sigma$ and that $I(m_i)_i \leq \bigoplus_i P_i$ for some $(m_i)_i \in \bigoplus_i M_i$. Then $Im_i \leq P_i$ for each i and, since P_i is σ-pure in M_i, for each nonzero m_i we can choose a $p_i \in P_i$ with $I(m_i - p_i) = 0$. Since there are only a finite number of nonzero p_i, $(p_i)_i \in \bigoplus_i P_i$ and $I((m_i)-(p_i)) = (0)_i$. Thus, $\bigoplus_i P_i$ is σ-pure in $\bigoplus_i M_i$.

Conversely, suppose that $\bigoplus_i P_i$ is σ-pure in $\bigoplus_i M_i$ and that $Im_i \leq P_i$ for some $m_i \in M_i$, $I \in \mathcal{L}_\sigma$.

Define $(n_i)_i \in \bigoplus_i M_i$ by $n_j = \begin{cases} m_i & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$. Then $I(n_i)_i \leq \bigoplus_i P_i$ and, by assumption, there is a $(p_i)_i \in \bigoplus_i P_i$ with $I((n_i)_i - (p_i)_i) = (0)_i$. Thus $I(m_i - p_i) = 0$ with $p_i \in P_i$ and P_i is σ-pure in M_i.

(ii) The proof is very similar to (i).

(iii) Let $\alpha : R/I \to (\prod_i P_i)/(\bigoplus_i P_i)$ be any R-homomorphism, and suppose that $\alpha(1+I) = (p_i)_i + \bigoplus_i P_i$. Then $I(p_i)_i \leq \bigoplus_i P_i$ and therefore $Ip_i = 0$ for all except at most a finite number of i, say $Ip_i \neq 0$ iff $i \in I_0$ where

I_0 is a finite set. Let $\pi: \prod_i P_i \to (\prod_i P_i)/(\bigoplus_i P_i)$ be the canonical epimorphism.

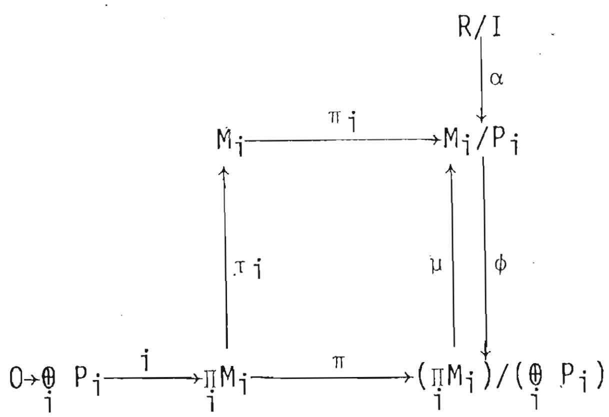
Define $\beta: R/I \to \prod_i P_i$ by $\beta(1+I) = (q_i)_i$ where $q_i = \begin{cases} p_i & \text{if } i \notin I_0 \\ 0 & \text{if } i \in I_0 \end{cases}$.

β is then a well defined homomorphism (since $Iq_i = 0$ for all i) and, moreover, $(p_i)_i - (q_i)_i \in \bigoplus_i P_i$, i.e. $(\pi \cdot \beta)(1+I) = (q_i)_i + \bigoplus_i P_i = (p_i)_i + \bigoplus_i P_i = \alpha(1+I)$. Hence $\pi \cdot \beta = \alpha$.

(iv) Suppose that P_i is σ-pure in M_i for all $i \in I$. Then, by (i), $\bigoplus_i P_i$ is σ-pure in $\bigoplus_i M_i$. By (iii), $\bigoplus_i M_i$ is σ-pure in $\prod_i M_i$ and by transitivity of σ-purity, $\bigoplus_i P_i$ is σ-pure in $\prod_i M_i$ (1.3.1.5(i)).

Conversely, suppose that $\bigoplus_i P_i$ is σ-pure in $\prod_i M_i$. Let $I \in \mathcal{L}_\sigma$ and let $\alpha: R/I \to M_i/P_i$ be an R-homomorphism for some $i \in I$.

Consider the diagram



where ϕ is the monomorphism $\phi(m_i + P_i) = (0,..0,m_i,0,.0) + \bigoplus_i P_i$, $\tau_i: \prod_i M_i \to M_i$ are the usual projections and π, π_i the canonical epimorphisms. ϕ has a left inverse, μ, defined by $\mu[(m_i)_i + \bigoplus_i P_i)] = m_i + P_i$.

It is easily verified that $\mu \cdot \pi = \pi_i \cdot \tau_i$. Since $\bigoplus_i P_i$ is σ-pure in $\prod_i M_i$, there is a homomorphism $\tau: R/I \to \prod_i M_i$ such that $\pi \cdot \tau = \phi \cdot \alpha$. Then $\tau_i \cdot \tau: R/I \to M_i$ satisfies $\pi_i \cdot (\tau_i \cdot \tau) = \mu \cdot \pi \cdot \tau = \mu \cdot \phi \cdot \alpha = \alpha$ and therefore P_i is σ-pure in M_i as required.

1.3.4 σ-Purity, Strong σ-Purity as Pure Theories

1.3.4.1 Theorem

If R is a ring and σ is a torsion radical on R-mod then σ-purity forms a (projectively generated) pure theory right generated by the set $\mathcal{Q}$ consisting of all short exact sequences of the form $0 \to I \xrightarrow{i} R \xrightarrow{\pi} R/I \to 0$ where $I \in \mathcal{L}_\sigma$. (This result was recorded, independently, in 1.23 of [57]).

Proof

See [41], Theorem 4.2.

1.3.4.2 Corollaries

(i) $0 \to P_1 \xrightarrow{\alpha} P_2$ is a σ-pure monomorphism iff for each $I \in \mathcal{L}_\sigma$ and each commutative square

$$\begin{array}{ccc} I & \xrightarrow{i} & R \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ P_1 & \xrightarrow[\alpha]{} & P_2 \end{array}$$

where i is the inclusion map, there is a homomorphism $\mu : R \to P_1$ such that $\mu \cdot i = f_1$ ([29], Corollary to Proposition 3.2).

(ii) If S,T are submodules of a module M and $S \cap T$ is σ-pure in T, then S is σ-pure in S+T ([40], Corollary 5.4).

1.3.4.3 Theorem

Let R be a ring and σ a torsion radical on R-mod. The strongly σ-pure exact sequences of R-mod are the pure exact sequences of the pure theory right generated by the family $\mathcal{Q} = \{0 \to Q_1 \to Q_2 \to Q_3 \to 0 \mid Q_3 \in \mathcal{T}_\sigma$ and Q_2 is projective$\}$ or equivalently by $\mathcal{Q}' = \{0 \to Q_1 \to Q_2 \to Q_3 \to 0 \mid Q_3 \in \mathcal{T}_\sigma$ and Q_2 is free$\}$.

Proof

Suppose that $P: 0 \to P_1 \to P_2 \xrightarrow{\beta} P_3 \to 0$ is in $\mathcal{Q}^*$ (or $(\mathcal{Q}')^*$) and let $f : Q_3 \to P_3$ be a homomorphism where $Q_3 \in \mathcal{T}_\sigma$. Let $\alpha : Q_2 \to Q_3$ be an epimorphism where Q_2 is free.

If $Q_1 = \ker\alpha$ then $0 \to Q_1 \xrightarrow{i} Q_2 \xrightarrow{\alpha} Q_3 \to 0$ is in $\mathcal{Q}(\mathcal{Q}')$ and we can form a commutative diagram

$$\begin{array}{lccccccccc} Q: & 0 & \longrightarrow & Q_1 & \xrightarrow{i} & Q_2 & \xrightarrow{\alpha} & Q_3 & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\ P: & 0 & \longrightarrow & P_1 & \longrightarrow & P_2 & \xrightarrow[\beta]{} & P_3 & \longrightarrow & 0 \end{array}$$

By assumption $Q \Theta P$ so there exists a homomorphism $\tau: Q_3 \to P_2$ with $\beta \cdot \tau = f$. Hence every σ-torsion module is projective with respect to P and P is therefore strongly σ-pure exact.

Conversely, suppose that $0 \to P_1 \longrightarrow P_2 \xrightarrow{\alpha} P_3 \to 0$ is strongly σ-pure exact, that the sequence $0: 0 \to Q_1 \to Q_2 \to Q_3 \to 0$ is in $\mathcal{Q}$ (or $\mathcal{Q}'$) and that we are given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_1 & \longrightarrow & Q_2 & \longrightarrow & Q_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & P_1 & \longrightarrow & P_2 & \xrightarrow[\beta]{} & P_3 & \longrightarrow & 0 \end{array}$$

By assumption, $Q_3 \epsilon \mathcal{T}_\sigma$ and hence there is a homomorphism $\tau: Q_3 \to P_2$ with $\beta \cdot \tau = f$.

Thus $Q \Theta P$. Since Q was arbitrary in $\mathcal{Q}$(or $\mathcal{Q}'$), $P \epsilon Q^*$ (or $(\mathcal{Q}')^*$).

1.3.4.4 <u>Definition</u>

Let R be a ring and σ a torsion radical on R-mod. A module P is called *σ-pure projective* iff for every σ-pure exact sequence $0 \to P_1 \xrightarrow{\alpha} P_2 \xrightarrow{\beta} P_3 \to 0$ and every homomorphism $\phi: P \to P_3$ there exists a homomorphism $\Psi: P \to P_2$ such that $\beta \cdot \Psi = \phi$.

In the usual way one can show that direct sums and summands of σ-pure projective modules are σ-pure projective.

1.3.4.5 Remark

It follows from the discussion in §1.3.1.3 that a short exact sequence $0 \to Q_1 \xrightarrow{\alpha_1} Q_2 \xrightarrow{\beta_1} Q_3 \to 0$ is co-σ-pure exact iff for every commutative diagram

$$
\begin{array}{lccccccccc}
Q: & 0 & \longrightarrow & Q_1 & \xrightarrow{\alpha_1} & Q_2 & \xrightarrow{\beta_1} & Q_3 & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
P: & 0 & \longrightarrow & P_1 & \xrightarrow[\alpha_1]{} & P_2 & \xrightarrow[\beta_2]{} & P_3 & \longrightarrow & 0
\end{array}
$$

where P is σ-pure exact, there is a homomorphism $\tau : Q_3 \to P_2$ such that $\beta_2 \cdot \tau = f_3$.

This raises the question of the relationship between σ-pure projectives and co-σ-pure exact sequences.

1.3.4.6 Theorem

Let R be a ring and σ a torsion radical on R-mod. A module M is σ-pure projective iff every exact sequence of the form $0 \to X \to Y \to M \to 0$ is co-σ-pure exact.

Proof

Suppose that the sequence $Q : 0 \to X \xrightarrow{\alpha_1} Y \xrightarrow{\beta_1} M \to 0$ is exact where M is σ-pure projective.

Let $P : 0 \to P_1 \xrightarrow{\alpha_2} P_2 \xrightarrow{\beta_2} P_3 \to 0$ be σ-pure exact and consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\beta_1} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
0 & \longrightarrow & P_1 & \xrightarrow[\alpha_2]{} & P_2 & \xrightarrow[\beta_2]{} & P_3 & \longrightarrow & 0
\end{array}
$$

Since M is σ-pure projective there exists a homomorphism $\tau : M \to P_2$ such that $\beta_2 \cdot \tau = f_3$, and $0 \to X \to Y \to M \to 0$ is therefore co-σ-pure exact.

Conversely, suppose every short exact sequence with M in the third nonzero position is co-σ-pure exact. Let P: $0\to P_1\to P_2\to P_3\to 0$ be a σ-pure exact sequence and let $f_3:M\to P_3$ be a homomorphism. Form the pullback diagram

$$\begin{array}{ccc} Y & \overset{\beta_1}{\dashrightarrow} & M \\ \downarrow & & \downarrow f_3 \\ P_2 & \underset{\beta_2}{\longrightarrow} & P_3 \end{array}$$

and hence the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \overset{\alpha_1}{\longrightarrow} & Y & \overset{\beta_1}{\longrightarrow} & M & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\ 0 & \longrightarrow & P_1 & \underset{\alpha_2}{\longrightarrow} & P_2 & \underset{\beta_2}{\longrightarrow} & P_3 & \longrightarrow & 0 \end{array}$$

By assumption, the top sequence is co-σ-pure exact and therefore there is a homomorphism $\tau:M\to P_2$ such that $\beta_2\cdot\tau = f_3$, whence M is σ-pure projective.

1.3.4.7 __Lemma__ **([64], Proposition 2.3(i))**

Let R be a ring and σ a torsion radical on R-mod. For any R-module M there exists a direct sum S of projective and cyclic, σ-torsion modules such that for some homomorphism $\alpha:S\to M$ the sequence $0\to\ker\alpha\longrightarrow S\overset{\alpha}{\longrightarrow}M\to 0$ is σ-pure exact.

1.3.4.8 __Theorem__ **([64], Proposition 2.4(iii))**

Let R be a ring, σ a torsion radical on R-mod and let M be an R-module. The following are equivalent:

(i) M is σ-pure projective.

(ii) Every σ-pure exact sequence of the form $0\to X\to Y\to M\to 0$ splits.

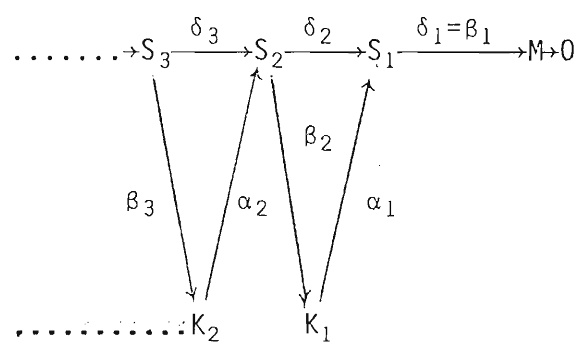
(iii) M is a direct summand of a direct sum of projective and cyclic, σ-torsion modules.

1.3.4.9 **Lemma**

Let R be a ring and σ a torsion radical on R-mod. Then there exists an exact sequence $\ldots\to S_2\xrightarrow{\delta_2}S_1\xrightarrow{\delta_1}M\to 0$ such that $\ker\delta_i$ is σ-pure in S_i, for all i.

Proof

Let M be any R-module. By 1.3.4.7, there is a σ-pure exact sequence $0\to K_1\xrightarrow{\alpha_1}S_1\xrightarrow{\beta_1}M\to 0$ where S_1 is σ-pure projective. Similarly, there is a σ-pure projective module S_2 and a σ-pure exact sequence $0\to K_2\xrightarrow{\alpha_2}S_2\xrightarrow{\beta_2}K_1\to 0$ and so on. (Note that α_1,α_2 are the inclusion maps). We may join these sequences to form the commutative diagram



It is then easy to verify that $\ker\delta_i$ ($\simeq K_i$) is σ-pure in S_i for all i and that $\ldots\to S_2\xrightarrow{\delta_2}S_1\xrightarrow{\delta_1}M\to 0$ is an exact sequence.

1.3.4.10 **Definitions**

(i) An exact sequence $\ldots S_2\xrightarrow{\delta_2}S_1\xrightarrow{\delta_1}M\to 0$ such that $\ker\delta_i$ is σ-pure in S_i and each S_i is a σ-pure projective module is called a *σ-pure projective resolution* for M .

(ii) The smallest n (if it exists) for which $\ker\delta_n$ is σ-pure projective in a σ-pure projective resolution will be called the *σ-pure projective dimension* of M (σppdM).

1.3.4.11 Remarks

(i) If $\ldots\ldots\rightarrow S_n \xrightarrow{\delta_n} S_{n-1} \rightarrow \ldots\ldots \rightarrow S_1 \xrightarrow{\delta_1} M \rightarrow 0$ is a σ-pure projective resolution of a module M and $\ker\delta_i$ is σ-pure projective, for some i, then $\ker\delta_{i+j}$ will be σ-pure projective for all $j \geq 1$.

Proof

$0 \rightarrow \ker\delta_{i+1} \rightarrow S_{i+1} \xrightarrow{\delta_{i+1}} \operatorname{im}\delta_{i+1} \rightarrow 0$ is σ-pure exact and, since $\operatorname{im}\delta_{i+1} = \ker\delta_i$ is σ-pure projective, this sequence splits. Hence $\ker\delta_{i+1}$ is σ-pure projective.

(ii) σppdM is independent of the particular σ-pure-projective resolution that is used to calculate it.

Proof

An easy extension of Theorem 3.5 of [13].

(iii) If $M = \bigoplus_i M_i$ for some modules M_i, $(i \in I)$, then $\sigma ppdM = \sup\{\sigma ppdM_i \mid i \in I\}$.

Proof Easy.

1.3.5 **σ-Purity, Strong σ-Purity as S-Purities**

1.3.5.1 In [69] C.P. Walker discusses S-purity where S is any class of modules closed under quotients. She defines a submodule L of a module M to be S-pure in M iff L is a direct summand of every module K such that $L \leq K \leq M$ and $K/L \in S$.

1.3.5.2 Theorem ([57], 1.48)

σ-purity is an S-purity if S is taken as the class of all cyclic, σ-torsion modules, (i.e. L is a σ-pure submodule of a module M iff L is a direct summand of every submodule K of M containing L, such that K/L is cyclic, σ-torsion).

1.3.5.3 Theorem

Strong σ-purity is an S-purity, if S is the class of all σ-torsion modules.

Proof

Suppose L is strongly σ-pure in M and K is a submodule of M containing L such that $K/L \in \mathcal{T}_\sigma$.

Form the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{i_1} & K & \xrightarrow{\pi_1} & K/L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_L} & & \downarrow{\scriptstyle i_3} & & \downarrow{\scriptstyle i_4} & & \\ 0 & \longrightarrow & L & \xrightarrow[i_2]{} & M & \xrightarrow[\pi_2]{} & M/L & \longrightarrow & 0 \end{array}$$

with inclusions i_j and canonical epimorphisms π_i.

Since K/L is σ-torsion and L is strongly σ-pure in M, there is a homomorphism $\tau: K/L \to M$ with $\pi_2 \cdot \tau = i_4$. By Lemma 1.3.1.2, this is equivalent to the existence of a left inverse for i_1, hence the top sequence splits as required.

Conversely, suppose that L is a direct summand of every submodule K of M containing L, for which $K/L \in \mathcal{T}_\sigma$. If T is σ-torsion and $\alpha: T \to M/L$ a homomorphism with $\alpha(T) = K/L (K \leq M)$, then $K/L \in \mathcal{T}_\sigma$ and, by assumption, L is a direct summand of K. Let $\beta: K/L \to K$ be a right inverse for the canonical epimorphism $\pi: K \to K/L$. Then $\beta \cdot \alpha: T \to M$ satisfies $\pi \cdot (\beta \cdot \alpha) = 1_{K/L} \cdot \alpha = \alpha$ and therefore L is strongly σ-pure in M.

1.3.5.4 Remark

σ-purity and strong σ-purity are therefore examples of some well-established generalizations of the concept of purity.

1.3.6 Some Relationships between our Three Pure Theories

1.3.6.1 Remark

The concepts of σ-purity, strong σ-purity and Cohn purity are distinct in general.

Clearly every strongly σ-pure submodule is σ-pure. This is in fact the only implication which holds for all R, σ:

The following example is found on page 596 of [59].

Example 1

Let $(\mathcal{T}_\sigma, \mathcal{F}_\sigma)$ be the usual torsion theory on Abelian groups. Let $G = \prod_{n=1}^{\infty} C(p^n)$ where $C(p^i)$ is the cyclic group of order p^i, p a fixed prime and let $M = \{(x_i)_i \in G \mid p^k(x_i)_i = (0)_i \text{ for some } k\}$. Further, let $N = \bigoplus_{n=1}^{\infty} C(p^n)$. Then N is σ-pure in M but not strongly σ-pure in M.

Example 2

Let R be a non-von Neumann regular ring and let $\mathcal{L}_\sigma = \{R\}$. Then every R-module is σ-torsion free (apply 1.2.3.5).

If $0 \to N \xrightarrow{i} M \xrightarrow{\pi} M/N \to 0$ is any short exact sequence and $T \in \mathcal{T}_\sigma$ then $\text{Hom}(T, M/N) = 0$ (1.2.3.4(iii)) and the sequence is strongly σ-pure exact, vacuously. Hence every submodule of every R-module is strongly σ-pure. Since R is not regular, there is a left ideal I of R which is not Cohn pure in R (1.3.2.7). This is therefore a strongly σ-pure submodule of R which is not Cohn pure.

Example 3

Let R be any ring which is not left Noetherian and $\mathcal{L}_\sigma$ the filter of a torsion radical which contains at least one left ideal I which is not f.g. (e.g. take $\mathcal{L}_\sigma$ as the set of all left ideals in R).

By [61], Corollary 3.42, R/I is not f.p. Since R/I is f.g. it is not Cohn pure projective (1.3.2.3(iv)). There is therefore a Cohn pure exact sequence $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} K \to 0$ with respect to which R/I is not projective. Since R/I is cyclic, σ-torsion, this sequence is not σ-pure exact.

1.3.6.2 **Remark**

The examples of §1.3.6.1 show that there are no universally valid relationships between σ-purity, strong σ-purity and Cohn purity, other than the fact that strong σ-purity implies σ-purity. Lemma 1.3.3.1 shows, however, that Cohn purity and σ-purity are in fact equivalent for Abelian groups if σ is the usual torsion radical.

This makes one ask what restrictions on R and/or σ have to be imposed in order to get equivalence of these purities. The rest of this paragraph is devoted to addressing this question.

1.3.6.3 **Lemma**

Let R be a ring and σ a torsion radical on R-mod. Then every $I \in \mathcal{L}_\sigma$ is f.g. iff every Cohn pure submodule is also a σ-pure submodule. (The fact that Cohn purity implies σ-purity if every $I \in \mathcal{L}_\sigma$ is f.g. is mentioned on page 170 (§14) of [64]).

Proof

Suppose that every $I \in \mathcal{L}_\sigma$ is f.g.

If $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} P \to 0$ is a Cohn pure exact sequence and R/I is a cyclic, σ-torsion module, then, since I is f.g., R/I is f.p. and hence projective with respect to this sequence, and the sequence is therefore σ-pure exact.

Conversely, suppose Cohn pure submodules are also σ-pure and let $I \in \mathcal{L}_\sigma$. Then R/I is cyclic, σ-torsion and therefore projective with respect to σ-pure exact sequences. By assumption any Cohn pure exact sequence is also σ-pure exact and R/I is therefore projective with respect to such a sequence. That is R/I is, f.g., Cohn pure projective. By 1.3.2.3(iv), R/I is f.p. and by 1.2.1.3(a), I is f.g.

1.3.6.4 Definitions ([61])

(i) An integral domain R is called *a Prüfer ring* iff it is semi-hereditary (i.e. every f.g. ideal is projective).

(ii) A Noetherian Prüfer ring is called a *Dedekind ring*.

1.3.6.5 Warfield in [71], Proposition 5 (page 706) shows that an integral domain is a Prüfer ring iff every f.p. module is a direct summand of a direct sum of cyclic modules.

1.3.6.6 Lemma

For any R-module M and any torsion radical σ on R-mod there is a direct sum S of cyclic submodules of M and a homomorphism $\alpha: S \to M$ such that the sequence $0 \to \ker\alpha \xrightarrow{i} S \xrightarrow{\alpha} M \to 0$ is σ-pure exact.

Proof

Let $M = \{m_i : i \in A\}$, where A is a suitable index set.

Put $S = \bigoplus_{i \in A} Rm_i$ and define $\alpha: S \to M$ by $\alpha: (r_i m_i)_i \mapsto \sum_{i \in A} r_i m_i$, where $r_i \in R$. Clearly α is an epimorphism. Suppose that $I \in \mathcal{L}_\sigma$ with $Is \leq \ker\alpha$ for some $s \in S$. If $\alpha(s) = m_i \in M$, then $\alpha(s) = \alpha(t)$ where $t = (t_j)_j \in S$ is given by
$$t_j = \begin{cases} m_i & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

Hence $s - t \in \ker\alpha$ and $Im_i = \alpha(Is) = 0$ (since $Is \leq \ker\alpha$), and thus $I(s-(s-t)) = It = 0$ and, since $s - t \in \ker\alpha$, this proves that $\ker\alpha$ is

σ-pure in S (1.3.3.4).

1.3.6.7 Theorem

Let R be a ring. For any torsion radical σ on R-mod, strong σ-purity and σ-purity are equivalent iff every σ-torsion module is a direct summand of a direct sum of cyclic, σ-torsion modules.

Proof

Suppose that the stated condition holds and that $0 \to N \to M \to K \to 0$ is a σ-pure exact sequence. It is then clear that any σ-torsion module T (being a direct summand of a direct sum of cyclic, σ-torsion modules) is projective with respect to this sequence, which is therefore strongly σ-pure exact.

Conversely, suppose that σ-pure exact sequences are also strongly σ-pure exact. Let M be any σ-torsion module. By 1.3.6.6, there is a σ-pure exact sequence: $0 \to \ker\alpha \xrightarrow{i} S \xrightarrow{\alpha} M \to 0$ where S is a direct sum, $\bigoplus_{i \in A} Rm_i$, of cyclic submodules of M, which are therefore σ-torsion. This sequence is, by assumption, strongly σ-pure exact and since M is σ-torsion the sequence splits, as required.

1.3.6.8 Theorem (c f. [64], Proposition 14.1)

Let R be an integral domain. The following pairs of conditions are equivalent (where σ is a torsion radical on R-mod):

I: (a) Cohn purity and σ-purity are equivalent

and **(b)** $\mathcal{L}_\sigma$ contains every nonzero projective ideal of R.

II: (a) R is a Dedekind ring

and **(b)** $\mathcal{L}_\sigma$ contains every nonzero ideal of R.

Proof

Suppose that I holds and let M be a f.p. module.

By 1.3.6.6, there is a direct sum S of cyclic modules and a σ-pure exact sequence $0 \to \ker\alpha \longrightarrow S \xrightarrow{\alpha} M \to 0$. M is Cohn pure projective and therefore σ-pure projective, by I(a). Therefore this sequence splits and M is a direct summand of a direct sum of cyclic modules. By 1.3.6.5, R is a Prüfer ring.

Let J be any nonzero ideal of R. J contains a nonzero f.g. ideal K (say) which is projective since R is a Prüfer ring. By I(b), $K \in L_\sigma$ hence $J \in L_\sigma$, and II(b) holds. By I(a), Cohn pure submodules are also σ-pure and, by 1.3.6.3, every ideal in L_σ is f.g. That is, every ideal of R is f.g., R is Noetherian and II(a) holds.

Conversely, suppose that II holds. It only remains to prove I(a).

Since R is Noetherian, Cohn purity implies σ-purity, by 1.3.6.3.

Every cyclic module is σ-pure projective by II(b). If M is any f.p. module, M is a direct summand of a direct sum of cyclic modules (1.3.6.5) and therefore M is σ-pure projective. It follows now that σ-pure exact sequences are also Cohn pure exact and I(a) holds.

§1.4 THE σ-PURE INJECTIVE HULL

1.4.1 In this paragraph we show that results in [24] and [64] can be used to construct a "σ-pure injective hull" of any module M, provided every $I \in L_\sigma$ is f.g. We adapt definitions given in those two papers to the category R-mod:

1.4.2 Definition

Let R be a ring and σ a torsion radical on R-mod. A module M is called *σ-pure injective* iff for each σ-pure monomorphism $\alpha: X \to Y$, any homomorphism $\phi: X \to M$ can be extended to a homomorphism $\psi: Y \to M$.

1.4.3 Lemma

Let R be a ring and σ a torsion radical on R-mod. The class of σ-pure monomorphisms is closed under pushouts and the class of σ-pure epimorphisms is closed under pullbacks.

(This fact has been noted, independently, by B. Stenstrom on page 160 of [64]).

Proof

See [41], Lemma 6.1.

1.4.4 Remark

By 1.3.1.5 and 1.4.3 the class of σ-pure monomorphisms is a "proper class" in the sense of [24] and [64].

1.4.5 Definitions

Let R be a ring and σ a torsion radical on R-mod.

(i) A module M is called a *σ-pure essential extension* of a submodule L iff L is σ-pure in M and there are no nonzero submodules N of M such that $N \cap L = 0$ and $(L+N)/N$ is σ-pure in M/N.

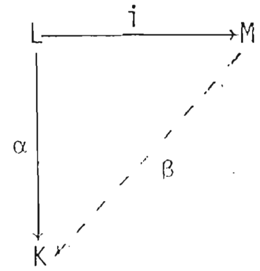
(ii) A σ-pure essential extension M of a module L is called a *maximal σ-pure essential extension* iff it is not properly contained in any σ-pure essential extension of M.

1.4.6 Lemma ([41], 7.2.3)

Let R be a ring and σ a torsion radical on R-mod. If $\{M_i\}_{i \in A}$ (where A is some suitable index set) is a chain of σ-pure essential extensions of a module L, then $M = \bigcup_i M_i$ is also a σ-pure essential extension of L.

1.4.7 Lemma ([64], Proposition 4.1)

Let R be a ring and σ a torsion radical on R-mod. Suppose M is a σ-pure essential extension of a submodule L. For each σ-pure injective module K and σ-pure monomorphism $\alpha : L \to K$, there is a monomorphism $\beta : M \to K$ such that the diagram



commutes, where i is the inclusion map.

1.4.8 Lemma ([24], Proposition 2)

Let R be a ring and σ a torsion radical on R-mod such that every $I \in \mathcal{L}_\sigma$ is f.g. The following are equivalent for an R-module module Q:

(i) Q is σ-pure injective.

(ii) If $\alpha : Q \to M$ is a σ-pure monomorphism then α has a left inverse.

(iii) Q has no non-trivial σ-pure essential extension.

(iv) Q is a maximal σ-pure essential extension of some submodule L.

1.4.9 Definition

Let R be a ring and σ a torsion radical on R-mod. A maximal σ-pure essential extension of a module M will be called a *σ-pure injective hull* of M.

1.4.10 Lemma ([57], 1.51)

Let R be a ring and σ a torsion radical on R-mod such that every $I \in \mathcal{L}_\sigma$ is f.g. Then every module may be embedded, as a σ-pure submodule, in a σ-pure injective module.

1.4.11 **Theorem** ([24], Proposition 3 and [64], Proposition 4.3)

Let R be a ring and σ a torsion radical on R-mod such that every $I \in \mathcal{L}_\sigma$ is f.g. Then every module M has a σ-pure-injective hull, which is unique up to an isomorphism that fixes M pointwise.

1.4.12 **Corollary** ([57], 1.18)

Let R be a ring and σ a torsion radical on R-mod such that every $I \in \mathcal{L}_\sigma$ is f.g. An exact sequence $0 \to K \xrightarrow{i} L$ is σ-pure exact iff every σ-pure injective module M is injective with respect to it.

1.4.13 **Remark**

Let R be a ring and σ a torsion radical on R-mod such that every $I \in \mathcal{L}_\sigma$ is f.g. If M is a σ-pure injective R-module and $L \leq M$ then M contains the σ-pure injective hull of L. In particular the σ-pure-injective hull of every module M is a submodule of its injective hull.

Proof

Follows easily from 1.4.6 and 1.4.7.

1.4.14 **Example**

Suppose R is σ-injective (as an R-module) for some torsion radical σ on R-mod. Then the injective and σ-pure injective hulls of R coincide. (For example, any self-injective ring R and any torsion radical σ on R-mod).

Proof

By [57], 1.7, 1.30 and 1.31, R is σ-pure in its injective hull, E(R). By 1.4.13, E(R) contains a maximal σ-pure essential extension S, (say), of R (which is the σ-pure injective hull of R). Since R is essential in E(R), E(R) is a σ-pure essential extension of R and therefore E(R) = S.

1.4.15 Theorem

Let R be a commutative ring and σ a torsion radical on R-mod such that every $I\epsilon\mathcal{L}_\sigma$ is f.g. If M is a σ-torsion free R-module, then the σ-pure injective hull of M is σ-torsion free.

Proof

Let P denote the σ-pure injective hull of M and let $S = \sigma(P) = \{x\epsilon P | Ix = 0 \text{ for some } I\epsilon\mathcal{L}_\sigma\}$ (1.2.3.5). $S \cap M = 0$ by assumption. Suppose $I\epsilon\mathcal{L}_\sigma$ and $I(p+S)\leq(M+S)/S$ for some $p\epsilon P$. Let I be generated by $a_1,a_2...a_n$. For each $i\epsilon\{1,2,...n\}$, there is an $m_i\epsilon M$ such that $a_i.p-m_i\epsilon S$. Thus there exist ideals $J_i\epsilon\mathcal{L}_\sigma$ such that $J_i(a_i.p-m_i) = 0$, $i = 1,2...n$.

$J = J_1.J_2 ...J_n\epsilon\mathcal{L}_\sigma$ (1.2.3.6(v)) and $J.I.p\leq M$. Since M is σ-pure in P and $J.I\epsilon\mathcal{L}_\sigma$, it follows from 1.3.3.4 that there is an $x\epsilon M$ with $J.I(p-x) = 0$. Hence $p-x\epsilon S$ and $p+S = x+S\epsilon(M+S)/S$. Hence, $(M+S)/S$ is σ-pure in P/S. Since P is a σ-pure essential extension of M, it follows that $S = 0$.

1.4.16 Remark

If R is a ring and σ a torsion radical on R-mod such that every module M can be embedded, as a σ-pure submodule, in a σ-pure injective module, then we say "(R,σ) *admits σ-pure injective hulls*" (This holds, in particular, if every $I\epsilon\mathcal{L}_\sigma$ is f.g., by 1.4.10).

§1.5 σ-PURE INJECTIVE DIMENSION AND GLOBAL DIMENSION

1.5.1 Definition

Let R be a ring and σ a torsion radical on R-mod.

Let $0\rightarrow M\xrightarrow{d_0}E_1\xrightarrow{d_1}E_2\rightarrow$ .... be a σ-pure exact sequence and suppose that each E_i is σ-pure injective.

This sequence is then called a *σ-pure injective resolution* of M. The smallest n for which imd_n is σ-pure injective (if it exists) will be called the *σ-pure injective dimension* (σpidM) of M.

1.5.2 Lemma

Let R be a ring and σ a torsion radical on R-mod such that every $I \in L_\sigma$ is f.g. Then every module M has a σ-pure injective resolution.

Proof

Follows easily from 1.4.11.

The proofs of the results which are listed below are standard and shall be omitted (see [13]).

Let R be a ring and σ a torsion radical on R-mod.

1.5.3

Suppose that in a σ-pure injective resolution $0 \to M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \to \ldots$ we have that imd_i is σ-pure injective for some i. Then imd_{i+j} is σ-pure injective for all $j \geq 1$.

1.5.4

σpidM is uniquely determined (i.e. it is independent of the particular σ-pure injective resolution used).

1.5.5

$\sigma pid(\prod_i M_i) = \inf\{\sigma pidM_i\}$ for any collection of modules $\{M_i\}_{i \in I}$.

1.5.6 Theorem

Let R be a ring and σ a torsion radical on R-mod. Then the following are equivalent for fixed R-modules A and B:

(i) Every σ-pure exact sequence of the form $0\to A\to M\to B\to 0$ splits.

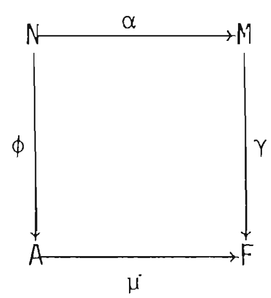
(ii) For any σ-pure exact sequence of the form $0\to N\to M\to B\to 0$, the induced sequence $0\to Hom(B,A)\to Hom(M,A)\to Hom(N,A)\to 0$ is exact.

(iii) For any σ-pure exact sequence of the form $0\to N\to M\to B\to 0$, where M is σ-pure projective, the sequence $0\to Hom(B,A)\to Hom(M,A)\to Hom(N,A)\to 0$ is exact.

Proof

(i) implies (ii): Let the sequence $0\to N\xrightarrow{\alpha}M\xrightarrow{\beta}B\to 0$ be σ-pure exact. We only need to prove exactness of $Hom(M,A)\to Hom(N,A)\to 0$. Let $\phi:N\to A$ be a homomorphism. We need to find a homomorphism $\psi:M\to A$ such that $\psi\cdot\alpha = \phi$.

Form the pushout diagram



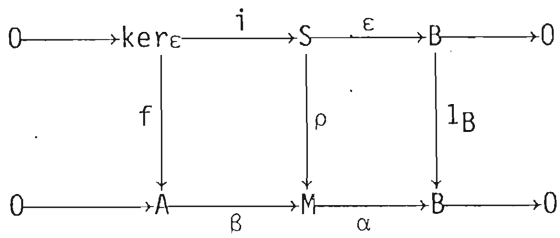
(where $F = (M\oplus A)/T$ for $T = \{(\alpha(n),-\phi(n))\,|\,n\in N\}$). Define $\theta:F\to B$ by $\theta((m,a)+T) = \beta(m)$ (for $m\in M$, $a\in A$).

It is routine to verify that θ is a well-defined R-homomorphism and that the sequence $0\to A\xrightarrow{\mu}F\xrightarrow{\theta}B\to 0$ is exact. By 1.4.3, this sequence is σ-pure exact. By (i), the sequence splits and hence there is a homomorphism $\delta:F\to A$ such that $\delta\cdot\mu = 1_A$. Define $\psi:M\to A$ by $\psi = \delta\cdot\gamma$, then $\psi\cdot\alpha = \phi$, proving (ii).

That (ii) implies (iii) is obvious.

(iii) implies (i): Let $0\to A\xrightarrow{\alpha}M\xrightarrow{\beta}B\to 0$ be a σ-pure exact sequence. (For simplicity we may take α to be the inclusion map). By 1.3.4.7, there is a σ-pure projective module S and a σ-pure exact sequence $0\to \ker\varepsilon\xrightarrow{i}S\xrightarrow{\varepsilon}B\to 0$. Hence there is a homomorphism $\rho:S\to M$ such that $\beta\cdot\rho = \varepsilon$.

Complete the commutative diagram



By (iii), there is a homomorphism $\mu:S\to A$ such that $\mu\cdot i = f$. By 1.3.1.2, there is a homomorphism $\mu:B\to M$ such that $\alpha\cdot\mu = 1_B$. This proves (i).

1.5.7 Theorem

Let R be a ring and σ a torsion radical on R-mod such that R-mod admits σ-pure injective hulls. Then the following are equivalent for modules A and B:

(i) Any σ-pure exact sequence of the form $0\to A\to K\to B\to 0$ splits.

(ii) For any σ-pure exact sequence of the form $0\to A\to N\to M\to 0$, the induced sequence $0\to \mathrm{Hom}(B,A)\to \mathrm{Hom}(B,N)\to \mathrm{Hom}(B,M)\to 0$ is exact.

(iii) For any σ-pure exact sequence of the form $0\to A\to N\to M\to 0$, where N is σ-pure injective, the induced sequence $0\to \mathrm{Hom}(B,A)\to \mathrm{Hom}(B,N)\to \mathrm{Hom}(B,M)\to 0$ is exact.

Proof

Dual to 1.5.6

1.5.8 Definition

Let R be a ring and σ a torsion radical on R-mod. Let A,B be any two R-modules. We call A and B σ-*pure projectively equivalent* iff there exist σ-pure projective modules P_1, P_2 such that $A\oplus P_1 \approx B\oplus P_2$. (We then write $A \underset{\sigma}{\sim} B$).

1.5.9 **Remark**

It is clear that $\underset{\sigma}{\sim}$ is an equivalence relation. The equivalence class of a module M under $\underset{\sigma}{\sim}$ will be denoted by $[M]_\sigma$.

1.5.10 **Lemma**

Let R be a ring and σ a torsion radical on R-mod. If $0\to K\xrightarrow{\alpha_1}P\xrightarrow{\beta_1}A\to 0$ and $0\to K'\xrightarrow{\alpha_2}P'\xrightarrow{\beta_2}A'\to 0$ are two σ-pure exact sequences, where P and P' are σ-pure projective and $A\underset{\sigma}{\sim}A'$ then $K\underset{\sigma}{\sim}K'$.

Proof

By assumption, there are σ-pure projective modules S and S' such that $A\oplus S\simeq A'\oplus S'$. Construct the exact sequences $0\to K\xrightarrow{\bar{\alpha}_1}P\oplus S\xrightarrow{\bar{\beta}_1}A\oplus S\to 0$ and $0\to K'\xrightarrow{\bar{\alpha}_2}P'\oplus S'\xrightarrow{\bar{\beta}_2}A'\oplus S'\to 0$ where $\bar{\alpha}_1(k) = (\alpha_1(k),0)$ and $\bar{\alpha}_2(k') = (\alpha_2(k'),0)$, for all $k\in K$, $k'\in K'$, $\bar{\beta}_1(p,s) = (\beta_1(p),s)$ for all $(p,s)\in P\oplus S$ and $\bar{\beta}_2(p',s') = (\beta_2(p'),s')$ for all $(p',s')\in P'\oplus S'$. (Exactness is routinely verified).

Let $I\in L_\sigma$ and $(p,s)\in P\oplus S$ such that $I(p,s)\leq\bar{\alpha}_1(K)$. Then $Is = 0$ and $Ip\leq\alpha_1(K)$. Since $\alpha_1(K)$ is σ-pure in P there is, by 1.3.3.4, an element $k\in K$ such that $I(p-\alpha_1(k)) = 0$. Then $I((p,s)-\bar{\alpha}_1(k)) = I((p,s)-(\alpha_1(k),0)) = 0$ and therefore $\bar{\alpha}_1(K)$ is σ-pure in $P\oplus S$. Similarly, the second sequence is σ-pure exact.

Let $\psi:A'\oplus S'\to A\oplus S$ be the given isomorphism. Then the sequences $0\to K\xrightarrow{\bar{\alpha}_1}P\oplus S\xrightarrow{\bar{\beta}_1}A\oplus S\to 0$ and $0\to K'\xrightarrow{\bar{\alpha}_2}P'\oplus S'\xrightarrow{\psi\cdot\bar{\beta}_2}A\oplus S\to 0$ are σ-pure exact. Thus there are homomorphisms $\alpha:P\oplus S\to P'\oplus S'$ and $\alpha':P'\oplus S'\to P\oplus S$ such that $\psi\cdot\bar{\beta}_2\cdot\alpha = \bar{\beta}_1$ and $\bar{\beta}_1\cdot\alpha' = \psi\cdot\bar{\beta}_2$. Let $\beta = 1_{P\oplus S}-\alpha'\cdot\alpha$ and $\beta' = 1_{P'\oplus S'}-\alpha\cdot\alpha'$. It is easy to see that $\mathrm{im}\beta\leq\bar{\alpha}_1(K)$ and $\mathrm{im}\beta'\leq\bar{\alpha}_2(K')$. Define, further, $\gamma:P\oplus S\oplus\bar{\alpha}_2(K')\to P'\oplus S'\oplus\bar{\alpha}_1(K)$ by $\gamma((p,s),\bar{\alpha}_2(k')) = (\alpha(p,s)+\bar{\alpha}_2(k'),\beta(p,s)-\alpha'\bar{\alpha}_2(k'))$ and $\gamma':P'\oplus S'\oplus\bar{\alpha}_1(K)\to P\oplus S\oplus\bar{\alpha}_2(K')$ by $\gamma'((p',s'),\bar{\alpha}_1(k)) = (\alpha'(p',s')+\bar{\alpha}_1(k), \beta'(p',s')-\alpha\cdot\bar{\alpha}_1(k))$.

It is then routine to verify that γ and γ' are well-defined R-homomorphisms, which are mutual inverses. Hence it follows that $P'\oplus S'\oplus K \simeq P\oplus S\oplus K'$ and therefore $K \underset{\sigma}{\sim} K'$, as required.

1.5.11 Definition

Let R be a ring and σ a torsion radical on R-mod. If the sequence $0\to K\to P\to A\to 0$ is σ-pure exact and P is σ-pure projective, then we denote the σ-pure projective equivalence class of K by $P_\sigma(A)$ (P_σ is well-defined by 1.5.10).

1.5.12 Definition

Let R be a ring and σ a torsion radical on R-mod. Let A be an R-module. We define $P_\sigma{}^{o}(A) = A$ and $P_\sigma{}^{n}(A) = P_\sigma(X)$ where $X\in[P_\sigma{}^{n-1}(A)]_\sigma$, n = 1,2,......

1.5.13 Remarks

(1) Given a module A, $P_\sigma{}^{n}(A)$ is the σ-pure projective equivalence class of $\ker\delta_n$ in any σ-pure projective resolution $\ldots\to S_i\xrightarrow{\delta_i}S_{i-1}\to\ldots\ldots\to S_1\xrightarrow{\delta_1}A\to 0$ for A, $n\geq 1$.

(2) In such a σ-pure projective resolution, $\ker\delta_n$ is σ-pure projective iff every element of $P_\sigma{}^{n}(A)$ is σ-pure projective.

(3) It follows therefore that σppdA is the smallest natural number n such that every element of $P_\sigma{}^{n}(A)$ is σ-pure projective.

1.5.14 Definition

Let R be a ring and σ a torsion radical on R-mod. Let A,B be R-modules. We call A and B σ-*pure injectively equivalent* iff there exist σ-pure injective modules I_1 and I_2 such that $A\oplus I_1\simeq B\oplus I_2$. If $0\to A\to E\to K\to 0$ is σ-pure exact and E is σ-pure injective, we denote by $I_\sigma(A)$ the σ-pure injective equivalence class of K and by $I_\sigma{}^{n}(A)$ the σ-pure injective equivalence class of imd_n in a σ-pure injective resolution $0 \to A \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \ldots$ of A.

1.5.15 Remark

Dual versions of 1.5.10, 1.5.11, 1.5.12 and 1.5.13 can be formulated.

1.5.16 Lemma

Let R be a ring, σ a torsion radical on R-mod and let B and C be R-modules such that every σ-pure exact sequence of the form $0 \to C \to X \to B \to 0$ splits. If D is σ-pure injectively equivalent to C, then every σ-pure-exact sequence of the form $0 \to D \to Y \to B \to 0$ splits.

Proof

Suppose I_1 and I_2 are σ-pure injective modules such that $D \oplus I_1 \simeq C \oplus I_2$. Let $0 \to A \xrightarrow{\alpha} Z \xrightarrow{\beta} B \to 0$ be σ-pure exact and $f: A \to C \oplus I_2$ a homomorphism. Let π_1, π_2 be the projection maps from $C \oplus I_2$ onto C and I_2 respectively.

By our assumption, every σ-pure exact sequence of the form $0 \to C \to X \to B \to 0$ splits and, by 1.5.6, there is a homomorphism $g_1: Z \to C$ such that $g_1 \cdot \alpha = \pi_1 \cdot f$. Since I_2 is σ-pure injective, there is a homomorphism $g_2: Z \to I_2$ such that $g_2 \cdot \alpha = \pi_2 \cdot f$. Define $g: Z \to C \oplus I_2$ by $g(z) = (g_1(z), g_2(z))$.

Then if $a \in A$, $(g \cdot \alpha)(a) = (g_1(\alpha(a)), g_2(\alpha(a))) = ((\pi_1 \cdot f)(a), (\pi_2 \cdot f)(a))$ $= f(a)$. Hence $\operatorname{Hom}(Z, C \oplus I_2) \to \operatorname{Hom}(A, C \oplus I_2) \to 0$ is exact and therefore $\operatorname{Hom}(Z, D \oplus I_1) \to \operatorname{Hom}(A, D \oplus I_1) \to 0$ is exact. It now follows easily that $\operatorname{Hom}(Z, D) \to \operatorname{Hom}(A, D) \to 0$ is exact and, by 1.5.6, every σ-pure exact sequence of the form $0 \to D \to Y \to B \to 0$ splits.

1.5.17 Lemma

Let R be a ring and σ a torsion radical on R-mod such that R-mod admits σ-pure injective hulls. Then the following are equivalent for R-modules A and B:

(i) Every σ-pure exact sequence of the form $0\to A\to M\to C\to 0$, where $C\in\mathcal{P}_\sigma(B)$, splits.

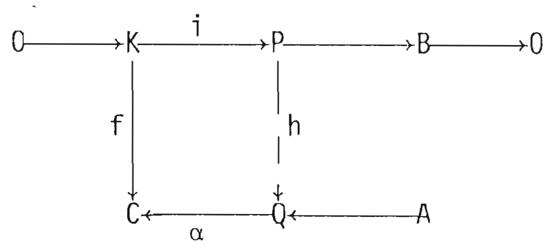
(ii) Every σ-pure exact sequence of the form $0\to D\to N\to B\to 0$, where $D\in\mathcal{I}_\sigma(A)$, splits.

Proof

We prove only that (i) implies (ii) since the converse is dual.

Suppose (i) holds and choose σ-pure exact sequences $0\to K\xrightarrow{i}P\to B\to 0$ and $0\to A\to Q\xrightarrow{\alpha}C\to 0$ where P is σ-pure projective and Q is σ-pure injective (see 1.3.4.7).

Let $f:K\to C$ be a homomorphism and consider the diagram



Then $K\in P_\sigma(B)$ and, by (i), every σ-pure exact sequence of the form $0\to A\to M\to K\to 0$ splits. By 1.5.7, there is a homomorphism $g:K\to Q$ such that $\alpha\cdot g = f$.

Since Q is σ-pure injective there is a homomorphism $h:P\to Q$ such that $h\cdot i = g$. Hence $\phi = \alpha\cdot h:P\to C$ satisfies $\phi\cdot i = \alpha\cdot g = f$ and we have therefore shown that $0\to \mathrm{Hom}(B,C)\to\mathrm{Hom}(P,C)\to\mathrm{Hom}(K,C)\to 0$ is exact for any σ-pure exact sequence $0\to K\to P\to B\to 0$, where P is σ-pure projective. By 1.5.6, it follows that every σ-pure exact sequence of the form $0\to C\to X\to B\to 0$ will split. Further, $C\in\mathcal{I}_\sigma(A)$ since $0\to A\to Q\to C\to 0$ is σ-pure exact with Q σ-pure injective, and from 1.5.16 it follows that (ii) holds.

1.5.18 Remark

A simple inductive argument will show that 1.5.17 holds when $\mathcal{P}_\sigma(B)$ and $\mathcal{I}_\sigma(A)$ are replaced by $\mathcal{P}_\sigma{}^n(B)$ and $\mathcal{I}_\sigma{}^n(A)$ respectively, $n\geq 1$.

1.5.19 Definition

Let R be a ring and σ a torsion radical on R-mod. We define global dimensions $g\sigma pidR$ = sup{σpidM|M is a (left) module} and $g\sigma ppd$ = sup{σppdM|M is a (left) R-module}. (If either of these suprema does not exist we set the global dimension equal to ∞).

1.5.20 Theorem

Let R be a ring and suppose that R-mod admits σ-pure injective hulls (for a torsion radical σ on R-mod). Then gσpidR = gσppdR.

Proof

Suppose gσppdR = n. We show that gσpidR$\leq$n (the converse is dual). We may assume n<∞. Let A and B be arbitrary R-modules. Then σppdB$\leq$n and hence $P_\sigma{}^n(B)$ consists of the class of all σ-pure projective modules (see 1.3.4.11(i) and 1.5.13). Hence every σ-pure exact sequence of the form $0\to A\to M\to C\to 0$, where $C\in P_\sigma{}^n(B)$, splits and by 1.5.18 every σ-pure exact sequence of the form $0\to D\to N\to B\to 0$ where $D\in I_\sigma{}^n(A)$ splits.

Since R-mod admits σ-pure injective hulls it follows that every element of $I_\sigma{}^n(A)$ is σ-pure injective, for all modules A. That is, in any σ-pure injective resolution $0\to A\xrightarrow{d_0}E_1\xrightarrow{d_1}E_2\to\ldots.$ (of an arbitrary module A), we have that $\text{im}d_n$ is σ-pure injective and therefore gσpidR$\leq$n.

1.5.21 Remarks

Let R be a ring and σ a torsion radical on R-mod such that (R,σ) admits σ-pure injective hulls. Then

(a) The following are equivalent:

(i) gσpidR = 0 (= gσppdR).

(ii) Every σ-pure submodule of every R-module is a direct summand.

(iii) Every R-module is σ-pure projective.

(iv) Every R-module is σ-pure injective.

(b) The following are equivalent:

(i) $g\sigma pidR \leqq 1$.

(ii) Epimorphic images of σ-pure injective R-modules are σ-pure injective.

(iii) Submodules of σ-pure projective modules are σ-pure projective.

1.5.22 **Example**

Let R be any uniserial ring and let L_σ consist of all the left ideals of R. Then any left R-module M is a direct sum of cyclic, (σ-torsion), modules ([52]) and hence is σ-pure projective. Thus $g\sigma ppdR = 0$.

Note, further, this example shows that $g\sigma ppdR$ is not necessarily equal to the global projective dimension of R, since, not every uniserial ring is semisimple (see [21]).

CHAPTER TWO

RELATIVE INJECTIVITY

§2.1 INTRODUCTION

Maddox ([55]) and Megibben ([56]) have studied the concept of absolute purity for Cohn purity. In the first part of this chapter we study absolutely σ-pure modules, which are the same as the σ-injective modules of Goldman ([37]), from the point of view of purity.

An exact sequence $0 \to M \to Q_0 \xrightarrow{d_0} Q_1 \xrightarrow{d_1} \ldots$ where each Q_i is injective is called an *injective resolution.* The smallest n for which $\operatorname{im} d_{n-1}$ is injective (if it exists) is uniquely determined and is called the *injective dimension* of M, *injM* ([49]). Requiring instead that $\operatorname{im} d_{n-1}$ be σ-injective, we obtain a new dimension, called the *absolutely σ-pure dimension*, $ad_\sigma(M)$ of M. We show that $ad_\sigma(M) =$ $\inf\{n \geq 0 \mid \operatorname{Ext}^{n-1}(X,M) = 0$ for all (cyclic,) σ-torsion (left) R-modules $X\}$ and study the corresponding global dimension briefly.

We also introduce semi-σ-injective modules, defined by suitably restricting conditions pertaining to σ-injective modules. These are characterized in various ways and we investigate the rings R for which every R-module is semi-σ-injective. As a result, a Quasi-Frobenius ring R (together with any torsion radical σ on R-mod) is shown to have certain equivalent properties, related to σ-injectivity.

§2.2 ABSOLUTELY σ-PURE MODULES:

2.2.1 Definition

A module M is called *absolutely σ-pure* iff M is σ-pure in every module M' containing M as a submodule.

The following theorem shows that absolute σ-purity is equivalent to σ-injectivity and also to the absolute σ-purity of Golan ([34]).

2.2.2 **Theorem (c f. [57], 1.7, 1.30, and 1.31; [34], Proposition 4.1)**

Let R be a ring and σ a torsion radical on R-mod. Then the following are equivalent for a module M:

(i) M is absolutely σ-pure.

(ii) M is σ-pure in any injective module containing M.

(iii) M is σ-pure in E(M).

(iv) M is σ-injective.

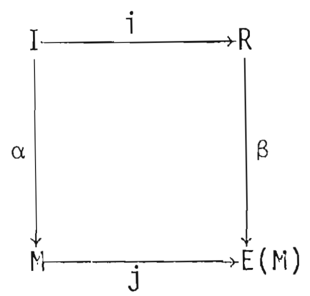
(v) $\sigma(E(M)/M) = 0$.

(vi) M is σ-neat in any module containing M.

Proof:

The facts that (i) implies (ii) and (ii) implies (iii) are obvious.

(iii) implies (iv): Suppose (iii) holds, let $I \in \mathcal{L}_\sigma$ and suppose that $\alpha: I \to M$ is a homomorphism. Complete the diagram



commutatively using the injectivity of E(M) (where i,j are the inclusion maps). From 1.3.4.2(i) we have, since M is σ-pure in E(M), that there is a homomorphism $\theta: R \to M$ such that $\theta \cdot i = \alpha$. Thus M is σ-injective.

(iv) implies (v): $E_\sigma(M)/M = \sigma(E(M)/M)$ by 1.2.4.4. But, by (iv), $M = E_\sigma(M)$, and therefore (v) follows.

(v) implies (i): Suppose (v) holds and M' is any module containing M. Since E(M)/M is σ-torsion free, Hom $(R/I, E(M)/M) = 0$ for all $I \in \mathcal{L}_\sigma$ (1.2.3.4). Hence M is σ-pure in E(M), vacuously.

$E(M)$ is a direct summand of $E(M')$ and therefore is σ-pure in $E(M')$. By 1.3.1.5(i), M is σ-pure in $E(M')$ and hence (by 1.3.1.5(ii)), M is σ-pure in M' as required to prove (i).

If (i) holds then (vi) follows from Proposition 7 of [44]. Conversely, if (vi) holds, then M is σ-neat in its injective hull and, by Proposition 2 (3) of [44], M is σ-injective. This completes the proof.

A proof in terms of σ-purity simplifies the following result considerably:

2.2.3 Remark

The class of absolutely σ-pure modules is closed under injective hulls and (module) extensions ([35], Proposition 8.4).

Proof:

Since any injective module is obviously absolutely σ-pure, by 2.2.2, closure under injective hulls is clear.

For closure under extensions, suppose that N is a submodule of a module M and that both N and M/N are absolutely σ-pure. Let M' be any module containing M. Then M/N is σ-pure in M'/N and N is σ-pure in M'. By 1.3.1.5(iv), M is σ-pure in M'. Thus M is absolutely σ-pure.

2.2.4 Definition ([55])

A module M is called *absolutely pure* iff it is Cohn pure in every module containing it as a submodule.

2.2.5 **Remark**

Let R be a ring and σ a torsion radical on R-mod such that every $I\epsilon L_\sigma$ is f.g. Then an absolutely pure module is σ-injective.

Proof:

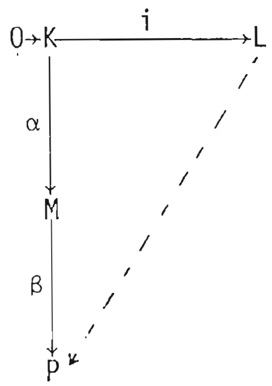
Follows easily from 1.3.6.3 and 2.2.2.

§2.3 A FURTHER CHARACTERIZATION OF σ-INJECTIVITY

2.3.1 **Theorem**

Let R be a ring and σ a torsion radical on R-mod such that (R,σ) admits σ-pure injective hulls. Then the following are equivalent for a module M:

(i) Every diagram of the form



where K is an arbitrary submodule of the module L, i is the inclusion map and P is σ-pure injective, can be completed commutatively as indicated.

(ii) M is σ-injective.

(iii) There is a σ-pure exact sequence $0 \to M \xrightarrow{i} I \xrightarrow{\pi} I/M \to 0$, in which I is σ-injective.

(iv) There is a σ-pure exact sequence $0 \to M \xrightarrow{i} I \xrightarrow{\pi} I/M \to 0$, in which I is injective.

(v) Every homomorphism $\alpha: M \to P$, where P is σ-pure injective, factors through an injective module.

Proof

(i) implies (ii): Suppose (i) holds, let $A \in L_\sigma$ and let $\alpha : A \to M$ be a homomorphism.

Let $P(M)$ be a σ-pure injective module containing M as a σ-pure submodule and let $j : M \rightarrowtail P(M)$ be the inclusion map. If $i : A \to R$ is the inclusion map, there is, by (i), a homomorphism $\beta : R \to P(M)$ such that $\beta \cdot i = j \cdot \alpha$.

Hence the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad i \quad} & R \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ M & \xrightarrow[\quad j \quad]{} & P(M) \end{array}$$

commutes and, since M is σ-pure in $P(M)$, there is a homomorphism $\phi : R \to M$ such that $\phi \cdot i = \alpha$ (1.3.4.2(i)). Thus M is σ-injective, as required.

That (ii) implies (iii) is trivial (take $I = M$).

(iii) implies (iv): Suppose that there is a σ-injective module I and a σ-pure exact sequence $0 \to M \xrightarrow{i} I \xrightarrow{\pi} I/M \to 0$. Let $E(M)$ be the injective hull of M, suppose that $A \in L_\sigma$ and that the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad k \quad} & R \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ M & \xrightarrow[\quad j \quad]{} & E(M) \end{array}$$

is given, where k, j are the inclusion maps. Since I is σ-injective there is a homomorphism $\phi : R \to I$ such that $\phi \cdot k = i \cdot \alpha$.

Thus the diagram

$$\begin{array}{ccc} A & \xrightarrow{k} & R \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\phi} \\ M & \xrightarrow[i]{} & I \end{array}$$

commutes and since M is σ-pure in I, there is a homomorphism $\mu:R\to M$ such that $\mu\cdot k = \alpha$. Referring back to the previous diagram, we see that this means that M is σ-pure in $E(M)$ (1.3.4.2(i)). Thus $0\to M\to E(M)\to E(M)/M\to 0$ is σ-pure exact, proving (iv).

(iv) implies (v): Suppose that (iv) holds and let $\alpha:M\to P$ be a homomorphism, where P is σ-pure-injective. Let I be an injective module containing M as a σ-pure submodule. If $i:M\to I$ is the inclusion map there is, by σ-pure injectivity of P, a homomorphism $\mu:I\to P$ such that $\mu\cdot i = \alpha$. Thus α factors through I and (v) holds.

(v) implies (i): Suppose (v) holds and K is an arbitrary submodule of a module L in the diagram

$$\begin{array}{ccc} K & \xrightarrow{i} & L \\ {\scriptstyle\alpha}\downarrow & & \\ M & & \\ {\scriptstyle\beta}\downarrow & & \\ P & & \end{array}$$

where i is the inclusion map and P is σ-pure injective. By assumption, there is an injective module I and homomorphisms $\mu:M\to I$ and $\Theta:I\to P$ such that $\beta = \Theta\cdot\mu$. By injectivity of I, there is a homomorphism $\phi:L\to I$ such that $\phi\cdot i = \mu\cdot\alpha$. Then $\Psi = \Theta\cdot\phi:L\to P$ satisfies $\Psi\cdot i = \Theta\cdot\phi\cdot i = \Theta\cdot\mu\cdot\alpha = \beta\cdot\alpha$. That is, (i) holds and the theorem is proved.

§2.4 ABSOLUTE σ-PURITY, EXT AND THE TENSOR PRODUCT

Let R be a ring. For any left R-module M we denote by M^* the *character module* $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$. M^* is then a right R-module under $(\mu r)(m) = \mu(rm)$ for $\mu\in M^*$, $r\in R$ and $m\in M$. It also follows that $M^* = 0$ iff $M = 0$ (see [54]). A famous result of Lambek ([54], page 239) states (essentially) that M^* is injective iff for every exact sequence $0\to N_1\to N_2$ the sequence $0\to N_1\otimes M\to N_2\otimes M$ is exact. If σ is a torsion radical on R-mod, we show that a similar relationship exists between σ-injectivity and the tensor product (2.4.2).

2.4.1 Lemma

Let R be a ring and σ a torsion radical on R-mod. Then a module M is σ-injective iff $\mathrm{Ext}^1(R/L,M) = 0$ for all $L\in\mathcal{L}_\sigma$ (see [66], page 29).

2.4.2 Theorem

Let R be a commutative ring and σ a torsion radical on R-mod. For a left R-module M, M^* is σ-injective iff $0\to L\otimes M\to R\otimes M$ is exact for all $L\in\mathcal{L}_\sigma$.

Proof

M^* is σ-injective iff $\mathrm{Ext}^1(R/L,M^*) = 0$ for all $L\in\mathcal{L}_\sigma$ (2.4.1) iff $[\mathrm{Tor}_1(R/L,M)]^* = 0$ (see §1.2.1.7) iff $\mathrm{Tor}_1(R/L,M) = 0$ for all $L\in\mathcal{L}_\sigma$. Hence the result follows from the exactness of the sequence $\mathrm{Tor}_1(R/L,M)\to L\otimes M\to R\otimes M$.

2.4.3 Remarks

1. If we take $\mathcal{L}_\sigma$ to be the set of all ideals of R in 2.4.2 then we get the following special case: M is flat iff M^* is injective

([54]) iff M^* is σ-injective iff $0\to L\otimes M\to R\otimes M$ is exact for all ideals L of R. This is just Theorem 14.6 of [45].

2. Suppose the class of σ-injective modules is closed under homomorphic images (e.g. if every $I\in\mathcal{L}_\sigma$ is projective - see 1.2.4.6(ii))

Then the following are equivalent:

(i) M is σ-injective.

(ii) $\mathrm{Ext}^n(S,M) = 0$ for all σ-torsion modules S and n = 1,2.

(iii) $\mathrm{Ext}^n(S,M) = 0$ for all cyclic, σ-torsion modules S and n = 1,2.

Proof:

Suppose (i) holds and let S be a σ-torsion module.

The exact sequence $0\to M\xrightarrow{i}E(M)\xrightarrow{\pi}E(M)/M\to 0$ induces an exact sequence $\mathrm{Hom}(S,E(M)/M)\to\mathrm{Ext}^1(S,M)\to\mathrm{Ext}^1(S,E(M))$. Since E(M) is injective, $\mathrm{Ext}^1(S,E(M)) = 0$. By (i), E(M)/M is σ-torsion free (2.2.2) and hence $\mathrm{Hom}(S,E(M)/M) = 0$. The exactness of the above sequence thus implies that $\mathrm{Ext}^1(S,M) = 0$.

We have therefore shown that $\mathrm{Ext}^1(S,M) = 0$ for all σ-torsion modules S, whenever M is σ-injective.

Suppose now that the class of σ-injective modules is closed under epimorphic images. Then, E(M)/M is σ-injective and, by the above, $\mathrm{Ext}^1(S,E(M)/M) = 0$ for all σ-torsion modules S.

Let S be a σ-torsion module. In the exact sequence $\mathrm{Ext}^1(S,E(M))\to\mathrm{Ext}^1(S,E(M)/M)\to\mathrm{Ext}^2(S,M)\to\mathrm{Ext}^2(S,E(M))$, the fact that E(M) is injective forces $\mathrm{Ext}^2(S,E(M)) = \mathrm{Ext}^1(S,E(M)) = 0$. Thus $\mathrm{Ext}^1(S,E(M)/M)\simeq\mathrm{Ext}^2(S,M)$ and hence $\mathrm{Ext}^2(S,M) = 0$. Thus (i) implies (ii).

That (ii) implies (iii) is obvious.

The fact that (iii) implies (i) is in Lemma 2.4.1.

§2.5 ABSOLUTE σ-PURE DIMENSION

2.5.1 In [27] D.J. Fieldhouse defines an *absolutely pure dimension* by apd M = $\inf\{n\geq 0 \mid Ext^{n+1}(X,M) = 0$ for all f.p. modules $X\}$. The f.p. modules are the modules in the third nonzero position of the co-σ-pure exact sequences for Cohn purity, corresponding to the cyclic, σ-torsion modules for σ-purity. It is therefore natural to define an *absolutely σ-pure dimension* as follows (where σ is a torsion radical):

$ad_\sigma(M) = \inf\{n\geq 0 \mid Ext^{n+1}(X,M) = 0$ for all cyclic, σ-torsion $X\}$. We define further $ad^*_\sigma(M) = \inf\{n\geq 0 \mid Ext^{n+1}(X,M) = 0$ for all σ-torsion $X\}$. (If no such n exists we define $ad_\sigma(M) = \infty$ or $ad^*_\sigma(M) = \infty$).

Fieldhouse also defines a *weak injective dimension* w-inj M = $\inf\{n\geq 0 \mid Ext^{n+1}(X,M) = 0$ for all cyclic, f.p.$X\}$. It follows that for all M, w-inj M$\leq$inj M where inj M = $\inf\{n\geq 0 \mid Ext^{n+1}(X,M) = 0$ for all left R-modules $X\}$ is the well known *injective dimension* of M (see [49]).

2.5.2 **Remarks**

(i) $ad_\sigma(M) = 0$ iff $ad^*_\sigma(M) = 0$ iff M is σ-injective.

(ii) $ad_\sigma(M)\leq ad^*_\sigma(M)\leq$inj M for all M.

(iii) If every $I\epsilon\mathcal{L}_\sigma$ is f.g., then $ad_\sigma(M)\leq$apd M and $ad_\sigma(M)\leq$w-inj M, for all M. Thus $ad_\sigma(M)$ is seen to be not greater than many of the well known dimensions .

(iv) From (i) above, it follows that M is σ-injective iff M is strongly σ-pure in its injective hull.

Proof

(i) $ad^*_\sigma(M) = 0$ iff $Ext^1(S,M) = 0$ for all σ-torsion S iff $Ext^1(S,M) = 0$ for all cyclic, σ-torsion S (see the proof of 2.4.3(2)) iff M is σ-injective (2.4.1).

(ii) If $ad_\sigma^*(M) = n$, then $Ext^{n+1}(S,M) = 0$ for all σ-torsion modules S and hence $Ext^{n+1}(S,M) = 0$ for all cyclic, σ-torsion modules S and therefore $ad_\sigma(M) \leq n$.

Similarly $ad_\sigma^*(M) \leq \text{inj } M$.

(iii) If every $I \in L_\sigma$ is f.g., then every cyclic, σ-torsion module is f.p., cyclic and therefore, as before, we get that $ad_\sigma(M) \leq \text{apd } M$ and $ad_\sigma(M) \leq \text{w-inj } M$.

(iv) If M is strongly σ-pure in E(M), then it follows from 2.2.2 that M is σ-injective.

Conversely, suppose M is σ-injective. By (i), $ad_\sigma^*(M) = 0$ and hence $Ext^1(S,M) = 0$ for all σ-torsion modules S.

The exact sequence $0 \to M \to E(M) \xrightarrow{\pi} E(M)/M \to 0$ induces an exact sequence: $\ldots \to Hom(S,E(M)) \xrightarrow{\theta} Hom(S,E(M)/M) \to Ext^1(S,M) = 0$, where $\theta(\phi) = \pi \cdot \phi$ for $\phi \in Hom(S,E(M))$. Thus if S is σ-torsion, θ is epic and therefore M is strongly σ-pure in E(M).

2.5.3 **Theorem**

Let R be a ring and σ a torsion radical on R-mod. Then the following are equivalent for any R-module M:

(i) In any injective resolution $0 \to M \to Q_0 \xrightarrow{d_0} Q_1 \xrightarrow{d_1} \ldots$ of M, im d_{n-1} is σ-injective.

(ii) There exists an injective resolution, as above, in which im d_{n-1} is σ-injective.

(iii) $Ext^{n+1}(X,M) = 0$ for all σ-torsion modules X.

(iv) $Ext^{n+1}(X,M) = 0$ for all cyclic, σ-torsion modules X.

Proof

That (i) implies (ii) is clear.

(ii) implies (iii): Suppose that (ii) holds and let $0 \to M \to Q_0 \xrightarrow{d_0} Q_1 \xrightarrow{d_1} \ldots \to Q_{n-1} \to K_n \to 0$ be an exact sequence, where Q_i is injective, $i = 0,1,\ldots n-1$ and $K_n = \text{im } d_{n-1}$ is σ-injective. By The Shifting Theorem for Injectives, ([49], page 54), we have that $\text{Ext}^{n+1}(X,M) = \text{Ext}^1(X,K_n)$ for all modules X. If X is σ-torsion then $\text{Ext}^1(X,K_n) = 0$ (see 2.4.3(2)) and therefore (iii) follows.

That (iii) implies (iv) is clear.

(iv) implies (i): Suppose that (iv) holds and let $0 \to M \to Q_0 \xrightarrow{d_0} Q_1 \to \ldots$ be any injective resolution of M. By another application of The Shifting Theorem for Injectives we have that, for any module X, $\text{Ext}^{n+1}(X,M) = \text{Ext}^1(X, \text{im } d_{n-1})$. Hence it follows from (iv) that $\text{Ext}^1(X, \text{im } d_{n-1}) = 0$ for all cyclic, σ-torsion modules X and hence $\text{im } d_{n-1}$ is σ-injective (2.4.1), as required to prove (i).

2.5.4 Corollary 1

$\text{ad}_\sigma(M)$ (= $\text{ad}^*_\sigma(M)$) is the (uniquely determined) least positive integer n such that in any injective resolution $0 \to M \to Q_0 \xrightarrow{d_0} \ldots$ of M, $\text{im } d_{n-1}$ is σ-injective.

2.5.5 Corollary 2:

Let R be a ring and σ a torsion radical on R-mod. Suppose, further, that $0 \to K \xrightarrow{\alpha} L \xrightarrow{\beta} M \to 0$ is an exact sequence, where L is injective.

(i) If K is not σ-injective, then $\text{ad}_\sigma(K) = \text{ad}_\sigma(M)+1$.

(ii) If M is σ-torsion, but not σ-injective, then $\text{ad}_\sigma(K) = \text{ad}_\sigma(M)+1$.

(iii) $\text{ad}_\sigma(K)$ is finite iff $\text{ad}_\sigma(M)$ is finite.

Proof

(i) Let $0 \to M \xrightarrow{w} Q_0 \xrightarrow{d_0} Q_1 \to \ldots$ be any injective resolution of M. Then $\ker d_0 = \text{im } w \simeq M = \text{im } \beta \simeq \text{im } w \cdot \beta$. Further $\ker w \cdot \beta = \ker \beta = \text{im } \alpha$

and hence the sequence $0 \to K \xrightarrow{\alpha} L \xrightarrow{\omega \cdot \beta} Q_0 \xrightarrow{d_0} Q_1 \to \ldots$ is an injective resolution of K.

Clearly (i) now follows from 2.5.4.

(ii) If $ad_\sigma(K) = 0$, the sequence splits by 1.2.4.6 (v). This would be contrary to the assumption that M is not σ-injective. Thus K is not σ-injective and the result follows from (i).

(iii) By The Shifting Theorem for Injectives $Ext^{n+1}(X,K) = Ext^n(X,M)$ for all left R-modules X.

2.5.6 Definitions

Let R be a ring and σ a torsion radical on R-mod. We define the *global absolutely σ-pure dimension* $AD_\sigma(R) = \sup\{ad_\sigma(M) \mid M$ is a left R-module$\}$. (If no supremum exists we write $AD_\sigma(R) = \infty$). We also define a ring R to be (left) *σ-regular* iff every submodule of every (left) R-module is a σ-pure submodule.

2.5.7 Note

Let R be a ring. In the Dimension Theorem ([49], page 48) we find the following: If M is a (left) R-module then the projective dimension of M is given by pr M = $\inf\{n \geq 0 \mid Ext^{n+1}(M,X) = 0$ for all left R-modules X$\}$.

2.5.8 Theorem

Let R be a ring and σ a torsion radical on R-mod. Then

(i) $AD_\sigma(R) = 0$ iff R is σ-regular iff every R-module is σ-injective.

(ii) $AD_\sigma(R) \leq \sup\{$pr S$\mid$S is a cyclic, σ-torsion R-module$\}$.

(iii) If $AD_\sigma(R) \neq 0$, then $AD_\sigma(R) = 1 + \sup\{ad_\sigma(E(M)/M) \mid M$ is not σ-injective$\}$.

(iv) $AD_\sigma(R) \leq 1$ iff E(M)/M is σ-injective for all modules M such that M is not σ-injective. (This is true, for example, if every $I \in \mathcal{L}_\sigma$ is projective - see 1.2.4.6 (ii)).

Proof

(i) R is σ-regular iff every exact sequence of R-modules $0\to N\to M\to M/N\to 0$ is σ-pure exact iff every R-module N is absolutely σ-pure iff $ad_\sigma N = 0$ for all (left) R-modules N (2.2.2) iff $AD_\sigma(R) = 0$.

(ii) Suppose that $\sup\{pr\ S \mid S \text{ is cyclic, } \sigma\text{-torsion}\} = n$. Then $pr\ S \leq n$ for all cyclic, σ-torsion S.

Then $Ext^{n+1}(S,X) = 0$ for all cyclic, σ-torsion modules S and all (left) R-modules X (see §1.2.1.4(a)) and hence $ad_\sigma(X) \leq n$, for all modules X, proving that $AD_\sigma(R) \leq n$.

(iii) If M is not σ-injective it follows from exactness of the sequence $0\to M\to E(M)\to E(M)/M\to 0$ that $ad_\sigma(M) = 1 + ad_\sigma(E(M)/M)$ (2.5.5).

Since $AD_\sigma(R)$ is nonzero, there is at least one module M such that $ad_\sigma(M)$ is nonzero, and hence $AD_\sigma(R) = \sup\{ad_\sigma(M) \mid ad_\sigma(M) \text{ is nonzero}\}$ $= \sup\{1 + ad_\sigma(E(M)/M) \mid M \text{ is not } \sigma\text{-injective}\}$.

(iv) If $AD_\sigma(R) = 1$, then it follows from (iii) above that $\sup\{ad_\sigma(E(M)/M) \mid M \text{ is not } \sigma\text{-injective}\} = 0$. If $AD_\sigma(R) = 0$, then all R-modules are σ-injective, by (i) above.

Conversely, suppose the stated condition holds and let M be any R-module which is not σ-injective. There is an injective resolution of the form $0\to M\to E(M)\xrightarrow{d_0} Q_1\to \ldots$ and, by assumption, $E(M)/M \simeq \operatorname{im} d_0$ is σ-injective. Thus $ad_\sigma(M) \leq 1$.

2.5.9 **Example**

If R is a left semi-hereditary ring, but not σ-regular, where σ is a torsion radical on R-mod such that every $I \in \mathcal{L}_\sigma$ is f.g., (e.g. $R = \mathbb{Z}$ with the usual torsion theory), then $AD_\sigma(R) = 1$.

Proof

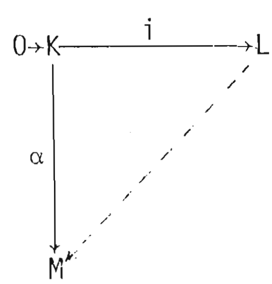
If S is cyclic, σ-torsion then $S \simeq R/I$ for some $I \in \mathcal{L}_\sigma$, where, by assumption, I is f.g. and therefore projective. Hence $0 \to I \xrightarrow{d_1} R \xrightarrow{d_0} R/I \to 0$ is a projective resolution of R/I such that $I \simeq \text{im } d_1$ is projective.

Thus pr $S \leq 1$ and since S was an arbitrary cyclic, σ-torsion module, $AD_\sigma(R) \leq \sup\{\text{pr } S \mid S \text{ is cyclic, } \sigma\text{-torsion}\} \leq 1$. But R is not σ-regular and by 2.5.8(i) $AD_\sigma(R) \neq 0$. That is, $AD_\sigma(R) = 1$.

§2.6 SEMI-σ-INJECTIVE MODULES:

2.6.1 Definition

A module M is called *semi-σ-injective* iff every diagram of the form



where K is a projective submodule of L, L/K is σ-torsion, (and i is the inclusion map), can be completed commutatively as indicated.

Obviously every σ-injective module is semi-σ-injective. It follows in the usual way that direct summands, finite direct sums and direct products of semi-σ-injective modules are semi-σ-injective.

2.6.2 Lemma

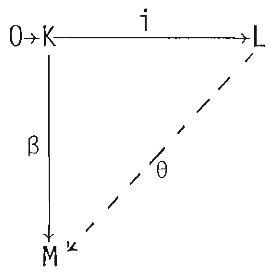
Let R be a ring and σ a torsion radical on R-mod. Epimorphic images of semi-σ-injective R-modules are semi-σ-injective.

Proof

Suppose that M is a semi-σ-injective R-module and that N is a submodule of M. Suppose further that K is projective, L/K is σ-torsion and $\alpha: K \to M/N$ is an R-homomorphism.

Let $\pi: M \to M/N$ be the canonical epimorphism. Since K is projective there is a homomorphism $\beta: K \to M$ such that $\pi \cdot \beta = \alpha$.

Since M is semi-σ-injective, we can complete the diagram



commutatively.

Hence $(\pi \cdot \theta) \cdot i = \alpha$ and M/N is therefore semi-σ-injective.

2.6.3 **Remark**

Let R be a ring and σ a torsion radical on R-mod such that every $I \in \mathcal{L}_\sigma$ is projective (e.g. if R is hereditary). Then semi-σ-injective modules are σ-injective.

Proof

Easy.

2.6.4 **Lemma**

Let R be a ring and σ a torsion radical on R-mod. Then projective, semi-σ-injective R-modules are σ-injective.

Proof

Let M be such a module, let $E_\sigma(M)$ be the σ-injective hull of M and let $i:M\to E_\sigma(M)$ be the inclusion map.

Since M is projective and $E_\sigma(M)/M$ is σ-torsion (1.2.4.4), there is a homomorphism $\mu:E_\sigma(M)\to M$ such that $\mu\cdot i = 1_M$ (M is semi-σ-injective). That is, M is a direct summand of $E_\sigma(M)$ and is therefore σ-injective.

2.6.5 Theorem

Let R be a ring and σ a torsion radical on R-mod and let M be an R-module. Then the following are equivalent:

(i) M is semi-σ-injective.

(ii) Every homomorphism $\alpha:P\to M$ where P is a projective module, can be factored through a σ-injective module.

(iii) There is an exact sequence of the form $0\to M\xrightarrow{i}S\xrightarrow{\pi}S/M\to 0$, where S is semi-$\sigma$-injective and $\sigma(S/M) = 0$.

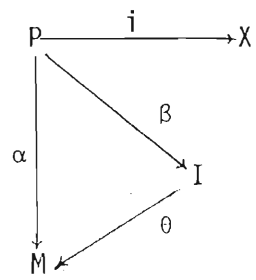
Proof

(i) implies (ii): Suppose that M is semi-σ-injective, P is projective and that $\alpha:P\to M$ is a homomorphism. Let $E_\sigma(P)$ be the σ-injective hull of P and let $i:P\to E_\sigma(P)$ be the inclusion map.

Since M is semi-σ-injective, a homomorphism $\mu:E_\sigma(P)\to M$ exists such that $\mu\cdot i = \alpha$. Hence α factors through $E_\sigma(P)$.

(ii) implies (i): Suppose, conversely, that every homomorphism $\alpha:P\to M$, where P is projective, factors through a σ-injective module. Let P be projective, X/P σ-torsion and $\alpha:P\to M$ a homomorphism. Then α factors through a σ-injective module, I (say). Let $i:P\to X$ be the inclusion map.

Consider the diagram



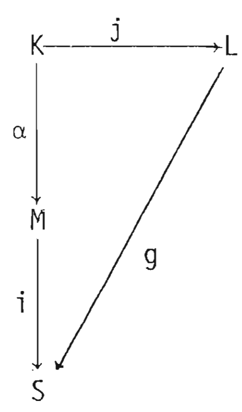
where α factors as $\theta \cdot \beta$. Since I is σ-injective, there is a homomorphism $\mu : X \to I$ such that $\mu \cdot i = \beta$. Hence $\theta \cdot \mu : X \to M$ satisfies $(\theta \cdot \mu) \cdot i = \theta \cdot \beta = \alpha$ and M is semi-σ-injective, as required.

That (i) implies (iii) is clear. (Take S = M).

(iii) implies (i): Suppose $0 \to M \xrightarrow{i} S \xrightarrow{\pi} S/M \to 0$ is an exact sequence where S is semi-σ-injective and $\sigma(S/M) = 0$.

Let K be a projective submodule of a module L such that L/K is σ-torsion and suppose that $\alpha : K \to M$ is a homomorphism. Let $j : K \to L$ be the inclusion map.

Since S is semi-σ-injective, there is a homomorphism $g : L \to S$ such that the diagram



commutes. Let $x \in L$. Let $A = \text{Ann}\,(g(x)+M)$ and let $B = \text{Ann}(x+K)$. $B \in L_\sigma$, because L/K is σ-torsion and it is easy to see that $B \leq A$. Hence $A \in L_\sigma$ and, since $\sigma(S/M) = 0$, it follows that $g(x) \in M$. Thus $g(L) \leq M$, which proves (i).

2.6.6 **Remark**

The equivalence of (i) and (iii) in 2.6.5 is the semi-σ-injective version of "${}_RM$ is σ-injective iff $\sigma(E/(M)/M) = 0$" (see 2.2.2).

2.6.7 **Definition**

Let R be a ring and σ a torsion radical on R-mod. An R-module F is called *Σ-flat* iff whenever we have a commutative diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{i_1} & L & \xrightarrow{\pi_1} & L/K & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
0 & \longrightarrow & N & \xrightarrow[i_2]{} & M & \xrightarrow[\pi_2]{} & F & \longrightarrow & 0
\end{array}$$

where K is projective and L/K is σ-torsion, there exists a homomorphism $\phi: L/K \to M$ such that $\pi_2 \cdot \phi = f_3$.

2.6.8 **Theorem**

Let R be a ring and σ a torsion radical on R-mod. The following are equivalent for an R-module M:

(i) M is semi-σ-injective.

(ii) There is a σ-pure exact sequence $0 \to M \to S \to S/M \to 0$, where S is semi-$\sigma$-injective and S/M is Σ-flat.

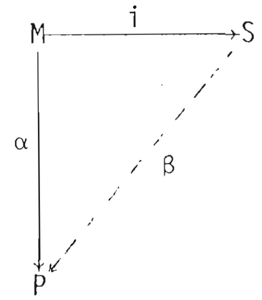
(iii) Every homomorphism $\alpha: M \to P$, where P is a σ-pure injective module, factors as $M \xrightarrow{i} S \xrightarrow{\beta} P$, where M is a σ-pure submodule of the semi-σ-injective module S and S/M is Σ-flat (i is the inclusion map).

(iv) There is a strongly σ-pure exact sequence of the form $0 \to M \to S \to S/M \to 0$, where S is semi-$\sigma$-injective.

Proof

That (i) implies (ii) is clear (take $S = M$).

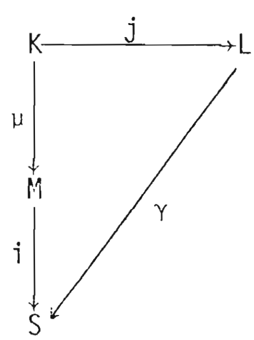
(ii) implies (iii): Let $0 \to M \xrightarrow{i} S \to S/M \to 0$ be a σ-pure exact sequence where S is semi-σ-injective and S/M is Σ-flat (i is the inclusion map). Further, let $\alpha: M \to P$ be a homomorphism where P is σ-pure injective. Since M is σ-pure in S and P is σ-pure-injective, there is a homomorphism $\beta: S \to P$ such that the diagram



commutes, proving (iii).

(iii) implies (i): Let K be projective, L/K σ-torsion and $\mu: K \to M$ a homomorphism. Let P be the injective hull of M. By (iii), there is a semi-σ-injective module S such that the inclusion map $\alpha: M \to P$ factors through S, as $M \xrightarrow{i} S \xrightarrow{\beta} P$, where i is the inclusion map, M is σ-pure in S and S/M is Σ-flat. Let $j: K \to L$ be the inclusion map and $\pi: S \to S/M$ the canonical epimorphism.

Since S is semi-σ-injective, there is a homomorphism $\gamma: L \to S$ such that the diagram



commutes.

Complete the commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{j} & L & \longrightarrow & L/K \\ \big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle \theta} \\ M & \xrightarrow[i]{} & S & \xrightarrow[\pi]{} & S/M \end{array}$$

Since S/M is Σ-flat, there is a homomorphism ϕ:L/K→S such that $\pi\cdot\phi = \theta$. By 1.3.1.2, there is a homomorphism ψ:L→M such that $\psi\cdot j = \mu$. That is, M is semi-σ-injective, proving (i).

The proof of the fact that (i) is equivalent to (iv) follows similar lines and is left to the reader.

2.6.9 Remarks

1. Let E(M) be the injective hull of a module M. If E(M)/M is Σ-flat that it is easy to see that M will be semi-σ-injective.

2. When working with R-modules one is often aware of a two-way relationship between R-mod and R. Conditions on R affect characteristics that R-modules may have and (since R is an R-module) properties of R-mod carry back onto the ring R. A famous example of this is the result ([67] , Proposition 3.5) that R is a Noetherian ring iff every f.g. R-module is a Noetherian module. Some of our theorems are of this type. R is, of course, projective as an R-module but not necessarily injective. The next theorem provides *inter alia*, conditions on R-mod which hold iff R is σ-injective (when every $I \in L_\sigma$ is f.g.). See also 3.4.4.

3. A module F is called *σ-flat* iff every exact sequence of the form 0→K→L→F→0 is σ-pure exact ([29]). It is then clear from definition 2.6.7 that whenever strong σ-purity is equivalent to σ-purity, σ-flatness implies Σ-flatness.

2.6.10 Theorem

Let R be a ring and σ a torsion radical on R-mod. Then the following are equivalent:

(i) Every R-module is Σ-flat.

(ii) Every projective R-module is σ-injective.

(iii) If K is projective and L/K is σ-torsion, then K is a direct summand of L.

(iv) Every R-module is semi-σ-injective.

Suppose, further, that every $I \in \mathcal{L}_\sigma$ is f.g. Then the above conditions are equivalent to:

(v) R is σ-injective.

(vi) Every R-module is an epimorphic image of a σ-injective module.

Proof

(i) implies (ii): Let M be a projective R-module and $E_\sigma(M)$ its σ-injective hull. $E_\sigma(M)/M$ is σ-torsion and Σ-flat, by (i). It then follows easily from Definition 2.6.7 that M is a direct summand of $E_\sigma(M)$, proving (ii).

(ii) implies (iii): If K is projective and L/K is σ-torsion, then K is σ-injective, by (ii). Hence, if $i:K \to L$ is the inclusion map, there is a homomorphism $\alpha:L \to K$ such that $\alpha \cdot i = 1_K$. That is, K is a direct summand of L.

That (iii) implies (iv) is clear.

(iv) implies (i): Let F be any R-module, and suppose that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & L/K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

commutes where K is projective and L/K is σ-torsion. K is semi-σ-injective, by (iv), and hence the top sequence is split exact. This proves (i).

That (iv) implies (v) follows from 2.6.4.

Suppose now that every $I \in L_\sigma$ is f.g.

(v) implies (vi): By 1.2.5.1, direct sums of σ-injective modules are σ-injective and hence, by (v), every free R-module is σ-injective. Since every R-module is an epimorphic image of a free R-module, (vi) follows.

That (vi) implies (iv) follows directly from 2.6.2.

2.6.11 Remarks

1. A ring R satisfying the equivalent conditions (i) to (iv) of 2.6.10 will be called Σ-*regular*.

2. An example of a ring R satisfying all the conditions of 2.6.10 is any Quasi-Frobenius ring R (together with any torsion radical σ on R-mod). (Since a Q.F. ring is Noetherian and has the property that projective R-modules are injective - see [49]).

3. If every $I \in L_\sigma$ is f.g. and projective then R is σ-injective iff R is σ-regular.

Proof

If R is σ-regular then it is σ-injective as an R-module by 2.5.8(i).

Conversely, suppose R is σ-injective. Let M be any R-module. By 2.6.10, M is semi-σ-injective. By 2.6.3, M is σ-injective. That is, every R-module is σ-injective and R is σ-regular, by 2.5.8(i).

4. It is easy to show that a weaker form of 2.6.10(ii), namely the condition "σ-pure projective R-modules are semi-σ-injective" is equivalent to the conditions of 2.6.10 when every $I \epsilon L_\sigma$ is f.g.

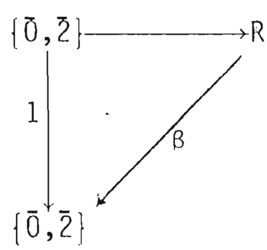
5. Semi-σ-injective modules need not be σ-injective:

(i) Let $R = \mathbb{Z}_4 = \{\bar{0},\bar{1},\bar{2},\bar{3}\}$ - the ring of congruence classes mod 4. Consider $\{\{\bar{0}\},\{\bar{0},\bar{2}\},R\}$. This is the filter of a torsion radical σ on R-mod, as is easily verified.

R is σ-injective: Suppose $I \epsilon L_\sigma$ and $\alpha : I \to R$ is a homomorphism.

The cases $I = \{\bar{0}\}$ or $I = R$ are trivial so suppose $I = \{\bar{0},\bar{2}\}$. The only nonzero homomorphism α is the inclusion map and therefore 1_R extends α.

By 2.6.10(iv), $\{\bar{0},\bar{2}\}$ is semi-σ-injective. It is not, however, σ-injective, since there is no homomorphism β making the diagram



commute, ($\{\bar{0},\bar{2}\}$ not being a direct summand of R).

(ii) In view of 2.5.8(i), it follows from [35], Proposition 8.10 that a ring R is σ-regular (for a torsion radical σ on R-mod) iff every $I \epsilon L_\sigma$ is a direct summand of R.

Let R be a ring and σ a torsion radical on R-mod such that R is σ-injective, every $I \epsilon L_\sigma$ is f.g. but R is not σ-regular. (We may take any Noetherian, self-injective ring R and any torsion radical σ on R-mod such that L_σ contains ideals which are non-direct summands of R). Then, by 2.6.10, every R-module is semi-σ-injective but, by 2.5.8(i), not every R-module is σ-injective.

6. If R is σ-regular then the first four conditions of 2.6.10 hold (since every R-module is then σ-injective). That is, if R is σ-regular then it is Σ-regular. The converse is not necessarily true: Let $R = Z_4 = \{\bar{0},\bar{1},\bar{2},\bar{3}\}$ and $\mathcal{L}_\sigma = \{\{\bar{0}\},\{\bar{0},\bar{2}\},R\}$. It was shown, in (5) above, that R is σ-injective. Since every $I \in \mathcal{L}_\sigma$ is f.g. it follows by 2.6.10 that every R-module is semi-σ-injective, and therefore R is Σ-regular. R is not, however, σ-regular since otherwise $\{\bar{0},\bar{2}\}$ would have to be a direct summand of R, which it is not.

7. R is σ-regular iff R is Σ-regular and every $I \in \mathcal{L}_\sigma$ is projective.

Proof

If R is σ-regular then it is Σ-regular (see 6 above) and every $I \in \mathcal{L}_\sigma$ is a direct summand of R and is therefore projective.

Conversely, suppose R is Σ-regular and every $I \in \mathcal{L}_\sigma$ is projective. Then every R-module is semi-σ-injective and hence σ-injective (by 2.6.3). Hence R is σ-regular (2.5.8(i)).

8. The following are equivalent for a ring R and a torsion radical σ on R-mod:

(i) A module M is semi-σ-injective and σ-torsion-free.

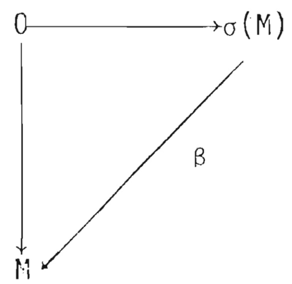
(ii) If K is a projective submodule of a module L and L/K is σ-torsion, then a homomorphism $\alpha: K \to M$ extends to a unique homomorphism $\beta: L \to M$.

Proof

(i) implies (ii): Let K, L and M be as given and let $\alpha: K \to M$ be a homomorphism. By definition there exists a homomorphism β which extends α. We only need to show uniqueness of β.

Suppose, therefore, that β_1 and β_2 both extend α. Define $\beta':L/K\to M$ by $\beta'(x+K) = \beta_1(x)-\beta_2(x)$, for $x\in L$. β' is a homomorphism and since L/K is σ-torsion and M is σ-torsion-free, $\beta' = 0$. That is, $\beta_1 = \beta_2$.

(ii) implies (i): By (ii), the zero map from 0 to M has a unique extension to a homomorphism $\beta:\sigma(M)\to M$ in the diagram



Clearly $\beta = 0$ and since the inclusion map $i:\sigma(M)\to M$ also makes this diagram commute, $i = 0$. That is, $\sigma(M) = 0$, proving (i).

CHAPTER THREE

RELATIVE REGULARITY AND FLATNESS

§3.1 INTRODUCTION

We have already noted that a ring R is von Neumann regular iff every ideal is Cohn pure in R. This result inspired Fieldhouse to define an *R-module* M to be *regular* iff every submodule is Cohn pure ([32]). In this chapter we extend this to σ-purity in the obvious way.

Some of our results are extensions of basic properties of regular modules found in [29] and [32]. We further characterize σ-regular rings for the case of an arbitrary ring, a commutative ring and a commutative, Quasi-Frobenius ring. We also show how σ-regularity of rings is related to their von Neumann regularity (3.4.3) and briefly consider properties of modules over σ-regular rings (3.4.6). We note, lastly, that σ-regularity is equivalent to the concept of σ-semisimplicity, defined in [62], for rings (but not arbitrary R-modules).

In the second part of this chapter we collect together the known properties of σ-flatness which are important for our purposes (including proofs in our terminology) and derive some new properties. More specifically, we extend the main result (Theorem 2.4) of [13] to σ-flatness (see 3.5.1.7). We also use σ-flatness to characterize σ-regular rings (3.5.1.10).

Lastly, we introduce semi-σ-flat modules, show that they have properties analogous to those of σ-flat modules, in many instances, and characterize rings R for which every R-module is semi-σ-flat (3.5.3.2).

§3.2 σ-REGULAR MODULES

3.2.1 Definition

Let R be a ring and σ a torsion radical on R-mod. We define an R-module M to be *σ-regular* iff every exact sequence of the form $0 \to N \xrightarrow{i} M \xrightarrow{\pi} M/N \to 0$ is σ-pure exact.

3.2.2 Theorem

Let R be a ring and σ a torsion radical on R-mod. If N is a submodule of an R-module M, then M is σ-regular iff N and M/N are σ-regular and N is σ-pure in M. In particular, thus, submodules and homomorphic images of σ-regular modules are σ-regular.

Proof

A direct extension of Theorem 6 of [32].

3.2.3 Lemma

Let R be a ring and let σ be a torsion radical on R-mod such that every $I \in L_\sigma$ is f.g. If P is a submodule of an R-module M such that every f.g. submodule of P is σ-pure in M, then P is σ-pure in M.

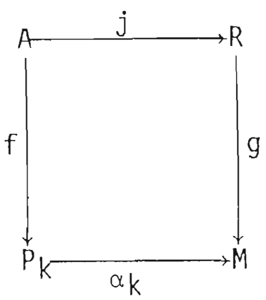
Proof

Let $\{P_i \mid i \in I\}$ be the collection of f.g. submodules of P. Obviously $P = \bigcup_i P_i$. Let $\alpha_i : P_i \to M$ be the inclusion map for each i. Suppose that $A \in L_\sigma$ and that the diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & R \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow[k]{} & M \end{array}$$

commutes, where j,k are the inclusion maps.

Suppose A is generated by the elements $a_1, a_2, \ldots\ldots a_n$. Then $f(a_i) \in P_{j_i}$ for some $j_i \in I$, $i = 1,2\ldots.n$. $P_k = P_{j_1} + P_{j_2} + \ldots\ldots + P_{j_n}$ is a f.g. submodule of P, $f(A) \leq P_k$ and the diagram



commutes.

Since P_k is σ-pure in M, there is a homomorphism $\rho: R \to P_k \leq P$ such that $\rho \cdot j = f$. Thus P is σ-pure in M, by 1.3.4.2(i).

3.2.4 Theorem

Let R be a ring and σ a torsion radical on R-mod such that every $A \in \mathcal{L}_\sigma$ is f.g. Then $\underset{i}{\oplus} M_i$ (and hence $\underset{i}{\Sigma} M_i$) is σ-regular iff every M_i is σ-regular.

Proof

A direct extension of Theorem 7 of [32].

§3.3 σ-REGULAR SOCLE

3.3.1 Definition

Let R be a ring and σ a torsion radical on R-mod. For any R-module M, denote by $\Sigma(M)$ the sum of all the σ-regular submodules of M.

3.3.2 Theorem

Let R be a ring and σ a torsion radical on R-mod such that every $A \in \mathcal{L}_\sigma$ is f.g.. If M is an R-module, $\Sigma(M)$ is the maximal σ-regular submodule of M and Σ is a torsion socle in the sense of Fieldhouse ([32]) (i.e. it satisfies 1.2.3.1(i) and (ii)).

Proof

Routine.

3.3.3 Remarks

1. There is a smallest torsion radical, which we will denote by Σ' such that $\Sigma(M) \leq \Sigma'(M)$ for all modules M.

It follows that $\Sigma'(M) = \cap \{N \leq M \mid \Sigma(M/N) = 0 \text{ for all } M\}$ (see [37], Proposition 1.1, Theorem 1.6).

2. A module M is Σ-torsion iff it is σ-regular.

§3.4 σ-REGULAR RINGS

3.4.1 Definition

Let R be a ring and σ a torsion radical on R-mod. A *ring* R is called (left) *σ-regular* iff it is σ-regular as a (left) R-module i.e. iff every (left) ideal of R is σ-pure in R.

3.4.2

The following theorem and its corollary shows the analogy between the σ-regularity and the von Neumann regularity of rings:

Theorem

Let R be a ring and σ a torsion radical on R-mod. Then the following are equivalent:

(i) Every cyclic, σ-torsion module is σ-regular.

(ii) Every cyclic, σ-torsion module is semisimple.

(iii) Given cyclic, σ-torsion modules F and G, and a homomorphism $\alpha: F \to G$, there is a homomorphism $\beta: G \to F$ such that $\alpha \cdot \beta \cdot \alpha = \alpha$.

Moreover, if every $A \in L_\sigma$ is f.g., then the above conditions are equivalent to:

(iv) f.g. submodules of cyclic, σ-torsion modules are direct summands.

Proof

(i) implies (ii): Suppose that (i) holds. Let G be a cyclic, σ-torsion R-module and F a submodule of G. Then G/F is cyclic, σ-torsion and G is σ-regular, by (i), hence $0 \to F \xrightarrow{i} G \xrightarrow{\pi} G/F \to 0$ is σ-pure exact. It follows that the above sequence splits and G is therefore semisimple, as required to prove (ii).

(ii) implies (iii): Suppose that (ii) holds, F and G are cyclic, σ-torsion modules and $\alpha: F \to G$ is a homomorphism. Say $G \simeq R/L_1$, $F \simeq R/L_2$ for some $L_1, L_2 \in L_\sigma$. Factor α through $\mathrm{im}\alpha$, i.e. put $\alpha = j \cdot \bar{\alpha}$ where $\bar{\alpha}: F \to \mathrm{im}\alpha$ and j is the inclusion map from $\mathrm{im}\alpha$ to G.

By (ii), $\mathrm{im}\alpha$ is a direct summand of G. Suppose that $\mu: G \to \mathrm{im}\alpha$ is the left inverse of j and that $\mathrm{im}\alpha \simeq F/\ker\alpha \simeq R/T$, where $\ker\alpha \simeq T/L_2$ (for a left ideal T of R containing L_2). Then, by (ii), the exact sequence $0 \to \ker\alpha \to F \xrightarrow{\alpha} \mathrm{im}\alpha \to 0$ splits and thus there is a homomorphism $\gamma: \mathrm{im}\alpha \to F$ such that $\alpha \cdot \gamma = 1_{\mathrm{im}\alpha}$. Hence $\gamma \cdot \mu: G \to F$ satisfies $\alpha \cdot \gamma \cdot \mu \cdot \alpha = 1_{\mathrm{im}\alpha} \cdot \mu \cdot \alpha$ $= 1_{\mathrm{im}\alpha} \cdot \mu \cdot j \cdot \bar{\alpha} = 1_{\mathrm{im}\alpha} \cdot 1_{\mathrm{im}\alpha} \cdot \bar{\alpha} = \alpha$. Thus (iii) holds, with $\beta = \gamma \cdot \mu$.

(iii) implies (i): Suppose that (iii) holds and that F is a cyclic, σ-torsion R-module with submodule L. Let $\pi: F \to F/L$ be the canonical epimorphism. Since F and F/L are cyclic, σ-torsion, there is a homomorphism $\beta: F/L \to F$ such that $\pi \cdot \beta \cdot \pi = \pi$ (by (iii)). Since π is epic, this means that $1_{F/L} = \pi \cdot \beta$ and hence the sequence $0 \to L \xrightarrow{i} F \xrightarrow{\pi} F/L \to 0$ splits. Thus L is σ-pure in F, as required to prove (i).

That (ii) implies (iv) is clear.

(iv) implies (iii): Suppose that (iv) holds, every $A \in \mathcal{L}_\sigma$ is f.g., and F,G are cyclic, σ-torsion modules with $\alpha: F \to G$ a homomorphism. Say $G \simeq R/L_1$, $F \simeq R/L_2$ for some $L_1, L_2 \in \mathcal{L}_\sigma$. As before, we put $\alpha = j \cdot \bar{\alpha}$ where $\bar{\alpha}: F \to \mathrm{im}\alpha$ and j is the inclusion map from $\mathrm{im}\alpha$ to G.

Imα is f.g. since it is an epimorphic image of the cyclic module F. Thus, by (iv), it is a direct summand of G. Suppose that $\mu:G\to\text{im}\alpha$ is the left inverse of j and that $\text{im}\alpha\simeq F/\ker\alpha\simeq R/T$ where $\ker\alpha\simeq T/L_2$ as a submodule of F. Then R/T is σ-torsion, $T\in\mathcal{L}_\sigma$ and therefore T is f.g. Hence $\ker\alpha$ is f.g. and, by (iv), the exact sequence $0\to\ker\alpha\to F\xrightarrow{\alpha}\text{im}\alpha\to 0$ splits and, as before, this implies the existence of a homomorphism $\beta:G\to F$ such that $\alpha\cdot\beta\cdot\alpha = \alpha$, proving (iii).

3.4.3 Corollary

Let R be a ring and σ a torsion radical on R-mod. If R is a σ-regular ring then every cyclic, σ-torsion R-module has a von Neumann regular endomorphism ring.

Proof

Suppose R/A is a cyclic, σ-torsion module with submodule L/A, (for a left ideal L of R). Since R is σ-regular, L is σ-pure in R and, by 1.3.1.5(iii), L/A is σ-pure in R/A. Thus every cyclic, σ-torsion module is σ-regular and taking F = G = R/A in 3.4.2(iii), we see that for any $\alpha\in$Hom(R/A,R/A) there is a $\beta\in$Hom(R/A,R/A) such that $\alpha\cdot\beta\cdot\alpha = \alpha$. Hence Hom(R/A,R/A) is a von Neumann regular ring.

The following theorem shows that a torsion theory for which the ring R is σ-regular is, in a sense, at the opposite extreme from the Goldie torsion theory, in that every ideal in $\mathcal{L}_\sigma$ is a direct summand. The filter $\mathcal{L}_{\mathcal{G}}$ of the Goldie torsion theory contains all the essential ideals of R, (see [1]) and e.g. if $\mathcal{G}(R) = 0$ it consists exactly of all the essential ideals of R (see [34], Example 8, page 312).

3.4.4 Theorem

The following are equivalent for a ring R (and a torsion radical σ on R-mod):

(i) R is σ-regular.

(ii) Every $A \in \mathcal{L}_\sigma$ is a direct summand of R.

(iii) Every R-module is σ-regular.

(iv) Every co-σ-pure exact sequence of R-modules is σ-pure exact.

(v) Every cyclic, σ-torsion (σ-pure projective) R-module is projective.

(vi) Every maximal ideal of R is σ-pure in R.

(vii) Every maximal ideal of R which lies in $\mathcal{L}_\sigma$ is a direct summand of R.

(viii) For every cyclic, σ-torsion module R/I, exact sequence $B \xrightarrow{\beta} C \to 0$ and a homomorphism $\alpha: R/I \to C$ such that $\alpha(R/I)$ is a σ-pure submodule of C, there is a homomorphism $\gamma: R/I \to B$ such that $\beta \cdot \gamma = \alpha$.

Proof

(i) implies (ii): Suppose (i) holds and $A \in \mathcal{L}_\sigma$.

Since R is σ-regular, A is σ-pure in R and therefore $1_{R/A}$ lifts to a homomorphism $\alpha: R/A \to R$. That is, A is a direct summand of R, as required.

(ii) implies (iii): If (ii) holds then every cyclic, σ-torsion module is projective and (iii) follows easily.

(iii) implies (iv): If (iii) holds then every short exact sequence is σ-pure exact. In particular, (iv) is true.

(iv) implies (v): If (iv) holds and M is a σ-pure-projective module, then, by 1.3.4.6, every exact sequence of the form $0 \to X \to Y \to M \to 0$ is co-σ-pure exact, therefore σ-pure exact, by (iv), and hence split exact. That is, M is projective, which proves (v).

That (v) implies (vi) follows easily.

That (vi) implies (vii) is clear.

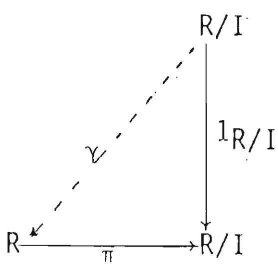
(vii) implies (ii): Suppose that (vii) holds but (ii) does not hold. Then there is an $I \in \mathcal{L}_\sigma$, which is not a direct summand of R. Let $\mathcal{S} = \{J \in \mathcal{L}_\sigma \mid I \leq J$ and J is not a direct summand of R$\}$. $\mathcal{S}$ is inductive: Suppose $\{A_i\}_i$ is an ascending chain of elements of $\mathcal{S}$ and suppose that there exists a (left) ideal K of R such that $(\bigcup_i A_i) \oplus K = R$. Then $1 = a_i + k$ for some $a_i \in A_i$ and $k \in K$. This implies that $R = A_i \oplus K$, contrary to the fact that $A_i \in \mathcal{S}$.

By Zorn's Lemma, $\mathcal{S}$ has a maximal element, M (say). If $M \lneqq M' \lneqq R$ for a (left) ideal M' of R, then, by maximality of M in $\mathcal{S}$, $M' \oplus K = R$ for some (left) ideal K of R. Since $K \neq 0$, $M \oplus K$ is a direct summand of R by maximality of M in $\mathcal{S}$, again. Hence M is a direct summand of R, which is contradictory. Thus M is a maximal ideal and, by (vii), a direct summand of R. This is a contradiction and therefore (vii) implies (ii).

(ii) implies (viii): If (ii) holds then every cyclic, σ-torsion module is projective and (viii) follows trivially.

(viii) implies (ii): Suppose (viii) holds and $I \in \mathcal{L}_\sigma$.

Consider the diagram



(where π is the canonical epimorphism).

By (viii), there is a homomorphism $\gamma : R/I \to R$ such that $\pi \cdot \gamma = 1_{R/I}$. This proves (ii).

(ii) implies (i): If (ii) holds then every cyclic, σ-torsion module is projective and hence every short exact sequence is σ-pure exact.

3.4.5 Corollary

Let R be a ring and σ a torsion radical on R-mod such that (R,σ) admits σ-pure injective hulls. Then the following are equivalent:

(i) R is σ-regular.

(ii) Every σ-pure-injective R-module is injective.

(iii) Every σ-pure-injective R-module is σ-injective.

Proof

(i) implies (ii): Suppose (i) holds and let P be a σ-pure-injective R-module.

Since any left ideal L of R is σ-pure in R, a homomorphism $\alpha : L \to P$ will extend to a homomorphism $\beta : R \to P$. Thus P is injective.

That (ii) implies (iii) is trivial.

(iii) implies (i): Suppose (iii) holds, let M be an arbitrary module and N a submodule of M.

Let P(N) be a σ-pure-injective R-module containing N as a σ-pure submodule. Suppose that $A \in \mathcal{L}_\sigma$ and that the diagram

$$\begin{array}{ccc} A & \xrightarrow{i_1} & R \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ N & \xrightarrow[i_2]{} & M \end{array}$$

commutes, (where i_1, i_2 are the inclusion maps). Let $i_3 : N \to P(N)$ be the inclusion map.

By (iii), $P(N)$ is σ-injective and hence $i_3 \cdot f_1 : A \to P(N)$ extends to a homomorphism $\phi : R \to P(N)$. That is, the diagram

$$\begin{array}{ccc} A & \xrightarrow{i_1} & R \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \phi} \\ N & \xrightarrow[i_3]{} & P(N) \end{array}$$

commutes and, since N is σ-pure in $P(N)$, there is a homomorphism $\mu : R \to N$ such that $\mu \cdot i_1 = f_1$ (1.3.4.2(i)).

Referring back to our first diagram we see that N is σ-pure in M (by 1.3.4.2(i)). Thus R is σ-regular, proving (i).

3.4.6 Theorem

Let R be a ring and σ a torsion radical on R-mod such that R is σ-regular.

Then **(i)** Every σ-torsion R-module is semisimple ([62]).

(ii) Every f.g., σ-torsion R-module with n generators is a direct sum of n cyclic, σ-torsion R-modules.

(iii) An R-module M is σ-torsion iff it is a direct sum of cyclic, σ-torsion R-modules (each of which is isomorphic to a direct summand of R, and is therefore projective). In particular, it follows that for a σ-regular ring, σ-purity is equivalent to strong σ-purity.

((iii) extends Theorem 8.7 of [29]).

Proof

(i) Let $M = \sum_i Rx_i$ be any σ-torsion module. Since R is σ-regular, every cyclic, σ-torsion module is σ-regular (by 3.4.4(iii)) and by 3.4.2, every cyclic, σ-torsion module is semisimple. Thus M is a sum of semisimple modules, proving (i).

(ii) Let $M = \sum_{i=1}^{n} Rx_i$ be a f.g., σ-torsion module (with n generators). By (i) above, M is semisimple.

We argue by induction. If M has only one generator, the result is clear. Suppose that all σ-torsion modules with n-1 generators can be written as a direct sum of n-1 cyclic, σ-torsion modules. Consider $M = I+Rx_n$ where $I = \sum_{i=1}^{n-1} Rx_i$.

Since M is semisimple, I is a direct summand. Let $\mu : M \mapsto I$ be the canonical projection.

$$\text{Then } I+Rx_n \leq I + \mu(Rx_n) + (1_M-\mu)(Rx_n)$$
$$\leq I + (1_M-\mu)(Rx_n)$$
$$\leq I + Rx_n$$

Hence $I + Rx_n = I + (1_M-\mu)(Rx_n) = I \oplus (1_M-\mu)(Rx_n)$

$(1_M-\mu)(Rx_n)$ is cyclic, σ-torsion and, by our induction assumption, I is a direct sum of n-1 cyclic, σ-torsion modules; hence the result follows.

(iii) If M is a σ-torsion module then, by (i) above, M is semisimple and is therefore a direct sum of cyclic, σ-torsion modules. Since R is σ-regular, every cyclic, σ-torsion module is isomorphic to a direct summand of R. The converse is clear.

The second statement follows from Theorem 1.3.6.7.

3.4.7 Theorem

Let R be a *commutative* ring and σ a torsion radical on R-mod. Then the following are equivalent:

(i) R is σ-regular

(ii) Every semiprime ideal in L_σ is a direct summand of R.

(iii) Every prime ideal in L_σ is a direct summand of R.

(iv) Every prime ideal of R is σ-pure in R.

(v) Every semiprime ideal of R is σ-pure in R.

Proof

The facts that (i) implies (v) and (v) implies (iv) are clear.

(iv) implies (iii): Let P be a prime ideal in L_σ. Then P is σ-pure in R, by (iv), and since R/P is σ-pure projective, P is a direct summand of R.

(iii) implies (i): Suppose (iii) holds but R is not σ-regular. Then there is an $I \in L_\sigma$ which is not a direct summand of R. Let $S = \{J \in L_\sigma | I \leq J$ and J is not a direct summand of R$\}$. $S \neq \phi$, since $I \in S$. S is inductive (see the proof of 3.4.4.) and by Zorn's Lemma has a maximal element M, (say).

M is not prime, by (iii), and hence there exist ideals A,B such that $A.B \leq M$ while $A \nleq M$, $B \nleq M$. By maximality of M in S, A+M and B+M are direct summands of R. Say $(A+M) \oplus X = R = (B+M) \oplus Y$ for ideals X,Y of R.

If X = Y = 0, then A+M = R = B+M and hence $R = R^2 = (A+M)(B+M) \leq M$ which is contrary to the fact that M is not a direct summand of R.

Suppose $X \neq 0$. Then $M+X = M \oplus X$ is a direct summand of R by maximality of M. This means that M is a direct summand of R, which is contradictory. Similarly if $Y \neq 0$ we get a contradiction. Therefore (iii) implies (i).

Clearly (i) implies (ii) and (ii) implies (iii) and hence the result follows.

3.4.8 Theorem

Suppose R is commutative, Quasi-Frobenius ring and σ a torsion radical on R-mod. Then R is σ-regular iff $0 \to I \otimes N \to R \otimes N$ is exact for all R-modules N and all $I \in \mathcal{L}_\sigma$.

Proof

If R is σ-regular, then every $I \in \mathcal{L}_\sigma$ is a direct summand of R and hence the one implication is easy.

Conversely, suppose $0 \to I \otimes N \to R \otimes N$ is exact for all left R-modules N and all $I \in \mathcal{L}_\sigma$.

Let M be any left R-module. By Corollary 1.2 of [50], $M \simeq M^{**}$.

By Theorem 2.4.2, N^* is σ-injective for all (left) R-modules N. Since R is commutative, M is also a right R-module and hence $N = M^*$ is a left R-module with $N^* = M^{**} (\simeq M)$ σ-injective. That is, every R-module is σ-injective and R is σ-regular (2.5.8(i)).

3.4.9 Examples

One wonders, especially in view of the equivalence of (i) and (ii) in 3.4.2, whether all σ-regular rings will be semisimple and/or von Neumann regular. The following examples of σ-regular rings show these conjectures to be false:

1. Let S be any commutative ring with unit, which is not von Neumann regular. Let F be a field and form $R = S \oplus F$ with componentwise operations. If M is the maximal ideal $M = \{(s,0) \mid s \in S\}$, then $\mathcal{L}_\sigma = \{M;R\}$ is the filter of a torsion radical on R-mod (see §1.2.5.3).

Since every ideal of L_σ is a direct summand, R is σ-regular. Since S is not von Neumann regular, there is an $s \in S$ such that no $x \in S$ exists with $sxs = s$. Thus $(s,0) \in R$ and there is no $(x,f) \in R$ with $(s,0)(x,f)(s,0) = (s,0)$. Hence R is not von Neumann regular. (In particular, R is not semisimple).

2. It follows from 1.3.6.3 that if R is a von Neumann regular ring and σ a torsion radical on R-mod such that every $I \in L_\sigma$ is f.g., then R is σ-regular. Such a ring will not necessarily be semisimple:

If we take, in Example 1 above, S to be von Neumann regular (but not semisimple), then R will be a von Neumann regular ring and σ a torsion radical on R-mod such that every $I \in L_\sigma$ is f.g., while R is not semisimple.

3. Lastly, we note that any ring R will be σ-regular if we let $L_\sigma = \{R\}$.

The next paragraph (*inter alia*) further clarifies the relationship between σ-regularity and semisimplicity. (Proofs are easy and are omitted):

3.4.10 <u>Remarks</u>

1. The following are equivalent for a ring R and a torsion radical σ on R-mod:

(i) Every σ-regular, σ-pure projective module is semisimple.

(ii) Homomorphic images of σ-regular, σ-pure projective modules are (σ-regular,) σ-pure projective.

2. The following are equivalent for a ring R:

(i) R is semisimple.

(ii) R is σ-regular and every epimorphic image of R is σ-pure projective for all torsion radicals σ on R-mod.

3 J.S. Golan has shown (in [35], Proposition 8.10) that for a ring R and a torsion radical, σ, on R-mod the following equivalent conditions *inter alia* are valid:

(i) Every $I \in L_\sigma$ is a direct summand of R.

(ii) Every (left) R-module is σ-injective.

(iii) Every σ-torsion (left) R-module is projective.

(iv) Every simple, σ-torsion (left) R-module is projective.

3.5. σ-FLAT AND SEMI-σ-FLAT MODULES

3.5.1 Properties of σ-Flat Modules

3.5.1.1 Definition ([29])

Let R be a ring and σ a torsion radical on R-mod. A module F is called *σ-flat* iff every short exact sequence of the form $0 \to N \to M \to F \to 0$ is σ-pure exact.

We can characterize σ-flatness as follows when every $I \in L_\sigma$ is f.g.:

3.5.1.2 Theorem

Let R be a ring and σ a torsion radical on R-mod such that every $I \in L_\sigma$ is f.g.

A nonzero module F is σ-flat iff in every exact sequence of the form $0 \to K \to G \to F \to 0$, where K is nonzero, K must contain a nonzero σ-pure submodule of G.

Proof

Suppose F is σ-flat and that $0\to K\to G\to F\to 0$ is exact where K is nonzero. Since F is σ-flat, K is σ-pure in G and the proof in one direction is complete.

Conversely, suppose that in every exact sequence of the form $0\to K\to G\to F\to 0$ where K is nonzero, K contains a nonzero σ-pure submodule of G. Let $0\to K\to G\xrightarrow{\alpha} F\to 0$ be exact with $K\neq 0$. We only need to show that K is σ-pure in G to prove that F is σ-flat. By assumption, K must contain a σ-pure submodule $N\neq 0$ say, of G. Using Zorn's Lemma and 1.3.3.5, we may choose N to be maximal among the σ-pure submodules of G contained in K. Define $\beta:G/N\to F$ by $\beta(g+N) = \alpha(g)$, for $g\in G$. It follows easily that $0\to K/N\to G/N\xrightarrow{\beta} F\to 0$ is exact.

If $K/N = 0$ then $K = N$ is σ-pure in G, proving our result. Suppose K/N is nonzero. By assumption, K/N contains a nonzero σ-pure submodule of G/N, S/N say. Since N is σ-pure in G, S is σ-pure in G, by 1.3.1.5(iv). By the maximality of N, S = N, a contradiction.

3.5.1.3 Example

A σ-flat module which is not flat:

Put $R = S\oplus F$ where S is any non-regular, commutative ring with identity (e.g. the ring of integers) and F is a field. Put $\mathcal{L}_\sigma = \{M;R\}$ where $M = \{(s,0)|s\in S\}$, as in §1.2.5.3.

R is σ-regular since every $I\in\mathcal{L}_\sigma$ is a direct summand and therefore every R-module is σ-flat. Let I be an ideal of S, not Cohn pure in S, then $(S\oplus F)/(I\oplus F)$ is a left R-module, σ-flat by the above, but not flat.

The following lemma relates σ-flat modules and projective modules. In particular it shows that any projective R-module is σ-flat (for all torsion radicals σ on R-mod):

3.5.1.4 Lemma

Let R be a ring. The following are equivalent for any R-module M and for any torsion radical σ on R-mod:

(i) M is σ-flat.

(ii) There is a σ-pure exact sequence $0\to\ker\alpha\to P\xrightarrow{\alpha}M\to 0$ with P projective.

(iii) Every homomorphism $\beta:R/I\to M$, where $I\in\mathcal{L}_\sigma$, factors through a projective module.

(The equivalence of (i) and (ii) was proved independently in 1.12 of [57]).

Proof

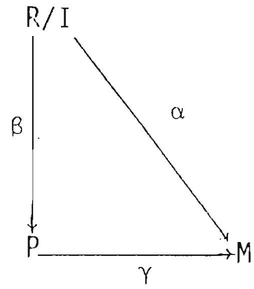
The proof of the fact that (i) implies (ii) is trivial.

(ii) implies (iii): Suppose (ii) holds and let $I\in\mathcal{L}_\sigma$ with $\beta:R/I\to M$ a homomorphism.

Let $0\to\ker\alpha\to P\xrightarrow{\alpha}M\to 0$ be a σ-pure exact sequence with P projective. Then there is a homomorphism $\gamma:R/I\to P$ with $\alpha\cdot\gamma = \beta$ and therefore β factors through P, proving (iii).

(iii) implies (i): Suppose (iii) holds and $0\to K\to N\xrightarrow{\phi}M\to 0$ is any exact sequence with M in the third nonzero position. Suppose $I\in\mathcal{L}_\sigma$ and $\alpha:R/I\to M$ is a homomorphism. By (iii), α factors through a projective module P, (say).

Suppose that the diagram



commutes.

Since P is projective $\gamma:P\to M$ lifts to $\rho:P\to N$ such that $\phi\cdot\rho = \gamma$. Thus $\mu = \rho\cdot\beta:R/I\to N$ satisfies $\phi\cdot\mu = \alpha$ and it follows that M is σ-flat.

3.5.1.5 Example

If $\sigma(R) = 0$, then a module F is σ-flat iff it is σ-torsion free (see [57], 2.21).

If σ is the usual torsion radical on Abelian groups then σ-purity is equivalent to Cohn purity (see Lemma 1.3.3.1). Since $\sigma(Z) = 0$, a module is σ-flat iff it is flat iff it is torsion-free. (Compare [45], Theorem 14.9). In this case then, homomorphic images of σ-flat modules are not σ-flat. Quotients of σ-flat modules by σ-pure submodules are, however, σ-flat :

3.5.1.6 Lemma ([29], Theorem 6.2)

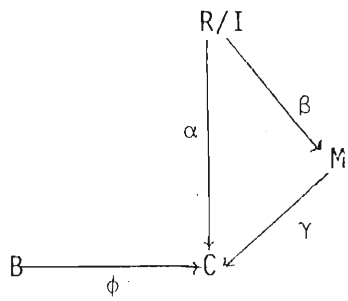
Suppose $0\to A\to B\to C\to 0$ is exact and B is σ-flat. Then A is σ-pure in B iff C is σ-flat.

3.5.1.7 Theorem

Let R be a ring and σ a torsion radical on R-mod. An R-module M is σ-flat iff for every epimorphism $B\xrightarrow{\phi}C$, every $I\in\mathcal{L}_\sigma$ and every homomorphism $\alpha:R/I\to C$ which factors through M, there is a homomorphism $\mu:R/I\to B$ such that $\phi\cdot\mu = \alpha$.

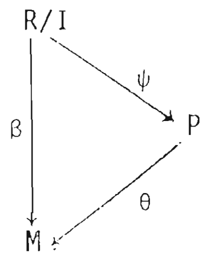
Proof

Suppose M is σ-flat and the diagram



is given.

By Lemma 3.5.1.4, β factors through a projective module P, say in the diagram



Since P is projective there is a homomorphism $\delta : P \to B$ such that $\phi \cdot \delta = \gamma \cdot \Theta$. Then $\mu = \delta \cdot \psi$ satisfies $\phi \cdot \mu = \alpha$, as required.

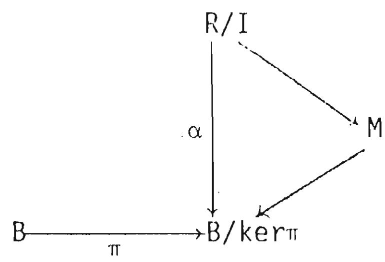
Conversely, suppose the stated condition is satisfied, and let any exact sequence $0 \to X \to Y \xrightarrow{\phi} M \to 0$ and a homomorphism $\alpha : R/I \to M$ be given, where $I \varepsilon L_\sigma$. α factors trivially through M and hence there is a $\mu : R/I \to Y$ such that $\phi \cdot \mu = \alpha$. This proves that M is σ-flat, as required.

3.5.1.8 Remark

In Theorem 3.5.1.7 it is sufficient to test epimorphisms $\phi : B \to C$ only for injective B.

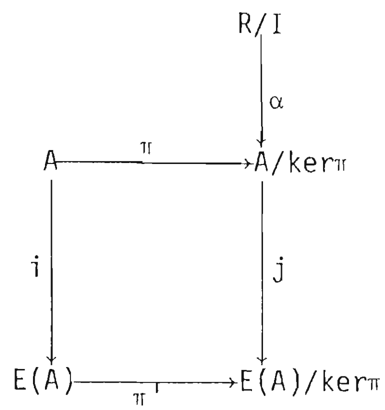
Proof

Suppose that $I \varepsilon L_\sigma$ and suppose that every diagram of the form



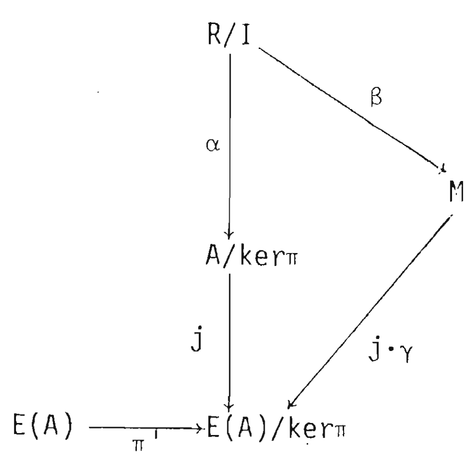
where B is injective, (and α factors through M), can be completed commutatively as indicated. Suppose further that we have an arbitrary epimorphism $A \xrightarrow{\pi} A/\ker\pi$ and a homomorphism $\alpha: R/I \to A/\ker\pi$ which factors through M as $\alpha = \gamma \cdot \beta$.

Let E(A) be the injective hull of A and construct the commutative diagram



(where i,j are the inclusion maps and π' is the canonical epimorphism).

$j \cdot \alpha$ factors through M in the diagram



By assumption, there is a homomorphism $\mu: R/I \to E(A)$ such that $\pi' \cdot \mu = j \cdot \alpha$. It is then easy to verify that $\mu(R/I) \leq A$ and the result follows.

3.5.1.9 <u>Corollaries</u>

1. A σ-flat, cyclic, σ-torsion module is projective.

2. The class of σ-flat modules is closed under module extensions, i.e. if N, M/N are σ-flat, then so is M ([57], 1.14).

3. If G is a σ-pure submodule of F and F and P are σ-flat, then the pushout of the diagram

$$\begin{array}{ccc} G & \xrightarrow{\;i\;} & F \\ {\scriptstyle\beta}\big\downarrow & & \\ P & & \end{array}$$

is also σ-flat (where i is the inclusion map).

Proof

We may take $X = (F\oplus P)/S$ where $S = \{(x,-\beta(x))|x\epsilon G\}$. Let $h:P\to X$ be the monomorphism defined by $h(p) = (0,p)+S$ (for $p\epsilon P$). Define $\Psi:X/h(P)\to F/G$ by $\Psi([(f,p)+S]+h(P)) = f+G$ (for $f\epsilon F$, $p\epsilon P$.

Ψ is 1-1: Suppose $f\epsilon G$. Then $(f,p)-(0,p+\beta(f))\epsilon S$ and therefore $(f,p)+S = (0,p+\beta(f))+S\epsilon h(P)$.

Ψ is clearly an R-epimorphism hence $X/h(P)\simeq F/G$. Since F is σ-flat and G is σ-pure in F, $F/G\simeq X/h(P)$ is σ-flat (3.5.1.6). Since $h(P)\simeq P$ is σ-flat it follows from (2) above that X is σ-flat.

4. Let $\{M_i\}_i$ be a collection of R-modules. Then $\bigoplus_i M_i$ is σ-flat iff M_i is σ-flat for all i. ([7], Theorem 9).

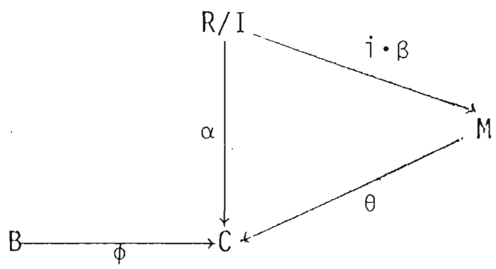
Suppose every $I\epsilon L_\sigma$ is projective. Then certainly every $I\epsilon L_\sigma$ is σ-flat. Thus if R/I is σ-torsion then I is σ-flat. The following shows that (more generally) if every $I\epsilon L_\sigma$ is projective, M is an arbitrary σ-flat module and N is a submodule N for which M/N is σ-torsion, then N is σ-flat:

5. Suppose that every $I\epsilon L_\sigma$ is projective. Then a module M is σ-flat iff every submodule N of M with M/N σ-torsion is σ-flat.

Proof

The "if" statement is trivial. Suppose therefore that M is σ-flat and let N be a submodule of M such that M/N is σ-torsion. Let $B\xrightarrow{\phi}C\rightarrow 0$ be exact, where B is injective, let $I\epsilon L_\sigma$ and let $\alpha:R/I\rightarrow C$ be a homomorphism, which factors through N. Then B is σ-injective and, by 1.2.4.6(ii), C is σ-injective. Suppose α factors through N as $\gamma\cdot\beta$. Since M/N is σ-torsion, $\gamma:N\rightarrow C$ extends to a homomorphism $\theta:M\rightarrow C$, by the σ-injectivity of C. Let $i:N\rightarrow M$ be the inclusion map.

Then the diagram



commutes and since M is σ-flat, there is a homomorphism $\mu:R/I\rightarrow B$ with $\phi\cdot\mu = \alpha$. By 3.5.1.8, N is σ-flat.

3.5.1.10 Theorem

Let R be a ring and σ a torsion radical on R-mod. Then the following are equivalent:

(i) R is σ-regular.

(ii) Every (simple) R-module is σ-flat.

(iii) Every countably generated, σ-torsion R-module is projective.

(iv) Every f.g., σ-torsion R-module is projective.

(v) Every σ-torsion R-module is σ-flat.

If, further, R is commutative then the above conditions are equivalent to:

(vi) (a) Every $I \in \mathcal{L}_\sigma$ is idempotent and

(b) Every semiprime ring, which is also an R-module, is σ-flat.

Proof

If (i) holds then, by Theorem 3.4.4, every R-module is σ-regular, hence every short exact sequence is σ-pure exact and therefore every R-module is σ-flat, proving (ii).

Conversely, if every simple R-module is σ-flat then every maximal ideal of R is σ-pure in R and R is σ-regular by 3.4.4. again. Thus (i) and (ii) are equivalent.

(i) implies (iii): Let $M = \sum_{i=1}^{\infty} Rx_i$ be a countably generated, σ-torsion module.

By Theorem 3.4.6, M is semisimple, if (i) holds. Thus M is a direct sum of cyclic, σ-torsion modules. But any cyclic, σ-torsion module is projective, by 3.4.4(v). This proves (iii).

That (iii) implies (iv) is clear.

(iv) implies (v): Let $M = \sum_i Rx_i$ be σ-torsion.

For any submodule N of Rx_i, Rx_i/N is cyclic, σ-torsion, σ-flat by (iv) and therefore N is a direct summand of Rx_i.

Thus we have shown that each Rx_i is semisimple and hence M is semisimple. But then M is a direct sum of cyclic, σ-torsion modules which are projective and therefore σ-flat by (iv) and (v) follows from 3.5.1.9(4).

(v) implies (i): If (v) holds then for every $A \epsilon L_\sigma$, R/A is σ-flat and hence A is σ-pure in R. Since R/A is σ-pure-projective, A is a direct summand of R and (i) follows.

(i) implies (vi): Suppose R is σ-regular. Then every $I \epsilon L_\sigma$ is a direct summand of R and hence I = Re for an idempotent element e of R ([2], Proposition 7.1). Since R has an identity, (vi)(a) holds. (vi)(b) is clear since if R is σ-regular every R-module is σ-flat.

Conversely, suppose (vi) holds. Suppose $M \epsilon L_\sigma$. Since R is commutative, R/M is a ring. If I/M is an ideal of the ring R/M with $(I/M)^n = 0$ then $I \epsilon L_\sigma$ and $I^n \leq M$. By (vi)(a), $I = I^n$ and therefore I/M = 0. Thus R/M has no nonzero nilpotent ideals (i.e. it is a *semiprime ring*), and by (vi)(b) it is σ-flat as an R-module. Hence M is a direct summand of R and R is σ-regular.

3.5.2 Semi-σ-Flat Modules

3.5.2.1 Definition

Let R be a ring and σ a torsion radical on R-mod. A module M is called *semi-σ-flat* iff for any exact sequence $B \to C \to 0$ where C is injective, given $I \epsilon L_\sigma$, every homomorphism $\alpha : R/I \to C$ which factors through M lifts to a homomorphism from R/I to B. (This extends the concept of a semiflat module introduced by Döman in [13]).

3.5.2.2 Remark

It follows from Theorem 3.5.1.7 that a σ-flat module is also semi-σ-flat. If R is hereditary, so that epimorphic images of injective modules are injective ([11]) then the two concepts are equivalent (by 3.5.1.8).

3.5.2.3 Lemma

Submodules of semi-σ-flat modules are semi-σ-flat.

Proof

Easy

3.5.2.4 Theorem

Let R be a ring and σ a torsion radical on R-mod.

A module M is semi-σ-flat iff for every $I \in \mathcal{L}_\sigma$ and every homomorphism $\alpha : R/I \to M$, there is an embedding of $\alpha(R/I)$ into a projective module P.

Proof

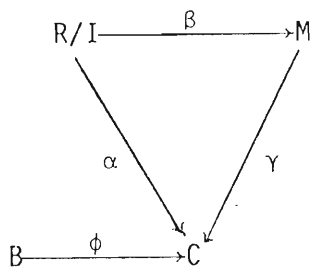
Suppose M is semi-σ-flat, $I \in \mathcal{L}_\sigma$ and $\alpha : R/I \to M$ is a homomorphism. $R/J \cong \alpha(R/I)$ is cyclic, σ-torsion and R/J is semi-σ-flat, since it is a submodule of M. Let E(R/J) be its injective hull, and let $j : R/J \to E(R/J)$ be the inclusion map. There is a projective module P and an epimorphism $\phi : P \to E(R/J)$.

Since R/J is semi-σ-flat, there is a homomorphism $\mu : R/J \to P$ such that the diagram

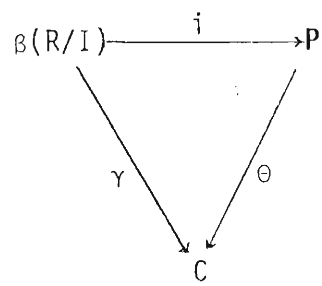
$$\begin{array}{ccc} R/J & \xrightarrow{\;1\;} & R/J \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle j} \\ P & \xrightarrow[\;\phi\;]{} & E(R/J) \end{array}$$

commutes. It then follows that μ is a monomorphism.

Conversely, suppose the stated condition holds, C is injective, $I \in \mathcal{L}_\sigma$ and $\alpha : R/I \to C$ is a homomorphism, which factors through M in the diagram



where $B \xrightarrow{\phi} C \to 0$ is exact. By assumption, there is a projective module P and an embedding $i:\beta(R/I)\to P$. Since C is injective, there is a homomorphism $\Theta:P\to C$ such that the diagram



commutes. Since P is projective, Θ lifts to a homomorphism $\mu:P\to B$ such that $\phi\cdot\mu = \Theta$. Then $\mu\cdot i\cdot\beta:R/I\to B$ satisfies $\phi\cdot\mu\cdot i\cdot\beta = \alpha$. Hence M is semi-$\sigma$-flat.

The following is the semi-σ-flat equivalent of 3.5.1.6:

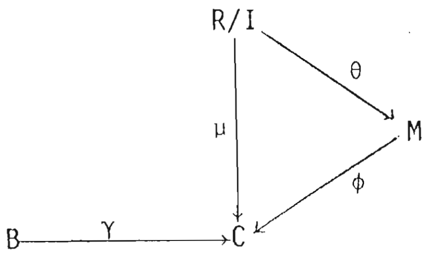
3.5.2.5 Theorem

Let R be a ring and σ a torsion radical on R-mod. An R-module M is semi-σ-flat iff there is a σ-pure exact sequence of the form $0\to K\to L\to M\to 0$, where L is semi-σ-flat.

Proof:

Only one direction is non-trivial.

Suppose $0\to K\xrightarrow{\alpha} L\xrightarrow{\beta} M\to 0$ is σ-pure exact, where L is semi-σ-flat. Let C be injective, $B\xrightarrow{\gamma} C\to 0$ exact, $I\in\mathcal{L}_\sigma$ and $\mu:R/I\to C$ a homomorphism such that the diagram



is commutative.

Since K is σ-pure in L, there is a homomorphism $\varepsilon: R/I \to L$ such that $\beta \cdot \varepsilon = \theta$. Since L is semi-$\sigma$-flat, there is a homomorphism $\delta: R/I \to B$ such that $\gamma \cdot \delta = \phi \cdot \beta \cdot \varepsilon$. But $\phi \cdot \beta \cdot \varepsilon = \phi \cdot \theta = \mu$ and it follows that M is semi-σ-flat.

3.5.2.6 **Remarks**

Let R be a ring and σ a torsion radical on R-mod.

1. For a collection $\{M_i : i \in I\}$ of R-modules, $\bigoplus_i M_i$ is semi-σ-flat iff M_i is semi-σ-flat for all i.

Proof

By 3.5.2.3, only one direction is non-trivial.

Suppose M_i is semi-σ-flat for each $i \in I$. Let $I \in \mathcal{L}_\sigma$ and suppose that $\alpha: R/I \to \bigoplus_i M_i$ is a homomorphism. Let $\delta_i: \bigoplus_i M_i \to M_i$ be the projection maps.

By 3.5.2.4, there are projective modules P_i and embeddings $k_i: (\delta_i \cdot \alpha)(R/I) \to P_i$, $i \in I$. Thus $\bigoplus_i k_i: \alpha(R/I) \to \bigoplus_i P_i$ is an embedding and it follows from 3.5.2.4 that $\bigoplus_i M_i$ is semi-σ-flat.

2. The following are equivalent for a module M:

(i) M is semi-σ-flat.

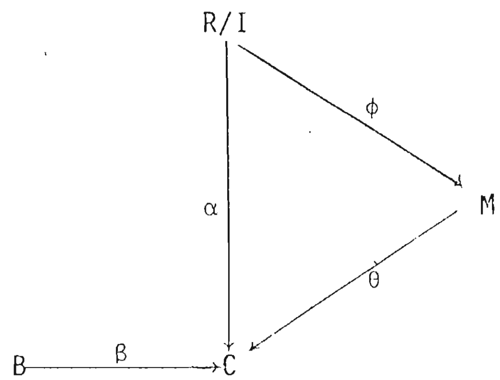
(ii) Every f.g. submodule of M is semi-σ-flat.

(iii) Every cyclic submodule of M is semi-σ-flat.

(iv) Every cyclic, σ-torsion submodule of M is semi-σ-flat.

Proof

(iv) implies (i): Let $I \in \mathcal{L}_\sigma$, suppose C is injective, $B \xrightarrow{\beta} C \to 0$ is exact and $\alpha: R/I \to C$ is a homomorphism which factors through M as follows:



Then $\phi(R/I)$ is a cyclic, σ-torsion submodule of M and, by assumption, is semi-σ-flat. That is, α factors through a semi-σ-flat module and hence there is a homomorphism $\psi: R/I \to B$ such that $\beta \cdot \psi = \alpha$. Hence (i) is proved.

The other implications are clear.

3. It follows from 3.5.2.2 and 3.5.2.3 that if R is hereditary then submodules of σ-flat modules are σ-flat. This is a remark of Rohlina mentioned on page 29 of [57].

3.5.2.7 Examples

1. Let R be a ring and σ a torsion radical on R-mod such that $\sigma(R) = 0$. Then M is semi-σ-flat iff $\sigma(M) = 0$ (iff M is σ-flat).

Proof

In view of 3.5.1.5 it only remains to prove that if M is semi-σ-flat then $\sigma(M) = 0$.

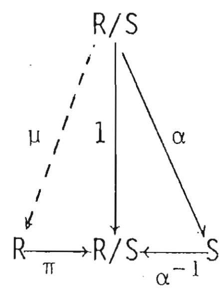
Suppose M is semi-σ-flat, T is σ-torsion and $\alpha \in \text{Hom}(T,M)$. If $x \in T$ then Rx is cyclic and σ-torsion. Let $\beta: Rx \to M$ be the restriction of α to Rx.

By 3.5.2.4, there is an embedding of $\beta(Rx)$ into a projective module P. Since $\sigma(R) = 0$, P is σ-torsion-free and hence so is $\beta(Rx)$. Since $\beta(Rx)$ is σ-torsion it follows that $\beta(Rx) = 0$ and hence $\alpha(x) = 0$. Since x was an arbitrary element of T, Hom(T,M) = 0 and therefore M is σ-torsion free.

2. A semi-σ-flat module which is not σ-flat:

Consider $R = Z_4 = \{\bar{0},\bar{1},\bar{2},\bar{3}\}$, the ring of congruence classes mod 4. $L_\sigma = \{\{\bar{0}\},\{\bar{0},\bar{2}\},R\}$ is the filter of a torsion radical, σ. $S = \{\bar{0},\bar{2}\}$, as a submodule of (the σ-flat R-module) R, is semi-σ-flat.

R/S is cyclic, σ-torsion and if $\alpha: R/S \to S$ is the isomorphism defined by mapping $\bar{1}+S$ to $\bar{2}$, there is no homomorphism $\mu: R/S \to R$ making the diagram



commute, since otherwise S would be a direct summand of R, which it is not. This shows that S is not σ-flat (3.5.1.7). (In particular this also shows that submodules of σ-flat modules need not always be σ-flat).

3.5.3 Semi-σ-Regular Rings

3.5.3.1 Definitions

1. If $A \leq B$ then A is called a *copure submodule* of B iff B/A is injective. A module M is called *copure injective* iff M is injective with respect to exact sequences of the form $0 \to A \to B$ where A is copure in B ([17]).

2. An R-module M will be called *torsionless* iff the map $\alpha: M \to M^{**}$ defined by $[\alpha(m)](\phi) = \phi(m)$ (for $\phi \in M^*$), is monic ([49]).

${}_RM$ is torsionless iff it can be embedded in a direct product of copies of R. (See [49], Chapter 5).

3. Let R be a ring and σ a torsion radical on R-mod.

Since R is σ-regular iff every R-module is σ-flat (see 3.5.1.10), the question of whether it is possible to characterize rings R (and torsion radicals σ on R-mod) for which every R-module is semi-σ-flat suggests itself. Such a ring will be called *semi-σ-regular*.

3.5.3.2 Theorem

The following are equivalent for a ring R and a torsion radical σ on R-mod:

(i) R is semi-σ-regular.

(ii) Injective R-modules are σ-flat.

(iii) The injective hull of every cyclic, σ-torsion module is σ-flat.

(iv) Direct sums of injective R-modules are semi-σ-flat.

(v) Every R-module can be embedded in a σ-flat R-module.

(vi) Every f.g. R-module can be embedded in a semi-σ-flat R-module.

(vii) Every f.g. R-module is semi-σ-flat.

(viii) Every copure submodule of an R-module M is σ-pure in M.

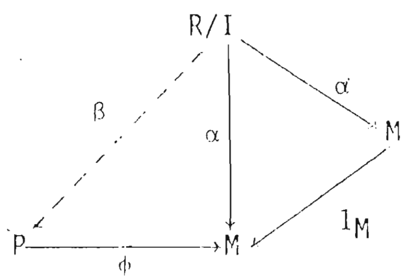
(ix) Every cyclic, σ-torsion module is semi-σ-flat.

(x) Every cyclic, σ-torsion module can be embedded in a projective module.

(xi) Epimorphic images of semi-σ-flat modules are semi-σ-flat.

Proof

(i) implies (ii): Suppose M is injective, let $I \in L_\sigma$, and let $\alpha: R/I \to M$ be a homomorphism. Let P be projective such that $P \xrightarrow{\phi} M \to 0$ is exact for some ϕ. In the diagram



(since M is semi-σ-flat, by (i)) there is a homomorphism $\beta: R/I \to P$ such that $\phi \cdot \beta = \alpha$. That is, α factors through a projective module and, by 3.5.1.4, M is σ-flat.

(ii) implies (i): Every R-module is a submodule of its injective hull which, by (ii), is semi-σ-flat. By Lemma 3.5.2.3, (i) holds.

That (ii) implies (iii) is clear.

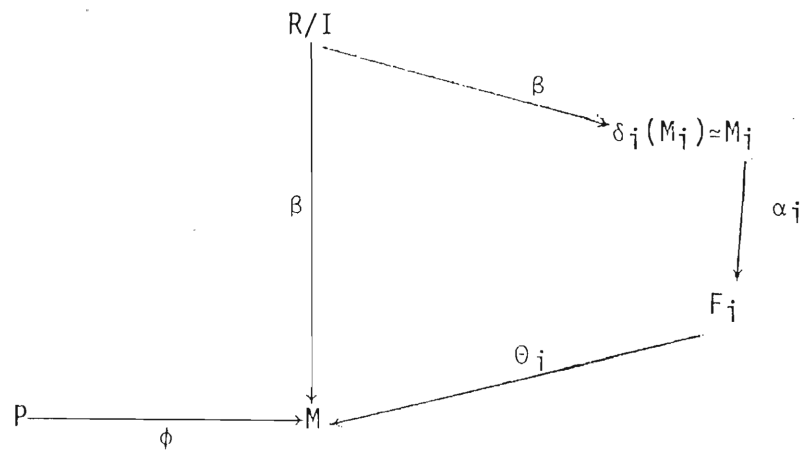
(iii) implies (ii): Suppose C is any injective R-module, and let $0 \to A \to B \xrightarrow{\pi} C \to 0$ be exact. Suppose $I \in L_\sigma$ and let $\alpha: R/I \to C$ be a homomorphism. If $i: R/I \to E(R/I)$ is the inclusion map, then there is a homomorphism $\alpha': E(R/I) \to C$ such that $\alpha' \cdot i = \alpha$. Thus α factors through E(R/I), which is σ-flat, by (iii), and hence there is, by 3.5.1.7, a homomorphism $\beta: R/I \to B$ such that $\pi \cdot \beta = \alpha$. $0 \to A \to B \xrightarrow{\pi} C \to 0$ is therefore σ-pure exact, proving that C is σ-flat.

The facts that (iv) implies (i) and that (ii) implies (v) follow from the fact that every R-module may be embedded in an injective R-module. The facts that (i) implies (iv), that (v) implies (vi), that (vi) implies (vii) and that (vii) implies (ix) are immediate.

(vi) implies (ii): Let M be an injective R-module. M is the direct limit of its f.g. submodules, $\{M_i\}_i$, say ([63]). Let F_i be semi-σ-flat modules and $\alpha_i:M_i\to F_i$ the embeddings given by (vi). Let $\delta_i:M_i\to M$ be the canonical monomorphisms.

Suppose P is projective, $P\xrightarrow{\phi}M\to 0$ is exact and R/I is cyclic, σ-torsion with $\beta:R/I\to M$ a homomorphism. $\beta(R/I)\leq\delta_i(M_i)$ for some i. Since M is injective, there are homomorphisms $\Theta_i:F_i\to M$ such that $\Theta_i\cdot\alpha_i = \delta_i$, for each i.

Hence the diagram



is commutative.

Since F_i is semi-σ-flat, there is a homomorphism $\mu:R/I\to P$ such that $\phi\cdot\mu = \beta$. By 3.5.1.4, M is σ-flat proving (ii).

(ii) implies (viii): Let A be a copure submodule of a module B. Then B/A is injective and, by (ii), B/A is σ-flat. Hence A is σ-pure in B.

(viii) implies (ix): Suppose $I \in \mathcal{L}_\sigma$. Embed R/I in its injective hull E, say. Let $0 \to K \to L \to E \to 0$ be any exact sequence with E in the third nonzero position. Since E is injective, K is copure in L and therefore K is σ-pure in L, by (viii). That is, E is σ-flat and since R/I is isomorphic to a submodule of E, (ix) follows.

(ix) implies (x): Let R/I be cyclic, σ-torsion. Then R/I is semi-σ-flat, by (ix), and applying 3.5.2.4 to the identity map on R/I we see that there is an embedding of R/I into a projective module P.

That (x) implies (i) follows from 3.5.2.6(2).

That (i) implies (xi) is clear and the fact that (xi) implies (i) follows, since every R-module is an epimorphic image of a projective module.

3.5.3.3 <u>Remarks</u>

1. 3.5.3.2(viii) is the semi-σ-regular version of 3.4.4(iv).

2. A ring R is Quasi-Frobenius iff injective R-modules are projective ([23] , Theorem 5.3). Every Quasi-Frobenius ring R, with any torsion radical σ on R-mod, is therefore an example of a ring satisfying the conditions of 3.5.3.2 since 3.5.3.2(ii) is satisfied.

3. Since not every Quasi-Frobenius ring is semisimple (see, for example, exercises 17 and 21 on page 82 of [49]), it is easy to find examples of semi-σ-regular rings which are not σ-regular.

For such a ring R, every R-module is semi-σ-flat but not every R-module is σ-flat. Hence every such ring provides us with examples of semi-σ-flat modules which are not σ-flat.

4. Every σ-regular ring is semi-σ-regular. Conversely, if R is (left) semi-hereditary and semi-σ-regular, then it is σ-regular.

Proof

Suppose that $I \in \mathcal{L}_\sigma$ and let $\alpha: R/I \to M$ be a homomorphism, where M is an arbitrary R-module. By assumption, M is semi-σ-flat. By Theorem 3.5.2.4, there is an embedding of $\alpha(R/I)$ into a projective R-module. $\alpha(R/I)$ is a f.g. submodule of a projective module and since R is semi-hereditary, $\alpha(R/I)$ is projective. Hence α factors through a projective R-module and by Lemma 3.5.1.4, M is σ-flat. Hence R is σ-regular (3.5.1.10).

5. Let R be a ring with unit and without divisors of zero. Then the following are equivalent (where σ is a torsion radical on R-mod):

(a) R is σ-regular.

(b) R is semi-σ-regular.

(c) $\mathcal{L}_\sigma = \{0,R\}$ or $\mathcal{L}_\sigma = \{R\}$.

(d) R is a division ring or every R-module is σ-torsion free.

Proof

That (a) implies (b) is obvious.

(b) implies (c): Suppose R is semi-σ-regular. By 3.5.3.2(x), if $I \in \mathcal{L}_\sigma$ then R/I is torsionless. Let $\alpha: R/I \to R^{(i)}$ be an embedding of R/I into a direct product of copies of R. Suppose $I \neq R$, then $\alpha(R/I)$ is nonzero and hence $\alpha(1+I) = (a_i)_i \neq 0$. If $s \in I$ then, $\alpha(s+I) = 0 = (sa_i)_i$. Since R has no divisors of zero, s = 0 and hence I = 0. This proves (c).

(c) implies (d): If $\mathcal{L}_\sigma = \{0,R\}$ is to be the filter of a torsion radical, then clearly R has only the two trivial left ideals 0 and R.

Alternatively, if $\mathcal{L}_\sigma = \{R\}$, then every R-module is σ-torsion-free. This proves (d).

(d) implies (a): If the conditions of (d) hold then there are only two possible radical filters, on R-mod: $\mathcal{L}_\sigma = \{0,R\}$ or $\mathcal{L}_\sigma = \{R\}$. Thus in all possible cases, every $I \in \mathcal{L}_\sigma$ is a direct summand and (a) follows.

CHAPTER FOUR

σ-PERFECT RINGS AND A RELATIVE JACOBSON RADICAL

§4.1. INTRODUCTION

Let R be a ring and σ a torsion radical on R-mod. In this chapter we introduce two related constructions: σ-perfect rings, and a new "radical", denoted by J_σ.

σ-perfect rings are an extension of the concept of a perfect ring (defined in [4]).

In 4.2.9 and 4.2.11 we generalize a famous result of H. Bass (Theorem P of [4]).

Secondly, we introduce a new relativized form of the Jacobson Radical, (defined in terms of σ-flatness). We show that many of the properties of the Jacobson Radical as well as those of a radical of J.S. Golan ([35], Chapter 24), extend to J_σ and consider, *inter alia*, the case when $J_\sigma(M) = 0$ for all modules M. (4.3.20).

§4.2 σ-PERFECT RINGS

4.2.1 Definitions

We call a submodule S of a module P, *small* in P, ($S << P$), iff whenever $K+S = P$ for a submodule K of P, then $K = P$ ([11]). H. Bass (see [4]) calls an exact sequence $0 \to S \to P \xrightarrow{\alpha} M \to 0$, where P is projective and $S = \ker\alpha$ is small in P, a *projective cover* of M. A ring is called *perfect* iff every R-module has a projective cover ([4]).

It is well known that R is perfect iff every flat R-module is projective ([30], Theorem 3.1, page 7). We use this characterization of a perfect ring to extend the concept to σ-purity:

4.2.2 Definition

Let R be a ring and σ a torsion radical on R-mod. A ring R is called σ-*perfect* iff every σ-flat module is projective.

4.2.3 Remark

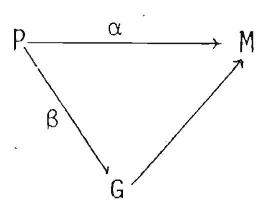
In Theorem 3 of [28] D. J. Fieldhouse shows that a projective module can have no nonzero small (Cohn) pure submodules. Thus if $0 \to S \to P \to M \to 0$ is a projective cover of M then S contains no nonzero (Cohn) pure submodules of P. We use this result to extend the concept of a projective cover, as follows:

4.2.4 Definition

An exact sequence $0 \to K \to P \to M \to 0$ where P is projective and K contains no nonzero σ-pure submodules of P is called a σ-*projective cover*.

4.2.5 Lemma

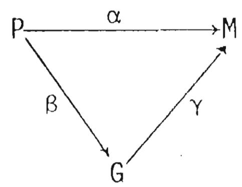
An exact sequence $0 \to \ker\alpha \to P \xrightarrow{\alpha} M \to 0$ is a σ-projective cover for M iff P is projective and for any commutative diagram



where β is epic and G is σ-flat, β must be an isomorphism.

Proof

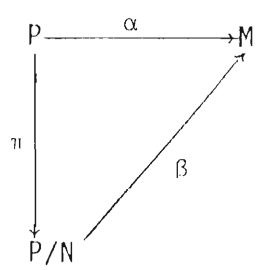
Suppose $0 \to \ker\alpha \to P \xrightarrow{\alpha} M \to 0$ is a σ-projective cover of M and the commutative diagram



is given where β is epic and G is σ-flat. Let $N = \ker\beta$, then clearly $N \leq \ker\alpha$. Since P/N is σ-flat, N is σ-pure in P. Since $0 \to \ker\alpha \to P \xrightarrow{\alpha} M \to 0$ is a σ-projective cover, $N = 0$ and β is an isomorphism.

Conversely, suppose the stated condition is satisfied by the exact sequence $0 \to \ker\alpha \to P \xrightarrow{\alpha} M \to 0$, where P is projective, and let N be a σ-pure submodule of P contained in $\ker\alpha$.

Then $\beta : P/N \to M$ defined by $\beta(p+N) = \alpha(p)$ (for $p \in P$) is a well-defined homomorphism. Since N is σ-pure in P and P is σ-flat, P/N is σ-flat, by 3.5.1.6. If $\pi : P \to P/N$ is the canonical epimorphism then the diagram



commutes and, by assumption, π is monic. Thus $N = 0$ and it follows that $0 \to \ker\alpha \to P \xrightarrow{\alpha} M \to 0$ is a σ-projective cover of M.

4.2.6 Definition

Let R be a ring and σ a torsion radical on R-mod. Let M be an R-module and $\ldots\ldots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ a projective resolution of M. The smallest integer $n \geq 0$ for which $\operatorname{im} d_n$ is σ-flat (if it exists) is called the *σ-flat dimension*, σfdM, of M. If no such integer n exists we say $\sigma\text{fd}M = \infty$.

4.2.7 Lemma

σfdM is uniquely determined.

Proof

Suppose $\ldots P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ and $\ldots \bar{P}_1 \xrightarrow{\bar{d}_1} \bar{P}_0 \xrightarrow{\bar{d}_0} M \to 0$ are two projective resolutions for a module M and $\mathrm{im}d_n = \mathrm{ker}d_{n-1}$ is σ-flat. (It is clear that $\mathrm{im}d_0$ is σ-flat iff M is σ-flat iff $\mathrm{im}\bar{d}_0$ is σ-flat and we may assume that $n \geq 1$). It is then easy to verify (see e.g. Theorem 8.8 of [45]) that $\mathrm{ker}\bar{d}_{n-1} \oplus P_{n-1} \oplus P_{n-2} \oplus \ldots \oplus P_0$ is isomorphic to $\mathrm{ker}d_{n-1} \oplus \bar{P}_{n-1} \oplus \ldots \oplus \bar{P}_0$. Since this second module is σ-flat, $\mathrm{ker}\bar{d}_{n-1} = \mathrm{im}\bar{d}_n$ is σ-flat (by 3.5.1.9(4)).

4.2.8 Remark

Let $0 \to K \to P \to M \to 0$ be exact where P is projective and M is not σ-flat. Then $\sigma\mathrm{fd}M = \sigma\mathrm{fd}K + 1$.(Dual of 2.5.5(i)).

4.2.9 Theorem

The following are equivalent for a ring R and a torsion radical σ on R-mod:

(i) R is σ-perfect.

(ii) Every σ-flat module has a σ-projective cover.

(iii) Every σ-flat module is σ-pure projective.

(iv) A σ-pure submodule of a σ-flat module is a direct summand.

(v) $\sigma\mathrm{fd}M = \mathrm{pr}M$ for all R-modules M.

(vi) A direct sum of modules each of which has σ-flat dimension $\leq n$ has projective dimension $\leq n$.

(vii) A direct limit of σ-flat modules, such that the canonical $\delta_i : M_i \to \varinjlim_I M_i$ are all monic, is projective.

Proof

(i) implies (ii): Suppose R is σ-perfect and let F be a σ-flat module. By assumption, F is projective and $0 \to 0 \to F \xrightarrow{1} F \to 0$ is a σ-projective cover for F.

(ii) implies (iii): Let F be a σ-flat module with a σ-projective cover $0\to K\to P\to F\to 0$. Since F is σ-flat, K is σ-pure in P and by definition of a σ-projective cover, K = 0. Thus $F\simeq P$ is projective (and therefore σ-pure projective).

(iii) implies (iv): Let F be a σ-flat module, and let K be a σ-pure submodule of F. By 3.5.1.6, F/K is σ-flat and therefore σ-pure projective by (iii). But then the sequence $0\to K\xrightarrow{i}F\xrightarrow{\pi}F/K\to 0$ splits, proving (iv).

(iv) implies (v): Suppose prM = n and σfdM = m for some module M. Then there is a projective resolution $\ldots P_2\xrightarrow{d_2}P_1\xrightarrow{d_1}P_0\xrightarrow{d_0}M\to 0$ with $\text{im}d_n$ projective: $\text{im}d_n$ is therefore σ-flat and it follows that $m\leq n$. Further, by 4.2.7, $\text{im}d_m$ is σ-flat. Consider the exact sequence $0\to\ker d_m\to P_m\xrightarrow{d_m}\text{im}d_m\to 0$. Since $\text{im}d_m$ is σ-flat, $\ker d_m$ is σ-pure in P_m and, by (iv), the sequence splits. Thus $\text{im}d_m$ is projective and it follows that $n\leq m$. Hence n = m, proving (v).

(v) implies (vi): Suppose (v) holds and $\{M_i\}$ is a collection of modules, where $\sigma\text{fd}M_i\leq n$ for all i. Let $\ldots\to P_{i_1}\xrightarrow{d_{i_1}}P_{i_0}\xrightarrow{d_{i_0}}M_i\to 0$ be a projective resolution for M_i. By (v), $\text{pr}M_i\leq n$ and hence $\text{im}d_{i_n}$ is projective for all i (see page 60 of [45]).

Thus $\ldots\to\bigoplus_i P_{i_1}\xrightarrow{d_1}\bigoplus_i P_{i_0}\xrightarrow{d_0}\bigoplus_i M_i\to 0$ is a projective resolution for $\bigoplus_i M_i$ (where $d_m = \bigoplus_i d_{i_m}$ for all m) and $\text{im}d_n = \bigoplus_i \text{im}d_{i_n}$ is projective. Hence $\text{pr}\bigoplus_i M_i\leq n$ as required.

(v) implies (vii): Let $\{M_i:\delta_i^j\}$ be a directed system of σ-flat modules where the canonical $\delta_i:M_i\to M = \varinjlim_I M_i$ are all monic. Let $I\in\mathcal{L}_\sigma$ and let $\alpha:R/I\to M$ be an R-homomorphism. Then $\text{im}\alpha\leq\delta_i(M_i)\simeq M_i$ for some i ([45], Lemma 4.3) and by 3.5.1.4, α factors through a projective module. By 3.5.1.4 again, M is σ-flat. Thus σfdM = 0 and M is projective by (v).

That (vi) or (vii) implies (i) follows easily since any given σ-flat module M is trivially a direct sum of σ-flat modules (and therefore also a direct limit of σ-flat modules, with the canonical δ_i monic) and hence if either (vi) or (vii) hold then M must be projective.

4.2.10 Examples

1. Any semisimple ring R is σ-perfect for all torsion radicals σ on R-mod.

2. Suppose that R-mod is Σ-*cyclic* (i.e. every R-module is a direct sum of cyclic R-modules). Examples are Z_n, n = 1,2.... (see [21], 25.0.1) and any Artinian principal ideal (i.e. *uniserial*) ring (see [52]). Let L_σ consist of all the (left) ideals of R. Then R is σ-perfect.

Proof

Let M be any (left) R-module. Then M is a direct sum of cyclic modules which are σ-torsion since R is. Thus M is σ-pure projective and by 4.2.9(iii), R is σ-perfect.

3. Let R be any perfect, Prüfer ring. (In particular, we may take R = Q, the ring of rational numbers. Since $_QQ$ is simple R is a Prüfer ring and the Jacobson radical J = 0. Since R/J is therefore (semi) simple and J is nilpotent, it follows from [2], Theorem 28.4 that R is a perfect ring). Take L_σ = {Left Ideals of R}, then R is σ-perfect.

Proof

If X is f.p. then X is a direct summand of a direct sum of cyclic, (σ-torsion) modules (Proposition 5 (page 706) of [71]). Thus X is σ-pure projective.

If M is σ-flat then every short exact sequence of the form $0 \to K \to L \to M \to 0$ is σ-pure exact and by the above, every f.p. module X is projective with respect to such a sequence which means that every such sequence is Cohn pure exact and hence M is flat. Since R is perfect, M is projective and the result follows.

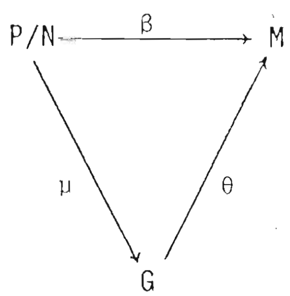
4.2.11 Theorem

Let R be a ring and σ a torsion radical on R-mod such that every $I \in \mathcal{L}_\sigma$ is f.g. Then R is σ-perfect iff every R-module has a σ-projective cover.

Proof

Let R be σ-perfect and let M be any R-module. Let P be projective with $\alpha: P \to M$ an epimorphism. Let $S = \{N \mid N$ is a σ-pure submodule of P contained in $\ker\alpha\}$. S is nonempty, since $0 \in S$. S is inductive, by 1.3.3.5, and has a maximal element N, by Zorn's Lemma. Let $\beta: P/N \to M$ be the epimorphism defined by $\beta(p+N) = \alpha(p)$, (for $p \in P$). Since N is σ-pure in P and P is σ-flat, it follows from 3.5.1.6 that P/N is σ-flat and therefore projective, by assumption. We use 4.2.5 to show that $0 \to \ker\beta \to P/N \xrightarrow{\beta} M \to 0$ is a σ-projective cover for M.

Suppose therefore that the diagram



commutes, where G is σ-flat and μ is an epimorphism. Then $G \simeq P/K$ where $\ker\mu = K/N$ (for some submodule K of P containing N). Since G is σ-flat, K is σ-pure in P and it is easy to verify that $K \leq \ker\alpha$. By the maximality of N, $K = N$ and μ is an isomorphism as required.

Conversely, if every R-module has a σ-projective cover then it follows from 4.2.9 that R is σ-perfect.

4.2.12 Remarks

1. If every $I \epsilon L_\sigma$ is f.g. and R is a σ-perfect ring then σ-projective covers are unique up to isomorphism (in the category of short exact sequences).

Proof

Let $A: 0 \to K_1 \xrightarrow{i_1} P_1 \xrightarrow{\pi_1} M \to 0$ and $B: 0 \to K_2 \xrightarrow{i_2} P_2 \xrightarrow{\pi_2} M \to 0$ be two σ-projective covers for the R-module M. Let F be a flat R-module and $0 \to K \to L \to F \to 0$ any short exact sequence with F in the third nonzero position. K is then Cohn pure in L and by 1.3.6.3 it is σ-pure in L. Thus F is σ-flat and therefore projective. Hence R is a perfect ring and M has a projective cover $C: 0 \to K \to P \xrightarrow{\pi} M \to 0$. Complete the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \xrightarrow{i_1} & P_1 & \xrightarrow{\pi_1} & M & \longrightarrow & 0 \\ & & \big\downarrow f_1 & & \big\downarrow f_2 & & \big\downarrow & & \\ 0 & \longrightarrow & K & \xrightarrow[i]{} & P & \xrightarrow[\pi]{} & M & \longrightarrow & 0 \end{array}$$

commutatively, using the projectivity of P_1. A diagram chase reveals that $K + \text{im} f_2 = P$ and since K is small in P, f_2 is epic.

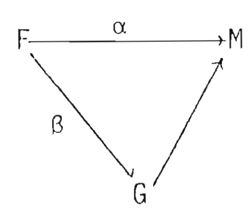
Consider the diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{\pi_1} & M \\ & {\scriptstyle f_2}\searrow \quad \nearrow{\scriptstyle \pi} & \\ & P & \end{array}$$

P is σ-flat and by Lemma 4.2.5, f_2 is an isomorphism. A simple diagram chase shows that f_1 is an isomorphism. Hence A and C are isomorphically equivalent short exact sequences. Similarly for B and C and hence the sequences A and B are isomorphic.

2. In the course of proving 4.2.11 we proved that if every $I \in L_\sigma$ is f.g. then for each module M there is an exact sequence $0 \to K \to F \to M \to 0$ such that F is σ-flat and K contains no nonzero σ-pure submodules of F. We call such an exact sequence a *σ-flat cover* for M.

3. A simple adaptation of 4.2.5 shows that an exact sequence $0 \to \ker\alpha \to F \xrightarrow{\alpha} M \to 0$ is a σ-flat cover iff F is σ-flat and for every commutative diagram:



where G is σ-flat and β is epic, β must be an isomorphism.

4. Suppose that every $I \in L_\sigma$ is f.g. Then a module M is σ-flat iff in every σ-flat cover $0 \to K \to F \to M \to 0$ of M, we must have $K = 0$.

Proof

Suppose the stated condition is valid for a module M. By (2) above, there is a σ-flat cover $0 \to K \to F \to M \to 0$ for M and, by assumption, $K = 0$ proving that M is σ-flat.

The converse follows by definition of a σ-flat cover.

5. Let R be a ring and σ a torsion radical on R-mod such that R is σ-perfect. Then the following are equivalent:

(a) Every $I \in L_\sigma$ is f.g. and every σ-pure injective module is copure injective.

(b) R is semi-σ-regular.

(c) R is Quasi-Frobenius.

(d) R is Noetherian and every R-module is copure injective.

Proof

(a) implies (b): Let M be any injective R-module. If $0 \to K \xrightarrow{i} L \to M \to 0$ is exact and $\alpha: K \to P$ is a homomorphism where P is σ-pure injective, then, by (a), P is copure injective and, since K is copure in L, there is a homomorphism $\beta: L \to P$ such that $\beta \cdot i = \alpha$. Since every $I \in \mathcal{L}_\sigma$ is f.g. it follows from 1.4.12 that K is σ-pure in L. That is, M is σ-flat.

We have therefore proved that every injective R-module is σ-flat. By 3.5.3.2, this is equivalent to (b).

(b) implies (c): If (b) holds, injective modules are σ-flat and, since R is σ-perfect, this means that every injective R-module is projective, whence R is Quasi-Frobenius.

(c) implies (d): If R is Quasi-Frobenius then it is Noetherian ([49]). Since every injective module is projective every copure exact sequence splits and therefore (d) holds. That (d) implies (a) is clear.

6. Let R be a Prüfer ring and let $\mathcal{L}_\sigma$ = {Left Ideals of R}. Then R is Quasi-Frobenius iff it is σ-perfect and semi-σ-regular.

Proof

If R is Q.F. it has minimum condition on right ideals ([49], Lemma 2, page 77) and hence it is a perfect ring ([2], Theorem 28.4). By 4.2.10(3), R is σ-perfect. By 3.5.3.3 (2), R is semi-σ-regular.

Conversely if R is σ-perfect and semi-σ-regular then, by (5) above, R is Quasi-Frobenius.

§4.3 A RELATIVIZED JACOBSON RADICAL

4.3.1 Introduction

J.S. Golan in Chapter 24 of [35] illustrates one way of defining a Jacobson Radical relative to a torsion theory. His radical is defined using a type of "purity" which does not form a Pure Theory (namely : $N \leq M$ is "pure" in M iff $\sigma(M/N) = 0$). For this reason we believe that it is relevant to relativize the Jacobson Radical in terms of σ-purity. It also turns out that our radical, $J_\sigma(M)$, contains the Jacobson Radical, RadM, of M.

4.3.2 Definitions

1. If M is an R-module we will denote by *RadM* the intersection of all the maximal submodules of M ([2]). (This is the classical "*Jacobson Radical*"). If R is a ring, the Jacobson Radical of R is defined to be $Rad({}_RR)$.

2. Let R be a ring and σ a torsion radical on R-mod. We define, for an R-module M, $J_\sigma(M) = \cap\{K \leq M \mid M/K \text{ is simple and } \sigma\text{-flat}\}$. (If there are no such submodules we set $J_\sigma(M) = M$).

4.3.3 Remarks

1. $RadM \leq J_\sigma(M)$ for all modules M.

2. If F is σ-flat then $J_\sigma(F) = \cap\{K \leq F \mid K \text{ is maximal and } \sigma\text{-pure in } F\}$ (apply 3.5.1.6).

3. If R is σ-regular then $J_\sigma(M) = RadM$ for all R-modules M.

4. If P is a f.g., projective module then $J_\sigma(P) = RadP$ iff $J_\sigma(P) << P$.

Proof

Suppose $J_\sigma(P)$ = RadP. By Proposition 17.10 of [2], RadP = J.P (where J = $\mathrm{Rad}_R R$) and by Nakayama's Lemma ([2], 15.13), RadP<<P, that is, $J_\sigma(P) << P$.

Conversely, if $J_\sigma(P) << P$ then $J_\sigma(P) \leq \sum \{K \leq P \mid K << P\}$ = RadP ([2], 9.13) and therefore RadP = $J_\sigma(P)$.

5. If R is σ-perfect and M is an R-module then $J_\sigma(M) = \cap\{K \leq M \mid K$ is maximal in M and M/K is projective$\}$. Hence $J_\sigma(R) = \cap\{I \leq R \mid I \oplus K = R$ for some minimal left ideal K of R$\}$.

6. $J_\sigma \neq$ Rad:

(i) Let R be any local ring such that the maximal ideal, M, is not a direct summand of R. Take L_σ as the set of all left ideals in R, then M is not σ-pure in R and hence $J_\sigma(R) = R$ whereas Rad R = M.

(ii) Let R = $\mathbb{Z}$ and let L_σ be the set of nonzero ideals of R. Since every nonzero ideal of R is essential no maximal ideal can be σ-pure in R and hence $J_\sigma(R) = R$ (while RadR = 0).

4.3.4 Lemma

Let R be a ring and σ a torsion radical on R-mod. If M is an R-module, $J_\sigma(M) = \cap\{\ker h \mid h \in \mathrm{Hom}_R(M,U),\ {}_RU$ simple and σ-flat$\}$.

Proof

Let N be any submodule of M such that M/N is simple and σ-flat.

If $\pi: M \to M/N$ is the canonical epimorphism then $\ker\pi$ = N and hence $\cap\{\ker h \mid h \in \mathrm{Hom}(M,U)$, U simple and σ-flat$\} \leq N$. Since $J_\sigma(M)$ is the intersection of all such N, $\cap\{\ker h \mid h \in \mathrm{Hom}(M,U)$, U simple and σ-flat$\} \leq J_\sigma(M)$.

Conversely, suppose $0 \neq h \in \mathrm{Hom}(M,U)$ where U is simple and σ-flat. Then im h = U and $M/\ker h \cong U$. Hence $M/\ker h$ is simple and σ-flat and $J_\sigma(M) \leq \ker h$, by definition. That is, $J_\sigma(M) \leq \cap\{\ker h \mid h \in \mathrm{Hom}(M,U)$, U is simple and σ-flat$\}$.

4.3.5 Theorem

Let R be a ring and σ a torsion radical on R-mod. For any R-module M, $J_\sigma(M)$ is the unique smallest submodule K of M such that M/K is cogenerated by $\mathcal{U} = \{{}_RU \mid U$ is simple and σ-flat$\}$.

Proof

Let the family $\{K \mid M/K \in \mathcal{U}\}$ be indexed by I.

If $x \in M$ we define $g(x+J_\sigma(M))$ to be that element of $\prod_{\alpha\in I} M/K_\alpha$ whose α-th component is $x+K_\alpha$, $\alpha \in I$ (g is therefore a homomorphism from $M/J_\sigma(M)$ to $\prod_{\alpha\in I} M/K_\alpha$). Since $J_\sigma(M) = \bigcap_\alpha K_\alpha$, g is a monomorphism. This shows that $M/J_\sigma(M)$ is cogenerated by $\mathcal{U}$.

To prove minimality, suppose $K \leq M$ and that $h: M/K \to \prod_\alpha U_\alpha$ is a monomorphism where $U_\alpha \in \mathcal{U}$ for all $\alpha \in I$. Let $\pi: M \to M/K$ be the canonical epimorphism and $\pi_\alpha: \prod_\alpha U_\alpha \to U_\alpha$ the projection maps, $\alpha \in I$. By 4.3.4,
$J_\sigma(M) \leq \bigcap_\alpha \ker(\pi_\alpha \cdot h \cdot \pi) \leq \ker h \cdot \pi = \ker \pi = K$.

4.3.6 Corollary

The following are equivalent for a ring R and a torsion radical σ on R-mod:

(i) ${}_RR$ is cogenerated by the class $\mathcal{U} = \{{}_RU \mid U$ is a simple, σ-flat R-module$\}$.

(ii) $J_\sigma(R) = 0$.

(iii) R is isomorphic to a subdirect sum of simple, σ-flat modules.

Proof

An easy consequence of 4.3.5.

4.3.7 Theorem

J_σ is a radical (in the sense of [67]). That is, J_σ is a subfunctor of the identity functor on R-mod and $J_\sigma(N/J_\sigma(N)) = 0$ for all modules N.

Proof

(i) Let $f:M\to N$ be an R-homomorphism. We show that $f(J_\sigma(M))\leq J_\sigma(N)$.

Suppose U is simple and σ-flat and $h\in Hom(N,U)$.

By 4.3.4, $J_\sigma(M) = \cap\{\ker g \mid g\in Hom(M,U),$ U simple and σ-flat$\}$. Thus if $x\in J_\sigma(M)$, $(h\cdot f)(x) = 0$ and therefore $f(x)\in \ker h$. That is, $f(x)\in \cap\{\ker h \mid h\in Hom(N,U),$ U is simple and σ-flat$\} = J_\sigma(N)$.

(ii) Let N be any R-module and let $M = N/J_\sigma(N)$. Then, $M/(0)$ is cogenerated by $\mathcal{U}$ = {Simple, σ-flat R-modules} (4.3.5). By the minimality clause of 4.3.5, $J_\sigma(M) = 0$. That is, $J_\sigma(N/J_\sigma(N)) = 0$.

4.3.8 **Definition**

Let R be a ring and σ a torsion radical on R-mod. We call a submodule N of a module M *σ-pure-small in M* iff whenever $N+N' = M$ for a σ-pure submodule N' of M we must have $N' = M$. (Thus every small submodule is σ-pure-small).

4.3.9 **Remarks**

1. $J_\sigma(M)$ contains all the σ-pure-small submodules of M.

Proof

Suppose N is σ-pure-small in M and K is a submodule of M such that M/K is σ-flat and simple. If $N\nleq K$ then by maximality of K, $N+K = M$. Since K is σ-pure in M, $K = M$. This is contradictory, hence $N\leq K$ and therefore $N\leq J_\sigma(M)$.

2. Suppose R is a local ring. Then R is σ-regular iff $J_\sigma(R) = \mathrm{Rad}R$.

Proof

Suppose $J_\sigma(R)$ = RadR. Then the unique maximal ideal of R is also σ-pure in R. By 3.4.4, R is σ-regular.

The other implication follows trivally.

4.3.10 Remark

$J_\sigma(R)$ is a two-sided ideal of R and hence if $J_\sigma(R) \neq R$ then $J_\sigma(R) = \cap \{Ann\ U | U$ is a simple, σ-flat R-module$\}$.

Proof

That $J_\sigma(R)$ is two-sided follows directly from 4.3.7 ([67]).

Let U be simple and σ-flat, then U is cyclic, of the form $U = Rx$ (say). By definition, $J_\sigma(R) \leq Ann\,x$ and hence $J_\sigma(R).U = J_\sigma(R).x = 0$.

Conversely if $x(\in R)$ annihilates every simple, σ-flat module and K is a left ideal of R such that R/K is simple and σ-flat, then $x(R/K) = 0$ and hence $x \in K$. Thus $x \in \cap \{K \leq R | R/K$ is simple and σ-flat$\} = J_\sigma(R)$.

4.3.11 Remark

Let $\{M_i\}_{i \in I}$ be a collection of R-modules. Then $J_\sigma(\bigoplus_i M_i) = \bigoplus_i J_\sigma(M_i)$.

Proof

Follows from 4.3.7 (see [67]).

4.3.12 **Remark**

Let R be a ring and σ a torsion radical on R-mod.

$J_\sigma(R).M \leq J_\sigma(M)$ for all R-modules M and if P is a f.g., projective module, then $J_\sigma(R).P = J_\sigma(P)$.

Proof

That $J_\sigma(R).M \leq J_\sigma(M)$ for all R-modules M follows from 4.3.7 ([67]).

Suppose, therefore, that P is a f.g., projective module. Then there exists a f.g., free module F such that $P \oplus P' = F$ for some $P' \leq F$. (Thus $F = R^{(A)}$ for some finite set A).

Hence $J_\sigma(P \oplus P') = J_\sigma(P) \oplus J_\sigma(P') = J_\sigma(R^{(A)}) = (J_\sigma(R))^{(A)}$ (4.3.11). Since $J_\sigma(R)$ is a two-sided ideal of R (4.3.10), $J_\sigma(R).R^{(A)} \leq (J_\sigma(R))^{(A)}$.

Conversely, suppose $(a_j)_j \in (J_\sigma(R))^{(A)}$ and let $in_j : R \to R^{(A)}$ be the canonical injections, $j \in A$. Then $a_i . in_i(1) \in J_\sigma(R).R^{(A)}$ for all i. Hence $(a_i)_i = \sum_i a_i . in_i(1) \in J_\sigma(R).R^{(A)}$. We have therefore shown that

$(J_\sigma(R))^{(A)} = J_\sigma(R).R^{(A)}(= J_\sigma(R).F)$. Hence, by the above, we have that $J_\sigma(P) \oplus J_\sigma(P') = J_\sigma(R).F \leq J_\sigma(R).P \oplus J_\sigma(R).P'$.

Let $x \in J_\sigma(P)$, then there are elements $a \in J_\sigma(R).P (\leq J_\sigma(P))$ and $b \in J_\sigma(R).P' (\leq J_\sigma(P'))$ such that $x = a+b$. Thus $x-a = b \in J_\sigma(P) \cap J_\sigma(P') = 0$. That is $x = a \in J_\sigma(R).P$, proving the result.

4.3.13 **Corollary**

Let R be a von Neumann regular ring and σ a torsion radical on R-mod such that every $I \in L_\sigma$ is f.g. Then $J_\sigma(R) = 0$.

Proof

By Theorem 4 of [32] every submodule of every R-module is Cohn pure and therefore σ-pure by 1.3.6.3. That is, R is σ-regular and hence $J_\sigma(R)$ = Rad R = 0 (since R is von Neumann regular (see [2])).

4.3.14 Definition

Let R be a ring and σ a torsion radical on R-mod. We call a module M *σ-local* iff M has a unique maximal submodule, which is also σ-pure in M. A ring R is called *σ-local* iff R is σ-local as a left R-module.

4.3.15 Theorem

The following are equivalent (for a ring R and a torsion radical σ on R-mod):

(i) $J_\sigma(R)$ is maximal and σ-pure in R.

(ii) R has one and only one ideal which is both maximal and σ-pure in R.

(iii) $J_\sigma(R)$ is σ-pure in R and every nonzero element of $R/J_\sigma(R)$ has a left inverse.

(iv) $J_\sigma(R)$ is σ-pure in R and $J_\sigma(R) = \{x \in R \mid x+J_\sigma(R)$ has no left inverse$\}$.

Proof

That (i) implies (ii) is clear from the definition of $J_\sigma(M)$.

(ii) implies (iii): If (ii) holds then clearly $J_\sigma(M)$ is the unique maximal, σ-pure ideal. If $x+J_\sigma(R) \neq 0$ then $x \notin J_\sigma(R)$ and by maximality, $J_\sigma(R)+Rx = R$. Hence there exist elements $r \in R$ and $j \in J_\sigma(R)$ such that $1 = rx+j$. From this it follows that $1+J_\sigma(R) = (r+J_\sigma(R))(x+J_\sigma(R))$, proving (iii).

(iii) implies (iv): Suppose $x+J_\sigma(R)$ has no left inverse in $R/J_\sigma(R)$. By (iii), $x+J_\sigma(R) = 0$ and $x\in J_\sigma(R)$. Conversely, if $x\in J_\sigma(R)$, then clearly $x+J_\sigma(R)$ (= 0) has no left inverse in $R/J_\sigma(R)$. Hence $J_\sigma(R) = \{x\in R \mid x+J_\sigma(R)$ has no left inverse in $R/J_\sigma(R)\}$.

(iv) implies (i): If $x\notin J_\sigma(R)$ then, by (iv), there is an $r\in R$ such that $rx+J_\sigma(R) = 1+J_\sigma(R)$. Hence $1\in Rx+J_\sigma(R)$, $J_\sigma(R)+Rx = R$ and we have proved that $J_\sigma(R)$ is maximal in R. $J_\sigma(R)$ is σ-pure in R by (iv).

4.3.16 Theorem

Let R be a ring and σ a torsion radical on R-mod. Then the following are equivalent for a projective R-module P:

(i) P is the projective cover of a simple, σ-flat R-module.

(ii) $J_\sigma(P)$ is the unique maximal, small and σ-pure submodule of P.

(iii) $J_\sigma(P)$ is small and maximal in P.

(iv) $J_\sigma(P)\neq P$ and if $x\in P-J_\sigma(P)$, then $Rx = P$.

(v) P is isomorphic to a direct summand of R, and P is σ-local.

(vi) P is f.g. and σ-local.

Proof

(i) implies (ii): Let $0\to K\to P\xrightarrow{\alpha} M\to 0$ be a projective cover, where M is simple and σ-flat.

Then $P/K\simeq M$ and by definition of $J_\sigma(P)$, $J_\sigma(P)\leq K$. Since $K<<P$, $K\leq\sum\{S\leq P \mid S<<P\} = \mathrm{Rad}P\leq J_\sigma(P)$. Thus $J_\sigma(P) = K$ and $J_\sigma(P)$ is therefore maximal, small and σ-pure in P.

If S is any other maximal, σ-pure submodule of P, then P/S is simple and σ-flat (by 3.5.1.6). Hence $J_\sigma(P)\leq S$ and by maximality of $J_\sigma(P)$, $J_\sigma(P) = S$. This proves uniqueness.

That (ii) implies (iii) is clear.

(iii) implies (iv): Since $J_\sigma(P) << P$, $J_\sigma(P) \neq P$. Let $x \in P - J_\sigma(P)$. Since $J_\sigma(P)$ is maximal in P, $J_\sigma(P)+Rx = P$. Since $J_\sigma(P) << P$, $Rx = P$.

(iv) implies (v): Let $x \in P - J_\sigma(P)$. By (iv), $P = Rx \simeq R/\text{Ann}x$. Since P is projective, Ann$x$ is a direct summand of R and hence P is isomorphic to a direct summand of R.

If $x \in P - J_\sigma(P)$ then (iv) implies that $J_\sigma(P)+Rx = P$, hence $J_\sigma(P)$ is maximal in P. Suppose $J_\sigma(P)+S = P$ for a submodule S of P. If $x \in S - J_\sigma(P)$, then it follows from (iv) that $P = Rx \leq S$. Hence $S = P$ and $J_\sigma(P) << P$. Thus $J_\sigma(P) \leq \sum\{K \leq P \mid K << P\} = \text{Rad}P$ and it follows that $J_\sigma(P) = \text{Rad}P$. This means that RadP is maximal in P and hence P has a unique maximal submodule (which is $\text{Rad}P = J_\sigma(P)$). Since $P \neq J_\sigma(P)$, this unique maximal submodule must be σ-pure in P and hence P is σ-local.

It is easy to see that (v) implies (vi).

(vi) implies (i): If (vi) holds then RadP is a maximal, σ-pure submodule of P. Since $\text{Rad}P \leq J_\sigma(P)$, it follows that $\text{Rad}P = J_\sigma(P)$. (Thus $J_\sigma(P)$ is maximal and σ-pure in P). Further, since P is f.g., it follows that $J_\sigma(P)$ (= RadP) is small in P (see 4.3.3(4)).

Hence $P/J_\sigma(P)$ is a simple, σ-flat module and $0 \to J_\sigma(P) \to P \to P/J_\sigma(P) \to 0$ is a projective cover, proving (i).

4.3.17 Corollary

The following are equivalent for a ring R and a torsion radical σ on R-mod:

(i) ${}_R R$ is σ-local.

(ii) R is σ-regular and local.

(iii) $J_\sigma(R)$ is maximal in R and if M is a f.g. R-module and I is a left ideal of R such that $I \leq J_\sigma(R)$, then $I.M << M$.

(iv) $J_\sigma(R)$ is small and maximal in R.

Proof

(i) implies (ii): If R is σ-local then it has a unique maximal ideal and is therefore local. Since this maximal ideal is σ-pure in R, R is σ-regular by 3.4.4.

(ii) implies (iii): If R is σ-regular then $J_\sigma(R)$ = RadR which is a maximal ideal of R by (ii). The second part of (iii) follows from Nakayama's Lemma ([2], 15.13), since RadR = $J_\sigma(R)$.

(iii) implies (iv): If (iii) holds then $J_\sigma(R) \leq J_\sigma(R).R << R$. $J_\sigma(R)$ is maximal by (iii).

That (iv) implies (i) follows from 4.3.16.

4.3.18 Examples

Let R be either a division ring and let $L_\sigma = \{0,R\}$ or a local ring and $L_\sigma = \{R\}$. Then R is a σ-regular, local ring and is therefore a σ-local ring. It also satisfies the equivalent conditions of 4.3.15 and 4.3.16 (with P = R in the latter).

4.3.19 Remark

Since it is easy to see that epimorphic images of local rings are local and epimorphic images of σ-regular rings are σ-regular, it follows from 4.3.17(ii) that epimorphic images of σ-local rings are σ-local.

4.3.20 Theorem

Let R be a ring and σ a torsion radical on R-mod. Then the following are equivalent:

(i) $J_\sigma(M) = 0$ for all (left) R-modules M.

(ii) (a) Injective hulls of simple, (σ-flat) R-modules are simple (and σ-flat) and

(b) Every nonzero R-module has a simple, σ-flat epimorphic image.

(iii) **(a)** RadM = 0 for every (left) R-module M (i.e. R is a (left) *V-ring* ([20])) and

(b) Every nonzero R-module has a simple, σ-flat epimorphic image.

(iv) **(a)** Every left ideal of R is the intersection of maximal, σ-pure left ideals of R and

(b) Every nonzero R-module has a simple, σ-flat epimorphic image.

Proof

(i) implies (iv): **(a)** Let I be a left ideal of R. By (i), $J_\sigma(R/I) = 0$. Let M_i/I be the ideals of R/I such that $(R/I)/(M_i/I) \simeq R/M_i$ is simple and σ-flat. Then each M_i is maximal and σ-pure in R and from $0 = J_\sigma(R/I) = \bigcap_i M_i/I$ it follows that $I = \bigcap_i M_i$, proving (iv)(a).

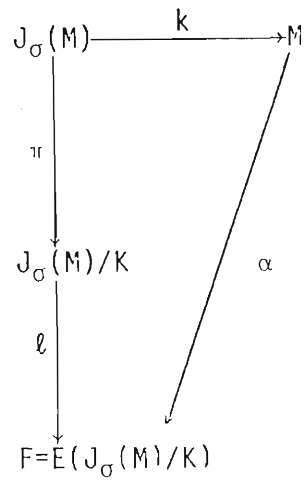
(b) Suppose ${}_RM \neq 0$. By (i), $J_\sigma(M) = 0$. By Lemma 4.3.5, there are simple, σ-flat modules U_j and an embedding $\alpha: M \rightarrowtail \prod_j U_j$. Let $\pi_k: \prod_j U_j \to U_k$ be the projection maps. Since $M \neq 0$, $(\pi_j \cdot \alpha)(M) \neq 0$ for at least one j. Since U_j is simple, $(\pi_j \cdot \alpha)(M) = U_j$ and U_j is therefore a simple, σ-flat epimorphic image of M.

(iv) implies (iii): If (iv) holds then certainly every ideal of R is the intersection of maximal ideals of R and by Theorem 7.32A of [20], RadM = 0 for all R-modules M.

(iii) implies (ii): By Theorem 7.32A of [20], if (iii) holds then every simple module is injective and this clearly means that (ii)(a) holds.

(ii) implies (i): Let M be any R-module and suppose $J_\sigma(M) \neq 0$. By (ii)(b), there is a (proper) submodule K of $J_\sigma(M)$ such that $J_\sigma(M)/K$ is simple and σ-flat. By (ii)(a), $F = E(J_\sigma(M)/K)$ is simple and σ-flat.

By injectivity of F, there is a map $\alpha: M \to F$ such that the diagram



commutes, (where k,ℓ are the inclusion maps and π is the canonical epimorphism).

Since $K \neq J_\sigma(M)$, $\alpha \neq 0$ and by simplicity of F, $\mathrm{im}\alpha = F$. Thus $M/\ker\alpha \simeq \mathrm{im}\alpha$ is simple and σ-flat. By definition, $J_\sigma(M) \leq \ker\alpha$. Thus $0 = \alpha(J_\sigma(M)) = \pi(J_\sigma(M))$ which is contradictory. Hence $J_\sigma(M) = 0$ proving (i).

4.3.21 **Remark**

The following are equivalent for a ring R and a torsion radical σ on R-mod:

(a) $J_\sigma(M) = 0$ for all σ-flat R-modules M.

(b) Every σ-pure submodule of a σ-flat R-module M is an intersection of maximal, σ-pure submodules of M.

Proof

Using 3.5.1.6 the proof follows lines similar to those in 4.3.20.

4.3.22 **Theorem**

Let R be a ring and σ a torsion radical on R-mod. Then the following are equivalent:

(i) $M/J_\sigma(M)$ is semisimple for all R-modules M.

(ii) $R/J_\sigma(R)$ is (left) Artinian.

(iii) $R/J_\sigma(R)$ is a direct sum of finitely many simple, σ-flat R-modules.

(iv) $R/J_\sigma(R)$ is semisimple.

(v) A direct product of simple, σ-flat R-modules is semisimple.

(vi) $S = R/J_\sigma(R)$ is a τ-perfect ring for all torsion radicals τ on S-mod such that every $I \in \mathcal{L}_\tau$ is f.g.

Proof

(i) implies (ii): If (i) holds then $R/J_\sigma(R)$ is semisimple and therefore Artinian.

(ii) implies (iii): Let $S = R/J_\sigma(R)$. By 4.3.5, S can be embedded in a direct product of simple, σ-flat R-modules, S_i (say). Since S is Artinian, by (ii), S is finitely cogenerated by Proposition 10.10 of [2]. Hence S can be embedded in a direct sum of finitely many of the S_i. It then follows that S is itself a direct sum of finitely many simple, σ-flat R-modules ([45], Theorem 5.4).

That (iii) implies (iv) is clear.

(iv) implies (v): Let $\{M_i : i \in I\}$ be a collection of simple, σ-flat R-modules, and let $M = \prod_i M_i$. By 4.3.10, $J_\sigma(R).M = 0$ and hence M is an $R/J_\sigma(R)$ module. (If (iv) holds then $J_\sigma(R) \neq R$). By (iv), $R/J_\sigma(R)$ is semisimple and hence M is a semisimple $R/J_\sigma(R)$ module ([45], Theorem 5.1). But any simple $R/J_\sigma(R)$ submodule of M is also simple as an R-module and hence M is semisimple as an R-module, proving (v).

(v) implies (i): Let M be any R-module. By 4.3.5, $M/J_\sigma(M)$ can be embedded in a direct product of simple, σ-flat R-modules, which is semisimple by (v). Hence (i) follows.

That (i) implies (vi) is clear.

(vi) implies (iv): Suppose τ is a torsion radical on S-mod such that every $I \in \mathcal{L}_\tau$ is f.g. (We may take $\mathcal{L}_\tau = \{S\}$, if necessary). By 1.3.6.3, flat S-modules are τ-flat and hence projective, (since S is τ-perfect, by assumption). That is, S is a perfect ring and by Theorem 28.4 of [2], S/RadS is a semisimple S-module. But $\text{RadS} \leq J_\sigma(S) = 0$ and therefore S is a semisimple S-module. Since $J_\sigma(R).S = 0$ it follows that S is also semisimple as an R-module, proving (iv).

4.3.23 Corollary

If R is a left Artinian ring and R-mod admits a torsion radical σ such that $J_\sigma(R) = 0$, then R is semisimple.

4.3.24 Examples

1. Let R be a commutative, von Neumann-regular ring and σ a torsion radical on R-mod such that every $I \in \mathcal{L}_\sigma$ is f.g. (see e.g. 3.4.9(2)). Then the equivalent conditions of 4.3.20 are valid for (R,σ).

Proof

By Theorem 6 of [60], R is a (left) V-ring. Hence if M is any nonzero module, $\text{RadM} = 0 = \cap \{K \leq M \mid K \text{ is maximal in } M\}$ and hence there is an embedding $\alpha: M \to \prod_i S_i$ where the S_i are simple R-modules. It follows as in the proof of 4.3.20 that one of the S_i is an epimorphic image of M. By 1.3.6.3, R is σ-regular and hence S_i is a simple, σ-flat epimorphic image of M. Hence 4.3.20(iii) is valid for (R,σ).

2. Let R be any perfect ring and σ any torsion radical on R-mod. Then the equivalent conditions of 4.3.22 are valid for (R,σ).

Proof

By Theorem 28.4 of [2], R/RadR is semisimple. But $R/J_{\sigma}(R)$ is an epimorphic image of R/RadR and hence $R/J_{\sigma}(R)$ is semisimple ([45], Theorem 5.4).

<u>INDEX OF DEFINITIONS</u>

Page

Page

Page

Page

BIBLIOGRAPHY

1. Alin, J.S. & Dickson, S.E. Goldie's Torsion Theory and its Derived Functor, *Pacific J. Math.* 24 No. 2, (1968), 195-203.

2. Anderson, F.W. & Fuller, K.R. *Rings and Categories of Modules,* Berlin-Heidelberg-New York, Springer-Verlag, (1973).

3. Arbib, M.A. & Manes, E.G. *Arrows, Structures and Functors,* New York, Academic Press, (1975).

4. Bass, H. Finitistic Dimension and a Homological Generalization of Semiprimary Rings, *Trans. Amer. Math. Soc.* 95, (1960), 466-488.

5. Bass, H. Injective Dimension in Noetherian Rings, *Trans. Amer. Math. Soc.* 102, (1962), 18-29.

6. Baumslag, B. & Chandler, B. *Group Theory,* New York, McGraw Hill, (1968).

7. Bican, L. Notes on Purities, *Czech. Math. J.* 22 (97), (1972), 525-534.

8. Bican, L. Pure Closures, *Czech. Math. J.* 22 (97), (1972), 78-82.

9. Bland, P.E. Divisible and Co-divisible Modules, *Math. Scand.* 34, (1974), 153-161.

10. Brown, B. & McCoy, N. The Maximal Regular Ideal of a Ring, *Proc. Amer. Math. Soc.* 1, (1950), 165-171.

11. Cartan, H. & Eilenberg, S. *Homological Algebra,* Princeton, Princeton University Press, (1956).

12. Cohn, P.M. On the Free Product of Associative Rings, *Math Z.* 71, (1959), 380-398.

13. Döman, D. *Aspekte van die Begrip Suiwerheid in die Teorie van Module,* Doctoral thesis, Johannesburg, R.A.U., (1975).

14. Döman, D. Plat en Semi-Plat-Module, *Algebra Simposium* (Proceedings of the Pretoria Conference, edited by Schoeman, M.J.), University of Pretoria Press, (1975), 84-93.

15. Döman, D. & Hauptfleisch, G.J. Filtered-Injective and Coflat Modules, *Quaestiones Math.* 3, (1978), 33-48.

16. van Dyk, T.J. *Suiwerheid van Ondergroepe in die Abelse Groep-Teorie,* Doctoral Thesis, Pretoria, University of Pretoria, (1979).

17. Enochs, E. & Jenda, O.M.G. Copure Injective Modules, *Quaestiones Mathematicae* 14(4), (1991), 401-409.

18 Enochs, E. & Jenda, O.M.G. Homological Properties of Pure Injective Resolutions, *Comm. Alg.* 16 (10), (1988), 2069-2082.

19 Facchini, A. Pure-injective Envelope of a Commutative Ring and Localizations, *Quart. J. Math. Oxford,* 39 (2), (1988), 307-321.

20. Faith, C. *Algebra: Rings, Modules and Categories* I, Berlin-Heidelberg-New-York, Springer-Verlag, (1973).

21. Faith, C. Algebra II: *Ring Theory*, Berlin-Heidelberg, Springer-Verlag, (1976).

22. Faith, C. Rings with Ascending Condition on Annihilators, *Nagoya Math. J.* 27, (1966), 179-191.

23. Faith, C. & Walker, E.A. Direct-Sum Representations of Injective Modules, *J. Algebra* 5, (1967), 203-221.

24. Fakir, S. On Relative Injective Envelopes, *J. Pure Appl. Algebra* 1, No.4, (1971), 377-383.

25. Fieldhouse, D.J. Absolute Purity, *Can. J. Math.* 27 No. 1, (1975), 6-10.

26. Fieldhouse, D.J. Aspects of Purity, *Ring Theory*: Proceedings of the Oklahoma Conference (edited by McDonald, R., Magid, B.R. & Smith K.C.), Marcel Dekker, (1974), 185-196.

27. Fieldhouse, D.J. Character Modules, Dimension and Purity, *Glasgow Math. J.* 13, (1972), 144-146.

28. Fieldhouse, D.J. Pure Simple and Indecomposable Rings, *Can. Math. Bull.* 13(1), (1970), 71-78.

29. Fieldhouse, D.J. Pure Theories, *Math. Ann.* 184, (1969), 1-18.

30. Fieldhouse, D.J. *Purity and Flatness*, Ph.D thesis, Montreal, McGill University, (1967).

31. Fieldhouse, D.J. Regular Modules and Rings, *Illinois J. Math.* 16, (1972), 217-221.

32. Fieldhouse, D.J. Regular Rings and Modules, *J. Austral. Math. Soc.* 13, (1972), 477-490.

33. Fuchs, L. *Infinite Abelian Groups*, New York and London, Academic Press, (1970).

34. Golan, J.S. *Localization of Noncommutative Rings*, New York, Springer-Verlag, (1975).

35. Golan, J.S. *Torsion Theories*, New York, John Wiley, (1986).

36. Golan J.S. & Teply, M.L. Finiteness Conditions on Filters of Left Ideals, *J. Pure Appl. Algebra* 3, (1973), 251-259.

37. Goldman, O. Rings and Modules of Quotients, *J. Algebra* 13, (1969), 10-47.

38. Goodearl, K.R. *Nonsingular Rings and Modules*, New York, Marcel Dekker, (1976).

39. Goodearl, K.R. *von Neumann Regular Rings*, London, San Francisco and Melbourne, Pitman, (1979).

40. Gray, D.J. *A Note on Purity Relative to an Idempotent Kernel Functor*, Unizulu Publications B63, (1987).

41. Gray, D.J. The σ-pure Injective Hull, for a Torsion Radical, σ., *Quaestiones Mathematicae*. 12(4), (1989), 397-413.

42 Gray, D.J. On Flatness Relative to a Torsion Theory, *Quaestiones Mathematicae* 14(4), (1991), 471-481.

43. Gray, D.J. On Regularity Relative to an Hereditary Torsion Theory, Preprint.

44. Harui, H. & Teply, M.L. Neat Submodules, *Comm. Alg.* 10(11), (1982), 1137-1153.

45. Hauptfleisch, G.J. *Inleiding tot die Teorie van Module*, Johannesburg, R.A.U. Publications, (1974).

46. Hauptfleisch, G.J. Plat Module in Terme van Kommutatiewe Diagramme, *Algebra Simposium* (Proceedings of the Pretoria Conference, edited by Schoeman, M.J.), University of Pretoria Press, (1975), 94-97.

47. Hauptfleisch, G.J. & Döman, D. Filtered-Projective and Semiflat Modules, *Quaestiones Math.* 1, (1976), 197-217.

48. Hungerford, T.W. *Algebra,* New York, Springer-Verlag, (1974).

49. Jans, J.P. *Rings and Homology,* New York, Holt,Rinehart and Winston, (1964).

50. Jans, J.P., Duality in Noetherian Rings, *Proc. A.M.S.* 12, (1961), 829-835.

51. Kaplansky, I. *Fields and Rings,* Chicago, University of Chicago Press, (1969).

52. Koethe, G. Verallgemeinerte Abelsche Gruppen mit Hyperkomplexem Operatorenring, *Math. Z.* 39, (1934), 31-44.

53. Lambek, J. *Torsion Theories, Additive Semantics and Rings of Quotients,* New York-Berlin-Heidelberg, Springer-Verlag, (1971).

54. Lambek, J. A Module is Flat if and only if its Character Module is Injective, *Can. Math. Bull.* 7, (1964), 237-243.

55. Maddox, B.H. Absolutely Pure Modules, *Proc. Amer. Math. Soc.* 18, (1967), 155-158.

56. Megibben, C. Absolutely Pure Modules, *Proc. Amer. Math. Soc.* 26, (1970), 561-566.

57 Mishina, A.P. & Skornjakov, L.A. *Abelian Groups and Modules*, Providence, A.M.S. Translations Ser. 2 Vol.107, (1976).

58. von Neumann, J. On Regular Rings, *Proc. Nat. Acad. Sci.* 22, (1936), 707-713.

59. Nishida, K. Divisible Modules, Codivisible Modules and Quasi-Divisible Modules, *Comm. Alg.* 5(6), (1977), 591-610.

60. Rosenberg, A. & Zelinsky, D. Finiteness of the Injective Hull, *Math. Z.* 70, (1959), 372-380.

61. Rotman, J.J. *Notes on Homological Algebra*, New York, van Nostrand Reinhold, (1968).

62. Rubin, R.A. Semisimplicity Relative to Kernel Functors, *Can. J. Math.* 26(6), (1974), 1405-1411.

63. Solian, A. *Theory of Modules*, New York, Wiley and Sons, (1977).

64. Stenstrom, B. Pure Submodules, *Ark. Mat.* 7 No. 10, (1967), 159-171.

65. Stenstrom, B. Purity in Functor Categories, *J. Algebra* 8, (1968), 352-361.

66. Stenstrom, B. *Rings and Modules of Quotients,* Berlin-Heidelberg-New York, Springer-Verlag, (1971).

67. Stenstrom, B. *Rings of Quotients,* Berlin-Heidelberg-New York, Springer-Verlag, (1975).

68. Teply, M. Some Aspects of Goldie's Torsion Theory, *Pacific J. Math.* 29 No. 2, (1969), 447-459.

69. Walker, C.P. Relative Homological Algebra and Abelian Groups, *Illinois J. Math.* 10, (1966), 186-209.

70. Ware, R. Endomorphism Rings of Projective Modules, *Trans. A.M.S.* 155(1), (1971), 233-256.

71. Warfield, R.B. Purity and Algebraic Compactness for Modules, *Pacific J. Math.* 28 No. 2, (1969), 699-719.

72. Zelmanowitz, J. Regular Modules, *Trans. A.M.S.* 163, (1972), 341-355.