A STUDY OF OPTIMIZATION PROBLEMS AND FIXED POINT ITERATIONS IN BANACH SPACES

by

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As the candidate's supervisor I have approved this thesis for submission.

Dr. O.T. Mewomo

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Dedication

This thesis is dedicated to the glory of Allah, the Most Beneficent and Most Merciful.

Abstract

The study of optimization and fixed point problems has remained as an attractive area of research due to its paramount importance in several areas of mathematics and other sciences. It constitutes a beautiful mixture of pure and applied analysis, topology, geometry, statistics and mechanics. It has also found several applications in solving nonlinear phenomena arising in diverse fields such as engineering, economics, biology, management science, transportation, game theory, physics, computer tomography, etc.

In this thesis, we present some inertial iterative schemes with strong convergence theorems for approximating solutions of certain optimization problems in real Hilbert spaces. We further analyze a parallel combination extragradient method for finite family of pseudo-monotone equilibrium problem and fixed point of demi-contractive mappings in real Hilbert spaces. By combining Mann and Krasnolselskii methods with inertial extrapolation term, we propose a new iterative method which converges strongly to a common solution of split variational inclusion problem and equilibrium problem with para-monotone equilibria.

More so, we introduce a projection-contraction method for approximating solution of split generalized equilibrium problem in real Hilbert space. We show that our projection-contraction method converges at a linear rate of convergence. Moreover, we extend the study of projection methods for solving variational inequality problem to reflexive Banach spaces. We introduce a projection algorithm and prove a strong convergence theorem for approximating solution of variational inequality problem in reflexive Banach spaces and give an application of our result to approximating solution of equilibrium problem in reflexive Banach space without prior knowledge of operator norms.

Furthermore, we introduce a totally relaxed subgradient extragradient method for approximating a common solution of variational inequality and fixed point of quasi-nonexpansive mapping in a 2-uniformly convex and uniformly smooth Banach space. We also study the approximation of solution of variational inequality problem using projection-contraction algorithm in real Hilbert space. Then, we extend the study of split equality monotone inclusion problem to p-uniformly convex and uniformly smooth real Banach spaces.

Ultimately, we consider the approximation of common fixed points of k-strictly pseudo-contractive mappings in a 2-uniformly smooth real Banach space. We introduce a class of N-generalized Bregman nonspreading mappings and propose an iterative method for approximating the common fixed points of this kind of mappings which is also a solution of equilibrium problem in a reflexive Banach space. Numerical experiments are presented to demonstrate the efficiency and performance of our algorithms in comparison with other existing algorithms in literature. We also achieve strong convergence results using our algorithms for approximating solutions of the underlying problems in each case.

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Declaration

I declare that this thesis, in its entirety or in part, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents my own work and I have made proper reference of the works of others wherever they are been used in the text.

Jolaoso Lateef Olakunle

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List of publications from the thesis

The following articles have been published from this thesis:

1. **L.O. Jolaoso, K.O. Oyewole, C.C. Okeke, O.T. Mewomo**; A unified algorithm for solving split generalized mixed equilibrium problem and fixed point of nonspreading mapping in Hilbert spaces, *Demonstratio Mathematica*, Volume: 51 (2018), 211–232. https://doi.org/10.1515/dema-2018-0015.

2. **L.O. Jolaoso, A. Taiwo, T.O. Alakoya, O.T. Mewomo**; A self adaptive inertial subgradient extragradient algorithm for variational inequality and common fixed point of multivalued mappings in Hilbert spaces, *Demostratio Mathematica*, Volume: 52, (2019), 183 - 203, https://doi.org/10.1515/dema-2019-0013.

3. **L.O. Jolaoso, F.U. Ogbuisi, O.T.Mewomo**; An intermixed algorithm for two -strict pseudocontraction mapping in q-uniformly smooth Banach space, *Nonlinear Studies*, Volume: 26 (1), (2019), 27-41.

4. **L.O. Jolaoso, T.O. Alakoya, A. Taiwo, O.T. Mewomo**; A parallel combination extragradient method with Armijo line searching for finding common solution of finite families of equilibrium and fixed point problems, *Rendiconti del Circolo Matematico di Palermo Series 2*, (First Online) https://doi.org/10.1007/s12215-019-00431-2, 2019.

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6. **L.O. Jolaoso, A. Taiwo, T.O. Alakoya, O.T. Mewomo**; A unified algorithm for solving variational inequality and fixed point problems with application to the split equality problem, *Computational and Applied Mathematics*. (Accepted to appear) 2019.

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1. **L.O. Jolaoso, F.U. Ogbuisi, O.T. Mewomo**; On split equality variation inclusion problems in Banach spaces without operator norms, Submitted to *Ukrainian Mathematical Journal*.

2. **L.O. Jolaoso, O.T. Mewomo**; Convergence theorem for finding common fixed points of N-genralized Bregman nonspreading mapping and solutions of equilibrium problems in Banach spaces, Submitted to *Kyungpook Mathematical Journal*.

3. **L.O. Jolaoso, A. Taiwo, T.O. Alakoya, O.T. Mewomo**; Strong convergence theorem for solving pseudo-monotone variational inequality problem using projection method in a reflexive Banach space, Submitted to *Journal of Optimization Theory and Applications*.

4. **L.O. Jolaoso, A. Taiwo, T.O. Alakoya, O.T. Mewomo, Q.L. Dong**; A Totally relaxed, self-adaptive subgradient extragradient method for variational inequality and fixed point problems in a Banach space, Submitted to *Mathematical Methods in the Applied Sciences*.

5. **L.O. Jolaoso, K.O. Oyewole, K.O. Aremu, O.T. Mewomo**; A new efficient algorithm for finding common fixed points and solutions of split generalized equilibrium problems in Hilbert spaces, Submitted to *International Journal of Computer Mathematics*.

6. **L.O. Jolaoso, A. A. Mebawondu, O.T. Mewomo**; Inertial Mann-Krasnoselskii algorithm with self adaptive stepsize for split variational inclusion problem and paramonotone equilibria, Submitted to *Bulletin of the Belgian Mathematical Society - Simon Stevin*.

CHAPTER 1

Introduction

1.1 General Introduction

Nonlinear analysis which includes optimization problems such as variational inequalities, Nash equilibrium problem, complementarity problem, convex minimization problem, vector optimization problem, minimax problem, saddle point problem and game theory, has recently been studied as an effective and powerful tool for studying many real life problems.

The classical Variational Inequality Problem (VIP) introduced by Fichera [105, 106] and Stampacchia [237] as an analytic tool for studying differential equations in finite dimensional spaces has played an important role as a modelling tool in diverse fields such as in economics, transportation, engineering, control theory, operation research, physics, mechanics and many others, see for example [3, 8, 28, 45, 91, 99, 102]. Let E be a Banach space with dual E^*, C be a nonempty, closed and convex subset of E and $A : C \to E^*$ be a nonlinear operator. The VIP is defined as finding a point $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall\ y \in E. \tag{1.1.1}$$

We denote the set of solutions of the VIP (1.1.1) by Ω_{VIP}.

The first general theorem for the existence and uniqueness of solution of VIP was proved by Lions and Stampacchia [163] in 1967. Since then, several authors have introduced various iterative methods for finding solutions of the VIP. One of the famous methods for solving the VIP is the *Extragradient Method (EM)* introduced by Korpelevich [157] in 1976 (in finite dimensional Euclidean space) and is given as follows:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \mu A x_n), \\ x_{n+1} = P_C(x_n - \mu A y_n), \quad \forall\ n \geq 1, \end{cases} \tag{1.1.2}$$

where $C \subseteq \mathbb{R}^N$, $A : C \to \mathbb{R}^N$ is a monotone, Lipschitz continuous operator with Lipschitz constant L, $\mu \in \left(0, \frac{1}{L}\right)$ and P_C is the metric projection onto C. If the solution set Ω_{VIP} is nonempty, then the sequence $\{x_n\}$ generated by *EM* converges weakly to an element in Ω_{VIP}. The EM has received a great attraction from many authors who have extended and generalized in both Hilbert and Banach spaces.

Note that the EM (1.1.2) requires two projections onto the set C and two evaluations of A per iteration. This makes the usage of EM (1.1.2) computationally expensive if the feasible set C is not so simple. A major improvement on the EM is to minimize the number of evaluations of P_C per iteration. An attempt in this direction was initiated by Y. Censor et al. [67, 69] who modified the EM by replacing the second projection with a projection onto a half-space. This new method which thus involves only one projection onto C is called the *Subgradient Extragradient Method (SEM)* and is given as follows:

Algorithm 1.1.1. *(The Subgradient Extragradient method (SEM)).*

$$\begin{cases} x_0 \in H, \\ y_n = P_C\left(x_n - \mu A x_n\right), \\ Q_n = \{z \in H : \langle x_n - \mu A x_n - y_n, z - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{Q_n}\left(x_n - \mu A y_n\right). \end{cases} \tag{1.1.3}$$

Censor et al. [69] showed that if the solution set Ω_{VIP} is nonempty, the sequence $\{x_n\}$ generated by SEM converges weakly to an element $p \in \Omega_{VIP}$, where $p = \lim_{n\to\infty} P_{\Omega_{VIP}}(x_n)$.

Also, using only a single projection onto C, Maingé and Gobinddass [173] (see also Maingé [170]) obtained a weak convergence result for solving the VIP in a real Hilbert space by means of a projected reflected gradient-type method [174] and inertial terms. Several other alternatives to the EM have further been introduced in the literature (see, for example, [67, 104, 113, 135, 136, 138, 145, 180, 235]).

Another important optimization problem which has found many applications in solving real life problems is the *Equilibrium Problem* (EP) introduced by Blum and Oettli [35] as a generalization of VIP. Let $F : C \times C \to \mathbb{R}$ be a bifunction, the EP is defined as finding a point $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall \; y \in C. \tag{1.1.4}$$

We shall denote the set of solutions of EP by Ω_{EP}. Blum and Oettli [35] discussed some existence theorems and variational principle for the EP and since then, various generalizations of EP have been introduced and studied by many authors. The theory of EP has also served as an important tool in studying a wide class of important nonlinear problems arising in several branches of pure and applied sciences in a unified and general framework (see, for instance [16, 32, 92, 110, 137, 149, 154, 184]).

The Generalized Mixed Equilibrium Problem (GMEP) is defined as finding a point $x \in C$ such that

$$F(x, y) + \langle hx, y - x \rangle + \phi(y) - \phi(x) \geq 0, \quad \forall \, y \in C, \tag{1.1.5}$$

where $h : C \to E$ is a nonlinear mapping and $\phi : C \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function.

If $h = 0$, GMEP (1.1.5) reduces to the Mixed Equilibrium Problem (MEP) which is to find a point $x \in C$ such that

$$F(x, y) + \phi(y) - \phi(x) \geq 0, \quad \forall\, y \in C. \tag{1.1.6}$$

If $\phi = 0$ in (1.1.5), the GMEP reduces to a Generalized Equilibrium Problem (GEP) which is to find a point $x \in C$ such that

$$F(x, y) + \langle hx, y - x \rangle \geq 0, \quad \forall\, y \in C. \tag{1.1.7}$$

In particular, if $h = 0$ and $\phi = 0$ in (1.1.5), the GMEP reduces to the classical equilibrium problem (1.1.4). The GMEP is very general in the sense that it includes as special cases, optimization problem, variational inequality problem, fixed point problem, Nash equilibrium problem in noncooperative games, etc, see [62, 90, 109].

The study of fixed point theory has also become a very powerful tool in nonlinear functional analysis. Recently, fixed point methods have found many applications in many fields of science such as biology, chemistry, economics, optimization theory, game theory, engineering, astrophysics and physics. Fixed point theorems are mainly used in the study of existence of solutions for nonlinear problems arising in physical science and biological science. They also play fundamental roles in establishing the existence theory for solutions of differential equations, integral equations, functional equations, partial differential equations, eigen-value problems and two-point boundary value problems, see for instance [4, 5, 100, 277].

Let E be a Banach space and $T : E \to E$ be a mapping. A point $x \in E$ is called a fixed point of T if

$$Tx = x. \tag{1.1.8}$$

The set of fixed points of T is denoted by $F(T)$. When T is a multi-valued mapping, e.g. $T : E \to 2^E$, then a point $x \in E$ is called a fixed point of T if $x \in Tx$.

The study of fixed point theory was initiated by Poincare [206] in 1886, followed by Brouwer [49] who proved a fixed point theorem for a square, a sphere and their n-dimensional counterparts. Brouwer [49] result was further extended by Kakutani [142]. *The Banach contraction mapping principle* by Stephan Banach [18] is also considered as one of the fundamental principle in this field. It shows that a contraction mapping on a complete metric space possesses a unique fixed point. The Banach contraction mapping principle is remarkable in its simplicity, yet it is perhaps the most widely used fixed point theorem. This is because the contraction condition on its mapping is easy to test and it requires only the structure of a complete metric space for its setting.

The study of fixed points for multivalued contractions and nonexpansive mappings was initiated by Nadler [190] and Markin [176] respectively, since then, there has been increasing effort on the study of fixed points of multivalued mappings. Also, there are many application of fixed point of multivalued mappings in convex optimization, differential inclusions, fractals, discontinuous differential equations, optimal control, computing homology of maps, computer-assisted proofs in dynamics, digital imaging and economics (e.g., [116, 141] and references cited therein).

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \to H_2$ be a linear operator. The *Split Feasibility Problem (SFP)* is defined as finding a point x^* such that

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \tag{1.1.9}$$

We shall denotes the set of solutions of the SFP by Ω_{SFP}. The SFP was originally introduced by Censor and Elfving [64] (in finite dimensional Euclidean space) for modelling phase retrieval, and later studied extensively as an extremely powerful tool for the treatment of a widely range of inverse problems, such as medical image reconstruction and intensity-modulated radiation therapy treatments; see e.g. [56, 63, 65, 71] for more details. When taking $C := \mathbb{R}^N$ and $Q = \{b\}$ $(b \in \mathbb{R}^M)$, the SFP (1.1.9) immediately reduces to the well known *Linear Inverse Problem (LIP)* which is to find $x^* \in \mathbb{R}^N$ such that

$$Ax^* = b. \tag{1.1.10}$$

The LIP has a long history and its theory and algorithms have extensively been developed in the literature, see the monographs [103, 259]. One of the most popular ways for solving the LIP is to reformulate it as a least squares problem, which greatly facilitates the employment of optimization algorithms for finding solutions of the resultant model. Similarly, the SFP can be solved by equivalently reformulating it as the following convex optimization problem:

$$\min \left\{ \frac{1}{2} ||Ax - P_Q(Ax)||^2 : x \in C \right\}, \tag{1.1.11}$$

where $P_Q(\cdot)$ is the projection onto the set Q defined by

$$P_Q(v) := argmin\{||z - y|| : z \in Q\}, \quad \forall\ v \in H_2.$$

In 2012, Ceng et al. [61] showed the following interesting relationship between VIP, fixed point problem and the SFP.

Proposition 1.1.2. *Suppose the SFP (1.1.9) is consistent, i.e., Ω_{SFP} is nonempty. Given $x^* \in H_1$, then the following statements are equivalent:*

(i) x^* solves the SFP (1.1.9);

(ii) x^* solves the fixed point equation

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*;$$

(iii) x^* solves the VIP of finding $x^* \in C$ such that

$$\langle \nabla f(x^*), y - x^* \rangle \geq 0, \quad \forall\ y \in C,$$

where $\nabla f = A^*(I - P_Q)A$ and A^* is the adjoint of A.

Because of this fact, many fixed point algorithms have been proposed for solving the SFP (1.1.9) in real Hilbert and Banach spaces.

1.2 Research Motivation

One of the popular methods used for accelerating the speed of convergence of iterative schemes is the multi-step method which can be viewed as the following discretization of the second-order dynamical system with friction:

$$\ddot{x}(t) + \gamma \dot{x}(t) + \nabla \varphi(x(t)) = 0,$$

where $\gamma > 0$ represents a friction parameter and $\varphi : H \to \mathbb{R}$ is a differentiable function. This can be formulated as a two-step heavy ball method, in which, given x_n and x_{n-1}, the next point x_{n+1} is determined via

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} + \gamma \frac{x_n - x_{n-1}}{h} + \nabla \varphi(x_n) = 0,$$

for $h > 0$, which results in an iterative algorithm of the form

$$x_{n+1} = x_n + \beta(x_n - x_{n-1}) - \alpha \nabla \varphi(x_n), \tag{1.2.1}$$

for each $n \geq 0$, where $\beta = 1 - \gamma h$ and $\alpha = h^2$. In 1964, Polyak [208] first used (1.2.1) to solve the optimization problem:

$$\min \varphi(x),$$

for all $x \in H$ and called it an *inertial type extrapolation algorithm*. In 1987, Polyak [207] also considered the relationship between the heavy ball method and the following conjugate gradient method

$$x_{n+1} = x_n + \beta_n(x_n - x_{n-1}) - \alpha_n \nabla \varphi(x_n), \tag{1.2.2}$$

for each $n \geq 0$, where α_n and β_n can be choosen through different ways. It is obvious that the only difference between the heavy ball method (1.2.1) and (1.2.2) is the choice of the parameters.

From Polyak's work, as an acceleration process, the inertial extrapolation algorithms were widely studied. Most especially, recent researchers have constructed many iterative algorithms by using inertial extrapolation, such as inertial extragradient method [97], inertial proximal method [9, 187], inertial forward-backward method [196], inertial proximal ADMM [73], fast iterative shrinkage thresholding algorithm FISTA [24, 72], inertial forward-backward-forward algorithm [40], inertial proximal-extragradient method [39], and inertial Mann method [254]. The inertial algorithm is a two-steps iterative method and its main feature is that the next iterate is defined by using the previous two iterates.

By using the technique of the inertial extrapolation, in 2008, Maingé introduced the classical inertial Mann algorithm as follows:

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \lambda_n)y_n + \lambda_n T y_n, \end{cases} \tag{1.2.3}$$

for each $n \geq 1$. He showed that the sequence $\{x_n\}$ converges weakly to a fixed point of T under the following conditions:

(A1) $\beta_n \in [0, \alpha)$ for each $n \geq 1$, where $\alpha \in [0, 1)$;

(A2) $\sum_{n=1}^{\infty} \beta_n ||x_n - x_{n-1}||^2 < +\infty$;

(A3) $0 < \inf \lambda_n \leq \sup \lambda_n < 1$.

Note that for the condition (A2) to be satisfied, one needs to first calculate β_n at each step of the iterations (see [186]). In 2015, Bot and Csetnek [41] removed the condition (A2) and substituted (A1) and (A3) with the following conditions:

(B1) for each $n \geq 1$, $\{\beta_n\} \subset [0, \alpha)$ is non-decreasing with $\beta_1 = 0$ and $0 \leq \alpha < 1$,

(B2) for each $n \geq 1$,

$$\delta > \frac{\alpha^2(1+\alpha) + \alpha\sigma}{1-\alpha^2}, \quad 0 \leq \lambda \leq \lambda_n \leq \frac{\delta - \alpha[\alpha(1+\alpha) + \alpha\delta + \sigma]}{\delta[1 + \alpha(1+\alpha) + \alpha\delta + \sigma]},$$

where $\lambda, \sigma, \delta > 0$.

Despite much effort been devoted on inertial algorithms, only weak convergence algorithms have mostly been achieved by many authors in the literature. It is important to note that strong convergence of iterative sequences for approximating solutions of optimization problems are more desirable than their weak convergence counterpart as pointed out by Bauschke and Combettes in [22]. Therefore, it is of great interest to develop inertial-type algorithms with strong convergence sequences. In this thesis, we propose some inertial algorithms with strong convergence properties for approximating solutions of certain optimization problems in real Hilbert spaces.

Furthermore, it was shown in [67] that the EM (1.1.2) and SEM (1.1.3) converge weakly to a solution of VIP (1.1.1) if the underlying operator A is monotone and Lipschitz continuous. When A is not monotone (say pseudo-monotone), both EM and SEM have failed to converge to a solution of VIP (1.1.1). Hence, there is need to find appropriate methods for solving VIP (1.1.1) when A is not monotone nor Lipschitz continuous. An attempt in this direction was made by Iusem and Svaiter [138] who introduced a new projection algorithm for approximating solution of VIP (1.1.1) where A is pseudo-monotone in a finite dimensional space. Their algorithm is unique in the sense that it uses an *Armijo line-searching technique* to determine the stepsize for the next iterate. The projection method involves taking an arbitrary stepsize β_n, compute $u_n = P_C(x_n - \beta_n A x_n)$ and then try vectors of the form $y(\alpha) = \alpha u_n + (1-\alpha)x_n$ with $\alpha \in (0, 1]$ until a value of α is reached such that

$$\langle Ay(\alpha), x_n - u_n \rangle \geq \frac{\delta}{\beta_n} ||x_n - u_n||^2, \tag{1.2.4}$$

for some fixed $\delta \in (0, 1)$. Then, take $y_n = y(\alpha)$ and compute the orthogonal projection w_n of x_n onto the hyperplane $Q_n = \{x \in \mathbb{R}^n : \langle Ay_n, x - y_n \rangle = 0\}$ and finally, take x_{n+1} as the orthogonal projection of w_n onto C. Note that along the search for appropriate α, the right hand-side of (1.2.4) is kept constant, and that, though A is evaluated at several points in the segment between u_n and x_n, no orthogonal projection onto C is required during the process. Iusem and Svaiter [138] further proved a weak convergence result using the projection algorithm for approximating solution of VIP (1.1.1) in a finite dimensional space.

The projection method was later extended to an infinite dimensional Hilbert space by Bello Cruz and Iusem [26]. Recently, Kanzow and Shehu [144] proved a strong convergence theorem for solving VIP (1.1.1) by combining the projection method with a Halpern method in a real Hilbert space H. They proposed the following scheme in particular.

Algorithm 1.2.1. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ and $A : C \to H$ be a monotone and uniformly continuous operator. Define $r(x) := x - P_C(x - Ax)$ for all $x \in C$.*
Step 0: *Given γ, $\sigma \in (0,1)$, $s > 0$, $x_1 \in C$ and set $n = 1$.*
Step 1: *Set*

$$w_n = (1-\alpha_n)x_n + \alpha_n x_1.$$

Step 2: *If $r(w_n) = 0$, stop. Else, let $y_n(\eta) = (1-\eta)w_n + \eta P_C(w_n - Aw_n)$, for $\eta \in \mathbb{R}$. Compute η_n as the maximum of the numbers $s, s\gamma, s\gamma^2, \dots$ such that*

$$\langle Ay_n(\eta_n), r(w_n)\rangle \geq \frac{\sigma}{2}||r(w_n)||^2,$$

and define $y_n = y_n(\eta_n)$.
Step 3: *Compute*

$$\begin{cases} \lambda_n = \dfrac{\langle Ay_n, w_n - y_n\rangle}{||Ay_n||^2}, \\ x_{n+1} = (1-\beta_n)w_n + \beta_n P_C(w_n - \lambda_n Ay_n). \end{cases} \tag{1.2.5}$$

Step 4: *Set $n \leftarrow n+1$ and go to Step 1.*

Motivated by the works of [26], [138] and [144], in this thesis, we extend the projection method for solving pseudo-monotone VIP to a real reflexive Banach space. We also propose other methods for approximating common solutions of VIP (1.1.1) and fixed point of nonlinear mappings in real Hilbert spaces and 2-uniformly convex and uniformly smooth Banach spaces.

One of the simplest methods for solving the SFP (1.1.9) is the *CQ-algorithm* introduced by Bryne [56] in 2002. The CQ-algorithm is given as follows: for $x_0 \in C$, compute

$$x_{n+1} = P_C(x_n - \tau_n A^*(I - P_Q)Ax_n), \quad n \geq 0, \tag{1.2.6}$$

where the stepsize τ_n is chosen in the interval $\left(0, \frac{2}{||A||^2}\right)$, A^* is the transpose of A and P_C and P_Q are the orthogonal projections onto C and Q respectively. Note that the determination of the stepsize τ_n depends on the operator norm $||A||$. This implies that in order to implement the CQ-algorithm, one has to first compute (or at least, estimate) the matrix norm of A, which is in general not an easy task in practice. In order to overcome this difficulty, there is a growing research on how to determine the best appropriate method for selecting the stepsize of the CQ-algorithm. Yang et al. [272] proposed the following adaptive stepsize selection:

$$\tau_n = \frac{\rho_n}{||A^*(I - P_Q)Ax_n||}, \tag{1.2.7}$$

where $\{\rho_n\}$ is a sequence of positive real numbers, and Lopez et al. [166] recently introduced the following adaptive stepsize method:

$$\tau_n = \frac{||(I - P_Q)Ax_n||^2}{||A^*(I - P_Q)Ax_n||^2}. \tag{1.2.8}$$

They showed that (1.2.7) and (1.2.8) are better selections compared to (1.2.6) of Bryne [56]. However, they all established weak convergence results for solutions of the SFP (1.1.9). Other notable modifications of CQ-algorithms can be found in [12, 13, 61] and the references therein.

Furthermore, Schöpfer [226, 227] recently extended the study of SFP (1.1.9) to Banach spaces such as the p-uniformly convex Banach spaces, which are also uniformly smooth. This has opened a growing research in this direction on the SFP in Banach spaces; see, for instance [168, 231, 232, 241, 246]. In this thesis, we introduce some iterative methods for approximating solutions of split equality monotone inclusion problem in p-uniformly convex real Banach spaces, which are also uniformly smooth. Our algorithms are designed in such a way that they do not require prior estimate of the norms of the bounded operators.

1.3 Objectives of the Study

The main objectives of this study are:

(i) To introduce new inertial-type iterative algorithms for solving certain optimization problems in real Hilbert spaces.

(ii) To extend the study of projection algorithm for solving the VIPs from a real Hilbert space to a real reflexive Banach space.

(iii) To introduce some other projection methods which are simpler to execute and faster than many existing algorithms for solving VIPs in the literature.

(iv) To introduce some projection methods for solving EPs in real Hilbert spaces.

(v) To propose some iterative methods which do not depend on the norms of the bounded linear operator for solving SFP and it generalizations in real Banach spaces.

(vi) To introduce a new class of N-generalized Bregman nonspreading mapping, and propose an iterative method for approximating its fixed point in real reflexive Banach spaces.

(vii) To compare the efficiency and performance of our algorithms with some existing ones in the literature.

1.4 Organization of the Thesis

The thesis is organized as follows:

Chapter 1 (General introduction): In this chapter, we present a brief introduction, research problem, motivation of our study and objectives of the research.

Chapter 2 (Preliminaries): In Chapter two, we give a background overview of some definitions and introduce some basic concepts which are needed to achieve our results.

Chapter 3 (Inertial algorithms and optimization problems): In Chapter 3, we first introduce an inertial gradient projection algorithm for approximating a common solution of a classical minimization problem and finding fixed point of δ-demimetric mapping in a real Hilbert space. We also introduce an inertial-viscosity subgradient extragradient method for approximating solution of VIP and fixed point of multi-valued demi-contractive mappings in real Hilbert spaces. Furthermore, we propose a modified Mann-inertial algorithm for finding a common solution of split generalized mixed equilibrium problem and fixed point of nonspreading mapping in real Hilbert spaces.

Chapter 4 (Equilibrium problems in Hilbert spaces): In this Chapter, we introduce a parallel combination extragradient method for solving a finite family of pseudo-monotone EPs and finding a common fixed point of a finite family of demi-contractive mappings in Hilbert space. We also present a new inertial Mann-Krasnolselskii algorithm for approximating a common solution of split variational inclusion problem and EP with paramonotone bifunction in real Hilbert space. Also, we introduce a projection-contraction algorithm for GEP and finding common fixed point of multi-valued demi-contractive mapping in real Hilbert spaces.

Chapter 5 (Variational inequality problems in Hilbert and Banach spaces): In this Chapter, we extend the projection algorithm for solving VIP from a real Hilbert space to a real reflexive Banach space using the Bregman distance technique. We also present a totally relaxed self-adaptive subgradient extragradient method with Halpern iterative method for finding a common solution of VIP and fixed point of quasi-nonexpansive mapping in a 2-uniformly convex and uniformly smooth Banach space. Then, we propose an extragradient method consisting of the Hybrid steepest descent method, a single projection method and an Armijo line searching technique for approximating a solution of VIP and finding the fixed point of demi-contractive mapping in a real Hilbert space.

Chapter 6 (Split feasibility problem in Banach spaces): In Chapter 6, we introduce an iterative algorithm and prove a strong convergence theorem without any prior estimation of operator norms for solving split equality variational inclusion problem in uniformly convex Banach spaces which are also uniformly smooth. We also present some application of our result and provide a numerical example to show the behaviour of the sequence generated by our algorithm.

Chapter 7 (Common fixed point problems in Banach spaces): In this Chapter, we introduce an intermixed algorithm and prove a strong convergence theorem for approximating individual fixed point of two strictly pseudocontractive mappings T and U in a q-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping j_p. Finally, we study some fixed point properties for N-generalized Bregman nonspreading mapping in reflexive Banach space. We introduce a hybrid iterative scheme for finding a common solution of countable family of EP and fixed point of N-generalized Bregman nonspreading mapping in a reflexive Banach space.

We stimulate our algorithms using `MATLAB` programming to establish the accuracy and efficiency of our algorithms in each chapter.

Chapter 8 (Conclusion and contribution to knowledge): In this chapter, we give a detailed summary of our results and also significant contribution of our research to the existing literature. We also suggest some possible problems we intend to consider in our future research work.

CHAPTER 2

Preliminaries

In this chapter, we present some definitions and basic concepts which are relevant to this study. Throughout this thesis, unless stated otherwise, E denotes a real Banach space with dual E^* where $\langle \cdot, \cdot \rangle$ is the duality pairing between E and E^*. Also, H denotes a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x.

2.1 Basic Definitions

We recall some basic definitions in functional analysis that are required for our work.

Definition 2.1.1. Let C be a nonempty, closed and convex subset of E and $f : C \to \mathbb{R} \cup \{+\infty\}$ be a mapping.

(i) The effective domain of f denoted by $dom\ f$ is defined by

$$dom\ f := \{v \in E : f(v) < +\infty\}.$$

(ii) The epigraph of f denoted by $epi\ f$ is defined by

$$epi\ f := \{(v, \beta) \in E \times \mathbb{R} : f(v) \leq \beta\}.$$

(iii) f is said to be convex if for all $u, v \in E$ and $\lambda \in [0, 1]$, we have

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v).$$

(iv) f is lower semicontinuous at $v_0 \in dom\ f$ if and only if

$$f(v_0) \leq \liminf_{v \to v_0} f(v).$$

(v) f is upper continuous at $v_0 \in dom\ f$ if and only if

$$f(v_0) \geq \limsup_{v \to v_0} f(v).$$

Remark 2.1.1. The function f is lower semicontinuous if and only if its epigraph $epi\ f$ is closed. Also, f is said to be lower (resp. upper) semicontinuous on its domain if it is lower (resp. upper) semicontinuous on every $v \in dom\ f$ (see [23]).

Definition 2.1.2. Let $x \in \text{int}(\text{dom} f)$, for any $y \in E$, the directional derivative of f at x is defined by

$$f^o(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.1.1}$$

If the limit in (2.1.1) exists as $t \to 0^+$ for each y, then the function f is said to be Gâteaux differentiable at x. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle \nabla f(x), y \rangle := f^o(x, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in \text{int}(\text{dom} f)$. When the limit as $t \to 0^+$ in (2.1.1) is attained uniformly for any $y \in E$ with $||y|| = 1$, we say that f is Fréchet differentiable at x. It is well known that f is Gâteaux (resp. Fréchet) differentiable at $x \in \text{int}(\text{dom} f)$ if and only if the gradient ∇f is norm-to-weak* (resp. norm-to-norm) continuous at x (see [21]).

Definition 2.1.3. Let f be a convex function. Then f is said to be differentiable at point $x \in E$ if the following set

$$\partial f(x) := \{e \in E : f(y) \geq f(x) + \langle e, y - x \rangle, \quad \forall\ y \in E\} \tag{2.1.2}$$

is nonempty. Each element $\partial f(x)$ is called a subgradient of f at x, $\partial f(x)$ is the subdifferential of f at x and the inequality in (2.1.2) is said to be the subdifferential inequality of f at x. We say that the function f is subdifferentiable on E, if f is subdifferentiable at each $x \in E$.

Definition 2.1.4. Let C be a nonempty, closed and convex subset of H. The normal cone to C at $x \in H$ is defined by

$$N_C(x) = \begin{cases} \{u \in H : \sup\langle C - x, u \rangle \leq 0\} & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases}$$

The indicator function of C is the function $i_C : H \to [-\infty, +\infty]$ such that

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

i_C is lower semicontinuous if and only if C is closed.

Proposition 2.1.2. *(Cioranescu [87]) Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function. Then*

(i) f is subdifferentiable on $int(domf)$;

(ii) f is Gâteaux differentiable at $x \in int(domf)$ if and only if its subgradient $\partial f(x) = \nabla f(x)$ is a singleton set.

The Frénchet conjugate of f is the function $f^* : E^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(y^*) = \sup\{\langle y^*, x\rangle - f(x) : x \in E\}.$$

Definition 2.1.5. The function f is called Legendre if it satisfies the following two conditions:

(L1) f is Gâteaux differentiable, $\text{int}(\text{dom } f) \neq \emptyset$ and $\text{dom } \nabla f = \text{int}(\text{dom } f)$,

(L2) f^* is Gâteaux differentiable, $\text{int}(\text{dom } f^*) \neq \emptyset$ and $\text{dom } \nabla f^* = \text{int}(\text{dom } f^*)$.

The notion of Legendre function in infinite dimensional spaces was first introduced by Bauschke, Borwein and Combettes in [21]. Their definition is equivalent to conditions (L1) and (L2) because the space E is assumed to be reflexive (see [21], Theorem 5.4 and 5.6, p. 634). It is also well known that in reflexive Banach space, $\nabla f = (\nabla f^*)^{-1}$ (see [36], p. 83). When this fact is combined with conditions (L1) and (L2), we obtain

$$ran\nabla f = dom\nabla f^* = int(domf)^*,$$
$$ran\nabla f^* = dom\nabla f = int(domf).$$

It also follows that f is Legendre if and only if f^* is Legendre (see [21], Corollary 5.5, p. 634) and that the functions f and f^* are Gâteaux differentiable and strictly convex in the interior of their respective domains. When the Banach space E is smooth and strictly convex, in particular, a Hilbert space, the function $\frac{1}{p}||.||^p$ with $p \in (1, \infty)$ is Legendre (cf. [19], Lemma 6.2, p. 639). For further details on Legendre functions, see, [19, 21].

Definition 2.1.6. Let X be a normed linear space. A mapping $T : X \to X$ is said to be

1. continuous at an arbitrary point $x_0 \in X$, if for each $\epsilon > 0$, there exist a real number $\delta > 0$ such that for $x \in X$

$$||x - x_0|| < \delta \Longrightarrow ||T(x) - T(x_0)|| \leq \epsilon, \qquad (2.1.3)$$

2. L-Lipschitz if there exists a real constant $L > 0$ such that

$$||T(x) - T(y)|| \leq L||x - y||, \quad \forall x, y \in X, \qquad (2.1.4)$$

3. contraction if it is $L-$Lipschitz with $L \in [0, 1)$,

4. strictly contractive if it is $L-$Lipschitz with $L \in (0,1)$.

Definition 2.1.7. Let $T : H \to H$ be a nonlinear mapping. Then T is called

(a) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall\, x, y \in H, \tag{2.1.5}$$

(b) α-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha||x - y||^2, \quad \forall\, x, y \in H, \tag{2.1.6}$$

(c) β-inverse strongly monotone (shortly, β-ism), if there exists a constant $\beta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \beta||Tx - Ty||^2, \quad \forall\, x, y \in H, \tag{2.1.7}$$

(d) firmly nonexpansive, if it is β-ism with $\beta = 1$.

Remark 2.1.3. It is easy to observe that every β-ism operator is monotone and $\frac{1}{\beta}-$Lipschitz.

Definition 2.1.8. A multi-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H$ such that $u \in Mx$ and $v \in My$, then

$$\langle x - y, u - v\rangle \geq 0. \tag{2.1.8}$$

Definition 2.1.9. A multi-valued monotone mapping $M : H \to 2^H$ is said to be maximal if the graph of M (denoted by $Gr(M)$) is not properly contained in the graph of any other monotone mapping. It is known that a multi-valued mapping M is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v\rangle \geq 0$ for every $(y, v) \in Gr(M)$ implies that $u \in Mx$.

Lemma 2.1.4. *(see, Rockafellar [220]): Let $A : C \to H$ be a monotone mapping and let $B : H \to 2^H$ be a mapping defined by*

$$Bq = \begin{cases} Aq + N_C(q), & q \in C, \\ \emptyset, & q \notin C. \end{cases} \tag{2.1.9}$$

Then B is maximal monotone and $x \in B^{-1}(0)$ if and only if $x \in \Omega_{VIP}$.

Definition 2.1.10. Let H be a real Hilbert space. The mapping $T : H \to H$ is said to be

(a) nonexpansive if

$$||Tx - Ty|| \leq ||x - y|| \quad \forall x, y \in H,$$

(b) quasi-nonexpansive if, $F(T) \neq \emptyset$ and

$$||Tx - Tp|| \leq ||x - p||, \quad \forall x \in H, \;\; p \in F(T),$$

(c) firmly nonexpansive, if

$$||Tx - Ty||^2 \leq ||x - y||^2 - ||(x - y) - (Tx - Ty)||^2, \ \forall x, y \in H, \qquad (2.1.10)$$

(d) nonspreading, if for all $x, y \in C$, we have

$$2||Tx - Ty||^2 \leq ||Tx - y||^2 + ||x - Ty||^2,$$

equivalently, T is nonspreading if for all $x, y \in C$,

$$||Tx - Ty||^2 \leq ||x - y||^2 + 2\langle x - Tx, y - Ty\rangle,$$

(e) k-strictly pseudo-contractive mapping if for $k \in [0, 1)$, we have

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(x - y) - (Tx - Ty)||^2, \quad \forall x, y \in H, \qquad (2.1.11)$$

(f) k demi-contractive if $F(T) \neq \emptyset$ and for $k \in [0, 1)$, we have

$$||Tx - Tp||^2 \leq ||x - p||^2 + k||x - Tx||^2, \quad \forall x \in H, \ \ p \in F(T). \qquad (2.1.12)$$

Remark 2.1.5.

(i) It is clear that in a real Hilbert space H, (2.1.10) is equivalent to the definition of firmly nonexpansive mapping in Definition 2.1.7 (d).

(ii) Also (2.1.12) is equivalent to

$$\langle Tx - p, x - p\rangle||x - p||^2 \geq \frac{1-k}{2}||x - Tx||^2, \quad \forall x \in H, \ p \in F(T).$$

We note that the following inclusions hold for the classes of the mappings:

firmly nonexpansive $\subset$ *nonexpansive* $\subset$ *quasi nonexpansive* $\subset$ k *strictly pseudo-contractive* $\subset$ k *demi-contractive.* (2.1.13)

More so, it is well known that the demi-contractive mappings has the following property.

Lemma 2.1.6 (see [171], Remark 4.2)**.** *Let $T : H \to H$ be a k demi-contractive mapping such that $F(T) \neq \emptyset$. Then*

(i) $T_v = (1 - v)I + vT$ is a quasi-nonexpansive mapping over C for every $v \in [0, 1 - k]$;

(ii) $F(T)$ is closed and convex.

Lemma 2.1.7. *[268] Let λ be a number in $(0, 1]$ and let $\mu > 0$. Let $B : H \to H$ be a k-Lipschitz and μ-strongly monotone mapping. Associating with a nonexpansive mapping $T : H \to H$, define a mapping $T^\lambda : H \to H$ by*

$$T^\lambda = Tx - \lambda BT(x), \qquad \forall \ x \in H.$$

Then T^λ is a contraction provided $\mu < \frac{2\eta}{k^2}$, that is,

$$||T^\lambda x - T^\lambda y|| \leq (1 - \lambda\tau)||x - y|| \quad \forall \ x, y \in H, \qquad (2.1.14)$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0, 1)$.

Definition 2.1.11. A mapping $T : H \to H$ is said to be an α-averaged mapping if $T = (1-\alpha)I + \alpha S$, where $\alpha \in (0,1)$, $S : H \to H$ is nonexpansive and I is the identity operator on H.

Many nonlinear operators belong to the class of averaged mapping. For instance, the class of firmly nonexpansive mapping is $\frac{1}{2}$-averaged.

The following lemmas will be used in the sequel.

Lemma 2.1.8. [55, 88] *Let $S, T, V : H \to H$ be given nonlinear operators.*

(i) If $T = (1-\alpha)S + \alpha V$, for some $\alpha \in (0,1)$, S is averaged and V is nonexpansive, then T is averaged.

(ii) The composition of finitely many averaged mapping is averaged. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then, the composition T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

(iii) If $\{T_i\}$ is a finite family of averaged mappings with a common fixed point, then

$$\bigcap_{i=1}^{N} F(T_i) = F(T_1 \dots T_N).$$

Lemma 2.1.9. [55, 179] *Let $U : H \to H$ be a given operator, we have*

(i) U is nonexpansive if and only if the complement $I - U$ is $\frac{1}{2}$-ism.

(ii) If U is κ-ism, then for $\gamma > 0$, κU is $\frac{\kappa}{\gamma}$-ism.

(iii) U is averaged if and only if the complement $I - U$ is κ-ism for some $\kappa > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, U is averaged if and only if $I - U$ is $\frac{1}{2\alpha}$-ism.

Definition 2.1.12. Let $T : H \to H$ be a nonlinear mapping. Then T is said to be a δ-demimetric mapping if there exists $\delta \in (-\infty, 1)$ such that

$$\langle x - p, x - Tx \rangle \geq \frac{1-\delta}{2}||x - Tx||^2, \quad \forall x \in dom(T) \quad \text{and} \quad p \in F(T). \tag{2.1.15}$$

Equivalently, T is δ-demimetric, if there exists $\delta \in (-\infty, 1)$ such that

$$||Tx - p||^2 \leq ||x - p||^2 + \delta||x - Tx||^2, \quad \forall x \in dom(T) \quad \text{and} \quad p \in F(T). \tag{2.1.16}$$

The class of δ-demimetric was recently introduced by Takahashi [249] as a generalization of k-strictly pseudo-contraction, firmly nonexpansive, quasi-nonexpansive and nonexpansive mappings in a real Hilbert space.

We give the following examples of δ-demimetric mapping in a real Hilbert space.

Example 2.1.10. Let $H=\mathbb{R}$ (the real line with usual metric). Define $T:\mathbb{R}\to\mathbb{R}$ by $Tx=\frac{x}{2}$, for all $x\in\mathbb{R}$. Clearly, $F(T)=\{0\}$. Thus

$$\begin{aligned}\langle x-p, x-Tx\rangle &= \langle x-0, x-\frac{x}{2}\rangle = \langle x, \frac{x}{2}\rangle = \frac{1}{2}\langle x, x\rangle\\ &= \frac{1}{2}|x|^2\\ &\geq \left|\frac{x}{2}\right|^2\\ &= \frac{1-\delta}{2}\left|\frac{x}{2}\right|^2 = \frac{1-\delta}{2}|x-Tx|^2,\end{aligned}$$

where $\delta=-1$. From (2.1.15), we see that T is -1 demimetric.

Example 2.1.11. Let H be the real line and $C=[-2,1]$. Define

$$Tx=\begin{cases}\dfrac{x+9}{10}, & x\in[0,1],\\[2ex] \dfrac{3+x}{4}, & x\in[-2,0).\end{cases}$$

Obviously, $F(T)=\{1\}$. We will show that there exists $\delta\in(-\infty,1)$ such that

$$|Tx-1|^2\leq|x-1|^2+\delta|x-Tx|^2, \qquad \forall x\in[-2,1].$$

Consider the following two cases:
Case (i): Let $x\in[0,1]$, then

$$|x-Tx|^2=\left|x-\frac{x+9}{10}\right|^2=\left|\frac{9}{10}(x-1)\right|^2=\frac{81}{100}|x-1|^2.$$

Also

$$\begin{aligned}|Tx-1|^2 &= \left|\frac{x+9}{10}-1\right|^2=\frac{1}{100}|x-1|^2\\ &= |x-1|^2-\frac{99}{100}|x-1|^2\\ &= |x-1|^2-\frac{99}{81}\times\frac{81}{100}|x-1|^2\\ &\leq |x-1|^2+\delta_1.\frac{81}{100}|x-1|^2,\end{aligned}$$

for any $\delta_1\in[-\frac{99}{81},1)$. Hence $|Tx-1|^2\leq|x-1|^2+\delta_1|x-Tx|^2$.
Case (ii): Let $x\in[-2,0)$, thus

$$|x-Tx|^2=\left|x-\frac{3+x}{4}\right|^2=\left|\frac{3(x-1)}{4}\right|^2=\frac{9}{16}|x-1|^2.$$

Then

$$\begin{aligned}
|Tx-1|^2 &= \left|\frac{3+x}{4}-1\right|^2 = \left|\frac{x-1}{4}\right|^2 = \frac{1}{16}|x-1|^2 \\
&= |x-1|^2 - \frac{15}{16}|x-1|^2 \\
&= |x-1|^2 - \frac{15}{9}.\frac{9}{16}|x-1|^2 \\
&\leq |x-1|^2 + \delta_2.\frac{9}{16}|x-1|^2,
\end{aligned}$$

for any $\delta_2 \in [-\frac{15}{9}, 1)$. Hence $|Tx-1|^2 \leq |x-1|^2 + \delta_1|x-Tx|^2$. In particular, choose $\delta = \min\{\delta_1, \delta_2\}$. Thus, T is $-\frac{15}{9}$-demimetric.

Definition 2.1.13. Let (X,d) be a complete metric space. A mapping $f : X \to X$ is called a Meir-Keeler contraction [182] if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x,y) < \epsilon + \delta \quad \text{implies} \quad d(f(x), f(y)) < \epsilon, \tag{2.1.17}$$

for all $x, y \in X$. It is easy to show that the Meir-Keeler contraction mapping is a generalization of the contraction mapping in Definition 2.1.6.

Lemma 2.1.12. [243] *Let f be a Meir-Keeler contraction on a convex subset C of a Banach space E. Then for every $\epsilon > 0$, there exists $r_\epsilon \in (0,1)$ such that*

$$||x-y|| \geq \epsilon \quad \Rightarrow \quad ||f(x) - f(y)|| \leq r_\epsilon ||x-y||$$

for all $x, y \in C$.

Lemma 2.1.13. [182] *A Meir-Keeler contraction defined on a complete metric space has a unique fixed point.*

Definition 2.1.14. [145] Let E be a real Banach space, then the operator $A : C \to E^*$ is said to be

(a) strongly monotone on C if there exists $\gamma > 0$ such that

$$\langle Au - Av, u - v\rangle \geq \gamma ||u-v||^2 \quad \forall \ u, v \in C;$$

(b) monotone on C if

$$\langle Au - Av, u - v\rangle \geq 0 \quad \forall \ u, v \in C;$$

(c) strongly pseudo-monotone on C if there exists $\gamma > 0$ such that

$$\langle Au, v - u\rangle \geq 0 \Rightarrow \langle Av, v - u\rangle \geq \gamma ||u-v||^2, \quad \forall \ u, v \in C;$$

(d) pseudo-monotone on C if for all $u, v \in C$

$$\langle Au, v - u\rangle \geq 0 \Rightarrow \langle Av, v - u\rangle \geq 0.$$

Remark 2.1.14. If E is a Hilbert space, then the definition of monotone operator (b) is the same as Definition 2.1.7(a). It is easy to see that the following implications hold: (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c), (c) $\Rightarrow$ (d) and (b) $\Rightarrow$ (d). We present the following example of a pseudo-monotone mapping which is neither strongly monotone nor monotone.

Example 2.1.15. [150] Let $E = \ell_2$, the real Hilbert space whose elements are the square summable sequences of real scalars, i.e.,

$$E = \{x = (x_1, x_2, \dots, x_k, \dots) \Big| \sum_{k=1}^{\infty} |x_k|^2 < +\infty\}.$$

The inner product and norm on E are given by

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k \quad \text{and} \quad ||x|| = \sqrt{\langle x, x\rangle},$$

where $x = (x_1, x_2, \dots, x_k, \dots)$, and $y = (y_1, y_2, \dots, y_k, \dots)$.
Let $\alpha, \beta \in \mathbb{R}$ such that $\beta > \alpha > \frac{\beta}{2} > 0$ and

$$C = \{x \in E : ||x|| \le \alpha\} \quad \text{and} \quad Ax = (\beta - ||x||)x.$$

It is easy to verify that $\Omega_{VIP} = \{0\}$. Now, let $x, y \in C$ such that $\langle Ax, y - x\rangle \ge 0$, i.e.

$$(\beta - ||x||)\langle x, y - x\rangle \ge 0.$$

Since $\beta > \alpha > \frac{\beta}{2} > 0$, the last inequality implies that $\langle x, y - x\rangle \ge 0$. Hence

$$\begin{aligned} \langle Ay, y - x\rangle &= (\beta - ||y||)\langle y, y - x\rangle \\ &\ge (\beta - ||y||)\langle y, y - x\rangle - (\beta - ||y||)\langle x, y - x\rangle \\ &= (\beta - ||y||)||y - x||^2 \ge 0. \end{aligned}$$

This means that A is pseudo-monotone on C. To show that A is not monotone on C, let us consider $x = \left(\frac{\beta}{2}, 0, \dots, 0, \dots\right)$, $y = (\alpha, 0, \dots, 0, \dots) \in C$. Then, we have

$$\langle Ax - Ay, x - y\rangle = \left(\frac{\beta}{2} - \alpha\right)^3 < 0.$$

Definition 2.1.15. A bifunction $f : C \times C \to \mathbb{R}$ is called

(a) strongly monotone on C if there exists a constant $\alpha > 0$ such that

$$f(x, y) + f(y, x) \le -\alpha||x - y||^2, \quad \forall\ x, y \in C;$$

(b) monotone on C if

$$f(x, y) + f(y, x) \le 0, \quad \forall\ x, y \in C;$$

(c) strongly pseudo-monotone on C if there is a constant $\alpha > 0$ such that

$$f(x, y) \ge 0 \Rightarrow f(y, x) \le -\alpha||x - y||^2, \quad \forall\ x, y \in C;$$

(d) pseudo-monotone on C if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0, \quad \forall\, x,y \in C.$$

It is easy to see that the following implications hold:

$$(a) \Rightarrow (b) \Rightarrow (d) \quad \text{and} \quad (a) \Rightarrow (c) \Rightarrow (d). \tag{2.1.18}$$

The converse implication of (2.1.18) is not true in general.

Pseudo-monotone operators in the sense of Karamardian were introduced back in 1976 as a generalization of monotone operators. This has been studied for the last 40 years and has found many applications in variational inequality and economics. In case of gradient maps, pseudo-monotonicity characterized the convexity of the underlying function [145]. Several algorithms have been introduced for solving the EP when the bifunction g is monotone on C (see, e.g. [33, 34, 90, 154, 155, 156]). However, when f is relaxed to be pseudo-monotone on C, these approaches fail to work. Hence there has been an increasing effort on finding suitable methods for solving EP where f is pseudo-monotone on C.

2.2 Metric Projection, Proximal and Resolvent Operators

In this section, we briefly look at the properties of some essential operators in functional analysis.

2.2.1 Metric projection operator

Definition 2.2.1. Let C be a nonempty, closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C denoted by $P_C(x)$ such that

$$||x - P_C(x)|| \leq ||x - y||, \quad \forall\, y \in C. \tag{2.2.1}$$

The operator $P_C : H \to C$ is called the metric projection of H onto C.

A very important inequality that characterizes the metric projection is stated below.

Proposition 2.2.1. [20] *Let C be a nonempty closed convex subset of a Hilbert space H. For arbitrary $x \in H$ and $z \in C$. Then, $z = P_C(x)$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C. \tag{2.2.2}$$

From Proposition 2.2.1, we deduce that:

(i) The metric projection is firmly nonexpansive, that is, for all $x, y \in H$,

$$||P_C(x) - P_C(y)||^2 \leq \langle x - y, P_C(x) - P_C(y)\rangle.$$

(ii) For all $x \in H$ and $y \in C$,

$$||x - P_C(x)||^2 + ||P_C(x) - y||^2 \leq ||x - y||^2. \tag{2.2.3}$$

(iii) If C is a closed subspace, then P_C coincides with the orthogonal projection from H onto C, that is, $x - P_C(x)$ is orthogonal to C. Thus, for any $y \in C$,

$$\langle x - P_C(x), y \rangle = 0.$$

If C is a closed convex subset with a particular simple structure, then the projection P_C has a closed form expression as describe below (see [177]):

1. If $C = \{x \in H : ||x - u|| \leq r\}$ is a closed ball centred at $u \in H$ with radius $r > 0$, then

$$P_C x = \begin{cases} u + \frac{r(x-u)}{||x-u||}, & \text{if } x \notin C, \\ \\ x, & \text{if } x \in C. \end{cases}$$

2. If $C = [a, b]$ is a closed rectangle in $\mathbb{R}^n$, where $a = (a_1, a_2, \ldots, a_n)^T$ and $b = (b_1, b_2 \ldots, b_n)^T$, then for $1 \leq i \leq n$, $P_C x$ has the i^{th} coordinate given by

$$(P_C x)_i = \begin{cases} a_i, & \text{if } x_i \leq a_i, \\ x_i, & \text{if } x_i \in [a_i, b_i], \\ b_i, & \text{if } x_i > b_i. \end{cases}$$

3. If $C = \{y \in H : \langle a, y \rangle = \alpha\}$is a hyperplane with $a \neq 0$ and $\alpha \in \mathbb{R}$, then

$$P_C x = x - \frac{\langle a, x \rangle - \alpha}{||a||^2} a.$$

4. If $C = \{y \in H : \langle a, y \rangle \leq \alpha\}$ is a closed halfspace, with $a \neq 0$ and $\alpha \in \mathbb{R}$, then

$$P_C x = \begin{cases} x - \frac{\langle a, x \rangle - \alpha}{||a||^2} a, & \text{if } \langle a, x \rangle > \alpha, \\ \\ x, & \text{if } \langle a, x \rangle \leq \alpha. \end{cases} \tag{2.2.4}$$

5. If C is the range of a $m \times n$ matrix A with full cloumn rank, then

$$P_C x = A(A^* A)^{-1} A^* x,$$

where A^* is the adjoint of A.

2.2.2 Proximal and resolvent operators

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. The proximal operator $prox_{\gamma f}$ of f with respect to parameter $\gamma > 0$ is defined by

$$prox_{\gamma f}(x) = \underset{y \in H}{argmin} \left(f(y) + \frac{1}{2\gamma} ||x - y||^2 \right). \tag{2.2.5}$$

The following is a useful property of the proximal operator.

Proposition 2.2.2. *[23] Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function and $\gamma > 0$. Then the following holds:*

(i) Let $x, p \in H$. Then

$$p = prox_{\gamma f}(x) \Leftrightarrow \langle y - p, x - p \rangle + f(p) \leq f(y) \quad \forall\, y \in H.$$

(ii) $prox_{\gamma f}$ and $I - prox_{\gamma f}$ are firmly nonexpansive.

(ii) $F(prox_{\gamma f}) = Argmin\ f$.

Definition 2.2.2. Let $M : H \to 2^H$ be a set-valued mapping and $\gamma > 0$.

1. The resolvent of M with respect to the parameter γ is the operator

$$J_\gamma^M := \frac{1}{\gamma}(I + \gamma M)^{-1}.$$

2. The Yosida approximation of M with respect to the parameter γ is define by

$$M_\gamma := \frac{1}{\gamma}(I - J_\gamma^M).$$

3. The zero set of M is the set $M^{-1}(0)$ define by

$$M^{-1}(0) := \{x \in H : 0 \in M(x)\}.$$

The following property of the resolvent operator will be used in this thesis.

Proposition 2.2.3 (Proposition 23.2 in [23])**.** *Let $M : H \to 2^H$ be a mapping, $\gamma > 0$ and $x, p \in H$. Then the following hold:*

(i) $dom\ (J_\gamma^M) = dom\ (M_\gamma) = ran(I + \gamma M)$ and $ran(J_\gamma^M) = dom\ (M)$;

(ii) $p \in J_\gamma^M(x) \Leftrightarrow x \in p + \gamma M p \Leftrightarrow x - p \in \gamma M p \Leftrightarrow (p, \gamma^{-1}(x - p)) \in Gr(M)$;

(iii) $p \in M_\gamma x \Leftrightarrow p \in M(x - \gamma p) \Leftrightarrow (x - \gamma p) \in Gr(M)$.

Next, we present some important examples of resolvent operator.

Example 2.2.4.

1. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function and $\gamma > 0$. Then
$$J_\gamma^{\partial f} = prox_{\gamma f}.$$

2. Let C be a nonempty closed convex subset of H and $\gamma > 0$. Setting $f = i_C$, then $\partial i_C = N_C$ and
$$J_\gamma^{\partial i_C} = prox_{\gamma \partial i_C} = P_C,$$
where P_C is the metric projection onto C.

3. Let $x_0 \in H$ and suppose $H = L^2([0,T]; H)$ and $\gamma = 1$. Let M be the time-derivative operator (see Example 2.9 and Example 23.5 in [23])
$$M : H \to 2^H : x \to \begin{cases} \{x'\}, & \text{if } \; x \in W^{1,2}([0,T]; H) \quad \text{and} \quad x(0) = x_0; \\ \emptyset, \quad \text{otherwise.} \end{cases}$$
Then $dom(J_1^M) = H$ and for every $x \in H$
$$J_1^M x : [0,T] \to H : t \mapsto \exp^{-t} x_0 + \int_0^t \exp^{s-t} x(s) ds.$$

2.3 Multivalued Mappings

A subset D of H is called proximal if for each $x \in H$, there exists $y \in D$ such that
$$||x - y|| = d(x, D).$$
We denote by $CB(H)$, $CC(H)$ and $P(H)$ the families of all nonempty closed bounded subsets of H, nonempty closed convex subset of H and nonempty proximal bounded subsets of H respectively. The Pompeiu-Hausdorff metric on $CB(H)$ is defined by
$$H(A, B) := \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$
for all $A, B \in CB(H)$. Let $S : H \to 2^H$ be a multivalued mapping. An element $p \in H$ is called a fixed point of S if $p \in Sp$. We say that S satisfies the endpoint condition if $Sp = \{p\}$ for all $p \in F(S)$. For multivalued mappings $S_i : H \to 2^H$ $(i \in \mathbb{N})$ with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, we say S_i satisfy the common endpoint condition if $S_i(p) = \{p\}$ for all $i \in N$, $p \in \cap_{i=1}^{\infty} F(S_i)$. We recall some basic definitions of multivalued mappings.

Definition 2.3.1. A multivalued mapping $S : H \to CB(H)$ is said to be

1. nonexpansive if
$$H(Sx, Sy) \leq ||x - y||, \quad \forall x, y \in H,$$

2. quasi-nonexpansive if $F(S) \neq \emptyset$ and
$$H(Sx, Sp) \leq ||x - p||, \quad \forall x \in H, \;\; p \in F(S),$$

3. λ-demi-contractive for $0 \leq \lambda < 1$ if $F(S) \neq \emptyset$, and

$$H(Sx, Sp)^2 \leq ||x - p||^2 + \lambda d(x, Sx)^2, \quad \forall x \in H,\ p \in F(S).$$

We note that the class of λ-demi-contractive mappings includes several other type of classes of nonlinear mappings such as nonexpansive and quasi-nonexpansive.

The best approximation operator P_S for a multivalued mapping $S : H \to P(H)$ is defined by

$$P_S(x) := \{y \in Sx : ||x - y|| = d(x, Sx)\}.$$

One can easily prove that $F(S) = F(P_S)$ and P_S satisfies the endpoint condition. However, Song and Cho [236] gave an example of a best approximation operator P_S which is nonexpansive but S is not necessarily nonexpansive.

2.4 Some Notions on Geometric Properties of Banach Spaces

We recall some important geometric properties of Banach spaces that relevant to this study.

Definition 2.4.1. A Banach space E is said to be uniformly convex if given any $\epsilon > (0, 2]$, there exist $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in E$ satisfying $||x|| = 1$, $||y|| = 1$ and $||x - y|| \geq \epsilon$, we have $\left\|\frac{1}{2}(x + y)\right\| < 1 - \delta$.

Let $dim(E) \geq 2$. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf \left\{1 - \left\|\frac{x + y}{2}\right\| : ||x|| = ||y|| = 1; \epsilon = ||x - y||\right\}.$$

E is said to be *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$ and p-*uniformly convex* if there exists a constant $C_p > 0$ such that $\delta_E(\epsilon) \geq C_p \epsilon^p$ for any $p \in (0, 2]$.

Definition 2.4.2. A normed linear space X is called strictly convex if for all $x, y \in X$ with $x \neq y$, $||x|| = ||y|| = 1$, we have $||\lambda x + (1 - \lambda)y|| < 1$, for all $\lambda \in (0, 1)$.

Proposition 2.4.1. [76] *Every uniformly convex Banach space is strictly convex.*

Remark 2.4.2. The space l_∞ is not strictly convex. To see this, if we consider $\bar{u} = (1, 1, 0, 0, 0, ...)$ and $\bar{v} = (-1, 1, 0, 0, 0, ...)$. Both $\bar{u}, \bar{v} \in l_\infty$. Taking $\epsilon = 1$, then $||\bar{u}||_\infty = 1 = ||\bar{v}||_\infty$ and $||\bar{u} - \bar{v}||_\infty = 2 > \epsilon$. However, $\left\|\frac{\bar{u} + \bar{v}}{2}\right\|_\infty = 1$. Thus l_∞ is not strictly convex.

Definition 2.4.3. A Banach space E is said to be smooth if for every $x \in E$, $||x|| = 1$, there exists a unique $x^* \in E^*$ such that $||x^*|| = 1$ and $\langle x, x^* \rangle = ||x||$.

The modulus of smoothness of E is the mapping $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup \left\{\frac{||x + y|| + ||x - y||}{2} - 1 : ||x|| = 1,\ ||y|| \leq t\right\}.$$

The Banach space E is said to be uniformly smooth if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t}=0,$$

see Chidume [76]. Suppose that $q>1$, then E is said to be q-uniformly smooth if there exists $c_q>0$ such that $\rho_E(t)\le c_q t^q$ for all $t>0$. It is well known that there is no Banach space which is q-uniformly smooth with $q>2$ (see [271, 247]). If E is q-uniformly smooth, then E is uniformly smooth. Also, each uniformly convex Banach space E is reflexive, strictly convex and every uniformly smooth Banach space E is a reflexive Banach space with uniformly Gâteaux differentiable norm (see [244]). Typical examples of both uniformly convex and uniformly smooth Banach spaces are L_p spaces, where $1<p<\infty$. Moreover, L_p is $\min\{p,2\}$-uniformly smooth for every $p>1$.

Definition 2.4.4. Let E^* be the dual space of a real Banach space E and $p>1$. The multi-valued mapping $J_p : E\to 2^{E^*}$ defined by

$$J_p^E x=\{x^*\in E^*:\langle x,x^*\rangle=||x||^p,||x^*||=||x||^{p-1}\}\quad \forall\, x\in E, \tag{2.4.1}$$

is called the generalized duality mapping of E. In particular, $J_2^E=J$ is called normalized duality mapping. The normalized duality mapping is known to have the following properties (see [87]):

(i) If E is smooth, then J is single-valued and denoted by j.

(ii) If E is strictly convex, then J is one-to-one and strictly monotone, i.e.,

$$\langle x-y,Jx-Jy\rangle>0\quad \forall\, x,y\in E.$$

(iii) If E is reflexive, then J is surjective.

(iv) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subset of E.

(v) If E^* is uniformly convex, then J is single-valued, one-to-one and uniformly continuous on bounded subsets of E.

The generalized duality mapping J_p^E is said to be weak-to-weak continuous if

$$x_n\rightharpoonup x\Rightarrow\langle J_p^E(x_n),y\rangle\to\langle J_p^E(x),y\rangle$$

holds true for any $y\in E$. It is worth noting that the l_p $(p>1)$ space has such property, but the L_p $(p>2)$ space does not share this property.

Let $1<q\le 2\le p$ with $\frac{1}{p}+\frac{1}{q}=1$. It is well known that E is p-uniformly convex and uniformly smooth if and only if its dual space E^* is q-uniformly smooth and uniformly convex. Moreover, if E is reflexive and strictly convex with a strictly convex dual, then $(J_p^E)^{-1}=J_q^{E^*}$ is single-valued, one-to-one, surjective and it is the duality mapping from E^* into E and thus $J_p^E J_q^{E^*}=I^{E^*}$ and $J_q^{E^*}J_p^E=I^E$, where I^E and I^{E^*} are the identity operators on E and E^* respectively. We note that in a real Hilbert space, the duality mappings reduce to the identity mapping. For more information on geometry of Banach spaces and duality mapping, see [76] and [87].

Definition 2.4.5. A mapping T with domain $D(T)$ and range $R(T)$ in E is called:

(i) λ-strictly pseudocontractive [50] if for all $x, y \in D(T)$, there exist $\lambda > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^q - \lambda||(I - T)x - (I - T)y||^q, \tag{2.4.2}$$

or equivalently

$$\langle (I - T)x - (I - T)y, j_q(x - y)\rangle \geq \lambda||(I - T)x - (I - T)y||^q, \tag{2.4.3}$$

(ii) accretive if for all $x, y \in C$ and $j_p(x - y) \in J_p(x - y)$, we have

$$\langle Tx - Ty, j_p(x - y)\rangle \geq 0, \tag{2.4.4}$$

(iii) μ-strongly accretive if for all $x, y \in C$, there exists $\mu > 0$ and $j_p(x - y) \in J_p(x - y)$, such that

$$\langle Tx - Ty, j_q(x - y)\rangle \geq \mu||x - y||^q. \tag{2.4.5}$$

By Definition 2.4.5, we know that every λ-strictly pseudocontractive mapping is $\frac{1+\lambda}{\lambda}$-Lipschitzian (see [75]). We also note that the class of λ-strict pseudocontractive mappings properly contains the class of nonexpansive mappings. If $\lambda \equiv 0$ in (2.4.2), then the mapping T is called pseudocontractive.
In a real Hilbert space H, it can easily be shown that for $\lambda \in (0, \frac{1}{2})$, (2.4.2) is equivalent to (2.1.11) with $k = 1 - 2\lambda < 1$.

Definition 2.4.6. Let C be a nonempty closed and convex subset of a real Banach space E and K be a nonempty subset of C. A mapping $Q_K : C \to K$ is called a retraction of C onto K if $Q_K x = x$ for all $x \in K$. We say that Q_K is sunny if, for each $x \in C$ and $t \geq 0$,

$$Q_K(tx + (1 - t)Q_K x) = Q_K x,$$

whenever $tx + (1 - t)Q_K x \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. It is well known that if $E := H$ is a Hilbert space, then the sunny nonexpansive retraction Q_K coincides with the metric projection from C onto K.

Proposition 2.4.3. *[212]: Let C be a closed and convex subset of a smooth Banach space E. Let K be a nonempty subset of C, $Q : C \to K$ be a retraction and let j, j_q be the normalized and generalized duality mappings on E respectively. Then the following statements are equivalent:*
(a) Q is sunny and nonexpansive,
(b) $||Qx - Qy||^2 \leq \langle x - y, j(Qx - Qy)\rangle$ for all $x, y \in C$,
(c) $\langle x - Qx, j(y - Qx)\rangle \leq 0$ for all $x \in C$ and $y \in K$,
(d) $\langle x - Qx, j_q(y - Qx)\rangle \leq 0$ for all $x \in C$ and $y \in K$.

2.5 The Bregman Distance and Some Related Notions

Definition 2.5.1. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex and Gâteaux differentiable function. The function $D_f : \text{dom} f \times \text{int}(\text{dom} f) \to [0, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle \tag{2.5.1}$$

is called the Bregman distance with respect to f, (see [44, 70]).

The Bregman distance does not satisfy the well-known properties of a metric, but it has the following important properties:

Proposition 2.5.1. [21](Basic properties of Bregman distance) *The following properties follow directly from the definition of Bregman distance: Let $u, v, x, y \in E$, then*

(i) $D_f(u, v) + D_f(v, u) = \langle u - v, \nabla f(u) - \nabla f(v) \rangle$;

(ii) $D_f(x, u) = D_f(x, y) + D_f(y, u) + \langle x - y, \nabla f(y) - \nabla f(u) \rangle$;

(iii) $D_f(x, u) + D_f(y, v) = D_f(x, v) + D_f(y, u) + \langle x - y, \nabla f(v) - \nabla f(u) \rangle$.

Definition 2.5.2. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex and Gâteaux differentiable function. The function f is called

(i) totally convex at x if its modulus of totally convexity at $x \in \text{int}(\text{dom} f)$, that is, the bifunction $v_f : \text{int}(\text{dom} f) \times [0, +\infty) \to [0, +\infty)$, defined by

$$v_f(x, t) := \inf\{D_f(y, x) : y \in dom f, ||y - x|| = t\}, \tag{2.5.2}$$

is positive for any $t > 0$,

(ii) totally convex if it is totally convex at every point $x \in \text{int}(\text{dom } f)$,

(iii) totally convex on bounded subset B of E, if $v_f(B, t)$ is positive for any nonempty bounded subset B, where the function $v_f : \text{int}(\text{dom } f) \times [0, +\infty) \to [0, +\infty]$ is defined by

$$v_f(B, t) := \inf\{v_f(x, t) : x \in B \cap int(dom f)\}, \quad t > 0. \tag{2.5.3}$$

For further details and examples on totally convex functions, see [38, 58, 59].

Definition 2.5.3. [58, 216] Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex and Gâteaux differentiable function. The function f is called

(i) cofinite if $\text{dom} f^* = E^*$,

(ii) coercive if $\lim\limits_{||x|| \to +\infty} \left| \frac{f(x)}{||x||} \right| = +\infty$,

(iii) sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{2.5.4}$$

Definition 2.5.4. Let E be a Banach space and let $B_r := \{z \in E : ||z|| \le r\}$ for all $r > 0$. Then, a function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E if $\rho_r(t) > 0$ for all $t > 0$, where $\rho_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_r(t) = \inf_{x,y\in B_r, ||x-y||=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}. \tag{2.5.5}$$

The function ρ_r is called the gauge of uniform convexity of f. More so, the function $f : E \to \mathbb{R} \cup \{+\infty\}$ is called strongly coercive if

$$\lim_{||x||\to+\infty} \left(\frac{f(x)}{||x||}\right) = +\infty.$$

Definition 2.5.5. Let $f : E \to \mathbb{R}\cup\{+\infty\}$ be a convex and Gâteaux differentiable function. The *Bregman projection* of $x \in \text{int}(\text{dom} f)$ onto the nonempty, closed and convex subset C $\subset \text{int}(\text{dom} f)$ is defined as the necessarily unique vector $\text{Proj}_C^f(x) \in C$ satisfying

$$D_f(Proj_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}. \tag{2.5.6}$$

It is known from [59] that $z = \text{Proj}_C^f(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z\rangle \le 0 \qquad for\ all\ y \in C. \tag{2.5.7}$$

We also have

$$D_f(y, Proj_C^f(x)) + D_f(Proj_C^f(x), x) \le D_f(y, x) \qquad for\ all\ x \in E,\ y \in C. \tag{2.5.8}$$

Lemma 2.5.2. [213] (Characterization of Bregman Projection)*: Let f be totally convex on int(dom$f$). Let C be a nonempty, closed and convex subset of int(dom$f$) and $x \in$int(dom$f$), if $\omega \in C$, then the following conditions are equivalent:*

(a) the vector ω is the Bregman projection of x onto C, with respect to f,

(b) the vector ω is the unique solution of the variational inequality

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \ge 0 \qquad \forall y \in C,$$

(c) the vector ω is the unique solution of the inequality

$$D_f(y, z) + D_f(z, x) \le D_f(y, x) \qquad \forall y \in C.$$

Definition 2.5.6. Let $T : C \to C$ be a mapping, a point $x^* \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to x^* and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$.

Definition 2.5.7. Let C be a nonempty, closed and convex subset of E. A mapping $T : C \to \text{int}(\text{dom } f)$ is called

1. Bregman Firmly Nonexpansive (BFNE for short) if
$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle \ \forall x, y \in C. \quad (2.5.9)$$

2. Bregman Strongly Nonexpansive (BSNE) with respect to a nonempty $\hat{F}(T)$ if
$$D_f(p, Tx) \leq D_f(p, x), \quad (2.5.10)$$
for all $p \in \hat{F}(T)$ and $x \in C$ and if whenever $\{x_n\}_{n=1}^{\infty} \subset C$ is bounded, $p \in \hat{F}(T)$ and
$$\lim_{n\to\infty} \Big(D_f(p, x_n) - D_f(p, Tx_n)\Big) = 0,$$
it follows that
$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0.$$

3. Bregman Relative Nonexpansive (BRNE) if $F(T) \neq \emptyset$,
$$D_f(p, Tx) \leq D_f(p, x) \quad \forall\ x \in C, p \in F(T) \quad \text{and} \quad \hat{F}(T) = F(T). \quad (2.5.11)$$

4. Quasi-Bregman Nonexpansive (QBNE) if $F(T) \neq \emptyset$ and
$$D_f(p, Tx) \leq D_f(p, x) \qquad \forall x \in C, p \in F(T). \quad (2.5.12)$$

From the Definition 2.5.1, it is clear that (2.5.9) is equivalent to
$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x). \quad (2.5.13)$$
We note that in the case where $\hat{F}(T) = F(T)$, the following inclusion holds
$$BFNE \subset BSNE \subset BRNE \subseteq QBNE. \quad (2.5.14)$$

It is worth noting that the duality mapping J_p^E is actually the derivative of the function $f_p(x) = \frac{1}{p}||x||^p$ for $2 \leq p < \infty$. If $f = f_p$, then the Bregman distance with respect to f_p now becomes
$$\begin{aligned} D_p(x, y) &= \frac{1}{q}||x||^p - \langle J_p^E x, y\rangle + \frac{1}{p}||y||^p \quad (2.5.15)\\ &= \frac{1}{p}(||y||^p - ||x||^p) + \langle J_p^E x, x - y\rangle \\ &= \frac{1}{q}(||x||^p - ||y||^p) - \langle J_p^E x - J_p^E y, y\rangle. \end{aligned}$$

For the p-uniformly convex Banach space E, the metric and Bregman distance has the following relation (see [227])
$$\tau||x - y||^p \leq D_p(x, y) \leq \langle x - y, J_p^E x - J_p^E y\rangle, \quad (2.5.16)$$

where $\tau > 0$ is some fixed number.

Similar to the metric projection, we define the Bregman projection with respect to D_p as

$$\Pi_C x := \underset{y \in C}{argmin}\ D_p(x,y),$$

for all $x \in E$, which is the unique minimizer of the Bregman distance. The Bregman projection is also characterized by the variational inequality:

$$\langle J_p^E(x) - J_p^E(\Pi_C x), z - \Pi_C x \rangle \leq 0.\ \forall\ z \in C, \tag{2.5.17}$$

which implies that

$$D_p(\Pi_C x, z) \leq D_p(x,z) - D_p(x, \Pi_C x), \tag{2.5.18}$$

for all $z \in C$.
Following [6, 66], we make use of the function $V_p : E^* \times E \to [0,\infty)$, defined by

$$V_p(x,y) := \frac{1}{q}||x||^q - \langle x,y \rangle + \frac{1}{p}||y||^p,\ \forall\ x \in E^*,\ y \in E. \tag{2.5.19}$$

Then V_p is nonnegative and $V_p(x,y) = D_p(J_p^{E^*}(x), y)$ for all $x \in E^*$ and $y \in E$. Moreover, by the subdifferential inequality

$$\langle f'(x), y - x \rangle \leq f(y) - f(x),$$

with $f(x) = \frac{1}{q}||x||^q$ and $x \in E^*$, then $f'(x) = J_q^{E^*}$. Then we have

$$\langle J_q^{E^*}(x), y \rangle \leq \frac{1}{q}||x+y||^q - \frac{1}{q}||x||^q, \tag{2.5.20}$$

and from (2.5.20), we obtain

$$\begin{aligned}
V_p(x^* + y^*, x) &= \frac{1}{q}||x^* + y^*||^q - \langle x^* + y^*, x \rangle + \frac{1}{q}||x||^p \\
&\geq \frac{1}{q}||x^*||^q + \langle y^*, J_p^{E^*}(x^*) \rangle - \langle x^* + y^*, x \rangle + \frac{1}{q}||x||^p \\
&= \frac{1}{q}||x^*||^q - \langle x^*, x \rangle + \frac{1}{p}||x||^p + \langle y^*, J_p^{E^*}(x^*) \rangle - \langle y^*, x \rangle \\
&= \frac{1}{q}||x^*||^q - \langle x^*, x \rangle + \frac{1}{p}||x||^p + \langle y^*, J_p^{E^*}(x^*) - x \rangle \\
&= V_p(x^*, x) + \langle y^*, J_p^{E^*}(x^*) - x \rangle,
\end{aligned} \tag{2.5.21}$$

for all $x \in E$ and $x^*, y^* \in E^*$. In addition, V_p is convex in the first variable. Thus, for all $z \in E$,

$$D_p(J_q^{E^*} \sum_{i=1}^N t_i J_p^E(x_i), w) \leq \sum_{i=1}^N t_i D_p(x_i, w), \tag{2.5.22}$$

where $\{x_i\} \subset E$ and $\{t_i\} \subset (0,1)$ with $\sum_{i=1}^{N} t_i = 1$.

Another important distance function we used in the thesis is the Lyapunov functional ϕ on $E \times E$ defined by (see [6])

$$\phi(x, y) = ||x||^2 - 2\langle x, J(y)\rangle + ||y||^2, \qquad \forall\, x, y \in E. \tag{2.5.23}$$

It is easy to see from the definition of ϕ that if $E = H$ a real Hilbert space, $\phi(x, y) = ||x - y||^2$.

Proposition 2.5.3. *The following properties clearly follows from the definition of* ϕ :

D1. $(||x|| - ||y||)^2 \leq \phi(y, x) \leq (||x|| + ||y||)^2$,

D2. $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$,

D3. for all $x, y, z \in E$ *and* $\alpha \in (0, 1)$

$$\phi(x, J^{-1}(\alpha Jy + (1-\alpha)Jz)) \leq \alpha\phi(x, y) + (1-\alpha)\phi(x, z).$$

We also note the following important relation.

Proposition 2.5.4. *[191] Let* E *be a 2-uniformly convex and smooth Banach space. Then for every* $x, y \in E$,

$$\phi(x, y) \geq c_1||x - y||^2, \tag{2.5.24}$$

where $c_1 > 0$ *is the 2-uniformly convexity constant of* E.

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E. Following Alber [6], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) := \underset{y \in C}{argmin}\ \phi(y, x), \qquad \forall\, x \in E.$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for instance [6, 244]). If E is a Hilbert space, then Π_C is the metric projection of H onto C.

Lemma 2.5.5. *[Characterization of Generalized Projection [6, 143]] Let* E *be a smooth, strictly convex and reflexive Banach space and* C *be a nonempty closed and convex subset of* E*. Then the following hold:*

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$, *for all* $x \in C$, $y \in E$,

(b) $z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz\rangle \geq 0$, *for all* $y \in C$.

Definition 2.5.8. A mapping $S : C \to C$ is said to be

(i) $\phi-$nonexpansive if

$$\phi(Sy, Sx) \leq \phi(y, x), \quad \forall\ x, y \in C,$$

(ii) $\phi-$quasi-nonexpansive if $F(S) \neq \emptyset$ and

$$\phi(p, Sx) \leq \phi(p, x), \quad \forall\ x \in C,\ p \in F(S),$$

(iii) $\phi-$relatively nonexpansive if $F(S) \neq \emptyset$,

$$\phi(p, Sx) \leq \phi(p, x), \quad \forall\ x \in C,\ p \in F(S) \ \text{ and } \ F(S) = \hat{F}(S).$$

Following Alber [6], we make use of the mapping $V : E \times E^* \to [0, \infty)$ defined by

$$V(x, x^*) = ||x||^2 - 2\langle x, x^* \rangle + ||x^*||^2, \tag{2.5.25}$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.5.6. *[6] Let E be a reflexive, strictly convex and smooth Banach space, and let V be as defined in (2.5.25). Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*), \tag{2.5.26}$$

for all $x \in E$ and $x^*, y^* \in E^*$

Remark 2.5.7. For a real Banach space E, the resolvent operator R_λ^M associated with M for $\lambda > 0$ is given as

$$R_\lambda^M(x) := \{z \in E : J_p^E x \in J_p^E z + \lambda M(z)\}.$$

Equivalently, $R_\lambda^M := (J_p^E + \lambda M)^{-1} J_p^E$. R_λ^M is single valued and also $M^{-1}0 = F(R_\lambda^M)$ (see Section 5 in [245]). It is well known that R_λ^M is relative nonexpansive, that is

$$0 \leq \langle R_\lambda^M(x) - R_\lambda^M(y), J_p^E(x) - J_p^E(R_\lambda^M(x)) - (J_p^E y - R_\lambda^M(y)) \rangle, \tag{2.5.27}$$

for all $x, y \in E$; see Theorem 5.2 of [245]. Also, for any $x \in E$, $u \in T^{-1}(0)$ and $\lambda > 0$, we have (see [213])

$$D_p(x, R_{\lambda T} x) + D_p(R_{\lambda T} x, u) \leq D_p(x, u). \tag{2.5.28}$$

2.6 Some other Important Results

In this section, we state some other important results which will be used in the sequel.

The following lemma is well known in Hilbert space; see for instance [22, 115].

Lemma 2.6.1. *Let H be a real Hilbert space. Then the following hold: for all $x, y \in H$,*

(i) $||x + y||^2 \leq ||y||^2 + 2\langle x, x + y \rangle$,

(ii) $||x-y||^2 = ||x||^2 + ||y||^2 - 2\langle x, y\rangle$,

(iii) $||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2$, *for* $\alpha \in [0, 1)$.

The following can easily be proved using Lemma 2.6.1 (ii).

Lemma 2.6.2. *Let H be a real Hilbert space and $a, b, c, d \in H$. Then*

$$\langle a-b, c-d\rangle = \frac{1}{2}\left(||a-d||^2 - ||a-c||^2\right) + \frac{1}{2}\left(||c-b||^2 - ||d-b||^2\right).$$

Lemma 2.6.3. *[79] Let H be a real Hilbert space, $x_i \in H$, $(1 \le i \le m)$ and $\{\alpha_i\} \subset (0, 1)$ with $\sum_{i=1}^{m} \alpha_i = 1$. Then the following identity holds:*

$$\Big\|\sum_{i=1}^{m} \alpha_i x_i\Big\|^2 = \sum_{i=1}^{m} \alpha_i ||x_i||^2 - \sum_{i,j=1, i\neq j}^{m} \alpha_i\alpha_j ||x_i - x_j||^2. \tag{2.6.1}$$

Lemma 2.6.4 (see [122])**.** *Let $\{h^i\}_{i=1}^m$ be a finite family of convex functions defined on H such that their level set is defined by $C^i = \{x \in H : h^i(x) \le 0\}$, $i = 1, 2, \ldots, N$, with nonempty intersection. Let $D = \{x \in H : \sum_{i=1}^m \beta_i h^i(x) \le 0\}$ with $\{\beta_i\}_{i=1}^m \subset (0, 1)$ such that $\sum_{i=1}^m \beta_i = 1$. Then, the following properties are satisfied:*

(i) If each C^i is a half space, i.e., $h^i(x) = \langle x, v_i\rangle - d_i$ with $d_i \in \mathbb{R}$ and $v_i \in H$ such that $v_i \neq 0$, in addition, if the vector group $\{v_i\}_{i=1}^m$ is also linearly independent, then D is a half space;

(ii) D is a closed ball if each C^i is a closed ball;

(iii) D is a closed ball if C^i is a closed ball or a half space and at least one of them is a closed ball.

Lemma 2.6.5 (Demiclosedness principle in Hilbert spaces [115])**.** *Let C be a closed and convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to p and if $\{(I-T)x_n\}$ converges strongly to q, then $(I-T)p = q$. In particular, if $q = 0$, then $p \in F(T)$.*

Lemma 2.6.6 (Demiclosedness principle in Banach spaces [242])**.** *Let C be a nonempty closed and convex subset of a q-uniformly smooth real Banach space E which admits weakly sequentially continuous generalized duality mapping j_p from E into E^*. Let $T : C \to C$ be a nonexpansive mapping. Then, for all $\{x_n\} \subset C$, if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.6.7. *[143] Let E be a uniformly convex and uniformly smooth real Banach space and $\{x_k\}, \{y_k\}$ be sequences in E such that either $\{x_k\}$ or $\{y_k\}$ is bounded. If*

$$\lim_{k\to\infty} \phi(x_k, y_k) = 0,$$

then $\lim_{k\to\infty} ||x_k - y_k|| = 0$.

Lemma 2.6.8. *[268] Let E be a uniformly convex real Banach space. Let $r > 0$, then there exists a strictly increasing continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and the following inequality holds:*

$$||\lambda x + (1 - \lambda)y||^2 \leq \lambda||x||^2 + (1 - \lambda)||y||^2 + \lambda(1 - \lambda)g(||x - y||), \tag{2.6.2}$$

for all $x, y \in B_r(0)$, where $B_r(0) := \{v \in E : ||v|| \leq r\}$ and $\lambda \in [0, 1]$.

Definition 2.6.1 (see [161, 180])**.** The Minty Variational Inequality Problem (MVIP) is defined as finding a point $\bar{x} \in C$ such that

$$\langle Ay, y - \bar{x}\rangle \geq 0, \quad \forall y \in C. \tag{2.6.3}$$

We denote by $M(C, A)$, the set of solution of (2.6.3). Some existence results for the MVIP has been presented in [161]. Also, the assumption that $M(C, A) \neq \emptyset$ has already been used for solving Ω_{VIP} in finite dimensional spaces (see e.g [235]). It is not difficult to prove that pseudo-monotonicity implies property $M(C, A) \neq \emptyset$, but the converse is not true. Indeed, let $A : \mathbb{R} \to \mathbb{R}$ be defined by $A(x) = \cos(x)$ with $C = [0, \frac{\pi}{2}]$. We have that $\Omega_{VIP} = \{0, \frac{\pi}{2}\}$ and $M(C, A) = \{0\}$. But if we take $x = 0$ and $y = \frac{\pi}{2}$ in Definition 2.1.14(d), we see that A is not pseudo-monotone.

Lemma 2.6.9 (see [180])**.** *Consider the VIP (1.1.1). If the mapping $h : [0, 1] \to E^*$ defined as $h(t) = A(tx + (1 - t)y)$ is continuous for all $x, y \in C$ (i.e., h is hemicontinuous), then $M(C, A) \subset \Omega_{VIP}$. Moreover, if A is pseudo-monotone, then Ω_{VIP} is closed, convex and $\Omega_{VIP} = M(C, A)$.*

Lemma 2.6.10. *[136] Let f be a totally convex and Gâteaux differentiable such that $domf = E$. Then for all $x^* \in E^* \setminus \{0\}$, $\tilde{y} \in E$, $x \in H^+$ and $\bar{x} \in H^-$, it holds that*

$$D_f(\bar{x}, x) \geq D_f(\bar{x}, z) + D_f(z, x),$$

where $z = argmin_{y \in H} D_f(y, x)$ and $H = \{y \in E : \langle x^*, y - \tilde{y}\rangle = 0\}$, $H^+ = \{y \in E : \langle x^*, y - \tilde{y}\rangle \geq 0\}$ and $H^- = \{y \in E : \langle x^*, y - \tilde{y}\rangle \leq 0\}$.

The following lemma was proved in $\mathbb{R}^n$ in [104] and it can easily be extended to a real Banach space.

Lemma 2.6.11. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed and convex subset of E. For any $x \in E$ and $\beta > 0$, we denote*

$$r_\beta(x) := x - \Pi_C J^{-1}(Jx - \beta Ax), \tag{2.6.4}$$

then

$$\min\{1, \beta\}||r_1(x)|| \leq ||r_\beta(x)|| \leq \max\{1, \beta\}||r_1(x)||.$$

Lemma 2.6.12. *[165] Let H be a real Hilbert space. Let $T : H \to 2^H$ be a maximal monotone operator and $S : H \to H$ be an α-inverse strongly monotone mapping on H. Define $K_r := (I + rT)^{-1}(x - rSx)$, $r > 0$, then we have*

$$F(K_r) = (S + T)^{-1}(0), \tag{2.6.5}$$

where $F(K_r)$ denotes the set of fixed points of K_r. Also, note that K_r is nonexpansive and $F(K_r)$ is closed and convex.

Lemma 2.6.13. *[86] Let H be a real Hilbert space and $B : H \to 2^H$ be a set-valued maximal monotone operator. For each $x \in H$, $\lambda > 0$ and $J_\lambda^B(x) = (I + \lambda B)^{-1}(x)$, then*

(i) J_λ^B *is single-valued and firmly nonexpansive;*

(ii) $D(J_\lambda^B) = H$ *and* $Fix(J_\lambda^B) = \{x \in H : 0 \in B(x)\}$;

(iii) $||x - J_\lambda^B x|| \leq ||x - J_\gamma^B||$ *for all* $0 < \lambda < \gamma$, $x \in H$;

(iv) Suppose $B^{-1}(0) \neq \emptyset$. *Then* $||x - J_\lambda^B x||^2 + ||J_\lambda^B x - y^*||^2 \leq ||x - y^*||^2$ *for each* $x \in H$ *and* $y^* \in B^{-1}(0)$;

(v) Suppose $B^{-1}(0) \neq \emptyset$. *Then* $\langle x - J_\lambda^B x, J_\lambda^B x - y\rangle \geq 0$ *for each* $x \in H$ *and* $y \in B^{-1}(0)$.

Lemma 2.6.14. *[223] Let C be a nonempty, closed and convex subset of a real reflexive Banach space E and let $f : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). For all $\lambda > 0$ be any given number and $x \in E$, there exists $z \in C$ such that*

$$g(z, y) + \frac{1}{r}\langle \nabla(z) - \nabla(x), y - z\rangle \geq 0, \ \forall \ y \in C. \tag{2.6.6}$$

Define the resolvent mapping $T_r : E \to 2^C$ as follows

$$Res_{\lambda,g}^f(x) = \{z \in C : g(z, y) + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0, \ \forall \ y \in C\}, \tag{2.6.7}$$

then, $Res_{\lambda,g}^f$ has the following properties:

1. $Res_{\lambda,g}^f$ *is single-valued;*

2. $Res_{\lambda,g}^f$ *is a firmly nonexpansive mapping, that is;*

$$\begin{aligned}\langle Res_{\lambda,g}^f z - Res_{\lambda,g}^f y, &\nabla f(Res_{\lambda,g}^f z) - \nabla f(Res_{\lambda,g}^f y)\rangle \\ &\leq \langle Res_{\lambda,g}^f z - Res_{\lambda,g}^f y, \nabla f(z) - \nabla f(y)\rangle \ \forall z, y \in E;\end{aligned} \tag{2.6.8}$$

3. $F(Res_{\lambda,g}^f) = \Omega_{EP(g)}$;

4. $\Omega_{EP(g)}$ *is closed and convex.*

It is easy to see that the resolvent operator satisfies the following inequality: for all $r > 0$, $u \in EP(g)$ and $x \in E$, then

$$D_f(x, Res_{\lambda,g}^f x) + D_f(Res_{\lambda,g}^f x, u) \leq D_f(x, u). \tag{2.6.9}$$

Lemma 2.6.15. *[270] Let E be a uniformly smooth Banach space, C a closed nonempty subset of E, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ a contraction mapping. For each $t \in (0, 1)$, define $z_t = tf(z_t) + (1 - t)Tz_t$, then, $\{z_t\}$ converges strongly to the unique fixed point $\bar{x}$ of T as $t \to 0$, where $\bar{x} = Q_{F(T)}f(\bar{x})$ and $Q_{F(T)} : C \to F(T)$ is the sunny nonexpansive retraction from C onto $F(T)$.*

Lemma 2.6.16. *[181] Suppose $q > 1$. Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \Big(\frac{q-1}{q}\Big)b^{\frac{q}{q-1}},$$

for arbitrary positive real numbers a, b.

Lemma 2.6.17. *[75] Let E be a real Banach space. Then for all $x, y \in E$ and $j_q(x+y) \in J_q(x+y)$, the following inequality holds:*

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x+y)\rangle. \tag{2.6.10}$$

Lemma 2.6.18. *[25] Let $q > 1$ and E be a real Banach space. Then the following are equivalent:*

(i) E is q-uniformly smooth.

(ii) There exists a constant c_q (called the best q-uniformly smoothness constant) such that for all $x, y \in E$,

$$||x+y||^q \leq ||x||^q + q\langle y, J_q(x)\rangle + c_q||y||^q, \tag{2.6.11}$$

(iii) There exists a constant $d_q > 0$ such that for all $x, y \in E$ and $\alpha \in [0,1]$,

$$||(1-\alpha)x + \alpha y||^q \geq (1-\alpha)||x||^q + \alpha||y||^q - \omega_q(\alpha)d_q||x-y||^q, \tag{2.6.12}$$

where $\omega_q(\alpha) := \alpha^q(1-\alpha) + \alpha(1-\alpha)^q$.

Lemma 2.6.19. *[279] Let C be a nonempty closed and convex subset of a q-uniformly smooth real Banach space E. Let $T : C \to C$ be a λ-strict pseudocontraction. For $\gamma \in (0,1)$, define $S_\gamma x = (1-\gamma)x + \gamma Tx$. Then, as $\gamma \in (0, a)$, $a = \min\left\{1, (\frac{q\lambda}{c_q})^{\frac{1}{q-1}}\right\}$, $S_\gamma : C \to C$ is nonexpansive and $F(S_\gamma) = F(T)$.*

Lemma 2.6.20. *[278] Let E be a real Banach space and C be a nonempty closed convex subset of E. For each $1 \leq i \leq N$, let $T_i : C \to C$ be a λ_i-strict pseudocontraction for some $0 \leq \lambda_i < 1$. Assume $\{\eta_i\}$ is a sequence of positive numbers such that $\sum\limits_{i=1}^{N} \eta_i = 1$. Then $\sum\limits_{i=1}^{N} \eta_i T_i$ is a λ-strict pseudocontraction with $\lambda = \min\{\lambda_i : 1 \leq i < N\}$.*
If in addition $\{T_i\}_{i=1}^N$ has a common fixed point, then

$$F\left(\sum_{i=1}^{N} \eta_i T_i\right) = \cap_{i=1}^N F(T_i).$$

Lemma 2.6.21. [193] *Let $r > 0$ be a constant and let $f : E \to \mathbb{R}$ be a continuous uniformly convex function on bounded subsets of E. Then*

$$f\left(\sum_{k=0}^{\infty} \alpha_k x_k\right) \leq \sum_{k=0}^{\infty} \alpha_k f(x_k) - \alpha_i\alpha_j\rho_r(||x_i - x_j||), \tag{2.6.13}$$

for all $i, j \in \mathbb{N} \cup \{0\}$, $x_k \in B_r$, $\alpha_k \in (0,1)$ and $k \in \mathbb{N} \cup 0$ with $\sum_{k=0}^{\infty} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Lemma 2.6.22. [203] *Let E be a real reflexive Banach space, let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper semicontinuous function, then $f^* : E^* \to \mathbb{R} \cup \{+\infty\}$ is a proper weak* lower semicontinuous and convex function. Thus, for all $z \in E$, one has*

$$D_f\Big(z, \nabla f^*(\sum_{i=1}^{N} t_i \nabla f(x_i))\Big) \leq \sum_{i=1}^{N} t_i D_f(z, x_i). \tag{2.6.14}$$

Lemma 2.6.23. [244] *Let E be a reflexive Banach space, let $f : E \to \mathbb{R}$ be a strong coercive Bregman function and $V_f : E \times E^* \to [0, +\infty)$ be defined by*

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad x \in E, \quad x^* \in E^*, \tag{2.6.15}$$

then the following assertions hold:

(i) $D_f(x, \nabla f(x^*)) = V_f(x, x^*)$ *for all* $x \in E$ *and* $x^* \in E^*$,

(ii) $V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*)$ *for all* $x \in E$ *and* x^*, $y^* \in E^*$.

Lemma 2.6.24. [59] *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex function whose domain contains at-least two points. Then the following statements holds:*

a. f is sequentially consistent if and only if it is totally convex on bounded subsets.

b. If f is lower semicontinuous, then f is sequential consistent if and only if it is uniformly convex on bounded subsets.

c. If f is uniformly strictly convex on bounded subsets, then it is sequentially consistent and the converse implication holds when f is lower semicontinuous, Fréchet differentiable on its domain, and the Fréchet derivative ∇f is uniformly continuous on bounded subsets.

Remark 2.6.25. [216] If f is Fréchet differentiable and totally convex, then f is cofinite.

Lemma 2.6.26. [213] *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.6.27. *[216] Let $A : E \to 2^{E^*}$ be a maximal monotone operator such that $A^{-1}(0) \neq \emptyset$ and the resolvent operator is defined by $Res_A^f := (\nabla f + A)^{-1} \circ \nabla f$. Then, for all $x \in E$ and $q \in A^{-1}(0)$, we have*

$$D_f(q, Res_{rA}^f x) + D_f(Res_{rA}^f x, x) \leq D_f(q, x).$$

Lemma 2.6.28. [216] *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.6.29 (see Lemma 2.1 of [267])**.** *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that *(i)* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n \alpha_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6.30 (see Lemma 2.5 of [269]). *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 1,$$

where

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$,

(iii) $\gamma_n \geq 0$, $(n \geq 1)$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then, $a_n \to 0$ *as* $n \to \infty$.

Lemma 2.6.31 (see Lemma 3.1 of [169]). *Let* $\{\alpha_n\}$ *and* $\{\gamma_n\}$ *be sequences of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\delta_n)\alpha_n + \beta_n + \gamma_n, \qquad n \geq 1,$$

where $\{\delta_n\}$ *is a sequence in* $(0,1)$ *and* $\{\beta_n\}$ *is a real sequence. Assume that* $\sum_{n=0}^{\infty} \beta_n < \infty$. *Then, the following results hold:*

(i) If $\beta_n \leq \delta_n M$ *for some* $M \geq 0$, *then* $\{\alpha_n\}$ *is a bounded sequence.*

(ii) If $\sum_{n=0}^{\infty} \delta_n = \infty$ *and* $\limsup_{n\to\infty} \frac{\beta_n}{\delta_n} \leq 0$, *then* $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.6.32 (see Lemma 2 of [207]). *Let* $\{v_n\}$ *and* $\{\delta_n\}$ *be nonnegative sequences of real numbers satisfying*

$$v_{n+1} \leq v_n + \delta_n$$

with $\sum_{n=1}^{\infty} \delta_n < +\infty$. *Then, the sequence* $\{v_n\}$ *is convergent.*

Lemma 2.6.33 (see Lemma 1.3 of [228]). *Let* H *be real Hilbert space,* $\{a_n\}$ *be a sequence of real numbers such that* $0 < a < a_n < b < 1$ *for all* $n \geq 1$ *and* $\{v_n\}, \{w_n\}$ *be the sequences in* H *such that*

$$\limsup_{n\to\infty} ||v_n|| \leq c, \quad \limsup_{n\to\infty} ||w_n|| \leq c, \tag{2.6.16}$$

and for some $c > 0$,

$$\limsup_{n\to\infty} ||a_n v_n + (1-a_n)w_n|| = c.$$

Then $\lim_{n\to\infty} ||v_n - w_n|| = 0$.

Lemma 2.6.34 (see Lemma 3.1 of [171]). *Let $\{\Gamma_n\}$ be a sequence of real numbers such that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$ with $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a non-decreasing sequence verifying $\lim_{n\to\infty} \tau(n) = \infty$, and for all $n \geq n_0$, the following estimates hold:

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \text{and} \quad \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

Let l^∞ be the Banach lattice of bounded real sequences with the supremum norm. It is well known that there exists a bounded linear functional μ on l^∞ such that the following three conditions hold:

1. if $\{t_n\}$ in l^∞ and $t_n \geq 0$ for every $n \in \mathbb{N}$, then $\mu(\{t_n\}) \geq 0$,
2. if $t_n = 1$ for every $n \in \mathbb{N}$, then $\mu(\{t_n\}) = 1$,
3. $\mu(\{t_{n+1}\}) = \mu(\{t_n\})$ for all $\{t_n\}$ in l^∞.

Here, $\{t_{n+1}\}$ denotes the sequence $(t_2, t_3, \dots, t_n, t_{n+1}, \dots)$ in l^∞. Such a functional μ is called a Banach limit and the value of μ at $\{t_n\}$ in l^∞ is denoted by $\mu_n t_n$. Therefore, condition (3) means $\mu_n t_n = \mu_n t_{n+1}$. If μ satisfies conditions (1) and (2), we call μ a mean on l^∞ (see, for example, [244] for more details).

Lemma 2.6.35. *[132] Let C be a nonempty, closed and convex subset of a real reflexive Banach space E. Let $f : E \to \mathbb{R}$ be strictly convex, continuous, strongly coercive, Gâteaux differentiable, bounded and locally uniformly convex on E. Let $T : C \to C$ be a mapping. Let $\{x_n\}$ be a bounded sequence of C and μ be a mean on l^∞. Suppose that*

$$\mu_n D_f(x_n, Ty) \leq \mu_n D_f(x_n, y) \quad \forall y \in C.$$

Then T has a fixed point in C.

CHAPTER 3

Inertial Algorithms and Optimization Problems

3.1 A Viscosity-Proximal Gradient Method with Inertial Extrapolation for Solving Minimization Problems in Hilbert Space

Consider the following Minimization Problem (shortly, MP)

$$minimize \ \{g(x) + h(x)\}, \tag{3.1.1}$$

where $h : H \to \mathbb{R} \cup \{+\infty\}$ is a proper, closed and convex function which is possibly nonsmooth and $g : H \to \mathbb{R}$ is a proper, closed, convex and continuously differentiable function with gradient $\nabla g(\cdot)$ which is Lipschitz continuous on H, i.e. there exists a constant $\alpha > 0$ such that

$$||\nabla g(x) - \nabla g(y)|| \leq \alpha ||x - y||, \quad \forall x, y \in H.$$

We shall assume that Problem (3.1.1) has a solution and denotes its set of solution by Ω_{MP}. One of the methods for approximating solutions of (3.1.1) is the Proximal Gradient Method (PGM) which is given as follows: pick an initial point $x_1 \in H$ and compute

$$x_{n+1} = prox_{\gamma_n h}(x_n - \gamma_n \nabla g(x_n)), \quad n \geq 1, \tag{3.1.2}$$

where $\gamma_n > 0$ is a stepsize. When $g \equiv 0$ in (3.1.2), the PGM reduces to the *classical proximal point algorithm*. The PGM can be shown to converge with rate $O(\frac{1}{k})$ when a fixed stepsize $\gamma_n = \gamma \in (0, \frac{1}{\alpha}]$ is used (see [199, 90]). If α is unknown, the stepsize γ_n can be found by using the line searching technique (see [25]). More so, if the condition

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \frac{2}{\alpha}$$

is satisfied, then the sequence $\{x_n\}$ generated by (3.1.2) converges weakly to a solution in Ω_{MP}. The PGM can as well be interpreted as a fixed point iteration. A point x^* is a solution of (3.1.1) if and only if it is a fixed point of the operator $prox_{\gamma h}(I - \gamma\nabla g)$ (see Section 4.2.1 in [199] and Proposition 3.2 in [269]).

When $h = I_C$ (the indicator function on a nonempty closed convex subset of H), the PGM reduces to the well known *Gradient Projection Algorithm (GPA)* which is defined as follows: for an initial guess $x_1 \in H$,

$$x_{n+1} = P_C(x_n - \gamma_n \nabla g(x_n)), \quad n \geq 1. \tag{3.1.3}$$

The convergence of algorithm (3.1.3) depends on the behaviour of the gradient ∇g. It is known that if ∇g is ν-strongly monotone operator, i.e. there exists $\nu > 0$ such that

$$\langle \nabla g(x) - \nabla g(y), x - y \rangle \geq \nu ||x - y||^2, \quad \forall \ x, y \in C,$$

then, the operator $P_C(I - \gamma\nabla g)$ is a contraction; hence, the sequence $\{x_n\}$ defined by GPA (3.1.3) converges strongly to a solution of (3.1.1). More general, if the sequence $\{\gamma_n\}$ is chosen to satisfy the property

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \frac{2\nu}{\alpha^2},$$

then the sequence $\{x_n\}$ defined by (3.1.3) converges in norm to the unique solution of (3.1.1). However, if the gradient ∇g fails to be strongly monotone, then the operator $P_C(I - \gamma\nabla g)$ would fail to be a contraction. Consequently, the sequence $\{x_n\}$ generated by (3.1.3) may fail to converge strongly (see Section 4 in [266]).

Recently, Xu [266] gave an alternative operator-oriented approach to the GPA (3.1.3). He also constructed a counter-example which shows that the GPA does not converge in norm in an infinite-dimensional space. He however, presented two modifications of the GPA which are shown to have strong convergence. Very recently, motivated by the work of Xu [266], Ceng et al. [62] proposed the following implicit algorithm:

$$x_k = P_C(\alpha_k \gamma V x_k + (1 - \alpha_k \mu B)T_k x_k),$$

and explicit formula

$$x_{n+1} = P_C(\alpha_n \gamma V x_n + (1 - \alpha\mu B)T_n x_n)$$

for finding the approximate minimizer of a constrained convex minimization problem and prove that the sequence generated by their algorithms converge strongly to a solution of the constrained convex minimization problem (see [62] for more details).

Also, Chembolle and Dossel [72] proved the weak convergence of the following modified PGM with inertial extrapolation term in a real Hilbert space

$$\begin{cases} x_n = T(y_{n-1}), \\ y_n = \left(1 - \frac{1}{t_{n+1}}\right)x_n + \frac{1}{t_{n+1}}u_n, \\ u_n = x_{n-1} + t_n(x_n - x_{n-1}), \qquad n \geq 1, \end{cases} \tag{3.1.4}$$

equivalently, (3.1.4) can be written as

$$\begin{aligned} x_n &= T(y_{n-1}), \\ y_n &= x_n + \alpha_n(x_n - x_{n-1}), \qquad \alpha_n = \frac{t_{n-1}}{t_{n+1}}, \quad \text{for} \quad n \geq 1, \end{aligned}$$

where $a > 2$ is a positive real number, $t_n = \dfrac{n+a-1}{a}$ for all $n \in \mathbb{N}$ and $Tx = prox_{\gamma h}(x - \gamma \nabla g(x))$.

Motivated by the works of Xu [266], Ceng et al. [62], Chembolle and Dossel [72], in Section 3.1, we present a viscosity-inertial proximal gradient algorithm for finding approximate solution of the convex minimization problem (3.1.1) in a real Hilbert space. We also establish a strong convergence theorem and provide some applications and numerical examples to show the relevance of our results in this section.

3.1.1 Main results

First, we prove the following lemma which plays a crucial role in the proof of the main theorem.

Lemma 3.1.1. *Assume that the minimization problem* (3.1.1) *is consistent and gradient* ∇g *is Lipschitz continuous with Lipschitz constant* $L > 0$. *Let* $\gamma > 0$ *such that* $0 < \gamma < \frac{2}{L}$, *then the following inequality holds:*

$$||prox_{\gamma h}(I - \gamma\nabla g)x - x||^2 \leq 2\langle x - y, x - prox_{\gamma h}(I - \gamma \nabla g)x\rangle, \tag{3.1.5}$$

for all $x \in C$ *and* $y \in \Omega_{MP}$.

Proof. Since $prox_{\gamma h}$ is firmly nonexpansive, then it is $\frac{1}{2}$-averaged. Also, the Lipschitz condition on ∇g implies that ∇g is $\frac{1}{L}$-ism and by Lemma 2.1.9(ii), $\gamma\nabla g$ is $\frac{1}{\gamma L}$-ism. Hence, by Lemma 2.1.9(iii), we have that $I - \gamma \nabla g$ is $\frac{\gamma L}{2}$-averaged. It follows from Lemma 2.1.8(ii) that the $prox_{\gamma h}(I - \gamma\nabla g)$ is averaged with constant $\frac{2+\gamma L}{4}$. In particular, $prox_{\gamma h}(I - \gamma \nabla g)$ is nonexpansive. Then, for any $x \in C$ and $y \in \Omega_{MP}$, we have

$$\begin{aligned} ||prox_{\gamma h}(I - \gamma\nabla g)x - y||^2 &= ||prox_{\gamma h}(I - \gamma\nabla g)x - prox_{\gamma h}(I - \gamma\nabla g)y||^2 \\ &\leq ||x - y||^2 \\ &= \langle x - y, x - prox_{\gamma h}(I - \gamma\nabla g)x + prox_{\gamma h}(I - \gamma\nabla g)x - y\rangle \\ &= \langle x - y, x - prox_{\gamma h}(I - \gamma\nabla g)x\rangle \\ &\quad + \langle x - y, prox_{\gamma h}(I - \gamma\nabla g)x - y\rangle. \end{aligned}$$

This implies that

$$\langle prox_{\gamma h}(I - \gamma\nabla g)x - x, prox_{\gamma h}(I - \gamma\nabla g)x - y\rangle \leq \langle x - y, x - prox_{\gamma h}(I - \gamma\nabla g)x\rangle.$$

Thus

$$\langle prox_{\gamma h}(I - \gamma\nabla g)x - x, prox_{\gamma h}(I - \gamma\nabla g)x - x + x - y\rangle \leq \langle x - y, x - prox_{\gamma h}(I - \gamma\nabla g)x\rangle,$$

which gives that

$$\begin{aligned}\langle prox_{\gamma h}(I-\gamma\nabla g)x-x, prox_{\gamma h}(I-\gamma\nabla g)x-x\rangle &\leq \langle x-y, x-prox_{\gamma h}(I-\gamma\nabla g)x\rangle \\ &\quad +\langle x-y, x-prox_{\gamma h}(I-\gamma\nabla g)x\rangle,\end{aligned}$$

therefore

$$||prox_{\gamma h}(I-\gamma\nabla g)x-x||^2 \leq 2\langle x-y, x-prox_{\gamma h}(I-\gamma\nabla g)x\rangle.$$

□

Next, we present our iterative algorithm and prove its strong convergence to the solution of MP (3.1.1).

Theorem 3.1.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $g, h : H \to \mathbb{R}\cup\{+\infty\}$ be two proper convex lower semicontinuous functions such that h is nonsmooth and the gradient ∇g is $\frac{1}{L}$-ism with $L > 0$. Let $f : C \to C$ be a Meir Keeler contraction mapping, $B : C \to H$ be a strongly positive bounded linear operator with coefficient $\tau > 0$ such that $0 < \xi < \frac{\tau}{2}$ and $T : C \to C$ be a δ-demimetric mapping for $\delta \in (-\infty, 1)$ and $\hat{F}(T) = F(T)$. Suppose $\Gamma = \Omega_{MP} \cap F(T) \neq \emptyset$, let $\alpha_n \in [0,1]$, $\beta_n \in [0,1)$, $w_n, \theta_n \in (0,1)$ and $\gamma_n > 0$. Choose initial points $x_0, x_1 \in H$ arbitrarily and let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be generated by*

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}), \\ u_n = (1-w_n)y_n + w_n prox_{\gamma_n h}(y_n - \gamma_n \nabla g(y_n)), \\ x_{n+1} = P_C\Big(\alpha_n \xi f(x_n) + \theta_n x_n + ((1-\theta_n)I - \alpha_n B)T_{\lambda_n} u_n\Big), \quad n \geq 1, \end{cases} \tag{3.1.6}$$

where $T_{\lambda_n} = (1-\lambda_n)I + \lambda_n T$ for $\lambda_n \in (0,1)$. Assume that the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n}||x_n - x_{n-1}|| = 0$,

(C3) $0 < \liminf_{n\to\infty} w_n \leq \limsup_{n\to\infty} w_n < 1$,

(C4) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \frac{2}{L}$,

(C5) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1-\delta$.

Then, $\{x_n\}$ converges strongly to a point $\bar{x}$, where $\bar{x} = P_\Gamma(I - B + \xi f)(\bar{x})$ is the unique solution of the variational inequality

$$\langle (B - \xi f)\bar{x}, \bar{x} - y\rangle \leq 0, \quad y \in \Gamma. \tag{3.1.7}$$

Proof. Firstly, we show that $\{x_n\}$ is bounded. Let $\varepsilon > 0$ and $x^* \in \Gamma$, since f is a Meir-Keeler contraction, there exists $\rho_\varepsilon \in (0,1)$ (by Lemma 2.1.12) such that

$$||f(x_n) - f(x^*)|| \leq \rho_\varepsilon ||x_n - x^*||. \tag{3.1.8}$$

From (3.1.6), we have

$$\begin{aligned} ||y_n - x^*|| &= ||x_n - x^* + \beta_n(x_n - x_{n-1})|| \\ &\leq ||x_n - x^*|| + \beta_n ||x_n - x_{n-1}||. \end{aligned} \tag{3.1.9}$$

Also

$$\begin{aligned} ||u_n - x^*||^2 &= ||(1-w_n)y_n + w_n prox_{\gamma_n h}(I - \gamma_n \nabla g)y_n - x^*||^2 \\ &= ||(y_n - x^*) + w_n(prox_{\gamma_n h}(I - \gamma_n \nabla g)y_n - y_n)||^2 \\ &= ||y_n - x^*||^2 + 2w_n\langle y_n - x^*, prox_{\gamma_n h}(I - \gamma_n \nabla g)y_n - y_n\rangle \\ &\quad + w_n^2 ||prox_{\gamma_n h}(I - \gamma_n \nabla g)y_n - y_n||^2. \end{aligned} \tag{3.1.10}$$

Using Lemma 3.1.1, we have that

$$\begin{aligned} ||u_n - x^*||^2 &\leq ||y_n - x^*||^2 - w_n(1-w_n)||prox_{\gamma_n h}(I - \gamma_n \nabla g)y_n - y_n||^2 \\ &\leq ||y_n - x^*||^2. \end{aligned} \tag{3.1.11}$$

Moreover, from the definition of δ-demimetric maps (2.1.15), we have

$$\begin{aligned} ||T_{\lambda_n} u_n - x^*||^2 &= ||(u_n - x^*) + \lambda_n(Tu_n - u_n)||^2 \\ &= ||u_n - x^*||^2 - 2\lambda_n\langle u_n - x^*, u_n - Tu_n\rangle + \lambda_n^2 ||u_n - Tu_n||^2 \\ &\leq ||u_n - x^*||^2 - \lambda_n(1-\delta)||u_n - Tu_n||^2 + \lambda_n^2||u_n - Tu_n||^2 \\ &= ||u_n - x^*||^2 - \lambda_n(1 - \delta - \lambda_n)||u_n - Tu_n||^2, \end{aligned} \tag{3.1.12}$$

and by condition (C5), we get

$$||T_{\lambda_n} u_n - x^*||^2 \leq ||u_n - x^*||^2. \tag{3.1.13}$$

Thus, we have from (3.1.6) that

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||P_C(\alpha_n \xi f(x_n) + \theta_n x_n + ((1-\theta_n)I - \alpha_n B)T_{\lambda_n} u_n) - P_C x^*|| \\ &\leq ||\alpha_n \xi f(x_n) + \theta_n x_n + ((1-\theta_n)I - \alpha_n B)T_{\lambda_n} u_n - x^*|| \\ &= ||\alpha_n(\xi f(x_n) - Bx^*) + \theta_n(x_n - x^*) + ((1-\theta_n)I - \alpha_n B)(T_{\lambda_n} u_n - x^*)|| \\ &\leq \alpha_n(\xi||f(x_n) - f(x^*)|| + ||\xi f(x^*) - Bx^*||) + \theta_n||x_n - x^*|| \\ &\quad + ((1-\theta_n)I - \alpha_n\tau)||T_{\lambda_n} u_n - x^*|| \\ &\leq \alpha_n \xi \rho_\varepsilon ||x_n - x^*|| + \alpha_n||\xi f(x^*) - Bx^*|| + \theta_n||x_n - x^*|| \\ &\quad + ((1-\theta_n)I - \alpha_n\tau)||u_n - x^*|| \\ &\leq \alpha_n \xi \rho_\varepsilon ||x_n - x^*|| + \alpha_n||\xi f(x^*) - Bx^*|| + \theta_n||x_n - x^*|| \\ &\quad + ((1-\theta_n)I - \alpha_n\tau)[||x_n - x^*|| + \beta_n||x_n - x_{n-1}||] \\ &= (1 - \alpha_n(\tau - \xi\rho_\varepsilon))||x_n - x^*|| + \alpha_n||\xi f(x^*) - Bx^*|| \\ &\quad + ((1-\theta_n)I - \alpha_n\tau)\beta_n||x_n - x_{n-1}|| \\ &= (1 - \alpha_n(\tau - \xi\rho_\varepsilon))||x_n - x^*|| + \alpha_n(\tau - \xi\rho_\varepsilon)\Big\{\frac{\xi f(x^*) - Bx^*||}{\tau - \xi\rho_\varepsilon} + \\ &\quad \frac{((1-\theta_n)I - \alpha_n\tau)\beta_n||x_n - x_{n-1}||}{\alpha_n(\tau - \xi\rho_\varepsilon)}\Big\}. \end{aligned} \tag{3.1.14}$$

Putting

$$\sigma_n = \Big(\frac{(1-\theta_n)I - \alpha_n\tau}{\tau - \xi\rho_\varepsilon}\Big)\frac{\beta_n}{\alpha_n}||x_n - x_{n-1}||,$$

from condition (C2), it is easy to see that $\lim_{n\to\infty}\sigma_n = 0$, which implies that the sequence $\{\sigma_n\}$ is bounded. Let

$$M = \max\Big\{\frac{||\xi f(x^*) - Bx^*||}{\tau - \xi\rho_\varepsilon}, \sup_{n\in\mathbb{N}}\sigma_n\Big\},$$

by using Lemma 2.6.31(i) and (3.1.14), we have that the sequence $\{||x_n - x^*||\}$ is bounded. This shows that $\{x_n\}$ is bounded and consequently, $\{u_n\}$ and $\{y_n\}$ are bounded.

Note that

$$\begin{aligned} ||y_n - x^*||^2 &= ||x_n - x^* + \beta_n(x_n - x_{n-1})||^2 \\ &= ||x_n - x^*|| + 2\beta_n\langle x_n - x^*, x_n - x_{n-1}\rangle + \beta_n^2||x_n - x_{n-1}||^2. \end{aligned} \quad (3.1.15)$$

From Lemma 2.6.1(ii), we have

$$2\langle x_n - x^*, x_n - x_{n-1}\rangle = ||x_n - x^*||^2 + ||x_n - x_{n-1}||^2 - ||x_{n-1} - x^*||^2, \quad (3.1.16)$$

substituting (3.1.16) into (3.1.15), we get

$$\begin{aligned} ||y_n - x^*||^2 &= ||x_n - x^*||^2 + \beta_n\Big[||x_n - x^*||^2 + ||x_n - x_{n-1}||^2 - ||x_{n-1} - x^*||^2\Big] \\ &\quad + \beta_n^2||x_n - x_{n-1}||^2 \\ &\leq ||x_n - x^*||^2 + \beta_n[||x_n - x^*||^2 - ||x_{n-1} - x^*||^2] \\ &\quad + 2\beta_n||x_n - x_{n-1}||^2. \end{aligned} \quad (3.1.17)$$

Now, put $m_n = \alpha_n\xi f(x_n) + \theta_n x_n + ((1-\theta_n)I - \alpha_n B)T_{\lambda_n}u_n$, using Lemma 2.6.1(i) and (3.1.6), we have

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq ||\alpha_n\xi f(x_n) + \theta_n x_n + ((1-\theta_n)I - \alpha_n B)T_{\lambda_n}u_n - x^*||^2 \\ &= ||\alpha_n(\xi f(x_n) - Bx^*) + \theta_n(x_n - x^*) + ((1-\theta_n)I - \alpha_n B)(T_{\lambda_n}u_n - x^*)||^2 \\ &\leq ||((1-\theta_n)I - \alpha_n B)(T_{\lambda_n}u_n - x^*) + \theta_n(x_n - x^*)||^2 \\ &\quad + 2\alpha_n\langle\xi f(x_n) - Bx^*, m_n - x^*\rangle \\ &= ||((1-\theta_n)I - \alpha_n B)(T_{\lambda_n}u_n - x^*)||^2 + \theta_n^2||x_n - x^*||^2 \\ &\quad + 2\theta_n\Big\langle((1-\theta_n)I - \alpha_n B)(T_{\lambda_n}u_n - x^*), x_n - x^*\Big\rangle \\ &\quad + 2\alpha_n\langle\xi f(x_n) - Bx^*, m_n - x^*\rangle \\ &\leq ((1-\theta_n)I - \alpha_n\tau)^2||T_{\lambda_n}u_n - x^*||^2 + \theta_n^2||x_n - x^*||^2 \\ &\quad + 2\theta_n((1-\theta_n)I - \alpha_n\tau)||T_{\lambda_n}u_n - x^*||||x_n - x^*|| \\ &\quad + 2\alpha_n\langle\xi f(x_n) - Bx^*, m_n - x^*\rangle \\ &\leq ((1-\theta_n)I - \alpha_n\tau)^2||T_{\lambda_n}u_n - x^*||^2 + \theta_n^2||x_n - x^*||^2 \\ &\quad + \theta_n((1-\theta_n)I - \alpha_n\tau)[||T_{\lambda_n}u_n - x^*||^2 + ||x_n - x^*||^2] \\ &\quad + 2\alpha_n\langle\xi f(x_n) - Bx^*, m_n - x^*\rangle \\ &\leq ((1-\theta_n)I - \alpha_n\tau)||T_{\lambda_n}u_n - x^*||^2 + \theta_n||x_n - x^*||^2 \\ &\quad + 2\alpha_n\langle\xi f(x_n) - Bx^*, m_n - x^*\rangle. \end{aligned} \quad (3.1.18)$$

Thus, from (3.1.13) and (3.1.17), we have

$$\begin{aligned}
||x_{n+1}-x^*||^2 &\leq ((1-\theta_n)I-\alpha_n\tau)[||u_n-x^*||^2-\lambda_n(1-\lambda_n-\delta)||u_n-Tu_n||^2] \\
&\quad +\theta_n||x_n-x^*||^2+2\alpha_n\langle \xi f(x_n)-Bx^*, m_n-x^*\rangle \\
&\leq ((1-\theta_n)I-\alpha_n\tau)\Big\{||x_n-x^*||^2+\beta_n[||x_n-x^*||^2-||x_{n-1}-x^*||^2] \\
&\quad +2\beta_n||x_n-x_{n-1}||^2\Big\}-\lambda_n(1-\lambda_n-\delta)||u_n-Tu_n||^2+\theta_n||x_n-x^*||^2 \\
&\quad +2\alpha_n\langle \xi f(x_n)-Bx^*, m_n-x^*\rangle \\
&\leq (1-\alpha_n\tau)||x_n-x^*||^2+\beta_n[||x_n-x^*||^2-||x_{n-1}-x^*||^2] \\
&\quad +2\beta_n||x_n-x_{n-1}||^2-\lambda_n(1-\lambda_n-\delta)||u_n-Tu_n||^2 \\
&\quad +2\alpha_n\langle \xi f(x_n)-Bx^*, m_n-x^*\rangle. \qquad (3.1.19)
\end{aligned}$$

Now we set $D_n=||x_n-x^*||^2$ and consider the following two cases.
<u>Case 1:</u> Suppose there exists a natural number N such that $D_{n+1}\leq D_n$ for all $n\geq \mathbb{N}$. In this case, $\{D_n\}$ is convergent. Since $\{x_n\}$ is bounded, it is easy to see that condition (C2) implies $\beta_n||x_n-x_{n-1}||\to 0$.

From (3.1.19), we have

$$\begin{aligned}
&\lambda_n(1-\lambda_n-\delta)||u_n-Tu_n||^2 \\
&\qquad \leq (1-\alpha_n\tau)||x_n-x^*||^2-||x_{n+1}-x^*||^2+\beta_n[||x_n-x^*||^2-||x_{n-1}-x^*||^2] \\
&\qquad + 2\beta_n||x_n-x_{n-1}||^2+2\alpha_n\langle \xi f(x_n)-Bx^*, m_n-x^*\rangle \\
&\qquad = (D_n-D_{n+1})+\beta_n(D_n-D_{n-1})+2\beta_n||x_n-x_{n-1}||^2 \\
&\qquad -\alpha_n\tau D_n+2\alpha_n\langle \xi f(x_n)-Bx^*, m_n-x^*\rangle.
\end{aligned}$$

Since $\{D_n\}$ is convergent and $\alpha_n\to 0$, we have

$$\lim_{n\to\infty}\lambda_n(1-\lambda_n-\delta)||u_n-Tu_n||^2=0,$$

using condition (C5), we obtain

$$\lim_{n\to\infty}||u_n-Tu_n||=0. \qquad (3.1.20)$$

This implies that

$$\begin{aligned}
\lim_{n\to\infty}||T_{\lambda_n}u_n-u_n|| &= \lim_{n\to\infty}||(1-\lambda_n)u_n+\lambda_nTu_n-u_n|| \\
&= \lim_{n\to\infty}\lambda_n||u_n-Tu_n||=0. \qquad (3.1.21)
\end{aligned}$$

Also, from (3.1.11) and (3.1.19), we see that

$$\begin{aligned}
||x_{n+1}-x^*||^2 &\leq ((1-\theta_n)I-\alpha_n\tau)[||u_n-x^*||^2-\lambda_n(1-\lambda_n-\delta)||u_n-Tu_n||^2] \\
&\quad +\theta_n||x_n-x^*||^2+2\alpha_n\langle \xi f(x_n)-Bx^*, m_n-x^*\rangle \\
&\leq ((1-\theta_n)I-\alpha_n\tau)||u_n-x^*||^2+\theta_n||x_n-x^*||^2 \\
&\quad +2\alpha_n\langle \xi f(x_n)-Bx^*, m_n-x^*\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq ((1-\theta_n)I-\alpha_n\tau)[||y_n-x^*||^2-w_n(1-w_n)||prox_{\gamma_n h}(I-\gamma_n\nabla g)y_n-y_n||^2] \\
&\quad +\theta_n||x_n-x^*||^2+2\alpha_n\langle \xi f(x_n)-Bx^*, m_n-x^*\rangle \\
&\leq ((1-\theta_n)I-\alpha_n\tau)\Big\{||x_n-x^*||^2+\beta_n[||x_n-x^*||^2-||x_{n-1}-x^*||^2]+2\beta_n||x_n-x_{n-1}||^2\Big\} \\
&\quad -w_n(1-w_n)||prox_{\gamma_n h}(I-\gamma_n\nabla g)y_n-y_n||^2+\theta_n||x_n-x^*||^2 \\
&\quad +2\alpha_n\langle \xi f(x_n)-Bx^*, m_n-x^*\rangle \\
&\leq (1-\alpha_n\tau)||x_n-x^*||^2+\beta_n[||x_n-x^*||^2-||x_{n-1}-x^*||^2]+2\beta_n||x_n-x_{n-1}||^2 \\
&\quad -w_n(1-w_n)||prox_{\gamma_n h}(I-\gamma_n\nabla g)y_n-y_n||^2+2\alpha_n\langle \xi f(x_n)-Bx^*, m_n-x^*\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&w_n(1-w_n)||prox_{\gamma_n h}(I-\gamma_n\nabla g)y_n-y_n||^2 \\
&\qquad\leq (1-\alpha_n\tau)||x_n-x^*||^2-||x_{n+1}-x^*||^2+\beta_n[||x_n-x^*||^2-||x_{n-1}-x^*||^2] \\
&\qquad\quad +2\beta_n||x_n-x_{n-1}||^2+2\alpha_n\langle \xi f(x_n)-Bx^*, m_n-x^*\rangle \\
&\qquad = (D_n-D_{n+1})+\beta_n(D_n-D_{n-1})+2\beta_n||x_n-x_{n-1}||^2-\alpha_n\tau||x_n-x^*||^2 \\
&\qquad\quad +2\alpha_n\langle \xi f(x_n)-Bx^*, m_n-x^*\rangle.
\end{aligned}$$

Since $\{D_n\}$ is convergent and $\alpha_n \to 0$, we have that

$$\lim_{n\to\infty} w_n(1-w_n)||prox_{\gamma_n h}(I-\gamma_n\nabla g)y_n-y_n||^2=0,$$

and by using condition (C3), we obtain

$$\lim_{n\to\infty}||prox_{\gamma_n h}(I-\gamma_n\nabla g)y_n-y_n||=0. \tag{3.1.22}$$

Clearly

$$||y_n-x_n||\leq \beta_n||x_n-x_{n-1}||\to 0, \quad \text{as } n\to\infty, \tag{3.1.23}$$

and

$$||u_n-y_n||\leq w_n||prox_{\gamma_n h}(I-\gamma_n\nabla g)y_n-y_n||\to 0, \quad \text{as } n\to\infty,$$

hence

$$\lim_{n\to\infty}||u_n-x_n||\leq \lim_{n\to\infty}(||u_n-y_n||+||y_n-x_n||)=0. \tag{3.1.24}$$

Also from (3.1.6), we have

$$\begin{aligned}
\lim_{n\to\infty}||m_n-u_n|| &\leq \lim_{n\to\infty}\Big(\alpha_n||\xi f(x_n)-Bu_n||+\theta_n||x_n-u_n|| \\
&\qquad +((1-\theta_n)I-\alpha_n\tau)||T_{\lambda_n}u_n-u_n||\Big) \\
&= 0,
\end{aligned}$$

then from (3.1.24), we have

$$||m_n-x_n||\leq ||m_n-u_n||+||u_n-x_n||\to 0, \quad \text{as } n\to\infty. \tag{3.1.25}$$

More so, by the firmly nonexpansivity of the P_C and (2.2.3), we have that

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||P_C m_n - P_C x^*||^2 \\ &\leq ||m_n - x^*||^2 - ||P_C m_n - m_n||^2. \end{aligned} \tag{3.1.26}$$

Substituting (3.1.19) into (3.1.26), we get

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq (1-\alpha_n\tau)||x_n - x^*||^2 + \beta_n[||x_n - x^*||^2 - ||x_{n-1} - x^*||^2] \\ &\quad +2\beta_n||x_n - x_{n-1}||^2 + 2\alpha_n\langle \xi f(x_n) - Bx^*, m_n - x^*\rangle - ||P_C m_n - m_n||^2, \end{aligned}$$

therefore

$$\begin{aligned} ||P_C m_n - m_n||^2 &\leq (1-\alpha_n\tau)||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + \beta_n[||x_n - x^*||^2 \\ &\quad -||x_{n-1} - x^*||^2] + 2\beta_n||x_n - x_{n-1}||^2 + 2\alpha_n\langle \xi f(x_n) - Bx^*, m_n - x^*\rangle \\ &= (D_n - D_{n+1}) + \beta_n(D_n - D_{n-1}) - \alpha_n\tau||x_n - x^*||^2 + 2\beta_n||x_n - x_{n-1}||^2 \\ &\quad +2\alpha_n\langle \xi f(x_n) - Bx^*, m_n - x^*\rangle, \end{aligned}$$

then

$$\lim_{n\to\infty} ||P_C m_n - m_n|| = 0. \tag{3.1.27}$$

Thus, we have from (3.1.25) and (3.1.27) that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty}(||x_{n+1} - m_n|| + ||m_n - x_n||) = 0. \tag{3.1.28}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x} \in C$. It follows from (3.1.23) and (3.1.24) that $y_{n_j} \rightharpoonup \bar{x}$ and $u_{n_j} \rightharpoonup \bar{x}$ respectively. Since $prox_{\gamma_n h}(I - \gamma_n \nabla g)$ is nonexpansive and $\lim_{n\to\infty} ||y_n - prox_{\gamma_n h}(I - \gamma_n\nabla g)y_n|| = 0$, so by the Demiclosedness principle (Lemma 2.6.5), we have that $\bar{x} \in F(prox_{\gamma_n h}(I - \gamma_n\nabla g))$. Hence, $\bar{x}$ is a solution of the minimization problem (3.1.1), that is, $\bar{x} \in \Omega_{MP}$. Also, since $\lim_{n\to\infty} ||u_n - Tu_n|| = 0$ and $\hat{F}(T) = F(T)$, we have that $\bar{x} \in F(T)$. Therefore $\bar{x} \in \Gamma := \Omega_{MP} \cap F(T)$.

We now show that $\limsup_{n\to\infty}\langle (B - \xi f)z, z - x_{n+1}\rangle \leq 0$, where $z = P_\Gamma(I - B + \xi f)z$. Since $x_{n_j} \rightharpoonup \bar{x}$ and from (2.2.2), we have

$$\begin{aligned} \limsup_{n\to\infty}\langle (B - \xi f)z, z - x_{n+1}\rangle &= \lim_{j\to\infty}\langle (B - \xi f)z, z - x_{n_j+1}\rangle \\ &= \langle (B - \xi f)z, z - \bar{x}\rangle \leq 0. \end{aligned} \tag{3.1.29}$$

Next, we show that $x_n \to z$ as $n \to \infty$. From (2.2.2), (3.1.6) and (3.1.8), we have

$$\begin{aligned} ||x_{n_k+1} - z||^2 &= \langle P_C m_{n_k} - z, P_C m_{n_k} - z\rangle \\ &= \langle P_C m_{n_k} - m_{n_k} + m_{n_k} - z, P_C m_{n_k} - z\rangle \\ &= \langle P_C m_{n_k} - m_{n_k}, P_C m_{n_k} - z\rangle + \langle m_{n_k} - z, x_{n_k+1} - z\rangle \\ &\leq \langle m_{n_k} - z, x_{n_k+1} - z\rangle \end{aligned}$$

$$
\begin{aligned}
&= \langle \alpha_{n_k}\xi f(x_{n_k}) + \theta_n x_{n_k} + ((1-\theta_{n_k})I - \alpha_{n_k}B)T\lambda_{n_k}u_{n_k} - z, x_{n_k+1} - z\rangle \\
&= \alpha_{n_k}\langle \xi f(x_{n_k}) - \xi f(z), x_{n_k+1} - z\rangle + \alpha_{n_k}\langle \xi f(z) - B(z), x_{n_k+1} - z\rangle \\
&\quad +\theta_{n_k}\langle x_{n_k} - z, x_{n_k+1} - z\rangle + \langle ((1-\theta_{n_k})I - \alpha_{n_k}B)(T_{\lambda_{n_k}}u_{n_k} - z), x_{n_k+1} - z\rangle \\
&\leq \alpha_{n_k}\xi r_\epsilon ||x_{n_k} - z||||x_{n_k+1} - z|| + \theta_{n_k}||x_{n_k} - z||||x_{n_k+1} - z|| \\
&\quad +((1-\theta_{n_k})I - \alpha_{n_k}\tau)||T_{\lambda_n}u_{n_k} - z||||x_{n_k+1} - z|| \\
&\quad +\alpha_{n_k}\langle \xi f(z) - Bz, x_{n_k+1} - z\rangle \\
&\leq \alpha_{n_k}\xi r_\epsilon ||x_{n_k} - z||||x_{n_k+1} - z|| + \theta_{n_k}||x_{n_k} - z||||x_{n_k+1} - z|| \\
&\quad +((1-\theta_{n_k})I - \alpha_{n_k}\tau)||u_{n_k} - z||||x_{n_k+1} - z|| + \alpha_{n_k}\langle \xi f(z) - Bz, x_{n_k+1} - z\rangle \\
&\leq \alpha_{n_k}\xi r_\epsilon ||x_{n_k} - z||||x_{n_k+1} - z|| + \theta_{n_k}||x_{n_k} - z||||x_{n_k+1} - z|| \\
&\quad +((1-\theta_{n_k})I - \alpha_{n_k}\tau)[||x_{n_k} - z|| + \beta_n||x_{n_k} - x_{n_k-1}||]||x_{n_k+1} - z|| \\
&\quad +\alpha_n\langle \xi f(z) - Bz, x_{n_k+1} - z\rangle \\
&= (1 - \alpha_{n_k}(\tau - \xi r_\epsilon))||x_{n_k} - z||||x_{n_k+1} - z|| \\
&\quad +((1-\theta_{n_k})I - \alpha_{n_k}\tau)\beta_n||x_{n_k} - x_{n_k-1}||||x_{n_k+1} - z|| \\
&\quad +\alpha_{n_k}\langle \xi f(z) - Bz, x_{n_k+1} - z\rangle \\
&\leq (1 - \alpha_{n_k}(\tau - \xi r_\epsilon))\frac{1}{2}(||x_{n_k} - z||^2 + ||x_{n_k+1} - z||^2) \\
&\quad +((1-\theta_{n_k})I - \alpha_{n_k}\tau)\beta_{n_k}||x_{n_k} - x_{n_k-1}||||x_{n_k+1} - z|| \\
&\quad +\alpha_{n_k}\langle \xi f(z) - Bz, x_{n_k+1} - z\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
||x_{n_k+1} - z||^2 &\leq \frac{(1 - \alpha_{n_k}(\tau - \xi r_\epsilon))}{1 + \alpha_{n_k}(\tau - \xi r_\epsilon)}||x_{n_k} - z||^2 + \frac{2((1-\theta_{n_k})I - \alpha_{n_k}\tau)\beta_{n_k}}{1 + \alpha_{n_k}(\tau - \xi r_\epsilon)}||x_{n_k} - x_{n_k-1}|| \times \\
&\quad ||x_{n_k+1} - z|| + \frac{2\alpha_{n_k}}{1 + \alpha_{n_k}(\tau - \xi r_\epsilon)}\langle \xi f(z) - Bz, x_{n_k+1} - z\rangle \\
&\leq (1 - \alpha_{n_k}(\tau - \xi r_\epsilon))||x_{n_k} - z||^2 + \frac{2\beta_{n_k}}{1 + \alpha_{n_k}(\tau - \xi r_\epsilon)}||x_{n_k} - x_{n_k+1}|| \times \\
&\quad ||x_{n_k+1} - z|| + \frac{2\alpha_{n_k}}{1 + \alpha_{n_k}(\tau - \xi r_\epsilon)}\langle \xi f(z) - Bz, x_{n_k+1} - z\rangle \\
&= (1 - \alpha_{n_k}(\tau - \xi r_\epsilon))||x_{n_k} - z||^2 + \frac{2\alpha_n(\tau - \xi r_\epsilon)}{(1 + \alpha_{n_k} \times (\tau - \xi r_\epsilon))(\tau - \xi r_\epsilon)} \times \\
&\quad \left(\frac{\beta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}||||x_{n_k+1} - z|| + \langle \xi f(z) - Bz, x_{n_k+1} - z\rangle\right) \\
&= (1 - p_{n_k})||x_{n_k} - z||^2 + p_{n_k}q_{n_k}, \qquad (3.1.30)
\end{aligned}
$$

where $p_{n_k} = \alpha_{n_k}(\tau - \xi r_\epsilon)$ and

$$
\begin{aligned}
q_{n_k} &= \left(\frac{2||x_{n_k+1} - z||}{(1 + \alpha_{n_k}(\tau - \xi r_\epsilon))(\tau - \xi r_\epsilon)}\right)\frac{\beta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}|| \\
&\quad + \frac{2}{(1 + \alpha_{n_k}(\tau - \xi r_\epsilon))(\tau - \xi r_\epsilon)}\langle \xi f(z) - Bz, x_{n_k+1} - z\rangle. \qquad (3.1.31)
\end{aligned}
$$

Applying Lemma 2.6.29 and using conditions (C1), (C2) and (3.1.29), we conclude that the sequence $\{x_{n_k}\}$ converges strongly to z. The contradiction permits us to conclude that $x_n \to z$, where $z = P_\Gamma(I - B + \xi f)z$ which is the unique solution to the variational inequality (3.1.7).

Case 2: Suppose there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $D_{n_i} \leq D_{n_i+1}$ for all $i \in \mathbb{N}$. Then, by Lemma 2.6.34, there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $D_{m_k} \leq D_{m_k+1}$, for all $k \in \mathbb{N}$. Let $\epsilon > 0$ and $||x_{m_k} - x^*|| > \epsilon$, then, by Lemma 2.1.12, there exists $r_\epsilon \in (0,1)$ such that

$$||f(x_{m_k}) - f(x^*)|| \leq r_\epsilon ||x_{m_k} - x^*||.$$

Following similar argument as in Case 1, we obtain $||y_{m_k} - prox_{\gamma_{m_k} h}(I - \gamma_{m_k}\nabla g)y_{m_k}|| \to 0$, $||u_{m_k} - Tu_{m_k}|| \to 0$, $||u_{m_k} - x_{m_k}|| \to 0$ and $||x_{m_k+1} - x_{m_k}|| \to 0$ as $k \to \infty$. Since $\{x_{m_k}\}$ is bounded, there exists a subsequence of $\{x_{m_k}\}$ still denoted by $\{x_{m_k}\}$ which converges weakly to $\bar{x}$. Suppose $\{x_{m_k}\}$ is such that

$$\limsup_{k\to\infty} \langle \xi f(x^*) - Bx^*, x_{m_k+1} - x^* \rangle = \lim_{k\to\infty} \langle \xi f(x^*) - Bx^*, x_{m_k+1} - x^* \rangle.$$

It follows from Lemma (2.2.2) that

$$\begin{aligned} \limsup_{k\to\infty} \langle \xi f(x^*) - Bx^*, x_{m_k+1} - x^* \rangle &= \lim_{k\to\infty} \langle \xi f(x^*) - Bx^*, x_{m_k+1} - x^* \rangle \\ &= \langle \xi f(x^*) - Bx^*, \bar{x} - x^* \rangle \leq 0. \end{aligned}$$

Hence

$$\limsup_{k\to\infty} \langle \xi f(x^*) - Bx^*, x_{m_k+1} - x^* \rangle \leq 0. \tag{3.1.32}$$

Similarly as in (3.1.30), we obtain

$$\begin{aligned} ||x_{m_k+1} - x^*||^2 \leq\ & (1 - \alpha_{m_k}(\tau - \xi r_\epsilon))||x_{m_k} - x^*||^2 + \frac{2\alpha_{m_k}}{(1 + \alpha_{m_k}(\tau - \xi r_\epsilon))} \times \\ & \Big(\frac{\beta_{m_k}}{\alpha_{m_k}} ||x_{m_k} - x_{m_k+1}|| ||x_{m_k+1} - x^*|| \\ & + \langle \xi f(x^*) - Bx^*, x_{m_k+1} - x^* \rangle \Big). \end{aligned} \tag{3.1.33}$$

Since $D_{m_k} \leq D_{m_k+1}$, then from (3.1.33), we have

$$\begin{aligned} 0 &\leq ||x_{m_k+1} - x^*||^2 - ||x_{m_k} - x^*||^2 \\ &\leq (1 - \alpha_{m_k}(\tau - \xi r_\epsilon))||x_{m_k} - x^*||^2 + \frac{2\alpha_{m_k}}{(1 + \alpha_{m_k}(\tau - \xi r_\epsilon))} \times \\ &\Big(\frac{\beta_{m_k}}{\alpha_{m_k}} ||x_{m_k} - x_{m_k+1}|| ||x_{m_k+1} - x^*|| + \langle \xi f(x^*) - Bx^*, x_{m_k+1} - x^* \rangle \Big) - ||x_{m_k} - x^*||^2. \end{aligned} \tag{3.1.34}$$

This implies that

$$\begin{aligned} \alpha_{m_k}(\tau - \xi r_\epsilon)||x_{m_k} - x^*||^2 \leq\ & \frac{2\alpha_{m_k}}{(1 + \alpha_{m_k}(\tau - \xi r_\epsilon))} \Big(\frac{\beta_{m_k}}{\alpha_{m_k}} ||x_{m_k} - x_{m_k+1}|| ||x_{m_k+1} - x^*|| \\ & + \langle \xi f(x^*) - Bx^*, x_{m_k+1} - x^* \rangle \Big). \end{aligned} \tag{3.1.35}$$

Hence, from condition (C2) and (5.2.32), we obtain

$$\lim_{n\to\infty} ||x_{m_k} - x^*|| = 0. \tag{3.1.36}$$

As a consequence, we obtain

$$||x_{m_k+1} - x^*|| \le ||x_{m_k+1} - x_{m_k}|| + ||x_{m_k} - x^*|| \to 0,$$

as $n \to \infty$. By Lemma 2.6.34, we have $D_n \le D_{m_k+1}$ and thus

$$D_n = ||x_n - x^*||^2 \le ||x_{m_k+1} - x^*||^2 \to 0, \tag{3.1.37}$$

as $n \to \infty$. This implies that $\{x_n\}$ converges strongly to x^*. This complete the proof.

□

3.1.2 Applications.

In this subsection, we present some applications of Theorem 3.1.2.

1. Application to Monotone Variational Inequality Problem

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The Variational Inequality Problem (VIP) (1.1.1) is equivalent to finding a point $x^* \in C$ such that (see [221])

$$0 \in (M + N_C)x^*,$$

where N_C is the normal cone operator of C and $M : C \to H$ is a monotone operator. Note that the resolvent of the normal cone is the projection operator and that if M is ν-ism, then the set Ω_{VIP} is closed and convex. Also, if $M : C \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function, then, the sugradient of M, i.e., ∂M is maximal monotone operator (see [222]). Thus, setting $M = g$ and $N_C = h$ in our Theorem 3.1.2, we get the following strong convergence theorem for finding a common solution of VIP (1.1.1) and fixed point of δ-demimetric mappings in a real Hilbert space.

Theorem 3.1.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $M, : C \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function such that the gradient ∇M is $\frac{1}{L}$-ism with $L > 0$. Let $f : C \to C$ be a Meir Keeler contraction mapping, $B : C \to H$ be a strongly positive bounded linear operator with coefficient $\tau > 0$ such that $0 < \xi < \frac{\tau}{2}$ and $T : C \to C$ be a δ-demimetric mapping for $\delta \in (-\infty, 1)$ and $\hat{F}(T) = F(T)$. Suppose $\Gamma = \Omega_{VIP} \cap F(T) \neq \emptyset$, let $\alpha_n \in [0, 1]$, $\beta_n \in [0, 1)$, $\{w_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1)$ and $\gamma_n > 0$. Choose initial points $x_0, x_1 \in H$ arbitrarily and let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be generated by*

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}), \\ u_n = (1 - w_n)y_n + w_n prox_{\gamma_n h}(y_n - \gamma_n \nabla M(y_n)), \\ x_{n+1} = P_C[\alpha_n \xi f(x_n) + \theta_n x_n + ((1 - \theta_n)I - \alpha_n B)T_{\lambda_n} u_n], \quad n \ge 1, \end{cases} \tag{3.1.38}$$

where $T_{\lambda_n} = (1 - \lambda_n)I + \lambda_n T$ for $\lambda_n \in (0, 1)$. Assume that the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n}||x_n - x_{n-1}|| = 0$,

(C3) $0 < \liminf_{n\to\infty} w_n \leq \limsup_{n\to\infty} w_n < 1$,

(C4) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \frac{2}{L}$,

(C5) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1 - \delta$.

Then, $\{x_n\}$ *converges strongly to a point* $\bar{x}$*, where* $\bar{x} = P_\Gamma(I - B + \xi f)(\bar{x})$ *is the unique solution of the variational inequality*

$$\langle (B - \xi f)\bar{x}, \bar{x} - y\rangle \leq 0, \quad y \in \Gamma. \tag{3.1.39}$$

2. Application to Proximal Split Feasibility Problem

Let H_1 and H_2 be real Hilbert spaces, C and Q be nonempty closed and convex subset of H_1 and H_2 respectively. Let $R : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $S : H_2 \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous functions, and $A : H_1 \to H_2$ be a bounded linear operator. The Proximal Split Feasibility Problem (PSFP) is to find a point x^* with the property

$$x^* \in argmin\ R \quad \text{such that} \quad Ax^* \in argmin\ S, \tag{3.1.40}$$

where

$$argmin\ S := \{x \in H_1 : S(x) \leq S(y), \quad \forall y \in H_1\},$$

and

$$argmin\ R := \{u \in H_2 : R(u) \leq R(v), \quad \forall v \in H_2\}.$$

We denote the solution set of the PSFP (3.1.40) by Ω_{PSFP}. The PSFP was first introduced by Moudafi and Thakur in [188]. By taking $S = i_C$ and $R = i_Q$, the indicator functions on C and Q respectively, the PSFP reduces to the Split Feasibility Problem (SFP) (1.1.9) introduced by Censor and Elfving [66]. To solve the PSFP, it is very important to investigate the following minimization problem: find a solution $x^* \in H_1$ such that

$$\underset{x\in H_1}{minimize}\{R(x) + S_\mu(Ax)\}, \tag{3.1.41}$$

where $S_\mu(y) = \underset{u\in H_2}{argmin}\{S(u) + \dfrac{1}{2\mu}||u - y||^2\}$ stands for the Moreau-Yosida approximation of S with parameter μ [188]. By the differentiability of the Yosida approximation S_μ (see [222]), we can add the subdifferentials and thus write

$$\begin{aligned} \partial(R(x) + S_\mu(Ax)) &= \partial R(x) + A^*\nabla S_\mu(Ax) \\ &= \partial R(x) + A^*\Big(\frac{I - prox_{\mu S}}{\mu}\Big)(Ax). \end{aligned}$$

This implies that the optimality condition of (3.1.41) can then be written as

$$0 \in \ \mu\partial R(x) + A^*(I - prox_{\mu S})Ax, \tag{3.1.42}$$

where $\partial R(x)$ stands for the subdifferential of R at x, i.e.

$$\partial R(x) := \{u \in H_1 : R(y) \geq R(x) + \langle u, y - x\rangle, \quad \forall y \in H_1\}.$$

This inclusion in (3.1.42) yields the following equivalent fixed point formulation

$$prox_{\gamma\mu R}(I - \gamma A^*(I - prox_{\mu S})A)x^* = x^*. \tag{3.1.43}$$

Hence, to solve (3.1.41), (3.1.43) suggest we consider the following split proximal algorithm:

$$x_{n+1} = prox_{\gamma\mu R}(x_n - \gamma_n A^*(I - prox_{\mu S}))Ax_n. \tag{3.1.44}$$

Setting $\nabla g(x) = A^*(I - prox_{\mu S})Ax$ in Theorem 3.1.2, then ∇g is $\frac{1}{\nu}$-ism with $\nu = ||A||$ (see [55], Page 113). This implies that we can apply Theorem 3.1.2 to obtain solution of PSFP in real Hilbert space. Thus, we give the following result which complement other results in literature on finding solution of PSFP.

Theorem 3.1.4. *Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, $R : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $S : H_2 \to \mathbb{R} \cup \{+\infty\}$ be two proper convex lower semicontinuous functions such that $\Lambda \neq \emptyset$. Let $f : C \to C$ be a Meir Keeler contraction mapping, $B : C \to H$ be a strongly positive bounded linear operator with coefficient $\tau > 0$ such that $0 < \xi < \frac{\tau}{\rho}$ and $T : C \to C$ be a δ-demimetric mapping for $\delta \in (-\infty, 1)$. Suppose $\Gamma = \Omega_{PSFP} \cap F(T) \neq \emptyset$, let $\alpha_n \in [0,1]$, $\beta_n \in [0,1)$, $w_n, \theta_n \in (0,1)$ and $\gamma_n > 0$. Choose initial points $x_0, x_1 \in H_1$ arbitrarily and let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be generated by*

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}), \\ u_n = (1 - w_n)y_n + w_n prox_{\mu_n\gamma_n R}(y_n - \gamma_n A^*(I - prox_{\mu_n S})Ay_n), \\ x_{n+1} = P_C[\alpha_n \xi f(x_n) + \theta_n x_n + ((1 - \theta_n)I - \alpha_n B)T_{\lambda_n} u_n], \quad n \geq 1, \end{cases} \tag{3.1.45}$$

where $T_{\lambda_n} = (1 - \lambda_n)I + \lambda_n T$ for $\lambda_n \in (0,1)$. Assume that the following conditions are satisfy:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\lim\limits_{n\to\infty} \frac{\beta_n}{\alpha_n} ||x_n - x_{n-1}|| = 0$,

(C3) $0 < \liminf\limits_{n\to\infty} w_n \leq \limsup_{n\to\infty} w_n < 1$,

(C4) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup\limits_{n\to\infty} \gamma_n < \frac{2}{||A||^2}$,

(C5) $0 < \liminf\limits_{n\to\infty} \lambda_n \leq \limsup\limits_{n\to\infty} \lambda_n < 1 - \delta$.

Then, $\{x_n\}$ converges strongly to a point $\bar{x}$, where $\bar{x} = P_\Gamma(I - B + \xi f)(\bar{x})$ is the unique solution of the variational inequality

$$\langle (B - \xi f)\bar{x}, \bar{x} - y\rangle \leq 0, \quad y \in \Gamma. \tag{3.1.46}$$

3.1.3 Numerical example

In this subsection, we give a numerical example by implementing our algorithm (3.1.45) for solving PSFP (3.1.40).

Example 3.1.5. Let $H_1 = \mathbb{R}^N = H_2$ and $S := ||.||_2$, the Euclidean norm on $\mathbb{R}^N$. It is obvious that we can project onto the Euclidean unit ball B_r as follows:

$$P_{B_r}(x) = \begin{cases} \frac{x}{||x||_2}, & if \quad ||x||_2 > 1, \\ x, & if \quad ||x||_2 \leq 1. \end{cases} \tag{3.1.47}$$

In this case, the proximal operator is given by

$$prox_S(x) = \begin{cases} \left(1 - \frac{1}{||x||_2}\right)x, & if \quad ||x||_2 \geq 1, \\ 0, & if \quad ||x||_2 < 1. \end{cases} \tag{3.1.48}$$

This proximal operator is called the block soft thresholding. Also, let $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$. Define

$$i_j(x_j) = \max\left\{|x_j| - 1, 0\right\}, \quad j = 1, 2, \ldots, N,$$

and

$$R(x) = \sum_{j=1}^{N} i_j(x_j).$$

Then (see [90])

$$prox_{i_j}(x_j) = \begin{cases} x_j, & if \quad |x_j| < 1, \\ sign(x_j), & if \quad 1 \leq |x_j| \leq 2, \\ sign(x_j - 1), & otherwise, \end{cases} \tag{3.1.49}$$

and

$$prox_R(x) = \Big(prox_{i_1}(x_1), prox_{i_2}(x_2), \ldots, prox_{i_N}(x_N)\Big).$$

Suppose $Ax = x \in \mathbb{R}^N$. We consider the following PSFP:

$$\text{find} \quad x^* \in argmin\ R \quad \text{such that} \quad Ax^* \in argmin\ S. \tag{3.1.50}$$

Chosen $\alpha_n = \dfrac{1}{n+1}$, $\beta_n = \dfrac{1}{(n+1)^3}$, $\theta_n = \dfrac{n}{2(n+3)}$, $w_n = \dfrac{1}{5(1+\frac{1}{n})}$ and $\lambda_n = \dfrac{n}{2n+3}$. Let $f(x) = \frac{x}{2}$, $B(x) = x$, $T(x) = \frac{x}{2}$, $\xi = 1$, $x_0 = 0.5 \times randn(50, N)$ and $x_1 = 2 \times randn(50, N)$ (randomly generated vectors in $\mathbb{R}^N$). Using $\dfrac{||x_{n+1} - x_n||_2}{||x_2 - x_1||_2} < 10^{-6}$ as the stopping criterion, we consider various values of N and choices of γ_n as follows:

Case (i): $N = 100$, Case (ii): $N = 500$, Case (iii): $N = 1000$, Case (iv): $N = 2000$,

and

Choice (i): $\gamma_n = \dfrac{n}{n+1}$, Choice (ii) $\gamma_n = \dfrac{n}{5n+7}$, Choice (iii) $\gamma_n = 0.7$.

Remark 3.1.6. The numerical results (see Table 3.1 and Figures 3.1) show that there is no significant change in the CPU time taken and the number of iterations for different values of N and the stepsizes.

Table 3.1: Table showing computation results for Example 3.1.5.

		Choice (i)	Choice (ii)	Choice (iii)
Case (i)	CPU time (sec)	0.0356	0.0399	0.0476
	No. of Iter.	25	27	27
Case (ii)	CPU time (sec)	0.2244	0.3913	0.5061
	No. of Iter.	27	29	29
Case (iii)	CPU time (sec)	0.4987	0.4095	0.5235
	No. of Iter.	29	30	30
Case (iv)	CPU time (sec)	1.0731	1.0912	0.8785
	No. of Iter.	30	30	30

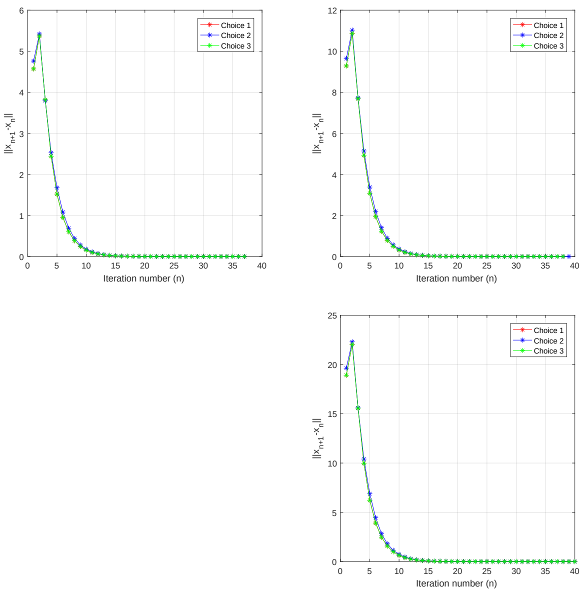


Figure 3.1: Example 3.1.5, Case (i); Case (ii), Case (iii); Case (iv).

3.2 A Self Adaptive Inertial Subgradient Extragradient Algorithm for Variational Inequality and Fixed Point of Multivalued Mappings in Hilbert Spaces

In this section, we consider a new subgradient extragradient iterative algorithm with inertial extrapolation for approximating a common solution of VIPs and fixed point problems of a multivalued demi-contractive mapping in a real Hilbert space.

In 2011, Y. Censor, A. Gibali and S. Reich [69] studied the approximation of common solution of a VIP and fixed point problem for a nonexpansive mapping. They proposed the following Subgradient Extragradient Algorithm (SEM) with Halpern method and proved its weak convergence to a solution $u^* \in F(S) \cap \Omega_{VIP}$:

$$\begin{cases} x_0 \in H, \quad \mu > 0, \\ y_n = P_C\left(x_n - \mu A x_n\right), \\ T_n = \langle z \in H : \langle x_n - \mu A x_n - y_n, z - y_n \rangle \leq 0\}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_{T_n}(x_n - \mu A y_n). \end{cases} \tag{3.2.1}$$

Also, Thong and Hieu [256] proposed the following two algorithms for finding a common element of the set of solutions of VIP and the fixed point of a demi-contractive mapping in a real Hilbert space.

Algorithm 3.2.1 (THSEgM(I))**.**

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \mu A x_n), \\ T_n = \{x \in H : \langle x_n - \mu A x_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = P_{T_n}(x_n - \mu A y_n), \\ x_{n+1} = (1 - \alpha_n - \beta_n) z_n + \beta_n S z_n. \end{cases} \tag{3.2.2}$$

Algorithm 3.2.2 (THSEgM(II))**.**

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \mu A x_n), \\ T_n = \{x \in H : \langle x_n - \mu A x_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = P_{T_n}\left(x_n - \mu A y_n\right), \\ x_{n+1} = (1 - \beta_n) \alpha_n z_n + \beta_n S z_n, \end{cases} \tag{3.2.3}$$

where $S : H \to H$ is a λ-demi-contractive mapping with $0 \leq \lambda < 1$, and where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$. Under suitable conditions on the parameters α_n and β_n, they proved that the sequence $\{x_n\}$ generated by (3.2.2) and (3.2.3) converges strongly to a solution $p \in \Omega_{VIP} \cap F(S)$.

Recently, Dong et al. [97] introduced the following inertial extragradient algorithm by incorporating the inertial term in the extragradient method (1.1.2).

Algorithm 3.2.3 (Inertia Extragradient Algorithm (iEgA))**.**

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \mu F(w_n)), \\ x_{n+1} = (1 - \lambda_n) w_n + \lambda_n P_C(w_n - \mu F(y_n)), \end{cases} \tag{3.2.4}$$

where $\{\alpha_n\}$ is a non-decreasing sequence with $\alpha_1 = 0$ and $0 \leq \alpha_n \leq \alpha < 1$ for any $n \geq 1$ and $\lambda, \sigma, \delta > 0$ are such that

$$\delta > \frac{\alpha[(1 + \mu L)^2 \alpha(1 + \alpha) + (1 - \mu^2 L^2)\alpha\sigma + \sigma(1 + \mu L)^2]}{1 - \mu^2 L^2}$$

and

$$0 < \lambda_n \leq \frac{\delta(1 - \mu^2 L^2) - \alpha[(1 + \mu L)^2 \alpha(1 + \alpha) + (1 - \mu^2 L^2)\alpha\sigma + \sigma(1 + \mu L^2)]}{\delta[(1 + \mu L)^2 \alpha(1 + \alpha) + (1 - \mu^2 L^2)\alpha\sigma + \sigma(1 + \mu L)^2]}.$$

Using the *iEgA* (3.2.4), Dong et al. [97] proved a weak convergence result for approximating the solution of the VIP (1.1.1) in a real Hilbert space.

Observe that the stepsize μ of the algorithms (3.2.1)-(3.2.4) plays an essential role in the convergence properties of the iterative methods. The Lipschitz constant L is typically assumed to be known, or at least estimated priorly. In many cases, this parameter is unknown or difficult to approximate. Moreover, the stepsize defined by this constant is often very small and deteriorates the convergence rate. In practice, a larger stepsize can often be used and yield better numerical results. It is thus natural to ask the following question:

> *Is it possible to have an inertial subgradient extragradient algorithm with self adaptive stepsize which converges strongly to a common solution of a variational inequality and fixed point a problem?*

It is our aim therefore to provide an affirmative answer to this question. Motivated by the work of Censor et al [69], Thong and Hieu [256] and Dong et al. [97], we introduce an inertial viscosity subgradient extragradient type algorithm with self adaptive stepsize.

3.2.1 Main results

In this subsection, we give a precise statement of our algorithm and discuss its strong convergence analysis.

Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to H$ be a monotone and L-Lipschitz continuous mapping. For $i = 1, 2, \dots, m$, let $S_i : H \to CB(H)$ be multivalued demi-contractive mappings with constant κ_i such that each $I - S_i$ are demiclosed at zero, $S_i(p) = \{p\}$ for all $p \in F(S_i)$, and $\kappa = max\{\kappa_i\}$. Suppose

$$\Gamma = \Omega_{VIP} \cap \bigcap_{i=1}^{m} F(S_i) \neq \emptyset,$$

and let $f : H \to H$ be a λ-contraction with constant $\lambda \in (0, 1)$. Let D be a bounded operator with coefficient $\rho > 0$ such that $0 < \xi < \frac{\rho}{\lambda}$ and let $\{\epsilon_n\}$, $\{\beta_{n,i}\}$ and $\{\delta_n\}$ be nonnegative sequences such that $0 < a \leq \epsilon_n, \beta_{n,i}, \delta_n \leq b < 1$, and let $\alpha \geq 3$.

Algorithm 3.2.4.

Step 0: *Select initial guesses $x_0, x_1 \in H$ and set $n = 1$.*
Step 1: *Given the $(n-1)$th and nth iterates, choose α_n such that we have $0 \leq \alpha_n \leq \tilde{\alpha}_n$ with $\tilde{\alpha}_n$ defined by*

$$\tilde{\alpha}_n = \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & if \quad x_n \neq x_{n-1}, \\ \frac{n-1}{n+\alpha-1}, & otherwise. \end{cases} \tag{3.2.5}$$

Step 2: *Compute*

$$\begin{aligned} u_n &= x_n + \alpha_n(x_n - x_{n-1}), \\ w_n &= P_C(u_n - \mu_n A(u_n)), \end{aligned} \tag{3.2.6}$$

where $\mu_n = \bar{\sigma}\delta^{m_n}$, $\bar{\sigma} > 0$, $\delta \in (0,1)$ *and* m_n *is the smallest nonnegative integer such that*

$$||A(u_n) - A(w_n)|| \leq \frac{\eta||u_n - w_n||}{\mu_n}, \qquad \eta \in (0,1). \tag{3.2.7}$$

Step 3: *Construct the set* Q_n *defined by*

$$Q_n = \{u \in H : \langle u_n - \mu_n A(u_n) - w_n, u - w_n\rangle \leq 0\},$$

and compute

$$\begin{cases} y_n &= P_{Q_n}(u_n - \mu_n A(w_n)), \\ z_n &= \beta_{n,0}y_n + \sum_{i=1}^{m} \beta_{n,i}v_{n,i}, \\ x_{n+1} &= \delta_n \xi f(x_n) + (1 - \delta_n D)z_n, \end{cases} \tag{3.2.8}$$

where $v_{n,i} \in S_i y_n$ *and* $\sum_{i=0}^{m} \beta_{n,i} = 1$. *Set* $n := n+1$ *and go to* **Step 1**.

Remark 3.2.5. Observe that if $w_n = u_n = x_n$ and $x_n \in S_i x_n$, then we are at a common solution of the variational inequality (1.1.1) and a common fixed point of S_i, for all $i = 1, 2, \ldots, m$. In our convergence analysis, we will implicitly assume that this does not occur after finitely many iterations so that our Algorithm 3.2.4 generates an infinite sequence. We will see in the following result that the stepsize rule defined by (3.2.7) is well defined.

Lemma 3.2.6. *[104] There exists a nonnegative integer* m_n *satisfying* (3.2.7). *In addition*

$$\mu^* \leq \mu_n \leq \bar{\sigma}, \quad \textit{where} \quad \mu^* = \min\left\{\bar{\sigma}, \frac{\eta\delta}{L}\right\}.$$

In order to establish our main result, we make the following assumption:

(C1) $\lim_{n\to\infty} \delta_n = 0$ and $\sum_{n=0}^{\infty} \delta_n = \infty$,

(C2) $\liminf_{n\to\infty} (\beta_{n,0} - \kappa)\beta_{n,i} > 0$ for all $i = 1, 2, ..., m$,

(C3) $\epsilon_n = o(\delta_n)$, i.e., $\lim_{n\to\infty} \frac{\epsilon_n}{\delta_n} = 0$ $\left(\textit{e.g.} \quad \epsilon_n = \frac{1}{(n+1)^2}, \ \delta_n = \frac{1}{n+1}\right)$.

Remark 3.2.7. Note that from (3.2.5) and Assumptions (C3), we have

$$\lim_{n\to\infty} \alpha_n||x_n - x_{n-1}|| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\alpha_n}{\delta_n}||x_n - x_{n-1}|| = 0. \tag{3.2.9}$$

Also, note that **Step 1** in our Algorithm 3.2.4 is easily implemented in numerical computation since the value of $||x_n - x_{n-1}||$ is known a prori before choosing α_n.

We proceed to prove the following lemmas before proving the convergence of our main Algorithm 3.2.4.

Lemma 3.2.8. *The sequence $\{x_n\}$ generated by Algorithm 3.2.4 is bounded.*

Proof. Let $p \in \Gamma$, then

$$\begin{aligned} ||u_n - p|| &= ||x_n + \alpha_n(x_n - x_{n-1}) - p|| \\ &\leq ||x_n - p|| + \alpha_n||x_n - x_{n-1}||. \end{aligned} \tag{3.2.10}$$

Also from (2.2.2) and (3.2.6), we get

$$\begin{aligned} ||y_n - p||^2 &= ||P_{Q_n}(u_n - \mu_n Aw_n) - p||^2 \\ &\leq ||u_n - \mu_n Aw_n - p||^2 - ||u_n - \mu_n Aw_n - y_n||^2 \\ &= ||u_n - \mu_n Aw_n||^2 - 2\langle u_n - \mu_n Aw_n, p\rangle + ||p||^2 \\ &\quad - \Big[||u_n - \mu_n Aw_n||^2 - 2\langle u_n - \mu_n Aw_n, y_n\rangle + ||y_n||^2\Big] \\ &= ||p||^2 - 2\langle u_n, p\rangle + ||u_n||^2 - ||u_n||^2 + 2\mu_n\langle Aw_n, p\rangle \\ &\quad + 2\langle u_n, y_n\rangle - 2\mu_n\langle Aw_n, y_n\rangle - ||y_n||^2 \\ &= ||u_n - p||^2 - ||u_n - y_n||^2 + 2\mu_n\langle Aw_n, p - y_n\rangle. \end{aligned} \tag{3.2.11}$$

Since A is monotone, then

$$\langle Aw_n - Ap, w_n - p\rangle \geq 0, \quad \text{for all} \quad n \geq 1,$$

and hence

$$\langle Aw_n, w_n - p\rangle \geq \langle Ap, w_n - p\rangle.$$

This implies that

$$\langle Aw_n, w_n - p\rangle \geq \langle Ap, w_n - p\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned} 0 &\leq \langle Aw_n, w_n - p\rangle \\ &= \langle Aw_n, w_n - y_n + y_n - p\rangle \\ &= \langle Aw_n, w_n - y_n\rangle + \langle Aw_n, y_n - p\rangle. \end{aligned}$$

Whence

$$\langle Aw_n, p - y_n\rangle \leq \langle Aw_n, w_n - y_n\rangle. \tag{3.2.12}$$

Substituting (3.2.12) into (3.2.11), we get

$$\begin{aligned} ||y_n - p||^2 &\leq ||u_n - p||^2 - ||u_n - y_n||^2 + 2\mu_n\langle Aw_n, w_n - y_n\rangle \\ &= ||u_n - p||^2 - ||u_n - w_n + w_n - y_n||^2 + 2\mu_n\langle Aw_n, w_n - y_n\rangle \\ &= ||u_n - p||^2 - \Big[||u_n - w_n||^2 + 2\langle u_n - w_n, w_n - y_n\rangle + ||w_n - y_n||^2\Big] \\ &\quad + 2\mu_n\langle Aw_n, w_n - y_n\rangle \\ &= ||u_n - p||^2 - ||u_n - w_n||^2 - ||w_n - y_n||^2 \\ &\quad + 2\langle u_n - \mu_n Aw_n - w_n, y_n - w_n\rangle. \end{aligned} \tag{3.2.13}$$

Using (3.2.7), we get

$$\begin{aligned}\langle u_n - \mu_n A w_n - w_n, y_n - w_n\rangle &= \langle u_n - \mu_n A u_n - w_n, y_n - w_n\rangle \\ &\quad + \langle \mu_n A u_n - \mu_n A w_n, y_n - w_n\rangle \\ &\leq \mu_n \langle A u_n - A w_n, y_n - w_n\rangle \\ &\leq \eta \|u_n - w_n\| \cdot \|y_n - w_n\|. \end{aligned} \tag{3.2.14}$$

Hence from (3.2.13) and (3.2.14), we get

$$\begin{aligned}\|y_n - p\|^2 &\leq \|u_n - p\|^2 - \|u_n - w_n\|^2 - \|w_n - y_n\|^2 + 2\eta\|u_n - w_n\| \cdot \|y_n - w_n\| \\ &\leq \|u_n - p\|^2 - \|u_n - w_n\|^2 - \|w_n - y_n\|^2 + \eta(\|u_n - w_n\|^2 + \|w_n - y_n\|^2) \\ &= \|u_n - p\|^2 - (1-\eta)\|u_n - w_n\|^2 - (1-\eta)\|w_n - y_n\|^2. \end{aligned} \tag{3.2.15}$$

Thus

$$\|y_n - p\|^2 \leq \|u_n - p\|^2. \tag{3.2.16}$$

Furthermore, using Lemma 2.6.3, we have

$$\begin{aligned}\|z_n - p\|^2 &= \Big\|\beta_{n,0} y_n + \sum_{i=1}^{m} \beta_{n,i} v_{n,i} - p\Big\|^2 \\ &\leq \beta_{n,0}\|y_n - p\|^2 + \sum_{i=1}^{m} \beta_{n,i}\|v_{n,i} - p\|^2 - \sum_{i=1}^{m} \beta_{n,0}\beta_{n,i}\|y_n - v_{n,i}\|^2 \\ &= \beta_{n,0}\|y_n - p\|^2 + \sum_{i=1}^{m} \beta_{n,i} d\big(v_{n,i}, S_i p\big)^2 - \sum_{i=1}^{m} \beta_{n,0}\beta_{n,i}\|y_n - v_{n,i}\|^2 \\ &\leq \beta_{n,0}\|y_n - p\|^2 + \sum_{i=1}^{m} \beta_{n,i} H\big(S_i y_n, S_i p\big)^2 - \sum_{i=1}^{m} \beta_{n,0}\beta_{n,i}\|y_n - v_{n,i}\|^2. \end{aligned}$$

Thus

$$\begin{aligned}\|z_n - p\|^2 &\leq \beta_{n,0}\|y_n - p\|^2 + \sum_{i=1}^{m} \beta_{n,i}\Big(\|y_n - p\|^2 + \kappa_i d(y_n, S_i y_n)^2\Big) \\ &\quad - \sum_{i=1}^{m} \beta_{n,0}\beta_{n,i}\|y_n - v_{n,i}\|^2 \\ &\leq \beta_{n,0}\|y_n - p\|^2 + \sum_{i=1}^{m} \beta_{n,i}\|y_n - p\|^2 + \sum_{i=1}^{m} \beta_{n,i}\kappa\|y_n - v_{n,i}\|^2 \\ &\quad - \sum_{i=1}^{m} \beta_{n,0}\beta_{n,i}\|y_n - v_{n,i}\|^2 \\ &= \|y_n - p\|^2 - \sum_{i=1}^{m} (\beta_{n,0} - \kappa)\beta_{n,i}\|y_n - v_{n,i}\|^2, \end{aligned} \tag{3.2.17}$$

and by condition (C2), we get

$$\|z_n - p\|^2 \leq \|y_n - p\|^2. \tag{3.2.18}$$

Therefore, from (3.2.10), (3.2.16) and (3.2.18), we have

$$\begin{aligned} ||x_{n+1}-p|| &= ||\delta_n(\xi f(x_n)-Dp)+(1-\delta_n D)(z_n-p)|| \\ &\leq \delta_n||\xi f(x_n)-Dp||+(1-\delta_n\rho)||z_n-p|| \\ &\leq \delta_n\Big[||\xi(f(x_n)-f(p))+(\xi f(p)-Dp)||\Big]+(1-\delta_n\rho)||z_n-p|| \\ &\leq \delta_n\xi\lambda||x_n-p||+\delta_n||\xi f(p)-Dp||+(1-\delta_n\rho)[||x_n-p||+\alpha_n||x_n-x_{n-1}||] \\ &= (1-\delta_n(\rho-\xi\lambda))||x_n-p||+\delta_n||\xi f(p)-Dp||+(1-\delta_n\rho)\alpha_n||x_n-x_{n-1}|| \\ &= (1-\delta_n(\rho-\xi\lambda))||x_n-p||+(\rho-\xi\lambda)\delta_n\Big\{\frac{||\xi f(p)-Dp||}{\rho-\xi\lambda} \\ &\quad +\left(\frac{1-\delta_n\rho}{\rho-\xi\lambda}\right)\frac{\alpha_n}{\delta_n}||x_n-x_{n-1}||\Big\}. \end{aligned} \tag{3.2.19}$$

Note that $\sup_{n\geq 1}\left(\frac{1-\delta_n\rho}{\rho-\xi\lambda}\right)\frac{\alpha_n}{\delta_n}||x_n-x_{n-1}||$ exists by Remark 3.2.7 and let

$$M := \max\left\{\frac{||\xi f(p)-Dp||}{||\rho-\xi\lambda||}, \sup_{n\geq 1}\left(\frac{1-\delta_n\rho}{\rho-\xi\lambda}\right)\frac{\alpha_n}{\delta_n}||x_n-x_{n-1}||\right\}.$$

Then we have

$$||x_{n+1}-p|| \leq (1-\delta_n(\rho-\xi\lambda))||x_n-p||+\delta_n(\rho-\xi\lambda)M. \tag{3.2.20}$$

Using Lemma 2.6.31(i) and (3.2.20), we have $\{||x_n-p||\}$ is bounded and thus $\{x_n\}$ is bounded. Consequently, $\{u_n\},\{Aun\},\{w_n\},\{y_n\}$ and $\{z_n\}$ are all bounded. □

Lemma 3.2.9. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2.4. Put* $s_n=||x_n-p||^2$, $\tilde{a}_n=\frac{2\delta_n(\rho-\xi\lambda)}{1-\delta_n\xi\lambda}$, $b_n=\frac{1}{2(\rho-\xi\lambda)}\Big(2\langle\xi f(p)-Dp,x_{n+1}-p\rangle+\delta_n M_1\Big)$, *for some* $M_1>0$ *and* $c_n=\frac{\alpha_n||x_n-x_{n-1}||}{1-\delta_n\xi\lambda}M_2$, *where* $M_2=\sup_{n\geq 1}\Big((1-\delta_n\rho)^2(||x_n-p||+||x_{n-1}-p||)+2(1-\delta_n\rho)^2||x_n-x_{n-1}||\Big)$ *and* $p\in\Gamma$. *Then, the following estimates hold:*

(i) $s_{n+1}\leq(1-\tilde{a}_n)s_n+\tilde{a}_n b_n+c_n$,

(ii) $-1\leq\limsup_{n\to\infty}b_n<+\infty$,

Proof. From (3.2.6), we have

$$\begin{aligned} ||u_n-p||^2 &= ||x_n+\alpha_n(x_n-x_{n-1})-p||^2 \\ &= ||x_n-p||^2+2\alpha_n\langle x_n-p,x_n-x_{n-1}\rangle+\alpha_n^2||x_n-x_{n-1}||^2. \end{aligned} \tag{3.2.21}$$

Using Lemma 2.6.1(ii), we have

$$2\langle x_n-p,x_n-x_{n-1}\rangle=-||x_{n-1}-p||^2+||x_n-p||^2+||x_n-x_{n-1}||^2, \tag{3.2.22}$$

thus, substituting (3.2.22) into (3.2.21), we get

$$\begin{aligned} ||u_n - p||^2 &= ||x_n - p||^2 + \alpha_n(-||x_{n-1} - p||^2 + ||x_n - p||^2 + ||x_n - x_{n-1}||^2) \\ &\quad +\alpha_n^2||x_n - x_{n-1}||^2 \\ &\leq ||x_n - p||^2 + \alpha_n(||x_n - p||^2 - ||x_{n-1} - p||^2) \\ &\quad +2\alpha_n||x_n - x_{n-1}||^2. \end{aligned} \tag{3.2.23}$$

Now, from Lemma 2.6.1(i), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||\delta_n(\xi f(x_n) - Dp) + (1 - \delta_n D)(z_n - p)||^2 \\ &\leq (1 - \delta_n\rho)^2||z_n - p||^2 + 2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle. \end{aligned} \tag{3.2.24}$$

It follows from (3.2.16), (3.2.18) and (3.2.21) that

$$\begin{aligned} ||x_{n+1} - p||^2 &\leq (1 - \delta_n\rho)^2||u_n - p||^2 + 2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \\ &= (1 - \delta_n\rho)^2\Big(||x_n - p||^2 + \alpha_n\left(||x_n - p||^2 - ||x_{n-1} - p||^2\right) + 2\alpha_n||x_n - x_{n-1}||^2\Big) \\ &\quad +2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \\ &= (1 - \delta_n\rho)^2||x_n - p||^2 + \alpha_n(1 - \delta_n\rho)^2\left(||x_n - p||^2 - ||x_{n-1} - p||^2\right) \\ &\quad +2\alpha_n(1 - \delta_n\rho)^2||x_n - x_{n-1}||^2 + 2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \\ &\leq (1 - \delta_n\rho)^2||x_n - p||^2 + \alpha_n(1 - \delta_n\rho)^2(||x_n - p|| + ||x_{n-1} - p||)||x_n - x_{n-1}|| \\ &\quad +2\alpha_n(1 - \delta_n\rho)^2||x_n - x_{n-1}||^2 + 2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle. \end{aligned} \tag{3.2.25}$$

Also

$$\begin{aligned} 2\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle &= 2\langle \xi(f(x_n) - f(p)) + \xi f(p) - Dp, x_{n+1} - p\rangle \\ &\leq 2\xi\lambda||x_n - p|| \cdot ||x_{n+1} - p|| + 2\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &\leq \xi\lambda(||x_n - p||^2 + ||x_{n+1} - p||^2) \\ &\quad +2\langle \xi f(p) - Dp, x_{n+1} - p\rangle. \end{aligned} \tag{3.2.26}$$

Substituting (3.2.26) into (3.2.25), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &\leq \Big[(1 - \delta_n\rho)^2 + \delta_n\xi\lambda\Big]||x_n - p||^2 + \alpha_n(1 - \delta_n\rho)^2(||x_n - p|| + ||x_{n-1} - p||) \times \\ &\quad ||x_n - x_{n-1}|| + 2\alpha_n(1 - \delta_n\rho)^2||x_n - x_{n-1}||^2 + \delta_n\xi\lambda||x_{n+1} - p||^2 \\ &\quad +2\delta_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &= (1 - \delta_n(2\rho - \xi\lambda))||x_n - p||^2 + (\delta_n\rho)^2||x_n - p||^2 + \alpha_n\Big[(1 - \delta_n\rho)^2 \times \\ &\quad (||x_n - p|| + ||x_{n-1} - p||) + 2(1 - \delta_n\rho)^2||x_n - x_{n-1}||\Big]||x_n - x_{n-1}|| \\ &\quad +\delta_n\xi\lambda||x_{n+1} - p||^2 + 2\delta_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &\leq (1 - \delta_n(2\rho - \xi\lambda))||x_n - p||^2 + \delta_n\xi\lambda||x_{n+1} - p||^2 + \alpha_n\Big[(1 - \delta_n\rho)^2 \times \\ &\quad (||x_n - p|| + ||x_{n-1} - p||) + 2(1 - \delta_n\rho)^2||x_n - x_{n-1}||\Big]||x_n - x_{n-1}|| \\ &\quad +\delta_n(2\langle \xi f(p) - Dp, x_{n+1} - p\rangle + \delta_n M_1) \end{aligned}$$

for some $M_1 \geq 0$. Hence

$$\begin{aligned}
||x_{n+1}-p||^2 &\leq \frac{(1-\delta_n(2\rho-\xi\lambda))}{1-\delta_n\xi\lambda}||x_n-p||^2+\frac{\alpha_n}{1-\delta_n\xi\lambda}||x_n-x_{n-1}||M_2 \\
&\quad+\frac{\delta_n(2\langle\xi f(p)-Dp,x_{n+1}-p\rangle+\delta_n M_1)}{1-\delta_n\xi\lambda} \\
&= \left(1-\frac{2\delta_n(\rho-\xi\lambda)}{1-\delta_n\xi\lambda}\right)||x_n-p||^2+\frac{\alpha_n}{1-\delta_n\xi\lambda}||x_n-x_{n-1}||M_2 \\
&\quad+\frac{2\delta_n(\rho-\xi\lambda)}{1-\delta_n\xi\lambda}\times\frac{(2\langle\xi f(p)-Dp,x_{n+1}-p\rangle+\delta_n M_1)}{2(\rho-\xi\lambda)}.
\end{aligned}$$

This establishes (i).

Next, we prove (ii). Since $\{x_n\}$ is bounded and $\delta_n \in (0,1)$, then we have

$$\sup_{n\geq 0} b_n \leq \sup_{n\geq 0}\frac{1}{2(\rho-\xi\lambda)}\Big(2||\xi f(p)-Dx^*||\cdot||x_{n+1}-p||+M_1\Big)<\infty.$$

We next show that $\limsup_{n\to\infty} b_n \geq -1$. Assume the contrary, i.e., suppose $\limsup_{n\to\infty} b_n < -1$, which implies that there exists $n_0 \in \mathbb{N}$ such that $b_n \leq -1$ for all $n \geq n_0$. Hence, it follows from (i) that

$$\begin{aligned}
s_{n+1} &\leq (1-\tilde{a}_n)s_n+\tilde{a}_n b_n+c_n \\
&< (1-\tilde{a}_n)s_n-\tilde{a}_n+c_n \\
&= s_n-\tilde{a}_n(s_n+1)+c_n \\
&\leq s_n-2(\rho-\xi\lambda)\delta_n+c_n.
\end{aligned}$$

By induction, we get

$$s_{n+1}\leq s_{n_0}-2(\rho-\xi\lambda)\sum_{i=n_0}^{n}\delta_i+c_n \qquad \text{for all } \ n\geq n_0.$$

Taking lim sup of both sides in the last inequality (noting that $c_n \to 0$), we have

$$\limsup_{n\to\infty} s_n \leq s_{n_0}-\lim_{n\to\infty}2(\rho-\xi\lambda)\sum_{i=n_0}^{n}\delta_i=-\infty.$$

This contradicts the fact that $\{s_n\}$ is a nonnegative real sequence. Therefore, $\limsup_{n\to\infty} b_n \geq -1$. ☐

Remark 3.2.10. Since $\delta_n \to 0$ as $n \to \infty$, it is easy to check that $\tilde{a}_n \to 0$, and by Remark 3.2.7, $c_n \to 0$ as $n \to \infty$.

We next state and prove our main theorem.

Theorem 3.2.11. *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $A : C \to H$ be a monotone and L-Lipschitz continuous mapping. For each*

$i = 1, 2, \ldots, m$, let $S_i : H \to CB(H)$ be multivalued demi-contractive mappings with constant κ_i such that each $I - S_i$ are demiclosed at zero, $S_i(p) = \{p\}$ for all $p \in F(S_i)$, $i = 1, 2, \ldots, m$ and $\kappa = max\{\kappa_i\}$. Suppose $\Gamma = \Omega_{VIP} \cap \bigcap_{i=1}^{m} F(S_i) \neq \emptyset$. Let $f : H \to H$ be a λ-contraction with constant $\lambda \in (0, 1)$ and D be a bounded operator with coefficient $\rho > 0$ such that $0 < \xi < \frac{\rho}{\lambda}$. Let $\{x_n\}$ be generated by Algorithm 3.2.4 and Assumptions (C1)-(C3) are satisfied. Then the sequence $\{x_n\}$ converges strongly to a point z, where $z = P_\Gamma(I - D + \xi f)(z)$ is a unique solution of the variational inequality

$$\langle (D - \xi f)z, z - x\rangle \leq 0, \quad x \in \Gamma. \tag{3.2.27}$$

Proof. Let $p \in \Gamma$ and denote $||x_n - p||^2$ by Φ_n. We consider the following two possible cases.
CASE A: Suppose there exists $n_0 \in \mathbb{N}$ such that Φ_n is monotonically non-increasing for all $n \geq n_0$. Since Φ_n is bounded, then it is convergent and so $\Phi_n - \Phi_{n+1} \to 0$ as $n \to \infty$. We first show that $||w_n - u_n|| \to 0$, $||v_{n,i} - y_n|| \to 0$ and $||x_{n+1} - x_n|| \to 0$, as $n \to \infty$. From (3.2.15), (3.2.18) and (3.2.23), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &\leq (1 - \delta_n\rho)^2||z_n - p||^2 + 2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \\ &\leq (1 - \delta_n\rho)^2\Big\{||u_n - p||^2 - (1 - \eta)||u_n - w_n||^2 - (1 - \eta)||w_n - y_n||^2\Big\} \\ &\quad +2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \\ &\leq (1 - \delta_n\rho)^2\Big\{||x_n - p||^2 + \alpha_n(||x_n - p||^2 - ||x_{n-1} - p||^2) + 2\alpha_n||x_n - x_{n-1}||^2 \\ &\quad -(1 - \eta)||u_n - w_n||^2 - (1 - \eta)||w_n - y_n||^2\Big\} \\ &\quad +2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle. \end{aligned} \tag{3.2.28}$$

Therefore

$$\begin{aligned} &(1 - \delta_n\rho)^2(1 - \eta)||u_n - w_n||^2 \\ &\quad \leq (1 - \delta_n\rho)^2||x_n - p||^2 + \alpha_n(1 - \delta_n\rho)^2(||x_n - p||^2 - ||x_{n-1} - p||^2) \\ &\quad + 2\alpha_n(1 - \delta_n\rho)^2||x_n - x_{n-1}||^2 + 2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle - ||x_{n+1} - p||^2 \\ &\quad \leq \Phi_n - \Phi_{n+1} + \delta_n M_3 + \alpha_n(1 - \delta_n\rho)^2(\Phi_n - \Phi_{n-1}) \\ &\quad + 2\alpha_n(1 - \delta_n\rho)^2||x_n - x_{n-1}|| + 2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \to 0, \end{aligned}$$

as $n \to \infty$ for some $M_3 > 0$. Since $\delta_n \to 0$ as $n \to \infty$ and $\eta \in (0, 1)$, then

$$\lim_{n\to\infty} ||u_n - w_n|| = 0. \tag{3.2.29}$$

Similarly, from (3.2.28), we can also show that

$$\lim_{n\to\infty} ||w_n - y_n|| = 0. \tag{3.2.30}$$

Clearly from (3.2.6), we get

$$\begin{aligned} ||u_n - x_n|| &= ||x_n + \alpha_n(x_n - x_{n-1}) - x_n|| \\ &= \alpha_n||x_n - x_{n-1}|| \to 0, \quad n \to \infty. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} ||w_n - x_n|| \le \lim_{n\to\infty} (||w_n - u_n|| + ||u_n - x_n||) = 0, \tag{3.2.31}$$

and

$$\lim_{n\to\infty} ||y_n - x_n|| \le \lim_{n\to\infty} (||y_n - w_n|| + ||w_n - x_n||) = 0. \tag{3.2.32}$$

Also from (3.2.16), (3.2.17), (3.2.23) and (3.2.24), we obtain

$$\begin{aligned}
||x_{n+1} - p||^2 &\le (1-\delta_n\rho)^2||z_n - p||^2 + 2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \\
&\le (1-\delta_n\rho)^2\Big\{||y_n - p||^2 - \sum_{i=1}^{m}(\beta_{n,0} - \kappa)\beta_{n,i}||y_n - v_{n,i}||^2\Big\} \\
&\quad +2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \\
&\le (1-\delta_n\rho)^2\Big\{||x_n - p||^2 + \alpha_n(||x_n - p||^2 - ||x_{n-1} - p||^2) + 2\alpha_n||x_n - x_{n-1}||^2 \\
&\quad - \sum_{i=1}^{m}(\beta_{n,0} - \kappa)\beta_{n,i}||y_n - v_{n,i}||^2\Big\} + 2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
&(1-\delta_n\rho)^2\sum_{i=1}^{m}(\beta_{n,0} - \kappa)\beta_{n,i}||y_n - v_{n,i}||^2 \\
&\le (1-\delta_n\rho)^2||x_n - p||^2 + \alpha_n(1-\delta_n\rho)^2(||x_n - p||^2 - ||x_{n-1} - p||^2) \\
&+ 2\alpha_n(1-\delta_n\rho)^2||x_n - x_{n-1}||^2 + +2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle - ||x_{n+1} - p||^2 \\
&\le \Phi_n - \Phi_{n+1} + \delta_n M_3 + \alpha_n(1-\delta_n\rho)^2(\Phi_n - \Phi_{n-1}) + 2\alpha_n(1-\delta_n\rho)^2||x_n - x_{n-1}||^2 \\
&+ 2\delta_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \to 0, \quad as \;\; n \to \infty.
\end{aligned}$$

Therefore, using condition (C2), it follows that

$$\lim_{n\to\infty} ||y_n - v_{n,i}|| = 0. \tag{3.2.33}$$

Also

$$\begin{aligned}
||z_n - y_n|| &= \Big\|\beta_{n,0}y_n + \sum_{i=1}^{m}\beta_{n,i}v_{n,i} - y_n\Big\| \\
&\le \beta_{n,0}||y_n - y_n|| + \sum_{i=1}^{m}\beta_{m,i}||v_{n,i} - y_n|| \to 0,
\end{aligned}$$

therefore

$$\lim_{n\to\infty} ||z_n - x_n|| = \lim_{n\to\infty} (||z_n - y_n|| + ||y_n - x_n||) = 0.$$

Now from (3.2.6) and condition (C1), we get

$$\begin{aligned}
||x_{n+1} - z_n|| &= ||\delta_n\xi f(x_n) + (1-\delta_n D)z_n - z_n|| \\
&= \delta_n||\xi f(x_n) - Dz_n|| \to 0,
\end{aligned}$$

and therefore

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - z_n|| + ||z_n - x_n|| \to 0, \quad n \to \infty.$$

Next, we show that $\Omega_w(x_n) \subset \Omega_{VIP} \cap \bigcap_{i=1}^m F(S_i)$, where $\Omega_w(x_n)$ is the weak subsequential limit of $\{x_n\}$. Let $\bar{x} \in \Omega_w(x_n)$, and observe that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x}$ as $j \to \infty$. Let $\{w_{n_j}\}$ and $\{u_{n_j}\}$ be subsequences of $\{w_n\}$ and $\{u_n\}$ respectively. Consequently from (3.2.31), $w_{n_j} \rightharpoonup \bar{x}$ as $j \to \infty$. Let B be a mapping defined by

$$Bv = \begin{cases} Av + N_C(v), & if \quad v \in C, \\ \emptyset & if \quad v \notin C. \end{cases}$$

By Lemma 2.1.4, B is maximal monotone and $B^{-1}(0) = \Omega_{VIP}$. If we let $(v, w) \in Gr(B)$, then $w \in Bv = Av + N_C(v)$ and thus $w - Av \in N_C(v)$. This implies that

$$\langle v - t, w - Av\rangle \geq 0, \quad \text{for all} \quad t \in C,$$

and in particular

$$\langle v - w_{n_j}, w - Av\rangle \geq 0. \tag{3.2.34}$$

Since $w_{n_j} = P_C(u_{n_j} - \mu_{n_j} Au_{n_j})$, by the characterization of P_C, we obtain

$$\langle w_{n_j} - v, u_{n_j} - \mu_{n_j} Au_{n_j} - w_{n_j}\rangle \geq 0, \quad \text{for all} \quad v \in C.$$

Hence

$$\left\langle v - w_{n_j}, \frac{w_{n_j} - u_{n_j}}{\mu_{n_j}} + Au_{n_j}\right\rangle \geq 0, \quad \text{for all} \quad v \in C. \tag{3.2.35}$$

Therefore, we have from (3.2.34) and (3.2.35) that

$$\begin{aligned}
\langle v - w_{n_j}, w\rangle &\geq \langle v - w_{n_j}, Av\rangle \\
&\geq \langle v - w_{n_j}, Av\rangle - \left\langle v - w_{n_j}, \frac{w_{n_j} - u_{n_j}}{\mu_{n_j}} + Au_{n_j}\right\rangle \\
&= \langle v - w_{n_j}, Av - Aw_{n_j}\rangle + \langle v - w_{n_j}, Aw_{n_j} - Au_{n_j}\rangle - \left\langle v - w_{n_j}, \frac{w_{n_j} - u_{n_j}}{\mu_{n_j}}\right\rangle \\
&\geq \langle v - w_{n_j}, Aw_{n_j} - Au_{n_j}\rangle - \left\langle v - w_{n_j}, \frac{w_{n_j} - u_{n_j}}{\mu_{n_j}}\right\rangle \\
&\geq \langle v - w_{n_j}, Aw_{n_j} - Au_{n_j}\rangle - ||v - w_{n_j}|| \left|\left|\frac{w_{n_j} - u_{n_j}}{\mu_{n_j}}\right|\right|.
\end{aligned} \tag{3.2.36}$$

Passing to the limit in the above inequality in (3.2.36) (using the continuity of A and noting that $\liminf_{j\to\infty} \mu_{n_j} > 0$), it follows from (3.2.29) that

$$\langle v - \bar{x}, w\rangle \geq 0.$$

Since B is maximal monotone, it follows that $\bar{x} \in B^{-1}(0)$, hence $\bar{x} \in \Omega_{VIP}$. On the other hand, let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$. Note that $y_{n_j} \rightharpoonup \bar{x}$ as $j \to \infty$ (by (3.2.32)).

For each $i = 1, 2, \ldots, m$, $I - S_i$ are demiclosed at zero, then it follows from (3.2.33) that $\bar{x} \in F(S_i)$. This implies that $\bar{x} \in \bigcap_{i=1}^m F(S_i)$. Hence

$$\bar{x} \in \Omega_{VIP} \cap \bigcap_{i=1}^{m} F(S_i).$$

Next, we show that $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Gamma(I - D + \xi f)x^*$ is a unique solution of the variational inequality

$$\langle (D - \xi f)x^*, x^* - x\rangle \leq 0, \quad x \in \Gamma.$$

To do this, we prove that $\limsup_{n\to\infty}\langle (D - \xi f)x^*, x^* - x_n\rangle \leq 0$. Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{j\to\infty}\langle (D - \xi f)x^*, x^* - x_n\rangle = \lim_{j\to\infty}\langle (D - \xi f)x^*, x^* - x_{n_j}\rangle.$$

Since $x_{n_j} \rightharpoonup \bar{x}$, and using (2.2.2), we have

$$\begin{aligned}\limsup_{j\to\infty}\langle (D - \xi f)x^*, x^* - x_n\rangle &= \lim_{j\to\infty}\langle (D - \xi f)x^*, x^* - x_{n_j}\rangle \\ &= \langle (D - \xi f)x^*, x^* - \bar{x}\rangle \\ &= \langle x^* - (I - (D - \xi f))x^*, x^* - \bar{x}\rangle \leq 0. \quad (3.2.37)\end{aligned}$$

Now using Lemma 2.6.30, Lemma 3.2.9(i) and (3.2.37), we obtain that $||x_n - x^*|| \to 0$, which implies that $\{x_n\}$ converges strongly to x^*. This conclude Case A.

CASE B: Suppose $\{||x_n - p||\}$ is not monotonically decreasing. Choose some n_0 large enough and for all $n \geq n_0$, we define $\phi : \mathbb{N} \to \mathbb{N}$ by

$$\phi(n) = \max\{k \in \mathbb{N} : k \leq n : \phi_k \leq \phi_{k+1}\}.$$

Clearly, ϕ is non-decreasing, where $\phi(n) \to \infty$ as $n \to \infty$ and

$$0 \leq ||x_{\phi(n)} - p|| \leq ||x_{\phi(n)+1} - p||, \quad \text{for all} \quad n \geq n_0.$$

A similar argument as in CASE A, we obtain

$$||w_{\phi(n)} - u_{\phi(n)}|| \to 0, \quad ||v_{\phi(n),i} - y_{\phi(n)}|| \to 0, \quad ||x_{\phi(n)+1} - x_{\phi(n)}|| \to 0,$$

as $n \to \infty$ and $\Omega_w(x_{\phi(n)}) \subset \Omega_{VIP} \cap \bigcap_{i=1}^m F(S_i)$, where $\Omega_w(x_{\phi(n)})$ is the weak subsequential limit of $\{x_{\phi(n)}\}$. Also, we have

$$\limsup_{n\to\infty}\langle (D - \xi f)p, p - x_{\phi(n)}\rangle \leq 0. \tag{3.2.38}$$

Now, from Lemma 3.2.7(i) we have

$$\begin{aligned}||x_{\phi(n)+1} - p||^2 \leq\ & \left(1 - \frac{2\delta_{\phi(n)}(\rho - \xi\lambda)}{1 - \delta_{\phi(n)}\xi\lambda}\right)||x_{\phi(n)} - p||^2 + \frac{2\delta_{\phi(n)}(\rho - \xi\lambda)}{1 - \delta_{\phi(n)}\xi\lambda} \times \\ & (2\langle \xi f(p) - Dp, x_{\phi(n)+1} - p\rangle + \delta_{\phi(n)}M) \\ & + \frac{\alpha_{\phi(n)}M_2||x_{\phi(n)} - x_{\phi(n)-1}||}{1 - \delta_{\phi(n)}\xi\lambda},\end{aligned} \tag{3.2.39}$$

for some $M > 0$, where

$$M_2 = \sup_{n\geq 1}\Big((1-\delta_{\phi(n)}\rho)^2(||x_{\phi(n)}-p||+||x_{\phi(n)-1}-p||)+2(1-\delta_{\phi(n)}\rho)^2||x_{\phi(n)}-x_{\phi(n)-1}||\Big).$$

Since $||x_{\phi(n)}-p||^2 \leq ||x_{\phi(n)+1}-p||^2$, then from (3.2.39), we obtain

$$\begin{aligned} 0 \;\leq\;& \left(1-\frac{2\delta_{\phi(n)}(\rho-\xi\lambda)}{1-\delta_{\phi(n)}\xi\lambda}\right)||x_{\phi(n)}-p||^2+\frac{2\delta_{\phi(n)}(\rho-\xi\lambda)}{1-\delta_{\phi(n)}\xi\lambda}\times \\ & (2\langle \xi f(p)-Dp, x_{\phi(n)+1}-p\rangle+\delta_{\phi(n)}M) \\ & +\frac{\alpha_{\phi(n)}M_2||x_{\phi(n)}-x_{\phi(n)-1}||}{1-\delta_{\phi(n)}\xi\lambda}-||x_{\phi(n)}-p||^2. \end{aligned}$$

Hence

$$\begin{aligned} \frac{2\delta_{\phi(n)}(\rho-\xi\lambda)}{1-\delta_{\phi(n)}\xi\lambda}||x_{\phi(n)}-p||^2 \;\leq\;& \frac{2\delta_{\phi(n)}(\rho-\xi\lambda)}{1-\delta_{\phi(n)}\xi\lambda}(2\langle \xi f(p)-Dp, x_{\phi(n)+1}-p\rangle+\delta_{\phi(n)}M) \\ & +\frac{\alpha_{\phi(n)}M_2||x_{\phi(n)}-x_{\phi(n)-1}||}{1-\delta_{\phi(n)}\xi\lambda}. \end{aligned}$$

Therefore,

$$\begin{aligned} ||x_{\phi(n)}-p||^2 \;\leq\;& 2\langle \xi f(p)-Dp, x_{\phi(n)+1}-p\rangle+\delta_{\phi(n)}M_4 \\ & +\frac{\alpha_{\phi(n)}M_2||x_{\phi(n)}-x_{\phi(n)-1}||}{2\delta_{\phi(n)}(\rho-\xi\lambda)}. \end{aligned} \tag{3.2.40}$$

Since $\{x_{\phi(n)}\}$ is bounded and $\delta_{\phi(n)} \to 0$, as $n \to \infty$, then it follows from (3.2.38) and Remark 3.2.7 that

$$\lim_{n\to\infty}||x_{\phi(n)}-p|| = 0. \tag{3.2.41}$$

As a consequence, we obtain that for all $n \geq n_0$,

$$0 \leq ||x_n-p||^2 \leq \max\{||x_{\phi(n)}-p||^2, ||x_{\phi(n)+1}-p||^2\} = ||x_{\phi(n)+1}-p||^2.$$

Hence, $\lim_{n\to\infty}||x_n-p|| = 0$. This implies that $\{x_n\}$ converges strongly to p. This completes the proof. □

Recall that the class of quasi-nonexpansive mappings is 0-demi-contractive. Thus, we can also obtain the following result for approximating a common solution of the VIP and a finite family of multivalued quasi-nonexpansive mappings.

Corollary 3.2.12. *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $A : C \to H$ be a monotone and L-Lipschitz mapping. For each $i = 1, 2, \ldots, m$ let $S_i : H \to CB(H)$ be multivalued quasi-nonexpansive mappings with constant such that $I - S_i$ are demiclosed at zero, $S_i(p) = \{p\}$ for all $p \in F(S_i)$ and $i = 1, 2, \ldots, m$. Suppose $\Gamma = \Omega_{VIP} \cap \bigcap_{i=1}^{m} F(S_i) \neq \emptyset$ and let $f : H \to H$ be a λ-contraction with constant $\lambda \in (0, 1)$ and let D be a bounded operator with coefficient $\rho > 0$ such that $0 < \xi < \frac{\rho}{\lambda}$. Let $\{x_n\}$ be* *generated by Algorithm 3.2.4 in which Assumptions (C1) and (C3) are satisfied. Then the sequence $\{x_n\}$ converges strongly to a point z, where $z = P_\Gamma(I - D + \xi f)(z)$ is a unique solution of the variational inequality*

$$\langle (D - \xi f)z, z - x\rangle \leq 0, \quad x \in \Gamma. \tag{3.2.42}$$

Remark 3.2.13. For suitable starting points, Algorithm 3.2.4 generates appropriate solutions which approximate the whole solution set Γ as guaranteed by Theorem 3.2.11. This is an interesting property which is different, for example, from the class of Tikhonov-type regularization approaches where the corresponding sequences always converge to the same solution.

3.2.2 Numerical example

In this subsection, we provide some numerical examples to compare our inertial viscosity subgradient extragradient Algorithm 3.2.4 with Algorithms (3.2.2) and (3.2.3) of Thong and Hieu [256] and, and with iEgA (3.2.4) of Dong et al. [97].

We start by giving the following example of multivalued λ-demi-contractive mapping given by Jailoka and Suntai in [139]. Let $H = \mathbb{R}$, and for each $i \in \mathbb{N}$ defined $S_i : \mathbb{R} \to 2^{\mathbb{R}}$ by

$$S_i = \begin{cases} \left[\dfrac{-(1+2i)x}{2}, -(1+i)x\right], & x \leq 0, \\ \left[-(1+i)x, \dfrac{-(1+2i)x}{2}\right], & x > 0. \end{cases} \tag{3.2.43}$$

Then S_i is λ_i-demi-contractive with $\lambda_i = \dfrac{4i^2 + 8i}{4i^2 + 12i + 9} \in (0, 1)$.

Example 3.2.14. Many problems arising in signal and image processing can be formulated as inverting the equation system

$$b = Bx + e, \tag{3.2.44}$$

where $x \in \mathbb{R}^N$ is the unknown original image or data to be recovered, $b \in \mathbb{R}^M$ is the vector of noisy observations, e is an additive noise with bounded variance and $B : \mathbb{R}^N \to \mathbb{R}^M$ is a bounded linear observation operator. In particular, we note that B is typically ill behaved because it models an acquisition process that encounters loss of information. When attempting to find sparse solutions to linear inverse problems of type (3.2.44), a successful model is the convex unconstrained minimization problem

$$\min_{x \in \mathbb{R}^N} \frac{1}{2}||b - Bx||^2 + \nu||x||_1, \tag{3.2.45}$$

where ν is a postive number, $||\cdot||$ is the Euclidean norm and $||\cdot||_1$ is the l_1 norm. The aim of the l_1 term, which is the convex sparsity-promoting penalty, is to make the small component of x become zero. By means of convex analysis, one is able to show that a minimizer to (3.2.45) is actually a solution to the LASSO problem

$$\min_{x \in \mathbb{R}^N} \frac{1}{2}||b - Bx||^2 \quad \text{subject to} \quad ||x||_1 < t, \tag{3.2.46}$$

for any nonnegative real number t (see [108]). It is easy to see that the optimization problem (3.2.46) is a special case of the variational inequality problem (1.1.1), where $A(x) = B^T(Bx - b)$ and $C = \{x : ||x||_1 \leq t\}$. Hence, we can use the proposed Algorithm (3.2.4) to approximate a solution of (3.2.44). The projection onto the closed l_1 ball in $\mathbb{R}^N$ is computed through the soft thresholding operator defined by

$$P_C(x) = \mathbb{S}(x) = argmin_{u\in\mathbb{R}^N}\left\{||x-u||^2 + \lambda||x||_1\right\},$$

for $\lambda > 0$. We set $f(x) = \frac{x}{16}$, $D(x) = x$, $\sigma = 6$, $\delta = 0.9$, $\eta = 0.7$, $\delta_n = \frac{1}{n+1}$, $\epsilon_n = \frac{1}{(n+1)^4}$ and $\alpha = 3$ in Algorithm 3.2.4, and for each $n \in \mathbb{N}$ and $i \geq 0$, define

$$\beta_{n,i} = \begin{cases} 0 & if \quad n < i, \\ 1 - \frac{n}{n+1}\sum_{k=1}^{n}\frac{1}{2^k} & if \quad n = i, \\ \frac{1}{2^{i+1}}\left(\frac{n}{n+1}\right) & if \quad n > i. \end{cases} \tag{3.2.47}$$

We set the image to go through a random blur and random noise and choose different values of starting point as follows: $x_0 = -0.5 * randn(N,1)$ and $x_1 = 2 * randn(N,1)$, where

$$Case(i) \quad N = 100, \quad Case(ii) \quad N = 200 \quad \text{and} \quad Case(iii) \quad N = 500.$$

We then plot the graphs of the error term ($||x_{n+1} - x_n||$) against number of iterations for Algorithm 3.2.4, THSEgM(i) and THSEgM(ii). The numerical result is shown in Table 3.2 and Figure 3.2. The stopping criterion used is $\dfrac{||x_{n+1} - x_n||}{||x_2 - x_1||} \leq 10^{-4}$.

Example 3.2.15. Suppose $H = L^2([0,1])$ with $||x||_{L^2} := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}$ and inner product
$\langle x, y\rangle := \int_0^1 x(t)y(t)dt$, for all $x, y \in H$. Define $A : H \to H$ by

$$A(x(t)) = \max\left\{0, x(t)\right\}. \tag{3.2.48}$$

It is easy to verify that A is 1-Lipschitz continuous and monotone on H. Let $C := \{x \in H : ||x|| \leq 1\}$ be the unit ball. It is known that

$$P_C(x) = \begin{cases} \dfrac{x}{||x||_{L^2}} & if \quad ||x||_{L^2} > 1, \\ x & if \quad ||x||_{L^2} \leq 1. \end{cases} \tag{3.2.49}$$

For $i = 1, 2, \dots, m$, let $S_i : L^2([0,1)]) \to L^2([0,1])$ be defined by

$$(S_i x)(t) = \int_0^1 x(t)dt, \quad t \in [0,1].$$

Clearly, S_i is 0-demi-contractive and $\Gamma = \Omega_{VIP} \cap \bigcap_{i\in\mathbb{N}} F(S_i) = \{0\}$. We choose the following starting points:
Case(i): $x_0(t) = t^2\exp(7t)$ and $x_1(t) = \frac{1}{6}\sin(-3t)$,
Case(ii): $x_0(t) = 0.5\cos(t)$ and $x_1(t) = \cos(-10t)$,
Case(iii): $x_0(t) = 5\exp(t)$ and $x_1(t) = \frac{2}{3}\cos(t)$.

Let $f(x) = \frac{x(t)}{2}$, $D(x) = \int_0^1 x(t)dt$, $\sigma = 4$, $\eta = 0.5$, $\delta_n = \frac{1}{n+1}$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\alpha = 3$, and for each $i \in \mathbb{N} \cup \{0\}$ $\beta_{n,i} = \frac{1}{n}$. Using $\dfrac{||x_{n+1} - x_n||}{||x_2 - x_1||} < 10^{-6}$ as a stopping criterion, we plot the garphs of $||x_{n+1} - x_n||$ against the number of iterations for both Algorithm 3.2.4 and iEgA (3.2.4). The numerical result is shown in Table 3.3 and Figures 3.3.

Table 3.2: Comparison of Algorithm 3.2.4, THSEgM(I) 3.2.2 and THSEgM(II) 3.2.3 for Example 3.2.14.

	Time Taken (Sec)		
	Case(i)	Case(ii)	Case(iii)
Algorithm 3.2.4	0.0052	0.0069	0.0096
THSEgM(I)	0.0155	0.0363	0.0312
THSEgM(II)	0.0367	0.0363	0.0383

Table 3.3: Comparison between Algorithm 3.2.4 and iEgA (3.2.4) for Example 3.2.15.

	Time taken (Secs)		
Algorithms	Case(i)	Case(ii)	Case(iii)
Algorithm 3.2.4	0.3641	0.1257	0.2523
iEgA (3.2.4)	0.5260	0.3291	1.7566

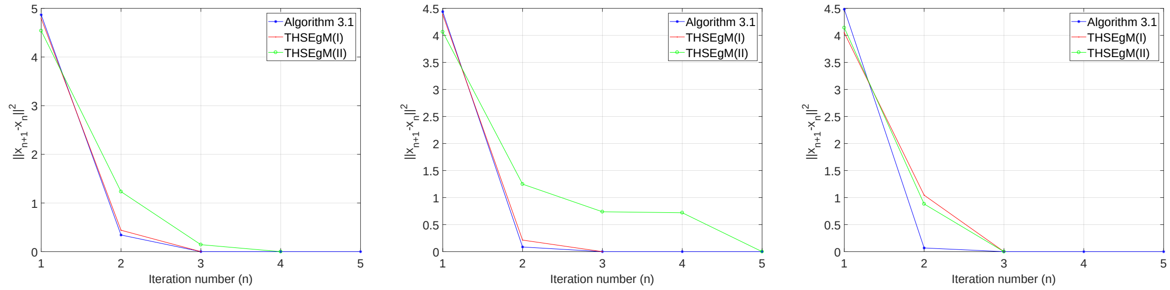


Figure 3.2: Example 3.2.14, Left: Case(i); Middle: Case(ii); Right: Case(iii).

3.3 An Inertial-Mann Algorithm for Split Generalized Mixed Equilibrium Problem and Fixed Point of Nonspreading Mapping in Hilbert Spaces

In this section, we study a split generalized mixed equilibrium problem and fixed point problem for nonspreading mapping in real Hilbert spaces.

Let H_1, H_2 be real Hilbert spaces and C and Q be nonempty closed convex subsets of H_1 and H_2 respectively. Let $\Theta_1 : C \times C \to \mathbb{R}$ and $\Theta_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions,

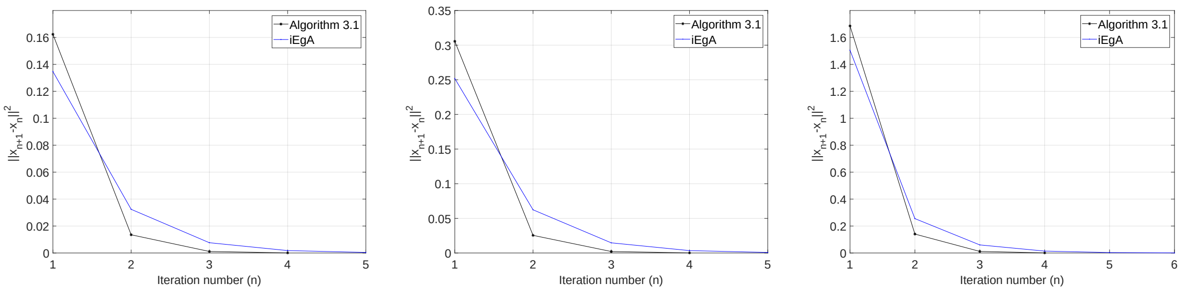


Figure 3.3: Example 3.2.15, Left: Case(i); Middle: Case(ii); Right: Case(iii).

$h_1 : C \to H_1$ and $h_2 : Q \to H_2$ be nonlinear mappings, $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions and $A : H_1 \to H_2$ a bounded linear operator. The Split Generalized Mixed Equilibrium Problem (SGMEP) is defined as follow: find a point $x^\dagger \in C$ such that

$$\begin{cases} \Theta_1(x^\dagger, x) + \langle h_1(x^\dagger), x - x^\dagger \rangle + \phi(x) - \phi(x^\dagger) \geq 0, \qquad \forall x \in C, \\ \text{ with} \\ y^\dagger = Ax^\dagger \quad \text{solves} \quad \Theta_2(y^\dagger, y) + \langle h_2(y^\dagger), y - y^\dagger \rangle + \varphi(y) - \varphi(y^\dagger) \geq 0, \qquad \forall y \in Q. \end{cases} \tag{3.3.1}$$

The set of solutions of the SGMEP is denoted by $\Omega_{SGMEP} := \{x^\dagger \in GMEP(\Theta_1, h_1, \phi) : Ax^\dagger \in GMEP(\Theta_2, h_2, \varphi)\}$.

We present the following examples to show that Ω_{SGMEP} is nonempty.

Example 3.3.1. Let $H_1 = H_2 = \mathbb{R}$, $C = [2, \infty)$ and $Q = (-\infty, -4]$. Let $A(x) = -2x$ for all $x \in \mathbb{R}$, then A is a bounded linear operator. Let $\Theta_1 : C \times C \to \mathbb{R}$ and $\Theta_2 : Q \times Q \to \mathbb{R}$ be define by $\Theta_1(x, y) = y - x$, $\Theta_2(u, v) = 3(u - v)$; $h_1 : C \to \mathbb{R}$ and $h_2 : Q \to \mathbb{R}$ be define by $h_1(x) = x$, $h_2(u) = 2u$; $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be defined by $\phi(x) = \dfrac{x^2}{2}$ and $\varphi(u) = 2u$. Clearly, $GMEP(\Theta_1, h, \phi) = \{2\}$ and $A(2) = -4 \in GMEP(\Theta_2, h_2, \varphi)$. Thus, $\Omega_{GMEP} = \{p \in GMEP(\Theta_1, h_1, \phi) : Ap \in GMEP(\Theta_2, h_2, \varphi)\} \neq \emptyset$.

Example 3.3.2. Let $H_1 = \mathbb{R}^2$ with the norm $||\bar{x}|| = \sqrt{x_1^2 + x_2^2}$ for $\bar{x} = (x_1, x_2) \in \mathbb{R}^2$ and $H_2 = \mathbb{R}$. Let $C := \{\bar{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 - x_1 \geq 1\}$ and $Q = [1, \infty)$. Define $\Theta_1(\bar{x}, \bar{y}) = y_2 - y_1 - x_2 + x_1$, where $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2) \in C$, then Θ_1 is a bifunction from $C \times C \to \mathbb{R}$. Let $h_1(\bar{x}) = \phi(\bar{x}) = x_2 - x_1$, then $GMEP(\Theta_1, h_1, \phi) = \{\bar{q} = (q_1, q_2) : q_2 - q_1 = 1\}$. Also define $\Theta_2(u, v) = v - u$ for all $u, v \in Q$, then Θ_2 is a bifunction from $Q \times Q$ to $\mathbb{R}$ and let $h_2(u) = 2u$, $\varphi(u) = u$. For each $\bar{x} = (x_1, x_2) \in H_1$, let $A(\bar{x}) = x_2 - x_1$, then A is bounded linear operator from H_1 into H_2. Clearly, when $\bar{q} \in GMEP(\Theta_1, h_1, \phi)$, we have $A\bar{q} = 1 \in GMEP(\Theta_2, h_2, \varphi)$. Thus $\Omega_{SGMEP} = \{\bar{q} \in GMEP(\Theta_1, h_1, \phi) : A\bar{q} \in GMEP(\Theta_2, h_2, \varphi)\} \neq \emptyset$.

Remark 3.3.3. We note that SGMEP in Example 3.3.1 lies in two different subsets of the same space, while SGMEP in Example 3.3.2 lies in two different subsets of different spaces.

In 2016, Suantai *et al.* [240] studied the Split Equilibrium Problem which is defined as: find a point $x^* \in C$ such that

$$\Theta_1(x^*, x) \geq 0, \quad \forall x \in C, \quad \text{and} \quad y^* = Ax^* \in Q \quad \text{solves} \quad \Theta_2(y^*, y) \geq 0, \quad \forall y \in C, \quad (3.3.2)$$

where $\Theta_1 : C \times C \to \mathbb{R}$ and $\Theta_2 : Q \times Q \to \mathbb{R}$ are nonlinear bifuncions. The set of solution of (3.3.2) is denoted by Ω_{SEqP}. The authors in [239] proposed the following iterative algorithm to solve the problem of finding a common element in Ω_{SEqP} and a fixed point of a nonspreading multi-valued mapping in Hilbert spaces: Given $\{x_n\}$ by

$$\begin{cases} x_1 \in C \quad \text{arbitrarily}, \\ u_n = T_{r_n}^{\Theta_1}(I - \gamma A^*(I - T_{r_n}^{\Theta_2})A)x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Su_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.3.3}$$

where $T_{r_n}^{\Theta_1}$ is the resolvent operator defined in Lemma 3.3.5, $\{\alpha_n\} \subset (0,1)$, $r_n \in (0, \infty)$ and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and A^* is the adjoint of A, $S : C \to K(C)$ is a nonspreading multi-valued mapping. Further, they proved that under certain conditions, the sequence $\{x_n\}$ converges weakly to an element of $F(S) \cap \Omega_{SEqP}$.

More recently, S.H. Rizvi [218] studied the following Split Mixed Equilibrium Problem $(SMEP)$ in real Hilbert spaces: find a point $x^* \in C$ such that

$$\begin{cases} \Theta_1(x^*, x) + \langle h_1 x^*, x - x^* \rangle \geq 0, \qquad \forall x \in C, \\ \text{with} \\ y^* = Ax^* \quad \text{solves} \quad \Theta_2(y^*, y) + \langle h_2 y^*, y - y^* \rangle \geq 0, \qquad \forall y \in Q, \end{cases} \tag{3.3.4}$$

where $h_1 : C \to C$ and $h_2 : Q \to Q$ are θ_1, θ_2-inverse strongly monotone mapping respectively with $\theta = \min(\theta_1, \theta_2)$. The set of solution of (3.3.4) is denoted by Ω_{SMEP}. Observe that when $\phi = \varphi = 0$ in (3.3.1), we obtain (3.3.4). Thus, Problem (3.3.1) is more general than Problem (3.3.4). Rizvi [218] introduced the following algorithm for solving (3.3.4) and fixed point problem for a nonexpansive mapping S in real Hilbert spaces:

$$\begin{cases} x_0 = x \in C, \\ y_n = T_{r_n}^{\Theta_1}(x_n - r_n \phi x_n), \\ v_n = T_{r_n}^{\Theta_2}(I - r_n \psi)Ay_n, \\ z_n = P_C(y_n + \delta A^*(v_n - Ay_n)), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S[\alpha_n u + (1-\alpha_n)z_n], \quad n \geq 0, \end{cases} \tag{3.3.5}$$

where P_C is the metric projection from H onto C, $\{r_n\} \subset (0, 2\theta)$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. The author also proved that under some mild conditions on α_n, β_n and r_n, the sequence $\{x_n\}$ converges strongly to a solution in $\Omega_{SMEP} \cap F(S)$.

By combining the Picard algorithm [205] and the conjugate gradient methods [195], Dong and Yaun [98] accelerated the Mann algorithm and obtained the following faster algorithm:

$$\begin{cases} d_{n+1} = \frac{1}{\lambda}(T(x_n) - x_n) + \beta_n d_n, \\ y_n = x_n + \lambda d_{n+1}, \\ x_{n+1} = \mu \alpha_n x_n + (1 - \mu \alpha_n) y_n, \end{cases} \tag{3.3.6}$$

for each $n \geq 0$, where $\mu \in (0,1]$ and $\lambda > 0$. They proved that the iterative sequence $\{x_n\}$ converges weakly to a fixed point of T provided that the nonnegative sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(BB1) $\sum_{n=0}^{\infty} \mu\alpha_n(1-\mu\alpha_n) = \infty$,

(BB2) $\sum_{n=0}^{\infty} \beta_n < \infty$.

More so, the sequence $\{x_n\}$ is assumed to satisfied the following:

$$\{T(x_n) - x_n\} \text{ is bounded.}$$

In this section, we introduce a modified-Mann algorithm for finding a common solution of SGMEP and fixed point of nonspreading mapping in real Hilbert spaces. It is easy to re-write (3.3.6) as the following inertial algorithm:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \mu\alpha_n w_n + (1-\mu\alpha_n)T(x_n). \end{cases} \tag{3.3.7}$$

Motivated by the works of Suantai et al. [240], Rizvi et al. [218], Dong and Yuan [98], it is our aim in this section to propose a new iterative algorithm for approximating a common solution of (3.3.1) and fixed point of a nonspreading mapping in real Hilbert spaces. Our algorithm is developed by modifying the accelerated Mann algorithm (3.3.6) combined with a modified viscosity approximation method to obtain a new faster iterative algorithm for finding a common solution of (3.3.1) and a fixed point of nonspreading mapping in real Hilbert spaces. Further, our algorithm does not require any prior knowledge of the operator norm.

For solving the SGMEP we make the following assumption:

Assumption 3.3.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H. We make the following assumptions on the bifunction $\Theta : C \times C \to \mathbb{R}$:*

LL1. $\Theta(x,x) = 0$, *for all* $x \in C$,

LL2. Θ *is monotone, i.e* $\Theta(x,y) + \Theta(y,x) \leq 0$, $\forall x, y \in C$,

LL3. for each $x, y, z \in C$, $\lim_{t\downarrow 0} \Theta(tz + (1-t)x, y) \leq \Theta(x,y)$,

LL4. for each $x \in C$, $y \mapsto \Theta(x,y)$ *is convex and lower semicontinuous,*

The following lemma will also be used in this section.

Lemma 3.3.5. [159] *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction which satisfies Assumption 3.3.4, $h : C \to H_1$ be a nonlinear mapping and let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. For $r > 0$ and $x \in H_1$, define a resolvent function*

$$T_r^{\Theta}(x) = \{z \in C : \Theta(z,y) + \langle h(z), y - z\rangle + \phi(y) - \phi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\},$$

for all $x \in H$. Then the following conclusions hold:

(i) for each $x \in H$, $T_r^{\Theta}(x) \neq \emptyset$,

(ii) T_r^{Θ} is single-valued,

(iii) T_r^{Θ} is firmly nonexpansive, i.e for any $x, y \in H$,

$$||T_r^{\Theta}x - T_r^{\Theta}y||^2 \leq \langle T_r^{\Theta}x - T_r^{\Theta}y, x - y\rangle,$$

(iv) $F(T_r^{\Theta}) = GMEP(\Theta, h, \phi)$,

(v) $GMEP(\Theta, h, \phi)$ is closed and convex.

3.3.1 Main results

Algorithm 3.3.6. *Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. Let $\Theta_1 : C \times C \to \mathbb{R}$ and $\Theta_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 3.3.4. Let $h_1 : C \to H_1$ and $h_2 : Q \to H_2$ be θ_1, θ_2-inverse strongly monotone operators, respectively, such that $\theta = \max\{\theta_1, \theta_2\}$. Let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be proper, lowersemicontinuous and convex functions, and let $S : C \to C$ be a nonspreading mapping such that $F(S) \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\beta \in (0, 1)$ and D be a bounded operator with coefficient $\bar{\gamma} \in (0, 1)$ such that $0 < \xi < \frac{\bar{\gamma}}{\beta}$. Choose an initial point $x_1 \in H_1$ arbitrarily and let $\alpha_n \in [0, 1]$, $\beta_n \in [0, 1]$, $w_n \in (0, 1)$, $r_n \in (0, 2\theta)$ and $\lambda > 0$. Assume that the nth iterate has been constructed, and set $m_1 = \frac{\gamma_1 A^*(T_{r_1}^{\Theta_2} - I)Ax_1}{\lambda}$. We then compute the $(n + 1)$th iterate via the formula*

$$\begin{cases} m_{n+1} = \dfrac{\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n}{\lambda} + \beta_n m_n, \\ y_n = x_n + \lambda m_{n+1}, \\ z_n = T_{r_n}^{\Theta_1}(I - r_n h_1)y_n, \\ x_{n+1} = \alpha_n \xi f(x_n) + (1 - \alpha_n D)[(1 - w_n)z_n + w_n S z_n], \end{cases} \tag{3.3.8}$$

for $n \geq 1$, where A^* is the adjoint operator of A. Further, we choose the stepsize γ_n such that, if $n \in O := \{n : (I - T_{r_n}^{\Theta_2})Ax_n \neq 0\}$, then

$$\gamma_n \in \left(0, \frac{2||(I - T_{r_n}^{\Theta_2})x_n||^2}{||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2}\right), \quad \forall n \in O. \tag{3.3.9}$$

Otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value).

Remark 3.3.7. Note that in (3.3.9), the choice of stepsize γ_n is independent of the norm $||A||$. The value of γ does not influence the considered algorithm but was introduced just for the sake of clarity. Furthermore, we will see from Lemma 3.3.8 that γ_n is well defined.

Lemma 3.3.8. *Assume that $\Omega_{SGMEP} := \{q \in GMEP(\Theta_1, h_1, \phi) : Aq \in GMEP(\Theta_2, h_2, \varphi)\}$ is nonempty. Then γ_n defined by (3.3.9) is well defined.*

Proof. We need to show that $||A^*(I - T_{r_n}^{\Theta_2})Ax_n|| > 0$. Take $x \in \Omega$, then $T_{r_n}^{\Theta_1}x = x$ and $T_{r_n}^{\Theta_2}Ax = Ax$, and observe the following:

$$\begin{aligned}
||(I - T_{r_n}^{\Theta_2})Ax_n||^2 &= \langle (I - T_{r_n}^{\Theta_2})Ax_n, (I - T_{r_n}^{\Theta_2})Ax_n \rangle \\
&= \langle (I - T_{r_n}^{\Theta_2})Ax_n, Ax_n - Ax + T_{r_n}^{\Theta_2}Ax - T_{r_n}^{\Theta_2}Ax_n \rangle \\
&= \langle (I - T_{r_n}^{\Theta_2})Ax_n, Ax_n - Ax \rangle + \langle (I - T_{r_n}^{\Theta_2})Ax_n, T_{r_n}^{\Theta_2}Ax - T_{r_n}^{\Theta_2}Ax_n \rangle \\
&= \langle A^*(I - T_{r_n}^{\Theta_2})Ax_n, x_n - x \rangle + \langle (I - T_{r_n}^{\Theta_2})Ax_n, T_{r_n}^{\Theta_2}Ax - T_{r_n}^{\Theta_2}Ax_n \rangle \\
&\leq ||A^*(I - T_{r_n}^{\Theta_2})Ax_n|| \times ||x_n - x|| + ||(I - T_{r_n}^{\Theta_2})Ax_n|| \times \\
&\quad ||T_{r_n}^{\Theta_2}Ax - T_{r_n}^{\Theta_2}Ax_n||.
\end{aligned}$$

Consequently, for $n \in O$, that is $||(I - T_{r_n}^{\Theta_2}Ax_n)Ax_n|| > 0$, we get $||A^*(I - T_{r_n}^{\Theta_2}Ax_n)Ax_n|| \times ||x_n - x|| > 0$ and $||(I - T_{r_n}^{\Theta_2})Ax_n|| \times ||T_{r_n}^{\Theta_2}Ax - T_{r_n}^{\Theta_2}Ax_n|| > 0$. Since $||A^*(I - T_{r_n}^{\Theta_2})Ax_n|| \times ||x_n - x|| > 0$, we obtain that $||A^*(I - T_{r_n}^{\Theta_2})Ax_n|| \neq 0$. This implies that γ_n is well defined. □

We make the following assumptions on the control sequences:

Assumption 3.3.9. *The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in Algorithm 3.3.6 satisfy the following:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\sum_{n=0}^{\infty} \beta_n < \infty$,

(C3) $\beta_n \leq \alpha_n^4$.

Furthermore, $\{x_n\}$ satisfies

(C4) $\{(T_{r_n}^{\Theta_2} - I)Ax_n\}$ *is bounded.*

Before giving the convergence analysis of Algorithm 3.3.6, we first prove the following result.

Lemma 3.3.10. *Suppose that $\Gamma := \Omega_{SGMEP} \cap F(S) \neq \emptyset$ and $\{x_n\}$ is generated by (3.3.8). Also, let Assumption 3.3.9 be satisfied and suppose r_n satisfies the following condition:*

(C5) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n \leq 2\theta$.

Then, $\{m_n\}$ and $\{x_n\}$ are bounded, and consequently $\{y_n\}$ is bounded.

Proof. It follows from (C2) that $\lim_{n\to\infty} \beta_n = 0$ and so there exists $n_0 \in \mathbb{N}$ such that $\beta_n \leq \frac{1}{2}$ for all $n \geq n_0$. Define a number $N_1 := \max\left\{\max_{1\leq k\leq n_0} ||m_k||, \frac{2}{\lambda} \sup_{n\geq 1} ||\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n||\right\}$.

Then (C4) implies that $N_1 < \infty$. Assume that $||m_n|| \le N_1$ for some $n \ge n_0$, then the triangle inequality ensures that

$$\begin{aligned} ||m_{n+1}|| &= \left|\left|\frac{\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n}{\lambda} + \beta_n m_n\right|\right| \\ &\le \frac{1}{\lambda}||\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n|| + \beta_n||m_n|| \le N_1, \end{aligned} \tag{3.3.10}$$

which means that $||m_{n+1}|| \le N_1$ for all $n \ge 0$, hence $\{m_n\}$ is bounded.

Also, the definition of $\{y_n\}$ implies that

$$\begin{aligned} y_n &= x_n + \lambda\Big(\frac{1}{\lambda}(\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n) + \beta_n m_n\Big) \\ &= x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n + \lambda\beta_n m_n. \end{aligned}$$

Let $p \in \Gamma$, then

$$\begin{aligned} ||y_n - p|| &= ||x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n + \lambda\beta_n m_n - p|| \\ &\le ||x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n - p|| + \lambda\beta_n||m_n||. \end{aligned} \tag{3.3.11}$$

Observe that

$$\begin{aligned} &||x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n - p||^2 \\ &= ||x_n - p||^2 - 2\gamma_n\langle x_n - p, A^*(I - T_{r_n}^{\Theta_2})Ax_n\rangle + \gamma_n^2||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 \\ &= ||x_n - p||^2 - 2\gamma_n\langle Ax_n - Ap, (I - T_{r_n}^{\Theta_2})Ax_n\rangle + \gamma_n^2||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 \\ &= ||x_n - p||^2 - 2\gamma_n\langle T_{r_n}^{\Theta_2}Ax_n - Ap, (I - T_{r_n}^{\Theta_2})Ax_n\rangle - 2\gamma_n||(I - T_{r_n}^{\Theta_2})Ax_n||^2 \\ &+ \gamma_n^2||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2. \end{aligned} \tag{3.3.12}$$

Since $T_{r_n}^{\Theta_2}$ is firmly nonexpansive, then

$$||T_{r_n}^{\Theta_2}Ax_n - Ap||^2 \le \langle T_{r_n}^{\Theta_2}Ax_n - Ap, Ax_n - Ap\rangle,$$

and so

$$\langle T_{r_n}^{\Theta_2}Ax_n - Ap, T_{r_n}^{\Theta_2}Ax_n - Ax_n\rangle \le 0. \tag{3.3.13}$$

It follows from (3.3.12) and (3.3.13) that

$$\begin{aligned} &||x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n - p||^2 \\ &\le ||x_n - p||^2 - 2\gamma_n||(I - T_{r_n}^{\Theta_2})Ax_n||^2 + \gamma_n^2||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 \\ &= ||x_n - p||^2 - \gamma_n\Big[2||(I - T_{r_n}^{\Theta_2})Ax_n||^2 - \gamma_n||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2\Big] \\ &\le ||x_n - p||^2. \end{aligned} \tag{3.3.14}$$

Therefore, from (3.3.11) and (3.3.14), we get

$$||y_n - p|| \le ||x_n - p|| + \lambda\beta_n N_1. \tag{3.3.15}$$

Again from (3.3.8), we use the fact that $T_{r_n}^{\Theta_1}$ is firmly nonexpansive to show that

$$\begin{aligned}||z_n - p||^2 &= ||T_{r_n}^{\Theta_1}(I - r_nh_1)y_n - T_{r_n}^{\Theta_1}(I - r_nh_1)p||^2 \\ &\leq ||(I - r_nh_1)y_n - (I - r_nh_1)p||^2 \\ &= ||(y_n - p) - r_n(h_1y_n - h_1p)||^2 \\ &= ||y_n - p||^2 - 2r_n\langle y_n - p, h_1y_n - h_1p\rangle + r_n^2||h_1y_n - h_1p||^2 \\ &\leq ||y_n - p||^2 - 2r_n\theta||h_1y_n - h_1p||^2 + r_n^2||h_1y_n - h_1p||^2 \\ &= ||y_n - p||^2 - r_n(2\theta - r_n)||h_1y_n - h_1p||^2. \end{aligned} \tag{3.3.16}$$

By condition (C5), we obtain

$$||z_n - p||^2 \leq ||y_n - p||^2. \tag{3.3.17}$$

Now define $U_n = (1 - w_n)I + w_nS$, and observe that

$$\begin{aligned}||U_nz_n - p|| &= ||(1 - w_n)(z_n - p) + w_n(Sz_n - p)|| \\ &\leq (1 - w_n)||z_n - p|| + w_n||Sz_n - p|| \\ &\leq (1 - w_n)||z_n - p|| + w_n||z_n - p|| \\ &= ||z_n - p||.\end{aligned}$$

Therefore, from (3.3.8), (3.3.15) and (3.3.17), we have

$$\begin{aligned}||x_{n+1} - p|| &= ||\alpha_n(\xi f(x_n) - Dp) + (1 - \alpha_nD)(U_nz_n - p)|| \\ &\leq \alpha_n||\xi f(x_n) - Dp|| + (1 - \alpha_n\bar{\gamma})||U_nz_n - p|| \\ &\leq \alpha_n\Big[||\xi(f(x_n) - f(p)) + (\xi f(p) - Dp)||\Big] + (1 - \alpha_n\bar{\gamma})||z_n - p|| \\ &\leq \alpha_n\xi\beta||x_n - p|| + \alpha_n||\xi f(p) - Dp|| + (1 - \alpha\bar{\gamma})[||x_n - p|| + \lambda\beta_nN_1] \\ &= (1 - \alpha_n(\bar{\gamma} - \xi\beta))||x_n - p|| + \alpha_n||\xi f(p) - Dp|| + \lambda\beta_nN_1 \\ &\leq \max\left\{||x_n - p||, \frac{||\xi f(p) - Dp||}{\bar{\gamma} - \xi\beta} + \frac{\lambda N_1}{\bar{\gamma} - \xi\beta}\right\} \\ &\vdots \\ &\leq \max\left\{||x_1 - p||, \frac{||\xi f(p) - Dp||}{\bar{\gamma} - \xi\beta} + \frac{\lambda N_1}{\bar{\gamma} - \xi\beta}\right\}.\end{aligned} \tag{3.3.18}$$

This implies that $\{x_n\}$ is bounded. It follows from (3.3.15) that $\{y_n\}$ is also bounded.

□

We now present the main theorem for the convergence analysis of Algorithm 3.3.6.

Theorem 3.3.11. *Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ a bounded linear operator. Let $\Theta_1 : C \times C \to \mathbb{R}$ and $\Theta_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 3.3.4. Let $h_1 : C \to H_1$ and $h_2 : Q \to H_2$ be θ_1, θ_2-inverse strongly monotone mappings, respectively, such that $\theta = \max\{\theta_1, \theta_2\}$. Let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be proper, lowersemicontinuous and convex functions, and let $S : C \to C$ be a nonspreading mapping* *such that $F(S) \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\beta \in (0,1)$, and let D a bounded operator with coefficient $\bar{\gamma} \in (0,1)$ such that $0 < \xi < \frac{\bar{\gamma}}{\beta}$. Choose an initial value $x_1 \in H_1$ arbitrarily and let $\alpha_n \in [0,1]$, $\beta_n \in [0,1]$, $w_n \in (0,1)$, $r_n \in (0, 2\theta)$ and $\lambda > 0$. Suppose $\Gamma := \Omega_{SGMEP} \cap F(S) \neq \emptyset$, Assumption 3.3.9, condition (C5) and the following are satisfied:*

(C6) $\liminf_{n\to\infty} r_n > 0$;

(C7) $0 < \liminf_{n\to\infty} w_n \leq \limsup_{n\to\infty} w_n < 1$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 3.3.6 converge strongly to a point z, where $z = P_\Gamma(I - D + \xi f)(z)$ is a unique solution of the variational inequality

$$\langle (D - \xi f)z, z - x\rangle \leq 0, \quad x \in \Gamma. \tag{3.3.19}$$

Proof. Let $p \in \Gamma$, then from Lemma 2.6.1(i) and (3.3.14), we have

$$\begin{aligned} ||y_n - p||^2 &= ||x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n - p + \lambda\beta_n m_n||^2 \\ &\leq ||x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n - p||^2 + 2\lambda\beta_n\langle y_n - p, m_n\rangle \\ &\leq ||x_n - p||^2 + \beta_n\rho_n, \end{aligned} \tag{3.3.20}$$

where $\rho_n := 2\lambda\langle y_n - p, m_n\rangle$. Using Lemma 3.3.10, it follows that $\{\rho_n\}$ is bounded. Thus, there exists $N_2 > 0$ such that $\rho_n \leq N_2$ for all $n \geq 1$. Hence, it follows from condition (C3) that

$$||y_n - p||^2 \leq ||x_n - p||^2 + 2\alpha_n^4 N_2. \tag{3.3.21}$$

Furthermore, from (3.3.17) and (3.3.21), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||\alpha_n(\xi f(x_n) - Dp) + (1 - \alpha_n D)(U_n z_n - p)||^2 \\ &\leq ||(1 - \alpha_n D)(U_n z_n - p)||^2 + 2\alpha_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \\ &\leq (1 - \alpha_n\bar{\gamma})^2||z_n - p||^2 + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle \\ &\quad + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &\leq (1 - \alpha_n\bar{\gamma})^2||y_n - p||^2 + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle \\ &\quad + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &\leq (1 - \alpha_n\bar{\gamma})^2\Big[||x_n - p||^2 + 2\alpha^4 N_2\Big] + 2\alpha_n\xi\beta||x_n - p|| \times ||x_{n+1} - p|| \\ &\quad + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle. \end{aligned} \tag{3.3.22}$$

We now divide the remaining proof of the theorem into two cases.
Case I: Suppose there exists $n_0 \in \mathbb{N}$ such that $\{||x_n - p||\}$ is monotonically decreasing for all $n \geq n_0$. Then $\{||x_n - p||\}$ converges as $n \to \infty$ and so

$$||x_n - p||^2 - ||x_{n+1} - p||^2 \to 0, \quad n \to \infty.$$

Note that, from (3.3.14), (3.3.20) and (3.3.21), we obtain

$$\begin{aligned} ||y_n - p||^2 &\leq ||x_n - p||^2 - \gamma_n\Big[2||(I - T_{r_n}^{\Theta_2})Ax_n||^2 - \gamma_n||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2\Big] \\ &\quad +2\alpha_n^4 N_2. \end{aligned} \tag{3.3.23}$$

Also from (3.3.22), we obtain

$$\begin{aligned} ||x_{n+1} - p||^2 &\leq (1 - \alpha_n\bar{\gamma})^2||y_n - p||^2 + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle \\ &\quad +2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &\leq ||y_n - p||^2 + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle \\ &\quad +2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle. \end{aligned} \tag{3.3.24}$$

Substituting (3.3.23) into (3.3.24), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &\leq ||x_n - p||^2 - \gamma_n\Big[2||(I - T_{r_n}^{\Theta_2})Ax_n||^2 - \gamma_n||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2\Big] \\ &\quad +2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &\quad +2\alpha_n^4 N_2. \end{aligned} \tag{3.3.25}$$

Putting $\Lambda_n := 2||I - T_{r_n}^{\Theta_2} Ax_n||^2 - \gamma_n||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2$, then since $\alpha_n \to 0$, as $n \to \infty$, it follows from (3.3.25) that

$$\begin{aligned} \gamma_n\Lambda_n &\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle \\ &\quad +2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle + 2\alpha_n^4 N_2 \to 0. \end{aligned} \tag{3.3.26}$$

From the condition on the stepsize given by (3.3.9), for a small $\epsilon > 0$, we know that

$$\gamma_n < \frac{2||(I - T_{r_n}^{\Theta_2})Ax_n||^2}{||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2} - \epsilon, \tag{3.3.27}$$

which implies

$$\gamma_n||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 < 2||(I - T_{r_n}^{\Theta_2})Ax_n||^2 - \epsilon||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2$$

and thus we have

$$\epsilon||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 < 2||(I - T_{r_n}^{\Theta_2})Ax_n||^2 - \gamma_n||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2.$$

This implies that

$$\epsilon||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 < \Lambda_n \to 0, \quad \text{as} \quad n \to \infty.$$

Hence

$$\lim_{n\to\infty} ||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 = 0. \tag{3.3.28}$$

Further, from (3.3.26) and (3.3.28), we get

$$
\begin{aligned}
0 < \epsilon||(I-T_{r_n}^{\Theta_2})Ax_n||^2 \leq \gamma_n||(I-T_{r_n}^{\Theta_2})Ax_n||^2 \\
\leq ||x_n-p||^2 - ||x_{n+1}-p||^2 + \gamma_n^2||A^*(I-T_{r_n}^{\Theta_2})Ax_n||^2 \\
+ 2\alpha_n\xi\langle f(x_n)-f(p), x_{n+1}-p\rangle + 2\alpha_n\langle \xi f(p)-Dp, x_{n+1}-p\rangle \\
+ 2\alpha_n^4 N_2 \to 0, \quad \text{as} \quad n\to\infty,
\end{aligned} \tag{3.3.29}
$$

and hence

$$
\lim_{n\to\infty} ||(I-T_{r_n}^{\Theta_2})Ax_n|| = 0. \tag{3.3.30}
$$

Also from (3.3.22), we obtain

$$
\begin{aligned}
||x_{n+1}-p||^2 &\leq (1-\alpha_n\bar{\gamma})^2||z_n-p||^2 + 2\alpha_n\xi\langle f(x_n)-f(p), x_{n+1}-p\rangle \\
&\quad +2\alpha_n\langle \xi f(p)-Dp, x_{n+1}-p\rangle \\
&\leq ||z_n-p||^2 + 2\alpha_n\xi\beta||x_n-p|| \times ||x_{n+1}-p|| \\
&\quad +2\alpha_n\langle \xi f(p)-Dp, x_{n+1}-p\rangle.
\end{aligned} \tag{3.3.31}
$$

Substituting (3.3.16) into (3.3.31), and from (3.3.21), we have

$$
\begin{aligned}
||x_{n+1}-p||^2 &\leq ||y_n-p||^2 - r_n(2\theta-r_n)||h_1y_n-h_1p||^2 + 2\alpha_n\xi\beta||x_n-p|| \times ||x_{n+1}-p|| \\
&\quad +2\alpha_n\langle \xi f(p)-Dp, x_{n+1}-p\rangle \\
&\leq ||x_n-p||^2 + 2\alpha_n^4N_2 - r_n(2\theta-r_n)||h_1y_n-h_1p||^2 + 2\alpha_n\xi\beta||x_n-p|| \times \\
&\quad ||x_{n+1}-p|| + 2\alpha_n\langle \xi f(p)-Dp, x_{n+1}-p\rangle.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
r_n(2\theta-r_n)||h_1y_n-h_1p||^2 &\leq ||x_n-p||^2 - ||x_{n+1}-p||^2 + 2\alpha_n\xi\beta||x_n-p|| \cdot ||x_{n+1}-p|| \\
&\quad +2\alpha_n\langle \xi f(p)-Dp, x_{n+1}-p\rangle \to 0, \quad \text{as} \quad n\to\infty.
\end{aligned}
$$

Since $\{r_n\} \subset (0, 2\theta)$, we conclude that

$$
\lim_{n\to\infty} ||h_1y_n-h_1p||^2 = 0. \tag{3.3.32}
$$

Further, observe that

$$
\begin{aligned}
||z_n-p||^2 &= ||T_{r_n}^{\Theta_1}(y_n-r_nh_1y_n) - T_{r_n}^{\Theta_1}(p-r_nh_1p)||^2 \\
&\leq \langle z_n-p, (y_n-r_nh_1y_n)-(p-r_nh_1p)\rangle \\
&\leq \frac{1}{2}\Big\{||z_n-p||^2 + ||(y_n-r_nh_1y_n)-(p-r_nh_1p)||^2 - \\
&\quad ||(z_n-p)-[(y_n-r_nh_1y_n)-(p-r_nh_1p)]||^2\Big\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
||z_n-p||^2 &\leq ||(y_n-r_nh_1y_n)-(p-r_nh_1p)||^2 - ||(z_n-y_n)+r_n(h_1y_n-h_1p)||^2 \\
&\leq ||y_n-p||^2 - ||z_n-y_n||^2 + 2r_n||z_n-y_n|| \times ||h_1y_n-h_1p||^2.
\end{aligned} \tag{3.3.33}
$$

From (3.3.31) and (3.3.33), we obtain

$$\begin{aligned} ||x_{n+1}-p||^2 &\leq ||y_n-p||^2-||z_n-y_n||^2+2r_n||z_n-y_n||\times||h_1y_n-h_1p||^2\\ &\quad +2\alpha_n\xi\beta||x_n-p||\times||x_{n+1}-p||+2\alpha_n\langle\xi f(p)-Dp,x_{n+1}-p\rangle\\ &\leq ||x_n-p||^2+2\alpha_n^4N_2-||z_n-y_n||^2+2r_n||z_n-y_n||\cdot|h_1y_n-h_1p||^2\\ &\quad +2\alpha_n\xi\beta||x_n-p||\times||x_{n+1}-p||+2\alpha_n\langle\xi f(p)-Dp,x_{n+1}-p\rangle. \end{aligned}$$

Therefore

$$\begin{aligned} ||z_n-y_n||^2 &\leq ||x_n-p||^2-||x_{n+1}-p||^2+2\alpha_n^4N_2+2r_n||z_n-y_n||\cdot||h_1y_n-h_1p||^2\\ &\quad +2\alpha_n\xi\beta||x_n-p||\times||x_{n+1}-p||+2\alpha_n\langle\xi f(p)-Dp,x_{n+1}-p\rangle. \end{aligned}$$

Since $\alpha_n\to 0$ as $n\to\infty$, and using (3.3.32), we obtain

$$\lim_{n\to\infty}||z_n-y_n||^2=0. \tag{3.3.34}$$

Moreover

$$\begin{aligned} ||U_nz_n-p||^2 &= ||(1-w_n)z_n+w_nSz_n-p||^2\\ &\leq (1-w_n)||z_n-p||^2+w_n||Sz_n-p||^2-w_n(1-w_n)||Sz_n-z_n||^2\\ &\leq (1-w_n)||z_n-p||^2+w_n||z_n-p||^2-w_n(1-w_n)||Sz_n-z_n||^2\\ &= ||z_n-p||^2-w_n(1-w_n)||Sz_n-z_n||^2\\ &\leq ||x_n-p||^2+2\alpha_n^4N_2-w_n(1-w_n)||Sz_n-z_n||^2. \end{aligned} \tag{3.3.35}$$

Note that from (3.3.22), we have

$$\begin{aligned} ||x_{n+1}-p||^2 &\leq (1-\alpha_n\bar{\gamma})^2||U_nz_n-p||^2+2\alpha_n\langle\xi f(x_n)-Dp,x_{n+1}-p\rangle\\ &\leq ||U_nz_n-p||^2+2\alpha_n\langle\xi f(x_n)-Dp,x_{n+1}-p\rangle, \end{aligned} \tag{3.3.36}$$

then from (3.3.35) and (3.3.36), we get

$$\begin{aligned} w_n(1-w_n)||Sz_n-z_n||^2 &\leq ||x_n-p||^2-||x_{n+1}-p||^2+2\alpha_n\langle\xi f(x_n)-Dp,x_{n+1}-p\rangle\\ &\quad +2\alpha_n^4N_2\to 0,\quad \text{as}\quad n\to\infty. \end{aligned}$$

By condition (C7), we have

$$\lim_{n\to\infty}||Sz_n-z_n||=0. \tag{3.3.37}$$

Also

$$||U_nz_n-z_n||=w_n||Sz_n-z_n||\to 0,\quad \text{as}\quad n\to\infty. \tag{3.3.38}$$

It is clear from (3.3.6) that

$$||x_{n+1}-U_nz_n||=\alpha_n||\xi f(x_n)-DU_nz_n||\to 0,\quad \text{as}\quad n\to\infty, \tag{3.3.39}$$

and

$$||y_n-x_n||\to 0,\quad n\to\infty, \tag{3.3.40}$$

then, it follows from (3.3.34) and (3.3.40) that

$$||z_n - x_n|| \leq ||z_n - y_n|| + ||y_n - x_n|| \to 0, \quad \text{as} \quad n \to \infty. \tag{3.3.41}$$

Furthermore, it follows from (3.3.38), (3.3.39) and (3.3.41) that

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - U_n z_n|| + ||U_n z_n - z_n|| + ||z_n - x_n|| \to 0, \quad \text{as} \quad n \to \infty.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x}$. It follows from (3.3.40) and (3.3.41) that $y_{n_j} \rightharpoonup \bar{x}$ and $z_{n_j} \rightharpoonup \bar{x}$, respectively. Since $\lim_{n\to\infty} ||Sz_n - z_n|| = 0$, and by Lemma 2.6.5, we have $\bar{x} \in F(S)$. Next, we show that $\bar{x} \in \Omega_{SGMEP}$. Since $z_n = T_{r_n}^{\Theta_1}(y_n - r_n h_1 y_n)$, then

$$\Theta_1(z_n, y) + \langle h_1 z_n, y - z_n\rangle + \phi(y) - \phi(z_n) + \frac{1}{r_n}\langle y - z_n, z_n - y_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from the monotonicity of Θ_1 that

$$\langle h_1 z_n, y - z_n\rangle + \phi(y) - \phi(z_n) + \frac{1}{r_n}\langle y - z_n, z_n - y_n\rangle \geq \Theta_1(y, z_n).$$

Replacing n by n_j, we get

$$\langle h_1 z_{n_j}, y - z_{n_j}\rangle + \frac{1}{r_{n_j}}\langle y - z_{n_j}, z_{n_j} - y_{n_j}\rangle \geq \Theta_1(y, z_{n_j}) + \phi(z_{n_j}) - \phi(y). \tag{3.3.42}$$

Further, for any $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1-t)\bar{x}$. Since $\bar{x} \in C$ and $y \in C$, then $y_t \in C$. So from (3.3.42), we have

$$\begin{aligned}
\langle y_t - z_{n_j}, h_1 y_t\rangle &\geq \langle y_t - z_{n_j}, h_1 y_t\rangle - \langle y_t - z_{n_j}, h_1 y_{n_j}\rangle - \left\langle y_t - z_{n_j}, \frac{z_{n_j} - y_{n_j}}{r_{n_j}}\right\rangle + \Theta_1(y_t, z_{n_j}) \\
&\quad + \phi(z_{n_j}) - \phi(y_t) \\
&= \langle y_t - z_{n_j}, h_1 y_t - h_1 z_{n_j}\rangle + \langle y_t - z_{n_j}, h_1 z_{n_j} - h_1 y_{n_j}\rangle \\
&\quad - \left\langle y_t - z_{n_j}, \frac{z_{n_j} - y_{n_j}}{r_{n_j}}\right\rangle + \Theta_1(y_t, z_{n_j}) + \phi(z_{n_j}) - \phi(y_t).
\end{aligned} \tag{3.3.43}$$

From the Lipschitz continuity of h_1 and $\lim_{n\to\infty} ||z_n - y_n|| = 0$, we obtain $||h_1 z_{n_j} - h_1 y_{n_j}|| \to 0$, as $n \to \infty$. Also since h_1 is monotone, we have $\langle y_t - z_{n_j}, h_1 y_t - h_1 z_{n_j}\rangle \geq 0$. Therefore, by LL4 and the weak lower semicontinuity of ϕ, taking the limit of (3.3.43) as $j \to \infty$, we have

$$\langle y_t - \bar{x}, h_1 y_t\rangle \geq \Theta_1(y_t, \bar{x}) + \phi(\bar{x}) - \phi(y_t). \tag{3.3.44}$$

Hence, from LL1 and (3.3.44), we get

$$\begin{aligned}
0 &= \Theta_1(y_t, y_t) + \phi(y_t) - \phi(y_t) \\
&\leq t\Theta_1(y_t, y) + (1-t)\Theta_1(y_t, \bar{x}) + t\phi(y) + (1-t)\phi(\bar{x}) - \phi(y_t) \\
&= t(\Theta_1(y_t, y) + \phi(y) - \phi(y_t)) + (1-t)(\Theta_1(y_t, \bar{x}) + \phi(\bar{x}) - \phi(y_t)) \\
&\leq t(\Theta_1(y_t, y) + \phi(y) - \phi(y_t)) + (1-t)\langle y_t - \bar{x}, h_1 y_t\rangle \\
&\leq t(\Theta_1(y_t, y) + \phi(y) - \phi(y_t)) + (1-t)t\langle y - \bar{x}, h_1 y_t\rangle,
\end{aligned}$$

which implies that

$$\Theta_1(y_t, y) + (1-t)\langle y - \bar{x}, h_1 y_t\rangle + \phi(y) - \phi(y_t) \geq 0.$$

Letting $t \to 0$, we have

$$\Theta_1(\bar{x}, y) + \langle y - \bar{x}, h_1\bar{x}\rangle + \phi(y) - \phi(\bar{x}) \geq 0, \quad y \in C,$$

which implies that $\bar{x} \in GMEP(\Theta_1, h, \phi)$.

Since A is a bounded linear operator, $Ax_{n_j} \rightharpoonup A\bar{x}$. It follows from (3.3.30) that

$$T^{\Theta_2}_{r_{n_j}} Ax_{n_j} \rightharpoonup A\bar{x}, \quad \text{as} \quad j \to \infty.$$

By the definition of $T^{\Theta_2}_{r_{n_j}} Ax_{n_j}$, we have

$$\begin{aligned} &\Theta_2(T^{\Theta_2}_{r_{n_j}} Ax_{n_j}, g) + \langle h_2(T^{\Theta_2}_{r_{n_j}} Ax_{n_j}), g - T^{\Theta_2}_{r_{n_j}} Ax_{n_j}\rangle + \varphi(g) - \varphi(T^{\Theta_2}_{r_{n_j}} Ax_{n_j}) \\ &+ \frac{1}{r_{n_j}}\langle y - T^{\Theta_2}_{r_{n_j}} Ax_{n_j}, T^{\Theta_2}_{r_{n_j}} Ax_{n_j} - Ax_{n_j}\rangle \geq 0, \qquad \forall g \in Q \quad \text{and} \quad y \in H_2. \end{aligned} \tag{3.3.45}$$

Since Θ_2 is upper semicontinuous in the first argument, taking limsup of the above inequality as $j \to \infty$, we get

$$\Theta_2(A\bar{x}, g) + \langle h_2(A\bar{x}), g - A\bar{x}\rangle + \varphi(g) - \varphi(A\bar{x}) \geq 0, \quad \forall g \in Q,$$

which implies $A\bar{x} \in GMEP(\Theta_2, h_2, \varphi)$ and thus $\bar{x} \in \Omega_{SGMEP}$. Therefore $\bar{x} \in \Gamma = \Omega_{SGMEP} \cap F(S)$.

We now show that $\{x_n\}$ converges strongly to $z = P_\Gamma(I - D + \xi f)(z)$ which is the unique solution of the variational inequality (3.3.19). To do this, we first prove that $\limsup_{n\to\infty}\langle (D - \xi f)z, z - x_n\rangle \leq 0$. Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup \langle (D - \xi f)z, z - x_n\rangle = \lim_{j\to\infty}\langle (D - \xi f)z, z - x_{n_j}\rangle.$$

Since $x_{n_j} \rightharpoonup \bar{x}$, we get

$$\begin{aligned} \limsup \langle (D - \xi f)z, z - x_n\rangle &= \lim_{j\to\infty}\langle (D - \xi f)z, z - x_{n_j}\rangle \\ &= \langle (D - \xi f)z, z - \bar{x}\rangle \leq 0. \end{aligned}$$

Now from (3.3.22), we have

$$\begin{aligned} ||x_{n+1} - z||^2 &\leq (1 - \alpha_n\bar{\gamma})^2\Big[||x_n - z||^2 + 2\alpha^4 N_2\Big] + 2\alpha_n\xi\beta||x_n - z|| \cdot ||x_{n+1} - z|| \\ &\quad + 2\alpha_n\langle \xi f(z) - Dz, x_{n+1} - z\rangle \\ &\leq (1 - \alpha_n\bar{\gamma})^2||x_n - z||^2 + \alpha_n\xi\beta(||x_n - z||^2 + ||x_{n+1} - z||^2) \\ &\quad + 2\alpha_n\langle \xi f(z) - Dz, x_{n+1} - z\rangle + 2\alpha_n^4 N_2 \\ &\leq (1 - \alpha_n\bar{\gamma})||x_n - z||^2 + \alpha_n\xi\beta(||x_n - z||^2 + ||x_{n+1} - z||^2) \\ &\quad + 2\alpha_n\langle \xi f(z) - Dz, x_{n+1} - z\rangle + 2\alpha_n^4 N_2 \\ &\leq \left(1 - \frac{\alpha_n(\bar{\gamma} - \xi\beta)}{1 - \alpha_n\xi\beta}\right)||x_n - z||^2 + \frac{2\alpha_n}{1 - \alpha_n\xi\beta}\Big(\langle \xi f(z) - Dz, x_{n+1} - z\rangle + \alpha_n^3 N_2\Big) \\ &= (1 - \nu_n)||x_n - z|| + \nu_n\delta_n, \end{aligned} \tag{3.3.46}$$

where

$$\nu_n = \frac{\alpha_n(\bar{\gamma} - \xi\beta)}{1 - \alpha_n\xi\beta} \quad \text{and} \quad \delta_n = \frac{2}{\bar{\gamma} - \xi\beta}[\langle \xi f(z) - Dz, x_{n+1} - z\rangle + \alpha_n^3 N_2].$$

It is easy to verify that $\sum_{n=0}^{\infty} \nu_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Therefore, from Lemma 2.6.29, we get $||x_n - z|| \to 0$, as $n \to \infty$ and hence $\{x_n\}$ converges strongly to z. From (3.3.40) and (3.3.41), it is easy to see that $\{y_n\}$ and $\{z_n\}$ converge strongly z.

Case II: Assume that $\{||x_n - p||\}$ is not monotonically decreasing. For all $n \geq n_0$ (for some n_0 large enough), let $\tau : \mathbb{N} \to \mathbb{N}$ be defined by

$$\tau(n) = \max\{k \in \mathbb{N} : k \leq n : \tau_k \leq \tau_{k+1}\}.$$

Clearly, τ is non-decreasing since $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq ||x_{\tau(n)} - p|| \leq ||x_{\tau(n)+1} - p||, \quad \forall n \geq n_0.$$

Following a similar argument as in Case I, we have $||(I - T^{\Theta_2}_{r_{\tau(n)}})Ax_{\tau(n)}|| \to 0$, $||Sz_{\tau(n)} - z_{\tau(n)}|| \to 0$, and $||x_{\tau(n)+1} - x_{\tau(n)}|| \to 0$. Also, we obtain

$$\limsup_{n\to\infty}\langle (D - \xi f)p, p - x_{\tau(n)}\rangle \leq 0.$$

Now since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$ denoted by $\{x_{\tau(n_j)}\}$ which converges weakly to $\bar{x}$. Suppose $\{x_{\tau(n_j)}\}$ is such that

$$\limsup_{n\to\infty}\langle \xi f(p) - Dp, x_{\tau(n)+1} - p\rangle = \lim_{j\to\infty}\langle \xi f(p) - Dp, x_{\tau(n_j)+1} - p\rangle.$$

Since $x_{\tau(n)} \rightharpoonup \bar{x}$, and from (3.3.19), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \xi f(p) - Dp, x_{\tau(n)+1} - p\rangle &= \lim_{j\to\infty}\langle \xi f(p) - Dp, x_{\tau(n_j)+1} - p\rangle \\ &= \langle \xi f(p) - Dp, \bar{x} - p\rangle \leq 0.\end{aligned}$$

Therefore

$$\limsup_{n\to\infty}\langle \xi f(p) - Dp, x_{\tau(n)+1} - p\rangle \leq 0. \tag{3.3.47}$$

Similarly, as in (3.3.46) we obtain

$$\begin{aligned}||x_{\tau(n)+1} - p||^2 &\leq (1 - \alpha_{\tau(n)}\bar{\gamma})^2\Big[||x_{\tau(n)} - p||^2 + 2\alpha^4 N_2\Big] + 2\alpha_{\tau(n)}\xi\beta||x_{\tau(n)} - p|| \times \\ &\quad ||x_{\tau(n)+1} - p|| + 2\alpha_{\tau(n)}\langle \xi f(p) - Dp, x_{\tau(n)+1} - p\rangle \\ &\leq \left(1 - \frac{\alpha_{\tau(n)}(\bar{\gamma} - \xi\beta)}{1 - \alpha_{\tau(n)}\xi\beta}\right)||x_{\tau(n)} - p||^2 \\ &\quad + \frac{2\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\xi\beta}[\langle \xi f(p) - Dp, x_{\tau(n)+1} - p\rangle + \alpha_n^3 N_2].\end{aligned} \tag{3.3.48}$$

Since $||x_{\tau(n)} - p||^2 \leq ||x_{\tau(n)+1} - p||^2$, then from (3.3.48), we have

$$\begin{aligned} 0 &\leq ||x_{\tau(n)+1} - p||^2 - ||x_{\tau(n)} - p||^2 \\ &\leq \left(1 - \frac{\alpha_{\tau(n)}(\bar{\gamma} - \xi\beta)}{1 - \alpha_{\tau(n)}\xi\beta}\right) ||x_{\tau(n)} - p||^2 + \frac{2\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\xi\beta}[\langle \xi f(p) - Dp, x_{\tau(n)+1} - p\rangle + \alpha_n^3 N_2] \\ &\quad - ||x_{\tau(n)} - p||^2. \end{aligned}$$

It follows that

$$\frac{\bar{\gamma} - \xi\beta}{1 - \alpha_{\tau(n)}\xi\beta} ||x_{\tau(n)} - p||^2 \leq \frac{2}{1 - \alpha_{\tau(n)}\xi\beta}[\langle \xi f(p) - Dp, x_{\tau(n)+1} - p\rangle + \alpha_n^3 N_2].$$

Since $\alpha_{\tau(n)} \to 0$, as $n \to \infty$ and from (3.3.47), we have

$$\lim_{n\to\infty} ||x_{\tau(n)} - p|| = 0.$$

As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq ||x_n - p||^2 \leq \max\{||x_{\tau(n)} - p||^2, ||x_{\tau(n)+1} - p||^2\} = ||x_{\tau(n)+1} - p||^2.$$

Hence, $\lim_{n\to\infty} ||x_n - p|| = 0$. This implies that $\{x_n\}$ converges strongly to p. This complete the proof.

$\square$

Remark 3.3.12. The condition that $\{(I - T_{r_n}^{\Theta_2})Ax_n\}$ is bounded is satisfied if the set of solutions Ω_{SGMEP} of SGMEP (3.3.1) is bounded. If Ω_{SGMEP} is not bounded, then we need to verify the condition that $\{(I - T_{r_n}^{\Theta_2})Ax_n\}$ is bounded before applying our algorithm.

3.3.2 Numerical example

In this subsection, we provide a numerical result to show the accuracy and efficiency of our proposed algorithm.

Example 3.3.13. Let $H_1 = H_2 = \mathbb{R}$ and $C = Q = [0, 2]$. Define $\Theta_1 : C \times C \to \mathbb{R}$ by $\Theta_1(x, y) = -\frac{1}{2}x^2 + \frac{1}{2}y^2$, $h_1 : C \to \mathbb{R}$ by $h_1(x) = x$ and $\phi : C \to \mathbb{R}$ by $\phi(x) = \frac{1}{2}x^2$. It is easy to see that

$$T_{r_n}^{\Theta_1}(z) = \frac{z}{3r_n + 1}, \quad \forall z \in \mathbb{R}.$$

Also, let $\Theta_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $\Theta_2(u, v) = -3u^2 + 2uv + v^2$, $h_2 : Q \to \mathbb{R}$ be defined by $h_2(u) = 2u$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi(u) = u^2$, then

$$T_{r_n}^{\Theta_2}(w) = \frac{w}{6r_n + 1}, \quad \forall w \in \mathbb{R}.$$

Let $A : \mathbb{R} \to \mathbb{R}$ be defined by $A(x) = 2x$ for all $x \in \mathbb{R}$. Then A is a bounded linear operator and $A^T(x) = 2x$ for all $x \in \mathbb{R}$. Clearly, $\Omega_{SGMEP} := \{p \in GMEP(\Theta_1, h_1, \phi) : Ap \in GMEP(\Theta_2, h_2, \varphi)\} = \{0\}$. This shows that Ω_{SGMEP} is bounded and thus, the sequence $\{(I - T_{r_n}^{\Theta_2})Ax_n\}$ is also bounded.

Define $S : \mathbb{R} \to \mathbb{R}$ by

$$Sx = \begin{cases} x, & if \quad x \in (-\infty, 1), \\ 1, & if \quad x \in [1, +\infty). \end{cases} \tag{3.3.49}$$

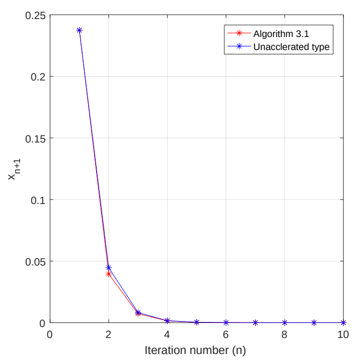
It is easy to see that S is nonspreading and $\Gamma = \{0\}$. Take $\xi = 1, D = I$, where I is an identity mapping and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x}{2}$. Choose $\alpha_n = \frac{1}{n+1}$, $w_n = \frac{1}{5(1+\frac{1}{n})}$, $r_n = \frac{2}{n+1}$, $\beta_n = \frac{1}{2(n+1)^4}$ and $\lambda = 1.5$, and set $m_1 = \frac{\gamma_1}{1.5}\left(\frac{-24r_n}{6r_n+1}\right)x_1$. Then Algorithm 3.2.4 gives the following:

$$\begin{cases} m_{n+1} = \frac{\gamma_n}{1.5}(\frac{-24r_n}{6r_n+1})x_n + \frac{m_n}{2(n+1)^4}, \\ y_n = x_n + 1.5m_{n+1}, \\ z_n = \frac{1}{3r_n+1}(\frac{n-1}{n+1})y_n, \\ x_{n+1} = \frac{1}{n+1}f(x_n) + \frac{n}{n+1}\left[\frac{4n+5}{5(n+1)}z_n + \frac{n}{5(n+1)}Sz_n\right], \quad n \geq 1. \end{cases} \tag{3.3.50}$$

We now make a different choice of the initial value x_1 and use $\epsilon < 10^{-6}$ for the stopping criterion.

Case 1: $x_1 = 0.0025$, Case 2: $x_1 = 0.01$, Case 3: $x_1 = 0.1$, Case 4: $x_1 = 1$.

We note that the choice of γ_n, as long as it is in the range, does not have any significant effect on either the number of iterations, nor the cpu time. We compare the computational result of Algorithm 3.3.6 with its unaccelerated form (i.e. taking $\beta_n = 0$) and plot the graphs of accuracy against number of iterations, and errors against number of iterations (see Figure 3.4-3.7 and Table 3.4). This shows that Algorithm 3.3.6 converges faster and is more efficient than its unaccelerated form (i.e. when $\beta_n = 0$).



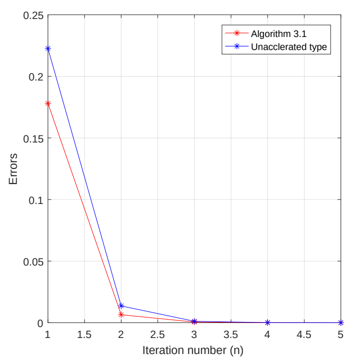


Figure 3.4: $x_1 = 0.0025$, Left: accuracy against number of iterations; Right: errors against numbers of iterations.

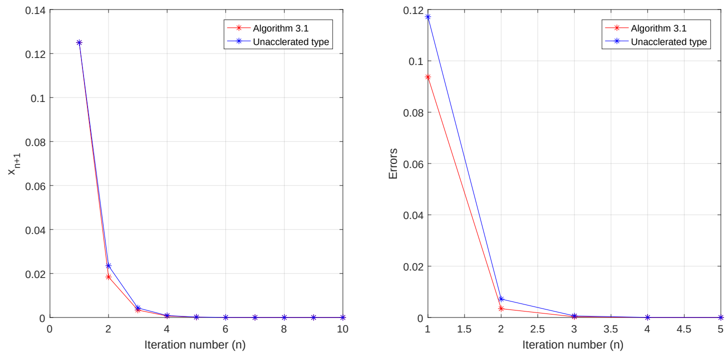


Figure 3.5: $x_1 = 0.01$, Left: accuracy against number of iterations; Right: errors against numbers of iterations.

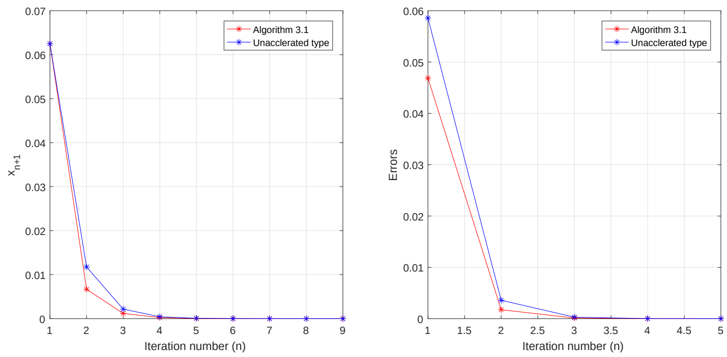


Figure 3.6: $x_1 = 0.1$, Left: accuracy against number of iterations; Right: errors against numbers of iterations.

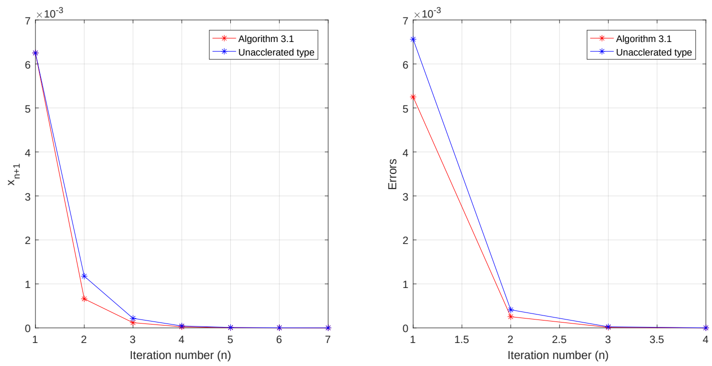


Figure 3.7: $x_1 = 1$, Left: accuracy against number of iterations; Right: errors against numbers of iterations.

Table 3.4: Computation result for Example 3.3.13

	Time Taken (Sec)	
	Algorithm 3.3.8	Unaccelerated alg.
$x_1 = 0.0025$	0.0011	0.0277
$x_1 = 0.01$	0.0020	0.0262
$x_1 = 0.1$	0.0026	0.0357
$x_1 = 1$	0.0038	0.1119

CHAPTER 4

Equilibrium Problems in Hilbert Spaces

4.1 A Parallel Combination Extragradient Method with Armijo Line Searching for Finding Common Solutions of Finite Families of Equilibrium and Fixed Point Problems

In this section, we focus on the approximation of a common solution of a Finite Family of Equilibrium Problems (FEP), i.e., finding $x^* \in C := \cap_{i=1}^N C_i$ such that

$$g_i(x^*, y) \geq 0, \quad \forall\, y \in C_i, \tag{4.1.1}$$

where C_i, $i = 1, 2, \ldots, N$ is a finite family of nonempty, closed and convex subsets of H, $g_i : C_i \times C_i \to \mathbb{R}$ is a finite family of bifunctions satisfying $g_i(x, x) = 0$. We denote the set of solution of (4.1.1) by Ω_{FEP}.
Clearly, the FEP (4.1.1) with $N = 1$ is the EP (1.1.4). The motivation and inspiration for studying the FEP originated from its importances and applications in Convex Feasibility Problem (CFP), i.e.,

$$\text{finding} \quad x^* \in C := \cap_{i=1}^N C_i \neq \emptyset. \tag{4.1.2}$$

The CFP has received great attention due to its broad applicability in many areas of applied mathematics such as image processing, computerized tomography and radiation therapy treatment. It is also worth mentioning that the FEP (4.1.1) has find applications in other areas of studies such as common fixed point problems, common solution of variational inequality problems and common solution of minimization problems.

The bifunction $g : C \times C \to \mathbb{R}$ is said to satisfy Lipschitz-type condition, if there exists two constants $c_1 > 0$ and $c_2 > 0$ such that

$$g(x, y) + g(y, z) \geq g(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2 \quad \forall\, x, y, z \in C. \tag{4.1.3}$$

Hieu et al. [130] introduced a parallel hybrid Mann-type extragradient method for solving FEP (4.1.1). This algorithm was defined as follows:

Algorithm 4.1.1.

Step 0: *Pick* $x_0 \in C$, $0 < \lambda < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right)$, $n = 0$ *and the sequence* $\{\alpha_n\} \subset (0,1)$ *such that* $\limsup_{n\to\infty} \alpha_n < 1$.

Step 1: *Solve* N *strong convex programs in parallel*

$$y_n^i = argmin\left\{\lambda g_i(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\right\}, \quad i = 1, 2, \ldots, N.$$

Step 2: *Solve* N *strong convex program in parallel*

$$z_n^i = argmin\left\{\lambda g_i(y_n^i, y) + \frac{1}{2}||x_n - y||^2 : y \in C\right\}, \quad i = 1, 2, \ldots, N. \tag{4.1.4}$$

Step 3: *Find among* z_n^i, $i = 1, 2, \ldots, N$, *the farthest element from* x_n, *i.e.,*

$$i_n = argmin\{||z_n^i - x_n|| : i = 1, 2, \ldots, N\}, \bar{z}_n := z_n^{i_n}.$$

Step 4: *Find intermediate* u_n^j *in parallel*

$$u_n^j = \alpha_n x_n + (1 - \alpha_n) S_j \bar{z}_n, \quad j = 1, 2, \ldots, M.$$

Step 5: *Find among* u_n^j, $j = 1, 2, \ldots, M$, *the farthest element* x_n, *i.e.,*

$$j_n = argmin\{||u_n^j - x_n|| : j = 1, 2, \ldots, M\}, \quad \bar{u}_n := u_n^{j_n}.$$

Step 6: *Construct two closed convex subsets of* C

$$\begin{aligned} C_n &= \{v \in C : ||\bar{u}_n - v|| \le ||x_n - v||\}, \\ Q_n &= \{v \in C : \langle x_0 - x_n, v - x_n \rangle \le 0\}. \end{aligned}$$

Step 7: *The next iteration* x_{n+1} *is defined as*

$$x_{n+1} = P_{C_n \cap Q_n}(x_0).$$

If $x_{n+1} = x_n$, *then Stop. Otherwise set* $n \leftarrow n + 1$ *and go to Step 1.*

Hieu et al. [130] proved that the sequence $\{x_n\}$ generated by the Algorithm 4.1.1 converges strongly to a solution $x \in \Omega_{FEP} \bigcap \cap_{j=1}^{M} F(T_i)$ where T_i are nonexpansive mappings. Other similar parallel methods which are modifications of Algorithm 4.1.1 can be found in (for instance) [11, 10, 127].

However, we note the following problems:

(p1) The convergence of Algorithm 4.1.1 requires the Lipschitz constants c_1 and c_2 to be known (or at least) estimated a priori. In practice, it is too difficult to approximate the Lipschitz constants.

(p2) Algorithm 4.1.1 needs to solve two or more strongly convex program in parallel at each iteration. This can be computationally costly and consumes large memory size if the feasible set is complex.

(p3) Also, Algorithm 4.1.1 requires at each step of the process, the computation of two subsets C_n and Q_n, their intersection, and the projection of x_0 onto $C_n \cap Q_n$. This can be computationally expensive if the feasible set is complex.

Recently, Hieu [128] proposed the following parallel hybrid extragradient-cutting method which does not require to solve many strongly convex problem at each iteration: Let C_i, $i = 1, 2, \ldots, N$ be family of nonempty closed convex subsets of H.

Algorithm 4.1.2.

Step 0: Pick $x_0 \in H$, $n = 0$, $0 < \lambda \leq \lambda_n^i < \mu < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\gamma_n^i \in [\epsilon, \frac{1}{2}]$ *for some* $\epsilon \in (0, \frac{1}{2}]$, $k = 1, 2, \ldots$ *and* $i = 1, 2, \ldots, N$.

Step 1: Solve N strongly convex program in parallel

$$y_n^i = argmin\left\{\lambda_n^i g_i(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C_i\right\}.$$

Step 2: Solve N strongly convex program in parallel

$$z_n^i = argmin\left\{\lambda_n^i g_i(y_n^i, y) + \frac{1}{2}||x_n - y||^2 : y \in C_i\right\}.$$

Step 3: Determine the next approximation via

$$x_{n+1} = P_{C_n \cap Q_n}(x_0),$$

where $C_n := \bigcap_{i=1}^N C_n^i$ *and*

$$C_n^i := \{z \in H : \langle x_n - z_n^i, z - x_n - \gamma_n^i(z_n^i - x_n)\rangle \leq 0\},$$

$$Q_n := \{z \in H : \langle x_0 - x_n, x_n - z\rangle \geq 0\}.$$

If $x_{n+1} = x_0$, *then Stop. Otherwise, set* $n \leftarrow n + 1$ *and go to Step 1.*

Note that, although Algorithm 4.1.2 improves Algorithm 4.1.1, but it still incurred some of the problems in Algorithm 4.1.1.

In order to address the problems (p1)-(p3), in this section, we introduce a parallel combination extragradient method with Armijo line search rule for finding a common solution

$x \in \Omega_{FEP} \cap \left(\bigcap_{j=1}^{M} F(S_j) \right)$, where $g_i : C \times C \to \mathbb{R}$ $i = 1, 2, \ldots, N$ are finite family of pseudo-monotone equilibrium bifunctions and $S_j : H \to CB$ $j = 1, 2, \ldots, M$ are finite family of multivalued demi-contractive mappings. The key of our new method is that for the current iterate x_n, the algorithm solves a single strongly convex program and determine an appropriate stepsize for the next step using an Armijo line searching rule. Then, a new level set D_n is constructed using the convex combination of finite convex functions and a projection P_{D_n} is made. The algorithm determines if a common solution is reached using a convex combination of the demi-contractive mappings, else, it updates the new iterate x_{n+1}. The simplicity and ease of implementation are two of the advantages of our method (in each iteration, a single strongly convex optimization program is solved and only one projection is made). Also, our method does not involve the projection on $P_{C_n \cap Q_n}$ in Algorithm 4.1.1 and 4.1.2 and other similar ones. We prove that the sequences generated by our algorithm also converge in norm to the unique solution x. This method improves many of the existing methods in the literature.

Throughout this section, we assume that the bifunction $g : C \times C \to \mathbb{R}$ satisfies the following assumptions:

A1. g is pseudo-monotone on C;

A2. g is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x_k\}, \{y_k\} \subset C$ converge weakly to x and y, respectively, then $g(x_k, y_k) \to g(x, y)$ as $k \to \infty$;

A3. $g(x, \cdot)$ is convex and subdifferentiable on C for every $x \in C$.

The following Lemmas would be useful for our result in this section.

Lemma 4.1.3. *[92] Let C be a convex subset of a real Hilbert space H and $\varphi : C \to \mathbb{R}$ be a convex and subdifferentiable function on C. Then x^* is a solution to the convex problem*

$$\textit{minimize}\{\varphi(x) : x \in C\}$$

if and only if $0 \in \partial\varphi(x^*) + N_C(x^*)$, where $\partial\varphi(x^*)$ denotes the subdifferential of φ and $N_C(x^*)$ is the normal cone of C at x^*.

Lemma 4.1.4. *[32] Let $C \subset H$ be a closed convex subset and $g : C \times C \to \mathbb{R} \cup \{+\infty\}$ be an equilibrium bifunction satisfying Assumption A1 - A3. If the solution set $\Omega_{EP(g)} \neq \emptyset$, then it is weakly closed and convex.*

Lemma 4.1.5. *[121] Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let h be a real-valued function on H and define $D := \{x \in C : h(x) \leq 0\}$. If D is nonempty and h is Lipschitz continuous on C with modulus $\theta > 0$, then*

$$d(x, D) \geq \theta^{-1} \max\{h(x), 0\}, \qquad \forall x \in C,$$

where $d(x, D)$ is the distance function from x to D.

4.1.1 Main results

In this subsection, we give a precise statement of our algorithm and discuss its strong convergence.

For $i = 1, 2, \ldots, N$, let C_i be nonempty, closed and convex subsets of a real Hilbert space H such that $C := \bigcap_{i=1}^{N} C_i$. Let $g_i : C_i \times C_i \to \mathbb{R}$ be bifunctions satisfying Assumptions A1 - A3. Let $S_j : H \to CB(H)$ $(j = 1, 2, \ldots, M)$ be multivalued demi-contractive mappings with constants κ_j such that $I - S_j$ are demiclosed at zero, $S_j p = \{p\}$ for all $p \in F(S_j)$ and $\kappa = \max\{\kappa_j\}$. Suppose

$$Sol = \Omega_{FEP} \bigcap \Big(\cap_{j=1}^{M} F(S_j) \Big) \neq \emptyset.$$

Let $\{\alpha_n\}$ and $\{\delta_{n_j}\}$ be nonnegative sequences in $(0, 1)$ such that $\sum\limits_{j=0}^{M} \delta_{n,j} = 1$.

Algorithm 4.1.6.

Step 0: Select the initial guess $x_1 \in C$ and let $\lambda > 0$, $\sigma \in (0, \frac{\lambda}{2})$ and $\gamma \in (0, 1)$. Set $n = 1$.

Step 1: Compute

$$w_n = (1 - \alpha_n)x_n + \alpha_n x_1, \tag{4.1.5}$$

$$z_n^i = argmin\Big\{ g_i(w_n, y) + \frac{\lambda}{2}||y - w_n||^2 : y \in C_i \Big\} \quad i = 1, 2, \ldots, N. \tag{4.1.6}$$

Set $r^i(w_n) = w_n - z_n^i$. If $r^i(w_n) = 0$, set $w_n = \bar{u}_n$ and go to Step 4. *Else, do* Step 2.

Step 2: Compute $y_n^i = w_n - \gamma^{m_n} r^i(w_n)$, where m_n is the smallest nonnegative integer satisfying

$$g_i(y_n^i, z_n^i) \leq -\sigma||r^i(w_n)||^2. \tag{4.1.7}$$

Step 3: Define $h_n^i(x) = \langle \bar{w}_n^i, x - y_n^i \rangle$, where $\bar{w}_n^i \in \partial g_i(y_n^i, \cdot)(y_n^i)$ for $i = 1, 2, \ldots, N$ and $x \in C$. Construct the set

$$D_n := \left\{ x \in H : \sum_{i=1}^{N} \beta_n^i h_n^i(w_n) \leq 0 \right\}, \tag{4.1.8}$$

where $\{\beta_n^i\}_{i=1}^{N} \subset (0, 1)$ such that $\sum_{i=1}^{N} \beta_n^i = 1$. Compute

$$\bar{u}_n = P_{D_n}(w_n).$$

Step 4: Compute

$$x_{n+1} = \delta_{n,0}\bar{u}_n + \sum_{j=1}^{M} \delta_{n,j} v_{n,j}, \tag{4.1.9}$$

where $v_{n,j} \in S_j \bar{u}_n$. Set $n \leftarrow n + 1$ and go back to Step 1.

Remark 4.1.7. Since $Sol \neq \emptyset$, then all the sets $EP(g_i)$ $i = 1, 2, \ldots, N$ and $F(S_j)$ $j = 1, 2, \ldots, M$ are nonempty. It follows from Lemma 3.2 of [263] and Lemma 4.1.4 that the sets $\Omega_{EP(g_i)}$ and $F(S_j)$ are closed and convex. Therefore, the solution set Sol is a nonempty, closed and convex subset of C. Hence, given any initial guess $x_1 \in C$, there exists a unique element $z = P_{Sol}x_1$.

In order to establish our main theorem, we make the following assumptions:

C1. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,

C2. $\liminf_{n\to\infty}(\delta_{n,0} - \kappa)\delta_{n,i} > 0$ for all $j = 1, 2, ..., M$.

Next, we prove some preliminary results which will be used to prove our main theorem. In the next result, we prove that the line searching rule define in Step 2 of Algorithm 4.1.6 is well defined.

Lemma 4.1.8. *Let w_n, y_n^i and z_n^i be as defined in Algorithm 4.1.6. If $w_n \neq z_n^i$ for each $i = 1, 2, \ldots, N$, then, there exists a smallest nonnegative integer m_n such that*

$$g_i(y_n^i, z_n^i) \leq -\sigma||r^i(w_n)||^2.$$

Proof. We suppose by contradiction that for every nonnegative integer m_n, we have

$$g_i(w_n - \gamma^{m_n} r^i(w_n), z_n^i) > -\sigma||r^i(w_n)||^2, \quad \forall\ i = 1, 2, \ldots, N.$$

Passing limit to the above inequality as $n \to \infty$, by continuity of $g_i(\cdot, y)$, we obtain

$$g_i(w_n, z_n^i) \geq -\sigma||r^i(w_n)||^2, \quad \forall\ i = 1, 2, \ldots, N,$$

and so

$$g_i(w_n, z_n^i) + \sigma||w_n - z_n^i||^2 \geq 0, \quad \forall\ i = 1, 2, \ldots, N. \tag{4.1.10}$$

On the other hand, since z_n^i is a solution to the strongly convex optimization problem (4.1.6), then we have

$$g_i(w_n, y) + \frac{\beta}{2}||y - w_n||^2 \geq g_i(w_n, z_n^i) + \frac{\beta}{2}||z_n^i - w_n||^2, \quad \forall\ y \in C_i, \quad i = 1, 2, \ldots, N.$$

Putting $y = w_n$ in the last inequality, we have

$$g_i(w_n, z_n^i) + \frac{\beta}{2}||z_n^i - w_n||^2 \leq 0, \quad \forall\ i = 1, 2, \ldots, N. \tag{4.1.11}$$

Combining (4.1.10) and (4.1.11), we obtain

$$\frac{\beta}{2}||z_n^i - w_n||^2 \leq \sigma||z_n^i - w_n||^2, \quad \forall\ i = 1, 2, \ldots, N.$$

Hence, we deduce that either $\frac{\beta}{2} \leq \sigma$ or $z_n^i = w_n$ for all $i = 1, 2, \ldots, N$. The first case contradicts $\sigma \in \left(0, \frac{\beta}{2}\right)$ while the second case contradicts the fact that $w_n \neq z_n^i$ for all $i = 1, 2, \ldots, N$. □

Lemma 4.1.9. *Let $x^* \in Sol$ and for all $i = 1, 2, \ldots, N$, let $h_n^i(x^*) = \langle \bar{w}_n^i, x^* - y_n^i \rangle$ where $\bar{w}_n^i$ and y_n^i are as defined in Algorithm 4.1.6. Then*

$$h_n^i(w_n) \geq \frac{\gamma^{m_n}\sigma}{1-\gamma^{m_n}}||r^i(w_n)||^2. \tag{4.1.12}$$

In addition, $h_n^i(x^*) \leq 0$ and if $w_n \neq z_n^i$, then $h_n^i(w_n) > 0$.

Proof. Since $y_n^i = w_n - r^i(w_n)$, then

$$w_n - y_n^i = \frac{\gamma^{m_n}}{1-\gamma^{m_n}}(y_n^i - z_n^i). \tag{4.1.13}$$

Select $\bar{w}_n^i \in \partial g_i(y_n^i, y_n^i)$, then it follows from (4.1.7) and (4.1.13) that

$$\begin{aligned} h_n^i(w_n) &= \langle \bar{w}_n^i, w_n - y_n^i \rangle \\ &= \frac{\gamma^{m_n}}{1-\gamma^{m_n}}\langle \bar{w}_n^i, y_n^i - z_n^i \rangle \\ &\geq \frac{\gamma^{m_n}}{1-\gamma^{m_n}}\Big(g_i(y_n^i, y_n^i) - g_i(y_n^i, z_n^i)\Big) \\ &\geq \frac{\gamma^{m_n}}{1-\gamma^{m_n}}\sigma||r^i(w_n)||^2. \end{aligned}$$

If $w_n \neq z_n^i$, then $h_n^i(w_n) > 0$. Furthermore, since $x^* \in Sol$, we have $g_i(x^*, y) \geq 0$ for all $y \in C_i$, $i = 1, 2, \ldots, N$. Since each g_i is pseudo-monotone on C_i, then $g_i(y, x^*) \leq 0$. Therefore

$$\begin{aligned} h_n^i(x^*) &= \langle \bar{w}_n^i, x^* - y_n^i \rangle \\ &\leq g_i(y_n^i, x^*) - g_i(y_n^i, y_n^i) \\ &\leq 0. \end{aligned}$$

□

Remark 4.1.10. Lemma 4.1.9 shows that $x^* \in D_n^i := \{x \in H : h_n^i(x) \leq 0\}$, for each $i = 1, 2 \ldots, N$, hence $x^* \in D_n$. Thus from Lemma 2.6.4, D_n is a nonempty, closed convex subset of H. In particular, D_n is a half space.

Next, we show that the sequence $\{x_n\}$ generated by Algorithm 4.1.6 is bounded.

Lemma 4.1.11. *Suppose Assumptions C1 and C2 are satisfied, then the sequence $\{x_n\}$ generated by Algorithm 4.1.6 is bounded. If in addition $\{\bar{w}_n\}$ is uniformly bounded and Algorithm 4.1.6 does not terminate, then*

$$||x_{n+1} - x^*||^2 \leq ||w_n - x^*||^2 - \left(\frac{\gamma^{m_n}\sigma}{K(1-\gamma^{m_n})}||w_n - z_n^i||^2\right)^2, \tag{4.1.14}$$

where $x^* \in Sol$, $K > 0$ and for all $i = 1, 2, \ldots, N$.

Proof. By the definition of D_n, it is easy to see from Lemma 4.1.9 that $Sol \subset D_n$ holds for all $n \geq 1$. Hence, for any $z \in Sol$, we have from (2.2.3) that

$$\begin{aligned} ||\bar{u}_n - z||^2 &= ||P_{D_n}(w_n) - z||^2 \\ &\leq ||w_n - z||^2 - ||P_{D_n}(w_n) - w_n||^2 \\ &= ||w_n - z||^2 - d(w_n, D_n) \quad i = 1, 2, \ldots, N. \end{aligned} \tag{4.1.15}$$

Using Lemma 2.6.3, (4.1.9) and (4.1.15), we get

$$\begin{aligned} ||x_{n+1} - z||^2 &= \left|\left|\delta_{n,0}(\bar{u}_n - z) - \sum_{j=1}^{M} \delta_{n,j}(v_{n,j} - z)\right|\right|^2 \\ &\leq \delta_{n,0}||\bar{u}_n - z||^2 + \sum_{j=1}^{M} \delta_{n,j}||v_{n,j} - z||^2 - \delta_{n,0} \sum_{n,j}^{M} \delta_{n,j}||v_{n,j} - \bar{u}_n||^2 \\ &\leq \delta_{n,0}||\bar{u}_n - z||^2 + \sum_{j=1}^{M} \delta_{n,j} H(S_j\bar{u}_n, S_j z) - \delta_{n,0} \sum_{n,j}^{M} \delta_{n,j}||v_{n,j} - \bar{u}_n||^2 \\ &\leq \delta_{n,0}||\bar{u}_n - z||^2 + \sum_{j=1}^{M} \delta_{n,j} \Big(||\bar{u}_n - z||^2 + \kappa_j||\bar{u}_n - v_{n,j}||^2 \Big) \\ &\quad -\delta_{n,0} \sum_{n,j}^{M} \delta_{n,j}||v_{n,j} - \bar{u}_n||^2 \\ &\leq ||\bar{u}_n - z||^2 - \sum_{n=1}^{M} \delta_{n,j}(\delta_{n,0} - \kappa)||\bar{u}_n - v_{n,j}||^2 \\ &\leq ||w_n - z||^2 - \sum_{n=1}^{M} \delta_{n,j}(\delta_{n,0} - \kappa)||\bar{u}_n - v_{n,j}||^2. \end{aligned} \tag{4.1.16}$$

Hence from (4.1.5) and (4.1.16), we have

$$\begin{aligned} ||x_{n+1} - z||^2 &\leq ||\alpha_n x_1 + (1 - \alpha_n)x_n - z||^2 \\ &\leq \alpha_n||x_1 - z||^2 + (1 - \alpha_n)||x_n - z||^2 \\ &\leq \max\{||x_1 - z||^2, ||x_n - z||^2\} \\ &\;\vdots \\ &\leq \max\{||x_1 - z||^2, ||x_1 - z||^2\} \\ &= ||x_1 - z||^2. \end{aligned} \tag{4.1.17}$$

This implies that $\{x_n\}$ is bounded. Consequently, for $i = 1, 2, \ldots, N$, $\{z_n^i\}$ and $\{y_n^i\}$ are bounded. Also, suppose the sequence $\{\bar{w}_n^i\}$ is uniformly bounded by $K > 0$ (cf. [219], Theorem 24.5), i.e.,

$$||\bar{w}_n^i|| \leq K \quad \forall\, n \in \mathbb{N}, \quad i = 1, 2, \ldots, N.$$

Combining Lemma 4.1.5, Lemma 4.1.9 and (4.1.15), we get

$$\begin{aligned} ||u_n^i - z||^2 &\leq ||w_n - z||^2 - \left(\frac{1}{K}h_n^i(w_n)\right)^2 \\ &\leq ||w_n - z||^2 - \left(\frac{\gamma^{m_n}\sigma}{K(1-\gamma^{m_n})}||r^i(w_n)||^2\right)^2 \\ &= ||w_n - z||^2 - \left(\frac{\gamma^{m_n}\sigma}{K(1-\gamma^{m_n})}||w_n - z_n^i||^2\right)^2. \end{aligned}$$

Therefore from (4.1.16), we have

$$||x_{n+1} - z||^2 \leq ||w_n - z||^2 - \left(\frac{\gamma^{m_n}\sigma}{K(1-\gamma^{m_n})}||w_n - z_n^i||^2\right)^2.$$

□

We are now in the position to prove the convergence of Algorithm 4.1.6. Note that if $x_n = w_n$, $r^i(w_n) = 0$ for $i = 1, 2, \dots, N$ and $w_n \in S_j w_n$ for $j = 1, 2, \dots, M$, we are at a common solution $x^* \in Sol$. In our convergence analysis, we will implicitly assume that this does not occur after finitely many iterations so that our Algorithm 4.1.6 generates an infinite sequence.

Theorem 4.1.12. *For $i = 1, 2, \dots, N$, let C_i be nonempty, closed and convex subsets of a real Hilbert space H such that $C := \bigcap_{i=1}^N C_i$. Let $g_i : C_i \times C_i \to \mathbb{R}$ be bifunctions satisfying Assumptions A1 - A3. Let $S_j : H \to CB(H)$ $(j = 1, 2, \dots, M)$ be multivalued demicontractive mappings with constants κ_j such that $I - S_j$ are demiclosed at zero, $S_j p = \{p\}$ for all $p \in F(S_j)$ and $\kappa = \max\{\kappa_j\}$. Suppose*

$$Sol = \Omega_{FEP} \bigcap \left(\cap_{j=1}^M F(S_j) \right) \neq \emptyset.$$

Let $\{\alpha_n\}$ and $\{\delta_{n_j}\}$ be nonnegative sequences in $(0, 1)$ and $\{x_n\}$ be generated by Algorithm 4.1.6 and Assumptions C1 and C2 are satisfied. Then, the sequence $\{x_n\}$ converges strongly to a point $p \in Sol$, where $p = P_{Sol}x_1$.

Proof. Let $z = P_{Sol}x_1$, using Lemma 2.6.1(i) and from (4.1.16), we have

$$\begin{aligned} ||x_{n+1} - z||^2 &\leq ||w_n - z||^2 \\ &= ||\alpha_n x_1 + (1-\alpha_n)x_n - z||^2 \\ &\leq (1-\alpha_n)||x_n - z||^2 + 2\alpha_n\langle x_1 - z, w_n - z\rangle \\ &= (1-\alpha_n)a_n + \alpha_n b_n, \end{aligned} \tag{4.1.18}$$

where $a_n := ||x_n - z||^2$ and $b_n = 2\langle x_1 - z, w_n - z\rangle$. We show that $\{b_n\}$ satisfies the following:

$$-1 \leq \limsup_{n\to\infty} b_n < +\infty. \tag{4.1.19}$$

Since $\{x_n\}$ and $\{w_n\}$ are bounded, we have

$$\sup_{n\geq 1} b_n \leq 2\sup_{n\geq 1} ||x_1 - z|| \cdot ||w_n - z|| < \infty.$$

This implies that $\limsup_{n\to\infty} b_n < \infty$. Next, we show that $\limsup_{n\to\infty} b_n \geq -1$. Assume the contrary, i.e. $\limsup_{n\to\infty} b_n < -1$. Then, there exists $n_0 \in \mathbb{N}$ such that $b_n < -1$ for all $n \geq n_0$. Thus we get from (4.1.18) that

$$\begin{aligned} a_{n+1} &\leq (1-\alpha_n)a_n + \alpha_n b_n \\ &< (1-\alpha_n)a_n - \alpha_n \\ &= a_n - \alpha_n(a_n+1) \leq a_n - \alpha_n. \end{aligned}$$

Taking lim sup of both sides of the inequality above, we have

$$\limsup_{n\to\infty} a_n \leq a_{n_0} - \lim_{n\to\infty}\sum_{i=n_0}^{n} \alpha_i = -\infty.$$

This contradicts the fact that $\{a_n\}$ is a non-negative real sequence. Therefore $\limsup_{n\to\infty} b_n \geq -1$.

Now from Lemma 4.1.11, we have

$$\begin{aligned} ||x_{n+1} - z||^2 &\leq ||w_n - z||^2 - \left(\frac{\gamma^{m_n}\sigma}{K(1-\gamma^{m_n})}||w_n - z_n^i||^2\right)^2 \\ &\leq \alpha_n||x_1 - z||^2 + (1-\alpha_n)||x_n - z||^2 - \left(\frac{\gamma^{m_n}\sigma}{K(1-\gamma^{m_n})}||w_n - z_n^i||^2\right)^2 \end{aligned} \tag{4.1.20}$$

We next consider the following two possible cases.

<u>CASE A:</u> Suppose there exists $n_0 \in \mathbb{N}$ such that $\{||x_n - z||^2\}$ is monotonically nonincreasing for all $n \geq n_0$. Since $\{x_n\}$ is bounded, then $\{||x_n - z||^2\}$ is also bounded and so it follows that $||x_n - z||^2 - ||x_{n+1} - z||^2 \to 0$ as $n \to \infty$. From (4.1.5) and using condition C1, we get

$$||w_n - x_n|| = \alpha_n||x_1 - x_n|| \to 0\ , \quad n \to \infty. \tag{4.1.21}$$

Also using (2.2.3), we have

$$||u_n^i - z||^2 \leq ||w_n - z||^2 - ||u_n^i - w_n|| \quad \forall\ i = 1, 2, \ldots, N. \tag{4.1.22}$$

This yields that

$$\begin{aligned} ||u_n^i - w_n||^2 &\leq ||w_n - z||^2 - ||u_n^i - z||^2 \\ &\leq ||\alpha_n(x_1 - z) + (1-\alpha_n)(x_n - z)||^2 - ||x_{n+1} - z||^2 \\ &\leq (1-\alpha_n)||x_n - z||^2 + 2\alpha_n\langle x_1 - z, w_n - z\rangle - ||x_{n+1} - z||^2 \\ &\to 0, \quad n \to \infty. \end{aligned} \tag{4.1.23}$$

Also from (4.1.20), we get

$$\left(\frac{\gamma^{m_n}\sigma}{K(1-\gamma^{m_n})}||w_n - z_n^i||^2\right)^2 \leq \alpha_n||x_1 - z||^2 + (1-\alpha_n)||x_n - z||^2 \\ -||x_{n+1} - z||^2 \to 0, \quad n \to \infty. \tag{4.1.24}$$

Hence

$$\lim_{n\to\infty} \frac{\gamma^{m_n}\sigma}{K(1-\gamma^{m_n})}||w_n - z_n^i||^2$$

exists. This implies that the sequence $\left\{\frac{\gamma^{m_n}\sigma}{K(1-\gamma^{m_n})}||w_n - z_n^i||^2\right\}$ is bounded. It is easy to see from (4.1.24) that

$$\lim_{n\to\infty} \gamma^{m_n}||w_n - z_n^i|| = 0, \quad \forall\ i = 1, 2, \ldots, N. \tag{4.1.25}$$

Next, we show that $\Omega_w(x_n) \subset Sol$, where $\Omega_w(x_n)$ is the set of weak subsequential limit of $\{x_n\}$. Let $\bar{x} \in \Omega_w(x_n)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. Let $\{w_{n_k}\}$ and $\{u_{n_k}^i\}$ be subsequences of $\{w_n\}$ and $\{u_n^i\}$ respectively for all $i = 1, 2, \ldots, N$. Consequently from (4.1.21) and (4.1.23), $w_{n_k} \rightharpoonup \bar{x}$ and $u_{n_k}^i \rightharpoonup \bar{x}$ for all $i = 1, 2, \ldots, N$.

From (4.1.25), we get

$$\lim_{k\to\infty} \gamma^{m_{n_k}}||w_{n_k} - z_{n_k}^i|| = 0, \quad \forall\ i = 1, 2, \ldots, N. \tag{4.1.26}$$

We claim that

$$\lim_{k\to\infty} ||w_{n_k} - z_{n_k}^i|| = 0, \quad \forall\ i = 1, 2, \ldots, N.$$

Indeed, let us consider two distinct cases depending on the behaviour of (the bounded) sequence $\{\gamma^{m_n}\}$.

`(i)`: If $\liminf_{k\to\infty} \gamma^{m_{n_k}} > 0$, then there exists $\bar{\gamma} > 0$ and a subsequence of $\{\gamma^{m_{n_k}}\}$ still denoted by $\{\gamma^{m_{n_k}}\}$ such that for some $k_0 > 0$, $\gamma^{m_{n_k}} > \bar{\gamma}$ for all $k \geq k_0$. Using this fact and (4.1.26), we have

$$\lim_{k\to\infty} ||w_{n_k} - z_{n_k}^i|| = 0, \quad \forall\ i = 1, 2, \ldots, N. \tag{4.1.27}$$

Recall that $w_{n_k} \rightharpoonup \bar{x}$, this together with (4.1.27) implies that $z_{n_k}^i \rightharpoonup \bar{x}$ as $k \to \infty$ for all $i = 1, 2, \ldots, N$. By the definition of $z_{n_k}^i$, i.e.

$$z_{n_k}^i = argmin\left\{g_i(w_{n_k}, y) + \frac{\lambda}{2}||y - w_{n_k}||^2 : y \in C\right\}, \quad i = 1, 2, \ldots, N,$$

we have

$$0 \in \partial g_i(w_{n_k}, z_{n_k}^i) + \lambda(z_{n_k}^i - w_{n_k}) + N_C(z_{n_k}^i), \quad \forall\ i = 1, 2, \ldots, N.$$

So there exists $\bar{e}_{n_k}^i \in \partial g_i(w_{n_k}, z_{n_k}^i)$, $i = 1, 2, \ldots, N$ such that

$$\langle \bar{e}_{n_k}^i, y - z_{n_k}^i\rangle + \lambda\langle z_{n_k}^i - w_{n_k}, y - z_{n_k}^i\rangle \geq 0, \quad \forall\ y \in C. \tag{4.1.28}$$

Combining this with

$$g_i(w_{n_k}, y) - g_i(w_{n_k}, z^i_{n_k}) \geq \langle \bar{e}^i_{n_k}, y - z^i_{n_k} \rangle, \quad \forall\, y \in C$$

yields

$$g_i(w_{n_k}, y) - g_i(w_{n_k}, z^i_{n_k}) + \lambda\langle z^i_{n_k} - w_{n_k}, y - z^i_{n_k} \rangle \geq 0, \quad \forall\, y \in C. \tag{4.1.29}$$

Since

$$\langle z^i_{n_k} - w_{n_k}, y - z^i_{n_k} \rangle \leq ||z^i_{n_k} - w_{n_k}|| \cdot ||y - z^i_{n_k}||,$$

from (4.1.29), we get that

$$g_i(w_{n_k}, y) - g_i(w_{n_k}, z^i_{n_k}) + \lambda||z^i_{n_k} - w_{n_k}|| \cdot ||y - z^i_{n_k}|| \geq 0, \quad \forall\, y \in C. \tag{4.1.30}$$

Letting $k \to \infty$, by the weak continuity of g_i and (4.1.27), from (4.1.30) we obtain

$$g_i(\bar{x}, y) - g_i(\bar{x}, \bar{x}) \geq 0, \quad \forall\, y \in C, i = 1, 2, \ldots, N.$$

Hence

$$g_i(\bar{x}, y) \geq 0, \quad \forall\, y \in C, i = 1, 2, \ldots, N.$$

This implies that $\bar{x} \in \Omega_{FEP}$.

`(ii):` Suppose $\lim_{k\to\infty} \gamma^{m_{n_k}} = 0$. From the boundedness of $\{z^i_{n_k}\}$, without loss of generality, we may assume that $z^i_{n_k} \rightharpoonup \bar{z}$ as $k \to \infty$. Replacing y by w_{n_k} in (4.1.29), we get

$$g_i(w_{n_k}, z^i_{n_k}) \leq -\lambda||z^i_{n_k} - w_{n_k}||^2, \quad \forall\, i = 1, 2, \ldots, N. \tag{4.1.31}$$

On the otherhand, by the stepsize rule (3.1.9), for $m_{n_k} - 1$ we have

$$g_i(w_{n_k} - \gamma^{m_{n_k}-1} r^i(w_{n_k}), z^i_{n_k}) > -\sigma||r^i(w_{n_k})||^2, \quad \forall\, i = 1, 2, \ldots, N. \tag{4.1.32}$$

Combining (4.1.31) and (4.1.32), we get

$$\frac{1}{\lambda} g_i(w_{n_k}, z^i_{n_k}) \leq -||z^i_{n_k} - w_{n_k}||^2 < \frac{1}{\sigma} g_i(w_{n_k} - \gamma^{m_{n_k}-1} r^i(w_{n_k}), z^i_{n_k}). \tag{4.1.33}$$

Taking limit of the above inequality (4.1.33) as $k \to \infty$, and using the weak continuity of g, we get

$$\frac{1}{\lambda} g_i(\bar{x}, \bar{z}) \leq -\lim_{k\to\infty} ||z^i_{n_k} - w_{n_k}||^2 \leq \frac{1}{\sigma} g_i(\bar{x}, \bar{z}), \quad \forall\, i = 1, 2, \ldots, N.$$

Therefore $g_i(\bar{x}, \bar{z}) = 0$ and $\lim_{k\to\infty} ||w_{n_k} - z^i_{n_k}|| = 0$. Following similar process as `(i)`, we have $\bar{x} \in \Omega_{FEP}$.

Next, we show that $\bar{x} \in \cap_{j=1}^M F(S_j)$. From the definition of $\bar{u}_n$, we have from (4.1.23) that

$$\lim_{n\to\infty} ||\bar{u}_n - w_n|| = 0. \tag{4.1.34}$$

Using (4.1.16) and Lemma 2.6.1(i), we have

$$\begin{aligned}||x_{n+1}-z||^2 &\leq ||w_n-z||^2-\sum_{n=1}^{M}\delta_{n,j}(\delta_{n,0}-\kappa)||\bar{u}_n-v_{n,j}||^2\\ &= ||\alpha_n(x_1-z)+(1-\alpha_n)(x_n-z)||^2-\sum_{n=1}^{M}\delta_{n,j}(\delta_{n,0}-\kappa)||\bar{u}_n-v_{n,j}||^2\\ &\leq (1-\alpha_n)||x_n-z||^2+2\alpha_n\langle x_1-z,w_n-z\rangle-\sum_{n=1}^{M}\delta_{n,j}(\delta_{n,0}-\kappa)||\bar{u}_n-v_{n,j}||^2.\end{aligned}$$

This implies that

$$\sum_{n=1}^{M}\delta_{n,j}(\delta_{n,0}-\kappa)||\bar{u}_n-v_{n,j}||^2 \leq (1-\alpha_n)||x_n-z||^2+2\alpha_n\langle x_1-z,w_n-z\rangle-||x_{n+1}-z||^2.$$

Hence

$$\lim_{n\to\infty}\sum_{n=1}^{M}\delta_{n,j}(\delta_{n,0}-\kappa)||\bar{u}_n-v_{n,j}||^2=0.$$

Using condition C2, we have

$$\lim_{n\to\infty}||\bar{u}_n-v_{n,j}||=0. \tag{4.1.35}$$

Let $\{\bar{u}_{n_k}\}$ be a subsequence of $\{\bar{u}_n\}$. It follows from (4.1.34) that $\bar{u}_{n_k}\rightharpoonup\bar{x}$, hence by the demiclosedness of S_j, $j=1,2,\dots,M$, we have that $\bar{x}\in F(S_j)$, for each $j=1,2,\dots,M$. This implies that $\bar{x}\in\bigcap_{j=1}^{M}F(S_j)$. Therefore $\bar{x}\in Sol$, which implies that $\Omega_w\{x_n\}\subset Sol$.

Now we show that $\{x_n\}$ converges strongly to an element $x^*=P_{Sol}x_1$. To do this, we first prove that $\limsup_{n\to\infty}\langle x_1-z,w_n-z\rangle\leq 0$. Choose a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty}\langle x_1-x^*,w_n-x^*\rangle=\lim_{k\to\infty}\langle x_1-x^*,w_{n_k}-x^*\rangle.$$

Since $||x_{n_k}-w_{n_k}||\to 0$, and $x_{n_k}\rightharpoonup\bar{x}$ as $n\to\infty$, then from (2.2.2), we get

$$\begin{aligned}\limsup_{n\to\infty}\langle x_1-x^*,w_n-x^*\rangle &= \lim_{k\to\infty}\langle x_1-x^*,w_{n_k}-x^*\rangle\\ &= \langle x_1-x^*,\bar{x}-x^*\rangle\leq 0.\end{aligned} \tag{4.1.36}$$

Combining (4.1.18), (4.1.36) and Lemma 2.6.29, we get that $\{x_n\}$ converges strongly to $x^*=P_{Sol}x_1$. Consequently from (4.1.21), (4.1.27) and (4.1.34), we obtain that the sequences $\{w_n\}$, $\{z_n^i\}$ and $\{\bar{u}_n\}$ converge strongly to x^*.

CASE B: Suppose $\{||x_n-p||^2\}$ is not monotonically decreasing. The technique of proof used here is adapted from [172, 171]. Put $\Gamma_n:=||x_n-z||^2$ and let $\tau:\mathbb{N}\to\mathbb{N}$ be a mapping defined for all $n\geq n_0$ (for some n_0 large enough) by

$$\tau(n):=\max\{k\in\mathbb{N}:k\leq n,\Gamma_k\leq\Gamma_{k+1}\}.$$

Clearly τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall\, n \geq n_0.$$

From (4.1.20), we have

$$\begin{aligned} ||x_{\tau(n)+1} - z||^2 &\leq \alpha_{\tau(n)}||x_1 - z||^2 + (1 - \alpha_{\tau(n)})||x_{\tau(n)} - z||^2 \\ &\quad - \left(\frac{\gamma^{m_{\tau(n)}}\sigma}{K(1-\gamma^{m_{\tau(n)}})}||w_{\tau(n)} - z^i_{\tau(n)}||^2\right)^2 \\ &\leq ||x_{\tau(n)} - z||^2 - \left(\frac{\gamma^{m_{\tau(n)}}\sigma}{K(1-\gamma^{m_{\tau(n)}})}||w_{\tau(n)} - z^i_{\tau(n)}||^2\right)^2 + \alpha_{\tau(n)}M^*, \end{aligned}$$

for some $M^* > 0$. Therefore

$$\begin{aligned} \left(\frac{\gamma^{m_{\tau(n)}}\sigma}{K(1-\gamma^{m_{\tau(n)}})}||w_{\tau(n)} - z^i_{\tau(n)}||^2\right)^2 &\leq ||x_{\tau(n)} - z||^2 - ||x_{\tau(n)+1} - z||^2 + \alpha_{\tau(n)}M^* \\ &\leq \alpha_{\tau(n)}M^* \to 0, \qquad n \to \infty. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \gamma^{m_{\tau(n)}}||w_{\tau(n)} - z^i_{\tau(n)}||^2 = 0.$$

Just as in CASE A, we can show that $\lim_{n\to\infty}||w_{\tau(n)} - x_{\tau(n)}|| = \lim_{n\to\infty}||u^i_{\tau(n)} - w_{\tau(n)}|| = 0$.

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$ still denoted by $\{x_{\tau(n)}\}$ which converges weakly to $\bar{x} \in C$. Similarly as in CASE A above, we can show that

$$\lim_{n\to\infty}||w_{\tau(n)} - z^i_{\tau(n)}|| = \lim_{n\to\infty}||w_{\tau(n)} - z^i_{\tau(n)}|| = \lim_{n\to\infty}||\bar{u}_{\tau(n)} - v_{\tau(n),j}|| = 0.$$

So $\bar{x} \in Sol$. Since $||w_{\tau(n)} - x_{\tau(n)}|| \to 0$, we get that $\limsup_{n\to\infty}\langle x_1 - z, w_{\tau(n)} - z\rangle \leq 0$. Following (4.1.18), we have

$$||x_{\tau(n)+1} - z||^2 \leq (1 - \alpha_{\tau(n)})||x_{\tau(n)} - z||^2 + 2\alpha_{\tau(n)}\langle x_1 - z, w_{\tau(n)} - z\rangle. \quad (4.1.37)$$

By Lemma 2.6.29 and using conditions C1 and C2, we have from (4.1.37) that $\lim_{n\to\infty}||x_{\tau(n)} - z|| = 0$. Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_n \leq \Gamma_{\tau(n)+1}$. As a consequence, we obtain for all sufficiently large n that $0 \leq \Gamma_n \leq \Gamma_{\tau(n)+1}$. Hence $\lim_{n\to\infty}\Gamma_n = 0$. Therefore $\{x_n\}$ converge strongly to z. Consequently $\{w_n\}$, $\{z^i_n\}$ and $\{\bar{u}_n\}$ converges strongly to z. This completes the proof. $\square$

4.1.2 Numerical examples

In this subsection, we present some numerical examples for Algorithm 4.1.6. All the tests were run using `MATLAB 2014b` programming on a HP personal computer with RAM 8gb.

Example 4.1.13. Let $H = \mathbb{R}^m$ and $C_i = C$, $i = 1, 2, ...N$, where the feasible set C is defined by

$$C = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : |x_k| \leq 1, \ k = 1, 2, \ldots, m\}$$

and consider the problem:

$$\text{Find } \ x \in Sol := \Omega_{FEP} \cap \Big(\bigcap_{j=1}^{M} F(S_j) \Big),$$

where $g_i : C \times C \to \mathbb{R}$ is defined by

$$g_i(x, y) = \sum_{k=1}^{m} \Big(p_{ik} y_k^2 - p_{ik} x_k^2 \Big), \quad i = 1, 2, \dots, N$$

with $p_{ik} \in (0,1)$ randomly generated for all $i = 1, 2, \dots, N$, $k = 1, 2, \dots, m$. Also, $S_j : \mathbb{R}^m \to CB(\mathbb{R}^m)$ is defined as

$$S_j x = \begin{cases} \left[\dfrac{-(1+2j)x}{2}, -(1+j)x \right], & x \leq 0, \\ \left[-(1+j)x, \dfrac{-(1+2j)x}{2} \right], & x > 0, \end{cases} \tag{4.1.38}$$

for all $j = 1, 2, \dots, M$ and $x \in \mathbb{R}^m$. It can be easily shown that S_j is demi-contractive with constant $\kappa_j = \frac{4j^2+8j}{4j^2+12j+9} \in (0,1)$. It is also easy to verify that conditions A1 - A3 are satisfied, $I - S_j$ is demiclosed at zero and S_j satisfies the end point condition. Also, $Sol = \{x^*\}$, where $x^* = (0, 0, \dots, 0)^T$. For each $n \in \mathbb{N}$, let $\alpha_n = \frac{1}{2(n+1)}$ and let

$$\delta_{n,j} = \begin{cases} \frac{1}{2^j} \left(\frac{n-1}{n} \right), & if \quad j < n, \\ 1 - \frac{n-1}{n} \left(\sum_{k=1}^{n} \frac{1}{2^{k-1}} \right) & if \quad n = j, \\ 0, & otherwise. \end{cases}$$

It is easy to see that conditions (C1) and (C2) are also satisfied. We choose $\lambda = 0.6$, $\sigma = 0.25$, $\gamma = 0.48$ and $\beta_n^i = \frac{1}{N}$. Using $\frac{||x_{n+1} - x_n||}{||x_2 - x_1||} < 10^{-4}$ as our stopping criterion, we choose $x_1 \in C$ generated randomly and take different values of N, M and m as follows:

Case I: $N = 5$, $M = 5$ and $m = 5$;
Case II: $N = 10, N = 15$ and $m = 10$;
Case III: $N = 20, N = 20$ and $m = 20$.

We compare the output of our Algorithm 4.1.6 with Algorithm 4.1.2 of Hieu [130]. We also plot the graphs $||x_{n+1} - x_n||$ against the number of iteration. The numerical results can be seen in Table 4.1 and Figure 4.1.

Example 4.1.14. Next, we consider a Nash-Cournot oligopolistic market equilibrium problem model taken from [110, 189]. Assume that there are n companies producing a common homogeneous commodity and that the price p_i of company i depends on the total quantity $\sigma_x = \sum_{i=1}^{n} x_i$ of the commodity. Let $\phi_i(x_i)$ denotes the cost (tax and fee) of company i for generating x_i. Suppose that the profit of company i is given by

$$g_i(x_1, x_2, \dots, x_n) := x_i p_i(\sigma_x) - \phi_i(x_i), \quad i = 1, 2, \dots, n.$$

Let $C_i = [x_i^{\min}, x_i^{\max}]$ be the strategy set of company i. Then, the strategy set of the model is $C := C_1 \times C_2 \times \cdots \times C_n$. Actually, each company seeks to maximize its profit by choosing the corresponding production level under the presumption that the production of the other companies is a parametric input. In this context, a Nash equilibrium is a production pattern in which no firm can increase its profit by changing its controlled variables. Thus under this equilibrium concept, each firm determines its best response given other firms' actions. One common approach use to solve this model is the Nash equilibrium concept.

Mathematically, a point $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in C$ is said to be a Nash equilibrium point if

$$g_i(x^*) \geq g_i(x^*[x_i]), \quad \forall\, x_i \in C_i, \quad i = 1, 2, \ldots, n,$$

where the vector $x^*[x_i]$ stands for the vector obtained from x^* by replacing x_i^* with x_i. By taking $g(x,y) := \Phi(x,y) - \Phi(x,x)$ with $\Phi(x,y) := -\sum_{i=1}^{n} g_i(x[y_i])$, the problem of finding a Nash equilibrium point of the model can be formulated as follows:

$$\text{Find} \quad x^* \in C: \quad g(x^*, x) \geq 0, \quad \forall\, x \in C. \tag{4.1.39}$$

Now, assume that the task-fee function $\phi_i(x_i)$ is increasing and affine for each $i \geq 1$. This means that both the tax and fee for producing a unit commodity are increasing as the quantity of the production is getting larger. In this situation, the bifunction g can be formulated in the form:

$$g(x,y) = \langle Px + Qy + q, y - x \rangle,$$

where $q \in \mathbb{R}^n$ and P, Q are two matrices of order n such that Q is symmetric positive semidefinite and $Q - P$ is symmetric negative semidefinite. This shows that g is pseudo-monotone. Using this model, our aim is to show the numerical behaviour of our proposed Algorithm 4.1.6. We take the feasible set C as a box defined by

$$C = \{x \in \mathbb{R}^n : -2 \leq x_l \leq 5, \quad l = 1, 2, \ldots, n\}.$$

Let $S : \mathbb{R}^n \to CB(\mathbb{R}^n)$ be defined by

$$Sx = \begin{cases} \{0\}, & if \;\; x < 0 \\ \left[\frac{x}{2}, x\right], & if \;\; x \geq 0. \end{cases}$$

It can easily be verified that S is 0-demi-contractive mapping and $I - S$ is demiclosed at zero. Let the matrices P and Q be generated randomly such that their conditions are satisfied and also the vector q be generated randomly. All the optimization subproblems are efficiently solved by the function `quadprog` in Matlab. We take $\alpha_n = \frac{1}{n+5}$, $\delta_n = \frac{n}{5n+3}$, $\sigma = 0.01$, $\gamma = 0.8$ and choose $x_1 \in \mathbb{R}^n$ randomly (with $n = 30, 50$ and 70), we compare the output of Algorithm 4.1.6 using the following values of the stepsize:

$$\text{Case I: } \lambda = 0.1, \quad \text{Case II: } \lambda = 0.25, \quad \text{Case III: } \lambda = 0.5.$$

The stopping criterion used is $\frac{||x_{n+1} - x_n||}{||x_2 - x_1||} < 10^{-4}$. The numerical results are reported in Table 4.2 and Figure 4.2.

Remark 4.1.15. The numerical results from Example 4.1.14 shows that Algorithm 4.1.6 is very efficient and easy to implement for solving the Nash Oligopolistic market equilibrium problem. Irrespective of the choice of λ, there is no significant difference in the number of iteration and CPU time taken.

Finally, we present an example in an infinite dimensional space setting. For simplicity, we take $N = M = 5$.

Example 4.1.16. Let $H = L^2([0,1])$ with the inner product $\langle x, y\rangle = \int_0^1 x(s)y(s)ds$ and the induced norm $||x||_L = \int_0^1 |x(s)|^2 ds$. For $i = 1, 2, \ldots, 5$, we define the feasible set as

$$C_i := \{x \in H : ||x||_L \leq 1\}.$$

Let $g_i(x, y)$ be of the form $\langle A_i x, y - x\rangle$ with the operator $A_i : H \to H$ define as $(A_i x)(t) = \max\{0, \frac{x(t)}{i}\}$ for $i = 1, 2...5$. Then it can easily be verified that each g_i is monotone (and so, pseudo-monotone) on C_i. For $j = 1, 2, \ldots, 5$, let $S_j : H \to CB(H)$ be defined by $S_j(x)(t) = [0, \frac{(x)(t)}{2j}]$, then $Sol = EP(g, C) \cap F(S) = \{0\}$. We take $\alpha_n = \frac{1}{n}$, $\delta_{n,j} = \frac{1}{M+1}$, $\beta_n^i = \frac{1}{N}$, $\lambda = 0.9$, $\sigma = 0.4$, $\gamma = 0.5$ and $\frac{||x_{n+1} - x_n||}{||x_2 - x_1||} < 5 \times 10^{-6}$ as the stopping criterion. We choose the following starting points: Case I: $x_1 = 4\sin(\frac{t}{2})$, Case II: $x_1 = \frac{1}{8}(\cos(2t) - \sin(3t))$, Case III: $x_1 = 2\cos(5t)\exp(4t)$, Case IV: $x_1 = t^2 \sin(5\pi t)$, and then plot the graphs of errors against the number of iterations in each case. The numerical results can be found in Figure 4.3. From this results, we conclude that the change in the initial values does not have any significant effect on the number of iterations and cpu time taken for execution by the algorithm.

Table 4.1: Computation results of Algorithm 4.1.6 and Algorithm 4.1.2 for Example 4.1.13.

		Algorithm 4.1.6	Algorithm 4.1.2
Case I	CPU time (sec)	0.4365	1.0796
	No. of Iter.	31	35
Case II	CPU time (sec)	0.6885	3.0243
	No. of Iter.	51	79
Case III	CPU time (sec)	1.2867	7.7517
	No. of Iter.	73	167

4.2 Inertial Mann-Krasnoselskii Algorithm with Self Adaptive Stepsize for Split Variational Inclusion Problem and Paramonotone Equilibria

In this section, we consider a Mann-Krasnoselskii algorithm with inertial extrapolation for approximating a common solution of split variational inclusion problem and equilibrium problem with paramonotone bifunction.

Table 4.2: Computational result of Algorithm 4.1.6 for Example 4.1.14.

		Case I	Case II	Case III
$m = 20$	CPU time (sec)	0.2232	0.2206	0.2135
	No. of Iter.	9	9	10
$m = 50$	CPU time (sec)	0.1739	0.1955	0.1250
	No. of Iter.	11	10	9
$m = 70$	CPU time (sec)	0.1368	0.1544	0.1153
	No. of Iter.	11	11	9

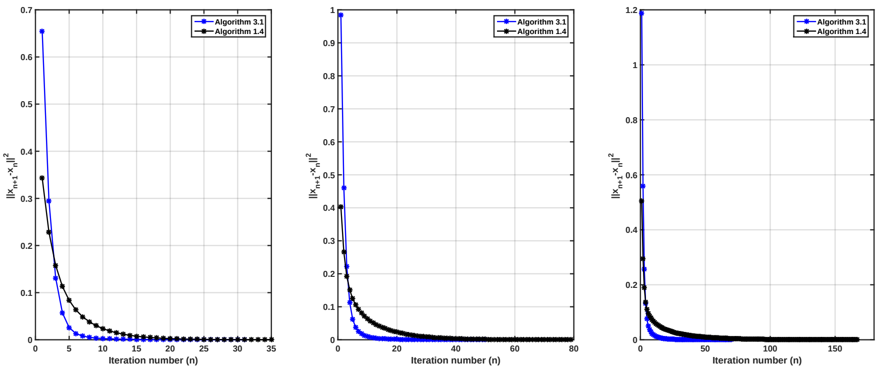


Figure 4.1: Example 4.1.13, Left: Case(i); Middle: Case(ii); Right: Case(iii).

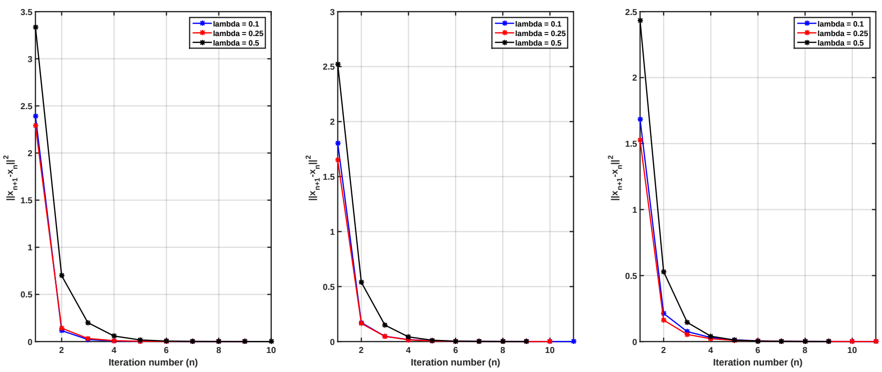


Figure 4.2: Example 4.1.14, Left: $m = 20$; Bottom: $m = 50$; Right $m = 70$.

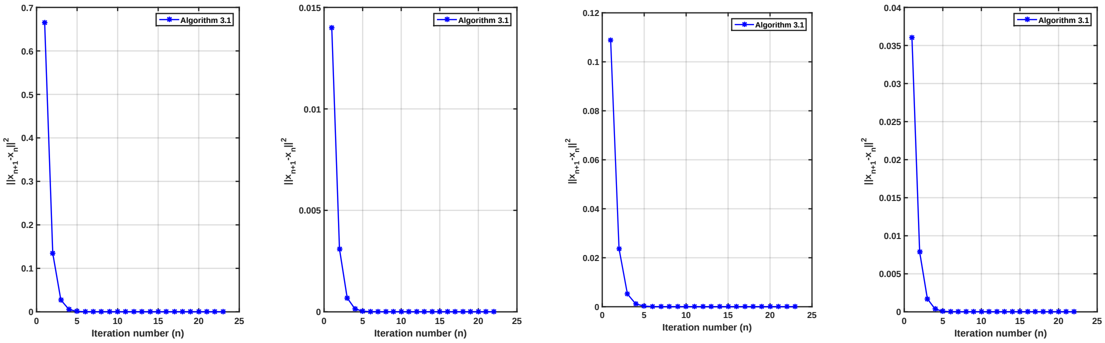


Figure 4.3: Example 4.1.16, Case I, Time: 10.5890sec; Case II, Time: 9.0166sec; Case III, Time: 9.2805sec; Case IV, Time:9.0381sec.

Let $B : H \to 2^H$ be a multi-valued maximal monotone mapping. The resolvent mapping $J_\lambda^B : H \to H$ associated with B is defined by

$$J_\lambda^B(x) := (I + \lambda B)^{-1}(x), \quad \forall\, x \in H,$$

for some $\lambda > 0$, where I is the identity operator on H. We note that for all $\lambda > 0$, the resolvent operator J_λ^B is single-valued, nonexpansive and firmly nonexpansive, see e.g [86].

In 2011, Moudafi [185] introduced the following Split Variational Inclusion Problem (shortly, SVIP): Find $x^\dagger \in H_1$ such that

$$0 \in B_1(x^\dagger) \quad \text{and} \quad 0 \in B_2(Ax^\dagger), \tag{4.2.1}$$

where H_1 and H_2 are real Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone operators, $A : H_1 \to H_2$ is a linear bounded operator. As noted in [66], the SVIP can be seen as generalization of split variational inequality problems, split feasibility problems, split common fixed point problems and split equilibrium problems. We denote the set of solutions of (4.2.1) by Ω_{SVIP}.

For solving the SVIP, Bryne et al. [57] introduced the following iterative algorithm with weak convergence property: For given $x_0 \in H_1$, compute iterative sequene $\{x_n\}$ by the following scheme

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n),$$

where $\lambda > 0$ and the stepsize γ is chosen such that $\gamma \in \left(0, \frac{2}{||A||^2}\right)$. It is noted that computation of the norm of the operator A may be difficult in practice.

In 2016, Chuang [84] studied the SVIP using the following descent projection method:

Algorithm 4.2.1. *Descent Projection Algorithm (DPA)*

Step 0: *Set $n = 1$ and choose $x_1 \in H_1$.*
Step 1: *Given $x_n \in H_1$, compute $\{y_n\}$ using*

$$y_n = J_{\lambda_n}^{B_1}[x_n - \gamma_n A^*(I - J_{\lambda_n}^{B_2})Ax_n],$$

where $\{\lambda_n\} \subset (0, \infty)$ *and* $\gamma_n > 0$ *satisfying*

$$\gamma_n||A^*(I - J_{\lambda_n}^{B_2})Ax_n - A^*(I - J_{\lambda_n}^{B_2})Ay_n|| \leq \delta||x_n - y_n||, \quad \delta \in (0, 1).$$

Step 2: *If* $x_n = y_n$, *STOP. Otherwise continue with Step 3.*
Step 3: *Compute* $x_{n+1} \in H_1$ *using*

$$x_{n+1} = J_{\lambda_n}^{B_1}(x_n - \alpha_n D(x_n, y_n)),$$

where

$$D(x_n, y_n) := x_n - y_n + \gamma_n[A^*(I - J_{\lambda_n}^{B_2})Ay_n - A^*(I - J_{\lambda_n}^{B_2})Ax_n],$$

$$\alpha_n = \frac{\langle x_n - y_n, D(x_n, y_n)\rangle}{||D(x_n, y_n)||^2}.$$

Then update $n := n + 1$ *and go to Step 1.*

For more details and recent results on SVIP, we refer the reader to [74, 85, 147, 253] and references therein.

Recently, Yen et al. [276] proposed a projection based algorithm for solving the split feasibility problem (SFP) (1.1.9) involving paramonotone equilibria and convex optimization. They considered the following problem:

$$\text{Find} \quad x^* \in C : f(x^*, y) \geq 0 \quad \forall\, y \in C \quad \text{and} \quad g(Ax^*) \leq g(u) \quad \forall\, u \in H_2, \tag{4.2.2}$$

where g is a proper lower semicontinuous convex function on H_2. They proposed the following algorithm and proved its strong convergence to a solution of problem (4.2.2).

Algorithm 4.2.2. *Mann-Krasnolselskii Proximal Algorithm (MKPA)*

Initialization: *Take positive parameters* δ, ξ *and real sequences* $\{a_n\}, \{\delta_n\}, \{\beta_n\}, \{\epsilon_n\}, \{\rho_n\}$ *satisfying*

$$0 < a < a_n < b < 1, \quad 0 < \xi < \rho_n \leq 4 - \xi, \quad \delta_n > \delta > 0, \quad \beta_n > 0, \quad \epsilon_n > 0, \quad \forall\, n \in \mathbb{N},$$

$$\lim_{n\to\infty} a_n = \frac{1}{2},$$

$$\sum_{n=1}^{\infty} \frac{\beta_n}{a_n} = +\infty, \quad \sum_{n=1}^{\infty} \beta_n^2 < +\infty, \quad \sum_{n=1}^{\infty} \frac{\beta_n \epsilon_n}{\delta_n} < +\infty.$$

Step 0: *Choose* $x_1 \in C$ *and let* $n = 1$.
Step n: *Having* $x_n \in C$, *take* $g_n \in \partial_2^{\epsilon_n} f(x_n, x_n)$ *and define*

$$\alpha_n = \frac{\beta_n}{\gamma_n} \quad \textit{where} \quad \gamma_n = \max\{\delta_n, ||g_n||\}.$$

Compute $y_n = P_C(x_n - \alpha_n g_n)$, *i.e.,*

$$\langle y_n - x_n + \alpha_n g_n, x - y_n\rangle \geq 0 \quad \forall\, x \in C.$$

Take

$$\mu_n = \begin{cases} 0 & \textit{if} \;\; \nabla h(y_n) = 0, \\ \rho_n \frac{h(y_n)}{||\nabla h(y_n)||^2} & \textit{if} \;\; \nabla h(y_n) \neq 0, \end{cases}$$

and compute

$$z_n = P_C(y_n - \mu_n A^*(I - prox_{\lambda g})Ay_n),$$

where

$$prox_{\lambda g}(u) := argmin \left\{ g(u) + \frac{1}{\lambda}||v-u||^2 : v \in H_2 \right\},$$

and $\nabla h(x) := A^*(I - prox_{\lambda g})Ax$.
Let

$$x_{n+1} = a_n x_n + (1-a_n) z_n.$$

Motivated by the works of Moudafi [185], Chuang [86] and Yen [276], in this section, we study the following problem:

$$\text{Find} \quad x^\dagger \in C : f(x^\dagger, y) \geq 0 \quad \forall\, y \in C, \quad 0 \in B_1(x^\dagger) \quad \text{and} \quad 0 \in B_2(Ax^\dagger), \tag{4.2.3}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone operators, $A : H_1 \to H_2$ is a bounded operator and $f : C \times C \to \mathbb{R}$ is a bifunction satisfying $f(x,x) = 0$. We denote the set of solutions of (4.2.3) with Γ, i.e., $\Gamma := \Omega_{EP(f)} \cap \Omega_{SVIP}$. It is easy to see that Problem (4.2.3) contain Problems (4.2.1), EP (1.1.4) and (4.2.2).

We propose an inertial Mann-Krasnoelskii algorithm which converges strongly to a solution of (4.2.3). The algorithm is designed in such a way that it stepsize is chosen self-adaptively, and its strong convergence analysis does not require a prior estimate of the norm of the bounded operator.

We make use of the following assumptions throughout this section.

Assumption 4.2.3.

(A1) H_1 *and* H_2 *are real Hilbert spaces, and* $A : H_1 \to H_2$ *is a bounded linear operator with adjoint* $A^* : H_2 \to H_1$.

(A2) $B_1 : H_1 \to 2^{H_1}$ *and* $B_2 : H_2 \to 2^{H_2}$ *are multi-valued maximal monotone operators.*

(A3) The bifunction $f : H \times H \to \mathbb{R}$ *satisfies the following:*

(B1) For each $x \in C$, $f(x,x) = 0$ *and* $f(x,\cdot)$ *is lower semicontinuous and convex on* C;

(B2) $\partial_2^\lambda f(x,x)$ *is nonempty for any* $\lambda > 0$ *and* $x \in C$ *and is bounded on any bounded subset of* C, *where* $\partial_2^\lambda f(x,x)$ *denotes* λ*-subdifferential of the convex function* $f(x,\cdot)$ *at* x, *that is*

$$\partial_2^\lambda(x,x) := \{\eta \in H_1 : \langle \eta, y - x\rangle + f(x,x) \leq f(x,y) + \lambda \quad \forall\, y \in C\}.$$

(B3) f *is pseudo-monotone on* C *with respect to every solution of the EP, that is* $f(x, x^*) \leq 0$ *for any* $x \in C$, $x^* \in \Omega_{EP(f)}$ *and* f *satisfies the following condition, which is called the para-monotonicity properly:*

$$x^* \in \Omega_{EP(f)}, \ y \in C, \quad f(x^*, y) = f(y, x^*) = 0 \Rightarrow y \in \Omega_{EP(f)}.$$

(B4) *For all* $x \in C$, $f(\cdot, x)$ *is weakly upper semicontinuous on* C.

(A4) *The problem* (4.2.3) *is consistent, i.e., its solution set* Γ *is nonempty.*

4.2.1 Main results

Here, we present an inertial Mann-Krasnolselskii algorithm with self adaptive stepsize for split variational inequality problem with para-monotone equilibria.

Algorithm 4.2.4. *Inertial Mann-Krasnolselskii algorithm*

Initialization: *Pick* $x_0, x_1 \in H_1$, $\theta \in [0, 1)$, $\{\epsilon_n\} \subset [0, \infty)$, $\{r_n\}$, $\{a_n\}, \{\rho_n\}, \{\beta_n\}$, $\{\lambda_n\}$ *satisfying the following condition for each* $n \in \mathbb{N}$:

$$\rho_n > \rho > 0, \quad 0 < a < a_n < b < 1, \quad \beta_n > 0, \quad r_n > 0, \quad \lambda_n \geq 0;$$

$$\sum_{n=1}^{\infty} \epsilon_n < \infty, \quad \lim_{n \to \infty} a_n = \frac{1}{2}, \quad \liminf_{n \to \infty} r_n > 0;$$

$$\sum_{n=1}^{\infty} \frac{\beta_n}{\rho_n} = +\infty \quad \sum_{n=1}^{\infty} \beta_n^2 = +\infty, \quad \sum_{n=1}^{\infty} \frac{\beta_n \lambda_n}{\rho_n} < +\infty. \tag{4.2.4}$$

*[***Step 1:***]* *Given* x_{n-1} *and* x_n, *choose* α_n *such that* $0 < \alpha_n \leq \bar{\alpha}_n$, *where*

$$\bar{\alpha}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{||x_n - x_{n-1}||^2}\right\} & \text{if} \quad x_n \neq x_{n-1}, \\ \theta \quad \text{otherwise.} \end{cases} \tag{4.2.5}$$

Set

$$w_n = x_n + \alpha_n(x_n - x_{n-1}) \tag{4.2.6}$$

*[***Step 2:***]* *Compute*

$$y_n = J_{r_n}^{B_1}[w_n - \xi_n A^*(I - J_{r_n}^{B_2})Aw_n], \tag{4.2.7}$$

where ξ_n *is chosen such that*

$$\xi_n = \begin{cases} \frac{2||(I - J_{r_n}^{B_2})Aw_n||^2}{||A^*(I - J_{r_n}^{B_2})Aw_n||^2}, & \text{if} \quad J_{r_n}^{B_2} Aw_n \neq Aw_n, \\ \xi, & \text{otherwise,} \end{cases} \tag{4.2.8}$$

where ξ *is any nonnegative value.*
*[***Step 3:***]* *Take* $\eta_n \in \partial_2^{\lambda_n} f(y_n, y_n)$ *and define*

$$\tau_n = \frac{\beta_n}{\gamma_n} \quad \text{where} \quad \gamma_n = \max\{\rho_n, ||\eta_n||\}.$$

Compute

$$z_n = P_C(y_n - \tau_n \eta_n). \tag{4.2.9}$$

*[***Step 4:***] Let*

$$x_{n+1} = a_n x_n + (1 - a_n) z_n. \tag{4.2.10}$$

The following lemma can be obtained from Lemma 3.2 of [224].

Lemma 4.2.5. *For every $n \geq 1$, the following inequalities hold:*

(i) $\tau_n ||\eta_n|| \leq \beta_n$;

(ii) $||z_n - y_n|| \leq \beta_n$.

Lemma 4.2.6. *The choice of the stepsize defined in (4.2.8) is well defined.*

Proof. Take $w \in SVIP(B_1, B_2)$, then $J_r^{B_1} w = w$ and $J_r^{B_2} Aw = Aw$. Observe that

$$\begin{aligned} ||(I - J_{r_n}^{B_2})Aw_n||^2 &= \langle (I - J_{r_n}^{B_2})Aw_n, (I - J_{r_n}^{B_2})Aw_n \rangle \\ &= \langle (I - J_{r_n}^{B_2})Aw_n, Aw_n - Aw + J_{r_n}^{B_2} Aw - J_{r_n}^{B_2} Aw_n \rangle \\ &= \langle (I - J_{r_n}^{B_2})Aw_n, Aw_n - Aw \rangle + \langle (I - J_{r_n}^{B_2})Aw_n, J_{r_n}^{B_2} Aw - J_{r_n}^{B_2} Aw_n \rangle \\ &= \langle A^*(I - J_{r_n}^{B_2})Aw_n, w_n - w \rangle + \langle (I - J_{r_n}^{B_2})Aw_n, J_{r_n}^{B_2} Aw - J_{r_n}^{B_2} Aw_n \rangle \\ &\leq ||A^*(I - J_{r_n}^{B_2})Aw_n|| \cdot ||w_n - w|| + ||(I - J_{r_n}^{B_2})Aw_n|| \times \\ &\quad ||J_{r_n}^{B_2} Aw - J_{r_n}^{B_2} Aw_n||. \end{aligned}$$

Consequently, for $n \in \mathbb{N}$, we get $||A^*(I - J_{r_n}^{B_2})Aw_n|| \cdot ||w_n - w|| \geq 0$ and $||(I - J_{r_n}^{B_2})Aw_n|| \cdot ||J_{r_n}^{B_2} Aw - J_{r_n}^{B_2} Aw_n|| \geq 0$. Since $J_{r_n}^{B_2} Aw_n \neq Aw_n$, then we obtain $||A^*(I - J_{r_n}^{B_2})Aw_n|| \cdot ||w_n - w|| > 0$ and hence $||A^*(I - J_{r_n}^{B_2})Aw_n|| > 0$. This implies that ξ_n defined in (4.2.8) is well defined. □

Lemma 4.2.7. *Let $x^* \in \Gamma$, then*

$$||y_n - x^*||^2 \leq ||x_n - x^*||^2 + \alpha_n c_1 ||x_n - x_{n-1}||,$$

where $c_1 = \sup_{n \geq 1}\{||x_n - x^*|| + ||x_{n-1} - x^*|| + 2||x_n - x_{n-1}||\}$.

Proof. Let $x^* \in \Gamma$, then

$$\begin{aligned} ||y_n - x^*||^2 &= ||J_{r_n}^{B_1}[w_n - \xi_n A^*(I - J_{r_n}^{B_2})Aw_n] - J_{r_n}^{B_1} x^*||^2 \\ &\leq ||w_n - x^* - \xi_n A^*(I - J_{r_n}^{B_2})Aw_n||^2 \\ &= ||w_n - x^*||^2 - 2\xi_n \langle (I - J_{r_n}^{B_2})Aw_n, Aw_n - Ax^* \rangle + \xi_n^2 ||A^*(I - J_{r_n}^{B_2})Aw_n||^2 \\ &\leq ||w_n - x^*||^2 - \xi_n[2||(I - J_{r_n}^{B_2})Aw_n||^2 - \xi_n ||A^*(I - J_{r_n}^{B_2})Aw_n||^2]. \end{aligned} \tag{4.2.11}$$

By the choice of ξ_n, we have

$$||y_n - x^*||^2 \leq ||w_n - x^*||^2. \tag{4.2.12}$$

Also from (4.2.7), we have

$$\begin{aligned}
||w_n - x^*||^2 &= ||x_n + \alpha_n(x_n - x_{n-1}) - x^*||^2 \\
&= ||x_n - x^*||^2 + 2\alpha_n\langle x_n - x^*, x_n - x_{n-1}\rangle + \alpha_n^2||x_n - x_{n-1}||^2 \\
&\leq ||x_n - x^*||^2 + \alpha_n(||x_n - x^*||^2 - ||x_{n-1} - x^*||^2) + 2\alpha_n||x_n - x_{n-1}||^2 \\
&= ||x_n - x^*||^2 + \alpha_n(||x_n - x^*|| + ||x_{n-1} - x^*||)||x_n - x_{n-1}|| + 2\alpha_n||x_n - x_{n-1}||^2 \\
&= ||x_n - x^*||^2 + \alpha_n(||x_n - x^*|| + ||x_{n-1} - x^*|| + 2||x_n - x_{n-1}||)||x_n - x_{n-1}|| \\
&\leq ||x_n - x^*||^2 + \alpha_n c_1||x_n - x_{n-1}||, \qquad (4.2.13)
\end{aligned}$$

where $c_1 = \sup_{n\geq 1}\{||x_n - x^*|| + ||x_{n-1} - x^*|| + 2||x_n - x_{n-1}||\}$. From (4.2.12) and (4.2.13), we have

$$||y_n - x^*||^2 \leq ||x_n - x^*||^2 + \alpha_n c_1||x_n - x_{n-1}||.$$

□

Lemma 4.2.8. *Let $x^* \in \Gamma$. Then for each $n \geq 1$, we have*

$$||z_n - x^*||^2 \leq ||w_n - x^*||^2 + 2\tau_n f(y_n, x^*) + 2\tau_n\lambda_n + 2\beta_n^2.$$

Proof. From Lemma 2.6.1(i), we get

$$\begin{aligned}
||z_n - x^*||^2 &= ||z_n - y_n + y_n - x^*||^2 \\
&\leq ||y_n - x^*||^2 + 2\langle y_n - z_n, x^* - z_n\rangle. \qquad (4.2.14)
\end{aligned}$$

From (2.2.2) and (4.2.9), we have

$$\langle z_n - y_n + \tau_n\eta_n, x - z_n\rangle \geq 0 \quad \forall\ x \in C.$$

Taking $x = x^*$, we have

$$\begin{aligned}
&\langle z_n - y_n + \tau_n\eta_n, x^* - z_n\rangle \geq 0 \\
&\quad \Leftrightarrow \langle \tau_n\eta_n, x^* - z_n\rangle \geq \langle y_n - z_n, x^* - z_n\rangle.
\end{aligned}$$

Hence from (4.2.14), we have

$$\begin{aligned}
||z_n - x^*||^2 &\leq ||y_n - x^*||^2 + 2\langle \tau_n\eta_n, x^* - z_n\rangle \\
&= ||y_n - x^*||^2 + 2\langle \tau_n\eta_n, x^* - y_n\rangle + 2\langle \tau_n\eta_n, y_n - z_n\rangle. \qquad (4.2.15)
\end{aligned}$$

Since $\eta_n \in \partial_2^{\lambda_n} f(y_n, y_n)$, we have

$$\begin{aligned}
&f(y_n, x^*) - f(y_n, y_n) \geq \langle \eta_n, x^* - y_n\rangle - \lambda_n \\
&\quad \Leftrightarrow f(y_n, x^*) + \lambda_n \geq \langle \eta_n, x^* - y_n\rangle. \qquad (4.2.16)
\end{aligned}$$

On the other hand, from Lemma 4.2.5 it holds that

$$\langle \tau_n\eta_n, y_n - z_n\rangle \leq \tau_n||\eta_n||||y_n - z_n|| \leq \beta_n^2. \qquad (4.2.17)$$

Combining (4.2.15), (4.2.16) and (4.2.17), we get

$$||z_n - x^*||^2 \leq ||y_n - x^*||^2 + 2\tau_n f(y_n, x^*) + 2\tau_n\lambda_n + 2\beta_n^2,$$

which together with (4.2.12) yields

$$||z_n - x^*||^2 \leq ||w_n - x^*||^2 + 2\tau_n f(y_n, x^*) + 2\tau_n\lambda_n + 2\beta_n^2.$$

□

We now give the convergence analysis of Algorithm 4.2.4 to solution of Problem (4.2.3).

Theorem 4.2.9. *Suppose Assumption 4.2.3 holds and the sequence $\{x_n\}$ is generated by Algorithm 4.2.4. Then, the sequence $\{x_n\}$ strongly converges to a solution of Problem (4.2.3).*

Proof. Claim 1: The sequence $\{||x_n - x^*||^2\}$ is convergent for all $x^* \in \Gamma$.

Since $x^* \in Sol(EP)$, and f is pseudomonotone on C with respect to every solution of EP, we have $f(y_n, x^*) \leq 0$. By the definition of x_{n+1}, we have

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||a_n x_n + (1-a_n)z_n - x^*||^2 \\ &\leq a_n||x_n - x^*||^2 + (1-a_n)||z_n - x^*||^2. \end{aligned} \tag{4.2.18}$$

From Lemma 4.2.8 and (4.2.13), we have

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq a_n||x_n - x^*||^2 + (1-a_n)[||w_n - x^*||^2 + 2\tau_n f(y_n, x^*) + 2\tau_n\lambda_n + 2\beta_n^2] \\ &\leq ||x_n - x^*||^2 + (1-a_n)\alpha_n c_1||x_n - x_{n-1}|| + \Lambda_n, \end{aligned} \tag{4.2.19}$$

where $\Lambda_n = 2(1-a_n)(\tau_n\lambda_n + \beta_n^2)$.

Since $\tau_n = \frac{\beta_n}{\gamma_n}$ with $\gamma_n = \max\{\rho_n, ||\eta_n||\}$,

$$\sum_{n=1}^{\infty} \tau_n\lambda_n = \sum_{n=1}^{\infty} \frac{\beta_n}{\gamma_n}\lambda_n \leq \sum_{n=1}^{\infty} \frac{\beta_n}{\rho_n}\lambda_n < +\infty.$$

Note that $\sum_{n=1}^{\infty} \beta_n^2 < +\infty$ and $0 < a < a_n < b < 1$ and thus, we have

$$\sum_{n=1}^{\infty} \Lambda_n < 2(1-a)\sum_{n=1}^{\infty}(\tau_n\lambda_n + \beta_n^2) < +\infty.$$

Also, we have from (4.2.6) that

$$\alpha_n||x_n - x_{n-1}||^2 \leq \bar{\alpha}_n||x_n - x_{n-1}||^2 \leq \epsilon_n,$$

and therefore

$$\sum_{n=1}^{\infty} \alpha_n||x_n - x_{n-1}||^2 < \infty.$$

Now using Lemma 2.6.32 and (4.2.19), we see that $\{||x_n - x^*||^2\}$ is convergent for all $x^* \in \Gamma$. Hence, the sequence $\{x_n\}$ is bounded. Consequently, the sequences $\{w_n\}, \{y_n\}$ and $\{z_n\}$ are bounded.

Claim 2: $\limsup_{n\to\infty} f(y_n, x^*) = 0$ for all $x^* \in \Gamma$.

From (4.2.19), we see that

$$\begin{aligned} -2(1-a_n)\tau_n f(y_n, x^*) &\leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + \Lambda_n \\ &\quad +(1-a_n)\alpha_n c_1 ||x_n - x_{n-1}||. \end{aligned} \tag{4.2.20}$$

Summing up (4.2.20), we get

$$\sum_{n=1}^{\infty} -2(1-a_n)\tau_n f(y_n, x^*) < +\infty.$$

On the otherhand, using Assumption (A2) and the fact that $\{x_n\}$ is bounded, we get that $||\eta_n||$ is bounded. Thus, there is a constant $L > \delta$ such that $||\eta_n|| \leq L$ for every $n \geq 1$, and hence

$$\frac{\gamma_n}{\rho_n} = \max\left\{1, \frac{||\eta_n||}{\rho_n}\right\} \leq \frac{L}{\rho}.$$

Therefore

$$\tau_n = \frac{\beta_n}{\gamma_n} \geq \frac{\rho}{L}\frac{\beta_n}{\rho_n}.$$

Since $x^* \in \Gamma$, it follows from the pseudo-monotonicity of f that $-f(y_n, x^*) \geq 0$ which together with $0 < a < a_n < b < 1$ implies

$$\sum_{n=1}^{\infty}(1-b)\frac{\beta_n}{\rho_n}\Big(-f(y_n, x^*)\Big) < +\infty.$$

Since $\sum_{n=1}^{\infty}\frac{\beta_n}{\rho_n} = \infty$, it implies that $\limsup_{n\to\infty} f(y_n, x^*) = 0$.

Claim 3: For any $x^* \in \Gamma$, let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$ such that

$$\limsup_{n\to\infty} f(y_n, x^*) = \lim_{j\to\infty} f(y_{n_j}, x^*) \tag{4.2.21}$$

and y^* be a weak cluster point of $\{y_{n_j}\}$. Then y^* belongs to $\Omega_{EP(f)}$.

Without loss of generality, we can assume that $y_{n_j} \rightharpoonup y^*$ as $j \to \infty$. Since $f(\cdot, x^*)$ is upper semi-continuous and by Claim 2, we have

$$f(y^*, x^*) \geq \limsup_{j\to\infty} f(y_{n_j}, x^*) = 0.$$

Since $x^* \in \Gamma$ and f is pseudo-monotone, we have $f(y^*, x^*) \leq 0$ and so $f(y^*, x^*) = 0$. Again, by the pseudo-monotonicity of f, $f(x^*, y^*) \leq 0$ and hence $f(y^*, x^*) = f(x^*, y^*) = 0$. Then, by the para-monotonicity of f (i.e., Assumption (A3)), we can conclude that y^* is also a solution of EP.

Claim 4: Every weak cluster point $\bar{x}$ belongs to the solution set Ω_{SVIP}.

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x}$. Observe that

$$\sum_{n=1}^{\infty} ||w_n - x_n|| = \sum_{n=1}^{\infty} \alpha_n ||x_n - x_{n-1}|| < \infty.$$

Hence

$$\lim_{n\to\infty} ||w_n - x_n|| = 0. \tag{4.2.22}$$

This implies that $w_{n_j} \rightharpoonup \bar{x}$, where $\{w_j\}$ is the subsequence of $\{w_n\}$. From (4.2.11) and (4.2.18), we have

$$\begin{aligned} ||x_{n+1} - x^*||^2 \quad &\leq \quad ||x_n - x^*||^2 - (1-a_n)\xi_n[2||(I - J_{r_n}^{B_2})Aw_n||^2 + \xi_n||A^*(I - J_{r_n}^{B_2})Aw_n||^2] \\ &\quad +(1-a_n)\alpha_n c_1 ||x_n - x_{n-1}|| + \Lambda_n, \end{aligned} \tag{4.2.23}$$

where c_1 and Λ_n are as defined in Lemma 4.2.7 and (4.2.19), respectively.
Put $\Theta_n = 2||(I - J_{r_n}^{B_2})Aw_n||^2 + \xi_n||A^*(I - J_{r_n}^{B_2})Aw_n||^2$. It follows that

$$(1-a_n)\xi_n\Theta_n \quad \leq \quad ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + (1-a_n)\alpha_n c_1||x_n - x_{n-1}|| + \Lambda_n.$$

This implies that

$$(1-b)\sum_{n=1}^{\infty} \xi_n\Theta_n < ||x_0 - x^*||^2 + (1-a)c_1 \sum_{n=1}^{\infty} \alpha_n||x_n - x_{n-1}|| + \sum_{n=1}^{\infty} \Lambda_n < +\infty.$$

Hence

$$\lim_{n\to\infty} \xi_n\Theta_n = 0. \tag{4.2.24}$$

Moreover, from the choice of ξ_n, for a small $\varepsilon > 0$, we have

$$\xi_n < \frac{2||(I - J_{r_n}^{B_2})Aw_n||^2}{||A^*(I - J_{r_n}^{B_2})Aw_n||^2} - \varepsilon.$$

This implies that

$$\xi_n||A^*(I - J_{r_n}^{B_2})Aw_n||^2 < 2||(I - J_{r_n}^{B_2})Aw_n||^2 - \varepsilon||A^*(I - J_{r_n}^{B_2})Aw_n||^2$$

and thus

$$\varepsilon||A^*(I - J_{r_n}^{B_2})Aw_n||^2 < 2||(I - J_{r_n}^{B_2})Aw_n||^2 - \xi_n||A^*(I - J_{r_n}^{B_2})Aw_n||^2.$$

Hence

$$\varepsilon||A^*(I - J_{r_n}^{B_2})Aw_n||^2 < \Theta_n \to 0, \quad \text{as} \quad n \to \infty.$$

Therefore

$$\lim_{n\to\infty} ||A^*(I - J_{r_n}^{B_2})Aw_n||^2 = 0. \tag{4.2.25}$$

Similarly from (4.2.23), we have

$$\lim_{n\to\infty} ||(I - J_{r_n}^{B_2})Aw_n||^2 = 0. \tag{4.2.26}$$

Furthermore from (4.2.7) and (4.2.11), we have

$$\begin{aligned}
||y_n - x^*||^2 &= ||J_{r_n}^{B_1}[w_n - \xi_n A^*(I - J_{r_n}^{B_2})Aw_n] - J_{r_n}^{B_1}x^*||^2 \\
&\leq \langle y_n - x^*, w_n - \xi_n A^*(I - J_{r_n}^{B_2})Aw_n - x^*\rangle \\
&= \frac{1}{2}\Big\{||y_n - x^*||^2 + ||w_n - x^*||^2 - ||y_n - w_n + \xi_n A^*(I - J_{r_n}^{B_2})Aw_n||^2\Big\} \\
&\leq \frac{1}{2}\Big\{||y_n - x^*||^2 + ||w_n - x^*||^2 - [||y_n - w_n||^2 \\
&\quad + \xi_n^2||A^*(I - J_{r_n}^{B_2})Aw_n||^2 - 2\xi_n||y_n - w_n|| \times ||A^*(I - J_{r_n}^{B_2})Aw_n||]\Big\}.
\end{aligned}$$

Hence

$$\begin{aligned}
||y_n - x^*||^2 &\leq ||w_n - x^*||^2 - ||y_n - x_n||^2 + 2\xi_n||y_n - w_n|| \times \\
&\quad ||A^*(I - J_{r_n}^{B_2})Aw_n||. \qquad (4.2.27)
\end{aligned}$$

From (4.2.12), (4.2.22) and (4.2.27), we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq a_n||x_n - x^*||^2 + (1 - a_n)||z_n - x^*||^2 \\
&\leq a_n||x_n - x^*||^2 + (1 - a_n)||y_n - x^*||^2 + \Lambda_n \\
&\leq a_n||x_n - x^*||^2 + (1 - a_n)[||w_n - x^*||^2 - ||y_n - w_n||^2 + \Lambda_n \\
&\quad + 2\xi_n||y_n - w_n|| \cdot ||A^*(I - J_{r_n}^{B_2})Aw_n||] + \Lambda_n \\
&\leq ||x_n - x^*||^2 - (1 - a_n)||y_n - w_n||^2 + (1 - a_n)\alpha_n c_1||x_n - x_{n-1}|| \\
&\quad + 2(1 - a_n)\xi_n||y_n - w_n|| \cdot ||A^*(I - J_{r_n}^{B_2})Aw_n|| + \Lambda_n.
\end{aligned}$$

This implies that

$$\begin{aligned}
(1 - a_n)||y_n - w_n||^2 &\leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + (1 - a_n)\alpha_n c_1||x_n - x_{n-1}|| \\
&\quad + 2(1 - a_n)\xi_n||y_n - w_n|| \cdot ||A^*(I - J_{r_n}^{B_2})Aw_n|| + \Lambda_n. \qquad (4.2.28)
\end{aligned}$$

It follows from (4.2.28) that

$$\begin{aligned}
(1 - b)\sum_{n=1}^{\infty}||y_n - w_n||^2 &< ||x_0 - x^*||^2 + (1 - a)c_1\sum_{n=1}^{\infty}\alpha_n||x_n - x_{n-1}||^2 \\
&\quad + 2(1 - a)\sum_{n=1}^{\infty}\xi_n||y_n - w_n||| \cdot ||A^*(I - J_{r_n}^{B_2})Aw_n|| + \sum_{n=1}^{\infty}\Lambda_n < \infty.
\end{aligned}$$

Hence

$$\lim_{n\to\infty}||y_n - w_n|| = 0. \qquad (4.2.29)$$

From (4.2.22) and (4.2.29), we have

$$\lim_{n\to\infty}||y_n - x_n|| \leq \lim_{n\to\infty}[||y_n - w_n|| + ||w_n - x_n||] = 0. \qquad (4.2.30)$$

Let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$, then $y_{n_j} \rightharpoonup \bar{x}$ as $j \to \infty$. Since $y_{n_j} = J_{r_{n_j}}^{B_1}(w_{n_j} - \xi_{n_j}A^*(I - J_{r_{n_j}}^{B_2})Aw_{n_j})$, we can write

$$\frac{(w_{n_j} - y_{n_j}) + A^*(I - J_{r_{n_j}}^{B_2})Aw_{n_j}}{r_{n_j}} \in B_1(y_{n_j}). \qquad (4.2.31)$$

By passing to limit $j \to \infty$ in (4.2.31) and by taking into account (4.2.25) and (4.2.29), using the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(\bar{x})$. Furthermore since A is linear, we know that $Aw_{n_j} \rightharpoonup A\bar{x}$. Again by (4.2.26) and the fact that the resolvet $J_{r_n}^{B_2}$ is nonexpansive and Lemma 2.6.5, we obtain $A\bar{x} \in B_2(A\bar{x})$. Hence $\bar{x} \in \Omega_{SVIP}$. This complete the proof of Claim 4.

Note that since $||y_n - x_n|| \to 0$, as $n \to \infty$, it follows from Claim 3 and Claim 4 that $\bar{x} \in \Gamma$.

<u>Claim 5:</u> Finally, we show that $\{x_n\}$ converges strongly to $\bar{x} \in \Gamma$.

By claim 1, we can assume that

$$\lim_{n\to\infty} ||x_n - \bar{x}|| = c < +\infty.$$

From Lemma 4.2.5(ii) and (4.2.12), we have

$$\begin{aligned} ||z_n - \bar{x}|| &\leq ||y_n - \bar{x}|| + ||z_n - y_n|| \\ &\leq ||w_n - \bar{x}|| + \beta_n \\ &\leq ||x_n - \bar{x}|| + |\alpha_n|||x_n - x_{n-1}|| + \beta_n. \end{aligned}$$

This implies that

$$\limsup_{n\to\infty} ||z_n - \bar{x}|| \leq \limsup_{n\to\infty}(||x_n - \bar{x}|| + |\alpha_n|||x_n - x_{n-1}|| + \beta_n) = c.$$

By applying Lemma 2.6.33, with $v_n = x_n - \bar{x}$, $u_n = z_n - \bar{x}$, we obtain

$$\lim_{n\to\infty} ||z_n - x_n|| = 0.$$

Following similar argument as in the proof of Theorem 1 in [276], we see that

$$\lim_{n\to\infty} x_n = \bar{x}. \tag{4.2.32}$$

Hence, the sequence $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof. □

4.2.2 Application

In this subsection, we give an application of the main result in Section 3 to approximating solutions of certain nonlinear optimization problem.

Split Minimization Problem:

Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Given some proper, lower semicontinuous and convex functions $g_1 : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g_2 : H_2 \to \mathbb{R} \cup \{+\infty\}$, the Split Minimization Problem (SMP) is define as

$$\text{find} \quad \bar{x} \in H_1 \quad \text{suh that} \quad \bar{x} \in argmin\ g_1 \quad \text{and} \quad A\bar{x} \in argmin\ g_2. \tag{4.2.33}$$

We denote the set of solution of the SMP (4.2.33) with Ω_{SMP}. The SMP was first introduced by Moudafi and Thakur [188] and has attracted lots of attention in recent years, see for instance [1, 2, 275] and references therein. Further, the SMP have being applied in the study of many applied science such as multi-resolution sparse regularization, Fourier regularization, hard-constrained inconsistent feasibility and alternating projection signal synthesis problems.

Recall that the subdifferential of $g_1 : H_1 \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\partial g_1(x) := \{\bar{x} \in H_1 : g_1(x) + \langle y - x, \bar{x} \rangle \leq g_1(y) \quad \text{for each} \quad y \in H_1\}$$

for each $x \in H_1$. The proximity operator with respect to g_1 is defined by

$$prox_{\lambda, g_1}(x) := argmin_{z \in H_1} \left\{ g_1(z) + \frac{1}{2\lambda}||x - z||^2 \} \right\},$$

for all $x \in H_1$ and $\lambda > 0$. It is well known that ∂g_1 is maximal monotone and

$$0 \in \partial g_1(\bar{x}) \Leftrightarrow \bar{x} = prox_{\lambda g_1}(\bar{x}).$$

By setting $B_1 = \partial g_1$ and $B_2 = \partial g_2$ in Algorithm 4.2.4, we see that Algorithm 4.2.4 reduces to the following algorithm for solving the SMP.

Algorithm 4.2.10.
Initialization: *Pick* $x_0, x_1 \in H_1$, $\theta \in [0, 1)$, $\{\epsilon_n\} \subset [0, \infty)$, $\{a_n\}, \{r_n\}, \{\rho_n\}, \{\beta_n\}$, $\{\lambda_n\}$ *satisfying the following condition for each* $n \in \mathbb{N}$:

$$\rho_n > \rho > 0, \quad 0 < a < a_n < b < 1, \quad \beta_n > 0, \quad r_n > 0, \quad \lambda_n \geq 0;$$

$$\sum_{n=1}^{\infty} \epsilon_n < \infty, \quad \lim_{n \to \infty} a_n = \frac{1}{2}, \quad \liminf_{n \to \infty} r_n > 0;$$

$$\sum_{n=1}^{\infty} \frac{\beta_n}{\rho_n} = +\infty \quad \sum_{n=1}^{\infty} \beta_n^2 = +\infty, \quad \sum_{n=1}^{\infty} \frac{\beta_n \lambda_n}{\rho_n} < +\infty.$$

*[***Step 1:***] Given* x_{n-1} *and* x_n*, choose* α_n *such that* $0 < \alpha_n \leq \bar{\alpha}_n$*, where*

$$\bar{\alpha}_n = \begin{cases} \min \left\{ \theta, \frac{\epsilon_n}{||x_n - x_{n-1}||^2} \right\} & \text{if} \quad x_n \neq x_{n-1}, \\ \theta \quad \text{otherwise.} \end{cases}$$

Set

$$w_n = x_n + \alpha_n(x_n - x_{n-1})$$

Step 2: *Compute*

$$y_n = prox_{r_n, g_1}[w_n - \xi_n A^*(I - prox_{r_n, g_2})Aw_n],$$

where ξ_n is chosen such that

$$\xi_n = \begin{cases} \frac{2||(I-prox_{r_n,g_2})Aw_n||^2}{||A^*(I-prox_{r_n,g_2})Aw_n||^2}, & \text{if } prox_{r_n,g_2}Aw_n \neq Aw_n, \\ \xi, & \text{otherwise}, \end{cases}$$

where ξ is any nonnegative value.
Step 3: *Take $\eta_n \in \partial_2^{\lambda_n} f(y_n, y_n)$ and define*

$$\tau_n = \frac{\beta_n}{\gamma_n} \quad \text{where} \quad \gamma_n = \max\{\rho_n, ||\eta_n||\}.$$

Compute

$$z_n = P_C(y_n - \tau_n \eta_n).$$

Step 4: *Let*

$$x_{n+1} = a_n x_n + (1 - a_n) z_n.$$

4.2.3 Numerical examples

In this subsection, we carry out some numerical experiments to test the accuracy and efficiency of our algorithm. All computational tests are carried out using `MATLAB 2019a` programming on a 8gb RAM personal computer.

Example 4.2.11. Let $H = \mathbb{R}^m$ and C be a box defined by $C = \{x \in \mathbb{R}^m : -1 \leq x_i \leq 1, i = 1, 2, \ldots, m\}$. Define the bifunction f on $C \times C$ by

$$f(x, y) = (Px + Qy + q)^T (y - x) \quad \forall\, x, y \in C,$$

where $q \in \mathbb{R}^m$ and P, Q are two matrices of order m such that Q is symmetric positive semidefinite and $Q - P$ is negative semidefinite. It is easy to check that f satisfies conditions (B1)-(B4). Precisely, in our example, we work with the Euclidean norm $\mathbb{R}^m$ (with $m = 50, 200, 500$ and 1000). The vector q is the zero vector in $\mathbb{R}^m$ and the two matrices P, Q are generated randomly such that their properties are satisfied using the `'gallery('gcdmat',m)'` function in MATLAB. The entries of matrix $A \in \mathbb{R}^m \times \mathbb{R}^m$ are randomly generated in the interval $[0, 1]$, $B_1 : \mathbb{R}^m \to 2^{\mathbb{R}^m}$, $B_2 : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ are define by $B_1(x) = 2x$ and $B_2(x) = -5x$. The sequences $\{\beta_n\}, \{a_n\}\{\rho_n\}, \{r_n\}, \{\epsilon_n\}, \{\lambda_n\}$ are chosen such that

$$\beta_n = \tfrac{5}{2n+1}, \quad a_n = \tfrac{n-1}{2n+5}, \quad r_n = \tfrac{1}{2}, \quad \epsilon_n = \tfrac{1}{(n+1)^4}, \quad \lambda_n = 0, \quad \rho_n = 4,$$
$$\tau_n = \max\{4, ||\eta_n||\},$$

for each $n \geq 1$. We compare the numerical results of Algorithm 4.2.4 and Algorithm 4.2.4 with $\alpha_n = 0$ choosing $m = 50, 200, 500$ and 1000. In each case, the initial vectors x_0 and x_1 are also generated using $rand(m, 1)$ and the stopping criteria used in each case is $\frac{||x_{n+1}-x_n||}{\max\{1,||x_n||\}} < 10^{-6}$. The computational results are shown in Figure 4.4 and 4.5. The horizontal and vertical axes show iteration n, as well as $error(n) := ||x_n - x_{n+1}||$ respectively.

Next, we give an example in an infinite dimensional Hilbert space.

Example 4.2.12. Let $H_1 = H_2 = L_2([a,b])$ with norm $||x||_{L_2} = \left(\int_b^a |x(t)|dt\right)^{\frac{1}{2}}$. Define $C \subseteq H_1$ and $Q \subseteq H_2$ by $C := \{x \in L_2([a,b]) : \langle u, x\rangle \leq z\}$, where $0 \neq u \in L_2([a,b])$ and $z \in \mathbb{R}$, $Q = \{y \in L_2([a,b]) : ||y-d||_{L_2} \leq r\}$, where $d \in L_2([a,b])$ and radius $r > 0$. The projection on C and Q are define by

$$P_C(x) = \begin{cases} \frac{z-\langle u,x\rangle}{||u||_{L_2}^2}u + x, & \langle u, x\rangle > z, \\ x, & \langle u, x\rangle \leq z, \end{cases}$$

and

$$P_Q(y) = \begin{cases} d + r\frac{y-d}{||y-d||}, & y \notin Q, \\ y, & y \in Q. \end{cases}$$

In this example, we consider $B_1 \equiv \partial i_C$ and $B_2 \equiv \partial i_Q$, where i_C and i_Q are the indicator functions on the sets C and Q respectively. Then, the resolvent operators with respect to B_1 and B_2 are the metric projections P_C and P_Q respectively.

In particular, we choose

$$C = \{x \in L_2([0,1]) : ||x(t)||_{L_2} \leq 1\},$$

and

$$Q = \{x \in L_2([0,1]) : \int_0^1 |x(t) - \cos(t)|^2 dt \leq 25\}.$$

Define an operator $F : C \to L^2([0,1])$ by $F(x)(t) = \int_0^1 (x(t) - B(t,s)p(x(s)))ds + q(t)$, for all $x \in C$ and $t \in [0,1]$, where

$$B(t,s) = \frac{2tse^{t+s}}{e\sqrt{e^2-1}}, \quad p(x) = \cos(x), \quad q(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

As shown in [233], F is monotone and L-Lipschitz continuous with $L = 2$. Let $f\Big(x(t), y(t)\Big) = \langle Fx(t), y(t) - x(t)\rangle$, and $Ax(t) = 3x(t)$. We consider the problem

$$\text{Find} \quad x^* \in C \quad \text{such that} \quad f(x^*, y) \geq 0 \quad \forall\, y \in C, \quad \text{and} \quad y^* = Ax^* \in Q. \qquad (4.2.34)$$

Clearly, Problem (4.2.34) is a subclass of (4.2.3), hence, we can apply Algorithm 4.2.4 to solving Problem (4.2.34). We choose the sequences $\{a_n\}, \{\epsilon_n\}, \{\beta_n\}, \{\lambda_n\}, \{\rho_n\}$ such that

$$a_n = \frac{1}{2}, \quad \lambda_n = 0.5, \quad \beta_n = \frac{2n}{7n+3}, \epsilon_n = \frac{1}{(n+1)^2}, \quad \rho_n = 3.$$

Using $\frac{||x_{n+1}-x_n||}{||x_2-x_1||} < 10^{-4}$ as stopping criterion with different choices of x_0 and x_1 given below, we compare the numerical results of Algorithm 4.2.4 with MKPA (4.2.2) and DPA (4.2.1):

(i) $x_1 = t^2 - 2t + 1$ and $x_0 = 3\sin(2t)$;

(ii) $x_1 = 2 - \exp(-2t)$ and $x_0 = 2t^2 - 3t$;

(iii) $x_1 = \frac{3t}{4} + \frac{5t}{2} + 1$ and $x_0 = \cos(5t)$;

(iv) $x_1 = \frac{12t^2}{5} - 2$ and $x_0 = \exp(-2t)/7$.

Remark 4.2.13. In conclusion, Example 4.2.11 shows that Algorithm 4.2.4 converges faster than its non-inertial version (that is, with $\alpha_n = 0$). Also from Example 4.2.12, we see that Algorithm 4.2.4 performs better than Algorithm 4.2.1 and Algorithm 4.2.2 in terms of number of iteration and cpu-time taken.

Table 4.3: Computation result for Example 4.2.11.

		Algorithm 4.2.4	Algorithm 4.2.4 with $\alpha_n = 0$
$m = 50$	CPU time (sec)	1.1185	1.15799
	No. of Iter.	22	32
$m = 200$	CPU time (sec)	1.7821	2.1582
	No. of Iter.	23	33
$m = 500$	CPU time (sec)	3.4738	10.7083
	No. of Iter.	24	35
$m = 1000$	CPU time (sec)	8.2317	12.5352
	No. of Iter.	24	35

Table 4.4: Computation result for Example 4.2.12.

		Algorithm 4.2.4	KKPA 4.2.2	DPA 4.2.1
Case I	CPU time (sec)	1.3210	2.9709	6.1351
	No. of Iter.	17	23	40
Case II	CPU time (sec)	10. 4288	20.2761	34.9238
	No. of Iter.	21	28	48
Case III	CPU time (sec)	1.5861	2.9477	6.1550
	No. of Iter.	22	30	48
Case IV	CPU time (sec)	2.3602	9.7439	17.3865
	No. of Iter.	19	26	45

4.3 A New Efficient Method for Finding Common Fixed Points and Solutions of Split Generalized Equilibrium Problems in Hilbert Spaces

In this section, we introduce a new iterative algorithm for approximating solutions of split generalized equilibrium problem and common fixed points of multivalued demi-contractive

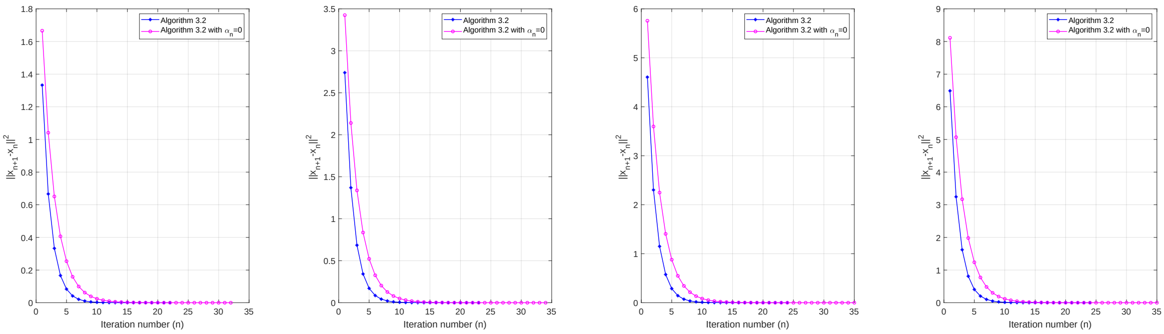


Figure 4.4: Example 4.2.11: $m = 50$; $m = 200$; $m = 500$; $m = 1000$.

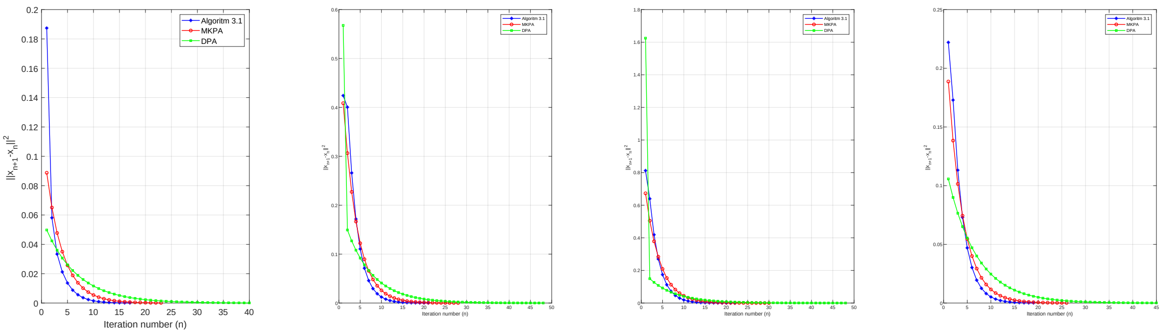


Figure 4.5: Example 4.2.12: Case I; Case II; Case III; Case IV.

mappings in real Hilbert spaces. We also study the rate of convergence of our proposed algorithm which is shown to be $O(1/t)$ convergence rate.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $\Phi : C \times C \to \mathbb{R}$ and $\varphi : C \times C \to \mathbb{R}$ be two bifunctions. The Generalized Equilibrium Problem (GEP) is defined as finding a point $x^* \in C$ such that

$$\Phi(x^*, x) + \varphi(x^*, x) \geq 0, \quad \forall\, x \in C. \tag{4.3.1}$$

We denote the set of solutions of the GEP by Ω_{GEP}. The GEP is very general in the sense that it includes as particular cases, minimization problems, variational inequality problems, fixed point problems, mixed equilibrium problems and Nash equilibrium problems in noncooperative games among others, see for instance [35, 62, 90, 109]. When $\varphi \equiv 0$, the GEP reduces to the classical Equilibrium Problem EP (1.1.4) introduced by Blum and Oettli [35].

Recently, Kazmi and Rizvi [148] introduced the Split Generalized Equilibrium Problem (SGEP) in Hilbert space. Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex sets and $A : H_1 \to H_2$ be bounded linear operator. Let $\Phi, \varphi_1 : C \times C \to \mathbb{R}$ and $\Psi, \varphi_2 : Q \times Q \to$

$\mathbb{R}$ be bifunctions. The SGEP is defined as finding $x^* \in C$ such that

$$\Phi(x^*, x) + \varphi_1(x^*, x) \geq 0, \quad \forall\, x \in C,$$

and

$$y^* = Ax^* \in Q \quad \text{solves} \quad \Psi(y^*, y) + \varphi_2(y^*, y) \geq 0, \quad \forall\, y \in Q.$$

We denote the set of solutions of the SGEP by Ω_{SGEP}. The authors in [148] proposed the following algorithm for approximating the solutions of the SGEP and fixed points of nonexpansive semigroup in real Hilbert spaces:

$$\begin{cases} u_n = T_{r_n}^{(\Phi,\varphi_1)}(x_n + \delta A^*(T_{r_n}^{(\Psi,\varphi_2)} - I)Ax_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \end{cases} \tag{4.3.2}$$

where $T_{r_n}^{(\Phi,\varphi_1)}$ is defined in Lemma 4.3.2, $r_n \in (0,\infty)$, $f : C \to C$ is a contraction with constant $\alpha \in (0,1)$ and B is a strongly positive linear bounded self-adjoint operator on H_1 with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$, $\{s_n\}$ is a positive real sequence diverging to $+\infty$, $\delta \in (0, \frac{1}{L})$, L being the spectral radius of the operator A^*A and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$. The authors obtained a strong convergence theorem for the sequence generated by algorithm (4.3.2) under some suitable conditions on α_n, β_n and s_n.

Also, Deepho et al. [93] studied the common solution of SGEP, variational inequality problem and fixed point of countable family of nonexpansive mapping in Hilbert space. They proposed the following algorithm and proved its strong convergence for approximating the underlying problem under some mild conditions on the control sequences:

$$\begin{cases} u_n = T_{r_n}^{(\Phi,\varphi_1)}(x_n + \delta A^*(T_{r_n}^{(\Psi,\varphi_2)} - I)Ax_n), \\ z_n = P_C(u_n - \xi_2 B_2 u_n), \\ y_n = \alpha_n x_n + (1-\alpha_n)S_n P_C(z_n - \xi_1 B_1 z_n), \\ C_{n+1} = \{z \in C_n : ||y_n - z|| \leq ||x_n - z||\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall\, n \in \mathbb{N} \end{cases} \tag{4.3.3}$$

where B_1, B_2 are β_1, β_2-inverse strongly monotone operators from C into H_1 respectively and L is the spectral radius of the operator A^*A. Also $\alpha_n \in (0,1)$, $\xi_1 \in [a_1, b_1] \subset (0, 2\beta_1)$, $\xi_2 \in [a_2, b_2] \subset (0, 2\beta_2)$, $\{r_n\} \subset (0,\infty)$, $\delta \in \left(0, \frac{1}{L}\right)$, satisfying

(AA1) $0 < a_1 \leq \xi_1 \leq b_1 < 2\beta_1$,

(AA2) $0 < a_2 \leq \xi_2 \leq b_2 < 2\beta_2$,

(AA3) $\liminf_{n\to\infty} r_n > 0$.

Very recently, Phuengrattana et al. [204] proposed the following hybrid algorithm and proved its strong convergence to a solution of SGEP and common fixed point of countable family of nonexpansive multivalued mapping S_i:

$$\begin{cases} x_1 \in C, \\ u_n = T_{r_n}^{(\Phi,\varphi_1)}(I - \delta A^*(I - T_{r_n}^{(\Psi,\varphi_2)})A)x_n, \\ z_n = \alpha_n^{(0)} x_n + \alpha_n^{(1)} y_n^1 + \cdots + \alpha_n^{(n)} y_n^{(n)}, \quad y_n^{(i)} \in S_i u_n, \\ C_{n+1} = \{p \in C_n : ||z_n - p|| \leq ||x_n - p||\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{4.3.4}$$

where $\{\alpha_n^{(i)}\} \subset (0,1)$ satisfying $\sum_{i=0}^{n} \alpha_n^i = 1$, $\{\gamma_n\} \subset (0,\infty)$ and $\delta \in (0, \frac{1}{L})$, where L is the spectral radius of A^*A.

However, we observe that the convergence of the iterative schemes (4.3.2), (4.3.3) and (4.3.4) depends on the spectral radius of A^*A which require a prior knowledge of the norm of the bounded operator A. This is very difficult to get in practice. Also, one can see that algorithms (4.3.3) and (4.3.4) slightly improved algorithm (4.3.2), but the projection onto C_{n+1} can be computationally expensive when the feasible set C is not simple. This may affect the usage and efficiency of algorithms (4.3.3) and (4.3.4).

In order to get an efficient method for approximating solution of SGEP, we introduce a new iterative scheme for approximating solution of SGEP and common fixed point of countable family of multivalued demi-contractive mappings in real Hilbert spaces. Our algorithm neither requires a prior knowledge of the operator norm nor the projection onto C_{n+1}. We prove a strong convergence theorem and show that our proposed method converges at a rate of $O(1/t)$. We also provide some numerical examples to show that our proposed iterative scheme performs better than some existing algorithms in the literature.

For solving the SGEP, we need the following lemmas.

Lemma 4.3.1. *[93] Let $\Phi : C \times C \to \mathbb{R}$ and $\varphi : C \times C \to \mathbb{R}$ be two bifunctions satisfying the following assumptions:*

(D1) $\Phi(x,x) \geq 0$ for all $x \in C$;

(D2) Φ is monotone, i.e. $\Phi(x,y) + \Phi(y,x) \leq 0$ for all $x, y \in C$;

(D3) Φ is upper hemicontinuous, i.e. for each $x, y, z \in C$,

$$\limsup_{t\to\infty} \Phi(tz + (1-t)x, y) \leq \Phi(x,y);$$

(D4) For each $x \in C$ fixed, the function $y \mapsto \Phi(x,y)$ is convex and lower semicontinuous;

(D5) $\varphi(x,x) \geq 0$ for all $x \in C$;

(D6) For each $y \in C$ fixed, the function $x \to \varphi(x,y)$ is upper semicontinuous;

(D7) For each $x \in C$ fixed, the function $y \to \varphi(x,y)$ is convex and lower semicontinuous,

and assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H and $x \in C \cap K$ such that

$$\Phi(y,x) + \varphi(y,x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \quad \forall\, y \in C \backslash K.$$

Lemma 4.3.2. *(see Lemma 3 in [93]) Let C be a nonempty, closed and convex subset of a real Hilbert space H. Assume that $\Phi, \varphi : C \times C \to \mathbb{R}$ are bifunctions satisfying the assumptions D1 - D7 in Lemma 4.3.1 and φ is monotone. For $r > 0$ and for all $x \in H$, define a mapping $T_r^{(\Phi,\varphi)} : H \to C$ as follows:*

$$T_r^{(\Phi,\varphi)}(x) = \left\{ z \in C : \Phi(z,y) + \varphi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall\, y \in C \right\}.$$

Then, the following hold:

(i) $T_r^{(\Phi,\varphi)}$ *is single-valued;*

(ii) $T_r^{(\Phi,\varphi)}$ *is firmly nonexpansive, i.e.*

$$||T_r^{(\Phi,\varphi)}x - T_r^{(\Phi,\varphi)}y||^2 \leq \langle T_r^{(\Phi,\varphi)}x - T_r^{(\Phi,\varphi)}y, x - y\rangle \quad \forall\ x, y \in H; \tag{4.3.5}$$

(iii) $F(T_r^{(\Phi,\varphi)}) = \Omega_{GEP}$;

(iv) Ω_{GEP} *is compact and convex.*

In addition, the following lemma is a consequence of Lemma 4.3.2. It will be used in establishing our main results.

Lemma 4.3.3. *Let* $\Phi, \varphi : C \times C \to \mathbb{R}$ *be bifunctions satisfying the assumptions D1 - D7 in Lemma 4.3.1 and* φ *be monotone. Suppose* Ω_{GEP} *is non-empty. For all* $x \in H$, $z \in F(T_r^{(\Phi,\varphi)})$ *and* $r > 0$, *we have*

$$\langle T_r^{(\Phi,\varphi)}x - z, T_r^{(\Phi,\varphi)}x - x\rangle \leq 0, \tag{4.3.6}$$

and

$$||x - T_r^{(\Phi,\varphi)}x||^2 + ||T_r^{(\Phi,\varphi)}x - z||^2 \leq ||x - z||^2. \tag{4.3.7}$$

Proof. It follows from Lemma 4.3.2 (ii) that

$$||T_r^{(\Phi,\varphi)}x - z||^2 \leq \langle T_r^{(\Phi,\varphi)}x - z, x - z\rangle.$$

This implies that

$$\langle T_r^{(\Phi,\varphi)}x - z, T_r^{(\Phi,\varphi)}x - z - (x - z)\rangle \leq 0,$$

hence

$$\langle T_r^{(\phi,\varphi)}x - z, T_r^{(\Phi,\varphi)}x - x\rangle \leq 0.$$

Also from Lemma 4.3.2 (ii), we have

$$\begin{aligned}
||T_r^{(\Phi,\varphi)}x - z||^2 &\leq \langle T_r^{(\Phi,\varphi)}x - z, x - z\rangle \\
&= \langle T_r^{(\Phi,\varphi)}x - x + x - z, x - z\rangle \\
&= \langle T_r^{(\phi,\varphi)}x - x, x - z\rangle + ||x - z||^2 \\
&= \langle T_r^{(\Phi,\varphi)}x - x, x - T_r^{(\Phi,\varphi)}x + T_r^{(\Phi,\varphi)}x - z\rangle + ||x - z||^2 \\
&= -||x - T_r^{(\Phi,\varphi)}x||^2 + \langle T_r^{(\Phi,\varphi)}x - x, T_r^{(\Phi,\varphi)}x - z\rangle + ||x - z||^2.
\end{aligned}$$

Therefore, from (4.3.6), we have

$$||T_r^{(\Phi,\varphi)}x - z||^2 + ||x - T_r^{(\Phi,\varphi)}x||^2 \leq ||x - z||^2.$$

□

4.3.1 Main results

In this subsection, we give a precise statement of our proposed algorithm and discuss its convergence analysis for approximating the solutions of SGEP and common fixed point of countable family of multivalued demi-contractive mappings in real Hilbert spaces.

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ be a bounded linear operator. Let $\Phi, \varphi_1 : C \times C \to \mathbb{R}$ and $\Psi, \varphi_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying assumption D1 - D7 in Lemma 4.3.1 and for each $i \in \mathbb{N}$, let $S_i : H_1 \to CB(H_1)$ be multivalued demi-contractive mappings with constants k_i such that $I - S_i$ are demiclosed at zero and for each $p \in F(S_i)$, $S_i(p) = \{p\}$. Suppose

$$Sol := \Omega_{SGEP} \cap \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset.$$

Let $f : H_1 \to H_1$ be a ρ-contraction with constant $\rho \in (0, 1)$ and D be a bounded operator on H_1 with coefficient $\bar{\tau} > 0$ such that $0 < \xi < \frac{\bar{\tau}}{\rho}$. Let $\{r_n\}$ be a sequence in $(0, \infty)$, $\{\alpha_n\}$ and $\{\beta_{n,i}\}$ be sequences in $(0, 1)$, $\{\lambda_n\}$ be a sequence in $(0, \infty)$ and $\{x_n\}$ be a sequence defined by the following algorithm.

Algorithm 4.3.4.

Step 0: Choose $\gamma > 0$, $\alpha, \theta \in (0, 1)$ *and* $\gamma \in (0, 2)$. *Choose* $x_1 \in H_1$ *and set* $n = 1$.

Step 1: Compute

$$y_n = T_{r_n}^{(\Phi,\varphi_1)}(x_n - \lambda_n A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ax_n), \tag{4.3.8}$$

where $\lambda_n = \sigma\eta^{m_n}$, $\sigma > 0$, $\eta \in (0, 1)$ *and* m_n *is the smallest nonnegative integer such that (see [151])*

$$\lambda_n||A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ax_n - A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n|| \leq \theta||x_n - y_n||. \tag{4.3.9}$$

Step 2: If $x_n = y_n$, *then go to Step 3. Else, compute*

$$d(x_n, y_n) = x_n - y_n - \lambda_n\Big[A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ax_n - A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n\Big], \tag{4.3.10}$$

and

$$w_n = x_n - \gamma\delta_n d(x_n, y_n) \tag{4.3.11}$$

where

$$\delta_n = \begin{cases} \frac{\langle x_n - y_n, d(x_n, y_n)\rangle}{||d(x_n, y_n)||^2}, & \text{if } \ d(x_n, y_n) \neq 0, \\ 0 & \text{if } \ d(x_n, y_n) = 0. \end{cases} \tag{4.3.12}$$

Step 3: Compute

$$x_{n+1} = \alpha_n\xi f(x_n) + (1 - \alpha_n D)\left(\beta_{n,0}w_n + \sum_{i=1}^{n}\beta_{n,i}v_{n,i}\right), \tag{4.3.13}$$

where $v_{n,i} \in S_i w_n$. *Set* $n = n + 1$ *and go to Step 1.*

Remark 4.3.5. Note that if $x_n = y_n$ and $x_n \in S_i x_n$ for each $i = 1, 2, \ldots,$ we are at a common solution of SGEP and fixed points of S_i. We will implicitly assume in our convergence analysis that this does not occur after finitely many iterations so that Algorithm 4.3.4 generates an infinite sequence.

In order to establish the convergence of Algorithm 4.3.4, we make the following assumptions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(C2) $\liminf_{n\to\infty}(\beta_{n,0} - k)\beta_{n,i} > 0$ for all $i = 1, 2, ...,$

(C3) $\liminf_{n\to\infty} r_n > 0$.

First, we show that the Algorithm 4.3.4 is well defined. To this end, it suffices to show that the inner loop in the calculation of the stepsize in Step 2 is well defined.

Lemma 4.3.6. *[104] The Armijo-line search rule (4.3.9) is well defined. More so*

$$\bar{\lambda} \leq \lambda_n \leq \sigma, \tag{4.3.14}$$

where $\bar{\lambda} = \min\left\{\sigma, \frac{\theta\eta}{||A||^2}\right\}$.

We proceed to prove the following lemmas before proving our main theorem.

Lemma 4.3.7. *Let $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ be sequences generated by Algorithm 4.3.4, then*

$$||x_n - y_n||^2 \leq \frac{(1+\theta)^2||x_n - w_n||^2}{(1-\theta)^2\gamma^2}. \tag{4.3.15}$$

Proof. From Algorithm 4.3.4, we have

$$\begin{aligned}
\langle x_n - y_n, d(x_n, y_n)\rangle &= \langle x_n - y_n, x_n - y_n - \lambda_n[A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ax_n - A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n]\rangle \\
&= ||x_n - y_n||^2 - \lambda_n\langle x_n - y_n, A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ax_n - A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n\rangle \\
&\geq ||x_n - y_n||^2 - \lambda_n||x_n - y_n|| \times ||A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ax_n \\
&\quad - A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n|| \\
&\geq (1-\theta)||x_n - y_n||^2. \qquad (4.3.16)
\end{aligned}$$

Also from the definitions of d_n and w_n, we obtain

$$\begin{aligned}
\delta_n\langle x_n - y_n, d(x_n, y_n)\rangle &= ||\delta_n d(x_n, y_n)||^2 \\
&= \frac{1}{\gamma^2}||w_n - x_n||^2. \qquad (4.3.17)
\end{aligned}$$

Note that

$$\begin{aligned}
||d(x_n,y_n)||^2 &= ||x_n - y_n - \lambda_n[A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ax_n - A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n]||^2\\
&= ||x_n - y_n||^2 + \lambda_n^2||A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ax_n - A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n||^2\\
&\quad + 2\lambda_n\langle x_n - y_n, A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ax_n - A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n\rangle\\
&\leq ||x_n - y_n||^2 + \theta^2||x_n - y_n||^2 + 2\lambda_n||x_n - y_n||\cdot||A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ax_n -\\
&\quad A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n||\\
&\leq ||x_n - y_n||^2 + \theta^2||x_n - y_n||^2 + 2\theta||x_n - y_n||^2\\
&= (1+\theta)^2||x_n - y_n||^2. \qquad (4.3.18)
\end{aligned}$$

Therefore from (4.3.12), (4.3.16) and (4.3.18), we get

$$\delta_n = \frac{\langle x_n - y_n, d(x_n,y_n)\rangle}{||d(x_n,y_n)||^2} \geq \frac{1-\theta}{(1+\theta)^2}. \qquad (4.3.19)$$

Consequently, we have from (4.3.16), (4.3.17) and (4.3.19)

$$\begin{aligned}
||x_n - y_n||^2 &\leq \frac{\langle x_n - y_n, d(x_n,y_n)\rangle}{(1-\theta)} = \frac{1}{\delta_n\gamma^2(1-\theta)}||w_n - x_n||^2\\
&\leq \frac{(1+\theta)^2}{(1-\theta)^2\gamma^2}||w_n - x_n||^2.
\end{aligned}$$

□

Lemma 4.3.8. *Let $z \in Sol$. Then from Algorithm 4.3.4, we get*

$$||w_n - z||^2 \leq ||x_n - z||^2 - \frac{(2-\gamma)}{\gamma}||w_n - x_n||^2. \qquad (4.3.20)$$

Proof. Since $T_{r_n}^{(\Psi,\varphi_2)}$ is firmly nonexpansive, it is $\frac{1}{2}$- averaged, and thus $I - T_{r_n}^{(\Psi,\varphi_2)}$ is 1-ism (see [55, 179]), i.e.

$$\langle (I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n - (I - T_{r_n}^{(\Psi,\varphi_2)})Az, Ay_n - Az\rangle \geq ||(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n - (I - T_{r_n}^{(\Psi,\varphi_2)})Az||^2. \qquad (4.3.21)$$

From (4.3.21), we have

$$\begin{aligned}
&\langle A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n - A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Az, y_n - z\rangle\\
&\qquad = \langle (I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n - (I - T_{r_n}^{(\Psi,\varphi_2)})Az, Ay_n - Az\rangle\\
&\qquad \geq ||(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n - (I - T_{r_n}^{(\Psi,\varphi_2)})Az||^2. \qquad (4.3.22)
\end{aligned}$$

Since $z \in Sol$, then $T_{r_n}^{(\Psi,\varphi_2)}Az = Az$, and thus $(I - T_{r_n}^{(\Psi,\varphi_2)})Az = 0$. Therefore we have from (4.3.22)

$$\lambda_n\langle A^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n, y_n - z\rangle \geq \lambda_n||(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n||^2. \qquad (4.3.23)$$

However, observe from (4.3.6) that

$$\langle y_n - z, x_n - \lambda_nA^*(I - T_{r_n}^{(\Psi,\varphi_2)})Ax_n - y_n\rangle \geq 0. \qquad (4.3.24)$$

On adding (4.3.23) and (4.3.24), we get

$$\langle y_n - z, d(x_n, y_n)\rangle \geq \lambda_n ||(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n||^2. \tag{4.3.25}$$

Also by Lemma 2.6.1 (ii), we have

$$\begin{aligned} ||w_n - z||^2 &= ||x_n - \gamma\delta_n d(x_n, y_n) - z||^2 \\ &\leq ||x_n - z||^2 - 2\gamma\delta_n\langle x_n - z, d(x_n, y_n)\rangle + \gamma^2\delta_n^2||d(x_n, y_n)||^2. \end{aligned} \tag{4.3.26}$$

But

$$\langle x_n - z, d(x_n, y_n)\rangle = \langle x_n - y_n, d(x_n, y_n)\rangle + \langle y_n - z, d(x_n, y_n)\rangle. \tag{4.3.27}$$

Then from (4.3.12) and (4.3.25), we have

$$\begin{aligned} \langle x_n - z, d(x_n, y_n)\rangle &= \delta_n||d(x_n, y_n)||^2 + \langle y_n - z, d(x_n, y_n)\rangle \\ &\geq \delta_n||d(x_n, y_n)||^2 + \lambda_n||(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n||^2. \end{aligned}$$

Hence, we get from (4.3.26) that

$$\begin{aligned} ||w_n - z||^2 &\leq ||x_n - z||^2 - \gamma\delta_n^2(2-\gamma)||d(x_n, y_n)||^2 - \gamma\delta_n\lambda_n||(I - T_{r_n}^{(\Psi,\varphi_2)})Ay_n||^2 \\ &\leq ||x_n - z||^2 - \gamma\delta_n^2(2-\gamma)||d(x_n, y_n)||^2. \end{aligned} \tag{4.3.28}$$

From (4.3.11), we have that $\delta_n^2||d(x_n, y_n)||^2 = \frac{1}{\gamma^2}||w_n - x_n||^2$. Then it follows from (4.3.28) that

$$||w_n - z||^2 \leq ||x_n - z||^2 - \frac{(2-\gamma)}{\gamma}||w_n - x_n||^2. \tag{4.3.29}$$

Consequently since $\gamma \in (0, 2)$, we get

$$||w_n - z||^2 \leq ||x_n - z||^2. \tag{4.3.30}$$

□

Now, we show that the sequences generated by Algorithm 4.3.4 are bounded.

Lemma 4.3.9. *The sequence $\{x_n\}$ generated by Algorithm 4.3.4 is bounded. Consequently, $\{f(x_n)\}$, $\{y_n\}$, $\{w_n\}$, $\{d(x_n, y_n)\}$ and $\{v_{n,i}\}$ are bounded, for each $i = 1, 2, \ldots$.*

Proof. Let us put $z_n = \beta_{n,0}w_n + \sum_{i=1}^{n} \beta_{n,i}v_{n,i}$, and $z \in Sol$. Then using Lemma 2.6.3, we have

$$
\begin{aligned}
||z_n - z||^2 &= \Big|\Big|\beta_{n,0}w_n + \sum_{i=1}^{n}\beta_{n,i}v_{n,i} - z\Big|\Big|^2 \\
&\leq \beta_{n,0}||w_n - z||^2 + \sum_{i=1}^{n}\beta_{n,i}||v_{n,i} - z||^2 - \sum_{i=1}^{n}\beta_{n,0}\beta_{n,i}||w_n - v_{n,i}||^2 \\
&\leq \beta_{n,0}||w_n - z||^2 + \sum_{i=1}^{n}\beta_{n,i}d\big(v_{n,i}, S_i z\big)^2 - \sum_{i=1}^{n}\beta_{n,0}\beta_{n,i}||w_n - v_{n,i}||^2 \\
&\leq \beta_{n,0}||w_n - z||^2 + \sum_{i=1}^{n}\beta_{n,i}H\big(S_i w_n, S_i z\big)^2 - \sum_{i=1}^{n}\beta_{n,0}\beta_{n,i}||w_n - v_{n,i}||^2 \\
&\leq \beta_{n,0}||w_n - z||^2 + \sum_{i=1}^{n}\beta_{n,i}\Big(||w_n - z||^2 + \kappa_i d(w_n, S_i w_n)^2\Big) \\
&\quad - \sum_{i=1}^{n}\beta_{n,0}\beta_{n,i}||w_n - v_{n,i}||^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \beta_{n,0}||w_n - z||^2 + \sum_{i=1}^{n}\beta_{n,i}||w_n - z||^2 + \sum_{i=1}^{n}\beta_{n,i}\kappa||w_n - v_{n,i}||^2 \\
&\quad - \sum_{i=1}^{n}\beta_{n,0}\beta_{n,i}||w_n - v_{n,i}||^2 \\
&= ||w_n - z||^2 - \sum_{i=1}^{n}(\beta_{n,0} - \kappa)\beta_{n,i}||w_n - v_{n,i}||^2. \qquad (4.3.31)
\end{aligned}
$$

It follows from (4.3.20) that

$$
||z_n - z||^2 \leq ||x_n - z||^2 - \sum_{i=1}^{n}(\beta_{n,0} - \kappa)\beta_{n,i}||w_n - v_{n,i}||^2, \qquad (4.3.32)
$$

and using condition (C2), we get

$$
||z_n - z||^2 \leq ||x_n - z||^2. \qquad (4.3.33)
$$

Therefore from (4.3.33), we get

$$
\begin{aligned}
||x_{n+1} - z|| &= ||\alpha_n(\xi f(x_n) - Dz) + (1 - \alpha_n D)(z_n - z)|| \\
&\leq \alpha_n||\xi f(x_n) - Dz|| + (1 - \alpha_n\bar{\tau})||z_n - z|| \\
&\leq \alpha_n\Big[||\xi(f(x_n) - f(z)) + (\xi f(z) - Dz)||\Big] + (1 - \alpha_n\bar{\tau})||z_n - z|| \\
&\leq \alpha_n\Big[||\xi(f(x_n) - f(z)) + (\xi f(z) - Dz)||\Big] + (1 - \alpha_n\bar{\tau})||w_n - z|| \\
&\leq \alpha_n\xi\rho||x_n - z|| + \alpha_n||\xi f(z) - Dz|| + (1 - \alpha_n\bar{\tau})||x_n - z|| \\
&= (1 - \alpha_n(\bar{\tau} - \xi\rho))||x_n - z|| + \alpha_n||\xi f(z) - Dz|| \\
&= (1 - \alpha_n(\bar{\tau} - \xi\rho))||x_n - z|| + (\bar{\tau} - \xi\rho)\alpha_n\frac{||\xi f(z) - Dz||}{\bar{\tau} - \xi\rho}.
\end{aligned}
$$

Hence

$$\begin{aligned} ||x_{n+1} - z|| &\leq \max\left\{||x_n - z||, \frac{||\xi f(z) - Dz||}{\bar{\tau} - \xi\rho}\right\} \\ &\vdots \\ &\leq \max\left\{||x_1 - z||, \frac{||\xi f(z) - Dz||}{\bar{\tau} - \xi\rho}\right\}. \end{aligned}$$

Hence $||x_n - z||$ is bounded. This implies that $\{x_n\}$ is bounded.
Consequently, $\{f(x_n)\}$, $\{y_n\}$, $\{d(x_n, y_n)\}$, $\{w_n\}$ and $\{v_{n,i}\}$ are bounded. □

Lemma 4.3.10. *The sequence $\{x_n\}$ generated by Algorithm 4.3.4 satisfies the following estimates:*

(i) $s_{n+1} \leq (1 - \tilde{a}_n)s_n + \tilde{a}_n b_n,$

(ii) $-1 \leq \limsup_{n\to\infty} b_n < +\infty,$

where $s_n = ||x_n - z||^2$, $\tilde{a}_n = \frac{2\alpha_n(\bar{\tau}-\xi\rho)}{1-\alpha_n\xi\rho}$, $b_n = \frac{\langle \xi f(z) - Dz, x_{n+1} - z\rangle}{\bar{\tau}-\xi\rho} + \frac{\alpha_n M}{2(\bar{\tau}-\xi\rho)}$, *for* $z \in Sol$ *and some* $M > 0$.

Proof. From (4.3.13) and (4.3.33), we have

$$\begin{aligned} ||x_{n+1} - z||^2 &= ||\alpha_n(\xi f(x_n) - Dz) + (1 - \alpha_n D)(z_n - z)||^2 \\ &\leq (1 - \alpha_n\bar{\tau})^2||z_n - z||^2 + 2\alpha_n\langle \xi f(x_n) - D(z), x_{n+1} - z\rangle \\ &= (1 - \alpha_n\bar{\tau})^2||x_n - z||^2 + 2\alpha_n\langle \xi(f(x_n) - f(z)) + \xi f(z) - Dz, x_{n+1} - z\rangle \\ &\leq (1 - \alpha_n\bar{\tau})^2||x_n - z||^2 + 2\alpha_n\xi\rho||x_n - z||||x_{n+1} - z|| \\ &\quad +2\alpha_n\langle \xi f(z) - Dz, x_{n+1} - z\rangle \\ &\leq (1 - 2\alpha_n\bar{\tau} + (\alpha_n\bar{\tau})^2)||x_n - z||^2 + \alpha_n\xi\rho(||x_n - z||^2 + ||x_{n+1} - z||^2) \\ &\quad + 2\alpha_n\langle \xi f(z) - Dz, x_{n+1} - z\rangle. \end{aligned}$$

This implies that

$$\begin{aligned} ||x_{n+1} - z||^2 &\leq \frac{1 - \alpha_n(2\bar{\tau} - \xi\rho)}{1 - \alpha_n\xi\rho}||x_n - z||^2 + \frac{(\alpha_n\bar{\tau})^2}{1 - \alpha_n\xi\rho}||x_n - z||^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n\xi\rho}\langle \xi f(z) - Dz, x_{n+1} - z\rangle \\ &= \left(1 - \frac{2\alpha_n(\bar{\tau} - \xi\rho)}{1 - \alpha_n\xi\rho}\right)||x_n - z||^2 + \frac{2\alpha_n(\bar{\tau} - \xi\rho)}{1 - \alpha_n\xi\rho}\Bigg(\frac{\langle \xi f(z) - Dz, x_{n+1} - z\rangle}{\bar{\tau} - \xi\rho} \\ &\quad + \frac{\alpha_n M}{2(\bar{\tau} - \xi\rho)}\Bigg), \end{aligned}$$

for some $M > 0$. This establishes (i).

Next, we prove (ii). Since $\{x_n\}$ is bounded and $\alpha_n \in (0, 1)$, then we have

$$\sup_{n\geq 0} b_n \leq \sup_{n\geq 0} \frac{1}{\bar{\tau} - \xi\rho}\Big(||\xi f(z) - Dz|| \cdot ||x_{n+1} - z|| + M\Big) < \infty.$$

We show that $\limsup_{n\to\infty} b_n \geq -1$. Assume the contrary, i.e suppose $\limsup_{n\to\infty} b_n < -1$, which implies that there exists $n_0 \in \mathbb{N}$ such that $b_n \leq -1$ for all $n \geq n_0$. Hence, it follows from (i) that

$$\begin{array}{rcl} s_{n+1} & \leq & (1-\tilde{a}_n)s_n + \tilde{a}_n b_n \\ & < & (1-\tilde{a}_n)s_n - \tilde{a}_n \\ & = & s_n - \tilde{a}_n(s_n+1) \\ & \leq & s_n - 2\alpha_n(\bar{\tau} - \xi\rho). \end{array}$$

By induction, we get

$$s_{n+1} \leq s_{n_0} - 2(\bar{\tau} - \xi\rho)\sum_{i=n_0}^{n} \alpha_i \qquad \text{for all} \quad n \geq n_0.$$

Taking lim sup of both sides in the last inequality, we have

$$\limsup_{n\to\infty} s_n \leq s_{n_0} - \lim_{n\to\infty} 2(\bar{\tau} - \xi\rho)\sum_{i=n_0}^{n} \alpha_i = -\infty.$$

This contradicts the fact that $\{s_n\}$ is a non-negative real sequence. Therefore, $\limsup_{n\to\infty} b_n \geq -1$. □

Lemma 4.3.11. *Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ defined by Algorithm 4.3.4 such that $x_{n_k} \rightharpoonup q \in C$. Suppose $||x_n - y_n|| \to 0$ and $||(I - T_{r_n}^{(\Psi,\varphi_2)})Ax_n|| \to 0$ as $n \to \infty$. Then $q \in \Omega_{SGEP}$.*

Proof. Since $y_{n_k} = T_{r_{n_k}}^{(\Phi,\varphi_1)}(x_{n_k} - \lambda_n A^*(I - T_{r_{n_k}}^{(\Psi,\varphi_2)})Ax_{n_k})$, we have

$$\Phi(y_{n_k}, y) + \varphi_1(y_{n_k}, y) + \frac{1}{r_{n_k}}\langle y - y_{n_k}, y_{n_k} - x_{n_k} - \lambda_{n_k} A^*(I - T_{r_{n_k}}^{(\Psi,\varphi_2)})Ax_{n_k}\rangle \geq 0, \quad \forall\, y \in C.$$

This implies that

$$\Phi(y_{n_k}, y) + \varphi_1(y_{n_k}, y) + \frac{1}{r_{n_k}}\langle y - y_{n_k}, y_{n_k} - x_{n_k}\rangle - \frac{1}{r_{n_k}}\langle y - y_{n_k}, \lambda_{n_k} A^*(I - T_{r_{n_k}}^{(\Psi,\varphi_2)})Ax_{n_k}\rangle \geq 0,$$

for all $y \in C$. It follows from the monotonicity of Φ and φ_1 that

$$\frac{1}{r_{n_k}}\langle y - y_{n_k}, y_{n_k} - x_{n_k}\rangle - \frac{1}{r_{n_k}}\langle y - y_{n_k}, \lambda_{n_k} A^*(I - T_{r_{n_k}}^{(\Psi,\varphi_2)})Ax_{n_k}\rangle \geq \Phi(y, y_{n_k}) + \varphi_1(y, y_{n_k}).$$

Since $||x_{n_k} - y_{n_k}|| \to 0$, then $y_{n_k} \rightharpoonup q$ as $k \to \infty$. Taking limit of the inequality above, we get

$$\Phi(y, q) + \varphi_1(y, q) \leq 0, \quad \forall\, y \in C. \tag{4.3.34}$$

Let $y_t = ty + (1-t)q$ for any $t \in (0,1]$ and $y \in C$. Consequently, we have $y_t \in C$ and hence $\Phi(y_t, q) + \varphi_1(y_t, q) \leq 0$. Using assumptions D1 and D4, we get

$$\begin{array}{rcl} 0 & \leq & \Phi(y_t, y_t) - \varphi_1(y_t, y_t) \\ & \leq & t(\Phi(y_t, y) + \varphi_1(y_t, y)) + (1-t)(\Phi(y_1, q) + \varphi_1(y_t, q)) \\ & \leq & t(\Phi(y_t, y) + \varphi_1(y_t, y)) + (1-t)(\Phi(q, y_t) + \varphi_1(q, y_t)) \\ & \leq & \Phi(y_t, y) + \varphi_1(y_t, y). \end{array}$$

Letting $t \to 0$ and using assumption D3, we have the upper semicontinuity of φ_1, we get

$$\Phi(q, y) + \varphi_1(q, y) \geq 0, \quad \forall\, y \in C.$$

This implies that $q \in \Omega_{GEP}$.

Furthermore, since A is a bounded operator, then $Ay_{n_k} \rightharpoonup Aq$. Then, it follows from the assumption that $||(I - T_{r_{n_k}}^{(\Psi,\varphi_2)})Ax_{n_k}|| \to 0$ that $T_{r_{n_k}}^{(\Psi,\varphi_2)})Ax_{n_k} \rightharpoonup Aq$. By the definition of $T_{r_{n_k}}^{(\Psi,\varphi_2)})Ax_{n_k}$, we have

$$\Psi(T_{r_{n_k}}^{(\Psi,\varphi_2)}Ax_{n_k}, y) + \varphi_2(T_{r_{n_k}}^{(\Psi,\varphi_2)}Ax_{n_k}, y) + \frac{1}{r_{n_k}}\langle y - T_{r_{n_k}}^{(\Psi,\varphi_2)}Ax_{n_k}, T_{r_{n_k}}^{(\Psi,\varphi_2)}Ax_{n_k} - Ax_{n_k}\rangle \geq 0,$$

for all $y \in Q$. Since both Ψ and φ_2 are upper semicontinuous in the first argument, it follows from the above inequality that

$$\Psi(Aq, y) + \varphi_2(Aq, y) \geq 0, \quad \forall\, y \in Q.$$

This shows that $Aq \in \Omega_{GEP}$. Therefore $q \in \Omega_{SGEP}$. □

We now present the main convergence theorem in this section.

Theorem 4.3.12. *Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ be a bounded linear operator. Let $\Phi, \varphi_1 : C \times C \to \mathbb{R}$ and $\Psi, \varphi_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying assumption D1 - D7 in Lemma 4.3.1 and for each $i \in \mathbb{N}$, let $S_i : H_1 \to CB(H_1)$ be multivalued demi-contractive mappings with constants k_i such that $I - S_i$ are demiclosed at zero and for each $p \in F(S_i)$, $S_i(p) = \{p\}$. Suppose $Sol := \Omega_{SGEP} \cap \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $f : H_1 \to H_1$ be a ρ-contraction with constant $\rho \in (0, 1)$ and D be a bounded operator on H_1 with coefficient $\bar{\tau} > 0$ such that $0 < \xi < \frac{\bar{\tau}}{\rho}$. Let $\{r_n\}$ be a sequence in $(0, \infty)$, $\{\alpha_n\}$ and $\{\beta_{n,i}\}$ be sequences in $(0, 1)$ and $\{\lambda_n\}$ be a sequence in $[a, b] \subset (0, \frac{1}{||A||^2})$ such that conditions (C1) - (C3) are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 4.3.4 converges strongly to a point x^*, where $x^* = P_{Sol}(I - D + \xi f)(x^*)$ is a unique solution of the variational inequality*

$$\langle (D - \xi f)x^*, x^* - z\rangle \leq 0, \quad z \in Sol. \tag{4.3.35}$$

Proof. Let $x^* \in Sol$ and put $\Gamma_n = ||x_n - x^*||^2$. We divide the proof into two cases.

Case 1: Suppose there exists $n_0 \in \mathbb{N}$ such that $\{\Gamma_n\}$ is decreasing. Then $\{\Gamma_n\}$ converges and

$$\Gamma_n - \Gamma_{n+1} \to 0, \quad \text{as} \quad n \to \infty. \tag{4.3.36}$$

From (4.3.32), we have

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||\alpha_n \xi f(x_n) + (1 - \alpha_n D)z_n - x^*||^2 \\ &\leq (1 - \alpha_n\bar{\tau})^2||z_n - x^*||^2 + 2\alpha_n\langle \xi f(x_n) - Dx^*, x_{n+1} - x^*\rangle \\ &\leq (1 - \alpha_n\bar{\tau})^2\left[||x_n - x^*||^2 - \sum_{i=1}^{n}(\beta_{n,0} - k)\beta_{n,i}||w_n - v_{n,i}||^2\right] \\ &\quad + 2\alpha_n\langle \xi f(x_n) - Dx^*, x_{n+1} - x^*\rangle. \end{aligned}$$

This implies that

$$\begin{aligned}(1-\alpha_n\bar{\tau})^2\sum_{i=1}^{n}(\beta_{n,0}-k)\beta_{n,i}||w_n-v_{n,i}||^2 &\leq (1-\alpha_n\bar{\tau})^2||x_n-x^*||^2-||x_{n+1}-x^*||^2\\ &\quad +2\alpha_n\langle \xi f(x_n)-Dx^*,x_{n+1}-x^*\rangle\\ &\leq \Gamma_n-\Gamma_{n+1}+2\alpha_n\langle \xi f(x_n)-Dx^*,x_{n+1}-x^*\rangle\\ &\quad +\alpha_n^2M,\end{aligned}$$

for some $M>0$. Since $\alpha_n\to 0$, we have from (4.3.36) that

$$\lim_{n\to\infty}\sum_{i=1}^{n}(\beta_{n,0}-k)\beta_{n,i}||w_n-v_{n,i}||^2=0.$$

Using condition (C2), we have

$$\lim_{n\to\infty}||w_n-v_{n,i}||^2=0. \tag{4.3.37}$$

Also from (4.3.31), we get

$$\begin{aligned}||x_{n+1}-x^*||^2 &= ||\alpha_n\xi f(x_n)+(1-\alpha_nD)z_n-x^*||^2\\ &\leq (1-\alpha_n\bar{\tau})^2||z_n-x^*||^2+2\alpha_n\langle \xi f(x_n)-Dx^*,x_{n+1}-x^*\rangle\\ &\leq (1-\alpha_n\bar{\tau})^2||w_n-z||^2+2\alpha_n\langle \xi f(x_n)-Dx^*,x_{n+1}-x^*\rangle\\ &\leq (1-\alpha_n\bar{\tau})^2\left[||x_n-z||^2-\frac{(2-\gamma)}{\gamma}||w_n-x_n||^2\right]\\ &\quad +2\alpha_n\langle \xi f(x_n)-Dx^*,x_{n+1}-x^*\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}(1-\alpha_n\bar{\tau})^2\frac{(2-\gamma)}{\gamma}||w_n-x_n||^2 &\leq (1-\alpha_n\bar{\tau})^2||x_n-z||^2-||x_{n+1}-x^*||^2\\ &\quad +2\alpha_n\langle \xi f(x_n)-Dx^*,x_{n+1}-x^*\rangle.\end{aligned}$$

Since $\alpha_n\to 0$ as $n\to\infty$ and $\gamma\in(0,2)$, we have

$$\lim_{n\to\infty}||w_n-x_n||=0. \tag{4.3.38}$$

Clearly from (4.3.15), we have

$$||x_n-y_n||^2\leq\frac{(1+\theta)^2}{(1-\theta)^2\gamma^2}||x_n-w_n||^2\to 0 \quad \text{as} \quad n\to\infty.$$

Therefore

$$\lim_{n\to\infty}||x_n-y_n||=0. \tag{4.3.39}$$

More so from (4.3.28), we have

$$||w_n-x^*||^2 \leq ||x_n-x^*||^2-\gamma\delta_n\lambda_n||(I-T_{r_n}^{(\Psi,\varphi_2)})Ay_n||^2.$$

This means that

$$\begin{aligned}\gamma\delta_n a||(I-T_{r_n}^{(\Psi,\varphi_2)})Ay_n||^2 &\leq ||x_n-x^*||^2-||w_n-x^*||^2\\ &= ||w_n-x_n||(||x_n-x^*||+||w_n-x^*||)\to 0, \quad \text{as} \quad n\to\infty.\end{aligned}$$

Hence

$$\lim_{n\to\infty}||(I-T_{r_n}^{(\Psi,\varphi_2)})Ay_n||=0. \tag{4.3.40}$$

Again, using Lemma 4.3.2(ii), we have

$$\begin{aligned}||Ax_n-T_{r_n}^{(\Psi,\varphi_2)}Ax_n|| &\leq ||Ax_n-T_{r_n}^{(\Psi,\varphi_2)}Ax_n-Ay_n+T_{r_n}^{(\Psi,\varphi_2)}Ay_n||+||Ay_n-T_{r_n}^{(\Psi,\varphi_2)}Ay_n||\\ &\leq 2||A||\cdot||x_n-y_n||+||(I-T_{r_n}^{(\Psi,\varphi_2)})Ay_n||.\end{aligned}$$

From (4.3.39) and (4.3.40), we get

$$\lim_{n\to\infty}||Ax_n-T_{r_n}^{(\Psi,\varphi_2)}Ax_n||=0. \tag{4.3.41}$$

Furthermore

$$\begin{aligned}||x_{n+1}-z_n|| &= ||\alpha_n\xi f(x_n)+(1-\alpha_n D)z_n-z_n||\\ &= \alpha_n||\xi f(x_n)-Dz_n||\to 0 \quad \text{as} \quad n\to\infty,\end{aligned}$$

and from (4.3.37), we have

$$\begin{aligned}||z_n-w_n|| &= \left\|\beta_{n,0}w_n+\sum_{i=1}^{n}\beta_{n,i}v_{n,i}-w_n\right\|\\ &\leq \sum_{i=1}^{n}\beta_{n,i}||v_{n,i}-w_n||\to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Therefore

$$||x_{n+1}-x_n||\leq||x_{n+1}-z_n||+||z_n-w_n||+||w_n-x_n||\to 0 \quad \text{as} \quad n\to\infty. \tag{4.3.42}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup q$. From (4.3.38), we have that $w_{n_k}\rightharpoonup q$. Now, using the fact that for each $i=1,2,\ldots$, $I-S_i$ is demiclosed at zero and since $\lim_{k\to\infty}||w_{n_k}-v_{n_k,i}||=0$ (from (4.3.37)), we obtain that $q\in F(S_i)$ for each $i=1,2,\ldots$. Hence $q\in\bigcap_{i=1}^{\infty}F(S_i)$. Also from (4.3.39), we have $y_{n_k}\to q$. Using Lemma 4.3.11 and (4.3.41), we have $q\in\Omega_{SGEP}$.
Therefore

$$q\in Sol:=\Omega_{SGEP}\cap\bigcap_{i=1}^{\infty}F(S_i).$$

Next, we show that $\{x_n\}$ converges strongly to x^*, where $x^*=P_{Sol}(I-D+\xi f)x^*$. To do this, it suffices to show that $\limsup_{n\to\infty}\langle\xi f(x^*)-Dx^*,x_{n+1}-x^*\rangle\leq 0$. Choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\xi f(x^*)-Dx^*,x_{n+1}-x^*\rangle=\lim_{k\to\infty}\langle\xi f(x^*)-Dx^*,x_{n_k+1}-x^*\rangle.$$

Since $||x_{n+1}-x_n|| \to 0$ as $n \to \infty$, we have from (2.2.2) that

$$\begin{aligned}\limsup_{n\to\infty}\langle \xi f(x^*) - Dx^*, x_{n+1} - x^*\rangle &= \lim_{k\to\infty}\langle \xi f(x^*) - Dx^*, x_{n_k+1} - x^*\rangle \\ &= \langle \xi f(x^*) - Dx^*, q - x^*\rangle \\ &\leq 0. \end{aligned} \tag{4.3.43}$$

Using Lemma 2.6.30, Lemma 4.3.10 (i) and (4.3.43), we obtain

$$\lim_{n\to\infty} ||x_n - x^*|| = 0. \tag{4.3.44}$$

This implies that $\{x_n\}$ converges strongly to x^*.

Case II: Assume that $\{\Gamma_n\}$ is not monotonically decreasing. For some n_0 large enough, define a mapping $\tau : \mathbb{N} \to \mathbb{N}$ for all $n \geq n_0$ defined by

$$\tau(n) := \max\{j \in \mathbb{N} : j \leq n, \Gamma_j \leq \Gamma_{j+1}\}.$$

By Lemma 2.6.34, $\tau(n)$ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall\ n \geq n_0.$$

Following same argument as in Case I, we have

$$\lim_{n\to\infty} ||w_{\tau(n)} - x_{\tau(n)}|| = \lim_{n\to\infty} ||x_{\tau(n)} - y_{\tau(n)}|| = \lim_{n\to\infty} ||Ax_{\tau(n)} - T_{r_{\tau(n)}}^{(\Psi,\varphi_2)} Ax_{\tau(n)}|| = \lim_{n\to\infty} ||x_{\tau(n)+1} - x_{\tau(n)}|| = 0,$$

and

$$\limsup_{n\to\infty}\langle (D - \xi f)x^*, x^* - x_{\tau(n)+1}\rangle \leq 0. \tag{4.3.45}$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$ still denoted by $\{x_{\tau(n)}\}$ which converges weakly to $z \in C$. By similar argument as in Case I, we obtain $z \in Sol := \Omega_{SGEP} \cap \bigcap_{i=1}^{\infty} F(S_i)$.

Now from Lemma 4.3.10(i), we have

$$\Gamma_{\tau(n)+1} \leq (1 - a_{\tau(n)})\Gamma_{\tau(n)} + a_{\tau(n)} b_{\tau(n)}, \tag{4.3.46}$$

where $a_{\tau(n)} = \frac{2\alpha_{\tau(n)}(\bar{\tau} - \xi\rho)}{1 - \alpha_{\tau(n)}\xi\rho}$, $b_{\tau(n)} = \frac{\langle \xi f(x^*) - Dx^*, x_{\tau(n)+1} - x^*\rangle}{\bar{\tau} - \xi\rho} + \frac{\alpha_{\tau(n)}M}{2(\bar{\tau} - \xi\rho)}$, for some $M > 0$. Note that $a_{\tau(n)} \to 0$ as $n \to \infty$ and $\limsup_{n\to\infty} b_{\tau(n)} \leq 0$.

Since $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $a_{\tau(n)} > 0$, we have

$$||x_{\tau(n)} - x^*||^2 \leq b_{\tau(n)}.$$

This implies that

$$\limsup_{n\to\infty} ||x_{\tau(n)} - x^*||^2 = 0,$$

and thus

$$\lim_{n\to\infty} ||x_{\tau(n)} - x^*|| = 0.$$

Also from (4.3.46) we obtain

$$\limsup_{n\to\infty} ||x_{\tau(n)+1} - x^*||^2 \le \limsup_{n\to\infty} ||x_{\tau(n)} - x^*||^2,$$

and thus

$$\lim_{n\to\infty} ||x_{\tau(n)+1} - x^*|| = 0.$$

Furthermore, for $n \ge n_0$, it is easy to see that $\Gamma_n \le \Gamma_{\tau(n)+1}$. As a consequence, we obtain that for all $n \ge n_0$

$$0 \le \Gamma_n \le \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\Gamma_n \to 0$ as $n \to \infty$. That is, $\{x_n\}$ converges strongly to x^*. This completes the proof. □

The following is a direct consequence of Theorem 4.3.12 by taking $\varphi_1 = \varphi_2 = 0$.

Corollary 4.3.13. *Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ be a bounded linear operator. Let $\Phi : C \times C \to \mathbb{R}$ and $\Psi : Q \times Q \to \mathbb{R}$ be bifunctions satisfying assumption D1 - D4 in Lemma 4.3.1 and for each $i \in \mathbb{N}$, let $S_i : H_1 \to CB(H_1)$ be multivalued demi-contractive mappings with constants k_i such that $I - S_i$ are demiclosed at zero and for each $p \in F(S_i)$, $S_i(p) = \{p\}$. Suppose $Sol := \Omega_{SEqP} \cap \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $f : H_1 \to H_1$ be a ρ-contraction with constant $\rho \in (0,1)$ and D be a bounded operator on H_1 with coefficient $\bar{\tau} > 0$ such that $0 < \xi < \frac{\bar{\tau}}{\rho}$. Let $\{r_n\}$ be a sequence in $(0,\infty)$, $\{\alpha_n\}$ and $\{\beta_{n,i}\}$ be sequences in $(0,1)$ and $\{\lambda_n\}$ be a sequence in $[a,b] \subset (0, \frac{1}{||A||^2})$ such that conditions (C1) - (C3) are satisfied. Then the sequence $\{x_n\}$ generated by the following Algorithm 4.3.14 converges strongly to a point x^*, where $x^* = P_{Sol}(I - D + \xi f)(x^*)$ is a unique solution of the variational inequality*

$$\langle (D - \xi f)x^*, x^* - z\rangle \le 0, \quad z \in Sol. \tag{4.3.47}$$

Algorithm 4.3.14.

Step 0: *Choose $\gamma > 0$, $\alpha \in (0,1)$, $\theta \in (0,1)$ and $\gamma \in (0,2)$. Let $x_1 \in H_1$ and set $n = 1$.*

Step 1: *Compute*

$$y_n = T_{r_n}^{\Phi}(x_n - \lambda_n A^*(I - T_{r_n}^{\Psi})Ax_n),$$

where $\lambda_n = \sigma\eta^{m_n}$, $\sigma > 0$, $\eta \in (0,1)$ and m_n is the smallest nonnegative integer such that

$$\lambda_n ||A^*(I - T_{r_n}^{\Psi})Ax_n - A^*(I - T_{r_n}^{\Psi})Ay_n|| \le \theta ||x_n - y_n||.$$

Step 2: *If $x_n = y_n$, then go to Step 3. Else, compute*

$$d(x_n, y_n) = x_n - y_n - \lambda_n\Big[A^*(I - T_{r_n}^{\Psi})Ax_n - A^*(I - T_{r_n}^{\Psi})Ay_n\Big],$$

and

$$w_n = x_n - \gamma\delta_n d(x_n, y_n)$$

where

$$\delta_n = \begin{cases} \frac{\langle x_n - y_n, d(x_n,y_n)\rangle}{||d(x_n,y_n)||^2}, & \text{if } d(x_n, y_n) \neq 0, \\ 0 & \text{if } d(x_n, y_n) = 0. \end{cases}$$

Step 3: Compute

$$x_{n+1} = \alpha_n \xi f(x_n) + (1 - \alpha_n D)\left(\beta_{n,0} w_n + \sum_{i=1}^{n} \beta_{n,i} v_{n,i}\right),$$

where $v_{n,i} \in S_i w_n$. Set $n = n + 1$ and go to Step 1.

4.3.2 Convergence rate

In this subsection, we study the rate of convergence of Algorithm 4.3.4.

Motivated by Nemirovski [194] and Tseng [261], Dong et al. [95] studied the convergence rate for an extragradient method for solving variational inequality problems. they proved that such sequence achieves $O(1/t)$ convergence rate. In this section, we shall also show that the proposed Algorithm 4.3.4 converges at the rate $O(1/t)$. To the best of our knowledge, no author have proved the convergence rate of an algorithm for solving the split equilibrium problem.

For the sake of simplicity, we take the sequence $\{r_n\}$ to be $r > 0$.

Theorem 4.3.15. *Let $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ be the sequences generated by Algorithm 4.3.4. For any $t > 0$ and $\gamma \in (0, 2]$, we have a $y_t \in C$ which satisfies*

$$\langle A^*(I - T_r^{\Psi,\varphi_2})A^*u, y_t - u\rangle \leq \frac{||x_1 - u||^2}{2\gamma\Upsilon_t}, \quad \forall u \in C, \tag{4.3.48}$$

where

$$y_t = \frac{1}{\Upsilon_t}\sum_{n=1}^{t} \gamma\lambda_n\delta_n y_n \quad \text{and} \quad \Upsilon_t = \sum_{n=1}^{t} \lambda_n\delta_n. \tag{4.3.49}$$

Proof. From (4.3.8) and (4.3.10), we have

$$y_n = T_r^{(\Phi,\varphi_1)}(y_n - [\lambda_n A^*(I - T_r^{\Psi\varphi_2})Ay_n - d(x_n, y_n)]). \tag{4.3.50}$$

We deduce from (4.3.6) and (4.3.50) that

$$\langle u - y_n, \lambda_n A^*(I - T_r^{\Psi\varphi_2})Ay_n - d(x_n, y_n)\rangle \geq 0.$$

This implies that

$$\langle u - y_n, \lambda_n A^*(I - T_r^{\Psi\varphi_2})Ay_n\rangle \geq \langle u - y_n, d(x_n, y_n)\rangle. \tag{4.3.51}$$

Also from (4.3.11), we get

$$\gamma\delta_n d(x_n, y_n) = x_n - w_n. \tag{4.3.52}$$

It follows from (4.3.51) and (4.3.52) that

$$\langle \gamma\delta_n\lambda_n A^*(I - T_r^{\Psi\varphi_2})Ay_n, u - y_n\rangle \geq \langle x_n - w_n, u - y_n\rangle. \tag{4.3.53}$$

However, by Lemma 2.6.2, we have

$$\langle x_n - w_n, u - y_n\rangle = \frac{1}{2}\left(||x_n - y_n||^2 - ||x_n - u||^2\right) + \frac{1}{2}\left(||u - w_n||^2 - ||y_n - w_n||^2\right). \quad (4.3.54)$$

Since $w_n = x_n - \gamma\delta_n d(x_n, y_n)$, then

$$\begin{aligned} ||x_n - y_n||^2 - ||y_n - w_n||^2 &= ||x_n - y_n||^2 - ||y_n - x_n - \gamma\delta_n d(x_n, y_n)||^2 \\ &= 2\gamma\delta_n\langle y_n - x_n, d(x_n, y_n)\rangle - \gamma^2\delta_n^2||d(x_n, y_n)||^2 \\ &= 2\gamma\delta_n^2||d(x_n, y_n)||^2 - \gamma^2\delta_n^2||d(x_n, y_n)||^2 \\ &= \gamma(2-\gamma)\delta_n^2||d(x_n, y_n)||^2. \end{aligned} \quad (4.3.55)$$

Putting (4.3.55) into (4.3.54), we obtain

$$\langle x_n - w_n, u - y_n\rangle + \frac{1}{2}\left(||x_n - u||^2 - ||u - w_n||^2\right) = \frac{\gamma(2-\gamma)\delta_n^2}{2}||d(x_n, y_n)||^2.$$

That is

$$\langle \gamma\delta_n\lambda_n A^*(I - T_r^{\Psi\varphi_2})Ay_n, u - y_n\rangle + \frac{1}{2}\left(||x_n - u||^2 - ||u - w_n||^2\right) \geq \frac{\gamma(2-\gamma)\delta_n^2}{2}||d(x_n, y_n)||^2.$$

Using the fact that $T_r^{(\Psi,\varphi_2)}$ is firmly nonexpansive and (4.3.22), we get

$$\langle \gamma\delta_n\lambda_n A^*(I - T_r^{\Psi\varphi_2})Au, u - y_n\rangle + \frac{1}{2}\left(||x_n - u||^2 - ||u - w_n||^2\right) \geq \frac{\gamma(2-\gamma)\delta_n^2}{2}||d(x_n, y_n)||^2.$$

This means that

$$\langle \gamma\delta_n\lambda_n A^*(I - T_r^{\Psi\varphi_2})Au, y_n - u\rangle \leq \frac{1}{2}||x_n - u||^2. \quad (4.3.56)$$

Summing the inequality (4.3.56) over $n = 1, \ldots, t$, we have

$$\sum_{n=1}^{t}\delta_n\lambda_n\langle A^*(I - T_r^{\Psi\varphi_2})Au, y_n - u\rangle \leq \frac{1}{2\gamma}||x_1 - u||^2.$$

Then by using the notation of Υ_t and y_t, we have

$$\langle A^*(I - T_r^{\Psi\varphi_2})Au, y_t - u\rangle \leq \frac{1}{2\gamma\Upsilon_t}||x_1 - u||^2.$$

□

Remark 4.3.16. From Lemma 4.3.6, it follows that

$$\Upsilon_t \geq (t+1)\bar{\lambda},$$

thus Algorithm 3.2.4 has $O(1/t)$ convergence rate (see [95]). In fact, for any bounded subset $D \subset C$ and given accuracy $\epsilon > 0$, Algorithm 4.3.4 will achieve

$$\langle A^*(I - T_r^{\Psi\varphi_2})Au, y_t - u\rangle \leq \epsilon, \quad \forall u \in D$$

in at most $t = \left[\frac{\alpha^2}{2\gamma\bar{\lambda}c\epsilon}\right]$ iterations, where y_t is defined by (4.3.49), $\alpha = \sup\{||u - x_1|| : u \in D\}$ and $c = \frac{1-\theta}{(1+\theta)^2}$.

4.3.3 Numerical examples

In this subsection, we provide some numerical examples to demonstrate the performance of our algorithm. All codes were written in `MATLAB 2014(b)` on a HP-Elitebook 6930 PC with 8gb RAM.

Example 4.3.17. Let $H_1 = H_2 = \mathbb{R}^3$ and $C = Q = [0,10] \times [0,10] \times [0,10]$. Let $A : H_1 \to H_2$ be defined by $Ax = \frac{x}{2}$ for each $x \in H_1$. For $x \in C$, $i = 1, 2, \ldots,$ we define $S_i : C \to CB(C)$ as

$$S_i x = \left[0, \frac{x}{10i}\right] \quad \text{for all} \quad i \in \mathbb{N}.$$

Then S_i is 0-demi-contractive mapping for all $i \in \mathbb{N}$ and $\bigcap_{i=1}^{\infty} F(S_i) = \{0\}$. Define the bifunctions $\Phi, \varphi_1 : C \times C \to \mathbb{R}$ by $\Psi(x,y) = 2x^2 + xy - 3y^2$ and $\varphi_1(x,y) = x - y$ for each $x, y \in C$. Also define $\Psi, \varphi_2 : Q \times Q \to \mathbb{R}$ by $\Psi(u,z) = u^2 + 3uz - 4z^2$ and $\varphi_2(u,z) = z^2 - u^2$ for each $u, z \in Q$. It is easy to check that

$$T_r^{(\Phi,\varphi_1)} x = \frac{x}{7r+1},$$

and

$$T_r^{(\Psi,\varphi_2)} z = \frac{z+r}{5r+1}.$$

For each $n \in \mathbb{N}$, and $i \geq 0$, let $\{\beta_{n,i}\}$ be defined by

$$\beta_{n,i} = \begin{cases} \frac{1}{b^{i+1}}\left(\frac{n}{n+1}\right), & n \geq i+1, \\ 1 - \frac{n}{n+1}\sum_{k=1}^{n} \frac{1}{b^k} & n = i, \\ 0 & n < i, \end{cases}$$

where $b > 1$. In this example, we set $b = 5$, then obtain

$$\beta_{n,i} = \begin{pmatrix} \frac{1}{10} & \frac{9}{10} & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{2}{15} & \frac{2}{75} & \frac{63}{75} & 0 & 0 & \cdots & 0 & \cdots \\ \frac{3}{20} & \frac{3}{100} & \frac{3}{500} & \frac{407}{500} & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ \frac{n}{5(n+1)} & \frac{n}{5^2(n+1)} & \frac{n}{5^3(n+1)} & \frac{n}{5^4(n+1)} & \frac{n}{5^4(n+1)} & \cdots & \frac{n}{5^i(n+1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \end{pmatrix}. \tag{4.3.57}$$

For Algorithm 4.3.4, we set $f(x) = \frac{x}{5}$ and $Dx = x$, $\xi = 0.7$, $r_n = \frac{n}{n+1}$, $\alpha_n = \frac{1}{\sqrt{n+1}}$, $\theta = 0.3$, $\lambda_n = \frac{\theta}{||A||^2}$, $\gamma = 0.3$. It can easily be checked that $\Phi, \Psi, \varphi_1, \varphi_2$ and $\{r_n\}$ satisfy all conditions in Theorem 4.3.12 with $Sol = \{0\}$. Let $\epsilon > 0$, the algorithm stops if $||x_n - x^*|| \leq \epsilon$. For algorithm (4.3.2), we take $T(s)x = \frac{x}{1+2s}$, $f(x) = \frac{x}{5}$, $Bx = x$ for all $x \in H_1$ and set $s_n = n$, $\alpha_n = \frac{1}{\sqrt{n+1}}$, $\beta_n = \frac{3n-2}{7n+9}$, $\xi = 1, \delta = \frac{1}{2}$. In Table 4.5, for a given tolerance level and randomly chosen initial points, we collect data of the number of iterations and time required to execute both Algorithm 4.3.4 and (4.3.2).

From the given table, we deduce that for a given tolerance, Algorithm 4.3.4 takes significant less number of iteration and CPU time compare to Algorithm (4.3.2).

Example 4.3.18. In this example, we take $H_1 = H_2 = \mathbb{R}$ and $C = Q = [-5, 5]$. Define $\Phi, \varphi_1 : C \times C \to \mathbb{R}$ by $\Phi(x, y) = -3x + xy + 2y^2$ and $\varphi_1(x, y) = x^2 - xy$ for each $x, y \in C$. Also, let $\Psi, \varphi_2 : Q \to \mathbb{R}$ by $\Psi(u, w) = -4u^2 + uw + 3w^2$ and $\varphi_2(u, w) = 2u(u - w)$ for each $u, w \in Q$. It is easy to see that

$$T_r^{(\Phi,\varphi_1)}x = \frac{1-r}{5r+1}x,$$

and

$$T_r^{(\Psi,\varphi_2)}z = \frac{1-2r}{7r+1}z.$$

For $i \in \mathbb{N}$, we define $S_i : C \to CB([-5, 5])$ by

$$S_i x = \begin{cases} \left[0, \frac{|x|}{i+2}\right], & \text{if} \quad x < i+2, \\ [1, i+1] & \text{if} \quad x \geq i+2. \end{cases} \tag{4.3.58}$$

It can easily be seen that S_i is 0-demi-contractive for all $i \in \mathbb{N}$ and $F(S_i) = \{0\}$. We also define a bounded linear operator $A : H_1 \to H_2$ by $Ax = 3x$. Thus, $A^*x = 3x$ and $||A|| = 3$. It is clear that $0 \in Sol$. For Algorithm 4.3.4, define $f, D : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{1}{2}$ and $Dx = 4$ and take $\xi = 1$, $\gamma = 0.5$, and $\lambda_n = \frac{\theta}{||A||^2}$, $\alpha_n = \frac{1}{10(n+1)}$ and $\{\beta_{n,i}\}$ is as defined in Example 4.3.17. In Algorithm (4.3.2), we take $B_1(x) = x$, $B_2(x) = 2x$, $\xi_1 = \frac{1}{2}$, $\xi_2 = 1$, $S_n(x) = \frac{x}{2n}$, $\alpha_n = \frac{1}{10(n+1)}$, and $\delta = 1.2$. Also in Algorithm (4.3.4), we choose $\alpha_n^{(i)} = \beta_{n,i}$ defined in (4.3.57) and take $S_i(x)$ to be as defined in (4.3.58). Using four various initial points and $\frac{||x_{n+1}-x_n||}{||x_2-x_1||} < 10^{-4}$ as stopping criterion, we plot the graphs of error ($||x_{n+1} - x_n||$) against number of iterations in each case for the three algorithms. The numerical results can be seen in Table 4.6, Figure 4.6.

Table 4.5: Comparison between Algorithm (4.3.2) and Algorithm 4.3.4

Tolerance Level	Initial Points	Algorithm (4.3.2)		Algorithm (4.3.4)	
	x_1	Iter.	CPU time (secs)	Iter.	CPU time (secs)
$\epsilon = 10^{-4}$	$(4, 0, 5)$	24	0.0272	8	0.0091
$\epsilon = 10^{-5}$	$(4, 0, 5)$	29	0.0294	9	0.0051
$\epsilon = 10^{-6}$	$(4, 0, 5)$	34	0.0510	11	0.0082
$\epsilon = 10^{-4}$	$(3, 2, 1)$	23	0.0210	8	0.0016
$\epsilon = 10^{-5}$	$(3, 2, 1)$	28	0.0275	9	0.0067
$\epsilon = 10^{-6}$	$(3, 2, 1)$	33	0.0324	10	0.0067
$\epsilon = 10^{-4}$	$(5, 8, 10)$	25	0.0203	8	0.0061
$\epsilon = 10^{-5}$	$(5, 8, 10)$	30	0.0305	9	0.0067
$\epsilon = 10^{-6}$	$(5, 8, 10)$	35	0.0512	11	0.0134

Table 4.6: Numerical result for for Example 4.3.18.

		Alg. 4.3.4	Alg. (4.3.3)	Alg. (4.3.4)
$x_1 = -1$	CPU time (sec)	0.0032	0.0465	0.0109
	No. of Iter.	7	33	13
$x_1 = 2.5$	CPU time (sec)	0.0040	0.0408	0.0126
	No. of Iter.	7	32	16
$x_1 = -3$	CPU time (sec)	0.0096	0.0384	0.0103
	No. of Iter.	8	33	15
$x_1 = 5$	CPU time (sec)	0.0071	0.0532	0.0112
	No. of Iter.	8	35	17

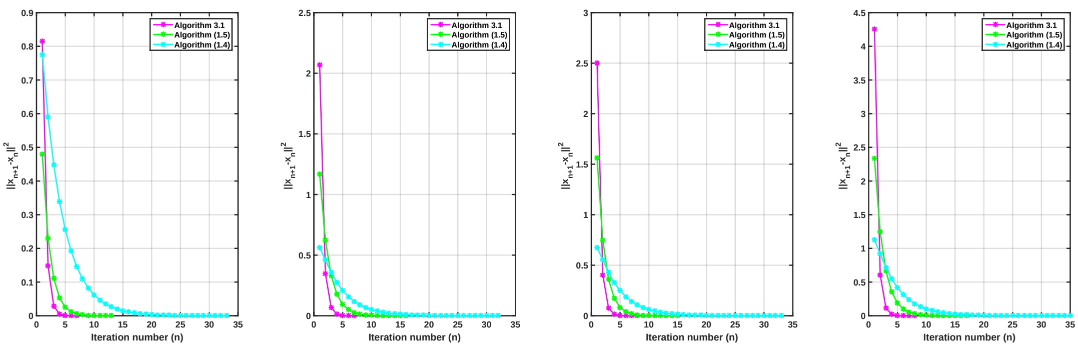


Figure 4.6: Example 4.3.18, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

CHAPTER 5

Variational Inequality Problems in Hilbert and Banach Spaces

5.1 Strong Convergence Theorem for Solving Pseudo-Monotone Variational Inequality Problem using Projection Method in a Reflexive Banach Space

Very recently, Gibali [112] proposed a new Bregman projection method for solving the VIP in a Hilbert space. Gibali's algorithm is an extension of the subgradient extragradient method of [69, 158, 256] with Bregman projection which makes only one projection per iteration.

In this section, we introduce a projection-type algorithm for finding a common solution of the variational inequality problem and fixed point problem in reflexive Banach space, where A is pseudo-monotone and not necessarily Lipschitz continuous. Also, we present an application of our result to approximating solution of pseudo-monotone equilibrium problem in reflexive Banach space.

5.1.1 Main results

In this subsection, we give a precise statement of our projection-type method and discuss some of its convergence analysis.

Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a coercive, Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$. Let $A : E \to E^*$ be a continuous pseudo-monotone operator and $T : C \to C$ be a Bregman quasi-nonexpansive mapping such that $\Gamma := \Omega_{VIP} \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be nonnegative sequences in $(0,1)$.

Algorithm 5.1.1.
Step 0: *Select the initial points* $x_1, u \in E$*, let* $\gamma, \sigma \in (0,1)$ *and* $s > 0$*. Choose* $\lambda_n \in [a,b]$ *such that* $0 < a \leq b$ *and set* $n = 1$*.*
Step 1: *Compute*

$$z_n = \nabla f^*(\nabla f(x_n) - \lambda_n A x_n). \tag{5.1.1}$$

Step 2: *If* $x_n = Proj_C^f(z_n)$ *and* $x_n = Tx_n$*: STOP. Else, let* $y_n(t) := (1-t)x_n + tProj_C^f(z_n)$ *for* $t \in \mathbb{R}$*. Compute* t_n *as the maximum of the numbers* $s, s\gamma, s\gamma^2, \dots$ *such that*

$$\langle Ay_n(t_n), x_n - Proj_C^f(z_n)\rangle \geq \frac{\sigma D_f(Proj_C^f z_n, x_n)}{\lambda_n}, \tag{5.1.2}$$

and define $y_n = y_n(t_n)$*.*
Step 3: *Construct the set* Q_n *define by* $Q_n = \{y \in E : \langle Ay_n, y - y_n\rangle = 0\}$ *and compute*

$$\begin{cases} u_n = Proj_{Q_n}^f(\nabla f(x_n) - \lambda_n A y_n), \\ v_n = Proj_C^f(u_n), \\ x_{n+1} = \nabla f^*\Big(\alpha_n \nabla f(u) + (1-\alpha_n)(\beta_n \nabla f(v_n) + (1-\beta_n)\nabla f(Tv_n))\Big). \end{cases} \tag{5.1.3}$$

Set $n \leftarrow n+1$ *and go to Step 1.*

Remark 5.1.2. Note that if $x_n - Proj_C^f(z_n) = 0$ and $x_n - Tx_n = 0$, then we are at a common solution of the VIP (1.1.1) and fixed point of the Bregman quasi-nonexpansive mapping. In our convergence analysis, we will implicitly assume that this does not occur after finitely many iterations so that Algorithm 5.1.1 generates an infinite sequences.

We first show that Algorithm 5.1.1 is well defined. To do this, it is sufficient to show that the inner loop in the stepsize rule in **Step 2** is well defined.

Lemma 5.1.3. *(i) The stepsize process in **Step 2** of Algorithm 5.1.1 is well defined.*

(ii) Let $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by Algorithm 5.1.1, then*

$$\langle Ay_n, x_n - y_n\rangle > 0.$$

Proof. (i) Assume that (5.1.2) does not hold for $n \in \mathbb{N}$. This implies that

$$\langle Ay_n(t_n), x_n - Proj_C^f z_n\rangle < \frac{\sigma D_f(Proj_C^f z_n, x_n)}{\lambda_n} \quad for \quad n \in \mathbb{N}.$$

Then, we have

$$\langle A((1-s\gamma^m)x_n + s\gamma^m Proj_C^f z_n), x_n - Proj_C^f z_n\rangle < \frac{\sigma D_f(Proj_C^f z_n, x_n)}{\lambda_n} \quad \forall m \geq 0.$$

Since A is continuous and $y_n(t_n) \to x_n$ as $m \to \infty$, it follows that

$$\langle \lambda_n A x_n, x_n - Proj_C^f z_n\rangle < \sigma D_f(Proj_C^f z_n, x_n),$$

equivalently, by (5.1.1), we have

$$\langle \nabla f(x_n) - \nabla f(z_n), x_n - Proj_C^f z_n \rangle < \sigma D_f(Proj_C^f z_n, x_n).$$

Applying the three point identity (Proposition 2.5.1 (ii)) to the left hand side of the above inequality, we obtain

$$D_f(Proj_C^f z_n, x_n) + D_f(x_n, z_n) - D_f(Proj_C^f z_n, z_n) < \sigma D_f(Proj_C^f z_n, x_n).$$

Since f is strictly convex and $\sigma \in (0,1)$, then

$$D_f(x_n, z_n) < D_f(Proj_C^f z_n, z_n).$$

This contradicts the definition of Bregman projection (see Definition 2.5.5). Hence, the stepsize rule in Step 2 of Algorithm 5.1.1 is well defined.

(ii) Furthermore, from (5.1.2), we have

$$\begin{aligned} \langle Ay_n, x_n - y_n \rangle &= \langle Ay_n, x_n - (1-t_n)x_n - t_n Proj_C^f z_n \rangle \\ &= t_n \langle Ay_n, x_n - Proj_C^f z_n \rangle \\ &\geq \frac{\sigma t_n D_f(Proj_C^f z_n, x_n)}{\lambda_n} > 0. \end{aligned}$$

□

In order to establish our main result, we make the following assumptions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

We proceed to prove the following lemmas before proving the convergence of our main Algorithm 5.1.1.

Lemma 5.1.4. *The sequence $\{x_n\}$ generated by Algorithm 5.1.1 is bounded.*

Proof. For each $n \in \mathbb{N}$, define the sets:

$$Q_n^- := \{u \in E : \langle Ax_n, u - x_n \rangle \leq 0\},$$

$$Q_n := \{u \in E : \langle Ax_n, u - x_n \rangle = 0\},$$

and

$$Q_n^+ := \{u \in E : \langle Ax_n, u - x_n \rangle \geq 0\}.$$

Let $p \in \Gamma$, then since A is pseudo-monotone, we have

$$\langle Ap, x - p \rangle \geq 0 \Rightarrow \langle Ax, x - p \rangle \geq 0 \quad \forall x \in E.$$

This implies that $p \in Q_n^-$ for all $n \in \mathbb{N}$. Furthermore, since we implicitly assumed that Algorithm 5.1.1 does not terminate after finitely many steps with an exact solution, we have from Lemma 5.1.3(ii) that $\langle Ay_n, x_n - y_n \rangle > 0$. This implies that $x_n \in Q_n^+$ and $x_n \notin Q_n^-$ for all $n \in N$. Therefore, using Lemma 2.6.10, we obtain

$$D_f(p, x_n) \geq D_f(p, u_n) + D_f(u_n, x_n). \tag{5.1.4}$$

Now, since $v_n = Proj_C^f(u_n)$, then from (2.5.8), we have

$$D_f(p, u_n) \geq D_f(p, v_n) + D_f(v_n, u_n). \tag{5.1.5}$$

Combining (5.1.4) and (5.1.5), we have

$$D_f(p, x_n) \geq D_f(p, v_n) + D_f(v_n, u_n) + D_f(u_n, x_n).$$

This implies that

$$D_f(p, v_n) \leq D_f(p, x_n) - D_f(v_n, u_n) - D_f(u_n, x_n). \tag{5.1.6}$$

From (5.1.3) and (5.1.6), we have

$$\begin{aligned}
D_f(p, x_{n+1}) &\leq D_f\Big(p, \nabla f^*\Big(\alpha_n \nabla f(u) + (1-\alpha_n)(\beta_n \nabla f(v_n) + (1-\beta_n)\nabla f(Tv_n))\Big)\Big), \\
&= V_f\Big(p, \alpha_n \nabla f(u) + (1-\alpha_n)(\beta_n \nabla f(v_n) + (1-\beta_n)\nabla f(Tv_n))\Big) \\
&= V_f\Big(p, \alpha_n \nabla f(u) + (1-\alpha_n)\beta_n \nabla f(v_n) + (1-\alpha_n)(1-\beta_n)\nabla f(Tv_n)\Big) \\
&= f(p) - \Big\langle p, \alpha_n \nabla f(u) + (1-\alpha_n)\beta_n \nabla f(v_n) + (1-\alpha_n)(1-\beta_n)\nabla f(Tv_n)\Big\rangle \\
&\quad + f^*\Big(\alpha_n \nabla f(u) + (1-\alpha_n)\beta_n \nabla f(v_n) + (1-\alpha_n)(1-\beta_n)\nabla f(Tv_n)\Big) \\
&\leq \alpha_n f(p) - \alpha_n \langle p, \nabla f(u)\rangle + \alpha_n f^*(\nabla f(u)) + (1-\alpha_n)\beta_n f(p) \\
&\quad - (1-\alpha_n)\beta_n \langle p, \nabla f(v_n)\rangle + (1-\alpha_n)\beta_n f^*(\nabla f(v_n)) + (1-\alpha_n)(1-\beta_n) f(p) \\
&\quad - (1-\alpha_n)(1-\beta_n)\langle p, \nabla f(Tv_n)\rangle + (1-\alpha_n)(1-\beta_n) f^*(\nabla f(Tv_n)) \\
&= \alpha_n V_f(p, \nabla f(u)) + (1-\alpha_n)\beta_n V_f(p, \nabla f(v_n)) \\
&\quad + (1-\alpha_n)(1-\beta_n) V_f(p, \nabla f(Tv_n)) \\
&= \alpha_n D_f(p, u) + (1-\alpha_n)\beta_n D_f(p, v_n) + (1-\alpha_n)(1-\beta_n) D_f(p, Tv_n) \\
&\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, v_n) \\
&\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n) \\
&\leq \max\{D_f(p, u), D_f(p, x_n)\} \\
&\ \vdots \\
&\leq \max\{D_f(p, u), D_f(p, x_1)\}.
\end{aligned}$$

Hence $\{D_f(p, x_n)\}$ is bounded. Then by Lemma 2.6.28, we obtain that $\{x_n\}$ is bounded. □

Remark 5.1.5. Since $\{x_n\}$ is bounded and A is continuous, it follows that $\{Ax_n\}$ and $\{z_n\}$ are bounded. Consequently, by the nonexpansivity of the projection operator and T, we have that $\{y_n\}$, $\{Ay_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{Tv_n\}$ are bounded.

Lemma 5.1.6. *Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by Algorithm 5.1.1. Suppose there exist subsequences $\{x_{n_k}\}$ and $\{u_{n_k}\}$ of $\{x_n\}$ and $\{u_n\}$ respectively such that $\lim_{k\to\infty} ||x_{n_k} - u_{n_k}|| = 0$. Let $\{y_{n_k}\}$ and $\{z_{n_k}\}$ be subsequences of $\{y_n\}$ and $\{z_n\}$ respectively, then*

(a) $\lim_{k\to\infty}\langle Ay_{n_k}, x_{n_k} - y_{n_k}\rangle = 0$,

(b) $\lim_{k\to\infty} ||Proj_C^f(z_{n_k}) - x_{n_k}|| = 0$,

(c) $0 \leq \liminf_{k\to\infty}\langle Ax_{n_k}, x - x_{n_k}\rangle$, *for all* $x \in C$.

Proof. (a) Since $u_n \in Q_n$, then we have

$$0 = \langle Ay_{n_k}, u_{n_k} - y_{n_k}\rangle = \langle Ay_{n_k}, u_{n_k} - x_{n_k}\rangle + \langle Ay_{n_k}, x_{n_k} - y_{n_k}\rangle,$$

which implies that

$$\begin{aligned} \langle Ay_{n_k}, x_{n_k} - y_{n_k}\rangle &= \langle Ay_{n_k}, x_{n_k} - u_{n_k}\rangle \\ &\leq ||Ay_{n_k}||_*||x_{n_k} - u_{n_k}||. \end{aligned}$$

Taking the limit of the above inequality as $k \to \infty$ yields

$$\lim_{k\to\infty}\langle Ay_{n_k}, x_{n_k} - y_{n_k}\rangle = 0.$$

(b) Let $\{t_{n_k}\}$ be a subsequence of $\{t_n\}$. We consider the following two cases based on the behaviour of t_{n_k}.

<u>Case I:</u> Suppose $\lim_{k\to\infty} t_{n_k} \neq 0$; i.e., there exists some $\delta > 0$ such that $t_{n_k} \geq \delta > 0$ for all $k \in \mathbb{N}$. It follows from Step 2 of Algorithm 5.1.1 that

$$\langle Ay_{n_k}, x_{n_k} - y_{n_k}\rangle \geq \frac{\sigma D_f(Proj_C^f z_{n_k}, x_{n_k})}{\lambda_n}.$$

Hence, from Lemma 5.1.6(a), we have

$$\lim_{k\to\infty} D_f(Proj_C^f z_{n_k}, x_{n_k}) = 0.$$

Therefore using Lemma 2.6.24 (a), we obtain that

$$\lim_{k\to\infty} ||Proj_C^f z_{n_k} - x_{n_k}|| = 0.$$

<u>Case II:</u> On the other hand, suppose $t_{n_k} \to 0$ as $k \to \infty$. Let $t_{n_k} < s$ so that the stepsize get reduced at least once for all iterations belonging to this subsequence. This implies that the trial stepsize does not satisfy the test from Step 2 of Algorithm 5.1.1. Assume that $\lim_{k\to\infty} D_f(Proj_C^f z_{n_k}, x_{n_k}) \neq 0$, i.e., there exists a positive constant $\delta < +\infty$ such that $\limsup_{k\to\infty}(Proj_C^f z_{n_k}, x_{n_k}) = \delta$.

Define $\bar{y}_k = (1 - t_{n_k})x_{n_k} + t_{n_k} Proj_C^f(z_{n_k})$. Then

$$\bar{y}_k - x_{n_k} = t_{n_k}(Proj_C^f z_{n_k} - x_{n_k}).$$

Since $\{Proj_C^f z_{n_k} - x_{n_k}\}$ is bounded and $t_{n_k} \to 0$ as $k \to \infty$, it follows that $\lim_{k\to\infty} ||\bar{y}_k - x_{n_k}|| = 0$. From the stepsize rule in Step 2 and the definition of $\bar{y}_k$, we have

$$\langle A\bar{y}_k, x_{n_k} - Proj_C^f z_{n_k}\rangle < \frac{\sigma D_f(Proj_C^f z_{n_k}, x_{n_k})}{\lambda_{n_k}} \quad \forall k \in \mathbb{N}.$$

Since A is uniformly continuous on bounded subsets of C and $\sigma \in (0,1)$, we obtain that there exists $N \in \mathbb{N}$ such that

$$\langle \lambda_{n_k} Ax_{n_k}, x_{n_k} - Proj_C^f z_{n_k}\rangle < D_f(Proj_C^f z_{n_k}, x_{n_k}) \quad \forall k \in \mathbb{N}, \quad k \geq N.$$

Therefore

$$\langle \nabla f(x_{n_k}) - \nabla f(z_{n_k}), x_{n_k} - Proj_C^f z_{n_k}\rangle < D_f(Proj_C^f z_{n_k}, x_{n_k}), \quad \forall k \in \mathbb{N}, \quad k \geq N.$$

Using the three points identity (Proposition 2.5.1 (ii)) in the last inequality, we get

$$D_f(Proj_C^f z_{n_k}, x_{n_k}) + D_f(x_{n_k}, z_{n_k}) - D_f(Proj_C^f z_{n_k}, z_{n_k}) < D_f(Proj_C^f z_{n_k}, z_{n_k}) \quad \forall k \geq N.$$

Hence

$$D_f(x_{n_k}, z_{n_k}) < D_f(Proj_C^f z_{n_k}, z_{n_k}) \quad \forall k \geq N.$$

This contradicts the definition of the Bregman projection. Hence $\lim_{k\to\infty} D_f(Proj_C^f z_{n_k}, x_{n_k}) = 0$. Therefore, by using Lemma 2.6.24 (a), we obtain thaat $\lim_{k\to\infty} ||Proj_C^f z_{n_k} - x_{n_k}|| = 0$.

(c) From (2.5.7), we have that

$$\langle \nabla f(z_{n_k}) - \nabla f(Proj_C^f z_{n_k}), y - Proj_C^f z_{n_k}\rangle \leq 0 \quad \forall y \in C.$$

This implies from (5.1.1) that

$$\langle \nabla f(x_{n_k}) - \nabla f(Proj_C^f z_{n_k}), y - Proj_C^f z_{n_k}\rangle \leq \langle \lambda_{n_k} Ax_{n_k}, y - Proj_C^f z_{n_k}\rangle \quad \forall y \in C.$$

Therefore

$$\begin{aligned}\langle \nabla f(x_{n_k}) - \nabla f(Proj_C^f z_{n_k}), &y - Proj_C^f z_{n_k}\rangle + \langle \lambda_{n_k} Ax_{n_k}, Proj_C^f z_{n_k} - x_{n_k}\rangle \\ &\leq \langle \lambda_{n_k} Ax_{n_k}, y - x_{n_k}\rangle \quad \forall y \in C. \end{aligned} \tag{5.1.7}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, by Lemma 2.6.26, ∇f is norm-to-norm uniformly continuous on bounded subsets of E^* and therefore, from (b), we get

$$\lim_{k\to\infty} ||\nabla f(Proj_C^f(z_{n_k})) - \nabla f(x_{n_k})||_* = 0. \tag{5.1.8}$$

Taking the limit of the inequality in (5.1.7) and noting that $\{\lambda_{n_k}\} \subset [a, b]$, we have

$$0 \leq \liminf_{k\to\infty} \langle Ax_{n_k}, y - x_{n_k}\rangle \quad \forall y \in C.$$

This completes the proof. □

Lemma 5.1.7. *The sequence $\{x_n\}$ generated by Algorithm 5.1.1 satisfies the following estimates:*

(i) $s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n b_n$,

(ii) $-1 \leq \limsup_{n\to\infty} b_n < +\infty$,

where $p \in \Gamma$, $s_n = D_f(p, x_n)$, $\quad b_n = \langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle$.

Proof. (i) Let $w_n = \nabla f^*(\beta_n \nabla f(v_n) + (1-\beta_n)\nabla f(Tv_n))$ and $p \in \Gamma$, then from (5.1.3), we have

$$\begin{aligned}
D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(w_n))) \\
&\leq V_f\Big(p, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(w_n) - \alpha_n(\nabla f(u) - \nabla f(p))\Big) \\
&\quad + \langle \alpha_n(\nabla f(u) - \nabla f(p)), x_{n+1} - p\rangle \\
&= V_f\Big(p, \alpha_n \nabla f(p) + (1-\alpha_n)\nabla f(w_n)\Big) + \alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)D_f(p, w_n) + \alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \\
&= (1-\alpha_n)\Big(D_f(p, \nabla f^*(\beta_n \nabla f(v_n) + (1-\beta_n)\nabla f(Tv_n)))\Big) \\
&\quad + \alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)\beta_n D_f(p, v_n) + (1-\alpha_n)(1-\beta_n)D_f(p, Tv_n) \\
&\quad + \alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)D_f(p, v_n) + \alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle. \qquad (5.1.9)
\end{aligned}$$

Therefore from (5.1.6), we have

$$\begin{aligned}
D_f(p, x_{n+1}) &\leq (1-\alpha_n)\Big(D_f(p, x_n) - D_f(v_n, u_n) - D_f(u_n, x_n)\Big) \\
&\quad + \alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle. \qquad (5.1.10)
\end{aligned}$$

Since $\{\alpha_n\} \subset (0,1)$, then

$$D_f(p, x_{n+1}) \leq (1-\alpha_n)D_f(p, x_n) + \alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle. \qquad (5.1.11)$$

This established (i).
(ii) Since $\{x_n\}$ is bounded, then we have

$$\sup_{n\geq 0} b_n \leq \sup ||\nabla f(u) - \nabla f(p)||_* ||x_{n+1} - p|| < \infty.$$

This implies that $\limsup_{n\to\infty} b_n < \infty$. Next, we show that $\limsup_{n\to\infty} b_n \geq -1$. Assume the contrary, i.e. $\limsup_{n\to\infty} b_n < -1$. Then there exists $n_0 \in \mathbb{N}$ such that $b_n < -1$, for all $n \geq n_0$. Then for all $n \geq n_0$, we get from (i) that

$$\begin{aligned}
s_{n+1} &\leq (1-\alpha_n)s_n + \alpha_n b_n \\
&< (1-\alpha_n)s_n - \alpha_n \\
&= s_n - \alpha_n(s_n + 1) \leq s_n - \alpha_n.
\end{aligned}$$

Taking lim sup of both sides of the last inequality, we have

$$\limsup_{n\to\infty} s_n \leq s_{n_0} - \lim_{n\to\infty}\sum_{i=n_0}^{n} \alpha_i = -\infty.$$

This contradicts the fact that $\{s_n\}$ is a nonnegative real sequence. Therefore $\limsup_{n\to\infty} b_n \geq -1$. □

We are now in position to state and prove our main theorem.

Theorem 5.1.8. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a coercive, Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$. Let $A : E \to E^*$ be a continuous pseudo-monotone operator and $T : C \to C$ be a Bregman quasi-nonexpansive mapping such that $\hat{F}(T) = F(T)$ and $\Gamma := \Omega_{VIP} \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be nonnegative sequences in $(0,1)$ and such that conditions (C1) and (C2) are satisfied. Let $\{x_n\}$ be generated by Algorithm 5.1.1. Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} = Proj^f_\Gamma(u)$, where $Proj^f_\Gamma$ is the Bregman projection from C onto Γ.*

Proof. Let $p \in \Gamma$, and denote $D_f(p, x_n)$ by Φ_n. We consider the following two possible cases.
<u>CASE A:</u> Suppose there exists $n_0 \in \mathbb{N}$ such that Φ_n is monotonically non-increasing for all $n \geq n_0$. Since Φ_n is bounded, then it is convergent and so $\Phi_n - \Phi_{n+1} \to 0$ as $n \to \infty$.

We first show that $||x_n - u_n|| \to 0$, $||v_n - Tv_n|| \to 0$ and $||x_{n+1} - x_n|| \to 0$ as $n \to \infty$. Since $\{\alpha_n\} \subset (0,1)$, we obtain from (5.1.10) that

$$(1-\alpha_n)D_f(u_n, x_n) \leq (1-\alpha_n)D_f(p, x_n) - D_f(p, x_{n+1}) + \alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle.$$

Using condition(C1), we obtain that $D_f(u_n, x_n) \to 0$ as $n \to \infty$, hence from Lemma 2.6.24(a), we have

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{5.1.12}$$

Similarly from (5.1.10), we can obtain

$$\lim_{n\to\infty} ||v_n - u_n|| = 0. \tag{5.1.13}$$

Hence

$$\lim_{n\to\infty} ||v_n - x_n|| = 0. \tag{5.1.14}$$

Recall that $w_n = \nabla f^*(\beta_n \nabla f(v_n) + (1-\beta_n)\nabla f(Tv_n))$, from Lemma 2.6.21, we have

$$\begin{aligned}
D_f(p, w_n) &= D_f(p, \nabla f^*(\beta_n \nabla f(v_n) + (1-\beta_n)\nabla f(Tv_n))) \\
&= V_f(p, \beta_n \nabla f(v_n) + (1-\beta_n)\nabla f(Tv_n)) \\
&= f(p) - \langle p, \beta_n \nabla f(v_n) + (1-\beta_n)\nabla f(Tv_n)\rangle + f^*(\beta_n \nabla f(v_n) \\
&\quad +(1-\beta_n)\nabla f(Tv_n)) \\
&\leq \beta_n f(p) - \beta_n\langle p, \nabla f(v_n)\rangle + \beta_n f^*(\nabla f(Tv_n)) + (1-\beta_n)f(p) \\
&\quad -(1-\beta_n)\langle p, \nabla f(Tv_n)\rangle + (1-\beta_n)f^*(\nabla f(Tv_n)) \\
&\quad -\beta_n(1-\beta_n)\rho_r(||\nabla f(v_n) - \nabla f(Tv_n)||_*) \\
&\leq \beta_n D_f(p, v_n) + (1-\beta_n)D_f(p, Tv_n) - \beta_n(1-\beta_n)\rho_r(||\nabla f(v_n) - \nabla f(Tv_n)||_*) \\
&\leq D_f(p, v_n) - \beta_n(1-\beta_n)\rho_r(||\nabla f(v_n) - \nabla f(Tv_n)||_*).
\end{aligned} \tag{5.1.15}$$

Thus from (5.1.6), (5.1.10) and (5.1.15), we have

$$\begin{aligned} D_f(p, x_{n+1}) &\leq (1-\alpha_n)D_f(p, v_n) - (1-\alpha_n)\beta_n(1-\beta_n)\rho_r(||\nabla f(v_n) - \nabla f(Tv_n)||) \\ &\quad +\alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \\ &\leq (1-\alpha_n)D_f(p, x_n) - (1-\alpha_n)\beta_n(1-\beta_n)\rho_r(||\nabla f(v_n) - \nabla f(Tv_n)||_*) \\ &\quad +\alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle. \end{aligned}$$

Hence

$$\begin{aligned} (1-\alpha_n)\beta_n(1-\beta_n)\rho_r(||\nabla f(v_n) - \nabla f(Tv_n)||) &\leq (1-\alpha_n)D_f(p, x_n) - D_f(p, x_{n+1}) \\ &\quad +\alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle. \end{aligned}$$

It follows from conditions (C1), (C2) and the properties of ρ_r that

$$\lim_{n\to\infty} ||\nabla f(v_n) - \nabla f(Tv_n)||_* = 0. \tag{5.1.16}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, by Lemma 2.6.26, it is also uniformly continuous and ∇f is norm-to-norm uniformly continuous on bounded subsets of E, hence from (5.1.16), we have

$$\lim_{n\to\infty} ||f(v_n) - f(Tv_n)|| = 0, \tag{5.1.17}$$

and

$$\lim_{n\to\infty} ||v_n - Tv_n|| = 0. \tag{5.1.18}$$

In addition, it is easy to see from definition of Bregman distance that $D_f(v_n, Tv_n) \to 0$ as $n \to \infty$. Thus

$$D_f(v_n, x_{n+1}) \leq \alpha_n D_f(v_n, u) + (1-\alpha_n)\beta_n D_f(v_n, v_n) + (1-\alpha_n)(1-\beta_n)D_f(v_n, Tv_n).$$

This implies that

$$\lim_{n\to\infty} ||v_n - x_{n+1}|| = 0. \tag{5.1.19}$$

Therefore from (5.1.14) and (5.1.19), we obtain

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - v_n|| + ||v_n - x_n|| \to 0, \quad as \quad n \to \infty. \tag{5.1.20}$$

Next, we show that $\Omega_w(x_n) \subset \Omega_{VIP} \cap F(T)$, where $\Omega_w(x_n)$ is the weak subsequential limit of $\{x_n\}$. Let $\bar{x} \in \Omega_w(x_n)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. Consequently from (5.1.17), $v_{n_k} \rightharpoonup \bar{x}$. Since $||v_{n_k} - Tv_{n_k}|| \to 0$, then $\bar{x} \in \hat{F}(T) = F(T)$. Furthermore, let $z \in C$ be an arbitrary point and $\{\varepsilon_k\}$ be a sequence of decreasing non-negative numbers such that $\varepsilon_k \to 0$ as $k \to \infty$. Using Lemma 5.1.6(c), we can find a large enough N_k such that

$$\langle Ax_{n_k}, z - x_{n_k}\rangle + \varepsilon_k \geq 0, \quad \forall k \geq N_k.$$

This implies that

$$\langle Ax_{n_k}, z + \varepsilon_k t_k - x_{n_k}\rangle \geq 0, \quad \forall k \geq N_k, \tag{5.1.21}$$

for some $t_k \in E$ satisfying $1 = \langle Ax_{n_k}, t_k\rangle$ (since $Ax_{n_k} \neq 0$). Since A is pseudo-monotone, then we have from (5.1.21) that

$$\langle A(z + \varepsilon_k t_k), z + \varepsilon_k t_k - x_{n_k}\rangle \geq 0, \quad \forall k \geq N_k. \tag{5.1.22}$$

This implies that

$$\langle Az, z - x_{n_k}\rangle \geq \langle Az - A(z + \varepsilon_k t_k), z + \varepsilon_k t_k - x_{n_k}\rangle - \varepsilon_k\langle Az, t_{n_k}\rangle \quad \forall k \geq N_k. \tag{5.1.23}$$

Since $\epsilon_k \to 0$ and A is continuous, then the right-hand side of (5.1.23) tends to zero. Thus, we obtain that

$$\liminf_{k\to\infty}\langle Az, z - x_{n_k}\rangle \geq 0, \quad \forall z \in C.$$

In view of Lemma 5.1.6(c), we have that

$$\langle Az, z - \bar{x}\rangle = \lim_{k\to\infty}\langle Az, z - x_{n_k}\rangle \geq 0, \quad \forall z \in C.$$

Hence, from Lemma 2.6.9 we obtain that $\bar{x} \in \Omega_{VIP}$. Therefore $\bar{x} \in \Gamma := \Omega_{VIP} \cap F(T)$.

We now show that $\{x_n\}$ converges strongly to $x^* = Proj_{\Gamma}^{f}u$. To do this, we first show that $\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(x^*), x_{n+1} - x^*\rangle \leq 0$. Choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(x^*), x_{n+1} - x^*\rangle = \lim_{k\to\infty}\langle \nabla f(u) - \nabla f(x^*), x_{n_k+1} - x^*\rangle.$$

Since $||x_{n_k+1} - x_{n_k}|| \to 0$ and $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$, then we have from Lemma 2.5.2(b) that

$$\begin{aligned}\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(x^*), x_{n+1} - x^*\rangle &= \lim_{k\to\infty}\langle \nabla f(u) - \nabla f(x^*), x_{n_k+1} - x^*\rangle \\ &= \langle \nabla f(u) - \nabla f(x^*), \bar{x} - x^*\rangle \leq 0. \end{aligned} \tag{5.1.24}$$

Now using Lemma 2.6.29, Lemma 5.1.7(i) and (5.1.24), we obtain that $D_f(x^*, x_n) \to 0$ as $n \to \infty$. It follows from Lemma 2.6.24(a) that $\lim_{n\to\infty}||x_n - x^*|| = 0$. Therefore, $\{x_n\}$ converges strongly to $x^* = Proj_{\Gamma}^{f}u$.

<u>CASE B:</u> Suppose $\{D_f(p, x_n)\}$ is not monotonically decreasing. Let $\phi : \mathbb{N} \to \mathbb{N}$ for all $n \geq n_0$ (for some n_0 large enough) be defined by

$$\phi_n = \max\{k \in \mathbb{N} : \phi_k \leq \phi_{k+1}\}.$$

Clearly, ϕ is non-decreasing, $\phi(n) \to \infty$ as $n \to \infty$ and

$$0 \leq D_f(p, x_{\phi(n)}) \leq D_f(p, x_{\phi(n)+1}), \quad \forall n \geq n_0.$$

Following similar argument as in CASE A, we obtain

$$||x_{\phi(n)} - u_{\phi(n)}|| \to 0, \quad ||v_{\phi(n)} - Tv_{\phi(n)}|| \to 0, \quad ||x_{\phi(n)+1} - x_{\phi(n)}|| \to 0$$

as $n \to \infty$ and $\Omega_w(x_{\phi(n)}) \subset \Omega_{VIP} \cap F(T)$, where $\Omega_w(x_{\phi(n)})$ is the weak subsequential limit of $\{x_{\phi(n)}\}$. Also,

$$\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), x_{\phi(n)+1} - p \rangle \leq 0. \tag{5.1.25}$$

From Lemma 5.1.7(i), we have that

$$D_f(p, x_{\phi(n)+1}) \leq (1 - \alpha_{\phi(n)}) D_f(p, x_{\phi(n)}) + \alpha_{\phi(n)} \langle \nabla f(u) - \nabla f(p), x_{\phi(n)+1} - p \rangle.$$

Since $D_f(p, x_{\phi(n)}) \leq D_f(p, x_{\phi(n)+1})$, then

$$\begin{aligned} 0 &\leq D_f(p, x_{\phi(n)+1}) - D_f(p, x_{\phi(n)}) \\ &\leq (1 - \alpha_{\phi(n)}) D_f(p, x_{\phi(n)}) + \alpha_{\phi(n)} \langle \nabla f(u) - \nabla f(p), x_{\phi(n)+1} - p \rangle - D_f(p, x_{\phi(n)}). \end{aligned}$$

Hence from (5.1.25), we obtain

$$D_f(p, x_{\phi(n)}) \leq \langle \nabla f(u) - \nabla f(p), x_{\phi(n)+1} - p \rangle \to 0, \quad as \quad n \to \infty.$$

As a consequence, we obtain that for all $n \geq n_0$,

$$0 \leq D_f(p, x_n) \leq \max\{D_f(p, x_{\phi(n)}), D_f(p, x_{\phi(n)+1})\} = D_f(p, x_{\phi(n)+1}).$$

Hence

$$D_f(p, x_n) \to 0 \quad as \quad n \to \infty.$$

Therefore, from Lemma 2.6.24(a),

$$\lim_{n\to\infty} ||x_n - p|| = 0.$$

This implies that $\{x_n\}$ converges strongly to p. This completes the proof. □

Remark 5.1.9. For a suitable starting point x_1, Algorithm 5.1.1 generates appropriate solution which approximates the whole solution set Γ as guaranteed by Theorem 5.1.8. This is an interesting property which is different (for example) from the class of Tikhonov-type regularization approaches where the corresponding sequences always converge to the same solution. With this fact, one can get an idea of the geometric shape of the whole solution set by using various starting point x_1. In fact, if one has some a priori knowledge regarding the location of a solution and is, therefore, interested in computing a particular solution which is as close as possible to this prior knowledge, Algorithm 5.1.1 allows one to take this knowledge into account by a suitable choice of x_1.

The following is a direct consequence of our result.

Corollary 5.1.10. *Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let $A : C \to H$ be a continuous pseudo-monotone operator and $T : C \to C$ be a quasi-nonexpansive mapping such that $\Gamma := \Omega_{VIP} \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be nonnegative sequences in $(0, 1)$ and such that conditions (C1) and (C2) are satisfied. Let $\{x_n\}$ be generated by the following Algorithm:*

Algorithm 5.1.11.
Step 0: *Select the initial points* $x_1, u \in H$*, let* $\gamma, \sigma \in (0,1)$ *and* $s > 0$*. Choose* $\lambda_n \in [a,b]$ *such that* $0 < a \leq b$ *and set* $n = 1$*.*
Step 1: *Compute*

$$z_n = x_n - \lambda_n A x_n. \tag{5.1.26}$$

Step 2: *If* $x_n = P_C(z_n)$ *and* $x_n = Tx_n$*: STOP. Else, let* $y_n(t) := (1-t)x_n + tP_C(z_n)$ *for* $t \in \mathbb{R}$*. Compute* t_n *as the maximum of the numbers* $s, s\gamma, s\gamma^2, \ldots$ *such that*

$$\langle Ay_n(t_n), x_n - P_C(z_n)\rangle \geq \frac{\sigma||P_C z_n - x_n||^2}{2\lambda_n}, \tag{5.1.27}$$

and define $y_n = y_n(t_n)$*.*
Step 3: *Construct the set* Q_n *define by* $Q_n = \{y \in E : \langle Ay_n, y - y_n\rangle = 0\}$ *and compute*

$$\begin{cases} u_n = P_{Q_n}(x_n - \lambda_n A y_n), \\ v_n = P_C(u_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(\beta_n v_n + (1 - \beta_n T v_n). \end{cases} \tag{5.1.28}$$

Set $n \leftarrow n+1$ *and go to Step 1.*

Then, the sequence $\{x_n\}$ *generated by Algorithm 5.1.11 converges strongly to a point* $\bar{x} = P_\Gamma(u)$*, where* P_Γ *is the metric projection from* C *onto* Γ*.*

Remark 5.1.12. Corollary 5.1.10 extends the work of Kanzow and Shehu [144] from monotone VIP to common solution of pseudo-monotone VIP and fixed point of quasi-nonexpansive mapping in a real Hilbert space.

5.1.2 Application to Equilibrium Problem

For solving the EP, we assume that the bifunction g satisfies the following:

Assumption 5.1.13.

(A1) g *is weakly continuous on* $C \times C$*,*

(A2) $g(x,\cdot)$ *is convex lower semicontinuous and subdifferentiable on* C *for every fixed* $x \in C$*,*

(A3) for each $x, y, z \in C$*,* $\limsup_{t\downarrow 0} g(tx + (1-t)y, z) \leq g(y,z)$*.*

Lemma 5.1.14. *[16] Let* E *be a nonempty convex subset of a Banach space* E *and* $f : E \to \mathbb{R}$ *be a convex and subdifferentiable function, then* f *is minimal at* $x \in E$ *if and only if*

$$0 \in \partial f(x) + N_C(x),$$

where $N_C(x)$ *is the normal cone of* C *at* x*, that is,* $N_C(x) := \{x^* \in E^* : \langle x^*, x - z\rangle \geq 0, \quad \forall z \in C\}$*.*

Lemma 5.1.15. *[87] Let E be a real reflexive Banach space. If f and g are two convex functions such that there is a point $x_0 \in dom\ f \cap dom\ g$ where f is continuous, then*

$$\partial(f+g)(x) = \partial f(x) + \partial g(x) \quad \forall x \in E.$$

Proposition 5.1.16. *Let E be a real reflexive Banach space and C be a nonempty, closed and convex subset of E. Let $g : C \times C \to \mathbb{R}$ be a bifunction such that $g(x,x) = 0$ and $f : E \to \mathbb{R}$ be a Legendre and totally coercive function. Then a point $x^* \in EP(C,g)$ if and only if x^* solves the following minimization problem:*

$$\min\Big\{\lambda g(x,y) + D_f(y,x) : y \in C\Big\},$$

where $x \in C$ and $\lambda > 0$.

Proof. Let $x^* = argmin_{y\in C}\Big\{\lambda g(x,y) + D_f(y,x)\Big\}$, then from Lemma 5.1.14 and 5.1.15, we have

$$0 \in \partial\lambda g(x,x^*) + \nabla D_f(x^*,x) + N_C(x^*).$$

Hence, there exist $w \in \partial g(x,x^*)$ and $\bar{w} \in N_C(x^*)$ such that

$$\lambda w + \nabla f(x^*) - \nabla f(x) + \bar{w} = 0. \tag{5.1.29}$$

Since $\bar{w} \in N_C(x^*)$, then $\langle \bar{w}, z - x^*\rangle \le 0$ for all $z \in C$. This together with (5.1.29) implies that

$$\langle \lambda w + \nabla f(x^*) - \nabla f(x), z - x^*\rangle \ge 0 \quad \forall \ \ z \in C,$$

and hence

$$\lambda\langle w, z - x^*\rangle \ge \langle \nabla f(x^*) - \nabla f(x), x^* - z\rangle \quad \forall \ \ z \in C. \tag{5.1.30}$$

Also, since $w \in \partial g(x,x^*)$, then

$$g(x,z) - g(x,x^*) \ge \langle w, z - x^*\rangle \quad \forall \ \ z \in C. \tag{5.1.31}$$

Therefore from (5.1.30) and (5.1.31), we obtain

$$\lambda\Big(g(x,z) - g(x,x^*)\Big) \ge \langle \nabla f(x^*) - \nabla f(x), x^* - z\rangle \quad \forall \ \ z \in C. \tag{5.1.32}$$

Replacing x with x^* in (5.1.32) yields

$$g(x^*,z) \ge 0, \quad \forall z \in C. \tag{5.1.33}$$

Hence, $x^* \in EP(C,g)$. The converse follows clearly. □

Proposition 5.1.17. *Let C be a nonempty closed convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ be a Legendre and totally coercive function. Let $A : C \to E^*$ be a nonlinear mapping such that $x \in \Omega_{VIP}$. Then x is the unique solution of the minimization problem*

$$\min\Big\{\lambda\langle Au, y - u\rangle + D_f(y,u) : y \in C\Big\},$$

where $u \in C$ and $\lambda > 0$.

Proof. Since $x \in \Omega_{VIP}$, then $x = Proj_C^f(\nabla f^*(\nabla f(x) - \lambda Ax))$. By the definition of Bregman projection, we have

$$\begin{aligned} Proj_C^f(\nabla f^*(\nabla f(x) - \lambda Ax)) &= \min\{D_f\big(y, \nabla f^*(\nabla f(x) - \lambda Ax)\big) : \quad y \in C\} \\ &= \min\{f(y) - f(\nabla f^*(\nabla f(x) - \lambda Ax)) \\ &\quad -\langle y - \nabla f^*(\nabla f(x) - \lambda Ax), \nabla f(x) - \lambda Ax\rangle : \quad y \in C\}. \end{aligned}$$

Using the four point identity (Proposition 2.5.1(iii)), we get

$$\begin{aligned} &Proj_C^f(\nabla f^*(\nabla f(x) - \lambda Ax)) \\ &= \min\{f(y) - f(\nabla f^*(\nabla f(x) - \lambda Ax)) - D_f(y, \nabla f^*(\lambda Ax)) - D_f(\nabla f^*(\nabla f(x) - \lambda Ax), x) \\ &\quad + D_f(y, x) + D_f(\nabla f^*(\nabla f(x) - \lambda Ax), \nabla f^*(\lambda Ax)) \ : y \in C\} \\ &= \min\{f(y) - f(\nabla f^*(\nabla f(x) - \lambda Ax)) - f(y) + f(\nabla f^*(\lambda Ax)) + \langle y - \nabla f^*(\lambda Ax), \lambda Ax\rangle \\ &\quad - D_f(\nabla f^*(\nabla f(x) - \lambda Ax), x) + D_f(y, x) + D_f(\nabla f^*(\nabla f(x) - \lambda Ax), \nabla f^*(\lambda Ax)) \ : y \in C\} \\ &= \min\{\lambda\langle Ax, y - x\rangle + \lambda\langle Ax, x - \nabla f^*(\lambda Ax)\rangle - f(\nabla f^*(\nabla f(x) - \lambda Ax)) + f(\nabla f^*(\lambda Ax)) \\ &\quad - D_f(\nabla f^*(\nabla f(x) - \lambda Ax, x)) + D_f(y, x) + D_f(\nabla f^*(\nabla f(x) - \lambda Ax), \nabla f^*(\lambda Ax)) \ : y \in C\} \\ &= \min\{\lambda\langle Ax, y - x\rangle + D_f(y, x) : \ y \in C\}. \end{aligned} \tag{5.1.34}$$

Therefore

$$x^* = argmin_{y \in C}\{\lambda\langle Ax, y - x\rangle + D_f(y, x)\}.$$

□

Recall that a mapping $A : C \to E^*$ is pseudo-monotone if and only if the bifunction $g(x, y) = \langle Ax, y - x\rangle$ is pseudo-monotone on C. Then, setting $\langle Ax, y - x\rangle = g(x, y)$ in Theorem 5.1.8, by Proposition 5.1.16 and 5.1.17, we have the following result for approximating solution of pseudo-monotone equilibrium problem.

Theorem 5.1.18. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a coercive, Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$. Let $g : C \times C \to \mathbb{R}$ be a pseudo-monotone bifunction such that $g(x, x) = 0$ for all $x \in C$ and satisfying Assumption 5.1.13. Let $T : C \to C$ be a Bregman quasi-nonexpansive mapping with $\hat{F}(T) = F(T)$ such that $\Gamma := \Omega_{EP(g)} \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be nonnegative sequences in $(0, 1)$ and such that conditions (C1) and (C2) are satisfied. Let $\{x_n\}$ be generated by the following algorithm:*

Algorithm 5.1.19.
Step 0: *Select the initial points $x_1, u \in E$, let $\gamma, \sigma \in (0, 1)$ and $s > 0$. Choose $\lambda_n \in [a, b]$ such that $0 < a \leq b$ and set $n = 1$.*
Step 1: *Compute*

$$z_n = argmin\left\{\lambda_n g(x_n, y) + D_f(y, x_n) : \quad y \in C\right\}.$$

Step 2: *If $x_n = z_n$ and $x_n = Tx_n$: STOP. Otherwise, let $y_n(t) := (1 - t)x_n + tz_n$ for $t \in \mathbb{R}$. Compute t_n as the maximum of the numbers $s, s\gamma, s\gamma^2, \ldots$ such that*

$$g(y_n(t_n), x_n - z_n) \geq \frac{\sigma D_f(z_n, x_n)}{\lambda_n},$$

and define $y_n = y_n(t_n)$.
Step 3: *Set* $w_n = \nabla f(x_n) - \lambda_n y_n$. *Compute* $u_n = Proj^f_{Q_n}(w_n)$ *where* $Q_n := \{x \in E : \langle \bar{w}_n, x - w_n\rangle = 0\}$, $\bar{w}_n \in \partial g(w_n, x - w_n)$. *Then compute*

$$\begin{cases} v_n = Proj^f_C(u_n), \\ x_{n+1} = \nabla f^*\Big(\alpha_n \nabla f(u) + (1-\alpha_n)(\beta_n \nabla f(v_n) + (1-\beta_n)\nabla f(Tv_n))\Big). \end{cases} \tag{5.1.35}$$

Set $n \leftarrow n+1$ *and go to Step 1.*

Then, the sequence $\{x_n\}$ *converges strongly to a point* $\bar{x} = Proj^f_\Gamma(u)$, *where* $Proj^f_\Gamma$ *is the Bregman projection from* C *onto* Γ.

5.1.3 Numerical examples

In this subsection, we present two numerical examples which demonstrate the performance of our Algorithm 5.1.1.

Example 5.1.20. Let $E = \mathbb{R}^n$ with standard topology and $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Tx = -\frac{1}{2}x$. Consider an operator $A : \mathbb{R}^m \to \mathbb{R}^m$ $(m = 20, 50, 100, 200)$ define by $Ax = Mx + q$ where

$$M = NN^T + S + D,$$

N is a $m \times m$ matrix, S is a $m \times m$ skew-symmetric matrix, D is a $m \times m$ diagonal matrix, whose diagonal entries are nonnegative so that M is positive definite and q is a vector in $\mathbb{R}^m$. The feasible set $C \subset \mathbb{R}^m$ is closed and convex (polyhedron) which is defined as $C = \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : Qx \leq b\}$, where Q is a $l \times m$ matrix and b is a nonnegative vector. It is clear that A is monotone (hence, pseudo-monotone) and L-Lipschitz continuous with $L = ||M||$. For experimental purpose, all the entries of N, S, D and b are generated randomly as well as the starting point $x_1 \in [0,1]^m$ and q is equal to the zero vector. In this case, the solution to the corresponding variational inequality is $\{0\}$ and thus, $\Gamma := \Omega_{VIP} \cap F(T) = \{0\}$. We fix the stopping criterion as $\frac{|x_{n+1} - x_n|}{|x_2 - x_1|} = \epsilon < 10^{-5}$, $\sigma = 0.7$, $\gamma = 0.9$, $s = 10$, $\lambda_n = 0.15$ and let $\alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{1}{4}$. The projection onto the feasible set C is carry-out by using the MATLAB solver `'fmincon'` and the projection onto an hyperplane $Q = \{x \in \mathbb{R}^m : \langle a, x\rangle = 0\}$ is defined by

$$P_Q(x) = x - \frac{\langle a, x\rangle}{||a||^2} a.$$

Since A is monotone, we compare the output of our Algorithm 5.1.1 with Algorithm 1.2.1. The numerical result is reported in Figure 5.1 and Table 5.1. We see that our Algorithm 5.1.1 converges faster than Algorithm 1.2.1. This is expected because the stepsize rule in STEP 2 of our algorithm tends to determine a larger stepsize closer to the solution of the problem.

Next, we give an example of a pseudo-monotone VIP which is not a monotone VIP.

Example 5.1.21. Let $E = L^2([0,1])$ and $C = \{u \in E : ||u|| \leq 2\}$. Let $B : C \to \mathbb{R}$ be an operator defined by $B(u) = \frac{1}{1+||u||^2}$ and $F : L^2([0,1]) \to L^2([0,1])$ be the Volterra integral operator defined by $F(u)(t) = \int_0^t u(s)ds$ for all $u \in L^2([0,1])$ and $t \in [0,1]$. F is bounded, linear and monotone (cf. Exercise 20.12 in [23]). Now define $A : C \to L^2([0,1])$ by $A(u)(t) = (B(u)F(u))(t)$. Suppose $\langle Au, v-u\rangle \geq 0$ for all $u, v \in C$, then $\langle Fu, v-u\rangle \geq 0$. Hence

$$\begin{aligned} \langle Av, v-u\rangle &= \langle BvFv, v-u\rangle \\ &= Bv\langle Fv, v-u\rangle \\ &\geq Bv(\langle Fv, v-u\rangle - \langle Fu, v-u\rangle) \\ &= Bv\langle Fv - Fu, v-u\rangle \geq 0. \end{aligned} \tag{5.1.36}$$

Thus, A is pseudo-monotone. To see that that A is not monotone, choose $v = 1$ and $u = 2$, then

$$\langle Av - Au, v-u\rangle = -\frac{1}{10} < 0.$$

Now consider the VIP in which the underlying operator A is as defined above. Let $T : L^2([0,1]) \to L^2([0,1])$ be define by $T(x)(t) = \int_0^1 x(t)dt$, it is easy to verify that T is quasi-nonexpansive and $\Gamma := \Omega_{VIP} \cap F(T) = \{0\}$. Choosing $\sigma = 0.5$, $\gamma = 0.7$, $s = 5$, $\lambda = 0.34$ and $\epsilon < 10^{-4}$. We plot the graph of $||x_{n+1} - x_n||$ against number of iteration for Algorithm 5.1.1 using the following initial points:
Case I: $x_1 = t + 0.5 * \cos(t)$, $u = \cos(5t)$,
Case II: $x_1 = 2t\exp(-t)$, $u = 1/\exp(t^2-1)$,
Case III: $x_1 = \frac{1}{6}\sin(-3t) + \cos(t)$, $u = \cos(-2t)$,
Case IV: $x_1 = \exp(-4t) + \cos(12t)$, $u = \sin(5t)$.

The numerical result is reported in Figure 5.2. This shows that the change in the initial points does not have significant effect on the number of iteration nor CPU time for Algorithm 5.1.1.

Finally, we give a concrete example in ℓ_p space ($1 \leq p < \infty$ with $p \neq 2$) which is not a Hilbert space. It is well known that the dual space $(\ell_p)^*$ is isomorphic to ℓ_q provided that $\frac{1}{q} + \frac{1}{p} = 1$ (see for instance [43], Lemma 2.2, Page 11). Also, the ℓ_p space is a reflexive Banach space and in this case, we take $f(x) = \frac{1}{p}||x||^p$.

Example 5.1.22. Let $E = \ell_3(\mathbb{R})$ define by

$$\ell_3(\mathbb{R}) := \{\bar{x} = (x_1, x_2, x_3, \dots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^3 < \infty\},$$

with norm $||\cdot||_{\ell_3} : \ell_3 \to [0,\infty)$ defined by

$$||\bar{x}||_{\ell_3} = \left(\sum_{i=1}^{\infty} |x_i|^3\right)^{\frac{1}{3}},$$

for arbitrary $\bar{x} = (x_1, x_2, x_3, \dots)$ in ℓ_3.

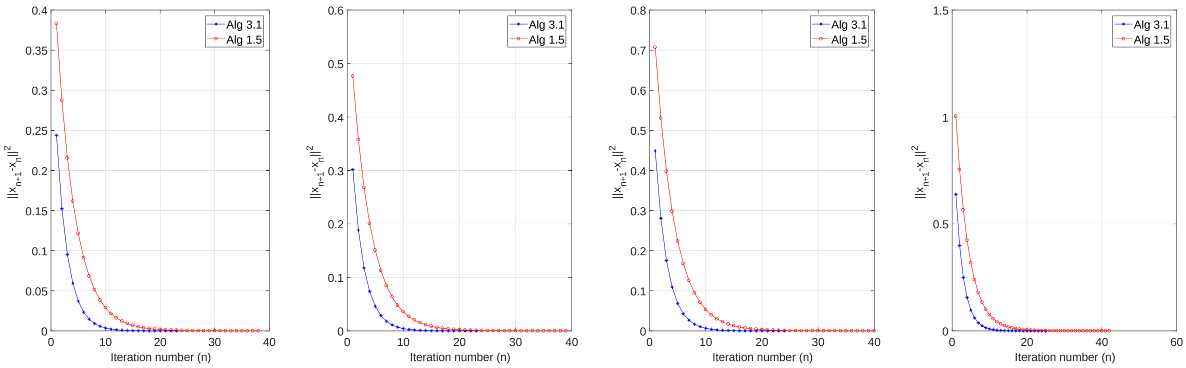


Figure 5.1: Example 5.1.20, $m = 20$; $m = 50$; $m = 100$; $m = 200$.

Let $C := \{x \in E : ||x||_{\ell_3} \leq 1\}$ and define the mapping $A : C \to (\ell_3)^*$ by

$$Ax = 2x + (1, 1, 1, 0, 0, 0, \dots),$$

with $(x_1, x_2, x_3, \dots) \in \ell_3(\mathbb{R})$. It is easy to show that A is monotone (hence, pseudo monotone). Take $Tx = \frac{x}{2}$, $\alpha_n = \frac{1}{100n+1}, \beta_n = \frac{3n+5}{7n+8}, \sigma = 0.14, \gamma = 0.4, s = 3, \lambda = 0.78$. The projections onto the feasibility set is carried out using optimization tool box in MATLAB. We carried out two numerical tests for approximating the common solution of the VIP and FPP using Algorithm 5.1.1. The initial value of x_1 and fixed u used are

Case I: $x_1 = (0.3241, 0.5387, -0.1256, 0, 0, 0, \dots)$ and $u = (-0.0988, 0.2679, 0.2890, 0, 0, 0, \dots)$

Case II: $x_1 = (-4.5289, -1.2345, 5.2238, 0, 0, 0 \dots)$ and $u = (1.3268, -5.3420, 3.2890, 0, 0, 0, \dots)$,

with stopping criterion $\frac{||x_{n+1}-x_n||_{\ell_3}}{||x_2-x_1||_{\ell_3}} < 10^{-7}$ in each case. The computational results obtain for these tests can be seen in Table 5.2 and Table 5.3.

Remark 5.1.23. The numerical experiments showed that the performance of the algorithm is essentially independent of the value of x_1 used in the computation.

Table 5.1: Comparison between Algorithm 5.1.1 and Algorithm 1.2.1 for Example 5.1.20.

		Algorithm 5.1.1	Algorithm 1.2.1
$m = 20$	CPU time (sec)	0.0065	0.0105
	No. of Iter.	23	38
$m = 50$	CPU time (sec)	0.0118	0.0178
	No. of Iter.	24	39
$m = 100$	CPU time (sec)	0.0189	0.0263
	No. of Iter.	25	40
$m = 200$	CPU time (sec)	0.0160	0.0306
	No. of Iter.	25	42

Table 5.2: Computation result for Example 5.1.22, Case I; Time: 0.1336sec.

Iter.	x_{n+1}	$\|\|x_{n+1} - x_n\|\|_{l_3}$
1	$(0.3241, 0.5387, -0.1256, 0, 0, 0, \dots)$	
2	$(0.4549, 1.0860, -0.4436, 0, 0, 0, \dots)$	0.5831
3	$(0.6304, 2.1364, -1.6952, 0, 0, 0, \dots)$	1.4617
4	$(0.3343, 1.3639, -2.1382, 0, 0, 0, \dots)$	0.1507
5	$(0.4774, 1.2958, -2.1483, 0, 0, 0, \dots)$	0.1481
10	$(0.8247, 1.2461, -2.1254, 0, 0, 0, \dots)$	0.0335
20	$(0.9056, 1.2781, -2.1054, 0, 0, 0, \dots)$	0.0015
30	$(0.9101, 1.2793, -2.1043, 0, 0, 0, \dots)$	0.0001
40	$(0.9104, 1.2794, -2.1042, 0, 0, 0, \dots)$	$9.6527\ e^{-6}$
50	$(0.9105, 1.2794, -2.1042, 0, 0, 0, \dots)$	$8.1868\ e^{-7}$
59	$(0.9105, 1.2794, -2.1042, 0, 0, 0, \dots)$	$8.8898\ e^{-8}$

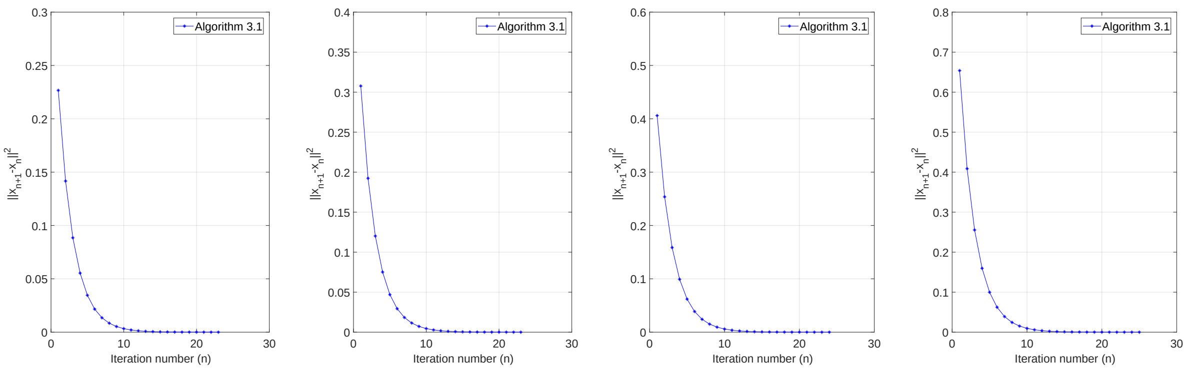


Figure 5.2: Example 5.1.21, Case I (CPU time: 1.4539sec); Case II (CPU time: 2.9472sec); Case III (CPU time: 2.7043sec); Case IV (CPU time: 2.9142sec).

Table 5.3: Computation result for Example 5.1.22, Case 2; Time: 0.2182sec.

Iter.	x_{n+1}	$\|\|x_{n+1} - x_n\|\|_{l_3}$
1	$(-4.5289, -1.2345, 5.2238, 0, 0, 0 \dots)$	
2	$(2.1415, -5.7883, 3.9968, 0, 0, 0, \dots)$	5.3096
3	$(2.8089, -5.6600, 3.4229, 0, 0, 0, \dots)$	2.0383
4	$(2.9175, -5.6352, 3.0466, 0, 0, 0, \dots)$	0.7875
5	$(2.9970, -5.5380, 3.0342, 0, 0, 0, \dots)$	0.3794
10	$(2.9923, -5.5568, 2.9463, 0, 0, 0, \dots)$	0.0333
20	$(2.9978, -5.5481, 2.9573, 0, 0, 0, \dots)$	0.0045
30	$(2.9985, -5.5470, 2.9588, 0, 0, 0, \dots)$	0.0006
40	$(2.9986, -5.5468, 2.9590, 0, 0, 0, \dots)$	0.0001
50	$(2.9986, -5.5470, 2.9573, 0, 0, 0, \dots)$	$1.1574\ e^{-5}$
60	$(2.9986, -5.5470, 2.9573, 0, 0, 0, \dots)$	$1.5821\ e^{-5}$
70	$(2.9986, -5.5470, 2.9573, 0, 0, 0, \dots)$	$2.1626\ e^{-7}$
74	$(2.9986, -5.5470, 2.9573, 0, 0, 0, \dots)$	$9.7559\ e^{-8}$

5.2 A Totally Relaxed, Self-Adaptive Subgradient Extragradient Method for Variational Inequality and Fixed Point Problems in a Banach Space

Recently, Chidume and Nnakwe [80] extends the subgradient extragradient method (3.2.1) to a 2-uniformly convex and uniformly smooth Banach space E. This extension was presented as follow:

$$\begin{cases} x_0 \in E, \ \beta > 0, \\ y_k = \Pi_C J^{-1}(Jx_k - \beta A x_k), \\ Q_k = \{x \in E : \langle x - y_k, Jx_k - \beta A x_k - J y_k \rangle \leq 0\}, \\ x_{k+1} = \Pi_{Q_k} J^{-1}(Jx_k - \beta A y_k), \qquad \forall\ k \geq 0, \end{cases} \tag{5.2.1}$$

where Π_C is the generalized projection from E onto C and J is the normalized duality mapping from E to 2^{E^*}. They proved the weak convergence of algorithm (5.2.1) to a solution of VIP (1.1.8). In order to obtain strong convergence of the subgradient extragradient method in Banach space, Ying Liu [164] combined the Halpern method [117] with (5.2.1) and introduced the following scheme:

$$\begin{cases} y_k = \Pi_C J^{-1}(Jx_k - \beta_k A x_k), \\ Q_k = \{w \in E : \langle w - y_k, Jx_k - \beta_k A x_k - J y_k \rangle \leq 0\}, \\ w_k = \Pi_{Q_k} J^{-1}(Jx_k - \beta_k A y_k), \\ x_{k+1} = J^{-1}(\alpha_k J x_0 + (1 - \alpha_k) J w_k), \qquad k \geq 0, \end{cases} \tag{5.2.2}$$

where $\{\alpha_k\} \subset [0, 1]$ satisfying $\lim\limits_{k\to\infty} \alpha_k = 0$ and $\sum\limits_{k=1}^{\infty} \alpha_k = \infty$, and $\{\beta_k\} \subset (0, \infty)$.

However, we note the following problems:

P1. Although, the subgradient extragradient algorithms (5.2.1), (5.2.2) improved the extragradient method (1.1.2), but they still preserved some of the weakness of the extragradient method since there is need to calculate one projection onto C, that is y_k, per each iteration;

P2. The stepsize β, β_k of the subgradient extragradient algorithms (5.2.1) and (5.2.2) respectively require the condition

$$\beta, \beta_k \in \left(0, \frac{c_1}{L}\right), \tag{5.2.3}$$

to be satisfied, which require at least a prior estimate of the Lipschitz constant L, where c_1 is the 2-uniformly convexity constant of E. In practice, it is too difficult to approximate the Lipschitz constant L.

As an attempt to solve Problem P1, He and Wu [123] introduced the *Relaxed Subgradient Extragradient Method (RSEM)* in a Hilbert space. This RSEM is presented as follows: Suppose C has the form $C := \{x \in H : c(x) \leq 0\}$ where $c : H \to \mathbb{R}$ is an approximate convex and lower semicontinuous function. Choose an arbitrary starting point $x_0 \in H$, given the current iterate x_k, calculate the next iteration x_{k+1} via

$$\begin{aligned} y_k &= P_{C_k}(x_k - \beta_k A x_k), \\ x_{k+1} &= P_{Q_k}(x_k - \beta_k A y_k), \end{aligned} \tag{5.2.4}$$

where C_k and Q_k are given by

$$\begin{aligned} C_k &= \{w \in H : c(x_k) + \langle \nabla c(x_k), w - x_k \rangle \leq 0\}, \\ a_k &= x_k - \beta_k A x_k - y_k, \end{aligned}$$

$$Q_k = \begin{cases} \{w \in H : \langle a_k, w - y_k \rangle \leq 0\}, & if \quad a_k \neq 0, \\ H, & if a_k = 0. \end{cases} \tag{5.2.5}$$

Motivated by the work of He and Wu [123] and the fact that in real-world application in which the feasible set of the VIP (1.1.8) might has a compex structure, He et al. [124] modified the RSEM (5.2.4) and introduced a *Totally Relaxed and Self adaptive Subgradient Extragradient Method (TRSSEM)* for solving the VIP (1.1.8) in a Hilbert space. Let $C^i := \{x \in H : h_i(x) \leq 0\}$, where $h_i : H \to \mathbb{R}$ for all $i \in I = \{1, 2, \ldots, m\}$ are convex functions. In the TRSSEM, the feasible set is defined as

$$C := \bigcap_{i=1}^{m} C^i.$$

Motivated by the TRSSEM of He et al. [124], Chidume and Nnakwe [80] and Ying Liu [164], in this section, we propose a new TRSSEM with Halpern iteration for approximating a common solution of VIP (1.1.8) and fixed point of quasi-nonexpansive mapping in a 2-uniformly convex and uniformly smooth Banach space. Comparing with the existing subgradient extragradient algorithms for solving VIP (1.1.1) in Banach space, the following are the advantages of the algorithm presented in this section:

(a) the simplicity of calculating the projection onto C and Q_k make our algorithm attractive for computation;

(b) the introduction of an Armijo line search rule which makes the stepsize not to depend on the Lipschitz constant makes our algorithm simple and easy for computation;

(c) the strong convergence guaranteed by our algorithm makes it a good candidate scheme for finding common solution of VIP (1.1.1) and fixed point of quasi-nonexpansive mapping.

5.2.1 Main results

In this subsection, we give a precise statement of our algorithm and discuss its strong convergence.

Let E be a 2-uniformly convex and uniformly smooth Banach space and C be a nonempty, closed convex subset of E. For $i = 1, 2, \dots, m$, let $h_i : E \to \mathbb{R}$ be family of convex, weakly lower semicontinuous and Gâteaux differentiable functions. Let $A : C \to E^*$ be a monotone operator which is uniformly continuous on bounded subsets of C and $S : C \to C$ be a quasi-nonexpansive mapping such that $Sol := \Omega_{VIP} \cap F(S)$ is nonempty. Let $\{\alpha_k\}$ and $\{\nu_k\}$ be nonnegative real sequences in $(0, 1)$ and $I = \{1, 2, \dots, m\}$.

Algorithm 5.2.1.

Step 1: (Initialization) Pick $x_1 \in E$, $\eta, \rho \in (0, 1)$ *and set* $k = 1$.

Step 2: Given the current iterate x_k*, construct the family of half-spaces*

$$C_k^i := \{w \in E : h_i(x_k) + \langle h_i'(x_k), w - x_k\rangle \le 0\}, \qquad i = 1, 2, .., m, \tag{5.2.6}$$

and set

$$C_k := \bigcap_{i=1}^{m} C_k^i, \tag{5.2.7}$$

then compute

$$y_k = \Pi_{C_k} J^{-1}(Jx_k - \beta_k A(x_k)), \tag{5.2.8}$$

where

$$\beta_k = \rho^{l_k}, \tag{5.2.9}$$

and l_k *is the smallest nonnegative integer such that*

$$\beta_k ||A(x_k) - A(y_k)|| \le \eta ||x_k - y_k||. \tag{5.2.10}$$

Step 3: If $x_k = y_k$ *(i.e.,* $x_k \in \Omega_{VIP}$*), then set* $x_k = w_k$ *and go to Stop 4. Otherwise, compute the next iterate by*

$$w_k = \Pi_{Q_k} J^{-1}(Jx_k - \beta_k A(y_k)), \tag{5.2.11}$$

where

$$Q_k = \{w \in E : \langle w - y_k, Jx_k - \beta_k A(x_k) - Jy_k\rangle \le 0\}. \tag{5.2.12}$$

Step 4: Compute

$$x_{k+1} = J^{-1}(\alpha_k Jx_1 + (1-\alpha_k)((1-\nu_k)Jw_k + \nu_k JSw_k)). \tag{5.2.13}$$

Set $k := k+1$ and go to Step 2.

Assumption 5.2.2. *For the convergence of Algorithm 5.2.1, we make the following assumption.*

(C1) The feasible set C is defined by

$$C := \bigcap_{i=1}^{m} C^i, \tag{5.2.14}$$

where $C^i := \{x \in E : h_i(x) \leq 0\}$;

(C2) $\lim_{k\to\infty} \alpha_k = 0$ *and* $\sum_{k=1}^{\infty} \alpha_i = \infty$;

(C3) $0 < \liminf_{k\to\infty} \nu_k \leq \limsup_{k\to\infty} \nu_k < 1$.

Remark 5.2.3. From (5.2.7) and (5.2.14), it is easy to see that $C \subset C_k$. Indeed, for each $i \in I$ and $x \in C^i$, we have by the subdifferential inequality that

$$h_i(x_k) + \langle h_i'(x_k), x - x_k\rangle \leq h_i(x) \leq 0.$$

By the definition of C_k^i in (5.2.6), we have that $x \in C_k^i$. Hence $C^i \subset C_k^i$ for all $i \in I$ and therefore $C \subset C_k$ for all $k \geq 1$.

Lemma 5.2.4. *If $x_k = y_k$ for some $k \geq 0$ in Algorithm 5.2.1 happened, then $x_k \in \Omega_{VIP}$.*

Proof. If $x_k = y_k$, then $x_k = \Pi_{C_k} J^{-1}(Jx_k - \beta_k Ax_k)$. We first show that $x_k \in C_k$, that is, $x_k \in C_k^i$ for each $i \in I$. By the definitions of C_k^i, we have $h_i(x_k) + \langle h_i'(x_k), x_k - x_k\rangle \leq 0$. So $h_i(x_k) \leq 0$ for each $i \in I$. This means that $x_k \in C$.

By the variational characterization of the generalized projection Π_C onto C, we have

$$\langle x_k - w, Jx_k - \beta_k Ax_k - Jx_k\rangle \geq 0, \qquad \forall\, w \in C.$$

This implies that

$$\beta_k \langle Ax_k, w - x_k\rangle \geq 0, \qquad \forall\, w \in C.$$

Since $\beta_k > 0$, we have $x_k \in \Omega_{VIP}$. ◻

Remark 5.2.5. Note that if $x_k = y_k$ and $x_k = Sx_k$, we are at a common solution of the VIP (1.1.1) and fixed point of S. In our convergence analysis, we will implicitly assume that this does not occur after finitely many iterations so that our Algorithm 5.2.1 generates an infinite sequence satisfying, in particular, $x_k - y_k \neq 0$ and $x_k - Sx_k \neq 0$ for all $k \in \mathbb{N}$.

We will see in the following result that the Armijo line search rule define in (5.2.10) is well defined.

Lemma 5.2.6. *There exists a nonnegative integer l_k satisfying* (5.2.10).

Proof. Let $r_{\beta_k}(x_k) := x_k - \Pi_{C_k} J^{-1}(Jx_k - \beta_k Ax_k)$ and suppose $r_{k_0}(x_k) = 0$ for some $k_0 \geq 1$. Take $l_k = k_0$ which satisfies (5.2.10). Suppose $r_{\rho^{k_1}}(x_k) \neq 0$ for some $k_1 \geq 1$ and assume the contrary, that is,

$$\rho^l ||Ax_k - A(\Pi_{C_k} J^{-1}(Jx_k - \rho^l Ax_k))|| > \eta ||x_k - \Pi_{C_k} J^{-1}(Jx_k - \rho^l Ax_k)||.$$

Then by Lemma 2.6.11 and the fact that $\rho \in (0,1)$, we obtain

$$\begin{aligned} ||Ax_k - A(\Pi_{C_k} J^{-1}(Jx_k - \rho^l Ax_k))|| &> \frac{\eta}{\rho^l} ||r_{\rho^l}(x_k)|| \\ &\geq \frac{\eta}{\rho^l} \min\{1, \rho^l\} ||r_1(x_k)|| \\ &= \eta ||r_1(x_k)||. \end{aligned} \tag{5.2.15}$$

Using the fact that J and Π_{C_k} are continuous, we have

$$\Pi_{C_k} J^{-1}(Jx_k - \rho^l Ax_k) \to \Pi_{C_k} x_k, \quad l \to \infty.$$

We now consider two cases; namely, when $x_k \in C$ and when $x_k \notin C$.

(i) If $x_k \in C$, then $x_k \in C_k$ and so $x_k = \Pi_{C_k} x_k$. Now, since $r_{\rho^{k_1}} \neq 0$ and $\rho^{k_1} \leq 1$, it follows from Lemma 2.6.11 that

$$\begin{aligned} 0 &< ||r_{\rho^{k_1}}(x_k)|| \leq \max\{1, \rho^{k_1}\} ||r_1(x_k)|| \\ &= ||r_1(x_k)||. \end{aligned}$$

Letting $l \to \infty$ in (5.2.15), we have that

$$0 = ||Ax_k - Ax_k|| \geq \eta ||r_1(x_k)|| > 0.$$

This is a contradiction and so, (5.2.10) is valid.

(ii) $x_k \notin C$, then

$$\rho^l ||Ax_k - y_k|| \to 0, \quad l \to \infty,$$

while

$$\lim_{l \to \infty} \eta ||r_{\rho^l}(x_k)|| = \lim_{l \to \infty} \eta ||x_k - \Pi_C J^{-1}(Jx_k - \rho^l Ax_k)|| = \eta ||x_k - \Pi_{C_k}(x_k)|| > 0.$$

This is a contradiction. Therefore, the stepsize rule in (5.2.10) is well defined.

□

Remark 5.2.7. We note that if A is L-Lipschitz continuous on E, then $\sup_{k \geq 1} l_k < \infty$. Indeed, for all $x, y \in E$, we have that $\rho^l ||Ax - Ay|| \leq \rho^l L ||x - y||$ and it suffices to take l such that $\rho^l \leq \frac{\eta}{L}$. This does not depend on x and y. Also, note that $\sup_{k \geq 1} l_k < \infty$ implies that $\inf_{k \geq 1} \beta_k > 0$. This is important for our convergence analysis.

We proceed to prove the following lemmas before proving the convergence of our main Algorithm 5.2.1.

Lemma 5.2.8. *The sequence $\{x_k\}$ generated by Algorithm 5.2.1 is bounded.*

Proof. Let $x^* \in Sol$, then we have from Lemma 2.5.25(a) that

$$\begin{aligned}
\phi(x^*, w_k) &= \phi(x^*, \Pi_{Q_k} J^{-1}(Jx_k - \beta_k Ay_k)) \\
&\leq \phi(x^*, J^{-1}(Jx_k - \beta_k Ay_k)) - \phi(w_k, J^{-1}(Jx_k - \beta_k Ay_k)) \\
&= ||x^*||^2 - 2\langle x^*, Jx_k - \beta_k Ay_k\rangle - ||w_k||^2 + 2\langle w_k, Jx_k - \beta_k Ay_k\rangle \\
&= \phi(x^*, x_k) - \phi(w_k, x_k) + 2\beta_k\langle x^* - w_k, Ay_k\rangle \\
&= \phi(x^*, x_k) - \phi(w_k, x_k) + 2\beta_k[\langle x^* - y_k, Ay_k - Ax^*\rangle + \\
&\quad \langle x^* - y_k, Ax^*\rangle + \langle y_k - w_k, Ay_k\rangle] \\
&\leq \phi(x^*, x_k) - \phi(w_k, x_k) + 2\beta_k[\langle x^* - y_k, Ax^*\rangle + \langle y_k - w_k, Ay_k\rangle] \\
&= \phi(x^*, x_k) - \phi(w_k, x_k) - \phi(y_k, x_k) + 2\langle w_k - y_k, Jx_k - Jy_k\rangle \qquad (5.2.16) \\
&\quad +2\beta_k[\langle x^* - y_k, Ax^*\rangle + \langle y_k - w_k, Ay_k\rangle] \\
&= \phi(x^*, x_k) - \phi(w_k, y_k) - \phi(y_k, x_k) \\
&\quad +2\langle w_k - y_k, Jx_k - \beta_k Ay_k - Jy_k\rangle, \qquad (5.2.17)
\end{aligned}$$

where the inequality in (5.2.16) follows from property D2 of Proposition 2.5.3. By the definition of Q_k and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
2\langle w_k - y_k, Jx_k - \beta_k Ay_k - Jy_k\rangle &= 2\langle w_k - y_k, Jx_k - \beta_k Ax_k - Jy_k\rangle + \\
&\quad 2\beta_k\langle w_k - y_k, Ax_k - Ay_k\rangle \\
&\leq 2\beta_k||w_k - y_k||||Ax_k - Ay_k||. \qquad (5.2.18)
\end{aligned}$$

Using (5.2.10) and Lemma 2.5.24 in (5.2.18), we have

$$\begin{aligned}
2\langle w_k - y_k, Jx_k - \beta_k Ay_k - Jy_k\rangle &\leq 2\eta||w_k - y_k||||y_k - x_k|| \\
&\leq 2\eta\sqrt{\frac{\phi(w_k, y_k)}{c_1}}\sqrt{\frac{\phi(y_k, x_k)}{c_1}} \\
&\leq \frac{\eta}{c_1}(\phi(w_k, y_k) + \phi(y_k, x_k)). \qquad (5.2.19)
\end{aligned}$$

Therefore from (5.2.17) and (5.2.19), we have

$$\phi(x^*, w_k) \leq \phi(x^*, x_k) - \left(1 - \frac{\eta}{c_1}\right)(\phi(w_k, y_k) + \phi(y_k, x_k)). \qquad (5.2.20)$$

From (5.2.13) and using property D3 of Proposition 2.5.3, we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\alpha_k Jx_1 + (1-\alpha_k)((1-\nu_k)Jw_k + \nu_k JSw_k))) \\
&= \phi(x^*, J^{-1}(\alpha_k Jx_1 + (1-\alpha_k)(1-\nu_k)Jw_k + (1-\alpha_k)\nu_k JSw_k)) \\
&\leq \alpha_k\phi(x^*, x_1) + (1-\alpha_k)(1-\nu_k)\phi(x^*, w_k) + (1-\alpha_k)\nu_k\phi(x^*, Sw_k) \\
&\leq \alpha_k\phi(x^*, x_1) + (1-\alpha_k)\phi(x^*, w_k). \qquad (5.2.21)
\end{aligned}$$

It follows from (5.2.20) and (5.2.21) that

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq \alpha_k \phi(x^*, x_1) + (1-\alpha_k)\phi(x^*, x_k) \\ &\leq \max\{\phi(x^*, x_1), \phi(x^*, x_k)\} \\ &\vdots \\ &\leq \max\{\phi(x^*, x_1), \phi(x^*, x_1)\} = \phi(x^*, x_1). \end{aligned} \tag{5.2.22}$$

This implies that $\{\phi(x^*, x_k)\}$ is bounded. Therefore $\{x_k\}$ is bounded. Consequently, since A is uniformly continuous on bounded subsets of C, then $\{Aw_k\}$ is bounded and by the nonexpansiveness of the projection operator and mapping S, the sequence $\{y_k\}$, $\{w_k\}$ and $\{Sw_k\}$ are bounded. □

Note that within the proofs of our subsequent results, we define some auxiliary sequences whose boundedness is stated without explicit proof, but that the corresponding proofs are more or less the same as the proof given in Lemma 5.2.8

Lemma 5.2.9. *Let $\{x_k\}$ and $\{y_k\}$ be two sequences generated by Algorithm 5.2.1 and suppose that $||x_k - y_k|| \to 0$, $k \to \infty$. Let $p \in C$ denotes the weak limit of the subsequence $\{x_{k_j}\}$ of the sequence $\{x_k\}$ for $j \in \mathbb{N}$. Then $p \in \Omega_{VIP}$.*

Proof. For all $x \in C$, using Lemma 2.5.5(b) and by the monotonicity of A, we have

$$\begin{aligned} 0 &\leq \langle x - y_{k_j}, Jy_{k_j} - Jx_{k_j} + \beta_{k_j} Ax_{k_j}\rangle \\ &= \langle x - y_{k_j}, Jy_{k_j} - Jx_{k_j}\rangle + \beta_{k_j}\langle x - x_{k_j}, Ax_{k_j}\rangle + \beta_{k_j}\langle x_{k_j} - y_{k_j}, Ax_{k_j}\rangle \\ &\leq \langle x - x_{k_j}, Jy_{k_j} - Jx_{k_j}\rangle + \beta_{k_j}\langle x - x_{k_j}, Ax_{k_j}\rangle + \beta_{k_j}\langle x_{k_j} - y_{k_j}, Ax_{k_j}\rangle. \end{aligned} \tag{5.2.23}$$

Passing limit to the inequality in (5.2.23), we have

$$\langle Ap, x - p\rangle \geq 0, \qquad \forall\, x \in C. \tag{5.2.24}$$

Therefore $p \in \Omega_{VIP}$. □

Lemma 5.2.10. *The sequence $\{x_k\}$ generated by Algorithm 5.2.1 satisfies the following estimates:*

(i) $t_{k+1} \leq (1-\alpha_k)t_k + \alpha_k b_k$,

(ii) $-1 \leq \limsup_{k\to\infty} b_k < +\infty$,

where $t_k = \phi(x^*, x_k)$, $b_k = \langle Jx_1 - Jx^*, x_{k+1} - x^*\rangle$ *and* $x^* = \Pi_{Sol}x_1$.

Proof. Let $z_k = J^{-1}((1-\nu_k)Jw_k + \nu_k JSw_k)$, then from Lemma 2.5.6, we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, \alpha_k Jx_1 + (1-\alpha_k)Jz_k) \\
&\leq V(x^*, \alpha_k Jx_1 + (1-\alpha_k)Jz_k - \alpha_k(Jx_1 - Jx^*)) \\
&\quad -2\langle -\alpha_k(Jx_1 - Jx^*), J^{-1}(\alpha_k Jx_1 + (1-\alpha_k)Jz_k) - x^*\rangle \\
&= V(x^*, \alpha_k Jx^* + (1-\alpha_k)Jz_k) + 2\alpha_k\langle Jx_1 - Jx^*, x_{k+1} - x^*\rangle \\
&\leq \alpha_k\phi(x^*, x^*) + (1-\alpha_k)\phi(x^*, z_k) + 2\alpha_k\langle Jx_1 - Jx^*, x_{k+1} - x^*\rangle \\
&= (1-\alpha_k)\phi(x^*, z_k) + 2\alpha_k\langle Jx_1 - Jx^*, x_{k+1} - x^*\rangle \\
&\leq (1-\alpha_k)(1-\nu_k)\phi(x^*, w_k) + (1-\alpha_k)\nu_k\phi(x^*, Sw_k) \\
&\quad +2\alpha_k\langle Jx_1 - Jx^*, x_{k+1} - x^*\rangle \\
&\leq (1-\alpha_k)\phi(x^*, w_k) + 2\alpha_k\langle Jx_1 - Jx^*, x_{k+1} - x^*\rangle \\
&\leq (1-\alpha_k)\phi(x^*, x_k) + 2\alpha_k\langle Jx_1 - Jx^*, x_{k+1} - x^*\rangle.
\end{aligned}$$

This established (i). Next we prove (ii). Since $\{x_k\}$ is bounded, then we have

$$\sup_{k\geq 0} b_k \leq \sup 2||Jx_1 - Jx^*||||x_{k+1} - x^*|| < \infty.$$

This implies that $\limsup_{k\to\infty} b_k < \infty$. Next, we show that $\limsup_{k\to\infty} b_k \geq -1$. Assume the contrary, i.e., $\limsup_{k\to\infty} b_k < -1$. Then there exists $k_0 \in \mathbb{N}$ such that $b_k < -1$, for all $k \geq k_0$. Then for all $k \geq k_0$, we get from (i) that

$$\begin{aligned}
t_{k+1} &\leq (1-\alpha_k)t_k + \alpha_k b_k \\
&< (1-\alpha_k)t_k - \alpha_k \\
&= t_k - \alpha_k(t_k + 1) \leq t_k - \alpha_k.
\end{aligned}$$

Taking lim sup of both sides of the last inequality, we have

$$\limsup_{k\to\infty} t_k \leq t_{k_0} - \lim_{k\to\infty}\sum_{i=k_0}^{k}\alpha_i = -\infty.$$

This contradicts the fact that $\{t_k\}$ is a nonnegative real sequence. Therefore $\limsup_{k\to\infty} b_k \geq -1$. $\square$

We now presents our main theorem.

Theorem 5.2.11. *Let C be a nonempty, closed convex subset of a 2-uniformly convex and uniformly smooth real Banach space E and $h_i : E \to \mathbb{R}$ be family of convex, weakly lower semicontinuous and Gâteaux differentiable functions, $i = 1, 2, \dots, m$. Let $A : C \to E^*$ be a monotone operator which is uniformly continuous on bounded subsets of C, $S : C \to C$ be a quasi-nonexpansive mapping and let $\{\alpha_k\}$ and $\{\nu_k\}$ be nonnegative real sequences in $(0, 1)$. Suppose $Sol = \Omega_{VIP} \cap F(S)$ is nonempty and Assumption 5.2.2 is satisfied, then the sequence $\{x_k\}$ generated by Algorithm 5.2.1 converges strongly to a unique solution $p = \Pi_{Sol}x_1$.*

Proof. Let $x^* \in Sol$, we divide the proof into two cases.

Case I: Suppose that there exists $k_0 \in \mathbb{N}$ such that $\{\phi(x^*, x_k)\}$ is non-increasing. Since $\{\phi(x^*, x_k)\}$ is bounded, then it is convergent and so

$$\phi(x^*, x_k) - \phi(x^*, x_{k+1}) \to 0, \quad n \to \infty. \tag{5.2.25}$$

Since $z_k = J^{-1}((1-\nu_k)Jw_k + v_k JSw_k)$, then we have from Lemma 2.6.8 that

$$\begin{aligned}
\phi(x^*, z_k) &= \phi(x^*, J^{-1}((1-\nu_k)Jw_k + \nu_k JSw_k)) \\
&= V(x^*, (1-\nu_k)Jw_k + \nu_k JSw_k) \\
&= ||x^*||^2 - 2\langle x^*, (1-\nu_k)Jw_k + \nu_k JSw_k\rangle + ||(1-\nu_k)Jw_k + \nu_k JSw_k||^2 \\
&\leq ||x^*||^2 - 2(1-\nu_k)\langle x^*, Jw_k\rangle - 2\nu_k\langle x^*, JSw_k\rangle + (1-\nu_k)||Jw_k||^2 \\
&\quad +\nu_k||JSw_k||^2 - g(||Jw_k - JSw_k||) \\
&= (1-\nu_k)\phi(x^*, w_k) + \nu_k\phi(x^*, Sw_k) - \nu_k(1-\nu_k)g(||Jw_k - JSw_k||) \\
&\leq \phi(x^*, w_k) - \nu_k(1-\nu_k)g(||Jw_k - JSw_k||).
\end{aligned} \tag{5.2.26}$$

Therefore from (5.2.13), (5.2.20) and (5.2.26), we have

$$\begin{aligned}
\phi(x^*, x_{k+1}) &= \phi(x^*, J^{-1}(\alpha_k Jx_1 + (1-\alpha_k)Jz_k)) \\
&\leq \alpha_k\phi(x^*, x_1) + (1-\alpha_k)\phi(x^*, z_k) \\
&\leq \alpha_k\phi(x^*, x_1) + (1-\alpha_k)\phi(x^*, w_k) - (1-\alpha_k)\nu_k(1-\nu_k)g(||Jw_k - JSw_k||) \\
&\leq \alpha_k\phi(x^*, x_1) + (1-\alpha_k)\phi(x^*, x_k) - (1-\alpha_k)\nu_k(1-\nu_k)g(||Jw_k - JSw_k||).
\end{aligned}$$

Hence

$$(1-\alpha_k)\nu_k(1-\nu_k)g(||Jw_k - JSw_k||) \leq \alpha_k\phi(x^*, x_1) + (1-\alpha_k)\phi(x^*, x_k) - \phi(x^*, x_{k+1}).$$

Using the fact that $\alpha_k \to 0$ and (5.2.25), we have

$$\nu_k(1-\nu_k)g(||Jw_k - JSw_k||) \to 0, \quad n \to \infty.$$

Therefore by condition (C3) and the property of g, we get

$$\lim_{k\to\infty} ||Jw_k - JSw_k|| = 0. \tag{5.2.27}$$

Since J^{-1} is norm-to-norm continuous on bounded subsets of E, then

$$\lim_{k\to\infty} ||w_k - Sw_k|| = 0. \tag{5.2.28}$$

Furthermore from (5.2.20), we have

$$\phi(x^*, w_k) \leq \phi(x^*, x_k) - \left(1 - \frac{\eta}{c_1}\right)(\phi(w_k, y_k) + \phi(y_k, x_k)),$$

therefore, it follows from (5.2.26) that

$$\begin{aligned}
\phi(x^*, x_{k+1}) &\leq \alpha_k\phi(x^*, x_1) + (1-\alpha_k)\phi(x^*, z_k) \\
&\leq \alpha_k\phi(x^*, x_1) + (1-\alpha_k)\phi(x^*, x_k) - (1-\alpha_k)\left(1 - \frac{\eta}{c_1}\right)(\phi(w_k, y_k) + \phi(y_k, x_k)).
\end{aligned}$$

This implies that

$$(1-\alpha_k)\left(1-\frac{\eta}{c_1}\right)(\phi(w_k,y_k)+\phi(y_k,x_k)) \leq \alpha_k\phi(x^*,x_1)+(1-\alpha_k)\phi(x^*,x_k)-\phi(x^*,x_{k+1}).$$

Similarly, since $\alpha_k \to 0$ and from (5.2.25), we have

$$\phi(w_k,y_k)+\phi(y_k,x_k)\to 0, \quad n\to\infty,$$

which means that

$$\lim_{k\to\infty}\phi(w_k,y_k)=\lim_{k\to\infty}\phi(y_k,x_k)=0. \tag{5.2.29}$$

Since $\{x_k\}$, $\{y_k\}$ and $\{w_k\}$ are bounded, then it follows from Lemma 2.6.7 that

$$\lim_{k\to\infty}||w_k-y_k||=\lim_{k\to\infty}||y_k-x_k||=0. \tag{5.2.30}$$

Also, it is easy to see from (5.2.28) that

$$\phi(w_k,z_k)=\phi(w_k,J^{-1}((1-\nu_k)Jw_k+\nu_kJSw_k))\to 0,\ n\to\infty.$$

Hence, by Lemma 2.6.7 we have

$$\lim_{k\to\infty}||w_k-z_k||=0. \tag{5.2.31}$$

Furthermore

$$||Jx_{k+1}-Jz_k||=\alpha_k||Jx_1-Jz_k||\to 0, \quad n\to\infty.$$

Since J^{-1} is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$||x_{k+1}-z_k||\to 0,\ n\to\infty.$$

Therefore

$$||x_{k+1}-x_k||\leq||x_{k+1}-z_k||+||z_k-w_k||+||w_k-x_k||\to 0,\ n\to\infty. \tag{5.2.32}$$

Now let $\Omega_w(x_k)$ denotes the set of all weak cluster point of $\{x_k\}$, since $||x_k-y_k||\to 0$ as $k\to\infty$, it follows from Lemma 5.2.13 that $\Omega_w(x_k)\subset\Omega_{VIP}$. Also, since $||w_k-Sw_k||\to 0$ and $||w_k-x_k||\to 0$ as $k\to\infty$, then we have that $\Omega_w(x_k)\subset\hat{F}(S)=F(S)$. Therefore $\Omega_w(x_k)\subset Sol:=\Omega_{VIP}\bigcap F(S)$.

We now show that the sequence $\{x_k\}$ converges strongly to a point $p=\Pi_{Sol}x_1$. Let $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$ such that $x_{k_j}\rightharpoonup\bar{x}$ and

$$\limsup_{k\to\infty}\langle Jx_1-Jp,x_{k+1}-p\rangle=\lim_{j\to\infty}\langle Jx_1-Jp,x_{k_j+1}-p\rangle.$$

Since $||x_{k+1}-x_k||\to 0$ as $k\to\infty$, we have from Lemma 2.5.5 (b) that

$$\begin{aligned}\limsup_{k\to\infty}\langle Jx_1-Jp,x_{k+1}-p\rangle &= \lim_{j\to\infty}\langle Jx_1-Jp,x_{k_j+1}-p\rangle\\ &= \langle Jx_1-Jp,\bar{x}-p\rangle\leq 0.\end{aligned} \tag{5.2.33}$$

It follows from Lemma 2.6.30, Lemma 5.2.10(i) and (5.2.33) that $\phi(p, x_k) \to 0$ as $k \to \infty$. Therefore by Lemma 2.6.7,

$$\lim_{k\to\infty} ||p - x_k|| = 0.$$

This implies that $\{x_k\}$ converges strongly to $p = \Pi_{Sol} x_1$.

Case II: Suppose there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that

$$\phi(x^*, x_{k_j+1}) > \phi(x^*, x_{k_j}) \quad \forall\ k \in \mathbb{N}.$$

From Lemma 2.6.34, there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following inequality hold for all $k \in \mathbb{N}$:

$$\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_k+1}) \quad \text{and} \quad \phi(x^*, x_k) \leq \phi(x^*, x_{m_k+1}). \tag{5.2.34}$$

Note that from (5.2.20) and (5.2.21), we have

$$\begin{aligned}
\phi(x^*, x_{m_k}) &\leq \phi(x^*, x_{m_k+1}) \leq \alpha_{m_k}\phi(x^*, x_1) + (1 - \alpha_{m_k})\phi(x^*, w_{m_k}) \\
&\leq \alpha_{m_k}\phi(x^*, x_1) + (1 - \alpha_{m_k})\phi(x^*, x_{m_k}) \\
&\quad -(1 - \alpha_{m_k})\left(1 - \frac{\eta}{c_1}\right)(\phi(w_{m_k}, y_{m_k}) + \phi(y_{m_k}, x_{m_k})).
\end{aligned}$$

Since $\alpha_{m_k} \to 0$, as $k \to \infty$, it follows that

$$\left(1 - \frac{\eta}{c_1}\right)(\phi(w_{m_k}, y_{m_k}) + \phi(y_{m_k}, x_{m_k})) \to 0, \quad n \to \infty,$$

hence

$$\lim_{k\to\infty} \phi(w_{m_k}, y_{m_k}) = \lim_{k\to\infty} \phi(y_{m_k}, x_{m_k}) = 0.$$

Since $\{x_{m_k}\}, \{y_{m_k}\}$ and $\{w_{m_k}\}$ are bounded, we have from Lemma 2.6.7 that

$$\lim_{k\to\infty} ||x_{m_k} - y_{m_k}|| = \lim_{k\to\infty} ||w_k - y_{m_k}|| = 0. \tag{5.2.35}$$

Following similar method as in Case I, we have

$$\lim_{k\to\infty} ||w_{m_k} - Sw_{m_k}|| = \lim_{k\to\infty} ||x_{m_k+1} - x_{m_k}|| = 0. \tag{5.2.36}$$

By Lemma 5.2.9 and (5.2.36), we have that $\Omega_w(x_{m_k}) \subset Sol := \Omega_{VIP} \bigcap F(S)$, where $\Omega_w(x_{m_k})$ is the set of all weak subsequential limit of $\{x_{m_k}\}$.

Since $\{x_{m_k}\}$ is bounded, we can choose a subsequence of $\{x_{m_k}\}$ still denoted by $\{x_{m_k}\}$ such that $x_{m_k} \rightharpoonup q$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \langle Jx_1 - Jx^*, x_{m_k+1} - x^* \rangle = \lim_{k\to\infty} \langle Jx_1 - Jx^*, x_{m_k+1} - x^* \rangle.$$

Hence, from Lemma 2.5.5(b), we have

$$\limsup_{k\to\infty} \langle Jx_1 - Jx^*, x_{m_k+1} - x^* \rangle = \langle Jx_1 - Jx^*, q - x^* \rangle \leq 0. \tag{5.2.37}$$

From (5.2.34), we have

$$\begin{aligned} 0 &\leq \phi(x^*, x_{m_k+1}) - \phi(x^*, x_{m_k}) \\ &\leq (1-\alpha_{m_k})\phi(x^*, x_{m_k}) + 2\alpha_{m_k}\langle Jx_1 - Jx^*, x_{m_k+1} - x^*\rangle - \phi(x^*, x_{m_k}). \end{aligned}$$

Since $\alpha_{m_k} > 0$, we have that

$$\phi(x^*, x_{m_k}) \leq 2\langle Jx_1 - Jx^*, x_{m_k+1} - x^*\rangle.$$

Hence by (5.2.37), we have

$$\phi(x^*, x_{m_k}) \to 0, \quad n \to \infty,$$

and by Lemma 2.6.7, we have that $\lim_{k\to\infty} ||x_{m_k} - x^*|| = 0$. Consequently, we obtain

$$||x_k - x^*|| \to 0, \quad n \to \infty. \tag{5.2.38}$$

Therefore, the sequence $\{x_k\}$ converges strongly to $x^* = \Pi_{Sol}x_1$. This completes the proof. □

The following result can be obtained as a direct consequence of Theorem 5.2.11.

Theorem 5.2.12. *Let H be a real Hilbert space and C be a nonempty, closed convex subset H. Let $h_i : H \to \mathbb{R}$ be families of convex, weakly lower semicontinuous and Gâteaux differentiable functions, $i = 1, 2, \ldots, m$ let $A : C \to H$ be a monotone operator which is uniformly continuous on bounded subsets of C, $S : C \to C$ be a quasi-nonexpansive mapping and let $\{\alpha_k\}$ and $\{\nu_k\}$ be nonnegative real sequences in $(0,1)$. Suppose $Sol = \Omega_{VIP} \cap F(S)$ is nonempty and Assumption 5.2.2 is satisfied, then the sequence $\{x_k\}$ generated by the following Algorithm 5.2.13 converges strongly to a unique solution $p = P_{Sol}x_1$, where $P_{Sol}x_1$ is the metric projection onto Sol.*

Algorithm 5.2.13.

Step 1: (Initialization) Pick $x_1 \in H$, $\eta, \rho \in (0,1)$ and set $k = 1$.

Step 2: Given the current iterate x_k, construct the family of half-spaces

$$C_k^i := \{w \in E : h_i(x_k) + \langle h_i'(x_k), w - x_k\rangle \leq 0\}, \qquad i \in I,$$

and set

$$C_k := \bigcap_{i \in I} C_k^i,$$

then compute

$$y_k = P_{C_k}(x_k - \beta_k A(x_k)),$$

where

$$\beta_k = \rho^{l_k},$$

and l_k is the smallest nonnegative integer such that

$$\beta_k||A(x_k) - A(y_k)|| \leq \eta||x_k - y_k||.$$

Step 3: If $x_k = y_k$ (i.e., $x_k \in \Omega_{VIP}$), then set $x_k = w_k$ and go to Stop 4. Otherwise, compute the next iterate by

$$w_k = P_{Q_k}(x_k - \beta_k A(y_k)),$$

where

$$Q_k = \{w \in E : \langle w - y_k, x_k - \beta_k A(x_k) - y_k \rangle \leq 0\}.$$

Step 4: Compute

$$x_{k+1} = \alpha_k x_1 + (1 - \alpha_k)((1 - \nu_k)w_k + \nu_k S w_k).$$

Set $k := k + 1$ and go to Step 2.

5.2.2 Application to nonlinear Hammerstein integral equations

We now present a strong convergence theorem for approximating solution of a nonlinear Hammerstein type integral equation.

A nonlinear integral equation of Hammerstein type (see for instance [119]) is one of the form

$$u(x) + \int_\Omega K(x, y) f(y, u(y)) dy = h(x), \tag{5.2.39}$$

where dy is a σ-finite measure on the measure space Ω; the real kernel K is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \Omega$ and is in general nonlinear and h is a given function on Ω. The nonlinear equations of Hammerstein type have proved to be one of the areas in which the ideas and techniques of nonlinear functional analysis found vast applications. This has drawn the attention of many authors who have studied its existence and approximation of its solutions. In fact, several differential equation problems can be recast into (5.2.39). This equation also plays crucial role in the study of theory of optimal control systems and in automation and in network theory. See [46, 47, 48, 107, 77, 78, 133, 273] and references therein.

If we now define an operator T by

$$Tv(x) = \int_\Omega K(x, y) v(y) dy, \quad x \in \Omega,$$

and denote by F the so-called *superposition* or *Nemytskii operator* corresponding to f, i.e., $Fu(y) := f(y; u(y))$ then, the integral equation (5.2.39) can be put in the operator theoretic form as follows:

$$u + TFu = 0, \tag{5.2.40}$$

where, without loss of generality, we have taken $h \equiv 0$.

Several problems that arises in differential equations, for instance, elliptic boundary value problems whose linear parts admits Green's functions can be transformed into the form of (5.2.40). Example is the problem of forced oscillations of finite amplitude of pendulum given below.

Example 5.2.14. (see [200]) The amplitude of oscillation $x(t)$ is a solution of the problem

$$\begin{cases} \frac{d^2x}{dt^2} + a^2 \sin(x)(t) = y(t), & t \in [0,1] \\ x(0) = x(1) = 0, \end{cases}$$

where the driving force $y(t)$ is periodical and odd. The constant $a \neq 0$ depends on the length of the pendulum and on gravity. Since the Green's function for the problem

$$x''(t) = 0, \quad x(0) = x(1) = 0$$

is a triangular function

$$h(u,t) = \begin{cases} t(1-u), & 0 \leq t \leq u, \\ u(1-t), & u \leq t \leq 1, \end{cases}$$

the problem (5.2.41) is equivalent to the nonlinear integral equation

$$x(t) - \int_0^1 h(u,t)[x(u) - a^2 \sin(x)(u)]du. \tag{5.2.41}$$

If

$$\int_0^1 h(u,t)x(u)du = g(t) \quad \text{and} \quad x(t) + g(t) = v(t),$$

then (5.2.41) can be written as the Hammerstein equation

$$v(t) + \int_0^1 h(u,t)f(u,v(u))du = 0,$$

where $f(u,v(u)) = a^2 \sin(v(u) - g(u))$.

Several existence and uniqueness results for equations of the Hammerstein type emphasize the monotonicity of the operators T and F. A monotone operator describes any system that grows with time-evolution equations. Typical examples where such evolution equations occur can be found in the heat, wave, or Schrödinger equations. It follows from the existence results for equations of the Hammerstein type that for iterative approximation of solutions of equations of the Hammerstein type, the monotonicity of operators T and F is crucial.

The following lemmas will be needed in the sequel.

Lemma 5.2.15. *[230] Let X be a real reflexive Banach space with dual X^*. Let $E := X \times X^*$ and with norm*

$$||x||_E := \left(||x_1||_X^2 + ||x_2||_{X^*}^2\right)^{\frac{1}{2}}, \quad for \;\; x = [x_1, x_2] \in E.$$

Let $E^* := (X \times X^*)^* = X^* \times X$ denotes the dual space of E. For arbitrary $x = [x_1, x_2] \in E$, define the map $j^E : E \to E^*$ by

$$j^E(x) = j^E[x_1, x_2] := [j^X(x_1), j^{X^*}(x_2)],$$

so that for arbitrary $x = [x_1, x_2]$, $y = [y_1, y_2]$ in E, the duality pairing $\langle ., . \rangle$ is given by

$$\langle x, j^E(y) \rangle = \langle x_1, j^X(y_1) \rangle + \langle x_2, j^{X^*}(y_2) \rangle.$$

Then

(i) j^E is a duality mapping on E;

(ii) E is a real reflexive Banach space.

Lemma 5.2.16. *[230] Let X be a real reflexive Banach space with dual X^*. Let $F : X \to X^*$ and $T : X^* \to X$ be continuous monotone mappings. Let $E := X \times X^*$ and $A : E \to E^*$ be a mapping defined by*

$$Aw = (Fu - v, Tv + u), \qquad \forall\ w = (u, v) \in E.$$

Then A is a monotone and continuous mapping.

Lemma 5.2.17. *Let X be a real reflexive Banach space with dual X^* and $E := X \times X^*$. Let $S_1 : X \to X$ and $S_2 : X^* \to X^*$ be quasi-nonexpansive mappings. Define the mapping $S : E \to E$ by $Sw = (S_1(u), S_2(v))$, $\forall\ w = (u, v) \in E$. Then, S is quasi-nonexpansive mapping.*

Proof. Let $x = (x_1, x_2) \in E$ and $p = (p_1, p_2) \in F(S)$, where $p_1 \in F(S_1)$ and $p_2 \in F(S_2)$, then

$$\begin{aligned} ||Sx - p||_E &= ||(S_1(x_1), S_2(x_2)) - (p_1, p_2)||_E \\ &= ||(S_1(x_1) - p_1, S_2(x_2) - p_2)||_E \\ &= \Big(||S_1x_1 - p_1||_X^2 + ||S_2x_2 - p_2||_{X^*}^2 \Big)^{\frac{1}{2}} \\ &\leq \Big(||x_1 - p_1||_X^2 + ||x_2 - p_2||_{X^*}^2 \Big)^{\frac{1}{2}} \\ &= ||x - p||_E. \end{aligned}$$

Hence, S is quasi-nonexpansive mapping on E. □

The following remark is very important for establishing our result.

Remark 5.2.18. Suppose $u + TFu = 0$ in X and $A : E \to E^*$ is defined by $Aw = (Fu - v, Tv + u)$, for all $w = (u, v) \in E$. Note that for $w^* = (u^*, v^*) \in E$,

$$\begin{aligned} 0 = Aw^* &\iff (0, 0) = (Fu^* - v^*, Tv^* + u^*) \\ &\iff v^* = Fu^* \quad \text{and} \quad 0 = u^* + Tv^* \\ &\iff 0 = u^* + TFu^*. \end{aligned} \tag{5.2.42}$$

This implies that u^* is a solution of $u + TFu = 0$ if and only if w^* is a solution of $Aw = 0$ for $v^* = Fu^*$.

Moreover, w^* is a solution of $Aw = 0$ if and only if it is a solution of the variational inequality [76]:

$$\text{find} \quad w \in E \quad \text{such that} \quad \langle Aw, y - w \rangle \geq 0, \qquad \forall\, y \in E.$$

Hence, we can apply Theorem 5.2.11 to solve (5.2.40). Hence using Theorem 5.2.11, we have the following result for approximating solutions of the nonlinear Hammerstein integral equation (5.2.40).

Theorem 5.2.19. *Let X be a 2-uniformly convex and uniformly smooth Banach space with dual X^*. Let $F : X \to X^*$ and $T : X^* \to X$ be continuous monotone mapping. Let $E := X \times X^*$ and $h_i : E \to \mathbb{R}$ be families of convex, weakly lower semicontinuous and Gâteaux differentiable functions, $i = 1, 2, \dots, m$. Let $A : E \to E^*$ be a mapping defined by $Aw = (Fu - v, Tv + u), \quad \forall\, w = (u, v) \in E$. Let $S_1 : X \to X$ and $S_2 : X^* \to X^*$ be quasi-nonexpansive mappings and define $S : E \to E$ by $Sw = (S_1(u), S_2(v)), \forall\, w = (u, v) \in E$. Assume Ω is the set of solutions of (5.2.40) and $Sol := \Omega \bigcap F(S) \neq \emptyset$. Let $\{\alpha_k\}$ and $\{\beta_k\}$ be nonnegative real sequences in $(0, 1)$ and Assumption 5.2.2 be satisfied. Then, the sequence $\{x_k\} = \{(u_k, v_k)\}$ generated by the following algorithm converges strongly to a point $p \in Sol := \Omega \bigcap F(S)$.*

Algorithm 5.2.20.

Step 1: (Initialization) Pick $x_1 = (u_1, v_1) \in E$, $\eta, \rho \in (0, 1)$ and set $k = 1$.

Step 2: Given the current iterate x_k, construct the family of half-spaces

$$C_k^i := \{w = (u, v) \in E : h_i(x_k) + \langle h_i'(x_k), w - x_k \rangle \leq 0\}, \qquad i \in I, \tag{5.2.43}$$

and set

$$C_k := \bigcap_{i \in I} C_k^i, \tag{5.2.44}$$

then compute

$$y_k = \Pi_{C_k} J^{-1}(Jx_k - \beta_k A(x_k)), \tag{5.2.45}$$

where

$$\beta_k = \rho^{l_k}, \tag{5.2.46}$$

and l_k is the smallest nonnegative integer such that

$$\beta_k ||A(x_k) - A(y_k)|| \leq \eta ||x_k - y_k||. \tag{5.2.47}$$

Step 3: If $x_k = y_k$, then set $x_k = w_k$ and go to Stop 4. Otherwise, compute the next iterate by

$$w_k = \Pi_{Q_k} J^{-1}(Jx_k - \beta_k A(y_k)), \tag{5.2.48}$$

where

$$Q_k = \{w = (u, v) \in E : \langle w - y_k, Jx_k - \beta_k A(x_k) - Jy_k \rangle \leq 0\}. \tag{5.2.49}$$

Step 4: Compute

$$x_{k+1} = J^{-1}(\alpha_k Jx_1 + (1 - \alpha_k)((1 - \nu_k)Jw_k + \nu_k JSw_k)). \tag{5.2.50}$$

Set $k := k + 1$ and go to Step 2.

5.2.3 Numerical Examples

In this subsection, we consider two examples related to equations of Hammerstein type to illustrate the performance of our Algorithm. All codes are written using `MATLAB 2014b` on a HP Elitebook personal computer. We choose different choices of the initial values in each example and test Algorithm 3.1 for solving the Hammerstein equation (5.2.40). The stopping criterion used in both test is $\frac{||x_{k+1}-x_k||^2}{||x_2-x_1||^2} = \frac{||(u_k+1,v_k+1)-(u_k,v_k)||^2}{||(u_2,v_2)-(u_1,v_1)||^2} < \epsilon$, where ϵ is chosen appropriately. The projections onto the half-spaces C_k and Q_k can be easily calculated since they have the specific form which can be found in Chapter 2.

Example 5.2.21. Let $E = \mathbb{R} \times \mathbb{R}$ and $C = C^1 \cap C^2 \subseteq \mathbb{R} \times \mathbb{R}$, where

$$C^1 := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : h_1(x_1, x_2) = x_1^2 + x_2^2 - 4 \leq 0\},$$

and

$$C^2 := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : h_2(x_1, x_2) = x_1^2 - x_2 \leq 0\}.$$

Consider the Hammerstein equation

$$u + TFu = 0$$

where $Tu = \max\{0, u\}$ for all $u \in \mathbb{R}$ and

$$Fu = \begin{cases} u, & ||u|| \leq 1, \\ \frac{u}{||u||}, & ||u|| > 1. \end{cases}$$

Define the mapping $A : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by

$$Aw = (Fu - v, Tv + u), \quad \forall\ w = (u, v) \in \mathbb{R} \times \mathbb{R},$$

and $S : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by

$$Sw = \left(\frac{-u}{2}, \frac{-v}{2}\right), \quad \forall w = (u, v) \in \mathbb{R} \times \mathbb{R}.$$

Clearly, F and T are continuous monotone, thus A is continuous and monotone (by Lemma 5.2.16). Also $Sol := \Omega \cap F(S) = \{0\}$. We choose $\alpha_k = \frac{1}{\sqrt{k+1}}$, $\nu_k = \frac{3k}{5k+1}$, $\eta = 0.5$ and $\rho = 0.07$ as our parameters which satisfy the desired requirements. Let $\epsilon = 10^{-5}$ and the initial values as follows:

Case I: $(10, 10)$, Case II: $(-20, 5)$, Case III: $(5, -10)$, Case IV: $(-5, -5)$.

We plot the graphs of Error against number of iterations in each case. The numerical results are reported in Figure 5.3.

Example 5.2.22. In this second example, we consider the infinite-dimensional space. Let $E = L^2([0,1]) \times L^2([0,1])$ with norm $||x||^2 = \int_0^1 |x(t)|dt$ and inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$, $x, y \in E$. We define $C^i := \{x \in E : ||x|| - 1 \leq 0\}$. Consider the Hammerstein equation $u + TFu = 0$ in $L^2([0,1])$ with F being the Volteral integral which is defined by

$$(Fu)(t) = \int_0^1 u(s)ds, \forall\ u \in L^2([0,1]),$$

and

$$(Tu)(t) = \begin{cases} (u)(t), & ||u|| \leq 1, \\ \frac{(u)(t)}{||u||}, & ||u|| > 1. \end{cases}$$

It is known that F is continuous and monotone (cf. Exercise 20.12 in [23], Page 308). Define the mapping $A : L^2([0,1]) \times L^2([0,1]) \to L^2([0,1]) \times L^2([0,1])$ by

$$Aw = (Fu - v, Tv + u), \quad \forall\, w = (u,v) \in L^2([0,1]) \times L^2([0,1]),$$

and $S : L^2([0,1]) \times L^2([0,1]) \to L^2([0,1]) \times L^2([0,1])$ by

$$Sw = S(u,v)(t) = \int_0^1 \left(\frac{u(t)}{4}, \frac{v(t)}{8} \right) dt.$$

Hence A is monotone and $Sol := \Omega \cap F(S) = \{0\}$ in $L^2([0,1]) \times L^2([0,1])$. Choose $\alpha_k = \frac{1}{5k+1}$, $\nu_k = \frac{8}{9(k+1)}$, $\eta = \frac{0.09}{4}$ and $\rho = 0.05$. We also take $\epsilon = 2 \times 10^{-3}$ and chose the following input values:

Case I: $(\exp(-3t),\ t^2 + 5t - 9)$,
Case II: $(t^2 - 1,\ \cos(2t))$,
Case III: $(-(3t-5)^2,\ 4t^2 + 3t + 1)$.

We then plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 5.4.

5.3 A Unified Algorithm for Solving Variational Inequality and Fixed Point Problems with Application to the Split Equality Problem

In this section, we propose a new extragradient method consisting of the Hybrid steepest descent method, a single projection method and an Armijo line searching technique for approximating a solution of variational inequality problem and finding the fixed point of demi-contractive mapping in a real Hilbert space. The essence of this algorithm is that a single projection is required in each iteration and the stepsize for the next iterate is determined in such a way that there is no need for a prior estimate of the Lipschitz constant of the underlying operator.

It is well known that $x^\dagger$ solves the VIP (1.1.1) if and only if $x^\dagger$ solves the fixed point equation

$$x^\dagger = P_C(x^\dagger - \lambda A x^\dagger), \quad \lambda > 0, \tag{5.3.1}$$

or equivalently, $x^\dagger$ solves the residual equation

$$r_\lambda(x^\dagger) = 0, \quad \text{where} \quad r_\lambda(x^\dagger) := x^\dagger - P_C(x^\dagger - \lambda A x^\dagger), \tag{5.3.2}$$

for an arbitrary positive constant λ, see [114] for details.

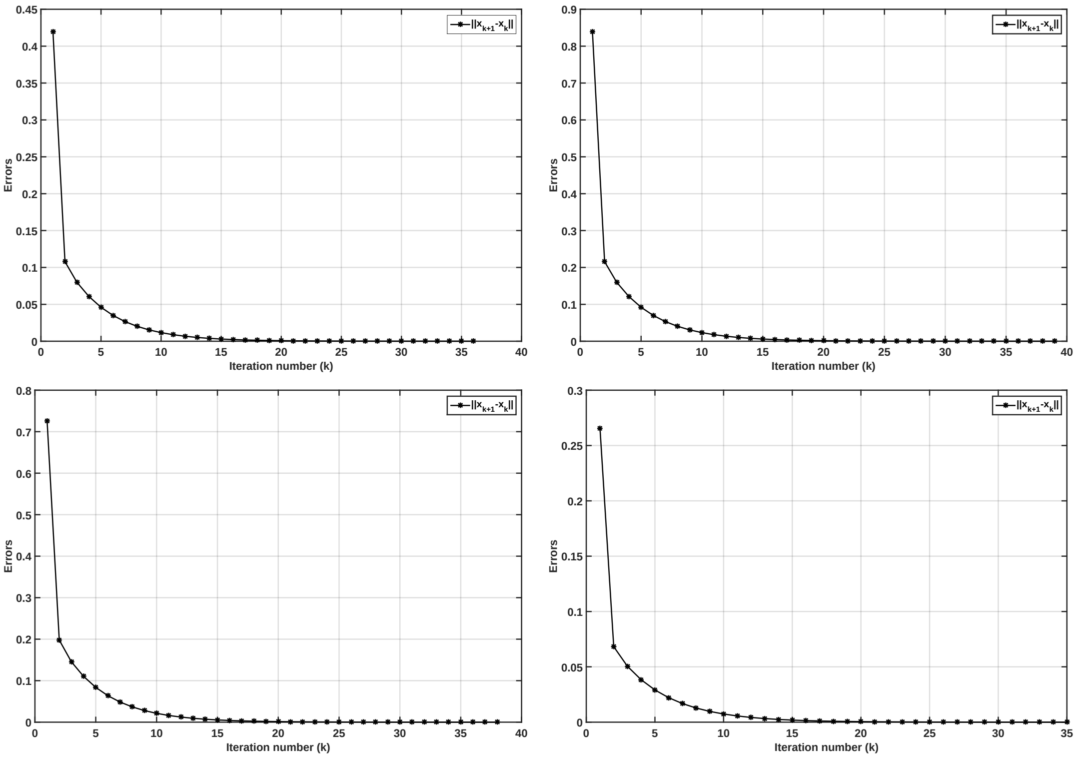


Figure 5.3: Example 5.2.21; Top Left $x_1 = (10, 10)$, Time = 0.0416sec; Top Right: $x_1 = (-20, 5)$, Time = 0.0436sec; Bottom Left $x_1 = (5, -10)$, Time = 0.0422sec; Bottom Right: $x_1 = (-5, -5)$, Time = 0.0533sec..

In order to obtain strong convergence of the Subgradrient Extragradient Algorithm (3.2.1), Censor et al. [68] combined the subgradient extragradient method and the hybrid method to obtain the following effective scheme for solving the VIP (1.1.1) and finding fixed point of a nonexpansive mapping T.

$$\begin{cases} y_k = P_C(x_k - \lambda A x_k), \\ D_k = \{w \in H : \langle x_k - \lambda A x_k - y_k, w - y_k\rangle \leq 0\}, \\ z_k = P_{D_k}(x_k - \lambda A y_k), \\ t_k = \alpha_k x_k + (1-\alpha_k)[\beta_k z_k + (1-\beta_k) T z_k], \\ C_k = \{z \in H : ||t_k - z|| \leq ||x_k - z||\}, \\ Q_k = \{z \in H : \langle x_k - z, x_k - x_0\rangle \leq 0\}, \\ x_{k+1} = P_{C_k \cap Q_k}(x_0). \end{cases} \tag{5.3.3}$$

As an improvement on (5.3.3), Maingé [171] further introduced the following hybrid extragradient viscosity method which does not involve computing the projection onto the intersection $C_k \cap Q_k$:

$$\begin{cases} y_k = P_C(x_k - \lambda_k A x_k), \\ z_k = P_C(x_k - \lambda_k A y_k), \\ x_{k+1} = [(1-w)I + wT]t_k, \qquad t_k = z_k - \alpha_k B z_k, \end{cases} \tag{5.3.4}$$

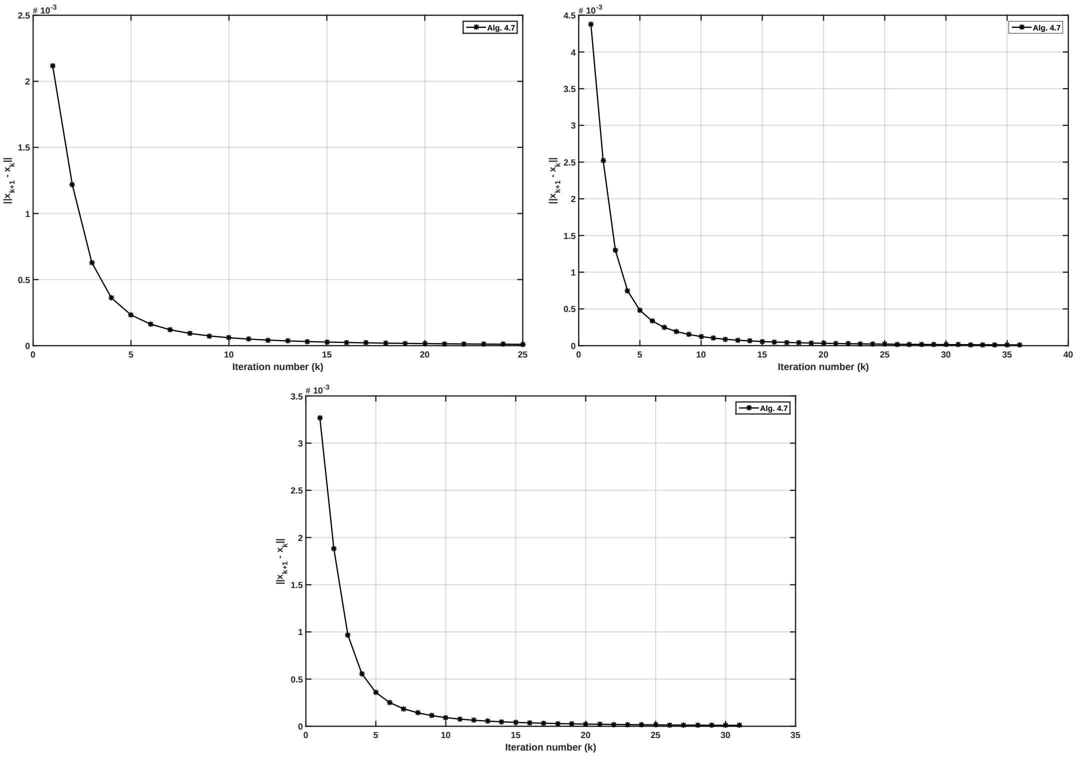


Figure 5.4: Example 5.2.22; Top Left Case I, Time = 62.1391sec; Top Right: Case II, Time = 45.6325sec; Bottom: Case III, Time = 40.5352.

where $\lambda_k > 0$, $\alpha_k > 0$ and $w \in [0, 1]$ are suitable parameters, $T : H \to H$ is β-demi-contractive mapping, $A : C \to H$ is a monotone and L-Lipschitz continuous mapping and $B : H \to H$ is η-strongly monotone and κ-Lipschitz continuous mapping. Maingé [171] proved that the sequence $\{x_k\}$ generated by (5.3.4) converges strongly to the unique solution $x^* \in \Omega_{VIP} \cap F(T)$.

Recently, Hieu et. al. [131] modified algorithm (5.3.4) and proposed a two-step extragradient viscosity method for solving similar problem in a Hilbert space. This method was presented as follows:

$$\begin{cases} y_k = P_C(x_k - \lambda_k A x_k), \\ z_k = P_C(y_k - \rho_k A y_k), \\ t_k = P_C(x_k - \rho_k A z_k), \\ x_{k+1} = (1 - \beta_k) v_k + \beta_k T v_k, \qquad v_k = t_k - \alpha_k B t_k, \end{cases} \tag{5.3.5}$$

where $\rho_k > 0$, $0 \le \lambda_k \le \rho_k$, $\beta_k \in [0, 1]$, A, T and B are as defined for (5.3.4). We observe that, although algorithm (5.3.5) does not contain (3.2.1), but the algorithm (5.3.5) requires computation of more projections onto the feasible set. This can be costly if the feasible set has a complex structure which may affects the usage of the algorithm.

Motivated by the above results, in this section, we present a unified algorithm which consist of the combination of hybrid steepest descent method (also called general viscosity method

[257]) and a projection method with an Armijo line searching rule for finding a common solution of VIP (1.1.1) and fixed point of β-demi-contractive mapping in a Hilbert space. Our contributions in this section is highlighted as follow:

(i) Our proposed algorithm requires only one projection onto the feasible set and no other projection along each iteration process. This is in contrast to the above-mentioned methods as well as many other recent results (such as [97, 144, 255, 256, 264]) which require more than one projection onto the feasible set in each iteration process.

(ii) The underlying operator A of the VIP considered in our result is pseudo-monotone. This extends the above results where the operator is assumed to be monotone.

(iii) In our result, the stepsize λ_k is determined via an Armijo line search rule. This is very important because it helps us to avoid a prior estimate of the Lipschitz constant L of the operator A used in the above mentioned results. In practice, it is too difficult to approximate this Lipschitz constant.

(iv) The strong convergence guaranteed by our algorithm makes it a good candidate method for approximating a common solution of VIP (1.1.1) and fixed point problem.

5.3.1 Main results

In this subsection, we give a precise statement of our algorithm and discuss its strong convergence.

Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a pseudo-monotone and L-Lipschitz continuous operator and $T : C \to C$ be a β-demi-contractive mapping with constant $\beta \in [0,1)$ and demiclosed at zero. Suppose $Sol := \Omega_{VIP} \bigcap F(T) \neq \emptyset$, let $B : H \to H$ be a k-Lipschitzian and η-strongly monotone mapping with $k > 0$ and $\eta > 0$ and $f : H \to H$ be a ρ-Lipschitz mapping with $\rho > 0$. Let $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \xi\rho < \tau$, where $\tau = \frac{1}{2}\mu(2\eta - \mu k^2)$. Let $\{\alpha_k\}$ and $\{v_k\}$ be sequences in $(0,1)$ and $\{x_k\}$ be generated by the following algorithm:

Algorithm 5.3.1.

Step 0: Choose the initial guess $x_1 \in H$ and parameters $\theta, \gamma \in (0,1)$, $\sigma \in (0,2)$. Set $k = 1$.

Step 1: Compute

$$y_k = P_C(x_k - \lambda_k A x_k), \tag{5.3.6}$$

where $\lambda_k = \gamma^{l_k}$, and l_k is the smallest nonnegative integer satisfying

$$\lambda_k ||A(x_k) - A(y_k)|| \leq \theta ||x_k - y_k||. \tag{5.3.7}$$

Step 2: Compute

$$d(x_k, y_k) = x_k - y_k - \lambda_k (A x_k - A y_k), \tag{5.3.8}$$

$$w_k = x_k - \sigma \delta_k d(x_k, y_k), \tag{5.3.9}$$

where

$$\delta_k = \begin{cases} \frac{\langle x_k - y_k, d(x_k, y_k)\rangle}{||d(x_k, y_k)||^2}, & if \quad d(x_k, y_k) \neq 0, \\ 0, & if \quad d(x_k, y_k) = 0. \end{cases} \tag{5.3.10}$$

Step 3: Compute

$$x_{k+1} = \alpha_k \xi f(x_k) + (1 - \alpha_k \mu B)(v_k T w_k + (1 - v_k) w_k). \tag{5.3.11}$$

Set $k := k + 1$ and go to Step 1.

In order to establish the convergence of Algorithm 5.3.1, we make the following assumption:

(C1) $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$;

(C2) $\liminf_{k\to\infty} \lambda_k > 0$;

(C3) $\liminf_{k\to\infty} (v_k - \beta) v_k > 0$.

Remark 5.3.2. Observe that if $x_k = y_k$ and $x_k - T x_k = 0$, then we are at a common solution of the variational inequality (1.1.1) and fixed point of the demi-contractive mapping T. In our convergence analysis, we will implicitly assume that this does not occur after finitely many iterations so that our Algorithm 5.3.1 generates an infinite sequences. We will see in the following result that the Algorithm 5.3.1 is well defined. In order to do this, it suffice to show that the Armijo line searching rule define by (5.3.7) is well defined and $\delta_k \neq 0$.

Lemma 5.3.3. *There exists a nonnegative integer l_k satisfying (5.3.7). In addition*

$$\delta_k \geq \frac{(1 - \theta)}{(1 + \theta)^2}. \tag{5.3.12}$$

Proof. Let $r_{\lambda_k}(x_k) = x_k - P_C(x_k - \lambda_k A x_k)$ and suppose $r_{\gamma^{k_0}}(x_k) = 0$ for some $k_0 \geq 1$. Take $l_k = k_0$ which satisfy (5.3.7). Suppose $r_{\gamma^{k_1}}(x_k) \neq 0$ for some $k_1 \geq 1$ and assume the contrary, that is

$$\gamma^l ||A x_k - A(P_C(x_k - \gamma^l A x_k))|| > \theta ||r_{\gamma^l}(x_k)||.$$

Then it follow from Lemma 2.6.11 and the fact that $\gamma \in (0, 1)$ that

$$\begin{aligned} ||A x_k - A(P_C(x_k - \gamma^l A x_k))|| &> \frac{\theta}{\gamma^l} ||r_{\gamma^l}(x_k)|| \\ &\geq \frac{\theta}{\gamma^l} \min\{1, \gamma^l\} ||r_1(x_k)|| \\ &= \theta ||r_1(x_k)||. \end{aligned} \tag{5.3.13}$$

Since P_C is continuous, we have that

$$P_C(x_k - \gamma^l A x_k) \to P_C(x_k), \qquad l \to \infty.$$

We now consider two cases, namely when $x_k \in C$ and $x_k \notin C$.

(i) If $x_k \in C$, then $x_k = P_C x_k$. Now since $r_{\gamma^{k_1}}(x_k) \neq 0$ and $\gamma^{k_1} \leq 1$, it follows from Lemma 2.6.11 that

$$\begin{aligned} 0 &< ||r_{\gamma^{k_1}}(x_k)|| \leq \max\{1, \gamma^{k_1}\}||r_1(x_k)|| \\ &= ||r_1(x_k)||. \end{aligned}$$

Letting $l \to \infty$ in (5.3.13), we have that

$$0 = ||Ax_k - Ax_k|| \geq \theta ||r_1(x_k)|| > 0.$$

This is a contradiction and so (5.3.7) is valid.

(ii) $x_k \notin C$, then

$$\gamma^l ||Ax_k - y_k|| \to 0, \quad l \to \infty, \tag{5.3.14}$$

while

$$\lim_{l\to\infty} \theta ||r_{\gamma^l}(x_k)|| = \lim_{l\to\infty} \theta ||x_k - P_C(x_k - \gamma^l Ax_k)|| = \theta ||x_k - P_C x_k|| > 0.$$

This is a contradiction. Therefore, the Armijo line searching rule in (5.3.7) is well defined.

On the other hand, since A is Lipschitz continuous, then, we have from (5.3.7) and (5.3.8)

$$\begin{aligned} \langle x_k - y_k, d(x_k, y_k)\rangle &= \langle x_k - y_k, x_k - y_k - \lambda_k(Ax_k - Ay_k)\rangle \\ &= ||x_k - y_k||^2 - \lambda_k \langle x_k - y_k, Ax_k - Ay_k\rangle \\ &\geq ||x_k - y_k||^2 - \lambda_k ||x_k - y_k|| ||Ax_k - Ay_k|| \\ &\geq ||x_k - y_k||^2 - \theta ||x_k - y_k||^2 \\ &= (1-\theta)||x_k - y_k||^2. \end{aligned} \tag{5.3.15}$$

Also,

$$\begin{aligned} ||d(x_k, y_k)|| &= ||x_k - y_k - \lambda_k(Ax_k - Ay_k)|| \\ &\leq ||x_k - y_k|| + \lambda_k ||Ax_k - Ay_k|| \\ &\leq (1+\theta)||x_k - y_k||. \end{aligned} \tag{5.3.16}$$

Therefore from (3.1.15) and (5.3.16), we get

$$\begin{aligned} \delta_k &= \frac{\langle x_k - y_k, d(x_k, y_k)\rangle}{||d(x_k, y_k)||^2} \\ &\geq \frac{(1-\theta)}{(1+\theta)^2}. \end{aligned}$$

□

Now, we prove that the sequences $\{x_k\}, \{y_k\}$ and $\{w_k\}$ generated by Algorithm 5.3.1 are bounded.

Lemma 5.3.4. *The sequence $\{x_k\}$ generated by Algorithm 5.3.1 is bounded. In addition, the following inequality is satisfied*

$$||w_k - x^*||^2 \leq ||x_k - x^*||^2 - \frac{(2-\sigma)}{\sigma}||w_k - x_k||^2, \tag{5.3.17}$$

where $x^* \in Sol$.

Proof. Let $x^* \in Sol$, then by Lemma 2.6.1(ii), we obtain

$$\begin{aligned} ||w_k - x^*||^2 &= ||x_k - x^* - \sigma\delta_k d(x_k, y_k)||^2 \\ &= ||x_k - x^*||^2 - 2\sigma\delta_k\langle x_k - x^*, d(x_k, y_k)\rangle + \sigma^2\delta_k^2||d(x_k, y_k)||^2 \end{aligned} \tag{5.3.18}$$

Observe that

$$\langle x_k - x^*, d(x_k, y_k)\rangle = \langle x_k - y_k, d(x_k, y_k)\rangle + \langle y_k - x^*, d(x_k, y_k)\rangle. \tag{5.3.19}$$

Since $y_k = P_C(x_k - \lambda_k A x_k)$ and $x^* \in Sol$, then by the variational characterization of P_C, we have

$$\langle x_k - \lambda_k A x_k - y_k, y_k - x^*\rangle \geq 0, \tag{5.3.20}$$

and from the pseudo-monotonicity of A, we have

$$\langle Ay_k, y_k - x^*\rangle \geq 0. \tag{5.3.21}$$

Hence, combining (5.3.20) and (5.3.21), with the fact that $\lambda_k > 0$, we get

$$\langle d(x_k, y_k), y_k - x^*\rangle \geq 0. \tag{5.3.22}$$

Thus from (5.3.22) and (5.3.19) , we get

$$\langle x_k - x^*, d(x_k, y_k)\rangle \geq \langle x_k - y_k, d(x_k, y_k)\rangle. \tag{5.3.23}$$

Therefore (5.3.18) yields

$$\begin{aligned} ||w_k - x^*||^2 &\leq ||x_k - x^*||^2 - 2\sigma\delta_k\langle x_k - y_k, d(x_k, y_k)\rangle + \sigma^2\delta_k^2||d(x_k, y_k)||^2 \\ &= ||x_k - x^*||^2 - 2\sigma\delta_k\langle x_k - y_k, d(x_k, y_k)\rangle + \sigma^2\delta_k\langle x_k - y_k, d(x_k, y_k)\rangle \\ &= ||x_k - x^*||^2 - \sigma(2-\sigma)\delta_k\langle x_k - y_k, d(x_k, y_k)\rangle. \end{aligned} \tag{5.3.24}$$

From the definition of δ_k and w_k, we have

$$\begin{aligned} \delta_k\langle x_k - y_k, d(x_k, y_k)\rangle &= ||\delta_k d(x_k, y_k)||^2 \\ &= \frac{1}{\sigma^2}||w_k - x_k||^2. \end{aligned} \tag{5.3.25}$$

Substituting (5.3.25) into (5.3.24), we have

$$||w_k - x^*||^2 \leq ||x_k - x^*||^2 - \frac{(2-\sigma)}{\sigma}||w_k - x_k||^2.$$

Hence

$$||w_k - x^*||^2 \leq ||x_k - x^*||^2. \tag{5.3.26}$$

Furthermore, observe that for any $x, y \in H$,

$$\begin{aligned} ||(I-\mu B)x - (I-\mu B)y||^2 &= ||x-y||^2 - 2\mu\langle x-y, Bx - By\rangle + \mu^2||Bx - By||^2 \\ &\leq (1 - 2\mu\eta + \mu^2 k^2)||x-y||^2 \\ &= (1-\tau)^2||x-y||^2, \end{aligned}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Also

$$\begin{aligned} &||P_{Sol}(I - \mu B + \xi f)x - P_{Sol}(I - \mu B + \xi f)y|| \\ &\qquad \leq ||(I - \mu B + \xi f)x - (I - \mu B + \xi f)y|| \\ &\qquad \leq ||(I-\mu B)x - (I - \mu B)y|| + \xi||f(x) - f(y)|| \\ &\qquad \leq (1-\tau)||x-y|| + \xi\rho||x-y|| \\ &\qquad = (1 - (\tau - \xi\rho))||x-y||. \end{aligned}$$

This implies that $P_{Sol}(I - \mu B + \xi f)$ is a contraction mapping which means that there exists a unique element $x* \in H$ such that $x^* = P_{Sol}(I - \mu B + \xi f)x^*$.

Now let $T_v = vT + (1-v)I$, then by Lemma 2.1.6, T_v is quasi-nonexpansive and therefore

$$\begin{aligned} ||x_{k+1} - x^*|| &= ||\alpha_k \xi f(x_k) + (1 - \alpha_k\mu B)T_{v_k}w_k - x^*|| \\ &= ||\alpha_k(\xi f(x_k) - \mu Bx^*) + (I - \alpha_k\mu B)T_{v_k}w_k - (I - \alpha_k\mu B)x^*|| \\ &= ||(I - \alpha_k\mu B)(T_{v_k}w_k - x^*) + \alpha_k(\xi f(x_k) - \mu Bx^* + \xi f(x^*) - \xi f(x^*))|| \\ &\leq ||(I - \alpha_k\mu B)(T_{v_k}w_k - x^*)|| + \alpha_k\xi||f(x_k) - f(x^*)|| + \alpha_k||\xi f(x^*) - \mu Bx^*|| \\ &\leq (1 - \alpha_k\tau)||T_{v_k}w_k - x^*|| + \alpha_k\xi\rho||x_k - x^*|| + \alpha_k||\xi f(x^*) - \mu Bx^*|| \\ &\leq (1 - \alpha_k\tau)||w_k - x^*|| + \alpha_k\rho||x_k - x^*|| + \alpha_k||\xi f(x^*) - \mu Bx^*|| \\ &\leq (1 - \alpha_k\tau)||x_k - x^*|| + \alpha_k\xi\rho||x_k - x^*|| + \alpha_k||\xi f(x^*) - \mu Bx^*|| \\ &= (1 - \alpha_k(\tau - \xi\rho))||x_k - x^*|| + \alpha_k(\tau - \xi\rho)\frac{||\xi f(x^*) - \mu B(x^*)||}{\tau - \xi\rho} \\ &\leq \max\left\{||x_k - x^*||, \frac{||\xi f(x^*) - \mu B(x^*)||}{\tau - \xi\rho}\right\} \\ &\vdots \\ &\leq \max\left\{||x_1 - x^*||, \frac{||\xi f(x^*) - \mu B(x^*)||}{\tau - \xi\rho}\right\}. \end{aligned} \tag{5.3.27}$$

This implies that $\{x_k\}$ is bounded in H. Consequently, from (5.3.26), $\{w_k\}$ is bounded and since A is continuous, then $\{Ax_k\}$ is bounded and therefore $\{y_k\}$ is bounded too. □

Lemma 5.3.5. *The sequence $\{x_n\}$ generated by Algorithm 5.3.1 satisfies the following estimates:*

(i) $s_{k+1} \leq (1 - a_k)s_k + a_k b_k$,

(ii) $-1 \leq \limsup_{k\to\infty} b_k < +\infty$,

where $s_k = ||x_k - x^*||^2$, $\quad a_k = \frac{2\alpha_k(\tau-\xi\rho)}{1-\alpha_k\xi\rho}$, $b_k = \frac{\alpha_k\tau^2 M_1}{2(\tau-\xi\rho)} + \frac{1}{\tau-\xi\rho}\langle \xi f(x^*) - \mu B(x^*), x_{k+1} - x^*\rangle$, *for some* $M_1 > 0$, $x^* \in Sol$.

Proof. Let $x^* \in Sol$, then from Lemma 2.6.1(i) and (5.3.11), we have

$$\begin{aligned}
||x_{k+1} - x^*||^2 &= ||\alpha_k \xi f(x_k) + (1-\alpha_k\mu B)T_{v_k}w_k - x^*||^2\\
&= ||\alpha_k(\xi f(x_k) - \mu Bx^*) + (I - \alpha_k\mu B)T_{v_k}w_k - (I-\alpha_k\mu B)x^*||^2\\
&\leq ||(1-\alpha_k\mu B)T_{v_k}w_k - (1-\alpha_k\mu B)x^*||^2 + 2\alpha_k\langle \xi f(x_k) - \mu B(x^*), x_{k+1} - x^*\rangle\\
&\leq (1-\alpha_k\tau)^2||w_k - x^*||^2 + 2\alpha_k\xi\langle f(x_k) - f(x^*), x_{k+1} - x^*\rangle\\
&\quad +2\alpha_k\langle \xi f(x^*) - \mu B(x^*), x_{k+1} - x^*\rangle\\
&\leq (1-\alpha_k\tau)^2||x_k - x^*||^2 + 2\alpha_k\xi\rho||x_k - x^*||||x_{k+1} - x^*||\\
&\quad +2\alpha_k\langle \xi f(x^*) - \mu B(x^*), x_{k+1} - x^*\rangle\\
&\leq (1-\alpha_k\tau)^2||x_k - x^*||^2 + \alpha_k\xi\rho(||x_k - x^*||^2 + ||x_{k+1} - x^*||)\\
&\quad +2\alpha_k\langle \xi f(x^*) - \mu B(x^*), x_{k+1} - x^*\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
||x_{k+1} - x^*||^2 &\leq \frac{(1-\alpha_k\tau)^2 + \alpha_k\xi\rho}{1-\alpha_k\xi\rho}||x_k - x^*||^2 + \frac{2\alpha_k}{1-\alpha_k\xi\rho}\langle \xi f(x^*) - \mu B(x^*), x_{k+1} - x^*\rangle\\
&= \left(1 - \frac{2\alpha_k(\tau-\xi\rho)}{1-\alpha_k\xi\rho}\right)||x_k - x^*||^2 + \frac{\alpha_k^2\tau^2}{1-\alpha_k\xi\rho}||x_k - x^*||^2\\
&\quad + \frac{2\alpha_k}{1-\alpha_k\xi\rho}\langle \xi f(x^*) - \mu B(x^*), x_{k+1} - x^*\rangle\\
&\leq \left(1 - \frac{2\alpha_k(\tau-\xi\rho)}{1-\alpha_k\xi\rho}\right)||x_k - x^*||^2\\
&\quad + \frac{2\alpha_k(\tau-\xi\rho)}{1-\alpha_k\xi\rho}\left\{\frac{\alpha_k\tau^2 M_1}{2(\tau-\xi\rho)} + \frac{1}{\tau-\xi\rho}\langle \xi f(x^*) - \mu B(x^*), x_{k+1} - x^*\rangle\right\}\\
&= (1-a_k)s_k + a_k b_k,
\end{aligned}$$

where the exists of M_1 follows from the boundedness of $\{x_k\}$. This established (i). Next, we proof (ii). Since $\{x_k\}$ is bounded and $\alpha_k \in (0,1)$, then we have that

$$\sup_{k\geq 0} b_k \leq \sup_{k\geq 0} \frac{1}{2(\tau-\xi\rho)}\Big(\tau^2 M_1 + 2||\xi f(x^*) - \mu B(x^*)||||x_{k+1} - x^*||\Big) < \infty.$$

We next show that $\limsup_{k\to\infty} b_k \geq -1$. Assume the contrary that $\limsup_{k\to\infty} b_k < -1$, which implies that there exists $k_0 \in \mathbb{N}$ such that $b_k \leq -1$ for all $k \geq k_0$. Hence, it follows from (i) that

$$\begin{aligned}
s_{k+1} &\leq (1-a_k)s_k + a_k b_k\\
&< (1-a_k)s_k - a_k\\
&= s_k - a_k(s_k + 1)\\
&\leq s_k - 2(\tau-\xi\rho)\alpha_k.
\end{aligned}$$

By induction, we get that

$$s_{k+1} \leq s_{k_0} - 2(\tau - \xi\rho)\sum_{i=k_0}^{k} \alpha_i \qquad \text{for all} \ \ k \geq k_0.$$

Taking lim sup of both sides in the last inequality, we have that

$$\limsup_{k\to\infty} s_k \leq s_{k_0} - \lim_{k\to\infty} 2(\tau - \xi\rho)\sum_{i=k_0}^{k} \alpha_i = -\infty.$$

This contradicts the fact that $\{s_k\}$ is a nonnegative real sequence. Therefore, $\limsup_{k\to\infty} b_k \geq -1$. □

Lemma 5.3.6. *Let $\{x_{k_j}\}$ be subsequence of the sequence $\{x_k\}$ generated by Algorithm 5.3.1 such that $x_{k_j} \rightharpoonup p \in C$. Suppose $||x_k - y_k|| \to 0$ as $k \to \infty$ and $\liminf_{j\to\infty} \lambda_{k_j} > 0$. Then*

(i) $0 \leq \liminf_{j\to\infty} \langle Ax_{k_j}, x - x_{k_j}\rangle$, *for all* $x \in C$;

(ii) $p \in \Omega_{VIP}$.

Proof. (i) Since $y_{k_j} = P_C(x_{k_j} - \lambda_{k_j} Ax_{k_j})$, from the variational characterization of P_C (2.2.2), we have

$$\langle x_{k_j} - \lambda_{k_j} Ax_{k_j} - y_{k_j}, x - y_{k_j}\rangle \leq 0, \qquad \forall\ x \in C.$$

Hence

$$\begin{aligned} \langle x_{k_j} - y_{k_j}, x - y_{k_j}\rangle &\leq \lambda_{k_j}\langle Ax_{k_j}, x - y_{k_k}\rangle \\ &= \lambda_{k_j}\langle Ax_{k_j}, x_{k_j} - y_{k_j}\rangle + \lambda_{k_j}\langle Ax_{k_j}, x - x_{k_k}\rangle \end{aligned}$$

This implies that

$$\langle x_{k_j} - y_{k_j}, x - y_{k_j}\rangle + \lambda_{k_j}\langle Ax_{k_j}, y_{k_j} - x_{k_j}\rangle \leq \lambda_{k_j}\langle Ax_{k_j}, x - x_{k_k}\rangle. \tag{5.3.28}$$

Fix $x \in C$ and let $j \to \infty$ in (5.3.28), since $||x_{k_j} - y_{k_j}|| \to 0$ and by condition (C2), $\liminf_{j\to\infty} \lambda_{k_j} > 0$, we have

$$0 \leq \liminf_{j\to\infty} \langle Ax_{k_j}, x - x_{k_j}\rangle, \quad \forall\ x \in C. \tag{5.3.29}$$

(ii) Let $\{\epsilon_j\}$ be a sequence of decreasing non-negative numbers such that $\epsilon_j \to 0$ as $j \to \infty$. For each ϵ_j, we denote by N the smallest positive integer such that

$$\langle Ax_{k_j}, x - x_{k_j}\rangle + \epsilon_j \geq 0, \quad \forall\ j \geq N$$

where the existence of N follows from (i). This implies that

$$\langle Ax_{k_j}, x + \epsilon_j t_{k_j} - x_{k_j}\rangle \geq 0, \quad \forall j \geq N,$$

for some $t_{k_j} \in H$ satisfying $1 = \langle Ax_{k_j}, t_{k_j}\rangle$ (since $Ax_{k_j} \neq 0$). Since A is pseudo-monotone, then we have from (i) that

$$\langle A(x + \epsilon_j t_{k_j}), x + \epsilon_j t_{k_j} - x_{k_j}\rangle \geq 0, \quad \forall j \geq N$$

which implies that

$$\begin{aligned} \langle Ax, x - x_{k_j}\rangle &\geq \langle Ax - A(x + \epsilon_j t_{k_j}), x + \epsilon_j t_{k_j} - x_{k_j}\rangle \\ &\quad -\epsilon_j \langle Ax, t_{k_j}\rangle \quad \forall\ j \geq N. \end{aligned} \tag{5.3.30}$$

Since $\epsilon_j \to 0$ and A is continuous, then the right hand side of (5.3.30) tends to zero. Thus, we obtain that

$$\liminf_{j\to\infty} \langle Ax, x - x_{k_j}\rangle \geq 0, \quad \forall\ x \in C.$$

Hence

$$\langle Ax, x - p\rangle = \lim_{j\to\infty} \langle Ax, x - x_{k_j}\rangle \geq 0, \quad \forall\ x \in C.$$

Therefore from Lemma 2.6.9, we obtain that $p \in VI(C, A)$.

□

We are now in position to prove the convergence of our Algorithm.

Theorem 5.3.7. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : H \to H$ be a pseudo-monotone and L-Lipschitz continuous operator and $T : C \to C$ be a β-demi-contractive mapping with constant $\beta \in [0, 1)$ and demiclosed at zero. Suppose $Sol := \Omega_{VIP} \bigcap F(T)$, let $B : H \to H$ be a k-Lipschitzian and η-strongly monotone mapping with $k > 0$ and $\eta > 0$ and $f : H \to H$ be a ρ-Lipschitz mapping with $\rho > 0$. Let $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \xi\rho < \tau$, where $\tau = \frac{1}{2}\mu(2\eta - \mu k^2)$. Let $\{\alpha_k\}$ and $\{v_k\}$ be sequences in $(0, 1)$, $\{x_k\}$ such that Assumptions (C1)-(C3) are satisfied. Then sequence $\{x_k\}$ generated by Algorithm 5.3.1 converges strongly to a point $x^\dagger$, where $x^\dagger = P_{Sol}(I - \mu B + \xi f)(x^\dagger)$ is a unique solution of the variational inequality*

$$\langle (\mu B - \xi f)x^\dagger, x^\dagger - x\rangle \leq 0, \quad \forall \ \ x \in Sol. \tag{5.3.31}$$

Proof. Let $x^* \in Sol$ and put $\Gamma_k := ||x_k - x^*||^2$. We divide the proof into two cases.
Case I: Suppose that there exists $k_0 \in \mathbb{N}$ such that $\{\Gamma_k\}$ is monotonically non-increasing for $k \geq k_0$. Then $\{\Gamma_k\}$ converges and therefore

$$\Gamma_k - \Gamma_{k+1} \to 0, \quad n \to \infty. \tag{5.3.32}$$

Let $z_k = (1 - v_k)w_k + v_k T w_k$, then using Lemma 2.6.1(iii), we have

$$\begin{aligned} ||z_k - x^*||^2 &= ||(1 - v_k)(w_k - x^*) + v_k(Tw_k - x^*)||^2 \\ &= (1 - v_k)||w_k - x^*||^2 + v_k||Tw_k - x^*||^2 - v_k(1 - v_k)||w_k - Tw_k||^2 \\ &\leq (1 - v_k)||w_k - x^*||^2 + v_k(||w_k - x^*||^2 + \beta||w_k - Tw_k||^2) \\ &\quad -v_k(1 - v_k)||w_k - Tw_k||^2 \\ &= ||w_k - x^*||^2 - v_k(1 - v_k - \beta)||w_k - Tw_k||^2. \end{aligned} \tag{5.3.33}$$

Then, from Lemma (2.6.1)(i) and (5.3.17), we have

$$\begin{aligned}
||x_{k+1} - x^*||^2 &= ||\alpha_k \xi f(x_k) + (1 - \alpha_k \mu B) z_k - x^*||^2 \\
&= ||\alpha_k (\xi f(x_k) - \mu B(x^*)) + (1 - \alpha_k \mu B)(z_k - x^*)||^2 \\
&\leq (1 - \alpha_k \mu B)^2 ||z_k - x^*|| + 2\alpha_k \langle \xi f(x_k) - \mu B x^*, x_{k+1} - x^* \rangle \\
&\leq (1 - \alpha_k \tau)(||w_k - x^*||^2 - v_k(1 - v_k - \beta)||w_k - Tw_k||^2) \\
&\quad + 2\alpha_k \langle \xi f(x_k) - \mu B x^*, x_{k+1} - x^* \rangle \\
&\leq (1 - \alpha_k \tau) \left(||x_k - x^*||^2 - \frac{2 - \sigma}{\sigma} ||w_k - x_k||^2 \right) \\
&\quad - (1 - \alpha_k \tau) v_k (1 - v_k - \beta) ||w_k - Tw_k||^2 \\
&\quad + 2\alpha_k \langle \xi f(x_k) - \mu B x^*, x_{k+1} - x^* \rangle. \qquad (5.3.34)
\end{aligned}$$

Hence

$$\begin{aligned}
(1 - \alpha_k \tau) \left(\frac{2 - \sigma}{\sigma} \right) ||w_k - x_k||^2 &\leq (1 - \alpha_k \tau)||x_k - x^*||^2 - ||x_{k+1} - x^*||^2 \\
&\quad + 2\alpha_k \langle \xi f(x_k) - \mu B x^*, x_{k+1} - x^* \rangle \\
&\leq \Gamma_k - \Gamma_{k+1} - \alpha_k M + 2\alpha_k \langle \xi f(x_k) - \mu B x^*, x_{k+1} - x^* \rangle,
\end{aligned}$$

for some $M > 0$. Since $\alpha_k \to 0$ and from (5.3.32), we have

$$\left(\frac{2 - \sigma}{\sigma} \right) ||w_k - x_k||^2 \to 0, \quad n \to \infty.$$

Therefore

$$\lim_{k \to \infty} ||w_k - x_k|| = 0. \qquad (5.3.35)$$

From (5.3.25), we have

$$\langle x_k - y_k, d(x_k, y_k) \rangle \leq \frac{(1 + \theta)^2}{(1 - \theta)\sigma^2} ||w_k - x_k||^2. \qquad (5.3.36)$$

Using (5.3.15), we have

$$||x_k - y_k||^2 \leq \frac{(1 + \theta)^2}{(1 - \theta)^2 \sigma^2} ||w_k - x_k||^2. \qquad (5.3.37)$$

From (5.3.35) and (5.3.37), we have

$$||x_k - y_k|| \to 0, \qquad n \to \infty. \qquad (5.3.38)$$

Therefore

$$||w_k - y_k|| \leq ||w_k - x_k|| + ||x_k - y_k|| \to 0, \quad n \to \infty. \qquad (5.3.39)$$

Also from (5.3.34), we have

$$\begin{aligned}
(1 - \alpha_k \tau) v_k (1 - v_k - \beta) ||w_k - Tw_k||^2 &\leq (1 - \alpha_k \tau)||x_k - x^*||^2 - ||x_{k+1} - x^*||^2 \\
&\quad + 2\alpha_k \langle \xi f(x_k) - \mu B x^*, x_{k+1} - x^* \rangle \\
&\leq \Gamma_k - \Gamma_{k+1} - \alpha_k M \\
&\quad + 2\alpha_k \langle \xi f(x_k) - \mu B x^*, x_{k+1} - x^* \rangle,
\end{aligned}$$

for some $M > 0$. Since $\alpha_k \to 0$ and from (5.3.32), we have

$$v_k(1 - v_k - \beta)||w_k - Tw_k||^2 \to 0, \quad n \to \infty.$$

Therefore from condition (C3), we have

$$\lim_{k\to\infty} ||w_k - Tw_k|| = 0. \tag{5.3.40}$$

Furthermore, from (5.3.40)

$$\begin{aligned} ||z_k - w_k|| &= ||(1 - v_k)w_k + v_kTw_k - w_k|| \\ &= v_k||w_k - Tw_k|| \to 0, \quad n \to \infty, \end{aligned} \tag{5.3.41}$$

and

$$\begin{aligned} ||x_{k+1} - z_k|| &= ||\alpha_k\xi f(x_k) + (1 - \alpha_k\mu B)z_k - z_k|| \\ &= \alpha_k||\xi f(x_k) - \mu B(z_k)|| \to 0, \quad n \to \infty. \end{aligned} \tag{5.3.42}$$

Therefore from (5.3.35), (5.3.41) and (5.3.42), we have

$$||x_{k+1} - x_k|| \leq ||x_{k+1} - z_k|| + ||z_k - w_k|| + ||w_k - x_k|| \to 0, \quad n \to \infty. \tag{5.3.43}$$

Since $\{x_k\}$ is bounded, there exists $\{x_{k_l}\}$ of $\{x_k\}$ such that $x_{k_l} \rightharpoonup p \in H$. From (5.3.40) and the demiclosedness of $I - T$ at zero, we have that $p \in F(T)$. Also, since $||x_k - y_k|| \to 0$, we have from Lemma 5.3.6 that $p \in \Omega_{VIP}$. Therefore $p \in Sol := \Omega_{VIP} \cap F(T)$.

Next we show that $\limsup_{k\to\infty}\langle(\mu B - \xi f)x^*, x^* - x_k\rangle \leq 0$, where $x^* = P_{Sol}(I - \mu B + \xi f)x^*$ is the unique solution of the variational inequality

$$\langle(\mu B - \xi f)x^*, x - x^*\rangle \geq 0, \quad \forall\ x \in Sol.$$

We obtain from (2.2.2) and (5.3.43) that

$$\begin{aligned} \limsup_{k\to\infty}\langle(\mu B - \xi f)x^*, x^* - x_{k+1}\rangle &= \limsup_{l\to\infty}\langle(\mu B - \xi f)x^*, x^* - x_{k_l+1}\rangle \\ &= \lim_{l\to\infty}\langle(\mu B - \xi f)x^*, x^* - p\rangle \\ &\leq 0. \end{aligned} \tag{5.3.44}$$

Finally, we show that $\{x_k\}$ converges strongly to x^*. By Lemma 5.3.5(i), we obtain

$$\Gamma_{k+1} \leq (1 - a_k)\Gamma_k + a_kb_k, \tag{5.3.45}$$

where $a_k = \frac{2\alpha_k(\tau - \xi\rho)}{1 - \alpha_k\xi\rho}$, $b_k = \frac{\alpha_k\tau^2 M_1}{2(\tau - \xi\rho)} + \frac{1}{\tau - \xi\rho}\langle\xi f(x^*) - \mu B(x^*), x_{k+1} - x^*\rangle$, for some $M_1 > 0$. It is easy to see that $a_k \to 0$ and $\sum_{k=1}^{\infty} a_k = \infty$. Also by (5.3.44), $\limsup_{k\to\infty} b_k \leq 0$. Therefore, using Lemma 2.6.30 in (5.3.45), we obtain

$$\lim_{k\to\infty} ||x_k - x^*|| = 0,$$

and hence $\{x_k\}$ converges strongly to x^* as $k \to \infty$.

Case II: Assume that $\{\Gamma_k\}$ is not monotonically decreasing. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $k \geq k_0$ (for some k_0 large enough) defined by

$$\tau(k) := \max\{j \in \mathbb{N} : j \leq k, \Gamma_j \leq \Gamma_{j+1}\}.$$

Clearly, τ is a non decreasing sequence, $\tau(k) \to 0$ as $k \to \infty$ and

$$0 \leq \Gamma_{\tau(k)} \leq \Gamma_{\tau(k)+1}, \quad \forall\ k \geq k_0.$$

Following similar process as in Case I, we have

$$||w_{\tau(k)} - Tw_{\tau(k)}|| \to 0, \quad k \to \infty,$$

$$||x_{\tau(k)+1} - x_{\tau(k)}|| \to 0, \quad k \to \infty,$$

and

$$\limsup_{k\to\infty} \langle(\mu B - \xi f)x^*, x^* - x_{\tau(k)+1}\rangle. \tag{5.3.46}$$

Since $\{x_{\tau(k)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(k)}\}$ still denoted by $\{x_{\tau(k)}\}$ which converges weakly to $z \in C$. By similar argument as in Case I, we conclude that $z \in Sol := \Omega_{VIP} \cap F(T)$. From Lemma 5.3.5(i), we have

$$\Gamma_{\tau(k)+1} \leq (1 - a_{\tau(k)})\Gamma_{\tau(k)} + a_{\tau(k)}b_{\tau(k)}. \tag{5.3.47}$$

Also $a_{\tau(k)} \to 0$ as $k \to \infty$ and $\limsup_{k\to\infty} b_{\tau(k)} \leq 0$.

Since $\Gamma_{\tau(k)} \leq \Gamma_{\tau(k)+1}$ and $a_{\tau(k)} > 0$, we have

$$||x_{\tau(k)} - x^*|| \leq b_{\tau(k)}.$$

This implies that

$$\limsup_{k\to\infty} ||x_{\tau(k)} - x^*||^2 = 0,$$

and thus

$$\lim_{k\to\infty} ||x_{\tau(k)} - x^*|| = 0.$$

Also from (5.3.47) we obtain

$$\limsup_{k\to\infty} ||x_{\tau(k)+1} - x^*||^2 \leq \limsup_{k\to\infty} ||x_{\tau(k)} - x^*||^2.$$

Therefore

$$\lim_{k\to\infty} ||x_{\tau(k)+1} - x^*|| = 0.$$

Furthermore, for $k \geq k_0$, it is easy to see that $\Gamma_{\tau(k)} \leq \Gamma_{\tau(k)+1}$ if $k \geq \tau(k)$ (that is $\tau(k) < k$), because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(k) + 1 \leq j \leq k$. As a consequence, we obtain that for all $k \geq k_0$

$$0 \leq \Gamma_k \leq \max\{\Gamma_{\tau(k)}, \Gamma_{\tau(k)+1}\} = \Gamma_{\tau(k)+1}.$$

Hence $\Gamma_k \to 0$ as $k \to \infty$. That is, $\{x_k\}$ converges strongly to x^*. This completes the proof. □

5.3.2 Application to Split Equality Problem

Let H_1, H_2 and H_3 be real Hilbert spaces, let $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex sets, let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. The Split Equality Problem (shortly, SEP) is to find (see [183])

$$x \in C, y \in Q \quad \text{such that} \quad Ax = By. \tag{5.3.48}$$

The SEP allows asymmetric and partial relations between the variables x and y. If $H_2 = H_3$ and $B = I$ (the identity mapping), then the SEP reduces to the Split Feasibility Problem (SFP) (1.1.9). The SEP (5.3.48) covers many situations, such as for instance in domain decomposition for PDE's, game theory and intensity-modulated radiation therapy (IMRT) [15, 63].

In this subsection, we adapt our Algorithm 5.3.1 to solve the SEP (5.3.48). Before that, let us first prove some lemmas which will be of help.

Lemma 5.3.8. *[96] Let $S = C \times Q \subset H := H_1 \times H_2$. Define $K := [A, -B] : H_1 \times H_2 \to H_1 \times H_2$ and let K^* be the adjoint operator of K, then the SEP (5.3.48) can be modified as*

$$\textit{Find} \quad z = (x, y) \in S \quad \textit{such that} \quad Kw = 0, \tag{5.3.49}$$

where $w = \begin{bmatrix} x \\ y \end{bmatrix}$ is the vector associated with z.

Lemma 5.3.9. *Let $H = H_1 \times H_2$, define $M : H \to H$ by $M(w) = M(u, v) := (\phi_1(u), \phi_2(v))$, $w = (u, v) \in H$, where $\phi_i : H \to H$ are k_i-Lipschitz and η_i-strongly monotone mapping with $k_i > 0$ and $\eta_i > 0$, $i = 1, 2$. Then M is k-Lipschitz and η-strongly monotone where $k = \max\{k_1, k_2\}$ and $\eta = \min\{\eta_1, \eta_2\}$.*

Proof. Let $x = (x_1, y_1)$, $y = (x_2, y_2) \in H$, then we have

$$\begin{aligned}
\langle Mx - My, x - y\rangle &= \langle(\phi_1(x_1), \phi_2(y_1)) - (\phi_1(x_2), \phi_2(y_2)), (x_1 - x_2, y_1 - y_2)\rangle \\
&= \langle(\phi_1(x_1) - \phi_1(x_2), \phi_2(y_1) - \phi_2(y_2)), (x_1 - x_2, y_1 - y_2)\rangle \\
&= \langle\phi_1(x_1) - \phi_1(x_2), x_1 - x_2\rangle + \langle\phi_2(y_1) - \phi_2(y_2), y_1 - y_2\rangle \\
&\geq \eta_1||x_1 - x_2||^2 + \eta_2||y_1 - y_2||^2 \\
&\geq \min\{\eta_1, \eta_2\}(||x_1 - x_2||^2 + ||y_1 - y_2||^2) \\
&= \eta||x - y||^2.
\end{aligned}$$

Hence M is η-strongly monotone , where $\eta = \min\{\eta_1, \eta_2\}$. Also

$$\begin{aligned}
||Mx - My||^2 &= ||(\phi_1(x_1), \phi_2(y_1)) - (\phi_1(x_2), \phi_2(y_2))||^2 \\
&= ||(\phi_1(x_1) - \phi_1(x_2), \phi_2(y_1) - \phi_2(y_2))||^2 \\
&= ||\phi_1(x_1) - \phi_1(x_2)||^2 + ||\phi_2(y_1) - \phi_2(y_2)||^2 \\
&\leq k_1^2||x_1 - x_2||^2 + k_2^2||y_1 - y_2||^2 \\
&\leq \max\{k_1^2, k_2^2\}(||x_1 - x_2||^2 + ||y_1 - y_2||^2) \\
&= k^2||x - y||^2.
\end{aligned}$$

Hence M is k-Lipschitz with $k = \max\{k_1, k_2\}$. □

In a similar fashion as in Lemma 5.3.9, we can prove the following results.

Lemma 5.3.10. *Let $H := H_1 \times H_2$, let $f : H \to H$ be defined by $f(u,v) = (f_1(u), f_2(v))$, $w = (u,v) \in H$, $f_i : H_i \to H_i$ is ρ_i-Lipschitz mapping with $\rho_i > 0$, $i = 1,2$. Then f is ρ-Lipschitz mapping with $\rho = \sqrt{\max\{\rho_1, \rho_2\}}$.*

Lemma 5.3.11. *Let $H := H_1 \times H_2$, let $T : H \to H$ be defined by $T(u,v) = (T_1(u), T_2(v))$, $w = (u,v) \in H$, $T_i : H_i \to H_i$ is β_i-demi-contractive mapping with $\beta_i \in [0,1)$, $i = 1,2$. Then T is β-demi-contractive mapping with $\beta = \max\{\beta_1, \beta_2\}$.*

We now adapts our algorithm to solving the SEP.

Let H, S, and K be as defined in Lemma 5.3.8. Let T be as defined in Lemma 5.3.11 such that

$$\Omega_{SEP} := \{(x,y) \in F(T_1) \times F(T_2) : Ax = By\} \neq \emptyset.$$

Let M and f as defined in Lemma 5.3.9 and Lemma 5.3.10 respectively such that $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \xi\rho < \tau$, where $\tau = \frac{1}{2}\mu(2\eta - \mu k^2)$. Let $\{\alpha_k\}$ and $\{v_k\}$ be sequences in $(0,1)$ and $\{z_k\} = \{(x_k, y_k)\}$ be generated by the following Algorithm.

Algorithm 5.3.12.

Step 0: Choose initial guess $z_1 = (x_1, y_1) \in H$ and parameters $\theta, \gamma \in (0,1)$, $\sigma \in (0,2)$. Set $k = 1$.

Step 1: Compute

$$t_k = P_S(z_k - \lambda_k K^* K(z_k)), \tag{5.3.50}$$

where $\lambda_k = \gamma^{l_k}$, and l_k is the smallest non-negative integer satisfying

$$\lambda_k ||K^* K(z_k) - K^* K(t_k)|| \leq \theta ||z_k - t_k||.$$

Step 2: Compute

$$d(z_k, t_k) = z_k - t_k - \lambda_k (K^* K(z_k) - K^* K(t_k)),$$

$$w_k = z_k - \sigma \delta_k d(z_k, t_k),$$

where

$$\delta_k = \begin{cases} \dfrac{\langle z_k - t_k, d(z_k, t_k)\rangle}{||d(z_k, t_k)||^2} & if \quad d(z_k, t_k) \neq 0, \\ 0, & if \quad d(z_k, t_k) = 0. \end{cases}$$

Step 3: Compute

$$z_{k+1} = \alpha_k \xi f(z_k) + (1 - \alpha_k \mu M)(v_k T w_k + (1 - v_k) w_k).$$

Set $k \leftarrow k+1$ and go to Step 1.

Remark 5.3.13. Let $z = (x, y)$, we know that

$$P_S(z) = \big(P_C(x), P_Q(y)\big).$$

Also, since

$$K = [A, -B], \quad \text{and} \quad K^* = \begin{bmatrix} A^* \\ -B^* \end{bmatrix},$$

then

$$\begin{aligned} K^*Kw &= \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} A^*(Ax - By) \\ B^*(Ax - By) \end{bmatrix}. \end{aligned} \tag{5.3.51}$$

Define the function $F : H_1 \times H_2 \to H_1$ by

$$F(x, y) = A^*(Ax - By),$$

and $G : H_1 \times H_2 \to H_2$ by

$$G(x, y) = B^*(By - Ax).$$

Now, by setting $z_k = (x_k, y_k)$, $t_k = (u_k, v_k)$ and $w_k = (s_k, e_k)$ in Algorithm 5.3.12, then Algorithm 5.3.12 can be rewritten in the following simultaneous form:

Algorithm 5.3.14.

Step 0: Choose initial guess $(x_1, y_1) \in H_1 \times H_2$ *and parameters* $\theta, \gamma \in (0, 1)$, $\sigma \in (0, 2)$. *Set* $k = 1$.

Step 1: Compute

$$\begin{cases} u_k = P_C(x_k - \lambda_k F(x_k, y_k)), \\ v_k = P_Q(y_k - \lambda_k G(x_k, y_k)), \end{cases} \tag{5.3.52}$$

where $\lambda_k = \gamma^{l_k}$, *and* l_k *is the smallest non-negative number satisfying*

$$\begin{aligned} \lambda_k^2(||F(x_k, y_k) - F(u_k, v_k)||^2 + ||G(x_k, y_k) - G(u_k, v_k)||^2) \\ \leq \theta^2(||x_k - u_k||^2 + ||y_k - v_k||^2). \end{aligned} \tag{5.3.53}$$

Step 2: Compute

$$\begin{cases} c_k = (x_k - u_k) - \lambda_k(F(x_k, y_k) - F(u_k, v_k)) \\ d_k = (y_k - v_k) - \lambda_k(G(x_k, y_k) - G(u_k, v_k)), \end{cases}$$

and

$$\begin{cases} s_k = x_k - \sigma\delta_k c_k, \\ e_k = y_k - \sigma\delta_k d_k, \end{cases} \tag{5.3.54}$$

where

$$\delta_k = \frac{\langle x_k - u_k, c_k\rangle + \langle y_k - v_k, d_k\rangle}{||c_k||^2 + ||d_k||^2}. \tag{5.3.55}$$

Step 3: Compute

$$\begin{cases} x_{k+1} = \alpha_k \xi f_1(x_k) + (1 - \alpha_k \mu \phi_1)(v_k T_1 s_k + (1 - v_k) s_k), \\ y_{k+1} = \alpha_k \xi f_2(y_k) + (1 - \alpha_k \mu \phi_2)(v_k T_2 t_k + (1 - v_k) e_k). \end{cases} \tag{5.3.56}$$

Set $k \leftarrow k+1$ and go to Step 1.

We now prove the convergence of Algorithm 5.3.14 using Algorithm 5.3.1.

Let $(x^*, y^*) \in \Omega_{SEP}$. Observe that

$$\begin{aligned} ||s_k - x^*||^2 + ||t_k - y^*||^2 &= ||x_k - x^* - \sigma \delta_k c_k||^2 + ||y_k - y^* - \sigma \delta_k d_k||^2 \\ &\leq ||x_k - x^*||^2 + ||y_k - y^*||^2 - 2\sigma \delta_k (\langle x_k - x^*, c_k \rangle + \langle y_k - y^*, d_k \rangle) \\ &\quad + \sigma^2 \delta_k^2 (||c_k||^2 + ||d_k||^2). \end{aligned} \tag{5.3.57}$$

But

$$\begin{aligned} \langle x_k - x^*, c_k \rangle + \langle y_k - y^*, d_k \rangle &= \langle x_k - u_k, c_k \rangle + \langle u_k - x^*, c_k \rangle \\ &\quad \langle y_k - v_k, d_k \rangle + \langle v_k - y^*, d_k \rangle, \end{aligned}$$

and

$$\langle u_k - x^*, c_k \rangle + \langle v_k - y^*, d_k \rangle \geq 0.$$

Hence

$$\langle x_k - x^*, c_k \rangle + \langle y_k - y^*, d_k \rangle \geq \langle x_k - u_k, c_k \rangle + \langle y_k - v_k, d_k \rangle. \tag{5.3.58}$$

Therefore from (5.3.57) and (5.3.58), we have

$$\begin{aligned} ||s_k - x^*||^2 + ||y_k - y^*||^2 &\leq ||x_k - x^*||^2 + ||y_k - y^*||^2 - 2\sigma \delta_k (\langle x_k - u_k, c_k \rangle + \langle y_k - v_k, d_K \rangle) \\ &\quad + \sigma^2 \delta_k^2 (||c_k||^2 + ||d_k||^2). \end{aligned} \tag{5.3.59}$$

From the definition of δ_k and (5.3.54), (5.3.54), we have

$$\begin{aligned} \delta_k (\langle x_k - u_k, c_k \rangle + \langle y_k - v_k, d_k \rangle) &= \delta_k^2 (||c_k||^2 + ||d_k||^2) \\ &= \frac{1}{\sigma^2} (||s_k - x_k||^2 + ||t_k - y_k||^2). \end{aligned} \tag{5.3.60}$$

Hence from (5.3.59) and (5.3.60), we get

$$\begin{aligned} ||s_k - x^*||^2 + ||y_k - y^*||^2 &\leq ||x_k - x^*||^2 + ||y_k - y^*||^2 - \left(\frac{2 - \sigma}{\sigma} \right) (||s_k - x_k||^2 + ||t_k - y_k||^2). \\ &\leq ||x_k - x^*||^2 + ||y_k - y^*||^2. \end{aligned} \tag{5.3.61}$$

Following similar approach as in (5.3.27), we get

$$\begin{aligned} ||x_{k+1} - x^*|| + ||y_{k+1} - x^*|| &\leq \max \Bigg\{ ||x_1 - x^*|| + ||y_1 - y^*||, \\ &\quad \frac{||\xi_1 f_1(x^*) - \mu_1 \phi_1(x^*)||}{\tau_1 - \xi \rho_1} + \frac{||\xi_2 f_2(y^*) - \mu_2 \phi_2(y^*)||}{\tau_2 - \xi_2 \rho_2} \Bigg\}. \end{aligned}$$

Hence $\{||x_{k+1} - x^*|| + ||y_{k+1} - y^*||\}$ is bounded and consequently, $\{||x_k - x^*||\}$, $\{||y_k - y^*||\}$ are bounded. Thus, $\{x_k\}$ and $\{y_k\}$ are bounded.

Lemma 5.3.15. *Suppose $\Omega_{SEP} := \{(x,y) \in C \times Q : Ax = By\} \neq \emptyset$. Let λ_n be a sequence in $(0, \frac{2}{||A||^2+||B||^2})$, such that (5.3.53) holds and suppose $\liminf_{n\to\infty} \lambda_n(2-\lambda_n(||A||^2+||B||^2)) > 0$, $||x_k - u_k|| \to 0, ||y_k - v_k|| \to 0$ as $k \to \infty$. Then, there exist $(\bar{x}, \bar{y}) \in \Omega_{SEP}$ such that $x_{k_j} \rightharpoonup \bar{x}$ and $y_{k_j} \rightharpoonup \bar{y}$, where $\{x_{k_j}\}$ and $\{y_{k_j}\}$ are subsequences of $\{x_k\}$ and $\{y_k\}$ generated by Algorithm 5.3.14.*

Proof. Let $(x^*, y^*) \in \Omega_{SEP}$, then from (5.3.52), we have

$$\begin{aligned} ||u_k - x^*||^2 &= ||P_C(x_k - \lambda_k F(x_k, y_k)) - x^*||^2 \\ &\leq ||x_k - \lambda_k(A^*(Ax_k - By_k)) - x^*||^2 \\ &\leq ||x_k - x^*||^2 - 2\lambda_k\langle Ax_k - Ax^*, Ax_k - By_k\rangle \\ &\quad +\lambda_k^2||A||^2||Ax_k - By_k||^2. \end{aligned} \tag{5.3.62}$$

Similarly, we have

$$\begin{aligned} ||v_k - y^*||^2 &\leq ||y_k - y^*||^2 + 2\lambda_k\langle By_k - By^*, Ax_k - By_k\rangle \\ &\quad +\lambda_k^2||B||^2||Ax_k - By_k||^2. \end{aligned} \tag{5.3.63}$$

Adding (5.3.62) and (5.3.63) while noting that $Ax^* = By^*$, we have

$$\begin{aligned} ||u_k - x^*||^2 + ||v_k - y^*||^2 &\leq ||x_k - x^*||^2 + ||y_k - y^*||^2 - \lambda_k(2 - \lambda_k(||A||^2 + ||B||^2)) \times \\ &\quad ||Ax_k - By_k||^2. \end{aligned} \tag{5.3.64}$$

Also, note that

$$\begin{aligned} ||u_k - x^*||^2 + ||v_k - y^*||^2 &= ||u_k - x_k||^2 + 2\langle u_k - x_k, x_k - x_k - x^*\rangle \\ &\quad +||x_k - x^*||^2 + ||v_k - y_k||^2 + 2\langle v_k - y_k, y_k - y^*\rangle \\ &\quad +||y_k - y^*||^2. \end{aligned} \tag{5.3.65}$$

Then from (5.3.64) and (5.3.65), we have

$$\lim_{k\to\infty} ||Ax_k - By_k|| = 0. \tag{5.3.66}$$

Without loss of generality, we may assume that $x_{k_j} \rightharpoonup \bar{x}$ and $y_{k_j} \rightharpoonup \bar{y}$ for some $\bar{x} \in H_1$ and $\bar{y} \in H_2$. Since $\{x_k\}$ is a sequence in C, we know that $\bar{x} \in C$. Similarly, $\bar{y} \in Q$. Since $x_{k_j} \rightharpoonup \bar{x}$ and $y_{k_j} \rightharpoonup \bar{y}$, it follows that $Ax_{k_j} \rightharpoonup A\bar{x}$ and $By_{k_j} \rightharpoonup B\bar{y}$. Hence $Ax_{k_j} - By_{k_j} \rightharpoonup A\bar{x} - B\bar{y}$. By the lower semicontinuity of the squared norm, we have

$$||A\bar{x} - B\bar{y}||^2 \leq \liminf_{k\to\infty} ||Ax_{k_j} - By_{k_j}||^2 = \lim_{k\to\infty} ||Ax_k - By_k||^2 = 0.$$

Hence $A\bar{x} = B\bar{y}$. Therefore $(\bar{x}, \bar{y}) \in \Omega$. □

Now using Lemma 5.3.15 and following the line of argument in Theorem 5.3.7, we can prove the following result.

Theorem 5.3.16. *Let H, S, and K be as defined in Lemma 5.3.8. Let T be as defined in Lemma 5.3.11 such that $\Gamma := \{(x,y) \in F(T_1) \times F(T_2) : Ax = By\} \neq \emptyset$. Let M and f be as defined in Lemma 5.3.9 and Lemma 5.3.10 respectively such that $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \xi\rho < \tau$, where $\tau = \frac{1}{2}\mu(2\eta - \mu k^2)$. Let $\{\alpha_k\}$ and $\{v_k\}$ be sequences in $(0,1)$ satisfying condition (C1) and (C3) and let λ_n be a sequence in $(0, \frac{2}{||A||^2+||B||^2})$, such that (5.3.53) holds and $\liminf_{n\to\infty} \lambda_n(2 - \lambda_n(||A||^2 + ||B||^2)) > 0$. Then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 5.3.14 converges strongly to a solution $(u, v) \in \Gamma$.*

5.3.3 Numerical examples

In this subsection, we present three numerical examples which demonstrate the performance of our Algorithm 5.3.1. Let $T : H \to H$ be defined by

$$Tx = \begin{cases} -\frac{9}{2}x, & if \quad x \leq 0, \\ -2x, & if \quad x > 0. \end{cases} \tag{5.3.67}$$

It easy to see that T is demi-contractive mapping with $\beta = \frac{77}{121}$, and $F(T) = \{0\}$. We let $f = I$, $B = \frac{1}{2}I$, then $\rho = 1$ and $\eta = 1 = k$. Hence $0 < \mu < \frac{2\eta}{k^2} = 2$. Let us choose $\mu = 1$ so that $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} = 1$. As $0 < \xi\rho < \tau$, we have $\xi \in (0, 2)$. Without loss of generality, we choose $\xi = 1$.

In each example, we fix the stopping criterion as $||x_{k+1} - x_k|| = \epsilon < 10^{-5}$, $\sigma = 0.7$, $\gamma = 0.54$, $\lambda_k = 0.15$ and let $\alpha_k = \frac{1}{k+1}$ and $v_k = \frac{2k+3}{4k+12}$. The projection onto the feasible set C is carry-out by using the MATLAB solver 'fmincon' and the projection onto an hyperplane $Q = \{x \in H : \langle a, x\rangle = 0\}$ is defined by

$$P_Q(x) = x - \frac{\langle a, x\rangle}{||a||^2}a.$$

Example 5.3.17. First, we consider the Hp-Hard problem. Let $A : \mathbb{R}^m \to \mathbb{R}^m$ define by $Ax = Mx + q$ where

$$M = NN^T + S + D,$$

N is a $m \times m$ matrix, S is a $m \times m$ skew-symmetric matrix, D is a $m \times m$ diagonal matrix, whose diagonal entries are nonnegative so that M is positive definite and q is a vector in $\mathbb{R}^m$. The feasible set $C \subset \mathbb{R}^m$ is the closed and convex polyhedron which is defined as $C = \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : Qx \leq b\}$, where Q is a $l \times m$ matrix and b is a nonnegative vector. It is clear that A is monotone (hence, pseudo-monotone) and L-Lipschitz continuous with $L = ||M||$. For experimental purpose, all the entries of N, S, D and b are generated randomly as well as the starting point $x_1 \in [0, 1]^m$ and q is equal to the zero vector. In this case, the solution to the corresponding variational inequality is $\{0\}$ and also, $Sol := \Omega_{VIP} \cap F(T) = \{0\}$. We take $m = 50, 100, 200$ and compare the output of Algorithm 5.3.1 with Algorithm (5.3.5) and Algorithm (5.3.3). The numerical results are reported in Table 5.4 and Figure 5.5.

Example 5.3.18. Let $H = L^2([0, 2\pi])$ with norm $||x|| = (\int_0^{2\pi} |x(t)|^2 dt)^{\frac{1}{2}}$ and inner product $\langle x, y\rangle = \int_0^{2\pi} x(t)y(t)dt$, $x, y \in H$. The operator $A : H \to H$ is defined by $Ax(t) = \frac{1}{2}\max\{0, x(t)\}$, $t \in [0, 2\pi]$ for all $x \in H$. It can easily be verified that A is Lipschitz continuous and monotone. The feasible set $C = \{x \in H : \int_0^{2\pi}(t^2 + 1)x(t)dt \leq 1\}$. Observe that $Sol = \{0\}$. We choose the following starting points and compare the result of Algorithm 5.3.1 with Algorithm (5.3.5) and Algorithm (5.3.3).

$$(i)\ x_1 = \frac{1}{3}t^2\exp(-3t), \quad (ii)\ x_1 = \frac{1}{200}\sin(3\pi t)cos(2\pi t), \quad (iii)\ x_1 = \frac{1}{50}\cos(3t)\exp(2t).$$

The numerical results are shown in Table 5.5 and Figure 5.6.

Example 5.3.19. Finally, we consider the Kojima-Shindo nonlinear complementarity problem (NCP) which was considered in [174], where $n = 4$ and the mapping A is defined by

$$A(x_1, x_2, x_3, x_4) = \begin{bmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{bmatrix}. \tag{5.3.68}$$

The feasible set $C = \{x \in \mathbb{R}^4_+ : x_1 + x_2 + x_3 + x_4 = 4\}$. We choose the following starting points and test our Algorithm 5.3.1 with Algorithm (5.3.5).

$$(i)\ x_1 = (2, 0, 0, 2), \quad (ii)\ x_1 = (1, 1, 1, 1), \quad (iii)\ x_1 = (1, 2, 0, 1).$$

The results are summarized in Table 5.6 and Figure 5.7.

Table 5.4: Numerical results for Example 5.3.17.

		Alg. 5.3.1	Alg. 5.3.5	Alg. 5.3.3
$m = 50$	CPU time (sec)	3.7937	12.5925	9.5794
	No. of Iter.	8	8	30
$m = 100$	CPU time (sec)	4.7710	15.6752	12.5470
	No. of Iter.	9	9	31
$m = 200$	CPU time (sec)	5.2795	16.6502	13.5667
	No. of Iter.	10	10	33

Table 5.5: Numerical results for Example 5.3.18.

$x_1 =$		Alg. 5.3.1	Alg. 5.3.5	Alg. 5.3.3
$\frac{1}{3}t^2 \exp(-3t)$	CPU time (sec)	0.4405	0.9491	1.5811
	No. of Iter.	6	12	28
$\frac{1}{200} \sin(3\pi t) cos(2\pi t)$	CPU time (sec)	0.4423	0.5964	6.4044
	No. of Iter.	7	9	29
$\frac{1}{50} \cos(3t) \exp(2t)$	CPU time (sec)	3.6693	5.2286	7.0858
	No. of Iter.	7	14	34

Table 5.6: Numerical result for for Example 5.3.19.

$x_1 =$		Alg. 5.3.1	Alg. 5.3.5
$(2, 0, 0, 2)$	CPU time (sec)	2.4540	6.0885
	No. of Iter.	9	6
$(1, 1, 1, 1)$	CPU time (sec)	4.2861	16.6686
	No. of Iter.	10	18
$(1, 2, 0, 1)$	CPU time (sec)	6.8174	15.6381
	No. of Iter.	24	19

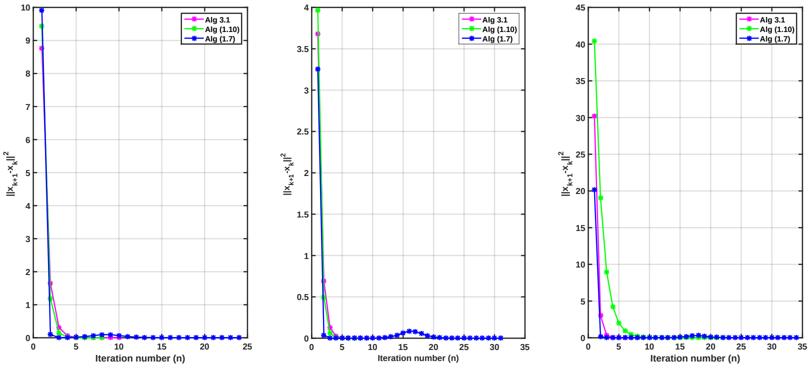


Figure 5.5: Example 5.3.17, Left: $m = 50$; Middle: $m = 100$; Right: $m = 200$.

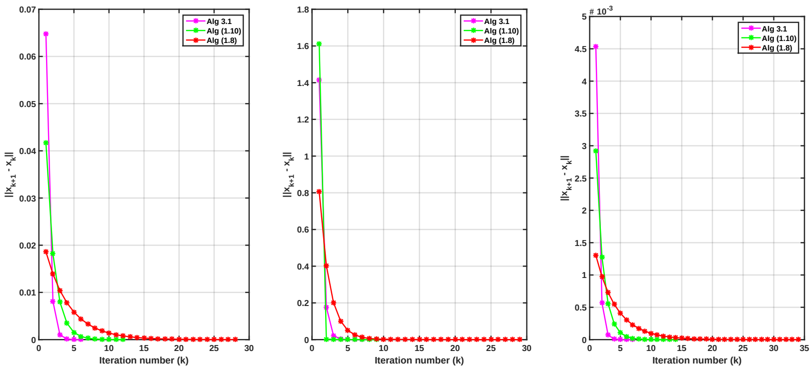


Figure 5.6: Example 5.3.18, Left: $x_1 = \frac{1}{3}t^2 \exp(-3t)$; Middle: $x_1 = \frac{1}{200}\sin(3\pi t)cos(2\pi t)$; Right: $x_1 = \frac{1}{50}\cos(3t)\exp(2t)$.

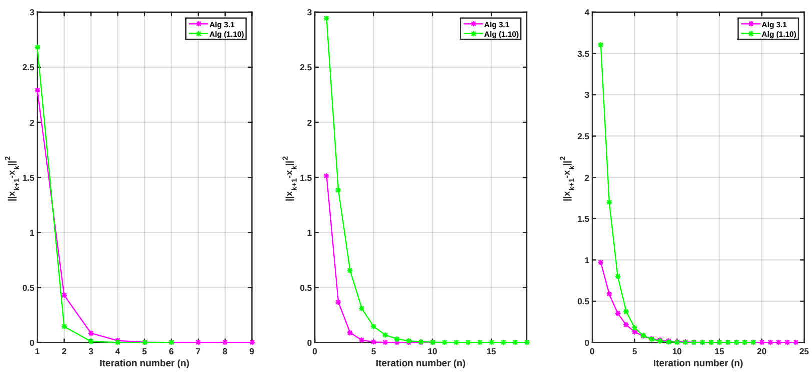


Figure 5.7: Example 5.3.18, Left: $x_1 = (2, 0, 0, 2)$; Middle: $x_1 = (1, 1, 1, 1)$; Right: $x_1 = (1, 2, 0, 1)$.

CHAPTER 6

Split Feasibility Problems in Banach Spaces

Let E_1 and E_2 be Banach spaces and let C and Q be nonempty closed convex subsets of E_1 and E_2 respectively. Let $A : E_1 \to E_2$ ba a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Recently, the study of SFP (1.1.9) in real Banach spaces has experienced an explosive attention after Schöpfer extend the SFP (1.1.9) from real Hilbert space to p-uniformly convex real Banach spaces which are also uniformly smooth. He introduced the following algorithm and proved its weak convergence to solution of SFP (1.1.9) in Banach space: for $x_1 \in E_1$, set

$$x_{n+1} = \Pi_C J^{E_1^*}\left[J^{E_1}(x_n) - t_n A^* J^{E_2}(Ax_n - P_Q(Ax_n))\right], \quad n \geq 1, \tag{6.0.1}$$

where Π_C denotes the Bregman projection from E_1 onto C and J^E is the duality mapping with the condition that J_p^E is weak-to-weak continuous. Based on an idea in Nakajo and Takahashi [192], Wang [265] introduced the following algorithm with strong convergence property: for any initial guess $x_0 \in E_1$, define $\{x_n\}$ recursively by

$$\begin{cases} y_n = Tx_n, \\ D_n = \{u \in E : D_p(y_n, u) \leq D_p(x_n, u)\}, \\ E_n = \{u \in E : \langle x_n - u, J_p^E x_0 - J_p^E x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{D_n \cap E_n}(x_0), \end{cases} \tag{6.0.2}$$

where Tx_n is defined for each $n \in \mathbb{N}$ by

$$T_n x = \begin{cases} \Pi_{C_{i(n)}}(x) & 1 \leq i(n) \leq r, \\ J_q^{E_1^*}[J_p^{E_1} x - t_n A^* J_p^{E_2}(I - P_{Q_{i(n)}})Ax], & \end{cases}$$

$i : \mathbb{N} \to I$ is the cyclic control mapping $i(n) - n \; mod(r + s) + 1$, and t_n satisfies

$$0 < t \leq t_n \leq \left(\frac{q}{C_q||A||^p}\right)^{\frac{1}{q-1}}.$$

Motivated by the ongoing effort on SFP (1.1.9) in Banach spaces, in this chapter, we introduce some new iterative methods for solving SFP (1.1.9) and its generalizations in real Banach spaces.

6.1 Split Equality Variation Inclusion Problems in Banach Spaces without Operator Norms

Let E_1, E_2, E_3 be Banach spaces, $M_1 : E_1 \to 2^{E_1^*}$ and $M_2 : E_2 \to 2^{E_2^*}$ be maximal monotone operators. The Split Equality Variational Inclusion Problem SEVIP is defined as: Find $x^* \in E_1$ and $y^* \in E_2$ such that

$$\begin{cases} 0 \in M_1(x^*) \text{ and } 0 \in M_2(y^*), \\ Ax^* = By^*, \end{cases} \tag{6.1.1}$$

where $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators.

In this section, we introduce an iterative algorithm with a self adaptive stepsize and prove a strong convergence theorem for approximating solution of SEVIP (6.1.1) in p-uniformly convex Banach spaces which are also uniformly smooth such that the ardours task of computing operator norms is avoided.

6.1.1 Main result

Theorem 6.1.1. *Let E_1, E_2 and E_3 be p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. Let $T_1 : E_1 \to 2^{E_1^*}$ and $T_2 \to 2^{E_2^*}$ be maximal monotone operators such that $\Gamma := \{(\bar{x}, \bar{y}) \in T_1^{-1}(0) \times T_2^{-1}(0); A\bar{x} = B\bar{y}\}$ is nonempty. For fixed $u \in E_1$ and $v \in E_2$, choose an initial guess $x_1 \in E_1$ and $y_1 \in E_2$ arbitrarily and let $\{\alpha_n\} \subset [0, 1]$. Assume that the nth iterate $(x_n, y_n) \in E_1 \times E_2$ has been constructed; then we calculate the $(n+1)$th iterate (x_{n+1}, y_{n+1}) via the formula*

$$\begin{cases} u_n = R_{\lambda T_1} J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)), \\ x_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(u_n)), \\ v_n = R_{\lambda T_2} J_q^{E_2^*}(J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)), \\ y_{n+1} = J_p^{E_2^*}(\alpha_n J_p^{E_2}(v) + (1-\alpha_n) J_p^{E_2}(v_n)), \end{cases} \tag{6.1.2}$$

where $\lambda > 0$, A^* and B^* are the adjoints of A and B respectively and the stepsize t_n is choosen in such a way that

$$t_n \in \Big(\epsilon, \Big(\frac{q||Ax_n - By_n||^p}{C_q||A^* J_p^{E_3}(Ax_n - By_n)||^q + Q_q||B^* J_p^{E_3}(Ax_n - By_n)||^q} - \epsilon\Big)^{\frac{1}{q-1}}\Big),\ n \in \Omega, \tag{6.1.3}$$

for small enough ϵ, otherwise $t_n = t$(t being any nonegative value), where the set of indices $\Omega = \{n : Ax_n - By_n \neq 0\}$. Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequence $\{(x_n, y_n)\}$ *strongly converges to* $(\bar{x}, \bar{y}) = (\Pi_{\Gamma_1} u, \Pi_{\Gamma_2} v)$*, where* $\Gamma_i = \{z \in E_i : 0 \in T_i(z)\}$ *for* $i = 1, 2$*, and* Π_{Γ_1} *and* Π_{Γ_2} *are the Bregman projections onto* Γ_1 *and* Γ_2 *respectively.*

Proof. We divide the proof into three steps:
STEP 1: We show that the step size (6.1.3) is well define. Observe that for any $(x, y) \in \Gamma$, we have

$$\langle A^* J_p^{E_3}(Ax_n - By_n), x_n - x\rangle = \langle J_p^{E_3}(Ax_n - By_n), Ax_n - Ax\rangle, \tag{6.1.4}$$

and

$$\langle B^* J_p^{E_3}(Ax_n - By_n), y - y_n\rangle = \langle J_p^{E_3}(Ax_n - By_n), By - By_n\rangle. \tag{6.1.5}$$

By adding (6.1.4) and (6.1.5) and taking into account the fact $Ax = By$, we have

$$\begin{aligned} ||Ax_n - By_n||^p &= \langle A^* J_p^{E_3}(Ax_n - By_n), x_n - x\rangle + \langle B^* J_p^{E_3}(Ax_n - By_n), y - y_n\rangle \\ &\leq ||A^* J_p^{E_3}(Ax_n - By_n)||||Ax_n - x|| + ||B^* J_p^{E_3}(Ax_n - By_n)||||y - y_n||. \end{aligned}$$

Therefore, for $n \in \Omega$, that is, $||Ax_n - By_n|| > 0$, we have $||A^* J_p^{E_3}(Ax_n - By_n)|| \neq 0$ and $||B^*(Ax_n - By_n)|| \neq 0$. Thus t_n is well defined.

STEP 2: We show that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Now let $(x^*, y^*) \in \Gamma$, then from (6.1.2), we have that

$$\begin{aligned} D_p(u_n, x^*) &= D_p(R_{\lambda T_1} J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)), x^*) \\ &\quad \leq D_p(J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)), x^*) \\ &\quad = \frac{1}{q}||J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)||^q - \langle J_p^{E_1}(x_n), x^*\rangle \\ &\quad + t_n\langle A^* J_p^{E_3}(Ax_n - By_n), x^*\rangle + \frac{1}{p}||x^*||^p \\ &\leq \frac{1}{q}||J_p^{E_1}(x_n)||^q - t_n\langle J_p^{E_3}(Ax_n - By_n), Ax_n\rangle + \frac{C_q}{q} t_n^q ||A^* J_p^{E_3}(Ax_n - By_n)||^q \\ &\quad - \langle J_p^{E_1}(x_n), x^*\rangle \\ &\quad + t_n\langle J_p^{E_3}(Ax_n - By_n), Ax^*\rangle + \frac{1}{p}||x^*||^p \\ &= \frac{1}{q}||x_n||^p - \langle J_p^{E_1}(x_n), x^*\rangle + \frac{1}{p}||x^*||^p - t_n\langle J_p^{E_3}(Ax_n - By_n), Ax_n - Ax^*\rangle \\ &\quad + \frac{C_q}{q} t_n^q ||A^* J_p^{E_3}(Ax_n - By_n)||^q \\ &= D_p(x_n, x^*) - t_n\langle J_p^{E_3}(Ax_n - By_n), Ax_n - Ax^*\rangle \\ &\quad + \frac{C_q t_n^q}{q}||A^* J_p^{E_3}(Ax_n - By_n)||^q. \end{aligned} \tag{6.1.6}$$

Following similar process as above, we obtain

$$
\begin{aligned}
D_p(v_n, y^*) &\leq D_p(J_q^{E_2^*}(J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)), y^*) &\quad (6.1.7)\\
&\leq D_p(y_n, y^*) - t_n \langle J_p^{E_3}(Ax_n - By_n), By^* - By_n \rangle \\
&\quad + \frac{Q_q t_n^q}{q} ||B^* J_p^{E_3}(Ax_n - By_n)||^q. &\quad (6.1.8)
\end{aligned}
$$

Adding (6.1.6) and (6.1.8), noting that $Ax^* = By^*$, we have

$$
\begin{aligned}
D_p(u_n, x^*) + D_p(v_n, y^*) &\leq D_p(x_n, x^*) + D_p(y_n, y^*) - t_n \Big[||Ax_n - By_n||^p \\
&\quad - \frac{t_n^{q-1}}{q} (C_q ||A^* J_p^{E_3}(Ax_n - By_n)||^q \\
&\quad + Q_q ||B^* J_p^{E_3}(Ax_n - By_n)||^q) \Big]. &\quad (6.1.9)
\end{aligned}
$$

Thus

$$D_p(u_n, x^*) + D_p(v_n, y^*) \leq D_p(x_n, x^*) + D_p(y_n, y^*). \quad (6.1.10)$$

Also from (6.1.2), we have

$$
\begin{aligned}
D_p(x_{n+1}, x^*) &= D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(u_n)), x^*) \\
&\leq \alpha_n D_p(u, x^*) + (1-\alpha_n) D_p(u_n, x^*).
\end{aligned}
$$

Similarly, we have

$$D_p(y_{n+1}, y^*) \leq \alpha_n D_p(v, y^*) + (1-\alpha_n) D_p(v_n, y^*). \quad (6.1.11)$$

Hence

$$
\begin{aligned}
D_p(x_{n+1}, x^*) + D_p(y_{n+1}, y^*) &\leq \alpha_n (D_p(u, x^*) + D_p(v, y^*)) + (1-\alpha_n) \times \\
&\quad (D_p(u_n, x^*) + D_p(v_n, y^*)) \\
&\leq \alpha_n (D_p(u, x^*) + D_p(v, y^*)) + (1-\alpha_n) \times \\
&\quad (D_p(x_n, x^*) + D_p(y_n, y^*)) \\
&\leq \max\{D_p(u, x^*) + (D_p(v, y^*), D_p(x_n, x^*) + D_p(y_n, y^*)\} \\
&\vdots \\
&\leq \max\{D_p(u, x^*) + (D_p(v, y^*), D_p(x_1, x^*) + D_p(y_1, y^*)\}.
\end{aligned}
$$

Thus $\{D_p(x_{n+1}, x^*) + D_p(y_{n+1}, y^*)\}$ is bounded. Consequently, $\{D_p(x_n, x^*)\}$ and $\{D_p(y_n, y^*)\}$ are bounded. It therefore, follows that $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ are bounded.

STEP 3: Next, we prove that $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{\Gamma_1} u$ and $\{y_n\}$ converges strongly to $\bar{y} = \Pi_{\Gamma_2} v$. From (6.1.2), we have that

$$\begin{aligned}
D_p(x_{n+1}, x^*) &= D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)(u_n)), x^*) \\
&= V_p(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)(u_n), x^*) \\
&= V_p(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)(u_n) - \alpha_n(J_p^{E_1}(u) - J_p^{E_1}(x^*)), x^*) \\
&\quad + \langle \alpha_n(J_p^{E_1}(u) - J_p^{E_1}(x^*)), J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)(u_n)) - x^* \rangle \\
&= V_p(\alpha_n J_p^{E_1}(x^*) + (1-\alpha_n)(u_n), x^*) + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^* \rangle \\
&= D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(x^*) + (1-\alpha_n)(u_n)), x^*) + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^* \rangle \\
&\leq \alpha_n D_p(x^*, x^*) + (1-\alpha_n) D_p(u_n, x^*) + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^* \rangle \\
&= (1-\alpha_n) D_p(u_n, x^*) + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^* \rangle. \qquad (6.1.12)
\end{aligned}$$

Similarly, we have

$$D_p(y_{n+1}, y^*) \leq (1-\alpha_n) D_p(v_n, y^*) + \alpha_n \langle J_p^{E_2}(u) - J_p^{E_2}(y^*), y_{n+1} - y^* \rangle. \qquad (6.1.13)$$

Therefore, from (6.1.10) we have

$$\begin{aligned}
D_p(x_{n+1}, x^*) + D_p(y_{n+1}, y^*) &\leq (1-\alpha_n)(D_p(u_n, x^*) + D_p(v_n, y^*)) \\
&\quad + \alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^* \rangle \\
&\quad + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^* \rangle) \\
&\leq (1-\alpha_n)(D_p(x_n, x^*) + D_p(y_n, y^*)) \\
&\quad + \alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^* \rangle \\
&\quad + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^* \rangle). \qquad (6.1.14)
\end{aligned}$$

Now, we set $\Theta_n(x^*, y^*) := D_p(x_n, x^*) + D_p(y_n, y^*)$, and divide the remaining part of the proof into two cases.
Case A: Suppose there exists $n_0 \in \mathbb{N}$ such that $\{\Theta_n(x^*, y^*)\}$ is monotonically non-increasing for all $n \geq n_0$. Then $\{\Theta_n(x^*, y^*)\}$ converges as $n \to \infty$ and so

$$\Theta_n(x^*, y^*) - \Theta_{n+1}(x^*, y^*) \to 0, \quad n \to \infty.$$

Let $M_n := C_q ||A^* J_p^{E_3}(Ax_n - By_n)||^q + Q_q ||B^* J_p^{E_3}(Ax_n - By_n)||^q$, then from (6.1.9), we have

$$\begin{aligned}
t_n \Big[||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q} M_n \Big] &\leq D_p(x_n, x^*) + D_p(y_n, y^*) \\
&\quad - (D_p(u_n, x^*) + (D_p(v_n, y^*)), \qquad (6.1.15)
\end{aligned}$$

and therefore,

$$\begin{aligned}
t_n \Big[||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q} M_n \Big] &\leq D_p(x_n, x^*) + D_p(y_n, y^*) - (D_p(u_n, x^*) + (D_p(v_n, y^*)) \\
&= \Theta_n(x^*, y^*) - \Theta_{n+1}(x^*, y^*) + \Theta_{n+1}(x^*, y^*) \\
&\quad - (D_p(u_n, x^*) + (D_p(v_n, y^*)). \qquad (6.1.16)
\end{aligned}$$

Moreover, it follows from (6.1.14) and (6.1.16) and the fact that $\alpha_n \to 0$ as $n \to \infty$ that

$$\begin{aligned} t_n\Big[||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q}M_n\Big] &\leq \Theta_n(x^*, y^*) - \Theta_{n+1}(x^*, y^*) + (1-\alpha_n)(D_p(u_n, x^*) + D_p(v_n, y^*)) \\ &\quad +\alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle \\ &\quad +\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle) \\ &\quad -(D_p(u_n, x^*) + (D_p(v_n, y^*)) \to 0, \quad n \to \infty. \end{aligned} \tag{6.1.17}$$

Again, by the condition on the stepsize t_n, we have that

$$t_n^{q-1} < \frac{q||Ax_n - By_n||^p}{M_n} - \epsilon,$$

which implies that

$$t_n^{q-1}M_n < q||Ax_n - By_n||^p - \epsilon M_n,$$

and thus

$$\frac{\epsilon M_n}{q} < ||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q}M_n \to 0, \quad n \to \infty.$$

Therefore,

$$C_q||A^*J_p^{E_3}(Ax_n - By_n)||^q + Q_q||B^*J_p^{E_3}(Ax_n - By_n)||^q \to 0, \quad n \to \infty.$$

It follows that

$$\lim_{n\to\infty} ||A^*J_p^{E^3}(Ax_n - By_n)||^q = 0 \text{ and } \lim_{n\to\infty} ||B^*J_p^{E^3}(Ax_n - By_n)||^q = 0. \tag{6.1.18}$$

Also, we have from (6.1.17) that

$$\begin{aligned} t_n||Ax_n - By_n||^p &\leq \alpha_n(D_p(u, x^*) + D_p(v, y^*)) - (1-\alpha_n)\Theta_n(x^*, y^*) - \Theta_{n+1}(x^*, y^*) \\ &\quad +\frac{t_n^q}{q}M_n \to 0, \quad n \to \infty. \end{aligned} \tag{6.1.19}$$

Hence,

$$\lim_{n\to\infty} ||Ax_n - By_n||^p = 0. \tag{6.1.20}$$

Let $a_n = J_q^{E_1^*}(J_p^{E_1}(x_n) - t_nA^*J_p^{E_3}(Ax_n - By_n))$ and $b_n = J_q^{E_2}(J_p^{E_2}(y_n) + t_nB^*J_p^{E_3}(Ax_n - By_n))$, then $u_n = R_{\lambda T_1}a_n$ and $v_n = R_{\lambda T_2}b_n$. Following similar argument as in (6.1.6), (6.1.7),(6.1.8) and (6.1.9) we obtain

$$D_p(a_n, x^*) + D_p(b_n, y^*) \leq D_p(x_n, x^*) + D_p(y_n, y^*).$$

It follows from (2.5.28) that

$$\begin{aligned} D_p(a_n, u_n) + D_p(b_n, v_n) &= D_p(a_n, R_{\lambda T_1}a_n) + D_p(b_n, R_{\lambda T_2}b_n) \\ &\leq (D_p(a_n, x^*) + D_p(b_n, y^*)) - (D_p(u_n, x^*) + D_p(v_n, y^*)) \\ &\leq (D_p(x_n, x^*) + D_p(y_n, y^*)) - (D_p(u_n, x^*) + D_p(v_n, y^*)) \\ &= (D_p(x_n, x^*) + D_p(y_n, y^*)) - (D_p(x_{n+1}, x^*) + D_p(y_{n+1}, y^*)) \\ &\quad +(D_p(x_{n+1}, x^*) + D_p(y_{n+1}, y^*)) - (D_p(u_n, x^*) + D_p(v_n, y^*)) \\ &\leq (D_p(x_n, x^*) + D_p(y_n, y^*)) - (D_p(x_{n+1}, x^*) + D_p(y_{n+1}, y^*)) \\ &\quad +\alpha_n(D_p(u, x^*) + D_p(v, y^*)) + (1-\alpha_n)(D_p(u_n, x^*) + D_p(v_n, y^*)) \\ &\quad -(D_p(u_n, x^*) + D_p(v_n, y^*)) \to 0, \ n \to \infty. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} D_p(a_n, u_n) = 0, \quad \text{and} \quad \lim_{n\to\infty} D_p(b_n, v_n) = 0.$$

Therefore,

$$\lim_{n\to\infty} ||a_n - u_n|| = 0, \quad \text{and} \quad \lim_{n\to\infty} ||b_n - v_n|| = 0. \tag{6.1.21}$$

Since E_1 and E_2 are uniformly smooth, then $J_p^{E_1}$ and $J_p^{E_2}$ are uniformly continuous on bounded subsets of E_1 and E_2, respectively. Thus

$$\lim_{n\to\infty} ||J_p^{E_1} a_n - J_p^{E_1} u_n|| = 0, \quad \text{and} \quad \lim_{n\to\infty} ||J_p^{E_2} b_n - J_p^{E_2} v_n|| = 0. \tag{6.1.22}$$

It follows from the definition of a_n that

$$\begin{aligned} 0 &\leq ||J_p^{E_1}(a_n) - J_p^{E_1}(x_n)|| \\ &\leq t_n||A^*||||J_p^{E_3}(Ax_n - By_n)|| \\ &= t_n||A^*||||Ax_n - By_n||^{p-1} \to 0, \ n \to \infty. \end{aligned}$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} ||a_n - x_n|| = 0, \ n \to \infty. \tag{6.1.23}$$

Similarly, we can show that

$$\lim_{n\to\infty} ||b_n - y_n|| = 0, \ n \to \infty. \tag{6.1.24}$$

It follows therefore from (6.1.21) that

$$||u_n - x_n|| \leq ||u_n - a_n|| + ||a_n - x_n|| \to 0, \ n \to \infty, \tag{6.1.25}$$

and

$$||v_n - y_n|| \leq ||v_n - b_n|| + ||b_n - y_n|| \to 0, \ n \to \infty.$$

Furthermore, from (6.1.2), we have

$$\begin{aligned} D_p(x_{n+1}, u_n) &\leq \alpha_n D_p(u, u_n) + (1 - \alpha_n) D_p(u_n, u_n) \\ &= \alpha_n D_p(u, u_n) \to 0, \ n \to \infty, \end{aligned}$$

and

$$\begin{aligned} D_p(y_{n+1}, v_n) &\leq \alpha_n D_p(v, v_n) + (1 - \alpha_n) D_p(v_n, v_n) \\ &\leq \alpha_n D_p(v, v_n) \to 0, \ n \to \infty. \end{aligned}$$

Hence

$$\lim_{n\to\infty} ||x_{n+1} - u_n|| = \lim_{n\to\infty} ||y_{n+1} - v_n|| = 0.$$

This together with (6.1.25) implies that

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - u_n|| + ||u_n - x_n|| \to 0, \tag{6.1.26}$$

and

$$||y_{n+1} - y_n|| \leq ||y_{n+1} - v_n|| + ||v_n - y_n|| \to 0, \ n \to \infty.$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, there exist subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{y_{n_i}\}$ of $\{y_n\}$ such that $x_{n_i} \rightharpoonup \bar{x} \in \omega(x_n)$ and $y_{n_i} \rightharpoonup \omega(y_n)$ respectively. Now, since $\lim_{n\to\infty} ||u_n - x_n|| = 0$ and $\lim_{n\to\infty} ||v_n - y_n|| = 0$, we obtain $u_{n_i} \rightharpoonup \bar{x}$ and $v_{n_i} \rightharpoonup \bar{y}$. Let $(z, u) \in G(T_1)$, that is $z \in T_1 u$. Since $u_{n_i} = R_{\lambda T_1} a_{n_i}$ for all $\lambda > 0$, we have

$$J_p^{E_1} a_{n_i} \in (J_p^{E_1} + \lambda T_1) u_{n_i},$$

which implies that

$$\frac{1}{\lambda}(J_p^{E_1} a_{n_i} - J_p^{E_1} u_{n_i}) \in T_1 u_{n_i}.$$

By the maximal monotonicity of T_1, we have

$$\langle z - \frac{1}{\lambda}(J_p^{E_1} a_{n_i} - J_p^{E_1} u_{n_i}), u - u_{n_i}\rangle \geq 0,$$

which implies that

$$\langle z, u - u_{n_i}\rangle \geq \frac{1}{\lambda}\langle u - n_{n_i}, J_p^{E_1} a_{n_i} - J_p^{E_1} u_{n_i}\rangle.$$

It follows from (6.1.22) and the fact that $u_{n_i} \rightharpoonup \bar{x}$ that

$$\langle z, u - \bar{x}\rangle \geq 0.$$

Since T_1 is maximal monotone, we have $0 \in T_1 \bar{x}$.
Following similar analysis as above, we obtain $0 \in T_2 \bar{y}$.

Now, since $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators, we have $Ax_{n_i} \rightharpoonup A\bar{x}$ and $By_{n_i} \rightharpoonup B\bar{y}$. By the weak lower semicontinuity of the norm and (6.1.20), we have

$$||A\bar{x} - B\bar{y}|| \leq \liminf_{i\to\infty} ||Ax_{n_i} - By_{n_i}|| = 0.$$

Hence, $A\bar{x} = B\bar{y}$.
We now show the sequence $\{(x_n, y_n)\}$ strongly converges to $(x^*, y^*) = (\Pi_{\Gamma_1} u, \Pi_{\Gamma_2} v)$. From (6.1.14), we have

$$\begin{aligned} D_p(x_{n+1}, x^*) + D(y_{n+1}, y^*) \ &\leq \ (1 - \alpha_n)(D_p(x_n, x^*) + D_p(y_n, y^*)) \\ &\quad + \alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle \\ &\quad + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle). \end{aligned} \tag{6.1.27}$$

Choose subsequences $\{x_{n_j}\}$ of $\{x_n\}$ and $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle = \lim_{j\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_j+1} - x^*\rangle,$$

and

$$\limsup_{n\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle = \lim_{j\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n_j+1} - y^*\rangle.$$

Since $x_{n_j} \rightharpoonup \bar{x}$ and $y_{n_j} \rightharpoonup \bar{y}$, it follows from (2.5.17) that

$$\begin{aligned}\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle &= \lim_{j\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_j+1} - x^*\rangle \\ &= \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), \bar{x} - x^*\rangle \leq 0, \quad (6.1.28)\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle &= \lim_{j\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n_j+1} - y^*\rangle \\ &= \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), \bar{y} - y^*\rangle \leq 0. \quad (6.1.29)\end{aligned}$$

Using Lemma 2.6.29 in (6.1.27), we conclude that

$$D_p(x_n, x^*) + D_p(y_n, y^*) \to 0, \; n \to \infty. \tag{6.1.30}$$

Thus, $D_p(x_n, x^*) \to 0$ and $D_p(y_n, y^*) \to 0$, $n \to \infty$. Therefore $x_n \to x^*$ and $y_n \to y^*$.

Case 2: Assume that $\{\Theta_n(x^*, y^*)\}$ is not monotonically decreasing. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by

$$\tau(n) = \max\{k \in \mathbb{N} : k \leq n, \tau_k \leq \tau_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$0 \leq \Theta_{\tau(n)}(x^*, y^*) \leq \Theta_{\tau(n)+1}(x^*, y^*), \quad \forall n \geq n_0.$$

Following similar analysis as in Case 1, we conclude that $\lim_{n\to\infty} ||Ax_{\tau(n)} - By_{\tau(n)}|| = 0$; $\lim_{n\to\infty} ||x_{\tau(n)+1} - x_{\tau(n)}|| = 0$ and $\lim_{n\to\infty} ||y_{\tau(n)+1} - y_{\tau(n)}|| = 0$. Also we have that

$$\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{\tau(n)+1} - x^*\rangle \leq 0 \text{ and } \limsup_{n\to\infty}\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{\tau(n)+1} - y^*\rangle \leq 0. \tag{6.1.31}$$

Now, since $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ are bounded, there exist subsequences of $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ still denoted as $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ which converge weakly to $\bar{x} \in E_1$ and $\bar{y} \in E_2$ respectively. From (6.1.14), we have

$$\begin{aligned}\Theta_{\tau(n)+1}(x^*, y^*) \leq\; &(1 - \alpha_{\tau(n)})\Theta_{\tau(n)}(x^*, y^*) + \alpha_{\tau(n)}(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{\tau(n)+1} - x^*\rangle \\ &+\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{\tau(n)+1} - y^*\rangle). \quad (6.1.32)\end{aligned}$$

Since $\Theta_{\tau(n)}(x^*, y^*) \leq \Theta_{\tau(n)+1}(x^*, y^*)$, it follows from (6.1.32) that

$$\Theta_{\tau(n)}(x^*, y^*) \leq \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{\tau(n)+1} - x^*\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{\tau(n)+1} - y^*\rangle.$$

Then from (6.1.31), we have that

$$\lim_{n\to\infty}\Theta_{\tau(n)}(x^*, y^*) = \lim_{n\to\infty}(D_p(x_{\tau(n)}, x^*) + D_p(y_{\tau(n)}, y^*)) = 0.$$

Hence, $\lim_{n\to\infty} D_p(x_{\tau(n)}, x^*) = 0$ and $\lim_{n\to\infty} D_p(y_{\tau(n)}, y^*) = 0$.
Thus we have $\lim_{n\to\infty} ||x_{\tau(n)} - x^*|| = 0$ and $\lim_{n\to\infty} ||y_{\tau(n)} - y^*|| = 0$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Theta_n(x^*, y^*) \leq \max\{\Theta_{\tau(n)}(x^*, y^*), \Theta_{\tau(n)+1}(x^*, y^*)\} = \Theta_{\tau(n)+1}(x^*, y^*).$$

Hence, $\lim_{n\to\infty} \Theta_n(x^*, y^*) = \lim_{n\to\infty} (D_p(x_n, x^*) + D_p(y_n, y^*)) = 0$.
Thus,

$$\lim_{n\to\infty} D_p(x_n, x^*) = \lim_{n\to\infty} D_p(y_n, y^*) = 0.$$

Therefore, we have

$$\lim_{n\to\infty} ||x_n - x^*|| = 0 \text{ and } \lim_{n\to\infty} ||y_n - y^*|| = 0.$$

This implies that the sequences $\{(x_n, y_n)\}$ strongly converges to $(x^*, y^*) = (\Pi_{\Gamma_1} u, \Pi_{\Gamma_2} v)$. □

6.1.2 Applications

Split Equality Feasibility Problem:

Let E be a p-uniformly real Banach space which is also uniformly smooth. Given a proper, convex and lower semicontinuous function $f : E \to \mathbb{R} \cup \{+\infty\}$, the subdifferential of such function is the mapping $\partial f : E \to 2^{E^*}$ defined by

$$\partial f(x) = \{x^* \in E^* : f(x) - f(u) \leq \langle x - u, x^* \rangle, \ \forall\ u \in E\}.$$

We define the resolvent $R_{\lambda \partial_{i_C}}$ of ∂_{i_C} for $\lambda > 0$ as

$$R_{\lambda \partial_{i_C}} x = (J_p^E + \lambda \partial_{i_C})^{-1} J_p^E x,$$

for all $x \in E$. By definitions, we obtain

$$\begin{aligned} \partial_{i_C} x &= \{x^* \in E^* : i_C x + \langle x^*, u - x \rangle \leq i_C u, \ \forall\ u \in E\} \\ &= \{x^* \in E^* : \langle x^*, u - x \rangle \leq 0, \ \forall\ u \in C\} \\ &= N_C x, \end{aligned}$$

for all $x \in C$. Hence, for $\lambda > 0$, we have that

$$\begin{aligned} u = R_{\lambda \partial_{i_C}} x &\Leftrightarrow J_p^E x \in J_p^E u + \lambda \partial i_C u \Leftrightarrow J_p^E (x - u) \in \lambda N_C u \\ &\Leftrightarrow \langle J_p^E (x - u), z - u \rangle \leq 0, \ \forall\ z \in C \\ &\Leftrightarrow u = \Pi_C x. \end{aligned}$$

Now, let E_1, E_2 and E_3 be Banach spaces, C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively and $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. The Split Equality Feasibility Problem (SEFP) is defined as

$$\text{find } x^* \in C \text{ and } y^* \in Q \text{ such that } Ax^* = By^*. \tag{6.1.33}$$

With $E_2 = E_3$ and $B = I$ the identity mapping in (6.1.33), the SEFP reduces to the SFP. Setting $T_1 = \partial_{i_C}$ and $T_2 = \partial_{i_Q}$ in Theorem 6.1.1, then the algorithm (6.1.2) becomes

$$\begin{cases} u_n = \Pi_C J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)), \\ x_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(u_n)), \\ v_n = \Pi_Q J_q^{E_2^*}(J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)), \\ y_{n+1} = J_p^{E_2^*}(\alpha_n J_p^{E_2}(v) + (1-\alpha_n) J_p^{E_2}(v_n)), \end{cases} \tag{6.1.34}$$

and so we obtain a strong convergence result for approximation of solution of SEFP in Banach spaces.

Split Equality Convex Minimization Problem:

Let E be a p-uniformly convex real Banach space which is also uniformly smooth and C be nonempty closed convex subset of E. Let $\phi : C \to \mathbb{R}$ be a proper convex lower semicontinuous function. We know that the subdifferential $\partial\phi$ is maximal monotone and the resolvent operator $R_{\lambda\partial\phi} = prox_{\lambda\phi}$ where

$$prox_{\lambda\phi} x = \underset{u\in E}{argmin}\{\phi(u) + \frac{1}{2\lambda} D_p(u,x)\},$$

for each $x \in E$ (see [216] for more details).
Let E_1, E_2 and E_3 be p-uniformly convex real Banach spaces which are also uniformly smooth and C and Q be nonempty closed convex subsets of E_1 and E_2 respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. The Split Equality Convex Minimization Problem (SECMP) is define as: find $x^* \in E_1$ and $y^* \in E_2$ such that

$$\begin{cases} x^* = \underset{x\in E_1}{argmin}\ \phi(x) \text{ and } y^* = \underset{y\in E_2}{argmin}\ \psi(y) \\ Ax^* = By^*, \end{cases} \tag{6.1.35}$$

where $\phi : C \to \mathbb{R}$ and $\psi : Q \to \mathbb{R}$ are proper convex lower semicontinuous functions. Now, by setting $T_1 = prox_{\partial\phi}$ and $T_2 = prox_{\partial\psi}$ in Theorem 6.1.1, then the Algorithm (6.1.2) becomes

$$\begin{cases} u_n = prox_{\lambda\phi} J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)), \\ x_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(u_n)), \\ v_n = prox_{\lambda\psi} J_q^{E_2^*}(J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)), \\ y_{n+1} = J_p^{E_2^*}(\alpha_n J_p^{E_2}(v) + (1-\alpha_n) J_p^{E_2}(v_n)), \end{cases} \tag{6.1.36}$$

and so we obtain a strong convergence result for approximating solutions of SECMP in Banach spaces.

Split Equality Equilibrium Problem:

Let $F : C \times C \to \mathbb{R}$ be a bifunction, we define a multi-valued mapping $A_F : E \to 2^{E^*}$ by

$$A_F(x) := \begin{cases} \{z \in E^* : F(x,y) \geq \frac{1}{r}\langle J_p^E y - J_p^E x, z\rangle,\ \forall\, y \in C\}; & x \in C, \\ \emptyset; & x \notin C. \end{cases} \tag{6.1.37}$$

Then, we know that $\Omega_{EP} = A_F^{-1}0$ and A_F is a maximal monotone operator with dom$(A_F) \subset C$ (see [248]). Further, for any $x \in E$ and $r > 0$, the resolvent T_r^F of F coincides with the resolvent of A_F, that is

$$T_r^F x = (J_p^E + rA_F)^{-1} J_p^E x.$$

Let E_1, E_2 and E_3 be p-uniformly convex real Banach spaces which are also uniformly smooth and C and Q be nonempty closed convex subsets of E_1 and E_2 respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be bifunctions. The Split Equality Equilibrium Problem (SEEP) is defined as: Find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{cases} F(x^*, x) \geq 0 \;\; \forall\, x \in C, \;\; G(y^*, y) \geq 0 \;\; \forall\, y \in Q \\ \text{and } Ax^* = By^*. \end{cases} \tag{6.1.38}$$

Setting $R_{\lambda T_1} = T_r^F$ and $R_{\lambda T_2} = T_r^G$ in Theorem 6.1.1, then the algorithm (6.1.2) becomes

$$\begin{cases} u_n = T_{r_n}^F J_q^{E_1^*}(J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)), \\ x_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(u_n)), \\ v_n = T_{r_n}^G J_q^{E_2^*}(J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)), \\ y_{n+1} = J_p^{E_2^*}(\alpha_n J_p^{E_2}(v) + (1-\alpha_n) J_p^{E_2}(v_n)), \end{cases} \tag{6.1.39}$$

for $r_n > 0$, and so we obtain a strong convergence result for approximation of solution of the SEEP in Banach spaces.

Application to Saddle Points Problem

Let X and Y be two Hilbert spaces and $E = X \times Y$. Let $L : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function such that $L(x, y)$ is convex in $x \in X$ and concave in $y \in Y$, (convex-concave function). To such a function, Rockafellar [222] associated the operator T_L defined by

$$T_L = \partial_1 L \times \partial_2(-L),$$

where ∂_1 (resp. ∂_2) stands for the subdifferential of L with respect to the first (resp. the second) variable. T_L is a maximal monotone operator if and only if L is closed and convex in Rockafellar sense (see [222]).
Moreover, it is well known that (x^*, y^*) is a saddle point of L, namely:

$$L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*), \; \forall \; (x, y) \in E,$$

if and only if the following inclusion holds

$$(0, 0) \in T_L(x^*, y^*).$$

The proximal operator associated with T_L is define by

$$prox_{\lambda L}(x, y) = arg \; \underset{(u,v)}{minmax}\{L(u, v) + \frac{1}{2\lambda}||x - u||^2 - \frac{1}{2\lambda}||y - v||^2\},$$

for all $(x, y) \in E$. Now, if in Problem (6.1.1), we set $E_1 = X_1 \times Y_1$, $E_2 = X_2 \times Y_2$, $E_3 = X_3 \times Y_3$ and $T_1 = T_{L_1}$, $T_2 = T_{L_2}$, where L_i $(i = 1, 2)$ are convex-concave functions on E_i for $i = 1, 2$, respectively. Then, we have the following split equality saddle point problem: find $(x_1^*, y_1^*) \in E_1$ and $(x_2^*, y_2^*) \in E_2$ such that

$$\begin{cases} (x_1^*, y_1^*) = arg \underset{(x_1,y_1)}{minmax}\, L_1(x_1, y_1) \\ (x_2^*, y_2^*) = arg \underset{(x_2,y_2)}{minmax}\, L_2(x_2, y_2) \\ \text{and } A(x_1^*, y_1^*) = B(x_2^*, y_2^*), \end{cases} \tag{6.1.40}$$

where $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators. Then we can obtain the following strong convergence result from Theorem 6.1.1.

Theorem 6.1.2. *Let X_i and Y_i be real Hilbert spaces for $i = 1, 2, 3$. Let $E_1 = X_1 \times Y_1$, $E_2 = X_2 \times Y_2$, $E_3 = X_3 \times Y_3$. Let C and Q be nonempty closed convex subset of E_1 and E_2 respectively, $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. Let $L_i : E_i \to \mathbb{R} \cup \{-\infty, +\infty\}$ be convex-concave functions, for $i = 1, 2, 3$. and $\Gamma := \{\bar{x} = (x_1, x_2) \in T_{L_1}^{-1}(0, 0),\ \bar{y} = (y_1, y_2) \in T_{L_2}^{-1}(0, 0)\ ; A\bar{x} = B\bar{y}\}$ is nonempty. For fixed $\bar{u} = (u_1, u_2) \in E_1$ and $\bar{v} = (v_1, v_2) \in E_2$, choose an initial guess $\bar{x}_1 \in E_1$ and $\bar{y}_1 \in E_2$ arbitrarily. Let $\{\alpha_n\} \subset [0, 1]$. Assume that the nth iterate $\bar{x}_n = (x_{n,1}, x_{n,2}) \in E_1$ and $\bar{y}_n = (y_{n,1}, y_{n,2}) \in E_2$ have been constructed; then we calculate the $(n + 1)$th iterate $(\bar{x}_{n+1}, \bar{y}_{n+1})$ via the formula*

$$\begin{cases} \bar{u}_n = prox_{\lambda L_1}(\bar{x}_n) - t_n A^*(A\bar{x}_n - B\bar{y}_n), \\ \bar{x}_{n+1} = \alpha_n(\bar{u}) + (1 - \alpha_n)\bar{u}_n, \\ \bar{v}_n = prox_{\lambda L_2}(\bar{y}_n) + t_n B^*(A\bar{x}_n - B\bar{y}_n), \\ \bar{y}_{n+1} = \alpha_n(\bar{v}) + (1 - \alpha_n)\bar{v}_n, \end{cases} \tag{6.1.41}$$

where $\lambda > 0$, A^* and B^* are the adjoints of A and B respectively and the stepsize t_n is choosen in such a way that

$$t_n \in \left(\epsilon, \frac{2||A\bar{x}_n - B\bar{y}_n||^2}{||A^*(A\bar{x}_n - B\bar{y}_n)||^2 + ||B^*(A\bar{x}_n - B\bar{y}_n)||^2} - \epsilon\right),\ n \in \Omega,$$

for small enough ϵ, otherwise $t_n = t$ (t being any nonegative value), where the set of indices $\Omega = \{n : A\bar{x}_n - B\bar{y}_n \neq 0\}$. Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequences $\{(\bar{x}_n, \bar{y}_n)\}$ strongly converges to $(\bar{x}, \bar{y}) = (P_{\Gamma_1}\bar{u}, P_{\Gamma_2}\bar{v})$, where $\Gamma_i = \{\bar{z} \in E_i : 0 \in T_{L_i}(\bar{z})\}$ for $(i = 1, 2)$, P_{Γ_1} and P_{Γ_2} are the metric projections onto Γ_1 and Γ_2 respectively.

6.1.3 Numerical Example

For simplicity, we take $E_1 = E_2 = E_3 = \mathbb{R}$, with $p = 2$. Let $A(x) = x,\ B(x) = 2x,\ T_1(x) = 2x$ and $T_2(x) = 3x$. Choose $\lambda = 2$ and $\alpha_n = \frac{1}{\sqrt{n}}$, then algorithm (6.1.2) becomes

$$\begin{cases} x_{n+1} = \frac{1}{\sqrt{n}}u + (\frac{\sqrt{n}-1}{5\sqrt{n}})(x_n - t_n(x_n - 2y_n)) \\ y_{n+1} = \frac{1}{\sqrt{n}}v + (\frac{\sqrt{n}-1}{7\sqrt{n}})(y_n + 2t_n(x_n - 2y_n)), \end{cases} \tag{6.1.42}$$

where the step size t_n is chosen in such a way that

$$t_n \in \Big(\epsilon, \frac{2||Ax_n - By_n||^2}{||A^T(Ax_n - By_n)||^2 + ||B^T(Ax_n - By_n)||^2} - \epsilon\Big),\ n \in \Omega,$$

for small enough ϵ, otherwise $t_n = t$ (t being any nonegative value), where the set of indices $\Omega = \{n : Ax_n - By_n \neq 0\}$.
We make different choice of u, v, x_1, and y_1 and use $\epsilon < 10^{-2}$, for the stopping criterion.
Case 1:
(i) Take $x_1 = 1, y_1 = -1, u = 0.5$ and $v = 1$.
(ii) Take $x_1 = 0.25, y_1 = 0.005, u = -0.0675$ and $v = 0.001$.
Case 2:
(i) Take $x_1 = -0.02, y_1 = -0.005, u = 0.1$ and $v = 1$.
(ii) Take $x_1 = -0.0005, y_1 = -0.12, u = 1$ and $v = 0.001$.
We note that the choice of t_n, as long as it is in the range, does not have any significant effect on both the number of iterations and cpu time. `Mathlab version R2014a` is used to obtain the graphs of errors against number of iterations, execution time against accuracy and number of iterations against accuracy.

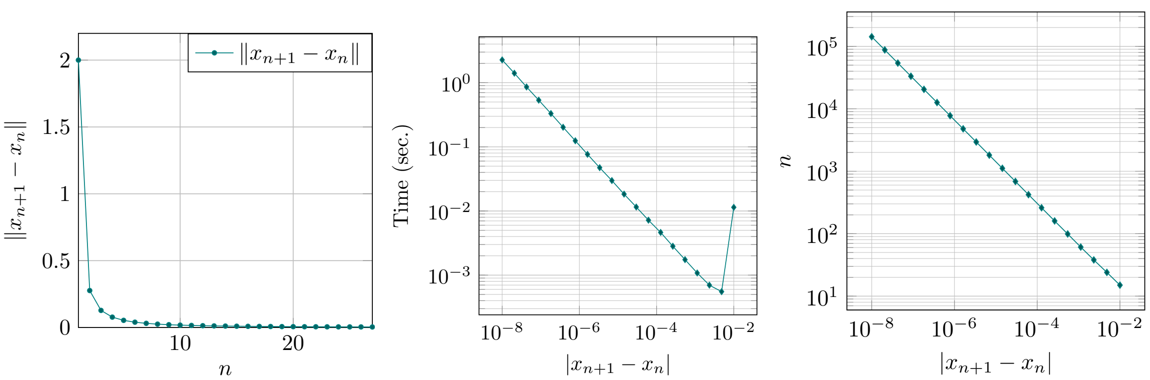


Figure 6.1: Case 1(i): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

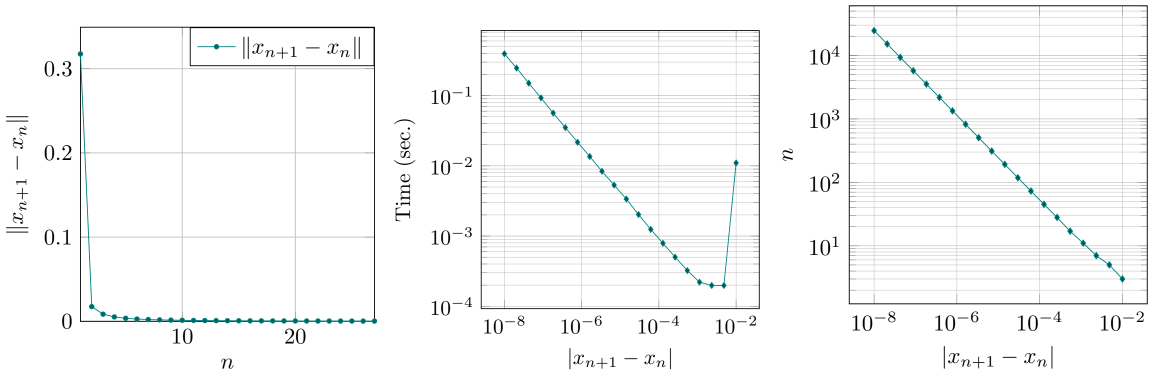


Figure 6.2: Case 1(ii): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

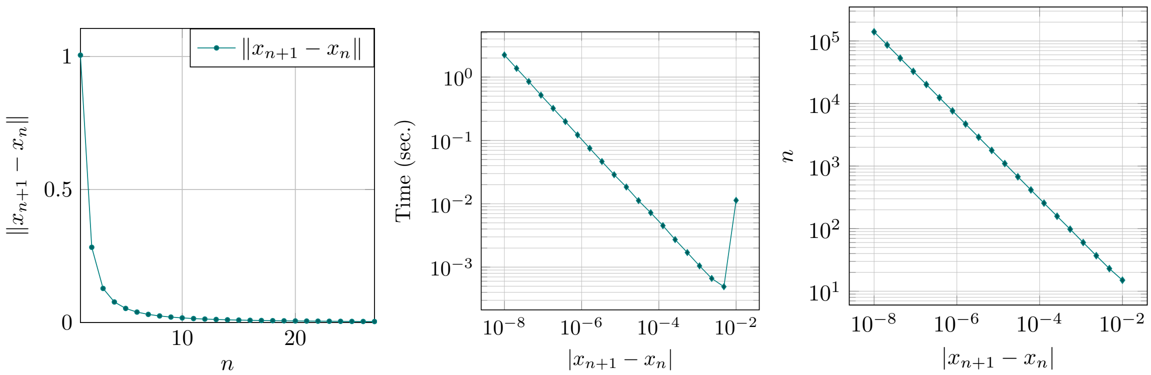


Figure 6.3: Case 2(i): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

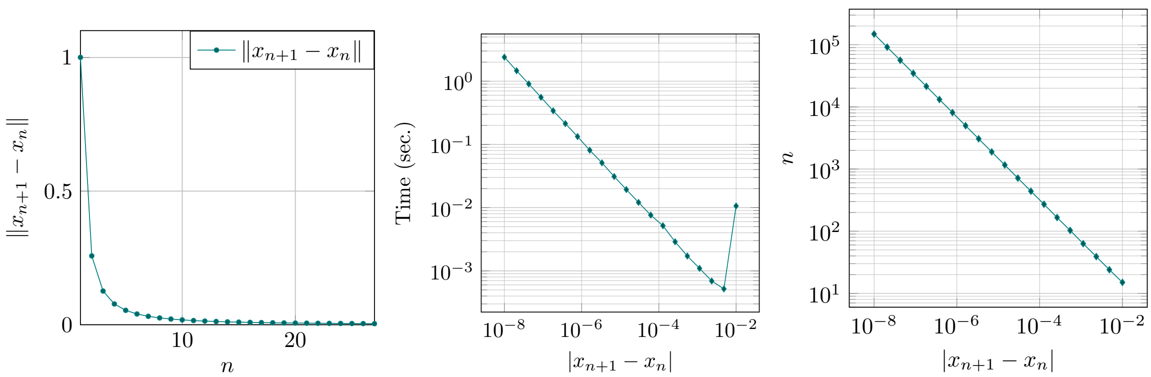


Figure 6.4: Case 2(ii): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

CHAPTER 7

Common Fixed Point Problems

7.1 An Intermixed Algorithm for Two λ-Strict Pseudocontraction Mappings in q-Uniformly Smooth Banach Space

In 2010, Chidume and Shahzad [81] established a weak convergence result for fixed point of pseudocontractive mapping in some uniformly smooth real Banach space which is also uniformly convex. In particular, they proved the following theorem.

Theorem 7.1.1. *Let E be a uniformly smooth real Banach space which is also uniformly convex and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a λ-strict pseudocontraction for some $0 \leq \lambda < 1$ with $F(T) \neq \emptyset$. For a fixed $x_0 \in C$, defined a sequence $\{x_n\}$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \tag{7.1.1}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying the following conditions: (i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T.

However, Cholamjiak and Suntai [82] improved and extended the result of [81] from a real-uniformly smooth Banach space which is also uniformly convex to a real uniformly convex Banach space which has a Fréchet differentiable norm.
Recently, Cholamjiak and Suntai [83] established a strong convergence result for a countable family of strictly pseudocontractive mappings in a q-uniformly smooth and uniformly convex real Banach space E with nonempty closed and convex subset C which admits a weakly sequentially continuous duality mapping j_q using the following algorithm: for

$x_1 \in C$,

$$\begin{cases} y_n = Q_C[(1-\alpha_n)x_n], \\ x_{n+1} = (1-\beta_n)x_n + \beta_n[(1-\gamma_n)y_n + \gamma_n T_n y_n], \quad n \geq 1, \end{cases} \tag{7.1.2}$$

where Q_C is a sunny nonexpansive retraction of E onto C, $T_n : C \to C$ is a countable family of strictly pseudocontractions, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $(0,1)$ which satisfy the following conditions:
(C1) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$,
(C2) $0 < \limsup_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(C3) $0 < a \leq \gamma_n \leq \mu$, $\mu = \min\{1, (\frac{q\lambda}{c_q})^{\frac{1}{q-1}}\}$.

Furthermore, Yao et. al. [274] first introduced the intermixed algorithm for approximating fixed points of two strict pseudocontractive mappings independently in a real Hilbert space. They proved the following strong convergence theorem:

Theorem 7.1.2. *(Theorem 3.3 of [274]): Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T, S : C \to C$ be a λ-strictly pseudocontraction with $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$. Let $f : C \to H$ be a ρ_1-contraction and $g : C \to H$ be a ρ_2-contraction. Let $k \in (0, 1-\lambda)$ be a constant. For arbitrarily given $x_0 \in C$, $y_0 \in C$, let the sequence $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

$$\begin{cases} x_{n+1} = (1-\beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n], \quad n \geq 0, \\ y_{n+1} = (1-\beta_n)y_n + \beta_n P_C[\alpha_n g(x_n) + (1-k-\alpha_n)y_n + kSy_n], \quad n \geq 0, \end{cases} \tag{7.1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0,1)$. Suppose the following conditions are satisfied

(C1). $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} = \infty$,

(C2). $\beta_n \in [\xi_1, \xi_2] \subset (0,1)$ for all $n \geq 0$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converges strongly to the fixed points $P_{F(T)}f(y^*)$ and $P_{F(S)}g(x^*)$ of T and S respectively, where $x^* \in F(T)$ and $y^* \in F(S)$.

We note that in (7.1.3), the definition of $\{x_n\}$ involves $\{y_n\}$ and as well the definition of $\{y_n\}$ involves $\{x_n\}$. Also, the intermixed algorithm can be use to find the common fixed point of the two strict pseudocontraction mappings T and S in a real Hilbert space.

Motivated by the result of [274], we introduce a new intermix algorithm for approximating fixed points of two strict pseudocontractions and further obtained a strong convergence result in q-uniformly smooth Banach space which admits weakly sequentially continuous duality mapping j_q.

7.1.1 Main result

Theorem 7.1.3. *Let C be a nonempty closed and convex subset of a real q-uniformly smooth Banach space E which admits a weakly sequentially continuous generalized duality* *mapping* j_q*. Let* $f : C \to C$ *be a* ρ_1*-contraction and* $g : C \to C$ *be a* ρ_2*-contraction. Let* $T : C \to C$ *be a* λ_1*-strict pseudocontaction and* $U : C \to C$ *be a* λ_2*-strict pseudocontraction such that* $F(T) \neq \emptyset$*,* $F(U) \neq \emptyset$ *and* $\lambda = \min\{\lambda_1, \lambda_2\}$*. For arbitrarily given* $x_1 \in C$ *and* $y_1 \in C$*, let the sequence* $\{x_n\}$ *and* $\{y_n\}$ *be generated iteratively by*

$$\begin{cases} x_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)[s_n T x_n + (1-s_n)x_n] & n \geq 1, \\ y_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)[s_n U y_n + (1-s_n)y_n] & n \geq 1, \end{cases} \tag{7.1.4}$$

where $\{\alpha_n\}$ *and* $\{s_n\}$ *are two real sequences in* $(0,1)$ *satisfying the following conditions:*

C1. $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} = \infty$*,*

C2. $0 < a \leq \liminf\limits_{n\to\infty} s_n \leq \limsup\limits_{n\to\infty} s_n \leq \min\{1, (\frac{\lambda q}{c_q})^{\frac{1}{q-1}}\}$.

Then, $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to fixed points* $Q_{F(T)}f(\bar{y})$ *and* $Q_{F(U)}g(\bar{x})$ *respectively, where* $\bar{x} \in F(T)$ *and* $\bar{y} \in F(U)$*,* $Q_{F(T)}$ *is the sunny nonexpansive retraction from* C *onto* $F(T)$ *and* $Q_{F(U)}$ *is the sunny nonexpansive retraction from* C *onto* $F(U)$*.*

Proof. Let $x^* \in F(T)$ and $y^* \in F(U)$, we define $T_s := (1-s)I + sT$ and $U_s := (1-s)I + sU$ where $I : C \to C$ is the identity mapping. Note that T_s and U_s are nonexpansive, $F(T_s) = F(T)$ and $F(U_s) = F(U)$. Let $\rho = \max\{\rho_1, \rho_2\}$, then

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||\alpha_n f(y_n) + (1-\alpha_n)T_{s_n}x_n - x^*|| \\ &= ||\alpha_n(f(y_n) - x^*) + (1-\alpha_n)(T_{s_n}x_n - x^*)|| \\ &\leq \alpha_n||f(y_n) - x^*|| + (1-\alpha_n)||T_{s_n}x_n - x^*|| \\ &\leq \alpha_n(||f(y_n) - f(y^*)|| + ||f(y^*) - x^*||) + (1-\alpha_n)||x_n - x^*|| \\ &\leq \alpha_n\rho_1||y_n - y^*|| + \alpha_n||f(y^*) - x^*|| + (1-\alpha_n)||x_n - x^*|| \\ &\leq \alpha_n\rho||y_n - y^*|| + \alpha_n||f(y^*) - x^*|| + (1-\alpha_n)||x_n - x^*||. \end{aligned} \tag{7.1.5}$$

Similarly, we obtain that

$$||y_{n+1} - y^*|| \leq \alpha_n\rho||x_n - x^*|| + \alpha_n||g(x^*) - y^*|| + (1-\alpha_n)||y_n - y^*||. \tag{7.1.6}$$

Therefore from (7.1.5) and (7.1.6), we have

$$\begin{aligned} ||x_{n+1} - x^*|| + ||y_{n+1} - y^*|| &\leq (1-(1-\rho)\alpha_n)(||x_n - x^*|| + ||y_n - y^*||) \\ &\quad +\alpha_n(||f(y^*) - x^*|| + ||g(x^*) - y^*||) \\ &\leq \max\left\{||x_n - x^*|| + ||y_n - y^*||, \frac{||f(y^*) - x^*|| + ||g(x^*) - y^*||}{1-\rho}\right\} \\ &\;\;\vdots \\ &\leq \max\left\{||x_1 - x^*|| + ||y_1 - y^*||, \frac{||f(y^*) - x^*|| + ||g(x^*) - y^*||}{1-\rho}\right\}. \end{aligned}$$

Hence, $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently, $\{Tx_n\}$ and $\{Uy_n\}$ are bounded. Furthermore, by (2.6.10) we have

$$\begin{aligned} ||x_{n+1} - x^*||^q &= ||\alpha_n(f(y_n) - x^*) + (1-\alpha_n)(T_{s_n}x_n - x^*)||^q \\ &\leq ||(1-\alpha_n)(T_{s_n}x_n - x^*)||^q + q\alpha_n\langle f(y_n) - x^*, j_q(x_{n+1} - x^*)\rangle. \end{aligned} \tag{7.1.7}$$

Moreover, by (2.6.11) we have

$$\begin{aligned}||(1-\alpha_n)(T_{s_n}x_n - x^*)||^q &\leq (1-\alpha_n)||T_{s_n}x_n - x^*||^q\\ &= (1-\alpha_n)||s_n(Tx_n - x_n) + (x_n - x^*)||^q\\ &\leq (1-\alpha_n)[||x_n - x^*||^q - qs_n\langle x_n - Tx_n, j(x_n - x^*)\rangle\\ &\quad + s_n^q c_q||x_n - Tx_n||^q]\\ &\leq (1-\alpha_n)[||x_n - x^*||^q - qs_n\lambda_1||x_n - Tx_n||^q\\ &\quad + s_n^q c_q||x_n - Tx_n||^q]\\ &\leq (1-\alpha_n)[||x_n - x^*||^q - qs_n\lambda||x_n - Tx_n||^q\\ &\quad + s_n^q c_q||x_n - Tx_n||^q]\\ &= (1-\alpha_n)||x_n - x^*||^q - s_n(1-\alpha_n)\times\\ &\quad (q\lambda - s_n^{q-1}c_q)||x_n - Tx_n||^q. \qquad (7.1.8)\end{aligned}$$

Also by Lemma 2.6.16, we have

$$\begin{aligned}q\alpha_n\langle f(y_n) - x^*, j_q(x_{n+1} - x^*)\rangle &= q\alpha_n\langle f(y_n) - f(y^*) + f(y^*) - x^*, j_q(x_{n+1} - x^*)\rangle\\ &\leq q\alpha_n\rho||y_n - y^*||||x_{n+1} - x^*||^{q-1}\\ &\quad + q\alpha_n\langle f(y^*) - x^*, j_q(x_{n+1} - x^*)\rangle\\ &\leq q\alpha_n\rho\Big(\frac{1}{q}||y_n - y^*||^q + \frac{q-1}{q}||x_{n+1} - x^*||^{\frac{q(q-1)}{q-1}}\Big)\\ &\quad + q\alpha_n\langle f(y^*) - x^*, j_q(x_{n+1} - x^*)\rangle\\ &= \alpha_n\rho(||y_n - y^*||^q + (q-1)||x_{n+1} - x^*||^q)\\ &\quad + q\alpha_n\langle f(y^*) - x^*, j_q(x_{n+1} - x^*)\rangle. \qquad (7.1.9)\end{aligned}$$

Substituting (7.1.8) and (7.1.9) into (7.1.7), we have that

$$\begin{aligned}||x_{n+1} - x^*||^q &\leq (1-\alpha_n)||x_n - x^*||^q - s_n(1-\alpha_n)(q\lambda - s_n^{q-1}c_q)||x_n - Tx_n||^q\\ &\quad + \alpha_n\rho(||y_n - y^*||^q + (q-1)||x_{n+1} - x^*||^q)\\ &\quad + q\alpha_n\langle f(y^*) - x^*, j_q(x_{n+1} - x^*)\rangle. \qquad (7.1.10)\end{aligned}$$

Following similar process as above, we obtain

$$\begin{aligned}||y_{n+1} - y^*||^q &\leq (1-\alpha_n)||y_n - y^*||^q - s_n(1-\alpha_n)(q\lambda - s_n^{q-1}c_q)||y_n - Uy_n||^q\\ &\quad + \alpha_n\rho(||x_n - x^*||^q + (q-1)||y_{n+1} - y^*||^q)\\ &\quad + q\alpha_n\langle g(x^*) - y^*, j_q(y_{n+1} - y^*)\rangle. \qquad (7.1.11)\end{aligned}$$

Therefore from (7.1.10) and (7.1.11), we have

$$\begin{aligned}&(1-\alpha_n\rho(q-1))(||x_{n+1} - x^*||^q + ||y_{n+1} - y^*||^q)\\ &\quad \leq (1-(1-\rho)\alpha_n)(||x_n - x^*||^q + ||y_n - y^*||^q)\\ &\quad + q\alpha_n(\langle f(y^*) - x^*, j_q(x_{n+1} - x^*)\rangle + \langle g(x^*) - y^*, j_q(y_{n+1} - y^*)\rangle)\\ &\quad - s_n(1-\alpha_n)(q\lambda - s_n^{q-1}c_q)(||x_n - Tx_n||^q + ||y_n - Uy_n||^q).\end{aligned}$$

Hence,

$$\begin{aligned} ||x_{n+1} - x^*||^q + ||y_{n+1} - y^*||^q \leq & \frac{1-(1-\rho)\alpha_n}{1-\alpha_n\rho(q-1)}(||x_n - x^*||^q + ||y_n - y^*||^q) \\ & + \frac{q\alpha_n}{1-\alpha_n\rho(q-1)}(\langle f(y^*) - x^*, j_q(x_{n+1} - x^*)\rangle \\ & + \langle g(x^*) - y^*, j_q(y_{n+1} - y^*)\rangle) \\ & - \frac{s_n(1-\alpha_n)(q\lambda - s_n^{q-1}c_q)}{1-\alpha_n\rho(q-1)}(||x_n - Tx_n||^q + ||y_n - Uy_n||^q). \end{aligned} \tag{7.1.12}$$

The rest of the proof will be divided into two cases.

CASE 1: Suppose $\{||x_n - x^*||^q + ||y_n - y^*||^q\}$ is monotonically nonincreasing, then

$$(||x_n - x^*||^q + ||y_n - y^*||^q) - (||x_{n+1} - x^*||^q + ||y_{n+1} - y^*||^q) \to 0, \quad n \to \infty.$$

This implies from (7.1.12) and condition (C2) that

$$||x_n - Tx_n||^q + ||y_n - Uy_n||^q \to 0, \quad n \to \infty.$$

Hence,

$$||x_n - Tx_n|| \to 0 \text{ and } ||y_n - Uy_n|| \to 0, \quad n \to \infty.$$

Moreover,

$$||T_{s_n}x_n - x_n|| = \beta_n||x_n - Tx_n|| \to 0, \quad n \to \infty, \tag{7.1.13}$$

and

$$||U_{s_n}y_n - y_n|| = \beta_n||y_n - Uy_n|| \to 0, \quad n \to \infty. \tag{7.1.14}$$

Also,

$$||x_{n+1} - x_n|| \leq \alpha_n||f(y_n) - x_n|| + (1-\alpha_n)||T_{s_n}x_n - x_n|| \to 0, \quad n \to \infty,$$

and

$$||y_{n+1} - y_n|| \leq \alpha_n||g(x_n) - y_n|| + (1-\alpha_n)||U_{s_n}y_n - y_n|| \to 0, \quad n \to \infty.$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $x_{n_k} \rightharpoonup \bar{x} \in C$ and $y_{n_k} \rightharpoonup \bar{y} \in C$. By (7.1.13), (7.1.14) and the demiclosedness principle of $I - T_s$ and $I - U_s$ at 0, we have that $\bar{x} \in F(T_s)$ and $\bar{y} \in F(U_s)$. Consequently, by Lemma 2.6.19, we have that $\bar{x} \in F(T)$ and $\bar{y} \in F(U)$.

Next we show that $x_n \to \hat{x}$ and $y_n \to \hat{y}$ where $\hat{x} = Q_{F(T)}f(\hat{y})$ and $\hat{y} = Q_{F(U)}g(\hat{x})$. Observe that from (7.1.12), we have

$$\begin{aligned}
||x_{n+1}-\hat{x}||^q+||y_{n+1}-\hat{y}||^q &\leq \Big[1-\frac{(1-\rho)\alpha_n-\alpha_n\rho(q-1)}{1-\alpha_n\rho(q-1)}\Big](||x_n-\hat{x}||^q+||y_n-\hat{y}||^q)\\
&\quad -\frac{s_n(1-\alpha_n)(q\lambda-s_n^{q-1}c_q)}{1-\alpha_n\rho(q-1)}(||x_n-Tx_n||^q+||y_n-Uy_n||^q)\\
&\quad +\frac{q\alpha_n}{1-\alpha_n\rho(q-1)}(\langle f(\hat{y})-\hat{x},j_q(x_{n+1}-\hat{x})\rangle\\
&\quad +\langle g(\hat{x})-\hat{y},j_q(y_{n+1}-\hat{y})\rangle)\\
&\leq \Big[1-\frac{(1-\rho)\alpha_n-\alpha_n\rho(q-1)}{1-\alpha_n\rho(q-1)}\Big](||x_n-\hat{x}||^q+||y_n-\hat{y}||^q)\\
&\quad +\frac{q\alpha_n}{1-\alpha_n\rho(q-1)}(\langle f(\hat{y})-\hat{x},j_q(x_{n+1}-\hat{x})\rangle\\
&\quad +\langle g(\hat{x})-\hat{y},j_q(y_{n+1}-\hat{y})\rangle). \qquad (7.1.15)
\end{aligned}$$

In view of Lemma 2.6.29, put

$$\begin{aligned}
c_n &= ||x_n-x^*||^q+||y_n-y^*||^q, \quad \Theta_n=\frac{(1-\rho)\alpha_n-\alpha_n\rho(q-1)}{1-\alpha_n\rho(q-1)},\\
\delta_n &= \frac{q\alpha_n}{(1-\rho)\alpha_n-\alpha_n\rho(q-1)}(\langle f(y^*)-x^*,j_q(x_{n+1}-x^*)\rangle+\langle g(x^*)-y^*,j_q(y_{n+1}-y^*)\rangle).
\end{aligned}$$

Then, it follows from (7.1.15) that

$$c_{n+1}\leq(1-\Theta_n)c_n+\Theta_n\delta_n.$$

Since $\{\alpha_n\}\subset(0,1)$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$, we have that $\Theta_n\in(0,1)$, $\sum_{n=1}^{\infty}\Theta_n=\infty$. In order to show that $c_n\to 0$, it is sufficient to prove that $\limsup_{k\to\infty}\delta_n\leq 0$.

Choose subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$ respectively, such that

$$\limsup_{n\to\infty}\langle f(\hat{y})-\hat{x},j_q(x_{n+1}-\hat{x})\rangle=\lim_{k\to\infty}\langle f(\hat{y})-\hat{x},j_q(x_{n_k+1}-\hat{x})\rangle$$

and

$$\limsup_{n\to\infty}\langle g(\hat{x})-\hat{y},j_q(y_{n+1}-\hat{y})\rangle=\lim_{k\to\infty}\langle g(\hat{x})-\hat{y},j_q(y_{n_k+1}-\hat{y})\rangle.$$

Since $x_{n_k}\rightharpoonup\bar{x}$ and $y_{n_k}\rightharpoonup\bar{y}$, it follows from Proposition 2.4.3 that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(\hat{y})-\hat{x},j_q(x_{n+1}-\hat{x})\rangle &= \lim_{k\to\infty}\langle f(\hat{y})-\hat{x},j_q(x_{n_k+1}-\hat{x})\rangle\\
&= \langle f(\hat{y})-\hat{x},j_q(\bar{x}-\hat{x})\rangle\leq 0, \qquad (7.1.16)
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{n\to\infty}\langle g(\hat{x})-\hat{y},j_q(y_{n+1}-\hat{y})\rangle &= \lim_{k\to\infty}\langle g(\hat{x})-\hat{y},j_q(y_{n_k+1}-\hat{y})\rangle\\
&= \langle g(\hat{x})-\hat{y},j_q(\bar{y}-\hat{y})\rangle\leq 0. \qquad (7.1.17)
\end{aligned}$$

Thus, it follows from Lemma 2.6.29 and (7.1.15) that $||x_n-\hat{x}||^q+||y_n-\hat{y}||^q\to 0, n\to\infty$, which implies that $x_n\to\hat{x}$ and $y_n\to\hat{y}$.

Case 2: Assume that $\{||x_n - x^*|| + ||y_n - y^*||\}$ is not monotonically decreasing. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_1$ (for some n_1 large enough) defined by

$$\tau(n) = \max\{k \in \mathbb{N} : k \leq n, \tau_k \leq \tau_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$0 \leq ||x_{\tau(n)} - x^*|| + ||y_{\tau(n)} - y^*|| \leq ||x_{\tau(n)+1} - x^*|| + ||y_{\tau(n)+1} - y^*||, \quad \forall n \geq n_1.$$

Now, since $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ are bounded, there exist subsequences of $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ still denoted as $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ which converge weakly to $\bar{x}$ and $\bar{y}$ respectively.

After a similar argument as in Case 1, we have $||x_{\tau(n)} - Tx_{\tau(n)}|| \to 0$, $||y_{\tau(n)} - Uy_{\tau(n)}|| \to 0$, $||x_{\tau(n)+1} - x_{\tau(n)}|| \to 0$ and $||y_{\tau(n)+1} - y_{\tau(n)}|| \to 0$ as $n \to \infty$.
Also following the same line of argument as in (7.1.16) and (7.1.17), we have

$$\limsup_{k\to\infty}\langle \hat{x} - f(\hat{y}), j_q(\hat{x} - x_{\tau(n)+1})\rangle \leq 0, \tag{7.1.18}$$

and

$$\limsup_{k\to\infty}\langle \hat{y} - g(\hat{x}), j_q(\hat{y} - y_{\tau(n)+1})\rangle \leq 0. \tag{7.1.19}$$

From (7.1.12), we have

$$\begin{aligned}
0 &\leq \Big(||x_{\tau(n)+1} - \hat{x}||^q + ||y_{\tau(n)+1} - \hat{y}||^q\Big) - \Big(||x_{\tau(n)} - \hat{x}||^q + ||y_{\tau(n)} - \hat{y}||^q\Big) \\
&\leq \Big[1 - \frac{(1-\rho)\alpha_{\tau(n)} - \alpha_{\tau(n)}\rho(q-1)}{1 - \alpha_{\tau(n)}\rho(q-1)}\Big](||x_{\tau(n)} - \hat{x}||^q + ||y_{\tau(n)} - \hat{y}||^q) \\
&\quad -(||x_{\tau(n)} - \hat{x}||^q + ||y_{\tau(n)} - \hat{y}||^q) \\
&\quad + \frac{q\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\rho(q-1)}(\langle f(\hat{y}) - \hat{x}, j_q(x_{\tau(n)+1} - \hat{x})\rangle + \langle g(\hat{x}) - \hat{y}, j_q(y_{\tau(n)+1} - \hat{y})\rangle) \\
&\quad - \frac{s_{\tau(n)}(1-\alpha_{\tau(n)})(q\lambda - s_{\tau(n)}^{q-1}c_q)}{1 - \alpha_{\tau(n)}\rho(q-1)}(||x_{\tau(n)} - Tx_{\tau(n)}||^q + ||y_{\tau(n)} - Uy_{\tau(n)}||^q).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
(1-\rho q)(||x_{\tau(n)} - \hat{x}||^q + ||y_{\tau(n)} - \hat{y}||^q) &\leq q(\langle f(\hat{y}) - \hat{x}, j_q(x_{\tau(n)+1} - \hat{x})\rangle \\
&\quad + \langle g(\hat{x}) - \hat{y}, j_q(y_{\tau(n)+1} - \hat{y})\rangle).
\end{aligned}$$

Since $\alpha_{\tau(n)} \to 0$, from (7.1.18) and (7.1.19), we have

$$||x_{\tau(n)} - \hat{x}||^q + ||y_{\tau(n)} - \hat{y}||^q \to 0, \quad n \to \infty. \tag{7.1.20}$$

Thus we have $\lim_{n\to\infty}||x_{\tau(n)} - \hat{x}|| = 0$ and $\lim_{n\to\infty}||y_{\tau(n)} - \hat{y}|| = 0$.
As a consequence, we obtain for all $n \geq n_1$

$$\begin{aligned}
0 \leq ||x_{\tau(n)} - \bar{x}||^q + ||y_{\tau(n)} - \hat{y}||^q &\leq \max\{||x_{\tau(n)} - \hat{x}||^q + ||y_{\tau(n)} - \hat{y}||^q, \\
&\quad ||x_{\tau(n)+1} - \hat{x}||^q + ||y_{\tau(n)+1} - \hat{y}||^q\} \\
&= ||x_{\tau(n)+1} - \hat{x}||^q + ||y_{\tau(n)+1} - \hat{y}||^q. \qquad (7.1.21)
\end{aligned}$$

Hence, $\lim_{n\to\infty} ||x_{\tau(n)} - \hat{x}||^q + ||y_{\tau(n)} - \hat{y}||^q = 0$. Therefore, we have

$$\lim_{n\to\infty} ||x_n - \hat{x}|| = 0 \quad \text{and} \quad \lim_{n\to\infty} ||y_n - \hat{y}|| = 0.$$

This implies that the sequences $x_n \to \hat{x}$ and $y_n \to \hat{y}$. This completes the proof. □

As consequence of Theorem 7.1.3, we consider the approximation of common fixed point of two λ-strict pseudocontractive mappings in q-uniformly smooth real Banach space which admits weakly sequentially continuous duality mapping j_q.
Suppose $F(T) \neq \emptyset$, $F(U) \neq \emptyset$ and let $\Gamma = F(T) \cap F(U) \neq \emptyset$. Now, putting $y_n = x_n$ and $g(x) = f(x)$ in Theorem 7.1.3, by adding x_{n+1} and y_{n+1}, we have

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\Big[s_n \frac{1}{2}(T+U)x_n + (1-s_n)x_n\Big]. \tag{7.1.22}$$

Let $G := \frac{1}{2}(T+U)$, then by Lemma 2.6.20, G is λ-strictly pseudocontraction where $\lambda = \min\{\lambda_1, \lambda_2\}$ and $F(G) = \Gamma$.
Thus, the following result is obtained from Theorem 7.1.3.

Corollary 7.1.4. *Let C be a nonempty closed and convex subset of a real q-uniformly smooth Banach space E which admits a weakly sequentially continuous generalized duality mapping j_q and with the best q-uniformly smoothness constant $c_q > 0$. Let $f : C \to C$ be a ρ-contraction. Let $G := \frac{1}{2}(T+U)$ where $T : C \to C$ is a λ_1-strict pseudocontaction and $U : C \to C$ is a λ_2-strict pseudocontraction such that $F(T) \neq \emptyset$, $F(U) \neq \emptyset$ and $\lambda = \min\{\lambda_1, \lambda_2\}$. Suppose $\Gamma := F(T) \cap F(U) \neq \emptyset$. For arbitrarily given $x_1 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$\Big\{ \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\Big[s_n G x_n + (1-s_n)x_n\Big] \quad n \geq 1, \tag{7.1.23}$$

where $\{\alpha_n\}$ and $\{s_n\}$ are two real sequences in $(0,1)$ satisfying the following conditions:

C1. $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

C2. $0 < a \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq \min\{1, (\frac{\lambda q}{c_q})^{\frac{1}{q-1}}\}$.

Then, $\{x_n\}$ converges strongly to the common fixed point $Q_\Gamma f(\hat{x})$ of T and U, where $\hat{x} \in \Gamma$ and Q_Γ is the sunny nonexpansive retraction from C onto Γ.

Corollary 7.1.5. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to C$ be a ρ_1-contraction and $g : C \to C$ be a ρ_2-contraction. Let $T : C \to C$ be a κ_1-strict pseudocontaction and $U : C \to C$ be a κ_2-strict pseudocontraction such that $F(T) \neq \emptyset$, $F(U) \neq \emptyset$ and $\kappa = \min\{\kappa_1, \kappa_2\}$. For arbitrarily given $x_1 \in C$ and $y_1 \in C$, let the sequence $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

$$\begin{cases} x_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)[s_n T x_n + (1-s_n)x_n] & n \geq 1, \\ y_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)[s_n U y_n + (1-s_n)y_n] & n \geq 1, \end{cases} \tag{7.1.24}$$

where $\{\alpha_n\}$ and $\{s_n\}$ are two real sequences in $(0,1)$ satisfying the following conditions:

C1. $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

C2. $0 < a \le \liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n \le 1$.

Then, $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to fixed points* $P_{F(T)}f(\hat{y})$ *and* $P_{F(U)}g(\hat{x})$ *respectively, where* $\hat{x} \in F(T)$ *and* $\hat{y} \in F(U)$, $P_{F(T)}$ *is the metric projection from* C *onto* $F(T)$ *and* $P_{F(U)}$ *is the metric projection from* C *onto* $F(U)$.

7.2 Convergence Theorem for the Class of of N-Generalized Bregman Nonspreading Mapping in Banach Spaces

In this section, we introduce the class of N-generalized Bregman nonspreading mapping in reflexive Banach spaces. We propose an hybrid iterative scheme for finding a common solution of a countable family of equilibrium problems and a fixed point of a mapping in reflexive Banach spaces.

Definition 7.2.1. Let C be a nonempty, closed and convex subset of a reflexive Banach space E. A mapping $T : C \to C$ is said to be

(a) Bregman nonexpansive [213] if

$$D_f(Tx, Ty) \le D_f(x, y) \quad \forall x, y \in C;$$

(b) Bregman nonspreading [153] if

$$D_f(Tx, Ty) + D_f(Ty, Tx) \le D_f(Tx, y) + D_f(Ty, x), \qquad \forall x, y \in C,$$

(c) $(\alpha, \beta, \gamma, \delta)$-generalized Bregman nonspreading [7] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha D_f(Tx, Ty) + (1-\alpha)D_f(x, Ty) + \gamma\{D_f(Ty, Tx) - D_f(Ty, x)\} \\ \le \beta D_f(Tx, y) + (1-\beta)D_f(x, y) + \delta\{D_f(y, Tx) - D_f(y, x)\}, \quad \forall x, y \in C.\end{aligned}$$

(d) n-generalized Bregman nonspreading mapping if there exist $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ such that

$$\begin{aligned}\sum_{k=1}^{n}\alpha_k D_f(T^{n+1-k}x, Ty) + (1 - \sum_{k=1}^{n}\alpha_k)D_f(x, Ty) + \sum_{k=1}^{n}\gamma_k\{D_f(Ty, T^{n+1-k}x) - D_f(Ty, x)\} \\ \le \sum_{k=1}^{n}\beta_k D_f(T^{n+1-k}x, y) + (1 - \sum_{k=1}^{n}\beta_k)D_f(x, y) + \sum_{k=1}^{n}\delta_k\{D_f(y, T^{n+1-k}x) - D_f(y, x)\},\end{aligned} \tag{7.2.1}$$

for all $x, y \in C$.

Remark 7.2.1. From Definition 7.2.1, it is obvious that

(i) every 1-generalized Bregman nonspreading mapping is $(\alpha, \beta, \gamma, \delta)$-generalized Bregman nonspreading.

(ii) The class of $(1,1,1,0)$-generalized Bregman nonspreading mappings is Bregman nonspreading.

(iii) Also, the class of $(1,0,0,0)$-generalized Bregman nonspreading mappings is Bregman nonexpansive.

(iv) If E is smooth and strictly convex and $f(x) = \frac{1}{2}||x||^2$, then the class of n-generalized Bregman nonspreading mapping reduces to the class of n-generalized nonspreading mapping introduced by Takahashi et al. [250].

We next present an example of a n-generalized Bregman nonspreading mapping with $n = 2$.

Example 7.2.2. Let $E = \mathbb{R}$ and $f(x) = \dfrac{x^4}{2}$, then the associated Bregman distance is given by $D_f(x, y) = \frac{1}{2}x^4 + \frac{3}{2}y^4 - 2xy^3, \quad \forall x, y \in \mathbb{R}$. Define $T : [0, 4] \to [0, 4]$ by

$$Tx = \begin{cases} 0, & if \quad x \in [0, 4), \\ 1, & if \quad x = 4. \end{cases} \tag{7.2.2}$$

It is easy to show that T is 2-generalized Bregman nonspreading with constants $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{3}{4}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{2}{3}$, $\gamma_1 = \frac{1}{4}$, $\gamma_2 = \frac{4}{5}$, $\delta_1 = \frac{1}{2}$, $\delta_2 = \frac{1}{2}$ and $F(T) = \{0\}$.

7.2.1 Main results

In this subsection, we present the existence and some properties of fixed points of n-generalized Bregman nonspreading mapping in a reflexive Banach space. This result extends the corresponding results of [250] and [161] to reflexive Banach space.

Proposition 7.2.3. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function and $T : C \to C$ be a n-generalized Bregman nonspreading mapping. Then, the following are equivalent*

(i) $F(T)$ is nonempty;

(ii) $\{T^m z\}$ is bounded for some $z \in C$ and $m \in \mathbb{N}$.

Proof. First we show that $(i) \Rightarrow (ii)$. Suppose $F(T) \neq \emptyset$, then $\{T^m z\} = \{z\}$ for $z \in F(T)$. So $\{T^m z\}$ is bounded. Next, we show that (ii) implies (i). Let $\{T^m z\}$ be bounded for some

$z \in C$. Since T is n-Bregman generalized nonspreading, then there exist $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$, such that

$$\begin{aligned}
&\sum_{k=1}^{n} \alpha_k D_f(T^{n+1-k}x, Ty) + (1 - \sum_{k=1}^{n} \alpha_k) D_f(x, Ty) \\
&+ \sum_{k=1}^{n} \gamma_k \{D_f(Ty, T^{n+1-k}x) - D_f(Ty, x)\} \\
&\leq \sum_{k=1}^{n} \beta_k D_f(T^{n+1-k}x, y) + (1 - \sum_{k=1}^{n} \beta_k) D_f(x, y) \\
&+ \sum_{k=1}^{n} \delta_k \{D_f(y, T^{n+1-k}x) - D_f(y, x)\}, \qquad (7.2.3)
\end{aligned}$$

for all $x, y \in C$. Replacing x by $T^{m-1}z$ in (7.2.3), we have that for any $y, z \in C$,

$$\begin{aligned}
&\sum_{k=1}^{n} \alpha_k D_f(T^{n+1-k}T^{m-1}z, Ty) + (1 - \sum_{k=1}^{n} \alpha_k) D_f(T^{m-1}z, Ty) \\
&+ \sum_{k=1}^{n} \gamma_k \{D_f(Ty, T^{n+1-k}T^{m-1}z) - D_f(Ty, T^{m-1}z)\} \\
&\leq \sum_{k=1}^{n} \beta_k D_f(T^{n+1-k}T^{m-1}z, y) + (1 - \sum_{k=1}^{n} \beta_k) D_f(T^{m-1}z, y) \\
&+ \sum_{k=1}^{n} \delta_k \{D_f(y, T^{n+1-k}T^{m-1}z) - D_f(y, T^{m-1}z)\}. \qquad (7.2.4)
\end{aligned}$$

Since $\{T^m z\}$ is bounded, we can apply Banach limit μ to both sides of (7.2.4), then we have

$$\begin{aligned}
&\mu_m \Big(\sum_{k=1}^{n} \alpha_k D_f(T^{m+n-k}z, Ty) + (1 - \sum_{k=1}^{n} \alpha_k) D_f(T^{m-1}z, Ty) \\
&+ \sum_{k=1}^{n} \gamma_k \{D_f(Ty, T^{m+n-k}z) - D_f(Ty, T^{m-1}z)\} \Big) \\
&\leq \mu_m \Big(\sum_{k=1}^{n} \beta_k D_f(T^{m+n-k}z, y) + (1 - \sum_{k=1}^{n} \beta_k) D_f(T^{m-1}z, y) \\
&+ \sum_{k=1}^{n} \delta_k \{D_f(y, T^{m+n-k}z) - D_f(y, T^{m-1}z)\} \Big).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n} \alpha_k \mu_m D_f(T^{m+n-k}z, Ty) + (1 - \sum_{k=1}^{n} \alpha_k)\mu_m D_f(T^{m-1}z, Ty) \\
&+ \sum_{k=1}^{n} \gamma_k \{\mu_m D_f(Ty, T^{m+n-k}z) - \mu_m D_f(Ty, T^{m-1}z)\} \\
&\le \sum_{k=1}^{n} \beta_k \mu_m D_f(T^{m+n-k}z, y) + (1 - \sum_{k=1}^{n} \beta_k)\mu_m D_f(T^{m-1}z, y) \\
&+ \sum_{k=1}^{n} \delta_k \{\mu_m D_f(y, T^{m+n-k}z) - \mu_m D_f(y, T^{m-1}z)\}. \qquad (7.2.5)
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{k=1}^{n} \alpha_k \mu_m D_f(T^{m}z, Ty) + (1 - \sum_{k=1}^{n} \alpha_k)\mu_m D_f(T^{m}z, Ty) \\
&+ \sum_{k=1}^{n} \gamma_k \{\mu_m D_f(Ty, T^{m}z) - \mu_m D_f(Ty, T^{m}z)\} \\
&\le \sum_{k=1}^{n} \beta_k \mu_m D_f(T^{m}z, y) + (1 - \sum_{k=1}^{n} \beta_k)\mu_m D_f(T^{m}z, y) \\
&+ \sum_{k=1}^{n} \delta_k \{\mu_m D_f(y, T^{m}z) - \mu_m D_f(y, T^{m}z)\}.
\end{aligned}$$

Hence

$$\mu_m D_f(T^m z, Ty) \le \mu_m D_f(T^m z, y).$$

Therefore by Lemma 2.6.35, T has a fixed point in C. This completes the proof. □

The following results follow as direct consequences of Theorem 7.2.3.

Corollary 7.2.4. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex Banach space E, let p be a real number such that $1 < p < +\infty$ and let f be a function defined by $f(x) = \frac{1}{p}||x||^p$ and $T : C \to C$ be a n-generalized Bregman nonspreading mapping. Then, the following assertions are equivalent:*

(i) $F(T)$ is nonempty;

(ii) $\{T^m z\}$ is bounded for some $z \in C$.

Corollary 7.2.5. *Let C be a nonempty bounded closed convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let $T : C \to C$ be a n-generalized Bregman nonspreading mapping. Then, T has a fixed point.*

Remark 7.2.6. Corollary 7.2.4 is a generalization of the corresponding result in Theorem 3.2 of [250], where the equivalence between the two assertions was shown for $p = 2$.

We now show another important property of the fixed points of n-generalized Bregman nonspreading mapping.

Proposition 7.2.7. *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let $T : C \to C$ be a n-generalized Bregman nonspreading mapping such that $F(T) \neq \emptyset$. Then $F(T)$ is closed and convex.*

Proof. Let $u \in F(T)$, then putting $u = x \in F(T)$ in (7.2.1), we have

$$\begin{aligned} &\sum_{k=1}^{n} \alpha_k D_f(u, Ty) + (1 - \sum_{k=1}^{n} \alpha_k) D_f(u, Ty) + \sum_{k=1}^{n} \gamma_k \{D_f(Ty, u) - D_f(Ty, u)\} \\ &\leq \sum_{k=1}^{n} \beta_k D_f(u, y) + (1 - \sum_{k=1}^{n} \beta_k) D_f(u, y) + \sum_{k=1}^{n} \delta_k \{D_f(y, u) - D_f(y, u)\}, \end{aligned}$$

which implies that

$$D_f(u, Ty) \leq D_f(u, y), \qquad \forall u \in F(T), y \in C. \tag{7.2.6}$$

This means that T is quasi-Bregman nonexpansive. Now let $\{x_n\} \subset F(T)$ such that $x_n \to p$. Then

$$D_f(p, Tp) = \lim_{n\to\infty} D_f(x_n, Tp) \leq D_f(x_n, p) = D_f(p, p) = 0.$$

Hence, $p \in F(T)$. Therefore $F(T)$ is closed.

Next, we show that $F(T)$ is convex. For any $x, y \in F(T)$ and $\lambda \in (0, 1)$, let $z = \lambda x + (1 - \lambda)y$. Then

$$\begin{aligned} D_f(z, Tz) &= f(z) - f(Tz) - \langle \nabla f(Tz), z - Tz \rangle \\ &= f(z) - f(Tz) - \langle \nabla f(Tz), \lambda x + (1 - \lambda)y - Tz \rangle \\ &= f(z) + \lambda D_f(x, Tz) + (1 - \lambda) D_f(y, Tz) - \lambda f(x) - (1 - \lambda) f(y) \\ &\leq f(z) + \lambda D_f(x, z) + (1 - \lambda) D_f(y, z) - \lambda f(x) - (1 - \lambda) f(y) \\ &= f(z) - f(z) - \langle \nabla f(z), \lambda x + (1 - \lambda)y - z \rangle \\ &= f(z) - f(z) - \langle \nabla f(z), z - z \rangle \\ &= 0. \end{aligned} \tag{7.2.7}$$

Hence, $z = Tz$. Therefore, $F(T)$ is convex. □

Using Corollary 7.2.5 and Proposition 7.2.7, we prove the following common fixed point theorem for a commutative family of n-generalized Bregman nonspreading mapping in a reflexive Banach space.

Theorem 7.2.8. *Let $f : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function, C be a nonempty bounded closed convex subset of a real reflexive Banach space E and let $\{T_\alpha\}_{\alpha \in I}$ be a commutative family of n-generalized Bregman nonspreading mappings from C into itself. Then $\{T_\alpha\}_{\alpha \in I}$ has a common fixed point.*

Proof. By Theorem 7.2.7, we know that $F(T_\alpha)$ is a closed convex subset of C. Since E is reflexive and C is a bounded closed and convex subset, C is weakly compact. To show that $\cap_{\alpha\in I}F(T_\alpha)$ is nonempty, it is sufficient to show that $\{F(T_\alpha)\}_{\alpha\in I}$ has a nonempty finite intersection property.

Now, let $\{T_1, T_2, \ldots, T_N\}$ be a commutative finite family of n-generalized Bregman nonspreading mapping from C into itself. We prove by induction that $\{T_1, T_2, \ldots, T_N\}$ has a common fixed point. To do this, we start by showing the case for $N = 2$. By Corollary 7.2.5 and Theorem 7.2.7, $F(T_1)$ is nonempty, bounded, closed and convex. Let $u \in F(T_1)$, since $T_1T_2 = T_2T_1$, then we have $T_1T_2u = T_2T_1u = T_2u$. This implies that $T_2u \in F(T_1)$. Hence, $F(T_1)$ is T_2-invariant. Thus, the restriction of T_2 to $F(T_1)$ is a n-generalized Bregman nonspreading self mapping. By Corollary 3.1.1, T_2 has a fixed point in $F(T_1)$, that is, we have $z \in F(T_1)$ such that $T_2z = z$. Hence, $z \in F(T_1) \cap F(T_2)$.

Suppose that for some $N \geq 2$, $\Gamma = \cap_{k=1}^{N} F(T_k)$ is nonempty. Then Γ is a nonempty, bounded, closed and convex subset of C and the restriction of T_{N+1} to Γ is a n-generalized Bregman nonspreading self mapping. By Corollary 3.1.1, T_{N+1} has a fixed point in Γ. This implies that $\Gamma \cap F(T_{N+1})$ is nonempty. Hence, $\cap_{k=1}^{N+1} F(T_k)$ is nonempty. This completes the proof. □

The following result will be used in the sequel.

Proposition 7.2.9. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T : C \to C$ be a n-generalized Bregman nonspreading mapping. Then, for any $x, y \in C$, $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$, for $i = 1, 2, \ldots, n$, we have*

$$\begin{aligned} 0 \;\leq\;& \sum_{k=1}^{n}(\beta_k - \alpha_k)\Big(D_f(T^{n+1-k}x, Ty) - D_f(x, Ty)\Big) + D_f(Ty, y) \\ &+\langle \nabla f(Ty) - \nabla f(y), \sum_{k=1}^{n}\beta_k(T^{n+1-k}x - x) + x - Ty\rangle \\ &+\sum_{k=1}^{n}\delta_k\{D_f(y, T^{n+1-k}x) - D_f(y, x)\} \\ &-\sum_{k=1}^{n}\gamma_k\{D_f(Ty, T^{n+1-k}x) - D_f(Ty, x)\}. \end{aligned} \tag{7.2.8}$$

Proof. From the definition of n-generalized Bregman nonspreading mapping, we have

$$\begin{aligned} &\sum_{k=1}^{n}\alpha_k D_f(T^{n+1-k}x, Ty) + (1 - \sum_{k=1}^{n}\alpha_k)D_f(x, Ty) + \sum_{k=1}^{n}\gamma_k\{D_f(Ty, T^{n+1-k}x) - D_f(Ty, x)\} \\ &\quad\leq \sum_{k=1}^{n}\beta_k D_f(T^{n+1-k}x, y) + (1 - \sum_{k=1}^{n}\beta_k)D_f(x, y) + \sum_{k=1}^{n}\delta_k\{D_f(y, T^{n+1-k}x) - D_f(y, x)\}, \end{aligned} \tag{7.2.9}$$

for all $x, y \in C$. This implies that

$$\begin{aligned}0 &\leq \sum_{k=1}^{n} \beta_k D_f(T^{n+1-k}x, y) + (1 - \sum_{k=1}^{n} \beta_k) D_f(x, y) + \sum_{k=1}^{n} \delta_k \{D_f(y, T^{n+1-k}x) - D_f(y, x)\} \\ &\quad - \sum_{k=1}^{n} \alpha_k D_f(T^{n+1-k}x, Ty) - (1 - \sum_{k=1}^{n} \alpha_k) D_f(x, Ty) \\ &\quad - \sum_{k=1}^{n} \gamma_k \{D_f(Ty, T^{n+1-k}x) - D_f(Ty, x)\}.\end{aligned}$$

Hence, from the three points identity (Proposition 2.5.1(ii)), we have

$$\begin{aligned}0 &\leq \sum_{k=1}^{n} \beta_k \Big(D_f(T^{n+1-k}x, Ty) + D_f(Ty, y) + \langle \nabla f(Ty) - \nabla f(y), T^{n+1-k}x - Ty \rangle \Big) \\ &\quad + (1 - \sum_{k=1}^{n} \beta_k) \Big(D_f(x, Ty) + D_f(Ty, y) + \langle \nabla f(Ty) - \nabla f(y), x - Ty \rangle \Big) \\ &\quad - \sum_{k=1}^{n} \alpha_k D_f(T^{n+1-k}x, Ty) - (1 - \sum_{k=1}^{n} \alpha_k) D_f(x, Ty) \\ &\quad - \sum_{k=1}^{n} \gamma_k \{D_f(Ty, T^{n+1-k}x) - D_f(Ty, x)\} + \sum_{k=1}^{n} \delta_k \{D_f(y, T^{n+1-k}x) - D_f(y, x)\}.\end{aligned}$$

Therefore

$$\begin{aligned}0 &\leq \sum_{k=1}^{n} (\beta_k - \alpha_k) \Big(D_f(T^{n+1-k}x, Ty) - D_f(x, Ty) \Big) + D_f(Ty, y) \\ &\quad + \langle \nabla f(Ty) - \nabla f(y), \sum_{k=1}^{n} \beta_k (T^{n+1-k}x - x) + x - Ty \rangle \\ &\quad + \sum_{k=1}^{n} \delta_k \{D_f(y, T^{n+1-k}x) - D_f(y, x)\} - \sum_{k=1}^{n} \gamma_k \{D_f(Ty, T^{n+1-k}x) - D_f(Ty, x)\}.\end{aligned}$$

□

The following result is another important property which characterized the n-generalized Bregman nonspreading mapping.

Proposition 7.2.10. *Let $T : C \to C$ be a n-generalized Bregman nonspreading mapping. Suppose $F(T) \neq \emptyset$, then T is Bregman relatively nonexpansive.*

Proof. It is clear that

$$D_f(p, Tx) \leq D_f(p, x) \quad \forall p \in F(T), x \in C.$$

We show that $\hat{F}(T) = F(T)$. It is easy to see that $F(T) \subset \hat{F}(T)$. Now let $p \in \hat{F}(T)$, that is, there exist a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $||x_n - Tx_n|| \to 0$. Since f is uniformly Fréhet differentiable on bounded subsets of E, then ∇f is uniformly continuous and thus

$$\lim_{n\to\infty} ||f(x_n) - f(Tx_n)|| = \lim_{n\to\infty} ||\nabla f(x_n) - \nabla f(Tx_n)|| = 0. \tag{7.2.10}$$

Putting $x = x_n$ and $y = q$ in Proposition 7.2.9, we have

$$\begin{aligned} 0 \ \leq\ & \sum_{k=1}^{n}(\beta_k - \alpha_k)\Big(D_f(T^{n+1-k}x_n, Tq) - D_f(x_n, Tq)\Big) + D_f(Tq, q) \\ & + \langle \nabla f(Tq) - \nabla f(q), \sum_{k=1}^{n} \beta_k (T^{n+1-k}x_n - x_n) + x_n - Tq\rangle \\ & + \sum_{k=1}^{n} \delta_k \{D_f(q, T^{n+1-k}x_n) - D_f(q, x_n)\} - \sum_{k=1}^{n} \gamma_k \{D_f(Tq, T^{n+1-k}x_n) - D_f(Tq, x_n)\} \end{aligned} \tag{7.2.11}$$

Observe that

$$\begin{aligned} D_f(T^{n+1-k}x_n, Tq) - D_f(x_n, Tq) &= f(T^{n+1-k}x_n) - f(Tq) - \langle \nabla f(Tq), T^{n+1-k}x_n - Tq\rangle \\ & \quad -f(x_n) + f(Tq) + \langle \nabla f(Tq), x_n - Tq\rangle \\ &= f(T^{n+1-k}x_n) - f(x_n) + \langle \nabla f(Tq), x_n - Tq\rangle \\ & \quad -\langle \nabla f(Tq), T^{n+1-k}x_n - Tq\rangle \\ &= f(T^{n+1-k}x_n) - f(x_n) + \langle \nabla f(Tq), x_n - T^{n+1-k}x_n\rangle. \end{aligned} \tag{7.2.12}$$

Similarly

$$\begin{aligned} D_f(q, T^{n+1-k}x_n) - D_f(q, x_n) &= f(x_n) - f(T^{n+1-k}x_n) + \langle \nabla f(x_n), T^{n+1-k}x_n - x_n\rangle \\ & \quad +\langle \nabla f(x_n) - \nabla f(T^{n+1-k}x_n), q - x_n\rangle, \end{aligned} \tag{7.2.13}$$

and

$$\begin{aligned} D_f(Tq, T^{n+1-k}x_n) - D_f(Tq, x_n) &= f(x_n) - f(T^{n+1-k}x_n) + \langle \nabla f(x_n), T^{n+1-k}x_n - x_n\rangle \\ & \quad +\langle \nabla f(x_n) - \nabla f(T^{n+1-k}x_n), Tq - x_n\rangle. \end{aligned} \tag{7.2.14}$$

Substituting (7.2.12), (7.2.13) and (7.2.14) into (7.2.11), we have

$$\begin{aligned} 0 \ \leq\ & \sum_{k=1}^{n}(\beta_k - \alpha_k)\Big(f(T^{n+1-k}x_n) - f(x_n) + \langle \nabla f(Tq), x_n - T^{n+1-k}x_n\rangle\Big) + D_f(Tq, q) \\ & + \langle \nabla f(Tq) - \nabla f(q), \sum_{k=1}^{n} \beta_k (T^{n+1-k}x_n - x_n) + x_n - Tq\rangle \\ & + \sum_{k=1}^{n} \delta_k \{f(x_n) - f(T^{n+1-k}x_n) + \langle \nabla f(x_n), T^{n+1-k}x_n - x_n\rangle \\ & + \langle \nabla f(x_n) - \nabla f(T^{n+1-k}x_n), q - x_n\rangle\} \\ & - \sum_{k=1}^{n} \gamma_k \{f(x_n) - f(T^{n+1-k}x_n) + \langle \nabla f(x_n), T^{n+1-k}x_n - x_n\rangle \\ & + \langle \nabla f(x_n) - \nabla f(T^{n+1-k}x_n), Tq - x_n\rangle\}. \end{aligned} \tag{7.2.15}$$

Taking limit as $n \to \infty$ in (7.2.15) and using (7.2.10), we have

$$0 \leq D_f(Tq, q) + \langle \nabla f(Tq) - \nabla f(q), q - Tq\rangle.$$

Using the four points identity (Proposition 2.5.1(iii)), we have

$$\begin{aligned} 0 &\leq D_f(Tq, q) + D_f(Tq, Tq) - D_f(Tq, q) - D_f(q, Tq) + D_f(q, q) \\ &= -D_f(q, Tq). \end{aligned}$$

Thus $D_f(q, Tq) \leq 0$ and then $D_f(q, Tq) = 0$. Since f is strictly convex, we have $q = Tq$. Hence, $q \in F(T)$. Therefore $\hat{F}(T) \subset F(T)$. This thus implies that $\hat{F}(T) = F(T)$. □

Convergence analysis

Let $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq \mathbb{N}\}$ be sequences of real numbers such that $\{\alpha_{n,i}\} \subset (0, 1)$. We define the following $W_n : C \to C$ mapping generated by T^i, $i = 1, 2, \ldots, N$ and $\{\alpha_{n,i}\}$, where $T^i : C \to C$ is a finite family of n-generalized Bregman nonspreading mappings.

$$\begin{aligned} S_{n,0}x &= x, \\ S_{n,1}x &= \nabla f^*[\alpha_{n,1}\nabla f(T^1x) + (1 - \alpha_{n,1})\nabla f(x)] \\ S_{n,2}x &= \nabla f^*[\alpha_{n,2}\nabla f(T^2S_{n,1}x) + (1 - \alpha_{n,2})\nabla f(S_{n,1}x)] \\ S_{n,3}x &= \nabla f^*[\alpha_{n,3}\nabla f(T^3S_{n,2}x) + (1 - \alpha_{n,3})\nabla f(S_{n,2}x)] \\ &\vdots \\ S_{n,N-1}x &= \nabla f^*[\alpha_{n,N-1}\nabla f(T^{N-1}S_{n,N-2}x) + (1 - \alpha_{n,N-1})\nabla f(S_{n,N-2}x)] \\ W_n = S_{n,N} &= \nabla f^*[\alpha_{n,N}\nabla f(T^NS_{n,N-1}x) + (1 - \alpha_{n,N})\nabla f(S_{n,N-1}x)]. \end{aligned} \tag{7.2.16}$$

Using the above definition, we have the following lemma.

Proposition 7.2.11. *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E and let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $\{T^i\}_{i=1}^N$ be a finite famiy of n-generalized Bregman nonspreading mapping of C into itself such that $\bigcap_{i=1}^N F(T^i) \neq \emptyset$. Let $\{\alpha_{n,i}\}$ be real sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_{n,i} > 0$, $\forall i \in \{1, 2, \ldots, N\}$. Let W_n be a Bregman W-mapping generated by $T^1, T^2, \ldots, T^N$ in (7.2.16). Then*

(i) $\cap_{i=1}^N F(T^i) = F(W_n)$,

(ii) W_n *is Bregman quasi-nonexpansive,*

(iii) If in addition, T^i is Bregman relatively nonexpansive mapping, for each i, then W_n is Bregman relatively nonexpansive.

Proof. Let $x \in \cap_{i=1}^N F(T^i)$. Then $T^ix = x$, $i = 1, 2, \ldots, N$. From (7.2.16), we have that $S_{n,1}x = x$, $S_{n,2}x = x, \ldots, S_{n,N}x = x$. Thus $\cap_{i=1}^N F(T^i) \subset F(W_n)$. Conversely, let $y \in$

$F(W_n)$ and $x \in \cap_{i=1}^N F(T^i)$. Then

$$\begin{aligned}
D_f(x,y) &= D_f(x, W_n y) \\
&= D_f(x, \nabla f^*(\alpha_{n,N}\nabla f(T^N S_{n,N-1}y) + (1-\alpha_{n,N})\nabla f(S_{n,N-1}y))) \\
&= f(x) - \langle x, \alpha_{n,N}\nabla f(T^N S_{n,N-1}y)\rangle + (1-\alpha_{n,N})\nabla f(S_{n,N-1}y)\rangle \\
&\quad + f^*(\alpha_{n,N}\nabla f(T^N S_{n,N-1}y) + (1-\alpha_{n,N})\nabla f(S_{n,N-1}y)) \\
&\leq \alpha_{n,N}(f(x) - \langle x, \nabla f(T^N S_{n,N-1}y) + f^*(\nabla f(T^N S_{n,N-1}y))) \\
&\quad +(1-\alpha_{n,N})(f(x) - \langle x, \nabla f(S_{n,N-1}y)\rangle + f^*(\nabla f(T^N S_{n,N-1}y))) \\
&\quad -\alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}y) - \nabla f(S_{n,N-1}y)||) \\
&= \alpha_{n,N} D_f(x, T^N S_{n,N-1}y) + (1-\alpha_{n,N}) D_f(x, S_{n,N-1}y) \\
&\quad -\alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}y) - \nabla f(S_{n,N-1}y)) \\
&\leq D_f(x, S_{n,N-1}y) - \alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}y) - \nabla f(S_{n,N-1}y)||) \\
&\;\;\vdots \\
&\leq D_f(x,y) - \alpha_{n,1}(1-\alpha_{n,1})\rho_r^*(||\nabla f(T^1 y) - \nabla f(y)||) \\
&\quad -\alpha_{n,2}(1-\alpha_{n,2})\rho_r^*(||\nabla f(T^2 S_{n,1}y) - \nabla f(S_{n,1}y)||) - \dots \\
&\quad -\alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}y) - \nabla f(S_{n,N-1}y)||).
\end{aligned} \tag{7.2.17}$$

This implies that

$$\begin{aligned}
\alpha_{n,1}(1-\alpha_{n,1})\rho_r^*(||\nabla f(T^1 y) - \nabla f(y)||) &= \alpha_{n,2}(1-\alpha_{n,2})\rho_r^*(||\nabla f(T^2 S_{n,1}y) - \nabla f(S_{n,1}y)||) \\
= \dots &= \alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}y) - \nabla f(S_{n,N-1}y)||) = 0.
\end{aligned}$$

Then by the property of ρ_r^* from Lemma 2.6.21 and the norm-to-norm continuity of ∇f^*, we have

$$\begin{aligned}
T^1 y &= y, \\
T^2 S_{n,1} y &= S_{n,1} y, \\
&\;\;\vdots \\
T^N S_{n,N-1} &= S_{n,N-1} y.
\end{aligned}$$

It follows that

$$\begin{aligned}
D_f(y, S_{n,1}y) &= D_f(y, \nabla f^*(\alpha_{n,1}\nabla f(T^1 y) + (1-\alpha_{n,1})\nabla f(y))) \\
&\leq \alpha_{n,1} D_f(y, T^1 y) + (1-\alpha_{n,1}) D_f(y,y) = 0.
\end{aligned}$$

Therefore $y \in F(S_{n,1})$ and consequently, $y \in F(T^1)$. Following similar argument, we have that $y \in F(T^i)$ for $i = 1, 2, \dots, N$ and hence $y \in \bigcap_{i=1}^N F(T^i)$.

(ii) Let $y \in F(W_n)$. Then

$$\begin{aligned}
D_f(y, W_n x) &= D_f(y, \nabla f^*(\alpha_{n,N}\nabla f(T^N S_{n,N-1}x) + (1-\alpha_{n,N})\nabla f(S_{n,N-1}x))) \\
&\leq \alpha_{n,N} D_f(y, T^N S_{n,N-1}x) + (1-\alpha_{n,N}) D_f(y, S_{n,N-1}x) \\
&\leq \alpha_{n,N} D_f(y, S_{n,N-1}x) + (1-\alpha_{n,N}) D_f(y, S_{n,N-1}x) \\
&= D_f(y, S_{n,N-1}x) \\
&= D_f(y, \nabla f^*(\alpha_{n,N-1}\nabla f(T^{N-1} S_{n,N-2}x) + (1-\alpha_{n,N-1})\nabla f(S_{n,N-2}x))) \\
&\leq \alpha_{n,N-1} D_f(y, T^{N-1} S_{n,N-2}x) + (1-\alpha_{n,N-1}) D_f(y, S_{n,N-2}x) \\
&\leq D_f(y, S_{n,N-2}x) \\
&\vdots \\
&\leq D_f(y, x).
\end{aligned}$$

(iii) Let $\{x_n\} \subset C$ such that $x_n \rightharpoonup \bar{x}$ and $||W_n x_n - x_n|| \to 0$ as $n \to \infty$. From (7.2.17), we have

$$\begin{aligned}
D_f(\bar{x}, W_n x_n) \leq\ & D_f(\bar{x}, x_n) - \alpha_{n,1}(1-\alpha_{n,1})\rho_r^*(||\nabla f(T^1 x_n) - \nabla f(x_n)||) \\
& -\alpha_{n,2}(1-\alpha_{n,2})\rho_r^*(||\nabla f(T^2 S_{n,1}x_n) - \nabla f(S_{n,1}x_n)||) - \dots \\
& -\alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}x_n) - \nabla f(S_{n,N-1}x_n)||).
\end{aligned} \tag{7.2.18}$$

Using three points identity (Proposition 2.5.1(ii)), we obtain

$$D_f(\bar{x}, x_n) - D_f(\bar{x}, W_n x_n) = \langle \bar{x} - x_n, \nabla f(W_n x_n) - \nabla f(x_n)\rangle - D_f(x_n, W_n x_n). \tag{7.2.19}$$

Since $x_n \rightharpoonup \bar{x}$ and $\lim_{n\to\infty} ||x_n - W_n x_n|| = 0$, we obtain

$$|D_f(\bar{x}, x_n) - D_f(\bar{z}, W_n x_n)| \leq ||\bar{x} - x_n|| ||\nabla f(W_n x_n) - \nabla f(x_n)|| - D_f(x_n, W_n x_n) \to 0 \tag{7.2.20}$$

as $n \to \infty$. Therefore from (7.2.18), we have

$$\begin{aligned}
&\alpha_{n,1}(1-\alpha_{n,1})\rho_r^*(||\nabla f(T^1 x_n) - \nabla f(x_n)||) + \\
&\alpha_{n,2}(1-\alpha_{n,2})\rho_r^*(||\nabla f(T^2 S_{n,1}x_n) - \nabla f(S_{n,1}x_n)||) + \dots \\
&+ \alpha_{n,N}(1-\alpha_{n,N})\rho_r^*(||\nabla f(T^N S_{n,N-1}x_n) - \nabla f(S_{n,N-1}x_n)||) \leq D_f(\bar{x}, x_n) - D_f(\bar{x}, x_n).
\end{aligned}$$

Taking limit as $n \to \infty$, using (7.2.20) and property of ρ_r^*, yields

$$\begin{aligned}
\lim_{n\to\infty} ||\nabla f(T^1 x_n) - \nabla f(x_n)|| &= \lim_{n\to\infty} ||\nabla f(T^2 S_{n,1}x_n) - \nabla f(S_{n,1}x_n)|| = \\
\dots &= \lim_{n\to\infty} ||\nabla f(T^N S_{n,N-1}x_n) - \nabla f(S_{n,N-1}x_n)|| = 0.
\end{aligned}$$

By the norm-to-norm uniform continuity of ∇f on bounded subset of E^*, it follows that

$$\begin{aligned}
\lim_{n\to\infty} ||T^1 x_n - x_n|| &= \lim_{n\to\infty} ||T^2 S_{n,1}x_n - S_{n,1}x_n|| = \dots \\
&= \lim_{n\to\infty} ||T^N S_{n,N-1}x_n - S_{n,N-1}x_n|| = 0.
\end{aligned} \tag{7.2.21}$$

We next prove that $S_{n,i}x_n - x_n \to 0$ for each $i = 1, 2, \dots, N-1$. From (7.2.16), we get

$$\begin{aligned}
D_p(x_n, S_{n,1}x_n) &= D_f(x_n, \nabla f^*[\alpha_{n,1}\nabla f(T^1 x_n) + (1-\alpha_{n,1})\nabla f(x_n)]) \\
&\leq \alpha_{n,1} D_f(x_n, T^1 x_n) + (1-\alpha_{n,1}) D_f(x_n, x_n).
\end{aligned}$$

Taking limit as $n \to \infty$ and using (7.2.21), we have

$$\lim_{n\to\infty} D_f(x_n, S_{n,1}x_n) = 0,$$

hence

$$\lim_{n\to\infty} ||S_{n,1}x_n - x_n|| = 0.$$

Thus

$$||T^2 S_{n,1}x_n - x_n|| \leq ||T^2 S_{n,1}x_n - S_{n,1}x_n|| + ||S_{n,1}x_n - x_n|| \to 0 \quad n \to \infty.$$

Similarly, we have

$$\begin{aligned} D_f(x_n, S_{n,2}x_n) &= D_f(x_n, \nabla f^*[\alpha_{n,2}\nabla f(T^2 S_{n,1}x_n) + (1-\alpha_{n,2})\nabla f(S_{n,1}x_n)]) \\ &\leq \alpha_{n,2} D_f(x_n, T^2 S_{n,1}x_n) + (1-\alpha_{n,2}) D_f(x_n, S_{n,1}x_n). \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} D_f(x_n, S_{n,2}x_n) = 0,$$

and hence

$$\lim_{n\to\infty} ||S_{n,2}x_n - x_n|| = 0.$$

Following similar approach as above, we have

$$\lim_{n\to\infty} ||S_{n,3}x_n - x_n|| = \lim_{n\to\infty} ||S_{n,4}x_n - x_n|| = \cdots = \lim_{n\to\infty} ||S_{n,N-1}x_n - x_n|| = 0.$$

Therefore

$$\lim_{n\to\infty} ||S_{n,i}x_n - x_n|| = 0 \quad \text{for each} \quad i = 1, 2, \ldots, N-1.$$

This together with the Bregman relative nonexpansiveness of each T^i for $i = 1, 2, \ldots, N$, implies that $\bar{x} \in F(S_{n,i})$ for $i = 1, 2, \ldots, N$. Hence $\bar{x} \in F(W_n)$. This therefore implies that W_n is Bregman relatively nonexpansive. □

We now present our iterative algorithm. In solving the EP(g) (1.1.4), we assume that the bifunction g satisfies the following assumptions:

(A1) $g(x, x) = 0$ for all $x \in C$;

(A2) g is monotone, that is $g(x, y) + g(y, x) \leq 0$ for all $x, y \in C$;

(A3) For all $x, y, z \in C$

$$\limsup_{t\downarrow 0^+} g(tz + (1-t)x, y) \leq g(x, y);$$

(A4) For all $x \in C$, $g(x, \cdot)$ is convex and lower semicontinuous.

Theorem 7.2.12. *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. For $i = 1, 2, \ldots, N$, let $\{\alpha_{n,i}\} \subset (0,1)$, $T^i : C \to C$ be a finite family of n-generalized Bregman nonspreading mappings and $W_n : C \to C$ be a Bregman W-mapping generated by $\{\alpha_{n,i}\}$ and $T^1, T^2, \ldots, T^N$ in (7.2.16). Let $g_j : C \times C \to \mathbb{R}$ be bifunctions satisfying assumptions (A1)-(A4) and suppose $\Gamma := \bigcap_{i=1}^N F(T^i) \cap \bigcap_{j=1}^\infty EP(g_j) \neq \emptyset$. Define the sequence $\{x_n\}$ by the following process*

$$\begin{cases} x_0 = x \in C, C_0 = Q_0 = C, \\ z_n = \nabla f^*[\beta_{n,0}\nabla f(x_n) + \sum_{j=1}^\infty \beta_{n,j}\nabla f(Res^f_{\lambda_n, g_j} x_n)], \\ y_n = \nabla f^*[\delta_n \nabla f(x_n) + (1-\delta_n)\nabla f(W_n z_n)], \\ C_n = \Big\{z \in C : D_f(z, y_n) \leq D_f(z, x_n)\Big\}, \\ Q_n = \Big\{z \in C : \langle \nabla f(x) - \nabla f(x_n), x_n - z\rangle \geq 0\Big\}, \\ x_{n+1} = Proj^f_{C_n \cap Q_n} x, \end{cases} \tag{7.2.22}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_{n,j}\}$ and $\{\delta_n\}$ are sequences in $[0,1)$ satisfying the following control conditions:

(i) $\sum_{j=0}^\infty \beta_{n,j} = 1$, $\forall\, n \in \mathbb{N} \cup \{0\}$;

(ii) There exists $k \in \mathbb{N}$ *such that* $\liminf_{n\to\infty} \beta_{n,j}\beta_{n,k} > 0$, $\forall j \in \mathbb{N} \cup \{0\}$;

(iii) $0 \leq \delta_n < 1$, $\forall n \in \mathbb{N}$ *and* $\liminf_{n\to\infty} \delta_n < 1$;

(iv) $\liminf_{n\to\infty} \lambda_n > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $Proj^f_\Gamma x$ as $n \to \infty$.

Proof. We divide the proof into several steps.

Step 1: We show that $\Gamma \subset C_n \cap Q_n$ and x_{n+1} is well defined.
It is clear that C_n and Q_n are closed and convex. Then $C_n \cap Q_n$ is closed and convex for $n \geq 0$. Obviously, $\Gamma \subset C_0 \cap Q_0$. Suppose $\Gamma \subset C_m \cap Q_m$ for some $m \in \mathbb{N}$. Let $p \in \Gamma$, then

$$\begin{aligned} D_f(p, y_m) &= D_f(p, \nabla f^*[\delta_m \nabla f(x_m) + (1-\delta_m)\nabla f(W_m z_m)]) \\ &= V_f(p, \delta_m \nabla f(x_m) + (1-\delta_m)\nabla f(W_m z_m)) \\ &= f(p) - \langle p, \delta_m \nabla f(x_m) + (1-\delta_m)\nabla f(W_m z_m)\rangle + f^*(\delta_m \nabla f(x_m) \\ &\quad + (1-\delta_m)\nabla f(W_m z_m)) \\ &\leq \delta_m[f(p) - \langle p, \nabla f(x_m)\rangle + f^*(x_m)] + (1-\delta_m)[f(p) - \langle p, \nabla f(W_m z_m)\rangle \\ &\quad + f^*(W_m z_m)] - \delta_m(1-\delta_m)\rho_r^*(||x_m - W_m z_m||) \\ &\leq \delta_m D_f(p, x_m) + (1-\delta_m)D_f(p, z_m) - \delta_m(1-\delta_m)\rho_r^*(||x_m - W_n z_m||) \\ &= \delta_n D_f(p, x_m) + (1-\delta_m)D_f(p, \nabla f^*[\beta_{m,0}\nabla f(x_m) + \sum_{j=1}^\infty \beta_{m,j}\nabla f(Res^f_{EP(g)} x_m)]) \\ &\quad - \delta_m(1-\delta_m)\rho_r^*(||x_m - W_m z_m||). \end{aligned}$$

Hence

$$\begin{aligned}
D_f(p, y_m) &\leq \delta_m D_f(p, x_m) + (1-\delta_m)[\beta_{m,0} D_f(p, x_m) + \sum_{j=1}^{\infty} \beta_{m,j} D_f(p, Res^f_{EP(g)} x_m) \\
&\quad -\beta_{m,0} \sum_{j=1}^{\infty} \beta_{m,j} \rho_r^*(||x_m - Res^f_{EP(g)} x_m||)] - \delta_m(1-\delta_m)\rho_r^*(||x_m - W_m z_m||) \\
&\leq \delta_m D_f(p, x_m) + (1-\delta_m)[\beta_{m,0} D_f(p, x_m) + \sum_{j=1}^{\infty} \beta_{m,j} D_f(p, x_m)] \\
&\quad -(1-\delta_m)\beta_{m,0} \sum_{j=1}^{\infty} \beta_{m,j} \rho_r^*(||x_m - Res^f_{EP(g)} x_m||) - \delta_n(1-\delta_m)\rho_r^*(||x_m - W_m z_m||) \\
&= D_f(p, x_m) - (1-\delta_m)\beta_{m,0} \sum_{j=1}^{\infty} \beta_{m,j} \rho_r^*(||x_m - Res^f_{EP(g)} x_m||) \\
&\quad -\delta_n(1-\delta_n)\rho_r^*(||x_m - W_m z_m||) \qquad (7.2.23) \\
&\leq D_f(p, x_m).
\end{aligned}$$

Hence $p \in C_m$, which implies that $\Gamma \in C_m$. Since $x_{m+1} = Proj^f_{C_m \cap Q_m} x$, then $\langle \nabla f(x) - \nabla f(x_{m+1}), z - x_{m+1} \rangle \leq 0 \; \forall \; z \in C_m \cap Q_m$. In particular, $\langle \nabla f(x) - \nabla f(x_{m+1}), p - x_{m+1} \rangle \leq 0$ $\forall p \in \Gamma$. Thus $p \in Q_{m+1}$. This proves that $\Gamma \subset C_{m+1} \cap Q_{m+1}$. Therefore $\Gamma \subset C_n \cap Q_n$ $\forall \; n \geq 0$. Consequently, since $C_n \cap Q_n$ is closed and convex, then $x_{n+1} = Prof^f_{C_n \cap Q_n} x$ is well-defined.

Step 2: We prove that $\{x_n\}, \{y_n\}, \{z_n\}, \{Res^f_{\lambda_n, g_j} x_n\}$ and $\{W_n z_n\}$ are bounded.
Since $\Gamma \subset C_n \cap Q_n$ for every $n \geq 0$ and $x_{n+1} = Proj^f_{C_n \cap Q_n} x$, then

$$D_f(p, x_{n+1}) \leq D_f(p, x) \quad \forall \; n \geq 0. \qquad (7.2.24)$$

So $\{D_f(p, x_n)\}$ is bounded and hence there exists a constant $M > 0$ such that

$$D_f(p, x_n) \leq M \quad \forall \; n \in \mathbb{N} \cup \{0\}.$$

In view of Lemma 2.6.28, we conclude that the sequence $\{x_n\}$ is bounded. Similarly, the sequences $\{y_n\}, \{z_n\}, \{Res^f_{\lambda_n, g_j} x_n\}$ and $\{W_n z_n\}$ are bounded.

Step 3: Next, we show that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$, $\lim_{n\to\infty} ||Res^f_{\lambda_n, g_j} x_n - x_n|| = 0$ and $\lim_{n\to\infty} ||W_n z_n - z_n|| = 0$.

Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = Proj^f_{Q_n}(x)$, we have

$$D_f(x_{n+1}, Proj^f_{Q_n}(x)) + D_f(Proj^f_{Q_n}(x_1), x) \leq D_f(x_{n+1}, x).$$

Thus

$$D_f(x_{n+1}, x_n) + D_f(x_n, x) \leq D_f(x_{n+1}, x). \qquad (7.2.25)$$

Therefore the sequence $\{D_f(x_n, x)\}$ is non-decreasing and thus $\lim_{n\to\infty} D_f(x_n, x)$ exists. Hence, it follows that $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$, and by Lemma 2.6.24, we have

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \qquad (7.2.26)$$

Also, since $x_{n+1} \in C_n$, we have

$$D_f(x_{n+1}, y_n) \leq D_f(x_{n+1}, x_n).$$

This yields that $\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0$ and thus

$$\lim_{n\to\infty} ||x_{n+1} - y_n|| = 0.$$

Therefore from (7.2.26) and (7.2.27), we get

$$\lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{7.2.27}$$

By the uniform continuity of f and ∇f on bounded subsets of E and E^* respectively, we have

$$\lim_{n\to\infty} ||f(y_n) - f(x_n)|| = 0 \tag{7.2.28}$$

and

$$\lim_{n\to\infty} ||\nabla f(y_n) - \nabla f(x_n)||_* = 0. \tag{7.2.29}$$

Furthermore,

$$\begin{aligned} D_f(p, x_n) - D_f(p, y_n) &= f(p) - f(x_n) - \langle p - x_n, \nabla f(x_n)\rangle - f(p) + f(y_n) + \langle p - y_n, \nabla f(y_n)\rangle \\ &= f(y_n) - f(x_n) + \langle p - y_n, \nabla f(y_n)\rangle - \langle p - x_n, \nabla f(x_n)\rangle \\ &= f(y_n) - f(x_n) + \langle x_n - y_n, \nabla f(y_n)\rangle - \langle p - x_n, \nabla f(y_n) - \nabla f(x_n)\rangle. \end{aligned}$$

Therefore from (7.2.27) - (7.2.29), we get

$$\lim_{n\to\infty} [D_f(p, x_n) - D_f(p, y_n)] = 0. \tag{7.2.30}$$

Note that from (7.2.23), we have

$$\begin{aligned} D_f(p, y_n) &\leq D_f(p, x_n) - (1 - \delta_n)\beta_{n,0} \sum_{j=1}^{\infty} \beta_{n,j} \rho_r^*(||x_n - Res^f_{\lambda_n, g_j} x_n||) \\ &\quad - \delta_n(1 - \delta_n)\rho_r^*(||x_n - W_n z_n||). \end{aligned}$$

Using the property of ρ_r^* and conditions (ii) and (iii) together with (7.2.30), we have

$$\lim_{n\to\infty} ||x_n - Res^f_{\lambda_n, g_j} x_n|| = 0 \tag{7.2.31}$$

and

$$\lim_{n\to\infty} ||x_n - W_n z_n|| = 0. \tag{7.2.32}$$

By the uniform continuity of ∇f on bounded subsets of E^*, we have

$$\lim_{n\to\infty} ||\nabla f(x_n) - \nabla f(Res^f_{\lambda_n, g_j} x_n)|| = 0.$$

Hence from (7.2.22), we get

$$\lim_{n\to\infty} ||\nabla f(z_n) - \nabla f(x_n)||' = \lim_{n\to\infty} \sum_{j=1}^{\infty} \beta_{n,j} ||\nabla f(Res^f_{\lambda_n,g_j} x_n) - \nabla f(x_n)||' = 0.$$

Furthermore, since f is Fréchet differentiable on bounded subset of E, then ∇f^* is uniformly continuous on bounded subsets of E^*. Thus

$$\lim_{n\to\infty} ||z_n - x_n|| = 0. \tag{7.2.33}$$

Therefore

$$\lim_{n\to\infty} ||W_n z_n - z_n|| = \lim_{n\to\infty} [||W_n z_n - x_n|| + ||x_n - z_n||] = 0. \tag{7.2.34}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to $q \in E$. Since $||W_n z_n - z_n|| \to 0$ and $||z_n - x_n|| \to 0$ as $n \to \infty$, then from Lemma 2.6.14 we have that $q \in F(W_n)$. Hence $q \in \bigcap_{i=1}^{N} F(T^i)$.

Also from Lemma 2.6.14, we have for each $j = 1, 2, \ldots$

$$g_j(Res^f_{\lambda_n,g_j} x_n, y) + \frac{1}{\lambda_n} \langle y - Res^f_{\lambda_n,g_j} x_n, \nabla f(Res^f_{\lambda_n,g_j} x_n) - \nabla f(x_n) \rangle \geq 0 \quad \forall y \in C.$$

Hence

$$g_j(Res^f_{\lambda_{n_k},g_j} x_{n_k}, y) + \frac{1}{\lambda_{n_k}} \langle y - Res^f_{\lambda_{n_k},g_j} x_{n_k}, \nabla f(Res^f_{\lambda_{n_k},g_j} x_{n_k}) - \nabla f(x_{n_k}) \rangle \geq 0 \quad \forall y \in C.$$

From the assumption (A2), we have

$$\begin{aligned} \frac{1}{\lambda_{n_k}} ||y - & Res^f_{\lambda_{n_k},g_j} x_{n_k}|| || \nabla f(Res^f_{\lambda_{n_k} g_j} x_{n_k}) - \nabla f(x_{n_k})|| \\ & \geq \frac{1}{\lambda_{n_k}} \langle y - Res^f_{\lambda_{n_k},g_j} x_{n_k}, \nabla f(Res^f_{\lambda_{n_k},g_j} x_{n_k}) - \nabla f(x_{n_k}) \rangle \\ & \geq -g_j(Res^f_{\lambda_{n_k} g_j} x_{n_k}, y) \geq g_j(y, Res^f_{\lambda_{n_k},g_j} x_{n_k}) \quad \forall y \in C. \end{aligned}$$

Taking the limit as $k \to \infty$ in the above inequality, from (A4) and condition (iv), we have $x_{n_k} \to q$, $||\nabla f(Res^f_{\lambda_{n_k},g_j} x_{n_k}) - \nabla f(x_{n_k})|| \to 0$, we have that $g_j(y, q) \leq 0$ for all $y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)q$. Noting that $y_t \in C$, which yields $g_j(y_t, q) \leq 0$. It therefore follows from (A1) that

$$0 = g_j(y_t, y_t) \leq t g_j(y_t, y) + (1-t) g_j(y_t, q) \leq t g_j(y_t, y).$$

That is $g_j(y_t, y) \geq 0$.

Let $t \downarrow 0$, from (A3), we obtain $g_j(q, y) \geq 0$ for any $y \in C$, $j = 1, 2, \ldots$. This implies that $q \in \bigcap_{j=1}^{\infty} EP(g_j)$. Therefore $q \in \Gamma := \bigcap_{i=1}^{N} F(T^i) \cap \bigcap_{j=1}^{\infty} EP(g_j)$.

Now since $x_{n+1} = Proj^f_{C_n \cap Q_n} x$, we have

$$\langle \nabla f(x) - \nabla f(x_{n+1}), x_{n+1} - z \rangle \geq 0, \quad \forall z \in C_n \cap Q_n.$$

Since $\Gamma \subset C_n \cap Q_n$, we have

$$\langle \nabla f(x) - \nabla f(x_{n+1}), x_{n+1} - z \rangle \geq 0 \quad \forall z \in \Gamma.$$

Taking the limit of the above inequality, we have

$$\langle \nabla f(x) - \nabla f(q), q - z \rangle \geq 0 \quad \forall z \in \Gamma.$$

Therefore $q = Proj_\Gamma^f x$. This completes the proof. □

7.2.2 Application to zeros of Maximal monotone operators

Sabach [223] showed that under some properties of the function f, the solution set of the equilibrium problem is equivalent to the set of zeros of a maximal monotone operator, that is the points $x^* \in dom\ A$ such that

$$0^* \in Ax^*, \tag{7.2.35}$$

where $A : E \to 2^{E^*}$ is a maximal monotone operator. We denotes the set of zeros of A by $A^{-1}(0^*)$. An operator $A : E \to 2^{E^*}$ is said to be monotone if for any $x, y \in dom\ A$, we have

$$\xi \in Ax \quad \text{and} \quad \mu \in Ay \Rightarrow \langle \xi - \mu, x - y \rangle \geq 0.$$

Let $g : C \times C \to \mathbb{R}$ be a bifunction and define the following operator $A_g : E \to 2^{E^*}$ in the following manner

$$A_g(x) = \begin{cases} \{\xi \in E^* : g(x,y) \geq \langle \xi, y - x \rangle \quad \forall y \in C\}, & x \in C, \\ \emptyset & x \notin C. \end{cases} \tag{7.2.36}$$

The following result was proved for the mapping A_g in [223].

Proposition 7.2.13. *(Sabach [223]) Let C be a nonempty, closed and convex subset of a reflexive Banach space E and let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Assume that the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions (A1)-(A4), then:*

(i) $EP(g) = A_g^{-1}(0^*)$;

(ii) A_g *is maximal monotone operator;*

(iii) $Res_g^f = Res_{A_g}^f$.

Based on the above result, we propose the following corollary which can be obtain from Theorem 7.2.12 for finding common fixed point of finite family of n-generalized Bregman nonspreading mapping and zeros of maximal monotone operators in reflexive Banach space.

Theorem 7.2.14. *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. For $i = 1, 2, \dots, N$, let $\{\alpha_{n,i}\} \subset (0, 1)$, $T^i : C \to C$ be finite family of n-generalized Bregman nonspreading mappings and $W_n : C \to C$ be a Bregman W-mapping generated by $\{\alpha_{n,i}\}$ and $T^1, T^2, \dots, T^N$ in (7.2.16). Let $g_j : C \times C \to \mathbb{R}$ be bifunctions satisfying assumptions (A1)-(A4), $A_{g_j} : E \to 2^{E^*}$ be as defined in (7.2.37) for $j = 1, 2, \dots$ and suppose $\Gamma := \bigcap_{i=1}^{N} F(T^i) \cap \bigcap_{j=1}^{\infty} A_{g_j}^{-1}(0^*) \neq \emptyset$. Define the sequence $\{x_n\}$ by the following process*

$$\begin{cases} x_0 = x \in C, C_0 = Q_0 = C, \\ z_n = \nabla f^*[\beta_{n,0}\nabla f(x_n) + \sum_{j=1}^{\infty} \beta_{n,j}\nabla f(Res^f_{A_{g_j}} x_n)], \\ y_n = \nabla f^*[\delta_n \nabla f(x_n) + (1 - \delta_n)\nabla f(W_n z_n)], \\ C_n = \Big\{ z \in C : D_f(z, y_n) \leq D_f(z, x_n) \Big\}, \\ Q_n = \Big\{ z \in C : \langle \nabla f(x) - \nabla f(x_n), x_n - z \rangle \geq 0 \Big\}, \\ x_{n+1} = Proj^f_{C_n \cap Q_n} x, \end{cases} \tag{7.2.37}$$

for all $n \geq 0$, where $\{\beta_{n,j}\}$ and $\{\delta_n\}$ are sequences in $[0, 1)$ satisfying the following control conditions:

(i) $\sum_{j=0}^{\infty} \beta_{n,j} = 1$, $\forall\ n \in \mathbb{N} \cup \{0\}$;

(ii) There exists $k \in \mathbb{N}$ such that $\liminf_{n\to\infty} \beta_{n,j}\beta_{n,k} > 0$, $\forall j \in \mathbb{N} \cup \{0\}$;

(iii) $0 \leq \delta_n < 1$, $\forall n \in \mathbb{N}$ *and* $\liminf_{n\to\infty} \delta_n < 1$.

Then, the sequence $\{x_n\}$ converges strongly to $Proj^f_{\Gamma} x$ as $n \to \infty$.

CHAPTER 8

Conclusion, Contribution to Knowledge and Future research

8.1 Conclusion

In this thesis, we presented some inertial-type iterative schemes with strong convergence properties for approximating solutions of certain optimization problems and finding fixed point of nonlinear mappings in real Hilbert spaces. We compared the performance of our algorithms against existing algorithms in the literature using MATLAB programming. In each case, we found that each of our proposed algorithms performs better than some related algorithms in the literature.

Also, we studied the approximation of common solution of non-monotone equilibrium problem and fixed point problem in real Hilbert spaces. We proved strong convergence theorems and provided numerical examples to show the accuracy and efficiency of our algorithms. We then introduced a new projection contraction type algorithm for solving split generalized equilibrium problem and finding common fixed point of finite family of nonlinear mappings in real Hilbert spaces. We also showed that our new projection algorithm converges at the rate of $O(1/t)$.

Furthermore, we extended the study of projection method for solving VIP from a real Hilbert space to a reflexive Banach space. We introduced a new projection method with Armijo-line search technique for solving pseudo-monotone VIPs in real reflexive Banach spaces. We also introduced a totally relaxed self-adaptive subgradient extragradient algorithm for finding common solution of VIP and fixed point problem in a 2-uniformly convex and uniformly smooth real Banach spaces. Then, we proposed another new projection contraction algorithm and proved strong convergence theorems for VIP and fixed point problems in real Hilbert spaces. We gave an application of our results to approximating solutions of split equality problem in Hilbert spaces. In each case, we provided some numerical examples to illustrate the performance of our algorithms.

Also, we extended the study of split equality monotone inclusion problem from real Hilbert spaces to p-uniformly convex and uniformly smooth real Banach spaces. We introduced a new iterative scheme and proved a strong convergence result for approximating solution of split equality monotone inclusion problem in p-uniformly convex and uniformly smooth real Banach spaces. We then presented some applications of our result to solving some other optimization problems in real Banach spaces.

More so, we introduced an intermixed iterative algorithm for approximating individual fixed point of two k-strictly pseudo-contractive mappings in a p-uniformly smooth Banach space. Using our result, we were proposed an algorithm for approximating a common fixed point for the two k-strictly pseudo-contractive mappings. Finally, we introduced a class of N-generalized Bregman nonspreading mapping in a reflexive Banach space. We also studied some fixed point properties for this new class of mapping. Then, we proposed an hybrid iterative scheme for approximating the common fixed points of finite family of N-generalized Bregman nonspreading mappings which is also a solution to an equilibrium problem in a reflexive Banach space. We gave an application of our result to approximating zeros of maximal monotone operators in a reflexive Banach space.

8.2 Contribution to Knowledge

We highlight some contributions in this thesis as follows:

(i) Our main theorem in Section 3.1 improved the corresponding results of Chembolle and Dossel [72], Cai and Shehu [60], Tian and Huang [258], Xu [266] and Shehu [234].

(ii) In Section 3.2, we improved the results of Kraikew and Saejung [158], Thong and Hieu [254, 255] and Dong et al. [97] by using a self-adaptive stepsize selection technique which does not require a prior estimate of the Lipschitz constant of the monotone operator.

(iii) Also in Section 3.3, our results generalized the results of Suantai et al. [240], [197] and Rizvi [218].

(iv) Our results in Section 4.1 generalized the results of Dinh and Muu [94], Hieu [127] and many other related results in the literature. It also improved the corresponding results of Hieu et al. [129] and [130] by constructing a sublevel set using convex combination of finite family of convex functions and does not involve the projection onto the intersection of C_n and Q_n.

(v) In Section 4.2, we improved and generalized the results of Chuang [85] and Yen [276] by introducing a simpler inertial Mann-Krasnoselskii algorithm in a real Hilbert space.

(vi) Our results in Section 4.3 extended and generalized the results of Kazmi and Rizvi [148], Deepho et al [93] and Phuengrattana et al. [204] in unified ways.

(vii) Section 5.1 extended the results of Kanzow and Shehu [144] from Hilbert space to a reflexive Banach space and from monotone variational inequality problem to pseudo-monotone variational inequality problem. It also improved many existing results such as [80, 113, 112, 158, 254] where the operator is required to satisfy Lipschitz and monotone conditions.

(viii) Our result in Section 5.2 improved the results of He et al. [124, 123], Chidume and Nnakwe [80].

(ix) Our results in Section 5.3 improved the results of [97, 144, 256, 254, 255] which requires more than one projection onto the the feasible set.

(x) In Section 6.1, we extended the results of [226, 227, 230, 232, 246] to a split equality monotone inclusion problem in p-uniformly convex and uniformly smooth Banach spaces.

(xi) Our results in Section 7.1 improved and generalized the results of Chidume and Shahzad [80], Cholamjiak and Suantai [239] in Banach spaces. We also extend the intermixed algorithm in [274] to a p-uniformly smooth Banach space.

(xii) In Section 7.2, we generalized the results of [146, 153, 217, 215] to N-generalized Bregman nonspreading mapping in reflexive Banach spaces. We also extend the results of [250, 251] to a reflexive Banach space.

8.3 Future Research

In our future research, we will like to study the approximation of solutions of optimization and fixed point problems in Hadamard spaces such as the CAT(0) spaces, CAT(k) spaces, p-uniformly convex metric spaces and $\mathbb{R}$-tree spaces.

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