This is to confirm that the student has carefully and thoroughly gone through the Examiners comments and reports and has carried out the corrections and comments by the examiners to my satisfaction as his supervision. I therefore approve the submission of the corrected and final version of her thesis.
Thanks

O.T. Mewomo (PhD) [redacted]
Supervisor
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Fixed Point Approach for Solving Optimization Problems in Hilbert, Banach and Convex Metric Spaces

by

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As the candidate's supervisor, I have approved this thesis for submission.

Prof. O. T. Mewomo

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Dedication

This thesis is dedicated to my father Mr. Ogwo Chukwu Okoroafor (of blessed memory) for all his sacrifices, support and words of encouragement.

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Abstract

In this thesis, we study the fixed point approach for solving optimization problems in real Hilbert, Banach and Hadamard spaces. These optimization problems include the variational inequality problem, split variational inequality problem, generalized variational inequality problem, split equality problem, monotone inclusion problem, split monotone inclusion problem, minimization problem, split equilibrium problem, among others. We consider some interesting classes of mappings such as the nonexpansive semigroup in real Hilbert spaces, strict pseudo-contractive mapping in real Hilbert spaces and 2-uniformly convex real Banach spaces, nonexpansive mapping between a Hilbert space and a Banach space, and quasi-pseudocontractive mapping in Hilbert spaces and Hadamard spaces. We introduce several iterative schemes for approximating the solutions of the various aforementioned optimization problems and fixed point problems and prove their convergence results. We adopt and implement several inertial methods such as the inertial-viscosity-type algorithm, relaxed inertial subgradient extragradient, modified inertial forward-backward splitting algorithm viscosity method, among others. Furthermore, we present several novel and practical applications of our results to solve other optimization problems, image restoration problem, among others. Finally we present several numerical examples in comparison with some results in the literature to illustrate the applicability of our proposed methods.

Contents

Declaration

This thesis in its entirety or in part, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used, proper reference has been made.

Grace Nnennaya Ogwo

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Contributed papers from the thesis

Papers published/accepted from the thesis.

(1) **G. N. Ogwo,** T. O. Alakoya, O. T. Mewomo, Inertial forward-backward method with self-adaptive step sizes for finding minimum-norm solutions of inclusion and split equilibrium problems, Applied Set-Valued Analysis and Optimization, **4** (2) (2022), 185-206.

(2) **G. N. Ogwo,** C. Izuchukwu, Y. Shehu, O. T. Mewomo, Convergence of Relaxed Inertial Subgradient Extragradient Methods for Quasimonotone Variational Inequality Problems, Journal of Scientific Computing, **90** (1) (2022), Paper No. 10, 35 pp.

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(1) O. T. Mewomo, **G. N. Ogwo,** T. O. Alakoya, Inertial iterative algorithm for approximating common solution of split equalities of some nonlinear optimization problems, Numerical Algebra, Control and Optimization.

(2) **G. N. Ogwo,** T. O. Alakoya, O. T. Mewomo, Strong convergent algorithm for finding minimum-norm solutions of quasimonotone variational inequalities with fixed point constraint and application, Journal of Scientific Computing.

(3) **G. N. Ogwo,** T. O. Alakoya, C. Izuchukwu, O. T. Mewomo, Strong convergent inertial projection and contraction methods for split variational inequality problem, Acta Mathematica Scientia.

(4) **G. N. Ogwo,** O. T. Mewomo, A generalized viscosity implict rule for quasi- pseudo-contractive mappings in Hadamard spaces, Topological Algebra and Its Applications.

CHAPTER 1

General Introduction

1.1 Background of study

Optimization theory has shown to be a very important research area with numerous applications in various fields. This is because many problems arising from these fields can be modeled as optimization problems. Optimization problems include; the variational inequality problem (VIP), split variational inequality problem (SVIP), generalized variational inequality problem (GVIP), variational inclusion problem (V_qIP), split monotone variational inclusion problem (SMV$_q$IP), split equality problem (SE_qp), split equilibrium problem (SEP), minimization problem (MP), among others. These optimization problems can be transformed into a fixed point problem (FPP) of a suitable nonlinear mapping. The fixed point problem is defined as follows:

$$\text{Find } x^* \in X \text{ such that } Tx^* = x^*, \tag{1.1.1}$$

where X is a metric space and $T : X \to X$ is a nonlinear mapping. We denote by $F(T)$ the fixed point set of T, that is

$$F(T) = \{x^* \in X : Tx^* = x^*\}. \tag{1.1.2}$$

The fixed point method is one of the most effective methods for solving optimization problems. Fixed point theory is a crucial area of research in nonlinear functional analysis which has continuously attracted the interest of numerous researchers due to its applications in various branches of mathematics and other related fields such as economics, physics, chemistry, biology, dynamical system theory, data science, among others (see [28, 87, 134, 172, 173, 212, 219, 236, 260] and other references therein).

Poincaré [207] introduced the study of the fixed point theory in 1886. Brouwer [46] in 1912 proved that a continuous mapping on a closed unit ball in finite dimensional space has a fixed point. Birkoff and Kellog [42] extended the result of Brouwer [46] to the infinite dimensional space.

The fixed points of nonlinear mappings are difficult and sometimes impossible to obtain. Hence, there is a need to study and develop iterative schemes to approximate the fixed points of nonlinear mappings. In 1922, Banach [38] proved the famous Banach contraction principle which is also known as the Banach fixed point theorem. He proved that a contraction mapping T defined on a complete metric space X has a unique fixed point. Furthermore, he proved that for any starting point $x_0 \in X$, the sequence defined by the Picard iteration

$$x_{n+1} = Tx_n, \qquad n \geq 1, \tag{1.1.3}$$

converges to the unique fixed point. The Banach fixed point theorem guarantees the existence and uniqueness of fixed points of certain self mappings. This has made it one of the main foundations of the theory of metric fixed points. However, for classes of mappings that are more general than the class of contraction mappings, the Banach contraction principle may fail to hold. Several examples in literature have shown that for a nonexpansive mapping T, the sequence generated by the Picard iteration (1.1.3) may fail to converge even when the fixed point exists. To overcome this limitation, several authors have studied and proposed iterative methods for approximating fixed points of mappings that are more general that the contraction mapping. The Mann iterative method proposed by Mann [171] is a more general iterative formula for approximating fixed points of nonlinear mappings. The Mann iteration generates a sequence $\{x_n\}$ as follows:

$$x_{n+1} = (1 - a_n)x_n + a_n Tx_n, \quad n \in \mathbb{N} \text{ and } \quad x_1 \in \mathcal{H}, \tag{1.1.4}$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$ and $\mathcal{H}$ is a real Hilbert space. The Mann iterative method converges weakly if T is nonexpansive, $\lim\limits_{n\to\infty} a_n = 0$ and $\sum\limits_{n=0}^{\infty} a_n = \infty$. However, when T is not nonexpansive, the Mann iterative method may fail to converge to a fixed point of T even when the algorithm converges. If $\{a_n\} = \{\lambda\}$ where λ is a positive real number, then (1.1.4) reduces to the Kransnoselskii iterative method proposed by Kransnoselskii [157], and if $\{a_n\} = 1$, then (1.1.4) reduces to (1.1.3). In 1974, Ishikawa [127] introduced a generalization of the Mann iterative method for approximating fixed points of pseudocontractive mappings in Hilbert spaces. Ishikawa called this generalization the Ishikawa method and defined it as follows:

$$x_{n+1} = (1 - a_n)x_n + a_n T[(1 - b_n)x_n + b_n Tx_n], \; n \in \mathbb{N} \text{ and } x_1 \in \mathcal{H}, \tag{1.1.5}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$. Writing (1.1.5) in a system form, we have

$$\begin{cases} y_n = (1 - b_n)x_n + b_n Tx_n \\ x_{n+1} = (1 - a_n)x_n + a_n Ty_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.1.6}$$

which implies that the Ishikawa iteration can be seen as a double Mann iteration and it reduces to Mann iteration when $b_n = 0$.

Agarwal *et al.* [2] introduced and studied the following S-iteration method defined as follows:

$$x_{n+1} = (1-a_n)Tx_n + a_nT[(1-b_n)x_n + b_nTx_n], \quad n \in \mathbb{N} \text{ and } x_1 \in \mathcal{H}, \tag{1.1.7}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences in $(0,1)$. It is well known that (1.1.7) has better convergence properties than (1.1.4) and (1.1.5) (see [3]).

We note that (1.1.4), (1.1.5) and (1.1.7) only generates sequences that converge weakly. However, strong convergence are more desirable than the weak convergence in infinite dimensional space. In an attempt to obtain strong convergence results, Halpern [117] introduced the Halpern iterative method which converges strongly to a fixed point of a nonexpansive mapping in real Hilbert spaces. Halpern [117] defined it as follows;

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad \text{for} \quad n \in \mathbb{N}, \tag{1.1.8}$$

where $\{\alpha_n\} \subset [0,1]$ and $u, x_1 \in \mathcal{H}$. The Halpern iterative method was generalized to the viscosity iterative method by Moudafi [181]. The viscosity iterative method is defined as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad \text{for} \quad n \in \mathbb{N}, \tag{1.1.9}$$

where $\{\alpha_n\} \subset [0,1]$ and f is a contraction mapping on $\mathcal{H}$.

The implicit midpoint rule is an important method for solving Ordinary Differential Equations (ODEs) because of its ability to eliminate stability errors of systems of ODEs (see [32, 203, 226] and other references therein). For an initial value problem of the ODE

$$x^{'}(t) = f(x(t)), \quad x(0) = x_0, \tag{1.1.10}$$

the implicit midpoint rule is a recursion procedure that generates the sequence $\{x_n\}$ by

$$x_{n+1} = x_n + hf\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.1.11}$$

where $h > 0$ is a stepsize and $f : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function. It is known that if f is Lipschitz continuous and sufficiently smooth, then the sequence $\{x_n\}$ converges to an exact solution of (1.1.10) as $h \to 0$ uniformly over $u \in [0, u]$ for any fixed point u. In 2015, Xu et. al. [263] combined (1.1.9) and (1.1.11) for nonexpansive mappings in Hilbert spaces and proposed a viscosity implicit midpoint rule defined as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.1.12}$$

where x_0 is an arbitrary point in the domain of T, f is a contraction on $\mathcal{H}$ and $\{\alpha_n\} \in [0,1]$. They proved that the sequence generated converge strongly under some suitable conditions to a fixed point of the nonexpansive mapping which is also a solution of a variational inequality. It is well known that (1.1.9) and (1.1.12) converge faster than (1.1.8) (see [263]).

Over the years, several researchers have studied these iterative methods (1.1.4)-(1.1.12) and their modifications for approximating fixed points of several nonlinear mappings and solutions of optimization problems in Hilbert, Banach and Hadamard spaces (see [5, 6, 23, 24, 51, 74, 105, 110, 152, 191, 194] and other references therein).

In this thesis, we study the VIP, V_qIP, SE_qP, SEP, MP, FPP and other related optimization problems in the frameworks of Hilbert, Banach and Hadamard spaces. We present several effective iterative methods for approximating the solutions of these problems. Furthermore, we analyze the convergence results of our proposed methods. Finally, we give where necessary some applications of our results and present numerical examples to show the applicability of our methods.

1.2 Research problems and motivation

In this section, we discuss the research problems and motivation for the study.

1.2.1 Research problems

In this thesis, we consider the following optimization problems; VIP, SVIP, GVIP, V$_q$IP, SMV$_q$IP, SE_qP, SEP, MP and FFP of certain nonlinear mappings in the frameworks of Hilbert, Banach and Hadamard spaces. A detailed review of these optimization problems and the spaces under consideration will be given in Chapter 2.

Let $\mathcal{C}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ and $A : \mathcal{H} \to \mathcal{H}$ be an operator. The VIP is defined as follows: Find $x \in \mathcal{C}$ such that

$$\langle Ax, y - x\rangle \geq 0, \; \forall y \in \mathcal{C}. \tag{1.2.1}$$

We denote the solution set of the problem (1.2.1) by $VI(\mathcal{C}, A)$. Many authors have proposed and analyzed several iterative algorithms for solving Problem (1.2.1) (see [9, 135], and references therein). To accelerate the rate of convergence of these iterative algorithms, authors often employ the inertial technique. Polyak [208] studied the convergence of the following inertial extrapolation algorithm

$$x_{n+1} = x_n + \beta_n(x_n - x_{n-1}) - \alpha_n A x_n, \; \forall n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences. Recently, there has been an increased interest in studying inertial type algorithm (see [5, 7, 8, 27, 40, 96, 246] and other references therein). On the other hand, the relaxation technique has also proven to be an essential ingredient in the resolution of optimization problems due to the improved convergence rate that it contributes to iterative schemes.

In this work, we propose and study new inertial viscosity Tseng's extragradient algorithms with self-adaptive step size for solving the VIP and the FPP in Hilbert spaces. Our main aim is to find a common solution of the VIP (1.2.1) and the FPP (1.1.1) for an infinite family of strict pseudo-contraction mappings T. That is, find a point $x^* \in \mathcal{C}$ such that

$$x^* \in \Gamma = VI(\mathcal{C}, A) \bigcap \cap_{i=1}^{\infty} F(T_i). \tag{1.2.2}$$

Furthermore, we present and study two new relaxed inertial subgradient extragradient methods for solving quasimonotone VIPs in a real Hilbert space. Also, we present a new inertial Tseng's extragradient method with self-adaptive step size for approximating the minimum-norm solutions of the quasimonotone VIP and FPP of a quasi-pseudocontractive mapping T in the framework of Hilbert space. That is, the problem of finding a point $x^* \in \mathcal{C}$ such that

$$x^* \in \Gamma = VI(\mathcal{C}, A) \bigcap F(T). \tag{1.2.3}$$

The VIP (1.2.1) was later generalized to the following SVIP: Find $x \in \mathcal{C}$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall\, y \in \mathcal{C}, \tag{1.2.4}$$

and $z = \mathcal{T}x \in \mathcal{Q}$ solves

$$\langle Fz, u - z \rangle \geq 0, \quad \forall\, u \in \mathcal{Q}, \tag{1.2.5}$$

where $\mathcal{C}$ and $\mathcal{Q}$ are nonempty, closed and convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, $A : \mathcal{H}_1 \to \mathcal{H}_1, F : \mathcal{H}_2 \to \mathcal{H}_2$ are two operators and $\mathcal{T} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator.

To study the SVIP, we present two new relaxed methods with inertial steps and two new inertial methods for solving the SVIP in real Hilbert spaces without any product space transformation, and for which the underlying operators are freed from the restrictive co-coercive assumption. Furthermore, we propose two new inertial projection and contraction methods for solving the SVIPs in real Hilbert spaces without the co-coercive condition and without the product space formulation, which does not fully exploit the attractive splitting structure of the SVIP. Also, we study an inertial viscosity-type method for solving a more generalized VIP (GVIP) in Hilbert spaces which is defined as follows: Find $x \in \mathcal{C}$ such that

$$\Big\langle Ax, y - x \Big\rangle \geq 0, \quad \forall y \in \mathcal{C} \ \text{ and } \mathcal{T}x \in F(T), \tag{1.2.6}$$

where $\mathcal{T}$ is a bounded linear operator and $F(T)$ is as defined in (1.1.2).

Furthermore, we consider the split equalities of EP, VIP and FPP.

Let $\mathcal{C}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ and $\Phi : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be a bifunction. The EP is defined as follows: Find $x \in \mathcal{C}$ such that

$$\Phi(x, y) \geq 0, \ \forall \ \ y \in \mathcal{C}. \tag{1.2.7}$$

The solution set of the Problem (1.2.7) is denoted by $EP(\Phi)$.

Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ be real Hilbert spaces. Let $\mathcal{C}, \mathcal{Q}$ be nonempty, closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $\mathcal{F}_1 : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{F}_2 : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. The SE_qP is defined as follows:

$$\text{Find } x \in \mathcal{C} \ \text{ and } \ y \in \mathcal{Q} \ \text{ such that } \ \mathcal{F}_1 x = \mathcal{F}_2 y. \tag{1.2.8}$$

We denote the solution set of (1.2.8) by

$$\Omega_{SEP} := \Big\{(x,y) \in \mathcal{C} \times \mathcal{Q} \, |\mathcal{F}_1 x = \mathcal{F}_2 y\Big\}.$$

In this study, we present an inertial Tseng's algorithm for approximating the common solution of split equalities of EP, VIP and FPP in Hilbert spaces. That is, find a point $(x,y) \in \Gamma = \Big\{x \in EP(\Phi_1) \cap VI(\mathcal{C}, A) \cap F(\mathcal{T}_a), \;\; y \in EP(\Phi_2) \cap VI(\mathcal{Q}, B) \cap F(\mathcal{T}_b) : \; \mathcal{F}_1 x = \mathcal{F}_2 y\Big\}$, where $\mathcal{T}_a$ and $\mathcal{T}_b$ are one-parameter nonexpansive semigroups and $F(\mathcal{T}_a), F(\mathcal{T}_b)$ are as defined in (1.1.2).

The next optimization problems considered in this thesis are the V$_q$IP and the SEP.

Let $\mathcal{H}$ be a real Hilbert space and $B : \mathcal{H} \to \mathcal{H}$ be an operator. The V$_q$IP is defined as: Find $x^* \in \mathcal{H}$ such that

$$0 \in (B + D)x^*, \tag{1.2.9}$$

where $D : \mathcal{H} \to 2^{\mathcal{H}}$ is a multivalued operator. The solution set of the Problem (1.2.9) is called the set of zero points of $(B + D)$ and it is denoted by $(B + D)^{-1}(0)$.

Let $\mathcal{C}, \mathcal{Q}$ be nonempty subsets of $\mathcal{H}_1, \mathcal{H}_2$, respectively, and $F_1 : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$, $F_2 : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ be bifunctions. Let $\mathcal{T} : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The SEP is defined as follows: Find $x \in \mathcal{C}$ such that

$$F_1(x, x^*) \geq 0, \; \forall \, x \in \mathcal{C} \tag{1.2.10}$$

and such that

$$y = \mathcal{T}x \in \mathcal{Q} \text{ solves } F_2(y, y^*), \; \forall \; y^* \in \mathcal{Q}. \tag{1.2.11}$$

The solution set of SEP (1.2.10)-(1.2.11) is denoted by $\Omega = \{z \in EP(F_1) : \; \mathcal{T}z \in EP(F_2)\}$. In this study, we propose a modified inertial forward-backward splitting algorithm with self-adaptive step size for approximating the common solution of the V$_q$IP (1.2.9) and SEP (1.2.10)-(1.2.11) in Hilbert spaces. That is, find a point $x^* \in \mathcal{C}$ such that

$$x^* \in \Gamma = (B + D)^{-1}(0) \cap \Omega. \tag{1.2.12}$$

Next, we extend the study of the VIP (1.2.1) from the framework of Hilbert spaces to the framework of Banach spaces. We propose an inertial subgradient extragradient algorithm with Armijo type step-size for solving the VIP in a 2-uniformly convex real Banach space.

The V$_q$IP was later generalized to the SMV$_q$IP formulated as follows:

$$\begin{cases} \text{Find} \quad x^* \in \mathcal{H}_1, \quad \text{such that} \quad 0 \in B_1(x^*) + D_1(x^*), \\ \text{and} \\ y^* = \mathcal{T}x^* \in \mathcal{H}_2 \quad \text{such that} \quad 0 \in B_2(y^*) + D_2(y^*), \end{cases} \tag{1.2.13}$$

where $D_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $D_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ are multivalued maximal monotone mappings, $B_1 : \mathcal{H}_1 \to \mathcal{H}_1$, $B_2 : \mathcal{H}_1 \to \mathcal{H}_2$ are operators and $\mathcal{T} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator.

In this study, we propose a new inertial iterative algorithm with self-adaptive step size for approximating a common solution of finite family of SMV$_q$IPs (1.2.13) and FPP (1.1.1) of a nonexpansive mapping between a Banach space and Hilbert space. That is,

$$\text{Find } x^* \in \mathcal{H} \text{ such that } 0 \in \bigcap_{i=1}^{m}(B_i + D_i)x^*, \quad Tx^* = x^*; \tag{1.2.14}$$

and

$$y^* = \mathcal{T}x^* \text{ such that } 0 \in \bigcap_{i=1}^{m} C_i y^*, \tag{1.2.15}$$

where $B_i : \mathcal{H} \to \mathcal{H}$ is a finite family of α_i-inversely strongly monotone operators, $D_i : \mathcal{H} \to 2^{\mathcal{H}}$ and $C_i : \mathcal{X} \to 2^{\mathcal{X}^*}$ are finite families of maximal monotone operators for each $i = 1, 2, \dots m$, $T : \mathcal{H} \to \mathcal{H}$ is a nonexpansive mapping and $\mathcal{T} : \mathcal{H} \to \mathcal{X}$ is a bounded linear operator.

We denote the solution set of Problem (1.2.14)-(1.2.15) by

$$\Gamma := \bigcap_{i=1}^{m}(B_i + D_i)^{-1}(0) \cap \bigcap_{i=1}^{m} \mathcal{T}^{-1}(C_i^{-1}0) \cap F(T).$$

Finally, we consider the MP in the framework of Hadamard spaces.

Let X be a metric space and $f : X \to (-\infty, \infty]$ be a proper and convex function. One of the major and interesting problems in optimization is the problem of finding minimizers of f. That is, find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y). \tag{1.2.16}$$

We study two new proximal point methods. Using the first method, we find a common solution of a finite family of MPs and FPP for a finite family of quasi-pseudocontractive mappings and using the second method, we find a common zero of a finite family of multivalued monotone operators which is also a common fixed point of a finite family of quasi-pseudocontractive mappings in an Hadamard space. Furthermore, we propose a generalized viscosity implicit rule for finding a common solution of a finite family of MPs and FPP for a finite family of quasi-pseudocontractive mappings in an Hadamard space.

1.2.2 Motivation

The motivation for our study will be discussed under the following headings:

(a) **Variational inequality and fixed point problems in real Hilbert spaces**

The VIP (1.2.1) has received a lot of research attention due to its wide range of applications in diverse fields (see, for example [5, 14, 104, 110, 123, 130, 135, 149, 235] and the references therein). Several authors have proposed and studied iterative methods for solving the VIPs (see [5, 9, 105, 106, 135, 136, 220, 221] and other references therein). The problem of finding the common solution of a VIP and FPP is motivated by the application it has in mathematical models whose constraints can be expressed as VIPs and FPPs. In particular, such a common solution problem has application in image recovery, signal processing, among others (see [126, 168] and other references therein). Recently, Nadeshkina and Takahashi [185] introduced an algorithm for finding a common solution of the FPP for a nonexpansive mapping and the VIP. They proved that the sequence generated by their proposed algorithm converges weakly to an element in the solution set of the problem under consideration. Since the strong convergence are more desirable than the weak convergence in infinite dimensional space, there is need to develop iterative methods that guarantees strong convergence. Also, since the pseudo-contractive mappings are more general than the nonexpansive mappings, we study the monotone VIP and FPP for a pseudo-contractive mapping in the real Hilbert space and introduce inertial iterative methods for approximating a common solution of the problem. We prove that our proposed method converges strongly to a point in the solution set. Considering the fact that the quasimonotone VIPs are more general than the monotone VIPs, we study the quasimonotone VIP in a real Hilbert space and introduce inertial and relaxed subgradient extragradient methods for solving the problem. We prove that the sequence generated by our proposed method converges weakly to the solution of the problem. Furthermore, we introduce an effective iterative method for finding a common solution of the quasimonotone VIP and FPP of quasi-pseudocontractive mappings in a real Hilbert space and prove that the sequence generated by our proposed method converge strongly to a point in the solution set of the aforementioned problem. Our results extend and generalize some existing results in the literature.

(b) **Split variational inequality problems in real Hilbert spaces**

The SVIP which can also be viewed as a combination of the classical VIP and the split feasibility problem (SFP) (introduced and studied by Censor *et al.* [57]) was introduced and studied by Censor *et al.* [61]. The first known results for SVIP is due to Censor *et al.* [63] (see also [61]). They studied the SVIP when the operators are monotone and Lipschitz continuous, by first transforming the problem into an equivalent Constrained VIP (CVIP) in the product space $\mathcal{H}_1 \times \mathcal{H}_2$ (see [63, Section 4]) and then employed the well-known subgradient extragradient method to solve it. To improve this result, Tian and Jiang [243] introduced a new class of SVIP which generalizes the class of SVIP considered by Censor *et al.* [63]. They proposed an algorithm for approximating a solution of the new class of SVIP and also proved the weak convergence of this algorithm when the mapping is nonexpansive and the cost operator is monotone and Lipschitz continuous. Motivated by the works of Censor *et al.* [63], Tian and Jiang [243] and other works in this direction, we propose two new relaxed inertial methods for solving the SVIP without any product space transformation where the underlying operator is monotone and Lipschitz continuous. Furthermore, we present two new methods with inertial steps for solving the SVIP in real Hilbert spaces without any product space formulation when the cost operators are pseudomonotone and Lipschitz continuous. We prove that the sequences generated by these methods converge strongly to a minimum-norm solution of the problem. We further propose two new inertial projection and contraction methods for solving the SVIPs in real Hilbert spaces without the co-coercive condition and without the product space formulation. Also, we propose a viscosity-type iterative method and prove its strong convergence to a solution of the GVIP in real Hilbert spaces. Our results extend and generalize some existing results in the literature.

(c) **Split equalities variational inclusion, split equilibrium and fixed point problems in real Hilbert spaces**

The problem of finding a common solution of the split equalities of EP, VIP and FPP of nonexpansive semigroups has applications to many nonlinear problems. Recently, Latif and Eslamian [159] studied and introduced a new algorithm for finding a common element of split equalities of EP, monotone VIP with Lipschitz operator and fixed point problem of nonexpansive semigroups satisfying the uniformly asymptotically regularity (u.a.r) condition in Hilbert spaces. Motivated by this result and other results in the literature, we propose and study an inertial Tseng's extragradient algorithm for the SE_qP for finding a common element of solution of the EP, non-Lipschitz pseudomonotone VIP and common fixed point of nonexpansive semigroups without the u.a.r condition. The problem of finding a common solution of the monotone V_qIP and SEP has possible applications to many nonlinear problems and mathematical models whose constraints can be expressed as V_qIPs and SEPs. Very recently, Cholamjiak *et al.* [81] proposed a modified inertial forward-backward splitting method for solving the SEP and the inverse strongly monotone V_qIP. Motivated by this, we propose a modified inertial forward-backward splitting algorithm with self-adaptive step sizes for approximating the solution of the aforementioned problem in Hilbert spaces without the inverse strongly monotone condition. Our result extend and generalize some existing results in the literature.

(d) **Variational inequality, variational inclusion and fixed point problems in Banach spaces**

It is well-known that many real life problems are generally defined in Banach spaces than Hilbert spaces, therefore there is need to extend the study of these optimization problems from Hilbert spaces to Banach spaces. Recently, Tan *et al.* [240] proposed an inertial iterative method for solving the VIP (1.2.1) in a real Hilbert space. Motivated by the work of Tan *et al.* [240], we propose and study an efficient method for solving the VIP (1.2.1) in a 2-uniformly convex real Banach space. The problem of finding the common solution of the SMV_qIP and FPP has possible applications to many nonlinear problems and mathematical models whose constraints can be expressed as SMV_qIPs and FPPs. Motivated by this and the ongoing research activities in this direction, we propose and study a new inertial iterative algorithm with self-adaptive step size for approximating a common solution of finite family of SMV_qIP and FPP for a nonexpansive mapping between a Banach space and Hilbert space.

(e) **Minimization and fixed point problems in Hadamard spaces**

The Hadamard space is an analogue of the Hilbert space with related geometry in the setting of the nonlinear space. Many results related to optimization problems have more applications in Hadamard spaces than in Hilbert spaces. This is due to the fact that the classical results on optimization and FPP in Hadamard spaces occur naturally in Hilbert spaces but the converse may not be true. This motivates the need to extend the results on optimization and FPP from the Hilbert and Banach spaces to Hadamard spaces. Just recently, Chang *et al.* [70] proposed an iterative algorithm for approximating a common solution of a finite family of MP and FPP for two demicontractive mappings in Hadamard spaces. Motivated by this, we propose two new proximal point methods involving quasi-pseudocontractive mappings in Hadamard spaces. Inspired by the viscosity technique, we propose a generalized viscosity implicit rule for approximating the common solution of a finite family of MPs and FPP for a finite family of quasi-pseudocontractive mapping in Hadamard spaces.

1.3 Objectives

The objectives we aim to achieve at the end of this study are:

(i) to introduce and study iterative algorithms for approximating the common solutions of VIPs and FPPs in Hilbert spaces;

(ii) to introduce efficient inertial and relaxed algorithms for approximating the solution of the VIP without any form of monotonicity in a real Hilbert space;

(iii) to introduce iterative algorithms for approximating the solution of the SVIP without any product space formation in real Hilbert spaces;

(iv) to introduce an efficient algorithm for approximating the solution of the GVIP in a real Hilbert space;

(v) to introduce an effective iterative algorithm for approximating the common solution of the split equalities of VIP, EP and FPP in real Hilbert spaces;

(vi) to introduce an iterative algorithm with self-adaptive step-size for approximating the common solution of the V_qIPs and SEPs in real Hilbert spaces;

(vii) to generalize the study of the VIPs from the framework of Hilbert space to the framework of Banach space and also introduce an iterative algorithm for approximating the solution of the VIPs in Banach spaces;

(viii) to introduce an iterative algorithm with self-adaptive step-size for approximating a common solution of finite family of SMV_qIPs and FPPs of a nonexpansive mapping between a Banach space and Hilbert space;

(ix) to introduce the concepts of MPs and FPPs in Hadamard spaces;

(x) to apply the obtained results to solve some optimization problems;

(xi) to give some numerical experiments to illustrate the applicability of our proposed methods and also compare them with some existing methods in the literature.

1.4 Organization of the thesis

The thesis is organized as follows;

Chapter 1 (General Introduction): In this chapter, we give a brief background study of our research interest. We discuss the research problems and motivations. Finally, we state the objectives of the study and give a detailed organization of the thesis.

Chapter 2 (Preliminaries and Literature Review): In this chapter, we recall some basic definitions, concepts, terms and results that will be needed in our study. Also, we give a detailed literature review of some important past and recent works that are related to our study.

Chapter 3 (Results on Variational Inequality and Fixed Point Problems in real Hilbert Spaces): The main results of this thesis starts in this chapter. The chapter consists of four (4) sections.
In Section 3.1, we give a brief introduction of the study of the chapter.
In Section 3.2, we propose and study new inertial Tseng's extragradient algorithms with self-adaptive step size to solve the monotone VIP and FPP in a real Hilbert space. We prove that the sequences generated by our proposed algorithms converge strongly to the solution set. We apply our result to solve the zero point problem (ZPP) in Hilbert spaces. Finally, we present numerical experiments to illustrate the applicability of our proposed methods and also compare them with some existing methods in the literature.
In Section 3.3, we introduce two new relaxed inertial subgradient extragradient methods for solving quasimonotone VIPs in a real Hilbert space. We prove that the sequences generated by our proposed methods converge weakly to a point in the solution set. Finally, we present numerical experiments to illustrate the applicability of our proposed methods and also compare them with some existing methods in the literature.
In Section 3.4, we study the quasimonotone VIP with constraint of FPP of a quasi-pseudocontractive mapping. We propose and study a new inertial Tseng's extragradient method with self-adaptive step size for approximating the minimum-norm solutions of aforementioned problem. Furthermore, we present some numerical experiments for our proposed method and compare it with other existing methods in the literature. Finally, we apply our result to image restoration problem.
Chapter 4 (Results on Split Variational Inequality in real Hilbert Spaces): This chapter consists of four (4) sections.
In Section 4.1, we give a brief introduction of the study of the chapter.
In Section 4.2, we propose two new relaxed inertial methods for solving the SVIP in real Hilbert spaces without any product space transformation when the cost operator is not co-coercive. We prove that the sequences generated by our proposed algorithms converge strongly to a point in the solution set. Finally, we present numerical experiments to illustrate the applicability of our proposed methods and also compare them with some existing methods in the literature.
In Section 4.3, we propose two new methods with inertial steps for solving the SVIP beyond monotonicity in real Hilbert spaces without any product space formulation. We prove that the sequences generated by these methods converge strongly to a minimum-norm solution of the aforementioned problem. Finally, we provide several numerical experiments of the

proposed methods in comparison with other related methods in literature.
In Section 4.4, we propose two new inertial projection and contraction methods for solving the SVIP in real Hilbert spaces. The sequences generated by these methods converge strongly to a point in the solution set of the SVIP in real Hilbert spaces. Furthermore, we present several numerical experiments for the proposed methods and compare their performances with other related methods in the literature.
In Section 4.5, we propose a viscosity-type method for solving a GVIP when the cost operator is pseudomonotone in a real Hilbert space. We prove that the sequence generated by our proposed algorithm converges strongly to the solution set of the GVIP. Finally, we present numerical experiments to illustrate the applicability of our proposed method and also compare it with some existing methods in the literature.

Chapter 5 (Results on Split equalities Variational Inclusion and Split Equilibrium Problems in real Hilbert Spaces): This chapter consists of three (3) sections.
In Section 5.1, we give a brief introduction of the study of the chapter.
In Section 5.2, we propose an inertial Tseng's extragradient algorithm with self adaptive step size for approximating a common solution of the split equalities of EP, non-Lipschitz pseudomonotone VIP and FPP of nonexpansive semigroups in real Hilbert spaces. We prove that the sequence generated by our proposed method converges strongly to a point in the solution set of the problem. Finally, we provide some numerical experiments for the proposed method in comparison with related methods in literature.
In Section, 5.3, we propose a modified inertial forward-backward splitting algorithm with self-adaptive step size for approximating a solution of the V_qIP and SEP in real Hilbert spaces. We prove that the sequence generated by our proposed method converges strongly to a minimum-norm solution of the aforementioned problem. Furthermore, we apply our result to study certain optimization problems. Finally, we provide some numerical experiments of our proposed method in comparison with other existing methods in the literature.

Chapter 6 (Results on Variational Inequality, Variational Inclusion and Fixed Point Problems in real Banach Spaces): This chapter consists of three (3) sections.
In Section 6.1, we give a brief introduction of the study of this chapter.
In Section 6.2, we study an inertial subgradient extragraident algorithm with Armijo-type step size for solving the VIP in a real Banach space. We prove that the sequence generated by our proposed method converges strongly to a solution of the VIP in a real Banach space. Finally, we provide some numerical experiments of the proposed method in comparison with other existing method in literature.
In Section 6.3, we propose and study a new inertial iterative algorithm with self-adaptive step size for approximating a common solution of finite family of SMV_qIP and FPP between a Hilbert and Banach space. Furthermore, we apply our result to study some optimization problems. Finally, we provide some numerical experiments to demonstrate the efficiency of our method in comparison with some well-known methods in the literature.

Chapter 7 (Results on Minimization Problems in Hadamard Spaces): This chapter consists of four (4) sections.
In Section 7.1, we give a brief introduction of the study of the chapter.
In Section 7.2, we prove some lemmas that will be required to establish our main result in this chapter.

In Section 7.3, we propose two new proximal point methods involving quasi-pseudocontractive mappings in an Hadamard space. We prove that the first method converges strongly to a common solution of a finite family of MP and FPP for a finite family of quasi-pseudocontractive mappings in an Hadamard space. We extend this method to monotone inclusion problems (MIPs) and prove that it converges strongly to a common zero of a finite family of multivalued monotone operator in an Hadamard space. Furthermore, we provide various nontrival numerical implementations of our methods in Hadamard spaces and compare them with some other recent methods in the literature.
In Section 7.4, we propose a generalized viscosity implicit rule involving quasi-pseudocontractive mappings in Hadamard spaces. We obtain a strong convergence result of our algorithm to the solution set.

Chapter 8 (Conclusion, Contributions to Knowledge and Future Research): In this chapter, we give a conclusion of our study and also highlight our contributions to existing knowledge. Furthermore, we discuss possible problems for future research.

CHAPTER 2

Preliminaries and Literature Review

In this section, we provide some basic definitions, terms, notations and concepts that will be relevant throughout our study. We also give a comprehensive literature review of some important past and recent works on optimization and fixed point problem. Furthermore, we state some important lemmas and results that will be required in establishing the proofs of our main results.

2.1 Geometric properties of Hilbert spaces

The Hilbert space which is an extension of the concept of Euclidean spaces to infinite dimensional spaces is known to have the most simplest and clearly discernible geometric properties. Let $\mathcal{C}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. We will denote the weak and strong convergence of a sequence $\{x_n\}$ to a point x by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Also, we denote the set of weak limits of $\{x_n\}$ by $w_\omega(x_n)$, that is

$$w_\omega(x_n) := \{x \in \mathcal{H} : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

Some of the geometric properties that characterize Hilbert spaces include: the availability of the inner product, the fact that the nearest point map of a real Hilbert space $\mathcal{H}$ onto a closed convex subset $\mathcal{C}$ of $\mathcal{H}$ is Lipschitzian with constant 1 and the following identities:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \tag{2.1.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{2.1.2}$$

which hold for $x, y \in \mathcal{H}$ and $\lambda \in [0,1]$. These geometric characteristics of Hilbert spaces makes certain problems posed in Hilbert spaces more manageable than those in general Banach spaces [74].

2.1.1 Some inequalities that characterize Hilbert spaces

Lemma 2.1.1. *[198, 201] Let $\mathcal{H}$ be a real Hilbert space, then the following assertions hold:*

(1) $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2, \quad \forall x, y \in \mathcal{H};$

(2) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \quad \forall x, y \in \mathcal{H}, \ \alpha \in \mathbb{R};$

(3) $\|x-y\|^2 \leq \|x\|^2 + 2\langle y, x-y\rangle, \quad \forall x, y \in \mathcal{H}.$

(4) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \forall\ x, y \in \mathcal{H}.$

Lemma 2.1.2. *[154] For $x, y, v, z \in \mathcal{H}$, we have*

$$2\langle x-y, v-z\rangle = \|x-z\|^2 + \|y-v\|^2 - \|x-v\|^2 - \|y-z\|^2.$$

Also,

$$\begin{aligned}\|x-y+v-z\|^2 &= \|x-y\|^2 + \|v-z\|^2 + 2\langle x-y, v-z\rangle \\ &= \|x-y\|^2 + \|v-z\|^2 + \|x-z\|^2 + \|y-v\|^2 - \|x-v\|^2 - \|y-z\|^2.\end{aligned}$$

2.1.2 Some nonlinear single-valued mappings in Hilbert spaces

Definition 2.1.3. *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{C}$ be a nonempty closed and convex subset of $\mathcal{H}$. A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be*

(i) L-Lipschitz if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x-y\|, \quad \forall \quad x, y \in \mathcal{H};$$

if $L = 1$, then T is called nonexpansive while T is called a contraction if $L \in [0, 1)$,

(ii) uniformly continuous, if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that

$$\|Tx - Ty\| < \epsilon \quad \text{whenever} \quad \|x-y\| < \delta, \quad \forall x, y \in \mathcal{H};$$

(iii) quasi-nonexpansive, if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x-y\|, \quad \forall\ x \in \mathcal{H}, \qquad \text{and} \quad y \in F(T);$$

(iv) firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 - \|(I-T)x - (I-T)y\|^2, \quad \forall\ x, y \in \mathcal{H};$$

(v) β-demicontractive with $0 \leq \beta < 1$, if $F(T) \neq \emptyset$ and

$$\|Tx - y\|^2 \leq \|x-y\|^2 + \beta\|(I-T)x\|^2, \quad \forall \quad x \in \mathcal{H}, \quad y \in F(T);$$

(vi) k-strictly pseudocontractive, if for $0 \leq k < 1$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \quad \forall \;\; x, y \in \mathcal{H}. \tag{2.1.3}$$

(vii) quasi-pseudocontractive, if $F(T) \neq \emptyset$ *and*

$$\|Tx - y\|^2 \leq \|x - y\|^2 + \|Tx - x\|^2, \quad \forall \; x \in \mathcal{H} \quad \textit{and} \quad x \in F(T);$$

(viii) η-strongly monotone, if there exists $\eta > 0$ *such that*

$$\Big\langle Tx - Ty, x - y \Big\rangle \geq \eta\|x - y\|^2 \;\; \forall \; x, y \in \mathcal{H};$$

(ix) η-inverse strongly monotone (η-ism), if there exists $\eta > 0$ *such that*

$$\Big\langle Tx - Ty, x - y \Big\rangle \geq \eta\|Tx - Ty\|^2 \;\; \forall \; x, y \in \mathcal{H};$$

if $\eta = 1$*, then* T *is called firmly nonexpansive,*

(x) monotone, if

$$\Big\langle Tx - Ty, x - y \Big\rangle \geq \; 0 \;\; \forall \; x, y \in \mathcal{H};$$

(xi) pseudo-monotone, if

$$\Big\langle Tx, y - x \Big\rangle \geq 0 \implies \Big\langle Ty, y - x \Big\rangle \geq 0 \;\; \forall \; x, y \in \mathcal{H};$$

(xii) quasimonotone, if

$$\langle Ty, x - y \rangle > 0 \implies \langle Tx, x - y \rangle \geq 0, \; \forall x, y \in \mathcal{H},$$

(xiii) sequentially weakly continuous on $\mathcal{H}$*, if for each sequence* $\{x_n\} \subset \mathcal{H}$ *we have that* $\{x_n\}$ *converges weakly to* $x \in \mathcal{H}$ *implies that* $\{Tx_n\}$ *converges weakly to* Tx*;*

(xiv) sequentially weakly-strongly continuous, if for every sequence $\{x_n\}$ *that converges weakly to a point* x*, the sequence* $\{Tx_n\}$ *converges strongly to* Tx*,*

(xv) α−averaged, if $f = (1-\alpha)I + \alpha T$*, where* $\alpha \in (0, 1)$ *and* $T : \mathcal{H} \to \mathcal{H}$ *is nonexpansive,*

From the above definition, we can deduce that firmly nonexpansive mappings are $\frac{1}{2}$*-averaged while averaged mappings are nonexpansive. This implies that every firmly nonexpansive mapping is nonexpansive. Clearly, the class of quasi-pseudocontractive mappings is more general and properly contains all the other classes of mappings defined in* $(iii) - (vi)$ *with nonempty fixed points set. It is also known that every* η*-inverse strongly monotone mapping is* $\frac{1}{\eta}$*-Lipschitz continuous. Also, if* T *is* η*-strongly monotone and* L*-Lipschitz continuous, then* T *is* η/L^2*-ism. Furthermore, both* η*-strongly monotone and* η*-inverse strongly monotone mappings are monotone while monotone mappings are pseudo-monotone. We also observe that monotone implies pseudo-monotone which implies quasi-monotone but the converses are not always true. Also, observe that uniform continuity is a weaker notion than Lipschitz continuity.*

Remark 2.1.4. It is well known that if D is a convex subset of $\mathcal{H}$, then $T : D \to \mathcal{H}$ is uniformly continuous if and only if, for every $\epsilon > 0$, there exists a constant $M < \infty$ such that

$$\|Tx - Ty\| \leq M\|x - y\| + \epsilon \quad \forall x, y \in D. \tag{2.1.4}$$

Example 2.1.5. Let $\mathcal{H} = l_2(\mathbb{R})$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \dots),\ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$. Then, the operator $A : \mathcal{H} \to \mathcal{H}$ defined by

$$A(x_1, x_2, x_3, \dots) = (x_1 e^{-x_1^2}, 0, 0, \dots)$$

is pseudomonotone, Lipschitz continuous and sequentially weakly continuous but not monotone.

Example 2.1.6. [145] Let $\mathbb{R}$ denote the real number with the usual norm and $\mathcal{C} = [0, 2]$. Let $T : \mathcal{C} \to \mathcal{C}$ be defined by

$$Tx = \frac{1}{3}\Big(x^2 + 2\Big)$$

for all $x \in \mathcal{C}$, where $x + 2 \leq 3,\ x \in [0, 1]$ and $x + 2 = 3,\ x > 1$. Clearly $F(T) = \{1, 2\}$. Then T is a quasi-nonexpansive mapping but not a nonexpansive mapping.

Example 2.1.7. Let X be a real line with the usual norm and $\mathbb{C} = \mathbb{R}$. Define $T : \mathcal{C} \to \mathcal{C}$ by

$$Tx = 7x.$$

T is k-strictly pseudo-contractive but not quasi-nonexpansive or nonexpansive.

Proof. It is clear that $F(T) = \{0\}$, thus for $x \in \mathbb{R}$, we have

$$|7x - 0|^2 = 49|x - 0|^2 > |x - 0|^2$$

which implies that T is not quasi-nonexpansive and hence, not nonexpansive.

Next we show that T is k- strictly pseudo-contractive.

$$|Tx - Ty|^2 = |7x - 7y|^2 = 49|x - y|^2.$$

Also,

$$\begin{aligned} |x - y - (Tx - Ty)|^2 &= |x - y - (7x - 7y)|^2 \\ &= 36|x - y|^2. \end{aligned}$$

$$\begin{aligned} |Tx - Ty|^2 &= |x - y|^2 + 48|x - y|^2 \\ &= |x - y|^2 + \frac{4}{3}|x - y - (Tx - Ty)|^2. \end{aligned}$$

Hence, T is $\frac{4}{3}$-strictly pseudo-contractive mapping. □

Lemma 2.1.8. *[276] Let $\mathcal{H}$ be a real Hilbert space and $S : \mathcal{H} \to \mathcal{H}$ be a κ-strictly pseudo-contractive mapping with $\kappa \in [0, 1)$. Let $S_\beta := \beta I + (1 - \beta)S$, where $\beta \in [\kappa, 1)$, then*

(i) $F(S) = F(S_\beta)$,

(ii) S_β *is a nonexpansive mapping.*

Lemma 2.1.9. *[238] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with $T \neq 0$, and $S : \mathcal{H}_2 \to \mathcal{H}_2$ be a nonexpansive mapping. Then $T^*(I - S)T$ is $\frac{1}{2\|A\|^2}$-inverse strongly monotone.*

Lemma 2.1.10. *[261] Let $\mathcal{H}$ be a real Hilbert space and $A : \mathcal{H} \to \mathcal{H}$ be a nonlinear mapping, then the following hold.*

(i) A is η-inverse strongly monotone and $\gamma > 0$, then γA is $\frac{\eta}{\gamma}$-inverse strongly monotone.

(ii) A is averaged if and only if the complement $I - A$ is η-inverse strongly monotone for some $\eta > \frac{1}{2}$. Indeed, for $\beta \in (0, 1)$, A is β-averaged if and only if $I - A$ is $\frac{1}{2\beta}$-inverse strongly monotone.

Definition 2.1.11. *A one-parameter family mapping $\mathcal{T} = \{T(s) : 0 \leq s < +\infty\}$ from $\mathcal{H}_1$ into itself is said to be a nonexpansive semigroup if it satisfies the following conditions:*

(i) $T(0)x = x, \ \forall \ x \in \mathcal{H}_1$;

(ii) $T(s + u) = T(s)T(u)$, *for all* $s, u \geq 0$;

(iii) for each $x \in \mathcal{H}_1$, the mapping $T(s)x$ is continuous;

(iv) $\|T(s)x - T(s)y\| \leq \|x - y\|$, *for all* $x, y \in \mathcal{H}_1$ *and* $s \geq 0$.

We denote the common fixed point set of the semigroup $\mathcal{T}$ by $F(\mathcal{T}) = \{x \in \mathcal{C} : T(s)x = x, \ \forall s \geq 0\}$. It is well known that $F(\mathcal{T})$ is closed and convex [47].

Lemma 2.1.12. *[240] Let $\mathcal{C}$ be a nonempty bounded closed and convex subset of a real Hilbert space $\mathcal{H}$. Let $\mathcal{T} = \{T(s) : s \geq 0\}$ from $\mathcal{C}$ be a nonexpansive semigroup on $\mathcal{C}$; then for all $h \geq 0$,*

$$\limsup_{t\to\infty, \ x\in\mathcal{C}} \left\| \frac{1}{t} \int_0^t T(s)x - T(h)\Big(\frac{1}{t} \int_0^t T(s)x dx\Big) \right\| = 0.$$

Lemma 2.1.13. *[240] Let $\mathcal{C}$ be a nonempty bounded closed and convex subset of a real Hilbert space $\mathcal{H}$. Let $\{x_n\}$ be a sequence and let $\mathcal{T} = \{T(s) : s \geq 0\}$ from $\mathcal{C}$ be a nonexpansive semigroup on $\mathcal{C}$, if the following conditions are satisfied*

(i) $x_n \rightharpoonup x$;

(ii) $\limsup_{s\to\infty} \limsup_{n\to\infty} \Big\| T(s)x_n - x_n \Big\| = 0,$

then, $x \in F(\mathcal{T})$.

Lemma 2.1.14. *[112] Let* $\mathcal{H}$ *be a real Hilbert space and let* $S : \mathcal{H} \to \mathcal{H}$ *be a nonexpansive mapping with* $F(S) \neq \emptyset$. *Then, the mapping* $I - S$ *is demiclosed at zero, that is, for any sequence* $\{x_n\}$ *in* $\mathcal{H}$ *such that* $x_n \rightharpoonup x \in \mathcal{H}$ *and* $\|x_n - Sx_n\| \to 0$ *implies* $x \in F(S)$.

Lemma 2.1.15. *[199] Each Hilbert space* $\mathcal{H}$ *satisfies the Opial condition, that is, for any sequence* $\{x_n\}$ *with* $x_n \rightharpoonup x$, *the inequality* $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ *holds for every* $y \in \mathcal{H}$ *with* $y \neq x$.

2.2 Geometric properties of Banach spaces

In this section, we discuss some of the geometric properties of Banach spaces that will be required in this study.

Let $\mathcal{X}$ be a real Banach space and $\mathcal{C}$ a nonempty, closed and convex subset of $\mathcal{X}$. We denote by $\mathcal{X}^*$ the dual of $\mathcal{X}$ and $\|\cdot\|$ the norm of $\mathcal{X}$ or $\mathcal{X}^*$.

Definition 2.2.1. *A Banach space* $\mathcal{X}$ *is said to be strictly convex if for all* $x, y \in \mathcal{X}$, $x \neq y$, $\quad \|x\| = \|y\| = 1$, *we have* $\|\lambda x + (1-\lambda)y\| < 1, \quad \forall \lambda \in (0,1)$.

Definition 2.2.2. *Let* $S(\mathcal{X}) := \{x \in \mathcal{X} : \|x\| = 1\}$. *A Banach space* $\mathcal{X}$ *is said to be smooth if*

$$\lim_{\sigma\to 0} \frac{\|x + \sigma y\| - \|x\|}{\sigma}$$

exists for $x, y \in S(\mathcal{X})$.

Definition 2.2.3. *Let* E *be a normed space with dim* $E \geq 2$. *The modulus of convexity* $\delta_E : (0,2] \to [0,1]$ *is defined by*

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = 1 = \|y\|, \|x - y\| = \epsilon\right\}.$$

Definition 2.2.4. *Let* E *be a normed space with dim* $E \geq 2$. *The modulus of smoothness* $\rho_{\mathcal{X}}(\sigma) : [0,\infty) \to [0,\infty)$ *is defined by*

$$\rho_{\mathcal{X}}(\sigma) = \sup\left\{\frac{\|x + \sigma y\| + \|x - \sigma y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}.$$

Let $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. A Banach space $\mathcal{X}$ is called uniformly convex if $\delta_{\mathcal{X}}(\epsilon) > 0$ for any $\epsilon \in (0,2]$, and p-uniformly convex if there exists a constant $c_p > 0$ such that $\delta_{\mathcal{X}}(\epsilon) > c_p\epsilon^p$ for any $\epsilon \in (0,2]$. $\mathcal{X}$ is called *uniformly smooth* if $\lim_{\sigma\to 0}\left(\frac{\rho_{\mathcal{X}}(\sigma)}{\sigma}\right) = 0$ and q-*uniformly smooth* if there is a constant $q > 0$ such that $\rho_{\mathcal{X}}(\sigma) \leq q\sigma^2$ for some $\sigma > 0$.

Theorem 2.2.5. *[74] Every uniformly smooth space is reflexive.*

Lemma 2.2.6. *[37] The space $\mathcal{X}$ is 2-uniformly convex if and only if there exists $\mu_{\mathcal{X}} \geq 1$ such that*

$$\frac{\|u+v\|^2 + \|u-v\|^2}{2} \geq \|u\|^2 + \|\mu_{\mathcal{X}}^{-1} v\|^2, \ \forall\ u, v \in \mathcal{X}. \tag{2.2.1}$$

The minimum value of the set of all $\mu_{\mathcal{X}} \geq 1$ satisfying (2.2.1) $\forall\ u, v \in \mathcal{X}$ is denoted by μ and is called the 2-uniform convexity constant of $\mathcal{X}$. It is easy to see that $\mu = 1$ whenever $\mathcal{X}$ is a Hilbert space. We denote the minimum value of the set of all $\mu_{\mathcal{X}} \geq 1$ by μ such that (2.2.1) holds for any $u, v \in \mathcal{X}$.

Lemma 2.2.7. *[262] Let $\mathcal{X}$ be a 2-uniformly smooth Banach space with the best smoothness constant $\mu > 0$. Then, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y,\ Jx\rangle + 2\|\mu y\|^2, \quad \forall\ x, y \in \mathcal{X}.$$

Remark 2.2.8. In Hilbert spaces the best smoothness constant is $\mu = \frac{1}{\sqrt{2}}$.

Remark 2.2.9. Every 2-uniformly convex Banach space is uniformly convex and every Hilbert space is uniformly smooth and 2-uniformly convex.

Definition 2.2.10. *The generalized duality mapping $J_p^{\mathcal{X}} : \mathcal{X} \to 2^{\mathcal{X}^*}$ is defined by*

$$J_p^{\mathcal{X}}(x) = \{x^* \in \mathcal{X}^* : \langle x,\ x^*\rangle = \|x\|^p, \ \ \|x^*\| = \|x\|^{p-1}, \ \ \forall\ x \in \mathcal{X}\}.$$

In particular, when $p = 2$ the duality map $J = J_2^{\mathcal{X}}$ is called the normalized duality map. The Hilbert spaces, L_p(or ℓ_p) spaces and the Sobolev spaces $W^{k,p}, (1 < p < 1)$ are uniformly convex and uniformly smooth Banach spaces. Furthermore, the L_p, ℓ_p and $W^{k,p}$ spaces are p-uniformly smooth if $1 < p < 2$ and 2-uniformly smooth if $p > 2$. If $\mathcal{X} = \mathcal{H}$, where $\mathcal{H}$ is a real Hilbert space, we have $J = I$.

Now, we state the relationship between uniformly convex and uniformly smooth space.

Theorem 2.2.11. *[74] Let $\mathcal{X}$ be a Banach space;*

(i) $\mathcal{X}$ is uniformly smooth if and only if $\mathcal{X}^*$ is uniformly convex.

(ii) $\mathcal{X}$ is uniformly convex if and only if $\mathcal{X}^*$ is uniformly smooth.

We give an example of a generalized duality mapping in Banach spaces.

Example 2.2.12. [3] Let $\mathcal{X} := L_p([\alpha, \beta])\ \ (1 < p < \infty)$, where $\alpha, \beta \in \mathbb{R}$ and let $f \in \mathcal{X}$. Then the generalized duality mapping $J_p^{\mathcal{X}}$ is given by

$$J_p^{\mathcal{X}}(f)(t) = |f(t)|^{p-1}\ \mathrm{sgn}(f(t)).$$

The normalized duality mapping is known to have the following properties [85]:

(i) If $\mathcal{X}$ is smooth, then J is single valued denoted by j.

(ii) Jx is nonempty closed, convex and bounded sub set of $\mathcal{X}^*$ for all $x \in \mathcal{X}$.

(iii) If $\mathcal{X}$ is reflexive, then J is surjective.

(iv) If $\mathcal{X}$ is strictly convex, then J is one-to-one and strictly monotone, ie $\langle x-y,\ Jx-Jy\rangle \geq 0,\ \forall\ x, y \in \mathcal{X}$ such that $x \neq y$.

(v) If $\mathcal{X}$ is reflexive and strictly convex, then J^{-1} is norm-to-weak continuous.

(vi) If $\mathcal{X}$ is uniformly smooth and uniformly convex, then J is norm-to-norm continuous on bounded subset of $\mathcal{X}$ and $J^{-1} = J^*$. Also the normalized duality mapping on $\mathcal{X}^*$ is also uniformly norm-to-norm continuous on bounded subsets of $\mathcal{X}^*$.

(vii) If $\mathcal{X}^*$ is uniformly convex, then J is single-valued, one-to-one and uniformly continuous on bounded subsets of $\mathcal{X}$.

If $\mathcal{X}$ is smooth, $J : \mathcal{X} \to \mathcal{X}^*$ is said to be weak-to-weak continuous if for every $y \in \mathcal{X}, \langle y, Jx_n\rangle \to \langle y, Jx\rangle$ as $x_n \rightharpoonup x$. It is known that the $l_p(p > 1)$ space has the weak-to-weak continuous property, but the $L_p(p > 2)$ space does not possess this property.

The Lyapunov functional $\phi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ [16] is defined by

$$\phi(u, v) := \|u\|^2 - 2\langle u,\ Jv\rangle + \|v\|^2,\ \forall\ u, v \in \mathcal{X}.$$

We can easily see that

$$\phi(u, v) \geq \Big(\|u\| - \|v\|\Big)^2 \geq 0.$$

Moreover, we observe that $\phi(u, u) = 0$ for each $u \in \mathcal{X}$. If $\mathcal{X}$ is strictly convex, then $\phi(u, v) = 0 \iff u = v$. If $\mathcal{X}$ is a Hilbert space, then $\phi(u, v) = \|u - v\|^2$ for all $u, v \in \mathcal{X}$.

2.2.1 Some inequalities that characterize Banach spaces

Lemma 2.2.13. *[16, 20] Suppose that $\mathcal{X}$ is a real, uniformly convex, smooth Banach space. Then, the Lyapunov functional satisfies the following properties for all $x, y, z \in \mathcal{X}$ and $\alpha \in (0, 1)$:*

(1) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z,\quad Jz - Jy\rangle$;

(2) $\phi(x, y) + \phi(y, x) = 2\langle x - y,\quad Jx - Jy\rangle$;

(3) $\phi(x, y) = \|x\|\|Jx - Jy\| + \|y\|\|x - y\|$;

(4) $0 \leq (||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2$;

(5) $\phi(x, J^{-1}(\alpha Jz + (1-\alpha)Jy)) \leq \alpha\phi(x, z) + (1-\alpha)\phi(x, y)$.

Alber [16] also studied the following functional $V : \mathcal{X} \times \mathcal{X}^* \to \mathbb{R}$ defined by

$$V(x,\ x^*) = \|x\|^2 - 2\langle x,\ x^*\rangle + \|x^*\|^2,$$

for all $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$. It is easy to see from the definition of the Lyapunov functional that

$$V(x,\ x^*) = \phi(x,\ J^{-1}x^*),$$

for all $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$. For each $x \in \mathcal{X}$, the mapping $f : \mathcal{X}^* \to \mathcal{R}$ defined by $f(x^*) = V(x,\ x^*)$ for all $x^* \in \mathcal{X}^*$ is a continuous and convex function. Moreover, the functional V is known to satisfy the following inequality [15]:

$$V(x, x^*) + 2\langle J^{-1}x^* - x,\ \ y^*\rangle \leq V(x, x^* + y^*),\ \forall\ x \in \mathcal{X},\ \ x^*, y^* \in \mathcal{X}^*. \tag{2.2.2}$$

2.2.2 Some nonlinear single-valued mappings in Banach spaces

Definition 2.2.14. *Let $\mathcal{X}$ be a real Banach space and $\mathcal{C}$ be a nonempty closed and convex subset of $\mathcal{X}$. A mapping $T : \mathcal{X} \to \mathcal{X}$ is said to be*

(i) *closed if for any sequence $\{x_n\} \subset \mathcal{C}$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$;*

(ii) *semi-compact if for any bounded sequence $\{x_n\}$ in $\mathcal{C}$ with $x_n - Tx_n \to 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to some $x \in \mathcal{X}$;*

(iii) *$\phi-$nonexpansive, if*
$$\phi(Tx, Ty) \leq \phi(x, y),\quad \forall\ x, y \in \mathcal{X};$$

(iv) *quasi ϕ-nonexpansive, if $F(T) \neq \emptyset$ and*
$$\phi(p, Tx) \leq \phi(p, x),\quad \forall\ x \in \mathcal{X},\ p \in F(T);$$

(v) *firmly nonexpansive, if for all $x, y \in \mathcal{X}$*
$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, y) + \phi(Ty, x)$$
or equivalently,
$$\Big\langle Tx - Ty, Jx - JTx - (Jy - JTy)\Big\rangle \geq 0.$$

Remark 2.2.15. Every firmly nonexpansive mapping is nonexpansive.

Example 2.2.16. [178] Consider $\mathbb{R}$ with the usual norm and let $\mathbb{R} \to \mathbb{R}$ be a function defined by
$$T(x, y) = (-y, x),\ \forall (x, y) \in \mathbb{R}^2.$$
Then, T is nonexpansive but not firmly nonexpansive.

2.3 Geometric properties of geodesic metric spaces

It is well known that the geometric structure of Hilbert spaces makes problems that occur in Hilbert spaces more manageable and easier to solve. However, most real life problems naturally occur in nonlinear spaces (for instance, in metric spaces). Therefore, there is the need to extend our study to nonlinear spaces. Sometimes, extending some known results to this space may be difficult because of the nonlinear structure of nonlinear spaces. In order to extend these results to nonlinear spaces, one needs to introduce some kind of convex properties which provide sufficient information that ensures the applications of such existing results. One of these properties is the existence of distance-preserving mapping, which provides the metric space (nonlinear space) with a structure that is similar to the linear structure of a normed linear space (in particular, Hilbert space). Metric spaces with this distance-preserving mappings are called geodesic metric spaces.

Definition 2.3.1. *[33] Let (X, d) be a metric space. A continuous mapping from the interval $[0, 1]$ to X is called a path.*

Definition 2.3.2. *Let $x, y \in X$ and $I = [0, d(x, y)]$, a geodesic path joining x to y is a distance-preserving mapping (isometry) $c : I \to X$, such that $c(0) = x$, $c(d(x, y)) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, d(x, y)]$.*

The image of a geodesic path is a geodesic segment and it is denoted by $[x, y]$ whenever it is unique. A metric space (X, d) is said to be a geodesic space if every pair of points x and y in X are connected by a geodesic. A subset C of a geodesic space X is said to be convex, if for all $x, y \in C$, the segment $[x, y]$ remains in C. For $x, y \in X$ and $t \in [0, 1]$ we write $tx \oplus (1 - t)y$ for the unique point z in the geodesic segment joining x to y such that

$$d(x, z) = (1 - t)d(x, y) \text{ and } d(z, y) = td(x, y). \tag{2.3.1}$$

2.3.1 Metric midpoints

Definition 2.3.3. *Let X be a metric space and $x, y \in X$. A point $z \in X$ is a metric midpoint of x and y if $d(x, z) = \frac{1}{2}d(x, y)$.*

Proposition 2.3.1. *Let X be a complete metric space. Then the following are equivalent;*

(a) The space X is a geodesic space.

(b) For every $x, y \in X$, there exists a point $z \in X$ such that

$$d^2(x, z) + d^2(y, z) = \frac{1}{2}d^2(x, y).$$

(c) Every pair of points in X has a metric midpoint.

Remark 2.3.4. If X is a geodesic metric space, then every pair of points in X has at least one midpoint which is also a metric midpoint.

Definition 2.3.5. *Let X be a geometric metric space. A subset C of X is said to be convex in C if it includes every geodesic segment joining two of its point. This means that C is convex if $x, y \in C$, we have that $tx \oplus (1-t)y \in C$.*

Remark 2.3.6. [45]

A geodesic segment in a space that is not uniquely geodesic may not necessarily be convex.

(b) A subset of a uniquely geodesic metric space which is endowed with the induced metric is geodesic if and only if it is convex.

2.3.2 Geodesic triangles

Definition 2.3.7. *A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space X consists of three points $x_1, x_2, x_3 \in X$ (which are also called the vertices of $\triangle$) and a geodesic segment between each pair of vertices (which are also known as edges of $\triangle$).*

Definition 2.3.8. *A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3) \in X$ is a triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in Euclidean space $\mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) \ \ \forall \ \ i, j \in \{1, 2, 3\}$.*

Definition 2.3.9. *Let $\{x_n\}$ be a bounded sequence in a geodesic metric space X. Then, the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined by $A(\{x_n\}) = \{\bar{v} \in X : \limsup_{n\to\infty} d(\bar{v}, x_n) = \inf_{v\in X} \limsup_{n\to\infty} d(v, x_n)\}$.*
The sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $\bar{v} \in X$ if $A(\{x_{n_k}\}) = \{\bar{v}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = \bar{v}$ and we say that $\bar{v}$ is the Δ-limit of $\{x_n\}$.

2.3.3 Geometric properties of Hadamard spaces

The Hadamard spaces which were named after Jacques Hadamard are known as complete uniquely geodesic metric spaces of nonpositive curvatures and they include Hilbert spaces, Euclidean spaces ($\mathbb{R}^n$), $\mathbb{R}$-trees, hyperbolic spaces, Hilbert ball, among others. The geometry of Hadamard spaces can be seen as the nonlinearization of the geometry of Hilbert spaces. Karl Menger [176] introduced the notion of a geodesic in metric spaces and he generalized classical results in geometry to his new metric space with geodesic. To improve the work of Menger [176], Wald [258] introduced the notion of two-dimensional curvature in metric spaces. Alexandror [17] also contributed by discovering some interesting characteristics of the spaces. Over the years, the Hadamard spaces has shown to be an appropriate framework for the study of optimization problems which has applications in diverse fields such as economics, engineering and science.

Now we are in the position to present some characterizations and conditions required for geodesic spaces to be CAT(0) spaces.

Characterizations of CAT(0) spaces

Gromov [114] coined the term CAT(0) from the initials of three mathematicians where C stands for Cartan, A stands for Alexandrov and T stands for Toponogov. Precisely, CAT(0) spaces are spaces of non-positive curvature bounded above by 0.

Definition 2.3.10. *A geodesic space is called a CAT(0) space if all geodesic triangles satisfy the comparison axiom.*

Definition 2.3.11. *Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then, $\triangle$ is said to satisfy the CAT(0) inequality, if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,*

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

Let x, y, z be points in X and y_0 be the midpoint of the segment $[y, z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z). \tag{2.3.2}$$

Thus, a geodesic space is a CAT(0) space if and only if it satisfies (2.3.2). It is generally known that a CAT(0) space is a uniquely geodesic space. Inequality (2.3.2) is known as the CN inequality of Bruhat and Titis [48].

Definition 2.3.12. *A geodesic metric space X is called a CAT(0) space if all geodesic triangles satisfy the CAT(0) inequality. Equivalently, X is called a CAT(0) space if and only if it satisfies the CN inequality.*

Theorem 2.3.13. *(see [33, Theorem 1.3.2]). Let X be a complete metric space. Then the following are equivalent.*

(i) The space X is a CAT(0) space.

(ii) For every pair of points $x, y \in X$, there exists $m \in X$ such that for each $z \in X$, we have that

$$d^2(m, z) \leq \frac{1}{2}d^2(x, z) + \frac{1}{2}d^2(y, z) - \frac{1}{4}d^2(x, y).$$

(iii) For every pair of points $x, y \in X$ and $\epsilon > 0$, there exists $m \in X$ such that for each $z \in X$, we have that

$$d^2(m, z) \leq \frac{1}{2}d^2(x, z) + \frac{1}{2}d^2(y, z) - \frac{1}{4}d^2(x, y) + \epsilon.$$

Next, we present the following Theorem which gives equivalent conditions for a geodesic space to be a CAT(0) space.

Theorem 2.3.14. *(see [33, Theorem 1.3.3]). Let X be a geodesic metric space. Then the following are equivalent:*

(i) The space X is a CAT(0) space.

(ii) For every pair of points $x, y, z \in X$, we have

$$d^2(m, x) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(x, z),$$

where m is the midpoint of $[y, z]$.

(iii) For every geodesic $x : [0, 1] \to X$ and every point $p \in X$, we have

$$d^2(p, x_1) \leq (1 - t)d^2(p, x_0) + td^2(p, x_1) - t(1 - t)d^2(x_0, x_1).$$

(iv) For every $x, y, u, v \in X$, we have

$$d^2(x, u) + d^2(y, v) \leq d^2(x, y) + d^2(u, v) + 2d(x, v)d(y, u).$$

(v) For every $x, y, u, v \in X$, we have

$$d^2(x, u) + d^2(y, v) \leq d^2(x, y) + d^2(y, u) + d^2(u, v) + d^2(v, x).$$

Definition 2.3.15. *A complete CAT(0) space is an Hadamard space.*

2.3.4 Examples of Hadamard spaces

Now, we present some examples of Hadamard spaces;

Example 2.3.16. (Hilbert space)[33]. Hilbert spaces are Hadamard. The geodesics are the line segments. It is also known that a Banach space is CAT(0) if and only if it is Hilbert.

Example 2.3.17. ($\mathbb{R}$- trees)[33]. A metric space (X, d) is an $\mathbb{R}$-tree if it is uniquely geodesic and for every $x, y, z \in X$, we have $[x, z] = [x, y] \cup [y, z]$ whenever $[x, y] \cap [y, z] = \{y\}$. Also, all triangles in an $\mathbb{R}$-tree are trivial.

Example 2.3.18. All simply connected Riemannian manifold with non-positive sectional curvature induced with the Riemannian metric.

2.3.5 Quasilinearization mapping and dual space

The concept of quasilinearization in Hadamard spaces was introduced by Berg and Nikolaev [41]. They denoted a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and called it a vector. Using this concept, they defined the quasilinearization as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right), \quad \forall a, b, c, d \in X. \tag{2.3.3}$$

It is easy to verify that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, for all $a, b, c, d, e \in X$.

Definition 2.3.19. *The space X is said to satisfy the Cauchy Schwartz inequality, if*

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d) \ \forall a,b,c,d \in X.$$

Moreover, a geodesic space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality (see [138]).

Using the idea of quasilinearization mapping, Kakavandi and Amini [138] introduced the concept of dual space of an Hadamard space X as follows:
Consider the map $\theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ defined by

$$\theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle \quad (t \in \mathbb{R}, a,b,x \in X),$$

where $C(X, \mathbb{R})$ denotes the space of all continuous real valued functions on X. The Cauchy-Schwartz inequality implies that $\theta(t,a,b)$ is a Lipschitz function with Lipschitz semi-norm $L(\theta(t,a,b)) = |t|d(a,b) \quad (t \in \mathbb{R}, a,b \in X)$, where

$$L(\varphi) = \sup\left\{\frac{\varphi(x) - \varphi(y)}{d(x,y)} : x,y \in X, x \neq y\right\}$$

is the Lipschitz semi-norm for any function $\varphi : X \to \mathbb{R}$.

Definition 2.3.20. *A pseudometric D on $\mathbb{R} \times X \times X$ is defined by*

$$D((t,a,b),(s,c,d)) = L(\theta(t,a,b) - \theta(s,c,d)), \quad (t,s \in \mathbb{R}, a,b,c,d \in X).$$

In an Hadamard space (X,d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subset of the pseudometric space of all real valued Lipschitz functions $(\text{Lip}(X,\mathbb{R}), L)$ (see [93, 211, 255]).
It is shown in [138] that $D((t,a,b),(s,c,d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for all $x,y \in X$. Thus, D induces an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t,a,b) is defined as

$$[\overrightarrow{tab}] := \{\overrightarrow{scd} : D((t,a,b),(s,c,d)) = 0\}.$$

Thus, the set $X^* = \{[\overrightarrow{tab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D([\overrightarrow{tab}], [\overrightarrow{scd}]) := D((t,a,b),(s,c,d))$.

Definition 2.3.21. *Let (X,d) be an Hadamard space. Then, the pair (X^*, D) is called the dual space of (X,d).*

Throughout this dissertation, we shall write X^* for the dual space of an Hadamard space X.

Remark 2.3.22. [138] The dual of a closed and convex subset of a Hilbert space H with nonempty interior is an Hadamard space and $t(b-a) \equiv [\overrightarrow{tab}]$ for all $t \in \mathbb{R}, \quad a,b \in H$. We also note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle, \quad (x^* = [\overrightarrow{tab}] \in X^*, \quad x,y \in X \text{ and } t \in \mathbb{R}).$$

2.3.6 Some inequalities that characterize Hadamard spaces

Lemma 2.3.23. *[194] Let X be a CAT(0) space, $x, y, z, \in X$ and $t \in [0,1]$. Then*

(a) $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$.

(b) $d^2(tx \oplus (1-t)y, z) \leq td^2(y,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$.

(c) $d^2(tx \oplus (1-t)y, z) \leq t^2 d^2(x,z) + (1-t)^2 d^2(y,z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$.

2.3.7 Some nonlinear single-valued mappings in Hadamard spaces

Let X be an Hadamard space and C be a nonempty closed and convex subset of X. A mapping $T : C \to C$ is said to be

(i) L-Lipschitzian, if there exists $L > 0$ such that

$$d(Tx, Ty) \leq Ld(x,y), \ \forall \ x, y \in C,$$

(ii) nonexpansive, if

$$d(Tx, Ty) \leq d(x,y) \quad \forall \ \ x, y \in C,$$

(iii) quasi-nonexpansive, if $F(T) \neq \emptyset$ and

$$d(Tx, y) \leq d(x,y), \ \ \forall x \in C \quad \text{ and } \quad y \in F(T),$$

(iv) demicontractive, if $F(T) \neq \emptyset$ and there exists $k \in (0,1)$ such that

$$d^2(Tx, y) \leq d^2(x,y) + kd^2(x, Tx), \ \ \forall \ \ x \in C, \quad \forall \ \ y \in F(T).$$

(v) quasi-pseudocontractive if $F(T) \neq \emptyset$ and

$$d^2(Tx, y) \leq d^2(x,y) + d^2(x, Tx), \quad \forall \ \ x \in C, \quad \forall \ \ y \in F(T). \tag{2.3.4}$$

Remark 2.3.24. The class of quasi-pseudocontractive mappings includes some nonlinear mappings like nonexpansive mappings (with nonempty fixed points set), quasi-nonexpansive mappings and demicontractive mappings.

Example 2.3.25. [186] Let C be the closed interval $[0,1]$ with the absolute value as norm. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x \in [0, \frac{1}{2}] \\ 0, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

T is quasi-pseudocontractive but not demicontractive.

2.3.8 Monotone operators and its resolvents in Hadamard spaces

One of the most important area of nonlinear and convex analysis is the monotone operator theory. This is due to the role it plays in optimization theory and related mathematical problems. In this section, we study the concept of monotone operators and its resolvent in Hadamard spaces.

Definition 2.3.26. *Let X be an Hadamard space and X^* be its dual space. A multivalued operator $A : X \to 2^{X^*}$ is monotone if and only if for all $x, y \in D(A)$, $x^* \in Ax$, $y^* \in Ay$, we have*

$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geq 0.$$

A monotone operator A is called a maximal monotone operator if the graph $G(A)$ of A defined by

$$G(A) := \{(x, x^*) \in X \times X^* : x^* \in A(x)\},$$

is not properly contained in the graph of any other monotone operator.

Definition 2.3.27. *The resolvent of a monotone operator A of order $\lambda > 0$ is the multi-valued mapping $J_\lambda^A : X \to 2^{X^*}$ defined by*

$$J_\lambda^A x := \left\{ z \in X \middle| \left[\frac{1}{\lambda}\overrightarrow{zx}\right] \in Az \right\}. \tag{2.3.5}$$

The operator A satisfies the range condition if for every $\lambda > 0$, $D(J_\lambda^A) = X$.

The resolvent of monotone operators plays a crucial role in the approximation of solutions of MIPs. We present some lemmas that relate the fixed points of a resolvent of a monotone operator and the set of solutions of the MIP (1.2.16).

Lemma 2.3.28. *[150] Let X be a CAT(0) space and J_λ^A be the resolvent of the operator A of order λ. Then we have that*

(a) For any $\lambda > 0$, $R(J_\lambda^A) \subset D(A)$ and $F(J_\lambda^A) = A^{-1}(0)$, where $R(J_\lambda^A)$ is the range of J_λ^A.

(b) If A is monotone, then J_λ^A is a single-valued and firmly nonexpansive mapping.

(c) If A is monotone and $0 < \lambda \leq \mu$, then $d^2(J_\lambda^A x, J_\mu^A x) \leq \frac{\mu-\lambda}{\mu+\lambda} d^2(x, J_\mu^A x)$, which implies that $d(x, J_\lambda^A x) \leq 2d(x, J_\mu^A x)$.

Lemma 2.3.29. *[255] Let X be an Hadamard space and $A : X \to 2^{X^*}$ be a monotone mapping. Then,*

$$d^2(u, J_\lambda^A x) + d^2(J_\lambda^A x, x) \leq d^2(u, x), \tag{2.3.6}$$

for all $u \in F(J_\lambda^A)$, $x \in X$ and $\lambda > 0$.

The Moreau-Yosida resolvent $J_\lambda^f : X \to X$ of a proper convex and lower semicontinuous function f in X is defined by

$$J_\lambda^f(x) = \arg\min_{y\in X}\left[f(y) + \frac{1}{2\lambda}d^2(y,x)\right] \;\; \forall x \in X,\; \lambda > 0. \tag{2.3.7}$$

Definition 2.3.30. *Let C be a nonempty closed and convex subset of an Hadamard space X. A mapping $T : C \to C$ is said to be Δ-demiclosed, if for any bounded sequence $\{x_n\}$ in C such that $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x = Tx$.*

2.4 Metric and generalized projections

In this section, we introduce and discuss some basic results on metric projection and present some examples.

Definition 2.4.1. *Let $\mathcal{C}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. For every point $x \in \mathcal{H}$, there exists a unique nearest point in $\mathcal{C}$ denoted by $P_{\mathcal{C}}x$ such that*

$$\|x - P_{\mathcal{C}}x\| \le \|x - y\| \quad \forall \;\; x \in \mathcal{H} \quad \text{and} \quad y \in \mathcal{C}.$$

The operator $P_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ is called the metric projection of $\mathcal{H}$ onto $\mathcal{C}$.

Lemma 2.4.2. *[16] The operator $\Pi_{\mathcal{C}} : \mathcal{X} \to \mathcal{C}$ is called the generalized projection operator if it is the solution to the minimization problem*

$$\Pi_{\mathcal{C}}x = x^* \iff \phi(x^*,\; x) = \inf_{u\in\mathcal{C}} \phi(u,\; x).$$

If $\mathcal{X}$ is a real Hilbert space then $\Pi_{\mathcal{C}} = P_{\mathcal{C}}$.

The metric projection $P_{\mathcal{C}}x$ is nonexpansive with the following properties;

Proposition 2.4.1. *The metric projection is characterized by*

$$\langle x - P_{\mathcal{C}}x, y - P_{\mathcal{C}}x\rangle \le 0 \quad \forall \;\; y \in \mathcal{C} \quad \text{and} \quad x \in \mathcal{H}. \tag{2.4.1}$$

The consequences of Proposition 2.4.1 are

(i) the metric projection is firmly nonexpansive, that is

$$\|P_{\mathcal{C}}x - P_{\mathcal{C}}y\|^2 \le \langle x - y, P_{\mathcal{C}}x - P_{\mathcal{C}}y\rangle \quad \forall \quad x, y \in \mathcal{H},$$

(ii)

$$\|x - P_{\mathcal{C}}x\|^2 + \|y - P_{\mathcal{C}}x\|^2 \le \|x - y\|^2 \quad \forall \quad x \in \mathcal{H} \quad \text{and} \quad y \in \mathcal{C}.$$

(iii) If $\mathcal{C}$ is a closed subspace, then $P_{\mathcal{C}}$ coincides with the orthogonal projection from $\mathcal{H}$ onto $\mathcal{C}$, that is, $x - P_{\mathcal{C}}x$ is orthogonal to $\mathcal{C}$. Hence, for any $y \in \mathcal{C}$,

$$\langle x - P_{\mathcal{C}}x, y\rangle = 0.$$

We know that the metric projections in Banach and Hadamard spaces have the same property.

Lemma 2.4.3. *[16] Let $\mathcal{C}$ be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space $\mathcal{X}$. If $x \in \mathcal{X}$ and $q \in \mathcal{C}$, then*

$$q = \Pi_{\mathcal{C}} x \iff \langle y - q,\ Jx - Jq \rangle \leq 0, \quad \forall\ y \in \mathcal{C} \tag{2.4.2}$$

and

$$\phi(y,\ \Pi_{\mathcal{C}} x) + \phi(\Pi_{\mathcal{C}} x, x) \leq \phi(y,\ x), \quad \forall\ y \in \mathcal{C},\ x \in \mathcal{X}. \tag{2.4.3}$$

We are now in the position to present some examples of the metric projection.

Example 2.4.4. If $\mathcal{C} = \{y \in \mathcal{H} : \|y - s\| \leq \alpha\}$ is a closed ball centered at $s \in \mathcal{H}$ with radius $\alpha \geq 0$,

$$P_{\mathcal{C}} x = \begin{cases} s + \frac{\alpha(x-s)}{\|x-s\|}, & \text{if } x \notin \mathcal{C} \\ x, \quad \text{if } x \in \mathcal{C}. \end{cases}$$

Example 2.4.5. Let $\mathcal{C} = [a, b]$ be a closed rectangle in $\mathbb{R}^n$, where $a = (a_1, a_2, \cdots, a_n)^T$ and $b = (b_1, b_2, \cdots, b_n)^T$, then for $1 \leq i \leq n$

$$(P_{\mathcal{C}} x)_i = \begin{cases} a_i, & x_i < a_i, \\ x_i, & x_i \in [a_i, b_i], \\ b_i, & x_i > b_i \end{cases} \tag{2.4.4}$$

is the metric projection with the i^{th} coordinate.

Example 2.4.6. Let $\mathcal{C} = \{y \in \mathcal{H} : \langle s, y \rangle \leq \alpha\}$ be a closed half space, with $s \neq 0$ and $\alpha \in \mathbb{R}$, then

$$P_{\mathcal{C}} x = \begin{cases} x - \frac{\langle s,x \rangle - \alpha}{\|s\|^2} s, & \text{if } \langle s, x \rangle > \alpha, \\ x, & \text{if } \langle s, x \rangle \leq \alpha \end{cases} \tag{2.4.5}$$

is the metric projection onto $\mathcal{C}$.

Example 2.4.7. Let $\mathcal{C} = \{y \in \mathcal{H} : \langle s, y \rangle = \alpha\}$ be a hyperplane, with $s \neq 0$ and $\alpha \in \mathbb{R}$, then

$$P_{\mathcal{C}} x = x - \frac{\langle s, x \rangle - \alpha}{\|s\|^2} s \tag{2.4.6}$$

is the metric projection onto $\mathcal{C}$.

Example 2.4.8. If $\mathcal{C}$ is the range of a $m \times n$ matrix A with full column rank, then $P_{\mathcal{C}} x = A(A^* A)^{-1} A^* X$, where A^* is the adjoint of A.

2.4.1 Convex functions

Definition 2.4.9. *A function $c : \mathcal{H} \to \mathbb{R}$ is said to be Gâteaux differential at $x \in \mathcal{H}$, if there exists an element denoted by $c'(x) \in \mathcal{H}$ such that*

$$\lim_{h\to 0} \frac{c(x+hv) - c(x)}{h} = \langle v, c'(x)\rangle, \quad \forall v \in \mathcal{H},$$

where $c'(x)$ is called the Gâteaux differential of c at x. We say that if for each $x \in \mathcal{H}$, c is Gâteaux differentiable at x, then c is Gâteaux differentiable on $\mathcal{H}$.

Definition 2.4.10. *A function $c : \mathcal{H} \to \mathbb{R}$ is called convex, if for all $t \in [0,1]$ and $x, y \in \mathcal{H}$,*

$$c(tx + (1-t)y) \le tc(x) + (1-t)c(y).$$

Remark 2.4.11. We note that in an Hadamard space X, the convex function is defined by

$$c(tx \oplus (1-t)y) \le tc(x) + (1-t)c(y)\ \forall x, y \in X,\ t \in (0,1);$$

Definition 2.4.12. *Let X be a geodesic metric space. The function $f : D(f) \subseteq X \to \mathbb{R} \cup \{+\infty\}$ is said to be uniformly convex (see [77]), if there exists a strictly increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$f\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) \le \frac{1}{2}f(x) + \frac{1}{2}f(y) - \phi(d(x,y)).$$

Now, we present an example of a convex function in an Hadamard space.

Example 2.4.13. [35] Let X be an Hadamard space. For a finite number of points $a_1, a_2, \dots, a_N$ and $(w_1, w_2, \dots, w_N) \in S$ (where S is the convex hull of the canonical basis $e_1, e_2, \dots, e_N \in \mathbb{R}^N$), the function $f : X \to \mathbb{R}$ defined by $f(x) = \sum\limits_{n=1}^{N} w_n d^2(x, a_n)$ is convex and continuous.

Definition 2.4.14. *A convex function $c : \mathcal{H} \to \mathbb{R}$ is said to be subdifferentiable at a point $x \in \mathcal{H}$ if the set*

$$\partial c(x) = \{u \in \mathcal{H} \mid c(y) \ge c(x) + \langle u, y - x\rangle,\ \forall y \in \mathcal{H}\} \tag{2.4.7}$$

is nonempty, where each element in $\partial c(x)$ is called a subgradient of c at x, $\partial c(x)$ is called the subdifferential of c at x and the inequality in (2.4.7) is called the subdifferential inequality of c at x. We say that c is subdifferentiable on $\mathcal{H}$ if c is subdifferentiable at each $x \in \mathcal{H}$.
We also note that if c is Gâteaux differentiable at x, then c is subdifferentiable at x, and $\partial c(x) = \{c'(x)\}$.

Remark 2.4.15. We note that in a Banach space $\mathcal{X}$ with a dual space $\mathcal{X}^*$, the subdifferential inequality (2.4.7) is given by

$$\partial c(x) = \{u \in \mathcal{X}^* \mid c(y) \ge c(x) + \langle u, y - x\rangle,\ \forall y \in \mathcal{X}\}. \tag{2.4.8}$$

Definition 2.4.16. *Let $\mathcal{H}$ be a Hilbert space. The domain of a function $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is defined by $D(f) = \{x \in \mathcal{H} : f(x) < +\infty\}$.*
The function $f : D(f) \subseteq \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is said to be

(a) proper, if $D(f) \neq \emptyset$;

(b) lower semicontinuous at a point $x \in D(f)$, if

$$f(x) \leq \liminf_{x_n \to x} f(x_n),$$

(c) weakly lower semicontinuous at a point $x \in D(f)$, if

$$\liminf_{n \to \infty} f(x_n) \geq f(x),$$

holds for an arbitrary sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ satisfying $x_n \rightharpoonup x$;

(d) weakly upper semicontinuous at a point $x \in D(f)$, if

$$f(x) \geq \limsup_{n \to \infty} f(x_n),$$

holds for an arbitrary sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ satisfying $x_n \rightharpoonup x$;

Definition 2.4.17. *A bifunction $f : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is called*

(i) strongly monotone on $\mathcal{C}$ if there exists a constant $\beta > 0$ such that

$$f(x, y) + f(y, x) \leq -\beta \|x - y\|^2, \ \forall x, y \in \mathcal{C};$$

(ii) monotone on $\mathcal{C}$ if

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in \mathcal{C};$$

(iii) pseudomonotone on $\mathcal{C}$ if

$$f(x, y) \geq 0 \implies f(y, x) \leq 0 \ \forall \ x, y \in \mathcal{C}.$$

It is easy to see that (i) $\implies$ (ii) $\implies$ (iii) but the converses are not generally true.

Definition 2.4.18. *A function $f : \mathcal{C} \to \mathbb{R}$ is said to be hemicontinuous at $y \in \mathcal{C}$, if and only if*

$$\lim_{t \to 0^+} f(tx + (1 - t)y) = f(y), \ \textit{for all } x \in \mathcal{C}.$$

Nonlinear Multivalued mappings

In this section, we shall denote by $P(\mathcal{C}), CB(\mathcal{C})$ and $2^{\mathcal{C}}$, the family of all nonempty proximinal subsets of $\mathcal{C}$, the family of all nonempty, closed and bounded subsets of $\mathcal{C}$ and the family of all nonempty subsets of $\mathcal{C}$ respectively. Let $\mathbf{H}$ denote the Hausdorff metric on $CB(\mathcal{C})$, then for all $A, B \in CB(\mathcal{C})$,

$$\mathbf{H}(A, B) = \max\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(b, A)\}, \tag{2.4.9}$$

where $d(a, B) = \inf_{b\in B} \|a-b\|$ is the distance from the point a to the subset B. Let $T : \mathcal{C} \to 2^{\mathcal{C}}$ be a multivalued mapping. A point $x \in \mathcal{C}$ is called a fixed point of T, if $x \in Tx$ while $x \in \mathcal{C}$ is called a strict fixed point of T, if $Tx = \{x\}$.

Definition 2.4.19. *The mapping $T : \mathcal{C} \to 2^{\mathcal{C}}$ is called*

- *L-Lipschitz, if there exists $L > 0$ such that*

$$\mathbf{H}(Tx, Ty) \leq L\|x - y\|\ \forall\ x, y \in \mathcal{C},$$

if $L = 1$, then T is called nonexpansive, while T is called a contraction if $L \in [0, 1)$;

- *quasinonexpansive, if $F(T) \neq \emptyset$ and*

$$\mathbf{H}(Tx, p) \leq \|x - p\|\ \forall\ x \in \mathcal{C} \text{ and } p \in F(T).$$

Definition 2.4.20. *Let $\mathcal{X}$ be a Banach space and B a mapping of $\mathcal{X}$ into $2^{\mathcal{X}^*}$. The effective domain of B denoted by $dom(B)$ is given as $dom(B) = \{x \in \mathcal{X} : Bx \neq \emptyset\}$. Let $B : \mathcal{X} \to 2^{\mathcal{X}^*}$ be a multivalued operator on $\mathcal{X}$. Then*

(i) The graph $G(B)$ is defined by

$$G(B) := \{(x, u) \in \mathcal{X} \times \mathcal{X} : u \in B(x)\},$$

(ii) the operator B is said to be monotone if $\langle x-y, u^*-v^*\rangle \geq 0$ for all $x, y \in dom(A), u^* \in Ax$ and $v^* \in Ay$.

(iii) A monotone operator B on $\mathcal{X}$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on $\mathcal{X}$.

Let $\mathcal{X}$ be a uniformly convex and smooth Banach space with a Gâteaux differential norm and let $B : \mathcal{X} \to 2^{\mathcal{X}^*}$ be a maximal monotone operator. We consider the metric resolvent of B,

$$Q^B_\mu = \left(I + \mu J^{-1}_{\mathcal{X}} B\right)^{-1},\ \ \mu > 0.$$

It is known that the operator Q^B_μ is firmly nonexpansive and the fixed points of the operator Q^B_μ are the null points of B (see [152, 153]). The resolvent plays an important role in the approximation theory for zero points of maximal monotone operators in Banach spaces.

The following are the properties of the resolvent (see [21])

$$\langle Q_\mu^B x - x^*, J(x - Q_\mu^B x)\rangle \geq 0, \;\; x \in \mathcal{X}, x^* \in B^{-1}(0), \tag{2.4.10}$$

in particular, if $\mathcal{X}$ is a real Hilbert space, then

$$\langle J_\mu^B x - x^*, x - J_\mu^B x\rangle \geq 0, \;\; x \in \mathcal{X}, x^* \in B^{-1}(0),$$

where $J_\mu^B = (I + \mu B)^{-1}$ is the general resolvent, $B^{-1}(0) = \{z \in \mathcal{X} : 0 \in Bz\}$ is the set of null points of B. Also, we know that $B^{-1}(0)$ is closed and convex (see [237]).

Lemma 2.4.21. *[44] Let $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone mapping, and $A : \mathcal{H} \to \mathcal{H}$ be a Lipschitz continuous and monotone mapping. Then the mapping $A + B$ is a maximal monotone.*

2.5 Review on some optimization problems

2.5.1 Variational inequality and fixed point problems

The VIP which was first introduced independently by Fichera [104] and Stampacchia [228] is a central problem in nonlinear analysis. It provides an effective tool for studying many interesting problems across several fields of study such as economics, physics, mathematical programming, among others (see [118, 197] and other references therein). Many authors have proposed and analyzed several iterative algorithms for solving the VIP (1.2.1) and other related optimization problems, (see [9, 59, 130, 132, 135, 191, 193, 233, 234, 245, 257], and references therein).

Definition 2.5.1. *[160, 175] Let $A : \mathcal{C} \to \mathcal{X}^*$ be an operator. The Minty variational inequalities (MVI) is a problem of finding a point $x^* \in \mathcal{C}$ such that*

$$\langle Ay, \; y - x^*\rangle \geq 0, \; \forall \; y \in \mathcal{C}. \tag{2.5.1}$$

We denote the set of solution of (2.5.1) by $M(\mathcal{C}, A)$. It is known that pseudomonotonicity implies $M(\mathcal{C}, A) \neq \emptyset$ but the converse is not true.

We have the following result showing the relationship between the VIP (1.2.1) and MVI (2.5.1).

Lemma 2.5.2. *[175] Consider the VIP (1.2.1). Suppose the mapping $g : [0, 1] \to \mathcal{X}^*$ defined by $g(t) = A(tx + (1 - t)y)$ for all $x, y \in \mathcal{C}$ and $t \in [0, 1]$ (i.e, g is hemicontinuous), then $M(\mathcal{C}, A) \subset VI(\mathcal{C}, A)$. However, if A is pseudomonotone, then $VI(\mathcal{C}, A)$ is well defined and $M(\mathcal{C}, A) = VI(\mathcal{C}, A)$.*

In recent years, authors have paid a lot of attention to developing efficient iterative algorithms for solving VIPs and other related optimization problems in Hilbert, Banach and Hadamard spaces. When approximating the solution of the VIP (1.2.1) under certain conditions, two common methods are used namely; the projection method and the regularized method. To use these methods, a certain level of monotonicity is required from the cost operator. In this study, we focus on approximating the solution of the VIP (1.2.1) using the projection method. Several authors have proposed and studied the projection type algorithms for approximating the solutions of VIPs (1.2.1) (see [9, 156] and other references therein).

In a real Hilbert space $\mathcal{H}$, the following fixed point theory characterizes the solution set of the VIP: For $\lambda > 0$, a point x^* is a solution of the VIP if and only if

$$x^* = P_{\mathcal{C}}(x^* - \lambda A x^*),$$

where $P_{\mathcal{C}}$ is the metric projection of $\mathcal{H}$ onto $\mathcal{C}$. The simplest algorithm for solving the VIP (1.2.1) is the gradient projection method given by

Algorithm 2.5.1.

$$x_{n+1} = P_{\mathcal{C}}(x_n - \lambda A x_n),\ n \geq 1,$$

where $\lambda > 0$. This method involves only one projection onto the feasible set $\mathcal{C}$ per iteration. The weak convergence result of this method was obtained under certain strict conditions. This method is only effective for solving VIP (1.2.1) when A is either strongly monotone or inverse strongly monotone. To circumvent this limitation, Korpelevich [155] proposed an extragradient method for solving the VIP (1.2.1) in Euclidean spaces when A is monotone and L-Lipschitz continuous. The extragradient method is defined as follows:

Algorithm 2.5.2.

$$\begin{cases} y_n = P_{\mathcal{C}}(x_n - \lambda A x_n),\ n \geq 1 \\ x_{n+1} = P_{\mathcal{C}}(x_n - \lambda A y_n), \end{cases} \tag{2.5.2}$$

where $\lambda \in (0, \frac{1}{L})$ and $P_{\mathcal{C}}$ is the metric projection from $\mathcal{H}$ onto $\mathcal{C}$. If the solution set $VI(\mathcal{C}, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (2.5.2) converges to an element in $VI(\mathcal{C}, A)$. The extragradient method involves two projections onto the feasible set $\mathcal{C}$ per iteration. Computing projection onto an arbitrary closed convex set might be computationally expensive, which could be a barrier to the implementation of the extragradient method and its variants. To improve on the extragradient method, authors try to minimize the number of evaluations of the projection map $P_{\mathcal{C}}$ per iteration. Censor et. al [58] initiated a study in this direction and proposed a new method called subgradient extragradient method. The authors replaced the second projection onto $\mathcal{C}$ in Algorithm (2.5.2) by a projection onto a specific constructible half space to come up with the following algorithm:

Algorithm 2.5.3.

$$\begin{cases} x_1 \in \mathcal{H}, \\ y_n = P_{\mathcal{C}}(x_n - \lambda_n A x_n), \\ T_n = \{w \in \mathcal{H} : \langle x_n - \lambda_n A x_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda_n A y_n), \; n \geq 1, \end{cases} \tag{2.5.3}$$

---

where $\mathcal{H}$ is a real Hilbert space. Algorithm (2.5.3) is less computationally expensive and more efficient than Algorithm (2.5.2). The authors proved that Algorithm (2.5.3) converges weakly to an element in $VI(\mathcal{C}, A) \neq \emptyset$. To improve on Algorithm (2.5.3), Tseng [253] proposed and studied a forward-backward method (also known as Tseng's extragradient method) which requires only one projection per iteration. The proposed method is presented as follow:

Algorithm 2.5.4.

$$\begin{cases} y_n = P_{\mathcal{C}}(x_n - \lambda A x_n) \\ x_{n+1} = y_n - \lambda(A y_n - A x_n), \qquad \forall\, n \geq 0, \end{cases} \tag{2.5.4}$$

---

where A is monotone, L-Lipschitz continuous and $\lambda \in \left(0, \frac{2}{L}\right)$. The author proved that the sequence $\{x_n\}$ generated by Algorithm (2.5.4) converges weakly to the solution set of VIP (1.2.1) under the assumption that $VI(\mathcal{C}, A) \neq \emptyset$.

The step size of all the algorithms above require prior knowledge of the Lipschitz constant of the monotone operator, which is often difficult to calculate or estimate. Recently, Yang and Liu [266] proposed and studied a Tseng's extragradient method combined with the Moudafi viscosity scheme, which does not require prior knowledge of the Lipschitz constant of the monotone operator. The authors proved strong convergence result for the proposed algorithm. Very recently, Shehu and Iyiola [223] proposed an algorithm which combines the viscosity method and the subgradient extragradient method for solving the VIP. They proved that the sequence generated by their algorithm converges strongly to a point in the solution set under appropriate conditions.

Motivated by the Tseng's method and the importance of studying the VIPs and FPPs, Yin et.al [270] proposed and studied a Tseng's type algorithm where $A : \mathcal{H} \to \mathcal{H}$ is quasimonotone, Lipschitz continuous and sequentially weakly continuous, and the mapping T is pseudocontractive. The authors were only able to obtain a weak convergence result of their method (see Appendix 3.5.17) to some point $x^* \in VI(\mathcal{C}, A) \cap F(T)$.

Chidume and Nnakwe [75] extended the study of the subgradient extragradient method from the framework of a real Hilbert space to the framework of a 2-uniformly convex and uniformly smooth Banach space. The authors proposed and studied the following method for solving the VIP (1.2.1):

Algorithm 2.5.5.

$$\begin{cases} x_0 \in \mathcal{X}, \\ y_n = \Pi_{\mathcal{C}} J^{-1}(Jx_n - \tau A x_n), \\ T_n = \{w \in \mathcal{X} : \langle w - y_n,\ Jx_n - \tau A x_n - Jy_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{T_n} J^{-1}(Jx_n - \tau A y_n), \end{cases} \tag{2.5.5}$$

where $\Pi_{\mathcal{C}} : \mathcal{X} \to \mathcal{C}$ is the generalized projection of the Banach space, $J : \mathcal{X} \to 2^{\mathcal{X}^*}$ the normalized duality mapping and $\tau > 0$. Using this proposed method, the authors obtained a weak convergence result.

Recently, Cai *et al.* [53] proposed an algorithm for solving the variational inequalities involving monotone and Lipschitz continuous mapping in Banach spaces. The proposed algorithm by the authors is presented as follows:

Algorithm 2.5.6.

Step 0: Let $u \in \mathcal{X}$ be a given starting point . Set $n = 1$.

Step 1: Given the current iterate x_n, compute

$$y_n = \Pi_{\mathcal{C}}(Jx_n - \lambda_n A x_n).$$

If $x_n = y_n$, Stop. Else, construct the set

$$T_n := \{z \in \mathcal{X} : \langle Jx_n - \lambda_n A x_n - Jy_n,\ z - y_n \rangle \leq 0\}$$

and compute

$$z_n = \Pi_{T_n}(Jx_n - \lambda_n A y_n)$$

and update the next iterate via

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1 - \alpha_n) Jz_n). \tag{2.5.6}$$

Step 2: Set $n := n + 1$ and go to **Step 1**.

Where $\mathcal{X}$ is a 2-uniformly convex Banach space and $\mathcal{X}^*$ the dual of $\mathcal{X}$. Under certain conditions, the authors obtained a strong convergence result for the proposed algorithm.

Recently, Tan *et al.* [240] proposed an inertial subgradient extragradient method with a new Armijo type step-size strategy to solve VIP (1.2.1) in real Hilbert spaces when the underlying operator is pseudomonotone and uniformly continuous. The proposed algorithm is presented as follows:

Algorithm 2.5.7.

Initialization: Given $\lambda_1 > 0,\quad \theta > 0,\quad \delta > 0, \ell \in (0, 1)$ and $\eta \in (0, 1)$. Let $x_0, x_1 \in \mathcal{H}$ be arbitrary. Set $n := 1$.

Iterative Steps: Given the iterates x_{n-1} and x_n for each $n \geq 1$, calculate x_{n+1} as follows;

Step 1: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}),$$

where

$$\theta_n := \begin{cases} \min\left\{\theta, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases} \tag{2.5.7}$$

Step 2: Compute

$$y_n = P_{\mathcal{C}}(w_n - \lambda_n A w_n).$$

If $w_n = y_n$ or $Ay_n = 0$, then **Stop** and y_n is a solution of the VIP. Otherwise, go to **Step 3**.

Step 3: Compute

$$z_n = P_{T_n}(w_n - \lambda_n A y_n)$$

where

$$T_n := \{x \in \mathcal{H} \ |\langle w_n - \lambda_n A w_n - y_n, \quad x - y_n\rangle \leq 0\},$$

$\lambda_n := \delta\ell^{m_n}$ and m_n is the smallest nonnegative integer m satisfying

$$\delta\ell^m\langle Ay_n - Aw_n, \quad y_n - z_n\rangle \leq \frac{\eta}{2}\Big[\|w_n - y_n\|^2 + \|y_n - z_n\|^2\Big].$$

Step 4: Calculate

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) z_n.$$

Set $n := n + 1$ and go to **Step 1**.

Under some mild conditions, the authors obtained strong convergence result for the proposed algorithm.

2.5.2 Split variational inequality problem

The SVIP which was introduced by Censor *et al.* [61] can be seen as a pair of VIPs in which a solution of one VIP occurs in the first space $\mathcal{H}_1$ whose image under a given bounded linear operator A is a solution of the second VIP in the second space $\mathcal{H}_2$. Hence, the SVIP (1.2.4)-(1.2.5) is an interesting combination of the classical VIP and the following SFP introduced and studied by Censor and Elfving [57]:

$$\text{Find} \quad x \in \mathcal{C} \quad \text{such that} \quad z = Tx \in \mathcal{Q}, \tag{2.5.8}$$

where $\mathcal{C}, \mathcal{Q}$ are nonempty, closed and convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. Thus, it has wide applications in many fields such as phase retrieval, medical image reconstruction, signal processing, radiation therapy treatment planning, among others (for example, see [52, 57, 61–63, 121, 132]).

Censor *et al.* [63] proposed an iterative algorithm to solve the SVIP when the cost operators A and F are monotone and Lipschitz continuous. They transformed the SVIP into an equivalent constrained VIP (CVIP) in the product space $\mathcal{H}_1 \times \mathcal{H}_2$ (see [63, Section 4]) and then solved the problem using the well-known subgradient extragradient method

[59]. This product space formation has some limitations which include: the difficulty encountered when computing the projection onto some new product subspace formulations, the difficulty encountered when translating the method back to the original spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and the fact that it does not fully exploit the splitting structure of the SVIP (1.2.4)-(1.2.5) (see, for example [63, Page 12]). To circumvent these limitations, Censor *et al.* [63] proposed a projection-based method that does not require any product space formulation. This makes the projection-based method easier to implement. The proposed projection-based method is presented as follows: For $x_1 \in \mathcal{H}_1$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = P_{\mathcal{C}}(I - \lambda A)(x_n + \eta T^*(P_{\mathcal{Q}}(I - \lambda F) - I)Tx_n), \quad n \geq 1, \tag{2.5.9}$$

where $\eta \in \left(0, \frac{1}{L}\right)$ with L being the spectral radius of T^*T and T^* is the adjoint of T. The identity operator is denoted by I and $P_{\mathcal{C}}, P_{\mathcal{Q}}$ are metric projections onto $\mathcal{C}, \mathcal{Q}$, respectively. They obtained a weak convergence of the sequence $\{x_n\}$ generated by (2.5.9) to a solution of (1.2.4)-(1.2.5) under the condition that the solution set of problem (1.2.4)-(1.2.5) is nonempty, A, F are L_1, L_2-co-coercive operators respectively, $\lambda \in [0, 2\alpha]$, where $\alpha := \min\{L_1, L_2\}$, and for all x which are solutions of (1.2.4),

$$\big\langle Ay, P_{\mathcal{C}}(I - \lambda A)(y) - x\big\rangle \geq 0, \quad \forall\, y \in \mathcal{H}. \tag{2.5.10}$$

Observe that Algorithm (2.5.9) does not require the product space formation, thus it fully exploits the attractive splitting structure of the SVIP (1.2.4)-(1.2.5). However, the authors obtained a weak convergence of this method under some strong assumptions which are the fact that both mappings are required to be co-coercive and (2.5.10). Many authors have studied several methods which do not rely on assumption (2.5.10) for solving SVIP and other related problems (see for example [131, 145, 183]), but their methods also relied on the co-coercivity of the cost operators.

In a quest to overcome these limitations, Tian and Jiang [243] proposed an iterative method and they defined it as follows:

Algorithm 2.5.8.

$$\begin{cases} y_n = P_{\mathcal{C}}(x_n - \tau_n A^*(I - S^{'})Tx_n) \\ v_n = P_{\mathcal{C}}(y_n - \lambda_n A y_n) \\ x_{n+1} = P_{\mathcal{C}}(y_n - \lambda_n A v_n) \end{cases} \tag{2.5.11}$$

where $\{\tau_n\} \subset [a, b]$, $\{\lambda_n\} \subset [c, d]$ for some $c, d \in \left(0, \frac{1}{L}\right)$, $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator, $S^{'} : \mathcal{H}_2 \to \mathcal{H}_2$ is a nonexpansive mapping and $A : \mathcal{C} \to \mathcal{H}_1$ is a monotone and Lipschitz continuous mapping. They obtained a weak convergence result of the sequence generated by Algorithm (2.5.11) to the following problem; Find

$$x^* \in \mathcal{C} \quad \text{such that} \quad \langle Ax^*,\ x - x^*\rangle, \quad \forall\, x \in \mathcal{C} \quad \text{such that} \quad Tx^* \in F(S^{'}) \tag{2.5.12}$$

where $F(S^{'})$ is the set of fixed points of $S^{'}$. Since strong convergence results are more desirable and more applicable than the weak convergence results in infinite dimensional spaces, there is need to develop algorithms that generate strong convergence results.

Tian and Jiang [244] modified Algorithm (2.5.11) into the following viscosity method and they defined it as follows:

Algorithm 2.5.9.

$$\begin{cases} y_n = P_{\mathcal{C}}(x_n - \tau_n A^*(I - S^{'})Tx_n) \\ v_n = P_{\mathcal{C}}(y_n - \lambda_n A y_n) \\ t_n = P_{\mathcal{C}}(y_n - \lambda_n A v_n) \\ x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n) t_n \end{cases} \tag{2.5.13}$$

where $\{\tau_n\} \subset [a, b]$, $\{\lambda_n\} \subset [c, d]$ for some $c, d \in \left(0, \frac{1}{L}\right)$, $\{\alpha_n\} \subset (0, 1)$, $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator, $S^{'} : \mathcal{H}_2 \to \mathcal{H}_2$ is a nonexpansive mapping, h is a contraction mapping and $A : \mathcal{C} \to \mathcal{H}_1$ is a monotone and Lipschitz continuous mapping. We observe that the conditions on the underlying operators in Algorithms (2.5.11)-(2.5.13) do not require the strong co-coercive assumption but involve computation of many projections which makes them computationally expensive and may affect the efficiency of Algorithms (2.5.11)-(2.5.13). Algorithms (2.5.11)-(2.5.13) can be used to solve the SVIP (1.2.4)-(1.2.5) if we set $S^{'} = P_{\mathcal{Q}}(I - \lambda A)$ and let A be co-coercive. This implies that when solving the SVIP (1.2.4)-(1.2.5), these methods (Algorithm (2.5.11)-(2.5.13)) still relies on the co-coercive assumption on the underlying operator A. To weaken the condition on the underlying operators, Pham *et al.* [206] combined the Halpern method with the subgradient extragradient method for solving the SVIP (1.2.4)-(1.2.5) in real Hilbert spaces when the underlying operators A and F are pseudomonotone and Lipschitz continuous. The authors obtained a strong convergence result of their proposed method (see Appendix (4.3.16)) to a solution of the SVIP (1.2.4)-(1.2.5) under the following conditions: $\limsup_{n\to\infty} \langle A(x_n), y - y_n\rangle \leq \langle A(\bar{x}), y - \bar{y}\rangle$, for every sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{H}_1$ converging weakly to $\bar{x}$ and $\bar{y}$ respectively, and $\limsup_{n\to\infty} \langle F(c_n), d - d_n\rangle \leq \langle F(\bar{c}), d - \bar{d}\rangle$ for every sequences $\{c_n\}$ and $\{d_n\}$ in $\mathcal{H}_2$ converging weakly to $\bar{c}$ and $\bar{d}$ respectively.

To accelerate the convergence of iterative methods for solving the SVIP (1.2.4)-(1.2.5), the inertial and relaxation techniques were employed. This dynamical approach leads to the following Relaxed Inertial Tseng's Forward-Backward-Forward (RITFBF) with parameter $\phi_n = \theta h_n^2$, $\forall n \geq 1$ (see, for example [25]):

Algorithm 2.5.10.

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}) \\ y_n = P_{\mathcal{C}}\left(w_n - \lambda_n A w_n\right) \\ x_{n+1} = (1 - \phi_n) w_n + \phi_n \left(y_n + \lambda_n (A w_n - A y_n)\right). \end{cases} \tag{2.5.14}$$

When $\phi_n = 1$ and $\alpha_n = 0$ for all $n \geq 1$, Algorithm (2.5.14) reduces to the well-known Tseng's forward-backward-forward method [253], which is known to converge weakly to a solution of the classical VIP.

Given the importance of these two techniques (inertial and relaxation), it is of interest to consider their combination for solving optimization problems. Attouch and Cabot [26, 27] employed both techniques into proximal algorithms (resulting to Relaxed Inertial Proximal Algorithms (RIPA)) for solving convex minimization and null point problems. Also, Iutzeler and Hendrickx [129] studied the influence of inertial and relaxation techniques on the numerical performance of iterative schemes.

Remark 2.5.3. Several of the existing methods for solving the SVIP require that the problem be transformed into a product space problem which does not fully exploit the splitting structure of the SVIP. Therefore, there is need to develop efficient algorithms for solving the SVIP which does not require the product space formulation and converges to the minimum-norm solution of the SVIP. Note that in many practical problems with physical and engineering backgrounds, it is very important if the minimum-norm solutions of such problems can be found (see for example, [121, Section 5] and [54, 261]).

2.5.3 Split equalities of equilibrium, variational inequality and fixed point problems

The EP (1.2.7) has received a lot of attention from several authors due to its application to problems arising in the field of optimization, economics, physics, variational inequalities, among others (see for example [194, 200] and other references therein). Several authors have analyzed and proposed various iterative algorithms for approximating the solution of the EP and other related optimization problems, (see for example [191, 205, 209] and other references therein).

The SEP was first proposed by Moudafi [180]. It is used in numerous practical problems such as game theory, medical image reconstruction, partial differential equation, decomposition method, among others (see [141, 224, 241]).

If $\mathcal{H}_2 = \mathcal{H}_3$ and $\mathcal{F}_2 = I$ (I is the identity operator), the equation (1.2.8) reduces to the SFP proposed by Censor et al. [64] and defined as follows:

$$\text{Find } \; x \in \mathcal{C} \text{ such that } \quad \mathcal{F}_1 x \in \mathcal{Q},$$

where $\mathcal{F}_1 : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. One of the most common method for solving (1.2.8) is the CQ projection method proposed and studied by Bryne et al. [49]. They defined it as follows:

Algorithm 2.5.11.

$$\begin{cases} x_{n+1} = P_{\mathcal{C}}(x_n - \eta_n \mathcal{F}_1^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)) \\ y_{n+1} = P_{\mathcal{Q}}(y_n + \eta_n \mathcal{F}_2^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)), \end{cases} \tag{2.5.15}$$

where $\eta_n \in \left(\epsilon, \frac{2}{\lambda_{\mathcal{F}_1} + \lambda_{\mathcal{F}_2}} - \epsilon\right)$, and $\lambda_{\mathcal{F}_1}$ and $\lambda_{\mathcal{F}_2}$ are the matrix operator norms $\|\mathcal{F}_1\|$ and $\|\mathcal{F}_2\|$, respectively. Note that the step size η_n in Algorithm (2.5.15) is dependent on the operator norms, which are difficult and sometimes impossible to compute.

If $\mathcal{C}$ and $\mathcal{Q}$ are the sets of fixed points of some nonlinear operators, the SEP (1.2.8) becomes the split equality common fixed point problem (SECFPP) which is defined as:

$$\text{Find } x \in F(T_1) \text{ and } y \in F(T_2) \text{ such that } \mathcal{F}_1 x = \mathcal{F}_2 y, \tag{2.5.16}$$

where $F(T_1) \neq \emptyset$ and $F(T_2) \neq \emptyset$ are the sets of fixed points of T_1 and T_2, respectively, $T_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $T_2 : \mathcal{H}_2 \to \mathcal{H}_2$ are nonlinear mappings and $\mathcal{F}_1 : \mathcal{H}_1 \to \mathcal{H}_3,\ \mathcal{F}_2 : \mathcal{H}_2 \to \mathcal{H}_3$ are bounded linear operators.

If $\mathcal{H}_2 = \mathcal{H}_3$ and $\mathcal{F}_2 = I$, then the SECFPP (2.5.16) reduces to the following split common fixed point problem (SCFPP) introduced by Censor et al. [66]

$$\text{Find } x \in F(T_1) \text{ such that } \mathcal{F}_1 x \in F(T_2).$$

The SECFPP was first studied by Moudafi et al. [180]. They introduced the following simultaneous iterative method for solving the SECFPP

Algorithm 2.5.12.

$$\begin{cases} x_{n+1} = T_1(x_n - \eta_n \mathcal{F}_1^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)) \\ y_{n+1} = T_2(y_n + \eta_n \mathcal{F}_2^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)), \end{cases} \tag{2.5.17}$$

where $\eta_n \in \left(\epsilon, \frac{2}{\lambda_{\mathcal{F}_1} + \lambda_{\mathcal{F}_2}} - \epsilon\right)$, $\lambda_{\mathcal{F}_1}$ and $\lambda_{\mathcal{F}_2}$ are the spectral radius of $\mathcal{F}_1^*\mathcal{F}_1$ and $\mathcal{F}_2^*\mathcal{F}_2$, respectively, and T_1 and T_2 are firmly quasi-nonexpansive mappings. We also observe that the step size of Algorithm (2.5.17) depends on the operator norms. Hence to implement Algorithm (2.5.17), one has to compute the operator norms of $\mathcal{F}_1$ and $\mathcal{F}_2$ which are difficult to compute. Several authors have studied and proposed modifications of Algorithm (2.5.17) for better implementation (see [177, 180, 272, 273] and other references there in).

Recently, Lopez et al. [115] studied and proposed a method for estimating the step size which does not require prior knowledge of the operator norms for solving the SFP. Dong et al. [119] and J. Zhao [271] also proposed new choices of step size which do not require prior knowledge of the operator norm for solving SECFPP. Zhao [271] studied the SEP and presented the following step size which guarantees convergence of the iterative method without prior knowledge of the operator norm of $\mathcal{F}_1$ and $\mathcal{F}_2$:

Algorithm 2.5.13.

$$\eta_n \in \left(0, \frac{2\|\mathcal{F}_1 x_n - \mathcal{F}_2 y_n\|^2}{\|\mathcal{F}_1^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)\|^2 + \|\mathcal{F}_2^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)\|^2}\right).$$

Recently, Latif and Eslamian [159] studied and introduced a new algorithm for finding a common element of split equalities of EP, monotone VIP with Lipschitz operator and fixed point problem of nonexpansive semigroups satisfying the uniformly asymptotically regularity (u.a.r) condition in Hilbert spaces. The authors obtained strong convergence result for the proposed algorithm. However, their proposed algorithm has certain drawbacks. For instance, their method requires computing two projections each per iteration onto $\mathcal{C}$

and $\mathcal{Q}$, which makes it computationally expensive to implement. Moreover, the associated cost operators for the VIP are required to be monotone and Lipschitz continuous and the step size of the algorithm depends on the Lipschitz constants of these operators. In addition, the authors needed to impose the uniformly asymptotically regularity condition on the nonexpansive semigroups to obtain their result. All of these drawbacks limit the scope of application of their proposed method. Therefore, there is need to develop and study effective iterative algorithms which does not depend on all the above mentioned weaknesses.

2.5.4 Variational inclusion and split equilibrium problems

The V_qIP has attracted the interest of many researchers due to its application to various problems arising in optimal control, mathematical economics, convex programming and other related optimization problems. Specifically, some problems in machine learning, image processing and linear inverse problems can be modeled mathematically as V_qIP (1.2.9) [1, 10, 65, 86, 90, 111, 195, 256]. Many authors have studied and proposed iterative algorithms for solving the V_qIP (1.2.9). An efficient method for solving the V_qIP is the forward-backward splitting method considered in [11, 13, 39, 82, 86, 132, 161, 165, 203] and is formulated as follows:

Algorithm 2.5.14.

$$x_{n+1} = (I + rD)^{-1}(I - rB)x_n, \quad n \geq 1,$$

where $r > 0$, $(I - rB)$ is the forward operator and $(I + rD)^{-1}$ is the resolvent operator introduced in [179] and it is also known as the backward operator. This forward-backward splitting method includes as special cases, the proximal point algorithm [215] and the gradient method. Several authors have studied and extended the forward-backward splitting methods. Lion and Mercier [161] introduced the Peaceman-Rachford splitting method [204]

Algorithm 2.5.15.

$$x_{n+1} = (2J_r^{D_1} - I)(2J_r^{D_2} - I)x_n, \quad n \geq 1$$

and the Douglas-Rachford splitting method [98]

$$x_{n+1} = J_r^{D_1}(2J_r^{D_1})x_n + (I - J_r^{D_1})x_n, \quad n \geq 1,$$

where $J_r^{D_i} = (I + rD_i)^{-1}$ $i = 1, 2$, with $r > 0$. They obtained weak convergence of these methods. Gibali *et al.* [108] introduced and studied two modifications of the forward-backward splitting method with a new step size rule for solving V_qIPs in real Hilbert spaces where the operators are Lipschitz continuous and monotone and maximal monotone. They obtained strong convergence results for the two methods.

Alvarez and Attouch [18] employed the heavy ball method which was studied by Polyak in [208] for maximal monotone operators using the proximal point algorithm. The proposed algorithm combines the inertial technique and the proximal point algorithm. The algorithm is called the inertial proximal point algorithm and it is defined as follows:

Algorithm 2.5.16.

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + r_n D)^{-1} y_n, \quad n \geq 1. \end{cases} \tag{2.5.18}$$

They obtained a weak convergence of Algorithm 2.5.18 to a zero of B when $\{r_n\}$ is non-decreasing and $\{\theta_n\} \subset [0, 1)$ with

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty. \tag{2.5.19}$$

Moudafi and Oliny [184] proposed an inertial proximal point algorithm for solving the zero problem of the sum of two monotone operators, which is presented as follows:

Algorithm 2.5.17.

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + r_n D)^{-1}(y_n - r_n B x_n), \quad n \geq 1 \end{cases} \tag{2.5.20}$$

They obtained a weak convergence of Algorithm 2.5.20 to the zero of $(B+D)$ when $r_n < \frac{2}{L}$, L is the Lipschitz constant of B and condition (2.5.19) is satisfied. Note that Algorithm 2.5.20 does not take the form of the forward-backward splitting algorithm if $\theta_n > 0$ because B is still evaluated at the point x_n.

Recently, Lorenz and Pock [165] proposed the following inertial forward-backward algorithm for monotone operators:

Algorithm 2.5.18.

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + r_n D)^{-1}(y_n - r_n B y_n), \quad n \geq 1 \end{cases} \tag{2.5.21}$$

where $r_n > 0$. They obtained a weak convergence analysis for this method. In 2018, Cholamjiak *et al.* [83] introduced the following inertial forward-backward splitting algorithm which combines Halpern and Mann iteration methods for solving V_qIPs in Hilbert spaces

Algorithm 2.5.19.

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = \beta_n u + \epsilon_n y_n + \mu_n J^D_{\lambda_n}(y_n - \lambda_n B y_n), \quad n \geq 1, \end{cases} \tag{2.5.22}$$

where $B : \mathcal{H} \to \mathcal{H}$ is k-inverse strongly monotone operator and $D : H \to 2^{\mathcal{H}}$ is a maximal monotone operator, $J^D_{\lambda_n} = (I + \lambda_n D)^{-1}$, $0 < \lambda_n \le 2k$, $\{\alpha_n\} \subset [0, \alpha]$ with $\alpha \in [0, 1)$ and $\{\beta_n\}, \{\epsilon_n\}$ and $\{u_n\}$ are sequences in $[0, 1]$ with $\beta_n + \epsilon_n + \mu_n = 1$ satisfying some certain conditions. They obtained a strong convergence of Algorithm 2.5.22 to an element in the solution set.

Kazmi and Rizvi [144] introduced and studied the SEP (1.2.10)-(1.2.11). Observe that problem (1.2.10) is the classical EP. When viewed separately, (1.2.10)-(1.2.11) are classical EPs. Thus, SEP constitutes a pair of EPs which have to be solved so that the image $y = \mathcal{T}x$ under a given bounded linear operator $\mathcal{T}$, of the solution of the EP in $\mathcal{H}_1$, is a solution of the other EP in $\mathcal{H}_2$.

Many authors have studied the problem of finding a common solution of the V_qIP and SEP due to its importance in solving real world problems whose constraints can be written as V_qIP and SEP. Very recently, Cholamjiak *et al.* [81] proposed a modified inertial forward-backward splitting method for solving the SEP and the V_qIP, which is presented as follows:

Algorithm 2.5.20.

$$\begin{cases} y_n = x_n + \delta_n(x_n - x_{n-1}) \\ z_n = \alpha_n y_n + (1 - \alpha_n)T^{F_1}_{r_n}(I - \gamma A^*(I - T^{F_2}_{r_n})A)y_n, \\ x_{n+1} = \beta_n z_n + (1 - \beta_n)J^D_{t_n}(I - t_n B)z_n, \ n \ge 1, \end{cases} \tag{2.5.23}$$

where $J^D_{t_n} = (I + t_n D)^{-1}$, $\{t_n\} \subset (0, 2\alpha)$, $\{\delta_n\} \subset [0, \delta]$, $\delta \in [0, 1)$, $\{r_n\} \subset (0, \infty)$ with $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$. Under some certain conditions they obtained a weak convergence of Algorithm 2.5.23 to an element in the solution set (1.2.12). However, it has been pointed out by authors that the summability condition used in Algorithm 2.5.22 and Algorithm 2.5.23 is a drawback in their implementation. Another computational weakness of Algorithm 2.5.23 is the fact that its step size γ depends on the norm of the operator $\|A\|$, which in most cases is unknown or very difficult to calculate or even estimate. Moreover, it is required that the operator B in Algorithm 2.5.23 and the other algorithms above is co-coercive (inverse strongly monotone) which is a very stringent condition. These are major drawbacks with the above algorithms and several existing algorithms in the literature. Therefore, there is need to develop and study effective iterative algorithms which does not depend on all the above mentioned weaknesses.

2.5.5 Split monotone variational inclusion and fixed point problems

In 2011, Moudafi [183] introduced the SMV_qIP (1.2.13). If $B_1 = B_2 = 0$ then the problem (1.2.13) reduces to the split variational inclusion problem (SV_qIP) defined as follows:

Algorithm 2.5.21.

$$\begin{cases} \text{Find} \quad x^* \in \mathcal{H}_1, \quad \text{such that} \quad 0 \in D_1(x^*), \\ \text{and} \\ y^* = \mathcal{T}x^* \in \mathcal{H}_2 \quad \text{such that} \quad 0 \in D_2(y^*), \end{cases} \tag{2.5.24}$$

where 0 is the zero vector, $\mathcal{H}_1$ and $\mathcal{H}_2$ are real Hilbert spaces, $D_1, D_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ are multi-valued mappings, $\mathcal{T} : H_1 \to H_2$ is a bounded linear operator. The SV$_q$IP is also known as the split common null point problem (SCNPP) or the split zero problem (SZP). It includes as special cases, the split common fixed point problem (SCFFP), the SVIP and SFP (see [169, 182, 213] and other references therein), and it has wide applications in different fields such as medical treatment of the intensity-modulated radiation therapy (IMRT), data compression, medical image reconstruction, signal processing, phase retrieval, among others (for example, see [52, 57, 63]). The SV$_q$IP is considered to be a central problem in optimization and nonlinear analysis and has attracted the attention of several researchers because the theory provides a simple, natural and unified framework for a general treatment of many important mathematical problems such as MPs, network equilibrium problems, complementary problems, systems of nonlinear equations and others (see [31, 36, 63, 86, 105, 107, 147, 151, 210, 243] and other references therein).

Byrne *et al.* [51] proposed and studied the following algorithms for solving the SV$_q$IP for two maximal monotone operators A and D in Hilbert spaces. Take $x_0 \in \mathcal{H}_1$ such that

Algorithm 2.5.22.

$$x_{n+1} = J_\mu^A(x_n + \lambda T^*(J_\mu^D - I)Tx_n) \tag{2.5.25}$$

and

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)J_\mu^A(x_n + \lambda T^*(J^D D_\mu - I)Tx_n) \tag{2.5.26}$$

for $\mu > 0, T^*$ the adjoint of T, $\lambda \in \left(0, \frac{2}{L}\right), L = \|T^*T\|$ and $J_\mu^A := (I + \mu A)^{-1}, J_\mu^D := (I + \mu D)^{-1}$ are the resolvent operators of A and D, respectively. Under certain conditions, the authors obtained a weak convergence result for Algorithm 2.5.25 and a strong convergence result for Algorithm 2.5.26.

Moudafi [181] first introduced the viscosity approximation method which is defined as : Choose $x_0 \in \mathcal{H}$ such that the sequence $\{x_n\}$ is generated by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Sx_n, \quad \forall n \geq 1, \tag{2.5.27}$$

where $\{\alpha_n\} \subset (0,1)$ and f is a contraction mapping. He proved that the sequence $\{x_n\}$ generated by (2.5.27) converges strongly to a fixed point of a nonexpansive mapping S under some suitable control conditions.

Very recently, Suantai *et al.* [229] proposed the following viscosity iterative scheme to approximate the solution of SV$_q$IP between a Banach space $\mathcal{X}$ and a Hilbert space $\mathcal{H}$:

Algorithm 2.5.23.

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J^A_{\lambda_n}(x_n + \lambda_n T^* J_{\mathcal{X}}(Q^D_\mu - I)Tx_n), \;\; \forall n \geq 1, \tag{2.5.28}$$

where $\{\mu_n\}, \{\lambda_n\} \subset (0, \infty), \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $\alpha_n + \beta_n + \gamma_n = 1$, $f : \mathcal{H} \to \mathcal{H}$ is a contraction, $J_{\mathcal{X}}$ is the duality mapping on $\mathcal{X}$, J^A_λ is the resolvent of A for $\lambda > 0, Q^D_\mu$ is the metric resolvent of D for $\mu > 0$. They proved that the sequence $\{x_n\}$ defined by (2.5.28) converges strongly to a solution of the SV$_q$IP under some certain conditions.

Many problems in sciences and engineering can be formulated as a problem of finding the solution of a FPP of a nonlinear mapping.

Motivated by the work of Byrne *et al.* [51], Kazmi and Rizvi [141] proposed the following algorithm for approximating a solution of SV$_q$IP which is also a fixed point of nonexpansive mapping.

Theorem 2.5.4. *[141] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction mapping with constant $\rho \in (0,1)$ and $S : \mathcal{H}_1 \to \mathcal{H}_1$ be a nonexpansive mapping. For $x_0 \in \mathcal{H}_1$, let the sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

Algorithm 2.5.24.

$$\begin{cases} u_n = J^B_r(x_n + \lambda T^*(J^C_r - I)Tx_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \; n \geq 0, \end{cases} \tag{2.5.29}$$

where $r > 0$ and $\lambda \in \left(0, \frac{1}{L}\right)$, L is the spectral radius of the operator T^*T and T^* is the adjoint of T, $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim\limits_{n\to\infty} \alpha_n = 0, \sum\limits_{n=0}^{\infty} \alpha_n = \infty$ and $\sum\limits_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$. Then the sequence $\{u_n\}$ and $\{x_n\}$ both converge strongly to an element in the solution set.

Remark 2.5.5. All the above algorithms for solving SV$_q$IP have a common feature, which is also their computational weakness. It is the fact that their step size λ (or λ_n) depends on the norm of the operator $\|T\|$, which in most cases is unknown or very difficult to calculate or even estimate. This is a major drawback of the algorithms and several existing algorithms in the literature. Therefore, there is need to develop efficient iterative algorithms (where the step size does not depend on the operator norm) for approximating the common solution of the SV$_q$IP and FPP.

2.5.6 Minimization and fixed point problems

A well-known and powerful method for finding solutions of MPs is the proximal point algorithm (PPA), which was introduced in Hilbert spaces by Martinet [174] and further developed by Rockafellar [215] who proved that the PPA converges weakly to a minimizer of f. Rockafellar [215] then posed a question as to whether the PPA converges strongly or not without additional assumptions, and this question was resolved in the negative by G*ü*ler's counter example (see [113]). To obtain strong convergence of the PPA, Kamimura and Takahashi [140] incorporated the Halpern iterative method into the PPA and proved a strong convergence of the resulting sequence to a minimizer of f.

The study of the PPA in spaces with nonlinearity plays crucial role in the branch of analysis and geometry. For instance, the PPA in Hadamard spaces (nonlinear version of Hilbert spaces) are known to have applications in computational phylogenetics, consensus algorithms, modeling of airway systems in human lungs and blood vessels, computing of medians and means, among others (see [33–35, 103] and the references therein). The PPA was first studied in Hadamard spaces by Ba*čák* [34] in 2013, as an extension from the classical linear spaces (for example, Euclidean and Hilbert spaces): For a starting point $x_1 \in X$, define $\{x_n\}$ by

Algorithm 2.5.25.

$$x_{n+1} = \arg\min_{y \in X} \left(f(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right), \tag{2.5.30}$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. Under the conditions that f has a minimizer in X and $\sum_{n=1}^{\infty} \lambda_n = \infty$, Ba*čák* [34] proved that $\{x_n\}$ Δ-converges to a minimizer of f. Later in 2015, Cholamjiak [79] modified the PPA (2.5.30) into the following Halpern-type PPA and proved that it converges strongly to a minimizer of f in an Hadamard space:

Algorithm 2.5.26.

$$\begin{cases} y_n = arg\min_{y \in X}(f(y) + \frac{1}{2\lambda} d^2(y, x_n)), \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) y_n, n \geq 1, \end{cases} \tag{2.5.31}$$

where $\lambda > 0, \{\alpha_n\} \subset (0, 1)$, $\lim\limits_{n \to \infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. In the same year, Cholamjiak *et al.* [80] proposed the following modified PPA by using the S-type iteration process for solving MP (2.4.4) and FPP for two nonexpansive mappings in CAT(0) spaces:

Algorithm 2.5.27.

$$\begin{cases} z_n = arg\min_{y \in X}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ y_n = (1 - \beta_n) x_n \oplus \beta_n T_1 z_n, \\ x_{n+1} = (1 - \eta_n) T_1 x_n \oplus \eta_n T_2 y_n, \end{cases} \tag{2.5.32}$$

where T_1 and T_2 are nonexpansive mappings on X, $\{\eta_n\}, \{\beta_n\}$ are sequences in $(0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$. They obtained a weak convergence result of Algorithm (2.5.32), and under some compactness conditions, they obtained some strong convergence results.

By combining the PPA with the Noor iteration process [189], Thounthong *et al.* [252] proposed the following method for finding a common element of the set of fixed points of three nonexpansive mappings and the set of minimizers of lower semicontinuous and convex function in Hadamard spaces:

Algorithm 2.5.28.

$$\begin{cases} z_n = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n}d^2(y,x_n)], \\ w_n = (1-\eta_n)x_n \oplus \eta_n T_1 z_n, \\ y_n = (1-\beta_n)x_n \oplus \beta_n T_2 w_n, \\ x_{n+1} = (1-\xi_n)x_n \oplus \xi_n T_3 y_n, \end{cases} \tag{2.5.33}$$

where $\lambda_n > 0$ for all $n \in \mathbb{N}, \{\eta_n\}, \{\beta_n\}, \{\xi_n\}$ are sequences in $(0,1)$ and $T_i(i = 1,2,3)$ are nonexpansive mappings. They also obtained a weak convergence result and some strong convergence results under some compactness conditions.

In 2017, Suparatulatorn *et al.* [232] proposed the following Halpern-type proximal point algorithm for approximating a common solution of MP and FFP for a nonexpansive mapping in an Hadamard space: For arbitrary $u, x_1 \in X$, define the sequence $\{x_n\}$ iteratively by

Algorithm 2.5.29.

$$\begin{cases} y_n = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n}d^2(y,x_n)], \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n)Ty_n \ n \geq 1, \end{cases} \tag{2.5.34}$$

where $\{\alpha_n\} \subset (0,1)$ and $\lambda_n \geq \lambda > 0$. They obtained a strong convergence result without the compactness condition. Later in 2018, Cuntavepanita and Phuengrattana [91] extended the results of Suparatulatorn *et al.* [232] from one nonexpansive mapping to a finite family of nonexpansive mappings. They obtained a weak convergence result and under some compactness assumption, they obtained some strong convergence results.

Just recently, Chang *et al.* [70] proposed the following iterative algorithm for approximating a common solution of a finite family of MPs and FFP for two demicontractive mappings in Hadamard spaces:

Algorithm 2.5.30.

$$\begin{cases} u_n = S_\lambda^m(x_n), \\ y_n = (1-\beta_n)u_n \oplus \beta_n K_1 u_n, \\ x_{n+1} = (1-\eta_n)u_n \oplus \eta_n K_2 y_n, \quad n \geq 1, \end{cases} \tag{2.5.35}$$

where $K_i(x) := \delta x \oplus (1-\delta)T_i x, \ \ x \in C, \ \ i = 1,2, \ \ S_\lambda^j := J_\lambda^{f_j} \circ J_\lambda^{f_{j-1}} \circ, \cdots, \circ J_\lambda^{f_2} \circ J_\lambda^{f_1}, \ \ j = 1,2,\cdots,m.$ $\lambda > 0$, $\{\eta_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ with $0 < a \leq \eta_n, \beta_n < b <$

1. Chang *et al.* [70] obtained a weak convergence result and under some compactness assumption, they obtained some strong convergence results.

Another effective iterative method for finding solutions of MPs is the viscosity implicit method (VIM). Many authors have obtained convergence results using the VIM in more general spaces (see [137, 264] and other references therein). This study has also been extended to the Hadamard space (see [265]) and like every other type of iterative method, the PPA type of VIM has also been studied in this setting. In 2015, Xu et. al. [263] applied the VIM to the implict midpoint rule (IMR) for nonexpansive mappings in Hilbert spaces and proposed a viscosity implicit midpoint rule (VIMR). They proved that the sequence generated converges strongly under suitable conditions to a fixed point of the nonexpansive mapping which is also a solution of the variational inequality.

Recently, Ahmad and Ahmad [4] proposed a VIM of IMR in Hadamard space and they defined it as follows;

Algorithm 2.5.31.

$$\begin{cases} w_n = \frac{x_n \oplus\ x_{n+1}}{2}, \\ y_n = \alpha_n(w_n) \oplus \beta_n g(w_n) \oplus \tau_n T(w_n), \\ x_{n+1} = T(y_n), \end{cases} \tag{2.5.36}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\tau_n\}$ are sequences in $(0, 1)$, g is a contraction with a coefficient $\theta \in [0, 1)$ and T is a nonexpansive mapping on D. They also obtained a strong convergence of (2.5.36) to a fixed point of the nonexpansive mapping.

Remark 2.5.6. Besides the above mentioned works which mainly motivated our study of MPs in Hadamard spaces, there are very few other results on MPs in Hadamard spaces. Thus, it is very important to further develop and generalize this study in Hadamard spaces.

2.5.7 Some important lemmas

Lemma 2.5.7. *[234] For each $x_1, \cdots, x_m \in \mathcal{H}$ and $\alpha_1, \cdots, \alpha_m \in [0, 1]$ with $\sum_{i=1}^m \alpha_i = 1$, the equality*

$$\|\alpha_1 x_1 + \cdots + \alpha_m x_m\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{1\le i<j\le m} \alpha_i\alpha_j \|x_i - x_j\|^2$$

holds.

Lemma 2.5.8. *[89] Assume that $A : \mathcal{H} \to \mathcal{H}$ is a continuous and pseudomonotone operator. Then, x is a solution of (1.2.1) if and only if $\langle Ay, y - x\rangle \ge 0, \quad \forall y \in \mathcal{C}$.*

Theorem 2.5.9. *[124, Theorem 2.3]. Let $p \in [1, \infty)$ be a rational number except for $p = 1, 2$. Unless $P = NP$, there is no algorithm which computes the p-norm of a matrix with entries in $\{-1, 0, 1\}$ to relative error with running time polynomial in the dimensions.*

Lemma 2.5.10. *[261, Theorem 4.1] Assume that the solution set $VI(A, \mathcal{C})$ of the classical VIP (1.2.1) is nonempty and $\mathcal{C}$ is defined as $\mathcal{C} := \{x \in \mathcal{H} \mid c(x) \leq 0\}$, where $c : \mathcal{H} \to \mathbb{R}$ is a continuously differential convex function. Let $p \in \mathcal{C}$. Then, $p \in VI(A, \mathcal{C})$ if and only if either*

(i) $Ap = 0$, *or*

(ii) $p \in \partial\mathcal{C}$ *and there exists* $\rho > 0$ *such that* $Ap = -\rho c'(p)$, *where* $\partial\mathcal{C}$ *denotes the boundary of* $\mathcal{C}$.

Following Attouch and Cabot [29, pages 5, 10], we note that if $x_{n+1} = x_n + \theta_n(x_n - x_{n-1})$, then for all $n \geq 1$, we have that

$$x_{n+1} - x_n = \left(\prod_{j=1}^{n} \theta_j\right)(x_1 - x_0),$$

which implies that

$$x_n = x_1 + \left(\sum_{j=1}^{n-1} \prod_{j=1}^{l} \theta_j\right)(x_1 - x_0).$$

Thus, $\{x_n\}$ converges if and only if $x_1 = x_0$ or if $\sum_{l=1}^{\infty} \prod_{j=1}^{l} \theta_j < \infty$.
Therefore, we assume henceforth that

$$\sum_{l=i}^{\infty}\left(\prod_{j=i}^{l} \theta_j\right) < \infty \quad \forall i \geq 1. \tag{2.5.37}$$

Hence, we can define the sequence $\{t_i\}$ in $\mathbb{R}$ by

$$t_i := \sum_{l=i-1}^{\infty}\left(\prod_{j=i}^{l} \theta_j\right) = 1 + \sum_{l=i}^{\infty}\left(\prod_{j=i}^{l} \theta_j\right), \tag{2.5.38}$$

with the convention $\prod_{j=i}^{i-1} \theta_j = 1 \ \forall i \geq 1$.

Remark 2.5.11. (See also [29]).
Assumption (2.5.37) ensures that the sequence $\{t_i\}$ given by (2.5.38) is well-defined, and

$$t_i = 1 + \theta_i t_{i+1}, \ \ \forall i \geq 1. \tag{2.5.39}$$

The following proposition provides a criterion for ensuring assumption (2.5.37).

Proposition 2.5.32. *[29, Proposition 3.1] Let $\{\theta_n\}$ be a sequence such that $\theta_n \in [0, 1)$ for every $n \geq 1$. Assume that*

$$\lim_{n\to\infty}\left(\frac{1}{1-\theta_{n+1}} - \frac{1}{1-\theta_n}\right) = c,$$

for some $c \in [0, 1)$. Then,

(i) assumption (2.5.37) *holds, and* $t_{n+1} \sim \frac{1}{(1-c)(1-\theta_n)}$ *as* $n \to \infty$,

(ii) the equivalence $1 - \theta_n \sim 1 - \theta_{n+1}$ *holds true as* $n \to \infty$. *Hence,* $t_{n+1} \sim t_{n+2}$ *as* $n \to \infty$.

Remark 2.5.12. Using Proposition 2.5.32, we can see that $\theta_n = 1 - \frac{\bar{\theta}}{n}, \bar{\theta} > 1$, is a typical example of a sequence satisfying assumption (2.5.37).
Indeed, we have that

$$\lim_{n\to\infty}\left(\frac{1}{1-\theta_{n+1}} - \frac{1}{1-\theta_n}\right) = \lim_{n\to\infty}\left(\frac{1}{\bar{\theta}}(n+1) - \frac{1}{\bar{\theta}}n\right) = \frac{1}{\bar{\theta}} \in [0,1),$$

which satisfies the assumption of Proposition 2.5.32. Hence by Proposition 2.5.32(i), assumption (2.5.37) holds.

It is worthy of note that the example $\theta_n = 1 - \frac{\bar{\theta}}{n}, \bar{\theta} > 1$, falls within the setting of Nesterov's extrapolation methods. In fact, many practical choices for θ_n satisfy assumption (2.5.37) (for instance, see [29, 40, 68, 187]).

The corresponding finite sum expression for $\{t_i\}$ is defined for $i, n \geq 1$, by

$$t_{i,n} := \begin{cases} \sum_{l=i-1}^{n-1}\left(\prod_{j=i}^{l}\theta_j\right) = 1 + \sum_{l=i}^{n-1}\left(\prod_{j=i}^{l}\theta_j\right), & i \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5.40}$$

In the same manner, we have that $\{t_{i,n}\}$ is well-defined and

$$t_{i,n} = 1 + \theta_i t_{i+1,n} \; \forall i \geq 1, \; n \geq i+1. \tag{2.5.41}$$

The sequences $\{t_i\}$ and $\{t_{i,n}\}$ are very crucial to our convergence analysis. In fact, their effect can be seen in the following lemma which also plays a crucial role in establishing our convergence results.

Lemma 2.5.13. *[29, page 42, Lemma B.1]. Let* $\{a_n\}, \{\theta_n\}$ *and* $\{b_n\}$ *be sequences of real numbers satisfying*

$$a_{n+1} \leq \theta_n a_n + b_n \quad \textit{for every} \quad n \geq 1.$$

Assume that $\theta_n \geq 0$ *for every* $n \geq 1$.

(a) For every $n \geq 1$, *we have*

$$\sum_{i=1}^{n} a_i \leq t_{1,n}a_1 + \sum_{i=1}^{n-1} t_{i+1,n}b_i,$$

where the double sequence $\{t_{i,n}\}$ *is defined by* (2.5.40).

(b) Under (2.5.37), assume that the sequence $\{t_i\}$ defined by (2.5.38) satisfies $\sum_{i=1}^{\infty} t_{i+1}[b_i]_+ < \infty$. Then, the series $\sum_{i\geq 1}[a_i]_+$ is convergent, and

$$\sum_{i=1}^{\infty}[a_i]_+ \leq t_1[a_1]_+ + \sum_{i=1}^{\infty} t_{i+1}[b_i]_+ \ ,$$

where $[t]_+ := \max\{t, 0\}$ for any $t \in \mathbb{R}$.

Lemma 2.5.14. *[29, page 7, Lemma 2.1]. Let $\{x_n\}$ be a sequence in $\mathcal{H}$ and $\{\theta_n\}$ be a sequence of real numbers. Given $z \in \mathcal{H}$, define the sequence $\{\Gamma_n\}$ by $\Gamma_n := \frac{1}{2}\|x_n - z\|^2$. Then*

$$\begin{aligned}\Gamma_{n+1} - \Gamma_n - \theta_n(\Gamma_n - \Gamma_{n-1}) &= \frac{1}{2}(\theta_n + \theta_n^2)\|x_n - x_{n-1}\|^2 + \langle x_{n+1} - w_n, w_n - z\rangle \\ &+\frac{1}{2}\|x_{n+1} - w_n\|^2, \end{aligned} \tag{2.5.42}$$

where $w_n = x_n + \theta_n(x_n - x_{n-1})$.

The following lemmas are well-known.

Lemma 2.5.15. *Let $\{x_n\}$ be any sequence in $\mathcal{H}$ such that $x_n \rightharpoonup x$. Then,*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \forall y \neq x.$$

Lemma 2.5.16. *[122] Let $\mathcal{C}$ be a closed convex set in $\mathcal{H}$, f be a real-valued function on $\mathcal{H}$ and define $K := \{x \in C : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on C with modulus $L > 0$, then*

$$dist(x, K) \geq L^{-1} \max\ \{f(x), 0\}, \quad \forall x \in \mathcal{C},$$

where $dist(x, K)$ denote the distance function from x to K.

Lemma 2.5.17. *[264] Let $\mathcal{H}$ be a real Hilbert space and $S : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping with $F(S) \neq \emptyset$. If $\{x_n\}$ is a sequence in $\mathcal{H}$ converging weakly to x^* and $\{(I-S)x_n\}$ converges strongly to y, then $(I - S)x^* = y$.*

Lemma 2.5.18. *[221] Let $\mathcal{C} \subseteq \mathcal{H}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. Let $u \in \mathcal{H}$ be arbitrarily given, $z := P_{\mathcal{C}}u$, and $\Omega := \{x \in \mathcal{H} : \langle x - u, x - z\rangle \leq 0\}$. Then $\Omega \cap \mathcal{C} = \{z\}$.*

Lemma 2.5.19. *Let $\mathcal{X}$ be a smooth, strictly convex and reflexive Banach space. Let $\mathcal{C}$ be a nonempty, closed and convex subset of $\mathcal{X}$, and let $x_1 \in \mathcal{X}$ and $z \in \mathcal{C}$. Then, $z = P_{\mathcal{C}}x_1$ if and only if*

$$\langle z - y, J_{\mathcal{X}}(x_1 - z)\rangle \geq 0, \ \forall y \in \mathcal{C}.$$

Lemma 2.5.20. *[164] Let $\mathcal{X}$ be a real Banach space. Let $B : \mathcal{X} \to 2^{\mathcal{X}}$ be a maximal monotone operator and $A : \mathcal{X} \to \mathcal{X}$ be a k-inverse strongly monotone mapping on $\mathcal{X}$. Define $T_\lambda = (I + \lambda B)^{-1}(I - \lambda A),\ \lambda > 0$. Then we have* *(i)* $F(T_\lambda) = (A+B)^{-1}(0)$;

(ii) for $0 < s \leq \lambda$ and $x \in \mathcal{X}$, $\|x - T_s x\| \leq 2\|x - T_\lambda x\|$.

Lemma 2.5.21. *[128] Let $\mathcal{X}$ be a 2-uniformly convex Banach space and $\mathcal{X}^*$ the dual space of $\mathcal{X}$. Suppose $A : \mathcal{X} \to \mathcal{X}^*$ is uniformly continuous on bounded subsets of $\mathcal{X}$ and B is a bounded subset of $\mathcal{X}$. Then $A(B)$ is bounded.*

Lemma 2.5.22. *[20] Suppose $\mathcal{X}$ is a 2- uniformly convex Banach space. Then there exists $\mu \geq 1$ such that*

$$\phi(x, y) \geq \frac{1}{\mu}\|x - y\|^2, \quad \forall \ x, y \in \mathcal{X},$$

where μ is the 2-uniform convexity constant of $\mathcal{X}$. If $\mathcal{X}$ is a Hilbert space, then $\mu = 1$.

Lemma 2.5.23. *[95] Let $x \in \mathcal{X}$ and $\psi \geq \sigma > 0$. The following inequality holds;*

$$\frac{\|x - \Pi_{\mathcal{C}} J^{-1}(x - \psi Ax)\|}{\psi} \leq \frac{\|x - \Pi_{\mathcal{C}} J^{-1}(x - \sigma Ax)\|}{\sigma}.$$

Lemma 2.5.24. *[139] Let $\mathcal{X}$ be a smooth and uniformly convex real Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $\mathcal{X}$. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$, as $n \to \infty$, then $\|x_n - y_n\| \to 0$, as $n \to \infty$.*

Lemma 2.5.25. *[196] Let X be an Hadamard space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, $d^2(J_\lambda^f x, x) \leq d^2(J_\mu^f x, x)$ for $0 < \lambda < \mu$ and $x \in X$.*

Lemma 2.5.26. *[196] Let X be an Hadamard space and $f_j : X \to (-\infty, \infty]$, $j = 1, 2, \cdots, m$ be a finite family of proper, convex and lower semicontinuous functions. If $0 < \lambda < \mu$ and $\left(\cap_{j=1}^m F\left(J_\mu^{(j)}\right)\right) \neq \emptyset$. Then,*

$$F\left(\Pi_{j=1}^m J_\mu^{(j)}\right) \subseteq \left(\cap_{j=1}^m F\left(J_\lambda^{(j)}\right)\right),$$

where, $\Pi_{j=1}^m J_\mu^{(j)} = J_\mu^{(1)} \circ J_\mu^{(2)} \circ \cdots \circ J_\mu^{(m)}$.

Lemma 2.5.27. *[158] Let X be an Hadamard space and $f : X \to (-\infty, \infty]$ be a proper, convex and lower semicontinuous function. Then, for all $x, y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda} d^2(J_\lambda^f x, y) - \frac{1}{2\lambda} d^2(x, y) + \frac{1}{2\lambda} d^2(x, J_\lambda^f x) + f(J_\lambda^f x) \leq f(y).$$

Lemma 2.5.28. *[93] Let X be a CAT(0) space, $\{v_1, v_2, \cdots, v_N\} \subset X$ and $\{\lambda_1, \lambda_2, \cdots, \lambda_N\} \subset (0, 1)$ with $\sum_{i=1}^N \lambda_i = 1$. Then,*

$$d\left(\sum_{i=1}^N \oplus \lambda_i v_i, x\right) \leq \sum_{i=1}^N \lambda_i d(v_i, x) \qquad \text{for} \qquad \text{each} \quad x \in X.$$

Remark 2.5.29. [93] For a CAT(0) space X, if $\{x_i, i = 1, 2, \ldots, N\} \subset X$, and $\alpha_i \in (0,1),\ i = 1, 2, \ldots, N$. Then by induction, we can write

$$\sum_{i=1}^{N} \oplus \alpha_i x_i \quad := \quad (1-\alpha_N) \sum_{i=1}^{N-1} \oplus \frac{\alpha_i}{1-\alpha_N} x_i \oplus \alpha_N x_N. \tag{2.5.43}$$

Lemma 2.5.30. *Let X be a CAT(0) space, $\{v_1, v_2, \cdots, v_N\} \subset X$ and $\{\lambda_1, \lambda_2, \cdots, \lambda_N\} \subset (0,1)$ with $\sum\limits_{i=1}^{N} \lambda_i = 1$. Then,*

$$d^2\left(\sum_{i=1}^{N} \oplus \lambda_i v_i, x\right) \leq \sum_{i=1}^{N} \lambda_i d^2(v_i, x) - \sum_{i,j=1, i\neq j}^{N} \lambda_i \lambda_j d^2(v_i, v_j).$$

Lemma 2.5.31. *[194] Every bounded sequence in an Hadamard space has a $\triangle$-convergence subsequence.*

Lemma 2.5.32. *[94] Let C be a nonempty convex subset of a CAT(0) space X and $x \in X$. Then, $u = P_C x$ if and only if $\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle \leq 0 \quad \forall y \in C$, where P_C is the metric projection of X onto C.*

Lemma 2.5.33. *[255] Let X be an Hadamard space and $T : X \to X$ be a nonexpansive mapping. Then T is a $\triangle$- demiclosed.*

Lemma 2.5.34. *[242] Suppose $\{\lambda_n\}$ and $\{\theta_n\}$ are two nonnegative real sequences such that*

$$\lambda_{n+1} \leq \lambda_n + \phi_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \phi_n < \infty$, then $\lim\limits_{n\to\infty} \lambda_n$ exists.

Lemma 2.5.35. *[166] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following*

$$a_{n+1} \leq (1-\beta_n)a_n + \tau_n + \gamma_n, \quad n \geq 1,$$

where $\{\beta_n\}$ is a sequence in $(0,1)$ and $\{\tau_n\}$ is a real sequence. Suppose that $\sum\limits_{n=1}^{\infty} \gamma_n < \infty$ and $\tau_n \leq \beta_n M$ for some $M > 0$. Then, $\{a_n\}$ is a bounded sequence.

Lemma 2.5.36. *[9] Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\gamma_n\}$ be a sequence of real numbers in $(0,1)$ with conditions $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n d_n, \quad n \geq 1.$$

If $\limsup\limits_{k\to\infty} d_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf\limits_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

CHAPTER 3

Results on Variational Inequality and Fixed Point Problems

3.1 Introduction

In this chapter, we present our results on monotone VIP and FPP of an infinite family of strict pseudo-contractive mappings in the real Hilbert space and apply these results to solve other nonlinear problems. Furthermore, using the projection technique, we present our results on VIP for a quasimonotone and Lipschitz operator in a real Hilbert space. Also, we present our result on quasimonotone VIP and FPP of a quasi pseudocontractive mapping in the real Hilbert space. We provide some numerical experiments of our proposed methods in comparison to other existing methods in the literature.

3.2 Preliminaries

Lemma 3.2.1. *[276] Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $S : \mathcal{C} \to \mathcal{C}$ be a k-strict pseudo-contractive mapping. Define a mapping $T : \mathcal{C} \to \mathcal{C}$ by $Tx = \alpha x + (1 - \alpha)Sx$ for all $x \in \mathcal{C}$ and $\alpha \in [k, 1)$. Then T is a nonexpansive mapping such that $F(T) = F(S)$.*

Lemma 3.2.2. *[239] Assume that $A : \mathcal{H} \to \mathcal{H}$ is a continuous and monotone operator. Then, x is a solution of the classical VIP (1.2.1) if and only if x is a solution of the following problem:*
Find $x \in \mathcal{C}$ such that

$$\langle Ay, y - x\rangle \geq 0, \quad \forall y \in \mathcal{C}.$$

Definition 3.2.3. *[259] Let $\{S_n\}$ be a sequence of k_n-strict pseudo-contractions. Define $S'_n = t_n I + (1 - t_n)S_n, t_n \in [k_n, 1)$. Then, by Lemma 3.2.1, S'_n is nonexpansive. In this* *work, we consider the mapping W_n defined by*

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \zeta_n S'_n U_{n,n+1} + (1-\zeta_n)I, \\ U_{n,n-1} = \zeta_{n-1} S'_{n-1} U_{n,n} + (1-\zeta_{n-1})I, \\ \cdots, \\ U_{n,k} = \zeta_k S'_k U_{n,k+1} + (1-\zeta_k)I, \\ U_{n,k-1} = \zeta_{k-1} S'_{k-1} U_{n,k} + (1-\zeta_{k-1})I, \\ \cdots, \\ U_{n,2} = \zeta_2 S'_2 U_{n,3} + (1-\zeta_2)I, \\ W_n = U_{n,1} = \zeta_1 S'_1 U_{n,2} + (1-\zeta_1)I. \end{cases} \tag{3.2.1}$$

where $\{\zeta_i\}$ is a sequence of real numbers such that $0 \le \zeta_i \le 1$ for all $i \ge 1$. For each $n \ge 1$, such a mapping W_n is nonexpansive.

We have the following lemmas relating to the mapping W_n, which are needed in proving our main results.

Lemma 3.2.4. *[225] Let $\mathcal{C}$ be a nonempty closed convex subset of a strictly convex Banach space $\mathcal{X}$. Let $\{S'_i\}$ be an infinite family of nonexpansive mappings of $\mathcal{C}$ into itself such that $\bigcap_{i=1}^{\infty} F(S'_i) \neq \emptyset$ and $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \le b < 1$ for all $i \ge 1$. Then we have the following:*

(1) W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(S'_i)$ for each $n \ge 1$;

(2) for each $x \in \mathcal{C}$ and for each positive integer k, then $\lim_{n\to\infty} U_{n,k}x$ exists;

(3) the mapping W of $\mathcal{C}$ into itself defined by

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x \quad \text{for all } x \in \mathcal{C} \tag{3.2.2}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(S'_i)$, which is called the modified W-mapping generated by $S'_1, S'_2, \cdots, \zeta_1, \zeta_2, \cdots$ and $t_1, t_2, \cdots$.

By combining Lemma 3.2.1 and Lemma 3.2.4, it follows that $F(W) = \bigcap_{i=1}^{\infty} F(S'_i) = \bigcap_{i=1}^{\infty} F(S_i)$.

Lemma 3.2.5. *[69] Let $\mathcal{C}$ be a nonempty closed convex subset of a strictly convex Banach space $\mathcal{X}$. Let $\{S'_i\}$ be an infinite family of nonexpansive mappings of $\mathcal{C}$ into itself such that $\bigcap_{i=1}^{\infty} F(S'_i) \neq \emptyset$ and $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \le b < 1$ for all $i \ge 1$, where b is a positive real number. If K is any bounded subset of $\mathcal{C}$, then*

$$\lim_{n\to\infty} \sup_{x\in K} ||Wx - W_n x|| = 0.$$

Lemma 3.2.6. *[276] If S is a k-strict pseudo-contraction on closed convex subset $\mathcal{C}$ of a real Hilbert space $\mathcal{H}$, then $I - S$ is demiclosed at any point $y \in \mathcal{H}$.*

Let $VI(\mathcal{C}, A)$ be the solution set of problem (1.2.1) and Γ be the solution set of the following problem:

$$\text{Find } x \in \mathcal{C} \text{ such that } \langle Ay, y - x\rangle \geq 0, \ \forall y \in \mathcal{C}. \tag{3.2.3}$$

Then, Γ is a closed and convex subset of $\mathcal{C}$, and since $\mathcal{C}$ is convex and A is continuous, we have the following relation

$$\Gamma \subseteq VI(\mathcal{C}, A). \tag{3.2.4}$$

Lemma 3.2.7. *[269] Let $\mathcal{C}$ be a nonempty closed and convex subset of $\mathcal{H}$. If either*

(i) A is pseudomonotone on $\mathcal{C}$ and $VI(\mathcal{C}, A) \neq \emptyset$,

(ii) A is the gradient of G, where G is a differential quasiconvex function on an open set $\mathcal{K} \supset \mathcal{C}$ and attains its global minimum on $\mathcal{C}$,

(iii) A is quasimonotone on $\mathcal{C}$, $A \neq 0$ on $\mathcal{C}$ and $\mathcal{C}$ is bounded,

(iv) A is quasimonotone on $\mathcal{C}$, $A \neq 0$ on $\mathcal{C}$ and there exists a positive number r such that, for every $x \in \mathcal{C}$ with $\|x\| \geq r$, there exists $y \in \mathcal{C}$ such that $\|y\| \leq r$ and $\langle Ax, y - x\rangle \leq 0$,

(v) A is quasimonotone on $\mathcal{C}$, int $\mathcal{C}$ is nonempty and there exists $x^* \in VI(\mathcal{C}, A)$ such that $Ax^* \neq 0$.

Then, Γ is nonempty.

Lemma 3.2.8. *[71] Let $\mathcal{H}$ be a real Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be an L-Lipschitz mapping with $L \geq 1$ such that for all $x \in \mathcal{H}$,*

$$K := (1 - \zeta)I + \zeta T((1 - \eta)I + \eta T). \tag{3.2.5}$$

If $0 < \zeta < \eta < \frac{1}{1+\sqrt{1+L^2}}$, then the following conditions hold:

(a) $F(T) = F(T((1 - \eta)I + \eta T)) = F(K)$;

(b) if $I - T$ is demiclosed at 0, then $I - K$ is also demiclosed at 0;

(c) if T is a quasi-pseudocontractive mapping, then K is a quasi-nonexpansive mapping.

3.3 Common solution of the variational inequality and the fixed point problems

In this section, we propose and study new inertial viscosity Tseng's extragradient algorithms with self-adaptive step size to solve the VIP and the FPP in Hilbert spaces. Our proposed methods involves a projection onto a halfspace and self-adaptive step size.

We prove that the sequence generated by our proposed methods converges strongly to a common solution of the VIP and FPP of an infinite family of strict pseudo-contractive mappings in Hilbert spaces under some mild assumptions when the underlying operator is monotone and Lipschitz continuous. Furthermore, we apply our results to find a common solution of VIP and ZPP for an infinite family of maximal monotone operators. Finally, we provide some numerical experiments of the proposed methods in comparison with other existing methods in the literature.

3.3.1 Proposed method

In this section, we present the following method with self-adaptive step size for solving the VIP (1.2.1) and the FPP (1.1.1).

Assumption 3.3.1. *Suppose that the following conditions hold:*

Condition A:

(A1) The feasible set $\mathcal{C}$ is given by

$$\mathcal{C} = \{x \in \mathcal{H} : h(x) \le 0\},$$

where $h : \mathcal{H} \to \mathbb{R}$ is a convex and subdifferentiable function on $\mathcal{C}$.

(A2) h is weakly lower semicontinuous.

(A3) For any $x \in \mathcal{H}$, at least one subgradient $\xi \in \partial h(x)$ can be calculated, where $\partial h(x)$ is defined as follows

$$\partial h(x) = \{z \in \mathcal{H} : h(u) \ge h(x) + \langle u - x, z\rangle, \quad \forall u \in \mathcal{H}\}.$$

In addition, $\partial h(x)$ is bounded on bounded sets.

(A4) Define the set $\mathcal{C}_n$ by the following half-space

$$\mathcal{C}_n = \{x \in \mathcal{H} : h(w_n) + \langle \xi_n, x - w_n\rangle \le 0\},$$

where $\xi_n \in \partial h(w_n)$. By the definition of the subgradient, it is clear that $\mathcal{C} \subseteq \mathcal{C}_n$.

Condition B:

(B1) $A : \mathcal{H} \to \mathcal{H}$ is monotone and Lipschitz continuous with Lipschitz constant $L > 0$.

(B2) The solution set $\Gamma := \{z \in VI(A, \mathcal{C}) \bigcap \cap_{i=1}^{\infty} F(\mathcal{S}_i)\}$ is nonempty, where $\mathcal{S}_i : \mathcal{H} \to \mathcal{H}$ is an infinite family of k_i-strict pseudo-contractions.

(B3) $D : \mathcal{H} \to \mathcal{H}$ is a strongly positive bounded linear operator with the coefficient γ_2.

(B4) $f : \mathcal{H} \to \mathcal{H}$ is a contraction with the coefficient $\rho \in (0, 1)$ and $0 < \gamma_1 < \frac{\gamma_2}{\rho}$.

(B5) $\{\beta_n\}_{n=1}^{\infty}, \{\theta_n\}_{n=1}^{\infty}$ *and* $\{\tau_n\}_{n=1}^{\infty}$ *are positive sequences satisfying the following conditions:*

$\{\beta_n\}, \{\theta_n\} \subset (0,1),\ 0 < c_1 \leq \beta_n,\ \lim_{n\to\infty} \theta_n = 0,\quad \sum_{n=1}^{\infty} \theta_n = \infty,\ \lim_{n\to\infty} \frac{\tau_n}{\theta_n} = 0.$

Algorithm 3.3.2. Inertial method with adaptive step size strategy.

Step 0: Choose sequences $\{\theta_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ such that condition (B5) holds and let $\lambda_1 > 0$, $\mu \in (0,1)$, $\alpha \geq 3$ and $x_0, x_1 \in \mathcal{H}$ be given arbitrarily. Set $n := 1$.

Step 1: Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n := \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases} \tag{3.3.1}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Step 3: Construct the halfspace

$$\mathcal{C}_n = \{w \in \mathcal{H}\ | c(w_n) + \langle \xi_n, w - w_n \rangle \leq 0\}$$

and compute

$$y_n = P_{\mathcal{C}_n}(w_n - \lambda_n A w_n).$$

If $y_n = w_n$ ($Aw_n = 0$), then set $w_n = z_n$ and go to Step 5. Else go to Step 4.

Step 4: compute

$$z_n = y_n - \lambda_n(Ay_n - Aw_n).$$

Step 5: Compute

$$x_{n+1} = \theta_n \gamma_1 f x_n + (I - \theta_n D) T_n z_n,$$

where

$T_n = (1-\beta_n)I + \beta_n W_n$ and W_n is the mapping defined by (3.2.1).

Step 6: Compute

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|},\ \lambda_n\right\}, & \text{if } Aw_n \neq Ay_n \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{3.3.2}$$

Set $n := n+1$ and go back to **Step 1**.

Remark 3.3.1. Unlike several other existing methods for solving the VIP (1.2.1), our methods involves a projection onto a half-space without any projection onto the feasible set. Note that while projection onto the feasible set may be difficult to execute, projections onto half-spaces have closed formulas, and hence, they are easy to execute. Thus, our methods are more easily implementable than several existing ones in the literature. We prove that the sequence $\{x_n\}$ generated by our proposed methods converges strongly to (1.2.2) (a common solution of the VIP (1.2.1) and the FPP (1.1.1) for an infinite family of strict pseudo-contractive mappings) in Hilbert spaces under some mild conditions when the cost operator is monotone and Lipschitz continuous.

Remark 3.3.2. From (3.3.2) in Algorithm 3.3.2, it is clear that $\lambda_{n+1} \leq \lambda_n$ $\forall n \geq 1$. Also, since A is L-Lipschitz continuous, we get in the case of $Aw_n \neq Ay_n$ in Algorithm 3.3.2, that

$$\lambda_{n+1} = \min\left\{\frac{\mu||w_n - y_n||}{||Aw_n - Ay_n||},\ \lambda_n\right\} \geq \min\left\{\frac{\mu}{L},\ \lambda_n\right\},$$

which by induction, implies that $\{\lambda_n\}$ is bounded below by $\min\left\{\frac{\mu}{L},\ \lambda_1\right\}$. Since $\{\lambda_n\}$ is also monotone nonincreasing, we have that the limit exists, and $\lim_{n\to\infty} \lambda_n \geq \min\{\frac{\mu}{L}, \lambda_1\} > 0$.

Remark 3.3.3. By condition (f), one can easily verify from (3.3.1) that

$$\lim_{n\to\infty} \alpha_n||x_n - x_{n-1}|| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\alpha_n}{\theta_n}||x_n - x_{n-1}|| = 0. \tag{3.3.3}$$

Remark 3.3.4. By the definition of the subgradient, it is clear that $\mathcal{C} \subseteq \mathcal{C}_n$.

3.3.2 Convergence analysis

Lemma 3.3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3.2 under Assumption 3.3.1. Then*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|y_n - w_n\|^2.$$

Proof. Let $p \in \Gamma$. Since $y_n = P_{\mathcal{C}_n}(w_n - \lambda_n w_n)$, we obtain by the characteristic property of $P_{\mathcal{C}_n}$ that $\langle y_n - w_n + \lambda_n Aw_n, y_n - p\rangle \leq 0$ and this implies that

$$\langle y_n - w_n, y_n - p\rangle \leq -\lambda_n\langle Aw_n, y_n - p\rangle. \tag{3.3.4}$$

Also, from the definition of z_n in **Step 4**, and Lemma 2.1.1, we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|y_n - \lambda_n(Ay_n - Aw_n) - p\|^2 \\
&= \|y_n - p\|^2 + \lambda_n^2\|Ay_n - Aw_n\|^2 - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|w_n - p\|^2 + \|y_n - w_n\|^2 + 2\langle y_n - w_n, w_n - p\rangle + \lambda_n^2\|Ay_n - Aw_n\|^2 \\
&- 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|w_n - p\|^2 + \|y_n - w_n\|^2 + \lambda_n^2\|Ay_n - Aw_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle \\
&+ 2\langle y_n - w_n, y_n - p\rangle \\
&- 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|w_n - p\|^2 - \|y_n - w_n\|^2 + \lambda_n^2\|Ay_n - Aw_n\|^2 + 2\langle y_n - w_n, y_n - p\rangle \\
&- 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle
\end{aligned} \tag{3.3.5}$$

From (3.3.4) and (3.3.5), we obtain

$$\begin{aligned}\|z_n - p\|^2 &\le \|w_n - p\|^2 - \|y_n - w_n\|^2 + \lambda_n^2\|Ay_n - Aw_n\|^2 - 2\lambda_n\langle Aw_n, y_n - p\rangle \\ &\quad - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle \\ &= \|w_n - p\|^2 - \|y_n - w_n\|^2 + \lambda_n^2\|Ay_n - Aw_n\|^2 - 2\lambda_n\langle Ay_n, y_n - p\rangle \\ &\le \|w_n - p\|^2 - \|y_n - w_n\|^2 + \lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2 - 2\lambda_n\langle Ay_n, y_n - p\rangle \\ &\le \|w_n - p\|^2 - \left(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|y_n - w_n\|^2. \end{aligned} \tag{3.3.6}$$

□

Consider the limit

$$\lim_{n\to\infty}\left(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right) = 1 - \mu^2 > 0.$$

Hence there exists $n_0 \ge 0$ such that for all $n \ge n_0$, we have that $\left(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right) > 0$. Thus, from (3.3.6), we have that

$$\|z_n - p\| \le \|w_n - p\|.$$

Lemma 3.3.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3.2 under Assumption 3.3.1. Then, $\{x_n\}$ is bounded.*

Proof. First, we show that $P_\Gamma(I - D + \gamma_1 f)$ is a contraction of $\mathcal{H}$. For all $x, y \in \mathcal{H}$, we have

$$\begin{aligned}\|P_\Gamma(I - D + \gamma_1 f)x - P_\Gamma(I - D + \gamma_1 f)y\| &\le \|(I - D + \gamma_1 f)x - (I - D + \gamma_1 f)y\| \\ &\le \|(I - D)x - (I - D)y\| + \gamma_1\|fx - fy\| \\ &\le (1 - \gamma_2)\|x - y\| + \gamma_1\rho\|x - y\| \\ &= (1 - (\gamma_2 - \gamma_1\rho))\|x - y\|.\end{aligned}$$

Hence, $P_\Gamma(I - D + \gamma_1 f)$ is a contraction. Let $p \in \Gamma$. Then from the definition of w_n in **Step 2**, we have

$$\begin{aligned}\|w_n - p\| &= \|x_n + \alpha_n(x_n - x_{n-1}) - p\| \\ &= \|x_n - p\| + \alpha_n\|x_n - x_{n-1}\|.\end{aligned}$$

Also, from **Step 2**, we observe that $\alpha_n\|x_n - x_{n-1}\| \le \tau_n, \forall n \ge 1$, which implies that

$$\frac{\alpha_n}{\theta_n}\|x_n - x_{n-1}\| \le \frac{\tau_n}{\theta_n} \to 0, \quad \text{as } n \to \infty. \tag{3.3.7}$$

Hence, there exists $M_1 > 0$ such that

$$\frac{\alpha_n}{\theta_n}\|x_n - x_{n-1}\| \le M_1, \quad \forall n \ge 1. \tag{3.3.8}$$

This implies that $\|w_n - p\| \leq \|x_n - p\| + \theta_n M_1, \forall n \geq 1$ and hence

$$\|z_n - p\| \leq \|w_n - p\| \leq \|x_n - p\| + \theta_n M_1. \tag{3.3.9}$$

From **Step 5**, we have

$$\begin{aligned} \|T_n z_n - p\| &\leq \|(1-\beta_n)z_n + \beta_n W_n z_n - p\| \\ &\leq (1-\beta_n)\|z_n - p\| + \beta_n\|W_n z_n - p\| \\ &\leq (1-\beta_n)\|z_n - p\| + \beta_n\|z_n - p\| \\ &= \|z_n - p\| \end{aligned} \tag{3.3.10}$$

From (3.3.9) and (3.3.10), we obtain for all $n \geq n_0$,

$$\begin{aligned} \|x_{n+1} - p\| &\leq \|\theta_n\gamma_1 f x_n + (1-\theta_n D)T_n z_n - p\| \\ &= \|\theta_n(\gamma_1 f x_n - Dp) + (1-\theta_n D)(T_n z_n - p)\| \\ &\leq \theta_n\|\gamma_1 f x_n - Dp\| + (1-\theta_n\gamma_2)\|T_n z_n - p\| \\ &\leq \theta_n\|\gamma_1(f x_n - fp) + (\gamma_1 fp - Dp)\| + (1-\theta_n\gamma_2)\|z_n - p\| \\ &\leq \theta_n\gamma_1\rho\|x_n - p\| + \theta_n\|\gamma_1 fp - Dp\| + (1-\theta_n\gamma_2)(\|x_n - p\| + \theta_n M_1) \\ &\leq (1-\theta_n(\gamma_2 - \gamma_1\rho))\|x_n - p\| + \theta_n\|\gamma_1 fp - Dp\| + \theta_n M_1 \\ &= (1-\theta_n(\gamma_2 - \gamma_1\rho))\|x_n - p\| + \theta_n(\gamma_2 - \gamma_1\rho)\frac{\|\gamma_1 fp - Dp\| + M_1}{\gamma_2 - \gamma_1\rho} \\ &\leq \max\Big\{\|x_n - p\|, \frac{\|\gamma_1 fp - Dp\| + M_1}{\gamma_2 - \gamma_1\rho}\Big\} \\ &\leq \max\Big\{\|x_{n_0} - p\|, \frac{\|\gamma_1 fp - Dp\| + M_1}{\gamma_2 - \gamma_1\rho}\Big\} \end{aligned}$$

Hence, the sequence $\{x_n\}$ is bounded. Consequently, $\{w_n\}, \{y_n\}$ and $\{z_n\}$ are also bounded. □

Lemma 3.3.7. *Assume that $\{w_n\}$ and $\{y_n\}$ are sequences generated by Algorithm 3.3.2 such that* $\lim_{n\to\infty} ||w_n - y_n|| = 0$. *If $\{w_{n_j}\}$ converges weakly to some $\hat{x} \in \mathcal{H}$ as $j \to \infty$, then $\hat{x} \in VI(\mathcal{C}, A)$.*

Proof. Since $w_{n_j} \rightharpoonup \hat{x}$, then by the hypothesis of the lemma it follows that $y_{n_j} \rightharpoonup \hat{x}$ as $j \to \infty$. Also, since $y_{n_j} \in \mathcal{C}_{n_j}$, then by the definition of $\mathcal{C}_n$ we get

$$h(w_{n_j}) + \langle \xi_{n_j}, y_{n_j} - w_{n_j}\rangle \leq 0. \tag{3.3.11}$$

By the boundedness of $\{w_n\}$ and by condition (A3), there exists a constant $M > 0$ such that $||\xi_{n_j}|| \leq M$ for all $j \geq 0$. Then, from (3.3.11) we obtain $h(w_{n_j}) \leq M||w_{n_j} - y_{n_j}|| \to 0$ as $j \to \infty$, and this in turn implies that $\liminf_{j\to\infty} h(w_{n_j}) \leq 0$. Applying condition (A2), we have $h(\hat{x}) \leq \liminf_{j\to\infty} h(w_{n_j}) \leq 0$. This implies that $\hat{x} \in \mathcal{C}$. From Lemma 2.4.1, we obtain

$$\langle y_{n_j} - w_{n_j} + \lambda_{n_j} A w_{n_j},\ z - y_{n_j}\rangle \geq 0, \quad \forall\ z \in \mathcal{C} \subseteq \mathcal{C}_{n_j}.$$

Since A is monotone, we have

$$\begin{aligned} 0 &\le \langle y_{n_j} - w_{n_j},\ z - y_{n_j}\rangle + \lambda_{n_j}\langle Aw_{n_j},\ z - y_{n_j}\rangle \\ &= \langle y_{n_j} - w_{n_j},\ z - y_{n_j}\rangle + \lambda_{n_j}\langle Aw_{n_j},\ z - w_{n_j}\rangle + \lambda_{n_j}\langle Aw_{n_j},\ w_{n_j} - y_{n_j}\rangle \\ &\le \langle y_{n_j} - w_{n_j},\ z - y_{n_j}\rangle + \lambda_{n_j}\langle Az,\ z - w_{n_j}\rangle + \lambda_{n_j}\langle Aw_{n_j},\ w_{n_j} - y_{n_j}\rangle. \end{aligned}$$

Letting $j \to \infty$, and since $\lim_{j\to\infty} ||y_{n_j} - w_{n_j}|| = 0$, we have

$$\langle Az,\ z - \hat{x}\rangle \ge 0, \quad \forall\ z \in \mathcal{C}.$$

Applying Lemma 3.2.2, we have that $\hat{x} \in VI(\mathcal{C}, A)$. □

Lemma 3.3.8. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3.2 under Assumption 3.3.1. Then,*

$$\begin{aligned} \|x_{n+1} - p\|^2 \le (1-\eta_n)\|x_n - p\|^2 + \eta_n\Big[\frac{\theta_n\gamma_2^2}{2(\gamma_2 - \gamma_1\rho)}M_3 + \frac{3M_2(1-\theta_n\gamma_2)^2}{2(\gamma_2 - \gamma_1\rho)}\frac{\alpha_n}{\theta_n}||x_n - x_{n-1}|| \\ + \frac{1}{(\gamma_2 - \gamma_1\rho)}\langle\gamma_1 fp - Dp, x_{n+1} - p\rangle\Big] - \frac{(1-\theta_n\gamma_2)^2}{(1-\theta_n\gamma_1\rho)}\Big[\left(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|y_n - w_n\|^2 \\ + \beta_n(1-\beta_n)\|W_n z_n - z_n\|^2\Big], \end{aligned}$$

where $\eta_n = \frac{2\theta_n(\gamma_2 - \gamma_1\rho)}{(1-\theta_n\gamma_1\rho)}$.

Proof. Let $p \in \Gamma$. Then from **Step 2**, by applying the Cauchy-Schwartz inequality and Lemma 2.1.1, we obtain

$$\begin{aligned} ||w_n - p||^2 &= ||x_n + \alpha_n(x_n - x_{n-1}) - p||^2 \\ &= ||x_n - p||^2 + \alpha_n^2||x_n - x_{n-1}||^2 + 2\alpha_n\langle x_n - p, x_n - x_{n-1}\rangle \\ &\le ||x_n - p||^2 + \alpha_n^2||x_n - x_{n-1}||^2 + 2\alpha_n||x_n - x_{n-1}||||x_n - p|| \\ &= ||x_n - p||^2 + \alpha_n||x_n - x_{n-1}||(\alpha_n||x_n - x_{n-1}|| + 2||x_n - p||) \\ &\le ||x_n - p||^2 + 3M_2\alpha_n||x_n - x_{n-1}|| \\ &= ||x_n - p||^2 + 3M_2\theta_n\frac{\alpha_n}{\theta_n}||x_n - x_{n-1}||, \end{aligned} \tag{3.3.12}$$

where $M_2 := \sup_{n\in\mathbb{N}}\{||x_n - p||, \alpha_n||x_n - x_{n-1}||\} > 0$.

Now, applying the last inequality Lemma 2.1.1 and (3.3.6), we have

$$\begin{aligned} \|T_n z_n - p\|^2 &= \|(1-\beta_n)(z_n - p) + \beta_n(W_n z_n - p)\|^2 \\ &\le (1-\beta_n)\|z_n - p\|^2 + \beta_n\|W_n z_n - p\|^2 - \beta_n(1-\beta_n)\|W_n z_n - z_n\|^2 \\ &\le (1-\beta_n)\|z_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\|W_n z_n - z_n\|^2 \\ &= \|z_n - p\|^2 - \beta_n(1-\beta_n)\|W_n z_n - z_n\|^2 \\ &\le \|w_n - p\|^2 - \left(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|y_n - w_n\|^2 - \beta_n(1-\beta_n)\|W_n z_n - z_n\|^2 \\ &\le ||x_n - p||^2 + 3M_2\theta_n\frac{\alpha_n}{\theta_n}||x_n - x_{n-1}|| - \left(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|y_n - w_n\|^2 \\ &- \beta_n(1-\beta_n)\|W_n z_n - z_n\|^2. \end{aligned} \tag{3.3.13}$$

Also, by applying Lemma 2.1.1 and (3.3.13), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\theta_n \gamma_1 f x_n + (1 - \theta_n D) T_n z_n - p\|^2 \\
&= \|\theta_n(\gamma_1 f x_n - Dp) + (1 - \theta_n D)(T_n z_n - p)\|^2 \\
&\leq (1 - \theta_n \gamma_2)^2 \|T_n z_n - p\|^2 + 2\theta_n \langle \gamma_1 f x_n - Dp, x_{n+1} - p\rangle \\
&\leq (1 - \theta_n \gamma_2)^2 \|T_n z_n - p\|^2 + 2\theta_n \gamma_1 \langle f x_n - fp, x_{n+1} - p\rangle \\
&+ 2\theta_n \langle \gamma_1 fp - Dp, x_{n+1} - p\rangle \\
&\leq (1 - \theta_n \gamma_2)^2 \Big[\|x_n - p\|^2 + 3M_2 \theta_n \frac{\alpha_n}{\theta_n} ||x_n - x_{n-1}|| \\
&- \left(1 - \frac{\lambda_n^2 \mu^2}{\lambda_{n+1}^2}\right) \|y_n - w_n\|^2 - \beta_n(1 - \beta_n) \|W_n z_n - z_n\|^2 \Big] \\
&+ 2\theta_n \langle \gamma_1 f x_n - fp, x_{n+1} - p\rangle + 2\theta_n \langle \gamma_1 fp - Dp, x_{n+1} - p\rangle \\
&\leq (1 - \theta_n \gamma_2)^2 \|x_n - p\|^2 + 3M_2 (1 - \theta_n \gamma_2)^2 \theta_n \frac{\alpha_n}{\theta_n} ||x_n - x_{n-1}|| \\
&- (1 - \theta_n \gamma_2)^2 \left(1 - \frac{\lambda_n^2 \mu^2}{\lambda_{n+1}^2}\right) \|y_n - w_n\|^2 - (1 - \theta_n \gamma_2)^2 \beta_n (1 - \beta_n) \|W_n z_n - z_n\|^2 \\
&+ 2\theta_n \gamma_1 \langle f x_n - fp, x_{n+1} - p\rangle + 2\theta_n \langle \gamma_1 fp - Dp, x_{n+1} - p\rangle \\
&\leq (1 - \theta_n \gamma_2)^2 \|x_n - p\|^2 + 3M_2 (1 - \theta_n \gamma_2)^2 \theta_n \frac{\alpha_n}{\theta_n} ||x_n - x_{n-1}|| \\
&- (1 - \theta_n \gamma_2)^2 \left(1 - \frac{\lambda_n^2 \mu^2}{\lambda_{n+1}^2}\right) \|y_n - w_n\|^2 \\
&- (1 - \theta_n \gamma_2)^2 \beta_n (1 - \beta_n) \|W_n z_n - z_n\|^2 + 2\theta_n \gamma_1 \rho \|x_n - p\| \|x_{n+1} - p\| \\
&+ 2\theta_n \langle \gamma_1 fp - Dp, x_{n+1} - p\rangle \\
&\leq (1 - \theta_n \gamma_2)^2 \|x_n - p\|^2 + 3M_2 (1 - \theta_n \gamma_2)^2 \theta_n \frac{\alpha_n}{\theta_n} ||x_n - x_{n-1}|| \\
&- (1 - \theta_n \gamma_2)^2 \Big[\left(1 - \frac{\lambda_n^2 \mu^2}{\lambda_{n+1}^2}\right) \|y_n - w_n\|^2 + \beta_n (1 - \beta_n) \|W_n z_n - z_n\|^2 \Big] \\
&+ \theta_n \gamma_1 \rho \Big(\|x_n - p\|^2 + \|x_{n+1} - p\|^2 \Big) + 2\theta_n \langle \gamma_1 fp - Dp, x_{n+1} - p\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= ((1 - \theta_n \gamma_2)^2 + \theta_n \gamma_1 \rho) \|x_n - p\|^2 + \theta_n \gamma_1 \rho \|x_{n+1} - p\|^2 \\
&+ 3M_2 (1 - \theta_n \gamma_2)^2 \theta_n \frac{\alpha_n}{\theta_n} ||x_n - x_{n-1}|| + 2\theta_n \langle \gamma_1 fp - Dp, x_{n+1} - p\rangle \\
&- (1 - \theta_n \gamma_2)^2 \Big[\left(1 - \frac{\lambda_n^2 \mu^2}{\lambda_{n+1}^2}\right) \|y_n - w_n\|^2 + \beta_n (1 - \beta_n) \|W_n z_n - z_n\|^2 \Big].
\end{aligned}
$$

From this, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \frac{(1-2\theta_n\gamma_2+(\theta_n\gamma_2)^2+\theta_n\gamma_1\rho)}{(1-\theta_n\gamma_1\rho)}\|x_n-p\|^2 \\
&+\frac{\theta_n}{(1-\theta_n\gamma_1\rho)}\Big[3M_2(1-\theta_n\gamma_2)^2\frac{\alpha_n}{\theta_n}||x_n-x_{n-1}||+2\langle\gamma_1 fp-Dp,x_{n+1}-p\rangle\Big] \\
&-\frac{(1-\theta_n\gamma_2)^2}{(1-\theta_n\gamma_1\rho)}\Big[\left(1-\frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2+\beta_n(1-\beta_n)\|W_nz_n-z_n\|^2\Big] \\
&=\frac{(1-2\theta_n\gamma_2+\theta_n\gamma_1\rho)}{(1-\theta_n\gamma_1\rho)}\|x_n-p\|^2+\frac{(\theta_n\gamma_2)^2}{(1-\theta_n\gamma_1\rho)}\|x_n-p\|^2 \\
&+\frac{\theta_n}{(1-\theta_n\gamma_1\rho)}\Big[3M_2(1-\theta_n\gamma_2)^2\frac{\alpha_n}{\theta_n}||x_n-x_{n-1}||+2\langle\gamma_1 fp-Dp,x_{n+1}-p\rangle\Big] \\
&-\frac{(1-\theta_n\gamma_2)^2}{(1-\theta_n\gamma_1\rho)}\Big[\left(1-\frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2+\beta_n(1-\beta_n)\|W_nz_n-z_n\|^2\Big] \\
&\le\left(1-\frac{2\theta_n(\gamma_2-\gamma_1\rho)}{(1-\theta_n\gamma_1\rho)}\right)\|x_n-p\|^2 \\
&+\frac{2\theta_n(\gamma_2-\gamma_1\rho)}{(1-\theta_n\gamma_1\rho)}\Big[\frac{\theta_n\gamma_2^2}{2(\gamma_2-\gamma_1\rho)}M_3+\frac{3M_2(1-\theta_n\gamma_2)^2}{2(\gamma_2-\gamma_1\rho)}\frac{\alpha_n}{\theta_n}||x_n-x_{n-1}|| \\
&+\frac{1}{(\gamma_2-\gamma_1\rho)}\langle\gamma_1 fp-Dp,x_{n+1}-p\rangle\Big] \\
&-\frac{(1-\theta_n\gamma_2)^2}{(1-\theta_n\gamma_1\rho)}\Big[\left(1-\frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2+\beta_n(1-\beta_n)\|W_nz_n-z_n\|^2\Big],
\end{aligned}$$

where $M_3=\sup\{\|x_n-p\|^2:n\in\mathbb{N}\}$ and thus we obtain the desired conclusion. □

We are now in the position to give the main theorem for Algorithm 3.3.2.

Theorem 3.3.9. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3.2 under Assumption 3.3.1. Suppose that $\{W_n\}$ is the sequence defined by (3.2.1). Then, the sequence $\{x_n\}$ converges strongly to a point $x^*\in\Gamma$, where $x^*=P_\Gamma(I-D+\gamma_1 f)x^*$ is a solution of the variational inequality*

$$\langle(D-\gamma_1 f)x^*,x^*-x\rangle\le 0,\ \forall x\in\Gamma.$$

Proof. Since $x^*=P_\Gamma(I-D+\gamma_1 f)x^*$, we obtain from Lemma 3.3.8 that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le(1-\eta_n)\|x_n-x^*\|^2 \\
&+\eta_n\Big[\frac{\theta_n\gamma_2^2}{2(\gamma_2-\gamma_1\rho)}M_3+\frac{3M_2(1-\theta_n\gamma_2)^2}{2(\gamma_2-\gamma_1\rho)}\frac{\alpha_n}{\theta_n}||x_n-x_{n-1}|| \\
&+\frac{1}{(\gamma_2-\gamma_1\rho)}\langle\gamma_1 fx^*-Dx^*,x_{n+1}-x^*\rangle\Big].
\end{aligned}\tag{3.3.14}$$

Now, we claim that the sequence $\{\|x_n-x^*\|\}$ converges to zero. To show this, by Lemma 2.5.36 it suffices to show that $\limsup_{k\to\infty}\langle\gamma_1 fx^*-Dx^*,x_{n_k+1}-x^*\rangle\le 0$ for every subsequence

$\{\|x_{n_k} - x^*\|\}$ of $\{\|x_n - x^*\|\}$ satisfying

$$\liminf_{k\to\infty} (\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|) \geq 0. \tag{3.3.15}$$

Suppose that $\{\|x_{n_k} - x^*\|\}$ is a subsequence of $\{\|x_n - x^*\|\}$ such that (3.3.15) holds. From Lemma 3.3.8, we obtain

$$\begin{aligned}\frac{(1-\theta_{n_k}\gamma_2)^2}{(1-\theta_{n_k}\gamma_1\rho)}\left(1 - \frac{\lambda_{n_k}^2\mu^2}{\lambda_{n_k+1}^2}\right)\|y_{n_k} - w_{n_k}\|^2 &\leq (1-\eta_{n_k})\|x_{n_k} - x^*\|^2 - \|x_{n_k+1} - x^*\|^2 \\ &+ \eta_{n_k}\Big[\frac{\theta_{n_k}\gamma_2^2}{2(\gamma_2 - \gamma_1\rho)}M_3 \\ &+ \frac{3M_2(1-\theta_{n_k}\gamma_2)^2}{2(\gamma_2-\gamma_1\rho)}\frac{\alpha_n}{\theta_{n_k}}||x_{n_k} - x_{n_k-1}|| \\ &+ \frac{1}{(\gamma_2-\gamma_1\rho)}\langle \gamma_1 f x^* - Dx^*, x_{n_k+1} - x^*\rangle\Big].\end{aligned}$$

By (3.3.15) and the fact that $\lim_{k\to\infty} \eta_{n_k} = 0$ (since $\lim_{k\to\infty} \theta_{n_k} = 0$), we obtain

$$\frac{(1-\theta_{n_k}\gamma_2)^2}{(1-\theta_{n_k}\gamma_1\rho)}\left(1 - \frac{\lambda_{n_k}^2\mu^2}{\lambda_{n_k+1}^2}\right)\|y_{n_k} - w_{n_k}\|^2 \to 0, \text{ as } k \to \infty.$$

Consequently, we have

$$\lim_{k\to\infty} \|y_{n_k} - w_{n_k}\| = 0. \tag{3.3.16}$$

Following similar argument, from Lemma 3.3.8 we have

$$\lim_{k\to\infty} \|W_{n_k} z_{n_k} - z_{n_k}\| = 0. \tag{3.3.17}$$

From the definition of z_{n_k} in **Step 4** and (3.3.16), we have

$$\begin{aligned}\|z_{n_k} - w_{n_k}\| &= \|y_{n_k} - \lambda_{n_k}(Ay_{n_k} - Aw_{n_k}) - w_{n_k}\| \\ &\leq \|y_{n_k} - w_{n_k}\| + \lambda_{n_k}\|Ay_{n_k} - Aw_{n_k}\| \\ &\leq \|y_{n_k} - w_{n_k}\| + \lambda_{n_k}\frac{\mu}{\lambda_{n_k+1}}\|w_{n_k} - y_{n_k}\| \\ &= \left(1 + \frac{\lambda_{n_k}\mu}{\lambda_{n_k+1}}\right)\|y_{n_k} - w_{n_k}\| \to 0, \text{ as } k \to \infty,\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \|z_{n_k} - w_{n_k}\| = 0. \tag{3.3.18}$$

From (3.3.16) and (3.3.18) we have

$$\lim_{k\to\infty} \|y_{n_k} - z_{n_k}\| = 0. \tag{3.3.19}$$

Also, from (3.3.17), (3.3.18) and (3.3.19), we get

$$\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\|=0 \qquad \lim_{k\to\infty}\|W_{n_k}z_{n_k}-w_{n_k}\|=0, \qquad \lim_{k\to\infty}\|W_{n_k}z_{n_k}-y_{n_k}\|=0. \tag{3.3.20}$$

Now, from **Step 2** and by Remark 3.3.3, we get

$$\lim_{k\to\infty}\|w_{n_k}-x_{n_k}\|=\lim_{k\to\infty}\alpha_{n_k}\|x_{n_k}-x_{n_k-1}\|=0. \tag{3.3.21}$$

From (3.3.18), (3.3.20) and (3.3.21), we obtain

$$\lim_{k\to\infty}\|z_{n_k}-x_{n_k}\|=0, \qquad \lim_{k\to\infty}\|W_{n_k}z_{n_k}-x_{n_k}\|=0. \tag{3.3.22}$$

By applying (3.3.17), we have

$$\begin{aligned}\|T_{n_k}z_{n_k}-z_{n_k}\| &= \|(1-\beta_{n_k})z_{n_k}+\beta_{n_k}W_{n_k}z_{n_k}-z_{n_k}\| \\ &\le (1-\beta_{n_k})\|z_{n_k}-z_{n_k}\|+\beta_{n_k}\|W_{n_k}z_{n_k}-z_{n_k}\|\to 0, \quad k\to\infty.\end{aligned} \tag{3.3.23}$$

Now, by using (3.3.18), (3.3.21) and (3.3.23) we obtain

$$\lim_{k\to\infty}||T_{n_k}z_{n_k}-w_{n_k}||=0, \qquad \lim_{k\to\infty}||T_{n_k}z_{n_k}-x_{n_k}||=0. \tag{3.3.24}$$

Consequently, by applying the fact that $\lim_{k\to\infty}\theta_{n_k}=0$ we get

$$\begin{aligned}\|x_{n_k+1}-x_{n_k}\| &= \|\theta_{n_k}\gamma_1 f x_{n_k}+(1-\theta_{n_k}D)T_{n_k}z_{n_k}-x_{n_k}\| \\ &= \|(\theta_{n_k}\gamma_1 f x_{n_k}-\theta_{n_k}Dx_{n_k})+(1-\theta_{n_k}D)(T_{n_k}z_{n_k}-x_{n_k})\| \\ &\le \theta_{n_k}\|\gamma_1 f x_{n_k}-Dx_{n_k}\|+(1-\theta_{n_k}\gamma_2)\|T_{n_k}z_{n_k}-x_{n_k}\|\to 0,\ k\to\infty.\end{aligned}$$

Hence

$$\lim_{k\to\infty}\|x_{n_k+1}-x_{n_k}\|=0. \tag{3.3.25}$$

Now, we show that $w_\omega(x_n)\subset\cap_{i=1}^{\infty}F(S_i)=F(W)$. Let $z\in w_\omega(x_n)$ and suppose that $z\notin F(W)$, that is, $Wz\neq z$. From (3.3.22), we have that $w_\omega(x_n)=w_\omega(z_n)$ and by Lemma 2.1.15 we have

$$\begin{aligned}\liminf_{k\to\infty}||z_{n_k}-z|| &< \liminf_{k\to\infty}|z_{n_k}-Wz||| \\ &\le \liminf_{k\to\infty}\{||z_{n_k}-Wz_{n_k}||+||Wz_{n_k}-Wz||\} \\ &\le \liminf_{k\to\infty}\{||z_{n_k}-Wz_{n_k}||+||z_{n_k}-z||\}.\end{aligned} \tag{3.3.26}$$

Since $x_{n_k}\in K$ for all $k\ge 1$ and $w_\omega(x_n)=w_\omega(z_n)$, we obtain

$$\begin{aligned}||Wz_{n_k}-z_{n_k}|| &\le ||Wz_{n_k}-W_{n_k}z_{n_k}||+||W_{n_k}z_{n_k}-z_{n_k}|| \\ &\le \sup_{x\in K}||Wx-W_{n_k}x||+||W_{n_k}z_{n_k}-z_{n_k}||.\end{aligned}$$

By applying Lemma 3.2.5 and (3.3.17), we have $\lim_{k\to\infty} ||Wz_{n_k} - z_{n_k}|| = 0$. Combining this with (3.3.26) yields

$$\liminf_{k\to\infty} ||z_{n_k} - z|| < \liminf_{k\to\infty} ||z_{n_k} - z||,$$

which is a contradiction. Hence, we have $z \in F(W) = \cap_{i=1}^{\infty} F(S_i)$, i.e., $w_\omega(x_n) \subset F(W) = \bigcap_{i=1}^{\infty} F(S_i)$.

Next, we show that $w_\omega(x_n) \subset VI(\mathcal{C}, A)$. By invoking Lemma 3.3.7, it follows from (3.3.20) that $z \in VI(\mathcal{C}, A)$. Thus, $w_\omega(x_n) \subset \Gamma$.

From the fact that $\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = 0$, we have that $w_\omega\{x_n\} = w_\omega\{z_n\}$. By Lemma 3.3.6, we have that $\{x_{n_k}\}$ is bounded which implies that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup \bar{x}$ and

$$\begin{aligned}\lim_{j\to\infty} \langle \gamma_1 f x^* - Dx^*, x_{n_{k_j}} - x^*\rangle &= \limsup_{k\to\infty} \langle \gamma_1 f x^* - Dx^*, x_{n_k} - x^*\rangle \\ &= \limsup_{k\to\infty} \langle \gamma_1 f x^* - Dx^*, z_{n_k} - x^*\rangle.\end{aligned}$$

Since $x^* = P_\Gamma(I - D + \gamma_1 f)x^*$, we have that

$$\begin{aligned}\limsup_{k\to\infty} \langle \gamma_1 f x^* - Dx^*, x_{n_k} - x^*\rangle &= \lim_{j\to\infty} \langle \gamma_1 f x^* - Dx^*, x_{n_{k_j}} - x^*\rangle \\ &= \langle \gamma_1 f x^* - Dx^*, \bar{x} - x^*\rangle \\ &\leq 0. \end{aligned} \tag{3.3.27}$$

From (3.3.25) and (3.3.27), we have

$$\begin{aligned}\limsup_{k\to\infty} \langle \gamma_1 f x^* - Dx^*, x_{n_k+1} - x^*\rangle &= \limsup_{k\to\infty} \langle \gamma_1 f x^* - Dx^*, x_{n_k} - x^*\rangle \\ &= \langle \gamma_1 f x^* - Dx^*, \bar{x} - x^*\rangle \\ &\leq 0. \end{aligned} \tag{3.3.28}$$

By applying Lemma 2.5.36 to (3.3.14) and using (3.3.28) together with the fact that $\lim_{n\to\infty} \theta_n = 0$ and $\lim_{n\to\infty} \frac{\alpha_n}{\theta_n} \|x_n - x_{n-1}\| = 0$, we have that $\lim_{n\to\infty} \|x_n - x^*\| = 0$. Therefore, $\{x_n\}$ converges strongly to x^*. □

Next, we propose our second algorithm which is a slight modification of Algorithm (3.3.2).

Algorithm 3.3.3. Inertial method with adaptive step size strategy.

Step 0: Choose sequences $\{\theta_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ such that condition (B5) holds and let $\lambda_1 > 0$, $\mu \in (0, 1)$, $\alpha \geq 3$ and $x_0, x_1 \in \mathcal{H}$ be given arbitrarily. Set $n := 1$.

Step 1: Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n := \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.}\end{cases} \tag{3.3.29}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Step 3: Construct the halfspace

$$\mathcal{C}_n = \{w \in \mathcal{H} \ |c(w_n) + \langle \xi_n, w - w_n\rangle \leq 0\}$$

and compute

$$y_n = P_{\mathcal{C}_n}(w_n - \lambda_n A w_n).$$

If $y_n = w_n$ ($Aw_n = 0$), then set $w_n = z_n$ and go to Step 5. Else go to Step 4.

Step 4: compute

$$z_n = y_n - \lambda_n(Ay_n - Aw_n).$$

Step 5: Compute

$$x_{n+1} = \theta_n \gamma_1 f w_n + (I - \theta_n D) T_n z_n,$$

where

$$T_n = (1-\beta_n)I + \beta_n W_n \quad \text{and} \quad W_n \ \text{ is the mapping defined by } \ (3.2.1).$$

Step 6: Compute

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu||w_n - y_n||}{||Aw_n - Ay_n||}, \ \lambda_n\right\}, & \text{if } Aw_n \neq Ay_n \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{3.3.30}$$

Set $n := n+1$ and go back to **Step 1**.

Theorem 3.3.10. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3.3 under Assumption 3.3.1. Suppose that $\{W_n\}$ is the sequence defined by (3.2.1). Then, the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$, where $x^* = P_\Gamma(I - D + \gamma_1 f)x^*$ is a solution of the variational inequality*

$$\langle (D - \gamma_1 f)x^*, x^* - x\rangle \leq 0, \ \forall x \in \Gamma.$$

Proof. The proof of this result follows similar argument with the proof of the result of Theorem (3.3.9). □

Taking $\gamma_1 = 1$ and $D = I$ in Theorem 3.3.9 (where I is the identity mapping), we obtain the following:

Algorithm 3.3.4. Inertial method with adaptive step size strategy.

Step 0: Choose sequences $\{\theta_n\}_{n=1}^\infty$ and $\{\tau_n\}_{n=1}^\infty$ such that condition (B5) holds and let $\lambda_1 > 0$, $\mu \in (0,1)$, $\alpha \geq 3$ and $x_0, x_1 \in \mathcal{H}$ be given arbitrarily. Set $n := 1$.

Step 1: Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n := \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Step 3: Construct the halfspace

$$\mathcal{C}_n = \{w \in \mathcal{H} \; |c(w_n) + \langle \xi_n, w - w_n \rangle \leq 0\}$$

and compute

$$y_n = P_{\mathcal{C}_n}(w_n - \lambda_n A w_n).$$

If $y_n = w_n$ ($Aw_n = 0$), then set $w_n = z_n$ and go to Step 5. Else go to Step 4.

Step 4: compute

$$z_n = y_n - \lambda_n(Ay_n - Aw_n).$$

Step 5: Compute

$$x_{n+1} = \theta_n f x_n + (I - \theta_n) T_n z_n,$$

where

$$T_n = (1 - \beta_n)I + \beta_n W_n \quad \text{and} \quad W_n \;\; \text{is the mapping defined by} \;\; (3.2.1).$$

Step 6: Compute

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu||w_n - y_n||}{||Aw_n - Ay_n||}, \; \lambda_n\right\}, & \text{if } Aw_n \neq Ay_n \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go back to **Step 1**.

Corollary 3.3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3.4 under Assumption 3.3.1. Suppose that $\{W_n\}$ is the sequence defined by (3.2.1). Then, the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$, where $x^* = P_\Gamma(f)x^*$ is a solution of the variational inequality*

$$\langle (I - f)x^*, x^* - x \rangle \leq 0, \; \forall x \in \Gamma.$$

Taking $\gamma_1 = 1$, $D = I$ (where I is the identity mapping) and $S_n = S$ for all $n \geq 1$ in Theorem 3.3.9, we obtain the following:

Algorithm 3.3.6. Inertial method with adaptive step size strategy.

Step 0: Choose sequences $\{\theta_n\}_{n=1}^\infty$ and $\{\tau_n\}_{n=1}^\infty$ such that condition (B5) holds and let $\lambda_1 > 0$, $\mu \in (0, 1)$, $\alpha \geq 3$ and $x_0, x_1 \in \mathcal{H}$ be given arbitrarily. Set $n := 1$.

Step 1: Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n := \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Step 3: Construct the halfspace

$$\mathcal{C}_n = \{w \in \mathcal{H} \ | c(w_n) + \langle \xi_n, w - w_n \rangle \leq 0\}$$

and compute

$$y_n = P_{\mathcal{C}_n}(w_n - \lambda_n A w_n).$$

If $y_n = w_n$ $(Aw_n = 0)$, then set $w_n = z_n$ and go to Step 5. Else go to Step 4.

Step 4: compute

$$z_n = y_n - \lambda_n(Ay_n - Aw_n).$$

Step 5: Compute

$$x_{n+1} = \theta_n f x_n + (I - \theta_n) T_n z_n,$$

where

$$T_n = (1 - \beta_n) I + \beta_n S.$$

Step 6: Compute

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu||w_n - y_n||}{||Aw_n - Ay_n||}, \ \lambda_n\right\}, & \text{if } Aw_n \neq Ay_n \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go back to **Step 1**.

Corollary 3.3.7. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3.6 under Assumption 3.3.1. Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$, where $x^* = P_\Gamma(f)x^*$ is a solution of the variational inequality*

$$\langle (I - f)x^*, x^* - x \rangle \leq 0, \ \forall x \in \Gamma.$$

3.3.3 Applications

The zero point problem

In this section, we apply our results to finding a common solution of VIP and ZPP for an infinite family of maximal monotone operators.

Let $F : \mathcal{H} \to \mathcal{H}$ be a single-valued nonlinear mapping and $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued mapping. The problem of finding a zero of the sum of two monotone operators, which is formulated as the following MIP is to find a point $x \in \mathcal{H}$ such that

$$0 \in (F + B)x. \tag{3.3.31}$$

This problem includes, as special cases, convex programming, VIP, SFP and MP. More precisely, some concrete problems in machine learning, image processing and linear inverse problem can be modeled mathematically as this form, see [86, 92, 99, 110]. We denote the zero point set $\{x \in \mathcal{H} : 0 \in (F + B)x\}$ of $F + B$ by $(F + B)^{-1}0$.

Definition 3.3.11. *Let $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a multivalued mapping. The effective domain of B denoted by $D(B)$ is given as $D(B) = \{x \in \mathcal{H} : Bx \neq \emptyset\}$.*

Definition 3.3.12. *Let $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued operator on $\mathcal{H}$. Then*

(1) the graph $G(B)$ is defined by

$$G(B) := \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in B(x)\};$$

(2) the operator B is said to be monotone if $\langle x-y, u-v\rangle \geq 0$ for all $x, y \in D(B), u \in Bx$, and $v \in By$;

(3) A monotone operator B on $\mathcal{H}$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on $\mathcal{H}$;

(4) For a maximal monotone multivalued mapping B on $\mathcal{H}$ and $r > 0$, the operator

$$J_r^B := (I + rB)^{-1} : \mathcal{H} \to D(B)$$

is called the resolvent of B.

Remark 3.3.13. In [101], it was shown that $F(J_r^B) = B^{-1}0 \equiv \{x \in \mathcal{H} : 0 \in Bx\}$ for all $r > 0$ and J_r^B is singled-valued and *firmly nonexpansive,* that is,

$$||J_r^B x - J_r^B y|| \leq \langle J_r^B x - J_r^B y, x - y\rangle, \quad \text{for all } x, y \in \mathcal{H}.$$

The following lemma will also be employed in establishing our results in this section.

Lemma 3.3.14. *[22] Let $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{H}, G : \mathcal{H} \to \mathcal{H}$ be a mapping and $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator. Then $F(J_r^B(I - rG)) = (G + B)^{-1}(0)$.*

Now, we have the following results.

Theorem 3.3.15. *Let $\mathcal{H}$ be a Hilbert space and suppose that $\{W_n\}$ is the sequence defined by (3.2.1). Let $B_i : \mathcal{H} \to 2^{\mathcal{H}}$ be an infinite family of maximal monotone mappings with $D(B_i) \neq \emptyset$ and $J_{r_i}^{B_i}$ is the resolvent of B_i for each $r_i > 0$. Suppose that $\{x_n\}$ is a sequence generated by Algorithm 3.3.2 such that Assumption 3.3.1 is satisfied. Then the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \Gamma = VI(\mathcal{C}, A) \bigcap \cap_{i=1}^{\infty}(B_i^{-1}0) \neq \emptyset$, where $\hat{x} = P_\Gamma(I - D + \gamma f)(\hat{x})$ is a solution of the variational inequality*

$$\langle (D - \gamma f)\hat{x}, \hat{x} - x\rangle \leq 0 \quad \textit{for all } x \in \Gamma.$$

Proof. Since $J_{r_i}^{B_i}$ is nonexpansive and $F(J_{r_i}^{B_i}) = B_i^{-1}0$, then the result follows from Theorem 3.3.9 by taking $J_{r_i}^{B_i} = S_i$ in Definition 3.2.3. □

Theorem 3.3.16. *Let $\mathcal{H}$ be a Hilbert space and suppose that $\{W_n\}$ is the sequence defined by (3.2.1). Let $B_i : \mathcal{H} \to 2^{\mathcal{H}}$ be an infinite family of maximal monotone mappings with $D(B_i) \neq \emptyset, J_{r_i}^{B_i}$ is the resolvent of B_i for each $r_i \in (0, 2\delta_i)$ and $F_i : \mathcal{H} \to \mathcal{H}$ be an infinite family of δ_i-inverse strongly monotone mappings. Suppose that $\{x_n\}$ is a sequence generated by Algorithm 3.3.2 such that Assumption 3.3.1 is satisfied. Then the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \Gamma = VI(\mathcal{C}, A) \bigcap \cap_{i=1}^{\infty}(B_i + F_i)^{-1}0 \neq \emptyset$, where $\hat{x} = P_\Gamma(I - D + \gamma f)(\hat{x})$ is a solution of the variational inequality*

$$\langle (D - \gamma f)\hat{x}, \hat{x} - x\rangle \leq 0 \quad \textit{for all } x \in \Gamma.$$

Proof. Since F_i is δ_i-inverse strongly monotone, $I - r_iF_i$ is nonexpansive. By the nonexpansiveness of $J_{r_i}^{B_i}$, it follows that $J_{r_i}^{B_i}(I - r_iF_i)$ is also nonexpansive. The proof follows from Theorem 3.3.9 by applying Lemma 3.3.14 and taking $J_{r_i}^{B_i}(I - r_iG_i) = S_i$ in Definition 3.2.3. □

3.3.4 Numerical experiments

In this section, using some test examples, we discuss the numerical behavior of Algorithm 3.3.2 as well as compare it with Appendix 3.3.19 proposed by Thong and Hieu [249], Appendix 3.3.20 proposed by Yang and Liu [266], the method of Shehu and Iyiola [223, Algorithm 1] (see Appendix 3.3.21) and the method of Fan and Qin [102, Algorithm 5] (see Appendix 3.3.22). We perform all implementations using Matlab 2016 (b), installed on a personal computer with Intel(R) Core(TM) i5-2600 CPU@2.30GHz and 8.00 Gb-RAM running on Windows 10 operating system. In Tables 3.3.1-3.3.2, “No. of Iter.” means the number of iterations.

In our computations, we define $h(x) = \|x\|^2$ and choose $\theta_n = \frac{1}{n+1}, \gamma_1 = 0.1, \beta_n = \frac{1}{2(1+\frac{1}{n})}$, $\tau_n = \frac{1}{(n+1)^3}, \lambda_1 = 0.8$, and $\mu = 0.7$. Furthermore, in the implementation, we define $\text{TOL}_n := \|x_{n+1} - x_n\|$ and use the stopping criterion $\text{TOL}_n < 10^{-2}$ for the iterative processes.

Example 3.3.17. Let $\mathcal{H} = \mathbb{R}^m$ with standard topology, $fx = \frac{x}{6}, Dx = x$ and $S_n : \mathcal{H} \to \mathcal{H}$ be a mapping defined by $S_nx := \frac{1}{5n}x \ \forall x \in \mathcal{H}$. Let $A : \mathbb{R}^m \to \mathbb{R}^m (m = 15, 30, 45, 60)$ be an operator defined as

$$Ax = Mx + q$$

and

$$M = NN^T + S^* + E$$

where N is an $m \times m$ matrix, S^* is an $m \times m$ skew-symmetric matrix, E is an $m \times m$ diagonal matrix with its diagonal entries being nonnegative and $q \in \mathbb{R}^m$. The feasible set $\mathcal{C}$ is given by

$$\mathcal{C} = \{x = (x_1, \cdots, x_m) \in \mathbb{R}^m : -2 \leq x_i \leq 5, i = 1, \cdots, m\}.$$

It is obvious that A is monotone and Lipschitz continuous with Lipschitz constant $L = \|M\|$. The entries of N, S^* are generated randomly in $[-2, 2]$ as well as the starting points

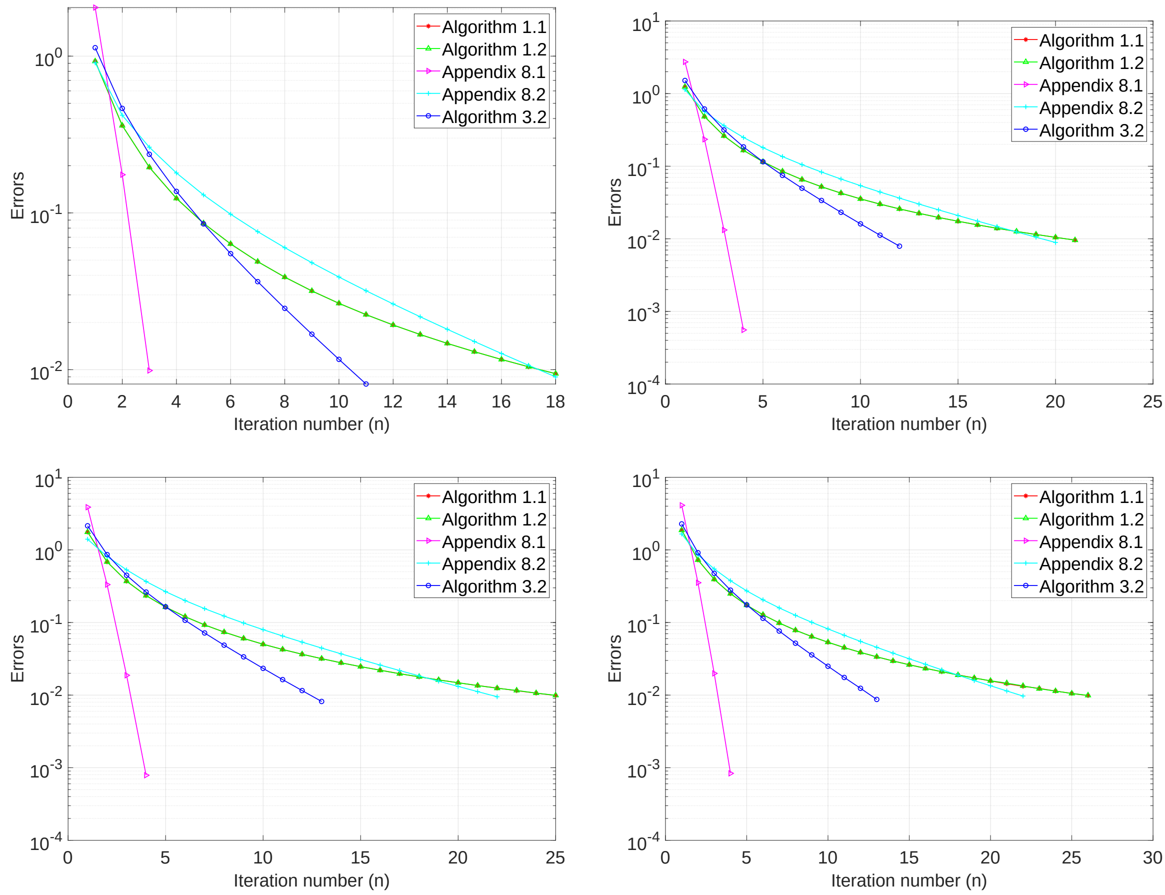


Figure 3.1: Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

x_0, x_1 and the diagonal entries of E are in $(0, 2)$, and q is taken as the zero vector in $\mathbb{R}^m$. We plot the graph of $\|x_{n+1} - x_n\|$ against number of iterations choosing $m = 15, 30, 45, 60$ and $\alpha = 7$. The numerical results are reported in Figure 3.1 and Table 3.3.1.

Table 3.3.1. Numerical results for Example 3.3.17.

		App. 3.3.19	App. 3.3.20	App. 3.3.21	App. 3.3.22	Alg. 3.3.2
m = 15	No. of Iter.	20	20	4	19	12
m = 30	No. of Iter.	20	20	4	19	12
m = 45	No. of Iter.	25	25	4	22	13
m = 60	No. of Iter.	13	21	4	21	13

Example 3.3.18. Let $\mathcal{H} = L^2([0,1])$ be endowed with inner product

$$\langle x, y \rangle = \int_0^1 x(t)y(t)dt, \quad \forall\ x, y \in \mathcal{H} \quad \text{and norm} \quad ||x|| := \Big(\int_0^1 |x(t)|^2 dt \Big)^{\frac{1}{2}}, \quad \forall\ x, y \in \mathcal{H}.$$

Now, define $A : \mathcal{H} \to \mathcal{H}$ by

$$A(x)(t) = \max\{0, x(t)\}, \quad \text{for all } t \in [0,1], x \in \mathcal{H}.$$

It is easy to see that A is monotone 1-Lipschitz continuous on $\mathcal{H}$. Let $f(x) = \frac{x}{2}(t), Dx = \int_0^1 tx(s)dx$ and $\mathcal{C} := \{x \in \mathcal{H} : \|x\| \leq 1\}$ be the unit ball. Then $VI(\mathcal{C}, A) = \{0\} \neq \emptyset$.
We define the mapping $S_n : L^2([0,1]) \to L^2([0,1])$ by

$$S_n x(t) = \int_0^1 t^n x(s)ds, \quad t \in [0,1].$$

Then, S_n is an infinite family of nonexpansive mappings. Indeed, we have

$$\begin{aligned}
|S_n x(t) - S_n y(t)|^2 &= |\int_0^1 t^n (x(s) - y(s))ds|^2 \\
&\leq \left(\int_0^1 t^n |x(s) - y(s)|ds \right)^2 \\
&\leq \int_0^1 |x(s) - y(s)|^2 ds \\
&= ||x - y||^2.
\end{aligned}$$

Thus, we obtain that

$$||S_n x - S_n y||^2 = \int_0^1 |S_n x(t) - S_n y(t)|^2 \leq ||x - y||^2.$$

Let $\{\zeta_n\}$ be a sequence of nonnegative real numbers defined by $\zeta_n = \{\frac{n}{n+1}\}$ for all $n \in \mathbb{N}$ and W_n be a mapping generated by $\{S_n\}$ and $\{\zeta_n\}$. We consider the following cases with $\alpha = 3$ for the numerical experiments of this example.
Case 1: Take $x_1(t) = 1 + t^2$ and $x_0(t) = t + 5$.

Case 2: Take $x_1(t) = t^2 + 1$ and $x_0(t) = \cos(t)$.

Case 3: Take $x_1(t) = t + 1$ and $x_0(t) = t + t^3$.

Case 4: Take $x_1(t) = t + 1$ and $x_0(t) = 2\sin(3t)$.
The numerical results are reported in Figure 3.2 and Table 3.3.2.

Table 3.3.2: Numerical results for Example 3.3.18

	App. 3.3.19	App. 3.3.20	App. 3.3.21	App. 3.3.22	Alg. 3.3.2
No. of Iter.	14	13	9	7	5
No. of Iter.	14	13	9	8	5
No. of Iter.	15	13	10	8	5
No. of Iter.	15	13	10	8	5

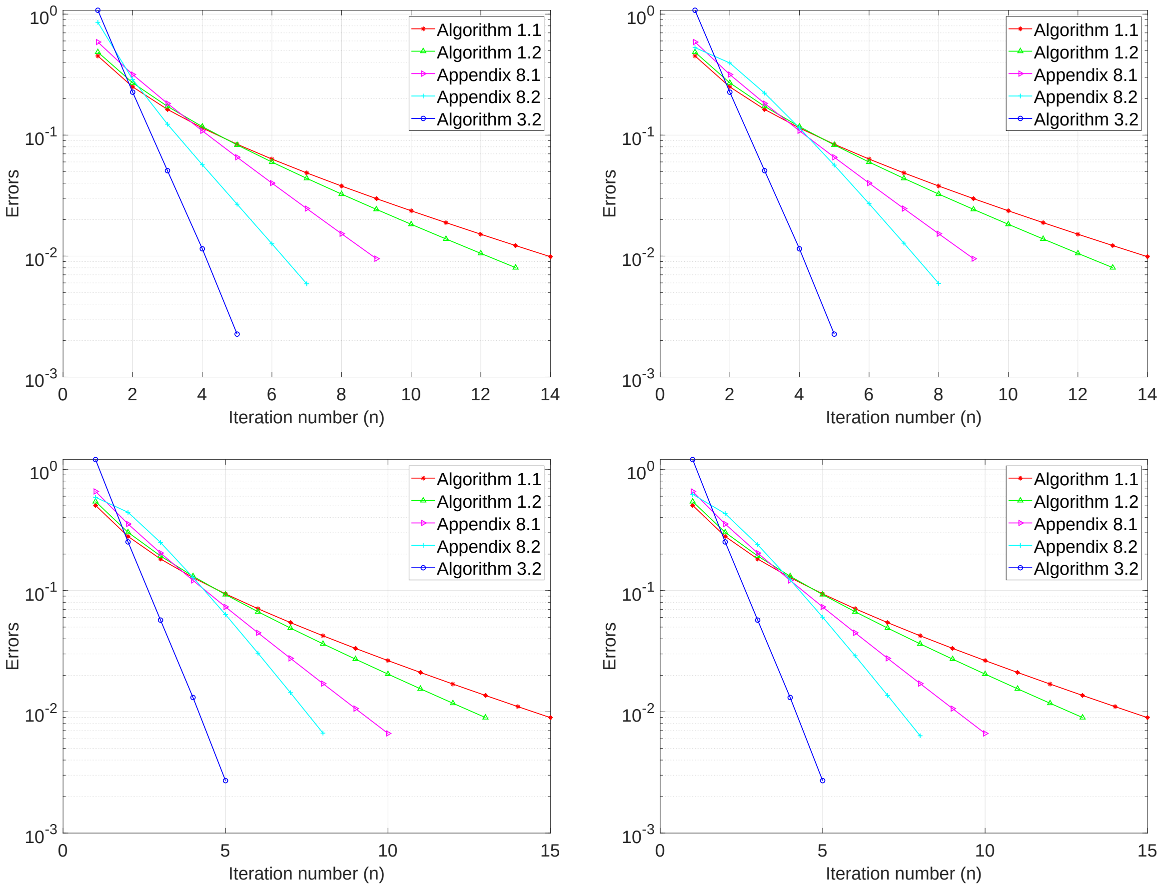


Figure 3.2: Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

Appendix 3.3.19. *The Algorithm in [249]*

Initialization: *Given* $\gamma > 0, l \in (0,1), \mu \in (0,1)$. *Let* $x_0 \in \mathcal{H}$ *be arbitrary.*

Iterative Steps: *Calculate* x_{n+1} *as follows:*

Step1. *Compute*

$$y_n = P_{\mathcal{C}}(x_n - \lambda_n A x_n),$$

where λ_n *is chosen to be the largest* $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ *satisfying*

$$\lambda \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|. \tag{3.3.32}$$

If $x_n = y_n$, *then stop and* x_n *is the solution of VIP* (1.2.1). *Otherwise,*

Step 2: *Compute*

$$x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n) z_n,$$

where $z_n = y_n - \lambda_n (Ay_n - Ax_n)$,
Set $n := n + 1$ *and return to* **Step 1**,

Appendix 3.3.20. *The Algorithm [266]*

Initialization: *Given* $\lambda_0 > 0, \mu \in (0,1)$. *Let* $x_0 \in \mathcal{H}$ *be arbitrary.*

Iterative Steps: *Calculate* x_{n+1} *as follows:*

Step1. *Given the current iterate* x_n, *compute*

$$y_n = P_{\mathcal{C}}(x_n - \lambda_n A x_n),$$

If $x_n = y_n$, *then stop and* x_n *is the solution of VIP* (1.2.1). *Otherwise,*

Step 2: *Compute*

$$x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n) z_n,$$

and

$$\lambda_{n+1} = \begin{cases} min\{\frac{\mu \|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n\}, & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n, & \text{otherwise} \end{cases}$$

where $z_n = y_n - \lambda_n (Ay_n - Ax_n)$,
Set $n := n + 1$ *and return to* **Step 1**

Appendix 3.3.21. *The Algorithm in [223].*

Initialization: *Given* $l \in (0,1), \mu \in (0,1)$. *Let* $x_0 \in \mathcal{H}$ *be arbitrary.*

Iterative Steps: *Calculate* x_{n+1} *as follows:*

Step1. *Compute*

$$y_n = P_{\mathcal{C}}(x_n - \lambda_n A x_n),$$

where $\lambda_n = l^{m_n}$, *where* m_n *is the smallest nonnegative integer* m *such that*

$$\lambda_n \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|. \tag{3.3.33}$$

If $x_n = y_n$, *then stop and* x_n *is the solution of VIP* (1.2.1). *Otherwise,*

Step 2: *Construct*

$$T_n = \{x \in \mathcal{H} : \langle x_n - \lambda_n A x_n - y_n, x - y_n \rangle \leq 0\}$$

and compute

$$z_n = P_{T_n}(x_n - \lambda_n A y_n).$$

Step 3: *Compute*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) z_n,$$

where $f : \mathcal{H} \to \mathcal{H}$ *is a contraction mapping with constant* $\rho \in [0, 1)$.

Set $n := n + 1$ *and return to* **Step 1**.

Appendix 3.3.22. *The Algorithm in [102].*

Initialization: *Given* $\gamma > 0, l \in (0, 1), \mu \in (0, 1)$. *Let* $x_0, x_1 \in \mathcal{H}$ *be arbitrary.*

Iterative Steps: *Calculate* x_{n+1} *as follows:*

Step1. *Set*

$$w_n = x_n + \alpha_n (x_n - x_{n-1})$$

and compute

$$y_n = P_{\mathcal{C}}(w_n - \lambda_n A x_n),$$

where λ_n *is chosen to be the largest* $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ *satisfying*

$$\lambda \|A w_n - A y_n\| \leq \mu \|w_n - y_n\|. \tag{3.3.34}$$

If $w_n = y_n$, *then stop and* y_n *is the solution of VIP* (1.2.1). *Otherwise,*

Step 2: *Compute*

$$x_{n+1} = \alpha_n f(x_n) + \gamma_n z_n + \beta_n w_n,$$

where $z_n = y_n - \lambda_n (A y_n - A w_n)$,
$A : \mathcal{H} \to \mathcal{H}$ *is monotone and Lipschitz continuous,* $f : \mathcal{H} \to \mathcal{H}$ *is a contraction with constant* $\rho \in [0, 1)$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Set $n := n + 1$ *and return to* **Step 1**.

3.4 On quasimonotone variational inequality problems

In this section, we present two new relaxed inertial subgradient extragradient methods for solving VIPs in a real Hilbert space. We establish the convergence of the sequence generated by these methods when the cost operator is quasimonotone and Lipschitz continuous, and when it is Lipschitz continuous without any form of monotonicity. The methods combine both the inertial and relaxation techniques in order to achieve high convergence speed. Furthermore, we present some experimental results to illustrate the profits gained from the relaxed inertial steps.

3.4.1 Proposed methods

In this section, we present our methods and discuss their features. We begin with the following assumptions under which we obtain our convergence results.

Assumption 3.4.1. *Conditions on the inertial and relaxation factors:*

Suppose that $\theta_n \in [0,1)$ *and* $\rho_n \in (0,1]$ *for all* $n \geq 1$ *such that* $\liminf_{n\to\infty} \rho_n > 0$. *Assume that there exists* $\epsilon \in (0,1)$ *such that for* n *large enough,*

$$\begin{cases} 2(1-\epsilon)\frac{1-\rho_n}{2\rho_n}(1-\theta_{n-1}) \geq \\ \theta_n t_{n+1}\Big(1+\theta_n+2\Big[\frac{1-\rho_n}{2\rho_n}(1-\theta_n)-\frac{1-\rho_{n-1}}{2\rho_{n-1}}(1-\theta_{n-1})\Big]_+\Big), & \text{if } \rho_n \in (0,1), \\ \\ (1-\epsilon)(1-\theta_{n-1}) \geq \theta_n t_{n+1}\Big(1+\theta_n+\Big[\theta_{n-1}-\theta_n\Big]_+\Big), & \text{if } \rho_n = 1. \end{cases} \tag{3.4.1}$$

Assumption 3.4.2. *We further make the following assumptions:*

(a) $\Gamma \neq \emptyset$,

(b) A *is Lipschitz-continuous on* $\mathcal{H}$ *with constant* $L > 0$,

(c) A *is sequentially weakly continuous on* $\mathcal{C}$,

(d) A *is quasimonotone on* $\mathcal{H}$,

(e) The set $\{z \in \mathcal{C} : Az = 0\} \setminus \Gamma$ *is a finite set.*

We now present the proposed methods of this section.

When the Lipschitz constant L is known, we present the following method for solving the VIP (1.2.1).

Algorithm 3.4.3. Relaxed inertial subgradient extragradient method with fixed stepsize.

Step 0: Choose sequences $\{\theta_n\}$ and $\{\rho_n\}$ such that $\theta_n \in [0,1)$ and $\rho_n \in (0,1]$ for all $n \geq 1$. Let $\lambda \in (0, \frac{1}{L})$ and $x_0, x_1 \in \mathcal{H}$ be given arbitrarily. Set $n := 1$.
Step 1: Given the current iterates x_{n-1} and x_n $(n \geq 1)$, compute

$$w_n = x_n + \theta_n(x_n - x_{n-1})$$

and

$$y_n = P_{\mathcal{C}}(w_n - \lambda A w_n).$$

If $w_n = y_n$: STOP. Otherwise, go to **Step 2.**
Step 2: Construct the half-space

$$T_n = \{x \in \mathcal{H} : \langle w_n - \lambda A w_n - y_n,\; x - y_n\rangle \leq 0\}.$$

Then, compute

$$z_n = P_{T_n}(w_n - \lambda A y_n)$$

and

$$x_{n+1} = (1-\rho_n)w_n + \rho_n z_n.$$

Set $n := n+1$ and return to **Step 1.**

In situations where the Lipschitz constant L is not known, we present the following method with adaptive stepsize for solving the VIP (1.2.1).

Algorithm 3.4.4. Relaxed inertial subgradient extragradient method with adaptive stepsize strategy.

Step 0: Choose sequences $\{\theta_n\}$ and $\{\rho_n\}$ such that $\theta_n \in [0,1)$ and $\rho_n \in (0,1]$ for all $n \geq 1$. Let $\lambda_1 > 0$, $\mu \in (0,1)$ and $x_0, x_1 \in \mathcal{H}$ be given arbitrarily. Choose a nonnegative real sequence $\{d_n\}$ such that $\sum\limits_{n=1}^{\infty} d_n < \infty$. Set $n := 1$.

Step 1: Given the current iterates x_{n-1} and x_n $(n \geq 1)$, compute

$$w_n = x_n + \theta_n(x_n - x_{n-1})$$

and

$$y_n = P_{\mathcal{C}}(w_n - \lambda_n A w_n).$$

If $w_n = y_n$: STOP. Otherwise, go to **Step 2.**

Step 2: Construct the half-space

$$T_n = \{x \in \mathcal{H} : \langle w_n - \lambda_n A w_n - y_n,\ x - y_n\rangle \leq 0\}.$$

Then, compute

$$z_n = P_{T_n}(w_n - \lambda_n A y_n)$$

and

$$x_{n+1} = (1-\rho_n)w_n + \rho_n z_n,$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right)}{2\langle Aw_n - Ay_n, z_n - y_n\rangle},\ \lambda_n + d_n\right\}, & \text{if } \langle Aw_n - Ay_n, z_n - y_n\rangle > 0, \\ \lambda_n + d_n, & \text{otherwise.} \end{cases} \tag{3.4.2}$$

Set $n := n+1$ and return to **Step 1.**

Remark 3.4.1. In a special case when $\rho_n = 1, \ \forall n \geq 1$, we have that $x_{n+1} = z_n$. Thus, Algorithms 3.4.3 and 3.4.4 reduce to the inertial version of the subgradient extragradient method of Censor *et al.* [58–60]. In such case, we have the following simple criteria which guarantee Assumption 3.4.1, as well as assumption (2.5.37).

Proposition 3.4.5. *Assume that $\{\theta_n\}$ is a nondecreasing sequence that satisfies $\theta_n \in [0,1) \ \forall n \geq 1$ with $\lim_{n\to\infty} \theta_n = \theta$ such that the following condition holds:*

$$1 - 3\theta > 0. \tag{3.4.3}$$

Then, assumption (2.5.37) and Assumption 3.4.1 hold.

Proof. Observe that $\theta_n \leq \theta, \quad \forall n \geq 1$. Thus, we have that assumption (2.5.37) is satisfied and $t_n \leq \frac{1}{1-\theta}, \ \forall n \geq 1$ (see [29]). Now, observe that $1-3\theta > 0$ implies that $(1-\theta) > \frac{\theta(1+\theta)}{1-\theta}$. This further implies that there exists $\epsilon \in (0,1)$ such that

$$(1-\epsilon)(1-\theta) \geq \frac{\theta(1+\theta)}{1-\theta}. \tag{3.4.4}$$

Since $\theta_n \leq \theta, \ \forall n \geq 1$, we obtain from (3.4.4) that

$$(1-\epsilon)(1-\theta_{n-1}) \geq \frac{\theta(1+\theta)}{1-\theta} \geq \theta_n t_{n+1}(1+\theta_n), \tag{3.4.5}$$

for some $\epsilon \in (0,1)$. Since $\theta_{n-1} \leq \theta_n, \ \forall n \geq 1$, we obtain that

$$\theta_n t_{n+1}(1+\theta_n) = \theta_n t_{n+1}(1+\theta_n + [\theta_{n-1} - \theta_n]_+).$$

Combining this with (3.4.5), we get that Assumption 3.4.1 is satisfied. □

Proposition 3.4.6. *Suppose that $\theta_n \in [0,1), \ \forall n \geq 1$ and there exists $c \in [0, \frac{1}{2})$ such that*

$$\lim_{n\to\infty} \left(\frac{1}{1-\theta_{n+1}} - \frac{1}{1-\theta_n} \right) = c \tag{3.4.6}$$

and

$$\liminf_{n\to\infty} (1-\theta_n)^2 > \limsup_{n\to\infty} \frac{\theta_n(1+\theta_n)}{1-2c}. \tag{3.4.7}$$

Then, assumption (2.5.37) and Assumption 3.4.1 hold.

Proof. It follows from Proposition 2.5.32(i) that assumption (2.5.37) holds.
Now, from (3.4.7), we obtain that

$$\liminf_{n\to\infty} (1-\theta_{n-1})^2 > \limsup_{n\to\infty} \frac{\theta_n(1+\theta_n)}{1-2c}. \tag{3.4.8}$$

Thus, there exists $\epsilon \in (0,1)$ sufficiently small enough such that

$$\liminf_{n\to\infty} (1-\theta_{n-1})^2 > \limsup_{n\to\infty} \frac{\theta_n(1+\theta_n)}{1-2c-\epsilon(1-c)} > \limsup_{n\to\infty} \frac{\theta_n(1+\theta_n)}{1-2c}. \tag{3.4.9}$$

This implies that

$$\begin{aligned}(1+o(1))\theta_n(1+\theta_n) &\le [1-2c-\epsilon(1-c)+o(1)](1-\theta_{n-1})^2\\ &= [(1-\epsilon)(1-c)-(2c-c+o(1))](1-\theta_{n-1})^2\\ &\le [(1-\epsilon)(1-c)-\theta_n(c+o(1))](1-\theta_{n-1})^2,\end{aligned}$$

which implies that

$$(1-\epsilon)(1-c)(1-\theta_{n-1})^2 \ge (1+o(1))\theta_n\left(1+\theta_n+(1-\theta_{n-1})^2+o\left((1-\theta_{n-1})^2\right)\right). \tag{3.4.10}$$

Now, observe from (3.4.6) that

$$\theta_{n-1}-\theta_n+c(1-\theta_{n-1})(1-\theta_n) = o\left((1-\theta_{n-1})(1-\theta_n)\right),$$

which implies from Proposition 2.5.32(ii) that

$$\begin{aligned}\theta_{n-1}-\theta_n &= -c(1-\theta_{n-1})(1-\theta_n)+o\left((1-\theta_{n-1})(1-\theta_n)\right)\\ &= -c(1-\theta_{n-1})^2+o(1-\theta_{n-1})^2 \quad \text{as} \quad n\to\infty.\end{aligned}$$

This implies that

$$\begin{aligned}|\theta_{n-1}-\theta_n| &= |-c(1-\theta_{n-1})^2+o(1-\theta_{n-1})^2|\\ &\le c(1-\theta_{n-1})^2+o(1-\theta_{n-1})^2 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{3.4.11}$$

Combining (3.4.10) and (3.4.11), we obtain that

$$(1-\epsilon)(1-c)(1-\theta_{n-1})^2 \ge (1+o(1))\,\theta_n\left(1+\theta_n+[\theta_{n-1}-\theta_n]_+\right). \tag{3.4.12}$$

By Proposition 2.5.32, we have that $t_{n+1}\sim t_n\sim \frac{1}{(1-c)(1-\theta_{n-1})}$ as $n\to\infty$.

Hence, (3.4.12) is equivalent to

$$(1-\epsilon)(1-c)(1-\theta_{n-1})^2 \ge \frac{\theta_n}{(1-c)(1-\theta_{n-1})}t_{n+1}\left(1+\theta_n+[\theta_{n-1}-\theta_n]_+\right),$$

which further implies Assumption 3.4.1. □

When $\rho_n\neq 1$, then we have the following proposition which provides some criteria for ensuring Assumption 3.4.1, as well as assumption (2.5.37).

Proposition 3.4.7. *(See for example, [29, Proposition 3.3]).*
Suppose that $\theta_n\in[0,1)$ and $\rho_n\in(0,1)$ for all $n\ge 1$. Assume that there exist $c\in[0,1)$ and $\bar{c}\in[c,1)$ such that

$$\lim_{n\to\infty}\left(\frac{1}{1-\theta_{n+1}}-\frac{1}{1-\theta_n}\right)=c,$$

$$\lim_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\gamma_n(1-\theta_n)} = \bar{c}$$

and

$$\liminf_{n\to\infty} \gamma_n(1-\theta_n)^2 > \limsup_{n\to\infty} \frac{\theta_n(1+\theta_n)}{1-\bar{c}},$$

where $\gamma_n = \frac{1-\rho_n}{2\rho_n}$*. Then, assumption* (2.5.37) *and Assumption 3.4.1 hold.*

Remark 3.4.2. It is worthy of note that many practical choices for the inertial and relaxation factors θ_n and ρ_n, respectively, satisfy Assumption 3.4.1. In fact, similar conditions as in Propositions 3.4.5-3.4.7 have already been used in the literature to ensure the convergence of inertial and relaxation methods (see [165, 246, 247] and the references therein). Thus, Assumption 3.4.1, as well as assumption (2.5.37) are much more weaker than the assumptions in those works in the literature.
Moreover, we shall give in Section 3.4.3, some typical examples of θ_n and ρ_n which satisfy the conditions in Propositions 3.4.5-3.4.7 (therefore, satisfying assumption (2.5.37) and Assumption 3.4.1). Then, we check the sensitivity of both θ_n and ρ_n in order to find numerically, the optimum choice of these parameters with respect to the convergence speed of our proposed methods.

3.4.2 Convergence analysis

Lemma 3.4.3. *Let* $\{x_n\}$ *be a sequence generated by Algorithm 3.4.3 such that Assumption 3.4.2(a)-(b) hold. Then,*

$$\begin{aligned}&\Gamma_{n+1} - \Gamma_n - \theta_n(\Gamma_n - \Gamma_{n-1})\\ &\le \frac{1}{2}(\theta_n + \theta_n^2)\|x_n - x_{n-1}\|^2 - \gamma_n\|x_{n+1} - w_n\|^2 - \frac{\rho_n}{2}(1-\lambda L)\left[||w_n - y_n||^2 + ||z_n - y_n||^2\right],\end{aligned}$$

where $\gamma_n := \frac{1-\rho_n}{2\rho_n}$ *and* $\Gamma_n := \frac{1}{2}\|x_n - z\|^2,\ \forall z \in \Gamma.$

Proof. Let $z \in \Gamma$. Since $x_{n+1} - w_n = \rho_n(z_n - w_n)$, we obtain

$$\begin{aligned}\langle x_{n+1} - w_n,\, w_n - z\rangle &= \rho_n\langle z_n - w_n,\, w_n - z_n\rangle + \rho_n\langle z_n - w_n,\, z_n - z\rangle\\ &= -\rho_n\|z_n - w_n\|^2 + \rho_n\langle z_n - w_n,\, z_n - z\rangle\\ &= -\rho_n^{-1}\|x_{n+1} - w_n\|^2 + \rho_n\langle z_n - w_n + \lambda A y_n,\, z_n - z\rangle\\ &\quad - \lambda\rho_n\langle Ay_n,\, z_n - z\rangle\\ &\le -\rho_n^{-1}\|x_{n+1} - w_n\|^2 + \lambda\rho_n\langle Ay_n,\, z - z_n\rangle, \qquad (3.4.13)\end{aligned}$$

where the last inequality follows from $z \in \Gamma \subseteq T_n$ and the characteristic property of P_{T_n} (see (2.4.1)).

Now, using (3.4.13) in Lemma 2.5.14, we obtain

$$
\begin{aligned}
&\Gamma_{n+1}-\Gamma_n-\theta_n(\Gamma_n-\Gamma_{n-1})\\
&\leq \frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2-\rho_n^{-1}\|x_{n+1}-w_n\|^2+\lambda\rho_n\langle Ay_n,\ z-z_n\rangle+\frac{1}{2}\|x_{n+1}-w_n\|^2\\
&=\frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2+\frac{\rho_n-1}{2\rho_n}\|x_{n+1}-w_n\|^2-\frac{1}{2\rho_n}\|x_{n+1}-w_n\|^2\\
&+\lambda\rho_n\langle Ay_n,\ z-z_n\rangle\\
&=\frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2+\frac{\rho_n-1}{2\rho_n}\|x_{n+1}-w_n\|^2+\lambda\rho_n\langle Ay_n,\ z-z_n\rangle\\
&-\frac{1}{2\rho_n}\cdot\rho_n^2\|w_n-z_n\|^2\\
&=\frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2+\frac{\rho_n-1}{2\rho_n}\|x_{n+1}-w_n\|^2+\lambda\rho_n\langle Ay_n,\ z-z_n\rangle \qquad (3.4.14)\\
&-\frac{\rho_n}{2}\|w_n-y_n\|^2-\frac{\rho_n}{2}\|y_n-z_n\|^2-\rho_n\langle w_n-y_n,\ y_n-z_n\rangle,
\end{aligned}
$$

where the last equality follows from Lemma 2.1.1.

Since $y_n\subset\mathcal{C}$ and $z\in\Gamma$, we get from (3.2.3) that $\langle Ay_n,\ y_n-z\rangle\geq 0,\forall n\geq 1$. That is $\langle Ay_n,\ y_n-z_n+z_n-z\rangle\geq 0,\forall n\geq 1$. This implies that

$$
\begin{aligned}
\lambda\langle Ay_n,\ z-z_n\rangle-\langle w_n-y_n,\ y_n-z_n\rangle &\leq \lambda\langle Ay_n,\ y_n-z_n\rangle-\langle w_n-y_n,\ y_n-z_n\rangle\\
&=\langle\lambda Ay_n-w_n+y_n,\ y_n-z_n\rangle. \qquad (3.4.15)
\end{aligned}
$$

Substituting (3.4.15) into (3.4.14), we obtain that

$$
\begin{aligned}
&\Gamma_{n+1}-\Gamma_n-\theta_n(\Gamma_n-\Gamma_{n-1})\\
\leq\ &\frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2+\frac{\rho_n-1}{2\rho_n}\|x_{n+1}-w_n\|^2\\
&-\frac{\rho_n}{2}\left(\|w_n-y_n\|^2+\|y_n-z_n\|^2\right)+\rho_n\langle\lambda Ay_n-w_n+y_n,\ y_n-z_n\rangle\\
=\ &\frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2+\frac{\rho_n-1}{2\rho_n}\|x_{n+1}-w_n\|^2-\frac{\rho_n}{2}\left(\|w_n-y_n\|^2+\|y_n-z_n\|^2\right)\\
&+\rho_n\langle w_n-\lambda Aw_n-y_n,\ z_n-y_n\rangle+\lambda\rho_n\langle Aw_n-Ay_n,\ z_n-y_n\rangle\\
\leq\ &\frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2+\frac{\rho_n-1}{2\rho_n}\|x_{n+1}-w_n\|^2 \qquad (3.4.16)\\
-\ &\frac{\rho_n}{2}\left(\|w_n-y_n\|^2+\|y_n-z_n\|^2\right)+\lambda\rho_n\langle Aw_n-Ay_n,\ z_n-y_n\rangle,
\end{aligned}
$$

where the last inequality follows from the definition of T_n.

Now, from the Lipschitz continuity of A, we obtain

$$
\begin{aligned}
\langle Aw_n-Ay_n,z_n-y_n\rangle &\leq L||w_n-y_n||||z_n-y_n||\\
&= \frac{L}{2}\Big[||w_n-y_n||^2+||z_n-y_n||^2\\
&- (||w_n-y_n||-||z_n-y_n||)^2\Big]\\
&\leq \frac{L}{2}\left(||w_n-y_n||^2+||z_n-y_n||^2\right). \qquad (3.4.17)
\end{aligned}
$$

Substituting (3.4.17) into (3.4.16), we obtain the desired conclusion. $\square$

Lemma 3.4.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.4.3 such that assumption (2.5.37) and Assumption 3.4.2(a)-(b) hold. Then, the following inequality holds:*

$$\sum_{i=1}^{n-1} t_{i+1,n}\Big[\big(2\gamma_i(1-\theta_i)^2 - (\theta_i+\theta_i^2)\big)\|x_i - x_{i-1}\|^2 + 2\gamma_i(1-\theta_i)\Big(\|x_{i+1}-x_i\|^2 - \|x_i - x_{i-1}\|^2\Big)\Big] \leq 2t_1|\Gamma_1 - \Gamma_0| + 2\Gamma_0,$$

where $t_{i,n}$ is as defined in (2.5.40).

Proof. By Lemma 2.1.1, we obtain

$$\begin{aligned}
\|x_{n+1}-w_n\|^2 &= \|x_{n+1}-x_n-(x_n-x_{n-1})+(1-\theta_n)(x_n-x_{n-1})\|^2\\
&= \|x_{n+1}-2x_n+x_{n-1}\|^2+(1-\theta_n)^2\|x_n-x_{n-1}\|^2\\
&\quad +2(1-\theta_n)\langle x_{n+1}-2x_n+x_{n-1}, x_n-x_{n-1}\rangle\\
&= \|x_{n+1}-2x_n+x_{n-1}\|^2+(1-\theta_n)^2\|x_n-x_{n-1}\|^2\\
&\quad +(1-\theta_n)\left[\|x_{n+1}-x_n\|^2-\|x_n-x_{n-1}\|^2-\|x_{n+1}-2x_n+x_{n-1}\|^2\right]\\
&= \theta_n\|x_{n+1}-2x_n+x_{n-1}\|^2+(1-\theta_n)^2\|x_n-x_{n-1}\|^2\\
&\quad +(1-\theta_n)\left[\|x_{n+1}-x_n\|^2-\|x_n-x_{n-1}\|^2\right]\\
&\geq (1-\theta_n)^2\|x_n-x_{n-1}\|^2\\
&\quad + (1-\theta_n)\left[\|x_{n+1}-x_n\|^2-\|x_n-x_{n-1}\|^2\right].
\end{aligned} \tag{3.4.18}$$

Using (3.4.18) in Lemma 3.4.3, and noting that $1-\lambda L>0$, we obtain

$$\begin{aligned}
\Gamma_{n+1}-\Gamma_n-\theta_n(\Gamma_n-\Gamma_{n-1}) &\leq \frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2-\gamma_n(1-\theta_n)^2\|x_n-x_{n-1}\|^2\\
&\quad -\gamma_n(1-\theta_n)\Big[\|x_{n+1}-x_n\|^2-\|x_n-x_{n-1}\|^2\Big]\\
&= \Big[\frac{1}{2}(\theta_n+\theta_n^2)-\gamma_n(1-\theta_n)^2\Big]\|x_n-x_{n-1}\|^2\\
&\quad -\gamma_n(1-\theta_n)\Big[\|x_{n+1}-x_n\|^2-\|x_n-x_{n-1}\|^2\Big].
\end{aligned}$$

This together with Lemma 2.5.13(a) imply that

$$\begin{aligned}
\Gamma_n-\Gamma_0 &= \sum_{i=1}^{n}(\Gamma_i-\Gamma_{i-1})\\
&\leq t_{1,n}(\Gamma_1-\Gamma_0)\\
&\quad +\sum_{i=1}^{n-1} t_{i+1,n}\Big[\Big(\frac{1}{2}(\theta_i+\theta_i^2)-\gamma_i(1-\theta_i)^2\Big)\|x_i-x_{i-1}\|^2\Big]\\
&\quad -\sum_{i=1}^{n-1} t_{i+1,n}\Big[\gamma_i(1-\theta_i)\Big(\|x_{i+1}-x_i\|^2-\|x_i-x_{i-1}\|^2\Big)\Big]
\end{aligned}$$

Noting that $t_{1,n} \le t_1$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n-1} t_{i+1,n}\Big[\left(2\gamma_i(1-\theta_i)^2 - (\theta_i+\theta_i^2)\right) \|x_i - x_{i-1}\|^2 \\
&+ 2\gamma_i(1-\theta_i)\Big(\|x_{i+1}-x_i\|^2 - \|x_i - x_{i-1}\|^2\Big)\Big] \\
&\le 2t_{1,n}(\Gamma_1 - \Gamma_0) + 2(\Gamma_0 - \Gamma_n) \\
&\le 2t_1|\Gamma_1 - \Gamma_0| + 2\Gamma_0 - 2\Gamma_n.
\end{aligned}$$

Using $\Gamma_n = \frac{1}{2}\|x_n - z\|^2 \ge 0, \ \forall n \ge 1$ in the above inequality, we obtain the desired conclusion. □

Lemma 3.4.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.4.3 such that assumption (2.5.37) and Assumption 3.4.2(a)-(b) hold. Then, the following inequality holds:*

$$\begin{aligned}
\sum_{i=1}^{n-1} 2\gamma_{i-1}(1-\theta_{i-1}) - \theta_i t_{i+1}\Big(1+\theta_i + 2\Big[\gamma_i(1-\theta_i) - \gamma_{i-1}(1-\theta_{i-1})\Big]_+\Big)\|x_i - x_{i-1}\|^2 \\
\le 2\Big[t_1|\Gamma_1 - \Gamma_0| + \Gamma_0 + t_1\gamma_0(1-\theta_0)\|x_1 - x_0\|^2\Big].
\end{aligned}$$

Proof. Observe that

$$\begin{aligned}
&\sum_{i=1}^{n-1} t_{i+1,n}\cdot 2\gamma_i(1-\theta_i)\Big[\|x_{i+1}-x_i\|^2 - \|x_i - x_{i-1}\|^2\Big] \\
&= 2\sum_{i=1}^{n-1}\Big(t_{i,n}\gamma_{i-1}(1-\theta_{i-1}) - t_{i+1,n}\gamma_i(1-\theta_i)\Big)\|x_i - x_{i-1}\|^2 \\
&+ 2t_{1,n}\gamma_{n-1}(1-\theta_{n-1})\|x_n - x_{n-1}\|^2 - 2t_{1,n}\gamma_0(1-\theta_0)\|x_1 - x_0\|^2 \\
&\ge 2\sum_{i=1}^{n-1}\Big(t_{i,n}\gamma_{i-1}(1-\theta_{i-1}) - t_{i+1,n}\gamma_i(1-\theta_i)\Big)\|x_i - x_{i-1}\|^2 \\
&- 2t_{1,n}\gamma_0(1-\theta_0)\|x_1 - x_0\|^2 \\
&\ge 2\sum_{i=1}^{n-1}\Big(t_{i,n}\gamma_{i-1}(1-\theta_{i-1}) - t_{i+1,n}\gamma_i(1-\theta_i)\Big)\|x_i - x_{i-1}\|^2 \\
&- 2t_1\gamma_0(1-\theta_0)\|x_1 - x_0\|^2,
\end{aligned} \tag{3.4.19}$$

where the last inequality follows from $t_{1,n} \le t_1$.

Now, using (3.4.19) in Lemma 3.4.4, we obtain that

$$\begin{aligned}
&\sum_{i=1}^{n-1} t_{i+1,n}\big(2\gamma_i(1-\theta_i)^2 - (\theta_i+\theta_i^2)\big)\|x_i - x_{i-1}\|^2 \\
&+2\sum_{i=1}^{n-1}\big(t_{i,n}\gamma_{i-1}(1-\theta_{i-1}) - t_{i+1,n}\gamma_i(1-\theta_i)\big)\|x_i - x_{i-1}\|^2 \\
&\le 2t_1|\Gamma_1 - \Gamma_0| + 2\Gamma_0 + 2t_1\gamma_0(1-\theta_0)\|x_1 - x_0\|^2.
\end{aligned}$$

That is

$$\sum_{i=1}^{n-1}\left[t_{i+1,n}(2\gamma_i(1-\theta_i)^2-(\theta_i+\theta_i^2)-2\gamma_i(1-\theta_i))+2t_{i,n}\gamma_{i-1}(1-\theta_{i-1})\right]\|x_i-x_{i-1}\|^2$$
$$\leq 2\Big[t_1|\Gamma_1-\Gamma_0|+\Gamma_0+t_1\gamma_0(1-\theta_0)\|x_1-x_0\|^2\Big]. \tag{3.4.20}$$

Now, recall from (2.5.41) that $t_{i,n}=1+\theta_i t_{i+1,n}$ for all $i\geq 1$ and $n\geq i+1$. Hence, we have

$$2t_{i,n}\gamma_{i-1}(1-\theta_{i-1})=2\Big[\gamma_{i-1}(1-\theta_{i-1})+\theta_i t_{i+1,n}\gamma_{i-1}(1-\theta_{i-1})\Big].$$

This implies that

$$\begin{aligned}
&t_{i+1,n}\Big[2\gamma_i(1-\theta_i)^2-(\theta_i+\theta_i^2)-2\gamma_i(1-\theta_i)\Big]+2t_{i,n}\gamma_{i-1}(1-\theta_{i-1})\\
&=t_{i+1,n}\big[2\gamma_i(1-\theta_i)^2-(\theta_i+\theta_i^2)-2\gamma_i(1-\theta_i)+2\theta_i\gamma_{i-1}(1-\theta_{i-1})\big]\\
&+2\gamma_{i-1}(1-\theta_{i-1})\\
&=2\gamma_{i-1}(1-\theta_{i-1})+t_{i+1,n}\Big(-2\gamma_i\theta_i(1-\theta_i)-(\theta_i+\theta_i^2)+2\theta_i\gamma_{i-1}(1-\theta_{i-1})\Big)\\
&=2\gamma_{i-1}(1-\theta_{i-1})-\theta_i t_{i+1,n}\Big(2\gamma_i(1-\theta_i)+1+\theta_i-2\gamma_{i-1}(1-\theta_{i-1})\Big)\\
&\geq 2\gamma_{i-1}(1-\theta_{i-1})-\theta_i t_{i+1,n}\Big(1+\theta_i+2\Big[\gamma_i(1-\theta_i)-\gamma_{i-1}(1-\theta_{i-1})\Big]_+\Big)\\
&\geq 2\gamma_{i-1}(1-\theta_{i-1})-\theta_i t_{i+1}\Big(1+\theta_i+2\big[\gamma_i(1-\theta_i)-\gamma_{i-1}(1-\theta_{i-1})\big]_+\Big),
\end{aligned} \tag{3.4.21}$$

where the last inequality follows from $t_{i+1,n}\leq t_{i+1}$.
Now, using (3.4.21) in (3.4.20), we obtain that the desired conclusion. □

Lemma 3.4.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.4.3 such that assumption (2.5.37), Assumption 3.4.1 and Assumption 3.4.2(a)-(b) hold. Then,*

$$\sum_{n=1}^{\infty}\theta_n t_{n+1}\|x_n-x_{n-1}\|^2<\infty.$$

Proof. Without loss of generality, we may assume that inequality (3.4.1) holds true for all $n\geq 1$. That is,

$$\begin{cases}2\epsilon\gamma_{n-1}(1-\theta_{n-1})\leq\\ 2\gamma_{n-1}(1-\theta_{n-1})-\theta_n t_{n+1}\Big(1+\theta_n+2\Big[\gamma_n(1-\theta_n)-\gamma_{n-1}(1-\theta_{n-1})\Big]_+\Big), & \text{if } \rho_n\in(0,1),\\[2ex] \epsilon(1-\theta_{n-1})\leq\\ (1-\theta_{n-1})-\theta_n t_{n+1}\Big(1+\theta_n+\Big[\theta_{n-1}-\theta_n\Big]_+\Big), & \text{if } \rho_n=1,\end{cases} \tag{3.4.22}$$

where $\gamma_n=\frac{1-\rho_n}{2\rho_n}$, for all $n\geq 1$.

At this point, we divide our proof into two cases.

Case 1: Suppose that $\rho_n \in (0,1)$, $\forall n \geq 1$. Then, using (3.4.22) in Lemma 3.4.5, we obtain

$$\sum_{i=1}^{n-1} \epsilon\gamma_{i-1}(1-\theta_{i-1})\|x_i - x_{i-1}\|^2 \leq t_1|\Gamma_1 - \Gamma_0| + \Gamma_0 + t_1\gamma_0(1-\theta_0)\|x_1 - x_0\|^2. \quad (3.4.23)$$

Now, taking limit as $n \to \infty$ in (3.4.23), we obtain that

$$\sum_{i=1}^{\infty} \gamma_{i-1}(1-\theta_{i-1})\|x_i - x_{i-1}\|^2 < \infty. \quad (3.4.24)$$

Again, from (3.4.22), we obtain that

$$\begin{aligned}\frac{1}{2}\theta_n t_{n+1} &\leq \gamma_{n-1}(1-\theta_{n-1}) - \frac{1}{2}\theta_n t_{n+1}\Big(\theta_n + 2\Big[\gamma_n(1-\theta_n) - \gamma_{n-1}(1-\theta_{n-1})\Big]_+\Big) \\ &\quad - \epsilon\gamma_{n-1}(1-\theta_{n-1}) \\ &\leq \gamma_{n-1}(1-\theta_{n-1}). \quad (3.4.25)\end{aligned}$$

Using (3.4.25) in (3.4.24), we obtain

$$\sum_{i=1}^{\infty} \theta_i t_{i+1}\|x_i - x_{i-1}\|^2 < \infty.$$

Case 2: Suppose that $\rho_n = 1$, $\forall n \geq 1$. Then, $x_{n+1} = z_n$ and $\gamma_n = 0$, $\forall n \geq 1$. Setting $x = w_n - y_n$ and $y = x_{n+1} - y_n$ in Lemma 2.1.1, and using its result in Lemma 3.4.3, we obtain

$$\begin{aligned}\Gamma_{n+1} - \Gamma_n - \theta_n(\Gamma_n - \Gamma_{n-1}) &\leq \frac{1}{2}(\theta_n + \theta_n^2)\|x_n - x_{n-1}\|^2 \\ &- \frac{1}{2}\left[||w_n - y_n + x_{n+1} - y_n||^2 + ||w_n - x_{n+1}||^2\right] \\ &\leq \frac{1}{2}(\theta_n + \theta_n^2)\|x_n - x_{n-1}\|^2 - \frac{1}{2}||w_n - x_{n+1}||^2. \quad (3.4.26)\end{aligned}$$

Now, using (3.4.18) in (3.4.26), and repeating the same line of proof as in Lemma 3.4.4, we get

$$\begin{aligned}&\sum_{i=1}^{n-1} t_{i+1,n}\Big[\left((1-\theta_i)^2 - (\theta_i + \theta_i^2)\right)\|x_i - x_{i-1}\|^2 + (1-\theta_i)\Big(\|x_{i+1} - x_i\|^2 - \|x_i - x_{i-1}\|^2\Big)\Big] \\ &\leq 2t_1|\Gamma_1 - \Gamma_0| + 2\Gamma_0. \quad (3.4.27)\end{aligned}$$

Similar to (3.4.19), we have

$$\begin{aligned}&\sum_{i=1}^{n-1} t_{i+1,n}(1-\theta_i)\Big[\|x_{i+1} - x_i\|^2 - \|x_i - x_{i-1}\|^2\Big] \\ \geq\ &\sum_{i=1}^{n-1}\Big(t_{i,n}(1-\theta_{i-1}) - t_{i+1,n}(1-\theta_i)\Big)\|x_i - x_{i-1}\|^2 \quad (3.4.28) \\ -\ &t_1(1-\theta_0)\|x_1 - x_0\|^2.\end{aligned}$$

Also, using (3.4.28) in (3.4.27), and repeating the same line of proof as in Lemma 3.4.5, we get

$$\begin{aligned}\sum_{i=1}^{n-1}(1-\theta_{i-1})-\theta_i t_{i+1}\Big(1+\theta_i+\Big[\theta_{i-1}-\theta_i\Big]_+\Big)\|x_i-x_{i-1}\|^2 \\ \leq 2t_1|\Gamma_1-\Gamma_0|+2\Gamma_0+t_1(1-\theta_0)\|x_1-x_0\|^2. \qquad (3.4.29)\end{aligned}$$

Using (3.4.22) in (3.4.29), and repeating the same line of proof as in **Case 1**, we obtain

$$\sum_{i=1}^{\infty}\theta_i t_{i+1}\|x_i-x_{i-1}\|^2<\infty,$$

which yields the desired conclusion. □

Lemma 3.4.7. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.4.3 such that assumption (2.5.37), Assumption 3.4.1 and Assumption 3.4.2(a)-(b) hold. Then,*

(a) $\lim_{n\to\infty}\|x_n-z\|$ *exists for all $z\in\Gamma$, and consequently, $\{x_n\}$ is bounded.*

(b) $\lim_{n\to\infty}[\|w_n-y_n\|^2+||z_n-y_n||^2]=0,$

(c) $\lim_{n\to\infty}\|x_n-w_n\|=0.$

Proof.

(a) From Lemma 3.4.3, we obtain

$$\begin{aligned}\Gamma_{n+1}-\Gamma_n &\leq \theta_n\Big(\Gamma_n-\Gamma_{n-1}\Big)+\frac{1}{2}\Big(\theta_n+\theta_n^2\Big)\|x_n-x_{n-1}\|^2 \\ &\quad -\frac{\rho_n}{2}(1-\lambda L)\left[||w_n-y_n||^2+||z_n-y_n||^2\right] \\ &\leq \theta_n\Big(\Gamma_n-\Gamma_{n-1}\Big)+\theta_n\|x_n-x_{n-1}\|^2-\frac{\rho_n}{2}(1-\lambda L)\left[||w_n-y_n||^2+||z_n-y_n||^2\right] \qquad (3.4.30)\\ &\leq \theta_n\Big(\Gamma_n-\Gamma_{n-1}\Big)+\theta_n\|x_n-x_{n-1}\|^2, \qquad (3.4.31)\end{aligned}$$

where the second inequality follows from $\theta_n^2\leq\theta_n$ and the third inequality follows from $1-\lambda L>0$.

Now, applying Lemma 2.5.13(b) and Lemma 3.4.6 in (3.4.31), we obtain that $\sum_{n=1}^{\infty}\Big[\Gamma_n-\Gamma_{n-1}\Big]_+<\infty$. Since $\Gamma_n=\frac{1}{2}\|x_n-z\|^2$, we get that $\lim_{n\to\infty}\|x_n-z\|^2$ exists. Hence, $\{x_n\}$ is bounded.

(b) By applying Lemma 2.5.13(a) in (3.4.30), we obtain that

$$
\begin{aligned}
\Gamma_n - \Gamma_0 &= \sum_{i=1}^{n} (\Gamma_i - \Gamma_{i-1}) \\
&\le t_{1,n}\Big(\Gamma_1 - \Gamma_0\Big) \\
&+ \sum_{i=1}^{n-1} t_{i+1,n}\Big[\theta_i\|x_i - x_{i-1}\|^2 - \frac{\rho_i}{2}(1-\lambda L)\left[||w_i - y_i||^2 + ||z_i - y_i||^2\right]\Big] \\
&\le t_1\big(\Gamma_1 - \Gamma_0\big) + \sum_{i=1}^{n-1} t_{i+1}\theta_i\|x_i - x_{i-1}\|^2 \\
&- \sum_{i=1}^{n-1} t_{i+1,n}\frac{\rho_i}{2}(1-\lambda L)\left[||w_i - y_i||^2 + ||z_i - y_i||^2\right]\Big],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\sum_{i=1}^{n-1} t_{i+1,n}\frac{\rho_i}{2}(1-\lambda L)\left[||w_i - y_i||^2 + ||z_i - y_i||^2\right]\Big] &\le \Gamma_0 - \Gamma_n + t_1(\Gamma_1 - \Gamma_0) \\
&+ \sum_{i=1}^{n-1} t_{i+1}\theta_i\|x_i - x_{i-1}\|^2 \\
&\le \Gamma_0 + t_1|\Gamma_1 - \Gamma_0| \\
&+ \sum_{i=1}^{n-1} t_{i+1}\theta_i\|x_i - x_{i-1}\|^2 \\
&< \infty,
\end{aligned}
$$

where the last inequality follows from Lemma 3.4.6.

Since $t_{i+1,n} = 0$ for $i \ge n$, letting n tend to ∞, the monotone convergence theorem then implies that

$$
\sum_{i=1}^{\infty} t_{i+1}\frac{\rho_i}{2}(1-\lambda L)\left[||w_i - y_i||^2 + ||z_i - y_i||^2\right]\Big] < \infty. \tag{3.4.32}
$$

Since $\liminf\limits_{n\to\infty} \rho_n > 0$, there exists $M > 0$ such that $\rho_n \ge M$ for n large enough. Now, replacing i with n in (3.4.32) and noting that $t_n \ge 1, \forall n \ge 1$, we obtain from (3.4.32) that $\sum\limits_{n=1}^{\infty} \frac{M}{2}(1-\lambda L)\left[||w_n - y_n||^2 + ||z_n - y_n||^2\right] < \infty$, which further gives that

$$
\lim_{n\to\infty}\left[||w_n - y_n||^2 + ||z_n - y_n||^2\right] = 0.
$$

(c) Since $w_n = x_n + \theta_n(x_n - x_{n-1})$, we obtain from Lemma 3.4.6 that

$$
\sum_{n=1}^{\infty} t_{n+1}\|w_n - x_n\|^2 \le \sum_{n=1}^{\infty} \theta_n t_{n+1}\|x_n - x_{n-1}\|^2 < \infty.
$$

Noting that $t_n \ge 1, \forall n \ge 1$, we conclude immediately that $\lim\limits_{n\to\infty}\|w_n - x_n\| = 0$. □

Lemma 3.4.8. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.4.3 such that assumption (2.5.37), Assumption 3.4.1 and Assumption 3.4.2(a)-(d) hold. If x^* is one of the weak cluster points of $\{x_n\}$, then we have at least one of the following: $x^* \in \Gamma$ or $Ax^* = 0$.*

Proof. By Lemma 3.4.7(a), we can choose a subsequence of $\{x_n\}$ denoted by $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup x^* \in \mathcal{H}$. Also, from Lemma 3.4.7(b),(c), we obtain that $\lim_{n\to\infty} \|y_n - x_n\| = 0$. Hence, we can choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightharpoonup x^*$. Note that $x^* \in \mathcal{C}$ since $\{y_{n_k}\} \subset \mathcal{C}$.

We now consider two possible cases.

Case 1: Suppose that $\limsup_{k\to\infty} \|Ay_{n_k}\| = 0$. Then, $\lim_{k\to\infty} \|Ay_{n_k}\| = \liminf_{k\to\infty} \|Ay_{n_k}\| = 0$. Also, by the sequentially weakly continuity of A on $\mathcal{C}$, we obtain that $Ay_{n_k} \rightharpoonup Ax^*$. Thus, by the weakly lower semicontinuity of $\|\cdot\|$, we have that

$$0 < \|Ax^*\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\| = 0, \tag{3.4.33}$$

which implies that $Ax^* = 0$.

Case 2: Suppose that $\limsup_{k\to\infty} \|Ay_{n_k}\| > 0$. Then, without loss of generality, we can choose a subsequence of $\{Ay_{n_k}\}$ still denoted by $\{Ay_{n_k}\}$ such that $\lim_{k\to\infty} \|Ay_{n_k}\| = M > 0$. Also by the characteristics property of $P_{\mathcal{C}}$, we obtain for all $x \in \mathcal{C}$ that

$$\langle w_{n_k} - \lambda A w_{n_k} - y_{n_k},\ x - y_{n_k}\rangle \leq 0.$$

This implies that

$$\frac{1}{\lambda}\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle Aw_{n_k},\ y_{n_k} - w_{n_k}\rangle \leq \langle Aw_{n_k}, x - w_{n_k}\rangle. \tag{3.4.34}$$

Thus, we obtain from Lemma 3.4.7(b) that

$$0 \leq \liminf_{k\to\infty}\langle Aw_{n_k}, x - w_{n_k}\rangle \leq \limsup_{k\to\infty}\langle Aw_{n_k}, x - w_{n_k}\rangle < \infty,\ \forall x \in \mathcal{C}. \tag{3.4.35}$$

Now, note that

$$\langle Ay_{n_k}, x - y_{n_k}\rangle = \langle Ay_{n_k} - Aw_{n_k}, x - w_{n_k}\rangle + \langle Aw_{n_k}, x - w_{n_k}\rangle + \langle Ay_{n_k}, w_{n_k} - y_{n_k}\rangle. \tag{3.4.36}$$

Moreover, since A is Lipschitz continuous on $\mathcal{H}$, we obtain from Lemma 3.4.7(b) that $\lim_{k\to\infty} \|Aw_{n_k} - Ay_{n_k}\| = 0$. Hence, we obtain from Lemma 3.4.7(b), (3.4.35) and (3.4.36) that

$$0 \leq \liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \leq \limsup_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle < \infty,\ \forall x \in \mathcal{C}. \tag{3.4.37}$$

Based on (3.4.37), we consider two cases under **Case 2**, as follows:

Case A: Suppose that $\limsup_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle > 0, \forall x \in \mathcal{C}$. Then, we can choose a subsequence of $\{y_{n_k}\}$ denoted by $\{y_{n_{k_j}}\}$ such that $\lim_{j\to\infty}\langle Ay_{n_{k_j}}, x - y_{n_{k_j}}\rangle > 0$. Thus, there exists

$j_0 \geq 1$ such that $\langle Ay_{n_{k_j}}, x - y_{n_{k_j}} \rangle > 0, \forall j \geq j_0$, which by the quasimonotonicity of A on $\mathcal{H}$, implies $\langle Ax, x - y_{n_{k_j}} \rangle \geq 0, \forall x \in \mathcal{C}, j \geq j_0$. Thus, letting $j \to \infty$, we obtain that $\langle Ax, x - x^* \rangle \geq 0, \forall x \in \mathcal{C}$. Hence, $x^* \in \Gamma$.

Case B: Suppose that $\limsup_{k\to\infty} \langle Ay_{n_k}, x - y_{n_k} \rangle = 0, \forall x \in \mathcal{C}$. Then, by (3.4.37), we obtain that

$$\lim_{k\to\infty} \langle Ay_{n_k}, x - y_{n_k} \rangle = 0, \forall x \in \mathcal{C}, \tag{3.4.38}$$

which implies that

$$\langle Ay_{n_k}, x - y_{n_k} \rangle + |\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1} > 0, \forall x \in \mathcal{C}. \tag{3.4.39}$$

Also, since $\lim_{k\to\infty} \|Ay_{n_k}\| = M > 0$, there exists $k_0 \geq 1$ such that $\|Ay_{n_k}\| > \frac{M}{2}, \forall k \geq k_0$. Therefore, we can set $q_{n_k} = \frac{Ay_{n_k}}{\|Ay_{n_k}\|^2}, \forall k \geq k_0$. Thus, $\langle Ay_{n_k}, q_{n_k} \rangle = 1, \forall k \geq k_0$. Hence, by (3.4.39), we obtain

$$\left\langle Ay_{n_k}, x + q_{n_k}\left[|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1} - y_{n_k}\right] \right\rangle > 0,$$

which by the quasimonotonicity of A on $\mathcal{H}$, implies

$$\left\langle A\Big(x + q_{n_k}\Big[|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}\Big]\Big), x + q_{n_k}[|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}] - y_{n_k} \right\rangle \geq 0.$$

This further implies that

$$\begin{aligned}
&\langle Ax, x + q_{n_k}\Big[|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}\Big] - y_{n_k} \rangle \\
&\geq \langle Ax, A(x + q_{n_k}[|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}]), x + q_{n_k}[|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}] - y_{n_k} \rangle \\
&\geq -\|Ax - A(x + q_{n_k}\Big[|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}\Big])\| \\
&\times \Big(\|x + q_{n_k}\Big[|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}\Big] - y_{n_k}\|\Big) \\
&\geq -L\|q_{n_k}\Big[|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}\Big]\| \cdot \|x + q_{n_k}\Big[|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}\Big] - y_{n_k}\| \\
&= \frac{-L}{\|Ay_{n_k}\|}\Big(|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}\Big) \cdot \|x + q_{n_k}\Big[|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}\Big] - y_{n_k}\| \\
&\geq \frac{-2L}{M}\Big(|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}\Big)M_1,
\end{aligned} \tag{3.4.40}$$

for some $M_1 > 0$, where the existence of M_1 follows from the boundedness of $\{x + q_{n_k}\Big[|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}\Big] - y_{n_k}\}$. Note from (3.4.38) that $\lim_{k\to\infty}\Big(|\langle Ay_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k+1}\Big) = 0$. Hence, letting $k \to \infty$ in (3.4.40), we get that $\langle Ax, x - x^* \rangle \geq 0, \forall x \in \mathcal{C}$. Therefore, $x^* \in \Gamma$. □

Lemma 3.4.9. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.4.3 such that assumption (2.5.37), Assumption 3.4.1 and Assumption 3.4.2(a)-(d) hold. Then $\{x_n\}$ has at most one weak cluster point in Γ.*

Proof. Suppose on the contrary that $\{x_n\}$ has at least two weak cluster points in Γ. Let $x^* \in \Gamma$ and $\bar{x} \in \Gamma$ be any two weak cluster points of $\{x_n\}$ such that $x^* \neq \bar{x}$. Also, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x}$, as $j \to \infty$. Then, by Lemma 3.4.7(a) and Lemma 2.5.15, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \bar{x}\| &= \lim_{j\to\infty} \|x_{n_j} - \bar{x}\| \\ &= \liminf_{j\to\infty} \|x_{n_j} - \bar{x}\| \\ &< \liminf_{j\to\infty} \|x_{n_j} - x^*\| \\ &= \lim_{n\to\infty} \|x_n - x^*\| \\ &= \liminf_{k\to\infty} \|x_{n_k} - x^*\| \\ &< \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| = \lim_{n\to\infty} \|x_n - \bar{x}\|,\end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ has at most one weak cluster point in Γ. □

Theorem 3.4.10. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.4.3 such that assumption (2.5.37), Assumption 3.4.1 and Assumption 3.4.2(a)-(e) hold. Then $\{x_n\}$ converges weakly to an element of $VI(\mathcal{C}, A)$.*

Proof. By Assumption 3.4.2(e), $\{x \in \mathcal{C} : Ax = 0\} \setminus \Gamma$ is a finite set. Hence, by Lemma 3.4.8 and Lemma 3.4.9, we have that $\{x_n\}$ has finite weak cluster points in $VI(\mathcal{C}, A)$. Let $x^1, x^2, \cdots, x^m$ be the weak cluster points of $\{x_n\}$, and let $\{x^i_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x^i_{n_k} \rightharpoonup x^i$, as $k \to \infty$. Then, we obtain

$$\lim_{k\to\infty} \left\langle x^i_{n_k}, x^i - x^j \right\rangle = \left\langle x^i, x^i - x^j \right\rangle, \forall j \neq i. \tag{3.4.41}$$

Now, for $j \neq i$, set $q = \beta^{-1}(x^i - x^j)$, where $\beta = \|x^i - x^j\|$. Then,

$$\begin{aligned}\langle x^i, q\rangle &= \beta^{-1}\langle x^i, x^i - x^j\rangle \\ &= \beta^{-1}\Big(\|x^i\|^2 - \langle x^i, x^j\rangle\Big) \\ &= \beta^{-1}\Big[\|x^i\|^2 - \frac{1}{2}\|x^i\|^2 - \frac{1}{2}\|x^j\|^2 + \frac{1}{2}\|x^i - x^j\|^2\Big] \\ &= \frac{1}{2\beta}\Big(\|x^i\|^2 - \|x^j\|^2\Big) + \frac{1}{2}\beta \\ &> \frac{1}{2\beta}\Big(\|x^i\|^2 - \|x^j\|^2\Big) + \frac{1}{4}\beta.\end{aligned} \tag{3.4.42}$$

For sufficiently large k, we obtain from (3.4.41) and (3.4.42) that

$$x^i_{n_k} \in \Big\{x : \langle x, q\rangle > \frac{1}{2\beta}\Big(\|x^i\|^2 - \|x^j\|^2\Big) + \frac{1}{4}\beta\Big\}. \tag{3.4.43}$$

Hence, there exists $N_1 > N$ $(N \in \mathbb{N})$ such that

$$x^i_{n_k} \in B^i := \bigcap_{j=1, j\neq i}^{m} \Big\{x : \Big\langle x, \frac{x^i - x^j}{\|x^i - x^j\|}\Big\rangle > \frac{1}{2\|x^i - x^j\|}\Big(\|x^i\|^2 - \|x^j\|^2\Big) + \epsilon_0\Big\}, \forall n \geq N_1, \tag{3.4.44}$$

where

$$\epsilon_0 = \min\Big\{\frac{1}{4}\|x^i - x^j\| : i, j = 1, 2, \cdots, m, i \neq j\Big\}.$$

Now, from Lemma 3.4.7(b), we obtain that $\lim_{n\to\infty} \|z_n - w_n\| = 0$. Since $x_{n+1} - w_n = \rho_n(z_n - w_n)$ and $\{\rho_n\}$ is bounded, we obtain that $\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0$. This together with Lemma 3.4.7(c), imply that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Hence, there exists $N_2 > N_1 > N$ such that $\|x_{n+1} - x_n\| < \epsilon_0, \forall n \geq N_2$.

Claim: $\{x_n\}$ has only one weak cluster point in $VI(\mathcal{C}, A)$.

Suppose on the contrary that $\{x_n\}$ has more than one weak cluster points in $VI(\mathcal{C}, A)$. Then, there exists $N_3 \geq N_2 > N_1 > N$ such that $x_{N_3} \in B^i$ and $x_{N_3+1} \in B^j$, where

$$B^j := \bigcap_{i=1, i\neq j}^{m} \Big\{x : \Big\langle -x, \frac{x^j - x^i}{\|x^j - x^i\|}\Big\rangle < \frac{1}{2\|x^j - x^i\|}\Big(\|x^i\|^2 - \|x^j\|^2\Big) - \epsilon_0\Big\},$$
$$i, j \in \{1, 2, \cdots, m\} \text{ and } m \geq 2. \tag{3.4.45}$$

In particular, we have

$$\|x_{N_3+1} - x_{N_3}\| < \epsilon_0. \tag{3.4.46}$$

Since $x_{N_3} \in B^i$ and $x_{N_3+1} \in B^j$, we obtain that

$$\Big\langle x_{N_3}, \frac{x^i - x^j}{\|x^i - x^j\|}\Big\rangle > \frac{1}{2\|x^i - x^j\|}\Big(\|x^i\|^2 - \|x^j\|^2\Big) + \epsilon_0 \tag{3.4.47}$$

and

$$\Big\langle -x_{N_3+1}, \frac{x^i - x^j}{\|x^j - x^i\|}\Big\rangle > \frac{1}{2\|x^i - x^j\|}\Big(\|x^j\|^2 - \|x^i\|^2\Big) + \epsilon_0. \tag{3.4.48}$$

Adding (3.4.47) and (3.4.48), and using (3.4.46), we obtain

$$2\epsilon_0 < \Big\langle x_{N_3} - x_{N_3+1}, \frac{x_i - x^j}{\|x^i - x^j\|}\Big\rangle \leq \|x_{N_3+1} - x_{N_3}\| < \epsilon_0, \tag{3.4.49}$$

which is not possible. Hence, our claim holds. That is, $\{x_n\}$ has only one weak cluster point in $VI(\mathcal{C}, A)$. Therefore, we conclude that $\{x_n\}$ converges weakly to an element of $VI(\mathcal{C}, A)$. □

Remark 3.4.11.

(i) The conclusion of Theorem 3.4.10 still hold even if $\lambda \in (0, \frac{1}{L})$ in Algorithm 3.4.3 is replaced with a variable stepsize λ_n such that $0 < \inf_{n\geq 1} \lambda_n \leq \sup_{n\geq 1} \lambda_n < \frac{1}{L}$. However, choosing this variable stepsize still requires the knowledge of the Lipschitz constant L.

(ii) When the Lipschitz constant is not known, we refer to Algorithm 3.4.4, where the choice of λ_n does not depend on its knowledge. Note from the stepsize λ_n in (3.4.2), that $\lim\limits_{n\to\infty} \lambda_n = \lambda$ and $\lambda \in [\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + d]$, where $d = \sum_{n=1}^{\infty} d_n$ (see also [162, Lemma 3.1]).

(iii) When $d_n = 0$, then the stepsize λ_n generated by Algorithm 3.4.4 is similar to that in [267]. We recall that the stepsize in [267] is monotone non-increasing, thus, their methods may depend on the choice of the initial stepsize λ_1. However, the stepsize given in (3.4.2) is non-monotonic and hence, the dependence on the initial stepsize λ_1 is reduced.

In the light of the above remark, we analyze the convergence of Algorithm 3.4.4 in what follows.

Lemma 3.4.12. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.4.4 such that Assumption 3.4.2(a)-(b) hold. Then,*

$$\begin{aligned}
&\Gamma_{n+1} - \Gamma_n - \theta_n(\Gamma_n - \Gamma_{n-1}) \\
&\leq \frac{1}{2}(\theta_n + \theta_n^2)\|x_n - x_{n-1}\|^2 - \gamma_n\|x_{n+1} - w_n\|^2 \\
&-\frac{\rho_n}{2}\left(1 - \lambda_n\frac{\mu}{\lambda_{n+1}}\right)\left[||w_n - y_n||^2 + ||z_n - y_n||^2\right],
\end{aligned}$$

where $\gamma_n := \frac{1-\rho_n}{2\rho_n}$ *and* $\Gamma_n := \frac{1}{2}\|x_n - z\|^2, \ \forall z \in \Gamma.$

Proof. By following the same line of proof used in obtaining (3.4.16), we have

$$\begin{aligned}
&\Gamma_{n+1} - \Gamma_n - \theta_n(\Gamma_n - \Gamma_{n-1}) \\
\leq \ &\frac{1}{2}(\theta_n + \theta_n^2)\|x_n - x_{n-1}\|^2 + \frac{\rho_n - 1}{2\rho_n}\|x_{n+1} - w_n\|^2 \\
&-\frac{\rho_n}{2}\left(\|w_n - y_n\|^2 + \|y_n - z_n\|^2\right) + \lambda_n\rho_n\langle Aw_n - Ay_n, \ z_n - y_n\rangle.
\end{aligned} \tag{3.4.50}$$

If $\langle Aw_n - Ay_n, z_n - y_n\rangle \leq 0$, then we obtain the conclusion of Lemma 3.4.12 immediately from (3.4.50).
In the case that $\langle Aw_n - Ay_n, z_n - y_n\rangle > 0$, we obtain from (3.4.2) that

$$\langle Aw_n - Ay_n, z_n - y_n\rangle \leq \frac{\mu}{2\lambda_{n+1}}\Big(\|w_n - y_n\|^2 + \|z_n - y_n\|^2\Big). \tag{3.4.51}$$

Substituting (3.4.51) into (3.4.50), we obtain the desired conclusion. □

Lemma 3.4.13. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.4.3 such that assumption (2.5.37) and Assumption 3.4.2(a)-(b) hold. Then, the following inequality holds:*

$$\begin{aligned}
\sum_{i=1}^{n-1} t_{i+1,n}\Big[&\left(2\gamma_i(1-\theta_i)^2 - (\theta_i + \theta_i^2)\right)\|x_i - x_{i-1}\|^2 \\
&+2\gamma_i(1-\theta_i)\Big(\|x_{i+1} - x_i\|^2 - \|x_i - x_{i-1}\|^2\Big)\Big] \\
&\qquad\qquad \leq 2t_1|\Gamma_1 - \Gamma_0| + 2\Gamma_0,
\end{aligned}$$

where $t_{i,n}$ is as defined in (2.5.40).

Proof. By following the same line of proof used in obtaining (3.4.18), we have

$$\begin{aligned}\|x_{n+1}-w_n\|^2 &\geq (1-\theta_n)^2\|x_n-x_{n-1}\|^2 \\ &+ (1-\theta_n)\left[\|x_{n+1}-x_n\|^2-\|x_n-x_{n-1}\|^2\right]. \end{aligned} \tag{3.4.52}$$

Also, by Remark 3.4.11(ii), we obtain that $\lim_{n\to\infty}\lambda_n\frac{\mu}{\lambda_{n+1}}=\mu\in(0,1)$. Thus, there exists $n_0\geq 1$ such that $\forall n\geq n_0$, $0<\lambda_n\frac{\mu}{\lambda_{n+1}}<1$. Hence, we get from Lemma 3.4.12 and (3.4.52) that

$$\begin{aligned}\Gamma_{n+1}-\Gamma_n-\theta_n(\Gamma_n-\Gamma_{n-1}) &\leq \frac{1}{2}(\theta_n+\theta_n^2)\|x_n-x_{n-1}\|^2-\gamma_n\|x_{n+1}-w_n\|^2 \\ &\leq \left[\frac{1}{2}(\theta_n+\theta_n^2)-\gamma_n(1-\theta_n)^2\right]\|x_n-x_{n-1}\|^2 \\ &\quad -\gamma_n(1-\theta_n)\left[\|x_{n+1}-x_n\|^2-\|x_n-x_{n-1}\|^2\right],\ \forall n\geq n_0.\end{aligned}$$

The remaining part of the proof is the same as the proof of Lemma 3.4.4. □

In similar manner, we have that Lemma 3.4.5 to Lemma 3.4.9 hold for Algorithm 3.4.4. Thus, we have the following theorem whose proof is the same as the proof of Theorem 3.4.10.

Theorem 3.4.14. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.4.4 such that assumption* (2.5.37)*, Assumption 3.4.1 and Assumption 3.4.2(a)-(e) hold. Then $\{x_n\}$ converges weakly to an element of $VI(\mathcal{C},A)$.*

Remark 3.4.15. From our analyses, one can see that Assumption 3.4.1 is mainly used to guarantee the summation:

$$\sum_{n=1}^{\infty}\theta_n t_{n+1}\|x_n-x_{n-1}\|^2<\infty \tag{3.4.53}$$

obtained in Lemma 3.4.6. Thus, if we assume that (3.4.53) directly, then we do not need Assumption 3.4.1 for the convergence of our methods.
In the case where $\rho_n=1$, $\forall n\geq 1$, our methods correspond to the inertial subgradient extragradient methods. Note that if $\theta_n\in[0,\theta]$ for every $n\geq 1$, where $\theta\in[0,1)$, then $t_n\leq\frac{1}{(1-\theta)}\quad\forall n\geq 1$. Under these settings, (3.4.53) is guaranteed by the condition

$$\sum_{n=1}^{\infty}\theta_n\|x_n-x_{n-1}\|^2<\infty. \tag{3.4.54}$$

Thus, instead of Assumption 3.4.1, we may assume directly that $\theta_n\in[0,\theta]$, $\forall n\geq 1$ and that condition (3.4.54) holds. Recall that this condition has been used by numerous authors to ensure convergence of inertial methods (see, for example, [18, 84, 165, 167, 184] and the references therein).

Remark 3.4.16. Note that we did not make use of Assumption 3.4.2(c)-(e) in the proof of Lemma 3.4.3-3.4.7. Suppose that $\mathcal{H}$ is a finite dimensional Hilbert space, then under Assumption 3.4.2(a)-(b), we get from Lemma 3.4.7(a) that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to some point x^*. By Lemma 3.4.7(b)(c), we get

$$\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$$

and

$$\lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = 0.$$

Thus, by the definition of y_n and the continuity of A, we have

$$x^* = \lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} P_{\mathcal{C}}(w_{n_k} - \lambda A w_{n_k}) = P_{\mathcal{C}}(x^* - \lambda A x^*),$$

which implies that $x^* \in VI(\mathcal{C}, A)$.

Now, replacing z by x^* in Lemma 3.4.7(a), we obtain that $\lim\limits_{n\to\infty} \|x_n - x^*\|^2$ exists. Since x^* is a cluster point of $\{x_n\}$, we obtain that $\{x_n\}$ converges to x^*.

In summary, in a finite dimensional Hilbert space, our methods require that $\Gamma \neq \emptyset$ and the operator A only needs to be Lipschitz continuous without any form of monotonicity.

To achieve this (convergence without any form of monotonicity) in infinite dimensional Hilbert space, we replace Assumption 3.4.2(d)-(e) with the following:

(d)* If $x_n \rightharpoonup x^*$ and $\limsup\limits_{n\to\infty} \langle Ax_n, x_n\rangle \leq \langle Ax^*, \bar{x}\rangle$, then $\lim\limits_{n\to\infty} \langle Ax_n, x_n\rangle = \langle Ax^*, x^*\rangle$.

(e)* The set $VI(\mathcal{C}, A) \setminus \Gamma$ is a finite set.

In fact, we have the following theorem.

Theorem 3.4.17. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.4.3 (or Algorithm 3.4.4) such that assumption (2.5.37), Assumption 3.4.1, Assumption 3.4.2(a)-(c) and conditions (d)*-(e)* hold. Then $\{x_n\}$ converges weakly to an element of $VI(\mathcal{C}, A)$.*

Proof. First notice that Assumption 3.4.2(d) was used only after (3.4.37) in order to establish the conclusion of Lemma 3.4.8.
But from (3.4.37), we have that

$$0 \leq \liminf_{k\to\infty} \langle Ay_{n_k}, x - y_{n_k}\rangle, \ \forall x \in \mathcal{C}.$$

Now, let $\{c_k\}$ be a sequence of positive numbers such that $\lim\limits_{k\to\infty} c_k = 0$ and $\langle Ay_{n_k}, x - y_{n_k}\rangle + c_k > 0, \ \forall k \geq 0, x \in \mathcal{C}$. Then,

$$\langle Ay_{n_k}, x\rangle + c_k > \langle Ay_{n_k}, y_{n_k}\rangle, \ \forall k \geq 0, x \in \mathcal{C}. \tag{3.4.55}$$

Note that $y_{n_k} \rightharpoonup x^*$ and $x^* \in \mathcal{C}$. Thus, we have in particular,

$$\langle Ay_{n_k}, x^*\rangle + c_k > \langle Ay_{n_k}, y_{n_k}\rangle, \ \forall k \geq 0. \tag{3.4.56}$$

Taking limit as $k \to \infty$ in (3.4.56), and using the sequentially weakly continuity of A, we obtain that

$$\langle Ax^*, x^* \rangle \geq \limsup_{k\to\infty} \langle Ay_{n_k}, y_{n_k} \rangle,$$

which by condition (d)* and (3.4.55) implies that

$$\begin{aligned} \langle Ax^*, x^* \rangle &= \lim_{k\to\infty} \langle Ay_{n_k}, y_{n_k} \rangle \\ &= \liminf_{k\to\infty} \langle Ay_{n_k}, y_{n_k} \rangle \\ &\leq \lim_{k\to\infty} (\langle Ay_{n_k}, x \rangle + c_k) = \langle Ax^*, x \rangle. \end{aligned}$$

This further implies that $x^* \in VI(\mathcal{C}, A)$.
Now, using condition (e)* and following similar line of proof as in Lemma 3.4.9 and Theorem 3.4.17, we get that $\{x_n\}$ converges weakly to x^*. □

Remark 3.4.18.

(i) If $x_n \rightharpoonup x^*$ and A is sequentially weakly-strongly continuous, then A satisfies condition (d)*.

(ii) In the numerical experiments, we do not need to consider condition (e) (or (e)*). First note that whenever $||y_n - w_n|| < \epsilon$, Algorithm 3.4.3 and Algorithm 3.4.4 terminate in a finite step of the iterations (and y_n is the solution of the VIP (1.2.1)). But from Lemma 3.4.7(b), $\lim_{n\to\infty} ||y_n - w_n|| = 0$ and condition (e) (or (e)*) was not used in establishing it.

We now give some remarks regarding the contributions in this work.

Remark 3.4.19.

(1) If we set the inertial factor $\theta_n = 0$ and relaxation factor $\rho_n = 1$, then our Algorithm 3.4.4 reduces to Algorithm 3.3 of [162]. Note that these parameters (factors) play vital role in improving the convergence rate of iterative methods. In fact, their influence with regards to the numerical performance of iterative schemes was discussed in [129]. Moreover, the benefits gained from incorporating the steps of these two parameters in our algorithms, are further verified in Section 3.4.3. Thus, bearing in mind the importance of these two parameters in iterative algorithms, we can see that our methods significantly improve the methods in [162].

(2) In the study of inertial methods for solving VIPs (even for monotone mappings), the inertia parameter is usually restricted in $[0, \frac{1}{3})$ and/or required to be nondecreasing

with an upper bound (see, for example, [18, 165, 246, 247]). In many cases, to ensure convergence, authors usually require the inertial parameter to depend on the knowledge of the Lipschitz constant of the cost operator which sometimes is very difficult to estimate in practice (see, for instance, [97]). Another condition usually imposed on the inertial parameter in the literature, is condition (3.4.54), which rely on the sequence $\{x_n\}$. One of the novelties of this work is that we derived a general condition (Assumption 3.4.1) which is weaker than the above conditions used in the literature for ensuring the convergence of inertial methods for VIPs. As a result, we developed a different technique to ensure the convergence of our methods.

(3) In addition to (2) above, bearing in mind the Nestrov's accelerated scheme ([187]), another novelty of this work, is that the assumptions on the inertial and relaxation paramters of this work, allow the case where θ_n converges to a point very close to 1 (see Remark 2.5.12 and the choices in **Experiment 1** of Section 3.4.3), which is very crucial in the study of inertial methods. This is actually where the relaxation effect and the parameter ρ_n come into play, as crucial ingredients of our methods. Thus, the novelty of this work is not only in improving the convergence rate of the methods in [162] but to provide weaker conditions on the inertial parameter in methods for solving VIPs and also to offer a different but unified technique in proving their convergence. Furthermore, we employed condition (Assumption 3.4.1) where joint adjustments of the inertial and relaxation parameters play crucial role (for instance, see **Experiment 1**). Indeed, this is a new perspective in the study of inertial methods for VIPs. Moreover, our study offers many possibilities for future research in this direction; like how to modify Assumption 3.4.1 so that θ_n is allowed to converge to 1?

3.4.3 Numerical experiments

In this section, we give some numerical examples to show the implementation of our proposed methods (Algorithm 3.4.3 and Algorithm 3.4.4). We also compare our new methods with Algorithms 3.1 and 3.3 in [162], and Algorithm 2.1 in [269].

The codes are written in Matlab 2016 (b) and performed on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30GHz and 8.00 Gb-RAM. In Tables 3.4.1-3.4.5, "Iter." means the number of iterations while "CPU" means the CPU time in seconds.
In our computations, we define $\text{TOL}_n := ||y_n - w_n||$ for Algorithm 3.4.3 and Algorithm 3.4.3. While for Algorithms 3.1 and 3.3 in [162], we define $\text{TOL}_n := \|y_n - x_n\|/\min\{\lambda_n, 1\}$ and for Algorithm 2.1 in [269], we define $\text{TOL}_n := \|x_n - z_n\| = \|x_n - P_{\mathcal{C}}(x_n - Ax_n)\|$ (as done in [162] and [269], respectively). Then, we use the stopping criterion $\text{TOL}_n < \varepsilon$ for the iterative processes, where ε is the predetermined error. These choices of stopping criterion for these methods are the best to be able to terminate the algorithms based on the examples we consider. As done in [162], we take $\varepsilon = 10^{-6}$ for all Algorithms.
We choose $\mu = 0.5$, $d_n = \frac{100}{(n+1)^{1.1}}$ and $\lambda_1 = 1$ for Algorithm 3.4.4 and Algorithms 3.1 and 3.3 in [162]; $\lambda = \frac{1}{2L}$ for Algorithm 3.4.3; $\gamma = 0.4$ and $\sigma = 0.99$ for Algorithm 2.1 in [269]. These choices are the same as in [162, 269] and are optimal values for these parameters.

Example 3.4.20. Let $\mathcal{C} = [-1,1]$ and

$$Ax = \begin{cases} 2x-1, & x > 1, \\ x^2, & x \in [-1,1], \\ -2x-1, & x < -1. \end{cases}$$

Then, A is quasimonotone and 2-Lipschitz continuous. Also, $\Gamma = \{-1\}$ and $VI(\mathcal{C}, A) = \{-1, 0\}$.

Example 3.4.21. Let $\mathcal{C} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1,\ 0 \leq x_1\}$ and $A(x_1, x_2) = (-x_1 e^{x_2}, x_2)$. Then, A may not be quasimonotone. We can also check that $(1,0) \in \Gamma$ and $VI(\mathcal{C}, A) = \{(1,0),(0,0)\}$ (see [162, Section 4]), which by Remark 3.4.16 satisfy our assumptions. This example was also tested in [162].

Example 3.4.22. We next consider the following problem which was also considered in [162, 170, 231].
Let $\mathcal{C} = [0,1]^m$ and $Ax = (f_1x, f_2x, \cdots, f_mx)$,
where $f_ix = x_{i-1}^2 + x_i^2 + x_{i-1}x_i + x_ix_{i+1} - 2x_{i-1} + 4x_i + x_{i+1} - 1,\ i = 1, 2, \cdots, m,$
$x_0 = x_{m+1} = 0$.

We test these examples under the following experiments.

Experiment 1
In this first experiment, we check the behavior of our methods by fixing the inertial parameter and varying the relaxation parameter. We do this in order to check the effects of the relaxation parameter on our methods.

For Example 3.4.20: We take $\theta_n = \frac{3n+1}{10n+5}$ with $\rho_n = 1$ (which by Proposition 3.4.5, satisfies assumption (2.5.37) and Assumption 3.4.1), and $\theta_n = \frac{19}{20} - \frac{1}{(n+1)^{\frac{1}{2}}}$ with $\rho_n \in \{\frac{1}{20} + \frac{1}{(n+1)^2}, \frac{1}{20} + \frac{2}{(n+1)^3}, \frac{1}{20} + \frac{3}{(n+1)^4}, \frac{1}{20} + \frac{5}{(n+1)^4}\}$ (which also satisfies assumption (2.5.37) and Assumption 3.4.1 by Proposition 3.4.7).
Also, we take $x_1 = 1$ and $x_0 = 0.5$ for this example. Since the Lipshitz constant is known for this example, we use Algorithm 3.4.3 and Algorithm 3.4.4 for the experiment and obtain the numerical results listed in Table 3.4.1 and Figure 3.3. From the table and graph, we can see that $\rho_n = 1$ performs better than other choices made for Algorithm 3.4.3 and Algorithm 3.4.4.

For Example 3.4.21 and Example 3.4.22: We take $\theta_n = \frac{9.9}{10} - \frac{1}{n+1}$ with $\rho_n = 1$ (which by Proposition 3.4.6 (see also Remark 2.5.12), satisfies assumption (2.5.37) and Assumption 3.4.1), and $\theta_n = \frac{9}{10} - \frac{1}{(n+1)^{\frac{1}{3}}}$ with $\rho_n \in \{\frac{1}{10} + \frac{1}{n+1}, \frac{1}{11} + \frac{1}{n+1}, \frac{1}{12} + \frac{1}{n+1}, \frac{1}{13} + \frac{1}{n+1}\}$.
Also, we take $x_1 = (0.1, 0.5)$ and $x_0 = (0.2, 0.1)$ for Example 3.4.21 while for Example 3.4.22, we choose x_1 and x_0 randomly with $m = 50$. For these examples, we use Algorithm 3.4.4 for the experiment and obtain the numerical results listed in Table 3.4.2 and Figure 3.4. From the table and graph, we can see that the $\rho_n = \frac{1}{11} + \frac{1}{n+1}$ performs better than other choices made for Example 3.4.21 while $\rho_n = \frac{1}{10} + \frac{1}{n+1}$ performs better for Example 3.4.22, which validates the benefits sometimes brought by the relaxation parameter.

Experiment 2
In this experiment, we compare our methods with Algorithms 3.1 and 3.3 in [162], and Algorithm 2.1 in [269]. Here, we randomly choose the θ_n and ρ_n from Experiment 1, and then, consider the following cases for the starting points in each example.

For Example 3.4.20: Case I: $x_1 = 0.2$, $x_0 = 0.1$ and Case II: $x_1 = 0.6$, $x_0 = 0.2$.

For Example 3.4.21: Case I: $x_1 = (1, 0.5)$, $x_0 = (0.2, 0.1)$ and Case II: $x_1 = (0.1, 0.2)$, $x_0 = (0.3, 0.5)$.

For Example 3.4.22: Case I: $m = 100$ and Case II: $m = 150$ (x_1 and x_0 are randomly taken).

We use Algorithm 3.4.1 and Algorithm 3.4.4 for the comparison in Example 3.4.20 while we use only Algorithm 3.4.4 for the comparison in Example 3.4.21 and Example 3.4.22. The numerical results are given in Tables 3.4.3-3.4.5 and Figures 3.5-3.7. The results show that our methods perform better than Algorithms 3.1 and 3.3 in [162], and Algorithm 2.1 in [269].

Table 3.4.1. Numerical results for Example 3.4.20 (Experiment 1).

ρ_n		Algorithm 3.4.3	Algorithm 3.4.4
1	CPU	0.0016	0.0015
	Iter.	7	3
$\frac{1}{20} + \frac{1}{(n+1)^2}$	CPU	0.0022	0.0027
	Iter.	13	6
$\frac{1}{20} + \frac{2}{(n+1)^3}$	CPU	0.0026	0.0020
	Iter.	8	9
$\frac{1}{20} + \frac{3}{(n+1)^4}$	CPU	0.0028	0.0028
	Iter.	11	12
$\frac{1}{20} + \frac{5}{(n+1)^4}$	CPU	0.0597	0.0146
	Iter.	21	20

Table 3.4.2. Numerical results for Algorithm 3.4.4 (Experiment 1).

ρ_n		Example 3.4.21	Example 3.4.22
1	CPU	0.0123	0.0232
	Iter.	20	68
$\frac{1}{10} + \frac{1}{n+1}$	CPU	0.0162	0.0107
	Iter.	23	39
$\frac{1}{11} + \frac{1}{n+1}$	CPU	0.0050	0.0304
	Iter.	16	45
$\frac{1}{12} + \frac{1}{n+1}$	CPU	0.0149	0.0346
	Iter.	30	51
$\frac{1}{13} + \frac{1}{n+1}$	CPU	0.0163	0.0257
	Iter.	45	71

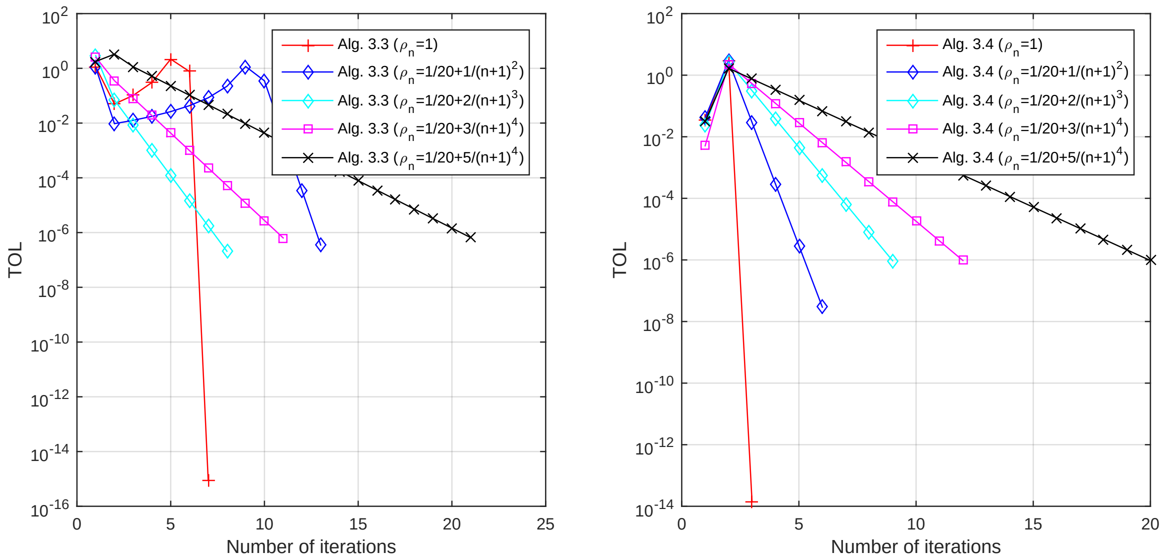


Figure 3.3: The behavior of Algorithms 3.4.3 and 3.4.4 for Example 3.4.20 (**Experiment 1**).

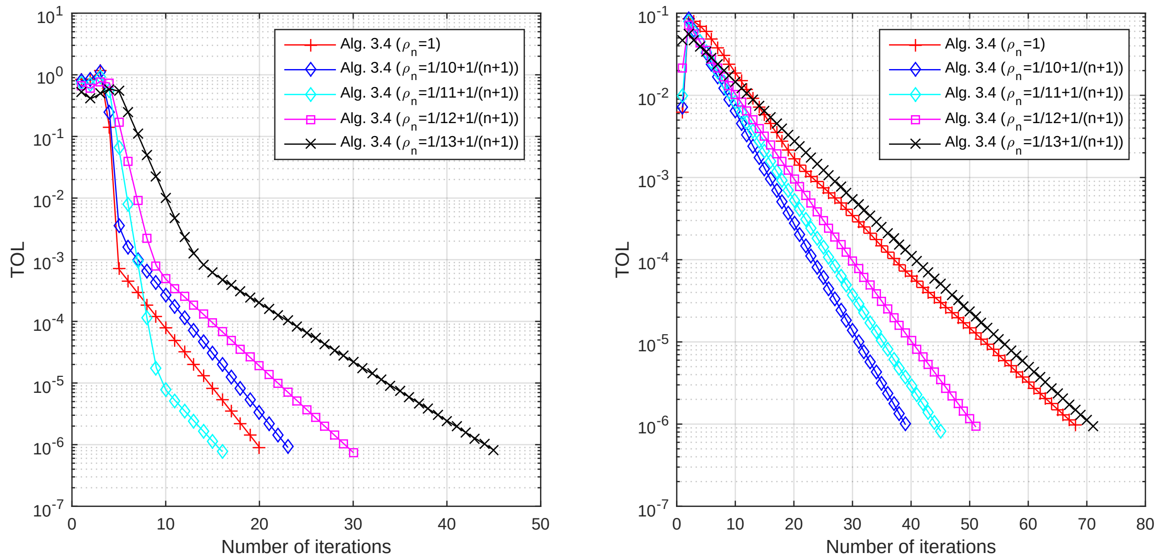


Figure 3.4: The behavior of Algorithm 3.4.4 for Examples 3.4.21 and 3.4.22 (**Experiment 1**).

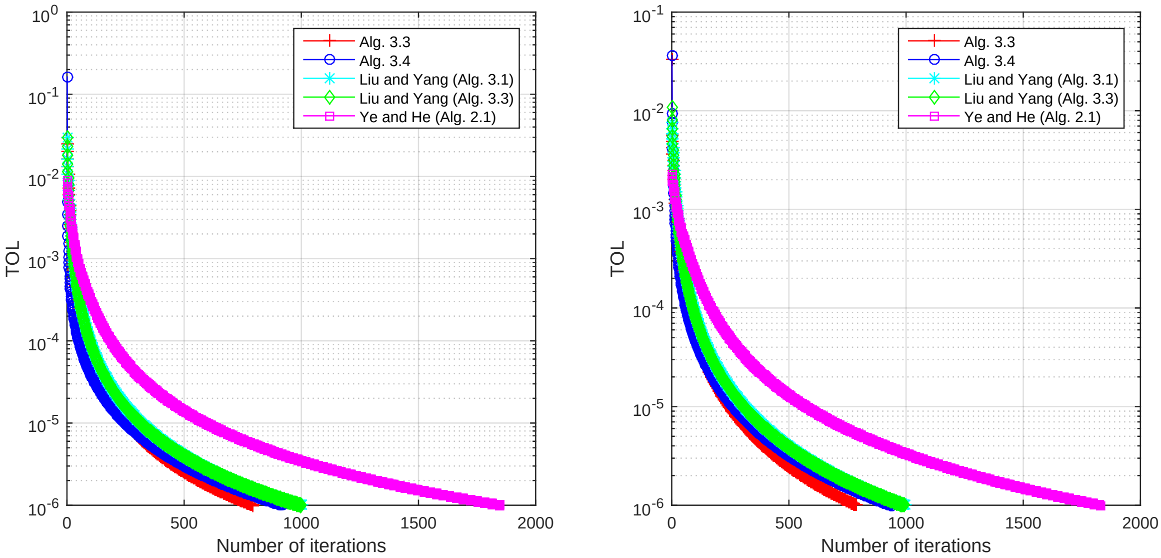


Figure 3.5: The behavior of TOL_n for Example 3.4.20 (**Experiment 2**): Left: **Case I**; Right: **Case II**.

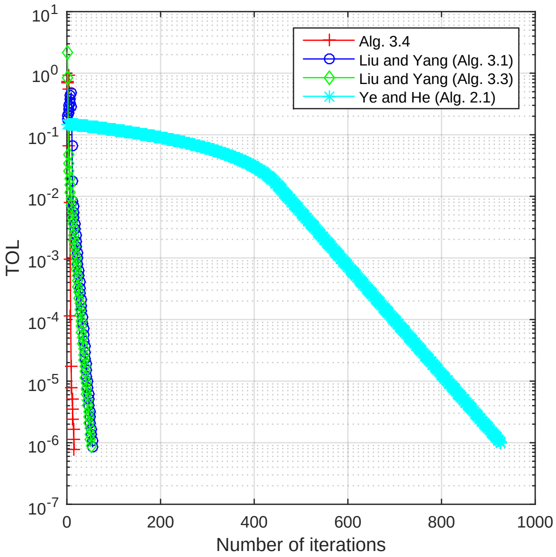
Table 3.4.3. Numerical results for Example 3.4.20 (Experiment 2).

Cases		Algorithm 3.4.3	Algorithm 3.4.4	Algorithm 3.1 in [162]	Algorithm 3.3 in [162]	Algorithm 2.1 in [269]
I	CPU	0.0223	0.0289	0.0612	0.0621	0.0923
	Iter.	795	922	1002	998	1848
II	CPU	0.0244	0.0272	0.0672	0.0662	0.0900
	Iter.	788	936	996	992	1828

Table 3.4.4. Numerical results for Example 3.4.21 (Experiment 2).

Cases		Algorithm 3.4.4	Algorithm 3.1 in [162]	Algorithm 3.3 in [162]	Algorithm 2.1 in [269]
I	CPU	0.0036	0.0200	0.0149	0.1780
	Iter.	16	55	52	927
II	CPU	0.0062	0.0185	0.0181	0.1289
	Iter.	44	80	77	1427

Table 3.4.5. Numerical results for Example 3.4.22 (Experiment 2).



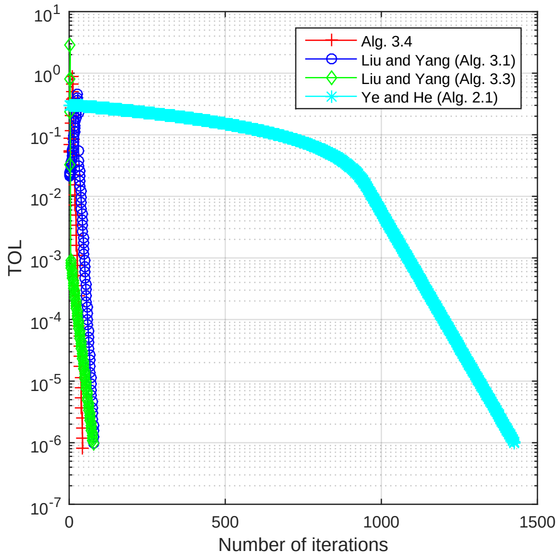
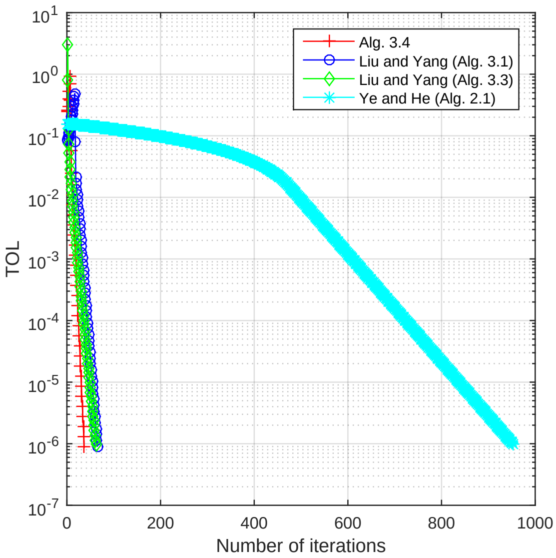


Figure 3.6: The behavior of TOL_n for Example 3.4.21 (**Experiment 2**): Left: **Case I**; Right: **Case II**.

Cases		Algorithm 3.4.4	Algorithm 3.1 in [162]	Algorithm 3.3 in [162]	Algorithm 2.1 in [269]
I	CPU	0.0049	0.0931	0.0911	0.2021
	Iter.	37	65	62	952
II	CPU	0.0041	0.0916	0.0900	0.2102
	Iter.	42	56	54	2427

3.5 On minimum-norm solutions of quasimonotone variational inequalities with fixed point constraint

The class of quasimonotone mappings are known to be more general and applicable than the classes of pseudomonotone and monotone mappings. However, only very few results can be found in the literature on quasimonotone VIPs and most of these results are on weak convergent algorithms. In this section, we study the quasimonotone VIP with constraint of FPP of quasi-pseudocontractive mappings. We introduce a new inertial Tseng's extragradient method with self-adaptive step size for approximating the minimum-norm solutions of the aforementioned problem in the framework of Hilbert spaces. We prove that the sequence generated by the proposed method converges strongly to a common (minimum-norm) solution of the quasimonotone VIP and FPP of quasi-pseudocontractive mappings without the Lipschitz continuity and sequentially weakly continuity conditions often assumed by authors when solving VIPs. We provide several numerical experiments for the proposed method in comparison with existing methods in the literature. Finally,



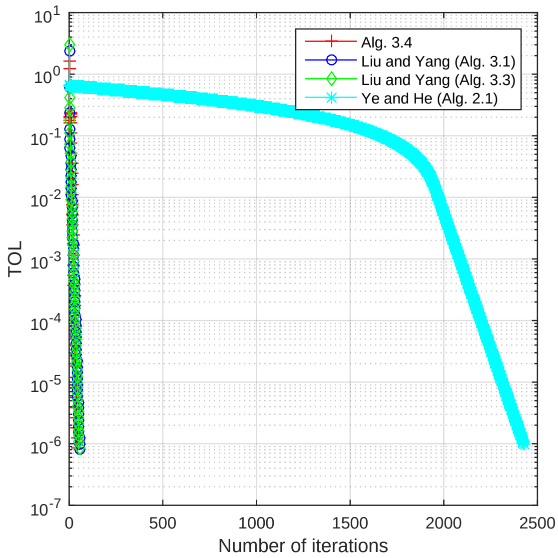


Figure 3.7: The behavior of TOL_n for Example 3.4.22 (**Experiment 2**): Left: **Case I**; Right: **Case II**.

we applied our result to image restoration problem. Our result improves, extends and generalizes several of the recently announced results in this direction.

3.5.1 Proposed method

In this section, we present our proposed method. We begin by giving the following assumptions under which our strong convergence result is obtained.

Assumption 3.5.1. *Suppose that the following conditions hold:*

(a) The feasible set $\mathcal{C}$ is a nonempty, closed and convex subsets of the real Hilbert space $\mathcal{H}$.

(b) $A : \mathcal{H} \to \mathcal{H}$ is quasimonotone and uniformly continuous on $\mathcal{H}$.

(c) The mapping $A : \mathcal{H} \to \mathcal{H}$ satisfies the following property: whenever $\{x_n\} \subset \mathcal{C}$, $\quad x_n \rightharpoonup x^*$, one has $\|Ax^*\| \le \liminf_{n\to\infty} \|Ax_n\|$.

(d) $T : \mathcal{H} \to \mathcal{H}$ is a quasi-pseudocontractive mapping such that $I - T$ is demiclosed at zero.

(e) The set $\Gamma_D = \Omega_D \cap F(T) \neq \emptyset$.

(f) $\{\alpha_n\} \subset (0,1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\quad \lim_{n\to\infty} \alpha_n = 0$, $\quad \{\tau_n\} \subset [a,b] \subset (0,1)$, $\quad 0 < \zeta < \eta < \frac{1}{1+\sqrt{1+L^2}}$.

(g) Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.

(h) Let $\{\rho_n\}$ be a nonnegative sequence such that $\sum_{n=1}^{\infty} \rho_n < +\infty$.

Algorithm 3.5.2.

Step 1: Select initial point $x_0, x_1 \in \mathcal{H}_1$, let $\lambda_1 > 0, \mu \in (0,1), \theta \geq 3$ and set $n = 1$. Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.} \end{cases} \tag{3.5.1}$$

Step 2: Compute

$$w_n = (1-\alpha_n)\Big(x_n + \theta_n(x_n - x_{n-1})\Big).$$

Step 3: Compute

$$y_n = P_{\mathcal{C}}(w_n - \lambda_n A w_n).$$

If $w_n = y_n$ and $Tw_n = w_n$, then stop: w_n is a solution of the Problem (1.2.3). Otherwise, go to **Step 4.**

Step 4: Compute

$$z_n = y_n - \lambda_n(Ay_n - Aw_n).$$

Step 5 Compute

$$x_{n+1} = (1-\tau_n)w_n + \tau_n\Big((1-\zeta)I + \zeta T((1-\eta)I + \eta T)\Big)z_n,$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n + \rho_n\right\} & \text{if} \quad Aw_n - Ay_n \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases} \tag{3.5.2}$$

Set $n := n+1$ and go back to **Step 1**.

Remark 3.5.1.

- Step 1 of Algorithm 3.5.2 is easily implemented since the knowledge of the value of $\|x_n - x_{n-1}\|$ is prior known before choosing θ_n.
- The algorithm involves only one projection onto the feasible set $\mathcal{C}$ per iteration, which makes the implementation of the algorithm computationally less expensive.
- Step 5 of Algorithm 3.5.2 guarantees the strong convergence of the algorithm to a point in the solution set of the problem.

- The step size (3.5.2) is self adaptive, and generates a non-monotonic sequence of step sizes. This makes the implementation of Algorithm 3.5.2 possible without the prior knowledge of the Lipschitz constant.
- The condition (c) is strictly weaker than the sequentially weakly continuity condition, which is commonly used in the literature when solving pseudomonotone VIP.

Remark 3.5.2. By conditions (f) and (g), from (3.5.1) we observe that

$$\lim_{n\to\infty} \theta_n||x_n - x_{n-1}|| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0. \tag{3.5.3}$$

First, we prove some lemmas which will be employed in proving our strong convergence theorem for the proposed algorithm.

3.5.2 Convergence analysis

Lemma 3.5.3. *Let $\{\lambda_n\}$ be a sequence generated by Algorithm 3.5.2. Then, we have $\lim_{n\to\infty} \lambda_n = \lambda$, where $\lambda \in \Big[min\{\frac{\mu}{\mathcal{K}}, \lambda_1\}, \lambda_1 + \Phi\Big]$ and $\Phi = \sum_{n=1}^{\infty} \rho_n$.*

Proof. Since A is uniformly continuous, then by (2.1.4) it follows that for any given $\epsilon > 0$, there exists $M < \infty$ such that $\|Aw_n - Ay_n\| \leq M\|w_n - y_n\| + \epsilon$. Thus, when $Aw_n - Ay_n \neq 0$ for all $n \geq 1$ we have

$$\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|} \geq \frac{\mu\|w_n - y_n\|}{M\|w_n - y_n\| + \epsilon} = \frac{\mu\|w_n - y_n\|}{(M + \epsilon_1)\|w_n - y_n\|} = \frac{\mu}{\mathcal{K}},$$

where $\epsilon = \epsilon_1\|w_n - y_n\|$ for some $\epsilon_1 \in (0, 1)$ and $\mathcal{K} = M + \epsilon_1$. Therefore, by the definition of λ_{n+1}, the sequence $\{\lambda_n\}$ has lower bound $\min\{\frac{\mu}{\mathcal{K}}, \lambda_1\}$ and upper bound $\lambda_1 + \Phi$. By Lemma 2.5.34, it follows that $\lim_{n\to\infty} \lambda_n$ exists and denoted by $\lambda = \lim_{n\to\infty} \lambda_n$. Clearly, we have $\lambda \in \big[\min\{\frac{\mu}{\mathcal{K}}, \lambda_1\}, \lambda_1 + \Phi\big]$.

□

Remark 3.5.4. The step size generated in Algorithm 3.5.2 is permitted to increase per iteration which reduces its dependence on the initial step size. The step size may not increase when n is large enough. In our convergence analysis, we assume that Algorithm 3.5.2 does not terminate in finite number of iterations.

Lemma 3.5.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.5.2 under Assumption 3.5.1. Then*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \Big(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\Big)\|y_n - w_n\|^2.$$

Proof. From (3.5.2), we obtain

$$\lambda_{n+1} = \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n + \rho_n\right\} \leq \frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|},$$

which implies that

$$\|Aw_n - Ay_n\| \leq \frac{\mu}{\lambda_{n+1}}\|w_n - y_n\|, \quad \forall n \geq 1. \tag{3.5.4}$$

Let $p \in \Gamma_D$. From Lemma 2.1.1 and the definition of z_n in **Step 4**, we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|y_n - \lambda_n(Ay_n - Aw_n) - p\|^2 \\
&= \|y_n - p\|^2 + \lambda_n^2\|Ay_n - Aw_n\|^2 - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|w_n - p\|^2 + \|y_n - w_n\|^2 + 2\langle y_n - w_n, w_n - p\rangle + \lambda_n^2\|Ay_n - Aw_n\|^2 \\
&- 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|w_n - p\|^2 + \|y_n - w_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle + 2\langle y_n - w_n, y_n - p\rangle \\
&- 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle + \lambda_n^2\|Ay_n - Aw_n\|^2 \\
&= \|w_n - p\|^2 - \|y_n - w_n\|^2 + 2\langle y_n - w_n, y_n - p\rangle \\
&- 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle + \lambda_n^2\|Ay_n - Aw_n\|^2.
\end{aligned} \tag{3.5.5}$$

Since $y_n = P_{\mathcal{C}}(w_n - \lambda_n w_n)$ and $p \in \Gamma_D \subset C$, we obtain from the characteristic property of $P_{\mathcal{C}}$ that

$$\langle y_n - w_n + \lambda_n Aw_n, y_n - p\rangle \leq 0,$$

which implies that

$$\langle y_n - w_n, y_n - p\rangle \leq -\lambda_n\langle Aw_n, y_n - p\rangle. \tag{3.5.6}$$

Also since $y_n \in \mathcal{C}$ and $p \in \Gamma_D \subset \Omega_D$, we have

$$\langle Ay_n, \quad y_n - p\rangle \geq 0, \quad \forall \ \ n \geq 0. \tag{3.5.7}$$

Applying (3.5.4), (3.5.6) and (3.5.7) in (3.5.5), we obtain

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|w_n - p\|^2 - \|y_n - w_n\|^2 - 2\lambda_n\langle Aw_n, y_n - p\rangle - 2\lambda_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&+ \lambda_n^2\|Ay_n - Aw_n\|^2 \\
&= \|w_n - p\|^2 - \|y_n - w_n\|^2 - 2\lambda_n\langle Ay_n, y_n - p\rangle + \lambda_n^2\|Ay_n - Aw_n\|^2 \\
&\leq \|w_n - p\|^2 - \|y_n - w_n\|^2 + \lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2 \\
&= \|w_n - p\|^2 - \left(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|y_n - w_n\|^2.
\end{aligned} \tag{3.5.8}$$

□

Since the limit

$$\lim_{n\to\infty}\left(1-\frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)=1-\mu^2>0, \tag{3.5.9}$$

there exists $n_0>0$ such that for all $n>n_0$, we have $\left(1-\frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)>0$.

Hence, applying (3.5.9) to (3.5.8), we have that

$$\|z_n-p\|\leq\|w_n-p\|. \tag{3.5.10}$$

Lemma 3.5.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.5.2 under Assumption 3.5.1. Then, $\{x_n\}$ is bounded.*

Proof. Let $p\in\Gamma_D$. From the definition of w_n in **Step 2**, we have

$$\begin{aligned}\|w_n-p\|&=\|(1-\alpha_n)(x_n+\theta_n(x_n-x_{n-1}))-p\|\\&=\|(1-\alpha_n)(x_n-p)+(1-\alpha_n)\theta_n(x_n-x_{n-1})-\alpha_np\|\\&\leq(1-\alpha_n)\|x_n-p\|+(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|+\alpha_n\|p\|\\&=(1-\alpha_n)\|x_n-p\|+\alpha_n\Big[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\|p\|\Big].\end{aligned}$$

By (3.5.3), we have

$$\lim_{n\to\infty}\Big[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\|p\|\Big]=\|p\|.$$

Thus, there exists a constant $M_1>0$ such that $(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\|p\|\leq M_1$ for all $n\in\mathbb{N}$. This implies that

$$\|w_n-p\|\leq(1-\alpha_n)\|x_n-p\|+\alpha_nM_1. \tag{3.5.11}$$

Hence, from (3.5.10) and (3.5.11) we have

$$\|z_n-p\|\leq\|w_n-p\|\leq(1-\alpha_n)\|x_n-p\|+\alpha_nM_1. \tag{3.5.12}$$

Let $K=(1-\zeta)I+\zeta T((1-\eta)I+\eta T)$ then, from the definition of x_{n+1} in **Step 5**, Lemma 2.5.27, Lemma 3.2.8 and (3.5.8), we have

$$\begin{aligned}\|x_{n+1}-p\|^2&=\|(1-\tau_n)w_n+\tau_nKz_n-p\|^2\\&=\|(1-\tau_n)(w_n-p)+\tau_n(Kz_n-p)\|^2\\&=(1-\tau_n)\|w_n-p\|^2+\tau_n\|Kz_n-p\|^2-\tau_n(1-\tau_n)\|Kz_n-w_n\|^2\\&\leq(1-\tau_n)\|w_n-p\|^2+\tau_n\|z_n-p\|^2-\tau_n(1-\tau_n)\|Kz_n-w_n\|^2\\&\leq(1-\tau_n)\|w_n-p\|^2+\tau_n\Big[\|w_n-p\|^2-\left(1-\frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2\Big]\\&-\tau_n(1-\tau_n)\|Kz_n-w_n\|^2\\&=\|w_n-p\|^2-\tau_n\left(1-\frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)\|y_n-w_n\|^2-\tau_n(1-\tau_n)\|Kz_n-w_n\end{aligned} \tag{3.5.13}$$

By the condition on τ_n, (3.5.9) and (3.5.11), we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq \|w_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n M_1 \\ &\leq \max \ \{\|x_n - p\|, \ \ M_1\} \\ &\ \vdots \\ &\leq \max \ \{\|x_{n_0} - p\|, \ \ M_1\}.\end{aligned}$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{w_n\}, \{y_n\}$ and $\{z_n\}$ are also bounded. $\square$

Lemma 3.5.7. *Assume that $\{w_n\}$ and $\{y_n\}$ are sequences generated by Algorithm 3.5.2 under Assumption 3.5.1 and $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$. Suppose there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\{w_{n_k}\}$ converges weakly to some $x^* \in \mathcal{H}$ as $k \to \infty$, then we having one of the following: $x^* \in \Omega_D$ or $Ax^* = 0$.*

Proof. Since $\{w_n\}$ is bounded, then $w_\omega(w_n)$ is nonempty. Let $x^* \in w_\omega(w_n)$ be an arbitrary element. Then, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \rightharpoonup x^*$ as $k \to \infty$. By the hypothesis of the lemma we have that $y_{n_k} \rightharpoonup x^* \in \mathcal{C}$ as $k \to \infty$. Now, we divide the proof into two cases.

Case 1: If $\limsup_{k\to\infty} \|Ay_{n_k}\| = 0$, then $\lim_{k\to\infty} \|Ay_{n_k}\| = \liminf_{k\to\infty} \|Ay_{n_k}\| = 0$. Since $\{y_{n_k}\}$ converges weakly to $x^* \in \mathcal{C}$ and A satisfies condition (c), we have

$$0 \leq \|Ax^*\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\| = 0.$$

Hence, we obtain $Ax^* = 0$.

Case 2: If $\limsup_{k\to\infty} \|Ay_{n_k}\| > 0$, without loss of generality, we let $\lim_{k\to\infty} \|Ay_{n_k}\| = N > 0$. Then there exists a $K' \in \mathbb{N}$ such that $\|Ay_{n_k}\| > \frac{N}{2}, \quad \forall\ k \geq K'$. By the characteristic property of $P_{\mathcal{C}}$, we have

$$\langle w_{n_k} - \lambda_{n_k} Aw_{n_k} - y_{n_k}, \ \ x - y_{n_k}\rangle \leq 0, \ \ x \in \mathcal{C},$$

which implies that

$$\frac{1}{\lambda_{n_k}} \langle w_{n_k} - y_{n_k}, \ \ x - y_{n_k}\rangle \leq \langle Aw_{n_k}, \ x - y_{n_k}\rangle, \qquad \forall\ x \in \mathcal{C}.$$

Consequently, we have

$$\frac{1}{\lambda_{n_k}} \langle w_{n_k} - y_{n_k}, \ x - y_{n_k}\rangle + \langle Aw_{n_k}, \ \ y_{n_k} - w_{n_k}\rangle \leq \langle Aw_{n_k}, \ \ x - w_{n_k}\rangle, \qquad \forall\ x \in \mathcal{C}. \tag{3.5.14}$$

Applying $\lim_{k\to\infty} \lambda_{n_k} = \lambda > 0$ and the hypothesis of the lemma to (3.5.14), we have

$$0 \leq \liminf_{k\to\infty} \langle Aw_{n_k}, \ \ x - w_{n_k}\rangle \leq \limsup_{k\to\infty} \langle Aw_{n_k}, \ x - w_{n_k}\rangle < +\infty, \qquad \forall\ x \in \mathcal{C}. \tag{3.5.15}$$

Observe that

$$\langle Ay_{n_k},\ x-y_{n_k}\rangle = \langle Ay_{n_k}-Aw_{n_k},\ x-w_{n_k}\rangle + \langle Aw_{n_k},\ x-w_{n_k}\rangle + \langle Ay_{n_k},\ w_{n_k}-y_{n_k}\rangle.$$

Since A is uniformly continuous on $\mathcal{H}$, we have from the hypothesis of the lemma that

$$\lim_{k\to\infty}\|Aw_{n_k}-Ay_{n_k}\| = 0. \tag{3.5.16}$$

From the hypothesis of the lemma, (3.5.14), (3.5.15) and (3.5.16), we have

$$0 \le \liminf_{k\to\infty}\langle Ay_{n_k},\ x-y_{n_k}\rangle \le \limsup_{k\to\infty}\langle Ay_{n_k},\ x-y_{n_k}\rangle < +\infty, \qquad \forall\, x\in\mathcal{C}. \tag{3.5.17}$$

If $\limsup_{k\to\infty}\langle Ay_{n_k},\ x-y_{n_k}\rangle > 0, \quad \forall\ x\in\mathcal{C}$, then there exists a subsequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ such that

$$\lim_{j\to\infty}\left\langle Ay_{n_{k_j}},\ x-y_{n_{k_j}}\right\rangle > 0.$$

Thus, there exists $j_0\in\mathbb{N}$ such that $\left\langle Ay_{n_{k_j}},\ x-y_{n_{k_j}}\right\rangle > 0, \quad \forall\ j\ge j_0$, which by the quasimonotonicity of A on $\mathcal{H}$ implies that $\left\langle Ax,\ x-y_{n_{k_j}}\right\rangle \ge 0,\ \forall x\in\mathcal{C},\ j\ge j_0$. Letting $j\to\infty$, we have $x^*\in\Omega_D$.

If $\limsup_{k\to\infty}\langle Ay_{n_k},\ x-y_{n_k}\rangle = 0, \quad \forall\ x\in\mathcal{C}$, we obtain from (3.5.17) that

$$\lim_{k\to\infty}\langle Ay_{n_k},\ x-y_{n_k}\rangle = \limsup_{k\to\infty}\langle Ay_{n_k},\ x-y_{n_k}\rangle = \liminf_{k\to\infty}\langle Ay_{n_k},\ x-y_{n_k}\rangle = 0. \tag{3.5.18}$$

Let $\delta_k = \Big|\langle Ay_{n_k},\ x-y_{n_k}\rangle\Big| + \frac{1}{k+1}$. Then, we have

$$\langle Ay_{n_k},\ x-y_{n_k}\rangle + \delta_k > 0. \tag{3.5.19}$$

Let $u_{n_k} = \frac{Ay_{n_k}}{\|Ay_{n_k}\|^2}\ \forall\ k\ge K'$, we have $\langle Ay_{n_k},\ u_{n_k}\rangle = 1$. Then, from (3.5.19) we have that for all $k\ge K'$

$$\langle Ay_{n_k},\ x+\delta_k u_{n_k}-y_{n_k}\rangle > 0.$$

By the quasimonotonicity of A, we have for all $k\ge K'$

$$\langle A(x+\delta_k u_{n_k}),\ x+\delta_k u_{n_k}-y_{n_k}\rangle \ge 0. \tag{3.5.20}$$

Consequently, by applying (2.1.4) and (3.5.20) we have $\forall\ x\in\mathcal{C}, k\ge K'$

$$\begin{aligned}
\langle Ax, x+\delta_k u_{n_k}-y_{n_k}\rangle &= \langle Ax - A(x+\delta_k u_{n_k}), x+\delta_k u_{n_k}-y_{n_k}\rangle \\
&\quad + \langle A(x+\delta_k u_{n_k}), x+\delta_k u_{n_k}-y_{n_k}\rangle \\
&\ge \langle Ax - A(x+\delta_k u_{n_k}), x+\delta_k u_{n_k}-y_{n_k}\rangle \\
&\ge -\|Ax - A(x+\delta_k u_{n_k})\|\|x+\delta_k u_{n_k}-y_{n_k}\| \\
&\ge -\delta_k(M'+{\epsilon_2}')\|u_{n_k}\|\|x+\delta_k u_{n_k}-y_{n_k}\| \\
&\quad (\text{where}\ \ \epsilon' = {\epsilon_2}'\|\delta_k u_{n_k}\|\ \ \text{for some}\ \ {\epsilon_2}'\in(0,1), M'<+\infty) \\
&= -\delta_k\frac{(M'+{\epsilon_2}')}{\|Ay_{n_k}\|}\|x+\delta_k u_{n_k}-y_{n_k}\| \\
&\ge -\delta_k\frac{2(M'+{\epsilon_2}')}{N}\|x+\delta_k u_{n_k}-y_{n_k}\|.
\end{aligned} \tag{3.5.21}$$

Letting $k \to \infty$ in (3.5.21), and applying the fact that $\lim_{k\to\infty} \delta_k = 0$ together with the boundedness of $\{\|x + \delta_k u_{n_k} - y_{n_k}\|\}$, we have

$$\langle Ax, x - x^* \rangle \geq 0, \quad \forall\, x \in \mathcal{C},$$

which implies that $x^* \in \Omega_D$ as desired.

□

Lemma 3.5.8. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.5.2 under Assumption 3.5.1. Then,*

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n d_n - \tau_n\Big(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\Big)\|y_n - w_n\|^2 \\ - \tau_n(1-\tau_n)\|Kz_n - w_n\|^2.\end{aligned}$$

Proof. Let $p \in \Gamma_D$. From Lemma 2.1.1 and the definition of w_n, we have

$$\begin{aligned}\|w_n - p\|^2 &= \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(x_n - x_{n-1}) - \alpha_n p\|^2 \\ &\leq \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(x_n - x_{n-1})\|^2 + 2\alpha_n\langle -p, \quad w_n - p\rangle \\ &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2(1-\alpha_n)\theta_n\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &\quad + 2\alpha_n\langle -p, \quad w_n - x_{n+1}\rangle + 2\alpha_n\langle -p, \quad x_{n+1} - p\rangle\end{aligned} \tag{3.5.22}$$

Applying (3.5.22) in (3.5.13), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2(1-\alpha_n)\theta_n\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &\quad + 2\alpha_n\langle -p, \quad w_n - x_{n+1}\rangle + 2\alpha_n\langle -p, \quad x_{n+1} - p\rangle - \tau_n\Big(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\Big)\|y_n - w_n\|^2 \\ &\quad - \tau_n(1-\tau_n)\|Kz_n - w_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 \\ &\quad + \alpha_n\Big[2(1-\alpha_n)\|x_n - p\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\quad + 2\|p\|\|w_n - x_{n+1}\| + 2\langle p, \quad p - x_{n+1}\rangle\Big] - \tau_n\Big(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\Big)\|y_n - w_n\|^2 \\ &\quad - \tau_n(1-\tau_n)\|Kz_n - w_n\|^2 \\ &= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n d_n - \tau_n\Big(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\Big)\|y_n - w_n\|^2 \\ &\quad - \tau_n(1-\tau_n)\|Kz_n - w_n\|^2,\end{aligned}$$

where

$$\begin{aligned}d_n = \Big[2(1-\alpha_n)\|x_n - p\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ +2\|p\|\|w_n - x_{n+1}\| + 2\langle p, \quad p - x_{n+1}\rangle\Big].\end{aligned}$$

This completes the proof. □

Theorem 3.5.9. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.5.2 such that Assumption 3.5.1 holds and $Ax \neq 0, \forall x \in C$. Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in \Gamma_D \subset \Gamma$, where $\bar{x} = P_{\Gamma_D}(0)$.*

Proof. Let $\bar{x} \in \Gamma_D \subset \Gamma$, where $\bar{x} = P_{\Gamma_D}(0)$. Then, it follows from Lemma 3.5.8 that

$$\|x_{n+1} - \bar{x}\|^2 \leq (1-\alpha_n)\|x_n - \bar{x}\|^2 + \alpha_n d_n \tag{3.5.23}$$

where

$$d_n = \Big[2(1-\alpha_n)\|x_n - \bar{x}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ +2\|\bar{x}\|\|w_n - x_{n+1}\| + 2\langle \bar{x}, \quad \bar{x} - x_{n+1}\rangle\Big].$$

Now, we claim that the sequence $\{\|x_n - \bar{x}\|\}$ converges to zero. To show this, by Lemma 2.5.36 it suffices to show that $\limsup_{k\to\infty} d_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - \bar{x}\|\}$ of $\{\|x_n - \bar{x}\|\}$ satisfying

$$\liminf_{k\to\infty} \left(\|x_{n_{k+1}} - \bar{x}\| - \|x_{n_k} - \bar{x}\|\right) \geq 0. \tag{3.5.24}$$

Suppose that $\{\|x_{n_k} - \bar{x}\|\}$ is a subsequence of $\{\|x_n - \bar{x}\|\}$ such that (3.5.24) holds. Again, from Lemma 3.5.8, we obtain

$$\tau_{n_k}\left(1 - \frac{\lambda_{n_k}^2\mu^2}{\lambda_{n_{k+1}}^2}\right)\|y_{n_k} - w_{n_k}\|^2 + \tau_{n_k}(1-\tau_{n_k})\|Kz_{n_k} - w_{n_k}\|^2 \leq (1-\alpha_{n_k})\|x_{n_k} - \bar{x}\|^2 \\ - \|x_{n_{k+1}} - \bar{x}\|^2 + \alpha_{n_k} d_{n_k}.$$

By (3.5.24) and the condition on α_{n_k} it follows that

$$\lim_{k\to\infty}\left[\tau_{n_k}\left(1 - \frac{\lambda_{n_k}^2\mu^2}{\lambda_{n_{k+1}}^2}\right)\|y_{n_k} - w_{n_k}\|^2 + \tau_{n_k}(1-\tau_{n_k})\|Kz_{n_k} - w_{n_k}\|^2\right] = 0.$$

By (3.5.9) and the conditions on the control parameters, we obtain

$$\lim_{k\to\infty}\|y_{n_k} - w_{n_k}\| = 0, \qquad \lim_{k\to\infty}\|Kz_{n_k} - w_{n_k}\| = 0. \tag{3.5.25}$$

From (3.5.25), we have

$$\lim_{k\to\infty}\|Kz_{n_k} - y_{n_k}\| = 0.$$

Moreover, from the definition of z_n and by applying (3.5.4) and (3.5.25), we get

$$\|z_{n_k} - y_{n_k}\| = \lambda_n\|Aw_{n_k} - Ay_{n_k}\| \leq \frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}}\|w_{n_k} - y_{n_k}\| \to 0, \quad k\to\infty. \tag{3.5.26}$$

From the definition of $x_{n_{k+1}}$ in **Step 5** and (3.5.25), we have

$$\begin{aligned}\|x_{n_{k+1}} - w_{n_k}\| &= \|(1-\tau_{n_k})w_{n_k} + \tau_{n_k} K z_{n_k} - w_{n_k}\| \\ &= \tau_{n_k}\|K z_{n_k} - w_{n_k}\| \to 0, \quad k \to \infty. \end{aligned} \tag{3.5.27}$$

From (3.5.25) and (3.5.27), we obtain

$$\lim_{k\to\infty} \|x_{n_{k+1}} - y_{n_k}\| = 0. \tag{3.5.28}$$

Now, from **Step 2** and by Remark 3.5.3, we get

$$\begin{aligned}\|w_{n_k} - x_{n_k}\| &= \|(1-\alpha_{n_k})(x_{n_k} + \theta_{n_k}(x_{n_k} - x_{n_k-1})) - x_{n_k}\| \\ &= \|(1-\alpha_{n_k})(x_{n_k} - x_{n_k}) + (1-\alpha_{n_k})\theta_{n_k}(x_{n_k} - x_{n_k-1}) - \alpha_{n_k}x_{n_k}\| \\ &\leq (1-\alpha_{n_k})\|x_{n_k} - x_{n_k}\| + (1-\alpha_{n_k})\theta_{n_k}\|x_{n_k} - x_{n_k-1}\| + \alpha_{n_k}\|x_{n_k}\| \\ &\to 0, \quad k\to\infty. \end{aligned} \tag{3.5.29}$$

From (3.5.25)-(3.5.29), we have

$$\lim_{k\to\infty}\|y_{n_k} - x_{n_k}\| = 0, \quad \lim_{k\to\infty}\|K z_{n_k} - z_{n_k}\| = 0, \quad \lim_{k\to\infty}\|z_{n_k} - x_{n_k}\| = 0. \tag{3.5.30}$$

Moreover, from (3.5.28) and (3.5.30) we obtain

$$\lim_{k\to\infty}\|x_{n_{k+1}} - x_{n_k}\| = 0. \tag{3.5.31}$$

Since $\{x_n\}$ is bounded, we have that $w_\omega(x_n) \neq \emptyset$. Let $z \in w_\omega(x_n)$ be arbitrary. Then, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z$ as $k \to \infty$. By (3.5.29) and (3.5.30) we have $w_\omega(x_n) = w_\omega(y_n) = w_\omega(w_n)$. Since $y_n \in C$ and C is weakly-closed, we have $z \in C$. By the assumption that $Ax \neq 0, \ \forall x \in C$, it follows that $Az \neq 0$. Thus, by Lemma 3.5.7 and (3.5.25), it follows that $z \in \Omega_D$. Consequently, we have $w_\omega(x_n) \subset \Omega_D$. Also, since $\lim_{n\to\infty}\|x_{n_k} - z_{n_k}\| = 0$, we have $z_{n_k} \rightharpoonup z$ as $k \to \infty$. Since $I - K$ is demiclosed at zero, it follows from (3.5.30) and Lemma 3.2.8(a) that $z \in F(K) = F(T)$. Consequently, we have $w_\omega(x_n) \subset F(T)$. It therefore follows that $w_\omega(x_n) \subset \Gamma_D$.

Next, by the boundedness of $\{x_{n_k}\}$, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to some $\hat{x} \in \mathcal{H}$, and such that

$$\lim_{j\to\infty}\left\langle \bar{x}, \ \bar{x} - x_{n_{k_j}}\right\rangle = \limsup_{k\to\infty}\left\langle \bar{x}, \ \bar{x} - x_{n_k}\right\rangle. \tag{3.5.32}$$

From (3.5.32) and the fact that $\bar{x} = P_{\Gamma_D}(0)$ we have

$$\limsup_{k\to\infty}\langle \bar{x}, \ \ \bar{x} - x_{n_k}\rangle = \lim_{j\to\infty}\left\langle \bar{x}, \ \ \bar{x} - x_{n_{k_j}}\right\rangle = \langle \bar{x}, \ \ \bar{x} - \hat{x}\rangle \leq 0. \tag{3.5.33}$$

From (3.5.31) and (3.5.33), it follows that

$$\limsup_{k\to\infty}\left\langle \bar{x}, \quad \bar{x}-x_{n_{k+1}}\right\rangle = \limsup_{k\to\infty}\langle \bar{x}, \quad \bar{x}-x_{n_k}\rangle = \langle \bar{x}, \quad \bar{x}-\hat{x}\rangle \leq 0. \tag{3.5.34}$$

Thus, by (3.5.34) and the fact that $\lim_{k\to\infty}\|x_{n_k+1}-w_{n_k}\| = \lim_{k\to\infty}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\| = 0$, we obtain $\limsup_{k\to\infty} d_{n_k} \leq 0$. Now, applying Lemma 2.5.36 to (3.5.23) we have $\lim_{n\to\infty}\|x_n-\bar{x}\| = 0$. Therefore, $\{x_n\}$ converges strongly to $\bar{x}$. □

3.5.3 Numerical experiments

In this section, we discuss the numerical behavior of our proposed Algorithm 3.5.2 and also compare it with the methods of Salahuddin [217] (see Appendix 3.5.15), Liu et.al [162] (see Appendix 3.5.16), Ogwo *et al.* [193] (see Appendix 3.5.3), Alakoya *et al.* [12] (see Appendix 3.5.4) and Yin et.al [270] (see Appendix 3.5.17). We perform all implementations using Matlab 2016 (b), installed on a personal computer with Intel(R) Core(TM) i5-2600 CPU@2.30GHz and 8.00 Gb-RAM running on Windows 10 operating system. In Tables 3.5.1-3.5.2, "No. of Iter." means the number of iterations.

In our computation, we choose $\theta = 3.5,\ \lambda_1 = \lambda = s = 0.65,\ \mu = 0.7,\ \epsilon_n = \frac{1}{(2n+1)^3},\ \alpha_n = \frac{1}{2n+1},\ \tau_n = \frac{n+1}{2(n+3)},\ \rho_n = \frac{10}{(n+1)^2},\ f(x) = \frac{x}{2}$ and $Sx = \frac{x}{3}$. For Appendix 3.5.3, we choose $\psi_n = \frac{3n+1}{10n+5}$ and for Appendix 3.5.17, we choose $\varphi_n = \frac{n}{2n+7},\quad \chi_n = \frac{n}{2n+5},\quad \phi_n = \frac{n}{10n+2}$.

We begin by presenting some examples in the finite dimensional Hilbert space.

Example 3.5.10. Let $\mathcal{C} = [-1, 1]$ and define

$$Ax = \begin{cases} 2x-1, & x > 1, \\ x^2, & x \in [-1,1], \\ -2x-1, & x < -1. \end{cases}$$

Then, A is quasimonotone and uniformly continuous but neither pseudomonotone nor monotone. Moreover, let $Tx = x$. Then, T is 1-Lipschitz quasi-pseudocontractive. Let $\zeta = 0.3, \eta = 0.4$.

We consider the following cases for the numerical experiments of this example.

Case 1: Take $x_1 = 0.3$ and $x_0 = 0.5$.

Case 2: Take $x_1 = 0.29$ and $x_0 = 0.66$.

Case 3: Take $x_1 = 0.28$ and $x_0 = 0.72$.

Case 4: Take $x_1 = 0.30$ and $x_0 = 0.48$.

Table 3.5.1: Numerical results for Example 3.5.10

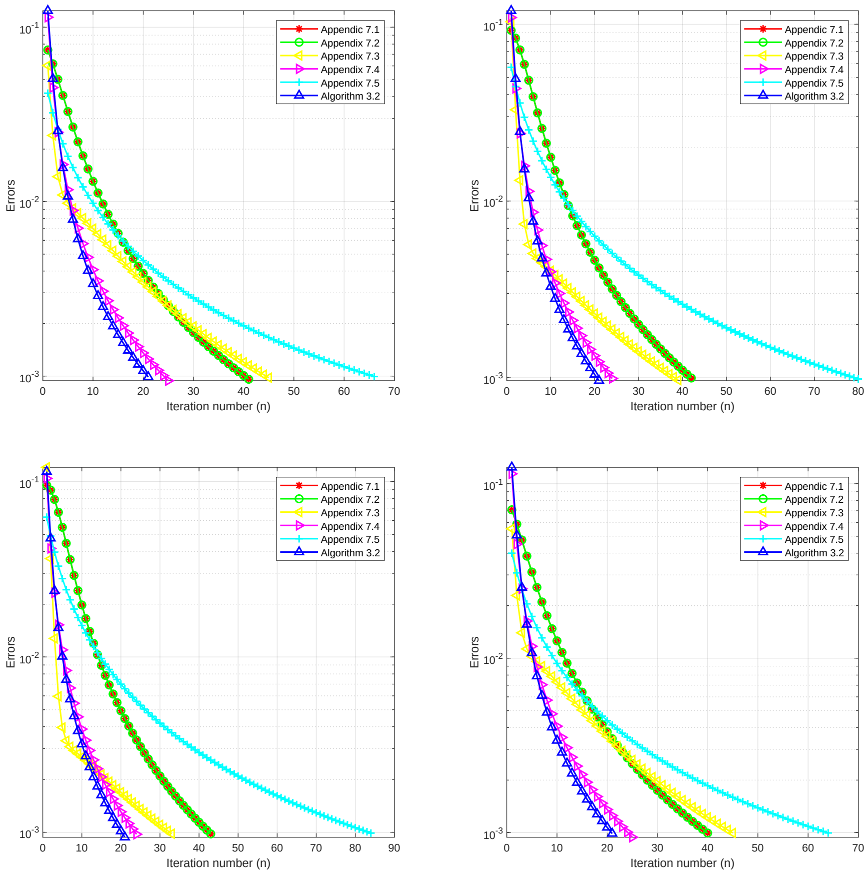


Figure 3.8: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

Cases		App. 3.5.15	App 3.5.16	App. 3.5.3	App. 3.5.4	App. 3.5.17	Alg. 3.5.2
Case 1	No. of Iter. CPU time (sec)	41 0.0063	41 0.0128	45 0.0238	25 0.0084	66 0.0091	21 0.0083
Case 2	No. of Iter. CPU time (sec)	42 0.0106	42 0.0067	39 0.0081	24 0.0057	80 0.0079	21 0.0067
Case 3	No. of Iter. CPU time (sec)	40 0.0106	40 0.0064	45 0.0087	25 0.0059	64 0.0056	21 0.0062
Case 4	No. of Iter. CPU time (sec)	43 0.0107	43 0.0059	33 0.0083	24 0.0059	84 0.0063	21 0.0056

Example 3.5.11. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$A(x_1, x_2) = (-x_1 e^{x_2}, x_2)$$

and

$$\mathcal{C} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1, \quad 0 < x_1\}.$$

Moreover, we define $Tx = -\frac{5}{3}x$. Then, T is $\frac{5}{3}$-Lipschitz quasi-pseudocontractive. Let $\zeta = 0.28, \eta = 0.32$. It can be easily verified that all the conditions imposed in Algorithm 3.5.2 are satisfied. We choose $T(x_1, x_2) = \Big(-\frac{5}{4}x_1, x_2\Big)$. One can easily show that T is $-\frac{5}{4}$-Lipschitz continuous and quasi-pseudocontractive. We consider the following cases for the numerical experiments of this example.

Case 1: Take $x_1 = (0.1, 0.2)^{\mathcal{T}}$ and $x_0 = (0.2, 0.9)^{\mathcal{T}}$.

Case 2: Take $x_1 = (0.1, 0.2)^{\mathcal{T}}$ and $x_0 = (0.3, 0.8)^{\mathcal{T}}$.

Case 3: Take $x_1 = (0.1, 0.1)^{\mathcal{T}}$ and $x_0 = (0.2, 0.7)^{\mathcal{T}}$.

Case 4: Take $x_1 = (0.1, 0.1)^{\mathcal{T}}$ and $x_0 = (0.3, 0.6)^{\mathcal{T}}$.

Table 3.5.2: Numerical results for Example 3.5.11

Cases		App. 3.5.15	App 3.5.16	App. 3.5.3	App. 3.5.4	App. 3.5.17	Alg. 3.5.2
Case 1	No. of Iter.	15	20	16	31	89	10
	CPU time (sec)	0.0129	0.0071	0.0090	0.0080	0.0099	0.0075
Case 2	No. of Iter.	14	20	14	31	85	10
	CPU time (sec)	0.0111	0.0066	0.0088	0.0077	0.0108	0.0078
Case 3	No. of Iter.	14	20	19	31	80	10
	CPU time (sec)	0.0117	0.0064	0.0087	0.0079	0.0100	0.0078
Case 4	No. of Iter.	14	19	18	31	75	10
	CPU time (sec)	0.0120	0.0062	0.0085	0.0079	0.0097	0.0075

Now, we present an example in the infinite dimensional Hilbert space.

Example 3.5.12. Let $\mathcal{H} = \left\{ x = (x_1, x_2, \cdots, x_i, \cdots) : \sum_{i=1}^{\infty} |x_i|^2 < +\infty \right\}$. Let $u, v \in \mathbb{R}$ such that $0 < \frac{v}{2} < u < v$. Let $\mathcal{C}_u = \left\{ x \in \mathcal{H} : \|x\| \le u \right\}$ and $A_v x = \Big(v - \|x\| \Big) x$. It is known that A is quasimonotone and Lipschitz continuous (see [217]). In this example, we take $u = 4, v = 5$. Furthermore, let $Tx = -2x$. Then, T is 2-Lipschitz quasi-pseudocontractive. Let $\zeta = 0.22, \eta = 0.27$. We consider the following cases for the numerical experiments of this example

Case 1: Take $x_1 = \Big(\frac{5}{7}, \frac{1}{7}, \frac{1}{35}, \cdots \Big)$ and $x_0 = \Big(\frac{1}{2}, \frac{1}{6}, \frac{1}{18} \Big)$.

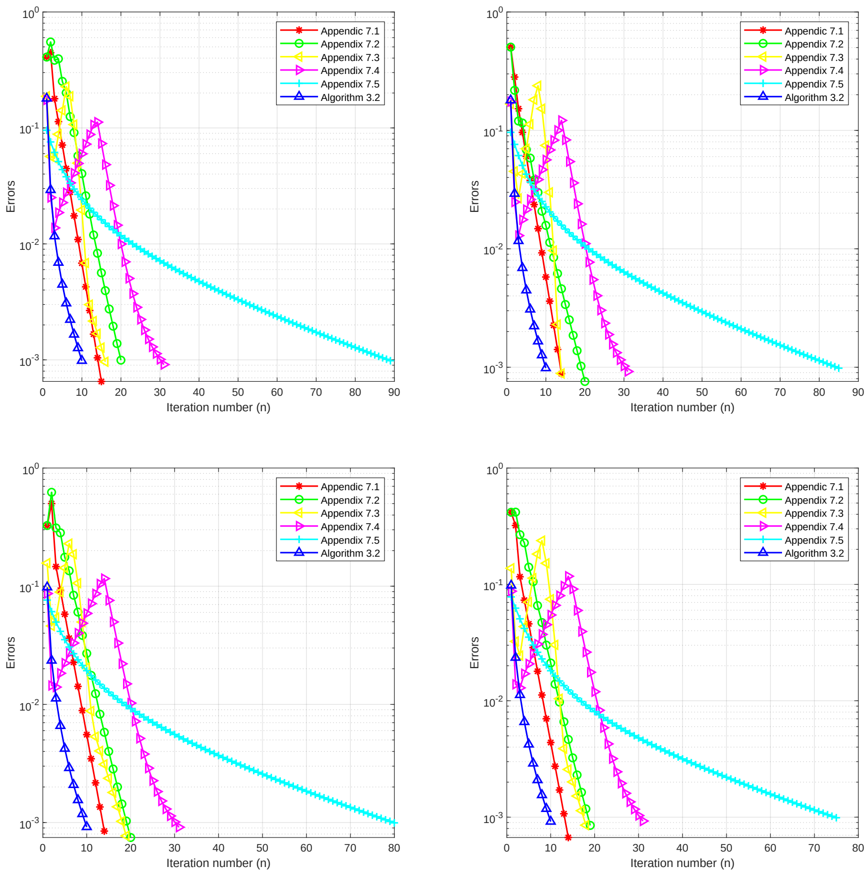


Figure 3.9: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

Case 2: Take $x_1 = \left(\frac{4}{7}, \frac{1}{7}, \frac{1}{28}, \cdots\right)$ and $x_0 = \left(\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots\right)$.

Case 3: Take $x_1 = \left(\frac{3}{4}, \frac{1}{4}, \frac{1}{12}, \cdots\right)$ and $x_0 = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{10}, \cdots\right)$.

Case 4: Take $x_1 = \left(\frac{3}{5}, \frac{1}{5}, \frac{1}{15}, \cdots\right)$ and $x_0 = \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \cdots\right)$.

Table 3.5.3: Numerical results for Example 3.5.12

Cases		App. 3.5.4	Alg. 3.5.2
Case 1	No. of Iter. CPU time (sec)	23 0.0378	18 0.0162
Case 2	No. of Iter. CPU time (sec)	23 0.0297	18 0.0116
Case 3	No. of Iter. CPU time (sec)	23 0.0243	18 0.0125
Case 4	No. of Iter. CPU time (sec)	23 0.0283	18 0.0123

Remark 3.5.13. Most of the results in the literature for approximating the solution of quasimonotone (or without monotonicity) VIP in infinite dimensional Hilbert spaces are weak convergence results. Hence, we can only compare our Algorithm 3.5.2 in infinite dimensional Hilbert spaces with Appendix 3.5.4 in Example 3.5.12.

Example 3.5.14. (**Application to Image Restoration Problem**).
In this last experiment, we consider an application to image restoration problem. We compare the performance of our Algorithm 3.5.2 with Appendix 3.5.15, Appendix 3.5.16 and Appendix 3.5.3.

We recall that the image restoration problem can be formulated as the following linear inverse problem:

$$v = Dx + e \tag{3.5.35}$$

where $x \in \mathbb{R}^N$ is the original image, $D \in \mathbb{R}^{M\times N}$ is the blurring matrix, $v \in \mathbb{R}^M$ is the observed blurred image while e is the Gaussian noise. Solving Problem (3.5.35) is equivalent to solving the convex MPs

$$\min_{x\in\mathbb{R}^N} \left\{\frac{1}{2}||Dx - v||_2^2 + \lambda||x||_1\right\}, \tag{3.5.36}$$

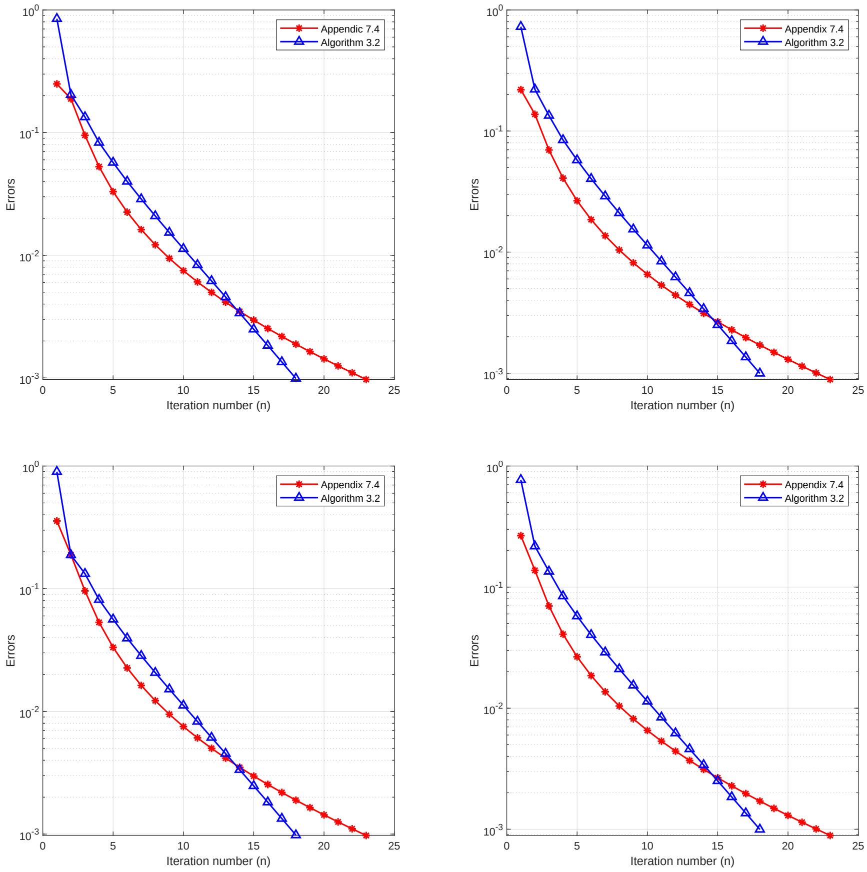


Figure 3.10: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

where $\lambda > 0$ is the regularization parameter, $\|\cdot\|_2$ denotes the Euclidean norm and $\|\cdot\|_1$ is the ℓ_1-norm. Our goal is to restore the original image x given the data of the blurred image v. We can express the MPs (3.5.36) as a VIP by taking $A := D^T(Dx - v)$. It is known in this case that the operator A is monotone and $\|D^T D\|$-Lipschitz continuous (thus A is quasimonotone and uniformly continuous). We consider the 291×240 Pout, 256×256 Cameraman 275×252 Circuit, and 128×128 Forest images from MATLAB Image Processing Toolbox. Moreover, we use the Gaussian blur of size 7×7 and standard deviation $\sigma = 4$ to create the blurred and noisy image (observed image) and use the algorithms to recover the original image from the blurred image. Also, we measure the quality of the restored image using the signal-to-noise ratio defined by

$$\text{SNR} = 20 \times \log_{10}\left(\frac{\|x\|_2}{\|x - x^*\|_2}\right),$$

where x is the original image and x^* is the restored image. Note that, the larger the SNR, the better the quality of the restored image. We choose the initial values as $x_0 = \mathbf{0} \in \mathbb{R}^N$ and $x_1 = \mathbf{1} \in \mathbb{R}^N$. The results are reported in Table 3.5.4, which shows the SNR values for each algorithm, and Figure 3.11 shows the original, blurred and restored images. The major advantages of our proposed Algorithm 3.5.2 over the algorithms in Appendix 3.5.15, Appendix 3.5.16 and Appendix 3.5.3 compared with are the higher SNR values for generating the recovered images.

Table 3.5.4: Numerical results for image restoration Problem 3.5.36

		Pout	Cameraman	Circuit	Forest
App. 3.5.15	SNR	5.91	5.82	5.82	5.91
App. 3.5.16	SNR	9.78	12.40	13.36	9.19
App. 3.5.3	SNR	9.78	9.52	9.55	9.79
Alg. 3.5.2	SNR	18.93	18.63	16.07	23.79

Appendix 3.5.15. *Algorithm 1 of Salahuddin [217]*

Initialization: *Let* $x_0 \in \mathcal{C}$ *and* $\lambda \in [a, b]$, *where* $0 < a \le b < \frac{1}{L}$ *be arbitrary.*

Iterative Steps: *Calculate* x_{n+1} *as follows:*

Step 0. *Set* $n = 0$.

Step 1. *If*

$$x_n = P_{\mathcal{C}}(x_n - \lambda A x_n),$$

then stop and x_n *is the solution of VIP* (1.2.1). *Otherwise, set*

Step 2:

$$y_n = P_{\mathcal{C}}(x_n - \lambda A x_n),$$

and

$$x_{n+1} = P_{\mathcal{C}}(x_n - \lambda A y_n),$$

Set $n := n + 1$ *and return to* **Step 1**.

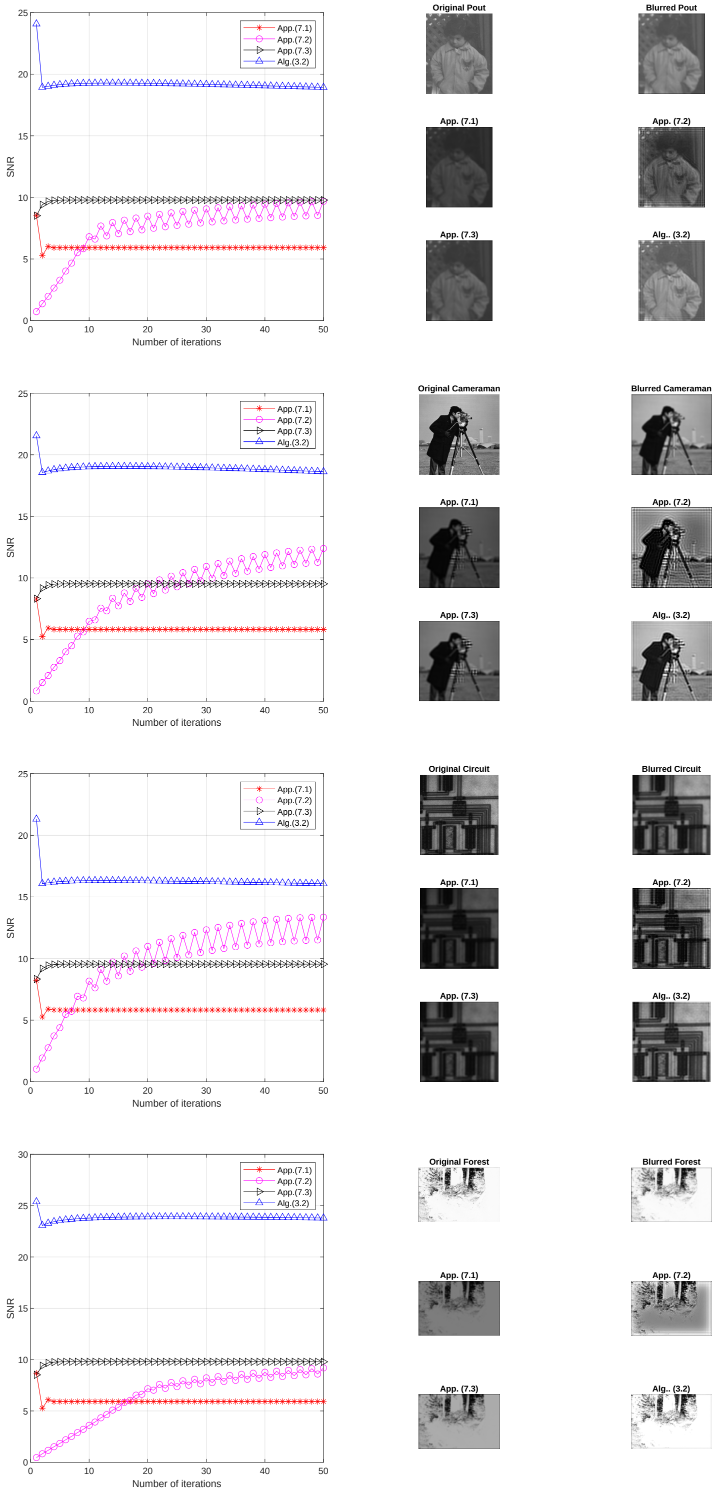


Figure 3.11: Comparison for numerical results for image restoration Problem (3.5.36) : Top: Pout; Next: Cameraman; Next:Circuit; Bottom: Forest.

Appendix 3.5.16. *Algorithm 1 of Liu et.al [162]*

Step 0: *Take* $\lambda_0 > 0, \quad x_0 \in \mathcal{H}, \quad 0 < \mu < 1$. *Choose a nonnegative real sequence* $\{\rho_n\}$ *such that* $\sum_{n=0}^{\infty} \rho_n < +\infty$.

Step 1: *Given the current iterate* x_n, *compute*

$$y_n = P_{\mathcal{C}}(x_n - \lambda_n A x_n).$$

If $x_n = y_n$, *(or* $Ay_n = 0$*), then stop and* y_n *is a solution of the VIP. Otherwise, go to* **Step 2.**

Step 2: *Compute*

$$x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n),$$

where

$$\lambda_{n+1} = \begin{cases} min\Big\{\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \rho_n\Big\} & if \quad Ax_n - Ay_n \neq 0, \\ \lambda_n + \rho_n, & otherwise. \end{cases} \tag{3.5.37}$$

Set $n := n + 1$ *and go back to* **Step 1**.

Algorithm 3.5.3. Algorithm 2 of Ogwo *et al.* [193]

Step 0: Choose sequences $\{\psi_n\}$ and $\{\tau_n\}$ such that $\psi_n \in [0, 1)$ and $\tau_n \in (0, 1]$ for all $n \geq 1$. Let $\lambda_1 > 0$, $\mu \in (0, 1)$ and $x_0, x_1 \in \mathcal{H}$ be given arbitrarily. Choose a nonnegative real sequence $\{\rho_n\}$ such that $\sum_{n=1}^{\infty} \rho_n < \infty$. Set $n := 1$.

Step 1: Given the current iterates x_{n-1} and $x_n \quad (n \geq 1)$, compute

$$w_n = x_n + \psi_n(x_n - x_{n-1})$$

and

$$y_n = P_C(w_n - \lambda_n A w_n).$$

If $w_n = y_n$: STOP. Otherwise, go to **Step 2.**

Step 2: Construct the half-space

$$T_n = \{x \in \mathcal{H} : \langle w_n - \lambda_n A w_n - y_n,\ x - y_n \rangle \leq 0\}.$$

Then, compute

$$z_n = P_{T_n}(w_n - \lambda_n A y_n)$$

and

$$x_{n+1} = (1 - \tau_n) w_n + \tau_n z_n,$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\left(\|w_n-y_n\|^2+\|z_n-y_n\|^2\right)}{2\langle Aw_n-Ay_n,z_n-y_n\rangle},\ \lambda_n+\rho_n\right\}, & \text{if } \langle Aw_n - Ay_n, z_n - y_n\rangle > 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases} \tag{3.5.38}$$

Set $n := n + 1$ and return to **Step 1.**

Algorithm 3.5.4. Algorithm 1 of Alakoya *et al.* [12]

Step 1: Select initial point $x_0, x_1 \in \mathcal{H}_1$. Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose ϑ_n such that $0 \leq \vartheta_n \leq \bar{\vartheta}_n$, where

$$\bar{\vartheta}_n := \begin{cases} \min\left\{\vartheta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \vartheta, & \text{otherwise.} \end{cases} \tag{3.5.39}$$

Step 2: Compute

$$w_n = x_n + \vartheta_n(x_n - x_{n-1}).$$

Step 3: Compute

$$y_n = P_{\mathcal{C}}(w_n - \lambda_n A w_n).$$

If $w_n = y_n$ (or $Ay_n = 0$) then stop: w_n is a solution of the Problem (1.2.3). Otherwise, go to **Step 4.**

Step 4: Compute

$$z_n = y_n + \lambda_n(Ay_n - Aw_n).$$

Step 5 Compute

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) z_n,$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n + \rho_n\right\} & \text{if} \quad Aw_n - Ay_n \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases} \tag{3.5.40}$$

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Set $n := n + 1$ and go back to **Step 1**.

Appendix 3.5.17. *Algorithm 1 of Yin et. al [270]*

Step 0: *Take* $s_0 > 0, \quad x_0 \in \mathcal{H}, \quad 0 < \mu < 1.$

Step 1: *Let the nth iterate* x_n *be given. Compute*

$$\bar{v_n} = (1 - \chi_n)x_n + \chi_n S x_n$$

and

$$v_n = (1-\varphi_n)x_n + \varphi_n S\bar{v_n}$$

Step 2: *Let the nth step size s_n be known. Compute*

$$y_n = P_{\mathcal{C}}(v_n - s_n A v_n)$$

and

$$x_{n+1} = (1-\phi_n)v_n + \phi_n \Big[y_n + s_n(Av_n - Ay_n)\Big]$$

where

$$s_{n+1} = \begin{cases} min\Big\{\frac{\mu\|v_n - y_n\|}{\|Av_n - Ay_n\|}, s_n\Big\} & if \quad Av_n - Ay_n \neq 0, \\ s_n, & otherwise. \end{cases} \tag{3.5.41}$$

Set $n := n+1$ and go back to **Step 1**.

CHAPTER 4

Results on Split Variational Inequality Problems

4.1 Introduction

In this chapter, we propose and study iterative algorithms for approximating solutions of SVIPs and GVIP in the framework of real Hilbert spaces. We present numerical examples of our proposed methods in comparison to other methods in literature to illustrate the applicability of our proposed methods.

4.2 On split variational inequality problems without product space formulation

Many methods have been proposed in the literature for solving the SVIP. Most of these methods either require that this problem is transformed into an equivalent VIP in a product space, or that the underlying operators are co-coercive. However, it has been discovered that such product space transformation may cause some potential difficulties during implementation and its approach may not fully exploit the attractive splitting nature of the SVIP. On the other hand, the co-coercive assumption of the underlying operators would preclude the potential applications of these methods. To avoid these setbacks, we propose two new relaxed inertial methods for solving the SVIP without any product space transformation, and for which the underlying operators are freed from the restrictive co-coercive assumption. The methods proposed, involve projections onto half-spaces only, and originate from an explicit discretization of a dynamical system, which combines both the inertial and relaxation techniques in order to achieve high convergence speed. Moreover, the sequence generated by these methods is shown to converge strongly to a minimum-norm solution of the problem in real Hilbert spaces. Furthermore, numerical implementations and comparisons are given to support our theoretical findings.

4.2.1 Proposed methods

In this section, we present our proposed methods and discuss their features. We begin with the following assumptions under which our strong convergent results are obtained.

Assumption 4.2.1. *Suppose that the following conditions hold:*

(a) *The feasible sets $\mathcal{C}$ and $\mathcal{Q}$ are nonempty closed and convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.*

(b) *The sets $\mathcal{C}$ and $\mathcal{Q}$ are defined as*

$$\mathcal{C} = \{x \in \mathcal{H}_1 \mid c(x) \leq 0\} \text{ and } \ Q = \{w \in \mathcal{H}_2 \mid q(w) \leq 0\},$$

where $c : \mathcal{H}_1 \to \mathbb{R}$ and $q : \mathcal{H}_2 \to \mathbb{R}$ are continuously differentiable convex functions such that $c'(\cdot)$ and $q'(\cdot)$ are Lipschitz continuous with constants K_1 and K_2, respectively.

(c) *$A : \mathcal{H}_1 \to \mathcal{H}_1$ and $F : \mathcal{H}_2 \to \mathcal{H}_2$ are monotone and Lipschitz continuous with Lipschitz constants L_1 and L_2, respectively.*

(d) *$T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator and the solution set $\Gamma := \{z \in VI(A, \mathcal{C}) : Tz \in VI(F, \mathcal{Q})\}$ of the SVIP (1.2.4)-(1.2.5) is nonempty, where $VI(A, \mathcal{C})$ is the solution set of the classical VIP (1.2.4).*

(e) *$\{\delta_n\}_{n=1}^{\infty}, \{\theta_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ are positive sequences satisfying the following conditions:*
$\delta_n \in (0, 1), \quad \lim\limits_{n\to\infty} \delta_n = 0, \quad \sum\limits_{n=1}^{\infty} \delta_n = \infty, \ \lim\limits_{n\to\infty} \frac{\tau_n}{\delta_n} = 0$ *and* $\{\theta_n\} \subset (a, 1 - \delta_n)$ *for some* $a > 0$.

(f) *$\{\phi_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that $\phi_n \in (0, 1]$ and $\lim\limits_{n\to\infty} \phi_n = \psi \in (0, 1]$.*

We present the following method for solving the SVIP (1.2.4)-(1.2.5) when L_1, L_2, K_1 and K_2 are known.

Algorithm 4.2.2. Relaxed inertial method with fixed stepsizes.

Step 0: Choose sequences $\{\delta_n\}_{n=1}^{\infty}, \{\theta_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ such that the conditions from Assumption 4.2.1 (e) hold and let $\eta \geq 0$, $\phi \in (0, 1]$, $\alpha \geq 3$ and $x_0, x_1 \in \mathcal{H}_1$ be given arbitrarily. Set $n := 1$.

Step 1: Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n := \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases} \tag{4.2.1}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}),$$

and construct the half-space

$$\mathcal{Q}_n = \{w \in \mathcal{H}_2 \mid q(Tw_n) + \langle q'(Tw_n), w - Tw_n\rangle \leq 0\}.$$

Then compute

$$y_n = P_{\mathcal{Q}_n}(Tw_n - \lambda FTw_n),$$

where

$$0 < \lambda < \frac{\sqrt{\rho_2^2 K_2^2 + 2\rho_2 K_2(1-\phi)L_2 + L_2^2} - [\rho_2 K_2 + (1-\phi)L_2]}{\phi L_2^2}, \quad \text{for some} \quad \rho_2 > 0. \tag{4.2.2}$$

Step 3: Compute

$$z_n = (1-\phi)Tw_n + \phi\Big(y_n + \lambda(FTw_n - Fy_n)\Big)$$

and

$$b_n = w_n + \eta_n T^*(z_n - Tw_n),$$

where the stepsize η_n is chosen such that for small enough $\epsilon > 0$,

$$\eta_n \in \left[\epsilon, \quad \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right],$$

if $z_n \neq Tw_n$; otherwise, $\eta_n = \eta$.
Step 4: Construct the half space

$$\mathcal{C}_n = \{x \in \mathcal{H}_1 \mid c(b_n) + \langle c'(b_n), x - b_n\rangle \leq 0\}.$$

Then compute

$$u_n = P_{\mathcal{C}_n}(b_n - \mu A b_n),$$

where

$$0 < \mu < \frac{\sqrt{\rho_1^2 K_1^2 + 2\rho_1 K_1(1-\phi)L_1 + L_1^2} - [\rho_1 K_1 + (1-\phi)L_1]}{\phi L_1^2}, \quad \text{for some} \quad \rho_1 > 0. \tag{4.2.3}$$

Step 5: Compute

$$x_{n+1} = (1 - \theta_n - \delta_n)b_n + \theta_n t_n,$$

where

$$t_n = (1-\phi)b_n + \phi\Big(u_n + \mu(Ab_n - Au_n)\Big).$$

Set $n := n+1$ and go back to **Step 1**.

In the case where the Lipschitz constants L_1, L_2, K_1 and K_2 are not known, we present the following method with adaptive stepsizes for solving the SVIP (1.2.4)-(1.2.5).

Algorithm 4.2.3. Relaxed inertial method with adaptive stepsize strategy.

Step 0: Choose sequences $\{\delta_n\}_{n=1}^\infty, \{\theta_n\}_{n=1}^\infty, \{\phi_n\}_{n=1}^\infty$ and $\{\tau_n\}_{n=1}^\infty$ such that the conditions from Assumption 4.2.1 (e)-(f) hold and let $\eta \geq 0$, $0 < a_i < \sqrt{(1-\psi)(2-\psi) + (1-\rho_i^2)} - (1-\psi)$ for some $\rho_i > 0$, $i = 1, 2$, $\lambda_1 > 0$, $\mu_1 > 0$, $\alpha \geq 3$ and $x_0, x_1 \in \mathcal{H}_1$ be given arbitrarily. Set $n := 1$.

Step 1: Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n := \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases} \tag{4.2.4}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}),$$

and construct the half-space

$$\mathcal{Q}_n = \{w \in \mathcal{H}_2 \mid q(Tw_n) + \langle q'(Tw_n), x - Tw_n\rangle \leq 0\}.$$

Then compute

$$y_n = P_{\mathcal{Q}_n}(Tw_n - \lambda_n FTw_n),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{a_2\|Tw_n - y_n\|}{\sqrt{\|FTw_n - Fy_n\|^2 + \|q'(Tw_n) - q'(y_n)\|^2}}, \lambda_n\right\}, \\ \text{if } \|FTw_n - Fy_n\|^2 + \|q'(Tw_n) - q'(y_n)\|^2 \neq 0, \\ \lambda_n, \quad \text{otherwise.} \end{cases} \tag{4.2.5}$$

Step 3: Compute

$$z_n = (1-\phi_n)Tw_n + \phi_n\Big(y_n + \lambda_n(FTw_n - Fy_n)\Big)$$

and

$$b_n = w_n + \eta_n T^*(z_n - Tw_n),$$

where the stepsize η_n is chosen such that for small enough $\epsilon > 0$,

$$\eta_n \in \left[\epsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right],$$

if $z_n \neq Tw_n$; otherwise, $\eta_n = \eta$.

Step 4: Construct the half-space

$$\mathcal{C}_n = \{x \in \mathcal{H}_1 \mid c(b_n) + \langle c'(b_n), x - b_n\rangle \leq 0\}.$$

Then compute

$$u_n = P_{\mathcal{C}_n}(b_n - \mu_n Ab_n),$$

where

$$\mu_{n+1} = \begin{cases} \min\left\{\frac{a_1||b_n-u_n||}{\sqrt{||Au_n-Ab_n||^2+\|c'(u_n)-c'(b_n)\|^2}},\ \mu_n\right\}, \\ \text{if } ||Au_n - Ab_n||^2 + \|c'(u_n) - c'(b_n)\|^2 \neq 0, \\ \mu_n, & \text{otherwise.} \end{cases} \tag{4.2.6}$$

Step 5: Compute

$$x_{n+1} = (1 - \theta_n - \delta_n)b_n + \theta_n t_n$$

where

$$t_n = (1 - \phi_n)b_n + \phi_n\Big(u_n + \mu_n(Ab_n - Au_n)\Big).$$

Set $n := n + 1$ and go back to **Step 1**.

Remark 4.2.1.

- Observe that Algorithms 4.2.2 and 4.2.3 can be viewed as modified RITFBF (2.5.14), involving only one projection onto the half-space $\mathcal{C}_n$ per iteration for solving the classical VIP in $\mathcal{H}_1$ and another RITFBF involving one projection onto the half-space $\mathcal{Q}_n$ per iteration under a bounded linear operator T for solving another VIP in another space $\mathcal{H}_2$, without further projections onto feasible sets, unlike other existing methods for solving SVIPs, where projections onto feasible sets are required; for instance, the method in [106, 221] for the case of classical VIP. A notable advantage of these methods (Algorithms 4.2.2 and 4.2.3) is that the co-coercive assumption on the operators A and F usually used in many works (see for example, [131, 142, 143, 146, 190]) to guarantee convergence is dispensed with, and no product space formulation of any sort is required under this setting, unlike in [63, 121].

- The stepsizes $\{\lambda_n\}$ and $\{\mu_n\}$ given by (4.2.5) and (4.2.6), respectively are generated at each iteration by some simple computations. Thus, Algorithm 4.2.3 is easily implemented without the prior knowledge of the Lipschitz constants L_1, L_2, K_1 and K_2 unlike in [106, Algorithm 4.1] where the knowledge of the Lipschitz constant is required and [221, Algorithm 3.3] where the stepsize is assumed to be 1.

- Step 1 of our methods is also easily implemented since the value of $||x_n - x_{n-1}||$ is a prior known before choosing α_n.

- Step 5 of both algorithms guarantee the strong convergence to the minimum-norm solution of the SVIP .

- It is easy to see that $\mathcal{C}_n \supset \mathcal{C}$ and $\mathcal{Q}_n \supset \mathcal{Q}$. Furthermore, since $\mathcal{C}_n$ and $\mathcal{Q}_n$ are half-spaces, the projections onto them have explicit formulas. In fact, u_n and y_n in our methods can be calculated as:

$$u_n = \begin{cases} b_n - \mu Ab_n, & \text{if } \ c(b_n) - \mu\langle c'(b_n), Ab_n\rangle \leq 0, \\ b_n - \mu Ab_n - \frac{c(b_n)-\mu\langle c'(b_n),Ab_n\rangle}{||c'(b_n)||^2}c'(b_n), & \text{otherwise} \end{cases} \tag{4.2.7}$$

and

$$y_n = \begin{cases} Tw_n - \lambda FTw_n, & \text{if } q(Tw_n) - \lambda\langle q'(Tw_n), FTw_n\rangle \leq 0, \\ Tw_n - \lambda FTw_n - \frac{q(Tw_n)-\lambda\langle q'(Tw_n), FTw_n\rangle}{\|q'(Tw_n)\|^2} q'(Tw_n), & \text{otherwise.} \end{cases} \quad (4.2.8)$$

Hence, our methods are computationally less expensive than other existing methods for solving the SVIP (1.2.4)-(1.2.5). This benefit (brought by the half-spaces) is further verified in our numerical analysis.

Remark 4.2.2. The choice of the stepsize η_n in Step 3 of Algorithms 4.2.2 and 4.2.3 do not require the prior knowledge of the operator norm $\|T\|$, unlike in [63, 121, 183, 244]. Note that algorithms whose implementation depends on the operator norm require the computation of the norm of the bounded linear operator, which in general is a very difficult task to accomplish as shown in [124, Theorem 2.3].
Also, we note that the value of the constant η in our methods does not influence the algorithms, but it was introduced for the sake of clarity. In what follows, we show that η_n is well-defined, and then, present an alternative choice for η_n without involving the constant η.

Proposition 4.2.4. *The stepsize η_n given in Step 3 of Algorithm 4.2.2 (or Algorithm 4.2.3) is well-defined.*

Proof. Let $z \in \Gamma$. Then, by Lemma 2.1.1 , we obtain

$$\begin{aligned} \|T^*(Tw_n - z_n)\|\|w_n - z\| &\geq \langle T^*(Tw_n - z_n), w_n - z\rangle \\ &= \langle Tw_n - z_n, Tw_n - Tz\rangle \\ &= \frac{1}{2}\Big[\|Tw_n - z_n\|^2 + \|Tw_n - Tz\|^2 - \|z_n - Tz\|^2\Big]. \end{aligned} \quad (4.2.9)$$

Furthermore, we claim that

$$\|z_n - Tz\|^2 \leq \|Tw_n - Tz\|^2 - \phi\left[2 - \phi - \phi\lambda^2 L_2^2 - 2\lambda(1-\phi)L_2 - 2\rho_2\lambda K_2\right]\|Tw_n - y_n\|^2. \quad (4.2.10)$$

We shall prove this claim (inequality (4.2.10)) later in Lemma 4.2.3 without involving the stepsize η_n.

Now, by the condition on λ, that is, from (4.2.2), we obtain

$$\begin{aligned} \Big(\phi L_2^2\lambda + \rho_2 K_2 + (1-\phi)L_2\Big)^2 &< \rho_2^2 K_2^2 + 2\rho_2 K_2(1-\phi)L_2 + L_2^2 \\ &= \rho_2^2 K_2^2 + 2\rho_2 K_2(1-\phi)L_2 + (1-\phi)^2 L_2^2 + \phi L_2^2(2-\phi). \end{aligned}$$

By some simple simplifications, we get

$$\lambda^2\phi^2 L_2^4 + 2\lambda\phi L_2^2\rho_2 K_2 + 2\lambda\phi L_2^2(1-\phi_2)L_2 - \phi L_2^2(2-\phi) < 0.$$

Since $L_2, \phi > 0$, dividing through by $L_2^2\phi$, we obtain

$$\lambda^2\phi L_2^2 + 2\lambda\rho_2 K_2 + 2\lambda(1-\phi)L_2 - (2-\phi) < 0,$$

which implies that

$$2 - \phi - \lambda^2 \phi L_2^2 - 2\lambda\rho_2 K_2 - 2\lambda(1-\phi)L_2 > 0. \tag{4.2.11}$$

Hence, we obtain from (4.2.10) that

$$\|z_n - Tz\|^2 \leq \|Tw_n - Tz\|^2. \tag{4.2.12}$$

Note that if λ is replaced with λ_n as in Algorithm 4.2.3, we will use inequality (4.2.65) (instead of (4.2.10)) and relation (4.2.67) (instead of (4.2.11)) to get (4.2.12). Therefore, substituting (4.2.12) into (4.2.9), we obtain that

$$\|T^*(Tw_n - z_n)\|\|w_n - z\| \geq \frac{1}{2}\|Tw_n - z_n\|^2. \tag{4.2.13}$$

Now, for $z_n \neq Tw_n$, we have that $\|Tw_n - z_n\| > 0$. This together with (4.2.13), imply that $\|T^*(Tw_n - z_n)\|\|w_n - z\| > 0$. Hence, we have that $\|T^*(Tw_n - z_n)\| \neq 0$. Therefore, η_n is well-defined. □

We now give an alternative way of choosing the stepsize η_n in our methods.
First, observe from Step 3 of Algorithms 4.2.2 and 4.2.3, that

$$\eta_n\|T^*(Tw_n - z_n)\|^2 \leq \|Tw_n - z_n\|^2 - \epsilon\|T^*(Tw_n - z_n)\|^2, \tag{4.2.14}$$

which implies that

$$\eta_n^2\|T^*(Tw_n - z_n)\|^2 - \eta_n\|Tw_n - z_n\|^2 \leq -\epsilon\eta_n\|T^*(Tw_n - z_n)\|^2. \tag{4.2.15}$$

Thus, η_n can be chosen such that for small enough $\epsilon > 0$,

$$\eta_n \in \left[\epsilon, \quad \frac{\|Tw_n - z_n\|^2}{1 + \|T^*(Tw_n - z_n)\|^2} - \epsilon\right]. \tag{4.2.16}$$

Then, we can see that

$$\eta_n \leq \frac{\|Tw_n - z_n\|^2 - \epsilon\big(1 + \|T^*(Tw_n - z_n)\|^2\big)}{1 + \|T^*(Tw_n - z_n)\|^2},$$

which implies that

$$\eta_n + \eta_n\|T^*(Tw_n - z_n)\|^2 \leq \|Tw_n - z_n\|^2 - \epsilon - \epsilon\|T^*(Tw_n - z_n)\|^2.$$

This further gives (4.2.14) and consequently (4.2.15).
As we shall see in our convergence analysis, (4.2.14) and (4.2.15) play crucial role in our proofs. Therefore, one can either choose the stepsize in Step 3 or that in (4.2.16), to ensure the convergence of Algorithms 4.2.2 and 4.2.3.

4.2.2 Convergence analysis

Lemma 4.2.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.2.2 under Assumption 4.2.1. Then*

$$\|z_n - Tp\|^2 \leq \|Tw_n - Tp\|^2 - \phi\left[2 - \phi - \phi\lambda^2 L_2^2 - 2\lambda(1-\phi)L_2 - 2\rho_2\lambda K_2\right]\|Tw_n - y_n\|^2 \tag{4.2.17}$$

and

$$\|t_n - p\|^2 \leq \|b_n - p\|^2 - \phi\left[2 - \phi - \phi\mu^2 L_1^2 - 2\mu(1-\phi)L_1 - 2\rho_1\mu K_1\right]\|b_n - u_n\|^2, \tag{4.2.18}$$

for all $p \in \Gamma$, where ρ_1, ρ_2 are positive constants.

Proof. Let $p \in \Gamma$. Then $Tp \in VI(F, \mathcal{Q}) \subset \mathcal{Q} \subset \mathcal{Q}_n$. Since $y_n = P_{\mathcal{Q}_n}(Tw_n - \lambda FTw_n)$ and $Tp \in \mathcal{Q}_n$, we obtain from the characteristic property of $P_{\mathcal{Q}_n}$, that

$$\langle Tw_n - \lambda FTw_n - y_n, y_n - Tp\rangle \geq 0$$

or equivalently

$$2\langle Tw_n - y_n, y_n - Tp\rangle - 2\lambda\langle FTw_n - Fy_n, y_n - Tp\rangle - 2\lambda\langle Fy_n, y_n - Tp\rangle \geq 0. \tag{4.2.19}$$

Also, from Lemma 2.1.1 , we obtain that

$$2\langle Tw_n - y_n, y_n - Tp\rangle = \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 - \|y_n - Tp\|^2. \tag{4.2.20}$$

Again, since F is monotone, we obtain that

$$\begin{aligned}\langle Fy_n, y_n - Tp\rangle &= \langle Fy_n - FTp, y_n - Tp\rangle + \langle FTp, y_n - Tp\rangle\\ &\geq \langle FTp, y_n - Tp\rangle.\end{aligned} \tag{4.2.21}$$

Substituting (4.2.20) and (4.2.21) into (4.2.19), we obtain

$$\begin{aligned}\|y_n - Tp\|^2 \leq{} & \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 - 2\lambda\langle FTw_n - Fy_n, y_n - Tp\rangle\\ & + 2\lambda\langle FTp, Tp - y_n\rangle.\end{aligned} \tag{4.2.22}$$

Also, since $z_n = (1-\phi)Tw_n + \phi\Big(y_n + \lambda(FTw_n - Fy_n)\Big)$, we get

$$\begin{aligned}\|z_n - Tp\|^2 &= \|(1-\phi)(Tw_n - Tp) + \phi(y_n - Tp) + \phi\lambda(FTw_n - Fy_n)\|^2\\ &= (1-\phi)^2\|Tw_n - Tp\|^2 + \phi^2\|y_n - Tp\|^2 + \phi^2\lambda^2\|FTw_n - Fy_n\|^2\\ &+ 2\phi(1-\phi)\langle Tw_n - Tp, y_n - Tp\rangle + 2\lambda\phi(1-\phi)\langle Tw_n - Tp, FTw_n - Fy_n\rangle\\ &+ 2\lambda\phi^2\langle y_n - Tp, FTw_n - Fy_n\rangle\\ &= (1-\phi)^2\|Tw_n - Tp\|^2 + \phi^2\|y_n - Tp\|^2 + \phi^2\lambda^2\|FTw_n - Fy_n\|^2\\ &+ \phi(1-\phi)\left[\|Tw_n - Tp\|^2 + \|y_n - Tp\|^2 - \|Tw_n - y_n\|^2\right]\\ &+ 2\lambda\phi(1-\phi)\langle Tw_n - Tp, FTw_n - Fy_n\rangle + 2\lambda\phi^2\langle y_n - Tp, FTw_n - Fy_n\rangle\\ &= (1-\phi)\|Tw_n - Tp\|^2 + \phi\|y_n - Tp\|^2 - \phi(1-\phi)\|Tw_n - y_n\|^2 \qquad (4.2.23)\\ &+ \phi_n^2\lambda^2\|FTw_n - Fy_n\|^2\\ &+ 2\lambda\phi(1-\phi)\langle Tw_n - Tp, FTw_n - Fy_n\rangle + 2\lambda\phi^2\langle y_n - Tp, FTw_n - Fy_n\rangle.\end{aligned}$$

Substituting (4.2.22) into (4.2.23), we obtain

$$
\begin{aligned}
\|z_n - Tp\|^2 &\leq (1-\phi)\|Tw_n - Tp\|^2 \\
&+ \phi\left[\|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 - 2\lambda\langle FTw_n - Fy_n, y_n - Tp\rangle\right] \\
&+ \phi 2\lambda\langle FTp, Tp - y_n\rangle - \phi(1-\phi)\|Tw_n - y_n\|^2 + \phi^2\lambda^2\|FTw_n - Fy_n\|^2 \\
&+ 2\lambda\phi(1-\phi)\langle Tw_n - Tp, FTw_n - Fy_n\rangle + 2\lambda\phi^2\langle y_n - Tp, FTw_n - Fy_n\rangle \\
&= \|Tw_n - Tp\|^2 - \phi(2-\phi)\|Tw_n - y_n\|^2 + \phi^2\lambda^2\|FTw_n - Fy_n\|^2 \qquad (4.2.24) \\
&+ 2\lambda\phi(1-\phi)\langle Tw_n - y_n, FTw_n - Fy_n\rangle + 2\lambda\phi\langle FTp, Tp - y_n\rangle \\
&\leq \|Tw_n - Tp\|^2 - \phi(2-\phi)\|Tw_n - y_n\|^2 + \phi^2\lambda^2 L_2^2\|Tw_n - y_n\|^2 \\
&+ 2\lambda\phi(1-\phi)L_2\|Tw_n - y_n\|^2 + 2\lambda\phi\langle FTp, Tp - y_n\rangle \\
&= \|Tw_n - Tp\|^2 - \phi\left[2 - \phi - \phi\lambda^2 L_2^2 - 2\lambda(1-\phi)L_2\right]\|Tw_n - y_n\|^2 \qquad (4.2.25) \\
&+ 2\lambda\phi\langle FTp, Tp - y_n\rangle.
\end{aligned}
$$

Now, if $FTp = 0$, then we obtain the desired conclusion from (4.2.25).
However, if $FTp \neq 0$, then we obtain from Lemma 2.5.10 that $Tp \in \partial Q$ and there exists $\rho_2 > 0$ such that $FTp = -\rho_2 q'(Tp)$. Since $Tp \in \partial Q$, $q(Tp) = 0$.
Also, since q is a differentiable convex function, it follows from (2.4.7) that

$$
\begin{aligned}
q(y_n) &\geq q(Tp) + \langle q'(Tp), y_n - Tp\rangle \\
&= \frac{-1}{\rho_2}\langle FTp, y_n - Tp\rangle,
\end{aligned}
$$

which implies that

$$\langle Tp - y_n, FTp\rangle \leq \rho_2 q(y_n). \qquad (4.2.26)$$

Since $y_n \in \mathcal{Q}_n$, we have

$$q(Tw_n) + \langle q'(Tw_n), y_n - Tw_n\rangle \leq 0. \qquad (4.2.27)$$

Again, since q is a differentiable convex function, we obtain from (2.4.7) that

$$\langle q'(y_n), Tw_n - y_n\rangle + q(y_n) \leq q(Tw_n). \qquad (4.2.28)$$

From (4.2.27), (4.2.28) and the Lipschitz continuity of $q'(\cdot)$, we obtain

$$q(y_n) \leq \langle q'(y_n) - q'(Tw_n), y_n - Tw_n\rangle \leq K_2\|y_n - Tw_n\|^2. \qquad (4.2.29)$$

By substituting (4.2.26) and (4.2.29) into (4.2.25), we obtain the desired conclusion (4.2.17).
Following the same line of argument, we obtain (4.2.18). □

Lemma 4.2.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.2.2 under Assumption 4.2.1. Then, $\{x_n\}$ is bounded.*

Proof. Let $p \in \Gamma$. Since $w_n = x_n + \alpha_n(x_n - x_{n-1})$, we have

$$\|w_n - p\| \leq \|x_n - p\| + \alpha_n\|x_n - x_{n-1}\|.$$

Also, from Step 1, we observe that $\alpha_n\|x_n - x_{n-1}\| \leq \tau_n, \quad \forall n \geq 1$, which implies that

$$\frac{\alpha_n}{\delta_n}\|x_n - x_{n-1}\| \leq \frac{\tau_n}{\delta_n} \to 0, \quad \text{as} \quad n \to \infty. \tag{4.2.30}$$

Hence, there exists $M_1 > 0$ such that

$$\frac{\alpha_n}{\delta_n}\|x_n - x_{n-1}\| \leq M_1, \quad \forall n \geq 1. \tag{4.2.31}$$

This implies that

$$\|w_n - p\| \leq \|x_n - p\| + \delta_n M_1, \ \forall n \geq 1. \tag{4.2.32}$$

Now, by Lemma 4.2.3 (inequality (4.2.17)) and (4.2.11), we obtain that

$$\|z_n - Tp\|^2 \leq \|Tw_n - Tp\|^2. \tag{4.2.33}$$

From Step 3, Lemma 2.1.1 and (4.2.33), we obtain

$$\begin{aligned}
\|b_n - p\|^2 &= \|w_n - p\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 + 2\eta_n\langle w_n - p, T^*(z_n - Tw_n)\rangle \\
&= \|w_n - p\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 + 2\eta_n\langle Tw_n - Tp, z_n - Tw_n\rangle \\
&= \|w_n - p\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 \\
&+ \eta_n\left[\|z_n - Tp\|^2 - \|Tw_n - Tp\|^2 - \|z_n - Tw_n\|^2\right] \\
&\leq \|w_n - p\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 - \eta_n\|z_n - Tw_n\|^2 \\
&= \|w_n - p\|^2 - \eta_n\left[\|z_n - Tw_n\|^2 - \eta_n\|T^*(z_n - Tw_n)\|^2\right].
\end{aligned} \tag{4.2.34}$$

Thus, by the condition on η_n, that is, from (4.2.14), we obtain that

$$\|b_n - p\|^2 \leq \|w_n - p\|^2,$$

which implies from (4.2.32) that

$$\|b_n - p\| \leq \|w_n - p\| \leq \|x_n - p\| + \delta_n M_1 \ \forall n \geq 1. \tag{4.2.35}$$

From Step 5, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \theta_n - \delta_n)(b_n - p) + \theta_n(t_n - p) - \delta_n p\| \\
&\leq \|(1 - \theta_n - \delta_n)(b_n - p) + \theta_n(t_n - p)\| + \delta_n\|p\|.
\end{aligned} \tag{4.2.36}$$

But from (4.2.18), condition (4.2.3) and (4.2.35), we get that

$$\begin{aligned}
\|(1 - \theta_n - \delta_n)(b_n - p) + \theta_n(t_n - p)\|^2 &= (1 - \theta_n - \delta_n)^2\|b_n - p\|^2 + \theta_n^2\|t_n - p\|^2 \\
&+ 2(1 - \theta_n - \delta_n)\theta_n\langle b_n - p, t_n - p\rangle \\
&\leq (1 - \theta_n - \delta_n)^2\|b_n - p\|^2 + \theta_n^2\|b_n - p\|^2 \\
&+ 2(1 - \theta_n - \delta_n)\theta_n\|b_n - p\|^2 \\
&\leq (1 - \delta_n)^2\|w_n - p\|^2.
\end{aligned} \tag{4.2.37}$$

Hence, (4.2.36) becomes

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - \delta_n)\|w_n - p\| + \delta_n\|p\| \\
&\leq (1 - \delta_n)\left[\|x_n - p\| + \delta_n M_1\right] + \delta_n\|p\| \\
&\leq (1 - \delta_n)\|x_n - p\| + \delta_n(M_1 + \|p\|),
\end{aligned}$$

which implies from Lemma 2.5.35 that $\{x_n\}$ is bounded. □

Lemma 4.2.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.2.2 under Assumption 4.2.1. Suppose that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to a point $z \in \mathcal{H}_1$ and $\lim_{k\to\infty} \|b_{n_k} - w_{n_k}\| = 0 = \lim_{k\to\infty} \|b_{n_k} - t_{n_k}\|$. Then $z \in \Gamma$.*

Proof. From Step 2 and (4.2.30), we get that

$$\lim_{n\to\infty} \|w_n - x_n\| = \lim_{n\to\infty} \delta_n \cdot \frac{\alpha_n}{\delta_n} \|x_n - x_{n-1}\| = 0. \tag{4.2.38}$$

Since the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is weakly convergent to a point $z \in \mathcal{H}_1$, it follows that the subsequence $\{w_{n_k}\}$ of $\{w_n\}$ is also weakly convergent to $z \in \mathcal{H}_1$. This, together with our hypothesis imply that $\{b_{n_k}\}$ converges weakly to z. Again, since T is a bounded linear operator, we obtain that $\{Tw_{n_k}\}$ converges weakly to Tz.

Now, without loss of generality, we may assume that $z_n \neq Tw_n$. Then,

$$\eta_n \in \left[\epsilon, \quad \frac{\|z_n - Tw_n\|^2}{\|T^*(z_n - Tw_n)\|^2} - \epsilon\right].$$

Thus, putting (4.2.15) into (4.2.34), we obtain that

$$\begin{aligned}\|b_{n_k} - p\|^2 &\leq \|w_{n_k} - p\|^2 - \eta_{n_k}\epsilon\|T^*(z_{n_k} - Tw_{n_k})\|^2 \\ &\leq \|w_{n_k} - p\|^2 - \epsilon^2\|T^*(z_{n_k} - Tw_{n_k})\|^2,\end{aligned} \tag{4.2.39}$$

which implies that

$$\begin{aligned}\epsilon^2\|T^*(z_{n_k} - Tw_{n_k})\|^2 &\leq \|w_{n_k} - p\|^2 - \|b_{n_k} - p\|^2 \\ &\leq \|w_{n_k} - b_{n_k}\|^2 + 2\|w_{n_k} - b_{n_k}\|\|b_{n_k} - p\|.\end{aligned}$$

Thus, by our hypothesis, we obtain

$$\lim_{k\to\infty} \|T^*(z_{n_k} - Tw_{n_k})\| = 0. \tag{4.2.40}$$

From (4.2.34) and (4.2.40), we obtain

$$\begin{aligned}\eta_{n_k}\|z_{n_k} - Tw_{n_k}\|^2 &\leq \|w_{n_k} - p\|^2 - \|b_{n_k} - p\|^2 + \eta_{n_k}^2\|T^*(z_{n_k} - Tw_{n_k})\|^2 \\ &\leq \|w_{n_k} - b_{n_k}\|^2 + 2\|w_{n_k} - b_{n_k}\|\|b_{n_k} - p\| + \eta_{n_k}^2\|T^*(z_{n_k} - Tw_{n_k})\|^2 \\ &\to 0, \quad \text{as} \quad k \to \infty.\end{aligned}$$

Since $\eta_{n_k} \geq \epsilon > 0$, we get that

$$\lim_{k\to\infty} \|z_{n_k} - Tw_{n_k}\| = 0. \tag{4.2.41}$$

From (4.2.17), we obtain

$$\begin{aligned}\phi\left[2 - \phi - \phi\lambda^2 L_2^2 - 2\lambda(1-\phi)L_2 - 2\rho_2\lambda K_2\right]\|Tw_{n_k} - y_{n_k}\|^2 &\leq \|Tw_{n_k} - Tp\|^2 - \|z_{n_k} - Tp\|^2 \\ &\leq \|Tw_{n_k} - z_{n_k}\|^2 \\ &\quad + 2\|Tw_{n_k} - z_{n_k}\|\|z_{n_k} - Tp\|.\end{aligned}$$

Thus, taking limit as $k \to \infty$ in the previous inequality, we obtain from (4.2.41) and (4.2.11) that

$$\lim_{k\to\infty} \|Tw_{n_k} - y_{n_k}\| = 0. \tag{4.2.42}$$

Now, by the monotonicity of F, the characteristic property of $P_{\mathcal{Q}_n}$ and $\mathcal{Q} \subset \mathcal{Q}_n$, we obtain for all $w \in \mathcal{Q}$ that

$$\begin{aligned} 0 &\leq \langle y_{n_k} - Tw_{n_k} + \lambda FTw_{n_k}, w - y_{n_k}\rangle \\ &= \langle y_{n_k} - Tw_{n_k}, w - y_{n_k}\rangle + \lambda\langle FTw_{n_k}, Tw_{n_k} - y_{n_k}\rangle + \lambda\langle FTw_{n_k}, w - Tw_{n_k}\rangle \\ &\leq \|y_{n_k} - Tw_{n_k}\|\|w - y_{n_k}\| + \lambda\|FTw_{n_k}\|\|Tw_{n_k} - y_{n_k}\| \\ &\quad +\lambda\left[\langle FTw_{m_k} - Fw, w - Tw_{n_k}\rangle + \langle Fw, w - Tw_{n_k}\rangle\right] \\ &\leq \|y_{n_k} - Tw_{n_k}\|\|w - y_{n_k}\| + \lambda\|FTw_{n_k}\|\|Tw_{n_k} - y_{n_k}\| \\ &\quad +\lambda\langle Fw, w - Tw_{n_k}\rangle. \end{aligned} \tag{4.2.43}$$

Thus, passing limit as $k \to \infty$ in the previous inequality, we obtain

$$\langle Fw, w - Tz\rangle \geq 0, \quad \forall w \in \mathcal{Q},$$

and from Lemma 3.2.2, we have that $Tz \in VI(F, \mathcal{Q})$.

Again, by our hypothesis and (4.2.18), we obtain that

$$\begin{aligned} \phi\left[2 - \phi - \phi\mu^2 L_1^2 - 2\mu(1-\phi)L_1 - 2\rho_1\mu K_1\right]\|b_{n_k} - u_{n_k}\|^2 &\leq \|b_{n_k} - p\|^2 - \|t_{n_k} - p\|^2 \\ &\leq \|b_{n_k} - t_{n_k}\|^2 \\ &\quad + 2\|b_{n_k} - t_{n_k}\|\|t_{n_k} - p\| \\ &\to 0, \text{ as } k \to \infty, \end{aligned}$$

which by condition (4.2.3), implies that

$$\lim_{k\to\infty} \|b_{n_k} - u_{n_k}\| = 0 \tag{4.2.44}$$

Similar to (4.2.43), we obtain from the monotonicity of A, the characteristic property of $P_{\mathcal{C}_n}$ and $\mathcal{C} \subset \mathcal{C}_n$, that

$$0 \leq \|u_{n_k} - b_{n_k}\|\|x - u_{n_k}\| + \mu\|Ab_{n_k}\|\|b_{n_k} - u_{n_k}\| + \mu\langle Ax, x - b_{n_k}\rangle, \ \forall x \in \mathcal{C}.$$

Passing limit as $k \to \infty$ in the previous inequality, we obtain from (4.2.44) that

$$\langle Ax, x - z\rangle \geq 0, \quad \forall x \in \mathcal{C}.$$

Therefore, we obtain from Lemma 3.2.2 that $z \in VI(A, \mathcal{C})$. Hence, $z \in \Gamma$. □

We now present the first theorem of this work.

Theorem 4.2.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.2.2 under Assumption 4.2.1. Then, $\{x_n\}$ converges strongly to $p \in \Gamma$, where $\|p\| = \min\{\|z\| : z \in \Gamma\}$.*

Proof. Let $p \in \Gamma$. Then, from Step 2, we obtain

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n - p\|^2 + \alpha_n^2 \|x_n - x_{n-1}\|^2 + 2\alpha_n \langle x_n - p, x_n - x_{n-1}\rangle \\
&\leq \|x_n - p\|^2 + \alpha_n^2 \|x_n - x_{n-1}\|^2 + 2\alpha_n ||x_n - p|| ||x_n - x_{n-1}|| \\
&= \|x_n - p\|^2 + \alpha_n \|x_n - x_{n-1}\| \left[\alpha_n \|x_n - x_{n-1}\| + 2\|x_n - p\|\right] \\
&\leq \|x_n - p\|^2 + 3\alpha_n \|x_n - x_{n-1}\| M_2, \qquad (4.2.45)
\end{aligned}$$

for some $M_2 > 0$. Thus, from (4.2.35), (4.2.18) and (4.2.45), we get

$$\begin{aligned}
\|(1-\theta_n)b_n + \theta_n t_n - p\|^2 &= \|(1-\theta_n)(b_n - p) + \theta_n(t_n - p)\|^2 \\
&= (1-\theta_n)^2\|b_n - p\|^2 + \theta_n^2\|t_n - p\|^2 + 2(1-\theta_n)\theta_n\langle b_n - p, t_n - p\rangle \\
&\leq (1-\theta_n)^2\|w_n - p\|^2 + \theta_n^2\|w_n - p\|^2 + 2(1-\theta_n)\theta_n\langle b_n - p, t_n - p\rangle \\
&\leq (1-\theta_n)^2\|w_n - p\|^2 + \theta_n^2\|w_n - p\|^2 + 2(1-\theta_n)\theta_n\|w_n - p\|^2 \\
&\leq \|x_n - p\|^2 + 3\alpha_n\|x_n - x_{n-1}\|M_2.
\end{aligned}$$

From Step 5, the previous inequality and (4.2.30), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\delta_n)\left[(1-\theta_n)b_n + \theta_n t_n - p\right] - \left[\delta_n\theta_n(b_n - t_n) + \delta_n p\right]\|^2 \\
&\leq (1-\delta_n)^2\|(1-\theta_n)b_n + \theta_n t_n - p\|^2 - 2\langle\delta_n\theta_n(b_n - t_n) + \delta_n p, x_{n+1} - p\rangle \\
&\leq (1-\delta_n)\|(1-\theta_n)b_n + \theta_n t_n - p\|^2 + 2\delta_n\theta_n\langle b_n - t_n, p - x_{n+1}\rangle \\
&+ 2\delta_n\langle p, p - x_{n-1}\rangle \\
&\leq (1-\delta_n)\|(1-\theta_n)b_n + \theta_n t_n - p\|^2 + 2\delta_n\theta_n\|b_n - t_n\|\|p - x_{n+1}\| \\
&+ 2\delta_n\langle p, p - x_{n+1}\rangle \\
&\leq (1-\delta_n)\left(\|x_n - p\|^2 + 3\alpha_n\|x_n - x_{n-1}\|M_2\right) + 2\delta_n\theta_n\|b_n - t_n\|\|p - x_{n+1}\| \\
&+ 2\delta_n\langle p, p - x_{n+1}\rangle \\
&\leq (1-\delta_n)\|x_n - p\|^2 + 3\alpha_n\|x_n - x_{n-1}\|M_2 + 2\delta_n\theta_n\|b_n - t_n\|\|p - x_{n+1}\| \\
&+ 2\delta_n\langle p, p - x_{n+1}\rangle \\
&= (1-\delta_n)\|x_n - p\|^2 + \delta_n d_n, \;\; n \geq 1,
\end{aligned}$$

where $d_n = 3\frac{\alpha_n}{\delta_n}\|x_n - x_{n-1}\|M_2 + 2\left(\theta_n\|b_n - t_n\|\|p - x_{n+1}\| + \langle p, p - x_{n+1}\rangle\right).$
According to Lemma 2.5.36, to conclude our proof, it suffices to show that $\limsup_{k\to\infty} d_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition;

$$\liminf_{k\to\infty}\left(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\right) \geq 0. \qquad (4.2.46)$$

To show that $\limsup_{k\to\infty} d_{n_k} \leq 0$, suppose that $\{\|x_{n_k} - p\|\}$ is a subsequence of $\{\|x_n - p\|\}$ such that (4.2.46) holds. Then,

$$\begin{aligned}
\liminf_{k\to\infty}\left(\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2\right) &= \liminf_{k\to\infty}\Big[\left(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\right) \\
&\quad\times\left(\|x_{n_k+1} - p\| + \|x_{n_k} - p\|\right)\Big] \\
&\geq 0. \qquad (4.2.47)
\end{aligned}$$

Now, observe that

$$
\begin{aligned}
\|(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p)\|^2 &= (1-\theta_n-\delta_n)^2\|b_n-p\|^2 \\
&+ 2(1-\theta_n-\delta_n)\theta_n\langle b_n-p, t_n-p\rangle + \theta_n^2\|t_n-p\|^2 \\
&\leq (1-\theta_n-\delta_n)^2\|b_n-p\|^2 \\
&+ (1-\theta_n-\delta_n)\theta_n\|b_n-p\|^2 \\
&+ (1-\theta_n-\delta_n)\theta_n\|t_n-p\|^2 + \theta_n^2\|t_n-p\|^2 \\
&\leq (1-\theta_n-\delta_n)(1-\delta_n)\|b_n-p\|^2 \\
&+ \theta_n(1-\delta_n)\|t_n-p\|^2.
\end{aligned} \tag{4.2.48}
$$

Then, from Step 5, (4.2.48) and (4.2.18), we get

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p)-\delta_n p\|^2 \\
&= \|(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p)\|^2 + \delta_n^2\|p\|^2 \\
&- 2\delta_n\langle(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p), p\rangle \\
&\leq (1-\theta_n-\delta_n)(1-\delta_n)\|b_n-p\|^2 + \theta_n(1-\delta_n)\|t_n-p\|^2 + \delta_n M_3 \\
&\leq (1-\theta_n-\delta_n)(1-\delta_n)\|b_n-p\|^2 \qquad (4.2.49) \\
&+ \theta_n(1-\delta_n)\|b_n-p\|^2 + \delta_n M_3 \\
&- \theta_n(1-\delta_n)\Big(\phi\left[2-\phi-\phi\mu^2L_1^2-2\mu(1-\phi)L_1-2\rho_1\mu K_1\right]\|b_n-u_n\|^2\Big) \\
&\leq (1-\theta_n-\delta_n)(1-\delta_n)\|w_n-p\|^2 + \delta_n M_3 \\
&+ \theta_n(1-\delta_n)\|w_n-p\|^2 \\
&- \theta_n(1-\delta_n)\Big(\phi\left[2-\phi-\phi\mu^2L_1^2-2\mu(1-\phi)L_1-2\rho_1\mu K_1\right]\|b_n-u_n\|^2\Big) \\
&\leq \|x_n-p\|^2 \qquad (4.2.50) \\
&- \theta_n(1-\delta_n)\phi\left[2-\phi-\phi\mu^2L_1^2-2\mu(1-\phi)L_1-2\rho_1\mu K_1\right]\|b_n-u_n\|^2 \\
&+ \delta_n\left(3\frac{\alpha_n}{\delta_n}\|x_n-x_{n-1}\|M_2+M_3\right),
\end{aligned}
$$

for some $M_3 > 0$.
Let $V := 2-\phi-\phi\mu^2L_1^2-2\mu(1-\phi)L_1-2\rho_1\mu K_1$. Then, from (4.2.50), (4.2.47) and (4.2.31), we obtain

$$
\begin{aligned}
\limsup_{k\to\infty}\left(\theta_{n_k}(1-\delta_{n_k})\phi V\|b_{n_k}-u_{n_k}\|^2\right) &\leq \limsup_{k\to\infty}\left(\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\right) \\
&+ \limsup_{k\to\infty}\delta_{n_k}(3M_1M_2+M_3) \\
&= -\liminf_{k\to\infty}\left(\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\right) \\
&\leq 0.
\end{aligned}
$$

This implies that

$$
\lim_{k\to\infty}\|b_{n_k}-u_{n_k}\| = 0. \tag{4.2.51}
$$

Since A is Lipschitz continuous on $\mathcal{H}_1$, we obtain from (4.2.51) that

$$\lim_{k\to\infty} \|Ab_{n_k} - Au_{n_k}\| = 0. \tag{4.2.52}$$

Also, from Step 5, we have that

$$\begin{aligned}\|t_{n_k} - b_{n_k}\| &= \|(1-\phi)b_{n_k} + \phi u_{n_k} + \phi\mu(Ab_{n_k} - Au_{n_k}) - b_{n_k}\| \\ &\le \phi\|u_{n_k} - b_{n_k}\| + \phi\mu\|Ab_{n_k} - Au_{n_k}\|,\end{aligned}$$

which implies from (4.2.51) and (4.2.52) that

$$\lim_{k\to\infty} \|t_{n_k} - b_{n_k}\| = 0. \tag{4.2.53}$$

From (4.2.49) and (4.2.39), we obtain

$$\begin{aligned}\|x_{n_k+1} - p\|^2 &\le (1-\theta_{n_k} - \delta_{n_k})(1-\delta_{n_k})\|b_{n_k} - p\|^2 + \theta_{n_k}(1-\delta_{n_k})\|b_{n_k} - p\|^2 + \delta_{n_k}M_3 \\ &\le \|b_{n_k} - p\|^2 + \delta_{n_k}M_3 \\ &\le \|w_{n_k} - p\|^2 - \epsilon^2\|T^*(z_{n_k} - Tw_{n_k})\|^2 + \delta_{n_k}M_3,\end{aligned}$$

which implies from (4.2.45) that

$$\begin{aligned}\limsup_{k\to\infty} \|T^*(z_{n_k} - Tw_{n_k})\|^2 &\le \frac{1}{\epsilon^2}\limsup_{k\to\infty}\left(\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2\right) \\ &\quad + \frac{1}{\epsilon^2}\limsup_{k\to\infty}\left(\delta_{n_k}\left(3\frac{\alpha_{n_k}}{\delta_{n_k}}\|x_{n_k} - x_{n_k-1}\|M_2 + M_3\right)\right) \\ &\le \frac{-1}{\epsilon^2}\liminf_{k\to\infty}\left(\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2\right) \le 0.\end{aligned}$$

Thus,

$$\lim_{k\to\infty} \|T^*(z_{n_k} - Tw_{n_k})\| = 0. \tag{4.2.54}$$

Hence, by Step 3, we obtain

$$\|b_{n_k} - w_{n_k}\| = \eta_{n_k}\|T^*(z_{n_k} - Tw_{n_k})\| \to 0, \quad \text{as} \quad k\to\infty. \tag{4.2.55}$$

From Step 5 and (4.2.53), we get

$$\|x_{n_k+1} - b_{n_k}\| \le \theta_{n_k}\|b_{n_k} - t_{n_k}\| + \delta_{n_k}\|b_{n_k}\| \to 0, \quad \text{as} \quad k\to\infty. \tag{4.2.56}$$

From (4.2.55) and (4.2.56), we get

$$\lim_{k\to\infty} \|x_{n_k+1} - w_{n_k}\| = 0. \tag{4.2.57}$$

From (4.2.38) and (4.2.57), we get

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{4.2.58}$$

By Lemma 4.2.4, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to z, and such that

$$\limsup_{k\to\infty}\langle p, p - x_{n_k}\rangle = \lim_{j\to\infty}\langle p, p - x_{n_{k_j}}\rangle = \langle p, p - z\rangle.$$

Also, since we have established (4.2.53) and (4.2.55), we can apply Lemma 4.2.5 to get that $z \in \Gamma$. Note also that $p = P_\Gamma 0$. Thus, we obtain from the previous inequality that

$$\limsup_{k\to\infty}\langle p, p - x_{n_k}\rangle = \langle p, p - z\rangle \leq 0,$$

which together with (4.2.58), imply that

$$\lim_{k\to\infty}\langle p, p - x_{n_k+1}\rangle \leq 0. \tag{4.2.59}$$

Now, recall that $d_{n_k} = 3\frac{\alpha_{n_k}}{\delta_{n_k}}\|x_{n_k} - x_{n_k-1}\|M_2 + 2\left(\theta_{n_k}\|b_{n_k} - t_{n_k}\|\|p - x_{n_k+1}\| + \langle p, p - x_{n_k+1}\rangle\right)$. So, by (4.2.30), (4.2.53) and (4.2.59), we obtain that $\limsup\limits_{k\to\infty} d_{n_k} \leq 0$. Hence, we get that $\lim\limits_{n\to\infty}\|x_n - p\| = 0$. Therefore, $\{x_n\}$ converges strongly to $p = P_\Gamma 0$.

□

Remark 4.2.7.

- We observe that if we set $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $F = 0$ and $T = I_{\mathcal{H}}$ (the identity operator on $\mathcal{H}$) in Algorithm 4.2.2, we obtain as a corollary, a relaxed inertial Tseng's forward-backward-forward with fixed stepsizes, involving only one projection onto the half-space $\mathcal{C}_n$ per iteration, for solving the classical VIP (1.2.4) when A is monotone and Lipschitz continuous. This method for solving the classical VIP (1.2.4) is also the first in the literature.

- The conclusion of Theorem 4.2.6 still holds even if the fixed stepsizes λ and μ given by (4.2.2) and (4.2.3), respectively, are replaced with variable stepsizes μ_n and λ_n, respectively such that

$$0 < \inf_{n\geq 1}\mu_n \leq \sup_{n\geq 1}\mu_n < \frac{\sqrt{\rho_2^2K_2^2 + 2\rho_2K_2(1-\phi)L_2 + L_2^2} - [\rho_2K_2 + (1-\phi)L_2]}{\phi L_2^2}$$

and

$$0 < \inf_{n\geq 1}\lambda_n \leq \sup_{n\geq 1}\lambda_n < \frac{\sqrt{\rho_1^2K_1^2 + 2\rho_1K_1(1-\phi)L_1 + L_1^2} - [\rho_1K_1 + (1-\phi)L_1]}{\phi L_1^2}.$$

Having concluded the convergence analysis of the first method of this work, we now study the convergence analysis of the second method (Algorithm 4.2.3), which do not depend on the availability of the Lipschitz constants. We begin with the following useful remarks and lemmas.

Remark 4.2.8. The stepsizes $\{\lambda_n\}$ and $\{\mu_n\}$ generated by Algorithm 4.2.3 are monotonically non-increasing sequences with lower bounds $\min\left\{\lambda_1, \frac{a_2}{\sqrt{L_2^2+K_2^2}}\right\}$ and $\min\left\{\mu_1, \frac{a_1}{\sqrt{L_1^2+K_1^2}}\right\}$, respectively.

Indeed, it is obvious that $\lambda_{n+1} \leq \lambda_n, \quad \forall n \in \mathbb{N}$. Also, since F is L_2-Lipschitz continuous and q' is K_2-Lipschitz continuous, we get in the case where $\|FTw_n - Fy_n\|^2 + \|q'(Tw_n) - q'(y_n)\|^2 \neq 0$ that

$$\begin{aligned}\lambda_{n+1} &= \min\left\{\frac{a_2\|Tw_n - y_n\|}{\sqrt{\|FTw_n - Fy_n\|^2 + \|q'(Tw_n) - q'(y_n)\|^2}},\ \lambda_n\right\}\\ &\geq \min\left\{\frac{a_2\|Tw_n - y_n\|}{\sqrt{L_2^2\|Tw_n - y_n\|^2 + K_2^2\|Tw_n - y_n\|^2}},\ \lambda_n\right\}\\ &= \min\left\{\frac{a_2}{\sqrt{L_2^2 + K_2^2}},\ \lambda_n\right\}.\end{aligned}$$

Hence, by induction, we obtain that $\{\lambda_n\}$ is bounded below by $\min\left\{\frac{a_2}{\sqrt{L_2^2+K_2^2}},\ \lambda_1\right\}$. Similar argument holds for $\{\mu_n\}$.

Note that Remark 4.2.8 implies that the limits of $\{\lambda_n\}$, $\{\mu_n\}$ exist, and $\lim_{n\to\infty} \lambda_n \geq \min\left\{\frac{a_2}{\sqrt{L_2^2+K_2^2}},\ \lambda_1\right\}$, $\lim_{n\to\infty} \mu_n \geq \min\left\{\frac{a_1}{\sqrt{L_1^2+K_1^2}},\ \mu_1\right\}$.

Remark 4.2.9. From (4.2.5), we observe that

$$\sqrt{\|FTw_n - Fy_n\|^2 + \|q'(Tw_n) - q'(y_n)\|^2} \leq \frac{a_2}{\lambda_{n+1}}\|Tw_n - y_n\|, \quad \forall n \in \mathbb{N}$$

holds for both $\|FTw_n - Fy_n\| + \|q'(Tw_n) - q'(y_n)\| \neq 0$ and otherwise. This implies that

$$\|FTw_n - Fy_n\|^2 + \|q'(Tw_n) - q'(y_n)\|^2 \leq \frac{a_2^2}{\lambda_{n+1}^2}\|Tw_n - y_n\|^2,\ \forall n \in \mathbb{N}, \tag{4.2.60}$$

which further implies

$$\|FTw_n - Fy_n\| \leq \frac{a_2}{\lambda_{n+1}}\|Tw_n - y_n\|,\ \forall n \in \mathbb{N}. \tag{4.2.61}$$

Lemma 4.2.10. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.2.3 under Assumption 4.2.1. Then, for all $p \in \Gamma$, there exists $n_0 \geq 1$ such that*

$$||Tw_n - y_n||^2 \leq K_n \left(||Tw_n - z_n||^2 + 2||Tw_n - z_n||||z_n - Tp||\right), \tag{4.2.62}$$

and

$$||b_n - u_n||^2 \leq L_n \left(||b_n - t_n||^2 + 2||b_n - t_n||||t_n - p||\right), \tag{4.2.63}$$

where

$$K_n = \left(\frac{1}{\phi_n\left[2 - \phi_n - \lambda_n^2 \frac{a_2^2}{\lambda_{n+1}^2} - \rho_2^2 - 2\lambda_n(1-\phi_n)\frac{a_2}{\lambda_{n+1}}\right]}\right)$$

and

$$L_n = \left(\frac{1}{\phi_n\left[2 - \phi_n - \mu_n^2 \frac{a_1^2}{\lambda_{n+1}^2} - \rho_1^2 - 2\mu_n(1-\phi_n)\frac{a_1}{\mu_{n+1}}\right]}\right),$$

with ρ_1, ρ_2 been positive constants, for all $n \geq n_0$.

Proof. By replacing λ with λ_n and ϕ with ϕ_n in (4.2.24), we have

$$\begin{aligned} \|z_n - Tp\|^2 &\leq \|Tw_n - Tp\|^2 - \phi_n(2-\phi_n)\|Tw_n - y_n\|^2 \\ &+ \phi_n^2\lambda_n^2\|FTw_n - Fy_n\|^2 + 2\lambda_n\phi_n(1-\phi_n)\langle Tw_n - y_n, FTw_n - Fy_n\rangle \\ &+ 2\lambda_n\phi_n\langle FTp, Tp - y_n\rangle. \end{aligned} \tag{4.2.64}$$

Suppose $FTp = 0$. Then, we obtain from (4.2.61) and (4.2.64) that

$$\begin{aligned} \|z_n - Tp\|^2 &\leq \|Tw_n - Tp\|^2 - \phi_n(2-\phi_n)\|Tw_n - y_n\|^2 + \phi_n\lambda_n^2\|FTw_n - Fy_n\|^2 \\ &\quad +2\lambda_n\phi_n(1-\phi_n)\langle Tw_n - y_n, FTw_n - Fy_n\rangle \\ &\leq \|Tw_n - Tp\|^2 - \phi_n(2-\phi_n)\|Tw_n - y_n\|^2 + \phi_n\lambda_n^2\frac{a_2^2}{\lambda_{n+1}^2}\|Tw_n - y_n\|^2 \\ &+ 2\lambda_n\phi_n(1-\phi_n)\frac{a_2}{\lambda_{n+1}}\|Tw_n - y_n\|^2 \\ &= \|Tw_n - Tp\|^2 - \phi_n\left[2 - \phi_n - \lambda_n^2\frac{a_2^2}{\lambda_{n+1}^2} - 2\lambda_n(1-\phi_n)\frac{a_2}{\lambda_{n+1}}\right]\|Tw_n - y_n\|^2 \\ &\leq \|Tw_n - Tp\|^2 \\ &- \phi_n\left[2 - \phi_n - \lambda_n^2\frac{a_2^2}{\lambda_{n+1}^2} - 2\lambda_n(1-\phi_n)\frac{a_2}{\lambda_{n+1}} - \rho_2^2\right]\|Tw_n - y_n\|^2, \end{aligned} \tag{4.2.65}$$

for some $\rho_2 > 0$.
Now, since the limit of $\{\lambda_n\}$ exists, then $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \lambda_{n+1}$. Hence, by Assumption 4.2.1 (f) and the condition on a_2, we obtain that

$$\lim_{n\to\infty}\left[2 - \phi_n - \lambda_n^2\frac{a_2^2}{\lambda_{n+1}^2} - \rho_2^2 - 2\lambda_n(1-\phi_n)\frac{a_2}{\lambda_{n+1}}\right] = \left(2 - \psi - a_2^2 - \rho_2^2 - 2a_2(1-\psi)\right) > 0. \tag{4.2.66}$$

Thus, there exists $n_0 \geq 1$ such that

$$2 - \phi_n - \lambda_n^2 \frac{a_2^2}{\lambda_{n+1}^2} - \rho_2^2 - 2\lambda_n(1-\phi_n)\frac{a_2}{\lambda_{n+1}} > 0, \ \forall n \geq n_0. \tag{4.2.67}$$

Hence, we obtain from (4.2.65) that

$$\begin{aligned} \|Tw_n - y_n\|^2 &\leq \left[\frac{1}{\phi_n \left(2 - \phi_n - \lambda_n^2 \frac{a_2^2}{\lambda_{n+1}^2} - \rho_2^2 - 2\lambda_n(1-\phi_n)\frac{a_2}{\lambda_{n+1}}\right)} \right] \\ &\cdot \left(\|Tw_n - Tp\|^2 - \|z_n - Tp\|^2 \right) \\ &= \left[\frac{1}{\phi_n \left(2 - \phi_n - \lambda_n^2 \frac{a_2^2}{\lambda_{n+1}^2} - \rho_2^2 - 2\lambda_n(1-\phi_n)\frac{a_2}{\lambda_{n+1}}\right)} \right] \\ &\cdot \left(\|(Tw_n - z_n) + (z_n - Tp)\|^2 - \|z_n - Tp\|^2 \right) \end{aligned} \tag{4.2.68}$$

for all $n \geq n_0$, which by some simple simplifications, gives (4.2.62).

Now, suppose $FTp \neq 0$. Then, from (4.2.26) and (4.2.29), we obtain

$$\langle FTp, Tp - y_n \rangle \leq \rho_2 \langle q'(y_n) - q'(Tw_n), y_n - Tw_n \rangle. \tag{4.2.69}$$

Also, observe that

$$2\lambda_n \rho_2 \langle q'(y_n) - q'(Tw_n), y_n - Tw_n \rangle \leq \lambda_n^2 \|q'(y_n) - q'(Tw_n)\|^2 + \rho_2^2 \|y_n - Tw_n\|^2. \tag{4.2.70}$$

Substituting (4.2.69) and (4.2.70) into (4.2.64), and using (4.2.61) and (4.2.60), we obtain

$$\begin{aligned} \|z_n - Tp\|^2 &\leq \|Tw_n - Tp\|^2 - \phi_n(2-\phi_n)\|Tw_n - y_n\|^2 + \phi_n \lambda_n^2 \|FTw_n - Fy_n\|^2 \\ &+ 2\lambda_n \phi_n (1-\phi_n) \langle Tw_n - y_n, FTw_n - Fy_n \rangle + \phi_n \lambda_n^2 \|q'(y_n) - q'(Tw_n)\|^2 \\ &+ \phi_n \rho_2^2 \|y_n - Tw_n\|^2 \\ &= \|Tw_n - Tp\|^2 - \phi_n(2-\phi_n)\|Tw_n - y_n\|^2 \\ &+ \phi_n \lambda_n^2 \left[\|FTw_n - Fy_n\|^2 + \|q'(Tw_n) - q'(y_n)\|^2 \right] \\ &+ 2\lambda_n \phi_n (1-\phi_n) \langle Tw_n - y_n, FTw_n - Fy_n \rangle + \phi_n \rho_2^2 \|y_n - Tw_n\|^2 \\ &\leq \|Tw_n - Tp\|^2 - \phi_n(2-\phi_n)\|Tw_n - y_n\|^2 + \phi_n \lambda_n^2 \frac{a_2^2}{\lambda_{n+1}^2} \|Tw_n - y_n\|^2 \\ &+ 2\lambda_n \phi_n (1-\phi_n) \frac{a_2}{\lambda_{n+1}} \|Tw_n - y_n\|^2 + \phi_n \rho_2^2 \|y_n - Tw_n\|^2 \\ &= \|Tw_n - Tp\|^2 \\ &- \phi_n \left[2 - \phi_n - \lambda_n^2 \frac{a_2^2}{\lambda_{n+1}^2} - 2\lambda_n(1-\phi_n)\frac{a_2}{\lambda_{n+1}} - \rho_2^2 \right] \|Tw_n - y_n\|^2. \end{aligned}$$

Notice that this is the same as (4.2.65). Hence, we obtain (4.2.62).

Also, we can follow the same line of proof as above to establish (4.2.63). □

We now state and prove the strong convergence theorem for Algorithm 4.2.3.

Theorem 4.2.11. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.2.3 under Assumption 4.2.1. Then, $\{x_n\}$ converges strongly to $p \in \Gamma$, where $\|p\| = \min\{\|z\| : z \in \Gamma\}$.*

Proof. Let $p \in \Gamma$. Then, by replacing μ and λ with μ_n and λ_n, respectively in Lemma 4.2.4, and using (4.2.65) and (4.2.67), we can easily get that $\{x_n\}$ is bounded. Furthermore, we can follow the same argument as in the proof of Theorem 4.2.6 to obtain (4.2.53), (4.2.54) and (4.2.55). Also, using (4.2.54) and (4.2.55), we can get (4.2.41).
Hence, from (4.2.41), Lemma 4.2.10 (that is, (4.2.62)) and (4.2.66), we obtain that

$$\lim_{k\to\infty} \|Tw_{n_k} - y_{n_k}\| = 0.$$

In a similar way, we can show that

$$\lim_{k\to\infty} \|b_{n_k} - u_{n_k}\| = 0.$$

Following the same arguments as in Lemma 4.2.5 and Theorem 4.2.6, we obtain the rest of the proof. □

Remark 4.2.12.

(1). Similar to Remark 4.2.7, we obtain a relaxed inertial Tseng's forward-backward-forward with adaptive stepsizes, involving only one projection onto the half-space $\mathcal{C}_n$ per iteration, for solving the classical VIP (1.2.4) when A is monotone and Lipschitz continuous as a corollary. This method is also new.

(2). Our proofs are independent of the usual "Two cases Approach" which have been used widely in many works (see, for example, [106, 221, 244]) to guarantee strong convergence of iterative methods. Hence, the techniques used in obtaining our strong convergence analysis are new for solving the SVIP (1.2.4)-(1.2.5).

(3). If Step 1 of Algorithm 4.2.2 and Algorithm 4.2.3 are replaced with the following: $0 \le \alpha_n \le \bar{\alpha}_n$, where

$$\bar{\alpha}_n := \begin{cases} \min\left\{\alpha, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \alpha, & \text{otherwise,} \end{cases}$$

with $\alpha \in [0,1)$, we still have valid conclusions of Theorem 4.2.6 and Theorem 4.2.11.

4.2.3 Numerical experiments

In this section, using various test examples, we discuss the numerical behavior of Algorithm 4.2.2 and Algorithm 4.2.3, and we compare them with the methods of He *et al.* [121, Algorithm 1] (see Appendix 4.2.17), Reich and Tuyen [213, Theorem 4.4] (see Appendix 4.2.18) and Tian and Jiang [244, Algorithm (4.8)] (see Appendix 4.2.19).

We perform all implementations using Matlab 2016 (b), installed on a personal computer with Intel(R) Core(TM) i5-2600 CPU@2.30GHz and 8.00 Gb-RAM running on Windows 10 operating system. In Tables 4.2.1-4.2.3, "Iter." means the number of iterations while "CPU" means the CPU time in seconds.
In our computations, we randomly choose the relaxation stepsize $\phi \in (0,1]$ and the starting points $x_0, x_1 \in \mathcal{H}_1$ (see the cases below). We also choose randomly, the parameters $\eta \geq 0$, $\lambda_1 > 0$ and $\mu_1 > 0$ (the choices of these parameters will be discussed in Remark 4.2.16). We choose $\delta_n = \frac{1}{n+1}$, $\theta_n = \frac{1}{2} - \delta_n$ and $\alpha_n = \bar{\alpha}_n$ with $\tau_n = \frac{\delta_n}{n^{0.01}}$ and different choices of $\alpha := 3, 6, 9, 12, 15$, which we will discuss in detail in our numerical analysis. Furthermore, in the implementation, we define

$$\text{TOL}_n := \frac{1}{2}\left(||x_n - P_{\mathcal{C}_n}(x_n - \mu A x_n)||^2 + ||Tx_n - P_{\mathcal{Q}_n}(Tx_n - \lambda F T x_n)||^2\right),$$

and use the stopping criterion $\text{TOL}_n < \varepsilon$ for the iterative processes, where ε is the predetermined error.

Firstly, we consider two examples in finite dimensional spaces. In these examples, we compare Algorithm 4.2.2 with the method of He *et al.* [121] since their method is given in finite dimension spaces.

Example 4.2.13. Consider the following separable, convex and quadratic programming problem, which has been considered by [121, Example 5.2] for their numerical experiments.

$$\min_{x,y} \{G_1(x) + G_2(y) \mid Tx = y, \ x \in \mathcal{C}, \ y \in \mathcal{Q}\}, \tag{4.2.71}$$

where

$$G_1(x) = \frac{1}{2}x^T M_1 x + q_1^T x \quad \text{and} \quad G_2(y) = \frac{1}{2}y^T M_2 y + q_2^T y \quad (x^T \text{ means the transpose of } x).$$

Problem (4.2.71) can also be rewritten as the SVIP (1.2.4)-(1.2.5), where

$$A(x) = M_1 x + q_1 \quad \text{and} \quad F(y) = M_2 y + q_2.$$

For $i = 1, 2$, the matrices M_i are formed as: $M_i = V_i \sum_i V_i^T$, where $V_i = I - \frac{2v_i v_i^T}{\|v_i\|^2}$ and $\sum_i = \text{diag}(\sigma_{i1}, \sigma_{i2}, \cdots, \sigma_{iN_i})$ are the Householder and the diagonal matrices, respectively, with $N_1 = N$ and $N_2 = m$ being the dimensional of x and y, respectively. Furthermore,

$$\sigma_{i,j} = \cos\frac{j\pi}{N_i + 1} + 1 + \frac{\cos\frac{\pi}{N_i+1} + 1 - \widehat{C}_i(\cos\frac{N_i\pi}{N_i+1} + 1)}{\widehat{C}_i - 1}, \qquad j = 1, 2, \cdots, N_i,$$

where $\widehat{C}_i$ is the preset condition number of M_i.

As done in He *et al.* [121, Example 5.2], we choose $\widehat{C}_i = 10^4$, $q_i = 0$, $i = 1, 2$ and uniformly take the vector $v_i \in \mathbb{R}^{N_i}$ $(i = 1, 2)$ in $(-1, 1)$. In this case, A and F are monotone and Lipschitz continuous operators with $L_i = ||M_i||$, $i = 1, 2$ (which can be computed in Matlab). Furthermore, we generate the bounded linear operator $T \in \mathbb{R}^{M\times N}$ with independent Gaussian components distributed in the interval $(0, 1)$, and then normalize each column of T with the unit norm.

Now, consider an ellipsoid in $\mathbb{R}^N$ as the feasible set $\mathcal{C}$. That is, $\mathcal{C} = \{x \in \mathbb{R}^N : (x - d)^T P(x - d) \le r^2\}$, where P is a positive definite matrix, $d \in \mathbb{R}^N$ and $r > 0$. Then, the projection onto $\mathcal{C}$ is difficult to compute (see, for example [119]). Consider the convex function $c : \mathbb{R}^N \to \mathbb{R}$ defined by $c(x) = \frac{1}{2}\left[(x - d)^T P(x - d) - r^2\right]$, $\forall x \in \mathbb{R}^N$. Then, $\mathcal{C}$ is a level set of c. That is, $\mathcal{C} := \{x \in \mathbb{R}^N : c(x) \le 0\}$. Also, it is easy to see that c is differentiable on $\mathbb{R}^N$. In fact, $c'(x) = P(x - d)$, $\forall x \in \mathbb{R}^N$. Hence, c' is K_1-Lipschitz continuous with $K_1 = ||P||$.
Now, let $\lambda_{\max}$ and $\lambda_{\min}$ be maximum and minimum eigenvalues of P, respectively.
Suppose $d = 0$, since

$$\frac{||A(x)||}{||c'(x)||} = \frac{||M_1x||}{||Px||} \le \frac{||M_1||||x||}{\lambda_{\min}||x||} = \frac{||M_1||}{\lambda_{\min}}$$

holds for arbitrary $x \in \mathbb{R}^N$, $x \ne 0$, we have that $\rho_1 = \frac{||M_1||}{\lambda_{\min}}$ (see [119]).
On the other hand, suppose $d \ne 0$, since

$$\frac{||A(x)||}{||c'(x)||} = \frac{||M_1x||}{||P(x-d)||} \le \frac{||M_1||\left(||x-d|| + ||d||\right)}{\lambda_{\min}||x-d||} \le \frac{||M_1||\sqrt{\lambda_{\max}}\left(r + ||d||\sqrt{\lambda_{\min}}\right)}{\lambda_{\min}^{\frac{3}{2}}r}$$

holds for all $x \in \partial C$, we have that $\rho_1 = \frac{||M_1||\sqrt{\lambda_{\max}}\left(r+||d||\sqrt{\lambda_{\min}}\right)}{\lambda_{\min}^{\frac{3}{2}}r}$.

Let $\mathcal{Q} = \{y \in \mathbb{R}^m : q(y) \le 0\}$, where $q(y) = y_1^2 - 2y_1 + \sum_{j=2}^m y_j^2$. Then, the explicit expression for projection onto $\mathcal{Q}$ may be difficult to find (see for example, [120]). Also, we note that q is a countinuously differentiable convex function, and q' is K_2-Lipschitz continuous with $K_2 = 2$. Moreover, $\rho_2 = ||M_2||$ (see [120]). Since the Lipschitz constants L_1, L_2, K_1, K_2 are known and we know ρ_1, ρ_2, we can choose in Algorithm 4.2.2, $\mu = \frac{\sqrt{\rho_1^2K_1^2+2\rho_1K_1(1-\phi)L_1+L_1^2} - [\rho_1K_1+(1-\phi)L_1]}{2\phi L_1^2}$ and $\lambda = \frac{\sqrt{\rho_2^2K_2^2+2\rho_2K_2(1-\phi)L_2+L_2^2} - [\rho_2K_2+(1-\phi)L_2]}{2\phi L_2^2}$, and take the starting point $x_1 = (1, 1, \ldots, 1)^T$ while the entries of x_0 are randomly generated in $[0, 1]$.
For Algorithm 1 of He *et al.* [121], we take

$\mu_1 = 5\left(||T^T\mathbb{H}T|| + L_1\right)/v, \mu_2 = 10\left(||T^T\mathbb{H}T|| + L_2\right)/v$, $v = 0.8$, $\mathbb{H} = \frac{2}{||T^TT||}I_N$, $\gamma = 1.2$ and $\rho_1 = -1.5$ (which is the optimum choice in their implementation), with starting points $x_1 = (1, 1, \ldots, 1)^T$, $y_1 = (0, 0, \ldots, 0)^T$ and $\lambda_1 = (0, 0, \ldots, 0)^T$. Since $P_{\mathcal{C}}$ and $P_{\mathcal{Q}}$ seem not to have explicit expressions, Algorithm 1 of He *et al.* [121] may be difficult to implement for this example. Therefore, we use the technique proposed in this study, that is, we replace $\mathcal{C}$ and $\mathcal{Q}$ in their algorithm with $\mathcal{C}_n$ and $\mathcal{Q}_n$, then use the formulas similar to (4.2.7) and (4.2.8) for the implementation. Furthermore, we consider different scenarios of

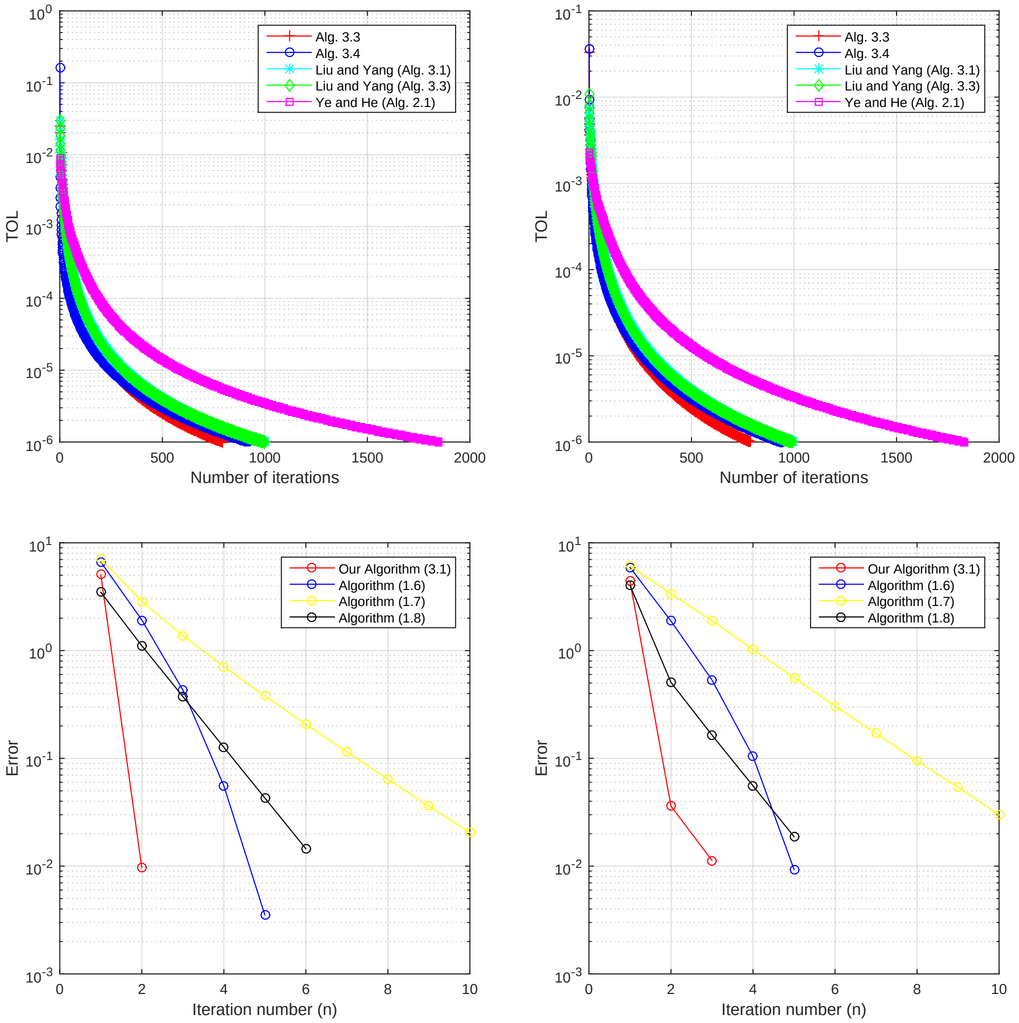

Figure 4.1: The behavior of TOL_n with $\varepsilon = 10^{-6}$ for Example 4.2.13: Top Left: $(N, m) = (100, 50)$; Top Right: $(N, m) = (300, 150)$; Bottom Left: $(N, m) = (500, 250)$; Bottom Right: $(N, m) = (1000, 500)$.

the problem's dimensions. That is, $N = 100, 300, 500, 1000$ and $m = N/2$. We also take $\varepsilon = 10^{-6}$ for the stopping criterion and obtain the numerical results reported in Table 4.2.1 and Figure 4.1. We stress that these choices are the same as in He *et al.* [121, Example 5.2].

Table 4.2.1. Numerical results for Example 4.2.13 with $\varepsilon = 10^{-6}$.

(N,m)		Alg.4.2.2 $\alpha = 3$	Alg.4.2.2 $\alpha = 6$	Alg.4.2.2 $\alpha = 9$	Alg.4.2.2 $\alpha = 12$	Alg.4.2.2 $\alpha = 15$	He *et al.*
$(100, 50)$	CPU	0.1917	0.0805	0.1205	0.0948	0.1337	10.5882
	Iter.	826	759	798	735	803	2482
$(300, 150)$	CPU	0.1378	0.1455	0.1609	0.1235	0.1217	15.6402
	Iter.	1211	1241	1238	1233	1251	8140
$(500, 250)$	CPU	0.2162	0.1814	0.1739	0.1953	0.1944	16.9445
	Iter.	1458	1480	1456	1468	1459	30043
$(1000, 500)$	CPU	0.4489	0.4561	0.5391	0.4120	0.4624	39.0961
	Iter.	1604	1622	1617	1620	1627	73092

Example 4.2.14. Let $\mathcal{C} = \{x \in \mathbb{R}^2 : c(x) := x_1^2 - x_2 \leq 0\}$ and $\mathcal{Q} = \{y \in \mathbb{R}^2 : q(y) := y_1^2 + y_2^2 - 2 \leq 0\}$,
where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Define $A, F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$A(x) = (3\Psi(x_1), 2x_1 + x_2), \;\; F(x) = (6\Psi(x_1), 4x_1 + 2x_2),$$

where

$$\Psi(s) = \begin{cases} e(s-1) + e, & \text{if } s > 1, \\ e^s, & \text{if } -1 \leq s \leq 1, \\ e^{-1}(s+1) + e^{-1}, & \text{if } s < -1. \end{cases}$$

Then, one can easily verify that $L_1 = \sqrt{9e^2 + 5}$ and $L_2 = 2\sqrt{9e^2 + 5}$. Also, it is easy to see that c and q are continuously differentiable convex functions with Lipschitz constants $K_1 = K_2 = 2$. Moreover, we have that $\rho_1 = 3\sqrt{e^2 + 1}$ and $\rho_2 = \frac{3}{2}\sqrt{e^2 + 1}$ (see for example, [120]). Furthermore, we choose the same parameters as in Example 4.2.13 and use the stopping criterion $\varepsilon = 10^{-3}$.

By considering the following cases for the numerical experiments, we obtain the numerical results displayed in Table 4.2.2 and Figure 4.2.
Case 1: Take $x_1 = (1,\ 0.5)^T$ and $x_0 = (2,\ 3)^T$.

Case 2: Take $x_1 = (2,\ 1)^T$ and $x_0 = (0.5,\ -1)^T$.

Case 3: Take $x_1 = (1,\ -1)^T$ and $x_0 = (2,\ -2)^T$.

Case 4: Take $x_1 = (0.5,\ 4)^T$ and $x_0 = (-1,\ -3)^T$.

Table 4.2.2. Numerical results for Example 4.2.14 with $\varepsilon = 10^{-3}$.

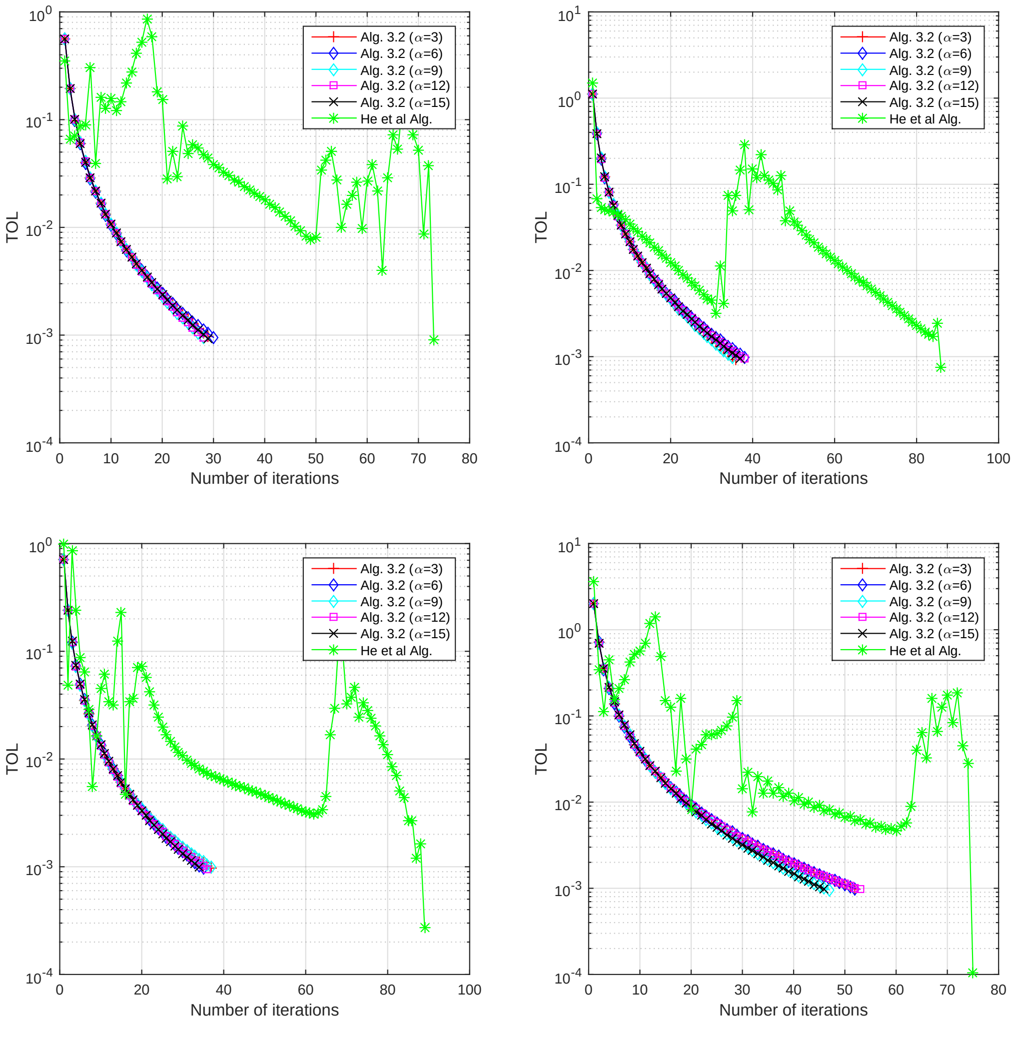

Figure 4.2: The behavior of TOL_n with $\varepsilon = 10^{-3}$ for Example 4.2.14: Top Left: **Case 1**; Top Right: **Case 2**; Bottom Left: **Case 3**; Bottom Right: **Case 4**.

Cases		Alg.4.2.2 $\alpha = 3$	Alg.4.2.2 $\alpha = 6$	Alg.4.2.2 $\alpha = 9$	Alg.4.2.2 $\alpha = 12$	Alg.4.2.2 $\alpha = 15$	He *et al.*
1	CPU	0.0203	0.0160	0.0168	0.0261	0.0202	1.1282
	Iter.	29	30	28	28	29	73
2	CPU	0.0229	0.0162	0.0168	0.0161	0.0147	1.0324
	Iter.	36	38	35	38	37	86
3	CPU	0.0240	0.0166	0.0167	0.0160	0.0153	1.1263
	Iter.	37	35	37	36	34	89
4	CPU	0.0216	0.0181	0.0169	0.0187	0.0165	1.1308
	Iter.	52	52	47	53	46	75

We present the next example in an infinite dimensional Hilbert space. In this example, we compare Algorithm 4.2.3 with the methods of Reich and Tuyen [213] and Tian and Jiang [244], since their methods are given in infinite dimensional Hilbert spaces.

Example 4.2.15. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{L}_2([0,1])$ be endowed with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt, \quad \forall\ x, y \in \mathcal{L}_2([0,1])$$

and norm

$$||x|| := \Big(\int_0^1 |x(t)|^2 dt\Big)^{\frac{1}{2}}, \quad \forall\ x, y \in \mathcal{L}_2([0,1]).$$

Let φ be an arbitrary fixed element in $\mathcal{C}([0,1])$ (the space of all real-valued continuous functions on $[0,1]$) and $\mathcal{C} = \mathcal{Q} = \{x \in \mathcal{L}_2([0,1]) : \|\varphi x\| \leq 1\}$. In particular, for all $t \in [0,1]$, we define $\varphi(t) = e^{-t}$ for $\mathcal{C}$ and $\varphi(t) = e^{-5t}$ for $\mathcal{Q}$. Then, $\mathcal{C}$ and $\mathcal{Q}$ are nonempty closed and convex subsets of $\mathcal{L}_2([0,1])$. However, we know from [119] that the explicit formulas for projections onto $\mathcal{C}$ and $\mathcal{Q}$ may be difficult to find.
Now, let $c, q : \mathcal{L}_2([0,1]) \to \mathbb{R}$ be defined by $c(x) = q(x) = \frac{1}{2}\Big(\|\varphi x\|^2 - 1\Big)$, $\forall x \in \mathcal{L}_2([0,1])$, where $\varphi(t) = e^{-t}$ for c and $\varphi(t) = e^{-5t}$ for q; $t \in [0,1]$. Then, c, q are convex functions with $\mathcal{C}$ and $\mathcal{Q}$ being level sets of c and q, respectively. Also, c and q are differentiable on $\mathcal{L}_2([0,1])$. In fact, $c'(x) = \varphi^2 x$, $\forall x \in \mathcal{L}_2([0,1])$. Furthermore, c' and q' are Lipschitz continuous.
Now, define $A : \mathcal{L}_2([0,1]) \to \mathcal{L}_2([0,1])$ by $Ax(t) = \max\{0,\ x(t)\}$, $\forall x \in \mathcal{L}_2([0,1])$. Then, A is monotone and Lipschitz continuous. Also, let $F : \mathcal{L}_2([0,1]) \to \mathcal{L}_2([0,1])$ be defined by $Fx(t) = \int_0^t x(s)ds$, $\forall x \in \mathcal{L}_2([0,1])$, $t \in [0,1]$. Then F is monotone and Lipschitz continuous.
Let $m\varphi = \min_{t\in[0,1]}|\varphi(t)| = e^{-1}$. Then,

$$\frac{\|Ax\|}{\|c'(x)\|} = \frac{\|x_+\|}{\|\varphi^2 x\|} \leq \frac{\|x\|}{m^2\varphi\|x\|} = e^2$$

holds for all $x \in \mathcal{L}_2([0,1])$, $x \neq 0$ (see [119]). Thus, we have that $\rho_1 = e^2$. Similarly, $\rho_2 = \frac{2}{\pi}e^{10}$.

We consider the following cases for the numerical experiments of this example. The results of the experiments are given in Table 4.2.3 and Figure 4.3.
Case 1: Take $x_1(t) = 1 + t^2$, $x_0(t) = t + 5$ and $\phi_n = \frac{n}{2n+5}$.

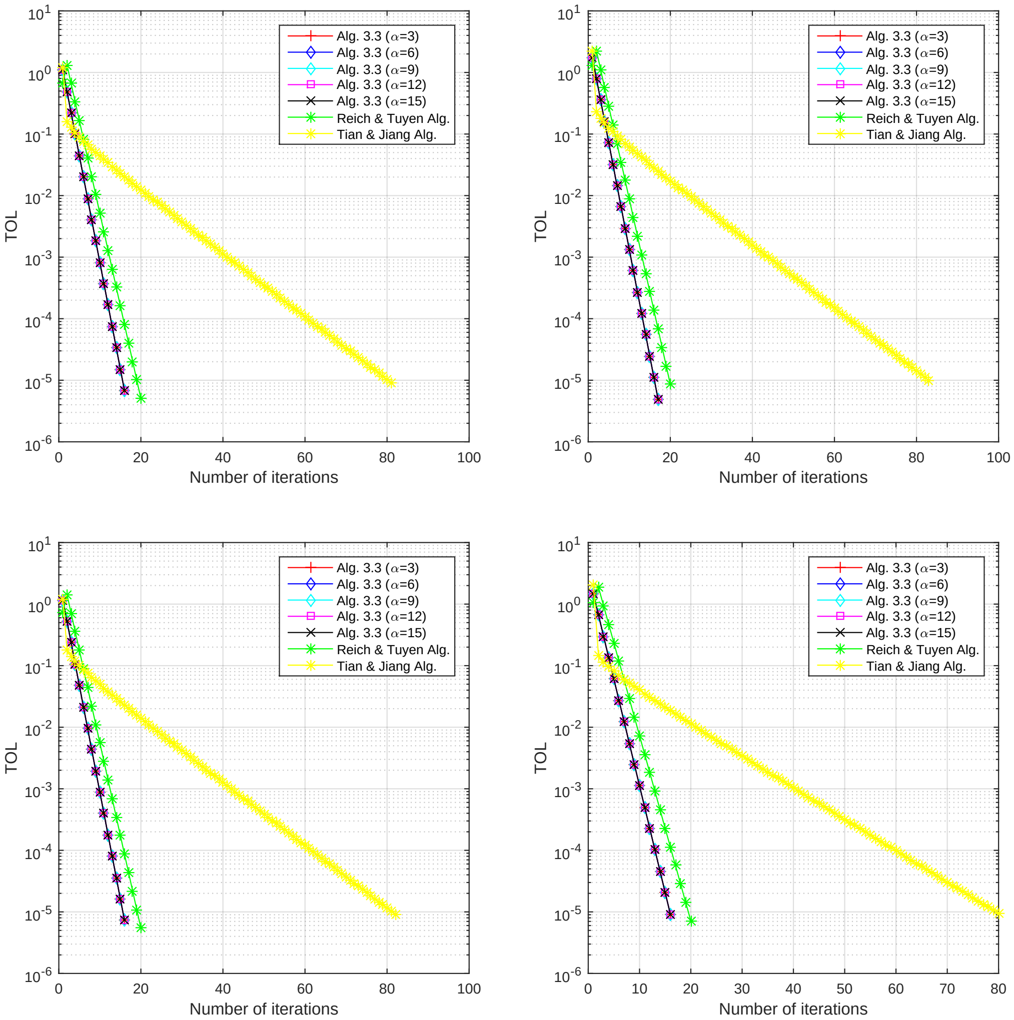

Figure 4.3: The behavior of TOL_n with $\varepsilon = 10^{-5}$ for Example 4.2.15: Top Left: **Case 1**; Top Right: **Case 2**; Bottom Left: **Case 3**; Bottom Right: **Case 4**.

Case 2: Take $x_1(t) = \sin(t)$, $x_0(t) = t + 1$ and $\phi_n = \frac{n}{n+10}$.

Case 3: Take $x_1(t) = t + 1$, $x_0(t) = t + t^3$ and $\phi_n = \frac{n}{n+10}$.

Case 4: Take $x_1(t) = 0.7e^{-t} + 1$, $x_0(t) = t + t^3$ and $\phi_n = \frac{2n}{18n+1}$.

Table 4.2.3. Numerical results for Example 4.2.15 with $\varepsilon = 10^{-5}$.

Cases		Alg.4.2.3 $\alpha = 3$	Alg.4.2.3 $\alpha = 6$	Alg.4.2.3 $\alpha = 9$	Alg.4.2.3 $\alpha = 12$	Alg.4.2.3 $\alpha = 15$	Reich Tuyen	Tian Jiang
1	CPU	2.9827	2.3552	2.3725	2.1437	2.3514	7.0796	12.3607
	Iter.	16	16	16	16	16	20	81
2	CPU	2.4189	1.9850	1.9034	1.8323	1.8094	5.4496	9.9440
	Iter.	17	17	17	17	17	20	83
3	CPU	2.2188	1.8285	1.8093	1.6663	1.6729	5.3220	9.9360
	Iter.	16	16	16	16	16	20	82
4	CPU	2.3943	2.0577	1.9658	1.7530	1.7740	5.6309	10.0070
	Iter.	16	16	16	16	16	20	80

Remark 4.2.16. In each example, using different starting points and varying both the inertial and relaxation stepsizes α and ϕ (ϕ_n), respectively, we obtain the numerical results displayed in Tables 4.2.1-4.2.3 and Figures 4.1-4.3. We compared our proposed Algorithm 4.2.2 with the method of He *et al.* [121, Algorithm 1] in Examples 4.2.13 and 4.2.14 while our proposed Algorithm 4.2.3 is compared with the methods of Reich and Tuyen [213, Theorem 4.4] and Tian and Jiang [244, Algorithm (4.8)] in Example 4.2.15.
Furthermore, the following were observed from our numerical experiments:

- We have seen from the numerical examples that the projections onto feasible sets are generally difficult to compute. Thus, the benefits brought from the half-space technique (introduced in our methods) are further verified by our numerical experiments.

- In the numerical experiments, we choose the parameters $\eta \geq 0$, $\lambda_1 > 0$ and $\mu_1 > 0$ randomly and observed that the number of iteration does not change and no significant difference in the CPU time irrespective of the choices of the parameters.

- In each of the examples, we check the sensitivity of α as we vary ϕ (ϕ_n) for each starting points, in order to know if these choices affect the efficiency of our methods. We can see from the tables and graphs that the gap between the best results and the worst results of Algorithm 4.2.2 and Algorithm 4.2.3 is acceptably small, especially in Example 4.2.14 and Example 4.2.15. We can also see from Table 4.2.1 and Figure 4.1 that for solving problem (4.2.71), the efficiency of Algorithm 4.2.2 is slightly affected by the dimensionality of the problem. These means that the proposed relaxed inertial technique has efficient and stable performance for each group of random data.

- It can also be inferred from the tables and figures that in terms of both CPU time and number of iterations, our proposed Algorithm 4.2.2 outperforms the method of He *et al.* [121, Algorithm 1] in Example 4.2.13 and Example 4.2.14, while our proposed Algorithm 4.2.3 outperforms the methods of Reich and Tuyen [213, Theorem 4.4] and Tian and Jiang [244, Algorithm (4.8)] in Example 4.2.15. Hence, our methods are more efficient and more implementable than these other methods for solving the SVIP.

Appendix 4.2.17. *Algorithm 1 of He et al. [121].*

Stop 0. *Given a symmetric positive definite matrix* $\mathbb{H} \in \mathbb{R}^{m\times m}$, $\gamma \in (0,2)$ *and* $\rho \in (\rho_{\min},\ 3)$, *where* $\rho_{\min} := \max\left\{-3, \frac{2(v-1)\mu_1}{4\zeta_{\max}(T^THT)}\right\}$ *and* $T : \mathbb{R}^N \to \mathbb{R}^m$ *is a linear operator (where* T^T *means the transpose of* T*). Set an initial point* $\boldsymbol{u}_1 := (x_1, y_1, \lambda_1) \in \Omega := \mathcal{C} \times \mathcal{Q} \times \mathbb{R}^m$, *where* $\mathcal{C}$ *and* $\mathcal{Q}$ *are nonempty closed and convex subsets of* $\mathbb{R}^N$ *and* $\mathbb{R}^m$, *respectively.*

Step 1. *Generate a predictor* $\tilde{\mathbf{u}}_n := (\tilde{x}_n, \tilde{y}_n, \tilde{\lambda}_n)$ *with appropriate parameters* μ_1 *and* μ_2 :

$$\begin{cases} \bar{\lambda}_n = \lambda_n - \mathbb{H}(Tx_n - y_n), \\ \tilde{x}_n = P_{\mathcal{C}}\left[x_n - \frac{1}{\mu_1}\left(A(x_n) - T^T\bar{\lambda}_n\right)\right], \\ \hat{\lambda}_n = \lambda_n - \mathbb{H}(T\hat{x}_n - y_n), \quad \textit{where} \quad \hat{x}_n := \rho x_n + (1-\rho)\tilde{x}_n, \\ \tilde{y}_n = P_{\mathcal{Q}}\left[y_n - \frac{1}{\mu_2}(F(y_n) + \hat{\lambda}_n)\right], \\ \tilde{\lambda}_n = \lambda_n - \mathbb{H}(T\tilde{x}_n - \tilde{y}_n). \end{cases} \tag{4.2.72}$$

Step 2. *Update the next iterative* $\boldsymbol{u}_{n+1} := (x_{n+1}, y_{n+1}, \lambda_{n+1})$ *via*

$$\boldsymbol{u}_{n+1} := \boldsymbol{u}_n - \gamma\alpha_k d(\mathbf{u_n}, \tilde{\mathbf{u}}_\mathbf{n}),$$

where

$$\begin{cases} \alpha_k := \frac{\varphi(\mathbf{u}_n,\tilde{\mathbf{u}}_n)}{\|d(\mathbf{u}_n,\tilde{\mathbf{u}}_n)\|^2}, \\ d(\mathbf{u}_n, \tilde{\mathbf{u}}_n) := G(\mathbf{u}_n - \tilde{\mathbf{u}}_n) - \xi_n, \\ \varphi(\mathbf{u}_n, \tilde{\mathbf{u}}_n) := \left\langle \lambda_n - \tilde{\lambda}_n, \rho T(x_n - \tilde{x}_n) - (y_n - \tilde{y}_n)\right\rangle + \langle \mathbf{u}_n - \tilde{\mathbf{u}}_n, d(\mathbf{u}_n, \tilde{\mathbf{u}}_n)\rangle, \end{cases}$$

$\xi_n := \begin{pmatrix} \xi_{n_x} \\ \xi_{n_y} \\ 0 \end{pmatrix} := \begin{pmatrix} A(x_n) - A(\tilde{x}_n) + T^T\mathbb{H}T(x_n - \tilde{x}_n) \\ F(y_n) - F(\tilde{y}_n) + \mathbb{H}(y_n - \tilde{y}_n) \\ 0 \end{pmatrix}$, *A and F are monotone and Lipschitz continuous with constants* L_1 *and* L_2 *respectively, and*

$G := \begin{pmatrix} \mu_1 I_N + \rho T^T\mathbb{H}T & 0 & 0 \\ 0 & \mu_2 I_m + \mathbb{H} & 0 \\ 0 & 0 & \mathbb{H}^{-1} \end{pmatrix}$ *is the block diagonal matrix, with identity matrices* I_N *and* I_m *of size* N *and* m*, respectively. The parameters* μ_1 *and* μ_2 *are chosen such that*

$$\|\xi_{n_x}\| \leq v\mu_1\|x_n - \tilde{x}_n\| \qquad \textit{and} \qquad \|\xi_{n_y}\| \leq v\mu_2\ \|y_k - \tilde{y}_n\|,$$

where $v \in (0,1)$.

Appendix 4.2.18. *The Algorithm in Reich and Tuyen [213, Theorem 4.4].*

For any initial guess $x_1 = x \in \mathcal{H}_1$, define the sequence $\{x_n\}$ by

$$\begin{aligned}
y_n &= VI(\mathcal{C},\ \lambda_n A + I_{\mathcal{H}_1} - x_n),\\
z_n &= VI(\mathcal{Q},\ \mu_n F + I_{\mathcal{H}_2} - Ty_n),\\
C_n &= \{z \in \mathcal{H}_1 : ||y_n - z|| \le ||x_n - z||\},\\
D_n &= \{z \in \mathcal{H}_1 : ||z_n - Tz|| \le ||Ty_n - Tz||\},\\
W_n &= \{z \in \mathcal{H}_1 : \langle z - x_n, x_1 - x_n\rangle \le 0\},\\
x_{n+1} &= P_{C_n \cap D_n \cap W_n}(x_1),\ n \ge 1,
\end{aligned}$$

where $I_{\mathcal{H}_1}$ and $I_{\mathcal{H}_2}$ are identity operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and $\{\lambda_n\}$ and $\{\mu_n\}$ are two given sequences of positive numbers satisfying the following condition:

$$\min\left\{\inf_n\{\lambda_n\},\quad \inf_n\{\mu_n\}\right\} \ge r > 0.$$

Appendix 4.2.19. *Algorithm (4.8) of Tian and Jiang [244].*

Let $\{x_n\}, \{y_n\}, \{t_n\}$ and $\{w_n\}$ be sequences generated by $x_1 = x \in \mathcal{C}$ and

$$\begin{cases}
y_n = P_{\mathcal{C}}(x_n - \eta_n T^*(I - P_{\mathcal{Q}}(I - \lambda F))Tx_n),\\
t_n = P_{\mathcal{C}}(y_n - \lambda_n A y_n),\\
T_n = \{w \in \mathcal{H}_1 : \langle y_n - \lambda_n A y_n - t_n, w - t_n\rangle \le 0\},\\
w_n = P_{T_n}(y_n - \lambda_n A t_n),\\
x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n) w_n, \quad n \ge 1,
\end{cases}$$

where $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator, $A : \mathcal{C} \to \mathcal{H}_1$ is a monotone and L-Lipschitz continuous operator with $L > 0$, $F : \mathcal{H}_2 \to \mathcal{H}_2$ is a k-inverse strongly monotone mapping and $h : \mathcal{H}_1 \to \mathcal{H}_1$ is a σ-contractive mapping with $0 \le \sigma < 1$. The sequence $\{\eta_n\}$ is in $[a, b]$, for some $a, b \in \left(0, \frac{1}{\|T\|^2}\right)$, $\{\lambda_n\}$ is in $[c, d]$, for some $c, d \in \left(0, \frac{1}{L}\right)$, $\{\alpha_n\}$ is in $(0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

4.3 On split variational inequality problem beyond monotonicity

In solving the SVIP, very few methods have been considered in the literature and most of these few methods require the underlying operators to be co-coercive. This restrictive co-coercive assumption has been dispensed with in some methods; many of which require a product space formulation of the problem. However, it has been discovered that this product space formulation may cause some potential difficulties during implementation and its approach may not fully exploit the attractive splitting structure of the SVIP. In this study, we present two new methods with inertial steps for solving the SVIP in real Hilbert spaces without any product space formulation. We prove that the sequence generated by these methods converges strongly to a minimum-norm solution of the problem when the operators are pseudomonotone and Lipschitz continuous. Also, we provide several numerical experiments of the proposed methods in comparison with other related methods in the literature.

4.3.1 Proposed methods

In this section, we present our proposed methods and discuss their features. We begin with the following assumptions under which our strong convergence results are obtained.

Assumption 4.3.1. *Suppose that the following conditions hold:*

(a) The feasible sets $\mathcal{C}$ and $\mathcal{Q}$ are nonempty closed and convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

(b) $A : \mathcal{H}_1 \to \mathcal{H}_1$ and $F : \mathcal{H}_2 \to \mathcal{H}_2$ are pseudomonotone, sequentially weakly continuous and Lipschitz continuous with Lipschitz constants L_1 and L_2, respectively.

(c) $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator and the solution set $\Gamma := \{z \in VI(A, \mathcal{C}) : Tz \in VI(F, \mathcal{Q})\}$ is nonempty, where $VI(A, \mathcal{C})$ is the solution set of the classical VIP (1.2.4).

(d) $\{\delta_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ are positive sequences satisfying the following conditions:
$$\delta_n \in (0,1), \quad \lim_{n\to\infty} \delta_n = 0, \quad \sum_{n=1}^{\infty} \delta_n = \infty \text{ and } \lim_{n\to\infty} \frac{\tau_n}{\delta_n} = 0.$$

(e) $\{\theta_n\} \subset (a, 1-\delta_n)$ for some $a > 0$.

We present the following method for solving the SVIP (1.2.4)-(1.2.5) when L_1 and L_2 are known.

Algorithm 4.3.2. Modified projection and contraction method with fixed stepsize.

Step 0: Choose sequences $\{\delta_n\}_{n=1}^{\infty}, \{\theta_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ such that the conditions from Assumption 4.3.1 (d)-(e) hold and let $\eta \geq 0, \gamma_i \in (0,2), i = 1,2,$ $\mu \in (0, \frac{1}{L_1})$, $\lambda \in (0, \frac{1}{L_2})$, $\alpha \geq 3$ and $x_0, x_1 \in \mathcal{H}_1$ be given arbitrarily. Set $n := 1$.

Step 1: Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n := \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases} \tag{4.3.1}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}),$$
$$y_n = P_{\mathcal{Q}}(Tw_n - \lambda FTw_n),$$
$$z_n = Tw_n - \gamma_2 \beta_n r_n,$$

where $r_n := Tw_n - y_n - \lambda(FTw_n - Fy_n)$ and $\beta_n := \frac{\langle Tw_n - y_n, r_n\rangle}{\|r_n\|^2}$, if $r_n \neq 0$; otherwise, $\beta_n = 0$.

Step 3: Compute

$$b_n = w_n + \eta_n T^*(z_n - Tw_n),$$

where the stepsize η_n is chosen such that for small enough $\epsilon > 0$,

$$\eta_n \in \left[\epsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right],$$

if $z_n \neq Tw_n$; otherwise, $\eta_n = \eta$.
Step 4: Compute

$$u_n = P_{\mathcal{C}}(b_n - \mu A b_n),$$

$$t_n = b_n - \gamma_1 \gamma_n v_n,$$

where $v_n := b_n - u_n - \mu(Ab_n - Au_n)$ and $\gamma_n := \frac{\langle b_n - u_n, v_n \rangle}{\|v_n\|^2}$, if $v_n \neq 0$; otherwise, $\gamma_n = 0$.
Step 5: Compute

$$x_{n+1} = (1 - \theta_n - \delta_n) b_n + \theta_n t_n.$$

Set $n := n + 1$ and go back to **Step 1**.

In the situation where L_1 and L_2 are not available, we present the following method with adaptive stepsize for solving the SVIP (1.2.4)-(1.2.5).

Algorithm 4.3.3. Modified projection and contraction method with adaptive stepsize strategy.

Step 0: Choose sequences $\{\delta_n\}_{n=1}^{\infty}$, $\{\theta_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ such that the conditions from Assumption 4.3.1 (d)-(e) hold and let $\eta \geq 0$, $\gamma_i \in (0,2)$, $a_i \in (0,1), i = 1,2$, $\lambda_1 > 0$, $\mu_1 > 0$, $\alpha \geq 3$ and $x_0, x_1 \in \mathcal{H}_1$ be given arbitrarily. Set $n := 1$.
Step 1: Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n := \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases} \tag{4.3.2}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}),$$

$$y_n = P_{\mathcal{Q}}(Tw_n - \lambda_n FTw_n),$$

$$z_n = Tw_n - \gamma_2 \beta_n r_n,$$

where $r_n := Tw_n - y_n - \lambda_n(FTw_n - Fy_n)$, $\beta_n := \frac{\langle Tw_n - y_n, r_n \rangle}{\|r_n\|^2}$, if $r_n \neq 0$; otherwise, $\beta_n = 0$; and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{a_2 \|Tw_n - y_n\|}{\|FTw_n - Fy_n\|}, \ \lambda_n\right\}, & \text{if } FTw_n \neq Fy_n \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{4.3.3}$$

Step 3: Compute

$$b_n = w_n + \eta_n T^*(z_n - Tw_n),$$

where the stepsize η_n is chosen such that for small enough $\epsilon > 0$,

$$\eta_n \in \left[\epsilon, \ \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right],$$

if $z_n \neq Tw_n$; otherwise, $\eta_n = \eta$.
Step 4: Compute

$$u_n = P_{\mathcal{C}}(b_n - \mu_n A b_n),$$

$$t_n = b_n - \gamma_1 \gamma_n v_n,$$

where $v_n := b_n - u_n - \mu_n(Ab_n - Au_n)$, $\gamma_n = \frac{\langle b_n - u_n, v_n \rangle}{\|v_n\|^2}$, if $v_n \neq 0$; otherwise, $\gamma_n = 0$; and

$$\mu_{n+1} = \begin{cases} \min\left\{\frac{a_1 ||b_n - u_n||}{||Au_n - Ab_n||},\ \mu_n\right\}, & \text{if } Ab_n \neq Au_n \\ \mu_n, & \text{otherwise.} \end{cases} \tag{4.3.4}$$

Step 5: Compute

$$x_{n+1} = (1 - \theta_n - \delta_n)b_n + \theta_n t_n.$$

Set $n := n + 1$ and go back to **Step 1**.

We now highlight some of the features of our proposed methods.

Remark 4.3.1.

- Observe that Algorithm 4.3.2 and Algorithm 4.3.3 can be viewed as modified projection and contraction methods involving one projection onto $\mathcal{C}$ per iteration for solving the classical VIP in $\mathcal{H}_1$ and another projection and contraction methods involving one projection onto $\mathcal{Q}$ per iteration under a bounded linear operator T for solving another VIP in another space $\mathcal{H}_2$, with no extra projections onto half-spaces or feasible sets unlike the method in [206] (see Appendix 4.3.16), where extra projections onto half-spaces are required. In fact, an interesting feature of the projection and contraction methods used here is that r_n of Step 2 in Algorithms 4.3.2 and 4.3.3 can be described as weighted average of $(Tw_n - y_n \sim \lambda FTw_n)$ and a hypothetical $(T\tilde{w}_n - \tilde{y}_n \sim \lambda FT\tilde{w}_n)$ in $\mathcal{H}_2$, where $T\tilde{w}_n = Tw_n - \lambda FTw_n$ and $\tilde{y}_n = y_n - \lambda F y_n$. We also have similar description for v_n of Step 4. This looks very similar to the Heun's method or modified Euler method from numerical methods for solving ordinary differential equations (see [230, page 328] for details). Furthermore, we can see that

  $$\beta_n ||r_n||^2 = \langle Tw_n - y_n, r_n \rangle, \ \forall n \geq 1 \tag{4.3.5}$$

  holds for both $r_n = 0$ and $r_n \neq 0$. Similarly, we have that

  $$\gamma_n ||v_n||^2 = \langle b_n - u_n, v_n \rangle, \ \forall n \geq 1 \tag{4.3.6}$$

  holds for both $v_n = 0$ and $v_n \neq 0$.

- Another notable advantage of Algorithm 4.3.2 and Algorithm 4.3.3 for solving the SVIP (1.2.4)-(1.2.5) is that the monotonicity assumption on the operators A and F usually used in many other works (see for example, [61, 63, 121, 131, 145, 190, 213, 243, 244]) to guarantee convergence, is dispensed with and no extra projections are required under this setting unlike in [206].

- The stepsizes $\{\lambda_n\}$ and $\{\mu_n\}$ given by (4.3.3) and (4.3.4), respectively are generated at each iteration by some simple computations. Thus, Algorithm 4.3.3 is easily implemented without the prior knowledge of the Lipschitz constants L_1 and L_2.

- Step 1 of our methods is also easily implemented since the value of $||x_n - x_{n-1}||$ is a priori known before choosing α_n. In our numerical analysis (Section 4.3.3), we shall check the sensitivity of α in order to find numerically, the optimum choice of α with respect to the convergence speed of our proposed methods.
- Step 5 of both algorithms guarantee the strong convergence to a minimum-norm solution of the problem.
- Unlike in [61, 63, 121], we can see that Algorithm 4.3.2 and Algorithm 4.3.3 do not require any product space formulation, thereby avoiding any potential difficulties that might be caused by the product space.

Remark 4.3.2. The choice of the stepsize η_n in Step 3 of Algorithms 4.3.2 and 4.3.3 do not require the prior knowledge of the operator norm $\|T\|$. Furthermore, the value of η does not influence the algorithms, but it was introduced for the sake of clarity. We show in the following lemma that η_n is well-defined (see also [275, Remark 2.3, Lemma 2.3] and [274, Lemma 3.3]).

Lemma 4.3.3. *The stepsize η_n given in Step 3 of Algorithms 4.3.2 and 4.3.3 is well-defined.*

Proof. Let $z \in \Gamma$. That is, $z \in VI(A, \mathcal{C})$ and $Tz \in VI(F, \mathcal{Q})$. Then, by Cauchy-Schwarz inequality and Lemma 2.1.1 , we obtain

$$\begin{aligned}\|T^*(Tw_n - z_n)\|\|w_n - z\| &\geq \langle T^*(Tw_n - z_n), w_n - z\rangle \\ &= \langle Tw_n - z_n, Tw_n - Tz\rangle \\ &= \frac{1}{2}\Big[\|Tw_n - z_n\|^2 + \|Tw_n - Tz\|^2 - \|z_n - Tz\|^2\Big]. \end{aligned} \tag{4.3.7}$$

Since $z_n = Tw_n - \gamma_2\beta_n r_n$, we obtain that

$$\begin{aligned}\|z_n - Tz\|^2 &= \|Tw_n - Tz\|^2 + \gamma_2^2\beta_n^2\|r_n\|^2 - 2\gamma_2\beta_n\langle Tw_n - Tz, r_n\rangle \\ &= \|Tw_n - Tz\|^2 + \gamma_2^2\beta_n^2\|r_n\|^2 - 2\gamma_2\beta_n\langle Tw_n - y_n, r_n\rangle - 2\gamma_2\beta_n\langle y_n - Tz, r_n\rangle. \end{aligned} \tag{4.3.8}$$

Since $y_n = P_{\mathcal{Q}}(Tw_n - \lambda FTw_n)$ and $Tz \in \mathcal{Q}$, we obtain from the characteristics property of $P_{\mathcal{Q}}$ that

$$\langle Tw_n - \lambda FTw_n - y_n, y_n - Tz\rangle \geq 0. \tag{4.3.9}$$

Also, since $Tz \in VI(F, \mathcal{Q})$ and $y_n \in \mathcal{Q}$, we have that

$$\langle FTz, y_n - Tz\rangle \geq 0 \ \ \text{(see Inequality (1.2.1))},$$

which by the pseudomonotonicity of F and $\lambda > 0$, implies

$$\langle \lambda Fy_n, y_n - Tz\rangle \geq 0. \tag{4.3.10}$$

Adding (4.3.9) and (4.3.10), we obtain

$$\langle Tw_n - y_n - \lambda(FTw_n - Fy_n), y_n - Tz\rangle \geq 0.$$

That is

$$\langle r_n, y_n - Tz\rangle \geq 0 \text{ (which is still true if } \lambda \text{ is replaced with } \lambda_n \text{ as in Algorithm 4.3.3).} \tag{4.3.11}$$

On the other hand, we have from the Lipschitz continuity of F and $\lambda \in (0, \frac{1}{L_2})$, that

$$\begin{aligned}\langle Tw_n - y_n, r_n\rangle &= \langle Tw_n - y_n, Tw_n - y_n - \lambda(FTw_n - Fy_n)\rangle \\ &= \|Tw_n - y_n\|^2 - \langle Tw_n - y_n, \lambda(FTw_n - Fy_n)\rangle \\ &\geq \|Tw_n - y_n\|^2 - \lambda\|Tw_n - y_n\|\|FTw_n - Fy_n\| \\ &\geq (1 - \lambda L_2)\|Tw_n - y_n\|^2 \geq 0,\end{aligned} \tag{4.3.12}$$

which by the definition of β_n, implies that $\beta_n \geq 0, \ \forall n \geq 1$ (this is still true even if we replace λ with λ_n as in Algorithm 4.3.3). Thus, using (4.3.11) and (4.3.5) in (4.3.8), and nothing that $\gamma_2 \in (0, 2)$, we obtain

$$\begin{aligned}\|z_n - Tz\|^2 &\leq \|Tw_n - Tz\|^2 + \gamma_2^2\beta_n^2\|r_n\|^2 - 2\gamma_2\beta_n\langle Tw_n - y_n, r_n\rangle \\ &= \|Tw_n - Tz\|^2 + \gamma_2^2\beta_n^2\|r_n\|^2 - 2\gamma_2\beta_n \cdot \beta_n\|r_n\|^2 \\ &= \|Tw_n - Tz\|^2 - \gamma_2(2 - \gamma_2)\beta_n^2\|r_n\|^2 \\ &\leq \|Tw_n - Tz\|^2.\end{aligned} \tag{4.3.13}$$

Substituting (4.3.13) into (4.3.7), we obtain that

$$\|T^*(Tw_n - z_n)\|\|w_n - z\| \geq \frac{1}{2}\|Tw_n - z_n\|^2. \tag{4.3.14}$$

Now, for $z_n \neq Tw_n$, we have that $\|Tw_n - z_n\| > 0$. This together with (4.3.14), imply that $\|T^*(Tw_n - z_n)\|\|w_n - z\| > 0$. Hence, we have that $\|T^*(Tw_n - z_n)\| \neq 0$. Therefore, η_n is well-defined. □

We also make the following observation regarding η_n. Note from Step 3 of Algorithms 4.3.2 and 4.3.3, that

$$\eta_n\|T^*(Tw_n - z_n)\|^2 \leq \|Tw_n - z_n\|^2 - \epsilon\|T^*(Tw_n - z_n)\|^2, \tag{4.3.15}$$

which implies that

$$\eta_n^2\|T^*(Tw_n - z_n)\|^2 - \eta_n\|Tw_n - z_n\|^2 \leq -\epsilon\eta_n\|T^*(Tw_n - z_n)\|^2. \tag{4.3.16}$$

Note also that, we can replace the choice of η_n in Step 3 of Algorithm 4.3.2 and Algorithm 4.3.3 with the following: For small enough $\epsilon > 0$,

$$\eta_n \in \left[\epsilon, \ \frac{\|Tw_n - z_n\|^2}{1 + \|T^*(Tw_n - z_n)\|^2} - \epsilon\right]. \tag{4.3.17}$$

Then, we can see that

$$\eta_n \leq \frac{\|Tw_n - z_n\|^2 - \epsilon\big(1 + \|T^*(Tw_n - z_n)\|^2\big)}{1 + \|T^*(Tw_n - z_n)\|^2},$$

which implies that

$$\eta_n + \eta_n\|T^*(Tw_n - z_n)\|^2 \leq \|Tw_n - z_n\|^2 - \epsilon - \epsilon\|T^*(Tw_n - z_n)\|^2.$$

This further gives (4.3.15) and consequently (4.3.16).
As we shall see in our convergence analysis, (4.3.15) and (4.3.16) play crucial role in our proofs. Therefore, one can either choose the stepsize in Step 3 or that in (4.3.17), to ensure the convergence of Algorithms 4.3.2 and 4.3.3.

4.3.2 Convergence analysis

Lemma 4.3.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.3.2 under Assumption 4.3.1. Then, $\{x_n\}$ is bounded.*

Proof. Let $p \in \Gamma$. Then $Tp \in VI(F, \mathcal{Q}) \subset \mathcal{Q}$. From the definition of w_n, we have

$$\begin{aligned} \|w_n - p\| &= \|x_n + \alpha_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \alpha_n\|x_n - x_{n-1}\|. \end{aligned}$$

Also, from Step 1, we observe that $\alpha_n\|x_n - x_{n-1}\| \leq \tau_n, \quad \forall n \geq 1$, which implies that

$$\frac{\alpha_n}{\delta_n}\|x_n - x_{n-1}\| \leq \frac{\tau_n}{\delta_n} \to 0, \quad \text{as} \quad n \to \infty. \tag{4.3.18}$$

Hence, there exists $M_1 > 0$ such that

$$\frac{\alpha_n}{\delta_n}\|x_n - x_{n-1}\| \leq M_1, \quad \forall n \geq 1. \tag{4.3.19}$$

This implies that

$$\|w_n - p\| \leq \|x_n - p\| + \delta_n M_1, \ \forall n \geq 1. \tag{4.3.20}$$

Since $y_n = P_{\mathcal{Q}}(Tw_n - \lambda FTw_n)$, we obtain by the characteristic property of $P_{\mathcal{Q}}$ that

$$\langle y_n - Tw_n + \lambda FTw_n, y_n - Tp\rangle \leq 0. \tag{4.3.21}$$

Again, since $Tp \in VI(F, \mathcal{Q})$ and $y_n \in \mathcal{Q}$, we obtain that

$$\langle FTp, y_n - Tp\rangle \geq 0,$$

which follows from the pseudomonotonicity of F that $\langle Fy_n, y_n - Tp\rangle \geq 0$. Since $\lambda > 0$, we get

$$\langle \lambda Fy_n, y_n - Tp\rangle \geq 0. \tag{4.3.22}$$

Combining (4.3.21) and (4.3.22), we have

$$\langle Tw_n - y_n - \lambda(FTw_n - Fy_n), y_n - Tp\rangle \geq 0. \tag{4.3.23}$$

From Step 2 and (4.3.23), we obtain

$$\begin{aligned}\langle Tw_n - Tp, r_n\rangle &= \langle Tw_n - y_n, r_n\rangle + \langle y_n - Tp, r_n\rangle \\ &= \langle Tw_n - y_n, r_n\rangle + \langle y_n - Tp, Tw_n - y_n - \lambda(FTw_n - Fy_n)\rangle \\ &\geq \langle Tw_n - y_n, r_n\rangle,\end{aligned}$$

which implies that

$$-\langle Tw_n - Tp, r_n\rangle \leq -\langle Tw_n - y_n, r_n\rangle. \tag{4.3.24}$$

Since $z_n = Tw_n - \gamma_2\beta_n r_n$, we have that $||\beta_n \cdot r_n||^2 = \gamma_2^{-2}||z_n - Tw_n||^2$.
Thus, from Lemma 2.1.1, (4.3.5) and (4.3.24) (nothing that β_n and γ_2 are nonnegative), we obtain

$$\begin{aligned}\|z_n - Tp\|^2 &= \|Tw_n - \gamma_2\beta_n r_n - Tp\|^2 \\ &= \|Tw_n - Tp\|^2 + \gamma_2^2\beta_n^2\|r_n\|^2 - 2\gamma_2\beta_n\langle Tw_n - Tp, r_n\rangle \\ &\leq \|Tw_n - Tp\|^2 + \gamma_2^2\beta_n^2\|r_n\|^2 - 2\gamma_2\beta_n\langle Tw_n - y_n, r_n\rangle \\ &= \|Tw_n - Tp\|^2 + \gamma_2^2\beta_n^2\|r_n\|^2 - 2\gamma_2\beta_n \cdot \beta_n\|r_n\|^2 \\ &= \|Tw_n - Tp\|^2 - \gamma_2(2-\gamma_2)\|\beta_n \cdot r_n\|^2 \\ &= \|Tw_n - Tp\|^2 - \gamma_2^{-1}(2-\gamma_2)||z_n - Tw_n||^2.\end{aligned} \tag{4.3.25}$$

Also, we obtain from Step 3, Lemma 2.1.1 and (4.3.25) that

$$\begin{aligned}\|b_n - p\|^2 &= \|w_n - p\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 + 2\eta_n\langle w_n - p, T^*(z_n - Tw_n)\rangle \\ &= \|w_n - p\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 + 2\eta_n\langle Tw_n - Tp, z_n - Tw_n\rangle \\ &= \|w_n - p\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 \\ &+ \eta_n\left[\|z_n - Tp\|^2 - \|Tw_n - Tp\|^2 - \|z_n - Tw_n\|^2\right] \\ &\leq \|w_n - p\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 - \eta_n\|z_n - Tw_n\|^2 \\ &= \|w_n - p\|^2 - \eta_n\left[\|z_n - Tw_n\|^2 - \eta_n\|T^*(z_n - Tw_n)\|^2\right].\end{aligned} \tag{4.3.26}$$

Thus, by (4.3.15), we obtain that

$$\|b_n - p\|^2 \leq \|w_n - p\|^2,$$

which implies from (4.3.20) that

$$\|b_n - p\| \leq \|w_n - p\| \leq \|x_n - p\| + \delta_n M_1, \ \forall n \geq 1. \tag{4.3.27}$$

Following similar argument used in obtaining (4.3.25), we get

$$\begin{aligned}\|t_n - p\|^2 &= \|b_n - \gamma_1\gamma_n v_n - p\|^2 \\ &\leq \|b_n - p\|^2 - \gamma_1^{-1}(2-\gamma_1)\|t_n - b_n\|^2.\end{aligned} \tag{4.3.28}$$

Also, by Step 5, we get

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p)-\delta_n p\| \\ &\leq \|(1-\theta_n-\delta_n))(b_n-p)+\theta_n(t_n-p)\|+\delta_n\|p\|. \end{aligned} \tag{4.3.29}$$

But from (4.3.27) and (4.3.28), we obtain that

$$\begin{aligned}\|(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p)\|^2 &= (1-\theta_n-\delta_n)^2\|b_n-p\|^2+\theta_n^2\|t_n-p\|^2 \\ &+2(1-\theta_n-\delta_n)\theta_n\langle b_n-p,t_n-p\rangle \\ &\leq (1-\theta_n-\delta_n)^2\|w_n-p\|^2+\theta_n^2\|b_n-p\|^2 \\ &+2(1-\theta_n-\delta_n)\theta_n\|b_n-p\|^2 \\ &\leq (1-\delta_n)^2\|w_n-p\|^2. \end{aligned} \tag{4.3.30}$$

Hence, (4.3.29) becomes

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-\delta_n)\|w_n-p\|+\delta_n\|p\| \\ &\leq (1-\delta_n)\left[\|x_n-p\|+\delta_n M_1\right]+\delta_n\|p\| \\ &\leq (1-\delta_n)\|x_n-p\|+\delta_n(M_1+\|p\|), \end{aligned}$$

which implies from Lemma 2.5.35 that $\{x_n\}$ is bounded. □

Lemma 4.3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.3.2 under Assumption 4.3.1. Suppose that, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to a point $z \in \mathcal{H}_1$ and $\lim_{k\to\infty}\|b_{n_k}-w_{n_k}\| = 0 = \lim_{k\to\infty}\|b_{n_k}-t_{n_k}\|$, then $z \in \Gamma$.*

Proof. Without loss of generality, we may assume that $z_n \neq Tw_n$. Then,

$$\eta_n \in \left[\epsilon, \quad \frac{\|z_n-Tw_n\|^2}{\|T^*(z_n-Tw_n)\|^2}-\epsilon\right].$$

Thus, putting (4.3.16) into (4.3.26), we obtain that

$$\begin{aligned}\|b_{n_k}-p\|^2 &\leq \|w_{n_k}-p\|^2-\eta_{n_k}\epsilon\|T^*(z_{n_k}-Tw_{n_k})\|^2 \\ &\leq \|w_{n_k}-p\|^2-\epsilon^2\|T^*(z_{n_k}-Tw_{n_k})\|^2, \end{aligned} \tag{4.3.31}$$

which implies that

$$\begin{aligned}\epsilon^2\|T^*(z_{n_k}-Tw_{n_k})\|^2 &\leq \|w_{n_k}-p\|^2-\|b_{n_k}-p\|^2 \\ &\leq \|w_{n_k}-b_{n_k}\|^2+2\|w_{n_k}-b_{n_k}\|\|b_{n_k}-p\|. \end{aligned}$$

Thus, by our hypothesis, we obtain

$$\lim_{k\to\infty}\|T^*(z_{n_k}-Tw_{n_k})\| = 0. \tag{4.3.32}$$

From (4.3.26) and (4.3.32), we obtain

$$\begin{aligned}\eta_{n_k}\|z_{n_k}-Tw_{n_k}\|^2 &\leq \|w_{n_k}-p\|^2-\|b_{n_k}-p\|^2+\eta_{n_k}^2\|T^*(z_{n_k}-Tw_{n_k})\|^2 \\ &\leq \|w_{n_k}-b_{n_k}\|^2+2\|w_{n_k}-b_{n_k}\|\|b_{n_k}-p\| \\ &+\eta_{n_k}^2\|T^*(z_{n_k}-Tw_{n_k})\|^2 \to 0, \quad \text{as} \quad k\to\infty. \end{aligned}$$

Since $\eta_{n_k} > \epsilon > 0$, we obtain that

$$\lim_{k\to\infty} \|z_{n_k} - Tw_{n_k}\| = 0. \tag{4.3.33}$$

From (4.3.12), we obtain

$$\langle Tw_{n_k} - y_{n_k}, r_{n_k}\rangle \geq (1-\lambda L_2)\|Tw_{n_k} - y_{n_k}\|^2. \tag{4.3.34}$$

Since $\lambda \in (0, \frac{1}{L_2})$, we have that $1-\lambda L_2 > 0$. Thus, we obtain from (4.3.34) and (4.3.5) that

$$\begin{aligned}
\|Tw_{n_k} - y_{n_k}\|^2 &\leq \left(\frac{1}{1-\lambda L_2}\right)\langle Tw_{n_k} - y_{n_k}, r_{n_k}\rangle \\
&= \left(\frac{1}{1-\lambda L_2}\right)\beta_{n_k}\|r_{n_k}\|^2 \\
&= \left(\frac{1}{1-\lambda L_2}\right)\beta_{n_k}\|r_{n_k}\|\|Tw_{n_k} - y_{n_k} - \lambda(FTw_{n_k} - Fy_{n_k})\| \\
&\leq \left(\frac{1}{1-\lambda L_2}\right)\beta_{n_k}\|r_{n_k}\|\left(\|Tw_{n_k} - y_{n_k}\| + \lambda\|FTw_{n_k} - Fy_{n_k}\|\right) \\
&\leq \left(\frac{1}{1-\lambda L_2}\right)\beta_{n_k}\|r_{n_k}\|\left(\|Tw_{n_k} - y_{n_k}\| + \lambda L_2\|Tw_{n_k} - y_{n_k}\|\right) \\
&= \left(\frac{1+\lambda L_2}{1-\lambda L_2}\right)\|Tw_{n_k} - y_{n_k}\|\beta_{n_k}\|r_{n_k}\| \\
&= \gamma_2^{-1}\left(\frac{1+\lambda L_2}{1-\lambda L_2}\right)\|Tw_{n_k} - y_{n_k}\|\|z_{n_k} - Tw_{n_k}\|,
\end{aligned}$$

where the last equality follows from $z_{n_k} = Tw_{n_k} - \gamma_2\beta_{n_k}r_{n_k}$, that is, $\beta_{n_k}||r_{n_k}|| = \gamma_2^{-1}||z_{n_k} - Tw_{n_k}||$.
Therefore, we obtain from (4.3.33) that

$$\|Tw_{n_k} - y_{n_k}\| \leq \gamma_2^{-1}\left(\frac{1+\lambda L_2}{1-\lambda L_2}\right)\|Tw_{n_k} - z_{n_k}\| \to 0, \quad \text{as} \quad k\to\infty. \tag{4.3.35}$$

By the characteristic property of $P_{\mathcal{Q}}$, we obtain for all $x \in \mathcal{Q}$ that

$$\langle Tw_{n_k} - \lambda FTw_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0.$$

This implies that

$$\frac{1}{\lambda}\langle Tw_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle FTw_{n_k}, y_{n_k} - Tw_{n_k}\rangle \leq \langle FTw_{n_k}, x - Tw_{n_k}\rangle. \tag{4.3.36}$$

Thus, we obtain from (4.3.35) that

$$0 \leq \liminf_{k\to\infty}\langle FTw_{n_k}, x - Tw_{n_k}\rangle, \ \forall x \in \mathcal{Q}. \tag{4.3.37}$$

Now, note that

$$\begin{aligned}
\langle Fy_{n_k}, x - y_{n_k}\rangle &= \langle Fy_{n_k} - FTw_{n_k}, x - Tw_{n_k}\rangle + \langle FTw_{n_k}, x - Tw_{n_k}\rangle \\
&+ \langle Fy_{n_k}, Tw_{n_k} - y_{n_k}\rangle.
\end{aligned}\tag{4.3.38}$$

Moreover, since F is Lipschitz continuous on $\mathcal{H}_2$, we obtain from (4.3.35) that

$$\lim_{k\to\infty} ||FTw_{n_k} - Fy_{n_k}|| = 0.$$

Hence, we obtain from (4.3.35), (4.3.37) and (4.3.38) that

$$0 \leq \liminf_{k\to\infty} \langle Fy_{n_k}, x - y_{n_k}\rangle, \ \forall x \in \mathcal{Q}. \tag{4.3.39}$$

We now show that $Tz \in VI(F, \mathcal{Q})$. Let $\{c_k\}$ be a sequence of positive numbers such that $c_{k+1} \leq c_k, \quad \forall k \geq 1$ and $c_k \to 0$ as $k \to \infty$. For each $k \geq 1$, denote by N_k, the smallest positive integer such that

$$\langle Fy_{n_j}, x - y_{n_j}\rangle + c_k \geq 0, \quad \forall j \geq N_k, \tag{4.3.40}$$

where the existence of N_k follows from (4.3.39). Note that $\{N_k\}$ is increasing since $\{c_k\}$ is decreasing. Also, since $\{y_{N_k}\} \subset \mathcal{Q}$ for all $k \geq 1$, we have that $Fy_{N_k} \neq 0$. Hence, we can set $q_{N_k} = \frac{Fy_{N_k}}{\|Fy_{N_k}\|^2}$, for each $k \geq 1$. Then, $\langle Fy_{N_k}, q_{N_k}\rangle = 1$, for each $k \geq 1$. Thus by (4.3.40), we have $\langle Fy_{N_k}, x + c_k q_{N_k} - y_{N_k}\rangle \geq 0$, which by the pseudomonotonicity of F on $\mathcal{H}_2$, yields

$$\langle F(x + c_k q_{N_k}), x + c_k q_{N_k} - y_{N_k}\rangle \geq 0. \tag{4.3.41}$$

This further implies that

$$\langle Fx, x - y_{N_k}\rangle \geq \langle Fx - F(x + c_k q_{N_k}), x + c_k q_{N_k} - y_{N_k}\rangle - c_k\langle Fx, q_{N_k}\rangle. \tag{4.3.42}$$

Now, from Step 2 and (4.3.18), we get that

$$\lim_{n\to\infty} \|w_n - x_n\| = \lim_{n\to\infty} \alpha_n\|x_n - x_{n-1}\| = \lim_{n\to\infty} \delta_n \cdot \frac{\alpha_n}{\delta_n}\|x_n - x_{n-1}\| = 0. \tag{4.3.43}$$

Since the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is weakly convergent to a point $z \in \mathcal{H}_1$, it follows that the subsequence $\{w_{n_k}\}$ of $\{w_n\}$ is also weakly convergent to $z \in \mathcal{H}_1$. Again, since T is a bounded linear operator, we obtain that $\{Tw_{n_k}\}$ converges weakly to Tz. Hence, by (4.3.35), we have that $\{y_{n_k}\}$ also converges weakly to Tz. Note that $Tz \in \mathcal{Q}$ since $\{y_{n_k}\} \subset \mathcal{Q}$. Thus, by the sequentially weakly continuity of F on $\mathcal{Q}$, we obtain that $\{Fy_{n_k}\}$ converges weakly to FTz. Now, if $FTz = 0$, then $Tz \in VI(F, \mathcal{Q})$. So, let $FTz \neq 0$. Then, by the weakly lower semicontinuity of $\|\cdot\|$, we obtain that

$$0 < \|FTz\| \leq \liminf_{k\to\infty} \|Fy_{N_k}\|.$$

Since $\{y_{n_k}\} \subset \{y_{N_k}\}$, we obtain that

$$0 \leq \limsup_{k\to\infty} \|c_k q_{N_k}\| = \limsup_{k\to\infty} \left(\frac{c_k}{\|Fy_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty} c_k}{\liminf_{k\to\infty} \|Fy_{n_k}\|} \leq \frac{0}{\|FTz\|} = 0.$$

Therefore, $\lim_{k\to\infty} c_k q_{N_k} = 0$. Thus, letting $k \to \infty$ in (4.3.42), we have

$$\langle Fx, x - Tz\rangle \geq 0, \ \ \forall x \in \mathcal{Q}, \tag{4.3.44}$$

which implies by Lemma 2.5.8 that $Tz \in VI(F, \mathcal{Q})$.

Next, we show that $z \in VI(A, \mathcal{C})$. Now, observe that, if we follow similar argument used in getting (4.3.35), and noting (by our hypothesis) that $\lim_{k\to\infty} \|b_{n_k} - t_{n_k}\| = 0$, we can also get

$$\|b_{n_k} - u_{n_k}\| \leq r_1^{-1} \left(\frac{1+\mu L_1}{1+\mu L_1}\right) \|b_{n_k} - t_{n_k}\| \to 0, \quad \text{as} \quad k \to \infty. \tag{4.3.45}$$

Thus, using (4.3.45), the characteristics property of $P_{\mathcal{C}}$ and the Lipschitz continuity of A on $\mathcal{H}_1$, we obtain (by following similar argument used in getting (4.3.39)) that

$$0 \leq \liminf_{k\to\infty} \langle Au_{n_k}, y - u_{n_k}\rangle, \ \forall y \in \mathcal{C}. \tag{4.3.46}$$

Now, since $\{w_{n_k}\}$ converges weakly to z, we obtain by our hypothesis and (4.3.45) that the subsequences $\{b_{n_k}\}$ and $\{u_{n_k}\}$ of $\{b_n\}$ and $\{u_n\}$ respectively, converge weakly to z. Again, since $\{u_{n_k}\} \subset \mathcal{C}$, we have that $z \in \mathcal{C}$. Thus, by the sequential weakly continuity of A on $\mathcal{C}$, we obtain that $\{Au_{n_k}\}$ converges weakly to Az.
Thus, we can follow similar proof used in obtaining (4.3.44), to get

$$\langle Ay, y - z\rangle \geq 0, \quad \forall\ y \in \mathcal{C}, \tag{4.3.47}$$

which by Lemma 2.5.8, implies $z \in \mathrm{V}I(A, \mathcal{C})$. Hence, we conclude that $z \in \Gamma$. □

We are now in position to give the main theorem for Algorithm 4.3.2.

Theorem 4.3.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.3.2 under Assumption 4.3.1. Then, $\{x_n\}$ converges strongly to $p \in \Gamma$, where $\|p\| = \min\{\|z\| : z \in \Gamma\}$.*

Proof. Let $p \in \Gamma$. Then, from Step 2 and (4.3.19), we obtain

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n - p\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle x_n - p, x_n - x_{n-1}\rangle \\
&\leq \|x_n - p\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n||x_n - p||||x_n - x_{n-1}|| \\
&= \|x_n - p\|^2 + \alpha_n\|x_n - x_{n-1}\| \left[\delta_n \cdot \frac{\alpha_n}{\delta_n}\|x_n - x_{n-1}\| + 2\|x_n - p\|\right] \\
&\leq \|x_n - p\|^2 + \alpha_n\|x_n - x_{n-1}\| \left[\delta_n M_1 + 2\|x_n - p\|\right].
\end{aligned} \tag{4.3.48}$$

Since by Lemma 4.3.4, $\{x_n\}$ is bounded, then, there exists $M_2 > 0$ such that

$$(\delta_n M_1 + \|x_n - p\|) \leq M_2, \ \forall n \geq 1.$$

Hence, we obtain from (4.3.48) that

$$\|w_n - p\|^2 \ \leq \ \|x_n - p\|^2 + 3\alpha_n\|x_n - x_{n-1}\|M_2. \tag{4.3.49}$$

Thus, from (4.3.27) and (4.3.28), we get

$$\begin{aligned}\|(1-\theta_n)b_n+\theta_n t_n-p\|^2 &= \|(1-\theta_n)(b_n-p)+\theta_n(t_n-p)\|^2\\ &= (1-\theta_n)^2\|b_n-p\|^2+\theta_n^2\|t_n-p\|^2\\ &+2(1-\theta_n)\theta_n\langle b_n-p,t_n-p\rangle\\ &\le (1-\theta_n)^2\|w_n-p\|^2+\theta_n^2\|w_n-p\|^2\\ &+2(1-\theta_n)\theta_n\langle b_n-p,t_n-p\rangle\\ &\le (1-\theta_n)^2\|w_n-p\|^2+\theta_n^2\|w_n-p\|^2+2(1-\theta_n)\theta_n\|w_n-p\|^2\\ &\le \|x_n-p\|^2+3\alpha_n\|x_n-x_{n-1}\|M_2.\end{aligned}$$

From Step 5, the previous inequality and (4.3.18), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\delta_n)\left[(1-\theta_n)b_n+\theta_n t_n-p\right]-\left[\delta_n\theta_n(b_n-t_n)+\delta_n p\right]\|^2\\ &\le (1-\delta_n)^2\|(1-\theta_n)b_n+\theta_n t_n-p\|^2-2\langle\delta_n\theta_n(b_n-t_n)+\delta_n p,x_{n+1}-p\rangle\\ &\le (1-\delta_n)\|(1-\theta_n)b_n+\theta_n t_n-p\|^2+2\delta_n\theta_n\langle b_n-t_n,p-x_{n+1}\rangle\\ &+2\delta_n\langle p,p-x_{n-1}\rangle\\ &\le (1-\delta_n)\|(1-\theta_n)b_n+\theta_n t_n-p\|^2+2\delta_n\theta_n\|b_n-t_n\|\|p-x_{n+1}\|\\ &+2\delta_n\langle p,p-x_{n+1}\rangle\\ &\le (1-\delta_n)\left(\|x_n-p\|^2+3\alpha_n\|x_n-x_{n-1}\|M_2\right)+2\delta_n\theta_n\|b_n-t_n\|\|p-x_{n+1}\|\\ &+2\delta_n\langle p,p-x_{n+1}\rangle\\ &\le (1-\delta_n)\|x_n-p\|^2+3\alpha_n\|x_n-x_{n-1}\|M_2+2\delta_n\theta_n\|b_n-t_n\|\|p-x_{n+1}\|\\ &+2\delta_n\langle p,p-x_{n+1}\rangle\\ &= (1-\delta_n)\|x_n-p\|^2+\delta_n d_n,\ \ n\ge 1,\end{aligned}$$

where $d_n = 3\frac{\alpha_n}{\delta_n}\|x_n-x_{n-1}\|M_2+2\left(\theta_n\|b_n-t_n\|\|p-x_{n+1}\|+\langle p,p-x_{n+1}\rangle\right)$.
According to Lemma 2.5.36, to conclude the proof, it sufficies to show that $\limsup_{k\to\infty} d_{n_k}\le 0$ for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ satisfying the condition;

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\right)\ge 0. \tag{4.3.50}$$

To show that $\limsup_{k\to\infty} d_{n_k}\le 0$, suppose that $\{\|x_{n_k}-p\|\}$ is a subsequence of $\{\|x_n-p\|\}$ such that (4.3.50) holds. Then,

$$\begin{aligned}\liminf_{k\to\infty}\left(\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\right) &= \liminf_{k\to\infty}\Big[\left(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\right)\\ &\times\left(\|x_{n_k+1}-p\|+\|x_{n_k}-p\|\right)\Big]\\ &\ge 0. \end{aligned}\tag{4.3.51}$$

Now, observe that

$$\begin{aligned}\|(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p)\|^2 &= (1-\theta_n-\delta_n)^2\|b_n-p\|^2\\ &+2(1-\theta_n-\delta_n)\theta_n\langle b_n-p,t_n-p\rangle+\theta_n^2\|t_n-p\|^2\\ &\le (1-\theta_n-\delta_n)^2\|b_n-p\|^2+(1-\theta_n-\delta_n)\theta_n\|b_n-p\|^2\\ &+(1-\theta_n-\delta_n)\theta_n\|t_n-p\|^2+\theta_n^2\|t_n-p\|^2\\ &\le (1-\theta_n-\delta_n)(1-\delta_n)\|b_n-p\|^2 \qquad (4.3.52)\\ &+\theta_n(1-\delta_n)\|t_n-p\|^2.\end{aligned}$$

Also, since $\{x_n\}$ and $\{\delta_n\}$ are bounded, there exists $M_3>0$ such that

$$(\delta_n\|p\|+2||x_{n+1}-(1-\delta_n)p||)\,||p||\le M_3$$

, for all $n\ge 1$. Hence, from Step 5, (4.3.28), (4.3.49) and (4.3.52), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p)-\delta_n p\|^2\\ &= \|(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p)\|^2+\delta_n^2\|p\|^2\\ &-2\delta_n\langle(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p),p\rangle\\ &\le \|(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p)\|^2\\ &+\delta_n\left(\delta_n\|p\|^2+2||(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p)||\;||p||\right)\\ &= \|(1-\theta_n-\delta_n)(b_n-p)+\theta_n(t_n-p)\|^2\\ &+\delta_n\left(\delta_n\|p\|+2||x_{n+1}-(1-\delta_n)p||\right)||p||\\ &\le (1-\theta_n-\delta_n)(1-\delta_n)\|b_n-p\|^2+\theta_n(1-\delta_n)\|t_n-p\|^2+\delta_n M_3\\ &\le (1-\theta_n-\delta_n)(1-\delta_n)\|b_n-p\|^2+\theta_n(1-\delta_n)(\|b_n-p\|^2 \qquad (4.3.53)\\ &-\gamma_1^{-1}(2-\gamma_1)\|t_n-b_n\|^2)+\delta_n M_3\\ &\le (1-\theta_n-\delta_n)(1-\delta_n)\|w_n-p\|^2+\theta_n(1-\delta_n)(\|w_n-p\|^2\\ &-\gamma_1^{-1}(2-\gamma_1)\|t_n-b_n\|^2)+\delta_n M_3\\ &\le \|w_n-p\|^2-\theta_n(1-\delta_n)\gamma_1^{-1}(2-\gamma_1)\|t_n-b_n\|^2+\delta_n M_3\\ &\le \|x_n-p\|^2-\theta_n(1-\delta_n)\gamma_1^{-1}(2-\gamma_1)\|t_n-b_n\|^2\\ &+\delta_n\left(3\frac{\alpha_n}{\delta_n}\|x_n-x_{n-1}\|M_2+M_3\right),\end{aligned}$$

Thus, from (4.3.51) and (4.3.19), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\left((1-\delta_{n_k})\theta_{n_k}\gamma_1^{-1}(2-\gamma_1)\|t_{n_k}-b_{n_k}\|^2\right) &\le \limsup_{k\to\infty}\left(\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\right)\\ &+\limsup_{k\to\infty}\left(\delta_{n_k}\left(3M_1M_2+M_3\right)\right)\\ &= -\liminf_{k\to\infty}\left(\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\right)\\ &\le 0,\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\|t_{n_k}-b_{n_k}\|=0. \qquad (4.3.54)$$

From (4.3.53) and (4.3.31), we obtain

$$\begin{aligned}\|x_{n_k+1}-p\|^2 &\le (1-\theta_{n_k}-\delta_{n_k})(1-\delta_{n_k})\|b_{n_k}-p\|^2+\theta_{n_k}(1-\delta_{n_k})\|b_{n_k}-p\|^2+\delta_{n_k}M_3\\ &\le \|b_{n_k}-p\|^2+\delta_{n_k}M_3\\ &\le \|w_{n_k}-p\|^2-\epsilon^2\|T^*(z_{n_k}-Tw_{n_k})\|^2+\delta_{n_k}M_3,\end{aligned}$$

which implies from (4.3.49) that

$$\begin{aligned}\limsup_{k\to\infty}\|T^*(z_{n_k}-Tw_{n_k})\|^2 &\le \frac{1}{\epsilon^2}\limsup_{k\to\infty}\left(\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\right)\\ &\quad+\frac{1}{\epsilon^2}\limsup_{k\to\infty}\left(\delta_{n_k}\left(3\frac{\alpha_{n_k}}{\delta_{n_k}}\|x_{n_k}-x_{n_k-1}\|M_2+M_3\right)\right)\\ &\le \frac{-1}{\epsilon^2}\liminf_{k\to\infty}\left(\|x_{n+1}-p\|^2-\|x_{n_k}-p\|^2\right)\le 0.\end{aligned}$$

Thus,

$$\lim_{k\to\infty}\|T^*(z_{n_k}-Tw_{n_k})\|=0. \tag{4.3.55}$$

Hence, by Step 3, we obtain

$$\|b_{n_k}-w_{n_k}\|=\eta_{n_k}\|T^*(z_{n_k}-Tw_{n_k})\|\to 0, \quad \text{as} \quad k\to\infty. \tag{4.3.56}$$

From Step 5 and (4.3.54), we get

$$\|x_{n_k+1}-b_{n_k}\|\le\theta_{n_k}\|b_{n_k}-t_{n_k}\|+\delta_{n_k}\|b_{n_k}\|\to 0, \quad \text{as} \quad k\to\infty. \tag{4.3.57}$$

From (4.3.56) and (4.3.57), we get

$$\lim_{k\to\infty}\|x_{n_k+1}-w_{n_k}\|=0. \tag{4.3.58}$$

From (4.3.43) and (4.3.58), we get

$$\lim_{k\to\infty}\|x_{n_k+1}-x_{n_k}\|=0. \tag{4.3.59}$$

By Lemma 4.3.4, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to z, such that

$$\limsup_{k\to\infty}\langle p,p-x_{n_k}\rangle=\lim_{j\to\infty}\langle p,p-x_{n_{k_j}}\rangle=\langle p,p-z\rangle.$$

Also, since we have established (4.3.54) and (4.3.56), then we can apply Lemma 4.3.5 to get that $z\in\Gamma$.
Thus, since $p=P_\Gamma 0$, we obtain from the previous inequality that

$$\limsup_{k\to\infty}\langle p,p-x_{n_k}\rangle=\langle p,p-z\rangle\le 0,$$

which implies by (4.3.59) that

$$\lim_{k\to\infty}\langle p,p-x_{n_k+1}\rangle\le 0. \tag{4.3.60}$$

Now, recall that $d_{n_k}=3\frac{\alpha_{n_k}}{\delta_{n_k}}\|x_{n_k}-x_{n_k-1}\|M_2+2\left(\theta_{n_k}\|b_{n_k}-t_{n_k}\|\|p-x_{n_k+1}\|+\langle p,p-x_{n_k+1}\rangle\right)$. Thus, by (4.3.18), (4.3.54) and (4.3.60), we obtain that $\limsup_{k\to\infty} d_{n_k}\le 0$. Hence, get that $\lim_{n\to\infty}\|x_n-p\|=0$. Therefore, $\{x_n\}$ converges strongly to $p=P_\Gamma 0$. □

Remark 4.3.7.

- Observe that by setting $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $F = 0$ and $T = I_{\mathcal{H}}$ (the identity operator on $\mathcal{H}$) in Theorem 4.3.6, we obtain as a corollary, an inertial projection and contraction method requiring only one projection onto the feasible set $\mathcal{C}$ per iteration, with fixed stepsize for solving the classical VIP (1.2.1) when A is pseudomonotone and Lipschitz continuous.

- The conclusions of Lemma 4.3.4, Lemma 4.3.5 and Theorem 4.3.6 still hold even if $\mu \in (0, \frac{1}{L_1})$ and $\lambda \in (0, \frac{1}{L_2})$ in Algorithm 4.3.2 are replaced with variable stepsizes μ_n and λ_n, respectively such that

$$0 < \inf_{n\geq 1} \mu_n \leq \sup_{n\geq 1} \mu_n < \frac{1}{L_1} \quad \text{and} \quad 0 < \inf_{n\geq 1} \lambda_n \leq \sup_{n\geq 1} \lambda_n < \frac{1}{L_2}.$$

We now turn to establish the strong convergence of Algorithm 4.3.3. We begin with the following useful results.

Remark 4.3.8.

(i) From (4.3.3) in Algorithm 4.3.3, it is clear that $\lambda_{n+1} \leq \lambda_n$, $\forall n \geq 1$. Also, since F is L_2-Lipschitz continuous, we get in the case of $FTw_n \neq Fy_n$ in Algorithm 4.3.3, that

$$\lambda_{n+1} = \min\left\{\frac{a_2||Tw_n - y_n||}{||FTw_n - Fy_n||},\ \lambda_n\right\} \geq \min\left\{\frac{a_2}{L_2},\ \lambda_n\right\},$$

which by induction, implies that $\{\lambda_n\}$ is bounded below by $\min\left\{\frac{a_2}{L_2},\ \lambda_1\right\}$. Since $\{\lambda_n\}$ is also monotone nonincreasing, we have that the limit exists, and $\lim_{n\to\infty} \lambda_n \geq \min\{\frac{a_2}{L_2}, \lambda_1\} > 0$.

(ii) Similar to (i), we have that the limit of the stepsize $\{\mu_n\}$ exists and $\lim_{n\to\infty} \mu_n \geq \min\{\frac{a_1}{L_1}, \mu_1\} > 0$.

Lemma 4.3.9. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.3.3 under Assumption 4.3.1. Then,*

$$||Tw_n - y_n|| \leq \gamma_2^{-1}\left(\frac{\lambda_{n+1} + \lambda_n a_2}{\lambda_{n+1} - \lambda_n a_2}\right)||Tw_n - z_n||,\ \forall n \geq 1 \tag{4.3.61}$$

and

$$||b_n - u_n|| \leq \gamma_1^{-1}\left(\frac{\mu_{n+1} + \mu_n a_1}{\mu_{n+1} - \mu_n a_1}\right)||b_n - t_n||,\ \forall n \geq 1. \tag{4.3.62}$$

Proof. From (4.3.3), we know that

$$||FTw_n - Fy_n|| \leq \frac{a_2}{\lambda_{n+1}}||Tw_n - y_n||,\ \forall n \geq 1, \tag{4.3.63}$$

holds for both $FTw_n = Fy_n$ and $FTw_n \neq Fy_n$. Thus, similar to (4.3.12), we obtain that

$$\begin{aligned}\langle Tw_n - y_n, r_n\rangle &\geq \|Tw_n - y_n\|^2 - \lambda_n \|Tw_n - y_n\| \|FTw_n - Fy_n\| \\ &\geq (1 - \lambda_n \frac{a_2}{\lambda_{n+1}}) \|Tw_n - y_n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\|Tw_n - y_n\|^2 &\leq \frac{1}{\left(1 - \lambda_n \frac{a_2}{\lambda_{n+1}}\right)} \langle Tw_n - y_n, r_n\rangle \\ &\leq \frac{1}{\left(1 - \lambda_n \frac{a_2}{\lambda_{n+1}}\right)} \beta_n \|r_n\| \left(\|Tw_n - y_n\| + \lambda_n \frac{a_2}{\lambda_{n+1}} \|Tw_n - y_n\|\right) \\ &= \left(\frac{1 + \frac{\lambda_n a_2}{\lambda_{n+1}}}{1 - \frac{\lambda_n a_2}{\lambda_{n+1}}}\right) \beta_n \|r_n\| \|Tw_n - y_n\| \\ &= \gamma_2^{-1} \left(\frac{\lambda_{n+1} + \lambda_n a_2}{\lambda_{n+1} - \lambda_n a_2}\right) \|Tw_n - z_n\| \|Tw_n - y_n\|.\end{aligned}$$

This further gives (4.3.61). In a similar manner, we get (4.3.62). ☐

In the light of Remark 4.3.8 and Lemma 4.3.9, we have the following result.

Theorem 4.3.10. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.3.3 under Assumption 4.3.1. Then, $\{x_n\}$ converges strongly to $p \in \Gamma$, where $\|p\| = \min\{\|z\| : z \in \Gamma\}$.*

Proof. Let $p \in \Gamma$. Then, by replacing μ and λ with μ_n and λ_n, respectively in Lemma 4.3.4, we can easily get that $\{x_n\}$ is bounded.
Also, from Remark 4.3.8 (i), we obtain that

$$\lim_{n\to\infty} \left(\frac{\lambda_{n+1} + \lambda_n a_2}{\lambda_{n+1} - \lambda_n a_2}\right) = \frac{1 + a_2}{1 - a_2}. \tag{4.3.64}$$

Similarly, from Remark 4.3.8 (ii), we obtain that

$$\lim_{n\to\infty} \left(\frac{\mu_{n+1} + \mu_n a_1}{\mu_{n+1} - \mu_n a_1}\right) = \frac{1 + a_1}{1 - a_1}. \tag{4.3.65}$$

Furthermore, we can follow the same argument as in the proof of Theorem 4.3.6 to obtain (4.3.54), (4.3.55) and (4.3.56). Also, using (4.3.55) and (4.3.56), we can get (4.3.33). Hence, from (4.3.33), (4.3.61) and (4.3.64), we obtain that

$$\lim_{k\to\infty} \|Tw_{n_k} - y_{n_k}\| = 0.$$

Again, from (4.3.54), (4.3.47) and (4.3.65), we obtain that

$$\lim_{k\to\infty} \|b_{n_k} - u_{n_k}\| = 0.$$

The rest of the proof follows the same arguments as in Lemma 4.3.5 and Theorem 4.3.6. ☐

Remark 4.3.11.

- Similar to Remark 4.3.7, we can obtain as corollary, an inertial projection and contraction method with adaptive stepsize, requiring only one projection onto the feasible set $\mathcal{C}$ per iteration, for solving the classical VIP (1.2.4) when A is pseudomonotone and Lipschitz continuous.

- As seen in our convergence analysis, we did not follow the usual "Two Cases Approach" (i.e., Case 1 and Case 2) used in many works to guarantee strong convergence of iterative methods. Thus, the techniques and ideas employed in our strong convergence analysis are new for solving the SVIP (1.2.4)-(1.2.5).

- If the operators A and F are monotone and Lipschitz continuous, then we do not need them to be sequentially weakly continuous. This is because the sequential weakly continuity assumption was only used after (4.3.36) to get the conclusion of Lemma 4.3.5. But, from (4.3.36), we obtain

$$\begin{aligned}
0 &\leq \langle FTw_{n_k}, x - Tw_{n_k}\rangle + \frac{1}{\lambda}\langle y_{n_k} - Tw_{n_k}, x - y_{n_k}\rangle + \langle FTw_{n_k}, Tw_{n_k} - y_{n_k}\rangle \\
&\leq \left(\langle FTw_{n_k} - Fx, x - Tw_{n_k}\rangle + \langle Fx, x - Tw_{n_k}\rangle\right) + \frac{1}{\lambda}\|y_{n_k} - Tw_{n_k}\|\|x - y_{n_k}\| \\
&+ \|FTw_{n_k}\|\|Tw_{n_k} - y_{n_k}\| \\
&\leq \langle Fx, x - Tw_{n_k}\rangle + \frac{1}{\lambda}\|y_{n_k} - Tw_{n_k}\|\|x - y_{n_k}\| + \|FTw_{n_k}\|\|Tw_{n_k} - y_{n_k}\|, \ \forall x \in \mathcal{Q},
\end{aligned}$$

  where the last inequality follows from the monotoncity of F. Thus, by passing limit as $k \to \infty$ and noting that $\{Tw_{n_k}\}$ converges weakly to z, we obtain from (4.3.35) that

$$\langle Fx, x - Tz\rangle \geq 0, \quad \forall x \in \mathcal{Q},$$

  which follows from Lemma 2.5.8 that $Tz \in VI(F, \mathcal{Q})$. Similarly, we also get that $z \in VI(A, \mathcal{C})$. Hence, the conclusion of Lemma 4.3.5 holds.

- In finite dimensional spaces, Theorem 4.3.6 and Theorem 4.3.10 are still true if the operators A and F are only required to be pseudomonotone and Lipschitz continuous. This is an improvement over the result of He *et al.* [121] since no product space formulation is required even with the relaxed pseudomonotonicity assumption.

- The conclusions of Theorem 4.3.6 and Theorem 4.3.10 are still valid if Step 1 of Algorithm 4.3.2 and Algorithm 4.3.3 is replaced with the following: $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n := \begin{cases} \min\left\{\alpha, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \alpha, & \text{otherwise,} \end{cases}$$

  with $\alpha \in [0, 1)$.

4.3.3 Numerical experiments

In this section, we discuss the numerical behavior of Algorithm 4.3.2 and Algorithm 4.3.3 using various test examples. In all of these examples, we give numerical comparison of our methods with the methods of He *et al.* [121, Algorithm 1] (see Appendix 4.2.17), Pham *et al.* [206, Algorithm 1] (see Appendix 4.3.16), Reich and Tuyen [213, Theorem 4.4] (see Appendix 4.2.18) and Tian and Jiang [244, Algorithm (3.1)] (see Algorithm (2.5.9)).

All codes are written in Matlab 2016 (b) and performed on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30GHz and 8.00 Gb-RAM. In Tables 4.3.1-4.3.3, "Iter." means the number of iterations while "CPU" means the CPU time in seconds. In our computations, we randomly choose $x_0, x_1 \in \mathcal{H}_1$ (see the cases below), $\eta \geq 0$, $\gamma_i \in (0,2)$, $a_i \in (0,1), i = 1,2$, $\lambda_1 > 0$ and $\mu_1 > 0$ (the choices of these parameters will be discussed in Remark 4.3.15). We choose $\delta_n = \frac{1}{n+1}$, $\theta_n = \frac{1}{2} - \delta_n$ and $\alpha_n = \bar{\alpha}_n$ with $\tau_n = \frac{\delta_n}{n^{0.01}}$ and different choices of $\alpha := 3, 6, 9, 12, 15$, which will be discussed in detail in our numerical analysis.

Furthermore, we define

$$\text{TOL}_n := \frac{1}{2}\left(||x_n - P_{\mathcal{C}}(x_n - \mu A x_n)||^2 + ||Tx_n - P_{\mathcal{Q}}(Tx_n - \lambda FTx_n)||^2\right)$$

for Algorithm 4.3.2, Algorithm 4.3.3, He *et al.* [121, Algorithm 1], Pham *et al.* [206, Algorithm 1] and Reich and Tuyen [213, Theorem 4.4]. While for Algorithm (2.5.9), we define

$$\text{TOL}_n := \frac{1}{2}\left(||x_n - P_{\mathcal{C}}(x_n - \mu A x_n)||^2 + ||Tx_n - STx_n||^2\right),$$

and use the stopping criterion $\text{TOL}_n < \varepsilon$ for the iterative processes, where ε is the predetermined error. Note that if $\text{TOL}_n = 0$, then $x_n \in \Gamma$, that is, x_n is a solution of the SVIP considered in this work.

We first consider an example in finite dimensional spaces. In this example, we carry out our comparsion with the method of He *et al.* [121] since their method is given in finite dimensional spaces.

Example 4.3.12. Following [121, Example 5.2] (see also [116]), we consider a separable, convex and quadratic programming problem

$$\min_{x,y} \{G_1(x) + G_2(y) \mid Tx = y,\ x \in \mathcal{C},\ y \in \mathcal{Q}\}, \tag{4.3.66}$$

where

$$G_1(x) = \frac{1}{2}x' M_1 x + q_1' x \quad \text{and} \quad G_2(y) = \frac{1}{2}y' M_2 y + q_2' y \quad (x' \text{ means the transpose of } x).$$

Problem (4.3.66) can also be rewritten as the SVIP (1.2.4)-(1.2.5), where

$$A(x) = M_1 x + q_1 \quad \text{and} \quad F(y) = M_2 y + q_2.$$

The matrices M_i $(i = 1, 2)$ are formed as: $M_i = V_i \sum_i V_i'$, where $V_i = I - \frac{2v_i v_i'}{\|v_i\|^2}$ and $\sum_i = \text{diag}(\sigma_{i1}, \sigma_{i2}, \cdots, \sigma_{iN_i})$ are the Householder and the diagonal matrix, respectively, with $N_1 = N$ and $N_2 = m$ being the dimensional of x and y, respectively. Furthermore,

$$\sigma_{i,j} = \cos\frac{j\pi}{N_i+1} + 1 + \frac{\cos\frac{\pi}{N_i+1} + 1 - \widehat{C}_i(\cos\frac{N_i\pi}{N_i+1} + 1)}{\widehat{C}_i - 1}, \qquad j = 1, 2, \cdots, N_i,$$

where $\widehat{C}_i$ is the present condition number of M_i.

As in He *et al.* [121, Example 5.2], we choose $\widehat{C}_i = 10^4$, $q_i = 0$, $i = 1, 2$ and uniformly take the vector $v_i \in \mathbb{R}^{N_i}$ $(i = 1, 2)$ in $(-1, 1)$. Thus, A, F are monotone and Lipschitz continuous operators with $L_i = ||M_i||$, $i = 1, 2$ (which can be computed in Matlab). Furthermore, we generate the bounded linear operator $T \in \mathbb{R}^{M\times N}$ with independent Gaussian components distributed in the interval $(0, 1)$, and then normalize each column of T with the unit norm. We set $\mathcal{C} = \{x \in \mathbb{R}^N : ||x|| \leq 1\}$ and $\mathcal{Q} = \{y \in \mathbb{R}^m : l \leq y \leq u\}$, where all the entries of $l \in \mathbb{R}^m$ and $u \in \mathbb{R}^m$ are the smallest and the largest components of $\tilde{y} = T\tilde{x}$, respectively, with $\tilde{x}$ been a sparse vector whose components are uniformly distributed in $(0, 1)$. The projections onto $\mathcal{C}$ and $\mathcal{Q}$ are effectively computed in Matlab. More so, we consider different scenarios of the problem's dimensions. That is, $N = 100, 300, 500, 1000$ and $m = N/2$.

Since the Lipschitz constants L_i $(i = 1, 2)$ can be known, we choose in Algorithm 4.3.2, $\mu = \frac{0.5}{L_1}$ and $\lambda = \frac{0.75}{L_2}$, and take the starting point $x_1 = (1, 1, \ldots, 1)'$ while the entries of x_0 are randomly generated in $[0, 1]$. For Algorithm 1 of He *et al.* [121], we take $\mu_1 = 5\left(||T'\mathbb{H}T|| + L_1\right)/v$, $\mu_2 = 10\left(||T'\mathbb{H}T|| + L_2\right)/v$, $v = 0.8$, $\mathbb{H} = \frac{2}{||T'T||}I_N$, $\gamma = 1.2$ and $\rho = -1.5$ (which is the optimum choice in their implementation), with starting points $x_1 = (1, 1, \ldots, 1)'$, $y_1 = (0, 0, \ldots, 0)'$ and $\lambda_1 = (0, 0, \ldots, 0)'$. Furthermore, we take $\varepsilon = 10^{-8}$ for the stopping criterion and obtain the numerical results reported in Table 4.3.1 and Figure 4.4. We stress that these choices as well as the stopping criterion are the same as in He *et al.* [121, Example 5.2].

Table 4.3.1. Numerical results for Example 4.3.12 with $\varepsilon = 10^{-8}$.

(N, m)		Alg.4.3.2 $\alpha = 3$	Alg.4.3.2 $\alpha = 6$	Alg.4.3.2 $\alpha = 9$	Alg.4.3.2 $\alpha = 12$	Alg.4.3.2 $\alpha = 15$	He *et al.*
$(100, 50)$	CPU	0.0305	0.0284	0.0195	0.0190	0.0180	0.2552
	Iter.	18	18	18	18	18	178
$(300, 150)$	CPU	0.0048	0.0033	0.0031	0.0030	0.0030	0.0388
	Iter.	19	19	19	19	19	216
$(500, 250)$	CPU	0.0055	0.0045	0.0042	0.0041	0.0041	0.0619
	Iter.	20	20	20	20	20	232
$(1000, 500)$	CPU	0.0081	0.0081	0.0080	0.0079	0.0074	0.1591
	Iter.	21	21	21	21	21	275

The next two examples are given in infinite dimensional Hilbert spaces. In these examples, we carry out our comparisons with the methods of Pham *et al.* [206], Reich and Tuyen [213] and Tian and Jiang [244] since their methods are given in infinite dimensional Hilbert spaces.

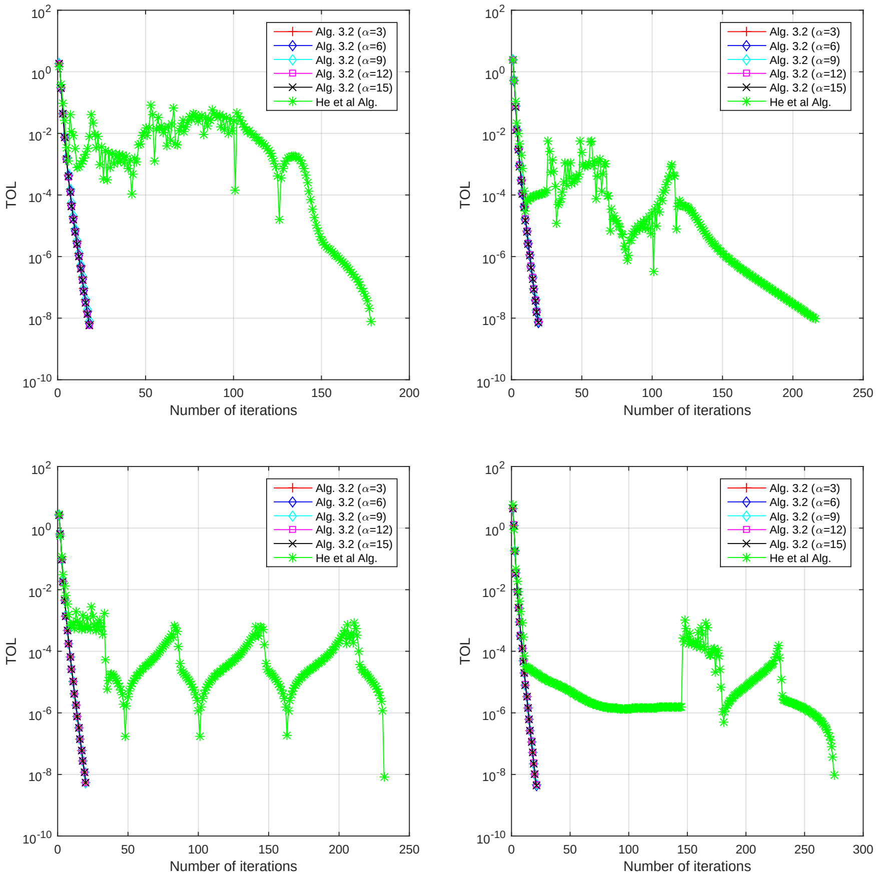

Figure 4.4: The behavior of TOL_n with $\varepsilon = 10^{-8}$ for Example 4.3.12: Top Left: $(N, m) = (100, 50)$; Top Right: $(N, m) = (300, 150)$; Bottom Left: $(N, m) = (500, 250)$; Bottom Right: $(N, m) = (1000, 500)$.

Example 4.3.13. Let $\mathcal{H}_1 = (l_2(\mathbb{R}), \ ||.||_{l_2}) = \mathcal{H}_2$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \dots), \ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $||x||_{l_2} := \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}, \quad \forall x \in l_2(\mathbb{R})$. Now, define the operator $T : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by

$$Tx = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right), \ \forall x \in l_2(\mathbb{R}).$$

Then, T is a bounded linear operator on $l_2(\mathbb{R})$ with adjoint

$$T^* y = \left(y_2, \frac{y_3}{2}, \frac{y_4}{3}, \dots\right), \ \forall y \in l_2(\mathbb{R}).$$

Indeed, let $x = (x_1, x_2, x_3, \dots)$, $y = (y_1, y_2, y_3, \dots)$ be arbitrary in $l_2(\mathbb{R})$ and α_1, α_2 be arbitrary in $\mathbb{R}$. Then,

$$\begin{aligned} T(\alpha_1 x + \alpha_2 y) &= \left(0, \ \alpha_1 x_1 + \alpha_2 y_1, \ \frac{\alpha_1 x_2 + \alpha_2 y_2}{2}, \ \frac{\alpha_1 x_3 + \alpha_2 y_3}{3}, \cdots\right) \\ &= \left(0, \ \alpha_1 x_1, \frac{\alpha_1 x_2}{2}, \ \frac{\alpha_1 x_3}{3}, \cdots\right) + \left(0, \ \alpha_2 y_1, \frac{\alpha_2 y_2}{2}, \ \frac{\alpha_2 y_3}{3}, \cdots\right) \\ &= \alpha_1 T(x) + \alpha_2 T(y). \end{aligned}$$

Therefore, T is linear. Furthermore, $||Tx||_{l_2} \leq ||x||_{l_2}, \ \forall x \in l_2(\mathbb{R})$. Thus, T is also bounded. The verification that T^* is the adjoint of T follows directly from definition.

Let $\mathcal{C} = \mathcal{Q} = \{x \in l_2(\mathbb{R}) : ||x - a||_{l_2} \leq r\}$, where $a = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$, $r = 3$ for $\mathcal{C}$ and $a = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$, $r = 1$ for $\mathcal{Q}$. Then $\mathcal{C}, \mathcal{Q}$ are nonempty closed and convex subsets of $l_2(\mathbb{R})$. Thus,

$$P_{\mathcal{C}}(x) = P_{\mathcal{Q}}(x) = \begin{cases} x, & \text{if } \ x \in ||x - a||_{l_2} \leq r, \\ \frac{x-a}{||x-a||_{l_2}} r + a, & \text{otherwise.} \end{cases}$$

Now, define the operators $A, F : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by

$$A(x_1, x_2, x_3, \dots) = (x_1 e^{-x_1^2}, 0, 0, \dots), \quad F(x_1, x_2, x_3, \dots) = (5x_1 e^{-x_1^2}, 0, 0, \dots).$$

Then, by Example 2.1.5, A, F are pseudomonotone, Lipschitz continuous and sequentially weakly continuous but not monotone.

More so, for Algorithm (2.5.9), we define the mappings $S, h : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by $Sx = (0, x_1, x_2, \dots)$ and $hx = \left(0, \ \frac{x_1}{2}, \ \frac{x_2}{2}, \cdots\right)$, for all $x \in l_2(\mathbb{R})$. Then, we consider the following cases for the numerical experiments.

Case 1: Take $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ and $x_0 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$.

Case 2: Take $x_1 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$ and $x_0 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.

Case 3: Take $x_1 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$ and $x_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$.

Case 4: Take $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$ and $x_0 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$.

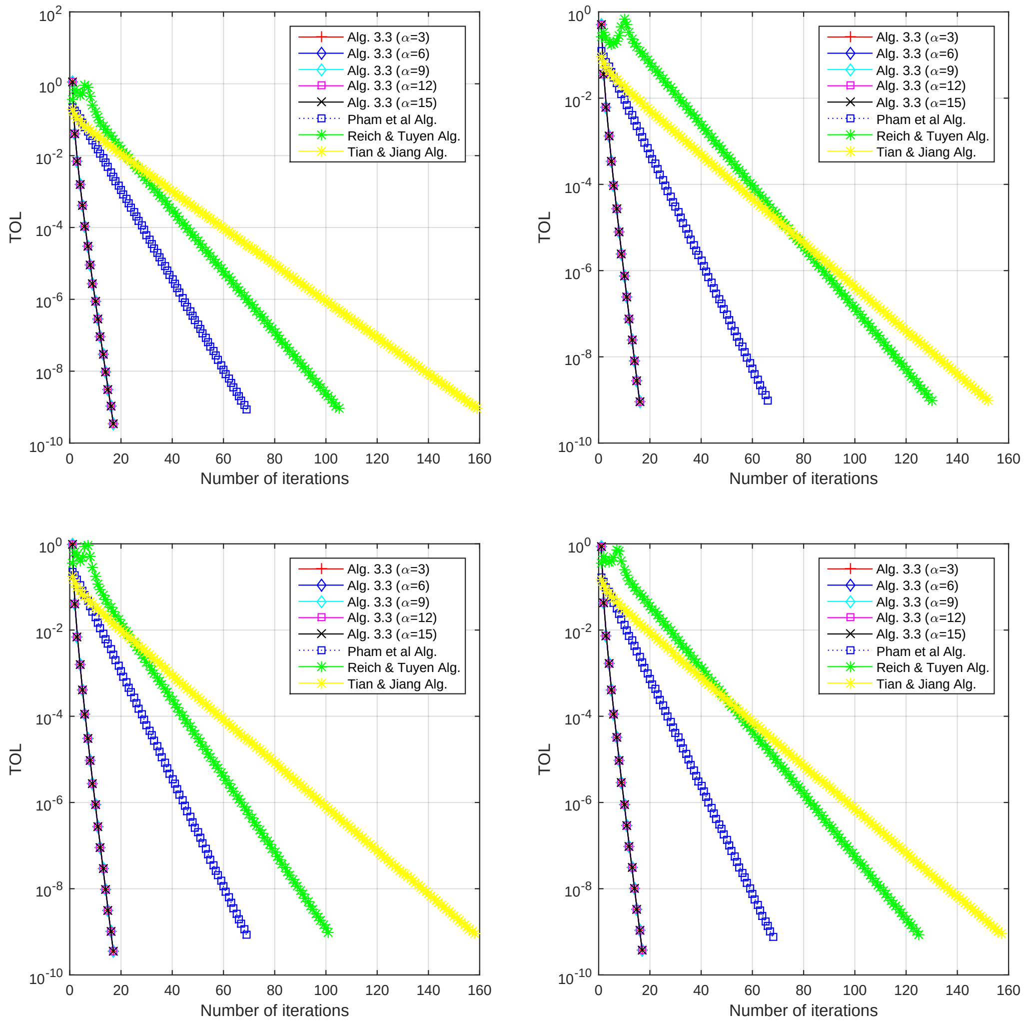

Figure 4.5: The behavior of TOL_n with $\varepsilon = 10^{-9}$ for Example 4.3.13: Top Left: **Case 1**; Top Right: **Case 2**; Bottom Left: **Case 3**; Bottom Right: **Case 4**.

Table 4.3.2. Numerical results for Example 4.3.13 with $\varepsilon = 10^{-9}$.

Cases		Alg.4.3.3 $\alpha = 3$	Alg.4.3.3 $\alpha = 6$	Alg.4.3.3 $\alpha = 9$	Alg.4.3.3 $\alpha = 12$	Alg.4.3.3 $\alpha = 15$	Pham *et al.*	Reich Tuyen	Tian Jiang
1	CPU	0.0040	0.0031	0.0025	0.0027	0.0027	0.0056	0.0113	0.1071
	Iter.	17	17	17	17	17	69	105	159
2	CPU	0.0039	0.0038	0.0033	0.0048	0.0051	0.0120	0.0671	0.1000
	Iter.	16	16	16	16	16	66	130	152
3	CPU	0.0029	0.0034	0.0029	0.0045	0.0051	0.0070	0.0133	0.1059
	Iter.	17	17	17	17	17	69	101	158
4	CPU	0.0030	0.0032	0.0027	0.0034	0.0037	0.0076	0.0142	0.1060
	Iter.	17	17	17	17	17	68	125	157

Example 4.3.14. Let $\mathcal{H}_1 = \mathcal{H}_2 = L_2([0,1])$ be endowed with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt, \quad \forall\ x, y \in L_2([0,1])$$

and norm

$$||x|| := \Big(\int_0^1 |x(t)|^2 dt\Big)^{\frac{1}{2}}, \quad \forall\ x, y \in L_2([0,1]).$$

Let $T : L_2([0,1]) \to L_2([0,1])$ be defined by

$$Tx(s) = \int_0^1 K(s,t)x(t)dt, \ \forall x \in L_2([0,1]),$$

where K is a continuous real-valued function on $[0,1]\times[0,1]$. Then, T is a bounded linear operator with adjoint

$$T^*x(s) = \int_0^1 K(t,s)x(t)dt, \ \forall x \in L_2([0,1]).$$

In particular, we define $K(s,t) = e^{-st}$ for all $s, t \in [0,1]$.

Let $\mathcal{C} = \{x \in L_2([0,1]) : \langle y, x\rangle \leq b\}$, where $y = t+1$ and $b = 1$, then $\mathcal{C}$ is a nonempty closed and convex subset of $L_2([0,1])$. Thus, we define the metric projection $P_{\mathcal{C}}$ as:

$$P_{\mathcal{C}}(x) = \begin{cases} \frac{b-\langle y,x\rangle}{||y||^2}y + x, & \text{if } \ \langle y, x\rangle > b, \\ x, & \text{if } \ \langle y, x\rangle \leq b. \end{cases}$$

Also, let $\mathcal{Q} = \{x \in L_2([0,1]) \ : \ ||x|| \leq r\}$, where $r = 2$, then $\mathcal{Q}$ is a nonempty closed and convex subset of $L_2([0,1])$. Thus, we define $P_{\mathcal{Q}}$ as:

$$P_{\mathcal{Q}}(x) = \begin{cases} x, & \text{if } \ x \in \mathcal{Q}, \\ \frac{x}{||x||^2}r, & \text{otherwise.} \end{cases}$$

Now, define $A : L_2([0,1]) \to L_2([0,1])$ by

$$A(x)(t) = e^{-||x||}\int_0^t x(s)ds, \quad \forall x \in L_2([0,1]), \quad t \in [0,1].$$

Then, A is pseudomonotone and Lipschitz continuous but not monotone on $L_2([0,1])$ (see [250]).

Also, define $F : \mathcal{Q} \to L_2([0,1])$ by

$$F(x)(t) := g(x)M(x)(t), \quad \forall\, x \in \mathcal{Q}, \quad t \in [0,1],$$

where $g : \mathcal{Q} \to \mathbb{R}$ is defined by $g(x) := \frac{1}{1+||x||^2}$ and $M : L_2([0,1]) \to L_2([0,1])$ is defined by $M(x)(t) := \int_0^t x(s)ds, \quad \forall x \in L_2([0,1]), \quad t \in [0,1]$. As given in [222], g is $\frac{16}{25}$-Lipschitz continuous and $\frac{1}{5} \leq g(x) \leq 1, \quad \forall\, x \in \mathcal{C}$. Also, M is the Volterra intergral mapping which is bounded and linear monotone. Hence, F is pseudomonotone and Lipschitz continuous but not monotone (see [222]).

For Algorithm (2.5.9), we define the mapping $S : L_2([0,1]) \to L_2([0,1])$ by

$$Sx(t) = \int_0^1 tx(s)ds, \quad t \in [0,1].$$

Then, S is nonexpansive. Indeed, we have

$$\begin{aligned} |Sx(t) - Sy(t)|^2 = |\int_0^1 t(x(s) - y(s))ds|^2 &\leq \left(\int_0^1 t|x(s) - y(s)|ds\right)^2 \\ &\leq \int_0^1 |x(s) - y(s)|^2 ds \\ &= ||x - y||^2. \end{aligned}$$

Thus, we obtain that

$$||Sx - Sy||^2 = \int_0^1 |Sx(t) - Sy(t)|^2 \leq ||x - y||^2.$$

We also define $h : L_2([0,1]) \to L_2([0,1])$ by

$$hx(t) = \int_0^1 \frac{t}{2} x(s)ds, \quad x \in [0,1].$$

Similar to above, it can also be shown that h is a contraction mapping.

We consider the following cases for the numerical experiments of this example.
Case 1: Take $x_1(t) = 1 + t^2$ and $x_0(t) = t + 5$.

Case 2: Take $x_1(t) = \sin(t)$ and $x_0(t) = t + 1$.

Case 3: Take $x_1(t) = t + 1$ and $x_0(t) = t + t^3$.

Case 4: Take $x_1(t) = 0.7e^{-t} + 1$ and $x_0(t) = t + t^3$.

Table 4.3.3. Numerical results for Example 4.3.14 with $\varepsilon = 10^{-10}$.

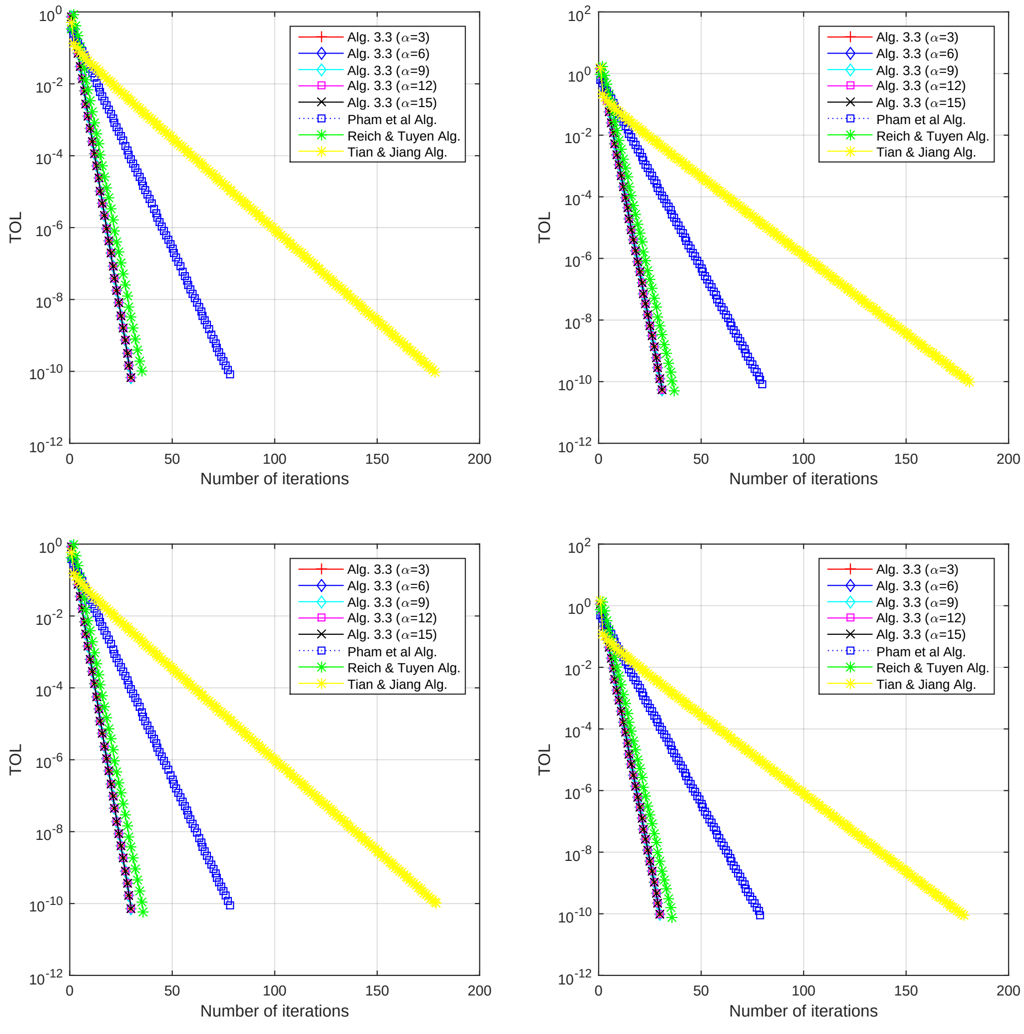

Figure 4.6: The behavior of TOL_n with $\varepsilon = 10^{-10}$ for Example 4.3.14: Top Left: **Case 1**; Top Right: **Case 2**; Bottom Left: **Case 3**; Bottom Right: **Case 4**.

Cases		Alg.4.3.3 $\alpha = 3$	Alg.4.3.3 $\alpha = 6$	Alg.4.3.3 $\alpha = 9$	Alg.4.3.3 $\alpha = 12$	Alg.4.3.3 $\alpha = 15$	Pham *et al.*	Reich Tuyen	Tian Jiang
1	CPU	3.6417	3.4282	3.4009	3.3256	3.3112	12.6238	9.7880	25.7461
	Iter.	30	30	30	30	30	77	35	178
2	CPU	3.9291	3.7850	3.6274	3.4970	3.4821	12.8552	10.3671	27.8420
	Iter.	31	31	31	31	31	80	37	181
3	CPU	3.6099	3.3950	3.3538	3.3495	3.3447	12.6815	10.1993	26.2060
	Iter.	30	30	30	30	30	78	36	179
4	CPU	3.6508	3.4405	3.3442	3.3112	3.2636	12.9326	10.0650	26.3111
	Iter.	30	30	30	30	30	79	36	178

Remark 4.3.15. By using different starting points and varying the inertial extrapolation factor α in each example (Examples 4.3.12-4.3.14), we obtain the numerical results displayed in Tables 4.3.1-4.3.3 and Figures 4.4-4.6. We compared our proposed Algorithm 4.3.2 with the method of He *et al.* [121, Algorithm 1] in Example 4.3.12 while our proposed Algorithm 4.3.3 is compared with the methods of Pham *et al.* [206, Algorithm 1], Reich and Tuyen [213, Theorem 4.4] and Tian and Jiang [244, Algorithm (3.1)] in Examples 4.3.13-4.3.14.
Furthermore, we note the following from our numerical experiments:

- In the numerical experiments, we randomly choose the parameters $\eta \geq 0$, $\gamma_i \in (0, 2)$, $a_i \in (0, 1), i = 1, 2$, $\lambda_1 > 0$ and $\mu_1 > 0$, and observed that irrespective of the choices made, the number of iteration does not change and no significant difference in the CPU time.

- In all the examples, we check the sensitivity of α for each starting points in order to know if the choices of α affect the efficiency of our methods. We can see from the tables and graphs that the number of iterations for our proposed methods remain consistent (well-behaved) for $\alpha = 3, 6, 9, 12, 15$. Also, there are no much significant difference in the CPU time as we vary α in $\{3, 6, 9, 12, 15\}$. However, it can be inferred from Table 4.3.1 that as α increases from 3 to 15, the convergence speed of Algorithm 4.3.2 increases. Therefore, the optimum choice of α for Example 4.3.12 is $\alpha = 15$. Similarly, we can see from Table 4.3.3 that as α increases from 3 to 15, the convergence speed of Algorithm 4.3.3 also increases. Hence, the optimum choice of α for Example 4.3.14 is again $\alpha = 15$. Unfortunately, we cannot follow similar pattern for Example 4.3.13, but we can see clearly that the the optimum choice of α for this example is $\alpha = 9$.

- It can also be inferred from Table 4.3.1 that in Example 4.3.12, the number of iterations for our proposed Algorithm 4.3.2 is almost the same for all starting points while the method of He *et al.* [121, Algorithm 1] depends on the different starting points. Also, from Tables 4.3.2-4.3.3, we see that for all starting points in Example 4.3.13-4.3.14, the number of iterations for our proposed Algorithm 4.3.3 is more consistent than every other methods.

- From the tables and figures, we can see clearly that in terms of both CPU time and number of iterations, our proposed Algorithm 4.3.2 outperforms the method of He *et al.* [121, Algorithm 1] in Example 4.3.12 while our proposed Algorithm 4.3.3 outperforms the methods of Pham *et al.* [206, Algorithm 1], Reich and Tuyen [213, Theorem 4.4] and Tian and Jiang [244, Algorithm (3.1)] in Examples 4.3.13-4.3.14. Hence, our methods are more efficient than these other methods.

Appendix 4.3.16. *Algorithm 1 of Pham et al. [206].*

Step 0. *Choose* $\mu_0, \lambda_0 > 0, \quad \mu, \lambda \in (0,1), \quad \{\tau_n\} \subset [\underline{\tau}, \; \bar{\tau}] \subset \left(0, \frac{1}{||T||^2+1}\right), \quad \{\alpha_n\} \subset (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.
Step 1. *Let* $x_1 \in \mathcal{H}_1$. *Set* $n = 1$.
Step 2. *Compute*

$$\begin{aligned} u_n &= Tx_n, \\ v_n &= P_{\mathcal{Q}}(u_n - \mu_n f u_n), \\ w_n &= P_{\mathcal{Q}_n}(u_n - \mu_n f v_n), \end{aligned}$$

where

$$\mathcal{Q}_n = \{w_2 \in \mathcal{H}_2 : \langle u_n - \mu_n f u_n - v_n, w_2 - v_n \rangle \leq 0\}$$

and

$$\mu_{n+1} = \begin{cases} \min\left\{\frac{\mu||u_n - v_n||}{||fu_n - fv_n||}, \; \mu_n\right\}, & \text{if } fu_n \neq fv_n, \\ \mu_n, & \text{otherwise.} \end{cases}$$

Step 3. *Compute*

$$\begin{aligned} y_n &= x_n + \tau_n T^*(w_n - u_n), \\ z_n &= P_{\mathcal{C}}(y_n - \lambda_n A y_n), \\ t_n &= P_{\mathcal{C}_n}(y_n - \lambda_n A z_n), \end{aligned}$$

where

$$\mathcal{C}_n = \{w_1 \in \mathcal{H}_1 : \langle y_n - \lambda_n A y_n - z_n, w_1 - z_n \rangle \leq 0\}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\lambda||y_n - z_n||}{||Ay_n - Az_n||}, \; \lambda_n\right\}, & \text{if } Ay_n \neq Az_n, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Step 4. *Compute*

$$x_{n+1} = \alpha_n x_1 + (1 - \alpha_n) t_n.$$

Set $n := n + 1$ *and go back to* **Step 2.**

4.4 Projection and contraction methods for split variational inequality problem

In the literature, several methods have been proposed for solving the SVIP and most of these methods require that the underlying operators be co-coercive while some of them requires product space formulation of the problem. These restrictive conditions affect the feasibility of these existing methods. In order to overcome these setbacks, we propose two new inertial projection and contraction methods for solving the SVIP in real Hilbert spaces without the co-coercive condition and without the product space formulation, which does not fully exploit the attractive splitting structure of the SVIP. The sequences generated by these methods converge strongly to the solution of the SVIP in real Hilbert spaces under the assumptions that the operators are pseudomonotone, Lipschitz continuous and without the sequentially weakly continuity condition. Furthermore, we present several numerical experiments for the proposed methods and compare their performance with other related methods in the literature.

4.4.1 Proposed methods

In this section, we present our proposed methods for solving the SVIP (1.2.4)-(1.2.5).

Assumption 4.4.1. *Suppose that the following conditions hold:*

(a) The feasible sets $\mathcal{C}$ and $\mathcal{Q}$ are nonempty closed and convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

(b) $A : \mathcal{H}_1 \to \mathcal{H}_1$ and $F : \mathcal{H}_2 \to \mathcal{H}_2$ are pseudomonotone and Lipschitz continuous with Lipschitz constants L_1 and L_2, respectively.

(c) $A : \mathcal{H}_1 \to \mathcal{H}_1$ and $F : \mathcal{H}_2 \to \mathcal{H}_2$ satisfy the following property whenever $\{x_n\} \subset \mathcal{C}$ and $\{y_n\} \subset \mathcal{Q}$, and $x_n \rightharpoonup x$, $y_n \rightharpoonup y$ one has $\|Ax\| \leq \liminf_{n\to\infty} \|Ax_n\|$ and $\|Fy\| \leq \liminf_{n\to\infty} \|Fy_n\|$.

(d) $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator and the solution set $\Gamma := \{z \in VI(A, \mathcal{C}) : Tz \in VI(F, \mathcal{Q})\}$ is nonempty, where $VI(A, \mathcal{C})$ is the solution set of the classical VIP (1.2.4).

(e) $\{\alpha_n\} \subset (0, 1]$ is non-increasing with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(f) $0 \leq \theta_n \leq \theta_{n+1} \leq \theta < \frac{1}{3}, \sigma \in (0, \frac{1}{2}]$.

(g) $\{\phi_n\}$ and $\{\psi_n\}$ are non-negative sequences such that $\sum_{n=1}^{\infty} \phi_n < +\infty$ and $\sum_{n=1}^{\infty} \psi_n < +\infty$.

When the Lipschitz constants L_1 and L_2 are known, we present the following method for solving the SVIP (1.2.4)-(1.2.5).

Algorithm 4.4.2. Inertial projection and contraction method with fixed step size.

Step 0: Choose sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ such that the conditions from Assumption 4.4.1 (e)-(f) hold and let $\eta \geq 0, \gamma_i \in (0,2), i = 1,2,\ \mu \in (0, \frac{1}{L_1}),\ \lambda \in (0, \frac{1}{L_2}),$ and $x_0, x_1 \in \mathcal{H}_1$ be given arbitrarily. Set $n := 1$.

Step 1: Given the iterates x_{n-1} and x_n $(n \geq 1)$, $\alpha_n \in (0,1)$ and $\theta_n \in [0, \frac{1}{3})$, compute

$$w_n = \alpha_n x_0 + (1-\alpha_n)x_n + \theta_n(x_n - x_{n-1}).$$

Step 2: Compute

$$y_n = P_{\mathcal{Q}}(Tw_n - \lambda FTw_n),$$

$$z_n = Tw_n - \gamma_2\beta_n r_n,$$

where $r_n := Tw_n - y_n - \lambda(FTw_n - Fy_n)$ and $\beta_n := \frac{\langle Tw_n - y_n, r_n\rangle}{\|r_n\|^2}$, if $r_n \neq 0$, otherwise $\beta_n = 0$.

Step 3: Compute

$$b_n = w_n + \eta_n T^*(z_n - Tw_n),$$

where the step size η_n is chosen such that for some $\epsilon > 0$, $\quad \eta_n \in \left(\epsilon,\ \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right),$ if $z_n \neq Tw_n$; otherwise $\eta_n = \eta$.

Step 4: Compute

$$u_n = P_{\mathcal{C}}(b_n - \mu A b_n),$$

$$t_n = b_n - \gamma_1\gamma_n v_n,$$

where $v_n := b_n - u_n - \mu(Ab_n - Au_n)$ and $\gamma_n := \frac{\langle b_n - u_n, v_n\rangle}{\|v_n\|^2}$, if $v_n \neq 0$, otherwise $\gamma_n = 0$.

Step 5: Compute

$$x_{n+1} = (1-\sigma)w_n + \sigma t_n.$$

Set $n := n+1$ and go back to **Step 1**.

When the Lipschitz constants L_1 and L_2 are not known, we present the following method with adaptive step size for solving the SVIP (1.2.4)-(1.2.5).

Algorithm 4.4.3. Inertial projection and contraction method with adaptive step size strategy.

Step 0: Choose the control parameters such that conditions (e)-(g) of Assumption 4.4.1 hold and let $\eta \geq 0, \gamma_i \in (0,2)\ a_i \in (0,1), i = 1,2,\ \lambda_1 > 0,\ \mu_1 > 0,\ \alpha \geq 3$ and $x_0, x_1 \in \mathcal{H}_1$ be given arbitrarily. Set $n := 1$.

Step 1: Given the iterates x_{n-1} and x_n $(n \geq 1)$, $\alpha_n \in (0,1)$ and $\theta_n \in [0, \frac{1}{3})$, compute

$$w_n = \alpha_n x_0 + (1-\alpha_n)x_n + \theta_n(x_n - x_{n-1}).$$

Step 2: Compute

$$y_n = P_{\mathcal{Q}}(Tw_n - \lambda_n FTw_n),$$

$$z_n = Tw_n - \gamma_2\beta_n r_n,$$

where $r_n := Tw_n - y_n - \lambda_n(FTw_n - Fy_n)$, $\beta_n := \frac{\langle Tw_n - y_n, r_n \rangle}{\|r_n\|^2}$, if $r_n \neq 0$, otherwise $\beta_n = 0$; and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{a_2||Tw_n - y_n||}{||FTw_n - Fy_n||}, \ \lambda_n + \phi_n\right\}, & \text{if } FTw_n \neq Fy_n \\ \lambda_n + \phi_n, & \text{otherwise.} \end{cases} \tag{4.4.1}$$

Step 3: Compute

$$b_n = w_n + \eta_n T^*(z_n - Tw_n),$$

where the step size η_n is chosen such that for some $\epsilon > 0$, $\quad \eta_n \in \left(\epsilon, \ \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right)$, if $z_n \neq Tw_n$; otherwise $\eta_n = \eta$.

Step 4: Compute

$$u_n = P_{\mathcal{C}}(b_n - \mu_n Ab_n),$$

$$t_n = b_n - \gamma_1 \gamma_n v_n,$$

where $v_n := b_n - u_n - \mu_n(Ab_n - Au_n)$, $\gamma_n = \frac{\langle b_n - u_n, v_n \rangle}{\|v_n\|^2}$, if $v_n \neq 0$, otherwise $\gamma_n = 0$; and

$$\mu_{n+1} = \begin{cases} \min\left\{\frac{a_1||b_n - u_n||}{||Au_n - Ab_n||}, \ \mu_n + \psi_n\right\}, & \text{if } Ab_n \neq Au_n \\ \mu_n + \psi_n, & \text{otherwise.} \end{cases} \tag{4.4.2}$$

Step 5: Compute

$$x_{n+1} = (1 - \sigma)w_n + \sigma t_n.$$

Set $n := n + 1$ and go back to **Step 1**.

We outline and discuss some of the properties of our proposed methods.

Remark 4.4.1.

- The choice of the inertial factor $\theta_n \in [0, \frac{1}{3})$ in Algorithms 4.4.2 and 4.4.3 is new and different from the choices in literature (see for example [55] and other references therein). As far as we know, this is the first time the inertial factor θ_n is chosen such that $\theta_n \in [0, \frac{1}{3})$ and solves the SVIP when the underlying operators are pseudomonotone and Lipschitz continuous.

- Algorithm 4.4.3 uses simple step size rules in (4.4.1) and (4.4.2), which generate non-monotonic sequences of step sizes. The step sizes are constructed such that the dependence of the algorithm on the initial step sizes λ_1 and μ_1 is reduced.

- We point out that if the pseudomonotone operators A and F are sequentially weakly continuous, then A and F satisfy condition (c) but the converse is not true. Hence, condition (c) is strictly weaker than the sequentially weakly continuity condition commonly employed in the literature (e.g., see [55, 191]).

- Algorithm 4.4.2 can be viewed as a modified inertial projection and contraction method involving one projection onto $\mathcal{C}$ per iteration for solving the classical VIP in $\mathcal{H}_1$. Algorithm 4.4.3 can be viewed as a modified inertial projection and contraction method involving one projection onto $\mathcal{Q}$ per iteration under a bounded linear operator T for solving VIP in $\mathcal{H}_2$. Our methods improves other methods in literature which requires extra projections onto half-spaces or feasible sets (see [206] (see Appendix 4.3.16) and other references therein). In Step 2 of Algorithms 4.4.2 and 4.4.3, r_n can be described as weighted average of $(Tw_n - y_n \sim \lambda FTw_n)$ and a hypothetical $(T\tilde{w}_n - \tilde{y}_n \sim \lambda FT\tilde{w}_n)$ in $\mathcal{H}_2$, where $T\tilde{w}_n = Tw_n - \lambda FTw_n$ and $\tilde{y}_n = y_n - \lambda Fy_n$. In Step 4 of Algorithms 4.4.3 and 4.4.3, v_n follows similar description. From Step 2 and Step 4 of Algorithms 4.4.3 and 4.4.3, we have

$$\beta_n||r_n||^2 = \langle Tw_n - y_n, r_n\rangle, \ \forall n \geq 1 \tag{4.4.3}$$

holds for both $r_n = 0$ and $r_n \neq 0$. Similarly, we have that

$$\gamma_n||v_n||^2 = \langle b_n - u_n, v_n\rangle, \ \forall n \geq 1 \tag{4.4.4}$$

holds for both $v_n = 0$ and $v_n \neq 0$.

- The step sizes $\{\lambda_n\}$ and $\{\mu_n\}$ given by (4.4.1) and (4.4.2), respectively are generated at each iteration by some simple computations which makes Algorithm 4.4.3 easier to implement since it does not require the prior knowledge of the Lipschitz constants L_1 and L_2.

- Algorithms 4.4.2 and 4.4.3 does not require any product space formulation unlike other algorithms in literature which require that the problem be transformed into a product space (see [63] and other references therein). This makes our algorithms easier to implement since they do not encounter the difficulties that might be caused by the product space.

Remark 4.4.2. [192] The choice of the step size η_n in Step 3 of Algorithms 4.4.2 and 4.4.3 do not require the prior knowledge of the operator norm $\|T\|$. Furthermore, the value of η does not influence the algorithms, but it was introduced for the sake of clarity.

Lemma 4.4.3. *[192] The step size η_n given in Step 3 of Algorithms 4.4.2 and 4.4.3 is well-defined.*

4.4.2 Convergence analysis

Lemma 4.4.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.4.2 under Assumption 4.4.1. Then, the following inequality holds:*

$$\|x_{n+1} - p\|^2 \leq \|w_n - p\|^2 - \|x_{n+1} - w_n\|^2, \ \forall \ p \in \Gamma. \tag{4.4.5}$$

Proof. Let $p \in \Gamma$. From the definition of y_n and the characteristic property of P_Q, we obtain

$$\langle y_n - Tw_n + \lambda FTw_n, \quad y_n - Tp\rangle \leq 0. \tag{4.4.6}$$

Since $Tp \in VI(F, Q)$ and $y_n \in Q$, we have

$$\langle FTp, y_n - Tp\rangle \geq 0$$

and from the pseudomonotonicity of F we have,

$$\langle Fy_n, y_n - Tp\rangle \geq 0.$$

Since $\lambda > 0$, we obtain

$$\langle \lambda Fy_n, y_n - Tp\rangle \geq 0. \tag{4.4.7}$$

Adding (4.4.6) and (4.4.7), we obtain

$$\langle Tw_n - y_n - \lambda(FTw_n - Fy_n), y_n - Tp\rangle \geq 0. \tag{4.4.8}$$

From (4.4.8) and the definition of r_n in Step 2, we obtain

$$\begin{aligned}
\langle Tw_n - Tp, r_n\rangle &= \langle Tw_n - y_n, r_n\rangle + \langle y_n - Tp, r_n\rangle \\
&= \langle Tw_n - y_n, r_n\rangle + \langle y_n - Tp, Tw_n - y_n - \lambda(FTw_n - Fy_n)\rangle \\
&\geq \langle Tw_n - y_n, r_n\rangle,
\end{aligned}$$

which implies that

$$-\langle Tw_n - Tp, r_n\rangle \leq -\langle Tw_n - y_n, r_n\rangle. \tag{4.4.9}$$

From the definition of z_n in Step 2, we have

$$||\beta_n \cdot r_n||^2 = \gamma_2^{-2}||z_n - Tw_n||^2. \tag{4.4.10}$$

Hence, from Lemma 2.1.1, (4.4.3), (4.4.9) and (4.4.10) we obtain

$$\begin{aligned}
\|z_n - Tp\|^2 &= \|Tw_n - \gamma_2\beta_n r_n - Tp\|^2 \\
&= \|Tw_n - Tp\|^2 + \gamma_2^2\beta_n^2\|r_n\|^2 - 2\gamma_2\beta_n\langle Tw_n - Tp, r_n\rangle \\
&\leq \|Tw_n - Tp\|^2 + \gamma_2^2\beta_n^2\|r_n\|^2 - 2\gamma_2\beta_n\langle Tw_n - y_n, r_n\rangle \\
&= \|Tw_n - Tp\|^2 + \gamma_2^2\beta_n^2\|r_n\|^2 - 2\gamma_2\beta_n \cdot \beta_n\|r_n\|^2 \\
&= \|Tw_n - Tp\|^2 - \gamma_2(2 - \gamma_2)\|\beta_n \cdot r_n\|^2 \\
&= \|Tw_n - Tp\|^2 - \gamma_2^{-1}(2 - \gamma_2)||z_n - Tw_n||^2. \qquad (4.4.11)
\end{aligned}$$

Also, from Step 3, Lemma 2.1.1 and (4.4.11) we obtain

$$\begin{aligned}
\|b_n - p\|^2 &= \|w_n - p\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 + 2\eta_n\langle w_n - p, T^*(z_n - Tw_n)\rangle \\
&= \|w_n - p\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 + 2\eta_n\langle Tw_n - Tp, z_n - Tw_n\rangle \\
&= \|w_n - p\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 \\
&+ \eta_n\left[\|z_n - Tp\|^2 - \|Tw_n - Tp\|^2 - \|z_n - Tw_n\|^2\right] \\
&\leq \|w_n - p\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 - \eta_n\|z_n - Tw_n\|^2 \qquad (4.4.12) \\
&= \|w_n - p\|^2 - \eta_n\left[\|z_n - Tw_n\|^2 - \eta_n\|T^*(z_n - Tw_n)\|^2\right].
\end{aligned}$$

Thus, by the condition on η_n, we obtain that

$$\|b_n - p\|^2 \leq \|w_n - p\|^2,$$

Following similar argument used in obtaining (4.4.11), we obtain

$$\begin{aligned}\|t_n - p\|^2 &= \|b_n - \gamma_1\gamma_n v_n - p\|^2 \\ &\leq \|b_n - p\|^2 - \gamma_1^{-1}(2-\gamma_1)\|t_n - b_n\|^2. \end{aligned} \tag{4.4.13}$$

From Step 5 we have,

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\sigma)w_n + \sigma t_n - p\|^2 \\ &= \|(1-\sigma)(w_n - p) + \sigma(t_n - p)\|^2 \\ &= (1-\sigma)\|w_n - p\|^2 + \sigma\|t_n - p\|^2 - (1-\sigma)\sigma\|w_n - t_n\|^2. \end{aligned} \tag{4.4.14}$$

Substituting (4.4.13) into (4.4.14), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\sigma)\|w_n - p\|^2 + \sigma\left(\|b_n - p\|^2 - \gamma_1^{-1}(2-\gamma_1)\|t_n - b_n\|^2\right) \\ &\quad - (1-\sigma)\sigma\|w_n - t_n\|^2 \\ &= (1-\sigma)\|w_n - p\|^2 + \sigma\|b_n - p\|^2 - \sigma\gamma_1^{-1}(2-\gamma_1)\|t_n - b_n\|^2 \\ &\quad - (1-\sigma)\sigma\|w_n - t_n\|^2 \\ &\leq (1-\sigma)\|w_n - p\|^2 + \sigma\|w_n - p\|^2 - \sigma\gamma_1^{-1}(2-\gamma_1)\|t_n - b_n\|^2 \\ &\quad - (1-\sigma)\sigma\|w_n - t_n\|^2 \\ &= \|w_n - p\|^2 - \sigma\gamma_1^{-1}(2-\gamma_1)\|t_n - b_n\|^2 - (1-\sigma)\sigma\|w_n - t_n\|^2. \end{aligned} \tag{4.4.15}$$

From Step 4, we have $t_n - w_n = \frac{1}{\sigma}(x_{n+1} - w_n)$. Substituting this into the previous equality we have,

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 - \sigma\gamma_1^{-1}(2-\gamma_1)\|t_n - b_n\| - (1-\sigma)\sigma \cdot \frac{1}{\sigma^2}\|x_{n+1} - w_n\|^2 \\ &= \|w_n - p\|^2 - \sigma\gamma_1^{-1}(2-\gamma_1)\|t_n - b_n\| - \left(\frac{1}{\sigma} - 1\right)\|x_{n+1} - w_n\|^2 \\ &\leq \|w_n - p\|^2 - \left(\frac{1}{\sigma} - 1\right)\|x_{n+1} - w_n\|^2 \\ &\leq \|w_n - p\| - \zeta\|x_{n+1} - w_n\|^2, \end{aligned}$$

where $\zeta := \left(\frac{1}{\sigma} - 1\right)$. □

Lemma 4.4.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.4.2 under Assumption 4.4.1. Then $\forall\, p \in \Gamma$, we have*

$$\begin{aligned}\|x_{n+1} - p\|^2 - \|x_n - p\|^2 &\leq \alpha_n\|x_n - x_0\|^2 + \theta_n\|x_n - p\|^2 - \theta_{n-1}\|x_{n-1} - p\|^2 \\ &\quad - (1 - 3\theta_{n+1} - \alpha_n)\|x_n - x_{n+1}\|^2 \\ &\quad - 2\alpha_n\langle x_n - p, x_n - x_0\rangle - 2\theta_{n+1}\|x_{n+1} - x_n\|^2 \\ &\quad + 2\theta_n\|x_n - x_{n-1}\|^2 - \alpha_{n+1}\|x_0 - x_{n+1}\|^2. \end{aligned} \tag{4.4.16}$$

Proof. From the definition of w_n and Lemma (2.1.1), we have

$$\begin{aligned}\|w_n - p\|^2 &= \|\alpha_n x_0 + (1-\alpha_n)x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\ &= \|(x_n - p) + \theta_n(x_n - x_{n-1}) - \alpha_n(x_n - x_0)\|^2 \\ &= \|x_n - p\|^2 + \|\theta_n(x_n - x_{n-1}) - \alpha_n(x_n - x_0)\|^2 \\ &+ 2\langle x_n - p, \theta_n(x_n - x_{n-1}) - \alpha_n(x_n - x_0)\rangle \\ &= \|x_n - p\|^2 + \|\theta_n(x_n - x_{n-1}) - \alpha_n(x_n - x_0)\|^2 \\ &+ 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle - 2\alpha_n\langle x_n - p, x_n - x_0\rangle.\end{aligned} \tag{4.4.17}$$

Now, replacing p with x_{n+1} in (4.4.17), we obtain

$$\begin{aligned}\|w_n - x_{n+1}\|^2 = \|x_n - x_{n+1}\|^2 + \|\theta_n(x_n - x_{n-1}) - \alpha_n(x_n - x_0)\|^2 \\ + 2\theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle - 2\alpha_n\langle x_n - x_{n+1}, x_n - x_0\rangle.\end{aligned} \tag{4.4.18}$$

Substituting (4.4.17) and (4.4.18) into (4.4.5) and from the condition on σ, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \|x_n - p\|^2 + \|\theta_n(x_n - x_{n-1}) - \alpha_n(x_n - x_0)\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle \\ &- 2\alpha_n\langle x_n - p, x_n - x_0\rangle - \|x_n - x_{n+1}\|^2 - \|\theta_n(x_n - x_{n-1}) \\ &- \alpha_n(x_n - x_0)\|^2 - 2\theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle + 2\alpha_n\langle x_n - x_{n+1}, x_n - x_0\rangle \\ &= \|x_n - p\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle - 2\alpha_n\langle x_n - p, x_n - x_0\rangle \\ &- 2\theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle - \|x_n - x_{n+1}\|^2 + 2\alpha_n\langle x_n - x_{n+1}, x_n - x_0\rangle \\ &= \|x_n - p\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle - 2\alpha_n\langle x_n - p, x_n - x_0\rangle \\ &+ \theta_n\|x_n - x_{n+1}\|^2 + \theta_n\|x_n - x_{n-1}\|^2 - \theta_n\|(x_n - x_{n+1}) + (x_n - x_{n-1})\|^2 \\ &- \|x_n - x_{n+1}\|^2 + 2\alpha_n\langle x_n - x_{n+1}, x_n - x_0\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_{n+1} - p\|^2 - \|x_n - p\|^2 &\le 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle - 2\alpha_n\langle x_n - p, x_n - x_0\rangle \\ &- (1-\theta_n)\|x_n - x_{n+1}\|^2 + \theta_n\|x_n - x_{n-1}\|^2 \\ &- \theta_n\|(x_n - x_{n+1}) + (x_n - x_{n-1})\|^2 + 2\alpha_n\langle x_n - x_{n+1}, x_n - x_0\rangle.\end{aligned} \tag{4.4.19}$$

Applying Lemma 2.1.1 to (4.4.19), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 - \|x_n - p\|^2 &\le \theta_n\|x_n - x_{n-1}\|^2 - (1-\theta_n)\|x_n - x_{n+1}\|^2 - 2\alpha_n\langle x_n - p, x_n - x_0\rangle \\ &+ 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle + 2\alpha_n\langle x_n - x_{n+1}, x_n - x_0\rangle \\ &= \theta_n\|x_n - x_{n-1}\|^2 - (1-\theta_n)\|x_n - x_{n+1}\|^2 - 2\alpha_n\langle x_n - p, x_n - x_0\rangle \\ &- \theta_n\|x_{n-1} - p\|^2 \\ &+ \theta_n\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|^2 - \alpha_n\|x_0 - x_{n+1}\| \\ &+ \alpha_n\|x_{n+1} - x_n\|^2 + \alpha_n\|x_n - x_0\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1} - p\|^2 - \|x_n - p\|^2 &\le \alpha_n\Big(\|x_n - x_0\|^2 - \|x_0 - x_{n+1}\|^2\Big) + \theta_n\Big(\|x_n - p\|^2 - \|x_{n-1} - p\|^2\Big) \\ &- \Big(1 - \theta_n - 2\theta_{n+1} - \alpha_n\Big)\|x_{n+1} - x_n\|^2 - 2\alpha_n\langle x_n - p, x_n - x_0\rangle \\ &- 2\theta_{n+1}\|x_{n+1} - x_n\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2.\end{aligned} \tag{4.4.20}$$

Using the fact that $\{\theta_n\}$ is non-decreasing and $\{\alpha_n\}$ is non-increasing on (4.4.20), we obtain (4.4.16), which is the desired conclusion. □

Lemma 4.4.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.4.2 under Assumption 4.4.1. Then $\{x_n\}$ is bounded.*

Proof. Let $p \in \Gamma$, then from (4.4.16) and Lemma 2.1.1, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 - \|x_n-p\|^2 &\le \alpha_n\|x_n-x_0\|^2 + \theta_n\|x_n-p\|^2 - \theta_{n-1}\|x_{n-1}-p\|^2 \\
&- (1-3\theta_{n+1}-\alpha_n)\|x_n-x_{n+1}\|^2 - 2\theta_{n+1}\|x_{n+1}-x_n\|^2 \\
&+ 2\theta_n\|x_n-x_{n-1}\|^2 - \alpha_{n+1}\|x_0-x_{n+1}\|^2 - 2\alpha_n\langle x_n-p, x_n-x_0\rangle \\
&= \alpha_n\|x_n-x_0\|^2 + \theta_n\|x_n-p\|^2 - \theta_{n+1}\|x_{n-1}-p\|^2 \\
&- (1-3\theta_{n+1}-\alpha_n)\|x_n-x_{n+1}\|^2 - 2\theta_{n+1}\|x_{n+1}-x_n\|^2 \\
&+ 2\theta_n\|x_n-x_{n-1}\|^2 - \alpha_{n+1}\|x_0-x_{n+1}\|^2 - \alpha_n\|x_n-p\|^2 \\
&- \alpha_n\|x_n-x_0\|^2 + \alpha_n\|x_0-p\|^2 \\
&\le \theta_n\|x_n-p\|^2 - \theta_{n-1}\|x_{n-1}-p\|^2 - (1-3\theta_{n+1}-\alpha_n)\|x_n-x_{n+1}\|^2 \qquad (4.4.21) \\
&- 2\theta_{n+1}\|x_{n+1}-x_n\|^2 \\
&+ 2\theta_n\|x_n-x_{n-1}\|^2 - \alpha_n\|x_n-p\|^2 + \alpha_n\|x_0-p\|^2.
\end{aligned}$$

From this we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|^2 - \|x_n-p\|^2 + \alpha_n\|x_n-p\|^2 \\
&\quad \le \theta_n\|x_n-p\|^2 - \theta_{n-1}\|x_{n-1}-p\|^2 - (1-3\theta_{n+1}-\alpha_n)\|x_n-x_{n+1}\|^2 \\
&\quad - 2\theta_{n+1}\|x_{n+1}-x_n\|^2 + 2\theta_n\|x_n-x_{n-1}\|^2 + \alpha_n\|x_0-p\|^2. \qquad (4.4.22)
\end{aligned}$$

Let $\rho_j := e^{\sum_{i=1}^{j}\alpha_i}, j \ge 1$. Since $e^x \ge x+1$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{1}{\rho_{n+1}}\Big(\rho_{n+1}\|x_{n+1}-p\|^2 - \rho_n\|x_n-p\|^2\Big) &= \|x_{n+1}-p\|^2 - \|x_n-p\|^2 \\
&+ \frac{1}{\rho_{n+1}}\Big(\rho_{n+1}-\rho_n\Big)\|x_n-p\|^2 \\
&\le \|x_{n+1}-p\|^2 - \|x_n-p\|^2 + \alpha_{n+1}\|x_n-p\|^2.
\end{aligned}$$

Since $\{\alpha_n\} \subset (0,1]$ is non-increasing, we have

$$\frac{1}{\rho_{n+1}}\Big(\rho_{n+1}\|x_{n+1}-p\|^2 - \rho_n\|x_n-p\|^2\Big) \le \|x_{n+1}-p\|^2 - \|x_n-p\|^2 + \alpha_n\|x_n-p\|^2. \qquad (4.4.23)$$

From (4.4.22) and (4.4.23), we obtain

$$\begin{aligned}\frac{1}{\rho_{n+1}}\Big[\rho_{n+1}\|x_{n+1}-p\|^2-\rho_n\|x_n-p\|^2\Big] \le\ & \theta_n\|x_n-p\|^2-\theta_{n-1}\|x_{n-1}-p\|^2\\ &-(1-3\theta_{n+1}-\alpha_n)\|x_{n+1}-x_n\|^2\\ &-2\theta_{n+1}\|x_{n+1}-x_n\|^2+2\theta_n\|x_n-x_{n-1}\|^2\\ &+\alpha_n\|x_0-p\|^2.\end{aligned}$$

Since $\rho_n \le \rho_{n+1}, \rho_{n+1}=\rho_n e^{\alpha_{n+1}}$ and $\{\alpha_n\}\subset(0,1]$ is non-increasing, we have

$$\begin{aligned}\rho_{n+1}\|x_{n+1}-p\|^2-\rho_n\|x_n-p\|^2 \le\ & \rho_{n+1}\theta_n\|x_n-p\|^2-\rho_n\theta_{n-1}\|x_{n-1}-p\|^2\\ &-\rho_{n+1}(1-3\theta_{n+1}-\alpha_n)\|x_{n+1}-x_n\|^2\\ &-2\rho_{n+1}\theta_{n+1}\|x_{n+1}-x_n\|^2+2\rho_n\theta_n e^{\alpha_{n+1}}\|x_n-x_{n-1}\|^2\\ &+\rho_{n+1}\alpha_n\|x_0-p\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\rho_{n+1}\|x_{n+1}-p\|^2-\rho_n\|x_n-p\|^2 \le\ & \rho_{n+1}\theta_n\|x_n-p\|^2-\rho_n\theta_{n-1}\|x_{n-1}-p\|^2\\ &-\rho_{n+1}\big[1-\theta_{n+1}\big(3+2(e^{\alpha_{n+1}}-1)\big)-\alpha_n\big]\|x_{n+1}-x_n\|^2\\ &-2\rho_{n+1}\theta_{n+1}e^{\alpha_{n+1}}\|x_{n+1}-x_n\|^2+2\rho_n\theta_n e^{\alpha_n}\|x_n-x_{n-1}\|^2\\ &+\rho_{n+1}\alpha_n\|x_0-p\|^2.\end{aligned}$$

Since $\{\theta_n\}\subset[0,\theta]$, we have

$$1-\theta_{n+1}(3+2(e^{\alpha_{n+1}}-1))-\alpha_n \ge 1-\theta(3+2(e^{\alpha_{n+1}}-1))-\alpha_n, \quad \forall n\in\mathbb{N}. \tag{4.4.24}$$

Since $\theta\in[0,\frac{1}{3})$ and $\lim\limits_{n\to\infty}\alpha_n=0$, it follows that the right-hand side of (4.4.24) is bounded below by a positive number, i.e., there exists a constant $\xi>0$ such that $1-\theta_{n+1}(3+2(e^{\alpha_{n+1}}-1))-\alpha_n\ge\xi$, for all $n\in\mathbb{N}$ sufficiently large, say for all $n\ge n_0$. Hence, we have

$$\begin{aligned}\rho_{n+1}\|x_{n+1}-p\|^2-\rho_n\|x_n-p\|^2 \le\ & \rho_{n+1}\theta_n\|x_n-p\|^2-\rho_n\theta_{n-1}\|x_{n-1}-p\|^2\\ &-\xi\|x_{n+1}-x_n\|^2\\ &-2\rho_{n+1}\theta_{n+1}e^{\alpha_{n+1}}\|x_{n+1}-x_n\|^2+2\rho_n\theta_n e^{\alpha_n}\|x_n-x_{n-1}\|^2\\ &+\rho_{n+1}\alpha_n\|x_0-p\|^2,\end{aligned}$$

which implies that for all $n\ge n_0$,

$$\begin{aligned}\|x_0-p\|^2\sum_{k=n_0+1}^{n}\rho_{k+1}\alpha_k \ge\ & \rho_{n+1}\|x_{n+1}-p\|^2+2\rho_{n+1}\theta_{n+1}e^{\alpha_{n+1}}\|x_{n+1}-x_n\|^2\\ &-\rho_{n+1}\theta_n\|x_n-p\|^2-\rho_{n_0+1}\|x_{n_0+1}-p\|^2\\ &-2\rho_{n_0+1}\theta_{n_0+1}e^{\alpha_{n_0+1}}\|x_{n_0+1}-x_{n_0}\|^2+\rho_{n_0+1}\theta_{n_0}\|x_{n_0}-p\|^2.\end{aligned} \tag{4.4.25}$$

Dividing the last inequality by ρ_{n+1} and omitting non-positive terms, we have

$$\begin{aligned}\|x_{n+1}-p\|^2-\theta_n\|x_n-p\|^2 \le e^{-t_n+1}\Big[&\rho_{n_0+1}\|x_{n_0+1}-p\|^2+2\rho_{n_0+1}\theta_{n_0+1}e^{\alpha_{n_0+1}}\|x_{n_0+1}-x_{n_0}\|^2\\ &-\rho_{n_0+1}\theta_{n_0}\|x_{n_0}-p\|^2\Big]\\ &+\|x_0-p\|^2e^{-t_n+1}\sum_{k=n_0+1}^{n}\alpha_k e^{t_k+1} \end{aligned} \tag{4.4.26}$$

where $t_n := \sum_{i=1}^{n}\alpha_i$. Since $\alpha_k\in(0,1]$ for all $k\in\mathbb{N}$, we observe that $\alpha_k e^{t_k+1}\le e^2(e^{t_k}-e^{t_k-1})$, for all $k\ge 2$, so that

$$\sum_{k=n_0+1}^{n}\rho_{k+1}\alpha_k=\sum_{k=n_0+1}^{n}\alpha_k e^{t_k+1}\le e^2(e^{t_n}-e^{t_{n_0}})\le e^2e^{t_n}.$$

Using (4.4.26), the fact that $\{\theta_n\}\subset[0,\theta]\subset[0,\frac{1}{3})$ and $e^{-t_n+1}\le 1$, we have

$$\begin{aligned}\|x_{n+1}-p\|^2\le \theta\|x_n-p\|^2+\rho_{n_0+1}\|x_{n_0+1}-p\|^2+2\rho_{n_0+1}\theta_{n_0+1}e^{\alpha_{n_0+1}}\|x_{n_0+1}-x_{n_0}\|^2\\ +e^2\|x_0-p\|^2.\end{aligned} \tag{4.4.27}$$

Applying (4.4.27), $\theta\in[0,1)$ and the convergence of the geometric series, we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2\le{}&\theta^{n-n_0}\|x_{n_0+1}-p\|^2\\ &+\frac{1}{1-\theta}\Big[\rho_{n_0+1}\|x_{n_0+1}-p\|^2+2\rho_{n_0+1}\theta_{n_0+1}e^{\alpha_{n_0+1}}\|x_{n_0+1}-x_{n_0}\|^2+e^2\|x_0-p\|^2\Big]\end{aligned} \tag{4.4.28}$$

Since $\theta<1$, it follows that $\{x_n\}$ is bounded.

□

Lemma 4.4.7. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.4.2 under Assumption 4.4.1. Suppose*

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$$

and

$$\lim_{n\to\infty}\left(\|x_{n+1}-p\|^2-\theta_n\|x_n-p\|^2\right)=0.$$

Then $\{x_n\}$ converges strongly to p.

Proof. By the hypothesis of the lemma we have that

$$\begin{aligned}0=\lim_{n\to\infty}\left(\|x_{n+1}-p\|^2-\theta_n\|x_n-p\|^2\right)=\lim_{n\to\infty}\Big[&\left(\|x_{n+1}-p\|+\sqrt{\theta_n}\|x_n-p\|\right)\\ &\times\left(\|x_{n+1}-p\|-\sqrt{\theta_n}\|x_n-p\|\right)\Big].\end{aligned} \tag{4.4.29}$$

We claim that this implies

$$\lim_{n\to\infty}\left(\|x_{n+1}-p\|+\sqrt{\theta_n}\|x_n-p\|\right)=0,$$

and from this it follows that $\{x_n\}$ converges strongly to p. On the contrary, assume that this limit does not hold. Then there exists a subset $K\subseteq\mathbb{N}$ and a constant $\beta>0$ such that

$$\|x_{n+1}-p\|+\sqrt{\theta_n}\|x_n-p\|\geq\beta,\forall\ n\in K. \tag{4.4.30}$$

Using (4.4.29) and the fact that $\theta_n\leq\theta<1$, we have

$$\begin{aligned}0&=\lim_{n\in K}\left(\|x_{n+1}-p\|-\sqrt{\theta_n}\|x_n-p\|\right)\\&=\limsup_{n\in K}\left(\|x_{n+1}-x_n+x_n-p\|-\sqrt{\theta_n}\|x_n-p\|\right)\\&\geq\limsup_{n\in K}\left(\|x_n-p\|-\|x_{n+1}-x_n\|-\sqrt{\theta_n}\|x_n-p\|\right)\\&\geq\limsup_{n\in K}\left((1-\sqrt{\theta})\|x_n-p\|-\|x_{n+1}-x_n\|\right)\\&=(1-\sqrt{\theta})\limsup_{n\in K}\|x_n-p\|-\lim_{n\in K}\|x_{n+1}-x_n\|\\&=(1-\sqrt{\theta})\limsup_{n\in K}\|x_n-p\|.\end{aligned}$$

Thus, we have $\limsup_{n\in K}\|x_n-p\|\leq 0$. Since $\liminf_{n\in K}\|x_n-p\|\geq 0$ holds, it follows that $\lim_{n\in K}\|x_n-p\|=0$.

Applying (4.4.30), we obtain

$$\begin{aligned}\|x_{n+1}-x_n\|&\geq\|x_{n+1}-p\|-\|x_n-p\|\\&=\|x_{n+1}-p\|+\sqrt{\theta_n}\|x_n-p\|-(1+\sqrt{\theta_n})\|x_n-p\|\\&\geq\frac{\beta}{2}\end{aligned}$$

for all $n\in K$ sufficiently large, which contradicts the assumption that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Hence, the result follows.

□

Lemma 4.4.8. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.4.2 under Assumption 4.4.1 such that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Suppose there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, which converges weakly to a point $z\in\mathcal{H}_1$ and $\lim_{k\to\infty}\|b_{n_k}-w_{n_k}\|=0=\lim_{k\to\infty}\|b_{n_k}-t_{n_k}\|$, then $z\in\Gamma$.*

Proof. From the definition of w_n in Step 1 and by the statement of the hypothesis together with the fact that $\lim_{n\to\infty} \alpha_n = 0$, we obtain

$$\begin{aligned}\|w_n - x_n\| &= \|\alpha_n(x_0 - x_n) + \theta_n(x_n - x_{n-1})\| \\ &\leq \alpha_n\|x_0 - x_n\| + \theta_n\|x_n - x_{n-1}\| \to 0, \quad \text{as } n \to \infty\end{aligned} \tag{4.4.31}$$

Since the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is weakly convergent to a point $z \in \mathcal{H}_1$, it follows that the subsequence $\{w_{n_k}\}$ of $\{w_n\}$ is also weakly convergent to $z \in \mathcal{H}_1$. Again, since T is a bounded linear operator, we obtain that $\{Tw_{n_k}\}$ converges weakly to Tz.

Without loss of generality, we may assume that $z_n \neq Tw_n$, then $\eta_n \in \left(\epsilon, \frac{\|z_n - Tw_n\|^2}{\|T^*(z_n - Tw_n)\|^2} - \epsilon\right)$.

Hence, we obtain from (4.4.12) that

$$\begin{aligned}\|b_{n_k} - p\|^2 &\leq \|w_{n_k} - p\|^2 - \eta_{n_k}\epsilon\|T^*(z_{n_k} - Tw_{n_k})\|^2 \\ &\leq \|w_{n_k} - p\|^2 - \epsilon^2\|T^*(z_{n_k} - Tw_{n_k})\|^2,\end{aligned} \tag{4.4.32}$$

which implies that

$$\begin{aligned}\epsilon^2\|T^*(z_{n_k} - Tw_{n_k})\|^2 &\leq \|w_{n_k} - p\|^2 - \|b_{n_k} - p\|^2 \\ &\leq \|w_{n_k} - b_{n_k}\|^2 + 2\|w_{n_k} - b_{n_k}\|\|b_{n_k} - p\|.\end{aligned}$$

From our hypothesis, we have

$$\lim_{k\to\infty} \|T^*(z_{n_k} - Tw_{n_k})\| = 0. \tag{4.4.33}$$

From (4.4.12) and (4.4.33), we have

$$\begin{aligned}\eta_{n_k}\|z_{n_k} - Tw_{n_k}\|^2 &\leq \|w_{n_k} - p\|^2 - \|b_{n_k} - p\|^2 + \eta_{n_k}^2\|T^*(z_{n_k} - Tw_{n_k})\|^2 \\ &\leq \|w_{n_k} - b_{n_k}\|^2 + 2\|w_{n_k} - b_{n_k}\|\|b_{n_k} - p\| + \eta_{n_k}^2\|T^*(z_{n_k} - Tw_{n_k})\|^2 \\ &\to 0, \quad \text{as} \quad k \to \infty.\end{aligned}$$

Since $\eta_{n_k} > \epsilon > 0$, we obtain

$$\lim_{k\to\infty} \|z_{n_k} - Tw_{n_k}\| = 0. \tag{4.4.34}$$

Using the definition of r_n in Step 2, we observe

$$\begin{aligned}\langle Tw_{n_k} - y_{n_k}, r_{n_k}\rangle &= \langle Tw_{n_k} - y_{n_k}, Tw_{n_k} - y_{n_k} - \lambda(FTw_{n_k} - Fy_{n_k})\rangle \\ &= \|Tw_{n_k} - y_{n_k}\|^2 - \langle Tw_{n_k} - y_{n_k}, \lambda(FTw_{n_k} - Fy_{n_k})\rangle \\ &\geq \|Tw_{n_k} - y_{n_k}\|^2 - \lambda\|Tw_{n_k} - y_{n_k}\|\|FTw_{n_k} - Fy_{n_k}\| \\ &\geq (1 - \lambda L_2)\|Tw_{n_k} - y_{n_k}\|^2.\end{aligned} \tag{4.4.35}$$

Since $\lambda \in (0, \frac{1}{L_2})$, we have that $1 - \lambda L_2 > 0$. Hence, from (4.4.35), (4.4.10) and (4.4.3) we obtain

$$
\begin{aligned}
\|Tw_{n_k} - y_{n_k}\|^2 &\leq \left(\frac{1}{1-\lambda L_2}\right) \langle Tw_{n_k} - y_{n_k}, r_{n_k}\rangle \\
&= \left(\frac{1}{1-\lambda L_2}\right) \beta_{n_k}\|r_{n_k}\|^2 \\
&= \left(\frac{1}{1-\lambda L_2}\right) \beta_{n_k}\|r_{n_k}\|\|Tw_{n_k} - y_{n_k} - \lambda(FTw_{n_k} - Fy_{n_k})\| \\
&= \left(\frac{1}{1-\lambda L_2}\right) \beta_{n_k}\|r_{n_k}\|\|(Tw_{n_k} - y_{n_k}) + \lambda(Fy_{n_k} - FTw_{n_k})\| \\
&\leq \left(\frac{1}{1-\lambda L_2}\right) \beta_{n_k}\|r_{n_k}\| \left(\|Tw_{n_k} - y_{n_k}\| + \lambda\|Fy_{n_k} - FTw_{n_k}\|\right) \\
&\leq \left(\frac{1}{1-\lambda L_2}\right) \beta_{n_k}\|r_{n_k}\| \left(\|Tw_{n_k} - y_{n_k}\| + \lambda L_2\|y_{n_k} - Tw_{n_k}\|\right) \\
&= \left(\frac{1+\lambda L_2}{1-\lambda L_2}\right) \|Tw_{n_k} - y_{n_k}\|\beta_{n_k}\|r_{n_k}\| \\
&= \gamma_2^{-1}\left(\frac{1+\lambda L_2}{1-\lambda L_2}\right) \|Tw_{n_k} - y_{n_k}\|\|z_{n_k} - Tw_{n_k}\|,
\end{aligned}
$$

which implies from (4.4.34) that

$$
\|Tw_{n_k} - y_{n_k}\| \leq \gamma_2^{-1}\left(\frac{1+\lambda L_2}{1-\lambda L_2}\right) \|Tw_{n_k} - z_{n_k}\| \to 0, \quad \text{as} \quad k \to \infty. \tag{4.4.36}
$$

Since $\{Tw_{n_k}\}$ converges weakly to Tz, then it follows from (4.4.36) that $\{y_{n_k}\}$ also converges weakly to Tz. Also, since $\{y_{n_k}\} \subset \mathcal{Q}$, we have that $Tz \in \mathcal{Q}$.

By the characteristic property of $P_\mathcal{Q}$, we obtain $\forall\ x \in \mathcal{Q}$ that

$$
\langle Tw_{n_k} - \lambda FTw_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0,
$$

which implies

$$
\frac{1}{\lambda}\langle Tw_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle FTw_{n_k}, y_{n_k} - Tw_{n_k}\rangle \leq \langle FTw_{n_k}, x - Tw_{n_k}\rangle. \tag{4.4.37}
$$

Hence, applying (4.4.36) in (4.4.37), we obtain that

$$
0 \leq \liminf_{k\to\infty}\langle FTw_{n_k}, x - Tw_{n_k}\rangle, \ \forall x \in \mathcal{Q}. \tag{4.4.38}
$$

Observe that

$$
\begin{aligned}
\langle Fy_{n_k}, x - y_{n_k}\rangle &= \langle Fy_{n_k} - FTw_{n_k}, x - Tw_{n_k}\rangle + \langle FTw_{n_k}, x - Tw_{n_k}\rangle \\
&+ \langle Fy_{n_k}, Tw_{n_k} - y_{n_k}\rangle.
\end{aligned} \tag{4.4.39}
$$

Since F is Lipschitz continuous on $\mathcal{H}_2$, we obtain from (4.4.36) that

$$
\lim_{k\to\infty} ||FTw_{n_k} - Fy_{n_k}|| = 0.
$$

Hence, from (4.4.36), (4.4.38) and (4.4.39), we obtain that

$$0 \leq \liminf_{k\to\infty} \langle Fy_{n_k}, x - y_{n_k}\rangle, \ \forall x \in \mathcal{Q}. \tag{4.4.40}$$

Next, we show that $Tz \in \mathrm{V}I(F, \mathcal{Q})$. Now, we choose a sequence $\{\delta_k\}$ of positive numbers such that $\delta_{k+1} \leq \delta_k, \quad \forall\ k \geq 1$ and $\delta_k \to 0$ as $k \to \infty$. From (4.4.40), we denote by N_k (for each $k \geq 1$), the smallest positive integer such that

$$\langle Fy_{n_j}, x - y_{n_j}\rangle + \delta_k \geq 0, \quad \forall j \geq N_k. \tag{4.4.41}$$

Since $\{\delta_k\}$ is decreasing, we have that $\{N_k\}$ is increasing. Also, since $\{y_{N_k}\} \subset \mathcal{Q}$ for all $k \geq 1$, we can suppose $Fy_{N_k} \neq 0$ (otherwise, y_{N_k} is a solution). Hence, we can set $q_{N_k} = \frac{Fy_{N_k}}{\|Fy_{N_k}\|^2}$ for each $k \geq 1$. Then, $\langle Fy_{N_k}, q_{N_k}\rangle = 1$ for each $k \geq 1$.

Therefore, from (4.4.41) we have

$$\langle Fy_{N_k}, x + \delta_k q_{N_k} - y_{N_k}\rangle \geq 0,$$

which implies from the pseudomonotonicity of F on $\mathcal{H}_2$ that

$$\langle F(x + \delta_k q_{N_k}), x + \delta_k q_{N_k} - y_{N_k}\rangle \geq 0. \tag{4.4.42}$$

This implies that

$$\langle Fx, x - y_{N_k}\rangle \geq \langle Fx - F(x + \delta_k q_{N_k}), x + \delta_k q_{N_k} - y_{N_k}\rangle - \delta_k\langle Fx, q_{N_k}\rangle. \tag{4.4.43}$$

Now, if $FTz = 0$, then $Tz \in VI(F, \mathcal{Q})$. So, we may suppose that $FTz \neq 0$. Since $\{y_{n_k}\}$ converges weakly to Tz, then by Condition (c) we obtain

$$0 < \|FTz\| \leq \liminf_{k\to\infty} \|Fy_{N_k}\|.$$

Since $\{y_{n_k}\} \subset \{y_{N_k}\}$, we obtain that

$$0 \leq \limsup_{k\to\infty} \|\delta_k q_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\delta_k}{\|Fy_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty} \delta_k}{\liminf_{k\to\infty} \|Fy_{n_k}\|} \leq \frac{0}{\|FTz\|} = 0.$$

Therefore, $\lim_{k\to\infty} \delta_k q_{N_k} = 0$. Thus, letting $k \to \infty$ in (4.4.43), we have

$$\langle Fx, x - Tz\rangle \geq 0, \ \ \forall x \in Q, \tag{4.4.44}$$

which implies by Lemma 2.5.8 that $Tz \in VI(F, Q)$.

Next, we show that $z \in VI(A, C)$. Following similar method of proof used in obtaining (4.4.36) and noting our hypothesis $\lim_{k\to\infty} \|b_{n_k} - t_{n_k}\| = 0$, we obtain

$$\|b_{n_k} - u_{n_k}\| \leq r_1^{-1}\left(\frac{1 + \mu L_1}{1 + \mu L_1}\right)\|b_{n_k} - t_{n_k}\| \to 0, \quad \text{as} \quad k \to \infty. \tag{4.4.45}$$

Following similar method of proof used in obtaining (4.4.40), we obtain from (4.4.45), the characteristic property of $P_{\mathcal{C}}$ and the Lipschitz continuity of A on $\mathcal{H}_1$ that

$$0 \leq \liminf_{k\to\infty} \langle Au_{n_k}, y - u_{n_k} \rangle, \ \forall y \in C. \tag{4.4.46}$$

From our hypothesis, (4.4.45) and the fact that $\{w_{n_k}\}$ converges weakly to z, we obtain that the subsequences $\{b_{n_k}\}$ and $\{u_{n_k}\}$ of $\{b_n\}$ and $\{u_n\}$ respectively, converge weakly to z. Also, since $\{u_{n_k}\} \subset \mathcal{C}$, we have that $z \in \mathcal{C}$. Following similar method of proof used in obtaining (4.4.44), we obtain

$$\langle Ay, y - z \rangle \geq 0, \quad \forall\, y \in C, \tag{4.4.47}$$

which implies by Lemma 2.5.8, that $z \in VI(A, C)$. Hence, we conclude that $z \in \Gamma$. □

Lemma 4.4.9. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.4.2 under Assumption 4.4.1. Then, for each $n \geq 1$*

$$v_n := \|x_n - p\|^2 + \alpha_n \|x_n - x_0\|^2 - \theta_{n-1}\|x_{n-1} - p\|^2 + 2\theta_n \|x_{n-1} - x_n\|^2 \geq 0.$$

Proof. Since $\{\theta_n\} \in [0, \frac{1}{3})$ is non-decreasing, we have from Lemma 2.1.1 that

$$\begin{aligned}
v_n &= \|x_n - p\|^2 + \alpha_n \|x_n - x_0\|^2 - \theta_{n-1}\|x_{n-1} - x_n + x_n - p\|^2 + 2\theta_n \|x_{n-1} - x_n\|^2 \\
&= \|x_n - p\|^2 + \alpha_n \|x_n - x_0\|^2 - \theta_{n-1}\|x_{n-1} - x_n\|^2 - \theta_{n-1}\|x_n - p\|^2 \\
&- 2\theta_{n-1}\langle x_{n-1} - x_n, x_n - p\rangle + 2\theta_n \|x_{n-1} - x_n\|^2 \\
&= \|x_n - p\|^2 + \alpha_n \|x_n - x_0\|^2 - \theta_{n-1}\|x_{n-1} - x_n\|^2 - \theta_{n-1}\|x_n - p\|^2 \\
&- \theta_{n-1}\Big[\|x_{n-1} - x_n\|^2 + \|x_n - p\|^2 - \|x_{n-1} - 2x_n + p\|^2\Big] + 2\theta_n \|x_{n-1} - x_n\|^2 \\
&= \|x_n - p\|^2 + \alpha_n \|x_n - x_0\|^2 - 2\theta_{n-1}\|x_{n-1} - x_n\|^2 - 2\theta_{n-1}\|x_n - p\|^2 \\
&+ \theta_{n-1}\|x_{n-1} - 2x_n + p\|^2 + 2\theta_n \|x_{n-1} - x_n\|^2 \\
&\geq \|x_n - p\|^2 + \alpha_n \|x_n - x_0\|^2 - 2\theta_n\|x_{n-1} - x_n\|^2 - \frac{2}{3}\|x_n - p\|^2 \\
&+ \theta_{n-1}\|x_{n-1} - 2x_n + p\|^2 + 2\theta_n \|x_{n-1} - x_n\|^2 \\
&= \frac{1}{3}\|x_n - p\|^2 + \alpha_n \|x_n - x_0\|^2 + \theta_{n-1}\|x_{n-1} - 2x_n + p\|^2 \\
&\geq \frac{1}{3}\|x_n - p\|^2 + \alpha_n \|x_n - x_0\|^2 \\
&\geq 0,
\end{aligned}$$

which is the desired conclusion. □

We are now in a position to prove the main theorem for Algorithm 4.4.2.

Theorem 4.4.10. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.4.2 under Assumption 4.4.1. Then, $\{x_n\}$ converges strongly to $p \in \Gamma$, where $p = P_\Gamma x_0$.*

Proof. From Lemma 4.4.9 and (4.4.16), we obtain

$$v_{n+1} - v_n + (1 - 3\theta_{n-1} - \alpha_n)\|x_n - x_{n+1}\|^2 \leq -2\alpha_n \langle x_n - p, x_n - x_0 \rangle. \tag{4.4.48}$$

We consider two cases for our proof.

CASE 1: Let $z \in \Gamma$. Suppose for some $n_0 \in \mathbb{N}$ large enough, we have $v_{n+1} \leq v_n$ for all $n \geq n_0$. Then by Lemma 4.4.9 we have $v_n \geq 0, \forall n \geq 1$ and $\lim_{n\to\infty} v_n = \lim_{n\to\infty} v_{n+1}$ exists. Since $\{x_n\}$ is bounded, there exists a constant $M > 0$ such that $2|\langle x_n - p, x_n - x_0\rangle| \leq M$. Hence, there exists $N \in \mathbb{N}$ and $\xi_1 > 0$ such that $(1 - 3\theta_{n+1} - \alpha_n) \geq \xi_1, \forall n \geq N$. Hence, from (4.4.48) we have that for all $n \geq N$

$$\xi_1 \|x_n - x_{n+1}\|^2 \leq v_n - v_{n+1} + \alpha_n M \to 0, \quad n \to \infty.$$

Thus,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.4.49}$$

From the definition of w_n in Step 1 and by applying (4.4.49) together with the fact that $\lim_{n\to\infty} \alpha_n = 0$, we have

$$\begin{aligned} \|w_n - x_n\| &= \|\alpha_n(x_0 - x_n) + \theta_n(x_n - x_{n-1})\| \\ &\leq \alpha_n\|x_0 - x_n\| + \theta_n\|x_n - x_{n-1}\| \to 0, \quad n \to \infty \end{aligned} \tag{4.4.50}$$

Consequently, we have

$$\|w_n - x_{n+1}\| \to 0, \quad n \to \infty. \tag{4.4.51}$$

From (4.4.15), we have

$$\begin{aligned} \sigma\gamma_1^{-1}(2 - \gamma_1)\|t_n - b_n\|^2 &\leq \|w_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &= \Big(\|w_n - p\| - \|x_{n+1} - p\|\Big)\Big(\|w_n - p\| + \|x_{n+1} - p\Big) \\ &\leq \|w_n - x_{n+1}\|\Big(\|w_n - p\| + \|x_{n+1} - p\Big) \\ &\leq \|w_n - x_{n+1}\| M_1 \end{aligned}$$

where $M_1 := \sup_{n\geq 1}\{\|w_n - p\| + \|x_{n+1} - p\|\}$. Hence

$$\|t_n - b_n\| \to 0, \ \ n \to \infty. \tag{4.4.52}$$

Similarly, we obtain from (4.4.15) that

$$\|w_n - t_n\| \to 0, \ \ n \to \infty.$$

Consequently, we have

$$\|b_n - w_n\| \to 0, \quad n \to \infty. \tag{4.4.53}$$

Using the fact that $\{x_n\}$ is bounded, $\{v_n\}$ is convergent and $\lim_{n\to\infty} \alpha_n = 0$, we obtain from Lemma 4.4.9 that

$$\lambda := \lim_{n\to\infty} \left(\|x_{n+1} - p\|^2 - \theta_n \|x_n - p\|^2 \right) < \infty, \tag{4.4.54}$$

which is the limit of $\lim_{n\to\infty} v_{n+1}$. Consequently, from Lemma 4.4.9, we have $\lambda \geq 0$. We show that $\lambda = 0$ holds. So that it follows from Lemma 4.4.7 that the sequence $\{x_n\}$ converges strongly to the solution p.

Suppose on the contrary $\lambda > 0$. Since $\{x_n\}$ is bounded by Lemma 4.4.6, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to z, such that

$$\liminf_{n\to\infty} \langle x_n - p, p - x_0 \rangle = \lim_{k\to\infty} \langle x_{n_k} - p, p - x_0 \rangle = \langle z - p, p - x_0 \rangle. \tag{4.4.55}$$

By applying (4.4.52) and (4.4.53), it follows from Lemma 4.4.8 that $z \in \Gamma$. Since $p = P_\Gamma x_0$, we obtain from (4.4.55)

$$\liminf_{n\to\infty} \langle x_n - p, p - x_0 \rangle = \langle z - p, p - x_0 \rangle \geq 0, \tag{4.4.56}$$

which follows from (4.4.55) that

$$\lim_{k\to\infty} \langle x_{n_k} - p, p - x_0 \rangle \geq 0.$$

From (4.4.54), we have

$$\liminf_{n\to\infty} \|x_{n+1} - p\|^2 \geq \lim_{n\to\infty} \left(\|x_{n+1} - p\|^2 - \theta_n \|x_n - p\|^2 \right) = \lambda,$$

and since $\lambda > 0$, we have

$$\|x_{n+1} - p\|^2 \geq \frac{1}{2}\lambda, \quad \forall n \geq n_1$$

for some sufficiently large $n_1 \in \mathbb{N}$. Observe that

$$\langle x_n - p, x_n - x_0 \rangle = \|x_n - p\|^2 + \langle x_n - p, p - x_0 \rangle.$$

Then, by applying (4.4.56) we have

$$\begin{aligned}
\liminf_{n\to\infty} \langle x_n - p, x_n - x_0 \rangle &= \liminf_{n\to\infty} \left(\|x_n - p\|^2 + \langle x_n - p, p - x_0 \rangle \right) \\
&\geq \liminf_{n\to\infty} \left(\frac{1}{2}\lambda + \langle x_n - p, p - x_0 \rangle \right) \\
&= \frac{1}{2}\lambda + \liminf_{n\to\infty} \langle x_n - p, p - x_0 \rangle \\
&\geq \frac{1}{2}\lambda.
\end{aligned}$$

Again, using the assumption that $\lambda > 0$, we have

$$\langle x_n - p, x_n - x_0 \rangle \geq \frac{1}{4}\lambda, \ \ \forall n \geq n_2,$$

for some sufficiently large $n_2 \in \mathbb{N}$ such that $n_2 \geq n_1$. From (4.4.48), we have

$$v_{n+1} - v_n \leq -\frac{1}{2}\alpha\lambda, \ \ \forall n \geq n_2.$$

Applying Lemma 4.4.9, it follows from the last inequality that

$$\frac{1}{2}\lambda \sum_{k=n_2}^{n} \alpha_k \leq v_{n_2} - v_n \leq v_{n_2}, \ \ \forall n \geq n_2.$$

Since $\lambda > 0$, this gives the summability of the sequence $\{\alpha_n\}$ which contradicts $\sum_{n=1}^{\infty} \alpha_n = \infty$. Therefore, we must have $\lambda = 0$, and it follows that the sequence $\{x_n\}$ converges strongly to $p = P_\Gamma x_0$ as required.

CASE 2: Suppose that $\{v_n\}$ is not monotonically decreasing. Let $\tau : \mathbb{N} \to \mathbb{N}$ be defined for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$ large enough by

$$\tau_{(n)} := \max\{k \in \mathbb{N} : k \leq n, \ v_k \leq v_{k+1}\}.$$

Observe that $\tau_{(n)}$ is a non-decreasing sequence such that $\tau_{(n)} \to \infty$ as $n \to \infty$ and $v_{\tau_{(n)}} \leq v_{\tau_{(n)}+1}$ for all $n \geq n_0$. Similar to **CASE 1,** for some constant $M > 0$, we obtain from (4.4.48) that

$$\xi_1 \|x_{\tau_{(n)+1}} - x_{\tau_{(n)}}\| \leq \alpha_{\tau(n)} M \to 0. \tag{4.4.57}$$

Consequently, we get

$$\|x_{\tau_{(n)+1}} - x_{\tau_{(n)}}\| \to 0, \ \ n \to \infty. \tag{4.4.58}$$

Also, following similar procedure as in **CASE 1,** we obtain

$$\begin{aligned} \|x_{\tau_{(n)}} - w_{\tau_{(n)}}\| &\to 0, \ \ n \to \infty. \\ \|t_{\tau_{(n)}} - b_{\tau_{(n)}}\| &\to 0, \ \ n \to \infty. \\ \|w_{\tau_{(n)}} - t_{\tau_{(n)}}\| &\to 0, \ \ n \to \infty. \\ \|b_{\tau_{(n)}} - w_{\tau_{(n)}}\| &\to 0, \ \ n \to \infty. \end{aligned} \tag{4.4.59}$$

Observe from (4.4.48) that for $j \geq 0$, we have $v_{j+1} < v_j$ when $x_j \notin \Omega := \{x \in \mathcal{H} : \langle x - x_0, x - p \rangle \leq 0\}$. Since $v_{\tau_{(n)}} \leq v_{\tau_{(n)}+1}$, we have that $x_{\tau_{(n)}} \in \Omega \ \forall \ n \geq n_0$. We have from

Lemma 4.4.6 that $\{x_{\tau(n)}\}$ is bounded, hence there exists a subsequence, again say $\{x_{\tau(n)}\}$ which converges weakly to some $z \in \mathcal{H}_1$. Since Ω is a closed and convex set, then it is weakly closed and it follows that $z \in \Omega$. By (4.4.59), it follows from Lemma 4.4.8 that $z \in \Gamma$. Hence, we have $z \in \Omega \cap \Gamma$. In view of Lemma 2.5.18, we know that $\Omega \cap \Gamma$ contains only p as its element. Consequently, we have $z = p$. Moreover, since $x_{\tau_n} \in \Omega$ we have

$$\begin{aligned}\|x_{\tau(n)} - p\|^2 &= \langle x_{\tau(n)} - x_0, x_{\tau(n)} - p\rangle - \langle p - x_0, x_{\tau(n)} - p\rangle \\ &\leq -\langle p - x_0, x_{\tau(n)} - p\rangle.\end{aligned}$$

Taking the lim sup of the above inequality, we get

$$\limsup_{n\to\infty} \|x_{\tau(n)} - p\| \leq 0.$$

Thus,

$$\|x_{\tau(n)} - p\| \to 0, \;\; n \to \infty. \tag{4.4.60}$$

We claim that this implies $\lim_{n\to\infty} v_{\tau(n)+1} = 0$. From the definition of $v_{\tau(n)+1}$, we have

$$\begin{aligned}v_{\tau(n)+1} &= \|x_{\tau(n)+1} - p\|^2 + \alpha_{\tau(n)+1}\|x_{\tau(n)+1} - x_0\|^2 - \theta_{\tau(n)}\|x_{\tau(n)} - p\|^2 \\ &+ 2\theta_{\tau(n)+1}\|x_{\tau(n)+1} - x_{\tau(n)}\|^2 \\ &= \|x_{\tau(n)+1} - x_{\tau(n)} + x_{\tau(n)} - p\|^2 + \alpha_{\tau(n)+1}\|x_{\tau(n)+1} - x_0\|^2 - \theta_{\tau(n)}\|x_{\tau(n)} - p\|^2 \\ &+ 2\theta_{\tau(n)+1}\|x_{\tau(n)+1} - x_{\tau(n)}\|^2.\end{aligned}$$

Using (4.4.58), (4.4.60), the boundedness of $\{\theta_n\}$ and $\{x_n\}$ and the fact that $\lim_{n\to\infty} \alpha_n = 0$, we obtain that $\lim_{n\to\infty} v_{\tau(n)+1} = 0$.

Next, we show that $\lim_{n\to\infty} v_n = 0$. Observe that for all $n \geq n_0$, we have $v_{\tau(n)} \leq v_{\tau(n)+1}$ if $n \neq \tau(n)$ since $v_j > v_{j+1}$ for $\tau(n) + 1 \leq j \leq n - 1$. It follows that $\forall n \geq n_0$, we have

$$v_n \leq \max\{v_{\tau(n)}, v_{\tau(n)+1}\} = v_{\tau(n)+1} \to 0.$$

Hence,

$$\limsup_{n\to\infty} v_n \leq 0.$$

From Lemma 4.4.9, we have that

$$\liminf_{n\to\infty} v_n \geq 0.$$

Thus,

$$\lim_{n\to\infty} v_n = 0.$$

Using the fact that $\{x_n\}$ is bounded, $\lim_{n\to\infty} \alpha_n = 0$ and by (4.4.48), we have

$$\|x_n - x_{n+1}\| \to 0, \;\; n \to \infty,$$

which implies from the definition of v_n that

$$\lim_{n\to\infty} \left(\|x_{n+1} - p\|^2 - \theta_n \|x_n - p\|^2 \right) = 0.$$

Thus, by Lemma 4.4.7 we obtain that $\{x_n\}$ converges strongly to $p = P_\Gamma x_0$, which completes the proof.

□

Remark 4.4.11. [192]

- Setting $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $F = 0$ and $T = I_\mathcal{H}$ (the identity operator on $\mathcal{H}$) in Theorem 4.4.10, we obtain an inertial projection and contraction method requiring only one projection onto the feasible set $\mathcal{C}$ per iteration with fixed step size for solving the classical VIP (1.2.4) when A is pseudomonotone and Lipschitz continuous as a corollary.
- The conclusions of Lemma 4.4.4, Lemma 4.4.8 and Theorem 4.4.10 still hold even if $\mu \in (0, \frac{1}{L_1})$ and $\lambda \in (0, \frac{1}{L_2})$ in Algorithm 4.4.2 are replaced with variable step sizes μ_n and λ_n, respectively such that

$$0 < \inf_{n\geq 1} \mu_n \leq \sup_{n\geq 1} \mu_n < \frac{1}{L_1} \;\; \text{and} \;\; 0 < \inf_{n\geq 1} \lambda_n \leq \sup_{n\geq 1} \lambda_n < \frac{1}{L_2}.$$

For the convergence analysis of Algorithm 4.4.3, which does not require the Lipschitz constants of the underlying cost operators to be known, we first state the following lemma on the step size rules derived from [162]. The proof of the lemma is similar to the method of proof in [162]. Hence, we omit the proof here.

Lemma 4.4.12. *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be the sequences generated by (4.4.1) and (4.4.2), respectively. Then the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are well defined, and $\lim_{n\to\infty} \lambda_n = \lambda$, $\lim_{n\to\infty} \mu_n = \mu$, where $\lambda \in \left[\min\left\{\frac{a_2}{L_2}, \lambda_1\right\}, \lambda_1 + \Phi\right]$, $\mu \in \left[\min\left\{\frac{a_1}{L_1}, \mu_1\right\}, \mu_1 + \Psi\right]$, and $\Phi = \sum_{n=1}^\infty \phi_n$, $\Psi = \sum_{n=1}^\infty \psi_n$.*

Lemma 4.4.13. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.4.3 under Assumption 4.4.1. Then,*

$$||Tw_n - y_n|| \leq \gamma_2^{-1} \left(\frac{\lambda_{n+1} + \lambda_n a_2}{\lambda_{n+1} - \lambda_n a_2} \right) ||Tw_n - z_n||, \;\; \forall n \geq 1 \tag{4.4.61}$$

and

$$||b_n - u_n|| \leq \gamma_1^{-1} \left(\frac{\mu_{n+1} + \mu_n a_1}{\mu_{n+1} - \mu_n a_1} \right) ||b_n - t_n||, \;\; \forall n \geq 1. \tag{4.4.62}$$

Proof. From (4.4.1), we have that

$$||FTw_n - Fy_n|| \leq \frac{a_2}{\lambda_{n+1}}||Tw_n - y_n||, \ \forall n \geq 1, \tag{4.4.63}$$

holds for both $FTw_n = Fy_n$ and $FTw_n \neq Fy_n$. Similar to (4.4.35), we obtain

$$\begin{aligned}\langle Tw_n - y_n, r_n\rangle &\geq \|Tw_n - y_n\|^2 - \lambda_n\|Tw_n - y_n\|\|FTw_n - Fy_n\| \\ &\geq (1 - \lambda_n\frac{a_2}{\lambda_{n+1}})\|Tw_n - y_n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\|Tw_n - y_n\|^2 &\leq \frac{1}{\left(1 - \lambda_n\frac{a_2}{\lambda_{n+1}}\right)}\langle Tw_n - y_n, r_n\rangle \\ &\leq \frac{1}{\left(1 - \lambda_n\frac{a_2}{\lambda_{n+1}}\right)}\beta_n\|r_n\|\left(\|Tw_n - y_n\| + \lambda_n\frac{a_2}{\lambda_{n+1}}\|Tw_n - y_n\|\right) \\ &= \left(\frac{1 + \frac{\lambda_n a_2}{\lambda_{n+1}}}{1 - \frac{\lambda_n a_2}{\lambda_{n+1}}}\right)\beta_n\|r_n\|\|Tw_n - y_n\| \\ &= \gamma_2^{-1}\left(\frac{\lambda_{n+1} + \lambda_n a_2}{\lambda_{n+1} - \lambda_n a_2}\right)\|Tw_n - z_n\|\|Tw_n - y_n\|,\end{aligned}$$

which reduces to (4.4.61) when simplified further. In a similar manner, we get (4.4.62).

□

Remark 4.4.14. Replacing μ and λ with μ_n and λ_n, respectively in Lemma 4.4.4, we obtain that $\{x_n\}$ is bounded.

By Lemma 4.4.12, it follows that

$$\lim_{n\to\infty}\left(\frac{\lambda_{n+1} + \lambda_n a_2}{\lambda_{n+1} - \lambda_n a_2}\right) = \frac{1 + a_2}{1 - a_2} \tag{4.4.64}$$

and

$$\lim_{n\to\infty}\left(\frac{\mu_{n+1} + \mu_n a_1}{\mu_{n+1} - \mu_n a_1}\right) = \frac{1 + a_1}{1 - a_1}. \tag{4.4.65}$$

Following the similar procedure used in Theorem 4.4.10, we obtain the following strong convergence theorem for Algorithm 4.4.3.

Theorem 4.4.15. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.4.3 under Assumption 4.4.1. Then, $\{x_n\}$ converges strongly to $p \in \Gamma$, where $p = P_\Gamma x_0$.*

Remark 4.4.16.

- The method of proof in this work is different from the method of proof used in obtaining strong convergence for SVIPs.
- Similar to Remark 4.4.11, we obtain an inertial projection and contraction method with adaptive step size, requiring only one projection onto the feasible set $\mathcal{C}$ per iteration, for solving the classical VIP (1.2.4) when A is pseudomonotone and Lipschitz continuous as a corollary.
- When the operators A and F are monotone and Lipschitz continuous, we do not need them to satisfy Condition (c). This is because Condition (c) was only used after (4.4.37) to get the conclusion of Lemma 4.4.8. But, from (4.4.37) and the monotonicity of F, we obtain

$$\begin{aligned}
0 &\leq \left\langle FTw_{n_k}, x - Tw_{n_k}\right\rangle + \frac{1}{\lambda}\left\langle y_{n_k} - Tw_{n_k}, x - y_{n_k}\right\rangle + \left\langle FTw_{n_k}, Tw_{n_k} - y_{n_k}\right\rangle \\
&\leq \left(\left\langle FTw_{n_k} - Fx, x - Tw_{n_k}\right\rangle + \left\langle Fx, x - Tw_{n_k}\right\rangle\right) + \frac{1}{\lambda}\|y_{n_k} - Tw_{n_k}\|\|x - y_{n_k}\| \\
&+ \|FTw_{n_k}\|\|Tw_{n_k} - y_{n_k}\| \\
&\leq \left\langle Fx, x - Tw_{n_k}\right\rangle + \frac{1}{\lambda}\|y_{n_k} - Tw_{n_k}\|\|x - y_{n_k}\| + \|FTw_{n_k}\|\|Tw_{n_k} - y_{n_k}\|, \ \forall x \in \mathcal{Q},
\end{aligned}$$

Passing limit as $k \to \infty$, noting that $\{Tw_{n_k}\}$ converges weakly to Tz and applying (4.4.36), it follows from the last inequality that

$$\langle Fx, x - Tz\rangle \geq 0, \quad \forall x \in \mathcal{Q}.$$

Consequently, by Lemma 2.5.8 we have that $Tz \in VI(F, \mathcal{Q})$. Similarly, we obtain that $z \in VI(A, \mathcal{C})$. Hence, we conclude that Lemma 4.4.8 holds.

- Theorem 4.4.10 and Theorem 4.4.15 are still true if the operators A and F in finite dimensional spaces are only required to be pseudomonotone and Lipschitz continuous which is an improvement over the results in literature since no product space formulation is required even with the relaxed pseudomonotonicity assumption.

4.4.3 Numerical experiments

In this section, using some test examples, we discuss the numerical behavior of our methods, Algorithm 4.4.2 and Algorithm 4.4.3, as well as compare them with the methods of Tian and Jiang [243] (Algorithm (2.5.11)), Tian and Jiang [244] (Algorithm (2.5.13)), Pham et. al [206] (see Appendix 4.3.16) and Ogwo et. al [192] (see Algorithm 4.3.3).

In our computations, we randomly choose $x_0, x_1 \in \mathcal{H}_1$, $\gamma_1 = 1.8$, $\gamma_2 = 1.1$, $a_1 = 0.6, a_2 = 0.4$, $\lambda_1 = 0.85$ and $\mu_1 = 0.9$. We choose $\alpha_n = \frac{1}{n+1}, \theta_n = 0.29, \sigma = 0.45, \phi_n = \frac{1}{(n+1)^2}, \psi_n = \frac{1}{(n+2)^2}$ in Algorithm 4.4.2 and Algorithm 4.4.3. Also, we choose $\delta_n = \frac{1}{n+1}$, $\theta_n = \frac{1}{2} - \delta_n$, $\alpha_n = \bar{\alpha}_n$, $\tau_n = \frac{\delta_n}{n^{0.01}}$ and $\alpha = 3$ in the method of Ogwo *et al.* [192, Algorithm 3.3]. Using

MATLAB 2019(b) and the stopping criterion $||x_{n+1}-x_n|| < 10^{-3}$, we plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 4.7, Figure 4.8, Table 4.4.1 and Table 4.4.2.

Example 4.4.17. Let $\mathcal{H}_1 = \mathcal{H}_2 = L_2([0, 2\pi])$ be equipped with inner product

$$\langle x, y\rangle = \int_0^{2\pi} x(t)y(t)dt, \quad \forall\ x, y \in L_2([0, 2\pi])$$

and norm

$$||x|| := \Big(\int_0^{2\pi} |x(t)|^2 dt\Big)^{\frac{1}{2}}, \quad \forall\ x, y \in L_2([0, 2\pi]).$$

Then we define $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ by

$$A(x)(t) = e^{-\|x\|}\int_0^t x(s)ds, \quad \forall x \in L_2([0, 2\pi]), \quad t \in [0, 2\pi].$$

From [250], we have that A is pseudomonotone and Lipschitz continuous but not monotone on $L_2([0, 1])$.

Let $\mathcal{C} = \{x \in L_2([0, 2\pi]) : \langle y, x\rangle \leq v\}$, where $y = t + e^t$ and $v = 1$, then $\mathcal{C}$ is a nonempty closed and convex subset of $L_2([0, 2\pi])$. We define the metric projection $P_{\mathcal{C}}$ as:

$$P_{\mathcal{C}}(x) = \begin{cases} x - \frac{\langle y,x\rangle - v}{||y||^2}y, & \text{if } \ \langle y, x\rangle > v, \\ x, & \text{if } \ \langle y, x\rangle \leq v. \end{cases}$$

Also, let $\mathcal{Q} = \{x \in L_2([0, 2\pi]) \ : \|x - a\|_{l_2} \leq d\}$, where $a = t + 3$ and $d = 2$, then $\mathcal{Q}$ is a nonempty closed and convex subset of $L_2([0, 2\pi])$. We define $P_{\mathcal{Q}}$ as:

$$P_{\mathcal{Q}}(x) = \begin{cases} x, & \text{if } \ x \in \mathcal{Q}, \\ \frac{x-a}{\|x-a\|^2}d + a, & \text{otherwise}. \end{cases}$$

We define the operator $F : \mathcal{Q} \to L_2([0, 2\pi])$ by

$$F(x)(t) := \mathcal{G}(x)\mathcal{M}(x)(t), \quad \forall\ x \in \mathcal{Q}, \quad t \in [0, 2\pi],$$

where $\mathcal{G} : \mathcal{Q} \to \mathbb{R}$ is defined by $g(x) := \frac{1}{1+\|x\|^2}$ and $\mathcal{M} : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ is defined by $\mathcal{M}(x)(t) := \int_0^t x(s)ds, \quad \forall \ \ x \in L_2([0, 2\pi]), \quad t \in [0, 2\pi]$. We have that $\mathcal{G}$ is $\frac{16}{25}$-Lipschitz continuous and $\frac{1}{5} \leq \mathcal{G}(x) \leq 1, \quad \forall\ x \in \mathcal{C}$ (see [222]). Hence, from [222], we have that F is pseudomonotone and Lipschitz continuous but not monotone since $\mathcal{M}$ is a Volterra intergral mapping which is bounded and linear monotone.

Let $T : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ be defined by

$$Tx(s) = \int_0^{2\pi} \mathcal{K}(s, t)x(t)dt, \ \forall\ x \in L_2([0, 2\pi]),$$

where $\mathcal{K}$ is a continuous real-valued function on $[0, 2\pi] \times [0, 2\pi]$. Then, T is a bounded linear operator with adjoint

$$T^*x(s) = \int_0^{2\pi} \mathcal{K}(t, s)x(t)dt, \ \forall\ x \in L_2([0, 2\pi]).$$

In particular, we define $\mathcal{K}(s,t) = e^{-st}$ for all $s, t \in [0, 2\pi]$. For Algorithms (2.5.11) and (2.5.13), we define the mapping $S : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ by

$$Sx(t) = \int_0^{2\pi} tx(s)ds, \quad t \in [0, 1].$$

Then, S is nonexpansive. For Algorithm (2.5.13), we define $h : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ by

$$hx(t) = \int_0^{2\pi} \frac{t}{2}x(s)ds, \quad x \in [0, 1].$$

Then, h is a contraction mapping.

We consider the following cases for the numerical experiments of this example.

Case 1: Take $x_0(t) = t + 2$ and $x_1(t) = 0.7e^{-t}$.

Case 2: Take $x_0(t) = 2t + 1$ and $x_1(t) = e^{-3t}$.

Case 3: Take $x_0(t) = 2t + 1$ and $x_1(t) = e^{-t}$.

Case 4: Take $x_0(t) = t^2 + 2t + 1$ and $x_1(t) = e^{-3t}$.

Table 4.4.1: Numerical results for Example 4.4.17

		Alg. (2.5.11)	Alg. (2.5.13)	App. 4.3.16	App. 4.3.3	Alg. 4.4.2	Alg. 4.4.3
Case 1	No. of Iter.	2	13	19	6	4	4
	CPU time (sec)	0.3973	1.2733	3.4674	8.6063	6.1012	4.0227
Case 2	No. of Iter.	2	12	19	6	4	4
	CPU time (sec)	0.3821	1.3016	3.3446	8.5436	4.9278	3.7949
Case 3	No. of Iter.	2	12	19	6	4	4
	CPU time (sec)	0.4129	1.2087	3.5745	8.7931	8.9403	7.7985
Case 4	No. of Iter.	2	13	19	6	4	4
	CPU time (sec)	0.3821	1.3016	3.6063	8.7372	11.3910	11.9466

Example 4.4.18. Let $\mathcal{H}_1 = (l_2(\mathbb{R}), \ ||.||_{l_2}) = \mathcal{H}_2$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \dots), \ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $||x||_{l_2} := \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}, \quad \forall x \in l_2(\mathbb{R})$.

Define the operators $A, F : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by $A(x_1, x_2, x_3, \dots) = (3x_1e^{-x_1^2}, 0, 0, \dots)$ and $F(x_1, x_2, x_3, \dots) = (7x_1e^{-x_1^2}, 0, 0, \dots)$ respectively. Then, A, F are pseudo-monotone, Lipschitz continuous and sequentially weakly continuous but not monotone. Let $T : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ be defined by $Tx = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right)$, for all $x \in l_2(\mathbb{R})$. Then, T is a bounded linear operator on $\ell_2(\mathbb{R})$ with adjoint $T^*y = \left(y_2, \frac{y_3}{2}, \frac{y_4}{3}, \dots\right)$ for all $y \in l_2(\mathbb{R})$.

Now, define $\mathcal{C} = \mathcal{Q} = \{x \in l_2(\mathbb{R}) : ||x - a||_{l_2} \le b\}$, where $a = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ and $b = 3$ for $\mathcal{C}$

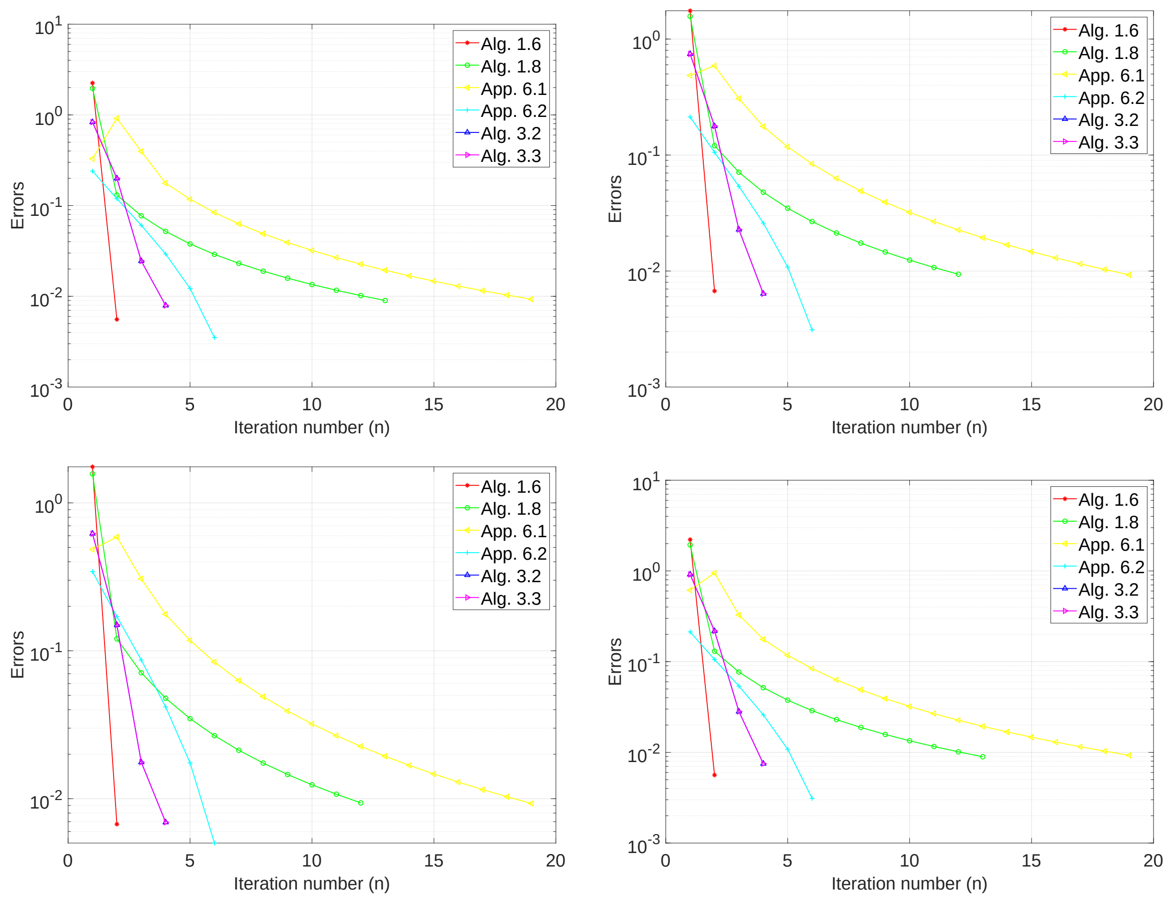

Figure 4.7: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

and $a = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$, $b = 1$ for $\mathcal{Q}$. Then $\mathcal{C}, \mathcal{Q}$ are nonempty closed and convex subsets of $l_2(\mathbb{R})$. Thus,

$$P_{\mathcal{C}}(x) = \begin{cases} x, & \text{if } ||x - y||_{\ell_2} \le b, \\ \frac{x-y}{||x-y||_{\ell_2}} b + a, & \text{otherwise.} \end{cases}$$

Furthermore, we define the mappings $S, h : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by $Sx = -4(x_1, x_2, x_3, \dots)$ and $hx = \left(\frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2}, \cdots\right)$ for all $x \in l_2(\mathbb{R})$, and consider the following cases for the starting point:

Case 1: Take $x_0 = (\frac{1}{5}, \frac{1}{15}, \frac{1}{45}, \cdots)$ and $x_1 = (1, \frac{1}{2}, \frac{1}{4}, \cdots)$.

Case 2: Take $x_0 = (\frac{2}{5}, \frac{2}{15}, \frac{2}{45}, \cdots)$ and $x_1 = (2, 1, \frac{1}{2}, \cdots)$.

Case 3: Take $x_0 = (\frac{1}{5} \frac{1}{15}, \frac{1}{45}, \cdots)$ and $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$.

Case 4: Take $x_0 = (-3, \frac{3}{2}, -\frac{3}{4}, \cdots)$ and $x_1 = (\frac{1}{3}, \frac{2}{9}, \frac{4}{27} \cdots)$.

Table 4.4.2 Numerical results for Example 4.4.18

		Alg. (2.5.11)	Alg. (2.5.13)	App. 4.3.16	App. 4.3.3	Alg. 4.4.2	Alg. 4.4.3
Case 1	No. of Iter.	2	6	14	7	4	4
	CPU time (sec)	0.0350	1.1215	0.0438	0.0512	0.1102	0.00186
Case 2	No. of Iter.	2	8	14	7	4	4
	CPU time (sec)	0.0093	0.7225	0.0331	0.0393	0.1011	0.0104
Case 3	No. of Iter.	2	6	14	6	4	4
	CPU time (sec)	0.0150	0.7402	0.0332	0.0378	0.1037	0.0119
Case 4	No. of Iter.	2	8	14	7	4	4
	CPU time (sec)	0.0091	0.7182	0.0333	0.0347	0.1051	0.0169

4.5 On a class of generalized variational inequality problem

The purpose of this study is to propose an inertial extrapolation method for solving a certain class of VIP more general than the classical VIP in real Hilbert spaces. Our proposed method is of viscosity-type and converges strongly to a solution of the aforementioned problem when the underlying/cost operator is pseudo-monotone and uniformly continuous; this makes our method to be potentially more applicable than most existing methods in the literature. To support our results numerically, we considered some examples in both finite and infinite dimensional Hilbert spaces and compared our results with other existing results in the literature.

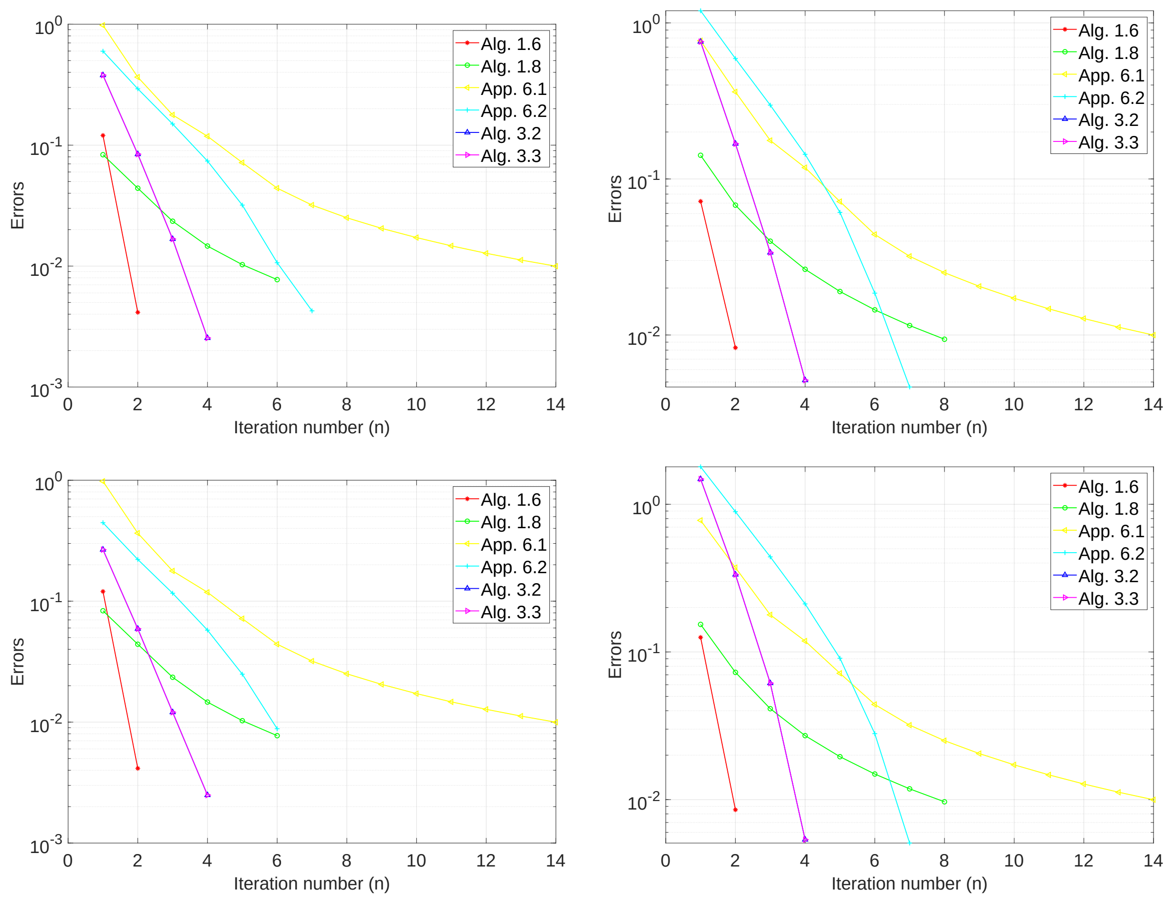

Figure 4.8: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

4.5.1 Main results

In this section, we present and study the proposed method for solving Problem (1.2.6). We begin with the following conditions: $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces, $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator, $S : \mathcal{H}_2 \to \mathcal{H}_2$ is a κ-strictly pseudocontractive mapping with $\kappa \in [0, 1)$, and $g : \mathcal{H}_1 \to \mathcal{H}_1$ is a contraction mapping with constant $\rho \in (0, 1)$. Furthermore, we make the following assumptions:

Assumption 4.5.1. *Suppose that the following conditions holds:*

(1) a. *The set $\mathcal{C}$ is a nonempty, closed and convex subset of $\mathcal{H}_1$.*

b. *$A : \mathcal{H}_1 \to \mathcal{H}_1$ is pseudo-monotone, uniformly continuous and sequentially weakly continuous on bounded subsets of $\mathcal{C}$.*

c. *The solution set $\Gamma := \{z \in VI(\mathcal{C}, A) : Tz \in F(S)\}$ is nonempty.*

(2). *$\{\gamma_n\}_{n=1}^{\infty}$ and $\{\epsilon_n\}_{n=1}^{\infty}$ are positive sequences satisfying the following conditions:*

a. $\gamma_n \in (0, 1), \quad \lim_{n\to\infty} \gamma_n = 0, \quad \sum_{n=1}^{\infty} \gamma_n = \infty.$

b. $\lim_{n\to\infty} \frac{\epsilon_n}{\gamma_n} = 0.$

Algorithm 4.5.2.

Initialization: Choose sequences $\{\gamma_n\}_{n=1}^{\infty}, \{\delta_n\}_{n=1}^{\infty}$ and $\{\varepsilon_n\}_{n=1}^{\infty}$ such that the condition from Assumption 3.1(2) hold and let $\gamma, \sigma \in (0, 1), \delta \in [0, 1)$ and $x_0, x_1 \in \mathcal{H}_1$ be given arbitrarily. Set $n := 1$.

Iterative Steps: Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose δ_n such that $0 \leq \delta_n \leq \bar{\delta}_n$, where

$$\bar{\delta}_n = \begin{cases} \min\{\delta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\}, & if\ x_n \neq x_{n-1} \\ \delta,\ otherwise \end{cases} \tag{4.5.1}$$

Step 2. Set $w_n := x_n + \delta_n(x_n - x_{n-1})$ and compute

$$u_n = w_n - \tau_n T^*(I - S_\beta)Tw_n, \tag{4.5.2}$$

where S_β is as defined in Lemma 2.1.8.
Compute $z_n := P_{\mathcal{C}}(u_n - A(u_n))$. If $u_n - z_n = 0$, then stop. Otherwise go to **Step 3**.
Step 3. Compute $y_n = u_n - \gamma^{m_n}(u_n - z_n)$, where m_n is the smallest nonnegative integer satisfying

$$\langle A(y_n), u_n - z_n\rangle \geq \frac{\sigma}{2}\|u_n - z_n\|^2. \tag{4.5.3}$$

Set $\eta_n := \gamma^{m_n}$.
Step 4. Compute

$$x_{n+1} = \gamma_n g(x_n) + (1 - \gamma_n)P_{\mathcal{C}_n}(u_n), \tag{4.5.4}$$

where $\mathcal{C}_n := \{x \in H_1 : f_n(x) \leq 0\}$ and

$$f_n(x) := \langle A(y_n), x - y_n \rangle. \tag{4.5.5}$$

Set $n := n + 1$ and go back to **Step 1**.

Remark 4.5.1. If $\{y_n\}$ is bounded, then f_n is Lipschitz continuous. Indeed, if $\{y_n\}$ is bounded, then there exists $\bar{M} > 0$ such that $\|A(y_n)\| \leq \bar{M}$ for all $n \geq 1$. Thus, for each $x, y \in \mathcal{C}$, we obtain

$$\begin{aligned} ||f_n(x) - f_n(y)|| &= ||\langle A(y_n), x - y_n \rangle - \langle A(y_n), y - y_n \rangle|| \\ &= ||\langle A(y_n), x - y \rangle|| \\ &\leq \|A(y_n)\| \|x - y\| \\ &\leq \bar{M} \|x - y\|. \end{aligned}$$

Therefore, f_n is Lipschitz continuous.

Lemma 4.5.2. *Suppose that Assumption 4.5.1 holds. Then, the linesearch (4.5.3) is well defined.*

Proof. Suppose $u_n = z_n$, then the method terminates at the solution of the GVIP (1.2.6). Let $u_n \neq z_n$, we prove by contradiction by assuming that the contrast of (4.5.3) holds, that is, for every nonnegative integer n

$$\langle A(y_n), u_n - z_n \rangle < \frac{\sigma}{2} \|u_n - z_n\|^2,$$

where $y_n = u_n - \eta_n(u_n - z_n)$. Since $y_n \to u_n$ as $n \to \infty$ and A is continuous, it follows from taking the limits as $n \to \infty$ that

$$\langle A(u_n), u_n - z_n \rangle \leq \frac{\sigma}{2} \|u_n - z_n\|^2. \tag{4.5.6}$$

Since $z_n = P_{\mathcal{C}}(u_n - A(u_n))$, we have

$$\langle u_n - A(u_n) - z_n, u - z_n \rangle \leq 0, \ \forall \ u \in \mathcal{C}.$$

Choose $u = u_n$ then, we have

$$\|u_n - z_n\|^2 \leq \langle A(u_n), u_n - z_n \rangle. \tag{4.5.7}$$

Combining (4.5.6) and (4.5.7), we obtain

$$\|u_n - z_n\|^2 \leq \frac{\sigma}{2} \|u_n - z_n\|^2$$

which contradicts the fact that $u_n \neq z_n$. Thus, the linesearch (4.5.3) is well defined. □

Lemma 4.5.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.5.2 such that Assumption 4.5.1 hold, then we have that $\{x_n\}$ is bounded.*

Proof. Let $x^* \in \Gamma$, then from Lemma 2.1.10 (i),(ii), Lemma 2.1.8 and Lemma 2.1.9, we obtain that $I - \tau_n T^*(I - S_\beta)T$ is $\tau_n\|T\|^2$-average. That is, $I - \tau_n T^*(I - S_\beta)T = (1-\beta_n)I + \beta_n S_n, \quad \forall n \geq 1$, where $\beta_n = \tau_n\|T\|^2$ and S_n is nonexpansive for all $n \geq 1$. Clearly, $\beta_n \in (0,1)$. Therefore, we can rewrite u_n from (4.5.2) as

$$u_n = (1-\beta_n)w_n + \beta_n S_n w_n, \ n \geq 1. \tag{4.5.8}$$

Hence, for any $x^* \in \Gamma$, we obtain that

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq (1-\beta_n)\|w_n - x^*\|^2 + \beta_n\|S_n w_n - x^*\|^2 - \beta_n(1-\beta_n)\|w_n - S_n w_n\|^2 \\
&\leq \|w_n - x^*\|^2 - \beta_n(1-\beta_n)\|w_n - S_n w_n\|^2 \\
&\leq \|w_n - x^*\|^2. \qquad (4.5.9)
\end{aligned}$$

Thus, from (4.5.4) and (4.5.9), we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \gamma_n\|g(x_n) - x^*\| + (1-\gamma_n)\|P_{C_n}(u_n) - x^*\| \\
&\leq \gamma_n\|g(x_n) - x^*\| + (1-\gamma_n)\|u_n - x^*\| \\
&\leq \gamma_n\|g(x_n) - x^*\| + (1-\gamma_n)\|w_n - x^*\| \\
&\leq \gamma_n\|g(x_n) - x^*\| + (1-\gamma_n)\left(\|x_n - x^*\| + \delta_n\|x_n - x_{n-1}\|\right) \\
&\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\rho\|x_n - x^*\| + \gamma_n\|g(x^*) - x^*\| \qquad (4.5.10) \\
&+ (1-\gamma_n)\delta_n\|x_n - x_{n-1}\| \\
&\leq (1-\gamma_n(1-\rho))\|x_n - x^*\| + \gamma_n\|g(x^*) - x^*\| + \delta_n\|x_n - x_{n-1}\|.
\end{aligned}$$

Now observe from (4.5.1) that

$$\delta_n\|x_n - x_{n-1}\| \leq \epsilon_n \quad \forall n \geq 1,$$

which implies that

$$\frac{\delta_n}{\gamma_n}\|x_n - x_{n-1}\| \leq \frac{\epsilon_n}{\gamma_n} \to 0, \quad \text{as} \quad n \to \infty. \tag{4.5.11}$$

Hence, there exists $M > 0$ such that

$$\frac{\delta_n}{\gamma_n}\|x_n - x_{n-1}\| \leq M, \quad \forall n \geq 1.$$

Therefore, (4.5.10) becomes

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1-\gamma_n(1-\rho))\|x_n - x^*\| + \gamma_n(\|g(x^*) - x^*\| + M) \\
&\leq \max\left\{\|x_n - x^*\|, \frac{\|g(x^*) - x\| + M}{1-\rho}\right\} \\
&\ \vdots \\
&\leq \max\left\{\|x_1 - x^*\|, \frac{\|g(x^*) - x\| + M}{1-\rho}\right\}.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. □

Lemma 4.5.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.5.2 such that Assumption 4.5.1 holds. If $\lim_{n\to\infty} \|P_{\mathcal{C}_n}(u_n) - u_n\| = 0$, then*

(i) $\lim_{n\to\infty} \eta_n \|u_n - z_n\|^2 = 0.$

(ii) $\lim_{n\to\infty} \|u_n - z_n\| = 0.$

Proof. Let $x^* \in \Gamma$. Since A is uniformly continuous on bounded subsets of $\mathcal{H}_1$, then $\{A(x_n)\}, \{A(y_n)\}, \{w_n\}$ and $\{u_n\}$ are bounded. Also, using (4.5.3) we have

$$\begin{aligned} f_n(u_n) &= \langle A(y_n), u_n - y_n \rangle \\ &= \eta_n \langle A(y_n), u_n - z_n \rangle \geq \eta_n \frac{\sigma}{2} \|u_n - z_n\|^2. \end{aligned} \tag{4.5.12}$$

Thus, applying Lemma 2.5.16 and Remark 4.5.1, we obtain

$$\begin{aligned} \|P_{\mathcal{C}_n}(u_n) - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|P_{\mathcal{C}_n}(u_n) - u_n\|^2 \\ &= \|u_n - x^*\|^2 - \operatorname{dist}^2(u_n, \mathcal{C}_n) \\ &\leq \|u_n - x^*\|^2 - \left(\frac{1}{M} f_n(u_n)\right)^2 \\ &\leq \|u_n - x^*\|^2 - \left(\frac{1}{2M} \sigma \eta_n \|u_n - z_n\|^2\right)^2. \end{aligned} \tag{4.5.13}$$

Since $\{x_n\}$ is bounded, we obtain from (4.5.13), (4.5.9) and (4.5.4) that

$$\begin{aligned} \left(\frac{1}{2M} \sigma \eta_n \|u_n - z_n\|^2\right)^2 &\leq \|u_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \gamma_n \|g(x_n) - x^*\|^2 \\ &\leq \|u_n - x_{n+1}\|^2 + \|x_{n+1} - x^*\|^2 + 2\|u_n - x_{n+1}\| \|x_{n+1} - x^*\| \\ &\quad - \|x_{n+1} - x^*\|^2 + \gamma_n \|g(x_n) - x^*\|^2 \\ &= \|u_n - x_{n+1}\|^2 + 2\|u_n - x_{n+1}\| \|x_{n+1} - x^*\| + \gamma_n \|g(x_n) - x^*\|^2 \\ &= \|u_n - x_{n+1}\| \left[\|u_n - x_{n+1}\| + 2\|x_{n+1} - x^*\|\right] + \gamma_n \|g(x_n) - x^*\|^2 \\ &\leq \left(\|u_n - P_{\mathcal{C}_n} u_n\| + \|P_{\mathcal{C}_n} u_n - x_{n+1}\|\right) M_1 + \gamma_n \|g(x_n) - x^*\|^2 \\ &\leq \|u_n - P_{\mathcal{C}_n} u_n\| M_1 + \gamma_n M_1 \|g(x_n) - P_{\mathcal{C}_n} u_n\| + \gamma_n \|g(x_n) - x^*\|^2, \end{aligned} \tag{4.5.14}$$

where $M_1 := \sup_{n\geq 1}\{\|u_n - x_{n+1}\| + 2\|x_{n+1} - x^*\|\}$. Hence $\lim_{n\to\infty} \eta_n \|u_n - z_n\|^2 = 0$.
We now show that $\lim_{n\to\infty} \|u_n - z_n\| = 0$.
From Algorithm 4.5.2, we obtain that $\liminf_{n\to\infty} \eta_n \geq 0$.
Case 1: Suppose that $\liminf_{n\to\infty} \eta_n > 0$. Then, we obtain that

$$\begin{aligned} \limsup_{n\to\infty} \|u_n - z_n\|^2 &\leq \limsup_{n\to\infty} \left(\eta_n \|u_n - z_n\|^2\right) \left(\limsup_{n\to\infty} \frac{1}{\eta_n}\right) \\ &\leq \left(\limsup_{n\to\infty} \eta_n \|u_n - z_n\|^2\right) \left(\frac{1}{\liminf_{n\to\infty} \eta_n}\right) \\ &= 0. \end{aligned}$$

Hence, $\limsup_{n\to\infty} \|u_n - z_n\| = 0$, which implies that

$$\lim_{n\to\infty} \|u_n - z_n\| = 0.$$

Case 2: Suppose that $\liminf_{n\to\infty} \eta_n = 0$. Without loss of generality, we may assume that there exists a subsequence of $\{\eta_n\}$ still denoted by $\{\eta_n\}$ such that $\lim_{n\to\infty} \eta_n = 0$ and $\lim_{n\to\infty} \|u_n - z_n\| = t \geq 0$.

Setting $v_n = u_n + \frac{1}{\gamma}\eta_n(z_n - u_n)$, we obtain that

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{4.5.15}$$

Also, by the definition of v_n and (4.5.3), we obtain

$$\langle A(v_n), u_n - z_n\rangle < \frac{\sigma}{2}\|u_n - z_n\|^2,$$

which implies that

$$2\langle A(v_n) - A(u_n), u_n - z_n\rangle + 2\langle A(u_n), u_n - z_n\rangle < \sigma\|u_n - z_n\|^2, \quad \forall n \geq 1. \tag{4.5.16}$$

Set $s_n := u_n - A(u_n)$, then (4.5.16) becomes

$$2\langle A(v_n) - A(u_n), u_n - z_n\rangle + 2\langle u_n - s_n, u_n - z_n\rangle < \sigma\|u_n - z_n\|^2,$$

which implies

$$\begin{aligned} \|s_n - u_n\|^2 - \|s_n - z_n\|^2 < (\sigma - 1)&\|u_n - z_n\|^2 \\ &- 2\langle A(v_n) - A(u_n), u_n - z_n\rangle, \quad \forall n \geq 1. \end{aligned} \tag{4.5.17}$$

Since A is uniformly continuous on bounded subsets of $\mathcal{H}_1$, if $t > 0$ then we obtain from (4.5.15) and (4.5.17) that

$$\limsup_{n\to\infty} \left(\|s_n - u_n\|^2 - \|s_n - z_n\|^2\right) \leq (\sigma - 1)t < 0.$$

For $\varepsilon = \frac{-(\sigma-1)}{2}t > 0$, there exists $N \in \mathbb{N}$ such that

$$\|s_n - u_n\|^2 - \|s_n - z_n\|^2 \leq (\sigma - 1)t + \varepsilon = \frac{(\sigma - 1)t}{2} < 0 \quad \forall n \in \mathbb{N}, \quad n \geq N.$$

Thus, we obtain that

$$\|s_n - u_n\| < \|s_n - z_n\| \quad \forall n \geq 1.$$

That is, $\|(u_n - A(u_n)) - u_n\| < \|(u_n - A(u_n)) - P_{\mathcal{C}}(u_n - A(u_n))\|$ which is a contradiction. Therefore, $t = 0$. Hence, $\lim_{n\to\infty} \|u_n - z_n\| = 0$. □

Lemma 4.5.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.5.2 such that Assumption 4.5.1 holds. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to a point $z \in \mathcal{H}_1$ and $\lim_{k\to\infty} \|u_{n_k} - w_{n_k}\| = 0 = \lim_{k\to\infty} \|P_{\mathcal{C}_{n_k}}(u_{n_k}) - u_{n_k}\|$, for some subsequences $\{u_{n_k}\}$ and $\{w_{n_k}\}$ of $\{u_n\}$ and $\{w_n\}$ respectively. Then, $z \in \Gamma$.*

Proof. By **Step 2** and (4.5.11), we obtain that

$$\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\delta_n\|x_n - x_{n-1}\| = \lim_{n\to\infty}\gamma_n \cdot \frac{\delta_n}{\gamma_n}\|x_n - x_{n-1}\| = 0. \tag{4.5.18}$$

Now, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that converges weakly to some $z \in \mathcal{H}_1$. Thus, $\{w_{n_k}\}$ also converges weakly to $z \in \mathcal{H}_1$. Hence, by our hypothesis, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ that converges weakly to $z \in \mathcal{H}_1$. We may also assume without loss of generality that, the subsequence $\{\tau_{n_k}\}$ of $\{\tau_n\}$ converges to a point say $\overline{\tau} \in \left(0, \frac{1}{\|T\|^2}\right)$. Also, by Lemma 2.1.9, $T^*(I - S_\beta)A$ is an inverse strongly monotone operator. Therefore, $\{T^*(I - S_\beta)Tw_{n_k}\}$ is bounded. Hence, we obtain that

$$\begin{aligned}\|(I - \tau_{n_k}T^*(I - S_\beta)T)w_{n_k} - (I - \overline{\tau}T^*(I - S_\beta)T)w_{n_k}\| &= |\tau_{n_k} - \overline{\tau}|\|T^*(I - S_\beta)Tw_{n_k}\| \\ &\to 0, \quad \text{as} \quad k \to \infty.\end{aligned}$$

That is,

$$\lim_{k\to\infty}\|u_{n_k} - (I - \overline{\tau}T^*(I - S_\beta)T)w_{n_k}\| = 0,$$

which implies from our hypothesis that

$$\lim_{k\to\infty}\|w_{n_k} - (I - \overline{\tau}T^*(I - S_\beta)T)w_{n_k}\| = 0. \tag{4.5.19}$$

Thus, by Lemma 2.5.17, we obtain that $z \in F(I - \overline{\tau}T^*(I - S_\beta)T)$, which together with Lemma 2.1.8 implies that

$$Tz \in F(S_\beta) = F(S). \tag{4.5.20}$$

Now, from the characteristics property of metric projection $P_{\mathcal{C}}$, we obtain for all $x \in \mathcal{C}$ that

$$\langle u_{n_k} - A(u_{n_k}) - z_{n_k}, x - z_{n_k}\rangle \le 0, \quad \forall x \in \mathcal{C},$$

which implies that

$$\begin{aligned}0 &\le \langle z_{n_k} - u_{n_k} + A(u_{n_k}), x - z_{n_k}\rangle \\ &= \langle z_{n_k} - u_{n_k}, x - z_{n_k}\rangle + \langle A(u_{n_k}), u_{n_k} - z_{n_k}\rangle + \langle A(u_{n_k}), x - u_{n_k}\rangle \\ &\le \|z_{n_k} - u_{n_k}\|\|x - z_{n_k}\| + \|A(u_{n_k})\|\|u_{n_k} - z_{n_k}\| + \langle A(u_{n_k}), x - u_{n_k}\rangle.\end{aligned} \tag{4.5.21}$$

Fix $x \in \mathcal{C}$ and let $k \to \infty$, since $\lim_{k\to\infty}\|u_{n_k} - z_{n_k}\| = 0$ (by Lemma 4.5.4), we have from (4.5.21) that

$$0 \le \liminf_{k\to\infty}\langle A(u_{n_k}), x - u_{n_k}\rangle \quad \forall x \in \mathcal{C}. \tag{4.5.22}$$

Now, choose a sequence $\{\delta_k\}$ of positive numbers such that $\delta_{k+1} \le \delta_k, \ \forall \ k \ge 1$ and $\delta_k \to 0$ as $k \to \infty$. Then, for each δ_k, we denote by N_k (which exists as a result of (4.5.22)) the smallest positive integer such that

$$\langle Au_{n_j}, x - u_{n_j}\rangle + \delta_k \ge 0 \ \ \forall j \ge N_k. \tag{4.5.23}$$

Since $\{\delta_k\}$ is decreasing, we have that $\{N_k\}$ is increasing. Furthermore, we set for each $k \geq 1$, $m_{N_k} = \frac{Au_{N_k}}{\|Au_{N_k}\|^2}$, provided $Au_{N_k} \neq 0$. Then it is easy to see that $\langle Au_{N_k}, m_{N_k}\rangle = 1$ for each $k \geq 1$. Thus, by (4.5.23), we have that

$$\langle Au_{N_k}, x + \delta_k m_{N_k} - u_{N_k}\rangle \geq 0,$$

which implies by the pseudo-monotonicity of A that

$$\langle A(x + \delta_k m_{N_k}), x + \delta_k m_{N_k} - u_{N_k}\rangle \geq 0. \tag{4.5.24}$$

Now, by the sequentially weakly continuity of A, we have that $\{Au_{n_k}\}$ converges weakly to Az. If $Az = 0$, then $z \in VI(\mathcal{C}, A)$. On the other hand, if we suppose that $Az \neq 0$, then by the weakly lower semicontinuity of $\|\cdot\|$, we obtain that

$$0 < \|Az\| \leq \liminf_{k\to\infty} \|Au_{n_k}\|.$$

Since $\{u_{N_k}\} \subset \{u_{n_k}\}$, we obtain that

$$\begin{aligned}
0 \leq \limsup_{k\to\infty} \|\delta_k m_{N_k}\| &= \limsup_{k\to\infty} \left(\frac{\delta_k}{\|Au_{n_k}\|}\right) \\
&\leq \frac{\limsup_{k\to\infty} \delta_k}{\liminf_{k\to\infty} \|Au_{n_k}\|} \\
&\leq \frac{0}{\|Az\|} = 0.
\end{aligned}$$

Therefore, $\lim_{k\to\infty} \|\delta_k m_{N_k}\| = 0$. Thus, letting $k \to \infty$ in (4.5.24) yields

$$\langle Ax, x - z\rangle \geq 0 \quad \forall x \in \mathcal{C}, \tag{4.5.25}$$

which implies by Lemma 2.5.2 that $z \in VI(C, A)$. This together with (4.5.20) gives that $z \in \Gamma$. □

We give our main result in the next theorem.

Theorem 4.5.6. *Let Assumption 4.5.1 hold. Then the sequence $\{x_n\}$ generated by Algorithm 4.5.2 converges strongly to $z^* := P_\Gamma g(z^*)$.*

Proof. Let $z^* = P_\Gamma g(z^*)$, then from (4.5.9) and Lemma 2.1.1, we obtain

$$\begin{aligned}
\|u_n - z^*\|^2 &\leq \|w_n - z^*\|^2 \\
&= \|x_n - z^* + \delta n(x_n - x_{n-1})\|^2 \\
&= \|x_n - z^*\|^2 + 2\delta_n\langle x_n - x_{n-1}, x_n - z^*\rangle + \delta_n\|x_n - x_{n-1}\|^2. \\
&= \|x_n - z^*\|^2 + \delta_n\left(-\|x_{n-1} - z^*\|^2 + \|x_n - z^*\|^2 + \|x_n - x_{n-1}\|^2\right) \\
&+ \delta_n\|x_n - x_{n-1}\|^2 \\
&= \|x_n - z^*\|^2 + \delta_n\left(\|x_n - z^*\|^2 - \|x_{n-1} - z^*\|^2\right) + 2\delta_n\|x_n - x_{n-1}\|^2. \qquad (4.5.26)
\end{aligned}$$

By (4.5.4), (4.5.26) and Lemma 2.1.1, we obtain that

$$\begin{aligned}
\|x_{n+1} - z^*\|^2 &= \|\gamma_n(g(x_n) - z^*) + (1-\gamma_n)(P_{\mathcal{C}_n}(u_n) - z^*)\|^2 \\
&\leq (1-\gamma_n)\|u_n - z^*\|^2 - (1-\gamma_n)\|P_{\mathcal{C}_n}(u_n) - u_n\|^2 \\
&+ 2\gamma_n\langle g(x_n) - z^*, x_{n+1} - z^*\rangle \\
&= (1-\gamma_n)\|x_n - z^*\|^2 - (1-\gamma_n)\|P_{\mathcal{C}_n}(u_n) - u_n\|^2 \qquad (4.5.27)\\
&+ \delta_n(1-\gamma_n)\left(\|x_n - z^*\|^2 - \|x_{n-1} - z^*\|^2\right) \\
&+ 2\delta_n(1-\gamma_n)\|x_n - x_{n-1}\|^2 + 2\gamma_n\langle g(x_n) - z^*, x_{n+1} - z^*\rangle. \\
&\leq (1-\gamma_n)\|x_n - z^*\|^2 + \gamma_n\left[2\langle g(x_n) - z^*, x_{n+1} - z^*\rangle\right] \\
&+ \gamma_n\left[\frac{\delta_n}{\gamma_n}(1-\gamma_n)\left[\|x_n - z^*\|^2 - \|x_{n-1} - z^*\|^2 + 2\|x_n - x_{n-1}\|^2\right]\right] \\
&= (1-\gamma_n)\|x_n - z^*\|^2 + \gamma_n\bar{v}_n \qquad (4.5.28)
\end{aligned}$$

where $\bar{v}_n := 2\langle g(x_n) - z^*, x_{n+1} - z^*\rangle + \frac{\delta_n}{\gamma_n}(1-\gamma_n)\left[\|x_n - z^*\|^2 - \|x_{n-1} - z^*\|^2 + \|x_n - x_{n+1}\|^2\right]$. To show that $\{x_n\}$ converges to z^*, we shall apply Lemma 2.5.36. That is, we show that $\limsup_{k\to\infty} \bar{v}_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - z^*\|\}$ of $\{\|x_n - z^*\|\}$ satisfying the condition

$$\liminf_{k\to\infty}\left(\|x_{n+1} - z^*\| - \|x_{n_k} - z^*\|\right) \geq 0. \qquad (4.5.29)$$

Now, suppose that $\{\|x_{n_k} - z^*\|\}$ is a subsequence of $\{\|x_n - z^*\|\}$ such that (4.5.29) holds. Then,

$$\begin{aligned}
&\liminf_{k\to\infty}\left(\|x_{n_k+1} - z^*\|^2 - \|x_{n_k} - z^*\|^2\right) \\
= \;&\liminf_{k\to\infty}\left[\left(\|x_{n_k+1} - z^*\| - \|x_{n_k} - z^*\|\right)\left(\|x_{n_k+1} - z^*\| + \|x_{n_k} - z^*\|\right)\right] \\
\geq \;&0. \qquad (4.5.30)
\end{aligned}$$

On the other hand, we obtain from (4.5.27) that

$$\begin{aligned}
(1-\gamma_{n_k})\|P_{\mathcal{C}_{n_k}}u_{n_k} - u_{n_k}\|^2 &\leq \|x_{n_k} - z^*\| - \|x_{n_k+1} - z^*\|^2 \\
&+ \delta_{n_k}(1-\gamma_{n_k})\left[\|x_{n_k} - z^*\|^2 - \|x_{n_k-1} - z^*\|^2\right] \\
&+ 2\delta_{n_k}\|x_{n_k} - x_{n_k-1}\|^2 + 2\gamma_{n_k}\langle g(x_{n_k}) - z^*, x_{n_k+1} - z^*\rangle \\
&= \|x_{n_k} - z^*\| - \|x_{n_k+1} - z^*\|^2 \\
&+ \delta_{n_k}(1-\gamma_{n_k})\left[(\|x_{n_k} - x_{n_k-1}\| + \|x_{n_k-1} - z^*\|)^2\right] \\
&- \delta_{n_k}(1-\gamma_{n_k})\left[\|x_{n_k-1} - z^*\|^2\right] \\
&+ 2\delta_{n_k}\|x_{n_k} - x_{n_k-1}\|^2 + 2\gamma_{n_k}\langle g(x_{n_k}) - z^*, x_{n_k+1} - z^*\rangle \\
&\leq \|x_{n_k} - z^*\| - \|x_{n_k+1} - z^*\|^2 \\
&+ \delta_{n_k}\left[\|x_{n_k} - x_{n_k-1}\|^2 + 2\|x_{n_k} - x_{n_k-1}\|\|x_{n_k-1} - z^*\|\right] \\
&+ 2\delta_{n_k}\|x_{n_k} - x_{n_k-1}\|^2 + 2\gamma_{n_k}\langle g(x_{n_k}) - z^*, x_{n_k+1} - z^*\rangle \\
&\to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

From (4.5.30) and Assumption 3.1, we have

$$\begin{aligned}\limsup_{k\to\infty}(1-\gamma_{n_k})\|P_{\mathcal{C}_{n_k}}u_{n_k}-u_{n_k}\|^2 &\le \limsup_{k\to\infty}\left[\|x_{n_k}-z^*\|-\|x_{n_k+1}-z^*\|^2\right]\\ &+\limsup_{k\to\infty}\left[\gamma_{n_k}\cdot\frac{\delta_{n_k}}{\gamma_{n_k}}\left[\|x_{n_k}-x_{n_k-1}\|^2\right]\right]\\ &+\limsup_{k\to\infty}\left[\gamma_{n_k}\cdot\frac{\delta_{n_k}}{\gamma_{n_k}}\left[2\|x_{n_k}-x_{n_{k-1}}\|\|x_{n_k-1}-z^*\|\right]\right]\\ &+\limsup_{n\to\infty}\left[2\gamma_{n_k}\cdot\frac{\delta_{n_k}}{\gamma_{n_k}}\|x_{n_k}-x_{n_k-1}\|^2\right]\\ &+\limsup_{n\to\infty}\left[2\gamma_{n_k}\langle g(x_{n_k})-z^*,x_{n_k+1}-z^*\rangle\right]\\ &=-\liminf_{k\to\infty}\left[\|x_{n_k+1}-z^*\|^2-\|x_{n_k}-z^*\|^2\right]\\ &\le 0.\end{aligned}$$

Hence,

$$\lim_{k\to\infty}\|P_{\mathcal{C}_{n_k}}u_{n_k}-u_{n_k}\|=0. \tag{4.5.31}$$

From (4.5.9) and (4.5.26), we obtain that

$$\begin{aligned}\beta_{n_k}(1-\beta_{n_k})\|w_{n_k}-S_{n_k}w_{n_k}\| &\le \|w_{n_k}-z^*\|^2-\|u_{n_k}-z^*\|^2\\ &\le \|x_{n_k}-z^*\|^2-\|u_{n_k}-z^*\|^2\\ &+\delta_{n_k}(\|x_{n_k}-z^*\|^2-\|x_{n_k-1}-z^*\|)\\ &+2\delta_{n_k}\|x_{n_k}-x_{n_k-1}\|^2.\end{aligned}\tag{4.5.32}$$

Again from (4.5.27), we obtain

$$-\|u_{n_k}-z^*\|^2\le-\|x_{n_k+1}-z^*\|^2+2\gamma_{n_k}\langle g(x_{n_k})-z^*,x_{n_k+1}-z^*\rangle,$$

which implies from (4.5.32) that

$$\begin{aligned}\beta_{n_k}(1-\beta_{n_k})\|w_{n_k}-S_{n_k}w_{n_k}\| &\le \|x_{n_k}-z^*\|^2-\|x_{n_k+1}-z^*\|^2\\ &+2\gamma_{n_k}\langle g(x_{n_k})-z^*,x_{n_k+1}-z^*\rangle\\ &+\delta_{n_k}\left(\|x_{n_k}-z^*\|^2-\|x_{n_k-1}-z^*\|^2\right)+2\delta_{n_k}\|x_{n_k}-x_{n_k-1}\|.\end{aligned}\tag{4.5.33}$$

By taking the lim sup as $k\to\infty$ in (4.5.33), we obtain that

$$\limsup_{k\to\infty}\beta_{n_k}(1-\beta_{n_k})\|w_{n_k}-S_{n_k}w_{n_k}\|=0.$$

Hence,

$$\lim_{k\to\infty}\|w_{n_k}-S_{n_k}w_{n_k}\|=0.$$

This implies that

$$\lim_{k\to\infty}\|w_{n_k}-u_{n_k}\|=\beta_{n_k}\lim_{n\to\infty}\|w_{n_k}-S_{n_k}w_{n_k}\|=0. \tag{4.5.34}$$

By similar method as in (4.5.18), we obtain

$$\lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = 0. \tag{4.5.35}$$

From (4.5.4), we obtain that

$$\lim_{k\to\infty} \|x_{n_k+1} - P_{C_{n_k}}(u_{n_k})\| = 0. \tag{4.5.36}$$

Using (4.5.36), (4.5.31), (4.5.34) and (4.5.35), we obtain that

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{4.5.37}$$

Since $\{x_{n_k}\}$ is bounded, we take a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to $z \in \mathcal{H}_1$ and

$$\begin{aligned}\limsup_{k\to\infty}\langle g(z^*) - z^*, x_{n_k} - z^*\rangle &= \lim_{j\to\infty}\langle g(z^*) - z^*, x_{n_{k_j}} - z^*\rangle \\ &= \langle g(z^*) - z^*, z - z^*\rangle.\end{aligned}$$

Also, we obtain from (4.5.31), (4.5.34) and Lemma 4.5.5 that $z \in \Gamma$. Since $z^* = P_\Gamma g(z^*)$, we obtain from the previous equality that

$$\limsup_{k\to\infty}\langle g(z^*) - z^*, x_{n_k} - z^*\rangle \leq 0, \tag{4.5.38}$$

which implies by (4.5.37) that

$$\limsup_{k\to\infty}\langle g(z^*) - z^*, x_{n_k+1} - z^*\rangle \leq 0. \tag{4.5.39}$$

Thus, $\limsup_{k\to\infty} \bar{v}_{n_k} \leq 0$. Hence from Lemma 2.5.36, we obtain that $\{x_n\}$ converges strongly to z^*. □

Remark 4.5.7. Observe that by setting $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $S = I$ and $T = I_{\mathcal{H}}$ (the identity operator on $\mathcal{H}$) in Theorem 4.5.6, we obtain as a corollary, an inertial extrapolation method for solving the classical VIP (1.2.1) when the cost operator is pseudo-monotone and Lipschitz continuous.

4.5.2 Numerical experiments

We give in this section, some numerical examples of Algorithm 4.5.2 in comparison with Algorithm 4.5.11 and related methods in the literature (Algorithm (4.5.41), Algorithm (4.5.42) and Algorithm (4.5.43)) for solving the class of split VIP (Problem (1.2.6)) considered in this work. All codes are written in Matlab 2016 (b) and performed on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30GHz and 8.00 Gb-RAM. In Tables 4.5.1-4.5.3, "Iter." means the number of iterations while "CPU" means the CPU time in seconds. In our computations, we choose $\gamma = \frac{1}{4} = \sigma$, $\gamma_n = \frac{1}{5n+1}$, $\delta_n = \bar{\delta}_n$, $\epsilon_n = \frac{\gamma_n}{n^{0.01}}$ and $\delta = 0.3$. Also, we define $g(x) = \frac{2x}{7}$ and $S(x) = -4x$. Then, g is a contraction with

coefficient $\rho = \frac{2}{7}$ and S is $\frac{3}{5}$-strictly pseudocontractive. Thus, we can choose $\beta = \frac{3}{5}$, so that $S_\beta(x) = -x$.
Furthermore, we define $\text{TOL}_n := \frac{1}{2}\left(||x_n - P_C(x_n - \lambda A x_n)||^2 + ||Sx_n - STx_n||^2\right)$, and use the stopping criterion $\text{TOL}_n < \varepsilon$ for the iterative processes, where ε is the predetermined error. Note that if $\text{TOL}_n = 0$, then x_n is a solution of problem (1.2.6).

Example 4.5.8. Let $\mathcal{H}_1 = \mathbb{R}^N$ and $\mathcal{H}_2 = \mathbb{R}^m$. Define $A : \mathbb{R}^N \to \mathbb{R}^N$ by $A(x) = Mx + q$, where the matrix M is formed as: $M = V \sum V'$, with $V = I - \frac{2vv'}{||v||^2}$ and $\sum = \text{diag}(\sigma_{11}, \sigma_{12}, \cdots, \sigma_{1N})$ been the Householder and the diagonal matrix, respectively, and

$$\sigma_{1j} = \cos\frac{j\pi}{N+1} + 1 + \frac{\cos\frac{\pi}{N+1} + 1 - \widehat{C}(\cos\frac{N\pi}{N+1} + 1)}{\widehat{C} - 1}, \qquad j = 1, 2, \cdots, N,$$

where $\widehat{C}$ is the present condition number of M ([121, Example 5.2]). In the numerical computation, we choose $\widehat{C} = 10^4$, $q = 0$ and uniformly take the vector $v \in \mathbb{R}^N$ in $(-1, 1)$. Thus, A is pseudo-monotone and uniformly continuous (see [121]). Furthermore, we generate the bounded linear operator $T \in \mathbb{R}^{M\times N}$ with independent Gaussian components distributed in the interval $(0, 1)$, and then normalize each column of T with the unit norm. We set $\mathcal{C} = \{x \in \mathbb{R}^N : ||x|| \leq 1\}$. The projection onto $\mathcal{C}$ is effectively computed in Matlab. We consider different scenarios of the problem's dimensions. That is, $N = 100, 300, 500, 1000$ and $m = N/2$, with starting points $x_1 = (1, 1, \ldots, 1)'$ and $x_0 = (0, 0, \ldots, 0)'$. For this example, we take $\varepsilon = 10^{-6}$ as the stopping criterion and obtain the numerical results reported in Table 4.5.1 and Figure 4.9.

Table 4.5.1. Numerical results for Example 4.5.8 with $\varepsilon = 10^{-6}$.

(N, m)		Algorithm 4.5.2	Algorithm 4.5.11	Algorithm (4.5.41)	Algorithm (4.5.42)	Algorithm (4.5.43)
$(100, 50)$	CPU	0.0545	0.1525	0.1435	0.1708	0.1762
	Iter.	412	785	633	1465	979
	CPU	0.0368	0.1510	0.1285	0.1523	0.1619
$(300, 150)$	Iter.	277	635	448	1465	865
	CPU	0.0314	0.1365	0.1280	0.1498	0.1505
$(500, 250)$	Iter.	283	823	501	1465	838
	CPU	0.0569	0.1414	0.1345	0.1747	0.1704
$(1000, 500)$	Iter.	473	795	578	1465	790

Example 4.5.9. Let $\mathcal{H}_1 = (l_2(\mathbb{R}), ||.||_{l_2}) = \mathcal{H}_2$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \ldots),\ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $||x||_{l_2} = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$, $\forall x \in l_2(\mathbb{R})$. Now, define the operator $T : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by $Tx = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$, $\forall x \in l_2(\mathbb{R})$. Then, T is a bounded linear operator on $l_2(\mathbb{R})$ with adjoint $T^*y = \left(y_2, \frac{y_3}{2}, \frac{y_4}{3}, \ldots\right)$, $\forall y \in l_2(\mathbb{R})$.

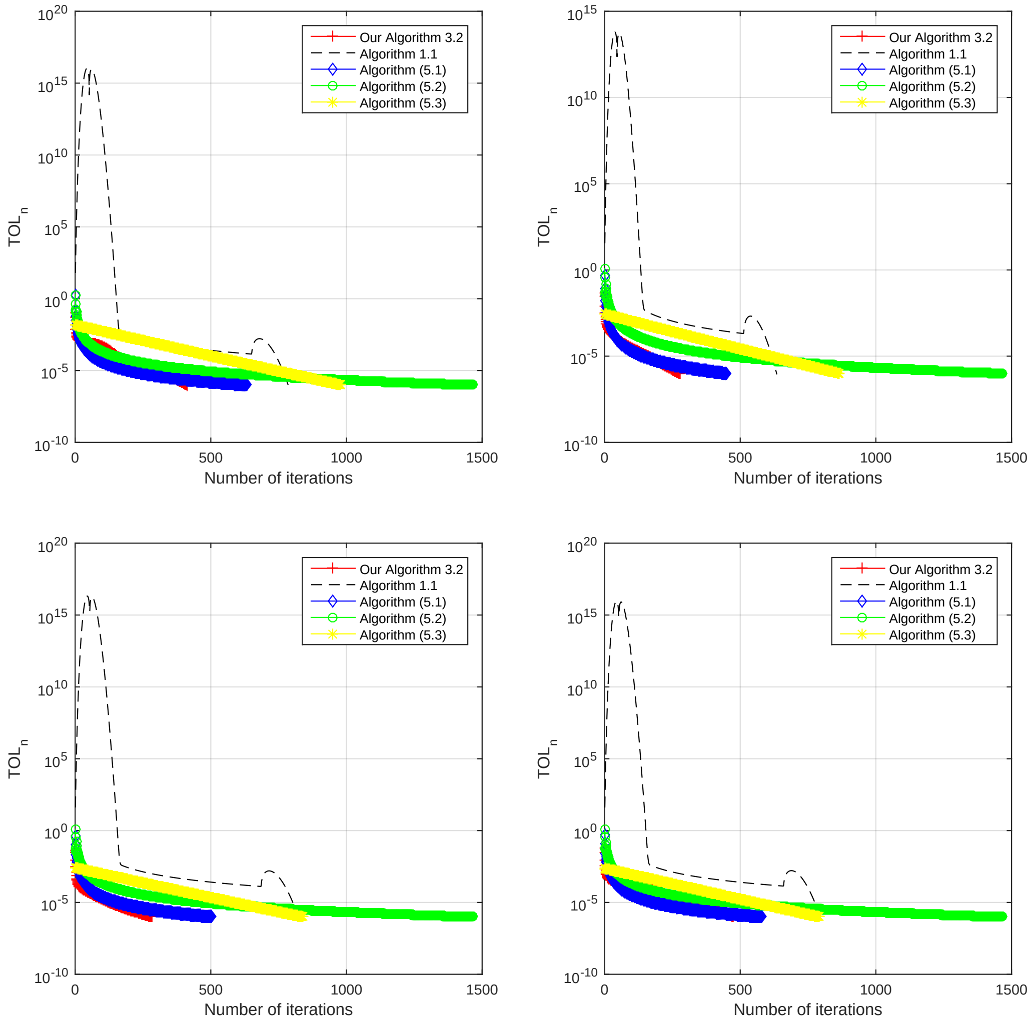

Figure 4.9: The behavior of TOL_n with $\varepsilon = 10^{-6}$ for Example 4.5.8: Top Left: $(N, m) = (100, 50)$; Top Right: $(N, m) = (300, 150)$; Bottom Left: $(N, m) = (500, 250)$; Bottom Right: $(N, m) = (1000, 500)$.

Let $\mathcal{C} = \{x \in l_2(\mathbb{R}) : |x_i| \leq \frac{1}{i},\ i = 1, 2, 3, \dots\}$. Then, we have explicit formula for $P_{\mathcal{C}}$. Now, define the operator $A : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by

$$Ax = \left(||x|| + \frac{1}{||x|| + \alpha} \right) \alpha,$$

for some $\alpha > 0$. Then, A is pseudomonotone on $l_2(\mathbb{R})$, uniformly continuous and sequentially weakly continuous on $\mathcal{C}$ but not Lipschitz continuous (see [248]). For this example, we take $\varepsilon = 10^{-8}$ as the stopping criterion and choose the starting points as follows:

Case 1: Take $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ and $x_0 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$.

Case 2: Take $x_1 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$ and $x_0 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.

Case 3: Take $x_1 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$ and $x_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$.

Case 4: Take $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$ and $x_0 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$.

The numerical results reported in Table 4.5.2 and Figure 4.10.

Table 4.5.2. Numerical results for Example 4.5.9 with $\varepsilon = 10^{-8}$.

Cases		Algorithm 4.5.2	Algorithm 4.5.11	Algorithm (4.5.41)	Algorithm (4.5.42)	Algorithm (4.5.43)
1	CPU	0.0136	0.1101	0.1110	0.1229	0.1334
	Iter.	37	44	58	145	118
2	CPU	0.0146	0.1103	0.1100	0.1238	0.1314
	Iter.	40	57	60	199	123
3	CPU	0.0190	0.1127	0.1212	0.1185	0.1283
	Iter.	38	50	58	148	118
4	CPU	0.0136	0.1183	0.1183	0.1253	0.1435
	Iter.	34	65	53	89	107

Example 4.5.10. Let $\mathcal{H}_1 = \mathcal{H}_2 = L_2([0, 2\pi])$ be endowed with inner product

$$\langle x, y \rangle = \int_0^{2\pi} x(t)y(t)dt \ \ \forall\ x, y \in L_2([0, 2\pi])$$

and norm

$$||x|| := \left(\int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} \ \ \forall\ x, y \in L_2([0, 2\pi]).$$

Let $\mathcal{C} = \{x \in L_2([0, 2\pi]) \ :\ ||x - e||_{L_2} \leq b\}$, where $e = 2 + \sin 2t$ and $b = 1$. Then, the metric projection $P_{\mathcal{C}}$ is defined as:

$$P_{\mathcal{C}}(x) = \begin{cases} x, & \text{if } \ x \in \mathcal{C}, \\ \frac{x-e}{||x-e||_{L_2}} b + e, & \text{otherwise.} \end{cases}$$

Now, define the operator $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ by

$$Ax(t) = L(x)D(x)(t), \ \ x \in L_2([0, 2\pi]), \ \ t \in [0, 2\pi],$$

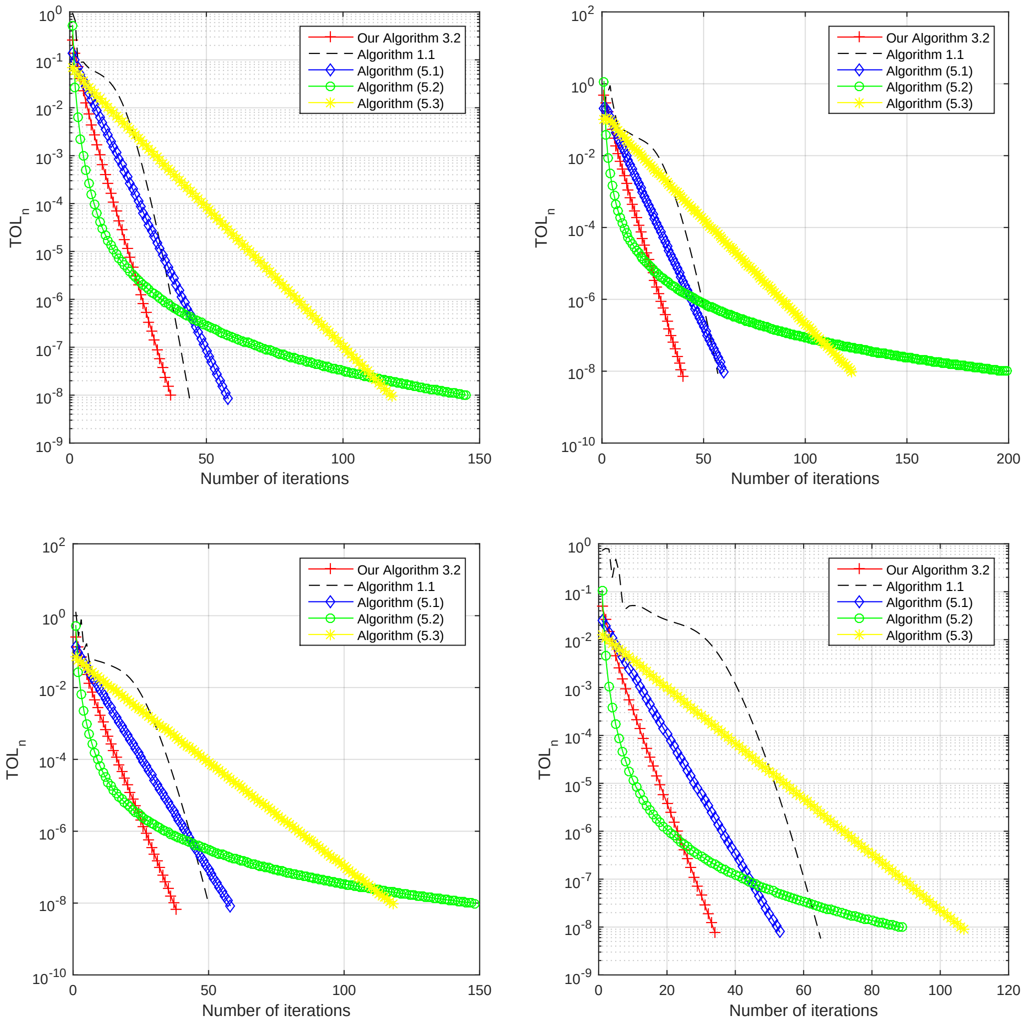

Figure 4.10: The behavior of TOL_n with $\varepsilon = 10^{-8}$ for Example 4.5.9: Top Left: **Case 1**; Top Right: **Case 2**; Bottom Left: **Case 3**; Bottom Right: **Case 4**.

where $L(x) := \frac{1}{1+||x||^2_{L_2}}$ and D is the Volterra integral mapping (which is monotone, bounded and linear, see [39]), defined by

$$Dx(t) = \int_0^t x(s)ds, \ x \in L_2([0, 2\pi]), \ t \in [0, 2\pi].$$

Then A is pseudo-monotone on $L_2([0, 2\pi])$. Let $T : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ be defined by

$$Tx(s) = \int_0^{2\pi} K(s,t)x(t)dt \ \forall x \in L_2([0, 2\pi]),$$

where K is a continuous real-valued function defined on $[0, 2\pi] \times [0, 2\pi]$. Then T is a bounded linear operator with adjoint

$$T^*x(s) \int_0^{2\pi} K(t,s)x(t)dt \ \forall x \in L_2([0, 2\pi]).$$

We take $\varepsilon = 10^{-10}$ as the stopping criterion and choose the starting points as follows:
Case 1: Take $x_1(t) = 1 + t^2$ and $x_0(t) = t + 5$.

Case 2: Take $x_1(t) = 2t + t^2$ and $x_0(t) = t + 1$.

Case 3: Take $x_1(t) = 0.7e^{-t} + 1$ and $x_0(t) = t + t^3$.

Case 4: Take $x_1(t) = \sin(t) + 1$ and $x_0(t) = e^t$.
The numerical results shown in Table 4.5.3 and Figure 4.11.

Table 4.5.3. Numerical results for Example 4.5.10 with $\varepsilon = 10^{-10}$.

Cases		Algorithm 4.5.2	Algorithm 4.5.11	Algorithm (4.5.41)	Algorithm (4.5.42)	Algorithm (4.5.43)
1	CPU	5.2616	12.2498	12.6245	13.4485	150.5827
	Iter.	56	95	76	68	158
2	CPU	5.8211	8.9063	8.5748	9.4896	124.5990
	Iter.	53	93	73	65	151
3	CPU	5.1099	11.1716	11.8042	14.2559	173.8901
	Iter.	55	95	76	68	157
4	CPU	6.5188	24.9020	29.0189	46.8546	167.9730
	Iter.	55	67	102	89	157

Appendix 4.5.11. *The Algorithm of [249]*

Initialization: *Given $\gamma > 0, l \in (0,1), \mu \in (0,1)$. Let $x_0 \in \mathcal{H}$ be arbitrary.*

Iterative Steps: *Calculate x_{n+1} as follows:*

Step1. *Compute*

$$y_n = P_{\mathcal{C}}(x_n - \lambda_n Ax_n),$$

where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda_n ||Ax_n - Ay_n|| \leq \mu ||x_n - y_n||. \tag{4.5.40}$$

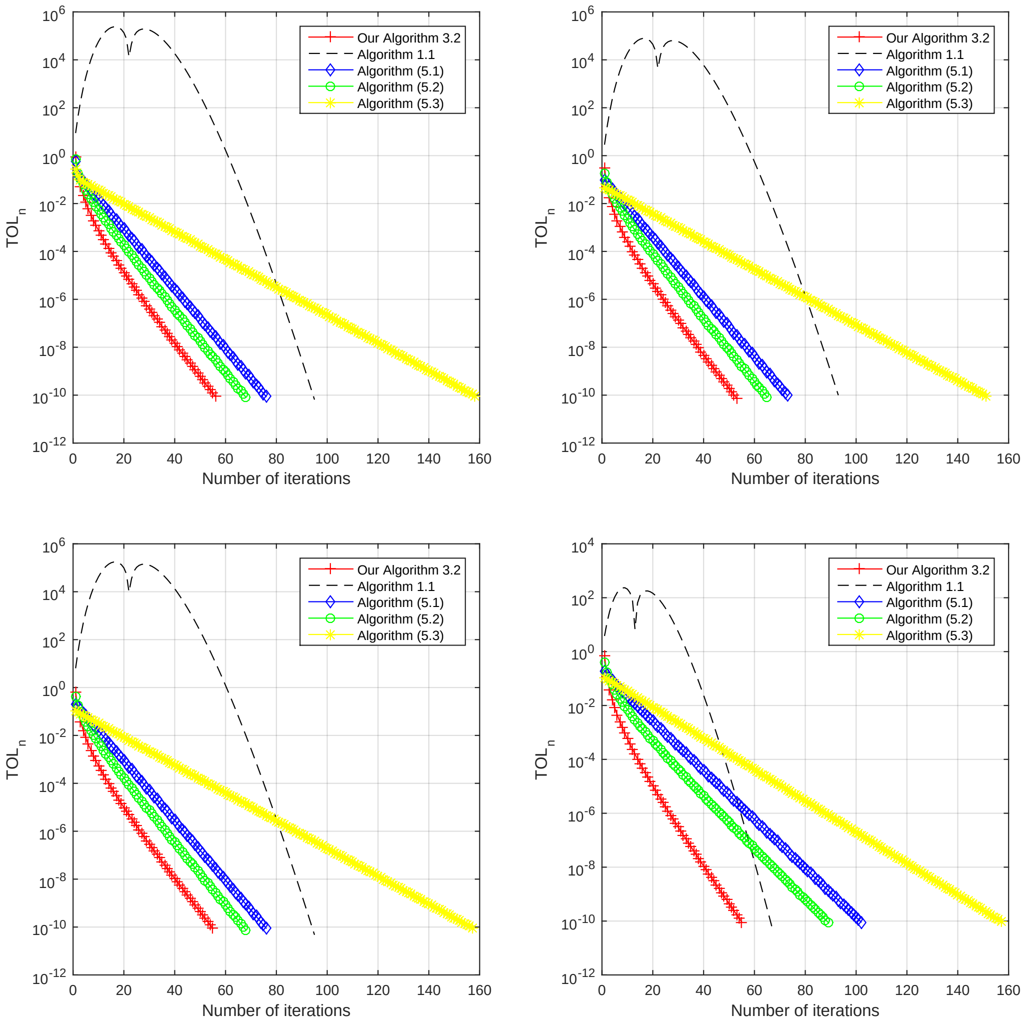

Figure 4.11: The behavior of TOL_n with $\varepsilon = 10^{-10}$ for Example 4.5.10: Top Left: **Case 1**; Top Right: **Case 2**; Bottom Left: **Case 3**; Bottom Right: **Case 4**.

If $x_n = y_n$, then stop and x_n is the solution of VIP. Otherwise,

Step 2: *Compute*

$$x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)z_n,$$

where $z_n = y_n - \lambda_n(Ay_n - Ax_n)$ and g is a contraction mapping on $\mathcal{H}$.

Appendix 4.5.12. *The Algorithm in [243].*

Let $x_1 \in \mathcal{C}$, define the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ by

$$\begin{cases} y_n = P_{\mathcal{C}}(x_n - \tau_n T^*(I-S)Tx_n), \\ t_n = P_{\mathcal{C}}(y_n - \lambda_n A(y_n)), \\ x_{n+1} = P_{\mathcal{C}}(y_n - \lambda_n A(t_n)), \quad n \geq 1, \end{cases} \tag{4.5.41}$$

where $\{\tau_n\} \subset [a,b]$ for some $a,b \in \left(0, \dfrac{1}{\|T\|^2}\right)$ and $\{\lambda_n\} \subset [c,d]$ for some $c,d \in \left(0, \dfrac{1}{L}\right)$, $S: \mathcal{H}_2 \to \mathcal{H}_2$ is a nonexpansive mapping, and $A: \mathcal{C} \to \mathcal{H}_1$ is a monotone and L-Lipschitz continuous operator.

Appendix 4.5.13. *The Algorithm in Tian and Jiang [244].*

Let $x_1 \in \mathcal{C}$, define the sequences $\{x_n\}$, $\{y_n\}$, $\{w_n\}$ and $\{t_n\}$ by

$$\begin{cases} y_n = P_{\mathcal{C}}(x_n - \tau_n T^*(I-S)Tx_n), \\ t_n = P_{\mathcal{C}}(y_n - \lambda_n A(y_n)), \\ w_n = P_{\mathcal{C}}(y_n - \lambda_n A(t_n)), \\ x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)w_n, \ n \geq 1, \end{cases} \tag{4.5.42}$$

where $\{\alpha_n\} \subset (0,1)$ with $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\tau_n\} \subset [a,b]$ for some $a,b \in \left(0, \dfrac{1}{\|T\|^2}\right)$ and $\{\lambda_n\} \subset [c,d]$ for some $c,d \in \left(0, \dfrac{1}{L}\right)$, $S: \mathcal{H}_2 \to \mathcal{H}_2$ is a nonexpansive mapping, $A: \mathcal{C} \to \mathcal{H}_1$ is a monotone and L-Lipschitz continuous operator and g is a contraction on $\mathcal{C}$.

Appendix 4.5.14. *The Algorithm in Chidume and Nnakwe [76].*

For $x_1 = x \in \mathcal{H}_1, \mathcal{C}_1 = \mathcal{H}_1$ and $W_1 = \mathcal{H}_1$, define the sequence $\{x_n\}$ by

$$\begin{cases} y_n^i = P_{\mathcal{C}_i}\Big(x_n - \tau T^*(I-S_i)Tx_n\Big), i = 1, \cdots, N \\ u_n^i = P_{\mathcal{C}_i}\Big(y_n^i - \lambda_n A_i(y_n^i)\Big), i = 1, \cdots, N \\ t_n^i = P_{\mathcal{C}_i}\Big(y_n^i - \lambda_n A_i(u_n^i)\Big), i = 1, \cdots, N \\ \mathcal{C}_n^i = \Big\{z \in \mathcal{H} : \|t_n^i - z\| \leq \|x_n - z\|\Big\} \\ W_n = \Big\{z \in \mathcal{H} : \langle z - x_n, x - x_n\rangle \leq 0\Big\} \\ x_{n+1} = P_{\mathcal{C}_n} \cap W_n x, \ \forall n \geq 1, \end{cases} \tag{4.5.43}$$

where $\mathcal{C}_n = \cap_{i=1}^{N}\mathcal{C}_n^i, \tau \in \left(0, \frac{1}{\|A\|^2}\right),\ \lambda_n \in \left(0, \frac{1}{L}\right), \mathcal{C}_i, i = 1, \cdots, N$ *are nonempty closed and convex subsets of* $\mathcal{H}_1$ *such that* $\mathcal{C} = \cap_{i=1}^{N}\mathcal{C}_i \neq \emptyset$, $T : \mathcal{H}_1 \to \mathcal{H}_2$ *is a bounded linear operator such that* $T \neq 0$ *and* T^* *is the adjoint of* T. $A_i : \mathcal{C}_i \to \mathcal{H}_1, i = 1, \cdots, N$ *is a finite family of monotone and L-Lipschitz mappings and* $S_i : \mathcal{H}_2 \to \mathcal{H}_2,\ i = 1, \cdots, N$ *is a finite family of nonexpansive mappings.*

CHAPTER 5

Results on Split Equalities Variational Inclusion and Split Equilibrium Problems

5.1 Introduction

In this chapter, we propose an inertial Tseng's extragradient algorithm for approximating a common solution of split equalities of VIP, EP and FPP of nonexpansive semigroups mapping. Furthermore, we propose and study an iterative algorithm for approximating the common solution of V_qIP and SEP in the framework of real Hilbert spaces and apply our result to study certain optimization problems. Finally, we present some numerical examples of our proposed methods in comparison with other methods in the literature to show the applicability of our proposed methods.

5.2 On split equalities of some nonlinear optimization problems

In this work, we introduce a new inertial Tseng's extragradient method with self-adaptive step sizes for approximating a common solution of split equalities of equilibrium problem (EP), non-Lipschitz pseudomonotone variational inequality problem (VIP) and fixed point problem (FPP) of nonexpansive semigroups in real Hilbert spaces. We prove that the sequence generated by our proposed method converges strongly to a common solution of the EP, pseudomonotone VIP and FPP of nonexpansive semigroups mapping without any linesearch procedure nor the sequential weak continuity condition often assumed by authors when solving non-Lipschitz VIPs. Finally, we provide some numerical experiments for the proposed method in comparison with related method in the literature. Our result improves, extends and generalizes several of the existing results in this direction.

The following assumptions are required in solving equilibrium problems.

Assumption 5.2.1. *[43] Let $F : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

1) $F(x, x) = 0, \quad \forall \quad x \in \mathcal{C};$

2) *F is monotone, i.e.,* $F(x, y) + F(y, x) \leq 0, \quad \forall x \in \mathcal{C};$

3) for each $x, y, z \in \mathcal{C}, \quad \limsup_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y);$

4) for each $x \in \mathcal{C}$, $y \to F(x, y)$ is convex and lower semi continuous.

Lemma 5.2.1. *[87] Let $F : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be a bifunction satisfying Assumption 5.2.1. For any $r > 0$ and $x \in \mathcal{H}$, define a mapping $T_r^F : \mathcal{H} \to \mathcal{C}$ as follows*

$$T_r^F(x) = \left\{ z \in \mathcal{C} : F(z, y) + \frac{1}{r}\Big\langle y - z, z - x \Big\rangle \geq 0, \ \forall y \in \mathcal{C} \right\}.$$

Then, we have the following

(1) T_r^F is nonempty and single valued;

(2) T_r^F is firmly nonexpansive, that is

$$\Big\langle T_r^F x - T_r^F y, \ x - y \Big\rangle \geq \|T_r^F x - T_r^F y\|^2;$$

equivalently

$$\|T_r^F x - T_r^F y\|^2 \leq \|x - y\|^2 - \|(I - T_r^F)x - (I - T_r^F)y\|^2, \quad \forall \ x, y \in \mathcal{H};$$

(3) $F(T_r^F) = EP(F)$ is closed and convex.

5.2.1 Proposed method

In this section, we present our proposed method and discuss its features. We begin with the following assumptions under which our strong convergence result is obtained.

Assumption 5.2.2. *Suppose that the following conditions hold:*

(a) The feasible sets $\mathcal{C}$ and $\mathcal{Q}$ are nonempty, closed and convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

(b) $A : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ are pseudomonotone and uniformly continuous.

(c) *The mapping* $A : \mathcal{H}_1 \to \mathcal{H}_1$ *and* $B : \mathcal{H}_2 \to \mathcal{H}_2$ *satisfies the following property: whenever* $\{x_n\} \subset \mathcal{C}$, $x_n \rightharpoonup x^*$, *one has* $\|Ax^*\| \leq \liminf_{n\to\infty} \|Ax_n\|$ *and whenever* $\{x_n\} \subset \mathcal{Q}$, $x_n \rightharpoonup x^*$, *one has* $\|Bx^*\| \leq \liminf_{n\to\infty} \|Bx_n\|$, *respectively.*

(d) $\mathcal{F}_1 : \mathcal{H}_1 \to \mathcal{H}_3$ *and* $\mathcal{F}_2 : \mathcal{H}_2 \to \mathcal{H}_3$ *are bounded linear operators.*

(e) $\Phi_1 : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$, $\Phi_2 : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ *are bifunctions satisfying Assumption* (5.2.1) *and* Φ_2 *is upper semi continuous in the first argument.*

(f) $\mathcal{T}_a = \{T_1(s) : 0 \leq s < \infty\}$ *and* $\mathcal{T}_b = \{T_2(u) : 0 \leq u < \infty\}$ *are one-parameter nonexpansive semigroups on* $\mathcal{H}_1$ *and* $\mathcal{H}_2$, *respectively.*

(g) *The solution set* $\Gamma = \Big\{x \in EP(\Phi_1) \cap VI(\mathcal{C}, A) \cap F(\mathcal{T}_a), \quad y \in EP(\Phi_2) \cap VI(\mathcal{Q}, B) \cap F(\mathcal{T}_b) : \mathcal{F}_1 x = \mathcal{F}_2 y\Big\} \neq \emptyset.$

(h) $\{\alpha_n\} \subset (0,1)$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$, $\lim_{n\to\infty} \alpha_n = 0$, $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$, $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.

(i) *Let* $\{\epsilon_n\}$ *and* $\{\zeta_n\}$ *be positive sequences such that* $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$ *and* $\lim_{n\to\infty} \frac{\zeta_n}{\alpha_n} = 0$, *respectively.*

(j) *Let* $\{\sigma_n\}$ *and* $\{\mu_n\}$ *be a nonnegative sequences such that* $\sum_{n=1}^{\infty} \sigma_n < +\infty$ *and* $\sum_{n=1}^{\infty} \mu_n < +\infty$, *respectively,* $\{t_{n,1}\}, \{t_{n,2}\} \subset (0, +\infty), \liminf r_{n,1} > 0, \liminf r_{n,2} > 0$

Algorithm 5.2.3.

Step 0: Choose sequences $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, $\{\theta_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ such that the conditions from Assumption 5.2.2 (h)-(i) hold. Select initial point $(x_0, y_0) \in \mathcal{H}_1 \times \mathcal{H}_2$, let $\eta \geq 0$, $a_i \in (0,1), i = 1,2$, $\lambda_1 > 0$, $\rho_1 > 0$, $\theta > 0$, $\tau > 0$ and set $n := 1$.
Step 1: Given the iterates x_{n-1}, y_{n-1} and x_n, y_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$ and τ_n such that $0 \leq \tau_n \leq \bar{\tau}_n$ where

$$\bar{\theta}_n := \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases} \tag{5.2.1}$$

Step 2: Compute

$$w_n = (1 - \alpha_n)\Big(x_n + \theta_n(x_n - x_{n-1})\Big).$$

Step 3: Compute

$$z_n = w_n - \eta_n \mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2 \varphi_n),$$
$$\phi_n = U_{r_{n,1}}^{\Phi_1} z_n,$$
$$u_n = P_{\mathcal{C}}(\phi_n - \lambda_n A\phi_n),$$

$$v_n = u_n - \lambda_n(Au_n - A\phi_n),$$
$$x_{n+1} = (1-\beta_n)v_n + \beta_n \frac{1}{t_{n,1}} \int_0^{t_{n,1}} T_1(s)v_n \, ds$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{a_1||u_n-\phi_n||}{||Au_n - A\phi_n||},\ \lambda_n + \sigma_n\right\}, & \text{if } Au_n \neq A\phi_n \\ \lambda_n + \sigma_n, & \text{otherwise.} \end{cases} \tag{5.2.2}$$

Step 4: Compute

$$\bar{\tau}_n := \begin{cases} \min\left\{\tau, \frac{\zeta_n}{\|y_n - y_{n-1}\|}\right\}, & \text{if } y_n \neq y_{n-1} \\ \tau, & \text{otherwise.} \end{cases} \tag{5.2.3}$$

Step 5 Compute

$$\varphi_n = (1-\alpha_n)\Big(y_n + \tau_n(y_n - y_{n-1})\Big),$$
$$k_n = \varphi_n + \eta_n \mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2 \varphi_n),$$

Step 6: Compute

$$\psi_n = U_{r_{n,2}}^{\Phi_2} k_n,$$
$$s_n = P_Q(\psi_n - \rho_n B\psi_n),$$
$$b_n = s_n - \rho_n(Bs_n - B\psi_n),$$
$$y_{n+1} = (1-\gamma_n)b_n + \gamma_n \frac{1}{t_{n,2}} \int_0^{t_{n,2}} T_2(u)b_n \, du$$

and

$$\rho_{n+1} = \begin{cases} \min\left\{\frac{a_2||s_n-\psi_n||}{||Bs_n - B\psi_n||},\ \rho_n + \mu_n\right\}, & \text{if } Bs_n \neq B\psi_n \\ \rho_n + \mu_n, & \text{otherwise,} \end{cases} \tag{5.2.4}$$

where the step size η_n is chosen such that for small enough $\epsilon > 0$,

$$\eta_n \in \left[\epsilon,\ \frac{2\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2}{\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2} - \epsilon\right],$$

if $\mathcal{F}_1 w_n \neq \mathcal{F}_2\varphi_n$; otherwise, $\eta_n = \eta$.

Set $n := n+1$ and go back to **Step 1**.

Remark 5.2.2. The step sizes generated in (5.2.2) and (5.2.4) are allowed to increase per iteration. This reduces their dependence on the initial step sizes. When n is large enough the step size may not increase. We assume that the Algorithm 5.2.3 does not terminate in finite number of iterations.

Remark 5.2.3. By conditions (h) and (i), from (5.2.1) we observe that

$$\lim_{n\to\infty} \theta_n||x_n - x_{n-1}|| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0. \tag{5.2.5}$$

Similarly, from (5.2.3) we have

$$\lim_{n\to\infty} \tau_n||y_n - y_{n-1}|| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\tau_n}{\alpha_n}||y_n - y_{n-1}|| = 0. \tag{5.2.6}$$

Remark 5.2.4. Since the limit

$$\lim_{n\to\infty} \left(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\right) = 1 - a_1^2 > 0, \tag{5.2.7}$$

there exists $n_{0_1} > 0$ such that for all $n > n_{0_1}$, we have $\left(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\right) > 0$.

Similarly, we have that

$$\lim_{n\to\infty} \left(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\right) = 1 - a_2^2 > 0, \tag{5.2.8}$$

and there exists $n_{0_2} > 0$ such that for all $n > n_{0_2}$, we have $\left(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\right) > 0$.

Now, we set $n_0 = \max\{n_{0_1}, n_{0_2}\}$.

Remark 5.2.5. From the definition of η_n, that is,

$$\eta_n \in \left[\epsilon, \quad \frac{2\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2}{\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2} - \epsilon\right]$$

we have

$$(\eta_n + \epsilon)\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2\Big] \le 2\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2.$$

Expanding the last inequality, we have

$$\begin{aligned}\eta_n \cdot \epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|)^2\Big] \le \eta_n\Big(&2\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2 \\ &- \eta_n\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \\ &\|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2\Big]\Big).\end{aligned} \tag{5.2.9}$$

5.2.2 Convergence analysis

Lemma 5.2.6. *Let $\{\lambda_n\}$ and $\{\rho_n\}$ be sequences generated by Algorithm 5.2.3. Then, we have $\lim_{n\to\infty} \lambda_n = \lambda$, where $\lambda \in \Big[min\{\frac{a_1}{K_1}, \lambda_1\}, \lambda_1 + b_1\Big], b_1 = \sum_{n=1}^{\infty} \sigma_n$, for some $K_1 > 0$ and $\lim_{n\to\infty} \rho_n = \rho$, where $\rho \in \Big[min\{\frac{a_2}{K_2}, \rho_1\}, \rho_1 + b_2\Big], b_2 = \sum_{n=1}^{\infty} \mu_n$, for some $K_2 > 0$.*

Proof. Since A is uniformly continuous, we obtain from (2.1.4) that for any given $\epsilon > 0$, there exists a constant $M < +\infty$ such that $\|Au_n - A\phi_n\| \le M\|u_n - \phi_n\| + \epsilon$. Thus, when $Au_n - A\phi_n \neq 0$ for all $n \ge 1$ we have

$$\frac{a_1\|u_n - \phi_n\|}{\|Au_n - A\phi_n\|} \ge \frac{a_1\|u_n - \phi_n\|}{M\|u_n - \phi_n\| + \epsilon} = \frac{a_1\|u_n - \phi_n\|}{(M + \epsilon_1)\|u_n - \phi_n\|} = \frac{a_1}{K_1},$$

where $\epsilon = \epsilon_1\|u_n - \phi_n\|$ for some $\epsilon_1 \in (0, 1)$ and $K_1 = M + \epsilon_1$. Hence, from the definition of λ_{n+1}, the sequence $\{\lambda_n\}$ is bounded below by $\min\{\frac{a_1}{K_1}, \lambda_1\}$ and above by $\lambda_1 + b_1$. By Lemma 2.5.34, it follows that $\lim_{n\to\infty} \lambda_n$ denoted by $\lambda = \lim_{n\to\infty} \lambda_n$ exists. Clearly, we have $\lambda \in \big[\min\{\frac{a_1}{K_1}, \lambda_1\}, \lambda_1 + b_1\big]$.

Similarly, we have $\lim_{n\to\infty} \rho_n = \rho$, and $\rho \in \Big[\min\{\frac{a_2}{K_2}, \rho_1\}, \rho_1 + b_2\Big]$.

□

Lemma 5.2.7. *Let $\{(x_n, y_n)\}$ be a sequence generated by Algorithm 5.2.3 under Assumption 5.2.2. Then*

$$\|z_n - x^*\|^2 + \|k_n - y^*\|^2 \le \|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2.$$

Proof. Let $(x^*, y^*) \in \Gamma$. Then, by applying Lemma 2.1.1, we have

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|w_n - \eta_n \mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n) - x^*\|^2 \\
&= \|w_n - x^*\|^2 + \eta_n^2\|\mathcal{F}_1^*(\mathcal{F}_1 w_n - F_2\varphi_n)\|^2 - 2\eta_n \langle w_n - x^*,\ \mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\rangle \\
&= \|w_n - x^*\|^2 + \eta_n^2\|\mathcal{F}_1^*(\mathcal{F}_1 w_n - F_2\varphi_n)\|^2 - 2\eta_n \langle \mathcal{F}_1 w_n - \mathcal{F}_1 x^*,\ \mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\rangle \\
&= \|w_n - x^*\|^2 + \eta_n^2\|\mathcal{F}_1^*(\mathcal{F}_1 w_n - F_2\varphi_n)\|^2 - \eta_n\|\mathcal{F}_1 w_n - \mathcal{F}_1 x^*\|^2 - \\
&\quad \eta_n\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2 + \eta_n\|\mathcal{F}_2\varphi_n - \mathcal{F}_1 x^*\|^2.
\end{aligned} \tag{5.2.10}$$

Similarly, we have

$$\begin{aligned}
\|k_n - y^*\|^2 &= \|\varphi_n + \eta_n \mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n) - y^*\|^2 \\
&= \|\varphi_n - y^*\|^2 + \eta_n^2\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - F_2\varphi_n)\|^2 - \eta_n\|\mathcal{F}_2\varphi_n - \mathcal{F}_2 y^*\|^2 \\
&\quad - \eta_n\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2 + \eta_n\|\mathcal{F}_1 w_n - \mathcal{F}_2 y^*\|^2
\end{aligned} \tag{5.2.11}$$

Adding (5.2.10) and (5.2.11), we have

$$\begin{aligned}
\|z_n - x^*\|^2 + \|k_n - y^*\|^2 &= \|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2 \\
&\quad + \eta_n^2\Big[\|\mathcal{F}_1^*(\mathcal{F}_1 w_n - F_2\varphi_n)\|^2 + \|\mathcal{F}_2^*(\mathcal{F}_1 w_n - F_2\varphi_n)\|^2\Big] \\
&\quad - \eta_n\Big[\|\mathcal{F}_1 w_n - \mathcal{F}_1 x^*\|^2 + \|\mathcal{F}_2\varphi_n - \mathcal{F}_2 y^*\|^2\Big] - 2\eta_n\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2 \\
&\quad + \eta_n\Big[\|\mathcal{F}_1 w_n - \mathcal{F}_2 y^*\|^2 + \|\mathcal{F}_2\varphi_n - \mathcal{F}_1 x^*\|^2\Big].
\end{aligned}$$

By (5.2.9) and the fact that $\mathcal{F}_1 x^* = \mathcal{F}_2 y^*$, we have

$$\begin{aligned}
\|z_n - x^*\|^2 + \|k_n - y^*\|^2 &= \|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2 \\
&\quad - \eta_n \Big[2\|\mathcal{F}_1 w_n - \mathcal{F}_2 \varphi_n\|^2 \\
&\quad - \eta_n \Big(\|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2 \varphi_n)\|^2 + \|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2 \varphi_n)\|^2 \Big) \Big] \\
&\leq \|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2 \\
&\quad - \eta_n \cdot \epsilon \Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2 \varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2 \varphi_n)\|^2 \Big] \\
&\leq \|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2, \qquad (5.2.12)
\end{aligned}$$

which is the desired result. □

Lemma 5.2.8. *Let $\{(x_n, y_n)\}$ be a sequence generated by Algorithm 5.2.3 under Assumption 5.2.2. Then*

$$\|v_n - x^*\|^2 \leq \|z_n - x^*\|^2 - \|z_n - \phi_n\|^2 - \Big(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\Big)\|u_n - \phi_n\|^2$$

and

$$\|b_n - y^*\|^2 \leq \|k_n - y^*\|^2 - \|k_n - \psi_n\|^2 - \Big(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\Big)\|s_n - \psi_n\|^2.$$

Proof. Let $(x^*, y^*) \in \Gamma$. Since $U_{r_{n,1}}^{\Phi_1}$ is firmly nonexpansive, it follows from Lemma 5.2.1 that

$$\|\phi_n - x^*\|^2 = \|U_{r_{n,1}}^{\Phi_1} z_n - x^*\| \leq \|z_n - x^*\|^2 - \|z_n - \phi_n\|^2. \qquad (5.2.13)$$

Similarly, we have

$$\|\psi_n - y^*\|^2 = \|U_{r_{n,2}}^{\Phi_2} k_n - y^*\| \leq \|k_n - y^*\|^2 - \|k_n - \psi_n\|^2. \qquad (5.2.14)$$

From (5.2.2), we obtain

$$\lambda_{n+1} = \min\Big\{ \frac{a_1\|u_n - \phi_n\|}{\|Au_n - A\phi_n\|}, \lambda_n + \sigma_n \Big\} \leq \frac{a_1\|u_n - \phi_n\|}{\|Au_n - A\phi_n\|},$$

which implies that

$$\|Au_n - A\phi_n\| \leq \frac{a_1}{\lambda_{n+1}}\|u_n - \phi_n\|, \quad \forall n \geq 1. \qquad (5.2.15)$$

Similarly, we have

$$\|Bs_n - B\psi_n\| \leq \frac{a_2}{\rho_{n+1}}\|s_n - \psi_n\|, \quad \forall n \geq 1. \qquad (5.2.16)$$

From the definition of v_n in **Step 3** and Lemma 2.1.1, we have

$$
\begin{aligned}
\|v_n - x^*\|^2 &\leq \|u_n - \lambda_n(Au_n - A\phi_n) - x^*\|^2 \\
&= \|u_n - x^*\|^2 + \lambda_n^2\|Au_n - A\phi_n\|^2 - 2\lambda_n\langle Au_n - A\phi_n, u_n - x^*\rangle \\
&= \|\phi_n - x^*\|^2 + \|u_n - \phi_n\|^2 + 2\langle u_n - \phi_n, \phi_n - x^*\rangle + \lambda_n^2\|Au_n - A\phi_n\|^2 \\
&- 2\lambda_n\langle Au_n - A\phi_n, u_n - x^*\rangle \\
&= \|\phi_n - x^*\|^2 + \|u_n - \phi_n\|^2 - 2\langle u_n - \phi_n, u_n - \phi_n\rangle + 2\langle u_n - \phi_n, u_n - x^*\rangle \\
&- 2\lambda_n\langle Au_n - A\phi_n, u_n - x^*\rangle + \lambda_n^2\|Au_n - A\phi_n\|^2 \\
&= \|\phi_n - x^*\|^2 - \|u_n - \phi_n\|^2 + 2\langle u_n - \phi_n, u_n - x^*\rangle \qquad (5.2.17) \\
&- 2\lambda_n\langle Au_n - A\phi_n, u_n - x^*\rangle + \lambda_n^2\|Au_n - A\phi_n\|^2.
\end{aligned}
$$

Since $u_n = P_{\mathcal{C}}(\phi_n - \lambda_n A\phi_n)$ and $x^* \in \mathcal{C}$, we obtain from the characteristic property of $P_{\mathcal{C}}$ that

$$\langle u_n - \phi_n + \lambda_n A\phi_n, u_n - x^*\rangle \leq 0.$$

This implies that

$$\langle u_n - \phi_n, u_n - x^*\rangle \leq -\lambda_n\langle A\phi_n, u_n - x^*\rangle. \tag{5.2.18}$$

Also since $u_n \in \mathcal{C}$ and $x^* \in \Gamma$ we have

$$\langle Au_n, \quad u_n - x^*\rangle \geq 0, \quad \forall \quad n \geq 0. \tag{5.2.19}$$

Applying (5.2.13), (5.2.15), (5.2.18) and (5.2.19) in (5.2.17), we obtain

$$
\begin{aligned}
\|v_n - x^*\|^2 &\leq \|\phi_n - x^*\|^2 - \|u_n - \phi_n\|^2 - 2\lambda_n\langle A\phi_n, u_n - x^*\rangle - 2\lambda_n\langle Au_n - A\phi_n, u_n - x^*\rangle \\
&+ \lambda_n^2\|Au_n - A\phi_n\|^2 \\
&= \|\phi_n - x^*\|^2 - \|u_n - \phi_n\|^2 - 2\lambda_n\langle Au_n, u_n - x^*\rangle + \lambda_n^2\|Au_n - A\phi_n\|^2 \\
&\leq \|\phi_n - x^*\|^2 - \|u_n - \phi_n\|^2 + \lambda_n^2\frac{a_1^2}{\lambda_{n+1}^2}\|u_n - \phi_n\|^2 \\
&= \|z_n - x^*\|^2 - \|z_n - \phi_n\|^2 - \left(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\right)\|u_n - \phi_n\|^2. \qquad (5.2.20)
\end{aligned}
$$

Following the same line of argument, we have

$$\|b_n - y^*\|^2 \leq \|k_n - y^*\|^2 - \|k_n - \psi_n\|^2 - \left(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\right)\|s_n - \psi_n\|^2, \tag{5.2.21}$$

which completes the proof.

□

Lemma 5.2.9. *Let* $\{(x_n, y_n)\}$ *be a sequence generated by Algorithm 5.2.3 satisfying Assumption 5.2.2. Then* $\{(x_n, y_n)\}$ *is bounded.*

Proof. Let $x^* \in \Gamma$. From the definition of w_n and Lemma 2.1.1, we have

$$\begin{aligned}
\|w_n - x^*\| &= \|(1-\alpha_n)(x_n + \theta_n(x_n - x_{n-1})) - x^*\| \\
&= \|(1-\alpha_n)(x_n - x^*) + (1-\alpha_n)\theta_n(x_n - x_{n-1}) - \alpha_n x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + (1-\alpha_n)\theta_n\|x_n - x_{n-1}\| + \alpha_n\|x^*\| \\
&= (1-\alpha_n)\|x_n - x^*\| + \alpha_n\Big[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|x^*\|\Big]. \qquad (5.2.22)
\end{aligned}$$

By (5.2.5), we have

$$\lim_{n\to\infty}\Big[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|x^*\|\Big] = \|x^*\|.$$

Thus, there exists a constant $M_1 > 0$ such that $(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|x^*\| \leq M_1$ for all $n \in \mathbb{N}$. Thus, from (5.2.22) it follows that

$$\|w_n - x^*\| \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n M_1.$$

Consequently, we have

$$\|w_n - x^*\|^2 \leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)M_1\|x_n - x^*\| + \alpha_n^2 M_1^2. \qquad (5.2.23)$$

Following similar procedure, we have

$$\|\varphi_n - y^*\|^2 \leq (1-\alpha_n)^2\|y_n - y^*\|^2 + 2\alpha_n(1-\alpha_n)M_2\|y_n - y^*\| + \alpha_n^2 M_2^2. \qquad (5.2.24)$$

Adding (5.2.23) and (5.2.24), we obtain

$$\begin{aligned}
\|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2 &\leq (1-\alpha_n)^2\Big[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\Big] \\
&\quad + 2\alpha_n(1-\alpha_n)\Big(M_1\|x_n - x^*\| + M_2\|y_n - y^*\|\Big) + \alpha_n^2(M_1^2 + M_2^2) \\
&\leq (1-\alpha_n)\Big[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\Big] \\
&\quad + 2\alpha_n\Big(M_1\|x_n - x^*\| + M_2\|y_n - y^*\|\Big) + \alpha_n(M_1^2 + M_2^2) \\
&= (1-\alpha_n)\Big[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\Big] + \alpha_n c_n, \qquad (5.2.25)
\end{aligned}$$

where $c_n = 2\Big(M_1\|x_n - x^*\| + M_2\|y_n - y^*\|\Big) + M_1^2 + M_2^2$.

From **STEP 3**, and by applying Lemma 2.1.1, (5.2.20) together with Remark 5.2.7, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \Big\|(1-\beta_n)v_n+\beta_n\frac{1}{t_{n,1}}\int_0^{t_{n,1}}T_1(s)v_nds-x^*\Big\|^2\\
&= \Big\|(1-\beta_n)(v_n-x^*)+\beta_n\Big(\frac{1}{t_{n,1}}\int_0^{t_{n,1}}T_1(s)v_nds-x^*\Big)\Big\|^2\\
&= (1-\beta_n)\|v_n-x^*\|^2+\beta_n\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}}T_1(s)v_nds-x^*\Big\|^2\\
&- \beta_n(1-\beta_n)\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}}T_1(s)v_nds-v_n\Big\|^2\\
&= (1-\beta_n)\|v_n-x^*\|^2+\beta_n\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}}T_1(s)v_nds-\frac{1}{t_{n,1}}\int_0^{t_{n,1}}T_1(s)x^*ds\Big\|^2\\
&- \beta_n(1-\beta_n)\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}}T_1(s)v_nds-v_n\Big\|^2\\
&\le (1-\beta_n)\|v_n-x^*\|^2+\beta_n\|v_n-x^*\|^2\\
&- \beta_n(1-\beta_n)\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}}T_1(s)v_nds-v_n\Big\|^2\\
&= \|v_n-x^*\|^2-\beta_n(1-\beta_n)\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}}T_1(s)v_nds-v_n\Big\|^2\\
&\le \|z_n-x^*\|^2-\|z_n-\phi_n\|^2-\Big(1-\frac{\lambda_n^2a_1^2}{\lambda_{n+1}^2}\Big)\|u_n-\phi_n\|^2 \qquad (5.2.26)\\
&- \beta_n(1-\beta_n)\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}}T_1(s)v_nds-v_n\Big\|^2\\
&\le \|z_n-x^*\|^2. \qquad (5.2.27)
\end{aligned}$$

Similarly, from **STEP 5**, and by applying Lemma 2.1.1, (5.2.21) together with Remark 5.2.7, we have

$$\begin{aligned}
\|y_{n+1}-y^*\|^2 &\le \|k_n-y^*\|^2-\|k_n-\psi_n\|^2-\Big(1-\frac{\rho_n^2a_2^2}{\rho_{n+1}^2}\Big)\|s_n-\psi_n\|^2 \qquad (5.2.28)\\
&- \gamma_n(1-\gamma_n)\Big\|\frac{1}{t_{n,2}}\int_0^{t_{n,2}}T_2(u)b_ndu-b_n\Big\|^2\\
&\le \|k_n-y^*\|^2. \qquad (5.2.29)
\end{aligned}$$

From (5.2.12), (5.2.25), (5.2.27) and (5.2.29), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 &\le \|z_n-x^*\|^2+\|k_n-y^*\|^2\\
&\le \|w_n-x^*\|^2+\|\varphi_n-y^*\|^2\\
&\le (1-\alpha_n)\Big[\|x_n-x^*\|^2+\|y_n-y^*\|^2\Big]+\alpha_nc_n\\
&\le \max\ \{\|x_n-x^*\|^2+\|y_n-y^*\|^2,\ \ c_n\}\\
&\ \vdots\\
&\le \max\ \{\|x_{n_0}-x^*\|^2+\|y_{n_0}-y^*\|^2,\ \ c_{n_0}\}
\end{aligned}$$

Thus, $\{(x_n, y_n)\}$ is bounded. Consequently, $\{z_n\}, \{v_n\}, \{k_n\}$ and $\{b_n\}$ are also bounded. □

Lemma 5.2.10. *Let $\{(x_n, y_n)\}$ be a sequence generated by Algorithm 5.2.3 under Assumption 5.2.2. Then,*

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \leq (1-\alpha_n)\Big[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\Big] + \alpha_n d_n \\
- \eta_n \cdot \epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2\Big] \\
- \|z_n - \phi_n\|^2 - \|k_n - \psi_n\|^2 - \Big(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\Big)\|u_n - \phi_n\|^2 \\
- \Big(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\Big)\|s_n - \psi_n\|^2 \\
- \beta_n(1-\beta_n)\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - v_n\|^2 \\
- \gamma_n(1-\gamma_n)\|\frac{1}{t_{n,2}}\int_0^{t_{n,2}} T_2(u)b_n du - b_n\|^2.
\end{aligned}$$

Proof. Let $(x^*, y^*) \in \Gamma$. From Lemma 2.1.1 and the definition of w_n, we have

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) + (1-\alpha_n)\theta_n(x_n - x_{n-1}) - \alpha_n x^*\|^2 \\
&\leq \|(1-\alpha_n)(x_n - x^*) + (1-\alpha_n)\theta_n(x_n - x_{n-1})\|^2 + 2\alpha_n\langle -x^*, \quad w_n - x^*\rangle \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2(1-\alpha_n)\theta_n\|x_n - x^*\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad + 2\alpha_n\langle -x^*, \quad w_n - x_{n+1}\rangle + 2\alpha_n\langle -x^*, \quad x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 \qquad (5.2.30)\\
&\quad + \alpha_n\Big[2(1-\alpha_n)\|x_n - x^*\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\| \cdot \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&\quad + 2\|x^*\|\|w_n - x_{n+1}\| + 2\langle x^*, x^* - x_{n+1}\rangle\Big].
\end{aligned}$$

Following the same line of argument, we have

$$\begin{aligned}
\|\varphi_n - y^*\|^2 &\leq (1-\alpha_n)\|y_n - y^*\|^2 \qquad (5.2.31)\\
&\quad + \alpha_n\Big[2(1-\alpha_n)\|y_n - y^*\|\frac{\tau_n}{\alpha_n}\|y_n - y_{n-1}\| + \tau_n\|y_n - y_{n-1}\| \cdot \frac{\tau_n}{\alpha_n}\|y_n - y_{n-1}\| \\
&\quad + 2\|y^*\|\|\varphi_n - y_{n+1}\| + 2\langle y^*, y^* - y_{n+1}\rangle\Big].
\end{aligned}$$

Adding (5.2.30) and (5.2.31) we have

$$\|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2 \leq (1-\alpha_n)\Big[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\Big] + \alpha_n d_n \qquad (5.2.32)$$

where $d_n = \Big[2(1-\alpha_n)\|x_n - x^*\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\| \cdot \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + 2\|x^*\|\|w_n - x_{n+1}\| + 2\langle x^*, x^* - x_{n+1}\rangle\Big] + \Big[2(1-\alpha_n)\|y_n - y^*\|\frac{\tau_n}{\alpha_n}\|y_n - y_{n-1}\| + \tau_n\|y_n - y_{n-1}\| \cdot \frac{\tau_n}{\alpha_n}\|y_n - y_{n-1}\| + 2\|y^*\|\|\varphi_n - y_{n+1}\| + 2\langle y^*, y^* - y_{n+1}\rangle\Big]$.

From (5.2.12), (5.2.26), (5.2.28) and (5.2.32), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 &\leq \|z_n - x^*\|^2 + \|k_n - y^*\|^2 - \|z_n - \phi_n\|^2 - \|k_n - \psi_n\|^2 \\
&- \Big(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\Big)\|u_n - \phi_n\|^2 - \Big(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\Big)\|s_n - \psi_n\|^2 \\
&- \beta_n(1-\beta_n)\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - v_n\|^2 \\
&- \gamma_n(1-\gamma_n)\|\frac{1}{t_{n,2}}\int_0^{t_{n,2}} T_2(u)b_n du - b_n\|^2 \\
&\leq \|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2 \\
&- \eta_n \cdot \epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2\Big] \\
&- \|z_n - \phi_n\|^2 - \|k_n - \psi_n\|^2 - \Big(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\Big)\|u_n - \phi_n\|^2 \\
&- \Big(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\Big)\|s_n - \psi_n\|^2 \\
&- \beta_n(1-\beta_n)\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - v_n\|^2 \\
&- \gamma_n(1-\gamma_n)\|\frac{1}{t_{n,2}}\int_0^{t_{n,2}} T_2(u)b_n du - b_n\|^2 \\
&\leq (1-\alpha_n)\Big[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\Big] + \alpha_n d_n \\
&- \eta_n \cdot \epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2\Big] \\
&- \|z_n - \phi_n\|^2 - \|k_n - \psi_n\|^2 - \Big(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\Big)\|u_n - \phi_n\|^2 \\
&- \Big(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\Big)\|s_n - \psi_n\|^2 \\
&- \beta_n(1-\beta_n)\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - v_n\|^2 \\
&- \gamma_n(1-\gamma_n)\|\frac{1}{t_{n,2}}\int_0^{t_{n,2}} T_2(u)b_n du - b_n\|^2,
\end{aligned}$$

which is the required result.

□

Theorem 5.2.11. *Let $\{(x_n, y_n)\}$ be a sequence generated by Algorithm 5.2.3 such that Assumption 5.2.2 holds. Then, the sequence $\{(x_n, y_n)\}$ converges strongly to $(\hat{x}, \hat{y}) \in \Gamma$, where $\hat{x} = P_D(0)$ and $\hat{y} = P_E(0), D := EP(\Phi_1) \cap VI(\mathcal{C}, A) \cap F(\mathcal{T}_a)$ and $E := EP(\Phi_2) \cap VI(\mathcal{Q}, B) \cap F(\mathcal{T}_b)$.*

Proof. Let $(\hat{x}, \hat{y}) \in \Gamma$, where $\hat{x} = P_D(0)$ and $\hat{y} = P_E(0)$. Then, it follows from Lemma

5.2.10 that

$$\|x_{n+1}-\hat{x}\|^2+\|y_{n+1}-\hat{y}\|^2 \le (1-\alpha_n)\Big[\|x_n-\hat{x}\|^2+\|y_n-\hat{y}\|^2\Big]+\alpha_n\hat{d}_n \tag{5.2.33}$$

where $\hat{d}_n = \Big[2(1-\alpha_n)\|x_n-\hat{x}\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\theta_n\|x_n-x_{n-1}\|\cdot\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+2\|\hat{x}\|\|w_n-x_{n+1}\|+2\langle\hat{x},\hat{x}-x_{n+1}\rangle\Big]+\Big[2(1-\alpha_n)\|y_n-\hat{y}\|\frac{\tau_n}{\alpha_n}\|y_n-y_{n-1}\|+\tau_n\|y_n-y_{n-1}\|\cdot\frac{\tau_n}{\alpha_n}\|y_n-y_{n-1}\|+2\|\hat{y}\|\|\varphi_n-y_{n+1}\|+2\langle\hat{y},\hat{y}-y_{n+1}\rangle\Big]$. Now, we claim that the sequence $\{\|x_n-\hat{x}\|+\|y_n-\hat{y}\|\}$ converges to zero. To show this, by Lemma 2.5.36 it suffices to show that $\limsup_{k\to\infty}\hat{d}_{n_k}\le 0$ for every subsequence $\{\|x_{n_k}-\hat{x}\|+\|y_{n_k}-\hat{y}\|\}$ of $\{\|x_n-\hat{x}\|+\|y_n-\hat{y}\|\}$ satisfying

$$\liminf_{k\to\infty}\big((\|x_{n_{k+1}}-\hat{x}\|+\|y_{n_{k+1}}-\hat{y}\|)-(\|x_{n_k}-\hat{x}\|+\|y_{n_k}-\hat{y}\|)\big)\ge 0. \tag{5.2.34}$$

Suppose that $\{\|x_{n_k}-\hat{x}\|+\|y_{n_k}-\hat{y}\|\}$ is a subsequence of $\{\|x_n-\hat{x}\|+\|y_n-\hat{y}\|\}$ such that (5.2.34) holds. Again, from Lemma 5.2.10, we obtain

$$\begin{aligned}
&\eta_{n_k}\cdot\epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|^2+\|\mathcal{F}_1^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|^2\Big]+\|z_{n_k}-\phi_{n_k}\|^2+\|k_{n_k}-\psi_{n_k}\|^2\\
&+\Big(1-\frac{\lambda_{n_k}^2 a_1^2}{\lambda_{n_{k+1}}^2}\Big)\|u_{n_k}-\phi_{n_k}\|^2+\Big(1-\frac{\rho_{n_k}^2 a_2^2}{\rho_{n_{k+1}}^2}\Big)\|s_{n_k}-\psi_{n_k}\|^2\\
&+\beta_{n_k}(1-\beta_{n_k})\|\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T_1(s)v_{n_k}ds-v_{n_k}\|^2+\gamma_{n_k}(1-\gamma_{n_k})\|\frac{1}{t_{n_{k,2}}}\int_0^{t_{n_{k,2}}}T_2(u)b_{n_k}du-b_{n_k}\|^2\\
&\le(1-\alpha_{n_k})\Big[\|x_{n_k}-\hat{x}\|^2+\|y_{n_k}-\hat{y}\|^2\Big]-\Big[\|x_{n_{k+1}}-\hat{x}\|^2+\|y_{n_{k+1}}-\hat{y}\|^2\Big]+\alpha_{n_k}\hat{d}_{n_k}.
\end{aligned}$$

From (5.2.34) and the condition on α_{n_k} we have

$$\begin{aligned}
\lim_{k\to\infty}\Big(&\eta_{n_k}\cdot\epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|^2+\|\mathcal{F}_1^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|^2\Big]+\|z_{n_k}-\phi_{n_k}\|^2+\|k_{n_k}-\psi_{n_k}\|^2\\
&+\Big(1-\frac{\lambda_{n_k}^2 a_1^2}{\lambda_{n_{k+1}}^2}\Big)\|u_{n_k}-\phi_{n_k}\|^2+\Big(1-\frac{\rho_{n_k}^2 a_2^2}{\rho_{n_{k+1}}^2}\Big)\|s_{n_k}-\psi_{n_k}\|^2\\
&+\beta_{n_k}(1-\beta_{n_k})\|\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T_1(s)v_{n_k}ds-v_{n_k}\|^2\\
&+\gamma_{n_k}(1-\gamma_{n_k})\|\frac{1}{t_{n_{k,2}}}\int_0^{t_{n_{k,2}}}T_2(u)b_{n_k}du-b_{n_k}\|^2\Big)=0.
\end{aligned}$$

From (5.2.7), (5.2.8) and the conditions on the control parameters, we have

$$\lim_{k\to\infty}\|z_{n_k}-\phi_{n_k}\|=0,\ \lim_{k\to\infty}\|k_{n_k}-\psi_{n_k}\|=0,\ \lim_{k\to\infty}\|u_{n_k}-\phi_{n_k}\|=0\ \lim_{k\to\infty}\|s_{n_k}-\psi_{n_k}\|=0 \tag{5.2.35}$$

$$\lim_{k\to\infty}\Big\|\frac{1}{t_{n_{k,2}}}\int_0^{t_{n_{k,2}}}T_2(u)b_{n_k}du-b_{n_k}\Big\|=0,\ \lim_{k\to\infty}\Big\|\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T_1(s)v_{n_k}ds-v_{n_k}\Big\|=0. \tag{5.2.36}$$

Also, we have

$$\lim_{k\to\infty}\left[\|\mathcal{F}_2^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|^2+\|\mathcal{F}_1^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|^2\right]=0$$

which implies that

$$\lim_{k\to\infty}\|\mathcal{F}_2^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|=0,\ \lim_{k\to\infty}\|\mathcal{F}_1^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|=0,\ \lim_{k\to\infty}\|\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k}\|=0.$$

From the definition of z_{n_k}, k_{n_k} and the previous inequality we have

$$\begin{aligned}\|z_{n_k}-w_{n_k}\|&=\eta_{n_k}\|\mathcal{F}_1^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|\to 0,\ \text{as}\ \ k\to\infty.\\ \|k_{n_k}-\varphi_{n_k}\|&=\eta_{n_k}\|\mathcal{F}_2^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|\to 0,\ \text{as}\ \ k\to\infty.\end{aligned}$$

Also, from the definition of v_{n_k}, b_{n_k} and (5.2.35), we have

$$\begin{aligned}\|v_{n_k}-u_{n_k}\|&=\lambda_{n_k}\|Au_{n_k}-A\phi_{n_k}\|\le\frac{\lambda_{n_k}a_1}{\lambda_{n_{k+1}}}\|u_{n_k}-\phi_{n_k}\|\to 0,\ \text{as}\ k\to\infty.\\ \|b_{n_k}-s_{n_k}\|&=\rho_{n_k}\|Bs_{n_k}-B\psi_{n_k}\|\le\frac{\rho_{n_k}a_2}{\rho_{n_{k+1}}}\|s_{n_k}-\psi_{n_k}\|\to 0,\ \text{as}\ k\to\infty.\end{aligned}$$

From (5.2.36) and Lemma 2.1.12 we have

$$\begin{aligned}\|v_{n_k}-T_1(v)v_{n_k}\|\le&\left\|v_{n_k}-\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T_1(s)v_{n_k}ds\right\|\\ &+\left\|\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T_1(s)v_{n_k}ds-T_1(v)\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T_1(s)v_{n_k}ds\right\|\\ &+\left\|T_1(v)\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T_1(s)v_{n_k}ds-T_1(v)v_{n_k}\right\|\to 0,\ \text{as}\ k\to\infty.\end{aligned}\tag{5.2.37}$$

Similarly, we have

$$\lim_{k\to\infty}\|b_{n_k}-T_2(b)b_{n_k}\|=0.\tag{5.2.38}$$

From the definition of $x_{n_{k+1}}$ and (5.2.35), we have

$$\begin{aligned}\|x_{n_{k+1}}-v_{n_k}\|&=\left\|(1-\beta_{n_k})v_{n_k}+\beta_{n_k}\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T_1(s)v_{n_k}-v_{n_k}\right\|\\ &=\beta_{n_k}\left\|\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T_1(s)v_{n_k}-v_{n_k}\right\|\to 0,\ \text{as}\ k\to\infty.\end{aligned}\tag{5.2.39}$$

Similarly, we have

$$\lim_{k\to\infty}\|y_{n_{k+1}}-b_{n_k}\|=0.\tag{5.2.40}$$

Now, from **Step 2** and by Remark 5.2.3, we get

$$\begin{aligned}\|w_{n_k}-x_{n_k}\| &= \|(1-\alpha_{n_k})(x_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k-1}))-x_{n_k}\|\\ &= \|(1-\alpha_{n_k})(x_{n_k}-x_{n_k})+(1-\alpha_{n_k})\theta_{n_k}(x_{n_k}-x_{n_k-1})-\alpha_{n_k}x_{n_k}\|\\ &\le (1-\alpha_{n_k})\|x_{n_k}-x_{n_k}\|+(1-\alpha_{n_k})\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|+\alpha_{n_k}\|x_{n_k}\|\to 0,\end{aligned} \tag{5.2.41}$$

as $k\to\infty$.

Similarly, we have

$$\lim_{k\to\infty}\|\varphi_{n_k}-y_{n_k}\|=0. \tag{5.2.42}$$

From (5.2.35)-(5.2.42) we have

$$\lim_{k\to\infty}\|x_{n_k}-\phi_{n_k}\|=0,\ \lim_{k\to\infty}\|x_{n_{k+1}}-w_{n_k}\|=0,\ \lim_{k\to\infty}\|y_{n_{k+1}}-\varphi_{n_k}\|=0.. \tag{5.2.43}$$

From (5.2.41) and (5.2.43) we have

$$\lim_{k\to\infty}\|x_{n_{k+1}}-x_{n_k}\|=0. \tag{5.2.44}$$

Similarly, from (5.2.42) and (5.2.43)

$$\lim_{k\to\infty}\|y_{n_{k+1}}-y_{n_k}\|=0.$$

To complete the proof, we show that $\Big(w_\omega(x_n),w_\omega(y_n)\Big)\subset\Gamma$. Since $\{x_n\}$ and $\{y_n\}$ are bounded we have that $w_\omega(x_n)$ and $w_\omega(y_n)$ are nonempty. Let $\hat{x}\in w_\omega(x_n)$ and $\hat{y}\in w_\omega(y_n)$ be arbitrary elements. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\hat{x}$ as $k\to\infty$. By (5.2.41), we have $w_\omega(x_n)=w_\omega(w_n)$. Since $\lim_{k\to\infty}\|x_{n_k}-\phi_{n_k}\|=0$, we have that $\phi_{n_k}\rightharpoonup\hat{x}\in\mathcal{C}$ as $k\to\infty$. From the characteristic property of $P_{\mathcal{C}}$, we have

$$\langle x-u_{n_k},\quad \phi_{n_k}-\lambda_{n_k}A\phi_{n_k}-u_{n_k},\rangle\le 0,\quad x\in\mathcal{C},$$

which implies that

$$\frac{1}{\lambda_{n_k}}\langle\phi_{n_k}-u_{n_k},\quad x-u_{n_k}\rangle\le\langle A\phi_{n_k},\ x-u_{n_k}\rangle,\qquad\forall\ x\in\mathcal{C}.$$

Consequently, we have

$$\frac{1}{\lambda_{n_k}}\langle\phi_{n_k}-u_{n_k},\ x-u_{n_k}\rangle+\langle A\phi_{n_k},\quad u_{n_k}-\phi_{n_k}\rangle\le\langle A\phi_{n_k},\quad x-\phi_{n_k}\rangle,\qquad\forall\ x\in\mathcal{C}. \tag{5.2.45}$$

Applying the fact that $\lim_{k\to\infty}\|\phi_{n_k}-u_{n_k}\|=0$ and $\lim_{k\to\infty}\lambda_{n_k}=\lambda>0$ to (5.2.45), we have

$$0\le\liminf_{k\to\infty}\langle A\phi_{n_k},\quad x-\phi_{n_k}\rangle,\quad\forall\ x\in\mathcal{C}. \tag{5.2.46}$$

Also, we have that

$$\langle Au_{n_k},\ x-u_{n_k}\rangle = \langle Au_{n_k}-A\phi_{n_k},\ x-\phi_{n_k}\rangle + \langle A\phi_{n_k},\ x-\phi_{n_k}\rangle + \langle Au_{n_k},\ \phi_{n_k}-u_{n_k}\rangle. \tag{5.2.47}$$

Since A is uniformly continuous on $\mathcal{H}$ and $\lim_{k\to\infty}\|\phi_{n_k}-u_{n_k}\|$, we have

$$\lim_{k\to\infty}\|A\phi_{n_k}-Au_{n_k}\| = 0. \tag{5.2.48}$$

From (5.2.46)-(5.2.48), we have

$$0 \le \liminf_{k\to\infty}\langle Au_{n_k},\ x-u_{n_k}\rangle\ \ \forall\ x\in\mathcal{C}. \tag{5.2.49}$$

Let $\{\delta_k\}$ be a sequence of positive numbers such that $\delta_{k+1}\le\delta_k,\ \ \forall\ \ k\ge 1$ and $\delta_k\to 0$ as $k\to\infty$. Then, for each $k\ge 1$, we denote by N_k the smallest positive integer such that

$$\langle Au_{n_j}, x-u_{n_j}\rangle + \delta_k \ge 0\ \ \forall j\ge N_k, \tag{5.2.50}$$

where the existence of N_k follows from (5.2.49). We have that $\{N_k\}$ is increasing since $\{\delta_k\}$ is decreasing. Furthermore, since $\{u_{n_k}\}\subset\mathcal{C}$ we can suppose $Au_{N_k}\neq 0$ (otherwise, u_{N_k} is a solution) and we set for each $k\ge 1$, $h_{N_k}=\frac{Au_{N_k}}{\|Au_{N_k}\|^2}$. Then we have that $\langle Au_{N_k}, h_{N_k}\rangle = 1$ for each $k\ge 1$. Thus, by (5.2.50), we have that

$$\langle Au_{N_k}, x+\delta_k h_{N_k}-u_{N_k}\rangle \ge 0,$$

which implies by the pseudo-monotonicity of A that

$$\langle A(x+\delta_k h_{N_k}), x+\delta_k h_{N_k}-u_{N_k}\rangle \ge 0. \tag{5.2.51}$$

Since $u_{n_k}\subset C$, then $\{u_{n_k}\}$ converges weakly to $\hat{x}\in\mathcal{C}$. If $A\hat{x}=0$, then $\hat{x}\in VI(\mathcal{C},A)$. On the contrary, we suppose $A\hat{x}\neq 0$. Since A satisfies condition (c), we have

$$0 < \|A\hat{x}\| \le \liminf_{k\to\infty}\|Au_{n_k}\|.$$

Since $\{u_{N_k}\}\subset\{u_{n_k}\}$, we obtain that

$$0 \le \limsup_{k\to\infty}\|\delta_k h_{N_k}\| = \limsup_{k\to\infty}\left(\frac{\delta_k}{\|Au_{n_k}\|}\right) \le \frac{\limsup_{k\to\infty}\delta_k}{\liminf_{k\to\infty}\|Au_{n_k}\|} = 0.$$

Therefore, $\lim_{k\to\infty}\|\delta_k h_{N_k}\| = 0$. Letting $k\to\infty$ in (5.2.51) gives

$$\langle Ax, x-\hat{x}\rangle \ge 0, \quad \forall x\in\mathcal{C}, \tag{5.2.52}$$

which implies by Lemma 2.5.2 that $\hat{x}\in VI(\mathcal{C},A)$. By similar argument, we have that $\hat{y}\in VI(\mathcal{Q},B)$.

Now, to show that $\hat{x} \in F(\mathcal{T}_a)$ and $\hat{y} \in F(\mathcal{T}_b)$. On the contrary, we suppose that $T_1(v)\hat{x} \neq \hat{x}$ and $T_2(b)\hat{x} \neq \hat{y}$, for all $v \geq 0$ and $b \geq 0$. Then, it follows from the Opial condition of Hilbert space and from (5.2.37) that

$$\begin{aligned}\liminf_{k\to\infty} \|v_{n_k} - \hat{x}\| &< \liminf_{k\to\infty} \|v_{n_k} - T_1(v)\hat{x}\| \\ &\leq \liminf_{k\to\infty} \Big\{ \|v_{n_k} - T_1(v)v_{n_k}\| + \|T_1(v)v_{n_k} - T_1(v)\hat{x}\| \Big\} \\ &\leq \liminf_{k\to\infty} \Big\{ \|v_{n_k} - T_1(v)v_{n_k}\| + \|v_{n_k} - \hat{x}\| \Big\} \\ &= \liminf_{k\to\infty} \|v_{n_k} - \hat{x}\|,\end{aligned}$$

which is a contradiction. Thus, it follows that $T_1(v)\hat{x} = \hat{x}$ for all $v \geq 0$ which implies that $\hat{x} \in F(\mathcal{T}_a)$. Similarly, $\hat{y} \in F(\mathcal{T}_b)$.

Next, from (5.2.35) we have that $\lim_{k\to\infty} \|\phi_{n_k} - z_{n_k}\| = \lim_{k\to\infty} \|U^{\Phi_1}_{r_{n_k},1} z_{n_k} - z_{n_k}\| = 0$, and since $z_{n_k} \rightharpoonup \hat{x}$ it follows from the demiclosed property of nonexpansive mappings that $\hat{x} \in EP(\Phi_1)$. Similarly, we have that $\hat{y} \in EP(\Phi_2)$. Since $\mathcal{F}_1\hat{x} - \mathcal{F}_2\hat{y} \in w_\omega(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)$, it follows from the weakly lower semi-continuity of the norm that

$$\|\mathcal{F}_1\hat{x} - \mathcal{F}_2\hat{y}\| \leq \liminf_{n\to\infty} \|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\| = 0.$$

Hence, we have that $(\hat{x}, \hat{y}) \in \Gamma$. Since $\hat{x} \in w_\omega(x_n)$ and $\hat{y} \in w_\omega(y_n)$ are arbitrary elements, it follows that $\Big(w_\omega(x_n), w_\omega(y_n)\Big) \subset \Gamma$.

To conclude, we show that

$$\limsup_{k\to\infty} \Big(\Big\langle \hat{x},\ \hat{x} - x_{n_{k+1}} \Big\rangle + \Big\langle \hat{y},\ \hat{y} - y_{n_{k+1}} \Big\rangle \Big) \leq 0.$$

By the boundedness of $\Big\{x_{n_k}\Big\}$, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to some $\bar{x} \in \mathcal{H}$, and such that

$$\lim_{j\to\infty} \Big\langle \hat{x},\ \hat{x} - x_{n_{k_j}} \Big\rangle = \limsup_{k\to\infty} \langle \hat{x},\ \hat{x} - x_{n_k} \rangle. \tag{5.2.53}$$

From (5.2.53) and the fact that $\hat{x} = P_D(0)$ we have

$$\limsup_{k\to\infty} \langle \hat{x},\ \ \hat{x} - x_{n_k} \rangle = \lim_{j\to\infty} \Big\langle \hat{x},\ \ \hat{x} - x_{n_{k_j}} \Big\rangle = \langle \hat{x},\ \ \hat{x} - \bar{x} \rangle \leq 0. \tag{5.2.54}$$

From (5.2.44) and (5.2.54), it follows that

$$\limsup_{k\to\infty} \Big\langle \hat{x},\ \ \hat{x} - x_{n_{k+1}} \Big\rangle = \limsup_{k\to\infty} \langle \hat{x},\ \ \hat{x} - x_{n_k} \rangle = \langle \hat{x},\ \ \hat{x} - \bar{x} \rangle \leq 0. \tag{5.2.55}$$

Following the same line of argument and the fact that $\hat{y} = P_E(0)$ we have

$$\limsup_{k\to\infty} \Big\langle \hat{y},\ \ \hat{y} - y_{n_{k+1}} \Big\rangle = \limsup_{k\to\infty} \langle \hat{y},\ \ \hat{y} - y_{n_k} \rangle = \langle \hat{y},\ \ \hat{y} - \bar{y} \rangle \leq 0. \tag{5.2.56}$$

Adding (5.2.55) and (5.2.56), we obtain

$$\limsup_{k\to\infty}\Big(\Big\langle \hat{x},\ \hat{x}-x_{n_{k+1}}\Big\rangle+\Big\langle \hat{y},\ \hat{y}-y_{n_{k+1}}\Big\rangle\Big)\leq 0. \tag{5.2.57}$$

Thus, by (5.2.43) and (5.2.57) we have $\limsup_{k\to\infty}\hat{d}_{n_k}\leq 0$. Now, applying Lemma 2.5.36 to (5.2.33) we have $\{\|x_n-\hat{x}\|+\|y_n-\hat{y}\|\}$ converges to zero, which implies that $\lim_{n\to\infty}\|x_n-\hat{x}\|=0$ and $\lim_{n\to\infty}\|y_n-\hat{y}\|=0$. Therefore, $\big(\{x_n\},\{y_n\}\big)$ converges strongly to $(\hat{x},\hat{y})$. □

5.2.3 Numerical experiments

In this section, we discuss the numerical behavior of our method, (Proposed Alg.) Algorithm 5.2.3 in comparison with the method in Appendix 5.2.2 proposed by Latif *et al.* (Latif *et al.* Alg.), which is the only related result we could find in the literature. We plot the graph of errors against the number of iterations in each case of both examples using $|x_{n+1}-x_n|<10^{-4}$ and $\|x_{n+1}-x_n\|<10^{-4}$ in Example 5.2.12 and Example 5.2.13 respectively as the stopping criterion. The numerical computations are reported in Figures 5.1 - 5.8 and Tables 5.2.1-5.2.2 with all implementations performed using Matlab 2021 (b).

In our computation, we choose $\theta=3.5,\ \tau=2.44,\ \lambda_1=1.5,\ \rho_1=1.8,\ a_1=0.8, a_2=0.9,\ \epsilon_n=\zeta_n=\frac{1}{(2n+1)^3},\ \alpha_n=\frac{3}{2n+1},\ \beta_n=\frac{1}{4},\ \gamma_n=\frac{1}{4},\ \rho_n=\sigma_n=\frac{100}{(n+1)^2},\eta=0.5, r_{n,1}=2.8, r_{n,2}=3.5, t_{n,1}=4.5, t_{n,2}=5.5, s=u=1.5$. For Appendix 5.2.14, we choose $\alpha=0.85, \varsigma_n=\kappa_n=\frac{1}{6},\ \xi_n=\delta_n=\frac{1-\alpha_n}{2}$.

Example 5.2.12. Let $\mathcal{H}_1=\mathcal{H}_2=\mathcal{H}_3=\mathbb{R}$ the set of all real numbers with the inner product $\langle x,y\rangle=xy,\ \forall\ x,y\in\mathbb{R}$ and induced norm $|\cdot|$. For $r_i>0,\ i=1,2$, consider $\mathcal{C}=[-10,10]$ and $\mathcal{Q}=[0,20]$. We define the bifunction $\Phi_1:\mathcal{C}\times\mathcal{C}\to\mathbb{R}$ and $\Phi_2:\mathcal{Q}\times\mathcal{Q}\to\mathbb{R}$ as follows:

$$U_{r_1}^{\Phi_1}(u)=\frac{u}{3r_1+1},\quad \forall\ x\in\mathcal{C}$$

and

$$U_{r_2}^{\Phi_2}(v)=\frac{v}{r_2+1},\quad \forall\ y\in\mathcal{Q}.$$

Let $\mathcal{F}_1x=2x$ and $\mathcal{F}_2x=5x$ which implies that $\mathcal{F}_1^*x=2x$ and $\mathcal{F}_2^*x=5x$. Next we define $A:\mathcal{H}_1\to\mathcal{H}_1$ as $Ax=2x$ and $B:\mathcal{H}_2\to\mathcal{H}_2$ as $Bx=3x$. We define the mappings $T_1(s):\mathbb{R}\to\mathbb{R}$ and $T_2(u):\mathbb{R}\to\mathbb{R}$ as follows; $T_1(s)x=10^{-s}x$ and $T_2(u)y=10^{-2u}y$. Clearly, we observe that $T_1(s)$ and $T_2(u)$ are nonexpansive semigroups.

We choose $\mathcal{V}_1=x_0, \mathcal{V}_2=y_0$ and consider the following cases for the numerical experiments of this example.
Case 1: Take $(x_0,y_0)=(-13.5,8.0)$ and $(x_1,y_1)=(5.7,-9.1)$.

Case 2: Take $(x_0,y_0)=(15.1,7.9)$ and $(x_1,y_1)=(6.4,81.3)$.

Case 3: Take $(x_0,y_0)=(10.9,-11.8)$ and $(x_1,y_1)=(-37.2,26.8)$.

Case 4: Take $(x_0,y_0)=(-14.9,-9.8)$ and $(x_1,y_1)=(-25.2,-17.7)$.

Table 5.2.1: Numerical Results for Example 5.2.12

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Latif *et al.* Alg.	86	0.0085	86	0.0062	89	0.0084	88	0.0064
Proposed Alg. 5.2.3	64	0.0132	64	0.0078	64	0.0093	64	0.0018

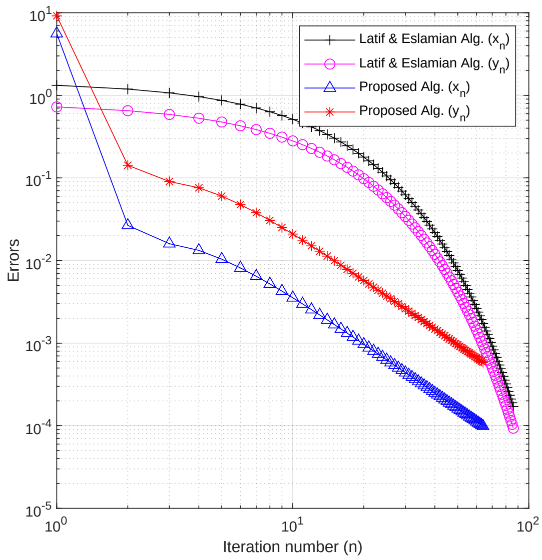

Figure 5.1: Example 5.2.12: Case 1

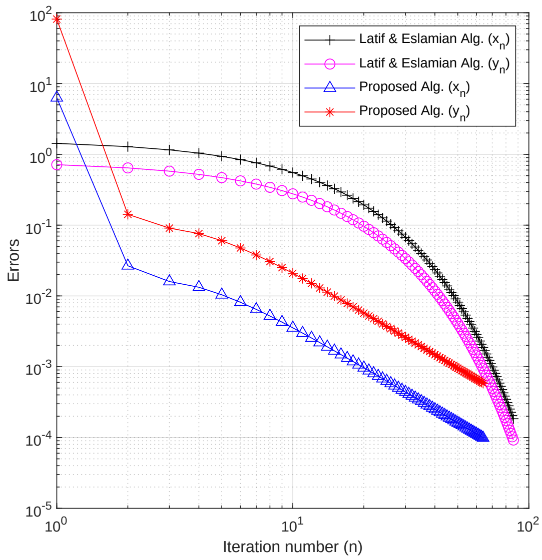

Figure 5.2: Example 5.2.12: Case 2

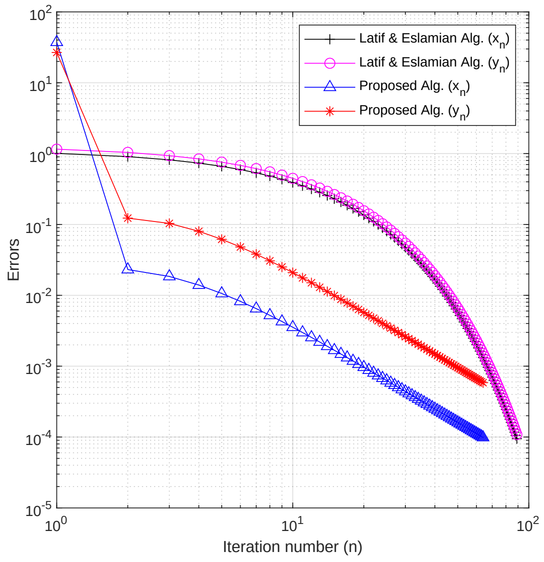

Figure 5.3: Example 5.2.12: Case 3

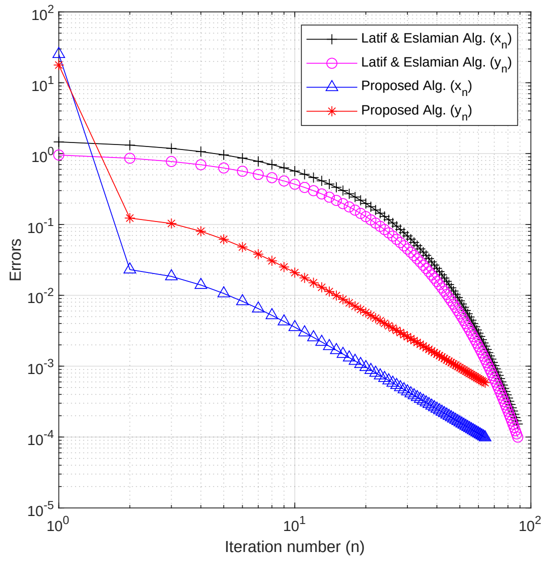

Figure 5.4: Example 5.2.12: Case 4

Example 5.2.13. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = (l_2(\mathbb{R}), \|\cdot\|_2)$, where

$$l_2(\mathbb{R}) := \{x = (x_1, x_2, \dots, x_n, \dots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < +\infty\},$$

$||x||_2 = \sqrt{(\sum_{i=1}^{\infty} |x_i|^2)}$ and $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$ for all $x \in \ell_2(\mathbb{R})$. For $r_i > 0,\ i = 1, 2$, we define the sets $\mathcal{C} := \{x \in \ell_2 : \|x\| \leq 1\}$ and $\mathcal{Q} := \{y \in \ell_2 : \|y\| \leq 1\}$. Let $\mathcal{F}_1 : \mathcal{H}_1 \to \mathcal{H}_2$, $\mathcal{F}_2 : \mathcal{H}_2 \to \mathcal{H}_3$ be defined by $\mathcal{F}_1 x = \frac{x}{3}$ and $\mathcal{F}_2 x = \frac{2x}{5}$ respectively which implies that $\mathcal{F}_1^* y = \frac{y}{3}$ and $\mathcal{F}_2^* y = \frac{2y}{5}$. Clearly, $\mathcal{F}_1$ and $\mathcal{F}_2$ are bounded linear operators. We define $\Phi_1 : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ and $\Phi_2 : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ by $\Phi_1(x, y) = \langle L_1 x, y - x\rangle$ and $\Phi_2(x, y) = \langle L_2 x, y - x\rangle$, where $L_1 x = \frac{x}{3}$ and $L_2 x = \frac{x}{2}$. Observe that Φ_1 and Φ_2 satisfy Assumption 5.2.2. After simple calculation and applying Lemma 5.2.1, we obtain

$$U_{r_1}^{\Phi_1}(u) = \frac{3u}{r_1 + 3}, \quad \forall\ x \in \mathcal{C},$$

and

$$U_{r_2}^{\Phi_2}(v) = \frac{2v}{r_2 + 2}, \quad \forall\ y \in \mathcal{Q}.$$

Let $A : \mathcal{H}_1 \to \mathcal{H}_1$ be defined by $A(x_1, x_2, x_3, \cdots) = (x_1 e^{-x_1^2}, 0, 0, \cdots)$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ as $B(x_1, x_2, x_3, \cdots) = (5x_1 e^{-x_1^2}, 0, 0, \cdots)$. Clearly, we see that A and B are pseudomonotone mappings. We define the mappings $T_1(s) : \mathbb{R} \to \mathbb{R}$ and $T_2(u) : \mathbb{R} \to \mathbb{R}$ as follows; $T_1(s)x = 10^{-5s}x$ and $T_2(u)y = 10^{-3u}y$. Clearly, we observe that $T_1(s)$ and $T_2(u)$ are nonexpansive semigroups.

We choose $\mathcal{V}_1 = x_0, \mathcal{V}_2 = y_0$ and consider different initial values as follows:
Case 1: $x_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$, $y_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$;

$x_1 = (\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots)$, $y_1 = (\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots)$;

Case 2: $x_0 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \cdots)$, $y_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$;
$x_1 = (-\frac{1}{3}, \frac{1}{6}, -\frac{1}{18}, \cdots)$, $y_1 = (-\frac{1}{3}, \frac{1}{6}, -\frac{1}{18}, \cdots)$;

Case 3: $x_0 = (\frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \cdots)$, $y_0 = (\frac{5}{9}, \frac{5}{18}, -\frac{5}{36}, \cdots)$;
$x_1 = (-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \cdots)$, $y_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \cdots)$;

Case 4: $x_0 = (\frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \cdots)$, $y_0 = (\frac{5}{9}, \frac{5}{18}, \frac{5}{36}, \cdots)$.
$x_1 = (\frac{1}{9}, \frac{1}{18}, \frac{1}{36}, \cdots)$, $y_1 = (-\frac{7}{12}, \frac{7}{24}, -\frac{7}{36})$.

Table 5.2.2: Numerical Results for Example 5.2.13

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Latif *et al.* Alg.	72	0.0134	72	0.0192	72	0.0147	72	0.0073
Proposed Alg. 5.2.3	58	0.0211	58	0.0171	58	0.0263	58	0.0100

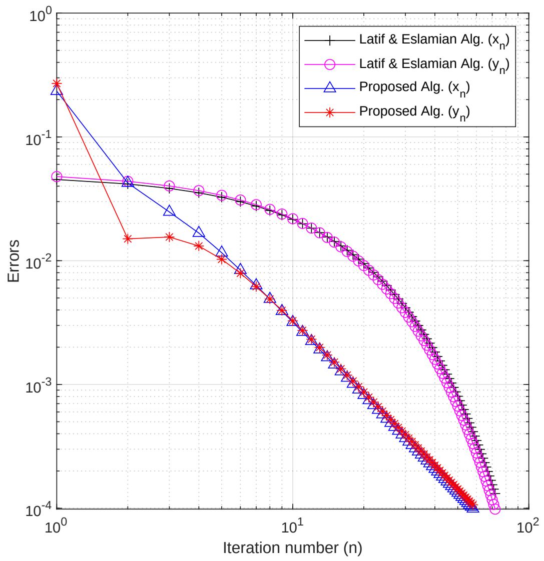

Figure 5.5: Example 5.2.13: Case 1

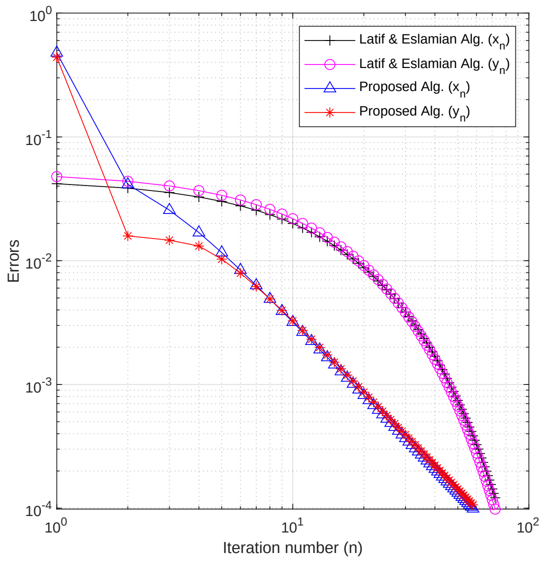

Figure 5.6: Example 5.2.13: Case 2

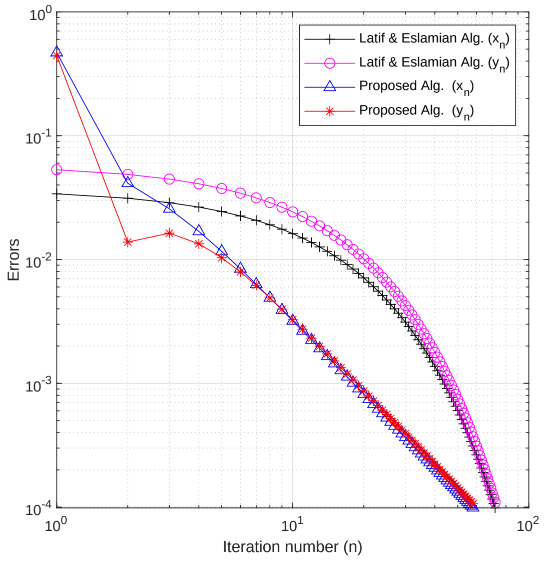

Figure 5.7: Example 5.2.13: Case 3

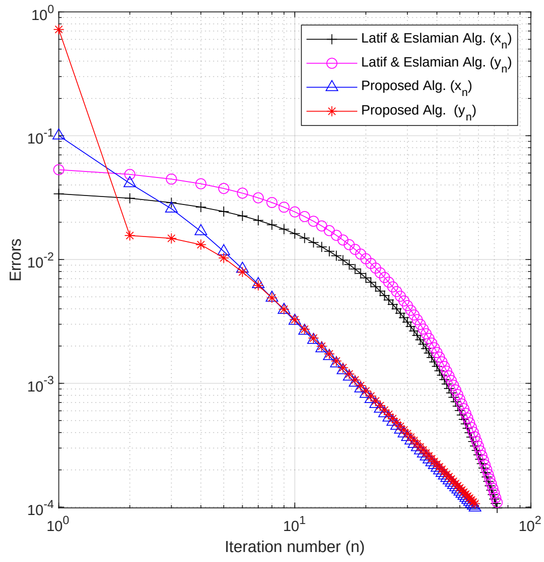

Figure 5.8: Example 5.2.13: Case 4

Appendix 5.2.14. *Algorithm 1 of Latif et al. [159]*

Choose sequences $\{\beta_n\}_{n=1}^{\infty}, \{\alpha_n\}_{n=1}^{\infty}, \{\delta_n\}_{n=1}^{\infty}$ *such that* $\beta_n + \alpha_n + \delta_n = 1$. *Select initial point* $x_0 \in \mathcal{H}_1, y_0 \in \mathcal{H}_2$, *let* $\vartheta \geq 0$. *Set* $n := 1$.

$$\begin{cases} z_n = x_n - \vartheta_n \mathcal{F}_1^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n), \\ \phi_n = U_{r_{n,1}}^{\Phi_1} z_n, \\ u_n = P_{\mathcal{C}}(\phi_n - \varsigma_n A \phi_n), \\ p_n = P_{\mathcal{C}}(\phi_n - \varsigma_n A u_n), \\ x_{n+1} = \alpha_n \mathcal{V}_1 + \xi_n p_n + \delta_n T_1(s) p_n \\ k_n = y_n + \vartheta_n \mathcal{F}_2^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n), \\ \psi_n = U_{r_{n,2}}^{\Phi_2} k_n, \\ s_n = P_{\mathcal{Q}}(\psi_n - \kappa_n B \psi_n), \\ l_n = P_{\mathcal{Q}}(\psi_n - \kappa_n B s_n), \\ y_{n+1} = \alpha_n \mathcal{V}_2 + \xi_n l_n + \delta_n T_2(u) l_n \end{cases} \tag{5.2.58}$$

where the step size ϑ_n *is chosen such that for small enough* $\epsilon > 0$,

$$\vartheta_n \in \left[\epsilon, \quad \frac{2\|\mathcal{F}_1 x_n - \mathcal{F}_2 y_n\|^2}{\|\mathcal{F}_2^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)\|^2} - \epsilon\right],$$

if $\mathcal{F}_1 x_n \neq \mathcal{F}_2 y_n$; *otherwise,* $\vartheta_n = \eta$.
Set $n := n + 1$ *and go back to* **Step 1**.

5.3 On minimum-norm solutions of inclusion and split equilibrium problems

In this work, we study the problem of finding the common solutions of V_qIP and the SEP. We propose a modified inertial forward-backward splitting algorithm with self-adaptive step size for approximating the solution of the problem in Hilbert spaces. Unlike several of the existing results in this direction, our proposed method only requires that the underlying single-valued operator be monotone and Lipschitz continuous. This relaxes the strong assumption of co-coerciveness (inverse strongly monotonicity) imposed in several of the existing results. Moreover, our method combines the inertial technique with the forward-backward splitting method and self-adaptive step sizes for solving the common solution problem. Unlike several of the existing methods, the self-adaptive step sizes ensure that knowledge of the norm of the bounded linear operator nor of the Lipschitz constant are not required to implement our algorithm. We prove that the sequence generated by our proposed method converges strongly to a minimum-norm solution of the aforementioned problem. Furthermore, we apply our result to study certain optimization problems. Finally, we provide some numerical experiments of our proposed method in comparison with other existing methods in the literature.

5.3.1 Proposed method

In this section, we present our proposed method and discuss its features. We begin by defining the following functions;

$$g(x) = \frac{1}{2}\|(I - T_r^{F_2})Ax\|^2, \quad h(x) = \frac{1}{2}\|(I - T_r^{F_1})x\|^2$$

and

$$G(x) = A^*(I - T_r^{F_2})Ax, \quad H(x) = (I - T_r^{F_1})x.$$

It can be easily verified from Aubin [30] that g and h are weakly lower semi-continuous, convex and differentiable functions. We now give the following assumptions under which our strong convergence result is obtained.

Assumption 5.3.1. *Suppose that the following conditions hold:*

(a) *The feasible sets $\mathcal{C}$ and $\mathcal{Q}$ are nonempty, closed and convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.*

(b) *$F_1 : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$, $F_2 : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ are bifunctions satisfying Assumption (5.2.1) and F_2 is upper semi continuous in the first argument.*

(c) *$A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator, $B : \mathcal{H}_1 \to \mathcal{H}_1$ is a Lipschitz continuous and monotone operator, and $D : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is a maximal monotone operator such that $\Gamma = (B+D)^{-1}(0) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in \mathcal{C} : z \in EP(F_1)$ and $Az \in EP(F_2)\}$.*

(d) $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *satisfying* $\beta_n + \gamma_n \leq 1, \lim_{n\to\infty} \gamma_n = 0, \sum_{n=1}^{\infty} \gamma_n = \infty, \inf(1-\beta_n-\gamma_n)\beta_n > 0, 0 < a \leq \alpha_n, \beta_n \leq b < 1, 0 < c \leq \tau_n \leq d < 4.$

(e) *Let* $\{\epsilon_n\}$ *be a positive sequence such that* $\lim_{n\to\infty} \frac{\epsilon_n}{\gamma_n} = 0$ *and* $\{r_n\} \subset (0,\infty)$ *such that* $\liminf_{n\to\infty} r_n > 0$.

Algorithm 5.3.2.

Step 1: Select initial point $x_0, x_1 \in \mathcal{H}_1$, let $s_1 > 0, \mu \in (0,1), \theta \geq 3$ and set $n = 1$. Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.} \end{cases} \tag{5.3.1}$$

Step 2: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3: Compute

$$z_n = T_{r_n}^{F_1}\big(I - \lambda_n A^*(I - T_{r_n}^{F_2})A\big)w_n$$

and

$$y_n = \alpha_n w_n + (1-\alpha_n)z_n.$$

Step 4: Compute

$$u_n = (I + s_n D)^{-1}(I - s_n B)y_n = J_{s_n}^D(I - s_n B)y_n$$

and

$$v_n = u_n - s_n(Bu_n - By_n).$$

Step 5 Compute

$$x_{n+1} = (1-\beta_n-\gamma_n)w_n + \beta_n v_n,$$

where

$$\lambda_n := \begin{cases} \tau_n \frac{g(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2}, & \text{if } \quad \|G(w_n)\|^2 + \|H(w_n)\|^2 \neq 0, \\ 0, & \text{otherwise} \end{cases} \tag{5.3.2}$$

and

$$s_{n+1} = \begin{cases} \min\left\{\frac{\mu\|y_n - u_n\|}{\|By_n - Bu_n\|}, s_n\right\} & \text{if } \quad By_n - Bu_n \neq 0, \\ s_n, & \text{otherwise.} \end{cases} \tag{5.3.3}$$

Set $n := n + 1$ and go back to **Step 1**.

Remark 5.3.1. • Different from all existing methods for finding the common solutions of the VIP (1.2.9) and SEP (1.2.10)-(1.2.11), our proposed method requires the underlying operator B to be monotone and Lipschitz continuous. This relaxes the strong assumption of co-coerciveness (inverse strongly monotonicity) on the operator.

- Contrast to the methods in the literature, the implementation of our method does not depend on the norm of the bounded linear operator nor the Lipschitz constant of the monotone operator. This makes our method easier to implement than several other methods in the literature, which depends on the norm of the bounded linear operator.

- Our method employs the inertial technique to improve the speed of convergence of the proposed algorithm. Unlike several other inertial methods in the literature, our method does not require the summability condition $\left(\sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\| < \infty\right)$, which makes our method easily implementable.

- The sequence generated by our proposed method was proved to converge strongly to a minimum-norm solution of the investigated problem under mild conditions. It is known that minimum-norm solutions of problems find applications in several practical problems. Moreover, our convergence analysis is not dependent on the usual "Two cases approach", which is widely used in many papers to guarantee strong convergence.

Remark 5.3.2. By conditions (d) and (e), one can easily verify from (5.3.1) that

$$\lim_{n\to\infty} \theta_n ||x_n - x_{n-1}|| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta_n}{\gamma_n} ||x_n - x_{n-1}|| = 0. \tag{5.3.4}$$

Remark 5.3.3. From (5.3.3) in Algorithm 5.3.2, we have that $s_{n+1} \leq s_n$ for all $n \geq 1$. Also, from our conditions we have that B is L-Lipschitz continuous, therefore when $By_n \neq Bu_n$ in Algorithm 5.3.2, we have

$$s_{n+1} = \min\left\{\frac{\mu ||y_n - u_n||}{||By_n - Bu_n||},\ s_n\right\} \geq \min\left\{\frac{\mu}{L},\ s_n\right\}.$$

By induction, we obtain that $\{s_n\}$ is bounded below by $\min\left\{\frac{\mu}{L},\ s_1\right\}$. Also, since $\{s_n\}$ is monotone nonincreasing, we have that the limit exists, and $\lim_{n\to\infty} s_n \geq \min\{\frac{\mu}{L}, s_1\} > 0$.

Remark 5.3.4. From the definition of $\{s_n\}$ in (5.3.3) we have

$$\|By_n - Bu_n\| \leq \frac{\mu}{s_{n+1}} \|y_n - u_n\|, \quad \forall\ n. \tag{5.3.5}$$

Inequality (5.3.5) holds if $By_n = Bu_n$. In the case where $By_n \neq Bu_n$, we have

$$s_{n+1} = \min\left\{\frac{\mu\|y_n - u_n\|}{\|By_n - Bu_n\|}, s_n\right\} \leq \frac{\mu\|y_n - u_n\|}{\|By_n - Bu_n\|},$$

which implies that

$$\|By_n - Bu_n\| \leq \frac{\mu}{s_{n+1}} \|y_n - u_n\|.$$

Hence, (5.3.5) holds for both cases when $By_n \neq Bu_n$ and $By_n = Bu_n$.

5.3.2 Convergence analysis

Lemma 5.3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 5.3.2 under Assumption 5.3.1. Then,*

$$\|v_n - p\|^2 \le \|w_n - p\|^2 - \tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2} - \alpha_n(1-\alpha_n)\|w_n - z_n\|^2$$
$$- \left(1 - s_n^2 \cdot \frac{\mu^2}{s_{n+1}^2}\right)\|y_n - u_n\|^2.$$

Proof. Let $p \in \Gamma$. Since $G(w_n) = A^*(I - T_{r_n}^{F_2})Aw_n,\ p = T_{r_n}^{F_2}p$ and $I - T_{r_n}^{F_2}$ is firmly nonexpansive, we have

$$\begin{aligned}
\langle G(w_n), w_n - p\rangle &= \langle A^*(I - T_{r_n}^{F_2})Aw_n,\ w_n - p\rangle \\
&= \langle (I - T_{r_n}^{F_2})Aw_n,\ Aw_n - Ap\rangle \\
&= \langle (I - T_{r_n}^{F_2})Aw_n - (I - T_{r_n}^{F_2})Ap,\ Aw_n - Ap\rangle \\
&\ge \|(I - T_{r_n}^{F_2})Aw_n\|^2 \\
&= 2g(w_n).
\end{aligned} \tag{5.3.6}$$

From the definition of z_n in Step 3, the nonexpansivity of $T_{r_n}^{F_1}$ and (5.3.6) we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|T_{r_n}^{F_1}(I - \lambda_n A^*(I - T_{r_n}^{F_2})A)w_n - p\|^2 \\
&\le \|(I - \lambda_n A^*(I - T_{r_n}^{F_2})A)w_n - p\|^2 \\
&= \|w_n - p - \lambda_n G(w_n)\|^2 \\
&= \|w_n - p\|^2 + \lambda_n^2\|G(w_n)\|^2 - 2\lambda_n\langle G(w_n), w_n - p\rangle \\
&\le \|w_n - p\|^2 + \lambda_n^2\|G(w_n)\|^2 - 4\lambda_n g(w_n) \\
&\le \|w_n - p\|^2 - \tau_n(4-\tau_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2}.
\end{aligned} \tag{5.3.7}$$

From the condition on τ_n we obtain

$$\|z_n - p\| \le \|w_n - p\| \tag{5.3.8}$$

From the definition of y_n and (5.3.7) we have,

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n w_n + (1-\alpha_n)z_n - p\|^2 \\
&= \alpha_n\|w_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|w_n - z_n\|^2 \\
&\le \alpha_n\|w_n - p\|^2 + (1-\alpha_n)\left[\|w_n - p\|^2 - \tau_n(4-\tau_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2}\right] \\
&\quad - \alpha_n(1-\alpha_n)\|w_n - z_n\|^2 \\
&= \|w_n - p\|^2 - \tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2} \\
&\quad - \alpha_n(1-\alpha_n)\|w_n - z_n\|^2.
\end{aligned} \tag{5.3.9}$$

From the condition on τ_n and α_n, we have

$$\|y_n - p\| \leq \|w_n - p\|. \tag{5.3.10}$$

Applying (5.3.5) and (5.3.9), we have

$$\begin{aligned}
\|v_n - p\|^2 &= \|u_n - s_n(Bu_n - By_n) - p\|^2 \\
&= \|u_n - p\|^2 + s_n^2\|Bu_n - By_n\|^2 - 2s_n\langle u_n - p, Bu_n - By_n\rangle \\
&= \|y_n - p\|^2 + \|u_n - y_n\|^2 + 2\langle u_n - y_n, \quad y_n - p\rangle + s_n^2\|Bu_n - By_n\|^2 \\
&- 2s_n\langle u_n - p, Bu_n - By_n\rangle \\
&= \|y_n - p\|^2 + \|u_n - y_n\|^2 + 2\langle u_n - y_n, \quad u_n - p\rangle - 2\langle u_n - y_n \quad u_n - y_n\rangle \\
&+ s_n^2\|Bu_n - By_n\|^2 - 2s_n\langle u_n - p, Bu_n - By_n\rangle \\
&= \|y_n - p\|^2 + \|u_n - y_n\|^2 + 2\langle u_n - y_n, \quad u_n - p\rangle - 2\|u_n - y_n\|^2 \\
&+ s_n^2\|Bu_n - By_n\|^2 - 2s_n\langle u_n - p, Bu_n - By_n\rangle \\
&= \|y_n - p\|^2 - \|u_n - y_n\|^2 + 2\langle u_n - y_n, \quad u_n - p\rangle + s_n^2\|Bu_n - By_n\|^2 \\
&- 2s_n\langle u_n - p, Bu_n - By_n\rangle \\
&= \|y_n - p\|^2 - \|u_n - y_n\|^2 + s_n^2\|Bu_n - By_n\|^2 \\
&- 2\langle u_n - p, \quad y_n - u_n - s_n(By_n - Bu_n)\rangle \\
&\leq \|w_n - p\|^2 - \tau_n(4 - \tau_n)(1 - \alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2} \\
&- \alpha_n(1 - \alpha_n)\|w_n - z_n\|^2 - \|y_n - u_n\|^2 + s_n^2 \cdot \frac{\mu^2}{s_{n+1}^2}\|y_n - u_n\|^2 \\
&- 2\langle u_n - p, \quad y_n - u_n - s_n(By_n - Bu_n)\rangle \\
&= \|w_n - p\|^2 - \tau_n(4 - \tau_n)(1 - \alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2} \\
&- \alpha_n(1 - \alpha_n)\|w_n - z_n\|^2 - \left(1 - s_n^2 \cdot \frac{\mu^2}{s_{n+1}^2}\right)\|y_n - u_n\|^2 \\
&- 2\langle u_n - p, \quad y_n - u_n - s_n(By_n - Bu_n)\rangle.
\end{aligned} \tag{5.3.11}$$

From **Step 4** we have that $u_n = (I+s_nD)^{-1}(I-s_nB)y_n$, hence $(I-s_nB)y_n \in (I+s_nD)u_n$. Since D is maximal monotone, there exists $k_n \in Du_n$ such that

$$(I - s_nB)y_n = u_n + s_nk_n$$

which implies that

$$k_n = \frac{1}{s_n}(y_n - s_nBy_n - u_n). \tag{5.3.12}$$

We have that $0 \in (B+D)p$ and $Bu_n + k_n \in (B+D)u_n$. Since $B+D$ is maximal monotone we obtain

$$\langle Bu_n + k_n, \quad u_n - p\rangle \geq 0. \tag{5.3.13}$$

Substituting (5.3.12) into (5.3.13) we have

$$\frac{1}{s_n}\langle s_n B u_n + y_n - s_n B y_n - u_n, \quad u_n - p\rangle \quad \geq 0$$

which implies that

$$\langle y_n - u_n - s_n(By_n - Bu_n), \;\; u_n - p\rangle \;\; \geq 0. \tag{5.3.14}$$

Applying (5.3.14) in (5.3.11) we have

$$\begin{aligned}\|v_n - p\|^2 \leq \|w_n - p\|^2 - \tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2} \\ - \alpha_n(1-\alpha_n)\|w_n - z_n\|^2 - \Big(1 - s_n^2 \cdot \frac{\mu^2}{s_{n+1}^2}\Big)\|y_n - u_n\|^2,\end{aligned} \tag{5.3.15}$$

which is the desired result. □

Lemma 5.3.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 5.3.2 under Assumption 5.3.1. Then, $\{x_n\}$ is bounded.*

Proof. Let $p \in \Gamma$. By applying the triangular inequality, from **Step 2** we have

$$\begin{aligned}\|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|.\end{aligned}$$

From Remark 5.3.2, $\lim\limits_{n\to\infty} \frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\| = 0$. Hence, there exists a constant $M_1 > 0$ such that $\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\| \leq M_1$ for all $n \geq 1$. This implies that

$$\|w_n - p\| \leq \|x_n - p\| + \gamma_n M_1. \tag{5.3.16}$$

Also, since $\lim\limits_{n\to\infty}\Big(1 - \mu^2\frac{s_n^2}{s_{n+1}^2}\Big) = 1 - \mu^2 > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$1 - \mu^2\frac{s_n^2}{s_{n+1}^2} > 0, \quad \forall \;\; n \geq n_0. \tag{5.3.17}$$

Applying (5.3.17) and the conditions on τ_n and α_n in (5.3.15), we have

$$\|v_n - p\| \leq \|w_n - p\|, \quad \forall \;\; n \geq n_0. \tag{5.3.18}$$

From the definition of x_{n+1} in **Step 4** and (5.3.18), we have for all $n \geq n_0$

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\beta_n-\gamma_n)(w_n - p) + \beta_n(v_n - p) - \gamma_n p\| \\ &\leq \|(1-\beta_n-\gamma_n)(w_n - p) + \beta_n(v_n - p)\| + \gamma_n\|p\|\end{aligned} \tag{5.3.19}$$

and

$$\begin{aligned}\|(1-\beta_n-\gamma_n)(w_n-p)+\beta_n(v_n-p)\|^2 &= (1-\beta_n-\gamma_n)^2\|w_n-p\|^2+\beta_n^2\|v_n-p\|^2 \\ &+ 2(1-\beta_n-\gamma_n)\beta_n\langle w_n-p, v_n-p\rangle \\ &\leq (1-\beta_n-\gamma_n)^2\|w_n-p\|^2+\beta_n^2\|w_n-p\|^2 \\ &+ 2(1-\beta_n-\gamma_n)\beta_n\|w_n-p\|^2 \\ &= (1-\gamma_n)^2\|w_n-p\|^2.\end{aligned}$$

Substituting the last inequality into (5.3.19) and applying (5.3.16), we have for all $n \geq n_0$

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-\gamma_n)\|w_n-p\|+\gamma_n\|p\| \\ &\leq (1-\gamma_n)\left[\|x_n-p\|+\gamma_n M_1\right]+\gamma_n\|p\| \\ &\leq (1-\gamma_n)\|x_n-p\|+\gamma_n(M_1+\|p\|) \\ &\leq \max\Big\{\|x_n-p\|, \quad M_1+\|p\|\Big\} \\ &\ \ \vdots \\ &\leq \max\Big\{\|x_{n_0}-p\|, \quad M_1+\|p\|\Big\},\end{aligned}$$

which implies that $\{x_n\}$ is bounded. Consequently, $\{w_n\}, \{z_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ are also bounded. □

Lemma 5.3.7. *Assume that $\{u_n\}$ and $\{y_n\}$ are sequences generated by Algorithm 5.3.2 under Assumption 5.3.1 such that $\lim_{j\to\infty}\|u_{n_j}-y_{n_j}\| = 0$ for some subsequences $\{u_{n_j}\}$ and $\{y_{n_j}\}$ of $\{u_n\}$ and $\{y_n\}$, respectively. If $\{y_{n_j}\}$ converges weakly to some $x^* \in \mathcal{H}_1$ as $j \to \infty$, then $x^* \in (B+D)^{-1}(0)$.*

Proof. Let $(u,v) \in G(B+D)$, that is $v - Bu \in Du$. Since

$$u_{n_j} = (I+s_{n_j}D)^{-1}(I-s_{n_j}B)y_{n_j},$$

then we have

$$(I-s_{n_j}B)y_{n_j} \in (I+s_{n_j}D)u_{n_j}.$$

This implies that

$$\frac{1}{s_{n_j}}\Big(y_{n_j}-u_{n_j}-s_{n_j}By_{n_j}\Big) \in Du_{n_j}.$$

Since D is maximal monotone, we have

$$\left\langle u-u_{n_j}, \quad v-Bu-\frac{1}{s_{n_j}}\Big(y_{n_j}-u_{n_j}-s_{n_j}By_{n_j}\Big)\right\rangle \geq 0,$$

which implies that

$$\left\langle u - u_{n_j}, \quad v \right\rangle - \left\langle u - u_{n_j}, \quad Bu + \frac{1}{s_{n_j}}\Big(y_{n_j} - u_{n_j} - s_{n_j} B y_{n_j}\Big) \right\rangle \geq 0.$$

Hence, we have

$$\begin{aligned}
\left\langle u - u_{n_j}, \quad v \right\rangle &\geq \left\langle u - u_{n_j}, \quad Bu + \frac{1}{s_{n_j}}\Big(y_{n_j} - u_{n_j} - s_{n_j} B y_{n_j}\Big) \right\rangle \\
&= \left\langle u - u_{n_j}, \quad Bu - By_{n_j} \right\rangle + \left\langle u - u_{n_j}, \quad \frac{1}{s_{n_j}}(y_{n_j} - u_{n_j}) \right\rangle \\
&= \left\langle u - u_{n_j}, \quad Bu - Bu_{n_j} \right\rangle + \left\langle u - u_{n_j}, \quad Bu_{n_j} - By_{n_j} \right\rangle \\
&+ \left\langle u - u_{n_j}, \quad \frac{1}{s_{n_j}}(y_{n_j} - u_{n_j}) \right\rangle \\
&\geq \left\langle u - u_{n_j}, \quad Bu_{n_j} - By_{n_j} \right\rangle + \left\langle u - u_{n_j}, \quad \frac{1}{s_{n_j}}(y_{n_j} - u_{n_j}) \right\rangle.
\end{aligned}$$

Now, since $\lim_{j\to\infty} \|u_{n_j} - y_{n_j}\| = 0$ and B is Lipschitz continuous, we have $\lim_{j\to\infty} \|Bu_{n_j} - By_{n_j}\| = 0$. Moreover, since $\lim_{n\to\infty} s_n = s \geq \min\{s_0, \frac{\mu}{L}\}$, we get

$$\left\langle u - x^*, \quad v \right\rangle = \lim_{j\to\infty} \left\langle u - u_{n_j}, \quad v \right\rangle \geq 0.$$

By the maximal monotonicity of $B + D$, it follows that $x^* \in (B + D)^{-1}(0)$. □

Lemma 5.3.8. *The following inequality holds for all $p \in \Gamma$ and $n \in \mathbb{N}$:*

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq (1 - \gamma_n)\|x_n - p\|^2 + \gamma_n\Big(3\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\|M_2 + \|p\|^2\Big) \\
- \beta_n \tau_n (4 - \tau_n)(1 - \alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2} - \beta_n \alpha_n (1 - \alpha_n)\|w_n - z_n\|^2 \\
- \beta_n\Big(1 - s_n^2 \cdot \frac{\mu^2}{s_{n+1}^2}\Big)\|y_n - u_n\|^2 - (1 - \beta_n - \gamma_n)\beta_n\|w_n - v_n\|^2
\end{aligned}$$

Proof. Let $p \in \Gamma$. Then, by applying Lemma 2.1.1 and the Cauchy-Schwartz inequality, from **Step 2** we obtain

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle \\
&\leq \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n||x_n - p||||x_n - x_{n-1}|| \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\| \left[\theta_n\|x_n - x_{n-1}\| + 2\|x_n - p\|\right] \\
&\leq \|x_n - p\|^2 + 3\theta_n\|x_n - x_{n-1}\|M_2 \\
&= \|x_n - p\|^2 + 3\gamma_n\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\|M_2, \qquad (5.3.20)
\end{aligned}$$

for some $M_2 > 0$.
From **Step 5** of Algorithm 5.3.2 and by applying Lemma 2.1.1, (5.3.15) and (5.3.20) we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\beta_n-\gamma_n)w_n+\beta v_n-p\|^2\\
&= \|(1-\beta_n-\gamma_n)(w_n-p)+\beta_n(v_n-p)+\gamma_n(-p)\|^2\\
&\leq (1-\beta_n-\gamma_n)\|w_n-p\|^2+\beta_n\|v_n-p\|^2+\gamma_n\|p\|^2\\
&- (1-\beta_n-\gamma_n)\beta_n\|w_n-v_n\|^2\\
&\leq (1-\beta_n-\gamma_n)\|w_n-p\|^2\\
&+ \beta_n\Big[\|w_n-p\|^2-\tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2+\|H(w_n)\|^2}\\
&- \alpha_n(1-\alpha_n)\|w_n-z_n\|^2-\Big(1-s_n^2\cdot\frac{\mu^2}{s_{n+1}^2}\Big)\|y_n-u_n\|^2\Big]+\gamma_n\|p\|^2\\
&- (1-\beta_n-\gamma_n)\beta_n\|w_n-v_n\|^2\\
&= (1-\gamma_n)\|w_n-p\|^2-\beta_n\tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2+\|H(w_n)\|^2}\\
&- \beta_n\alpha_n(1-\alpha_n)\|w_n-z_n\|^2-\beta_n\Big(1-s_n^2\cdot\frac{\mu^2}{s_{n+1}^2}\Big)\|y_n-u_n\|^2+\gamma_n\|p\|^2\\
&- (1-\beta_n-\gamma_n)\beta_n\|w_n-v_n\|^2\\
&\leq (1-\gamma_n)\Big[\|x_n-p\|^2+3\gamma_n\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|M_2\Big]\\
&- \beta_n\tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2+\|H(w_n)\|^2}\\
&- \beta_n\alpha_n(1-\alpha_n)\|w_n-z_n\|^2-\beta_n\Big(1-s_n^2\cdot\frac{\mu^2}{s_{n+1}^2}\Big)\|y_n-u_n\|^2+\gamma_n\|p\|^2\\
&- (1-\beta_n-\gamma_n)\beta_n\|w_n-v_n\|^2\\
&\leq (1-\gamma_n)\|x_n-p\|^2+\gamma_n\Big(3\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|M_2+\|p\|^2\Big)\\
&\quad - \beta_n\tau_n(4-\tau_n)(1-\alpha_n)\frac{g^2(w_n)}{\|G(w_n)\|^2+\|H(w_n)\|^2}-\beta_n\alpha_n(1-\alpha_n)\|w_n-z_n\|^2\\
&\quad - \beta_n\Big(1-s_n^2\cdot\frac{\mu^2}{s_{n+1}^2}\Big)\|y_n-u_n\|^2-(1-\beta_n-\gamma_n)\beta_n\|w_n-v_n\|^2,
\end{aligned}$$

which is the desired result. □

Lemma 5.3.9. *The following inequality holds for all $p \in \Gamma$, $M_2 > 0$ and $n \geq n_0$:*

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\gamma_n)\|x_n-p\|^2\\
&+ \gamma_n\Big[3\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|M_2+2\beta_n\|v_n-w_n\|\|p-x_{n+1}\|+2\langle p, \quad p-x_{n+1}\rangle\Big].
\end{aligned} \tag{5.3.21}$$

Proof. Let $p \in \Gamma$ and let $d_n = (1-\beta_n)w_n + \beta_n v_n$ then we have

$$\|d_n - w_n\| = \|(1-\beta_n)w_n + \beta_n v_n - w_n\| = \beta_n\|v_n - w_n\|. \tag{5.3.22}$$

Applying Lemma 2.1.1 and (5.3.18), we have

$$\begin{aligned}\|d_n - p\| &= \|(1-\beta_n)w_n + \beta_n v_n - p\| \\ &\leq (1-\beta_n)\|w_n - p\| + \beta_n\|v_n - p\| \\ &\leq (1-\beta_n)\|w_n - p\| + \beta_n\|w_n - p\| \\ &= \|w_n - p\|, \quad \forall \quad n \geq n_0. \end{aligned} \tag{5.3.23}$$

Also, from the definition of x_{n+1} in **Step 5**, (5.3.22) and (5.3.23), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\beta_n-\gamma_n)w_n + \beta_n v_n - p\|^2 \\ &= \|(1-\beta_n)w_n + \beta_n v_n - \gamma_n w_n - p\|^2 \\ &= \|d_n - \gamma_n w_n - p\|^2 \\ &= \|(1-\gamma_n)(d_n - p) - \gamma_n(w_n - d_n) - \gamma_n p\|^2 \\ &\leq (1-\gamma_n)^2\|d_n - p\|^2 - 2\langle\gamma_n(w_n - d_n) + \gamma_n p, \quad x_{n+1} - p\rangle \\ &= (1-\gamma_n)^2\|d_n - p\|^2 + 2\langle\gamma_n(w_n - d_n) + \gamma_n p, \quad p - x_{n+1}\rangle \\ &= (1-\gamma_n)\|d_n - p\|^2 + 2\langle\gamma_n(w_n - d_n), \quad p - x_{n+1}\rangle + 2\langle\gamma_n p, \quad p - x_{n+1}\rangle \\ &\leq (1-\gamma_n)\|d_n - p\|^2 + 2\gamma_n\|w_n - d_n\|\|p - x_{n+1}\| + 2\langle\gamma_n p, \quad p - x_{n+1}\rangle \\ &\leq (1-\gamma_n)\Big[\|x_n - p\|^2 + 3\gamma_n\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\|M_2\Big] \\ &+ 2\gamma_n\|w_n - d_n\|\|p - x_{n+1}\| + 2\gamma_n\langle p, \quad p - x_{n+1}\rangle \\ &\leq (1-\gamma_n)\|x_n - p\|^2 \\ &+ \gamma_n\Big[3\frac{\theta_n}{\gamma_n}\|x_n - x_{n+1}\|M_2 + 2\beta_n\|v_n - w_n\|\|p - x_{n+1}\| + 2\langle p, \quad p - x_{n+1}\rangle\Big],\end{aligned}$$

which is the desired result. ☐

We now prove the strong convergence theorem for Algorithm 5.3.2.

Theorem 5.3.10. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $\{x_n\}$ be generated by Algorithm 5.3.2 and suppose Assumption 5.3.1 is satisfied. Then, the sequence $\{x_n\}$ converges strongly to a point $q \in \Gamma$, where $\|q\| = \min\{\|z\| : z \in \Gamma\}$.*

Proof. Since $\|q\| = \min\{\|z\| : z \in \Gamma\}$, then $q = P_\Gamma(0)$. It follows that $q \in \Gamma$. Now, from Lemma 5.3.9 we obtain

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq (1-\gamma_n)\|x_n - q\|^2 \\ &+ \gamma_n\Big[3\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\|M_2 + 2\beta_n\|v_n - w_n\|\|q - x_{n+1}\| + 2\langle q, \quad q - x_{n+1}\rangle\Big] \\ &= (1-\gamma_n)\|x_n - q\|^2 + \gamma_n a_n \end{aligned} \tag{5.3.24}$$

where $a_n = 3\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\|M_2 + 2\beta_n\|v_n - w_n\|\|q - x_{n+1}\| + 2\langle q, \quad q - x_{n+1}\rangle$.

Now, we claim that $\{\|x_n - q\|\}$ converges to zero. To establish this, by Lemma 2.5.36 it suffices to show that $\limsup_{k\to\infty} a_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - q\|\}$ of $\{\|x_n - q\|\}$ satisfying the condition

$$\liminf_{k\to\infty} \Big(\|x_{n_{k+1}} - q\| - \|x_{n_k} - q\|\Big) \geq 0. \tag{5.3.25}$$

Suppose that $\{\|x_{n_k} - q\|\}$ is a subsequence of $\{\|x_n - q\|\}$ such that (5.3.25) holds. We obtain from Lemma 5.3.8 that

$$\begin{aligned}
\beta_{n_k}\tau_{n_k}(4 - \tau_{n_k})(1 - \alpha_{n_k})\frac{g^2(w_{n_k})}{\|G(w_{n_k})\|^2 + \|H(w_{n_k})\|^2} + \beta_{n_k}\alpha_{n_k}(1 - \alpha_{n_k})\|w_{n_k} - z_{n_k}\|^2 \\
+\beta_{n_k}\Big(1 - s_{n_k}^2 \cdot \frac{\mu^2}{s_{n_k+1}^2}\Big)\|y_{n_k} - u_{n_k}\|^2 + (1 - \beta_{n_k} - \gamma_{n_k})\beta_{n_k}\|w_{n_k} - v_{n_k}\|^2 \\
\leq (1 - \gamma_{n_k})\|x_{n_k} - q\|^2 - \|x_{n_k+1} - q\|^2 \\
+ \gamma_{n_k}\Big(3\frac{\theta_{n_k}}{\gamma_{n_k}}\|x_{n_k} - x_{n_k-1}\|M_2 + \|q\|^2\Big).
\end{aligned}$$

By applying (5.3.25) and the fact that $\lim_{k\to\infty} \gamma_{n_k} = 0$, we have that

$$\begin{aligned}
\lim_{k\to\infty} \beta_{n_k}\tau_{n_k}(4 - \tau_{n_k})(1 - \alpha_{n_k})\frac{g^2(w_{n_k})}{\|G(w_{n_k})\|^2 + \|H(w_{n_k})\|^2} &= 0; \\
\lim_{k\to\infty} \beta_{n_k}\alpha_{n_k}(1 - \alpha_{n_k})\|w_{n_k} - z_{n_k}\|^2 &= 0; \\
\lim_{k\to\infty} \beta_{n_k}\Big(1 - s_{n_k}^2 \cdot \frac{\mu^2}{s_{n_k+1}^2}\Big)\|y_{n_k} - u_{n_k}\|^2 &= 0; \\
\lim_{k\to\infty} (1 - \beta_{n_k} - \gamma_{n_k})\beta_{n_k}\|w_{n_k} - v_{n_k}\|^2 &= 0.
\end{aligned}$$

By the conditions on the control parameters, it follows that

$$\begin{aligned}
\lim_{k\to\infty} \frac{g^2(w_{n_k})}{\|G(w_{n_k})\|^2 + \|H(w_{n_k})\|^2} = \lim_{k\to\infty} \|w_{n_k} - z_{n_k}\| &= \lim_{k\to\infty} \|y_{n_k} - u_{n_k}\| \\
&= \lim_{k\to\infty} \|w_{n_k} - v_{n_k}\| = 0.
\end{aligned} \tag{5.3.26}$$

Since G and H are Lipschitz continuous, we have that

$$\lim_{k\to\infty} g(w_{n_k}) = 0. \tag{5.3.27}$$

From the definition of $g(w_{n_k})$ we have

$$\lim_{k\to\infty} g(w_{n_k}) = \lim_{k\to\infty} \frac{1}{2}\|(I - T_{r_{n_k}}^{F_2})Aw_{n_k}\|^2 = 0,$$

which implies that

$$\lim_{k\to\infty} \|(I - T_{r_{n_k}}^{F_2})Aw_{n_k}\| = 0. \tag{5.3.28}$$

Consequently, we get

$$\lim_{k\to\infty} \|A^*(I - T_{r_{n_k}}^{F_2})Aw_{n_k}\| \le \|A^*\|\|(I - T_{r_{n_k}}^{F_2})Aw_{n_k}\| = \|A\|\|(I - T_{r_{n_k}}^{F_2})Aw_{n_k}\| = 0. \tag{5.3.29}$$

From the definition of y_{n_k} and by applying (5.3.26) we have

$$\|y_{n_k} - z_{n_k}\| \le \alpha_{n_k}\|w_{n_k} - z_{n_k}\| + (1 - \alpha_{n_k})\|z_{n_k} - z_{n_k}\| \to 0, \quad k \to \infty. \tag{5.3.30}$$

From (5.3.26) and (5.3.30) we have

$$\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0; \quad \lim_{k\to\infty} \|w_{n_k} - u_{n_k}\| = 0; \quad \lim_{k\to\infty} \|z_{n_k} - u_{n_k}\| = 0. \tag{5.3.31}$$

From **Step 5** and by applying (5.3.26) together with the fact that $\lim_{k\to\infty} \gamma_{n_k} = 0$, we have

$$\begin{aligned} \|x_{n_{k+1}} - w_{n_k}\| &= \|(1 - \beta_{n_k} - \gamma_{n_k})w_{n_k} + \beta_{n_k}v_{n_k} - w_{n_k}\| \\ &= \|\beta_{n_k}(v_{n_k} - w_{n_k}) - \gamma_{n_k}w_{n_k}\| \\ &\le \beta_{n_k}\|v_{n_k} - w_{n_k}\| + \gamma_{n_k}\|w_{n_k}\| \to 0, \quad k \to \infty. \end{aligned} \tag{5.3.32}$$

Also, by Remark 5.3.2 we have

$$\|w_{n_k} - x_{n_k}\| = \theta_{n_k}\|x_{n_k} - x_{n_{k-1}}\| \to 0, \quad k \to \infty. \tag{5.3.33}$$

Also, from (5.3.32) and (5.3.33) we have

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0 \tag{5.3.34}$$

From (5.3.26)-(5.3.33) we obtain

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = 0, \ \lim_{k\to\infty} \|x_{n_k} - v_{n_k}\| = 0, \ \lim_{k\to\infty} \|x_{n_k} - u_{n_k}\| = 0, \ \lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0. \tag{5.3.35}$$

Since $\{x_n\}$ is bounded, $w_\omega(x_n)$ is nonempty. Let $x^* \in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$. From (5.3.35) we obtain $y_{n_k} \rightharpoonup x^*$ as $k \to \infty$. By invoking Lemma 5.3.7, it follows from (5.3.26) that $x^* \in (B+D)^{-1}(0)$. Since x^* is an arbitrary element in $w_\omega(x_n)$, it follows that $w_\omega(x_n) \subset (B + D)^{-1}(0)$.

Next, we show that $w_\omega(x_n) \subset \Omega$. First, we establish that $w_\omega(x_n) \subset EP(F_1)$. From $z_{n_k} = T^{F_1}_{r_{n_k}}(I - \lambda_{n_k} A^*(I - T^{F_2}_{r_{n_k}})A)w_{n_k}$, we have

$$F_1(z_{n_k}, y) + \frac{1}{r_{n_k}}\Big\langle y - z_{n_k},\ z_{n_k} - w_{n_k} + \lambda_{n_k} G(w_{n_k})\Big\rangle \geq 0$$

for all $y \in \mathcal{C}$ and this implies that

$$F_1(z_{n_k}, y) + \frac{1}{r_{n_k}}\Big\langle y - z_{n_k},\ z_{n_k} - w_{n_k}\Big\rangle + \frac{1}{r_{n_k}}\Big\langle y - z_{n_k},\ \lambda_{n_k} G(w_{n_k})\Big\rangle \geq 0,\ \forall\ y \in \mathcal{C}.$$

We have from Assumption 5.2.1 (2) that

$$\frac{1}{r_{n_k}}\Big\langle y - z_{n_k},\ z_{n_k} - w_{n_k}\Big\rangle + \frac{1}{r_{n_k}}\Big\langle y - z_{n_k},\ \lambda_{n_k} G(w_{n_k})\Big\rangle \geq F_1(y,\ z_{n_k}),\ \forall\ y \in \mathcal{C}.$$

Since $z_{n_k} \rightharpoonup x^*$, then by applying (5.3.26), (5.3.29), Assumption 5.2.1 (4) and the fact that $\liminf_{k\to\infty} r_{n_k} > 0$, we obtain

$$F_1(y, x^*) \leq 0,\ \forall\ y \in \mathcal{C}. \tag{5.3.36}$$

Let $y_t = ty + (1-t)x^*$, $\forall t \in (0,1]$ and $y \in \mathcal{C}$. This implies that $y_t \in \mathcal{C}$. Hence, from (5.3.36) we get $F_1(y_t, x^*) \leq 0$. Applying Assumption 5.2.1 (1)-(4), we have

$$\begin{aligned} 0 &= F_1(y_t, y_t) \\ &\leq tF_1(y_t, y) + (1-t)F_1(y_t, x^*) \\ &\leq tF_1(y_t, y). \end{aligned}$$

Hence, we have $F_1(y_t, y) \geq 0$, $\forall y \in \mathcal{C}$. Letting $t \to 0$ and applying Assumption 5.2.1 (3), we have

$$F_1(x^*, y) \geq 0,\ \forall y \in \mathcal{C}, \tag{5.3.37}$$

which implies that $x^* \in EP(F_1)$.

Next, we show that $Ax^* \in EP(F_2)$. Since A is a bounded linear operator and $w_{n_k} \rightharpoonup x^*$, we have $Aw_{n_k} \rightharpoonup Ax^*$. Consequently, it follows from (5.3.28) that

$$T^{F_2}_{r_{n_k}} Aw_{n_k} \rightharpoonup Ax^* \tag{5.3.38}$$

as $k \to \infty$. From the definition of $T^{F_2}_{r_{n_k}} Aw_{n_k}$, we have

$$F_2(T^{F_2}_{r_{n_k}} Aw_{n_k}, y) + \frac{1}{r_{n_k}}\Big\langle y - T^{F_2}_{r_{n_k}} Aw_{n_k},\ T^{F_2}_{r_{n_k}} Aw_{n_k} - Aw_{n_k}\Big\rangle \geq 0,\ \forall\ y \in \mathcal{Q}.$$

Since F_2 is upper semicontinuous in the first argument, it follows from (5.3.28), (5.3.38) and the fact that $\liminf_{k\to\infty} r_{n_k} > 0$, that

$$F_2(Ax^*, y) \geq 0, \; \forall \; y \in \mathcal{Q}, \tag{5.3.39}$$

which implies that $Ax^* \in EP(F_2)$. Thus, $w_\omega(x_n) \subset \Gamma$.

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to $\hat{x}$ such that

$$\limsup_{k\to\infty} \left\langle q, \; q - x_{n_k} \right\rangle = \lim_{j\to\infty} \langle q, \; q - x_{n_{k_j}} \rangle.$$

Since $q = P_\Gamma(0)$, we have

$$\limsup_{k\to\infty} \left\langle q, \; q - x_{n_k} \right\rangle = \lim_{j\to\infty} \langle q, \; q - x_{n_{k_j}} \rangle = \left\langle q, \; q - \hat{x} \right\rangle \leq 0,$$

which together with (5.3.34) implies that

$$\lim_{k\to\infty} \left\langle q, \; q - x_{n_{k+1}} \right\rangle \leq 0. \tag{5.3.40}$$

Using (5.3.26), (5.3.40) together with the fact that $\lim_{n\to\infty} \frac{\theta_n}{\gamma_n} \|x_n - x_{n-1}\| = 0$, we have $\limsup_{k\to\infty} a_{n_k} \leq 0$. Now, applying Lemma 2.5.36 to (5.3.24) we have that $\lim_{n\to\infty} \|x_n - q\| = 0$. Therefore, $\{x_n\}$ converges strongly to q.

□

5.3.3 Applications

In this section, we apply our result to study certain optimization problems.

Variational inclusion and split variational inequality problems

In this subsection, we apply our result to approximate the common solution of V_qIP and SVIP.

Here, we apply our result to study the following SVIP with constraint of V_qIP:

$$\text{Find} \;\; x^* \in (B + D)^{-1}(0) \;\; \text{such that} \;\; \langle x - x^*, M_1 x^* \rangle \geq 0, \quad \forall \; x \in \mathcal{C}. \tag{5.3.41}$$

and such that

$$y^* = Ax^* \in \mathcal{Q} \;\; \text{solves} \;\; \langle y - y^*, M_2 y^* \rangle \geq 0, \quad \forall \; y \in \mathcal{Q}. \tag{5.3.42}$$

where $\mathcal{C}$ and $\mathcal{Q}$ are nonempty, closed and convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $M : \mathcal{H} \to \mathcal{H}$ be a single-valued mapping, $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator, $B : \mathcal{H}_1 \to \mathcal{H}_1$ is a Lipschitz continuous and monotone operator and $D : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is a maximal monotone operator. We denote the solution set of problem (5.3.41)-(5.3.42) by Ω_1 and assume that $(B+D)^{-1}(0)\cap\Omega_1 \neq \emptyset$. Taking $F_i(x,y) := \langle y-x, M_i x\rangle, i = 1,2$, then the problem (5.3.41)-(5.3.42) becomes the problem of finding a solution of the SEP (1.2.10)-(1.2.11) which is also a solution of the V_qIP (1.2.9). Furthermore, all the conditions of Theorem 5.3.10 hold. Thus, Theorem 5.3.10 provides a strong convergence theorem for approximating a common solution of V_qIP (1.2.9) and the problem (5.3.41)-(5.3.42).

Convex minimization and split equilibrium problems

Let $\mathcal{H}$ be a real Hilbert space, $f : \mathcal{H} \to \mathbb{R}$ be a convex function and $F : \mathcal{H} \to \mathbb{R}$ be a proper convex, lower semicontinuous function. We apply our result to approximate the solution of the following *convex MP*:

$$\min_{x\in H} f(x) + F(x), \tag{5.3.43}$$

which is equivalent to finding $x \in \mathcal{H}$ such that

$$0 \in \nabla f(x) + \partial F(x), \tag{5.3.44}$$

where ∇f is the gradient of f and ∂F is the subdifferential of F. It is known that if ∇f is L-Lipschitz continuous, then it is $\frac{1}{L}$-inverse strongly monotone (co-coercive), and hence it is L-Lipschitz continuous and monotone. Also, it is known that ∂F is maximal monotone (see [216]). The solution set of (5.3.43) we denote by Ω_2.

So, by setting $B = \nabla f$ and $D = \partial F$ in Theorem 5.3.10, we obtain the following result for approximating a common solution of convex MP (5.3.43) and split equilibrium problem (1.2.10)-(1.2.11) in Hilbert spaces.

Theorem 5.3.11. *Let $\mathcal{C}$ and $\mathcal{Q}$ be nonempty, closed and convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Suppose $F_1 : \mathcal{C}\times\mathcal{C} \to \mathbb{R}$, $F_2 : \mathcal{Q}\times\mathcal{Q} \to \mathbb{R}$ are bifunctions satisfying Assumption (5.2.1) and such that F_2 is upper semi continuous in the first argument. Let $f : \mathcal{H} \to \mathbb{R}$ be a convex and differentiable function such that ∇f is L-Lipschitz continuous and suppose $G : \mathcal{H} \to \mathbb{R}$ is a proper convex, lower semicontinuous function. Let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 5.3.3.

Step 1: Select initial point $x_0, x_1 \in \mathcal{H}_1$, let $s_1 > 0, \mu \in (0,1), \theta \geq 3$ and set $n = 1$. Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.} \end{cases} \tag{5.3.45}$$

Step 2: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3: Compute

$$z_n = T_{r_n}^{F_1}\big(I - \lambda_n A^*(I - T_{r_n}^{F_2})A\big)w_n$$

and

$$y_n = \alpha_n w_n + (1-\alpha_n)z_n.$$

Step 4: Compute

$$u_n = J_{s_n}^{\partial F}(I - s_n \nabla f)y_n$$

and

$$v_n = u_n - s_n(\nabla f u_n - \nabla f y_n).$$

Step 5 Compute

$$x_{n+1} = (1-\beta_n-\gamma_n)w_n + \beta_n v_n,$$

where

$$\lambda_n := \begin{cases} \tau_n \frac{g(w_n)}{\|G(w_n)\|^2 + \|H(w_n)\|^2}, & \text{if} \quad \|G(w_n)\|^2 + \|H(w_n)\|^2 \neq 0, \\ 0, & \text{otherwise} \end{cases} \tag{5.3.46}$$

and

$$s_{n+1} = \begin{cases} \min\Big\{\frac{\mu\|y_n - u_n\|}{\|\nabla f y_n - \nabla f u_n\|}, s_n\Big\} & \text{if} \quad \nabla f y_n - \nabla f u_n \neq 0, \\ s_n, & \text{otherwise.} \end{cases} \tag{5.3.47}$$

Set $n := n+1$ and go back to **Step 1**.

Then, the sequence $\{x_n\}$ *converges strongly to a point* $q \in \Omega_2 \cap \Omega$, *where* $\|q\| = \min\{\|z\| : z \in \Omega_2 \cap \Omega\}$.

5.3.4 Numerical experiments

In this section, using some test examples, we discuss the numerical behavior of Algorithm 5.3.2 as well as compare it with the standard forward-backward method (i.e when $\theta_n = 0$), Algorithm 2.5.23 proposed by Cholamjiak *et al.* [81], the shrinking projection method of Cholamjiak *et al.* [81] (see Appendix 5.3.14), Appendix 5.3.15 and Appendix 5.3.16. We perform all implementations using Matlab 2016 (b), installed on a personal computer with Intel(R) Core(TM) i5-2600 CPU@2.30GHz and 8.00 Gb-RAM running on Windows 10 operating system. In Tables 5.3.1-5.3.2, "No. of Iter." means the number of iterations.

In our computations, we choose $\alpha_n = \frac{n+1}{2n+1}, \gamma_n = \frac{1}{n+2}, \beta_n = \frac{1}{2} - \gamma_n$, $\epsilon_n = \frac{1}{(n+2)^2}$, $\theta = 7$, $s_1 = 0.65$, $\mu = 0.8$, $\tau_n = \frac{6n+2}{2n+1}, r_n = \frac{3n+1}{2n+1}$, and we take $\delta_n = \frac{n+1}{2n+3}, t_n = 0.1$ in Algorithm 2.5.23 and Appendix 5.3.14. Also, for Appendix 5.3.16 we choose $f(x) = \frac{x}{2}$. Furthermore, in the implementation, we define $\text{TOL}_n := \|x_{n+1} - x_n\|$ and use the stopping criterion $\text{TOL}_n < 10^{-2}$ for the iterative processes.

Example 5.3.12. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$, the set of all real numbers with the inner product defined by $\langle x, y\rangle = xy, \quad \forall\ x, y \in \mathbb{R}$ and induced norm $|\cdot|$. For $r > 0$, Consider $\mathcal{C} = [-10, 10]$ and $\mathcal{Q} = [0, 20]$, we define the bifunctions $F_1 : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ and $F_2 : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ as follows;

$F_1 = -2x^2 + xy + y^2$, $F_2 = -x^2 + xy$. After simple calculation and applying Lemma 5.2.1, we get

$$T_r^{F_1}(u) = \frac{u}{3r+1}, \quad \forall\, u \in \mathcal{C}$$

and

$$T_r^{F_2}(v) = \frac{v}{r+1}, \quad \forall\, v \in \mathcal{Q}.$$

Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be defined by $Ax = 2x$, $B : \mathcal{H}_1 \to \mathcal{H}_1$ be defined by $Bx = x + \sin x$ and $D : \mathcal{H}_1 \to \mathcal{H}_1$ be defined by $Dx = 5x$, where $x \in \mathcal{H}_1$. Clearly, we see that B is $\frac{1}{3}$-inverse strongly monotone and D is maximal monotone.

We consider the following cases for the numerical experiments of this example and choose $\gamma = \frac{3}{20}$

Case I: Take $x_0 = -\frac{17}{33}$ and $x_1 = \frac{2}{15}$.

Case II: Take $x_0 = -\frac{19}{35}$ and $x_1 = \frac{1}{6}$.

Case III: Take $x_0 = \frac{11}{20}$ and $x_1 = \frac{1}{7}$.

Case IV: Take $x_0 = -\frac{10}{19}$ and $x_1 = -\frac{5}{34}$.

We compare the performance of our Algorithm 5.3.2 with Algorithm 2.5.23, Appendix 5.3.14, Appendix 5.3.15 and Appendix 5.3.16. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Table 5.3.1 and Figure 5.9.

Table 5.3.1: Numerical results for Example 5.3.12

		Alg. 2.5.23	App 5.3.14	App. 5.3.15	App. 5.3.16	Alg. 5.3.2
Case I	No. of Iter.	8	3	11	17	5
	CPU time (sec)	0.0099	0.0071	0.0090	0.0063	0.0114
Case II	No. of Iter.	8	3	11	17	5
	CPU time (sec)	0.0095	0.0075	0.0080	0.0072	0.0155
Case III	No. of Iter.	8	3	11	16	5
	CPU time (sec)	0.0103	0.0081	0.0093	0.0084	0.0125
Case IV	No. of Iter.	8	3	11	17	5
	CPU time (sec)	0.0098	0.0071	0.0082	0.0070	0.0111

Example 5.3.13. Let $\mathcal{H}_1 = \mathcal{H}_2 = (l_2(\mathbb{R}), \|\cdot\|_2)$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_n, \ldots), x_j \in \mathbb{R} : \sum_{j=1}^{\infty} |x_j|^2 < \infty\}$, $||x||_2 = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}}$ and $\langle x, y\rangle = \sum_{j=1}^{\infty} x_j y_j$ for all $x \in \ell_2(\mathbb{R})$. We define $F_1 : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ and $F_2 : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ by $F_1(x, y) = \langle L_1 x, y - x\rangle$ and $F_2(x, y) = \langle L_2 x, y - x\rangle$, where $L_1 x = \frac{x}{3}$ and $L_2 x = \frac{x}{2}$. One can easily verify that F_1 and F_2 satisfy Assumption 5.2.1. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be defined by $Ax = \frac{x}{3}$ and $A^* y = \frac{y}{3}$. Then,

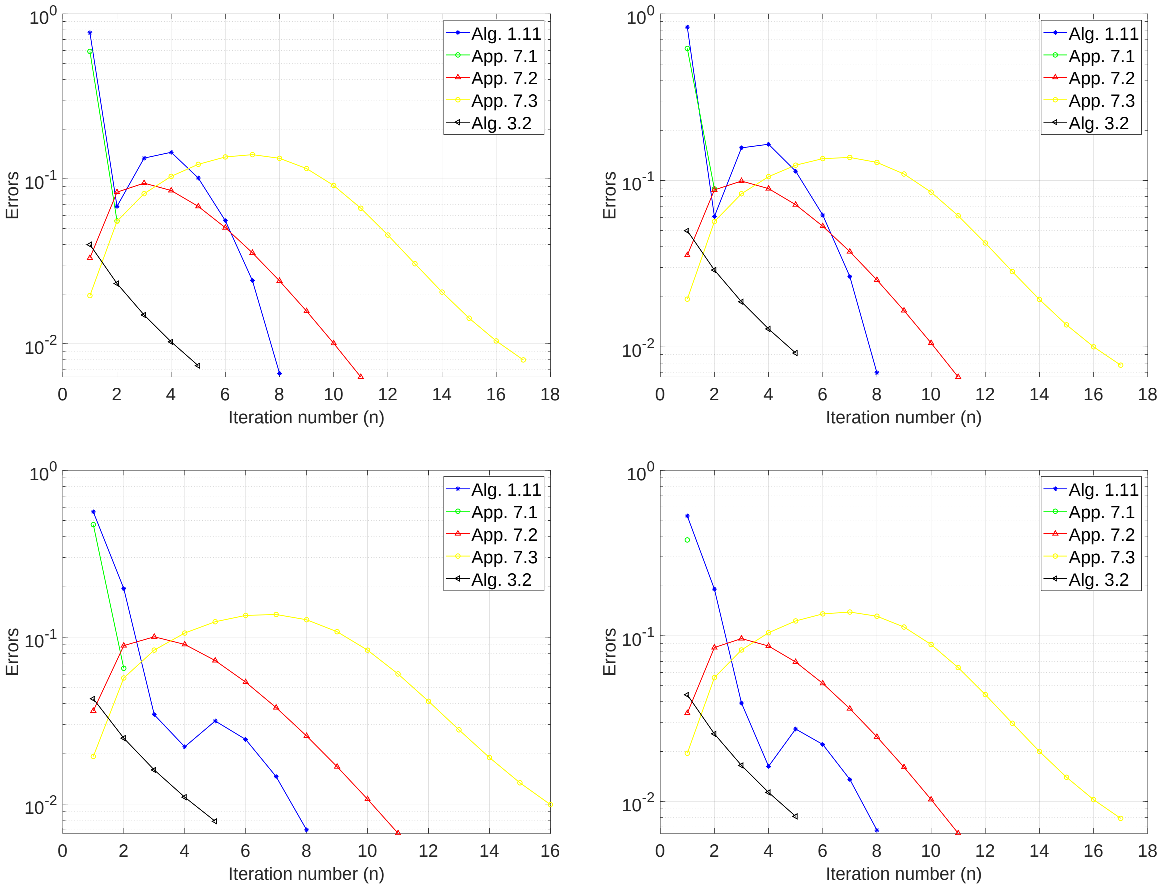

Figure 5.9: Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

A is a bounded linear operator. After simple calculation and applying Lemma 5.2.1, we obtain

$$T_r^{F_1}(u) = \frac{3u}{r+3}, \quad \forall\ u \in \mathcal{C},$$

and

$$T_s^{F_2}(v) = \frac{2v}{s+2}, \quad \forall\ v \in \mathcal{Q}.$$

Let $B : \mathcal{H}_1 \to \mathcal{H}_1$ be defined by $Bx = \frac{1}{3}x$ and $D : \mathcal{H}_1 \to \mathcal{H}_1$ be defined by $Dx = 3x$, where $x \in \mathcal{H}_1$. Clearly, we see that B is $\frac{1}{2}$-inverse strongly monotone and D is maximal monotone.

We consider different initial values as follows:
Case I: $x_0 = (-23, 1, -\frac{1}{23}, \cdots)$, $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$;
Case II: $x_0 = (37, 1, \frac{1}{37}, \cdots)$, $x_1 = (\frac{5}{18}, \frac{5}{36}, \frac{5}{72}, \cdots)$;
Case III: $x_0 = (-29, 1, -\frac{1}{29}, \cdots)$, $x_1 = (-\frac{3}{8}, \frac{3}{16}, -\frac{1}{32}, \cdots)$;
Case IV: $x_0 = (25, 1, \frac{1}{25}, \cdots)$, $x_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{18})$.

We compare the performance of our Algorithm 5.3.2 with Appendix 5.3.14, Appendix 5.3.15 and Appendix 5.3.16. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Table 5.3.2 and Figure 5.10.

Table 5.3.2: Numerical results for Example 5.3.13

		Alg. 2.5.23	App 5.3.14	App. 5.3.15	App. 5.3.16	Alg. 5.3.2
Case I	No. of Iter.	13	3	14	11	6
	CPU time (sec)	0.0137	0.0148	0.0145	0.0092	0.0175
Case II	No. of Iter.	13	3	14	11	6
	CPU time (sec)	0.0114	0.0123	0.0120	0.0097	0.0170
Case III	No. of Iter.	13	3	14	11	6
	CPU time (sec)	0.0113	0.0135	0.0154	0.0082	0.0150
Case IV	No. of Iter.	13	3	14	11	6
	CPU time (sec)	0.0126	0.0162	0.0139	0.0122	0.0245

Appendix 5.3.14. *The Algorithm in [81].*

Initialization: *Given* $\gamma \in (0, \frac{1}{L})$. *Let* $x_0, x_1 \in \mathcal{H}_1$ *be arbitrary.*

Iterative Steps: *Calculate* x_{n+1} *as follows:*

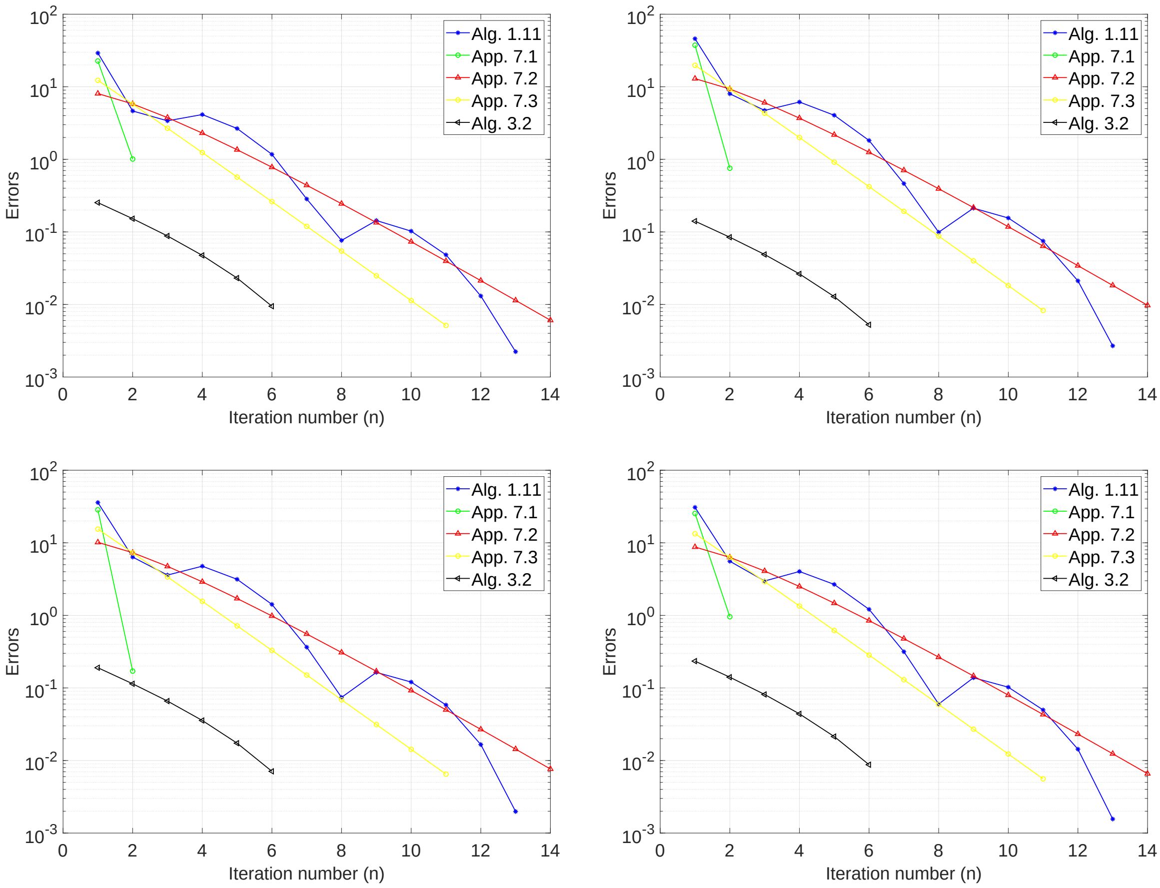

Figure 5.10: Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

$$\begin{cases} y_n = x_n + \delta_n(x_n - x_{n-1}) \\ z_n = \alpha_n y_n + (1-\alpha_n)T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)y_n, \\ w_n = \beta_n z_n + (1-\beta_n)J_{t_n}^D(I - t_nB)z_n, \\ C_{n+1} = \{z \in C_n : \|w_n - z\|^2 \le \|x_n - z\|^2 + 2{\delta_n}^2\|x_n - x_{n-1}\|^2 - 2\delta\langle x_n - z,\ x_{n-1} - x_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}}x_1,\ \ n \ge 1, \end{cases} \tag{5.3.48}$$

where $J_{t_n}^D = (I + t_nD)^{-1}$, $\{t_n\} \subset (0, 2\alpha)$, $\{\delta_n\} \subset [0, \delta]$, $\delta \in [0, 1)$, $\{r_n\} \subset (0, \infty)$ *with* $\gamma \in (0, \frac{1}{L})$ *such that* L *is the spectral radius of* A^*A *and* $\{\alpha_n\}, \{\beta_n\}$ *are sequences in* $[0, 1]$.

Set $n := n + 1$ *and return to* **Step 1**.

Appendix 5.3.15. *The Algorithm in [108].*

Initialization: *Given* $s_1 > 0, \mu \in (0, 1)$. *Let* $x_1 \in \mathcal{H}$ *be arbitrary.*

Iterative Steps: *Given the current iterates* x_n, *calculate the next iterate as follows:*
Step 1.

$$y_n = (I + s_nD)^{-1}(I - s_nB)x_n.$$

If $x_n = y_n$ *then stop and* y_n *is a solution of* (1.2.9). *Otherwise,*
Step 2. *Compute*

$$z_n = y_n - s_n(By_n - Bx_n)$$

and

$$x_{n+1} = (1 - \beta_n - \gamma_n)x_n + \gamma_n z_n.$$

Update

$$s_{n+1} = \begin{cases} min\left\{\frac{\mu\|x_n - y_n\|}{\|Bx_n - By_n\|}, s_n\right\} & \text{if} \quad Bx_n - By_n \ne 0, \\ s_n, & \text{otherwise.} \end{cases} \tag{5.3.49}$$

Set $n := n + 1$ *and return to* **Step 1**.

Appendix 5.3.16. *The Algorithm in [108].*

Initialization: *Given* $s_1 > 0, \mu \in (0, 1)$. *Let* $x_1 \in \mathcal{H}$ *be arbitrary.*

Iterative Steps: *Given the current iterates* x_n, *calculate the next iterate as follows:*
Step 1.

$$y_n = (I + s_nD)^{-1}(I - s_nB)x_n.$$

If $x_n = y_n$ *then stop and* y_n *is a solution of* (1.2.9). *Otherwise,*
Step 2. *Compute*

$$z_n = y_n - s_n(By_n - Bx_n)$$

and

$$x_{n+1} = \gamma_n f(x_n) + (1 - \gamma_n)z_n.$$

Update

$$s_{n+1} = \begin{cases} min\Big\{\frac{\mu\|x_n - y_n\|}{\|Bx_n - By_n\|}, s_n\Big\} & \textit{if} \quad Bx_n - By_n \neq 0, \\ s_n, & \textit{otherwise.} \end{cases} \tag{5.3.50}$$

Set $n := n + 1$ *and return to* **Step 1**.

CHAPTER 6

Results on Variational Inequality, Split Monotone Variational Inclusion and Fixed Point Problems in Banach Spaces

6.1 Introduction

The extension of known basis concepts from Hilbert spaces to Banach spaces has been of great interest to many researchers due to the fact that many real life problems can be posed in general Banach spaces than in Hilbert spaces. In this chapter, we propose and study an iterative algorithm for approximating the solution of VIPs in a real Banach space. Furthermore, between a Banach and Hilbert space, we propose and study an effective iterative algorithm for approximating a common solution of the SMV_qIP and FPP. We apply our result to study some optimization problems and present some numerical experiments of our methods in comparison to other methods in literature.

6.2 On pseudomonotone variational inequalities with non-Lipschitz operators

In this section, we study the pseudomonotone VIP with non-Lipschitz operators in Banach spaces. We propose an inertial subgradient extragradient method with Halpern technique and Armijo type step size for approximating the solution of the problem in the framework of 2-uniformly convex real Banach spaces. We prove that the sequence generated by our proposed method converges strongly to the solution of the problem under some mild conditions and without the weakly sequential continuity condition often assumed by authors in solving pseudomonotone VIP. Finally, we provide some numerical experiments for the proposed method in comparison with other existing methods in the literature. Our result extends and improves several of the existing results in the current literature in this direction.

6.2.1 Proposed method

In this section, we present the proposed method and highlight some of its features. We start by giving the following assumptions under which our strong convergence result is obtained.

Assumption 6.2.1. *Suppose that the following conditions hold:*

(a) $\mathcal{X}$ *is a real 2-uniformly convex and uniformly smooth Banach space with 2-uniform convexity constant* μ *and* $\mathcal{C}$ *is a nonempty, closed and convex subset of* $\mathcal{X}$.

(b) $A : \mathcal{X} \to \mathcal{X}^*$ *is pseudomonotone and uniformly continuous on* $\mathcal{X}$.

(c) The solution set $VI(\mathcal{C}, A)$ *is nonempty.*

(d) Let $\{\tau_n\}$ *be a positive sequence such that* $\lim_{n\to\infty} \frac{\tau_n}{\alpha_n} = 0$ *where* α_n *is a sequence in* $(0,1)$ *satisfying the following conditions:* $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$

Now, our proposed method for solving the VIP (1.2.1) is presented as follows:

Algorithm 6.2.2. Inertial method with new Armijo-type step size.

Initialization: Given $\lambda_1 > 0, \quad \theta > 0, \quad \delta > 0, \ell \in (0,1)$ and $\eta \in (0, \frac{1}{\mu})$. Let $u, \ x_0, x_1 \in \mathcal{X}$ be arbitrary. Set $n := 1$.

Step 1: Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\theta, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases} \tag{6.2.1}$$

Step 2: Compute

$$w_n = J^{-1}(Jx_n + \theta_n(Jx_{n-1} - Jx_n)).$$

Step 3: Compute

$$y_n = \Pi_{\mathcal{C}} J^{-1}(Jw_n - \lambda_n A w_n).$$

If $w_n = y_n$, or $Ay_n = 0$ then stop and y_n is a solution of the VIP. Otherwise, go to **Step 4**.

Step 4: Compute

$$z_n = \Pi_{T_n} J^{-1}(Jw_n - \lambda_n A y_n)$$

where

$$T_n := \{x \in \mathcal{X} \ |\langle Jw_n - \lambda_n A w_n - Jy_n, \ \ x - y_n\rangle \leq 0\},$$

$\lambda_n := \delta\ell^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\delta\ell^m\langle Ay_n - Aw_n, \quad y_n - z_n\rangle \leq \frac{\eta}{2}\Big[\|w_n - y_n\|^2 + \|y_n - z_n\|^2\Big]. \tag{6.2.2}$$

Step 5: Calculate

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n).$$

Set $n := n+1$ and go to **Step 1**.

Remark 6.2.1. If $w_n = y_n$ or $Ay_n = 0$ then $y_n \in VI(\mathcal{C}, A)$. From Lemma 2.5.23 and the fact that $0 < \lambda_n \leq \delta$, we have

$$0 = \frac{\|w_n - y_n\|}{\lambda_n} = \frac{\|w_n - \Pi_{\mathcal{C}}J^{-1}(Jw_n - \lambda_n Aw_n)\|}{\lambda_n} \geq \frac{\|w_n - \Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta Aw_n)\|}{\delta}$$

which shows that w_n is a solution of the VIP. Thus, $y_n \in VI(\mathcal{C}, A)$. Also, since $y_n \in \mathcal{C}$, we observe that if $Ay_n = 0$ then $y_n = \Pi_{\mathcal{C}}J^{-1}(Jy_n - \delta Ay_n)$, which implies that $y_n \in VI(\mathcal{C}, A)$.

6.2.2 Convergence analysis

Lemma 6.2.2. *Suppose that Assumption 6.2.1 holds. The Armijo-like criteria (6.2.2) is well defined. Furthermore, we have that $\lambda_n \leq \delta$.*

Proof. If $w_n \in VI(\mathcal{C}, \ A)$ then $w_n = \Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta Aw_n)$, which implies that $w_n = y_n$. Thus $m_n = 0$. If $w_n \notin VI(\mathcal{C}, A)$, we prove by contradiction by assuming that the contrast of (6.2.2) holds, that is,

$$\delta\ell^m\left\langle A\Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta\ell^m Aw_n) - Aw_n, \ \ \Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta\ell^m Aw_n) - \Pi_{T_n}J^{-1}(Jw_n - \delta\ell^m Ay_n)\right\rangle$$
$$> \frac{\eta}{2}\left[\|w_n - \Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta\ell^m Aw_n)\|^2 + \|\Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta\ell^m Aw_n) - \Pi_{T_n}J^{-1}(Jw_n - \delta\ell^m Ay_n)\|^2\right].$$

This implies that

$$\delta\ell^m\left\|A\Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta\ell^m Aw_n) - Aw_n\right\| \cdot \left\|\Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta\ell^m Aw_n) - \Pi_{T_n}J^{-1}(Jw_n - \delta\ell^m Ay_n)\right\|$$
$$> \eta\left\|w_n - \Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta\ell^m Aw_n)\right\| \cdot \left\|\Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta\ell^m Aw_n) - \Pi_{T_n}J^{-1}(Jw_n - \delta\ell^m Ay_n)\right\|.$$

Thus, we obtain

$$\left\|A\Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta\ell^m Aw_n) - Aw_n\right\| > \frac{\eta}{\delta\ell^m}\left\|w_n - \Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta\ell^m Aw_n)\right\|. \tag{6.2.3}$$

Next, we consider the following cases: when $w_n \in \mathcal{C}$ and $w_n \notin \mathcal{C}$.

First, suppose that $w_n \in \mathcal{C}$. By the continuity of A and $\Pi_{\mathcal{C}}$ we have

$$\lim_{m\to\infty}\left\|w_n - \Pi_{\mathcal{C}}J^{-1}(Jw_n - \delta\ell^m Aw_n)\right\| = 0.$$

Consequently, by the continuity of A we get

$$\lim_{m\to\infty} \left\|Aw_n - A\Pi_{\mathcal{C}} J^{-1}(Jw_n - \delta\ell^m Aw_n)\right\| = 0. \tag{6.2.4}$$

From (6.2.3) and (6.2.4), we obtain

$$\lim_{m\to\infty} \frac{\|w_n - \Pi_{\mathcal{C}} J^{-1}(Jw_n - \delta\ell^m Aw_n)\|}{\delta\ell^m} = 0.$$

By the continuity of J, we have

$$\lim_{m\to\infty} \frac{\|Jw_n - J\Pi_{\mathcal{C}} J^{-1}(Jw_n - \delta\ell^m Aw_n)\|}{\delta\ell^m} = 0. \tag{6.2.5}$$

Let $q_m = \Pi_{\mathcal{C}} J^{-1}(Jw_n - \delta\ell^m Aw_n)$. Then from (2.4.2), we have

$$\langle Jq_m - Jw_n + \delta\ell^m Aw_n,\ y - q_m\rangle \geq 0, \quad \forall y \in \mathcal{C},$$

which implies that

$$\langle Aw_n,\ y - q_m\rangle \geq \frac{\langle Jw_n - Jq_m,\ \ y - q_m\rangle}{\delta\ell^m}.$$

Letting $m \to \infty$ and using (6.2.5), we have

$$\langle Aw_n,\ y - w_n\rangle \geq 0, \quad \forall y \in \mathcal{C}.$$

This implies that $w_n \in VI(\mathcal{C}, A)$, which contradicts the hypothesis.

Next suppose $w_n \notin \mathcal{C}$, then we obtain

$$\lim_{m\to\infty} \left\|w_n - \Pi_{\mathcal{C}} J^{-1}(Jw_n - \delta\ell^m Aw_n)\right\| = \|w_n - \Pi_{\mathcal{C}} w_n\| > 0 \tag{6.2.6}$$

and

$$\lim_{m\to\infty} \delta\ell^m \left\|Aw_n - A\Pi_{\mathcal{C}} J^{-1}(Jw_n - \delta\ell^m Aw_n)\right\| = 0. \tag{6.2.7}$$

Combining (6.2.3), (6.2.6) and (6.2.7), we obtain a contradiction. Hence, the linesearch (6.2.2) is well defined. □

Lemma 6.2.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 6.2.2 under Assumption 6.2.1 and $p \in VI(\mathcal{C}, A)$. Then, we have*

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\Big(\phi(p, x_{n-1}) - \phi(p, x_n)\Big) = 0. \tag{6.2.8}$$

Proof. Let $p \in VI(\mathcal{C}, A)$. From (6.2.1), we obtain

$$\theta_n \|x_n - x_{n-1}\| \leq \tau_n \quad \text{for each} \quad n \geq 1. \tag{6.2.9}$$

Since $\lim_{n\to\infty} \frac{\tau_n}{\alpha_n} = 0$ and $\lim_{n\to\infty} \alpha_n = 0$, it follows that $\lim_{n\to\infty} \tau_n = 0$. Thus, we obtain

$$\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \tau_n = 0. \tag{6.2.10}$$

Since J is norm-to-norm continuous on subsets of $\mathcal{X}$, it follows that

$$\lim_{n\to\infty} \theta_n \|Jx_n - Jx_{n-1}\| = 0. \tag{6.2.11}$$

Again, from (6.2.9) we have

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\tau_n}{\alpha_n} = 0. \tag{6.2.12}$$

Since J is norm-to-norm continuous on $\mathcal{X}$, we have

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|Jx_n - Jx_{n-1}\| = 0. \tag{6.2.13}$$

Next, observe that

$$\begin{aligned} \phi(p, x_{n-1}) - \phi(p, x_n) &= \|p\|^2 - 2\langle p, Jx_{n-1}\rangle + \|x_{n-1}\|^2 - \big(\|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2\big) \\ &= \|x_{n-1}\|^2 - \|x_n\|^2 + 2\langle p, Jx_n - Jx_{n-1}\rangle \\ &\leq \|x_{n-1} - x_n\|\big(\|x_{n-1}\| + \|x_n\|\big) + 2\|p\|\|Jx_n - Jx_{n-1}\|. \end{aligned} \tag{6.2.14}$$

Now, by applying (6.2.12) and (6.2.13), it follows from (6.2.14) that

$$\begin{aligned} \lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\Big(\phi(p, x_{n-1}) - \phi(p, x_n)\Big) &\leq \lim_{n\to\infty} \left(\frac{\theta_n}{\alpha_n}\|x_{n-1} - x_n\|\big(\|x_{n-1}\| + \|x_n\|\big)\right) \\ &\quad + \lim_{n\to\infty} \left(2\|p\|\frac{\theta_n}{\alpha_n}\|Jx_n - Jx_{n-1}\|\right) \\ &= 0, \end{aligned}$$

which is the desired result.

□

Lemma 6.2.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 6.2.2 under Assumption 6.2.1. Then,*

$$\phi(p, \quad z_n) \leq \phi(p, \quad w_n) - (1 - \mu\eta)\Big[\phi(y_n, w_n) + \phi(z_n, y_n)\Big]. \tag{6.2.15}$$

Proof. Let $p \in VI(\mathcal{C}, A)$. and $Jd_n := Jw_n - \lambda_n Ay_n$, by (2.4.3) we have

$$\begin{aligned}
\phi(p,\ z_n) &\leq \phi(p,\ d_n) - \phi(z_n,\ d_n) \\
&= \|p\|^2 - 2\langle p,\ Jw_n - \lambda_n Ay_n\rangle + \|d_n\|^2 - \left(\|z_n\|^2 - 2\langle z_n,\ Jw_n - \lambda_n Ay_n\rangle + \|d_n\|^2\right) \\
&= \|p\|^2 + \|d_n\|^2 - \|z_n\|^2 - \|d_n\|^2 - 2\langle p,\ Jw_n - \lambda_n Ay_n\rangle + 2\langle z_n,\ \ Jw_n - \lambda_n Ay_n\rangle \\
&= \|p\|^2 - \|z_n\|^2 - 2\langle p,\ Jw_n - \lambda_n Ay_n\rangle + 2\langle z_n,\ \ Jw_n - \lambda_n Ay_n\rangle \\
&= \|p\|^2 - \|z_n\|^2 - 2\langle p,\ Jw_n\rangle + 2\langle p,\ \lambda_n Ay_n\rangle + 2\langle z_n,\ Jw_n\rangle - 2\langle z_n,\ \lambda_n Ay_n\rangle \\
&= \|p\|^2 - \|z_n\|^2 + \|w_n\|^2 - \|w_n\|^2 - 2\langle p,\ Jw_n\rangle + 2\langle p,\ \lambda_n Ay_n\rangle + 2\langle z_n,\ Jw_n\rangle \\
&\quad - 2\langle z_n,\ \lambda_n Ay_n\rangle \\
&= \left(\|p\|^2 - 2\langle p,\ Jw_n\rangle + \|w_n\|^2\right) - \left(\|w_n\|^2 - 2\langle z_n,\ \ Jw_n\rangle + \|z_n\|^2\right) - 2\langle z_n,\ \lambda_n Ay_n\rangle \\
&\quad + 2\langle p,\ \lambda_n Ay_n\rangle \\
&= \phi(p,\ w_n) - \phi(z_n, w_n) - 2\langle z_n,\ \lambda_n Ay_n\rangle + 2\langle p,\ \lambda_n Ay_n\rangle \\
&= \phi(p,\ w_n) - \phi(z_n, w_n) + 2\lambda_n\langle p - y_n,\ Ay_n\rangle + 2\lambda_n\langle y_n - z_n,\ Ay_n\rangle. \qquad (6.2.16)
\end{aligned}$$

Since $p \in VI(\mathcal{C}, A)$ and $y_n \in \mathcal{C}$, we have

$$\langle Ap,\ \ y_n - p\rangle \geq 0.$$

Also, since A is pseudomonotone, it follows that

$$\langle Ay_n,\ \ y_n - p\rangle \geq 0.$$

Applying the last inequality in (6.2.16), we have

$$\phi(p,\ z_n) \leq \phi(p,\ w_n) - \phi(z_n,\ w_n) + 2\lambda_n\langle y_n - z_n,\ Ay_n\rangle. \qquad (6.2.17)$$

Next, by applying Lemma 2.2.13, we get

$$\begin{aligned}
\phi(z_n, w_n) - 2\lambda_n\langle y_n - z_n,\ Ay_n\rangle &= \phi(z_n,\ y_n) + \phi(y_n,\ w_n) + 2\langle z_n - y_n,\ Jy_n - Jw_n\rangle \\
&\quad - 2\lambda_n\langle y_n - z_n,\ Ay_n\rangle \\
&= \phi(z_n,\ y_n) + \phi(y_n,\ w_n) - 2\langle z_n - y_n,\ Jw_n - Jy_n - \lambda_n Ay_n\rangle \\
&= \phi(z_n,\ y_n) + \phi(y_n,\ w_n) - 2\langle z_n - y_n,\ Jw_n - \lambda_n Aw_n - Jy_n\rangle \qquad (6.2.18) \\
&\quad - 2\lambda_n\langle y_n - z_n,\ Ay_n - Aw_n\rangle.
\end{aligned}$$

Now, since $z_n \in T_n$ we obtain from the definition of T_n that

$$\langle z_n - y_n,\ \ Jw_n - \lambda_n Aw_n - Jy_n\rangle \leq 0.$$

Applying the last inequality in (6.2.18), we have

$$\begin{aligned}
\phi(z_n,\ w_n) - 2\lambda_n\langle y_n - z_n,\ \ Ay_n\rangle &\geq \phi(y_n,\ w_n) + \phi(z_n,\ y_n) - 2\lambda_n\langle y_n - z_n,\ Ay_n - Aw_n\rangle \\
&\geq \phi(y_n,\ w_n) + \phi(z_n,\ y_n) - \eta\left[\|w_n - y_n\|^2 + \|y_n - z_n\|^2\right]. \qquad (6.2.19)
\end{aligned}$$

Applying (6.2.19) in (6.2.17) and using Lemma 2.5.22, we obtain

$$\begin{aligned}\phi(p,\ z_n) &\leq \phi(p,\ w_n) - \phi(y_n,\ w_n) - \phi(z_n,\ y_n) + \eta\left[\|w_n - y_n\|^2 + \|y_n - z_n\|^2\right]\\ &= \phi(p,\ w_n) - \phi(y_n,\ w_n) - \phi(z_n,\ y_n) + \eta\left[\|y_n - w_n\|^2 + \|z_n - y_n\|^2\right]\\ &\leq \phi(p,\ w_n) - \phi(y_n,\ w_n) - \phi(z_n,\ y_n) + \eta\left[\mu\phi(y_n,\ w_n) + \mu\phi(z_n,\ y_n)\right]\\ &= \phi(p, w_n) - (1-\mu\eta)\left[\phi(y_n, w_n) + \phi(z_n,\ y_n)\right],\end{aligned}$$

which is the desired result. □

Lemma 6.2.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 6.2.2 under Assumption 6.2.1. Then, $\{x_n\}$ is bounded.*

Proof. Let $p \in VI(\mathcal{C}, A)$. Applying the conditions on η, we obtain from Lemma 6.2.4 that

$$\phi(p,\ z_n) \leq \phi(p,\ w_n). \tag{6.2.20}$$

From the definition of w_n, we have

$$\begin{aligned}\phi(p,\ w_n) &= \phi(p,\ J^{-1}((1-\theta_n)Jx_n + \theta_n Jx_{n-1}))\\ &\leq (1-\theta_n)\phi(p,\ x_n) + \theta_n\phi(p,\ x_{n-1}).\end{aligned} \tag{6.2.21}$$

By the definition of x_{n+1}, and by applying (6.2.20) and (6.2.21), we have

$$\begin{aligned}\phi(p,\ x_{n+1}) &= \phi(p,\ J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n))\\ &\leq \alpha_n\phi(p,\ u) + (1-\alpha_n)\phi(p,\ z_n)\\ &\leq \alpha_n\phi(p,\ u) + (1-\alpha_n)\phi(p,\ w_n)\\ &\leq \alpha_n\phi(p,\ u) + (1-\alpha_n)\Big[(1-\theta_n)\phi(p,\ x_n) + \theta_n\phi(p, x_{n-1})\Big]\\ &\leq \max\Big\{\phi(p,\ u),\ \max\{\phi(p,\ x_n),\ \ \phi(p,\ x_{n-1})\}\Big\}\\ &\ \ \vdots\\ &\leq \max\Big\{\phi(p,\ u),\ \max\{\phi(p,\ x_1),\ \ \phi(p,\ x_0)\}\Big\}.\end{aligned}$$

Hence, $\phi(p, x_n)$ is bounded, and it follows that $\{x_n\}$ is bounded. Consequently, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$ are bounded. □

Lemma 6.2.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 6.2.2 under Assumption 6.2.1. Then, the following inequality holds for all $p \in VI(\mathcal{C}, A)$:*

$$\begin{aligned}\phi(p,\ x_{n+1}) &\leq (1-\alpha_n)\phi(p,\ x_n)\\ &\quad + \alpha_n\left[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\big[\phi(p, x_{n-1}) - \phi(p, x_n)\big] + 2\langle Ju - Jp,\ x_{n+1} - p\rangle\right]\\ &\quad - (1-\alpha_n)(1-\mu\eta)\big[\phi(y_n, w_n) + \phi(z_n, y_n)\big]\end{aligned}$$

Proof. Let $p \in VI(\mathcal{C}, A)$. From the definition of x_{n+1} and by applying (2.2.2), (6.2.15) and (6.2.21), we have

$$\begin{aligned}
\phi(p,\ x_{n+1}) &= \phi\Big(p, J^{-1}\Big(\alpha_n Ju + (1-\alpha_n)Jz_n\Big)\Big)\\
&= V(p,\ \alpha_n Ju + (1-\alpha_n)Jz_n)\\
&\leq V(p,\ \alpha_n Ju + (1-\alpha_n)Jz_n - \alpha_n(Ju - Jp)) + 2\alpha_n\langle Ju - Jp,\ x_{n+1} - p\rangle\\
&\leq \alpha_n V(p,\ Jp) + (1-\alpha_n)V(p,\ Jz_n) + 2\alpha_n\langle Ju - Jp,\ x_{n+1} - p\rangle\\
&= \alpha_n(p,\ p) + (1-\alpha_n)\phi(p,\ z_n) + 2\alpha_n\langle Ju - Jp,\ x_{n+1} - p\rangle\\
&\leq (1-\alpha_n)\Big[\phi(p,\ \ w_n) - (1-\mu\eta)\big[\phi(y_n, w_n) + \phi(z_n, y_n)\big]\Big]\\
&+ 2\alpha_n\langle Ju - Jp,\ x_{n+1} - p\rangle\\
&\leq (1-\alpha_n)\left[(1-\theta_n)\phi(p,\ x_n) + \theta_n\phi(p,\ x_{n-1})\right] + 2\alpha_n\langle Ju - Jp,\ x_{n+1} - p\rangle\\
&\quad - (1-\alpha_n)(1-\mu\eta)\big[\phi(y_n, w_n) + \phi(z_n, y_n)\big] \qquad (6.2.22)\\
&= (1-\alpha_n)\phi(p,\ x_n)\\
&+ \alpha_n\left[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\big[\phi(p, x_{n-1}) - \phi(p, x_n)\big] + 2\langle Ju - Jp,\ x_{n+1} - p\rangle\right]\\
&\quad - (1-\alpha_n)(1-\mu\eta)\big[\phi(y_n, w_n) + \phi(z_n, y_n)\big],
\end{aligned}$$

which is the required inequality.

□

Lemma 6.2.7. *Let $\{w_n\}$ and $\{y_n\}$ be sequences generated by Algorithm 6.2.2 under Assumption 6.2.1. If there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\{w_{n_k}\}$ converges weakly to $p \in \mathcal{C}$ and $\lim_{k\to\infty}\|w_{n_k} - y_{n_k}\| = 0$, then $p \in VI(\mathcal{C}, A)$.*

Proof. For all $x \in \mathcal{C}$, we obtain from (2.4.2) and the definition of $\{y_{n_k}\}$ that

$$\begin{aligned}
0 &\leq \langle Jy_{n_k} - Jw_{n_k} + \lambda_{n_k}Aw_{n_k},\ x - y_{n_k}\rangle\\
&= \langle Jy_{n_k} - Jw_{n_k},\ x - y_{n_k}\rangle + \lambda_{n_k}\langle Aw_{n_k},\ x - y_{n_k}\rangle\\
&= \langle Jy_{n_k} - Jw_{n_k},\ x - y_{n_k}\rangle + \lambda_{n_k}\langle Aw_{n_k},\ w_{n_k} - y_{n_k}\rangle + \lambda_{n_k}\langle Aw_{n_k},\ x - w_{n_k}\rangle \qquad (6.2.23)
\end{aligned}$$

Next, we show that $\liminf_{k\to\infty}\langle Aw_{n_k},\ x - w_{n_k}\rangle \geq 0,\quad \forall\ x \in \mathcal{C}$. We consider the following two cases for λ_{n_k}:

Case 1: Suppose that $\liminf_{k\to\infty}\lambda_{n_k} > 0$. Since $\{w_{n_k}\}$ is bounded and A is uniformly continuous, we obtain from Lemma 2.5.21 that $\{Aw_{n_k}\}$ is bounded. Since $\lim_{k\to\infty}\|w_{n_k} - y_{n_k}\| = 0$, $\{y_{n_k}\}$ is bounded. From the boundedness of $\{y_{n_k}\}$, (6.2.23), the facts that J is norm-to-norm uniformly continuous and $\liminf_{k\to\infty}\lambda_{n_k} > 0$, we have

$$\liminf_{k\to\infty}\langle Aw_{n_k},\ x - w_{n_k}\rangle \geq 0, \quad \forall\ \ x \in \mathcal{C}. \qquad (6.2.24)$$

Case 2: Suppose that $\liminf_{k\to\infty} \lambda_{n_k} = 0$. Let $t_{n_k} = \Pi_{\mathcal{C}} J^{-1}(Jw_{n_k} - \lambda_{n_k}\ell^{-1}Aw_{n_k})$. From Lemma 2.5.23 and the fact that $\lambda_{n_k}\ell^{-1} > \lambda_{n_k}$ we have,

$$\ell\|w_{n_k} - t_{n_k}\| \leq \|w_{n_k} - y_{n_k}\| \to 0, \quad \text{as} \quad k \to \infty.$$

Thus, we have that $t_{n_k} \rightharpoonup p \in \mathcal{C}$, which implies that $\{t_{n_k}\}$ is bounded.

Since $\lim_{k\to\infty} \|w_{n_k} - t_{n_k}\| = 0$ and A is uniformly continuous, we obtain

$$\lim_{k\to\infty} \|Aw_{n_k} - At_{n_k}\| = 0. \tag{6.2.25}$$

Also, since J is norm-to-norm continuous we have

$$\|Jw_{n_k} - Jt_{n_k}\| \to 0, \ k \to \infty.$$

From (6.2.2) we have,

$$\begin{aligned}&\lambda_{n_k}\ell^{-1}\langle A\Pi_{\mathcal{C}} J^{-1}(Jw_{n_k} - \lambda_{n_k}\ell^{-1}Aw_{n_k}) - Aw_{n_k}, \quad \Pi_{\mathcal{C}} J^{-1}(Jw_{n_k} - \lambda_{n_k}\ell^{-1}Aw_{n_k}) - z_{n_k}\rangle \\ &> \frac{\eta}{2}\Big[\|w_{n_k} - \Pi_{\mathcal{C}} J^{-1}(Jw_{n_k} - \lambda_{n_k}\ell^{-1}Aw_{n_k})\|^2 + \|\Pi_{\mathcal{C}} J^{-1}(Jw_{n_k} - \lambda_{n_k}\ell^{-1}Aw_{n_k}) - z_{n_k}\|^2\Big].\end{aligned}$$

Applying the Cauchy-Schwartz inequality to the last inequality, we have

$$\begin{aligned}&\lambda_{n_k}\ell^{-1}\|A\Pi_{\mathcal{C}} J^{-1}(Jw_{n_k} - \lambda_{n_k}\ell^{-1}Aw_{n_k}) - Aw_{n_k}\| \cdot \quad \|\Pi_{\mathcal{C}} J^{-1}(Jw_{n_k} - \lambda_{n_k}\ell^{-1}Aw_{n_k}) - z_{n_k}\| \\ &> \eta\Big[\|w_{n_k} - \Pi_{\mathcal{C}} J^{-1}(Jw_{n_k} - \lambda_{n_k}\ell^{-1}Aw_{n_k})\| \cdot \ \|\Pi_{\mathcal{C}} J^{-1}(Jw_{n_k} - \lambda_{n_k}\ell^{-1}Aw_{n_k}) - z_{n_k}\|\Big],\end{aligned}$$

which implies that

$$\lambda_{n_k}\ell^{-1}\|Aw_{n_k} - A\Pi_{\mathcal{C}} J^{-1}(Jw_{n_k} - \lambda_{n_k}\ell^{-1}Aw_{n_k})\| > \eta\|w_{n_k} - \Pi_{\mathcal{C}} J^{-1}(Jw_{n_k} - \lambda_{n_k}\ell^{-1}Aw_{n_k})\|.$$

From this, we have

$$\frac{1}{\eta}\|Aw_{n_k} - At_{n_k}\| > \frac{1}{\lambda_{n_k}\ell^{-1}}\|w_{n_k} - t_{n_k}\|. \tag{6.2.26}$$

Applying (6.2.25) to (6.2.26) we have

$$\lim_{k\to\infty} \frac{1}{\lambda_{n_k}\ell^{-1}}\|w_{n_k} - t_{n_k}\| = 0.$$

From the definition of t_{n_k} and Lemma 2.4.3, we have

$$0 \leq \left\langle Jt_{n_k} - Jw_{n_k} + \lambda_{n_k}\ell^{-1}Aw_{n_k}, \ x - t_{n_k}\right\rangle.$$

It follows that

$$\frac{1}{\lambda_{n_k}\ell^{-1}}\left\langle Jw_{n_k} - Jt_{n_k}, \ x - t_{n_k}\right\rangle + \left\langle Aw_{n_k}, \ t_{n_k} - w_{n_k}\right\rangle \leq \left\langle Aw_{n_k}, \ x - w_{n_k}\right\rangle, \quad \forall \ x \in \mathcal{C}.$$

Taking the limit of the last inequality as $k \to \infty$ and noting the fact that J is norm-to-norm continuous, we have

$$\liminf_{k\to\infty} \langle Aw_{n_k},\ x - w_{n_k} \rangle \geq 0, \quad \forall x \in \mathcal{C},$$

which is the desired result.

Now, to complete the proof we show that $p \in VI(\mathcal{C}, A)$. Observe that

$$\langle Ay_{n_k},\ x - y_{n_k} \rangle = \langle Ay_{n_k} - Aw_{n_k},\ x - w_{n_k} \rangle + \langle Aw_{n_k},\ x - w_{n_k} \rangle + \langle Ay_{n_k},\ w_{n_k} - y_{n_k} \rangle. \tag{6.2.27}$$

Since $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$ and A is uniformly continuous, we obtain

$$\lim_{k\to\infty} \|Aw_{n_k} - Ay_{n_k}\| = 0. \tag{6.2.28}$$

Then, it follows from (6.2.24) and (6.2.27) that

$$\liminf_{k\to\infty} \langle Ay_{n_k},\ x - y_{n_k} \rangle \geq 0, \quad \forall\ x \in \mathcal{C}.$$

Now, we show that $p \in VI(\mathcal{C}, A)$. We choose a decreasing sequence of positive numbers $\{\zeta_k\}$ such that $\{\zeta_k\} \to 0$ as $k \to \infty$. For each k, we denote the smallest positive integer by N_k such that

$$\langle Ay_{n_j},\ x - y_{n_j} \rangle + \zeta_n \geq 0,\ \forall\ j \geq N_k. \tag{6.2.29}$$

We observe that $\{N_k\}$ is increasing since $\{\zeta_n\}$ is decreasing. However, since $\{y_{n_k}\} \subset \mathcal{C}$ for each k, we assume that $Ay_{N_k} \neq 0$ (otherwise, y_{N_k} is a solution). Moreover, let $d_{N_k} \in E$ such that $\lim_{k\to\infty} d_{N_k} = d \in E$ and $\langle Ay_{N_k},\ d_{N_k} \rangle = 1$ for each k. Then, from (6.2.29), we have for each k that

$$\langle Ay_{N_k},\ x + \zeta_k d_{N_k} - y_{N_k} \rangle \geq 0, \quad \forall\ x \in \mathcal{C}.$$

Since A is pseudomonotone on $\mathcal{X}$, we have

$$\langle A(x + \zeta_k d_{N_k}),\ x + \zeta_k d_{N_k} - y_{N_k} \rangle \geq 0, \quad \forall\ x \in \mathcal{C}.$$

which implies that

$$\langle Ax,\ x - y_{N_k} \rangle \geq \langle Ax - A(x + \zeta_k d_{N_k}),\ x + \zeta_k d_{N_k} - y_{N_k} \rangle - \zeta_k \langle Ax,\ d_{N_k} \rangle, \quad \forall\ x \in \mathcal{C}. \tag{6.2.30}$$

Since $\{d_{N_k}\}$ is bounded and $\lim_{k\to\infty} \zeta_k = 0$, we have $\lim_{k\to\infty} \zeta_k d_{N_k} = 0$. Applying the fact that A is uniformly continuous, $\lim_{k\to\infty} \zeta_k d_{N_k} = 0$, $\{y_{N_k}\}$ and $\{d_{N_k}\}$ are bounded on (6.2.30), we have

$$\liminf_{k\to\infty} \langle Ax,\ x - y_{N_k} \rangle \geq 0.$$

Therefore,

$$\langle Ax,\ x - p \rangle = \lim_{k\to\infty} \langle Ax,\ x - y_{N_k} \rangle = \liminf_{k\to\infty} \langle Ax,\ x - y_{N_k} \rangle \geq 0, \quad \forall\ x \in \mathcal{C}.$$

From Lemma 2.5.2, we obtain that $p \in VI(\mathcal{C}, A)$. This completes the proof. □

Remark 6.2.8. Note that Lemma 6.2.7 holds if the mapping A in Assumption 6.2.1 (b) is Lipschitz continuous instead of uniformly continuous. Moreover, the result of Lemma 6.2.7 also holds if the step-size of our proposed Algorithm 6.2.2 is a sequence of positive numbers.

We are now in the position to present our main result.

Theorem 6.2.9. *Let $\{x_n\}$ be a sequence generated by Algorithm 6.2.2 under Assumption 6.2.1. Then, $\{x_n\}$ converges strongly to $p \in VI(\mathcal{C}, A)$, where $p = \Pi_{VI(\mathcal{C},A)}(u)$.*

Proof. Let $p = \Pi_{VI(\mathcal{C},A)}u$. From Lemma 6.2.6, we have

$$\begin{aligned}\phi(p,\ x_{n+1}) &\leq (1-\alpha_n)\phi(p,\ x_n)\\ &\quad + \alpha_n\left[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\big[\phi(p, x_{n-1}) - \phi(p, x_n)\big] + 2\langle Ju - Jp,\ x_{n+1} - p\rangle\right]\\ &= (1-\alpha_n)\phi(p,\ x_n) + \alpha_n v_n, \end{aligned} \tag{6.2.31}$$

where $v_n = (1-\alpha_n)\frac{\theta_n}{\alpha_n}\big[\phi(p, x_{n-1}) - \phi(p, x_n)\big] + 2\langle Ju - Jp,\ x_{n+1} - p\rangle$.
Now, we claim that $\{\phi(p,\ x_n)\}$ converges to zero. To establish this, by Lemma 2.5.36 it suffices to show that $\limsup_{k\to\infty} v_{n_k} \leq 0$ for every subsequence $\{\phi(p,\ x_{n_k})\}$ of $\{\phi(p,\ x_n)\}$ satisfying the condition

$$\liminf_{k\to\infty}\big[\phi(p,\ x_{n_{k+1}}) - \phi(p,\ x_{n_k})\big] \geq 0. \tag{6.2.32}$$

Suppose that $\{\phi(p,\ x_{n_k})\}$ is a subsequence of $\{\phi(p,\ x_n)\}$ such that (6.2.32) holds. Again, we obtain from Lemma 6.2.6 that

$$\begin{aligned}(1-\alpha_{n_k})(1-\mu\eta)\big[\phi(y_{n_k}, w_{n_k}) + \phi(z_{n_k}, y_{n_k})\big] &\leq (1-\alpha_{n_k})\phi(p,\ x_{n_k}) - \phi(p,\ x_{n_k+1})\\ &\quad + \alpha_{n_k}\left[(1-\alpha_{n_k})\frac{\theta_{n_k}}{\alpha_{n_k}}\big[\phi(p, x_{n_k-1}) - \phi(p, x_{n_k})\big] + 2\langle Ju - Jp,\ x_{n_k+1} - p\rangle\right].\end{aligned}$$

By (6.2.8), (6.2.32) and the fact that $\lim_{k\to\infty}\alpha_{n_k} = 0$, we obtain

$$\lim_{k\to\infty}(1-\alpha_{n_k})(1-\mu\eta)\big[\phi(y_{n_k}, w_{n_k}) + \phi(z_{n_k}, y_{n_k})\big] = 0.$$

By the conditions on the control sequences, it follows that

$$\lim_{k\to\infty}\phi(y_{n_k},\ w_{n_k}) = \lim_{k\to\infty}\phi(z_{n_k},\ y_{n_k}) = 0.$$

From Lemma 2.5.24, we have

$$\|w_{n_k} - y_{n_k}\| \to 0, \ \text{as} \ k \to \infty \tag{6.2.33}$$

and

$$\|y_{n_k} - z_{n_k}\| \to 0, \ \text{as} \ k \to \infty. \tag{6.2.34}$$

From (6.2.33) and (6.2.34) we have

$$\|w_{n_k} - z_{n_k}\| \to 0, \ \text{as} \ k \to \infty. \tag{6.2.35}$$

From the definition of $x_{n_{k+1}}$ in Algorithm 6.2.2, we have

$$\|Jx_{n_{k+1}} - Jz_{n_k}\| \le \alpha_{n_k}\|Ju - Jz_{n_k}\| + (1-\alpha_{n_k})\|Jz_{n_k} - Jz_{n_k}\| \to 0, \ \text{as} \ k \to \infty. \tag{6.2.36}$$

Since J^{-1} is norm-to-norm uniformly continuous on bounded subsets of $\mathcal{X}^*$, we obtain

$$\|x_{n_{k+1}} - z_{n_k}\| \to 0, \text{ as } k \to \infty. \tag{6.2.37}$$

Also, by applying (6.2.34) and (6.2.37) we obtain

$$\|x_{n_{k+1}} - y_{n_k}\| \le \|x_{n_{k+1}} - z_{n_k}\| + \|z_{n_k} - y_{n_k}\| \to 0, \text{ as } k \to \infty. \tag{6.2.38}$$

From the definition of w_{n_k} and by applying (6.2.11), we have

$$\|Jw_{n_k} - Jx_{n_k}\| = \theta_{n_k}\|Jx_{n_{k-1}} - Jx_{n_k}\| \to 0, \text{ as } k \to \infty.$$

Since J^{-1} is norm-to-norm uniformly continuous on bounded subsets of $\mathcal{X}^*$, we have

$$\|w_{n_k} - x_{n_k}\| \to 0, \text{ as } k \to \infty. \tag{6.2.39}$$

From (6.2.33)-(6.2.39), we have

$$\lim_{k\to\infty}\|y_{n_k} - x_{n_k}\| = 0, \ \lim_{k\to\infty}\|x_{n_k} - z_{n_k}\| = 0, \ \lim_{k\to\infty}\|w_{n_k} - y_{n_k}\| = 0. \tag{6.2.40}$$

Also, from (6.2.38) and (6.2.40),we have

$$\|x_{n_{k+1}} - x_{n_k}\| \le \|x_{n_{k+1}} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \to 0, \text{ as } k \to \infty. \tag{6.2.41}$$

Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $x^* \in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$. Then, from (6.2.39), we obtain $w_{n_k} \rightharpoonup x^*$ as $k \to \infty$. Now, by Lemma 6.2.7, it follows from (6.2.40) that $x^* \in VI(\mathcal{C}, A)$. Since $x^* \in w_\omega(x_n)$ is an arbitrary element, we have $w_\omega(x_n) \subset VI(\mathcal{C}, A)$.

Moreover, since the sequence $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to $z^* \in \mathcal{X}$. Since $p = \Pi_{VI(\mathcal{C},A)}(u)$, we have that

$$\begin{aligned} \limsup_{k\to\infty} \langle Ju - Jp,\ x_{n_k} - p\rangle &= \lim_{j\to\infty} \langle Ju - Jp,\ x_{n_{k_j}} - p\rangle \\ &= \langle Ju - Jp,\ z^* - p\rangle \\ &\leq 0. \end{aligned} \tag{6.2.42}$$

From (6.2.41) and (6.2.42), we obtain

$$\limsup_{k\to\infty} \langle Ju - Jp,\ x_{n_{k+1}} - p\rangle \leq 0. \tag{6.2.43}$$

From (6.2.8) and (6.2.43), we obtain $\limsup_{k\to\infty} v_{n_k} \leq 0$. Hence, by applying Lemma 2.5.36 to (6.2.31) we obtain that $\lim_{n\to\infty} \phi(p,\ x_n) = 0$. It follows from Lemma 2.5.24 that $\lim_{n\to\infty} \|x_n - p\| = 0$. Therefore, $\{x_n\}$ converges strongly to $p \in VI(\mathcal{C}, A)$, where $p = \Pi_{VI(\mathcal{C},A)}(u)$. □

Remark 6.2.10. In real Hilbert spaces, we have that $\phi(x, y) = \|x - y\|^2, \ \ \forall\, x, y \in \mathcal{C}$ and $\Pi_{\mathcal{C}} = P_{\mathcal{C}}$ where $P_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ is the metric projection of $\mathcal{H}$ onto $\mathcal{C}$.

A mapping $P_{\mathcal{C}}$ is said to be the metric projection of a real Hilbert space $\mathcal{H}$ onto $\mathcal{C}$ if for all $x \in \mathcal{H}$ there exists a unique nearest point in $\mathcal{C}$, denoted by $P_{\mathcal{C}}x$, such that

$$\|x - P_{\mathcal{C}}x\| \leq \|x - y\|, \qquad \forall\, y \in \mathcal{C}.$$

If $\mathcal{X}$ is a Hilbert space, then taking $J = I$ and $\phi(x,\ y) = \|x - y\|^2, \ \ \forall\, x, y \in \mathcal{C}$ in Theorem 6.2.9 (where I is the identity mapping on $\mathcal{X}$), we have the following consequent result:

Corollary 6.2.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 6.2.4 below under Assumption 6.2.1. Then, $\{x_n\}$ converges strongly to $p \in VI(\mathcal{C}, A)$, where $p = P_{VI(\mathcal{C},A)}(u)$.*

Algorithm 6.2.4. Inertial method with new Armijo-type stepsize.

Initialization: Given $\lambda_1 > 0, \ \ \theta > 0, \ \ \delta > 0, \ell \in (0, 1)$ and $\eta \in (0, \frac{1}{\mu})$. Let $u, \ x_0, x_1 \in \mathcal{X}$ be arbitrary. Set $n := 1$.

Step 1: Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\theta, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2: Compute

$$w_n = x_n + \theta_n(x_{n-1} - x_n)).$$

Step 3: Compute

$$y_n = P_{\mathcal{C}}(w_n - \lambda_n A w_n).$$

If $w_n = y_n$, or $Ay_n = 0$ then stop and y_n is a solution of the VIP. Otherwise, go to **Step 4**.

Step 4: Compute
$$z_n = P_{T_n}(w_n - \lambda_n Ay_n)$$
where
$$T_n := \{x \in \mathcal{X} \ |\langle w_n - \lambda_n Aw_n - y_n, \quad x - y_n\rangle \leq 0\},$$
$\lambda_n := \delta\ell^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\delta\ell^m\langle Ay_n - Aw_n, \quad y_n - z_n\rangle \leq \frac{\eta}{2}\Big[\|w_n - y_n\|^2 + \|y_n - z_n\|^2\Big].$$

Step 5: Calculate
$$x_{n+1} = \alpha_n u + (1-\alpha_n)z_n).$$
Set $n := n+1$ and go to **Step 1**.

6.2.3 Numerical experiments

In this section, using some test examples, we discuss the numerical behavior of Algorithm 6.2.2 and compare it with Algorithm 2.5.6 proposed by Cai et. al [53], Algorithm 2.5.7 proposed by Tan et. al [240], Appendix 6.2.13 proposed by Thong et.al [251] and Appendix 6.2.14 proposed by Reich et.al [214]. We perform all implementations using Matlab 2016 (b), installed on a personal computer with Intel(R) Core(TM) i5-2600 CPU@2.30GHz and 8.00 Gb-RAM running on Windows 10 operating system. In Tables 6.2.1-6.2.2, "No. of Iter." means the number of iterations.

In our computations, we choose $\tau_n = \frac{10}{(n+2)^2}$, $\alpha_n = \frac{1}{n+2}$, $\lambda_1 = \lambda = 0.9$, $\theta = 0.7$, $\delta = 2.7$, $\ell = 0.5$ and $\eta = 0.6$. Furthermore, in the implementation, we define $\text{TOL}_n := \|x_{n+1} - x_n\|$ and use the stopping criterion $\text{TOL}_n < 10^{-3}$ for the iterative processes.

Example 6.2.11. Consider the linear operator $A : \mathbb{R}^m \to \mathbb{R}^m$ $(m = 15, 30, 45, 60)$ defined by
$$A = Mx + q,$$
where
$$M = BB^T + P + Q$$
with $P, Q \in \mathbb{R}^{m\times m}$ randomly generated matrices such that P is skew-symmetric, Q is a diagonal matrix of nonnegative entries and q is a vector in $\mathbb{R}^m$. The feasible set $\mathcal{C} \subset \mathcal{R}^m$ is a closed and convex subset defined by
$$\mathcal{C} := \{x \in \mathbb{R}^m : \quad Gx \leq b\},$$
where G is an $k \times m$ matrix and b is a nonnegative vector. We easily observe that A is lipschitz continuous and monotone with $L = \|M\|$. Let $q = 0$ then, the solution set $VI(\mathbb{C}, \ A) := \{0\}$. The entries of B, P are generated randomly in $[-2, 2]$ as well as the

starting points x_0, x_1 and the diagonal entries of Q are in $(0, 2)$, and q is taken as the zero vector in $\mathbb{R}^m$. Moreover, for Algorithm 2.5.7, we define the contraction mapping $fx = 0.1x$ for all $x \in \mathbb{R}^m$. We plot the graph of $\|x_{n+1} - x_n\|$ against number of iterations choosing $m = 15, 30, 45, 60$. The numerical results are reported in Figure 6.1 and Table 6.3.1.

Table 6.3.1: Numerical results for Example 6.2.11

Cases		Alg. 2.5.6	Alg. 2.5.7	App. 6.2.13	App. 6.2.14	Alg. 6.2.2
m = 15	No. of Iter.	19	31	24	33	14
	CPU time (sec)	8.2375	5.3746	3.7576	5.3048	5.8052
m = 30	No. of Iter.	23	33	26	44	18
	CPU time (sec)	11.6430	6.3741	4.7479	7.8673	6.9292
m = 45	No. of Iter.	25	40	29	46	20
	CPU time (sec)	9.6496	8.3209	5.6280	9.5082	6.3000
m = 60	No. of Iter.	26	43	31	51	24
	CPU time (sec)	9.5374	10.1644	6.5321	11.4672	6.5011

Example 6.2.12. Let $\mathcal{H}_1 = (l_2(\mathbb{R}),\ ||.||_{l_2}) = \mathcal{H}_2$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \dots),\ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $||x||_{l_2} := \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$, $\forall x \in l_2(\mathbb{R})$. Let $\mathcal{C} = \{x \in l_2(\mathbb{R}) : ||x - a||_{l_2} \leq r\}$, where $a = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$, $r = 3$. Then $\mathcal{C}$ is a nonempty, closed and convex subset of $l_2(\mathbb{R})$. Thus,

$$P_{\mathcal{C}}(x) = \begin{cases} x, & \text{if } \ x \in ||x - a||_{l_2} \leq r, \\ \frac{x-a}{||x-a||_{l_2}} r + a, & \text{otherwise.} \end{cases}$$

Now, define the operator $A, : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by

$$Ax = \left(\|x - a\| + \frac{1}{\|x - a\| + 0.7} \right) x.$$

It is easy to see that the solution set $VI(\mathcal{C}, A) = \{0\}$, A is pseudomonotone on $l_2(\mathbb{R})$, uniformly continuous and sequentially weakly continuous on C but not Lipschitz on $l_2(\mathbb{R})$.

More so, for Algorithm 2.5.7, we define the mapping $fx = \frac{x}{2}$, for all $x \in l_2(\mathbb{R})$. Then, we take $u = (1, \frac{1}{5}, \frac{1}{25}, \cdots)$ and consider the following cases for the numerical experiments:
Case 1: Take $x_1 = (-4, 1, -\frac{1}{4}, \cdots)$ and $x_0 = (2, 1, \frac{1}{2}, \cdots)$.

Case 2: Take $x_1 = (-2, 1, -\frac{1}{2}, \cdots)$ and $x_0 = (-3, 1, -\frac{1}{3}, \cdots)$.

Case 3: Take $x_1 = (-3, 1, -\frac{1}{3}, \cdots)$ and $x_0 = (-2, 1, -\frac{1}{2}, \cdots)$.

Case 4: Take $x_1 = (-4, 1, -\frac{1}{4}, \cdots)$ and $x_0 = (3, 1, \frac{1}{3}, \cdots)$.

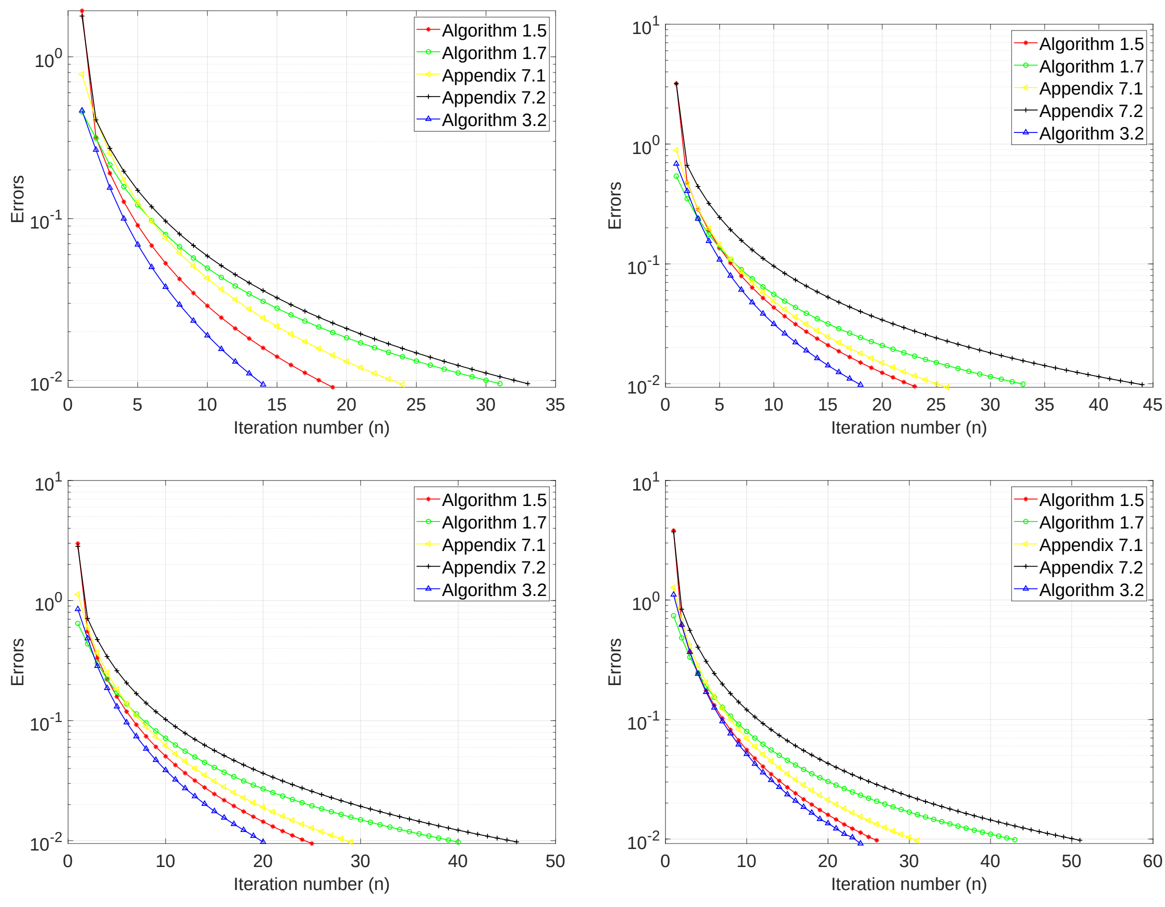

Figure 6.1: Top left: m = 15; Top right: m = 30; Bottom left: m = 45; Bottom right: m = 60.

We plot the graph of $\|x_{n+1} - x_n\|$ against number of iterations using the above four cases. The numerical results are reported in Figure 6.2 and Table 6.3.2.

Table 6.3.2: Numerical results for Example 6.2.12

		Alg. 2.5.6	Alg. 2.5.7	App. 6.2.13	App. 6.2.14	Alg. 6.2.2
Case I	No. of Iter.	30	20	39	26	9
	CPU time (sec)	0.0169	0.0104	0.0102	0.0102	0.0107
Case II	No. of Iter.	19	18	26	33	8
	CPU time (sec)	0.0164	0.0099	0.0091	0.0096	0.0111
Case III	No. of Iter.	16	19	33	26	9
	CPU time (sec)	0.0181	0.0106	0.0124	0.0132	0.0144
Case IV	No. of Iter.	35	20	39	32	9
	CPU time (sec)	0.0171	0.0093	0.0094	0.0098	0.0104

Appendix 6.2.13. *(Algorithm 3.3 in [251])*

Initialization: *Given* $\{\alpha_n\} \subset (0,1), \ell \in (0,1), \beta > 0, \lambda \in (0, \frac{1}{\beta})$. *Let* $x_1 \in \mathcal{C}$ *be arbitrary*

Iterative Steps: *Given the current iterate* x_n, *calculate* x_{n+1} *as follows:*

Step 1: *Compute*

$$z_n = P_{\mathcal{C}}(x_n - \lambda A x_n)$$

and $r_\lambda(x_n) := x_n - z_n$. *If* $r_\lambda(x_n) = 0$ *then stop and* x_n *is a solution of* $VI(\mathcal{C}, A)$. *Otherwise*

Step 2: *Compute*

$$y_n = x_n - \tau_n r_\lambda(x_n),$$

where $\tau_n := \ell^{j_n}$ *and* j_n *is the smallest non-negative integer* j *satisfying*

$$\langle Ax_n - Ay_n, r_\lambda(x_n)\rangle \leq \mu \|r_\lambda(x_n)\|^2.$$

Step 2: *Compute*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_{C_n}(x_n),$$

where

$$C_n := \{x \in \mathcal{C} : h_n(x) \leq 0\}$$

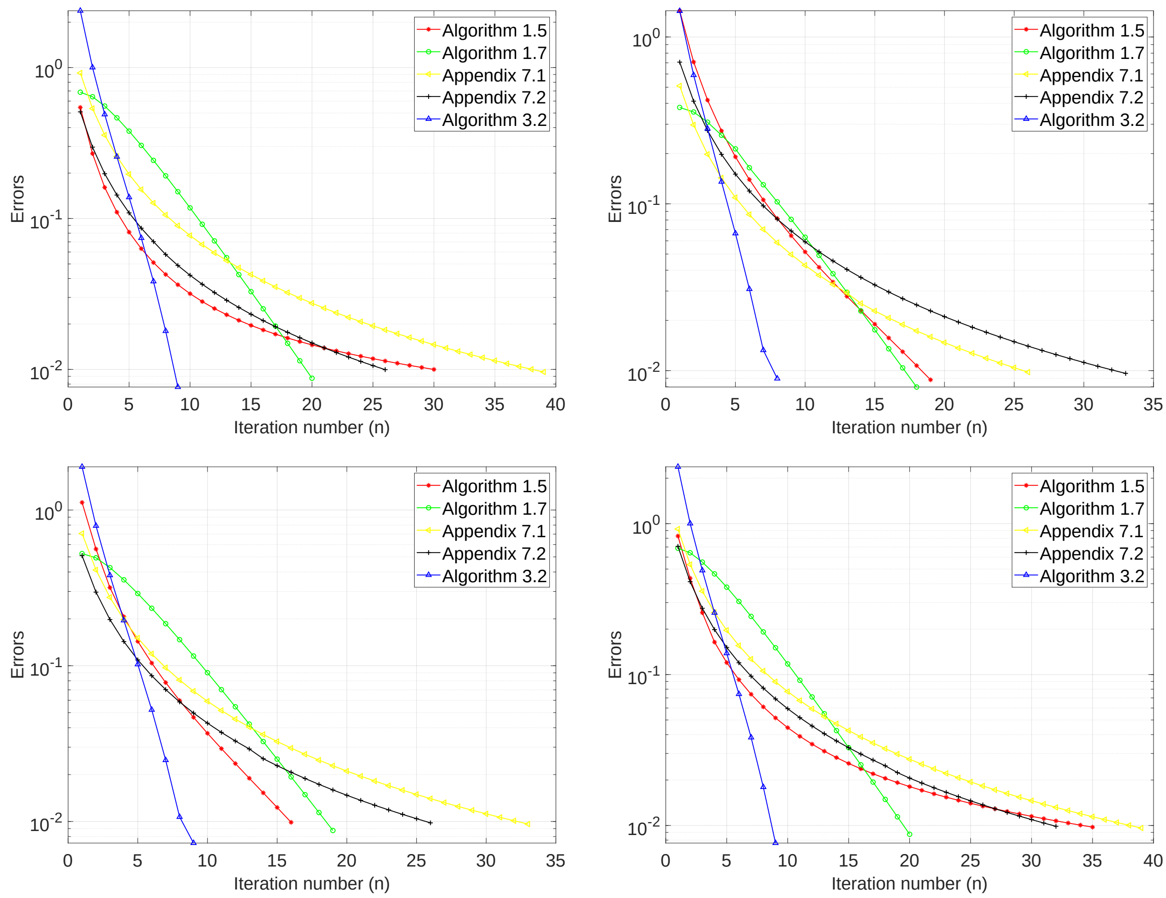

Figure 6.2: Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

and

$$h_n(x) = \langle Ay_n, x - y_n \rangle.$$

Set $n := n+1$ *and go to* ***Step 1.***

where $A : \mathcal{H} \to \mathcal{H}$ *is pseudomonotone, uniformly continuous and sequentially weakly continuous on bounded subsets of* $\mathcal{C}$.

Appendix 6.2.14. *(Algorithm 4 in [214])*

Initialization: *Given* $\{\alpha_n\} \subset (0,1), \ell \in (0,1), \beta > 0, \lambda \in (0, \frac{1}{\beta})$. *Let* $x_1 \in \mathcal{C}$ *be arbitrary*

Iterative Steps: *Given the current iterate* x_n, *calculate* x_{n+1} *as follows:*

Step 1: *Compute*

$$z_n = P_{\mathcal{C}}(x_n - \lambda A x_n)$$

and $r_\lambda(x_n) := x_n - z_n$. *If* $r_\lambda(x_n) = 0$ *then stop and* x_n *is a solution of the VIP. Otherwise*

Step 2: *Compute*

$$y_n = x_n - \tau_n r_\lambda(x_n),$$

where $\tau_n := \ell^{j_n}$ *and* j_n *is the smallest non-negative integer* j *satisfying*

$$\langle Ax_n - Ay_n, r_\lambda(x_n) \rangle \leq \frac{\mu}{2} \|r_\lambda(x_n)\|^2.$$

Step 3: *Compute*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_{C_n}(x_n),$$

where

$$C_n := \{x \in \mathcal{C} : h_n(x) \leq 0\}$$

and

$$h_n(x) = \langle Ay_n, x - x_n \rangle + \frac{\tau_n}{2} \|r_\lambda(x_n)\|^2.$$

Set $n := n+1$ *and go to* ***Step 1.***

where $A : \mathcal{H} \to \mathcal{H}$ *is pseudomonotone and uniformly continuous.*

6.3 On finite family of split monotone variational inclusion and fixed point problems

In this section, we propose and study a new inertial iterative algorithm with self-adaptive step size for approximating a common solution of finite family of SMV_qIPs and FPP of a nonexpansive mapping between a Banach space and Hilbert space. This method combines the inertial technique with viscosity method and self-adaptive step size for solving the common solution problem. We prove a strong convergence result for the proposed method under some mild conditions. Moreover, we apply our result to study the SFP and split minimization problem (SMP). Finally, we provide some numerical experiments to demonstrate the efficiency of our method in comparison with some well-known methods in the literature. Our method does not require prior knowledge or estimate of the operator norm, which makes it easily implementable unlike so many other methods in the literature which require prior knowledge of the operator norm for their implementation.

6.3.1 Proposed method

Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{X}$ be a uniformly convex and smooth Banach space. Let $J_{\mathcal{X}}$ be the duality mapping on $\mathcal{X}$. Let $T : \mathcal{H} \to \mathcal{X}$ be a bounded linear operator such that $T \neq 0$ and T^* is the adjoint operator of T. Let $f : \mathcal{H} \to \mathcal{H}$ be a contraction mapping with constant $\rho \in (0,1)$ and $0 < \gamma < \frac{\gamma}{\rho}$, and let $S : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping. For each $i = 1, 2, \cdots, m$, let $A_i : \mathcal{H} \to \mathcal{H}$ be a finite family of α_i-inverse strongly monotone operators, $B_i : \mathcal{H} \to 2^{\mathcal{H}}$ and $C_i : \mathcal{X} \to 2^{\mathcal{X}^*}$ be finite families of maximal monotone operators, and let $J_{r_i}^{B_i}$ be the resolvent of B_i for $r_i > 0$ and $Q_{\mu_i}^{C_i}$ be the metric resolvent of C_i for $\mu_i > 0$. Suppose that the solution set $\Gamma \neq \emptyset$. We establish the convergence of the algorithm under the following assumptions on the control parameters:

(A1) $\{\alpha_n\}, \{\delta_n\}, \{\gamma_n\} \subset (0,1)$ such that $\alpha_n + \delta_n + \gamma_n = 1$, $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;

(A2) $\liminf\limits_{n\to\infty} \delta_n > 0, \liminf\limits_{n\to\infty} \gamma_n > 0, \{\beta_{n,i}\} \subset (0,1)$ and $\sum\limits_{i=0}^{m} \beta_{n,i} = 1, \liminf\limits_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$;

(A3) $0 < a \leq \tau_n \leq b < 2, 0 < c \leq \mu_{n,i}, 0 < d \leq r_{n,i} \leq e < 2\alpha_i$ for each $i = 1, 2, \ldots, m$;

(A4) Let $\theta > 0$ and $\{\epsilon_n\}$ be a nonnegative sequence such that $0 < d \leq \epsilon_n$ and $\epsilon_n = \circ(\alpha_n)$, i.e., $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.

Algorithm 6.3.1.

Step 0: Select initial points $x_0, x_1 \in \mathcal{H}$ and set $n = 1$.
Step 1: Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases} \tag{6.3.1}$$

Step 2: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3: Compute

$$z_{n,i} = w_n - \lambda_{n,i} T^* J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}}) T w_n,$$

where

$$\lambda_{n,i} = \frac{\tau_n \| J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}}) T w_n\|^2}{\|T^* J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}}) T w_n\|^2}.$$

Step 4: Compute

$$u_n = \beta_{n,0} w_n + \sum_{i=1}^{m} \beta_{n,i} J^{B_i}_{r_{n,i}}(I - r_{n,i}A_i) z_{n,i}.$$

Step 5: Compute

$$x_{n+1} = \alpha_n f(x_n) + \delta_n x_n + \gamma_n S u_n.$$

Set $n := n+1$ and return to **Step 1**.

Remark 6.3.1. By conditions (A1) and (A4), one can easily verify from (6.3.1) that

$$\lim_{n\to\infty} \theta_n ||x_n - x_{n-1}|| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| = 0. \tag{6.3.2}$$

6.3.2 Convergence analysis

Lemma 6.3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 6.3.1 under Assumption (A1)-(A4). Then, $\{x_n\}$ is bounded.*

Let $p \in \Gamma$. Then, it follows that

$$p = J^{B_i}_{r_{n,i}}(I - r_{n,i}A_i)p, Tp = Q^{C_i}_{\mu_{n,i}} Tp, \quad \text{and} \quad p = Sp$$

for all $n \in \mathbb{N}$ and $i = 1, 2, \ldots, m$. From the definition of w_n in **Step 2** and by applying the triangle inequality, we have

$$\begin{aligned} ||w_n - p|| &= ||x_n + \theta_n(x_n - x_{n-1}) - p|| \\ &\le ||x_n - p|| + \theta_n ||x_n - x_{n-1}|| \\ &= ||x_n - p|| + \alpha_n \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}||. \end{aligned} \tag{6.3.3}$$

Since, by Remark 6.3.1, $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| = 0$, it follows that there exists a constant $M_1 > 0$ such that $\frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| \le M_1$ for all $n \ge 1$. Hence, it follows from (6.3.3) that

$$||w_n - p|| \le ||x_n - p|| + \alpha_n M_1. \tag{6.3.4}$$

From **Step 3** and the property of the resolvent (2.4.10), we have

$$\begin{aligned}
\|z_{n,i} - p\|^2 &= \|w_n - \lambda_{n,i} T^* J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n - p\|^2 \\
&= \|w_n - p\|^2 - 2\lambda_{n,i}\langle w_n - p, T^* J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\rangle \\
&+ \lambda^2_{n,i}\|T^* J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2 \\
&= \|w_n - p\|^2 - 2\lambda_{n,i}\langle Tw_n - Tp, J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\rangle \\
&+ \lambda^2_{n,i}\|T^* J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2 \\
&= \|w_n - p\|^2 - 2\lambda_{n,i}\langle Tw_n - Q^{C_i}_{\mu_{n,i}}Tw_n, J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\rangle \\
&- 2\lambda_{n,i}\langle Q^{C_i}_{\mu_{n,i}}Tw_n - Tp, J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\rangle + \lambda^2_{n,i}\|T^* J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2 \\
&= \|w_n - p\|^2 - 2\lambda_{n,i}\|J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2 \\
&- 2\lambda_{n,i}\langle Q^{C_i}_{\mu_{n,i}}Tw_n - Tp, J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\rangle + \lambda^2_{n,i}\|T^* J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2 \\
&\leq \|w_n - p\|^2 - 2\lambda_{n,i}\|J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2 + \lambda^2_{n,i}\|T^* J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2 \\
&= \|w_n - p\|^2 - \lambda_{n,i}\Big[2\|J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2 - \lambda_{n,i}\|T^* J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2\Big]
\end{aligned} \tag{6.3.5}$$

From the definition of $\lambda_{n,i}$ and (6.3.5), we have

$$\|z_{n,i} - p\|^2 \leq \|w_n - p\|^2 - \frac{\tau_n(2-\tau_n)\|J_{\mathcal{X}}(I - Q^{C_i}_{Q_{\mu_{n,i}}})Tw_n\|^4}{\|T^* J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2}. \tag{6.3.6}$$

Thus, by the assumption on τ_n we have that

$$\|z_{n,i} - p\|^2 \leq \|w_n - p\|^2. \tag{6.3.7}$$

From the definition of u_n in **Step 4**, the nonexpansivity of $J^{B_i}_{r_{n,i}}$ and Lemma 2.5.7 we get

$$\begin{aligned}
\|u_n - p\|^2 &= \|\beta_{n,0}w_n + \sum_{i=1}^{m}\beta_{n,i}J^{B_i}_{r_{n,i}}(I - r_{n,i}A_i)z_{n,i} - J^{B_i}_{r_{n,i}}(I - r_{n,i}A_i)p\|^2 \\
&= \beta_{n,0}\|w_n - J^{B_i}_{r_{n,i}}(I - r_{n,i}A_i)p\|^2 + \sum_{i=1}^{m}\beta_{n,i}\|J^{B_i}_{r_{n,i}}(I - r_{n,i}A_i)z_{n,i} \\
&- J^{B_i}_{r_{n,i}}(I - r_{n,i}A_i)p\|^2 - \beta_{n,0}\sum_{i=1}^{m}\beta_{n,i}\|J^{B_i}_{r_{n,i}}(I - r_{n,i}A_i)z_{n,i} - w_n\|^2 \\
&\leq \beta_{n,0}\|w_n - p\|^2 + \sum_{i=1}^{m}\beta_{n,i}\|z_{n,i} - p - r_{n,i}(A_i z_{n,i} - A_i p)\|^2 \\
&- \beta_{n,0}\sum_{i=1}^{m}\beta_{n,i}\|J^{B_i}_{r_{n,i}}(I - r_{n,i}A_i)z_{n,i} - w_n\|^2.
\end{aligned} \tag{6.3.8}$$

Next, by applying the inversely strongly monotonicity of the A_i's, Lemma 2.1.1 and (6.3.7), for each $i = 1, 2, \ldots, m$, we observe that

$$
\begin{aligned}
\|z_{n,i}-p-r_{n,i}(A_iz_{n,i}-A_ip)\|^2 &= \|z_{n,i}-p\|^2 - 2r_{n,i}\langle A_iz_{n,i}-A_ip, z_{n,i}-p\rangle \\
&+ r_{n,i}^2\|A_iz_{n,i}-A_ip\|^2 \\
&\le \|z_{n,i}-p\|^2 - 2r_{n,i}\alpha_i\|A_iz_{n,i}-A_ip\|^2 \\
&+ r_{n,i}^2\|A_iz_{n,i}-A_ip\|^2 \\
&= \|z_{n,i}-p\|^2 - (2\alpha_i - r_{n,i})r_{n,i}\|A_iz_{n,i}-A_ip\|^2. \quad (6.3.9)
\end{aligned}
$$

Now, by applying (6.3.6), it follows from (6.3.8) and (6.3.9) that

$$
\begin{aligned}
\|u_n-p\|^2 &\le \beta_{n,0}\|w_n-p\|^2 + \sum_{i=1}^m \beta_{n,i}\|z_{n,i}-p\|^2 - \sum_{i=1}^m \beta_{n,i}(2\alpha_i - r_{n,i})r_{n,i}\|A_iz_{n,i}-A_ip\|^2 \\
&- \beta_{n,0}\sum_{i=1}^m \beta_{n,i}\|J_{r_{n,i}}^{B_i}(I-r_{n,i}A_i)z_{n,i}-w_n\|^2 \\
&\le \beta_{n,0}\|w_n-p\|^2 + \sum_{i=1}^m \beta_{n,i}\|w_n-p\|^2 - \sum_{i=1}^m \beta_{n,i}\frac{\tau_n(2-\tau_n)\|J_{\mathcal{X}}(I-Q_{Q_{\mu_{n,i}}}^{C_i})Tw_n\|^4}{\|T^*J_{\mathcal{X}}(I-Q_{\mu_{n,i}}^{C_i})Tw_n\|^2} \\
&- \sum_{i=1}^m \beta_{n,i}(2\alpha_i - r_{n,i})r_{n,i}\|A_iz_{n,i}-A_ip\|^2 - \beta_{n,0}\sum_{i=1}^m \beta_{n,i}\|J_{r_{n,i}}^{B_i}(I-r_{n,i}A_i)z_{n,i}-w_n\|^2 \\
&= \|w_n-p\|^2 - \sum_{i=1}^m \beta_{n,i}\frac{\tau_n(2-\tau_n)\|J_{\mathcal{X}}(I-Q_{Q_{\mu_{n,i}}}^{C_i})Tw_n\|^4}{\|T^*J_{\mathcal{X}}(I-Q_{\mu_{n,i}}^{C_i})Tw_n\|^2} \quad (6.3.10) \\
&- \sum_{i=1}^m \beta_{n,i}(2\alpha_i - r_{n,i})r_{n,i}\|A_iz_{n,i}-A_ip\|^2 \\
&- \beta_{n,0}\sum_{i=1}^m \beta_{n,i}\|J_{r_{n,i}}^{B_i}(I-r_{n,i}A_i)z_{n,i}-w_n\|^2 \\
&\le \|w_n-p\|^2 \quad (6.3.11)
\end{aligned}
$$

Consequently, we have that

$$
\|u_n-p\| \le \|w_n-p\| \le ||x_n-p|| + \alpha_n M_1. \quad (6.3.12)
$$

From **Step 5**, (6.3.12) and the nonexpansivity of S, we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n f(x_n)+\delta_n x_n+\gamma_n Su_n-p\| \\
&\le \alpha_n\|f(x_n)-p\|+\delta_n\|x_n-p\|+\gamma_n\|Su_n-p\| \\
&\le \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+\delta_n\|x_n-p\|+\gamma_n\|u_n-p\| \\
&\le \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+\delta_n\|x_n-p\|+\gamma_n\Big(\|x_n-p\|+\alpha_n M_1\Big) \\
&\le \alpha_n\rho\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|x_n-p\|+\gamma_n\alpha_n M_1 \\
&\le \Big(1-\alpha_n(1-\rho)\Big)\|x_n-p\|+\alpha_n(1-\rho)\left[\frac{\|f(p)-p\|}{1-\rho}+\frac{M_1}{1-\rho}\right] \\
&\le \max\left\{\|x_n-p\|,\frac{\|f(p)-p\|+M_1}{1-\rho}\right\} \\
&\vdots \\
&\le \max\left\{\|x_0-p\|,\frac{\|f(p)-p\|+M_1}{1-\rho}\right\}.
\end{aligned}$$

Hence, the sequence $\{x_n\}$ is bounded. Consequently, $\{w_n\},\{z_{n,i}\}$ and $\{u_n\}$ are bounded.

Lemma 6.3.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 6.3.1 under Assumption (A1)-(A4). Then, for all $p\in\Gamma$ and $n\in\mathbb{N}$ we have:*

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le (1-\eta_n)\|x_n-p\|^2+\eta_n\Big[&\frac{\alpha_n}{2(1-\rho)}M_3+3M_2\frac{\gamma_n(1-\alpha_n)}{2(1-\rho)}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| \\
&+\frac{1}{(1-\rho)}\langle f(p)-p,x_{n+1}-p\rangle\Big] \\
&-\sigma_n\Big[\sum_{i=1}^m\beta_{n,i}\frac{\tau_n(2-\tau_n)\|J_{\mathcal{X}}(I-Q^{C_i}_{Q_{\mu_{n,i}}})Tw_n\|^4}{\|T^*J_{\mathcal{X}}(I-Q^{C_i}_{\mu_{n,i}})Tw_n\|^2} \\
&+\sum_{i=1}^m\beta_{n,i}(2\alpha_i-r_{n,i})r_{n,i}\|A_iz_{n,i}-A_ip\|^2+\beta_{n,0}\sum_{i=1}^m\beta_{n,i}\|J^{B_i}_{r_{n,i}}(I-r_{n,i}A_i)z_{n,i}-w_n\|^2\Big],
\end{aligned}$$

where $\eta_n=\frac{2\alpha_n(1-\rho)}{(1-\alpha_n\rho)}$ and $\sigma_n=\frac{\gamma_n(1-\alpha_n)}{(1-\alpha_n\rho)}$.

Proof. Let $p\in\Gamma$. Then from **Step 2**, by applying the Cauchy-Schwartz inequality and Lemma 2.1.1, we obtain

$$\begin{aligned}
||w_n-p||^2 &= ||x_n+\theta_n(x_n-x_{n-1})-p||^2 \\
&= ||x_n-p||^2+\theta_n^2||x_n-x_{n-1}||^2+2\theta_n\langle x_n-p,x_n-x_{n-1}\rangle \\
&\le ||x_n-p||^2+\theta_n^2||x_n-x_{n-1}||^2+2\theta_n||x_n-x_{n-1}||||x_n-p|| \\
&= ||x_n-p||^2+\theta_n||x_n-x_{n-1}||(\theta_n||x_n-x_{n-1}||+2||x_n-p||) \\
&\le ||x_n-p||^2+3M_2\theta_n||x_n-x_{n-1}|| \\
&= ||x_n-p||^2+3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||, \qquad (6.3.13)
\end{aligned}$$

where $M_2 := \sup_{n\in\mathbb{N}}\{||x_n - p||, \theta_n||x_n - x_{n-1}||\} > 0$.

By applying Lemma 2.1.1, (6.3.10) and (6.3.13) we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + \delta_n x_n + \gamma_n S u_n - p\|^2 \\
&= \|\alpha_n (f(x_n) - p) + \delta_n (x_n - p) + \gamma_n (S u_n - p)\|^2 \\
&\le \|\delta_n (x_n - p) + \gamma_n (S u_n - p)\|^2 + 2\alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle \\
&= \delta_n^2 \|x_n - p|^2 + \gamma_n^2 \|S u_n - p\|^2 + 2\delta_n \gamma_n \langle x_n - p, S u_n - p\rangle \\
&+ 2\alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle \\
&\le \delta_n^2 \|x_n - p|^2 + \gamma_n^2 \|S u_n - p\|^2 + 2\delta_n \gamma_n \|x_n - p\| \|S u_n - p\| \\
&+ 2\alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle \\
&\le \delta_n^2 \|x_n - p|^2 + \gamma_n^2 \|S u_n - p\|^2 + \delta_n \gamma_n \Big(\|x_n - p\|^2 + \|S u_n - p\|^2 \Big) \\
&+ 2\alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle \\
&\le \delta_n (\delta_n + \gamma_n) \|x_n - p\|^2 + \gamma_n (\gamma_n + \delta_n) \|u_n - p\|^2 + 2\alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle \\
&= \delta_n (1 - \alpha_n) \|x_n - p\|^2 + \gamma_n (1 - \alpha_n) \|u_n - p\|^2 + 2\alpha_n \langle f(x_n) - f(p), x_{n+1} - p\rangle \\
&+ 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle \\
&\le \delta_n (1 - \alpha_n) \|x_n - p\|^2 \\
&+ \gamma_n (1 - \alpha_n) \Big[\|w_n - p\|^2 - \sum_{i=1}^{m} \beta_{n,i} \frac{\tau_n (2 - \tau_n) \|J_{\mathcal{X}} (I - Q_{Q_{\mu_{n,i}}}^{C_i}) T w_n\|^4}{\|T^* J_{\mathcal{X}} (I - Q_{\mu_{n,i}}^{C_i}) T w_n\|^2} \\
&- \sum_{i=1}^{m} \beta_{n,i} (2\alpha_i - r_{n,i}) r_{n,i} \|A_i z_{n,i} - A_i p\|^2 \\
&- \beta_{n,0} \sum_{i=1}^{m} \beta_{n,i} \|J_{r_{n,i}}^{B_i} (I - r_{n,i} A_i) z_{n,i} - w_n\|^2 \Big] \\
&+ 2\alpha_n \rho \|x_n - p\| \|x_{n+1} - p\| + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle \\
&\le \delta_n (1 - \alpha_n) \|x_n - p\|^2 \\
&+ \gamma_n (1 - \alpha_n) \Big[||x_n - p||^2 + 3M_2 \alpha_n \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| \\
&- \sum_{i=1}^{m} \beta_{n,i} \frac{\tau_n (2 - \tau_n) \|J_{\mathcal{X}} (I - Q_{Q_{\mu_{n,i}}}^{C_i}) T w_n\|^4}{\|T^* J_{\mathcal{X}} (I - Q_{\mu_{n,i}}^{C_i}) T w_n\|^2} \\
&- \sum_{i=1}^{m} \beta_{n,i} (2\alpha_i - r_{n,i}) r_{n,i} \|A_i z_{n,i} - A_i p\|^2 \\
&- \beta_{n,0} \sum_{i=1}^{m} \beta_{n,i} \|J_{r_{n,i}}^{B_i} (I - r_{n,i} A_i) z_{n,i} - w_n\|^2 \Big] \\
&+ \alpha_n \rho (\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= ((1-\alpha_n)^2 + \alpha_n\rho)\|x_n - p\|^2 + \alpha_n\rho\|x_{n+1} - p\|^2 \\
&+ 3M_2\gamma_n(1-\alpha_n)\alpha_n\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&- \gamma_n(1-\alpha_n)\Big[\sum_{i=1}^{m}\beta_{n,i}\frac{\tau_n(2-\tau_n)\|J_{\mathcal{X}}(I - Q^{C_i}_{Q_{\mu_{n,i}}})Tw_n\|^4}{\|T^*J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2} \\
&+ \sum_{i=1}^{m}\beta_{n,i}(2\alpha_i - r_{n,i})r_{n,i}\|A_i z_{n,i} - A_i p\|^2 \\
&+ \beta_{n,0}\sum_{i=1}^{m}\beta_{n,i}\|J^{B_i}_{r_{n,i}}(I - r_{n,i}A_i)z_{n,i} - w_n\|^2\Big].
\end{aligned}
$$

From this we obtain

$$\|x_{n+1}-p\|^2 \le \frac{\Big(1-2\alpha_n+\alpha_n^2+\alpha_n\rho\Big)}{(1-\alpha_n\rho)}\|x_n-p\|^2 + \frac{3M_2\gamma_n(1-\alpha_n)}{(1-\alpha_n\rho)}\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||$$
$$+\frac{2\alpha_n}{(1-\alpha_n\rho)}\langle f(p)-p, x_{n+1}-p\rangle$$
$$-\frac{\gamma_n(1-\alpha_n)}{(1-\alpha_n\rho)}\Big[\sum_{i=1}^m \beta_{n,i}\frac{\tau_n(2-\tau_n)\|J_{\mathcal{X}}(I-Q^{C_i}_{Q_{\mu_{n,i}}})Tw_n\|^4}{\|T^*J_{\mathcal{X}}(I-Q^{C_i}_{\mu_{n,i}})Tw_n\|^2}$$
$$+\sum_{i=1}^m \beta_{n,i}(2\alpha_i-r_{n,i})r_{n,i}\|A_iz_{n,i}-A_ip\|^2$$
$$+\beta_{n,0}\sum_{i=1}^m \beta_{n,i}\|J^{B_i}_{r_{n,i}}(I-r_{n,i}A_i)z_{n,i}-w_n\|^2\Big]$$
$$=\frac{\Big(1-2\alpha_n+\alpha_n\rho\Big)}{(1-\alpha_n\rho)}\|x_n-p\|^2+\frac{\alpha_n^2}{(1-\alpha_n\rho)}\|x_n-p\|^2$$
$$+\frac{3M_2\gamma_n(1-\alpha_n)}{(1-\alpha_n\rho)}\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||+\frac{2\alpha_n}{(1-\alpha_n\rho)}\langle f(p)-p, x_{n+1}-p\rangle$$
$$-\frac{\gamma_n(1-\alpha_n)}{(1-\alpha_n\rho)}\Big[\sum_{i=1}^m \beta_{n,i}\frac{\tau_n(2-\tau_n)\|J_{\mathcal{X}}(I-Q^{C_i}_{Q_{\mu_{n,i}}})Tw_n\|^4}{\|T^*J_{\mathcal{X}}(I-Q^{C_i}_{\mu_{n,i}})Tw_n\|^2}$$
$$+\sum_{i=1}^m \beta_{n,i}(2\alpha_i-r_{n,i})r_{n,i}\|A_iz_{n,i}-A_ip\|^2$$
$$+\beta_{n,0}\sum_{i=1}^m \beta_{n,i}\|J^{B_i}_{r_{n,i}}(I-r_{n,i}A_i)z_{n,i}-w_n\|^2\Big]$$
$$\le\left(1-\frac{2\alpha_n(1-\rho)}{(1-\alpha_n\rho)}\right)\|x_n-p\|^2$$
$$+\frac{2\alpha_n(1-\rho)}{(1-\alpha_n\rho)}\Big[\frac{\alpha_n}{2(1-\rho)}M_3+3M_2\frac{\gamma_n(1-\alpha_n)}{2(1-\rho)}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|$$
$$+\frac{1}{(1-\rho)}\langle f(p)-p, x_{n+1}-p\rangle\Big]$$
$$-\frac{\gamma_n(1-\alpha_n)}{(1-\alpha_n\rho)}\Big[\sum_{i=1}^m \beta_{n,i}\frac{\tau_n(2-\tau_n)\|J_{\mathcal{X}}(I-Q^{C_i}_{Q_{\mu_{n,i}}})Tw_n\|^4}{\|T^*J_{\mathcal{X}}(I-Q^{C_i}_{\mu_{n,i}})Tw_n\|^2}$$
$$+\sum_{i=1}^m \beta_{n,i}(2\alpha_i-r_{n,i})r_{n,i}\|A_iz_{n,i}-A_ip\|^2$$
$$+\beta_{n,0}\sum_{i=1}^m \beta_{n,i}\|J^{B_i}_{r_{n,i}}(I-r_{n,i}A_i)z_{n,i}-w_n\|^2\Big],$$

where $M_3=\sup\{\|x_n-p\|^2 : n\in\mathbb{N}\}$. By taking $\eta_n=\frac{2\alpha_n(1-\rho)}{(1-\alpha_n\rho)}$ and $\sigma_n=\frac{\gamma_n(1-\alpha_n)}{(1-\alpha_n\rho)}$, we obtain the desired result. □

Lemma 6.3.4. *Let $p \in \Gamma$ and suppose $\{x_n\}$ is a sequence generated by Algorithm 6.3.1. Under Assumption (A1)-(A4), the following inequality holds for all $n \in \mathbb{N}$:*

$$\|x_{n+1}-p\|^2 \leq \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2+3M_2\gamma_n\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||-\delta_n\gamma_n\|x_n-Su_n\|^2.$$

Proof. Let $p \in \Gamma$. By applying Lemma 2.5.7, (6.3.11) and (6.3.13), from **Step 5** we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n f(x_n)+\delta_n x_n+\gamma_n Su_n-p\|^2\\
&\leq \alpha_n\|f(x_n)-p\|^2+\delta_n\|x_n-p\|^2+\gamma_n\|Su_n-p\|^2-\delta_n\gamma_n\|x_n-Su_n\|^2\\
&\leq \alpha_n\|f(x_n)-p\|^2+\delta_n\|x_n-p\|^2+\gamma_n\|u_n-p\|^2-\delta_n\gamma_n\|x_n-Su_n\|^2\\
&\leq \alpha_n\|f(x_n)-p\|^2+\delta_n\|x_n-p\|^2+\gamma_n\big(||x_n-p||^2+3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\big)\\
&-\delta_n\gamma_n\|x_n-Su_n\|^2\\
&= \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2+3M_2\gamma_n\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||)\\
&-\delta_n\gamma_n\|x_n-Su_n\|^2,
\end{aligned}$$

which is the desired inequality.

□

We are now in the position to give the strong convergence theorem for Algorithm 6.3.1.

Theorem 6.3.5. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{X}$ a uniformly convex and smooth Banach space and $J_{\mathcal{X}}$ the duality mapping on $\mathcal{X}$. Let $\{x_n\}$ be generated by Algorithm 6.3.1 and suppose Assumption (A1-A4) are satisfied. Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in \Gamma$, where $\bar{x} = P_\Gamma \circ f(\bar{x})$.*

Proof. Let $\bar{x} = P_\Gamma \circ f(\bar{x})$. From Lemma 6.3.3 we have

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 \leq (1-\eta_n)\|x_n-\bar{x}\|^2+\eta_n\Big[\frac{\alpha_n}{2(1-\rho)}M_3+3M_2\frac{\gamma_n(1-\gamma_n)}{2(1-\rho)}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\\
+\frac{1}{(1-\rho)}\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle\Big].
\end{aligned} \tag{6.3.14}$$

Now, we claim that the sequence $\{\|x_n-\bar{x}\|\}$ converges to zero. To establish this, by Lemma 2.5.36 it suffices to show that $\limsup_{k\to\infty}\langle f(\bar{x})-\bar{x}, x_{n_k+1}-\bar{x}\rangle \leq 0$ for every subsequence $\{\|x_{n_k}-\bar{x}\|\}$ of $\{\|x_n-\bar{x}\|\}$ satisfying

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-\bar{x}\|-\|x_{n_k}-\bar{x}\|\right) \geq 0.$$

Suppose that $\{\|x_{n_k}-\bar{x}\|\}$ is a subsequence of $\{\|x_n-\bar{x}\|\}$ such that

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-\bar{x}\|-\|x_{n_k}-\bar{x}\|\right) \geq 0. \tag{6.3.15}$$

From Lemma 6.3.3, we have

$$\sigma_{n_k}\sum_{i=1}^{m}\beta_{n_k,i}\frac{\tau_{n_k}(2-\tau_{n_k})\|J_{\mathcal{X}}\left(I-Q^{C_i}_{Q_{\mu_{n_k,i}}}\right)Tw_{n_k}\|^4}{\|T^*J_{\mathcal{X}}\left(I-Q^{C_i}_{\mu_{n_k,i}}\right)Tw_{n_k}\|^2}\leq(1-\eta_{n_k})\|x_{n_k}-\bar{x}\|^2$$
$$-\|x_{n_k+1}-\bar{x}\|^2$$
$$+\eta_{n_k}\left[\frac{\alpha_{n_k}}{2(1-\rho)}M_3+3M_2\frac{\gamma_{n_k}(1-\alpha_{n_k})}{2(1-\rho)}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|+\frac{1}{\alpha_{n_k}(1-\rho)}\langle f(\bar{x})-\bar{x},x_{n_k+1}-\bar{x}\rangle\right].$$

Since $\lim_{k\to\infty}\alpha_{n_k}=0$, then $\lim_{k\to\infty}\eta_{n_k}=0$. Hence, by applying (6.3.15) we obtain

$$\sigma_{n_k}\sum_{i=1}^{m}\beta_{n_k,i}\frac{\tau_{n_k}(2-\tau_{n_k})\|J_{\mathcal{X}}\left(I-Q^{C_i}_{Q_{\mu_{n_k,i}}}\right)Tw_{n_k}\|^4}{\|T^*J_{\mathcal{X}}\left(I-Q^{C_i}_{\mu_{n_k,i}}\right)Tw_{n_k}\|^2}\to 0,\ \ \text{as}\ \ k\to\infty.$$

By the conditions on the control parameters, it follows that

$$\beta_{n_k,i}\frac{\tau_{n_k}(2-\tau_{n_k})\|J_{\mathcal{X}}\left(I-Q^{C_i}_{Q_{\mu_{n_k,i}}}\right)Tw_{n_k}\|^4}{\|T^*J_{\mathcal{X}}\left(I-Q^{C_i}_{\mu_{n_k,i}}\right)Tw_{n_k}\|^2}\to 0,\ \ \text{as}\ \ k\to\infty\quad\forall i=1,2,\ldots,m.$$

Since $0<a\leq\tau_{n_k}\leq b<2, \liminf_{k\to\infty}\beta_{n_k,0}\beta_{n_k,i}>0$ and $\|T^*J_{\mathcal{X}}\left(I-Q^{C_i}_{\mu_{n_k,i}}\right)Tw_{n_k}\|$ is bounded for all $i=1,2,\cdots,m$, we have that

$$\lim_{k\to\infty}\|J_{\mathcal{X}}\left(I-Q^{C_i}_{\mu_{n_k,i}}\right)Tw_{n_k}\|=0\quad\forall i=1,2,\cdots,m. \tag{6.3.16}$$

Consequently, for $i=1,2,\cdots,m$ we get

$$\begin{aligned}\|T^*J_{\mathcal{X}}\left(I-Q^{C_i}_{\mu_{n_k,i}}\right)Tw_{n_k}\|&\leq\|T^*\|\|J_{\mathcal{X}}\left(I-Q^{C_i}_{\mu_{n_k,i}}\right)Tw_{n_k}\| \\ &=\|T\|\|J_{\mathcal{X}}\left(I-Q^{C_i}_{\mu_{n_k,i}}\right)Tw_{n_k}\|\to 0,\quad k\to\infty.\end{aligned} \tag{6.3.17}$$

Following similar argument, from Lemma 6.3.3 we obtain

$$\|A_iz_{n_k,i}-A_i\bar{x}\|\to 0\quad k\to\infty\quad\forall i=1,2,\ldots,m, \tag{6.3.18}$$

and

$$\|J^{B_i}_{r_{n_k},i}\left(I-r_{n_k,i}A_i\right)z_{n_k,i}-w_{n_k}\|\to 0\quad k\to\infty\quad\forall i=1,2,\ldots,m. \tag{6.3.19}$$

Also, from 6.3.4 we obtain

$$\delta_{n_k}\gamma_{n_k}\|x_{n_k}-Su_{n_k}\|^2\leq\alpha_{n_k}\|f(x_{n_k})-\bar{x}\|^2+(1-\alpha_{n_k})\|x_{n_k}-\bar{x}\|^2-\|x_{n_k+1}-\bar{x}\|^2$$
$$+3M_2\gamma_{n_k}\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}||.$$

By Remark 6.3.1, (6.3.15) and the conditions on the control parameters, we obtain

$$\|x_{n_k} - Su_{n_k}\| \to \quad k \to \infty. \tag{6.3.20}$$

From **Step 3** and (6.3.17), we have

$$\begin{aligned} \|z_{n_k,i} - w_{n_k}\| &= \|w_{n_k} - \lambda_{n_k,i} T^* J_{\mathcal{X}} \left(I - Q^{C_i}_{\mu_{n_k,i}}\right) T w_{n_k} - w_{n_k}\| \\ &= \lambda_{n_k,i} \|T^* J_{\mathcal{X}} \left(I - Q^{C_i}_{\mu_{n_k,i}}\right) T w_{n_k}\| \to 0, \qquad k \to \infty \quad \forall i = 1, 2, \dots, m. \end{aligned} \tag{6.3.21}$$

Also, from **Step 4** and (6.3.19) we get

$$\begin{aligned} \|u_{n_k} - w_{n_k}\| &\le \beta_{n_k,0} \|w_{n_k} - w_{n_k}\| \\ &\quad + \sum_{i=1}^{m} \beta_{n_k,i} \|J^{B_i}_{r_{n_k,i}} (I - r_{n_k,i} A_i) z_{n_k,i} - w_{n_k}\| \to 0, \ \text{ as } \ k \to \infty. \end{aligned} \tag{6.3.22}$$

From (6.3.21) and (6.3.22), we obtain

$$\|z_{n_k,i} - u_{n_k}\| \to 0, \text{as } \ k \to \infty.$$

Also from Remark 6.3.1, we have

$$\|w_{n_k} - x_{n_k}\| = \theta_{n_k} \|x_{n_k} - x_{n_k-1}\| \to 0, \ \text{as}\ k \to \infty. \tag{6.3.23}$$

From (6.3.21), (6.3.22) and (6.3.23) we have

$$\|z_{n_k,i} - x_{n_k}\| \to 0, \quad \|u_{n_k} - x_{n_k}\| \to 0, \quad \text{as}\ k \to \infty. \tag{6.3.24}$$

Also, from (6.3.20) and (6.3.24) we obtain

$$\|u_{n_k} - Su_{n_k}\| \to 0, \quad k \to \infty. \tag{6.3.25}$$

By applying (6.3.20) and the condition on α_n, from **Step 5** we obtain

$$\begin{aligned} \|x_{n_k+1} - x_{n_k}\| &= \|\alpha_{n_k} f(x_{n_k}) + \delta_{n_k} x_{n_k} + \gamma_{n_k} Su_{n_k} - x_{n_k}\| \\ &\le \alpha_{n_k} \|f(x_{n_k}) - x_{n_k}\| + \delta_{n_k} \|x_{n_k} - x_{n_k}\| + \gamma_{n_k} \|Su_{n_k} - x_{n_k}\| \to 0, \ \text{ as } \ k \to \infty. \end{aligned} \tag{6.3.26}$$

To complete the proof, we need to establish that $w_\omega(x_n) \subset \Gamma$. Let $x^* \in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$. From (6.3.23), we have that $w_{n_k} \rightharpoonup x^*$ as $k \to \infty$. Since T is bounded and linear we have

$$Tw_{n_k} \rightharpoonup Tx^*. \tag{6.3.27}$$

From (6.3.16), we have

$$\lim_{k\to\infty} \left\| \left(I - Q^{C_i}_{\mu_{n_k},i}\right) Tw_{n_k} \right\| = \lim_{k\to\infty} \left\| J_{\mathcal{X}} \left(I - Q^{C_i}_{\mu_{n_k},i}\right) Tw_{n_k} \right\| = 0 \quad \forall i = 1, 2, \ldots, m. \tag{6.3.28}$$

This together with (6.3.27) implies that $Q^{C_i}_{\mu_{n_k},i} Tw_{n_k} \rightharpoonup Tx^*$, as $k \to \infty$ for each $i = 1, 2, \ldots, m$. Since $Q^{C_i}_{\mu_{n_k},i}$ is the metric of C_i for $\mu_{n_k,i} > 0$, we have that

$$\frac{1}{\mu_{n_k,i}} J_{\mathcal{X}} \left(I - Q^{C_i}_{\mu_{n_k},i}\right) Tw_{n_k} \in C_i Q^{C_i}_{\mu_{n_k},i} Tw_{n_k}$$

for all $i = 1, 2, \cdots, m$. Let $v \in X$. By the monotonicity of each C_i, it follows that

$$0 \le \left\langle v - Q^{C_i}_{\mu_{n_k},i} Tw_{n_k}, v^* - \frac{1}{\mu_{n_k,i}} J_{\mathcal{X}} \left(I - Q^{C_i}_{\mu_{n_k},i}\right) Tw_{n_k} \right\rangle,$$

for all $v^* \in C_i(v)$. Passing limit as $k \to \infty$, and by applying (6.3.28) together with the fact that $0 < c \le \mu_{n,i}$, we obtain $0 \le \langle v - Tx^*, v^* - 0\rangle$ for all $v^* \in C_i(v)$, $i = 1, 2, \ldots, m$. Since each C_i is maximal monotone, we have that $Tx^* \in C_i^{-1}(0)$, for all $i = 1, 2, \ldots, m$. This implies that

$$x^* \in \bigcap_{i=1}^{m} T^{-1}(C_i^{-1}(0)). \tag{6.3.29}$$

Next, we show that $x^* \in \bigcap_{i=1}^{m}(A_i + B_i)^{-1}(0)$. Let $T_{r_{n,i}} = J^{B_i}_{r_{n,i}}(I - r_{n,i}A_i)$, then by applying (6.3.19) and (6.3.21) we have

$$\|T_{r_{n_k,i}} z_{n_k,i} - z_{n_k,i}\| \le \|T_{r_{n_k,i}} z_{n_k,i} - w_{n_k}\| + \|w_{n_k} - z_{n_k,i}\| \to 0, \quad k \to \infty \quad \forall i = 1, 2, \ldots, m. \tag{6.3.30}$$

for all $i = 1, 2, \ldots, m$.

By the condition on $r_{n,i}$, there exists $r_i > 0$ such that $r_{n_k,i} \ge r_i$ for all $k \ge 1$ and $i = 1, 2 \ldots, m$. Applying Lemma 2.5.20(ii), we have

$$\lim_{k\to\infty} \|T_{r_{n_k,i}} z_{n_k,i} - z_{n_k,i}\| \le 2 \lim_{k\to\infty} \|T_{r_{n_k,i}} z_{n_k,i} - z_{n_k,i}\| = 0 \quad \forall i = 1, 2, \ldots.$$

We know that T_{r_i} is nonexpansive and from (6.3.24) $z_{n_k,i} \rightharpoonup x^*$ for all $i = 1, 2, \ldots, m$. By the demiclosedness of $I - T_{r_i}$, we have that $x^* \in F(T_{r_i})$ for all $i = 1, 2, \ldots, m$. By Lemma 2.5.20(i) we have that $x^* \in (A_i + B_i)^{-1}(0)$ for all $i = 1, 2, \ldots, m$. This implies that

$$x^* \in \bigcap_{i=1}^{m}(A_i + B_i)^{-1}(0). \tag{6.3.31}$$

Next, we show that $x^* \in F(S)$. From (6.3.24) we have that $u_{n_k} \rightharpoonup x^*$. By (6.3.25) and the demiclosedness property of $I - S$ at zero, we have that $x^* \in F(S)$. Hence, by (6.3.29) and (6.3.31) we have that $w_\omega(x_n) \subset \Gamma$.

From (6.3.24) we have $w_\omega\{z_{n,i}\} = w_\omega\{x_n\}$. By the boundedness of $\{x_{n_k}\}$ there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup x^\dagger$ and

$$\begin{aligned}\lim_{j\to\infty}\langle f(\bar{x})-\bar{x}, x_{n_{k_j}}-\bar{x}\rangle &= \limsup_{k\to\infty}\langle f(\bar{x})-\bar{x}, x_{n_k}-\bar{x}\rangle \\ &= \limsup_{k\to\infty}\langle f(\bar{x})-\bar{x}, z_{n_k,i}-\bar{x}\rangle. \end{aligned} \tag{6.3.32}$$

Since $\bar{x} = P_\Gamma \circ f(\bar{x})$, then it follows that

$$\begin{aligned}\limsup_{k\to\infty}\langle f(\bar{x})-\bar{x}, x_{n_k}-\bar{x}\rangle &= \lim_{j\to\infty}\langle f(\bar{x})-\bar{x}, x_{n_{k_j}}-\bar{x}\rangle \\ &= f(\bar{x})-\bar{x}, x^\dagger-\bar{x}\rangle \\ &\leq 0. \end{aligned} \tag{6.3.33}$$

Thus, from (6.3.26) and (6.3.33) we have

$$\begin{aligned}\limsup_{k\to\infty}\langle f(\bar{x})-\bar{x}, x_{n_k+1}-\bar{x}\rangle &= \limsup_{k\to\infty}\langle f(\bar{x})-\bar{x}, x_{n_k+1}-x_{n_k}\rangle + \limsup_{k\to\infty}\langle f(\bar{x})-\bar{x}, x_{n_k}-\bar{x}\rangle \\ &= f(\bar{x})-\bar{x}, x^*-\bar{x}\rangle \\ &\leq 0. \end{aligned} \tag{6.3.34}$$

Applying Lemma 2.5.36 to (6.3.14), and by Remark 6.3.1, (6.3.34) together with the fact that $\lim_{n\to\infty}\alpha_n = 0$ we have $\lim\limits_{n\to\infty}\|x_n-\bar{x}\| = 0$ as required. This completes the proof. □

Taking $A_i = 0$ for all $i = 1, 2, \dots, m$, we obtain the following consequent result.

Algorithm 6.3.2.

Step 0: Select initial points $x_0, x_1 \in \mathcal{H}$ and set $n = 1$.
Step 1: Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases} \tag{6.3.35}$$

Step 2: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3: Compute

$$z_{n,i} = w_n - \lambda_{n,i}T^*J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n,$$

where

$$\lambda_{n,i} = \frac{\tau_n\|J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2}{\|T^*J_{\mathcal{X}}(I - Q^{C_i}_{\mu_{n,i}})Tw_n\|^2}.$$

Step 4: Compute

$$u_n = \beta_{n,0}w_n + \sum_{i=1}^m \beta_{n,i}J^{B_i}_{r_{n,i}}z_{n,i}.$$

Step 5: Compute

$$x_{n+1} = \alpha_n f(x_n) + \delta_n x_n + \gamma_n S u_n.$$

Set $n := n + 1$ and return to **Step 1**.

Corollary 6.3.3. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{X}$ a uniformly convex and smooth Banach space and $J_{\mathcal{X}}$ the duality mapping on $\mathcal{X}$. Let $\{x_n\}$ be generated by Algorithm 6.3.2 and suppose Assumption (A1-A4) are satisfied. Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in \Omega$, where $\bar{x} = P_\Omega \circ f(\bar{x})$ and $\Omega = \bigcap_{i=1}^m B_i^{-1}(0) \cap \bigcap_{i=1}^m T^{-1}(C_i^{-1}0) \cap F(S)$*

6.3.3 Application

In this section, we apply our result to SFP and SMP.

Split feasibility problem

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $\mathcal{D}$ and $\mathcal{Q}$ be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The SFP is defined as follows:

$$\text{find } x^* \in \mathcal{D} \text{ such that } Tx^* \in \mathcal{Q}, \tag{6.3.36}$$

where $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator.

Let Q be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and i_Q be the indicator function on Q, that is

$$i_Q(x) = \begin{cases} 0 & \text{if } \ x \in Q; \\ \infty & \text{if } \ x \notin Q. \end{cases}$$

Moreover, we define the *normal cone* $N_Q u$ of Q at $u \in Q$ as follows:

$$N_Q u = \{z \in \mathcal{H} : \langle z, v - u \rangle \leq 0, \forall v \in Q\}.$$

It is known that i_Q is a proper, lower semicontinuous and convex function on $\mathcal{H}$. Hence, the subdifferential ∂i_Q of i_Q is a maximal monotone operator. Therefore, we define the resolvent $J_r^{\partial i_Q}$ of ∂i_Q, $\forall r > 0$ as follows:

$$J_r^{\partial i_Q} x = (I + r\partial i_Q)^{-1} x, \forall x \in \mathcal{H}.$$

Moreover, for each $x \in Q$, we have

$$\begin{aligned} \partial i_Q x &= \{z \in \mathcal{H} : i_Q x + \langle z, u - x \rangle \leq i_Q u, \forall u \in \mathcal{H}\} \\ &= \{z \in \mathcal{H} : \langle z, u - x \rangle \leq 0, \forall u \in Q\} \\ &= N_Q x. \end{aligned}$$

Hence, for all $\alpha > 0$, we derive

$$\begin{aligned} u = J_r^{\partial i_Q} x &\Leftrightarrow x \in u + r\partial i_Q u \\ &\Leftrightarrow x - u \in r\partial i_Q u \\ &\Leftrightarrow \langle x - u, z - u \rangle \leq 0 \ \ \forall z \in Q \\ &\Leftrightarrow u = P_Q x. \end{aligned}$$

Now, by applying Corollary 6.3.3 using the case for which the sequences $\{\mu_{n,i}\}$ and $\{r_{n,i}\}$ are taken as constant sequences $\{\mu_i\}$ and $\{r_i\}$, respectively for each $i = 1, 2, \dots, m$. We obtain the following result for approximating a common solution of finite family of SFPs and FPP for a nonexpansive mapping between Hilbert and Banach spaces.

Theorem 6.3.6. *Let D_i and Q_i be finite families of nonempty, closed and convex subsets of a Hilbert space $\mathcal{H}$ and a uniformly convex and smooth Banach space $\mathcal{X}$, respectively for $i = 1, 2, \dots, m$. Let $J_{\mathcal{X}}$ be the duality mapping on $\mathcal{X}$. Let $T : \mathcal{H} \to \mathcal{X}$ be a bounded linear operator such that $T \neq 0$ and T^* is the adjoint operator of T. Let $f : \mathcal{H} \to \mathcal{H}$ be a contraction mapping with $\rho \in (0, 1)$ and $0 < \gamma < \frac{\gamma}{\rho}$. Let $S : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping and suppose that the solution set $\Gamma := \mathcal{F} \cap F(S) \neq \emptyset$, where $\mathcal{F} = \{x^* \in \bigcap_{i=1}^m D_i : Tx^* \in \bigcap_{i=1}^m Q_i\}$. Let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 6.3.4.

Step 0: Select initial points $x_0, x_1 \in \mathcal{H}$ and set $n = 1$.
Step 1: Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases} \tag{6.3.37}$$

Step 2: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3: Compute

$$z_{n,i} = w_n - \lambda_{n,i} T^* J_{\mathcal{X}}(I - P_{Q_i})Tw_n,$$

where

$$\lambda_{n,i} = \frac{\tau_n \|J_{\mathcal{X}}(I - P_{Q_i})Tw_n\|^2}{\|T^* J_{\mathcal{X}}(I - P_{Q_i})Tw_n\|^2}.$$

Step 4: Compute

$$u_n = \beta_{n,0} w_n + \sum_{i=1}^m \beta_{n,i} P_{D_i} z_{n,i}.$$

Step 5: Compute

$$x_{n+1} = \alpha_n f(x_n) + \delta_n x_n + \gamma_n S u_n.$$

Set $n := n + 1$ and return to **Step 1**.

Suppose Assumption (A1-A4) are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 6.3.4 converges strongly to a point $\bar{x} \in \Gamma$, where $\bar{x} = P_\Gamma \circ f(\bar{x})$.

Split minimization problem

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces, $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator.

Given some proper, lower semicontinuous and convex functions $f_1 : \mathcal{H}_1 \to \mathbb{R} \cup \{+\infty\}$ and $f_2 : \mathcal{H}_2 \to \mathbb{R} \cup \{+\infty\}$, the SMP is defined as: find $x^* \in \mathcal{H}_1$ such that

$$x^* \in \operatorname{argmin}_{x\in\mathcal{H}_1} f_1(x) \qquad \text{and} \qquad Tx^* \in \operatorname{argmin}_{y\in\mathcal{H}_2} f_2(y). \tag{6.3.38}$$

Moudafi and Thakur [182] first introduced the SMP which has attracted a lot of attention from researchers in recent years (see [1, 182, 268] and the references therein). The SMP has been applied to the study of many problems in applied science which includes; Fourier regularization, multi-resolution sparse regularization, alternating projection signal synthesis problems, amongst others.

In a real Hilbert space $\mathcal{H}$, the proximal operator of f is defined by

$$prox_{\lambda,f}(x) := \operatorname{argmin}_{z\in\mathcal{H}} \left\{ f(z) + \frac{1}{2\lambda}\|x-z\|^2 \right\} \quad \forall\, x \in \mathcal{H}, \quad \lambda > 0.$$

It is well known that

$$prox_{\lambda,f}(x) = (I + \lambda\partial f)^{-1}(x) = J_\lambda^{\partial f}(x), \tag{6.3.39}$$

where ∂f is the subdifferential of f defined by

$$\partial f(x) = \{z \in \mathcal{H} : f(x) - f(y) \le \langle z, x-y\rangle, \forall y \in \mathcal{H}\},$$

for each $x \in \mathcal{H}$. From [50], ∂f is a maximal monotone operator and $prox_{\lambda,f}$ is firmly nonexpansive.

By setting $B_i = \partial f_i$ and $C_i = \partial g_i$ in Corollary 6.3.3 for each $i = 1, 2, \dots, m$, we obtain the following result for approximating a common solution of finite family of SMP and FPP for nonexpansive mapping between Hilbert and Banach spaces.

Theorem 6.3.7. *Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{X}$ be a uniformly convex and smooth Banach space. Let $J_\mathcal{X}$ be the duality mapping on $\mathcal{X}$. Let $T : \mathcal{H} \to \mathcal{X}$ be a bounded linear operator such that $T \neq 0$ and T^* is the adjoint operator of T. Let $f : \mathcal{H} \to \mathcal{H}$ be a contraction mapping with $\rho \in (0,1)$ and $0 < \gamma < \frac{\gamma}{\rho}$, and Let $S : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping. Let $f_i : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ and $g_i : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be finite families of proper convex lower semicontinuous functions for $i =, 2, \dots, m$. Suppose that the solution set $\Gamma := \mathcal{F} \bigcap F(S) \neq \emptyset$, where $\mathcal{F} = \{x^* \in \bigcap_{i=1}^m \operatorname{argmin} f_i : Tx^* \in \bigcap_{i=1}^m \operatorname{argmin} g_i\}$. Let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 6.3.5.

Step 0: Select initial points $x_0, x_1 \in \mathcal{H}$ and set $n = 1$.
Step 1: Given the iterates x_{n-1} and x_n for each $n \ge 1$, choose θ_n such that $0 \le \theta_n \le \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases} \tag{6.3.40}$$

Step 2: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3: Compute

$$z_{n,i} = w_n - \lambda_{n,i} T^* J_{\mathcal{X}}(I - prox_{\mu_{n,i},g_i}) T w_n,$$

where

$$\lambda_{n,i} = \frac{\tau_n \| J_{\mathcal{X}}(I - prox_{\mu_{n,i},g_i}) T w_n \|^2}{\| T^* J_{\mathcal{X}}(I - prox_{\mu_{n,i},g_i}) T w_n \|^2}.$$

Step 4: Compute

$$u_n = \beta_{n,0} w_n + \sum_{i=1}^{m} \beta_{n,i} prox_{r_{n,i},f_i} z_{n,i}.$$

Step 5: Compute

$$x_{n+1} = \alpha_n f(x_n) + \delta_n x_n + \gamma_n S u_n.$$

Set $n := n + 1$ and return to **Step 1**.

Suppose Assumption (A1-A4) are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 6.3.5 converges strongly to a point $\bar{x} \in \Gamma$, where $\bar{x} = P_\Gamma \circ f(\bar{x})$.

6.3.4 Numerical experiments

In this section, we provide a numerical example in infinite dimensional spaces to demonstrate the performance of Algorithm 6.3.1 and then compare it with the method of Kazmi and Riziv [141, Theorem 3.1] (see Appendix 6.3.9), the method of Sitthithakerngkiet [218, Theorem 3.4] (see Appendix 6.3.10), the method of Long *et al.* [163, Algorithm 5] (see Appendix 6.3.11) and the method of Byrne *et al.* [51, Algorithm 4.4] (see Appendix 6.3.12).

We perform all implementations using Matlab 2016 (b), installed on a personal computer with Intel(R) Core(TM) i5-2600 CPU@2.30GHz and 8.00 Gb-RAM running on Windows 10 operating system. In Tables 6.3.1, "Iter." means the number of iterations while "CPU" means the CPU time in seconds.

Example 6.3.8. Let $H = \Big(l_2(\mathbb{R}, \|\cdot\|_2)\Big) = \mathcal{X}$, where

$$l_2(\mathbb{R}) := \Big\{x = (x_1, x_2, \cdots, x_n, \cdots,), x_j \in \mathbb{R} : \sum_{j=1}^{\infty} |x_j|^2 < \infty\Big\},$$

$\|x\|_2 = \Big(\sum_{j=1}^{\infty} |x_j|^2\Big)^{\frac{1}{2}}$ for all $x \in l_2(\mathbb{R})$. Let $T : \mathcal{H} \to \mathcal{X}$ be defined by $Tx = \frac{3}{2}x$. For $i = 1, 2, \cdots, 5$, define $A_i : \mathcal{H} \to \mathcal{H}$ as $A_i x = \frac{x}{2i}$, $B_i : \mathcal{H} \to \mathcal{H}$ by $B_i x = \frac{3}{2i}x$, $C_i : \mathcal{X} \to \mathcal{X}$ by $C_i x = \frac{5}{2i}x$. Then, A_i is a finite family of inverse strongly monotone operators, B_i and C_i are maximal monotone operators for each $i = 1, 2, \cdots, 5$. Let $S : \mathcal{H} \to \mathcal{H}$ be defined by $Sx = \frac{1}{2}x$. Observe that S is nonexpansive. We set $f(x) = \frac{x}{3}, \beta_{n,0} = \frac{5n}{2(3n+2)}, \beta_{n,i} = \frac{n+4}{10(3n+2)}, \alpha_n = \frac{1}{3n+1}, \delta_n = \frac{n}{3n+1}, \gamma_n = \frac{2n}{3n+1}, \epsilon_n = \frac{1}{(3n+1)^3}, \tau_n = \frac{n}{2n+1}, \mu_{n,i} = r_{n,i} = r = 0.01$

and $\theta = 0.6$ in Algorithm 6.3.1 for each $n \in \mathbb{N}$. We take $S_n(x) = \frac{1}{2^n}x, D = I$ and $\mathcal{E} = 1$ in Appendix 6.3.9, $\theta_n = \frac{1}{(3n+1)^2}$ in Appendix 6.3.11 and $\lambda = \lambda_n = 0.05$.

We consider the following cases for the numerical experiments of this example.

Case I: Take $x_0 = \Big(97, 74, -68, \cdots\Big)$ and $x_1 = \Big(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{19}\cdots\Big)$.

Case II: Take $x_0 = \Big(-66, 584, 99, \cdots\Big)$ and $x_1 = \Big(0.1, -0.03, 0.57, \cdots\Big)$.

Case III: Take $x_0 = \Big(79, 731, 49, \cdots\Big)$ and $x_1 = \Big(0.7, \frac{1}{43}, -\frac{7}{39}, \cdots\Big)$.

Case IV: Take $x_0 = \Big(513, -27, 88, \cdots\Big)$ and $x_1 = \Big(\frac{3}{190}, \frac{23}{77}, -\frac{1}{2}, \cdots\Big)$.

We compare the performance of the algorithms using the stopping criterion $\|x_{n+1} - x_n\| < 10^{-2}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Table 6.3.1 and Figure 6.3.

Table 6.3.1: Numerical results for Example 6.3.8

		App. 6.3.9	App. 6.3.10	App. 6.3.11	App. 6.3.12	Alg. 6.3.1
Case I	CPU time (sec)	0.0151	0.0062	0.0089	0.0091	0.0116
	No. of Iter.	14	6	9	13	6
Case II	CPU time (sec)	0.0102	0.0091	0.0079	0.0055	0.0102
	No. of Iter.	16	7	11	13	7
Case III	CPU time (sec)	0.0103	0.0055	0.0085	0.0061	0.0103
	No. of Iter.	16	7	12	13	7
Case IV	CPU time (sec)	0.0101	0.0056	0.0079	0.0058	0.0106
	No. of Iter.	15	7	11	13	7

Appendix 6.3.9. *The Algorithm in [141].*

Initialization: *Given* $\lambda \in (0, \frac{1}{L}), r > 0$. *Let* $x_0 \in \mathcal{H}_1$ *be arbitrary.*

Iterative Steps: *Calculate* x_{n+1} *as follows:*

Step1. *Compute*

$$u_n = J_r^B\Big(x_n + \lambda T^*(J_r^C - I)Tx_n\Big).$$

Step 2: *Compute*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n,$$

where $f : \mathcal{H} \to \mathcal{H}$ *is a contraction mapping with constant* $\rho \in (0, 1)$, S *is a nonexpansive mapping such that* $F(S) \cap \Gamma \neq \emptyset$.

Set $n := n + 1$ *and return to* **Step 1**.

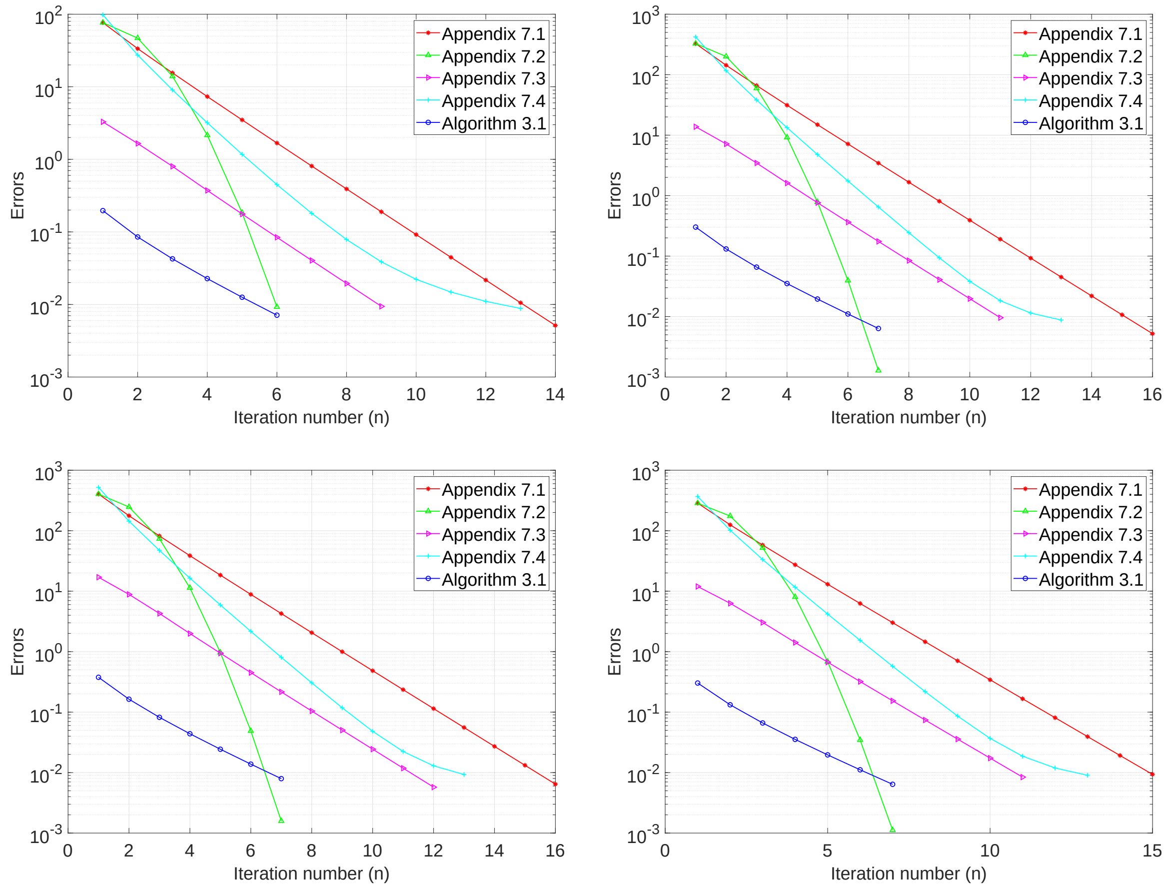

Figure 6.3: Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

Appendix 6.3.10. *The Algorithm in [218].*

Initialization: *Given* $\lambda \in (0, \frac{1}{L}), r > 0$. *Let* $x_0 \in \mathcal{H}_1$ *be arbitrary.*

Iterative Steps: *Calculate* x_{n+1} *as follows:*

Step1. *Compute*

$$y_n = J_r^B\Big(x_n + \lambda T^*(J_r^C - I)Tx_n\Big).$$

Step 2: *Compute*

$$x_{n+1} = \alpha_n \mathcal{E} f(x_n) + (I - \alpha_n D)S_n y_n,$$

where $f : \mathcal{H} \to \mathcal{H}$ *is a contraction mapping with constant* $\alpha \in (0,1), \{S_n\}$ *is a sequence of nonexpansive mappings such that* $F(S_n) \cap \Gamma \neq \emptyset$, D *is strongly positive bounded linear operator with coefficient* $\bar{\gamma} > 0$ *and* $0 < \mathcal{E} < \frac{\bar{\gamma}}{\beta^*}$.

Set $n := n + 1$ *and return to* **Step 1**.

Appendix 6.3.11. *The Algorithm in [163].*

Initialization: *Let* $r > 0$.

Iterative Steps: *Calculate* x_{n+1} *as follows: Let* $\{x_n\}$ *be a sequence in* $\mathcal{H}_1$ *defined by*

$$\begin{cases} x_0, x_1 \in \mathcal{H}, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_r^B(I - \lambda_n T^*(I - J_r^C)T)w_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n. \end{cases} \tag{6.3.41}$$

where $f : \mathcal{H} \to \mathcal{H}$ *is a contraction mapping with constant* $\alpha \in (0,1)$.

Set $n := n + 1$ *and return to* **Step 1**.

Appendix 6.3.12. *The Algorithm in [51].*

$$\begin{cases} v \in H_1, \\ x_{n+1} = \alpha_n v + (1 - \alpha_n)J_r^B(x_n - \lambda T^*(I - J_r^C)Tx_n), \ n \in \mathbb{N}, \end{cases} \tag{6.3.42}$$

CHAPTER 7

Results on Minimization Problems in Hadamard Spaces

7.1 Introduction

The extension of known concepts from Hilbert and Banach spaces to Hadamard spaces has been of great research interest to several researchers. In this chapter, we extend our results from the frameworks of the Hilbert and Banach spaces to the framework of an Hadamard space. We present our results on modified proximal point methods and viscosity implicit rule involving quasi-pseudocontractive mappings in Hadamard spaces. Furthermore, we present some numerical examples of our methods in Hadamard spaces which are not Hilbert spaces and compare them with other results in literature to show the applicability of our methods.

7.2 Preliminaries

Lemma 7.2.1. *Let X be an Hadamard space and $T : X \to X$ be an L-Lipschitz mapping with $L \geq 1$ such that for all $x \in X$,*

$$Kx := (1-\epsilon)x \oplus \epsilon T((1-\gamma)x \oplus \gamma Tx). \tag{7.2.1}$$

If $0 < \epsilon < \gamma < \frac{1}{1+\sqrt{1+L^2}}$, then the following conditions hold:

(a) $x = Tx \Longleftrightarrow x = T((1-\gamma)x \oplus \gamma Tx) \Longleftrightarrow x = Kx$;

(b) if T is $\triangle$-demiclosed, then K is also $\triangle$-demiclosed;

(c) if T is a quasi-pseudocontractive mapping, then K is a quasi-nonexpansive mapping.

Proof. (a) Let $x = Tx$. Then, one can easily see that $x = T((1-\gamma)x \oplus \gamma Tx)$. Conversely, let $x = T((1-\gamma)x \oplus \gamma Tx)$ and define $S_\gamma x := (1-\gamma)x \oplus \gamma Tx$. Then $TS_\gamma x = x$. Thus, we obtain from (2.3.1) that

$$d(x, S_\gamma x) = \gamma d(x, Tx) \leq \gamma L d(S_\gamma x, x).$$

Since $0 < L\gamma < 1$, we obtain that $x = S_\gamma x$. This implies that $x = TS_\gamma x = Tx$.
By similar argument, we obtain that $x = T((1-\gamma)x \oplus \gamma Tx) \iff x = Kx$.

(b) Let T be $\triangle$-demiclosed and $\{x_n\}$ be a sequence in X such that $\triangle\text{-}\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Kx_n) = 0$, then we have that

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, Kx_n) + d((1-\epsilon)x_n \oplus \epsilon T((1-\gamma)x_n \oplus \gamma Tx_n), Tx_n) \\ &\leq d(x_n, Kx_n) + (1-\epsilon)d(x_n, Tx_n) + \epsilon d(T((1-\gamma)x_n \oplus \gamma Tx_n), Tx_n) \\ &\leq d(x_n, Kx_n) + (1-\epsilon)d(x_n, Tx_n) + \epsilon L d((1-\gamma)x_n \oplus \gamma Tx_n, x_n) \\ &= d(x_n, Kx_n) + [(1-\epsilon) + \epsilon L\gamma]\, d(Tx_n, x_n), \end{aligned}$$

which implies that

$$d(x_n, Tx_n) \leq \frac{1}{\epsilon(1-L\gamma)} d(x_n, Kx_n) \to 0.$$

Since T is demiclosed, we get that $x = Tx$. By (a), we obtain the desired conclusion.

(c) Let $y \in F(K)$ and $x \in X$. Then $y \in F(T)$. Hence, we obtain from (2.3.4) that

$$\begin{aligned} d^2(T((1-\gamma)x \oplus \gamma Tx), y) &\leq d^2((1-\gamma)x \oplus \gamma Tx, y) + d^2((1-\gamma)x \oplus \gamma Tx, T((1-\gamma)x \oplus \gamma Tx)) \\ &\leq (1-\gamma)d^2(x,y) + \gamma d^2(Tx, y) - \gamma(1-\gamma)d^2(x, Tx) \\ &\quad + d^2((1-\gamma)x \oplus \gamma Tx, T((1-\gamma)x \oplus \gamma Tx)) \\ &\leq (1-\gamma)d^2(x,y) + \gamma d^2(x,y) + \gamma d^2(x, Tx) - \gamma(1-\gamma)d^2(x, Tx) \\ &\quad + d^2((1-\gamma)x \oplus \gamma Tx, T((1-\gamma)x \oplus \gamma Tx)) \\ &= d^2(x,y) + \gamma^2 d^2(x, Tx) \qquad (7.2.2) \\ &\quad + d^2((1-\gamma)x \oplus \gamma Tx, T((1-\gamma)x \oplus \gamma Tx)). \end{aligned}$$

But,

$$\begin{aligned} d^2((1-\gamma)x \oplus \gamma Tx, T((1-\gamma)x \oplus \gamma Tx)) &\leq (1-\gamma)d^2(x, T((1-\gamma)x \oplus \gamma Tx)) \\ &\quad + \gamma d^2(Tx, T((1-\gamma)x \oplus \gamma Tx)) - \gamma(1-\gamma)d^2(x, Tx) \\ &\leq (1-\gamma)d^2(x, T((1-\gamma)x \oplus \gamma Tx)) \\ &\quad + \gamma L^2 d^2(x, (1-\gamma)x \oplus \gamma Tx) - \gamma(1-\gamma)d^2(x, Tx) \\ &\leq (1-\gamma)d^2(x, T((1-\gamma)x \oplus \gamma Tx)) \qquad (7.2.3) \\ &\quad - (\gamma - \gamma^2 - \gamma^3 L^3)d^2(x, Tx). \end{aligned}$$

Now, from the condition on γ and L, we obtain that $1 - 2\gamma - \gamma^2 L^2 > 0$. Hence, we get from (7.2.2) and (7.2.3) that

$$\begin{aligned} d^2(T((1-\gamma)x \oplus \gamma Tx), y) &\leq d^2(x,y) + (1-\gamma)d^2(x, T((1-\gamma)x \oplus \gamma Tx)) \qquad (7.2.4) \\ &\quad - \gamma(1 - 2\gamma - \gamma^2 L^2)d^2(x, Tx). \\ &\leq d^2(x,y) + (1-\gamma)d^2(x, T((1-\gamma)x \oplus \gamma Tx)). \qquad (7.2.5) \end{aligned}$$

Hence, we obtain from (7.2.4) that

$$\begin{aligned} d^2(Kx,y) &\leq (1-\epsilon)d^2(x,y) + \epsilon d^2(T((1-\gamma)x \oplus \gamma Tx), y) \\ &- \epsilon(1-\epsilon)d^2(x, T((1-\gamma)x \oplus \gamma Tx)) \\ &\leq d^2(x,y) - \epsilon(\gamma - \epsilon)d^2(x, T((1-\gamma)x \oplus \gamma Tx)) \\ &\leq d^2(x,y). \end{aligned}$$

□

Lemma 7.2.2. *Let X be an Hadamard space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, $d^2(J_\lambda^f x, x) \leq d^2(J_\mu^f x, x)$ for $0 < \lambda < \mu$ and $x \in X$.*

Proof. Let $x, y \in X$, then we obtain from the definition of the resolvent of f that

$$f(J_\mu^f x) + \frac{1}{2\mu}d^2(J_\mu^f x, x) \leq f(J_\lambda^f x) + \frac{1}{2\mu}d^2(J_\lambda^f x, x), \tag{7.2.6}$$

and

$$f(J_\lambda^f x) + \frac{1}{2\lambda}d^2(J_\lambda^f x, x) \leq f(J_\mu^f x) + \frac{1}{2\lambda}d^2(J_\mu^f x, x). \tag{7.2.7}$$

Adding both sides of relations (7.2.6) and (7.2.7), we obtain that

$$\left(1 - \frac{\lambda}{\mu}\right) d^2(J_\lambda^f x, x) \leq \left(1 - \frac{\lambda}{\mu}\right) d^2(J_\mu^f x, x).$$

Since, $0 < \lambda < \mu$, we obtain that

$$d^2(J_\lambda^f x, x) \leq d^2(J_\mu^f x, x).$$

□

Lemma 7.2.3. *Let X be an Hadamard space and $f_j : X \to (-\infty, \infty]$, $j = 1, 2, \cdots, m$ be a finite family of proper, convex and lower semicontinuous functions. If $0 < \lambda < \mu$ and $\left(\cap_{j=1}^m F\left(J_\mu^{(j)}\right)\right) \neq \emptyset$. Then,*

$$F\left(\Pi_{j=1}^m J_\mu^{(j)}\right) \subseteq \left(\cap_{j=1}^m F\left(J_\lambda^{(j)}\right)\right),$$

where, $\Pi_{j=1}^m J_\mu^{(j)} = J_\mu^{(1)} \circ J_\mu^{(2)} \circ \cdots \circ J_\mu^{(m)}$.

Proof. Let $x \in F\left(\Pi_{j=1}^m J_\mu^{(j)}\right)$ and $y \in \left(\cap_{j=1}^m F\left(J_\mu^{(j)}\right)\right)$, then by the nonexpansivity of $J_\mu^{(1)}$, we have that

$$d^2(x,y) = d^2\left(\Pi_{j=1}^m J_\mu^{(j)} x, y\right) \leq d^2\left(\Pi_{j=2}^m J_\mu^{(j)} x, y\right). \tag{7.2.8}$$

More so, we obtain from Lemma 2.5.27 that

$$\frac{1}{2\mu}d^2\left(\Pi_{j=1}^m J^{(j)}x, y\right) - \frac{1}{2\mu}d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, y\right) + \frac{1}{2\mu}d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, \Pi_{j=1}^m J_\mu^{(j)}x\right) + f\left(\Pi_{j=1}^m J_\mu^{(j)}x\right) \leq f(y).$$

Since $f(y) \leq f\left(\Pi_{j=1}^m J^{(j)}x\right)$, we obtain from (7.2.8) that

$$\begin{aligned} d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, \Pi_{j=1}^m J_\mu^{(j)}x\right) &\leq d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, y\right) - d^2\left(\Pi_{j=1}^m J_\mu^{(j)}x, y\right) \\ &\leq d^2(x, y) - d^2\left(\Pi_{j=1}^m J_\mu^{(j)}x, y\right) \\ &= d^2\left(\Pi_{j=1}^m J_\mu^{(j)}x, y\right) - d^2\left(\Pi_{j=1}^m J_\mu^{(j)}x, y\right), \end{aligned}$$

which implies that

$$\Pi_{j=1}^m J_\mu^{(j)}x = \Pi_{j=2}^m J_\mu^{(j)}x. \tag{7.2.9}$$

Similarly, we obtain from Lemma 2.5.27 and (7.2.8) that

$$\Pi_{j=2}^m J_\mu^{(j)}x = \Pi_{j=3}^m J_\mu^{(j)}x = \Pi_{j=4}^m J_\mu^{(j)}x = \cdots = \Pi_{j=m-1}^m J_\mu^{(j)}x = J_\mu^{(m)}x = x. \tag{7.2.10}$$

From (7.2.10), we obtain that

$$x = J_\mu^{(m)}x = \Pi_{j=m-1}^m J_\mu^{(j)}x = J_\mu^{(m-1)}J_\mu^{(m)}x = J_\mu^{(m-1)}x. \tag{7.2.11}$$

By repeating the process, we obtain

$$J_\mu^{(1)}x = J_\mu^{(2)}x = \cdots = J_\mu^{(m-1)}x = J_\mu^{(m)}x = x. \tag{7.2.12}$$

Thus, by Lemma 7.2.2, we obtain

$$d^2(x, J_\lambda^{(j)}x) \leq d^2(x, J_\mu^{(j)}x) = 0, \quad j = 1, 2, \cdots, m,$$

which implies that $x \in F(J_\lambda^{(j)})$, $j = 1, 2, \cdots, m$. Hence, we get the desired conclusion. □

7.3 Modified proximal point methods

In this section, we propose two new proximal point methods involving quasi- pseudocontractive mappings in Hadamard spaces. We prove that the first method converges strongly to a common solution of a finite family of MPs and FPP for a finite family of quasi-pseudocontractive mappings in an Hadamard space. We extend this method to a more general method involving multivalued monotone operators to approximate the solution of MIP, which is an important optimization problem. We establish that this method converges strongly to a common zero of a finite family of multivalued monotone operators which is also a common fixed point of a finite family of quasi-pseudocontractive mappings in an Hadamard space. Furthermore, we provide various nontrivial numerical implementations of our method in Hadamard spaces (which are non-Hilbert) and compare them with some other recent methods in the literature.

7.3.1 Main results

Lemma 7.3.1. *Let X be an Hadamard space and $f_j : X \to (-\infty, +\infty]$, $j = 1, 2, \cdots, m$ be a finite family of proper, convex and lower semicontinuous functions. Let $T_i : X \to X$, $i = 1, 2, \cdots, N$ be a finite family of L-Lipschitzian and quasi-pseudocontractive mappings with $L \geq 1$ such that T_i is demiclosed. Suppose that $\Omega := \{\cap_{i=1}^{N} F(T_i) \cap (\cap_{j=1}^{m} argmin_{y\in X} f_j(y))\} \neq \emptyset$ and for arbitrary $x_1, u \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} w_n = \Pi_{j=1}^{m} J_{\lambda_n}^{(j)} x_n = J_{\lambda_n}^{(1)} \circ J_{\lambda_n}^{(2)} \circ \cdots \circ J_{\lambda_n}^{(m)} x_n, \\ y_n = \beta_{n,0} w_n \oplus (1-\beta_{n,0}) \sum_{i=1}^{N} \oplus \frac{\beta_{n,i}}{(1-\beta_{n,0})} \left((1-\epsilon_{n,i}) w_n \oplus \epsilon_{n,i} T_i((1-\gamma_{n,i}) w_n \oplus \gamma_{n,i} T_i w_n)\right) \\ x_{n+1} = (1-\alpha_n) y_n \oplus \alpha_n u, \quad n \geq 1, \end{cases} \tag{7.3.1}$$

where $\lambda_n > \lambda > 0$, $\{\alpha_n\}$ and $\{\beta_{n,i}\}$ are sequences in $(0, 1)$ satisfying $\sum_{i=0}^{N} \beta_{n,i} = 1$, and $0 < \epsilon_{n,i} < \gamma_{n,i} < \frac{1}{1+\sqrt{1+L^2}}$, $i = 1, 2, \cdots, N$. Then $\{x_n\}$ is bounded.

Proof. First observe that by Remark 2.5.29, Algorithm (7.3.1) is well defined.
Now, let $p \in \Omega$ and $K_i w_n := ((1-\epsilon_{n,i}) w_n \oplus \epsilon_{n,i} T_i((1-\gamma_{n,i}) w_n \oplus \gamma_{n,i} T_i w_n))$, then we have from (7.3.1) and Lemma 2.5.28 that

$$\begin{aligned} d(y_n, p) &= d\left(\beta_{n,0} w_n \oplus (1-\beta_{n,0}) \sum_{i=1}^{N} \oplus \frac{\beta_{n,i}}{(1-\beta_{n,0})} K_i w_n, p\right) \\ &\leq \beta_{n,0} d(w_n, p) + \sum_{i=1}^{N} \beta_{n,i} d(K_i w_n, p) \\ &\leq d(w_n, p) \qquad (7.3.2) \\ &\;\;\vdots \\ &\leq d(x_n, p). \qquad (7.3.3) \end{aligned}$$

Now, using (7.3.1) and (7.3.3), we obtain that

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n) y_n + \alpha_n u, p) \\ &\leq (1-\alpha_n) d(y_n, p) + \alpha_n d(u, p) \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n d(u, p) \\ &\leq \max\{d(x_n, p), d(u, p)\}. \end{aligned}$$

By induction, we obtain that $\{x_n\}$ is bounded. Consequently $\{w_n\}$ and $\{y_n\}$ are bounded. □

Theorem 7.3.2. *Let X be an Hadamard space and $f_j : X \to (-\infty, +\infty]$, $j = 1, 2, \cdots, m$ be a finite family of proper, convex and lower semicontinuous functions. Let $T_i : X \to X$, $i = 1, 2, \cdots, N$ be a finite family of L-Lipschitzian and quasi-pseudocontractive mappings with $L \geq 1$ such that T_i is demiclosed. Suppose that $\Omega := \{\cap_{i=1}^{N} F(T_i) \cap (\cap_{j=1}^{m} argmin_{y\in X} f_j(y))\} \neq \emptyset$ and the sequence $\{x_n\}$ is generated by (7.3.1), where $\lambda_n > \lambda > 0$, $\{\alpha_n\}$ and $\{\beta_{n,i}\}$ are sequences in $(0, 1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < a < \epsilon_{n,i} < \gamma_{n,i} < b < \frac{1}{1+\sqrt{1+L^2}}, \quad i = 1, 2, \cdots, N$;

(iii) $\sum_{i=0}^{N} \beta_{n,i} = 1$ *and* $0 < c \leq \beta_{n,i} \leq d < 1$, *for all* $n \geq 1$.

Then, $\{x_n\}$ *converges strongly to a point* $z = P_\Omega u$*, where* P_Ω *is the metric projection of* X *onto* Ω*.*

Proof. Let $p \in \Omega$, then we have from Lemma 2.5.30 and (7.3.3) that

$$\begin{aligned} d^2(y_n, p) &= d^2\left(\beta_{n,0} w_n \oplus (1-\beta_{n,0}) \sum_{i=1}^{N} \frac{\beta_{n,i}}{(1-\beta_{n,0})} K_i w_n, p\right) \\ &\leq \beta_{n,0} d^2(w_n, p) + \sum_{i=1}^{N} \beta_{n,i} d^2(K_i w_n, p) - \beta_{n,0} \sum_{i=1}^{N} \beta_{n,i} d^2(w_n, K_i w_n) \\ &\leq d^2(x_n, p) - \beta_{n,0} \sum_{i=1}^{N} \beta_{n,i} d^2(w_n, K_i w_n). \end{aligned} \tag{7.3.4}$$

Now, using (7.3.1) and (7.3.4), we have that

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n u \oplus (1-\alpha_n) y_n, p) \\ &\leq \alpha_n^2 d^2(u, p) + (1-\alpha_n)^2 d^2(y_n, p) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{up}, \overrightarrow{y_n p}\rangle \\ &\leq \alpha_n^2 d^2(u, p) + (1-\alpha_n) d^2(x_n, p) - (1-\alpha_n)\beta_{n,0} \sum_{i=1}^{N} \beta_{n,i} d^2(w_n, K_i w_n) \quad (7.3.5) \\ &+ 2\alpha_n(1-\alpha_n)\langle \overrightarrow{up}, \overrightarrow{y_n p}\rangle \\ &\leq \alpha_n^2 d^2(u, p) + (1-\alpha_n) d^2(x_n, p) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{up}, \overrightarrow{y_n p}\rangle \\ &= (1-\alpha_n) d^2(x_n, p) + \alpha_n c_n, \end{aligned} \tag{7.3.6}$$

where $c_n = \left[\alpha_n d^2(u, p) + 2(1-\alpha_n)\langle \overrightarrow{up}, \overrightarrow{y_n p}\rangle\right]$.
According to Lemma 2.5.36, to conclude our proof, it suffices to show that $\limsup_{k\to\infty} c_{n_k} \leq 0$ for every subsequence $\{d(x_{n_k}, p)\}$ of $\{d(x_n, p)\}$ satisfying the condition

$$\liminf_{k\to\infty} \left(d(x_{n_k+1}, p) - d(x_{n_k}, p)\right) \geq 0. \tag{7.3.7}$$

To show this, suppose that $\{d(x_{n_k}, p)\}$ is a subsequence of $\{d(x_n, p)\}$ such that (7.3.7) holds. Then,

$$\begin{aligned} \liminf_{k\to\infty} \left(d^2(x_{n_k+1}, p) - d^2(x_{n_k}, p)\right) &= \liminf_{k\to\infty} \left(d(x_{n_k+1}, p) - d(x_{n_k}, p)\right)\left(d(x_{n_k+1}, p) + d(x_{n_k}, p)\right) \\ &\geq 0. \end{aligned} \tag{7.3.8}$$

Now, by (7.3.5), we obtain that

$$\begin{aligned}\limsup_{k\to\infty}\left((1-\alpha_{n_k})\beta_{n_k,0}\sum_{i=1}^{N}\beta_{n_k,i}d^2(w_{n_k},K_iw_{n_k})\right) &\leq \limsup_{k\to\infty}\left(\alpha_{n_k}^2d^2(u,p)+(1-\alpha_{n_k})d^2(x_{n_k},p)\right)\\ &- \limsup_{k\to\infty}\left(d^2(x_{n_k+1},p)\right)\\ &+ \limsup_{k\to\infty}\left(2\alpha_{n_k}(1-\alpha_{n_k})\langle\overrightarrow{up},\overrightarrow{y_{n_k}p}\rangle\right)\\ &\leq \limsup_{k\to\infty}\left(d^2(x_{n_k},p)-d^2(x_{n_k+1},p)\right)\\ &+ \limsup_{k\to\infty}\left(\alpha_{n_k}^2d^2(u,p)\right)\\ &+ \limsup_{k\to\infty}\left(2\alpha_{n_k}(1-\alpha_{n_k})\langle\overrightarrow{up},\overrightarrow{y_{n_k}p}\rangle\right)\\ &= -\liminf_{k\to\infty}\left[d^2(x_{n_k+1},p)-d^2(x_{n_k},p)\right]\\ &\leq 0, \end{aligned} \tag{7.3.9}$$

which implies that

$$\lim_{k\to\infty} d(w_{n_k},K_iw_{n_k})=0,\ \ i=1,2,\dots,N. \tag{7.3.10}$$

Furthermore, we obtain from (7.3.1) that

$$d(x_{n_k+1},y_{n_k})=\alpha_{n_k}d(u,y_{n_k})\to 0,\quad k\to\infty. \tag{7.3.11}$$

Again, from (7.3.1) and (7.3.10), we obtain

$$\begin{aligned} d(y_{n_k},w_{n_k}) &= d\left(\beta_{n_k,0}w_{n_k}\oplus(1-\beta_{n_k,0})\sum_{i=1}^{N}\oplus\frac{\beta_{n_k,i}}{(1-\beta_{n_k,0})}K_iw_{n_k},w_{n_k}\right)\\ &\leq \sum_{i=1}^{N}\beta_{n_k,i}d(K_iw_{n_k},w_{n_k})\to 0,\ \text{ as } k\to\infty. \end{aligned} \tag{7.3.12}$$

From Lemma 2.5.27 and (7.3.3), we obtain that

$$\begin{aligned} d^2(w_{n_k},\Pi_{j=2}^{m}J_{\lambda_{n_k}}^{(j)}x_{n_k}) &= d^2(\Pi_{j=1}^{m}J_{\lambda_{n_k}}^{(j)}x_{n_k},\Pi_{j=2}^{m}J_{\lambda_{n_k}}^{(j)}x_{n_k})\\ &\leq d^2(\Pi_{j=2}^{m}J_{\lambda_{n_k}}^{(j)}x_{n_k},p)-d^2(w_{n_k},p)\\ &\ \vdots\\ &\leq d^2(x_{n_k},p)-d^2(w_{n_k},p)\\ &\leq d^2(x_{n_k},p)-d^2(y_{n_k},p)\\ &= d^2(x_{n_k},p)-d^2(x_{n_k+1},p)+d^2(x_{n_k+1},p)-d^2(y_{n_k},p)\\ &\leq d^2(x_{n_k},p)-d^2(x_{n_k+1},p)+\alpha_{n_k}d^2(u,p)\\ &+(1-\alpha_{n_k})d^2(y_{n_k},p)-d^2(y_{n_k},p). \end{aligned} \tag{7.3.13}$$

By taking limsup as $k\to\infty$ on both sides of (7.3.13), and following similar argument as in (7.3.9), we obtain that

$$\lim_{k\to\infty} d\left(w_{n_k},\Pi_{j=2}^{m}J_{\lambda_{n_k}}^{(j)}x_{n_k}\right)=d\left(\Pi_{j=1}^{m}J_{\lambda_{n_k}}^{(j)}x_{n_k},\Pi_{j=2}^{m}J_{\lambda_{n_k}}^{(j)}x_{n_k}\right)=0. \tag{7.3.14}$$

Following the same argument as in (7.3.14), we can show that

$$\lim_{k\to\infty} d\left(\Pi_{j=2}^m J_{\lambda_{n_k}}^{(j)} x_{n_k}, \Pi_{j=3}^m J_{\lambda_{n_k}}^{(j)} x_{n_k}\right) = \cdots = \lim_{k\to\infty} d\left(J_{\lambda_{n_k}}^{(m)} x_{n_k}, x_{n_k}\right) = 0. \qquad (7.3.15)$$

Hence, we obtain that

$$\begin{aligned} d(w_{n_k}, x_{n_k}) \leq\; & d\left(w_{n_k}, \Pi_{j=2}^m J_{\lambda_{n_k}}^{(j)} x_{n_k}\right) + d\left(\Pi_{j=2}^m J_{\lambda_{n_k}}^{(j)} x_{n_k}, \Pi_{j=3}^m J_{\lambda_{n_k}}^{(j)} x_{n_k}\right) \\ & + d\left(\Pi_{j=3}^m J_{\lambda_{n_k}}^{(j)} x_{n_k}, \Pi_{j=4}^m J_{\lambda_{n_k}}^{(j)} x_{n_k}\right) \\ & + \cdots + d\left(J_{\lambda_{n_k}}^{(m)} x_{n_k}, x_{n_k}\right) \to 0, \quad \text{as} \quad k \to \infty. \end{aligned} \qquad (7.3.16)$$

From (7.3.12) and (7.3.16), we obtain that

$$\lim_{k\to\infty} d(y_{n_k}, x_{n_k}) = 0. \qquad (7.3.17)$$

Since $\{x_{n_k}\}$ is bounded in an Hadamard space X, it follows from Lemma 2.5.31 that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\triangle - \lim_{j\to\infty} x_{n_{k_j}} = v$. From (7.3.16) and (7.3.17), we have that there exist subsequences $\{w_{n_{k_j}}\}$ and $\{y_{n_{k_j}}\}$ of $\{w_{n_k}\}$ and $\{y_{n_k}\}$ respectively such that $\triangle - \lim_{j\to\infty} w_{n_{k_j}} = v$ and $\triangle - \lim_{j\to\infty} y_{n_{k_j}} = v$. Since $T_i, \quad i = 1, 2, \cdots, N$ is demiclosed, we obtain from Lemma 7.2.1 and (7.3.10) that $v \in \cap_{i=1}^N F(T_i) = \cap_{i=1}^N F(K_i)$. More so, $J_{\lambda_{n_k}}^{(j)}$ is a nonexpansive mapping for each $j = 1, 2, \cdots, m$. Thus, we obtain from (7.3.16) and Lemma 2.5.33 that $v \in F\left(\Pi_{j=1}^m J_{\lambda_{n_k}}^{(j)}\right)$. Thus, by Lemma 7.2.3, we have that $v \in \cap_{j=1}^m F\left(J_\lambda^{(j)}\right) = \cap_{j=1}^m \arg min_{y\in X} f_j(y)$. Hence $v \in \Omega$.
Let $z = P_\Omega u$, since $\{y_{n_k}\}$ is bounded. We can choose a subsequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ that Δ-converges to v, and such that

$$\limsup_{k\to\infty} \langle \overrightarrow{uz}, \overrightarrow{y_{n_k}z}\rangle = \lim_{j\to\infty} \langle \overrightarrow{uz}, \overrightarrow{y_{n_{k_j}}z}\rangle \leq \langle \overrightarrow{uz}, \overrightarrow{vz}\rangle. \qquad (7.3.18)$$

Since $v \in \Omega$, we obtain from (7.3.18) and Lemma 2.5.32 that

$$\limsup_{k\to\infty} \langle \overrightarrow{uz}, \overrightarrow{y_{n_k}z}\rangle \quad \leq \quad \langle \overrightarrow{uz}, \overrightarrow{vz}\rangle \leq 0. \qquad (7.3.19)$$

Replacing p with z in (7.3.6), we obtain from (7.3.19) that $\limsup_{k\to\infty} c_{n_k} \leq 0$ (where $c_{n_k} = [\alpha_{n_k} d^2(u, z) + 2(1-\alpha_{n_k})\langle \overrightarrow{uz}, \overrightarrow{y_{n_k}z}\rangle]$). Thus, applying Lemma 2.5.36 to (7.3.6), we conclude that $d(x_n, z) \to 0, \quad$ as $\; n \to \infty$. Therefore, $\{x_n\}$ converges strongly to $z = P_\Omega u$.

□

Remark 7.3.3. Note that by setting $N = 2$ and letting $T_i, \; i = 1, \ldots, N$ to be a demicontractive mapping, we recover the main result of Chang *et al.* [70] as corollary of our main result. Also, by setting $N = 1 = m$ and letting T to be a nonexpansive mapping, we recover the main result of Suparatulatorn *et al.* [232] as corollary of our main result. Furthermore, we considered a finite family of MPs and FPP for a finite family of quasi-pseudocontractive mappings, while in [72], the authors considered one MP and FPP for a single nonexpansive mapping. Moreover, if $m = 1$, $N = 3$ and T is nonexpansive, we recover the main result of Thounthong *et al.* [252]. In a similar way, we can derive the results in [34, 79, 80, 91, 254] as corollaries of our main result.

7.3.2 Extension to monotone inclusion problems

In this section, we extend our study so far to the following MIP: Find $x \in X$ such that

$$0 \in Ax, \tag{7.3.20}$$

where the multivalued operator $A : X \to 2^{X^*}$ (X^* is the dual space of the Hadamard space X recently introduced in [138]) is monotone and may not necessarily be the subdifferential ∂f of a proper convex and lower semicontinuous function f. Note that if $A = \partial f$, then problem (7.3.20) is just equivalent to problem (1.2.16).

The MIPs are known to be one of the most important problems in optimization, nonlinear and convex analysis. The problem was first introduced in Hadamard spaces by Khatibzadeh and Ranjbar [150]. Since then many authors have studied this problem in Banach and Hadamard spaces (see for example [67, 100, 125, 133, 191, 233, 255] and the references therein). One striking shortcoming in all these works is that they fail to provide typical example(s) of multivlaued monotone operators which are not necessarily the subdifferential ∂f of a proper convex and lower semicontinuous function. It is worthy to note that despite that such examples are not easy to find, they are necessary to further motivate the study of problem (7.3.20) in Hadamard spaces (from application point of view) since the main results of all these works are about the multivalued monotone operator A. Thus, examples of A in the form of subdifferential operator would simply not give the full essence of studying problem (7.3.20). For this, we present in what follows, a typical example of a monotone operator defined on an Hadamard space and valued in the dual space. We also present a typical example of the associated resolvent operator. First, we present the definition of these concepts.

Definition 7.3.4. *(see [150]) Let X be an Hadamard space and X^* be its dual space. A multivalued operator $A : X \to 2^{X^*}$ with domain $D(A) := \{x \in X : Ax \neq \emptyset\}$ is monotone if and only if for all $x, y \in D(A)$, $x \neq y$, $x^* \in Ax$, $y^* \in Ay$,*

$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geq 0.$$

Definition 7.3.5. *[150] Let X be an Hadamard space and X^* be its dual. Let $A : X \to 2^{X^*}$ be a multivalued operator, then the resolvent of A of order $\lambda > 0$ is the multivalued operator $J_\lambda^A : X \to 2^X$ defined by*

$$J_\lambda^A(x) := \{z \in X \mid [\frac{1}{\lambda}\overrightarrow{zx}] \in Az\}. \tag{7.3.21}$$

Definition 7.3.6. *Let X be an Hadamard space and X^* be its dual. The subdifferential of a function $f : (-\infty, +\infty] \to \mathbb{R}$ is a multivalued function $\partial f : X \to 2^{X^*}$ defined by*

$$\partial f(x) = \begin{cases} \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz} \rangle, \ \forall z \in X\}, & \textit{if } x \in D(f), \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Example 7.3.7. Let $Y = \mathbb{R}^2$ be an $\mathbb{R}$-tree with the radial metric d_r, where $d_r(x, y) = d(x, y)$ if x and y are situated on a Euclidean straight line passing through the origin and

$d_r(x,y) = d(x,\mathbf{0}) + d(y,\mathbf{0}) := \|x\| + \|y\|$ otherwise (see [93, 148] and [202, page 65]). We put $p = (1,0)$ and $X = B \cup C$, where

$$B = \{(h,0) : h \in [0,1]\} \ \text{ and } \ C = \{(h,k) : h+k = 1, h \in [0,1)\}.$$

Note that X is closed and convex and so, (X, d_r) is an Hadamard space. To present an example of the dual space, first note that the dual space $(X^*, \mathcal{D})$ is defined as

$$X^* = \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$$

together with the associated metric $\mathcal{D}$ which defines an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of $t\overrightarrow{ab} := (t,a,b)$ is given by

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle \ \forall x,y \in X\}.$$

Note that $\mathcal{D}((t,a,b),(s,c,d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle$ for all $x, y \in X$ (see [138, 150] for more details and formulation of $(X^*, \mathcal{D})$).
Thus, we can define the dual space X^* as the space of elements $[t\overrightarrow{ab}]$ such that

$$[t\overrightarrow{ab}] = \begin{cases} \{s\overrightarrow{cd} : c,d \in B, s \in \mathbb{R},\ t(\|b\| - \|a\|) = s(\|d\| - \|c\|)\} & a,b \in B, \\ \{s\overrightarrow{cd} : c,d \in C \cup \{\mathbf{0}\}, s \in \mathbb{R},\ t(\|b\| - \|a\|) = s(\|d\| - \|c\|)\} & a,b \in C \cup \{\mathbf{0}\}, \\ \{t\overrightarrow{ab}\} & a \in B,\ b \in C. \end{cases}$$

Indeed, for each $[t\overrightarrow{ab}] \in X^*$, we calculate its equivalence class as follows: If $[t\overrightarrow{ab}] = [s\overrightarrow{cd}] \neq [\overrightarrow{pp}]$, we must have that $t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle$ for all $x, y \in X$.

Cases:

(1): If $\{a,b,c,d\} \subset C \cup \{\mathbf{0}\}$, we have that $d_r(e,z) = \|e\| + \|z\|$ for all $e \in A$ and $z \in X$. Thus, the equality $t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle$ is equivalent to $t(\|b\| - \|a\|) = s(\|d\| - \|c\|)$.

(2): If $\{a,b,c,d\} \subset B$, we obtain by similar argument as in Case 1 that the equality $t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle$ is equivalent to $t(\|b\| - \|a\|) = s(\|d\| - \|c\|)$.

(3): In Case 1 and Case 2, the equation does not depend on x and y. But in other cases, the equation depends on x and y, i.e., the equality for $x, y \in B$ is different from the equality for $x, y \in C$.

Thus, we conclude that X^* is the dual space of X.

Now that we have both the Hadamard space and its dual space, we can now define the monotone operator and corresponding resolvent operator.
Let $A : X \to 2^{X^*}$ be defined by

$$Ax = \begin{cases} \{[\overrightarrow{\mathbf{0}p}]\} & x \in B, \\ \\ \{[\overrightarrow{\mathbf{0}p}], [\overrightarrow{\mathbf{0}x}]\} & x \in C. \end{cases}$$

Then A is a multivalued monotone operator. Indeed, we consider the following cases.

Cases:

(1): If $x, y \in B$, then $Ax = Ay = \{[\overrightarrow{\mathbf{0}p}]\}$ and $x^* = y^* = [\overrightarrow{\mathbf{0}p}]$. So, $\langle x^* - y^*, \overrightarrow{yx}\rangle = 0 \geq 0$.

(2): If $x, y \in C$, then $Ax = \{[\overrightarrow{\mathbf{0}p}], [\overrightarrow{\mathbf{0}x}]\}$ and $Ay = \{[\overrightarrow{\mathbf{0}p}], [\overrightarrow{\mathbf{0}y}]\}$.

(i) If $x^* = y^* = [\overrightarrow{\mathbf{0}p}]$, then $\langle x^* - y^*, \overrightarrow{yx}\rangle = 0 \geq 0$.

(ii) If $x^* = [\overrightarrow{\mathbf{0}x}]$ and $y^* = [\overrightarrow{\mathbf{0}y}]$, then $\langle x^* - y^*, \overrightarrow{yx}\rangle = d_r^2(x, y) \geq 0$.

(iii) If $x^* = [\overrightarrow{\mathbf{0}p}]$ and $y^* = [\overrightarrow{\mathbf{0}y}]$, then

$$\begin{aligned}
\langle x^* - y^*, \overrightarrow{yx}\rangle &= \langle \overrightarrow{yp}, \overrightarrow{yx}\rangle \\
&= \frac{1}{2}(d_r^2(y, x) + d_r^2(p, y) - d_r^2(p, x)) \\
&= \frac{1}{2}((\|y\| + \|x\|)^2 + (1 + \|y\|)^2 - (1 + \|x\|)^2) \\
&\geq 0 \text{ (since } 1/\sqrt{2} \leq \|x\|, \|y\| \leq 1).
\end{aligned}$$

(iv) If $x^* = [\overrightarrow{\mathbf{0}x}]$ and $y^* = [\overrightarrow{\mathbf{0}p}]$, then $\langle x^* - y^*, \overrightarrow{yx}\rangle = \langle \overrightarrow{px}, \overrightarrow{yx}\rangle$, which is similar to (iii).

(3): If $x \in B, y \in C$, then $Ax = \{[\overrightarrow{\mathbf{0}p}]\}$, $Ay = \{[\overrightarrow{\mathbf{0}p}], [\overrightarrow{\mathbf{0}y}]\}$.

(i) If $x^* = y^* = [\overrightarrow{\mathbf{0}p}]$, then $\langle x^* - y^*, \overrightarrow{yx}\rangle = 0 \geq 0$.

(ii) If $x^* = [\overrightarrow{\mathbf{0}p}]$ and $y^* = [\overrightarrow{\mathbf{0}y}]$, then

$$\begin{aligned}
\langle x^* - y^*, \overrightarrow{yx}\rangle &= \langle \overrightarrow{yp}, \overrightarrow{yx}\rangle \\
&= \frac{1}{2}(d_r^2(y, x) + d_r^2(p, y) - d_r^2(p, x)) \\
&\geq 0 \text{ (since } d(p, x) \leq 1 \leq d(p, y)).
\end{aligned}$$

Thus, A is monotone. We now compute the resolvent of A as follows:

Cases:

(I) Let $x = (h, 0) \in B$.

(i) If $z = (k, 0) \in B$ and $z \in J_\lambda^A(x)$, then $Az = \{[\overrightarrow{\mathbf{0}p}]\}$ and $[\frac{1}{\lambda}\overrightarrow{zx}] = [\overrightarrow{\mathbf{0}p}]$. It follows from (7.3.22) that $\frac{1}{\lambda}(k - h) = 1$ or $k = h - \lambda$.

(ii) If $z = (h', k') \in C$ and $z \in J_\lambda^A(x)$, then $Az = \{[\overrightarrow{\mathbf{0}p}], [\overrightarrow{\mathbf{0}z}]\}$ and $[\frac{1}{\lambda}\overrightarrow{zx}] = [\overrightarrow{\mathbf{0}p}]$ or $[\frac{1}{\lambda}\overrightarrow{zx}] = [\overrightarrow{\mathbf{0}z}]$. Using (7.3.22) we see that both of these two cases are impossible.

(II) Let $x = (h, k) \in C$.

(i) If $z = (h', 0) \in B$ and $z \in J_\lambda^A(x)$, then $Az = \{[\overrightarrow{\mathbf{0}p}]\}$ and $[\frac{1}{\lambda}\overrightarrow{zx}] = [\overrightarrow{\mathbf{0}p}]$ which is impossible by (7.3.22).

(ii) If $z = (h', k') \in C$ and $z \in J_\lambda^A(x)$, then $Az = \{[\overrightarrow{\mathbf{0}p}], [\overrightarrow{\mathbf{0}z}]\}$ and $[\frac{1}{\lambda}\overrightarrow{zx}] \in Az$. The case $[\frac{1}{\lambda}\overrightarrow{zx}] = [\overrightarrow{\mathbf{0}p}]$ is impossible. For the case $[\frac{1}{\lambda}\overrightarrow{zx}] = [\overrightarrow{\mathbf{0}z}]$. Using (7.3.22), we see that $\frac{1}{\lambda}(\|x\| - \|z\|) = \|z\|$ or $\|z\| = \frac{1}{1+\lambda}\|x\|$. Note that there are at most two solutions for z.

Therefore, we derive the resolvent as:

$$J_\lambda^A(x) = \begin{cases} \{z = (h-\lambda, 0)\} & x = (h,0) \in B, \\ \{z = (h', k') \in C : (1+\lambda)^2(h'^2 + k'^2) = h^2 + k^2\} & x = (h,k) \in C. \end{cases}$$

Now that we have a typical example of a multivalued monotone operator (which is not the subdifferential of a proper convex and lower semicontinuous function) in a typical Hadamard space (which is non-Hilbert), we proceed to propose a new PPA similar to (7.3.1) and establish that it converges strongly to a common solution of a finite family of MIPs (7.3.20), which is also a common fixed point of a finite family of qausi-pseudocontractive mappings in an Hadamard space X. We begin with the following crucial lemmas.

Lemma 7.3.8. *[150] Let X be an Hadamard space and J_λ^A be the resolvent of a multivalued operator A of order λ. Then,*

(i) for any $\lambda > 0$, $R(J_\lambda^A) \subset D(A)$ and $F(J_\lambda^A) = A^{-1}(0)$, where $R(J_\lambda^A)$ is the range of J_λ^A (we say that the operator A satisfies the range condition if for every $\lambda > 0$, $D(J_\lambda^A) = X$).

(ii) if A is monotone then J_λ^A is a single-valued and nonexpansive mapping,

(iii) if A is monotone and $0 < \lambda_1 \leq \lambda_2$, then $d(x, J_{\lambda_1}^A x) \leq 2d(x, J_{\lambda_2}^A x)$.

Lemma 7.3.9. *[255] Let X be an Hadamard space and $A : X \to 2^{X^*}$ be a monotone operator. Then,*

$$d^2(u, J_\lambda^A x) + d^2(J_\lambda^A x, x) \leq d^2(u, x)$$

for all $u \in A^{-1}(0)$, $x \in X$ and $\lambda > 0$.

Lemma 7.3.10. *Let X be an Hadamard space and X^* be its dual space. Let $A_j : X \to 2^{X^*}$, $j = 1, 2, \dots, m$ be a finite family of multivalued monotone operators. If $0 < \lambda < \mu$ and $\left(\cap_{j=1}^m F\left(J_\mu^{(j)}\right)\right) \neq \emptyset$. Then,*

$$F\left(\Pi_{j=1}^m J_\mu^{(j)}\right) \subseteq \left(\cap_{j=1}^m F\left(J_\lambda^{(j)}\right)\right),$$

where, $\Pi_{j=1}^m J_\mu^{(j)} = J_\mu^{(1)} \circ J_\mu^{(2)} \circ \cdots \circ J_\mu^{(m)}$.

Proof. By careful observation, we can see that Lemma 7.3.8 (iii) and Lemma 7.3.9 can be used in place of Lemma 7.2.2 and Lemma 2.5.27 respectively as the monotone version of J_λ. Thus, the proof follows similar argument as in the proof of Lemma 7.2.3. □

Using Lemma 7.3.8 and Lemma 7.3.10, we can prove the following results by following similar line of arguments as in the proof of Lemma 7.3.1 and Theorem 7.3.2.

Lemma 7.3.11. *Let X be an Hadamard space and X^* be its dual space. Let $A_j : X \to 2^{X^*}$, $i = 1, 2, \ldots, m$ be a finite family of multivalued monotone operators that satisfy the range condition. Let $T_i : X \to X$, $i = 1, 2, \cdots, N$ be a finite family of L-Lipschitzian and quasi-pseudocontractive mappings with $L \geq 1$ such that T_i is demiclosed. Suppose that $\Omega := \{\cap_{i=1}^N F(T_i) \cap \left(\cap_{j=1}^m A_j^{-1}(0)\right)\} \neq \emptyset$ and for arbitrary $x_1, u \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} w_n = \Pi_{j=1}^m J_{\lambda_n}^{(j)} x_n = J_{\lambda_n}^{(1)} \circ J_{\lambda_n}^{(2)} \circ \cdots \circ J_{\lambda_n}^{(m)} x_n, \\ y_n = \beta_{n,0} w_n \oplus (1-\beta_{n,0}) \sum_{i=1}^N \oplus \frac{\beta_{n,i}}{(1-\beta_{n,0})} ((1-\epsilon_{n,i}) w_n \oplus \epsilon_{n,i} T_i ((1-\gamma_{n,i}) w_n \oplus \gamma_{n,i} T_i w_n)), \\ x_{n+1} = (1-\alpha_n) y_n \oplus \alpha_n u, \quad n \geq 1, \end{cases} \tag{7.3.22}$$

where $\lambda_n > \lambda > 0$, $\{\alpha_n\}$ and $\{\beta_{n,i}\}$ are sequences in $(0,1)$ satisfying $\sum_{i=0}^N \beta_{n,i} = 1$, and $0 < \epsilon_{n,i} < \gamma_{n,i} < \frac{1}{1+\sqrt{1+L^2}}$, $i = 1, 2, \cdots, N$. Then $\{x_n\}$ is bounded.

Theorem 7.3.12. *Let X be an Hadamard space and X^* be its dual space. Let $A_j : X \to 2^{X^*}$, $i = 1, 2, \ldots, m$ be a finite family of multivalued monotone operators that satisfy the range condition. Let $T_i : X \to X$, $i = 1, 2, \cdots, N$ be a finite family of L-Lipschitzian and quasi-pseudocontractive mappings with $L \geq 1$ such that T_i is demiclosed. Suppose that $\Omega := \{\cap_{i=1}^N F(T_i) \cap \left(\cap_{j=1}^m A_j^{-1}(0)\right)\} \neq \emptyset$ and the sequence $\{x_n\}$ is generated by (7.3.22), where $\lambda_n > \lambda > 0$, $\{\alpha_n\}$ and $\{\beta_{n,i}\}$ are sequences in $(0,1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $0 < a < \epsilon_{n,i} < \gamma_{n,i} < b < \frac{1}{1+\sqrt{1+L^2}}$, $i = 1, 2, \cdots, N$;

(iii) $\sum_{i=0}^N \beta_{n,i} = 1$ *and* $0 < c \leq \beta_{n,i} \leq d < 1$, *for all* $n \geq 1$.

Then, $\{x_n\}$ converges strongly to a point $z = P_\Omega u$, where P_Ω is the metric projection of X onto Ω.

We now give the following remark regarding Theorems 7.3.2 and 7.3.12.

Remark 7.3.13.

(i) In Theorems 7.3.2 and 7.3.12, we assumed that the solution set Ω is nonempty which is a strict assumption. Thus, it will be interesting to study some sufficient conditions which guarantee this assumption. This we will investigate in our future project.

(ii) In Theorems 7.3.2 and 7.3.12, the prior knowledge of the Lipschitz constant L is needed which is often not known in many applications. Part of our future research is to modify Algorithms (7.3.1) and (7.3.22) such that the prior knowledge of the constant L is not required.

7.3.3 Numerical experiments

In this section, we discuss the numerical behavior of Algorithm (7.3.1) using various non-trivial test examples. We compare our method with Algorithms (2.5.33)-(2.5.35).
The codes are implemented in Matlab 2016 (b). We perform all computations on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30GHz and 8.00 Gb-RAM. Throughout this section, we shall take $m = 4 = N$, $\gamma_{n,i} = \frac{n}{100in+1}$ $\epsilon_{n,i} = \frac{n}{100in+11}$ and $\beta_{n,i} = \frac{(i+1)n+1}{5(3n+1)}$ $\forall n \geq 1,\ i = 0,1,2,3,4$. Also, we choose $\alpha_n = \frac{3}{5n+7}$ for Algorithms (7.3.1) and (2.5.34). While we choose $\eta_n = \frac{3n}{7n+1}$, $\beta_n = \frac{n}{2n+3}$ and $\frac{8n}{10n+7}$ for Algorithms (2.5.33) and (2.5.35).
Algorithm (7.3.1) becomes:

$$\begin{cases} z_n = \arg\min_{y\in X} \left(f_4(y) + \frac{1}{2\lambda_n}d^2(y,x_n)\right), \\ v_n = \arg\min_{y\in X} \left(f_3(y) + \frac{1}{2\lambda_n}d^2(y,z_n)\right), \\ u_n = \arg\min_{y\in X} \left(f_2(y) + \frac{1}{2\lambda_n}d^2(y,v_n)\right), \\ w_n = \arg\min_{y\in X} \left(f_1(y) + \frac{1}{2\lambda_n}d^2(y,u_n)\right), \\ y_n = \beta_{n,0}w_n \oplus (1-\beta_{n,0})\sum_{i=1}^{4} \oplus \frac{\beta_{n,i}}{(1-\beta_{n,0})}\left((1-\epsilon_{n,i})w_n \oplus \epsilon_{n,i}T_i((1-\gamma_{n,i})w_n \oplus \gamma_{n,i}T_iw_n)\right) \\ x_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n u, \quad n \geq 1. \end{cases}$$

We now consider the following concrete examples of f_j and T_i.

Example 7.3.14. Let $X = \mathbb{R}^2$ be endowed with a metric $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ defined by

$$d(x,y) = \sqrt{(x_1-y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2} \quad \forall x,y \in \mathbb{R}^2.$$

Then, $(\mathbb{R}^2, d)$ is an Hadamard space (see [100]) where the geodosic joining x to y is given by

$$(1-t)x \oplus ty = \left((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)\right).$$

Now define $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ by $T_1(x_1,x_2) = (x_1, 2x_1^2 - x_2)$. We see that T_1 is quasi-pseudocontractive in $(\mathbb{R}^2, d)$ but T_1 is not quasi-pseudocontractive in the classical sense. Indeed, for all $x \in \mathbb{R}^2$ and $y \in F(T_1)$, we have

$$\begin{aligned} d^2(T_1x,y) &= d^2\left((x_1, 2x_1^2 - x_2),(y_1,y_2)\right) \\ &= (x_1-y_1)^2 + (x_1^2 - (2x_1^2 - x_2) - y_1^2 + 2y_1^2 - y_2) \\ &= (x_1-y_1)^2 + (-x_1^2 + x_2 + y_1^2 - y_2)^2 \\ &= (x_1-y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2 \\ &\leq d^2(x,y) + d^2(x,T_1x). \end{aligned}$$

Also, define $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_i(x_1,x_2) = -\left(\frac{2i+1}{2}\right)(x_1,x_2),\ i = 2,3,4.$$

Then, T_i is an L-Lipschitzian and quasi-pseudocontractive mapping with $L = (\frac{2i+1}{2})^2 > 1,\ i = 2, 3, 4$. Define $f_1 : \mathbb{R}^2 \to \mathbb{R}$ by $f_1(x_1, x_2) = 100\left((x_2 + 1) - (x_1 + 1)\right)^2 + x_1^2$. Then f_1 is a proper convex and lower semicontinuous function in $(\mathbb{R}^2, d)$ but not convex in the classical sense (see [100]). We also define $f_j : \mathbb{R}^2 \to \mathbb{R}$ by $f_j(x_1, x_2) = 50jx_1^2,\quad j = 2, 3, 4$. Then f_j is a proper convex and lower semicontinuous function for each $j = 2, 3, 4$.

We now consider the following 4 cases for our numerical experiments for Example 7.3.14 given in FIGURE 7.1 below.

Case 1: $x_1 = (0.5,\ 1)^T$, $u = (1,\ -0.5)^T$ and $\lambda_n = \frac{2n}{n+1}$.

Case 2: $x_1 = (1.5,\ 2)^T$, $u = (1.5,\ 2)^T$ and $\lambda_n = \frac{2n}{n+1}$.

Case 3: $x_1 = (0.5,\ e^2)^T$, $u = (-2,\ 2)^T$ and $\lambda_n = \frac{n}{3n+2}$.

Case 4: $x_1 = (-1,\ e^{-1})^T$, $u = (-2,\ e^{-1})^T$ and $\lambda_n = \frac{n}{3n+2}$.

Example 7.3.15. Let $Y := \{(x, e^x) : x \in \mathbb{R}\}$ and $X_n := \{(n, y) : y \geq e^n\}$ for each $n \in \mathbb{Z}$. Set $X := Y \cup \bigcup_{n \in \mathbb{Z}} X_n$ and equip it with a metric $d : X \times X \to [0, \infty)$, defined by (see [67])

$$d(x, y) = \begin{cases} \displaystyle\int_{x_1}^{y_1} ||\dot{\gamma}(t)||_2 dt + |x_2 - e^{x_1}| + |y_2 - e^{y_1}|, & \text{if } x_1 \neq y_1, \\ |x_2 - y_2|, & \text{if } x_1 = y_1, \end{cases} \tag{7.3.23}$$

where $\dot{\gamma}$ is the derivative of the curve $\gamma : \mathbb{R} \to X$, define by $\gamma(t) := (t, e^t)$ for each $t \in \mathbb{R}$. Then (X, d) is an Hadamard space. Let $f_j := j||.||_2^2 : X \to \mathbb{R},\ j = 1, 2, 3, 4$. Then, f_j is proper, convex and lower semicontinuous in (X, d) (see [67, Example 7.1]).

Now, define $T_i : X \to X$ by $T_1(x_1, x_2) = T_2(x_1, x_2) = (x_1, e^{x_1})$ and $T_3(x_1, x_2) = T_4(x_1, x_2) = (-x_1, e^{-x_1})$ for all $x = (x_1, x_2) \in X$. Then, we check that T_i is quasi-pseudocontractive for each $i = 1, 2, 3, 4$.

Indeed, for each $x, y \in X$, we have that

$$d^2(T_1x, T_1y) = d^2\big((x_1, e^{x_1}), (y_1, e^{y_1})\big)$$

$$= \begin{cases} \left(\displaystyle\int_{x_1}^{y_1} ||\dot{\gamma}(t)||_2 dt + |e^{x_1} - e^{x_1}| + |e^{y_1} - e^{y_1}|\right)^2 & \text{if } x_1 \neq y_1, \\ |e^{x_1} - e^{y_1}|^2 & \text{if } x_1 = y_1, \end{cases}$$

$$= \begin{cases} \left(\displaystyle\int_{x_1}^{y_1} ||\dot{\gamma}(t)||_2 dt\right)^2 & \text{if } x_1 \neq y_1, \\ |e^{x_1} - e^{y_1}|^2 & \text{if } x_1 = y_1, \end{cases}$$

$$\leq \begin{cases} \left(\displaystyle\int_{x_1}^{y_1} ||\dot{\gamma}(t)||_2 dt + |x_2 - e^{x_1}| + |y_2 - e^{y_1}|\right)^2 & \text{if } x_1 \neq y_1, \\ |x_2 - y_2|^2 & \text{if } x_1 = y_1, \end{cases}$$

$$\leq d^2(x, y) + d^2(x, Tx).$$

Therefore, T_1 is quasi-pseudocontractive. The proof that T_i, $i = 2, 3, 4$ are also quasi-pseudocontractive is very similar to that of T_1 above.

We now consider the following 4 cases for our numerical experiments for Example 7.3.15 given in FIGURE 7.2 below.

Case 1: $x_1 = (2,\ e^2)^T$, $u = (2,\ e^2)^T$ and $\lambda_n = \frac{n}{3n+2}$.

Case 2: $x_1 = (1,\ 3)^T$, $u = (0.5,\ e^{0.5})^T$ and $\lambda_n = \frac{n}{3n+2}$.

Case 3: $x_1 = (0.5,\ e^{0.5})^T$, $u = (3,\ 21)^T$ and $\lambda_n = \frac{n}{n+100}$.

Case 4: $x_1 = (-1,\ e^{-1})^T$, $u = (-2,\ e^{-1})^T$ and $\lambda_n = \frac{n}{n+100}$.

Example 7.3.16. Let $\mathbf{P}$(n) be the space of $n \times n$ Hermitian positive definite matrices. The geodesic distance between A and B in $\mathbf{P}$(n) (also called the Riemannian (trace) distance) $d_2 : \mathbf{P}(n) \times \mathbf{P}(n) \to [0, \infty)$ is defined by (see [88] [188, Chapter 2] [78, Example 5.2])

$$\begin{aligned} d_2(A, B) &= \inf\{L(c) | c : [0, 1] \to \mathbf{P}(n) \ \text{ is a solution curve with } \ c(0) = A \ \text{ and } \ c(1) = B\} \\ &= \| \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \|_2 \\ &= \left(\sum_{i=1}^{m} \log^2 \mu_i(A^{-1}B) \right)^{\frac{1}{2}}, \end{aligned}$$

where $\mu_i(A^{-1}B)$ are the eignevalues of $A^{-1}B$, $L(c) := \displaystyle\int_0^1 \|c(t)^{-\frac{1}{2}} c'(t) c(t)^{-\frac{1}{2}}\|_2 dt$, $\|A\|_2 := (tr|A|^2)^{\frac{1}{2}}$, tr is the usual trace functional and $|A| = \big(A^H A\big)^{\frac{1}{2}}$ (where A^H is the conjugate transpose of A). The pair $(\mathbf{P}(n), d_2)$ is an Hadamard space with geodesic joining A to B in $\mathbf{P}$(n) given by (see [73, 88, 188])

$$(1-t)x \oplus ty = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}, \quad 0 \leq t \leq 1.$$

Now, define $T_i : \mathbf{P}(n) \to \mathbf{P}(n)$ by $T_iA = D^HAD,\ i = 1, 2, 3, 4$ where $D \in GL(n)$ (the set of $n \times n$ invertible matrices). Then T_i is nonexpansive (see [188, Chapter 2]), and hence, quasi-pseudocontractive. Also, define $f_1 : \mathbf{P}(n) \to \mathbb{R}$ by $f_1A = \left(\sum_{i=1}^{m} \log^p \mu_i(A^{-1}e^A)\right)^{\frac{1}{p}}$, where $\mu_i(A^{-1}e^A)$ are the eigenvalues of $A^{-1}e^A$. Then f_1 is convex and lower semicontinuous (see [19]). Again, define $f_2, f_3, f_4 : \mathbf{P}(n) \to \mathbb{R}$ by $f_2A = -\log detA$, $f_3A = \text{tr}(A)$ and $f_4A = \text{tr}(e^A)$ respectively, then f_j is convex and lower semicontiunous for each $j = 2, 3, 4$ (see [19, 227]).

We now consider the following 4 cases for our numerical experiments for Example 7.3.16 given in FIGURE 7.3 below.

Case I: $x_1 = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}$, $u = \begin{bmatrix} 5 & 1+i \\ 1-i & 4 \end{bmatrix}$ and $\lambda_n = \frac{n}{n+100}$.

Case II: $x_1 = \begin{bmatrix} 2 & 2-i \\ 2+i & 4 \end{bmatrix}$, $u = \begin{bmatrix} 5 & 1+i \\ 1-i & 4 \end{bmatrix}$ and $\lambda_n = \frac{n}{n+100}$.

Case III: $x_1 = \begin{bmatrix} 2 & 2-i \\ 2+i & 4 \end{bmatrix}$, $u = \begin{bmatrix} 1 & 4+i \\ 4-i & 3 \end{bmatrix}$ and $\lambda_n = \frac{10n}{2n+5}$.

Case IV: $x_1 = \begin{bmatrix} 3 & -3-i \\ -3+i & 4 \end{bmatrix}$, $u = \begin{bmatrix} 2 & -i \\ i & 2 \end{bmatrix}$ and $\lambda_n = \frac{10n}{2n+5}$.

Remark 7.3.17. The numerical results for Examples 7.3.14, 7.3.15 and 7.3.16 are displayed in Figures 7.1, 7.2 and 7.3, respectively.
In Figures 7.1, 7.2 and 7.3, Error= $d(x_{n+1}, x_n)$. We can see from these figures that our algorithm requires the least number of iterations (in all the examples) when compared with Algorithms (2.5.33)-(2.5.35). Thus, our method is more efficient than these other methods.

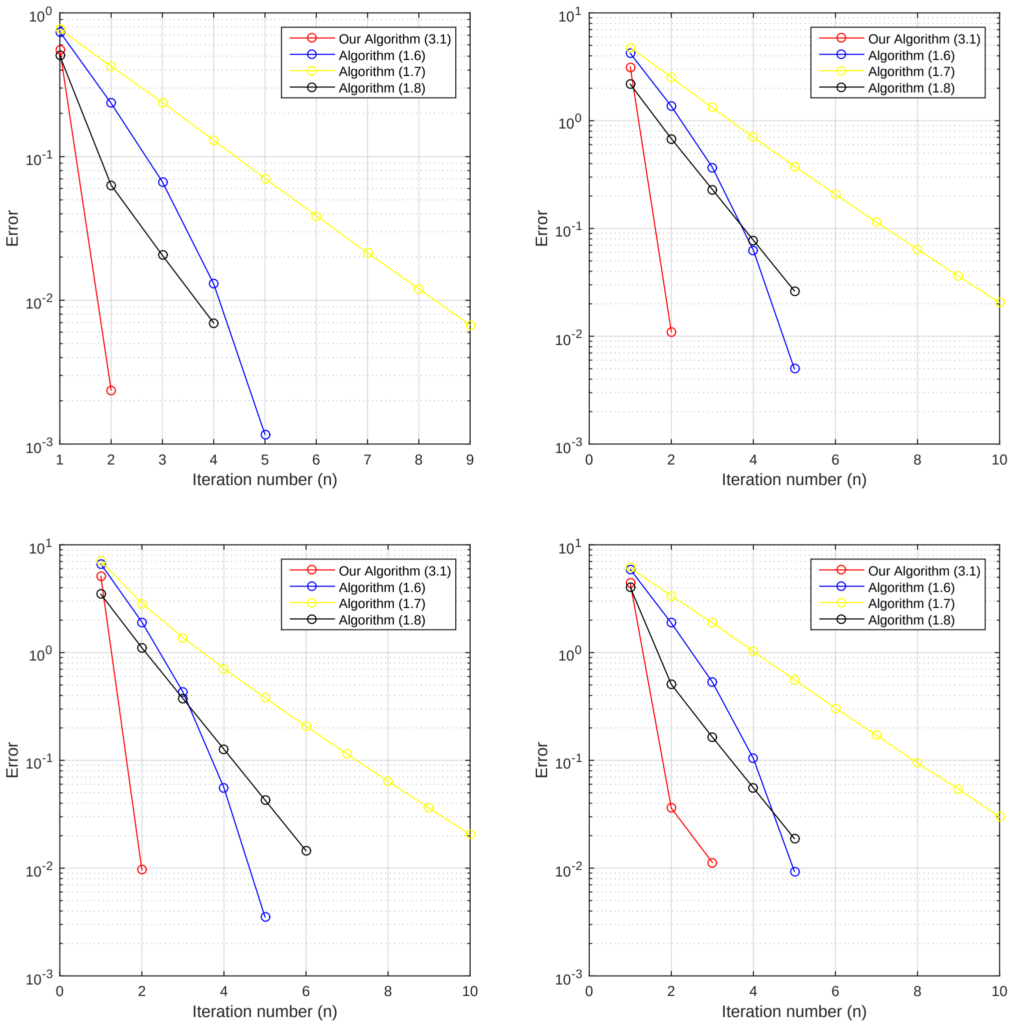

Figure 7.1: Errors vs Iteration numbers for **Example 7.3.14**: **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

7.4 On generalized viscosity implicit rule for quasi-pseudocontractive mappings

In this section, we propose a proximal point approach of a viscosity type iterative method for generalized viscosity implicit rule involving quasi-pseudocontractive mappings in Hadamard spaces. We prove that the sequence generated by our proposed algorithm converges strongly to a common solution of a finite family of MPs and FFP for a finite family of quasi-pseudocontractive mappings in an Hadamard space.

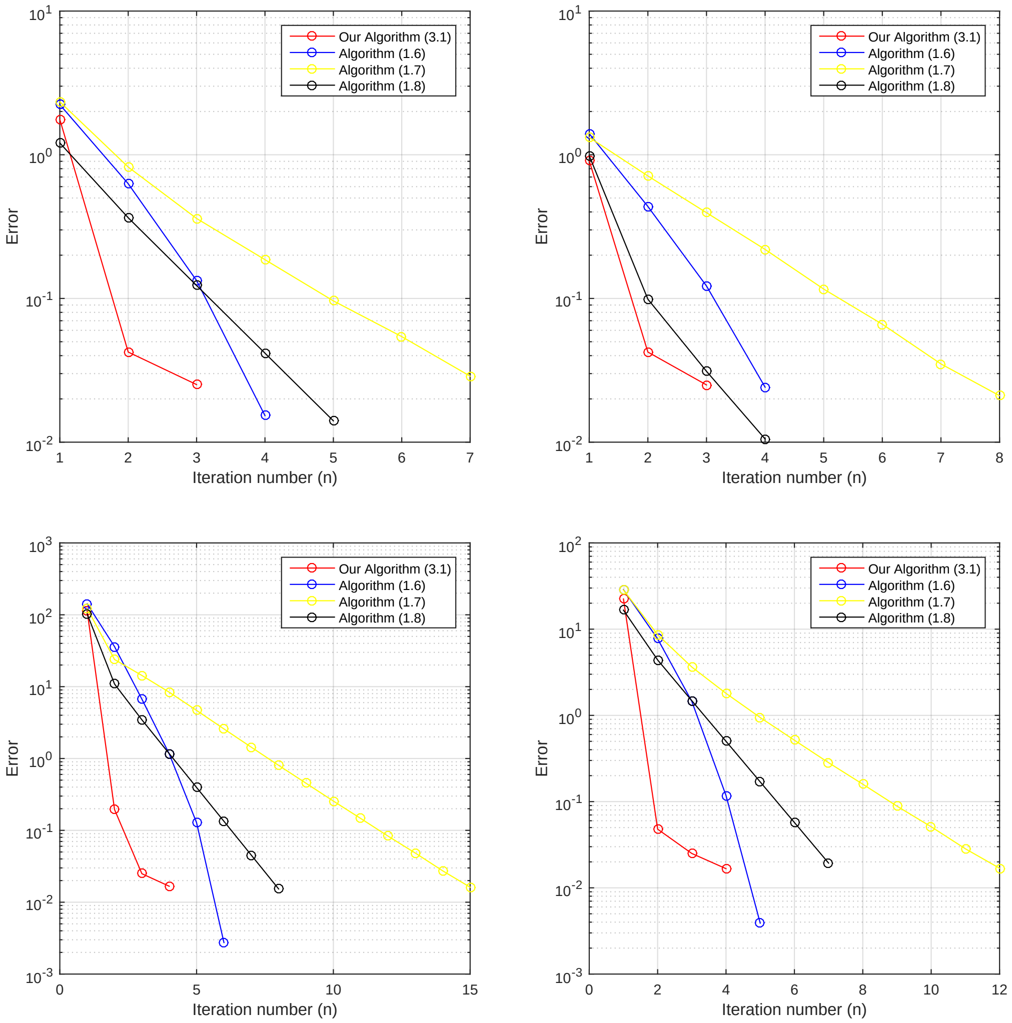

Figure 7.2: Errors vs Iteration numbers for **Example 7.3.15**: **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

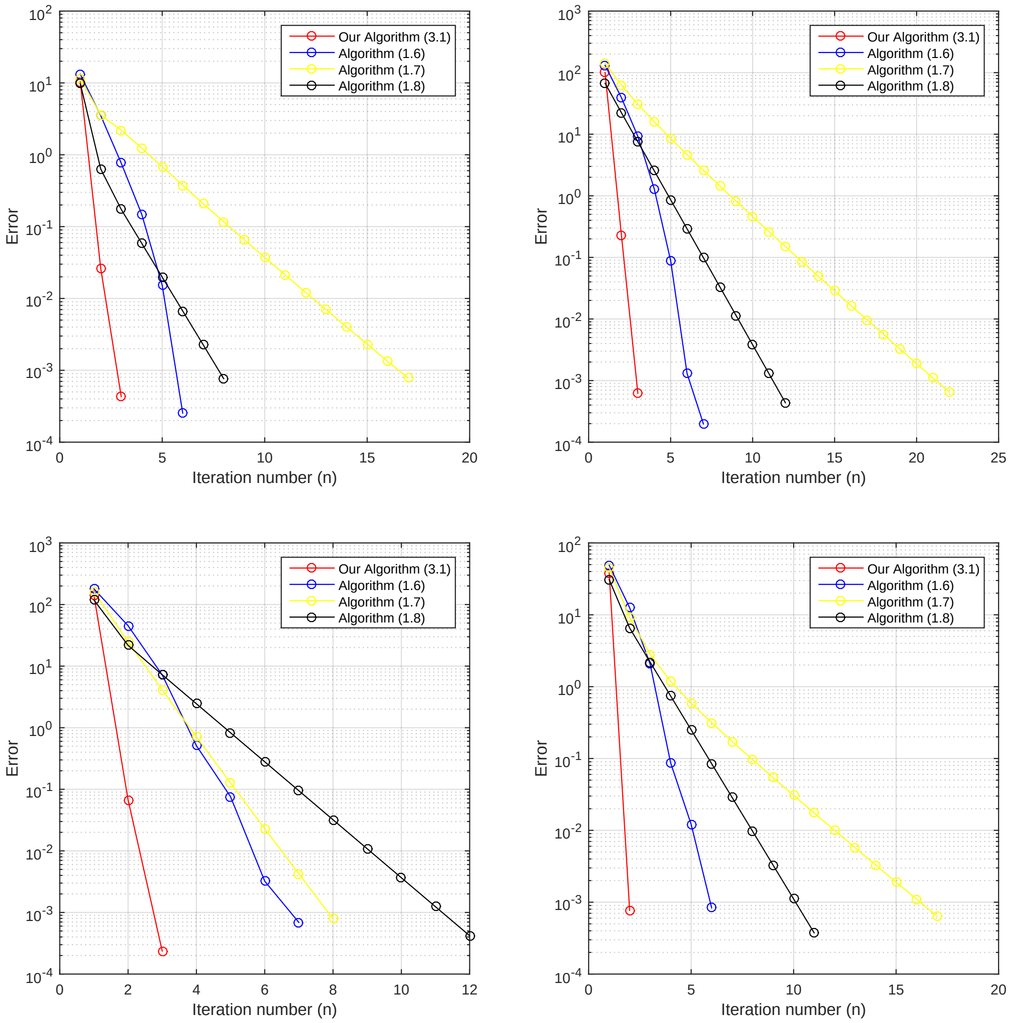

Figure 7.3: Errors vs Iteration numbers for **Example 7.3.16**: **Case I** (top left); **Case II** (top right); **Case III** (bottom left); **Case IV** (bottom right).

7.4.1 Main result

Lemma 7.4.1. *Let X be an Hadamard space and $f_j : X \to (-\infty, +\infty]$, $j = 1, 2, \cdots, m$ be a finite family of proper, convex and lower semicontinuous functions. Let $g : X \to X$ be a contraction with coefficient $\sigma \in [0, 1)$. Let $J_{\lambda_n}^{f_m} : X \to X$ be the resolvents of a finite family of proper, convex and lower semicontinuous functions. Let $T_i : X \to X$, $i = 1, 2, \cdots, N$ be a finite family of L-Lipschitzian and quasi-pseudocontractive mappings with $L \geq 1$ such that T_i is demiclosed. Suppose that $\Gamma := \{\cap_{i=1}^N F(T_i) \cap \left(\cap_{j=1}^m argmin_{y\in X} f_j(y)\right)\} \neq \emptyset$ and for arbitrary $x_1, u \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} w_n = \theta_n x_n \oplus (1-\theta_n)x_{n+1}, \\ y_n = \beta_0 w_n \oplus \beta_1 J_{\lambda_n}^{f_1} w_n \oplus \beta_2 J_{\lambda_n}^{f_2} w_n \oplus \cdots \oplus \beta_n J_{\lambda_n}^{f_m} w_n, \\ u_n = \beta_{n,0} y_n \oplus (1-\beta_{n,0}) \sum_{i=1}^{N} \oplus \frac{\beta_{n,i}}{(1-\beta_{n,0})} ((1-\epsilon_{n,i})y_n \oplus \epsilon_{n,i} T_i((1-\gamma_{n,i})y_n \oplus \gamma_{n,i} T_i y_n)), \\ x_{n+1} = \alpha_n g(x_n) \oplus \psi_n w_n \oplus \tau_n u_n, \end{cases} \tag{7.4.1}$$

where $\lambda_n > \lambda > 0$, $\{\alpha_n\}, \{\psi_n\}, \{\tau_n\}, \{\theta_n\}$ and $\{\beta_{n,i}\}$ are sequences in $(0, 1)$ satisfying;

(i) $\alpha_n + \psi_n + \tau_n = 1$

(ii) $\sum_{i=0}^{N} \beta_{n,i} = 1,$

(iii) $\sum_{i=0}^{N} \beta_i = 1,$

(iv) $0 < \epsilon_{n,i} < \gamma_{n,i} < \frac{1}{1+\sqrt{1+L^2}}, \quad i = 1, 2, \cdots, N.$

Then $\{x_n\}$ is bounded.

Proof. First observe that by Remark 2.5.29, Algorithm 7.4.1 is well defined.
let $p \in \Gamma$, then from the definition of y_n in Algorithm 7.4.1 and Lemma 2.3.23 we have

$$\begin{aligned} d(y_n, p) &= d(\beta_0 w_n \oplus \beta_1 J_{\lambda_n}^{f_1} \oplus \beta_2 J_{\lambda_n}^{f_2} \oplus \cdots \oplus \beta_n J_{\lambda_n}^{f_m} w_n, p) \\ &\leq \beta_0 d(w_n, p) + \beta_1 d(w_n, p) + \beta_2 d(w_n, p) \cdots + \beta_n d(w_n, p) \\ &\leq d(w_n, p). \end{aligned} \tag{7.4.2}$$

Let $K_i y_n := \Big((1-\epsilon_{n,i})y_n \oplus \epsilon_{n,i} T_i((1-\gamma_{n,i})y_n \oplus \gamma_{n,i} T_i y_n)\Big)$, then we have from Algorithm

7.4.1, Lemma 2.5.28, Remark 2.5.29 and (7.4.2) that

$$\begin{aligned} d(u_n,p) &= d(\beta_{n,0}y_n \oplus (1-\beta_{n,0})\sum_{i=1}^{N} \oplus \frac{\beta_{n,i}}{(1-\beta_{n,0})}K_iy_n, p) \\ &\leq \beta_{n,0}d(y_n,p) + \sum_{i=1}^{N}\beta_{n,i}d(K_iy_n,p) \\ &\leq \beta_{n,0}d(y_n,p) + \sum_{i=1}^{N}\beta_{n,i}d(y_n,p) \\ &= d(y_n,p) \\ &\leq d(w_n,p). \end{aligned} \tag{7.4.3}$$

We obtain from the definition of w_n in Algorithm 7.4.1 and Lemma 2.3.23 that

$$d(w_n,p) \leq \theta_n d(x_n,p) + (1-\theta_n)d(x_{n+1},p). \tag{7.4.4}$$

From the definition of x_{n+1} in Algorithm (7.4.1), Condition (i), (7.4.3) and (7.4.4), we have

$$\begin{aligned} d(x_{n+1},p) &= d(\alpha_n g(x_n) \oplus \psi_n w_n \oplus \tau_n u_n, p) \\ &\leq \alpha_n d(g(x_n),p) + \psi_n d(w_n,p) + \tau_n d(u_n,p) \\ &\leq \alpha_n \sigma d(x_n,p) + \alpha_n d(g(p),p) + \psi_n d(w_n,p) + \tau_n d(u_n,p) \\ &\leq \alpha_n \sigma d(x_n,p) + \alpha_n d(g(p),p) + \psi_n d(w_n,p) + \tau_n d(w_n,p) \\ &\leq \alpha_n \sigma d(x_n,p) + \alpha_n d(g(p),p) + (1-\alpha_n)\theta_n d(x_n,p) + (1-\alpha_n)(1-\theta_n)d(x_{n+1},p) \\ &= (\alpha_n\sigma + \theta_n(1-\alpha_n))d(x_n,p) + (1-\alpha_n)(1-\theta_n)d(x_{n+1},p) + \alpha_n d(g(p),p), \end{aligned}$$

which implies that

$$(1-((1-\alpha_n)(1-\theta_n)))d(x_{n+1},p) \leq (\alpha_n\sigma + \theta_n(1-\alpha_n))d(x_n,p) + \alpha_n d(g(p),p). \tag{7.4.5}$$

Since $\alpha_n \in (0,1)$ and $\sigma \in [0,1)$, we have from (7.4.5) that

$$\begin{aligned} d(x_{n+1},p) &\leq \frac{\alpha_n\sigma + \theta_n(1-\alpha_n)}{1-((1-\alpha_n)(1-\theta_n))}d(x_n,p) + \frac{\alpha_n}{1-((1-\alpha_n)(1-\theta_n))}d(g(p),p) \\ &\leq 1 - \frac{\alpha_n(1-\sigma)}{1-((1-\alpha_n)(1-\theta_n))}d(x_n,p) \\ &+ \frac{\alpha_n(1-\sigma)}{1-((1-\alpha_n)(1-\theta_n))}\left(\frac{1}{1-\sigma}d(g(p),p)\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} d(x_{n+1},p) &\leq \max\left\{d(x_n,p), \frac{1}{1-\sigma}d(g(p),p)\right\} \\ &\vdots \\ &\leq \max\left\{d(x_1,p), \frac{1}{1-\sigma}d(g(p),p)\right\}, \end{aligned}$$

which implies that $\{x_n\}$ is bounded. Consequently, $\{g(x_n)\}, \{w_n\}, \{y_n\}$ and $\{u_n\}$ are all bounded.

$\square$

Theorem 7.4.2. *Let X be an Hadamard space and $f_j : X \to (-\infty, +\infty]$, $j = 1, 2, \cdots, m$ be a finite family of proper, convex and lower semicontinuous functions. Let $g : X \to X$ be a contraction with coefficient $\sigma \in [0, 1)$. Let $J_{\lambda_n}^{f_m} : X \to X$ be the resolvents of a finite family of proper, convex and lower semicontinuous functions. Let $T_i : X \to X$, $i = 1, 2, \cdots, N$ be a finite family of L-Lipschitzian and quasi-pseudocontractive mappings with $L \geq 1$ such that T_i is demiclosed. Suppose that $\Gamma := \{\cap_{i=1}^N F(T_i) \cap (\cap_{j=1}^m argmin_{y \in X} f_j(y))\} \neq \emptyset$ and the sequence $\{x_n\}$ is generated by Algorithm (7.4.1), where $\lambda_n > \lambda > 0$, $\{\alpha_n\}, \{\psi_n\}, \{\tau_n\}, \{\theta_n\}$ and $\{\beta_{n,i}\}$ are sequences in $(0, 1)$ satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < a < \epsilon_{n,i} < \gamma_{n,i} < b < \frac{1}{1+\sqrt{1+L^2}}$, $i = 1, 2, \cdots, N$;

(iii) $\sum\limits_{i=0}^{N} \beta_{n,i} = 1, \sum\limits_{i=0}^{N} \beta_i = 1$ *and* $0 < c \leq \beta_{n,i} \leq d < 1$, *for all* $n \geq 1$,

(iv) $\alpha_n + \psi_n + \tau_n = 1$.

Then, $\{x_n\}$ converges strongly to a point $p = P_\Gamma g(p)$, where P_Γ is the metric projection of X onto Γ.

Proof. We can rewrite Algorithm 7.4.1 as follows;

$$\begin{cases} w_n = \theta_n x_n \oplus (1 - \theta_n) x_{n+1}, \\ y_n = \beta_0 w_n \oplus \beta_1 J_{\lambda_n}^{f_1} w_n \oplus \beta_2 J_{\lambda_n}^{f_2} w_n \oplus \cdots \oplus \beta_n J_{\lambda_n}^{f_m} w_n, \\ u_n = \beta_{n,0} y_n \oplus (1 - \beta_{n,0}) \sum\limits_{i=1}^{N} \oplus \frac{\beta_{n,i}}{(1-\beta_{n,0})} K_i y_n, \\ z_n = \left(\frac{\psi_n}{1-\alpha_n} w_n \oplus \frac{\tau_n}{1-\alpha_n} u_n \right), \\ x_{n+1} = \alpha_n g(x_n) \oplus (1 - \alpha_n) z_n. \end{cases} \tag{7.4.6}$$

Applying Lemma 2.5.30 to the definition of u_n in Algorithm 7.4.6 we obtain

$$\begin{aligned} d^2(u_n, p) &\leq \beta_{n,0} d^2(y_n, p) + \sum_{i=1}^{N} \beta_{n,i} d^2(K_i y_n, p) - \beta_{n,0} \sum_{i=1}^{N} \beta_{n,i} d^2(K_i y_n, y_n) \\ &\leq d^2(y_n, p) - \beta_{n,0} \sum_{i=1}^{N} \beta_{n,i} d^2(K_i y_n, y_n). \end{aligned} \tag{7.4.7}$$

We obtain from the definition of w_n in Algorithm 7.4.1 and Lemma 2.3.23 that

$$d^2(w_n, p) \leq \theta_n d^2(x_n, p) + (1 - \theta_n) d^2(x_{n+1}, p). \tag{7.4.8}$$

We obtain from the definition of x_{n+1} and z_n in Algorithm 7.4.6, (7.4.2), (7.4.7) and (7.4.8) that

$$\begin{aligned}
d^2(x_{n+1},p) &= d^2(\alpha_n g(x_n)\oplus(1-\alpha_n)z_n,p)\\
&\le \alpha_n^2 d^2(g(x_n),p)+(1-\alpha_n)^2 d^2(z_n,p)+2\alpha_n(1-\alpha_n)\langle\overrightarrow{g(x_n)p},\overrightarrow{z_np}\rangle\\
&\le \alpha_n^2 d^2(g(x_n),p)+\beta_n d^2(w_n,p)+\tau_n d^2(u_n,p)+2\alpha_n(1-\alpha_n)\langle\overrightarrow{g(x_n)p},\overrightarrow{z_np}\rangle\\
&\le \alpha_n^2 d^2(g(x_n),p)+\beta_n d^2(w_n,p)+\tau_n d^2(y_n,p)-\beta_{n,0}\sum_{i=1}^{N}\tau_n\beta_{n,i}d^2(y_n,K_iy_n)\\
&+2\alpha_n(1-\alpha_n)\langle\overrightarrow{g(x_n)p},\overrightarrow{z_np}\rangle \qquad (7.4.9)\\
&\le \alpha_n^2 d^2(g(x_n),p)+(1-\alpha_n)d^2(w_n,p)-\beta_{n,0}\sum_{i=1}^{N}\tau_n\beta_{n,i}d^2(y_n,K_iy_n)\\
&+2\alpha_n(1-\alpha_n)\langle\overrightarrow{g(x_n)p},\overrightarrow{z_np}\rangle \qquad (7.4.10)\\
&\le \alpha_n^2 d^2(g(x_n),p)+(1-\alpha_n)\Big(\theta_n d^2(x_n,p)+(1-\theta_n)d^2(x_{n+1},p)\Big)\\
&-\beta_{n,0}\sum_{i=1}^{N}\tau_n\beta_{n,i}d^2(y_n,K_iy_n)+2\alpha_n(1-\alpha_n)\langle\overrightarrow{g(x_n)p},\overrightarrow{z_np}\rangle \qquad (7.4.11)
\end{aligned}$$

which implies that

$$\begin{aligned}
\Big(1-((1-\alpha_n)(1-\theta_n))\Big)d^2(x_{n+1},p) &\le ((1-\alpha_n)\theta_n)d^2(x_n,p)\\
&+\alpha_n\left[\alpha_n d^2(g(x_n),p)+2(1-\alpha_n)\langle\overrightarrow{g(x_n)p},\overrightarrow{z_np}\rangle\right].
\end{aligned}$$

Hence, from the previous inequality we obtain

$$\begin{aligned}
d^2(x_{n+1},p) &\le \frac{(1-\alpha_n)\theta_n}{\theta_n+\alpha_n(1-\theta_n)}d^2(x_n,p)\\
&+\frac{\alpha_n}{\theta_n+\alpha_n(1-\theta_n)}\left[\alpha_n d^2(g(x_n),p)+2(1-\alpha_n)\langle\overrightarrow{g(x_n)p},\overrightarrow{z_np}\rangle\right]\\
&=\left[1-\frac{\alpha_n}{\theta_n+\alpha_n(1-\theta_n)}\right]d^2(x_n,p)+\frac{\alpha_n}{\theta_n+\alpha_n(1-\theta_n)}d_n, \qquad (7.4.12)
\end{aligned}$$

where

$$d_n=\alpha_n d^2(g(x_n),p)+2(1-\alpha_n)\langle\overrightarrow{g(x_n)p},\overrightarrow{z_np}\rangle$$

Using Lemma 2.5.36, we need to show that $\limsup_{k\to\infty} d_{n_k}\le 0$ for every subsequence $\{d(x_{n_k},p)\}$ of $\{d(x_n,p)\}$ satisfying the condition

$$\liminf_{k\to\infty}(d(x_{n_k+1},p)-d(x_{n_k},p))\ge 0. \qquad (7.4.13)$$

Now suppose that $\{d(x_{n_k}, p)\}$ is a subsequence of $\{d(x_n, p)\}$ such that (7.4.13) holds then $\liminf_{k\to\infty}(d^2(x_{n_k+1}, p) - d^2(x_{n_k}, p)) = \liminf_{k\to\infty}(d(x_{n_k+1}, p) - d(x_{n_k}, p))(d(x_{n_k+1}, p) + d(x_{n_k}, p))$.

From (7.4.9), we have

$$\begin{aligned}
\limsup_{k\to\infty}\left(\beta_{n_k,0}\sum_{i=1}^{N}\tau_{n_k}\beta_{n_k,i}d^2(y_{n_k}, K_i y_{n_k})\right) &\le \limsup_{k\to\infty}\left(\alpha_{n_k}^2 d^2(g(x_{n_k}), p)\right)\\
&+ \limsup_{k\to\infty}(1-\alpha_{n_k})\left[\theta_{n_k}d^2(x_{n_k}, p)\right]\\
&+ \limsup_{k\to\infty}(1-\alpha_{n_k})\left[(1-\theta_{n_k})d^2(x_{n_k+1}, p)\right]\\
&+ \limsup_{k\to\infty}\left(2\alpha_{n_k}(1-\alpha_{n_k})\langle\overrightarrow{g(x_{n_k}))p}, \overrightarrow{z_{n_k}p}\rangle\right)\\
&- \limsup_{k\to\infty}\left(d^2(x_{n_k+1}, p)\right)\\
&\le \limsup_{k\to\infty}\left(\alpha_{n_k}^2 d^2(g(x_{n_k}), p)\right)\\
&+ \limsup_{k\to\infty}\left(\theta_{n_k}[d^2(x_{n_k}, p) - d^2(x_{n_k+1}, p)]\right)\\
&+ \limsup_{k\to\infty}\left(2\alpha_{n_k}(1-\alpha_{n_k})\langle\overrightarrow{g(x_{n_k})p}, \overrightarrow{z_{n_k}p}\rangle\right)
\end{aligned}$$

which implies from the conditions on the control parameter that

$$\lim_{k\to\infty} d(y_{n_k}, K_i y_{n_k}) = 0, \quad i = 1, 2, \cdots, N. \tag{7.4.14}$$

We also have from Algorithm 7.4.6 and (7.4.14) that

$$\begin{aligned}
d(u_{n_k}, y_{n_k}) &= d\left(\beta_{n_k,0}y_{n_k} \oplus (1-\beta_{n_k,0})\sum_{i=1}^{N}\oplus\frac{\beta_{n_k,i}}{(1-\beta_{n_k,0})}K_i y_{n_k}, y_{n_k}\right)\\
&\le \sum_{i=1}^{N}\beta_{n_k,i}d(K_i y_{n_k}, y_{n_k}) \to 0 \quad \text{as} \quad k\to\infty.
\end{aligned}$$

Hence,

$$\lim_{k\to\infty} d(u_{n_k}, y_{n_k}) = 0. \tag{7.4.15}$$

Also,

$$\begin{aligned}
d(y_{n_k}, w_{n_k}) &= d(\beta_0 w_{n_k} \oplus \beta_1 J_{\lambda_{n_k}}^{f_1} w_{n_k} \oplus \beta_2 J_{\lambda_{n_k}}^{f_2} w_{n_k} \oplus \cdots \oplus \beta_{n_k} J_{\lambda_{n_k}}^{f_m} w_{n_k}, w_{n_k})\\
&\le d(w_{n_k}, w_{n_k}) \to 0, \quad \text{as} \quad k\to\infty.
\end{aligned}$$

Thus,

$$\lim_{k\to\infty} d(y_{n_k}, w_{n_k}) = 0. \tag{7.4.16}$$

From (7.4.15) and (7.4.16), we obtain that

$$\lim_{k\to\infty} d(u_{n_k}, w_{n_k}) = 0 \tag{7.4.17}$$

From Algorithm 7.4.6 and (7.4.17), we obtain

$$d(u_{n_k}, z_{n_k}) \leq \frac{\beta_{n_k}}{1-\alpha_{n_k}} d(u_{n_k}, w_{n_k}) + \frac{\tau_{n_k}}{1-\alpha_{n_k}} d(u_{n_k}, u_{nk}) \to 0, \quad \text{as} \quad k \to \infty. \tag{7.4.18}$$

Hence, from (7.4.15) and (7.4.18) we have

$$\lim_{k\to\infty} d(z_{n_k}, y_{n_k}) = 0 \tag{7.4.19}$$

From Algorithm 7.4.6 and the condition on θ_{n_k} we have

$$\lim_{k\to\infty} d(w_{n_k}, x_{n_k}) = 0. \tag{7.4.20}$$

From (7.4.16) and (7.4.20), we have

$$\lim_{k\to\infty} d(y_{n_k}, x_{n_k}) = 0. \tag{7.4.21}$$

Using (7.4.19) and (7.4.21), we have

$$\lim_{k\to\infty} d(x_{n_k}, z_{n_k}) = 0. \tag{7.4.22}$$

We obtain from Algorithm 7.4.6 and the condition on α_n that

$$d(x_{n_k+1}, z_{n_k}) = 0. \tag{7.4.23}$$

Hence, we obtain from (7.4.22) and (7.4.23) that

$$\lim_{k\to\infty} d(x_{n_k+1}, x_{n_k}) = 0.$$

Since $\{x_n\}$ is bounded it follows from Lemma 2.5.31 that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which $\triangle$- converges to a point v. From (7.4.22), we have that there exists a subsequence $\{z_{nk_j}\}$ which also $\triangle$- converges to v. Since T_i, $i = 1, 2, \cdots, N$ is demiclosed, we obtain from Lemma 7.2.1 and (7.4.14) that $v \in \cap_{i=1}^N F(T_i) = \cap_{i=1}^N F(K_i)$. Also, $J_{\lambda_{n_k}}$ is a nonexpansive mapping for each $j = 1, 2, \cdots, m$. We obtain from (7.4.21), Lemma 2.5.33 and Lemma 7.2.3 that $v \in \cap_{j=1}^N F(J_\lambda^{(j)}) = \cap_{j=1}^m \arg\min_{y\in X} f_j(y)$. Hence, $v \in \Gamma$.

Let $p = P_\Gamma g(p)$. Since $\{x_{n_k}\}$ is bounded, we choose a subsequence $\{x_{n_kj}\}$ which $\triangle$- converges to a point v. From (7.4.22), we have that $\{z_{n_k}\}$ is bounded and we chose subsequence $\{z_{n_{k_j}}\}$ which $\triangle$-converges to a point v such that

$$\begin{aligned} \limsup_{k\to\infty} \left\langle \overrightarrow{g(p)p}, \overrightarrow{z_{n_k}p} \right\rangle &= \lim_{j\to\infty} \left\langle \overrightarrow{g(p)p}, \overrightarrow{z_{n_kj}p} \right\rangle \\ &= \left\langle \overrightarrow{g(p)p}, \overrightarrow{vp} \right\rangle. \end{aligned}$$

From Lemma 2.5.32, we obtain that

$$\limsup_{k\to\infty}\left\langle\overrightarrow{g(p)p},\overrightarrow{z_{n_k}p}\right\rangle=\left\langle\overrightarrow{g(p)p},\overrightarrow{vp}\right\rangle\leq 0.$$

Hence, from (7.4.12) and the previous inequality, we obtain that $\limsup_{k\to\infty} d_{n_k}\leq 0$, where

$$d_{n_k}=\left[\alpha_{n_k}d^2(g(x_{n_k}),u)+2(1-\alpha_n)\left\langle\overrightarrow{g(p)p},\overrightarrow{z_{n_k}p}\right\rangle\right].$$

Applying Lemma 2.5.36 to (7.4.12), we have that $d(x_n,p)\to 0$, as $n\to\infty$. Therefore, $\{x_n\}$ converges strongly to $p=P_\Gamma g(p)$. □

If T is a nonexpansive mapping in Theorem 7.4.2, then we obtain the following corollary.

Corollary 7.4.1. *Let X be an Hadamard space and $f_j: X\to(-\infty,+\infty]$, $j=1,2,\cdots,m$ be a finite family of proper, convex and lower semicontinuous functions. Let $g: X\to X$ be a contraction with coefficient $\sigma\in[0,1)$. Let $J_{\lambda_n}^{f_m}: X\to X$ be the resolvents of a finite family of proper, convex and lower semicontinuous functions. Let $T_i: X\to X$, $i=1,2,\cdots,N$ be a finite family of L-Lipschitzian and nonexpansive mappings with $L\geq 1$ such that T_i is demiclosed. Suppose that $\Gamma:=\{\cap_{i=1}^N F(T_i)\cap\left(\cap_{j=1}^m argmin_{y\in X}f_j(y)\right)\}\neq\emptyset$ and for arbitrary $x_1,u\in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} w_n=\theta_n x_n\oplus(1-\theta_n)x_{n+1},\\ y_n=\beta_0 w_n\oplus\beta_1 J_{\lambda_n}^{f_1}w_n\oplus\beta_2 J_{\lambda_n}^{f_2}w_n\oplus\cdots\oplus\beta_n J_{\lambda_n}^{f_m}w_n,\\ u_n=\beta_{n,0}y_n\oplus(1-\beta_{n,0})\sum_{i=1}^N\oplus\frac{\beta_{n,i}}{(1-\beta_{n,0})}\left((1-\epsilon_{n,i})y_n\oplus\epsilon_{n,i}T_i((1-\gamma_{n,i})y_n\oplus\gamma_{n,i}T_iy_n)\right),\\ x_{n+1}=\alpha_n g(x_n)\oplus\psi_n w_n\oplus\tau_n u_n,\end{cases}\tag{7.4.24}$$

where $\lambda_n>\lambda>0$, $\{\alpha_n\},\{\psi_n\},\{\tau_n\},\{\theta_n\}$ and $\{\beta_{n,i}\}$ are sequences in $(0,1)$ satisfying conditions (i)-(iv) then, $\{x_n\}$ converges strongly to a point $p=P_\Gamma g(p)$, where P_Γ is the metric projection of X onto Γ.

By setting $\theta_n=\frac{1}{2}$ for $n\geq 1$ in Theorem 7.4.2, we obtain the following corollary.

Corollary 7.4.2. *Let X be an Hadamard space and $f_j: X\to(-\infty,+\infty]$, $j=1,2,\cdots,m$ be a finite family of proper, convex and lower semicontinuous functions. Let $g: X\to X$ be a contraction with coefficient $\sigma\in[0,1)$. Let $J_{\lambda_n}^{f_m}: X\to X$ be the resolvents of a finite family of proper, convex and lower semicontinuous functions. Let $T_i: X\to X$, $i=1,2,\cdots,N$ be a finite family of L-Lipschitzian and quasi-pseudocontractive mappings with $L\geq 1$ such that T_i is demiclosed. Suppose that $\Gamma:=\{\cap_{i=1}^N F(T_i)\cap\left(\cap_{j=1}^m argmin_{y\in X}f_j(y)\right)\}\neq\emptyset$ and for arbitrary $x_1,u\in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} w_n=\left(\frac{x_n\oplus x_{n+1}}{2}\right),\\ y_n=\beta_0 w_n\oplus\beta_1 J_{\lambda_n}^{f_1}w_n\oplus\beta_2 J_{\lambda_n}^{f_2}w_n\oplus\cdots\oplus\beta_n J_{\lambda_n}^{f_m}w_n,\\ u_n=\beta_{n,0}y_n\oplus(1-\beta_{n,0})\sum_{i=1}^N\oplus\frac{\beta_{n,i}}{(1-\beta_{n,0})}\left((1-\epsilon_{n,i})y_n\oplus\epsilon_{n,i}T_i((1-\gamma_{n,i})y_n\oplus\gamma_{n,i}T_iy_n)\right),\\ x_{n+1}=\alpha_n g(x_n)\oplus\psi_n w_n\oplus\tau_n u_n,\end{cases}\tag{7.4.25}$$

where $\lambda_n>\lambda>0$, $\{\alpha_n\},\{\psi_n\},\{\tau_n\},\{\theta_n\}$ and $\{\beta_{n,i}\}$ are sequences in $(0,1)$ satisfying conditions (i)-(iv), then, $\{x_n\}$ converges strongly to a point $p=P_\Gamma g(p)$, where P_Γ is the metric projection of X onto Γ.

By setting $g(x_n) = u$ for all $n \geq 1, u \in X$ fixed in Theorem 7.4.2, we obtain the following corollary with the Halpern-type algorithm

Corollary 7.4.3. *Let X be an Hadamard space and $f_j : X \to (-\infty, +\infty]$, $j = 1, 2, \cdots, m$ be a finite family of proper, convex and lower semicontinuous functions. Let $J_{\lambda_n}^{f_m} : X \to X$ be the resolvents of a finite family of proper, convex and lower semicontinuous functions. Let $T_i : X \to X$, $i = 1, 2, \cdots, N$ be a finite family of L-Lipschitzian and quasi-pseudocontractive mappings with $L \geq 1$ such that T_i is demiclosed. Suppose that $\Gamma := \{\cap_{i=1}^{N} F(T_i) \cap \left(\cap_{j=1}^{m} argmin_{y \in X} f_j(y)\right)\} \neq \emptyset$ and for arbitrary $x_1, u \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} w_n = \theta_n x_n \oplus (1 - \theta_n) x_{n+1}, \\ y_n = \beta_0 w_n \oplus \beta_1 J_{\lambda_n}^{f_1} w_n \oplus \beta_2 J_{\lambda_n}^{f_2} w_n \oplus \cdots \oplus \beta_n J_{\lambda_n}^{f_m} w_n, \\ u_n = \beta_{n,0} y_n \oplus (1 - \beta_{n,0}) \sum_{i=1}^{N} \oplus \frac{\beta_{n,i}}{(1-\beta_{n,0})} \left((1 - \epsilon_{n,i}) y_n \oplus \epsilon_{n,i} T_i((1 - \gamma_{n,i}) y_n \oplus \gamma_{n,i} T_i y_n)\right), \\ x_{n+1} = \alpha_n u \oplus \psi_n w_n \oplus \tau_n u_n, \end{cases} \tag{7.4.26}$$

where $\lambda_n > \lambda > 0$, $\{\alpha_n\}, \{\psi_n\}, \{\tau_n\}, \{\theta_n\}$ and $\{\beta_{n,i}\}$ are sequences in $(0, 1)$ satisfying conditions (i)-(iv) then, $\{x_n\}$ converges strongly to a point $p = P_\Gamma g(p)$, where P_Γ is the metric projection of X onto Γ.

CHAPTER 8

Conclusion, Contributions to Knowledge and Future research

8.1 Conclusion

In this thesis, we studied optimization and FPP in the frameworks of Hilbert, Banach and Hadamard spaces. We developed several effective algorithms for solving these problems. Our results complement several existing results in the literature. In Chapter 1 of this thesis, we gave a comprehensive background of our study which highlighted some of the importance of the optimization problems and FPP. We discussed some iterative schemes for solving these aforementioned problems. Furthermore, we discussed our research problems and motivation. Finally, we discussed the objectives of our study and presented the organization of our study. In Chapter 2 of this thesis, we discussed the geometry of the Hilbert, Banach and Hadamard spaces and presented some basic definitions, terms and concepts used in this study. We gave a detailed literature review of past works which motivated our study and presented some results that were important to our study. In Chapter 3, we studied the VIPs and FPP in the framework of a real Hilbert space. We studied the SVIPs and GVIP in the framework of real Hilbert spaces in Chapter 4. In Chapter 5 of this thesis, we presented our results on SE_qP, V_qIP and SEP in the framework of Hilbert spaces. Furthermore, we extended our results from the framework of Hilbert spaces to the framework work of Banach spaces. In Chapter 6 of this thesis, we presented our results on VIP, V_qIP and FPP in the framework of Banach space. Finally, we extended our results from the framework of Banach space to the framework of the Hadamard space. In Chapter 7 of this thesis, we presented our results on MPs in Hadamard spaces. In each of these chapters, we presented numerical examples of our algorithms in comparison with other existing algorithms in the literature to show the applicability of our methods. We also presented where necessary some applications of our results.

8.2 Contributions to Knowledge

Generally, the results obtained in this thesis extend and improve several other existing results in the literature in the frameworks of Hilbert, Banach and Hadamard spaces. In Chapter 3 of this thesis, we studied and improved several existing works on VIP and FPP in the framework of Hilbert spaces. In Section 3.3, we generalized the result of Nadeshkina and Takahashi [185] from approximating the common solution of the VIP and FPP of a nonexpansive mapping to approximating the common solution of VIP and FPP of an infinite family of strict pseudo-contractive mappings. We obtained a strong convergence result of our proposed method. Furthermore, we applied our result to find a common solution of VIP and ZPP for an infinite family of maximal monotone operators. In Section 3.4, we proposed two new relaxed inertial subgradient extragradient methods, and proved that they converge weakly to a solution of VIP when the operator A is quasimonotone and Lipschitz continuous, and when it is Lipschitz continuous without any form of monotonicity. The assumptions on the inertial and relaxation factors in this work, are weaker than those in many works in literature for solving VIPs. Our work generalizes the results of Liu *et al.* [162] and Ye *et al.* [269]. In Section 3.5, we generalized the result of Yin *et al.* [270] from approximating a common solution of VIP and FPP of a pseudocontractive mapping when the cost operator is quasimonotone, Lipschitz continuous and sequentially weakly continuous to approximating the common solution of VIP and FPP of a quasi-pseudocontractive mapping when the cost operator is quasimonotone, uniformly continuous (which is a weaker condition that the Lipschitz continuity) and without the sequentially weakly continuous condition. We obtained a strong convergence result of our proposed method to the minimum-norm solution of the aforementioned problem. Furthermore, we applied our result to image recovery.

In Chapter 4 of this thesis, we studied the SVIPs and GVIP in the framework of real Hilbert spaces. In Section 4.2, we introduced two new relaxed inertial Tseng's forward-backward-forward for solving the SVIP in real Hilbert spaces without any product space formulation when the underlying operator is monotone and Lipschitz continuous. We proved that the proposed methods converges strongly to the minimum-norm solution of the SVIP. In Section 4.3, our results extend and improve the works of Pham *et al.* [206] and Reich *et al.* [213] by introducing two new inertial projection and contraction methods for solving the SVIP when both underlying operators are pseudomonotone, Lipschitz continuous and without any product space reformulation of the original problem with minimized number of projections per iteration. Our results in Section 4.4 generalized the works of Tian and Jiang [244] and Ogwo *et al.* [192] by introducing two inertial projection and contraction methods for solving the SVIP when the underlying operators are pseudomonotone and Lipschitz continuous without the sequentially weakly continuity condition. In Section 4.5, we proposed a viscosity-type iterative method for solving the GVIP in real Hilbert spaces. We obtained a strong convergence result of our proposed method to a solution of the GVIP.

In Chapter 5, we studied the SE_qP, V_qIP and SEP in the framework of Hilbert spaces. In Section 5.2, we generalized the result of Latif and Eslamian [159] from approximating the common solution of the split equalities of EP, Lipschitz monotone VIP and FPP of nonexpansive semigroups satisfying the u.a.r condition to approximating the common solution of the split equalities of EP, non-Lipschitz pseudomonotone VIP and FPP of nonexpansive semigroups without the u.a.r condition. We obtained a strong convergence result of our proposed method to a common solution of the aforementioned problem. In Section 5.3, we improved the result of Cholamjiak *et al.* [70] from approximating the common solution of SEP and V_qIP in Hilbert spaces using the summability condition to approximating the common solution of SEP and V_qIP without the summability condition. Furthermore, we proved that the sequence generated by our proposed method converges strongly to a minimum solution (which finds application in several practical problems) of the aforementioned problem.

Next, we extended some of our results in the previous chapters to the framework of Banach spaces. In Section 6.2, we extended the result of Tan *et al.* [240] from approximating the solution of the VIP in a real Hilbert space to a 2-uniformly convex real Banach space. In Section 6.3, we extended the result of Long *et al.* [163] from approximating the SV_qIP in Hilbert spaces to approximating a common solution of finite family of SMV_qIP and FPP for nonexpansive mappings between a Banach and Hilbert space without the prior knowledge of the operator norm. Furthermore, we applied our results to study SFP and SMP.

Finally, we extended some of our earlier results to the framework of Hadamard spaces. We studied the MPs and FPP in the framework of Hadamard spaces. In Section 7.2, we generalized the result of Chang *et al.* [70] from approximating a common solution of a finite family of MPs and FPP for two demicontractive mappings in Hadamard spaces to approximating a common solution of a finite family of MPs and FPP of a finite family of quasi-pseudocontractive mapping in Hadamard spaces. Furthermore, we extended this method to a more general method involving the resolvents of multivalued monotone operators and established that it converges strongly to a common zero of a finite family of multivalued monotone operators which is also a common fixed point of a finite family of quasi-pseudocontractive mappings in Hadamard spaces. In Section 7.3, we generalized the result of Ahmad and Ahmad [4] for approximating a common solution of a MP and FPP of a nonexpansive mapping using the viscosity implicit midpoint rule in Hadamard spaces to approximating a common solution of MP and FPP of a quasi-pseudocontractive mapping using the proximal point approach of a viscosity type iterative method for a generalized implicit rule in Hadamard spaces.

8.3 Future research

Some of the possible research problems we will like to consider for future research are highlighted below.

In Chapter 3 of this thesis, the VIPs and FPP were studied in the framework of a real Hilbert space. In future research, we desire to extend our results from the framework of Hilbert spaces to more general Banach spaces and possibly, Hadamard spaces. Also, in Chapter 4, the SVIPs were studied when the cost operators are either pseudomonotone or monotone mappings in the framework of Hilbert spaces. It is known that the quasimonotone operators are more general and applicable than the pseudomonotone and monotone mappings. In future, we intend to extend our results on SVIPs to when the cost operators are quasimonotone mappings. Furthermore, we hope to extend our results on SVIPs to a more general Banach space.

Note that the split equalities of EP, VIP and FPP considered in this thesis were studied in the framework of Hilbert spaces when the cost operator is pseudomonotone and uniformly continuous. Also, the V_qIP and SEP considered in this thesis were studied in the framework of Hilbert spaces when the cost operators were monotone and Lipschitz continuous. In our future research, we desire to extend our results on SE_qP, V_qIP and SEP to the framework of more general Banach spaces when the cost operators are quasimonotone and uniformly continuous.

In Chapter 6, we studied the VIP in the framework of Banach space when the operator is pseudomonotone and non-Lipschitz. We also studied the SMV_qIPs and FPP of a non-expansive mapping between a Banach space and a Hilbert space. In future, we intend to extend these results to a more general mapping when the cost operator is quasimonotone and uniformly continuous in the framework of Hadamard space. Furthermore, we note that the MPs considered in this thesis were studied in the framework of the Hadamard spaces. In future, we desire to study these results in the framework of p-uniformly convex metric spaces.

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