Iterative algorithms for approximating solutions of some optimization problems in Hadamard spaces

by

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As the candidate's supervisor, I have approved this dissertation for submission.

Dr. O. T. Mewomo

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Dedication

This dissertation is dedicated to the Almighty God.

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Abstract

In this work, we introduce and study a modified extragradient algorithm for approximating solutions of a certain class of split pseudo-monotone variational inequality problems in real Hilbert spaces. Using our proposed algorithm, we establish a strong convergence result for the approximation of solutions of the aforementioned problem. Our strong convergence result is obtained without prior knowledge of the Lipschitz constant of the pseudo-monotone and Lipschitz continuous operator used in this work, and with minimized number of projections per iteration compared to other results on split variational inequality problems in the literature. More so, a numerical example of our algorithm in comparison with other algorithm is given to show the efficiency and advantage of our results. We further extend our study from the frame work of real Hilbert spaces to Hadamard spaces, and from variational inequality problems to monotone inclusion problems. Precisely, we introduce a viscosity-type proximal point algorithm which comprises of a finite sum of resolvents of monotone operators and a generalized asymptotically nonexpansive mapping. We prove that the algorithm converges strongly to a common solution of a finite family of monotone inclusion problems and fixed point problem for a generalized asymptotically nonexpansive mapping in an Hadamard space. Furthermore, we give two numerical examples of our algorithm in finite dimensional spaces and one numerical example in a non-Hilbert space setting, in order to show the applicability of our results. We then introduce and study the class of generalized demimetric mappings in Hadamard spaces. We also propose a Halpern-type proximal point algorithm comprising of a generalized demimetric mapping and a finite composition of resolvents of monotone operators, and prove that it converges strongly to a common solution of a finite family of monotone inclusion problems and fixed point problem for a demimetric mapping in an Hadamard space. More so, we apply our results to solve a finite family of convex minimization problems, variational inequality problems and convex feasibility problems in Hadamard spaces.

Contents

Declaration

This dissertation has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used, proper reference has been made.

Grace Nnennaya Ogwo

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Contributed papers from the dissertation

Papers accepted/submitted from the dissertation.

1. **G.N. Ogwo**, C. Izuchukwu, K.O. Aremu and O.T. Mewomo, A viscosity iterative algorithm for a family of monotone inclusion problems in an Hadamard space, The Bulletin of the Belgian Mathematical Society, (Accepted, to appear in **27** (2) (2020)).

2. **G.N. Ogwo**, C. Izuchukwu and O.T. Mewomo, A modified extragradient algorithm for a certain class of split pseudo-monotone variational inequality problem, Advance in Pure and Applied Mathematics (Submitted).

3. **G.N. Ogwo**, C. Izuchukwu, K.O. Aremu and O.T. Mewomo, On Generalized Demimetric Mappings and monotone operators in Hadamard spaces, Demonstratio Mathematica, (Submitted).

Chapter 1

Introduction

1.1 Background of study

Optimization theory is an important area of research in mathematics that has attracted the interest of many researchers due to its usefulness in nonlinear and convex analysis. Some of the important problems in optimization theory are the Variational Inequality Problem (VIP) and Monotone Inclusion Problem (MIP) since they include many other optimization and mathematical problems as special cases; namely, minimization problems, complementarity problems, fixed point problems, convex feasibility problems, among others.

Let C be a nonempty closed and convex subset of a real Hilbert space H and $f : C \to C$ be any nonlinear operator. The VIP is defined as: Find $x \in C$ such that

$$\left\langle fx, y - x \right\rangle \geq 0, \quad \forall\ y \in C. \tag{1.1.1}$$

VIP was first introduced by Stampacchia [76] for modeling problems arising from mechanics. To study the regularity problem for partial differential equations, Stampacchia [76] studied a generalization of the Lax-Milgram theorem and called all problems involving inequalities of such kind, the VIPs. The VIP is also known to have numerous applications in diverse fields such as, physics, engineering, economics, mathematical programming, among others. It can be considered as a central problem in optimization and nonlinear analysis since the theory of variational inequalities provides a simple, natural and unified frame work for a general treatment of many important mathematical problems such as, minimization problems, network equilibrium problems, complementary problems, systems of nonlinear equations and others (see [5, 9, 21, 31, 50, 51, 76, 86, 85] and the references therein). Thus, the theory has become an active area of research to numerous researchers. As a result of this, there has been an increased interest in developing efficient and implementable methods for solving VIPs.

In the same vein, the MIP which is an important generalization of the VIP is defined as follows:

$$\text{Find} \quad x \in D(A) \ \text{ such that } \quad 0 \in Ax, \tag{1.1.2}$$

where $A : X \to 2^{X^*}$ is a monotone operator, X^* is the dual of the Hadamard space X and $D(A)$ is the domain A (to be defined in Chapter 2).
The MIP is of central importance in nonlinear and convex analysis since many mathematical problems such as the equilibrium problems, VIPs, minimization problems among others can be posed as MIPs.

Various methods have been developed by numerous authors for solving VIPs and MIPs, which include fixed point method, proximal point method, gradient method, extragradent method, subgradient extragradient method, amongst others. One of the most effective methods for finding the solutions of VIPs and MIPs is the fixed point method. Fixed point theory is an important area of research in nonlinear functional analysis that has continued to attract the interest of numerous researchers in past years. Many important nonlinear problems in disciplines like engineering, physics, economics, life science and medical sciences reduces to nonlinear functional equations such as nonlinear integral equations and boundary value problems for nonlinear ordinary or partial differential equations. This nonlinear functional equations can be translated in terms of a fixed point equation,

$$Tx = x \tag{1.1.3}$$

for a given nonlinear mapping T on a nonempty set X, where $x \in X$ satisfying (1.1.3) is called the fixed point of T. Throughout this work, we will denote by $F(T)$ the set of fixed points of T.

The fixed point theory is used in proving the existence and uniqueness of solutions of different mathematical problems and as a result of this, it is sometimes referred to as the kernel of the modern nonlinear analysis. The existence of a fixed point is important in several areas of mathematics and many related areas. The design of fixed point iterative methods for solving nonlinear problems, in particular, nonlinear equations or systems, has gained a spectacular development in the last two decades. For example, if we consider the nonlinear ordinary differential equation

$$x^{'}(t) = f(t, x(t)), \qquad x(t_0) = x_0. \tag{1.1.4}$$

To find the solution of the differential equation (1.1.4), we solve

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds,$$

and to establish the existence of the solution of the posed problem, we consider the operator

$$T : C([a, b]) \to C([a, b]),$$

defined by

$$Tx = x_0 + \int_{t_0}^{t} f(s, x(s))ds,$$

where $C([a, b])$ is the space of continuous real valued functions on closed and bounded interval $[a, b]$.

Any x that solves the problem above is a fixed point of the operator T. Thus, finding a solution to the problem is the same as finding a fixed point of T. However, an existence theorem only involves the establishment of sufficient conditions under which a given problem has a solution, but does not tell us how to find the solution of such a problem. On the other hand, the iterative methods are concerned with the approximation of sequences that converge to fixed points of nonlinear operators and solutions of such problems (in particular, (1.1.1) and (1.1.2)). The iterative methods are our major concern in this dissertation.

1.2 Some important iterative schemes

Fixed points of nonlinear operators are not easily obtained so there is need for approximate solutions. To overcome this, different iterative schemes have been developed and used to approximate the fixed points of nonlinear mappings on suitable spaces. In this section, we recall some important iterative schemes in literature for approximating solutions of fixed point problems.

1.2.1 Picard iteration

Let $T : X \to X$ be an α-contraction mapping on a complete metric space (X, d), satisfying

$$d(Tx, Ty) \leq \alpha d(x, y) \qquad \forall \ \ x, y \in X, \tag{1.2.1}$$

with $\alpha \in [0, 1)$ fixed. Then by the contraction mapping theorem, we have

(i) $F(T) = \{x^*\}$, that is T has a unique fixed point;

(ii) the Picard iteration

$$x_n = T^n x_0, \quad n = 1, 2, \cdots \tag{1.2.2}$$

converges to x^* for $x_0 \in X$;

(iii) both the priori and the posterior error estimates

$$d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha} \cdot d(x_0, x_1), \quad n = 0, 1, 2, \cdots, \tag{1.2.3}$$

$$d(x_n, x^*) \leq \frac{\alpha}{1-\alpha} \cdot d(x_{n-1}, x_n), \quad n = 1, 2, \cdots \tag{1.2.4}$$

hold,

(iv) the rate of convergence is given by

$$d(x_n, x^*) \leq \alpha \cdot d(x_{n-1}, x^*) \leq \alpha^n \cdot d(x_0, x^*), \quad n = 1, 2, \cdots. \tag{1.2.5}$$

Remark 1.2.1. *[13] The errors $d(x_n, x^*)$ are decreasing as rapidly as the terms of geometric progression with ratio α, that is, $\{x_n\}_{n=0}^{\infty}$ converges to x^* at least as rapidly as the geometric series. The convergence is however linear as shown in (1.2.5).*

If T satisfies a weaker contractive condition, for example when T is nonexpansive, the Picard iteration may not converge and if it does converge, its limit may not be a fixed point of T.

1.2.2 Krasnoselskii iteration

Replacing (1.2.2) with

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n), \quad \text{for} \quad n = 0, 1, 2, \cdots \quad \text{and} \quad x_0 \in C, \tag{1.2.6}$$

the iterative sequence converges to the unique fixed point. In general, if X is a normed linear space and T is a nonlinear mapping, then (1.2.6) can be generalized as follows;

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, \quad n = 0, 1, 2, \cdots \quad \text{for} \quad x_0 \in C \quad \text{and} \quad \lambda \in (0, 1). \tag{1.2.7}$$

The formula (1.2.7) is called the Krasnoselskii iteration. Krasnoselskii iteration (1.2.7) reduces to the Picard iteration when $\lambda = 1$, and it is the Picard iteration corresponding to the averaged operator $T_\lambda = (1-\lambda)I + \lambda T$, where I is the identity operator.

1.2.3 Mann iteration

Mann iteration [59] formula is the most general iterative formula for the approximation of fixed points of nonlinear mappings, and it is given by

$$x_{n+1} = (1-a_n)x_n + a_n Tx_n, \quad n = 0, 1, 2, \cdots \quad \text{and} \quad x_0 \in C, \tag{1.2.8}$$

where $\{a_n\}_{n=0}^{\infty}$ is a sequence in $(0, 1)$ satisfying the following conditions

(i) $\lim_{n\to\infty} a_n = 0$,

(ii) $\sum_{n=0}^{\infty} a_n = \infty$.

If $\{a_n\} = \{\lambda\}$ then (1.2.8) reduces to (1.2.7) and if $\{a_n\} = 1$, (1.2.8) reduces to (1.2.2).

Example 1.2.2. *[13] Let $X = \mathbb{R}$ with the usual norm, $K = \left[\frac{1}{2}, 2\right]$ and $T : K \to K$ be the function given by $Tx = \frac{1}{x}$ for all $x \in K$. Then;*

(i) $F(T) = \{1\}$;

(ii) the Picard iteration associated to T does not converge to the fixed point of T for any $x_0 \in X \setminus \{1\}$;

(iii) the Krasnoselskii iteration associated to T converges to the fixed point $p = 1$, for any $x_0 \in K$ and $\lambda \in \left(0, \frac{1}{16}\right)$;

(iv) the Mann iteration associated to T with $\alpha_n = \frac{n}{2n+1}$, $n \geq 0$ and $x_0 = 2$ converges to 1, the unique fixed point of T.

1.2.4 Ishikawa iteration

The Mann iterative algorithm was improved by Ishikawa [36] to a new iterative algorithm for pseudocontractive mappings that generates the sequence $\{x_n\}$ by

$$x_{n+1} = (1 - a_n)x_n + a_n T[(1 - b_n)x_n + b_n T x_n], \quad n = 0, 1, 2, \cdots \quad \text{and} \quad x_0 \in C, \tag{1.2.9}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences satisfying the following conditions

(i) $0 \leq a_n \leq b_n \leq 1$,

(ii) $\lim\limits_{n\to\infty} b_n = 0$,

(iii) $\sum\limits_{n=1}^{\infty} a_n b_n = \infty$.

Writing (1.2.9) in a system form, we have

$$\begin{cases} y_n = (1 - b_n)x_n + b_n T x_n \\ x_{n+1} = (1 - a_n)x_n + a_n T y_n, \quad n = 0, 1, 2, \cdots \end{cases} \tag{1.2.10}$$

which implies that the Ishikawa iteration is regarded as a double Mann iteration and it reduces to Mann iteration when $b_n = 0$. The Mann and the Ishikawa iterations have been successfully used by various authors to approximate fixed points of various classes of mappings.

Replacing T by T^n in (1.2.10), we have the modified Ishikawa iterative algorithm defined as follows:

$$\begin{cases} y_n = (1 - b_n)x_n + b_n T^n x_n \\ x_{n+1} = (1 - a_n)x_n + a_n T^n y_n, \quad n = 0, 1, 2, \cdots . \end{cases} \tag{1.2.11}$$

1.2.5 Krik's iteration

Let H be a Hilbert space, $T : H \to H$ be a self map, $x_0 \in H$ and the sequence $\{x_n\}$ be defined by

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T x_n + \alpha_2 T^2 x_n + \cdots + \alpha_k T^k x_n, \tag{1.2.12}$$

where k is a fixed integer.

$$k \geq 1, \quad \alpha_i \geq 0, \quad \text{for} \quad i = 0, 1, \cdots, k, \quad \alpha_i > 0$$

and

$$\alpha_0 + \alpha_1 + \cdots + \alpha_k = 1.$$

The iterative processes of Kirk, Ishikawa, Mann and Krasnoselskii are mainly used to generate successive approximations for fixed points of nonlinear mappings for which the Picard iteration does not converge to.

1.2.6 Halpern iteration

Halpern [33] introduced the explicit iterative algorithm which generates a sequence using the recursive formula

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad \text{for} \quad n = 0, 1, 2, \cdots \tag{1.2.13}$$

with the initial guess $x_0 \in C$ and $u \in C$ arbitrarily fixed and the sequence $\{\alpha_n\}$ is contained in $(0,1)$. The iterative method is used for finding fixed points of a nonlinear mapping with $F(T) \neq \emptyset$. This iterative method is known as Halpern iteration.

1.2.7 Viscosity iteration

Moudafi [62] proposed the viscosity iterative method. Choose an arbitrary initial point $x_0 \in H$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_{n+1} = \frac{\epsilon_n}{1+\epsilon_n} g(x_n) + \frac{1}{1+\epsilon_n} Tx_n, \quad \forall \ n \geq 0, \tag{1.2.14}$$

where T is a nonexpansive self-mapping and g is a contraction with a coefficient $\alpha \in [0,1]$, the sequence $\{\epsilon_n\}$ in $(0,1)$, such that

(i) $\lim\limits_{n\to\infty} \epsilon_n = 0$,

(ii) $\sum\limits_{n=0}^{\infty} \epsilon_n = \infty$,

(iii) $\lim\limits_{n\to\infty} \left(\frac{1}{\epsilon_n} - \frac{1}{\epsilon_{n+1}}\right) = 0$,

then $\lim\limits_{n\to\infty} x_n = x^*$, where $x^* \in F(T)$ is the unique solution of the variational inequality

$$\langle (1-f)x^*, x - x^* \rangle \geq 0, \quad \forall \ x \in F(T). \tag{1.2.15}$$

1.3 Research motivation

Censor et al. [21] introduced and studied the Spilt Variational Inequality Problem (SVIP) which is a generalization of the spilt feasibility problems introduced by Censor and Elfving

[20]. They proposed an algorithm for approximating the solution of the SVIP and proved that the sequence generated by this algorithm converges weakly to a solution of SVIP when the associated operators are inverse strongly monotone. Also, Tian and Jian [85] introduced a new class of SVIP which generalizes the class of SVIP considered by Censor et al. [21]. They proposed an algorithm for approximating the solution of the new class of SVIP and also proved the weak convergence of this algorithm when the associated operator is monotone and Lipschitz continuous, and the mapping is nonexpansive.

Motivated by the works of Censor et al. [21] and Tian and Jian [85], we propose a new modified extragradient iterative algorithm for solving the new class of SVIP (introduced by Tian and Jian [85]). We also prove that the sequence generated by our algorithm converges strongly to a solution of the SVIP when the associated operator is pseudomonotone and Lipschitz continuous, and the mapping is strictly pseudocontractive. Our proposed algorithm has minimized number of projections per iteration and does not require prior knowledge of the Lipschitz constant unlike the algorithms of Censor et al. [21] and Tian and Jian [85]. Also, our numerical experiements show that our iterative algorithm performs better than that of Tian and Jian [85].

Rockafellar [71] studied the Proximal Point Algorithm (PPA) for approximating solutions of MIP in real Hilbert spaces and obtained a weak convergence result. Bačák [7] extended the study of the PPA from Hilbert spaces to Hadamard spaces, and obtained a $\triangle$-convergence result when the associated operator is the subdifferential of a proper, convex and lower semi-continuous function. Ranjbar and Khatibzadeh [69] proposed a Mann-type and a Halpern-type PPA in an Hadamard space for approximating solutions of MIP, and obtained a $\triangle$ and a strong convergence results respectively, when the associated operator is monotone.

Motivated by the works of Bač*á*k [7] and Ranijbar and Khatibzabeh [69], we introduce a viscosity-type PPA which comprises of a finite sum of resolvents of monotone operators, and a generalized asymptotically nonexpansive mapping. We prove that the algorithm converges strongly to a common solution of a finite family of MIPs and fixed point problem for a generalized asymptotically nonexpansive mapping in an Hadamard space.

In 2018, Aremu et al. [4] introduced the class of demimetric mappings in Hadamard spaces and established some fixed point results for this class of mappings. They proved a strong convergence theorem for approximating a common solution of finite family of minimization problems and fixed point problems for this class of mappings in Hadamard spaces. In the same year, Kawasaki and Takahashi [45] generalized the class of demimetric mappings to the class of generalized demimetric mappings in Banach space, and obtained strong convergence results. Motivated by the works of Aremu et al. [4], Kawasaki and Takahashi [45], we introduce and study the class of generalized demimetric mappings in Hadamard spaces. We also propose a Halpern-type PPA comprising of a generalized demimetric mappings and a finite composition of resolvents of monotone operators, and prove that it converges strongly to a common solution of a finite family of MIPs and fixed point problem for a demimetric mapping in an Hadamard space.

1.4 Statement of problem

The following problems have been studied in this dissertation:

1. Let H_1 and H_2 be two real Hilbert spaces, C be a nonempty closed and convex subset of H_1 and $f : H_1 \to H_1$ be a pseudo-monotone and Lipschitz continuous operator. Let $g : H_1 \to H_1$ be a contraction mapping with constant $\rho \in (0,1)$, $A : H_1 \to H_2$ a bounded linear operator and $T : H_2 \to H_2$ be a κ-strictly pseudocontractive mapping with $\kappa \in [0,1)$ and $F(T) \neq \emptyset$. We intend to find $x \in C$ such that

$$\big\langle fx, y - x \big\rangle \geq 0 \ \ \forall\, y \in C \ \text{ and } Ax \in F(T).$$

2. Let X be an Hadamard space and X^* be its dual space. Let $T : X \to X$ be a uniformly asymptotically regular and uniformly L-Lipschitzian generalized asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be a finite family of multivalued monotone mappings which satisfy the range condition and g be a contraction mapping on X with coefficient $\rho \in (0,1)$. We intend to find

$$\bar{v} \in F(T) \ \text{ such that } \ 0 \in \cap_{i=1}^{N} A(\bar{v}). \tag{1.4.1}$$

3. Let C be a nonempty closed and convex subset of an Hadamard space and $T : C \to C$ be a θ-generalized demimetric mapping with $\theta \neq 0$ and $F(T) \neq \emptyset$. Let $A_i : X \to 2^{X^*}, i = 1, 2, \cdots, N$ be multivalued monotone mappings that satisfy the range condition. We intend to solve problem (1.4.1) for the class of θ-generalized demimetric mappings.

1.5 Objectives

The objectives of this work are to:

(i) review some existing results on VIPs and MIPs in real Hilbert spaces and Hadamard spaces,

(ii) develop and study better implementable iterative algorithms for approximating solutions of VIPs and MIPs in Hilbert spaces and Hadamard spaces respectively,

(iii) establish strong convergence results of the proposed algorithms,

(iv) introduce and study the class of generalized demimetric mappings in Hadamard spaces,

(v) give nontrivial numerical experiments of our results in comparison with others, in order to illustrate the applicability and the competitive advantages of our results over existing iterative algorithms in the literature,

(vi) apply the obtained results to solve some important optimization problems.

1.6 Organization of dissertation

The rest of this dissertation is organized as follows:

Chapter 2: In this chapter, we give preliminaries of our study and also recall some basic definitions, concepts, theorems, lemmas and propositions that will be useful in the main results.

Chapter 3: In this chapter, we introduce and study a modified extragradient algorithm for approximating solutions of a certain class of split pseudo-monotone VIP in real Hilbert spaces. A numerical example of our algorithm in comparison with other algorithm is also given in this chapter to show the efficiency and advantage of our results.

Chapter 4: In this chapter, we introduce a viscosity-type PPA which comprises of a finite sum of resolvents of monotone operators, and a generalized asymptotically nonexpansive mapping. We prove that the algorithm converges strongly to a common solution of a finite family of MIPs and fixed point problem for a generalized asymptotically nonexpansive mapping in an Hadamard space. Numerical examples of our results are given to further illustrate the applicability of our results.

Chapter 5: In this chapter, we introduce a new class of nonlinear mappings, namely, the class of generalized demimetric mappings in Hadamard spaces. We propose a Halpern-type PPA comprising of our new class of mappings, and a finite composition of resolvents of monotone operators. Using our proposed algorithm, we prove that the sequence generated by it converges strongly to a common solution of MIPs and fixed point problem for the generalized demimetric mapping in an Hadamard space. Finally, we applied our results to solve a finite family of convex minimization problems, VIPs and convex feasibility problems in Hadamard spaces.

Chapter 6: In this chapter, we give the conclusion of our research. We also highlight the contributions of our research to existing knowledge. Furthermore, we discuss possible areas of future research.

Chapter 2

Preliminaries

In this chapter, we give preliminaries of our study and also recall some basic definitions and results that will be useful in our main results.

2.1 Hilbert spaces

The notion of Hilbert spaces was introduced by David Hilbert (between 1862-1943) and it is known to be an extension of the concept of Euclidean spaces to infinite dimensional spaces. The Hilbert space is known to have the most simplest and clearly discernible geometric structure compared to other Banach spaces. Hilbert space is our first space of interest in this dissertation.

Definition 2.1.1. *[25] Let H be a nonempty set. An inner product on H is a function $\langle \cdot, \cdot \rangle$ defined on $H \times H$ with values in $K = \mathbb{R}$ or $\mathbb{C}$ such that the following conditions hold:*

(i) $\langle x, x \rangle \geq 0 \quad \forall \;\; x \in H \quad \text{and} \quad \langle x, x \rangle = 0 \quad \iff \quad x = 0;$

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle} \;\; \forall \;\; x, y \in H$ *(the bar denotes the complex conjugate)*;

(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle;$

(iv) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle \quad \forall \;\; x, y, z \in H \quad \text{and} \quad \lambda, \mu \in K.$

The pair $(H, \langle \cdot, \cdot \rangle)$ is called an inner product space.

We note that if $K = \mathbb{R}$, then (ii) becomes $\langle x, y \rangle = \langle y, x \rangle$. In this case, $(H, \langle \cdot, \cdot \rangle)$ is called a real inner product space.

Definition 2.1.2. *An inner product space $(H, \langle \cdot, \cdot \rangle)$ is said to be complete if every Cauchy sequence in H converges to a point in H and a complete inner product space is called a Hilbert space.*

Proposition 2.1.3. *[25](Cauchy Schwartz inequality) Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space. For arbitrary $x, y \in H$*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle. \tag{2.1.1}$$

Lemma 2.1.4. *[25] The function $\|\cdot\| : H \to [0, \infty)$, defined by*

$$\|x\| = \sqrt{\langle x, x\rangle} \tag{2.1.2}$$

is a norm on H.

Remark 2.1.5. *Following (2.1.2), the Cauchy Schwartz inequality (2.1.1) can be written generally as*

$$\sqrt{|\langle x, y\rangle|^2} \leq \sqrt{\langle x, x\rangle} \cdot \sqrt{\langle y, y\rangle} \tag{2.1.3}$$

so that $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$ for arbitrary $x, y \in H$.

2.1.1 Examples of Hilbert spaces

(i) The space $\mathbb{R}^n$ is a Hilbert space with inner product defined by

$$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i,$$

where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ are in $\mathbb{R}^n$.

(ii) The space $L_2(\mathbb{R})$ is the space of real valued functions such that

$$\int_{\mathbb{R}} |f(x)|^2 dx < \infty,$$

and it is a real Hilbert space, with inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)dx.$$

(iii) Let ω be an open set in $\mathbb{R}^n$. The space $L^2(\omega)$ is the space of complex valued functions such that

$$\int_{\omega} |f(x)|^2 dx < \infty,$$

where $x = (x_1, \cdots, x_n) \in \omega$ and $dx = dx_1 \cdots dx_n$. $L^2(\omega)$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\omega} f^*(x)g(x)dx.$$

2.1.2 Geometric properties of Hilbert spaces

Some geometric properties that characterize Hilbert spaces include: the inner product, the fact that the nearest point map of a real Hilbert space H onto a closed convex subset C of H is lipschitzian with constant 1 and the following identities:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \tag{2.1.4}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{2.1.5}$$

which hold for $x, y \in H$ and $\lambda \in [0,1]$. These geometric characteristics of Hilbert spaces makes certain problems posed in Hilbert spaces more manageable than those in general Banach spaces [24].

We observe that (2.1.4) can be written as

$$2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2 \quad x, y \in H \tag{2.1.6}$$

and

$$2(\|x\|^2 + \|y\|^2) = \|x-y\|^2 + \|x+y\|^2, \tag{2.1.7}$$

where (2.1.7) is called the parallelogram identity.

2.1.3 Some inequalities that characterize Hilbert spaces

Lemma 2.1.6. *[24] Let H be a real Hilbert space, then for each $x, y \in H$, we have*

(i)

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \tag{2.1.8}$$

(ii)

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \tag{2.1.9}$$

Proof. (i)

$$\begin{aligned}\|x-y\|^2 &= \langle x-y, x-y\rangle \\ &= \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle \\ &= \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle.\end{aligned}$$

(ii)

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &\le \langle x, x\rangle + 2\langle y, x\rangle + 2\langle y, y\rangle \\ &= \langle x, x\rangle + 2\langle y, x+y\rangle.\end{aligned}$$

□

Remark 2.1.7. *It then follows from Lemma 2.1.6 (i) that*

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \quad \forall x, y \in H.$$

Lemma 2.1.8. *For $x, y \in H$ and $\lambda \in [0,1]$, we have*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Proof.

$$\begin{aligned}
\|\lambda x + (1-\lambda)y\|^2 &= \langle \lambda x + (1-\lambda)y, \lambda x + (1-\lambda)y\rangle \\
&= \lambda^2\langle x, x\rangle + \lambda(1-\lambda)\langle x, y\rangle + \lambda(1-\lambda)\langle y, x\rangle + (1-\lambda)^2\langle y, y\rangle \\
&= \lambda^2\langle x, x\rangle + 2\lambda(1-\lambda)\langle x, y\rangle + (1-\lambda)^2\langle y, y\rangle,
\end{aligned}$$

which implies from (2.1.8) that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

□

Lemma 2.1.9. *[52] For $x, y, m, z \in H$, we have*

$$2\langle x-y, m-z\rangle = \|x-z\|^2 + \|y-m\|^2 - \|x-m\|^2 - \|y-z\|^2.$$

We also have

$$\begin{aligned}
\|x-y+m-n\|^2 &= \|x-y\|^2 + \|m-z\|^2 + 2\langle x-y, m-z\rangle \\
&= \|x-y\|^2 + \|m-z\|^2 + \|x-z\|^2 + \|y-m\|^2 - \|x-m\|^2 - \|y-z\|^2.
\end{aligned}$$

2.1.4 Some nonlinear single-valued mappings in Hilbert spaces

Definition 2.1.10. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. A mapping $f : H \to H$ is said to be*

(i) L-Lipschitz if there exists $L > 0$ such that

$$\|fx - fy\| \le L\|x-y\|, \quad \forall \; x, y \in H;$$

if $L = 1$, then f is called nonexpansive while f is called a contraction if $L \in (0,1)$,

(ii) η-strongly monotone, if there exists $\eta > 0$ such that

$$\big\langle fx - fy, x-y\big\rangle \ge \eta\|x-y\|^2 \quad \forall \; x, y \in H;$$

(iii) η-inverse strongly monotone (η-ism), if there exists $\eta > 0$ such that

$$\big\langle fx - fy, x-y\big\rangle \ge \eta\|fx - fy\|^2 \quad \forall \; x, y \in H;$$

if $\eta = 1$, then f is called firmly nonexpansive, *(iv) monotone, if*

$$\langle fx - fy, x - y\rangle \geq \ 0 \ \ \forall \ x, y \in H;$$

(vi) pseudo-monotone, if

$$\langle fx, y - x\rangle \geq 0 \implies \langle fy, y - x\rangle \geq 0 \ \ \forall \ x, y \in H;$$

(vii) sequentially weakly continuous on H*, if for each sequence* $\{x_n\} \subset H$ *we have that* $\{x_n\}$ *converges weakly to* $x \in H$ *implies that* $\{fx_n\}$ *converges weakly to* fx*;*

(viii) $\alpha-$*averaged, if* $f = (1-\alpha)I + \alpha T$*, where* $\alpha \in (0,1)$ *and* $T : H \to H$ *is nonexpansive,*

(ix) k*-strictly pseudocontractive, if for* $0 \leq k < 1$

$$\|fx - fy\|^2 \leq \|x - y\|^2 + k\|(I - f)x - (I - f)y\|^2 \quad \forall \ \ x, y \in C. \tag{2.1.10}$$

When $k = 0$, (2.1.10) *becomes nonexpansive.*

It is well-known that firmly nonexpansive mappings are $\frac{1}{2}$*-averaged while averaged mappings are nonexpansive. It is also known that every* η*-ism mapping is* $\frac{1}{\eta}$*-Lipschitz continuous. Also, if* f *is* η*-strongly monotone and* L*-Lipschitz continuous, then* f *is* η/L^2*-ism. Furthermore, both* η*-strongly monotone and* η*-inverse strongly monotone mappings are monotone while monotone mappings are pseudo-monotone. However, there are pseudo-monotone mappings which are not monotone.*

Example 2.1.11. *[47] Let* $H = \mathbb{R}$ *be endowed with the usual metric and* $f : (0, \infty) \to (0, \infty)$ *be defined by*

$$f(x) = \frac{1}{1 + x}, \qquad x \in (0, \infty),$$

f *is a pseudo-monotone mapping but not a monotone mapping.*

2.2 Hadamard spaces

Although the geometric structure of Hilbert spaces makes problems that occur in Hilbert spaces more manageable and easier to solve. However, most real life problems naturally occur in nonlinear spaces (for instance, in Hadamard spaces). Thus, the need to extend our study to nonlinear spaces arises. However, the nonlinear structure of nonlinear spaces sometimes makes it difficult to extend some known results to this space. To guarantee the extension of such existing results to nonlinear spaces, there was need to introduce some kind of a convex structure or properties which provides sufficient information that ensures the applications of such existing results. One of these properties is the existence of distance-preserving mapping, which provides the metric space (nonlinear space) with a structure that is similar to the linear structure of a normed linear space (in particular, Hilbert space). Hadamard space is our second space of interest in this dissertation.

Definition 2.2.1. *[8] Let (X, d) be a metric space. A continuous mapping from the interval $[0, 1]$ to X is called a path.*

Let $x, y \in X$ and $I = [0, d(x, y)]$, a geodesic path joining x to y is an isometry $C : I \to X$, such that $C(0) = x$, $C(d(x, y)) = y$ and $d(C(t), C(t')) = |t - t'|$. The image of a geodesic path is a geodesic segment and it is denoted by $[x, y]$ whenever it is unique. A metric space (X, d) is said to be a geodesic space if every pair of points x and y in X are connected by a geodesic. A subset C of a geodesic space X is said to be convex, if for all $x, y \in C$, the segment $[x, y]$ remains in C. For $x, y \in X$ and $t \in [0, 1]$ we write $tx \oplus (1 - t)y$ for the unique point z in the geodesic segment joining x to y such that

$$d(x, z) = (1 - t)d(x, y) \;\text{ and }\; d(z, y) = td(x, y). \tag{2.2.1}$$

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space X consists of three points $x_1, x_2, x_3 \in X$ (which are also called the vertices of $\triangle$) and a geodesic segment between each pair of vertices (which are also known as edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3) \in X$ is a triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in Euclidean space $\mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) \;\; \forall \;\; i, j \in \{1, 2, 3\}$. Thus, a geodesic space is called a CAT(0) space if all geodesic triangles satisfy the comparison axiom. Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then, $\triangle$ is said to satisfy the CAT(0) inequality, if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

Let x, y, z be points in X and y_0 be the midpoint of the segment $[y, z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d(y, z). \tag{2.2.2}$$

Thus, a geodesic space is a CAT(0) space if and only if it satisfies (2.2.2). It is generally known that a CAT(0) space is a uniquely geodesic space.

Definition 2.2.2. *A complete CAT(0) space is an Hadamard space.*

2.2.1 Quasilinearization mapping and dual space

Berg and Nikolaev [11] introduced the concept of quasilinearization for Hadamard spaces. They denoted a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and called it a vector. Using this notation, they defined the quasilinearization as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right), \quad \forall a, b, c, d \in X. \tag{2.2.3}$$

One can easily see that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, for all $a, b, c, d, e \in X$.

The space X is said to satisfy the Cauchy Schwartz inequality, if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d)\ \forall a,b,c,d \in X$. Moreover, a geodesic space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality (see [43]).

Based on the concept of quasilinearlization mapping, Kakavandi and Amini [43] introduced the concept of dual space of an Hadamard space X as follows:
Consider the map $\theta : \mathbb{R} \times X \times X \to C(X,\mathbb{R})$ defined by

$$\theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle \quad (t \in \mathbb{R}, a,b,x \in X),$$

where $C(X,\mathbb{R})$ denotes the space of all continuous real valued functions on X. The Cauchy-Schwartz inequality implies that $\theta(t,a,b)$ is a Lipschitz function with Lipschitz semi-norm $L(\theta(t,a,b)) = |t|d(a,b) \quad (t \in \mathbb{R}, a,b \in X)$, where $L(\varphi) = \sup\left\{ \frac{\varphi(x)-\varphi(y)}{d(x,y)} : x,y \in X, x \neq y \right\}$ is the Lipschitz semi norm for any function $\varphi : X \to \mathbb{R}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t,a,b),(s,c,d)) = L(\theta(t,a,b) - \theta(s,c,d)), \quad (t,s \in \mathbb{R}, a,b,c,d \in X).$$

In an Hadamard space (X,d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subset of the pseudometric space of all real valued Lipschitz functions $(\mathrm{Lip}(X,\mathbb{R}), L)$ (see [27, 69, 87]).
It is shown in [43] that $D((t,a,b),(s,c,d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for all $x,y \in X$. Thus, D induces an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t,a,b) is defined as

$$[\overrightarrow{tab}] := \{\overrightarrow{scd} : D((t,a,b),(s,c,d)) = 0\}.$$

The set $X^* = \{[\overrightarrow{tab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D([\overrightarrow{tab}], [\overrightarrow{scd}]) := D((t,a,b),(s,c,d))$.

Definition 2.2.3. *Let (X,d) be an Hadamard space. Then, the pair (X^*, D) is called the dual space of (X,d).*

Throughout this dissertation, we shall write X^* for the dual space of an Hadamard space X.

It is shown in [43] that the dual of a closed and convex subset of a Hilbert space H with nonempty interior is an Hadamard space and $t(b-a) \equiv [\overrightarrow{tab}]$ for all $t \in \mathbb{R}, \quad a,b \in H$. We also note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle, \quad (x^* = [\overrightarrow{tab}] \in X^*, \quad x,y \in X \text{ and } t \in \mathbb{R}).$$

2.2.2 Examples of Hadamard spaces

Example 2.2.4. *(Hilbert space)[8]. Hilbert spaces are Hadamard. The geodesics are the line segments. It is also known that a Banach space is CAT(0) if and only if it is Hilbert.*

Example 2.2.5. *($\mathbb{R}$- trees)[8]. A metric space (X,d) is an $\mathbb{R}$-tree if it is uniquely geodesic and for every $x,y,z \in X$, we have $[x,z] = [x,y] \cup [y,z]$ whenever $[x,y] \cap [y,z] = \{y\}$. Also, all triangles in an $\mathbb{R}$-tree are trivial.*

Other examples of Hadamard spaces include, simply connected Riemannian manifolds of non-positive sectional curvature, Hilbert balls, hyperbolic spaces, to mention a few (see [8, 27, 41, 70]).

2.2.3 Some inequalities that characterize Hadamard space

Lemma 2.2.6. *Let X be a CAT(0) space, $w, x, y, z, \in X$ and $t \in [0,1]$. Then*

(a) $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$ *[29].*

(b) $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$ *[29].*

(c) $d^2(tx \oplus (1-t)y, z) \leq t^2d^2(x,z) + (1-t)^2d^2(y,z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$ *[28].*

(d) $d(tw \oplus (1-t)x, ty \oplus (1-t)z) \leq td(w,y) + (1-t)d(x,z)$ *[14].*

(e) $z = tx \oplus (1-t)y$ *implies* $\langle \overrightarrow{zy}, \overrightarrow{zw} \rangle \leq t\langle \overrightarrow{xy}, \overrightarrow{zw} \rangle \quad \forall\ w \in X$ *[28].*

(f) $d(tx \oplus (1-t)y, sx \oplus (1-s)y) \leq |t-s|d(x,y)$ *[23].*

Lemma 2.2.7. *[88] Let X be a CAT(0) space. For any $t \in [0,1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,*

(a) $\langle \overrightarrow{u_tx}, \overrightarrow{u_ty} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{u_ty} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_ty} \rangle,$

(b) $\langle \overrightarrow{u_tx}, \overrightarrow{uy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{ux} \rangle,$

(c) $\langle \overrightarrow{u_tx}, \overrightarrow{vy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle.$

2.2.4 Some nonlinear single-valued mappings in Hadamard spaces

Definition 2.2.8. *Let X be an Hadamard space and C be a nonempty closed and convex subset of X. A mapping $T : C \to C$ is said to be*

(i) a contraction, if there exists $\alpha \in [0,1)$ such that

$$d(Tx, Ty) \leq \alpha d(x,y) \quad \forall \quad x, y \in C;$$

(ii) nonexpansive, if

$$d(Tx, Ty) \leq d(x,y) \quad \forall \quad x, y \in C;$$

(iii) firmly nonexpansive, if

$$d^2(Tx,Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \quad \forall \quad x,y \in C; \tag{2.2.4}$$

(iv) uniformly L-Lipschitzian, if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L d(x,y) \; \forall n \geq 1, \;\; x,y \in C;$$

(v) asymptotically regular, if

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0 \quad \forall \; x \in C;$$

(vi) asymptotically nonexpansive, if there exists a sequence $\{u_n\} \subseteq [0,\infty)$ with $u_n \to 0$ as $n \to \infty$ such that

$$d(T^n x, T^n y) \leq (1 + u_n) d(x,y) \; \forall n \geq 1, \;\; x,y \in C;$$

(vii) generalized asymptotically nonexpansive, if there exist nonnegative sequences $\{u_n\}, \{v_n\}$ with $u_n \to 0, v_n \to 0$ as $n \to \infty$ such that

$$d(T^n x, T^n y) \leq (1 + u_n) d(x,y) + v_n, \quad \forall \quad n \geq 1, x,y \in C;$$

(viii) quasi-nonexpansive, if $F(T) \neq \emptyset$ and

$$d(p, Tx) \leq d(p,x) \quad \forall \quad p \in F(T), \;\; x \in C;$$

(ix) k-strictly pseudocontractive, if there exists $k \in [0,1)$ such that

$$d^2(Tx,Ty) \leq d^2(x,y) + k\left[d(x,Tx) + d(x,Ty)\right]^2, \quad \textit{for all} \quad x,y \in C;$$

(x) k-demicontractive, if $F(T) \neq \emptyset$ and there exists $k \in [0,1)$ such that

$$d^2(Tx,y) \leq d^2(x,y) + kd^2(Tx,x), \quad \forall \quad x \in C, \quad y \in F(T);$$

(xi) nonspreading if

$$2d^2(Tx,Ty) \leq d^2(Tx,y) + d^2(Ty,x) \quad \textit{for all} \;\; x,y \in C;$$

(xii) hybrid if

$$3d^2(Tx,Ty) \leq d^2(x,y) + d^2(Tx,y) + d^2(Ty,x) \quad \textit{for all} \quad x,y \in C;$$

(xiii) generalized hybrid, if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha d^2(Tx,Ty) + (1-\alpha) d^2(x,Ty) \leq \beta d^2(Tx,y) + (1-\beta) d^2(x,y) \; \forall x,y \in C. \tag{2.2.5}$$

The class of nonexpansive mappings with $F(T) \neq \emptyset$ is contained in the class of quasi-nonexpansive mappings, while the class of demicontractive mappings contains both the classes of nonexpansive and quasi-nonexpansive mappings.

Definition 2.2.9. *[4] Let X be a CAT(0) space and C be a nonempty, closed and convex subset of X. A mapping $T : C \to X$ is said to be k-demimetric if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$ such that*

$$\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle \geq \frac{1-k}{2} d^2(x, Tx) \tag{2.2.6}$$

for all $x \in C, y \in F(T)$.

We now state the relationship between generalized asymptotically nonexpansive mappings and other mappings. Also, we state the relationship between demimetric mappings and other mappings.

Firmly nonexpansive mappings $\subset$ nonexpansive mappings $\subset$ asymptotically nonexpansive mappings $\subset$ generalized asymptotically nonexpansive mappings.

Firmly nonexpansive mappings $\subset$ nonexpansive mappings (with $F(T) \neq \emptyset$) $\subset$ quasi-nonexpansive mapping $\subset$ strictly pseudo-contraction mappings (with $F(T) \neq \emptyset$) $\subset$ demi-contractive mappings $\subset$ demimetric mappings.

Example 2.2.10. *Let $X = \mathbb{R}$ with the usual norm and $T : \mathbb{R} \to \mathbb{R}$ be a mapping defined by*

$$Tx = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0. \end{cases} \tag{2.2.7}$$

T is quasi-nonexpansive but not nonexpansive.

Proof. We observe that $F(T) \neq \emptyset$ since 0 is a fixed point of T. We show that T is quasi-nonexpansive. From definition, we have

$$|Tx - 0| = \left| x \sin\left(\frac{1}{x}\right) - 0 \right| \leq |x - 0|$$

satisfying the quasi-nonexpansive condition, hence T is a quasi-nonexpansive mapping.

We now show that T is not nonexpansive. Let $x = \frac{2}{\pi}$ and $y = \frac{2}{3\pi}$, we have

$$|Tx - Ty| = \left| \frac{2}{\pi} \sin\left(\frac{\pi}{2}\right) - \frac{2}{3\pi} \sin\left(\frac{3\pi}{2}\right) \right| = \frac{8}{3\pi}.$$

For the right hand side we have,

$$|x - y| = \left| \frac{2}{\pi} - \frac{2}{3\pi} \right| = \frac{4}{3\pi} < \frac{8}{3\pi}.$$

Hence, T is not nonexpansive. □

Example 2.2.11. *Let X be a real line with the usual norm and $\mathbb{C} = \mathbb{R}$. Define $T : C \to C$ by*

$$Tx = 5x.$$

T is k-strictly pseudo-contractive but not quasi-nonexpansive or nonexpansive.

Proof. It is clear that $F(T) = \{0\}$, thus for $x \in \mathbb{R}$, we have

$$|5x - 0|^2 = 25|x - 0|^2 > |x - 0|^2$$

which implies that T is not quasi-nonexpansive and hence, not nonexpansive.

Next we show that T is k- strictly pseudo-contraction.

$$|Tx - Ty|^2 = |5x - 5y|^2 = 25|x - y|^2.$$

Also,

$$\begin{aligned} |x - y - (Tx - Ty)|^2 &= |x - y - (5x - 5y)|^2 \\ &= 16|x - y|^2. \end{aligned}$$

$$\begin{aligned} |Tx - Ty|^2 &= |x - y|^2 + 24|x - y|^2 \\ &= |x - y|^2 + \frac{3}{2}|x - y - (Tx - Ty)|^2. \end{aligned}$$

Hence, T is $\frac{3}{2}$-strictly pseudo-contractive mapping. □

The following are some examples of demimetric mappings.

Example 2.2.12. *Let X be a real line and $C = [-1, 1]$. Define T on C by*

$$Tx = \frac{x}{2}\cos x, \quad \text{if} \quad x \neq 0 \quad \text{and} \quad T(0) = 0.$$

Then, T is a demimetric mapping.

Proof. Clearly, 0 is the only fixed point of T. Also, for $x \in C$, $|Tx - 0|^2 = |Tx|^2 = |\frac{x}{2}cos\ x|^2 \leq |\frac{x}{2}|^2 \leq |x|^2 \leq |x - 0|^2 + k|Tx - x|^2$ for all $k \in [0, 1)$. Thus, T is demimetric. □

Let X be a real line and $C = [-1, 1]$. Define T on C by

$$Tx = \frac{2}{3}x\sin\frac{1}{x}, \quad \text{if} \quad x \neq 0 \quad \text{and} \quad T(0) = 0.$$

Thus, T is a demimetric mapping.

Example 2.2.13. *[4] Let $T : [0, 1] \to [0, 1]$ be defined by $Tx = x - x^j,\ j \geq 1$. Then T is k-demimetric with $k = -1$.*

Proof. Clearly, $F(T) = \{0\}$. Now, for all $x \in [0,1]$ and $j \geq 1$, we obtain that

$$\begin{aligned}\langle x - 0, x - Tx\rangle &= \langle x, x^j\rangle \\ &= \frac{1}{2}\left[|x|^2 + |x^j|^2 - |x - x^j|^2\right] \\ &= \frac{1}{2}\left[|x|^2 + |x^j|^2 - |x|^2 + 2|x||x^j|^2 - |x^j|^2\right] \\ &\geq \frac{1}{2}\left[2|x^j||x^j|\right] = |x^j|^2.\end{aligned}$$

That is,

$$\langle x - 0, x - Tx\rangle \ \geq \ \frac{1-(-1)}{2}|x^j|^2.$$

Hence, we have that $\langle x - 0, x - Tx\rangle \geq \frac{1-(-1)}{2}|x - Tx|^2$. □

The following is an example of a generalized asymptotically nonexpansive mapping.

Example 2.2.14. *[68] The mapping $T : \left[-\frac{2}{3}, \frac{2}{3}\right] \to \left[-\frac{2}{3}, \frac{2}{3}\right]$ defined by*

$$Tx = \begin{cases} x, & \textit{if} \quad x \in [-\frac{2}{3}, 0), \\ \frac{16}{81}, & \textit{if} \quad x = 0, \\ x^4, & \textit{if} \quad x \in (0, \frac{2}{3}] \end{cases} \tag{2.2.8}$$

is generalized asymptotically nonexpansive.

2.3 Relationship between Hilbert spaces and Hadamard spaces

As mentioned earlier (see Example 2.2.4), all Hilbert spaces are Hadamard spaces. However, there are some differences between both spaces. In this section, we shall briefly discuss some of them. We begin by highlighting some of the similarities that both spaces share.

Hadamard spaces share many properties with Hilbert spaces which include: the nonexpansivity of the metric projections onto convex closed sets (which we shall discuss in the next section), the Kadec-Klee property, the Opial property, the finite intersection property (reflexivity), analogs of weak convergence, the Banach-Saks property, among others (see [6, 7]).

On the other hand, Hadamard and Hilbert spaces have some differences which include:

- A convex continuous function on an Hadamard space does not have to be locally Lipschitz, while this function is known to be locally Lipschitz in Hilbert space.

- A weakly convergent sequence does not have to be bounded in an Hadamard space but known to be bounded in Hilbert spaces.

- There are nonconvex Chebyshev sets in Hadamard spaces (see for example [6]).

- It is not known whether there exists a topology corresponding to the weak convergence in Hadamard spaces, or whether the closed convex hull of a compact set is compact in Hadamard spaces (see [6, 7]). However, these results are known in Hilbert spaces.

More so, many non-convex problems in Hilbert spaces can be viewed as convex problems in the Hadamard spaces. For example, consider the following.

Example 2.3.1. *Let $X = \mathbb{R}^2$ be endowed with a metric $d : X \times X \to [0, \infty)$ defined by*

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2} \ \forall x, y \in X.$$

Then, (X, d) is an Hadamard space, and the geodesic joining x to y is given by

$$(1-t)x \oplus ty = \left((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)\right).$$

Define $f : X \to \mathbb{R}$ by $f(x_1, x_2) = 100((x_2 + 1) - (x_1 + 1)^2)^2 + x_1^2$. Then, f is not convex in Hilbert setting but convex in the Hadamard space so defined (see [94]).

Now consider the following convex minimization problem:

$$\text{Find } \ x \in X \ \text{ such that } \ f(x) = \min_{y \in X} f(y), \tag{2.3.1}$$

where X and f are as defined above. We see clearly that problem (2.3.1) is non-convex in Hilbert setting but convex in the Hadamard setting.

Henceforth, we shall denote the real Hilbert space by H and Hadamard space by X.

2.4 Metric projection

Definition 2.4.1. *The mapping $P_C : H \to C$ which assigns every point in H to it's unique point in C is called a metric projection of H onto C and it is defined by*

$$\|x - P_C x\| \leq \|x - y\| \quad \forall \ \ x \in H \quad \textit{and} \quad y \in C.$$

Proposition 2.4.2. *The metric projection is characterized by*

$$\langle x - P_C x, y - P_C x \rangle \leq 0 \quad \forall \ \ y \in C \quad \textit{and} \quad x \in H.$$

The consequences of Proposition (2.4.2) are

(i) the metric projection is firmly nonexpansive, that is

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle \quad \forall \quad x, y \in H,$$

(ii)

$$\|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2 \quad \forall \quad x \in H \quad \text{and} \quad y \in C.$$

It is well known that metric projections onto closed convex subsets of Hilbert spaces are well defined single-valued mappings which are nonexpansive. We know also that the metric projections in Hadamard spaces have the same property.

2.4.1 Examples of metric projections

Example 2.4.3. *Let $C = [a, b]$ be a closed rectangle in $\mathbb{R}^n$, where $a = (a_1, a_2, \cdots, a_n)^T$ and $b = (b_1, b_2, \cdots, b_n)^T$, then for $1 \leq i \leq n$*

$$(P_C x)_i = \begin{cases} a_i, & x_i < a_i, \\ x_i, & x_i \in [a_i, b_i], \\ b_i, & x_i > b_i \end{cases} \tag{2.4.1}$$

is the metric projection with the i^{th} coordinate.

Example 2.4.4. *Let $C = \{y \in H : \langle s, y\rangle \leq \alpha\}$ be a closed half space, with $s \neq 0$ and $\alpha \in \mathbb{R}$, then*

$$P_C x = \begin{cases} x - \frac{\langle s,x\rangle - \alpha}{\|s\|^2} s, & \text{if } \langle s, x\rangle > \alpha, \\ x, & \text{if } \langle s, x\rangle \leq \alpha \end{cases} \tag{2.4.2}$$

is the metric projection onto C.

Example 2.4.5. *Let $C = \{y \in H : \langle s, y\rangle = \alpha\}$ be a hyperplane, with $s \neq 0$ and $\alpha \in \mathbb{R}$, then*

$$P_C x = x - \frac{\langle s, x\rangle - \alpha}{\|s\|^2} s \tag{2.4.3}$$

is the metric projection onto C.

2.5 Monotone operators in Hadamard spaces

Monotone operator theory is one of the most important aspect of nonlinear and convex analysis due to the role it plays in optimization theory and related mathematical problems. In this section, we study the concept of monotone operators in Hadamard spaces. Let X be an Hadamard space and X^* be its dual space. A multivalued operator $A : X \to 2^{X^*}$ is monotone if and only if for all $x, y \in D(A), \quad x^* \in Ax, \quad y^* \in Ay$, we have

$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geq 0.$$

A monotone operator A is called a maximal monotone operator if the graph $G(A)$ of A defined by

$$G(A) := \{(x, x^*) \in X \times X^* : x^* \in A(x)\},$$

is not properly contained in the graph of any other monotone operator. The resolvent of a monotone operator A of order $\lambda > 0$ is the multivalued mapping $J_\lambda^A : X \to 2^X$ defined by

$$J_\lambda^A := \left\{ z \in X \middle| \left[\frac{1}{\lambda} \overrightarrow{zx} \right] \in Az \right\}. \tag{2.5.1}$$

The operator A satisfies the range condition if for every $\lambda > 0, \quad D(J_\lambda^A) = X$. For examples of monotone operators in Hadamard spaces, see [27].

The resolvent of monotone operators plays an important role in the approximation of solutions of MIPs. The following lemmas relates the fixed points of a resolvent of a monotone operator and the set of solutions of the MIP (1.1.2).

Lemma 2.5.1. *[48] Let X be a CAT(0) space and J_λ^A be the resolvent of the operator A of order λ. Then we have that*

(a) For any $\lambda > 0, \quad R(J_\lambda^A) \subset D(A)$ and $F(J_\lambda^A) = A^{-1}(0)$, where $R(J_\lambda^A)$ is the range of J_λ^A.

(b) If A is monotone, then J_λ^A is a single-valued and firmly nonexpansive mapping.

(c) If A is monotone and $0 < \lambda \leq \mu$, then $d^2(J_\lambda^A x, J_\mu^A x) \leq \frac{\mu - \lambda}{\mu + \lambda} d^2(x, J_\mu^A x)$, which implies that $d(x, J_\lambda^A x) \leq 2d(x, J_\mu^A x)$.

Lemma 2.5.2. *[87] Let X be an Hadamard space and $A : X \to 2^{X^*}$ be a monotone mapping. Then,*

$$d^2(u, J_\lambda^A x) + d^2(J_\lambda^A x, x) \leq d^2(u, x), \tag{2.5.2}$$

for all $u \in F(J_\lambda^A), \quad x \in X$ and $\lambda > 0$.

Remark 2.5.3. *We observe that inequality (2.5.2) is a property of any firmly nonexpansive mapping. That is, if T is a firmly nonexpansive mapping, then from (2.2.3) and (2.2.4), we obtain*

$$d^2(u, Tx) + d^2(Tx, x) \leq d^2(u, x), \quad \text{for all } u \in F(T) \text{ and } x \in X.$$

Lemma 2.5.4. *[87] Let X be an Hadamard space and X^* be its dual space. Let $T : X \to X$ be a nonexpansive mapping for each $i = 1, 2, \cdots, N$, let J_λ^i be the resolvent of monotone operators A_i of order $\lambda > 0$. Then*

$$F(T \circ J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1) = F(T) \cap F(J_\lambda^N) \cap F(J_\lambda^{N-1}) \cap \cdots \cap F(J_\lambda^2) \cap F(J_\lambda^1).$$

2.6 Variational inequality problems in Hilbert spaces

The VIP (1.1.1) is called a monotone VIP, if f is monotone and it is called a pseudo-monotone VIP, if f is pseudo-monotone. Henceforth, we shall denote the set of solutions of VIP (1.1.1) by $VI(C, f)$ irrespective of whether f is monotone or pseudo-monotone.

2.6.1 Examples of variational inequality problems

1. Let f and C associated with the VIP (1.1.1), be defined by

   $$f(x) = \begin{bmatrix} 22x_1 - 2x_2 + 6x_3 - 4 \\ 2x_2 - 2x_1 \\ 2x_3 + 6x_1 \end{bmatrix} \text{ and}$$

   $C = \{x \in \mathbb{R}^3 \mid x_1 - x_2 \geq 1, \; -3x_1 - x_3 \geq -4, \; 2x_1 + 2x_2 + x_3 = 0, \; l \leq x \leq h\}$, where $l = (-6, -6, -6)^T$ and $h = (6, 6, 6)^T$. The above VIP is linear and has only one solution $x^* = (2, 1, -6)^T$ (see [93]).

2. Consider the following VIP. The mapping f and the set C are defined by

   $$f(x) = \begin{bmatrix} 3x_1 - \frac{1}{x_1} + 3x_2 - 2 \\ 3x_1 + 3x_2 \\ 4x_3 + 4x_4 \\ 4x_3 + 4x_4 - \frac{1}{x_4} - 3 \end{bmatrix} \text{ and}$$

   $C = \{x \in \mathbb{R}^n \mid x_1 + x_2 = 1, \; x_3 + x_4 \geq 0, \; l \leq x \leq h\}$, where $l = (0.1, 0, 0, 1)^T$ and $h = (10, 10, 10, 10)^T$. The above VIP is nonlinear and has a unique solution $x^* = (1, 0, 0, 1)^T$ (see [93]).

2.6.2 Past works on variational inequality problems in Hilbert spaces

Many authors have studied and developed various iterative methods for approximating soltuions of VIPs when the underlying operator is strongly monotone, inverse strongly monotone, monotone or pseudo-monotone. A simple iterative method for solving VIP (1.1.1) is the gradient projection method which is only efficient for solving VIP (1.1.1) when f is either strongly monotone or inverse strongly monotone. To overcome this setback, Korpelevich [53] introduced the extragradient method for solving VIP (1.1.1) in the finite dimensional Euclidean space when f is monotone and L-Lipschitz continuous

$$\begin{cases} y_n = P_C(x_n - \lambda f x_n), \\ x_{n+1} = P_C(x_n - \lambda f y_n), \quad n \geq 1, \end{cases} \tag{2.6.1}$$

where $\lambda \in (0, \frac{1}{L})$. He also proved the convergence of a sequence $\{x_n\}$ to a solution of $VI(C, f)$ provided that $VI(C, f) \neq \emptyset$. Since then, many authors have studied the extragradient method in infinite dimensional spaces (see [3, 17, 34, 37, 63] and the references therein). It is worth mentioning that, in infinite dimensional Hilbert spaces, the extragradient method (2.6.1) only yeilds weak convergence results. However, we know that in such spaces, strong convergence results are often much more desirable than weak convergence results. For this reason, many authors modified Agorithm (2.6.1) by either strengthening the assumptions on the underline operator f or by adding more projections to it (see [63, 89] and the references therein) in order to obtain strong convergence results.

Note also that, Algorithm (2.6.1) needs two projections onto the feasible set C per iteration, thus making it very difficult to implement especially when the projection onto C is the dominating task in the iteration. In fact, to develop better implementable and efficient algorithms for solving problem (1.1.1), one important task would be to minimize the number of projections onto C per iteration. On this note, Censor et al. [21] modified the extragradient method to the following subgradient extragradient method:

$$\begin{cases} y_n = P_C(x_n - \lambda_n f x_n), \\ T_n := \{w \in H : \langle x_n - \lambda_n f x_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda_n f y_n), \quad n \geq 1. \end{cases} \tag{2.6.2}$$

Observe that Algorithm (2.6.2) is less computationally expensive than Algorithm (2.6.1), since the second projection in (2.6.2) is onto a subgradient half-space which is much more easier to compute. However, as in the case of Algorithm (2.6.1), Algorithm (2.6.2) converges only weakly.

By combining the subgradient extragradient method and the Halpern's method, Kraikaew and Saejung [54] obtained a strong convergence result for solving VIP (1.1.1) when f is monotone and Lipschitz continuous. Later, Maingé and Gobinddass [56] introduced a new algorithm for solving (1.1.1), which involves only one projection onto a feasible set C without any additional projection onto the half-space, as follows: For $x_0, x_1 \in C$, choose $\delta \in (0, 1]$ and $\lambda_n \in (0, \infty)$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = x_n + \frac{\lambda_n}{\delta \lambda_{n-1}}(x_n - x_{n-1}), \\ x_{n+1} = P_C(x_n - \lambda_n f y_n), \ n \geq 1. \end{cases} \tag{2.6.3}$$

Algorithm (2.6.3) is less computationally expensive than Algorithm (2.6.2) (as well as Algorithm (2.6.1)) and that considered by Kraikaew and Saejung [54]. However, the authors only obtained a weak convergence of Algorithm (2.6.3) to a solution of VIP (1.1.1) when the underline operator f is monotone and lipschitz continuous.

Very recently, Thong and Hieu [86] proposed the following iterative method for approximating solutions of (1.1.1):

Algorithm 1

Initialization: Given $\gamma > 0, l \in (0, 1), \mu \in (0, 1)$. Let $x_0 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

Step1. Compute

$$y_n = P_C(x_n - \lambda_n f x_n),$$

where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda_n \|f x_n - f y_n\| \leq \mu \|x_n - y_n\|. \tag{2.6.4}$$

If $x_n = y_n$, then stop and x_n is the solution of VIP. Otherwise,

Step 2: Compute

$$x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n) z_n,$$

where $z_n = y_n - \lambda_n (f y_n - f x_n)$,
$f : H \to H$ is monotone and Lipschitz continuous, and $g : H \to H$ is a contraction with constant $\rho \in [0, 1)$.

Set $n := n + 1$ and return to **Step 1**.
Furthermore, they proved that Algorithm 1 converges strongly to a solution of (1.1.1) in a real Hilbert space.

Now, we point out some interesting features of Algorithm 1.

1. In Algorithm 1, only one projection onto C is required to be computed per iteration just as in the case of Algorithm (2.6.3). Thus, Algorithm 1 is less computationally expensive than Algorithm (2.6.1) and that of Kraikaew and Saejung [54].

2. Unlike Algorithm (2.6.3), Algorithm 1 converges strongly to a solution of VIP (1.1.1), which is an important factor to consider while in an infinite dimensional space.

3. the Armijo-like search rule (2.6.4) which has been established in [86, Lemma 3.1] to be well-defined, can be seen as a local approximation of the Lipschitz constant L of the mapping f. Thus, the Lipschitz constant need not to be known. Hence, the stepsize $\{\lambda_n\}$ is given self-adaptively (see [31]) unlike other algorithms where $\{\lambda_n\}$ (or λ) depends on the knowledge of L.

Therefore, Algorithm 1 is very efficient for solving problem (1.1.1) when f is monotone and Lipschitz continuous.

The Split Variational Inequality Problem (SVIP) was introduced by Censor, Gibali and Reich [21] and they defined it as follows: Find $x \in C$ such that

$$\langle fx, y - x \rangle \geq 0 \quad \forall\, y \in C \tag{2.6.5}$$

and

$$\langle g(Ax), z - Ax \rangle \geq 0 \quad \forall\, z \in Q, \tag{2.6.6}$$

where C and Q are nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A : H_1 \to H_2$ is a bounded linear operator and f, g are nonlinear mappings on C and Q respectively. As observed in [21], the SVIP can be seen as a pair of VIPs in which a solution of one VIP occur in the first space H_1 whose image under a given bounded linear operator A is a solution of the second VIP in the second space H_2. They established the solutions of the SVIP (2.6.5) and (2.6.6) by considering two methods and in each of methods they established strong convergence results of the SVIP (2.6.5) and (2.6.6). Furthermore, SVIPs are very important in optimization, nonlinear and convex analysis. They can be viewed as an important generalization of the spilt feasibility problems introduced by Censor and Elfving [20] which are known to have applications in many fields such as phase retrieval, medical image reconstruction, signal processing, radiation therapy treatment planning among others (for example, see [1, 16, 18, 19, 20, 39] and the references therein). To solve SVIP (2.6.5) and (2.6.6), Censor [21] proposed the following algorithm:

$$x_{n+1} = P_C(I - \lambda f)(x_n + \tau A^*(P_Q(I - \lambda g) - I)Ax_n) \quad n \geq 1, \tag{2.6.7}$$

where $\tau \in \left(0, \frac{1}{L}\right)$, L being the spectral radius of A^*A. They proved that the sequence $\{x_n\}$ generated by (2.6.7) converges weakly to a solution of (2.6.5) and (2.6.6) provided that the solution set of problem (2.6.5) and (2.6.6) is nonempty, f, g are α_1, α_2- inverse strongly monotone mappings, $\lambda \in (0, 2\alpha)$, where $\alpha := \min\{\alpha_1, \alpha_2\}$, and for all x which are solutions of (2.6.5),

$$\langle fy, P_C(I - \lambda f)(y) - x\rangle \geq 0, \quad \forall y \in H. \tag{2.6.8}$$

We point out here that the weak convergence of Algorithm (2.6.7) is established under strong assumptions; namely assumption (2.6.8) and the fact that both mappings are inverse strongly monotone. To relax these assumptions, Tian and Jiang [85] proposed a new algorithm by combining Algorithm (2.6.1) and (2.6.7) as follows:

$$\begin{cases} y_n = P_C(x_n - \tau_n A^*(I - T)Ax_n), \\ x_n = P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} = P_C(y_n - \lambda_n f(x_n)), \quad n \geq 1. \end{cases} \tag{2.6.9}$$

They obtained the following results without assumption (2.6.8), and under the assumption that f is monotone and Lipschitz continuous.

Theorem 2.6.1. *Let H_1 and H_2 be real Hilbert spaces and C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$, and $T : H_2 \to H_2$ be a nonexpansive mapping. Let $f : C \to H_1$ be a monotone and L-Lipschitz continuous mapping. Suppose that $\Gamma := \{z \in VI(C, f) : Az \in F(T)\} \neq \emptyset$ and the sequence $\{x_n\}$ is defined for arbitrary $x_1 \in C$ by (2.6.9), where $\{\tau_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$ and $\{\lambda_n\} \subset [c, d]$ for some $c, d \in \left(0, \frac{1}{L}\right)$. Then $\{x_n\}$ converges weakly to $z \in \Gamma$.*

Observe that the class of SVIP considered by Tian and Jian [85], that is, find $x \in C$ such that

$$\langle fx, y - x\rangle \geq 0 \;\; \forall\, y \in C \;\text{ and } Ax \in F(T), \tag{2.6.10}$$

generalizes the class of SVIP considered by Censor et al. [21] (see [85, Theorem 3.3]). We also observe the following about the results of Tian and Jiang [85].

Remark 2.6.2. *1. The sequence generated by Algorithm (2.6.9) converges weakly to a solution of problem (2.6.10). However, we know that strong convergence results are more desirable than weak convergence results in infinite dimensional spaces.*

2. *For the weak convergence of Algorithm* (2.6.9) *to the solution of problem* (2.6.10), *one needs to compute three projections onto the feasible set C per iteration which is very difficult to do in practice when C does not posses a simple structure, and this could seriously affect the efficiency of the algorithm. Thus, for the sake of computation, it is more desirable to develop algorithms with minimized number of evaluations of P_C per iteration.*

3. *To implement Theorem 2.6.1, one needs to compute the Lipschitz constant L before the stepsize $\{\lambda_n\}$ can be computed. Thus, Theorem 2.6.1 is dependent on the knowledge of the Lipschitz constant L.*

4. *Problem* (2.6.10) *can be viewed as a class of SVIP for which a solution of a VIP occur in the first space H_1 (where the underline operator f is monotone and Lipschitz continuous) whose image under a given bounded linear operator A is a fixed point of a nonexpansive mapping T in the second space H_2. Thus, problem* (2.6.10) *will be more applicable if the underline operator f is pseudo-monotone and Lipschitz continuous and the mapping T is more general than nonexpansive mappings.*

From the above remarks, we see the need to further improve and generalize the works discussed above.

2.7 Monotone inclusion problem

An important problem in monotone operator theory is the MIP (1.1.2) (also known as the problem of finding a zero of a monotone operator). The solution set of problem (1.1.2) is denoted by $A^{-1}(0)$ and it is known to be closed and convex (see [69]). The MIP is of central importance in nonlinear and convex analysis since many mathematical problems such as optimization problems, equilibrium problems, VIPs among others can be modelled as MIP (1.1.2). For instance, the problem of finding a zero of a monotone operator describes the equilibrium or stable state of an evolution system controlled by the monotone operator. Also, the problem of finding a zero of a monotone operator is the problem of finding a minimizer of a proper convex and lower semicontinuous functional. Furthermore, a zero of a monotone operator is a solution of a VIP associated with the monotone operator (see [10, 12, 38, 48, 69] and the references therein). Thus, there has been an increase interest in the search of effective iterative methods that best approximate solutions of MIP by numerous researchers.

Martinet [60] introduced in the real Hilbert space, one of the most successful methods for finding solutions of MIP, which is the PPA. Rockafellar [71] further developed it for approximating solutions of (1.1.2) in a real Hilbert space. He defined it as follows;

$$\begin{cases} x_0 \in H, \\ x_{n+1} = J^A_{\lambda_n} x_n, \quad n \geq 0, \end{cases} \tag{2.7.1}$$

where $J^A_{\lambda_n} = (I + \lambda_n A)^{-1}$ is the resolvent of the monotone operator A and $\{\lambda_n\}$ is a sequence of positive real numbers.

Rockafellar [71] proved that the sequence $\{x_n\}$ generated by the PPA convergence weakly to a solution of MIP (1.1.2) under some given conditions. He then raised an important question as to whether the PPA converges strongly or not, which was later resolved in the negative by Güler [32] who provided a counterexample showing that the PPA may not converge strongly except additional conditions are imposed. Since then, numerous authors have modified the PPA so as to obtain strong convergence results. For example, Kamimura and Takahashi [44] introduced the Halpern-type PPA in real Hilbert spaces and established its strong convergence. For more interesting results on the construction of iterative techniques for finding solutions of (1.1.2) in both Hilbert and Banach spaces, see [15, 44, 46, 65, 78] and the references therein.

The study of the PPA for approximating solutions of MIP (1.1.2) was later extended to Hadamard spaces by Bačák [7], who proved its $\triangle$-convergence when the operator A is the subdifferential of a convex, proper and lower semicontinuous function. In 2016, Khatibzadeh and Ranjbar [48] introduced and studied the PPA in Hadamard spaces, when the operator A is a monotone operator.

$$\begin{cases} x_0 \in X \\ \left[\frac{1}{\lambda_n}\overrightarrow{x_n x_{n-1}}\right] \in Ax_n, \quad n \geq 0. \end{cases} \tag{2.7.2}$$

They obtained a $\triangle$-convergence result of the PPA to a solution of MIP (1.1.2). Furthermore, they established a strong convergence result when the operator A is strongly monotone. Later in 2017, Ranjbar and Khatibzabeh [69] proposed the following Mann-type and Halpern-type PPA in an Hadamard space for approximating solutions of MIP (1.1.2)

$$\begin{cases} x_0 \in X \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) J^A_{\lambda_n} x_n, \quad n \geq 0 \end{cases} \tag{2.7.3}$$

and

$$\begin{cases} x_0 \in X \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n) J^A_{\lambda_n} x_n, \quad n \geq 0, \end{cases} \tag{2.7.4}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$. Under some conditions, they obtained $\triangle$-convergence result using the Mann-type PPA (2.7.3) and a strong convergence result using the Halpern-type PPA (2.7.4). Many other authors have also studied the MIP in Hadamard spaces (see for example [27, 40, 66, 87, 94] and the references therein).

Chapter 3

A Modified Extragradient Algorithm for a Certain Class of Split Pseudo-monotone Variational Inequality Problem

3.1 Introduction

In this chapter, we introduce and study a modified extragradient algorithm for approximating solutions of a certain class of split pseudo-monotone VIP in real Hilbert spaces. Using our proposed algorithm, we establish a strong convergence result for the approximation of solutions of the aforementioned problem. Our strong convergence result is obtained without prior knowledge of the Lipschitz constant of the pseudo-monotone operator used in this paper, and with minimized number of projections per iteration compared to other results on SVIPs in the literature. Furthermore, a numerical example of our algorithm in comparison with other algorithm is given to show the the efficiency and advantage of our results. Our results extend and improve many recent results in this direction.

More precisely, we study the following problem: Let H_1 and H_2 be two real Hilbert spaces, C be a nonempty closed and convex subset of H_1 and $f : H_1 \to H_1$ be a pseudo-monotone and Lipschitz continuous operator. Let $g : H_1 \to H_1$ be a contraction mapping with constant $\rho \in (0,1)$, $A : H_1 \to H_2$ a bounded linear operator and $T : H_2 \to H_2$ be a κ-strictly pseudocontractive mapping with $\kappa \in [0,1)$ and $F(T) \neq \emptyset$. Our interest is to find $x \in C$ such that

$$\big\langle fx, y - x \big\rangle \geq 0 \;\; \forall\, y \in C \;\text{ and } Ax \in F(T).$$

3.2 Preliminaries

In this section, we present some lemmas which are required in establishing the main results of this chapter. We denote the strong and weak convergence of a sequence $\{x_n\}$ to a point

$x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$ respectively.

Lemma 3.2.1. *[90] Let H be a Hilbert space and $f : H \to H$ be a nonlinear mapping, then the following hold.*

(i) f is nonexpansive if and only if the complement $I - f$ is $\frac{1}{2}$-ism.

(ii) f is η-ism and $\gamma > 0$, then γf is $\frac{\eta}{\gamma}$-ism.

(iii) f is averaged if and only if the complement $I - f$ is η-ism for some $\eta > \frac{1}{2}$. Indeed, for $\beta \in (0, 1)$, f is β-averaged if and only if $I - f$ is $\frac{1}{2\beta}$-ism.

(iv) If f_1 is β_1-averaged and f_2 is β_2-averaged, where $\beta_1, \beta_2 \in (0, 1)$, then the composite $f_1 f_2$ is β-averaged, where $\beta = \beta_1 + \beta_2 - \beta_1\beta_2$.

(v) If f_1 and f_2 are averaged and have a common fixed point, then $F(f_1 f_2) = F(f_1) \cap F(f_2)$.

Lemma 3.2.2. *[80] Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq 0$, and $T : H_2 \to H_2$ be a nonexpansive mapping. Then $A^*(I - T)A$ is $\frac{1}{2\|A\|^2}$-ism.*

Lemma 3.2.3. *[85] Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty, closed and convex subset of H_1. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $C \cap A^{-1}F(T) \neq \emptyset$. Let $\gamma > 0$ and $x^* \in H_1$. Then the following are equivalent.*

(i) $x^* = P_C(I - \gamma A^*(I - T)A)x^*$;

(ii) $0 \in A^*(I - T)Ax^* + N_C x^*$;

(iii) $x^* \in C \cap A^{-1}F(T)$,

where N_C is the normal cone of C at a point $x^* \in H_1$ such that $N_C x^* = \{z \in H_1 : \langle z, y - x\rangle \leq 0 \forall y \in C\}$, if $x^* \in C$ and empty if otherwise.

Lemma 3.2.4. *[91] Let H be a real Hilbert space and $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in H converging weakly to x^* and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x^* = y$.*

Lemma 3.2.5. *[92] Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:
(i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$,
(iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 3.2.6. *[96] Let H be a real Hilbert space and $T : H \to H$ be a κ-strictly pseudo-contractive mapping with $\kappa \in [0, 1)$. Let $T_\beta := \beta I + (1 - \beta)T$, where $\beta \in [\kappa, 1)$, then*

(i) $F(T) = F(T_\beta)$,

(ii) T_β *is a nonexpansive mapping.*

Lemma 3.2.7. *[57] Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$such that*

$$\Gamma_{n_j} < \Gamma_{n_j+1} \quad \forall j \geq 0.$$

Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by

$$\tau(n) = \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\Gamma_{\tau(n)}\}_{n\geq n_0}$ is a nondecreasing sequence such that $\tau(n) \to \infty$, as $n \to 0$, and for all $n \geq n_0$, the following two estimates hold:

$$\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}, \quad \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

Lemma 3.2.8. *[86] The Armijo-like search rule (2.6.4) is well defined and $\min\{\gamma, \frac{\mu l}{L}\} \leq \lambda_n \leq \gamma$.*

Lemma 3.2.9. *[61] Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. If the mapping $h : [0, 1] \to H$ defined as $h(t) := f(tx+(1-t)y)$ is continuous for all $x, y \in C$ (i.e. h is hemicontinuous), then $M(f, C) := \{x \in C : \langle fy, y - x\rangle \geq 0, \ \forall \ y \in C\} \subset VI(C, f)$. Moreover, if f is pseudo-monotone, then $VI(C, f)$ is closed, convex and $M(C, f) = VI(C, f)$.*

3.3 Main results

In this section, we present and study our modified extragradient algorithm for solving the SVIP (2.6.10). Throughout this section, we assume that H_1 and H_2 are two real Hilbert spaces, C is a nonempty closed and convex subset of H_1 and $f : H_1 \to H_1$ is a pseudo-monotone, Lipschitz continuous and sequentially weakly continuous operator on bounded subsets of H_1 but the Lipschitz constant need not to be known. We also assume that $g : H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0, 1)$, $A : H_1 \to H_2$ is a bounded linear operator and $T : H_2 \to H_2$ is a κ-strictly pseudocontractive mapping with $\kappa \in [0, 1)$. Finally, we assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ and the solution set $\Gamma := \{z \in VI(C, f) : Az \in F(T)\} \neq \emptyset$.

Algorithm 3.1

Initialization: Let $\gamma > 0$, $l, \mu \in (0, 1)$ and $x_1 \in H$ be given arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Compute

$$w_n = P_C(x_n - \tau_n A^*(I - T_\beta)Ax_n) \text{ and } y_n = P_C(w_n - \lambda_n f w_n), \tag{3.3.1}$$

where T_β is as defined in Lemma 3.2.6 and λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda_n \|f w_n - f y_n\| \leq \mu \|w_n - y_n\|. \tag{3.3.2}$$

Step 2. Compute

$$x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n) z_n, \tag{3.3.3}$$

where $z_n = y_n - \lambda_n(f y_n - f w_n)$.

Set $n := n + 1$ and go back to **Step 1.**

Lemma 3.3.1. *Let $\{x_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by Algorithm 3.1, then for all $p \in \Gamma$, we have*

(i) $\|z_n - p\|^2 \leq \|w_n - p\|^2 - (1 - \mu^2)\|y_n - w_n\|^2$.

(ii) $\|x_{n+1} - p\|^2 \leq \alpha_n \|g(x_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - (1 - \alpha_n)(1 - \mu^2)\|y_n - w_n\|^2$.

Proof.

(i) Let $p \in \Gamma$, then by Proposition 2.4.2(i) , we obtain

$$\begin{aligned}
\|y_n - p\|^2 &\leq \langle y_n - p, (w_n - \lambda_n f w_n) - p \rangle \\
&= \frac{1}{2}\left(\|y_n - p\|^2 + \|w_n - \lambda_n f w_n - p\|^2 - \|y_n - w_n + \lambda_n f w_n\|^2\right) \\
&= \frac{1}{2}\left(\|y_n - p\|^2 + \|w_n - p\|^2 + {\lambda_n}^2 \|f w_n\|^2 - 2\lambda_n \langle w_n - p, f w_n \rangle\right) \\
&- \frac{1}{2}\left(\|y_n - w_n\|^2 + {\lambda_n}^2 \|f w_n\|^2 - 2\lambda_n \langle y_n - w_n, f w_n \rangle\right) \\
&= \frac{1}{2}\left(\|y_n - p\|^2 + \|w_n - p\|^2 - \|y_n - w_n\|^2 - 2\lambda_n \langle y_n - p, f w_n \rangle\right),
\end{aligned}$$

which implies that

$$\|y_n - p\|^2 \leq \|w_n - p\|^2 - \|y_n - w_n\|^2 - 2\lambda_n \langle y_n - p, f w_n \rangle. \tag{3.3.4}$$

Also, we obtain from (3.3.2), (3.3.3) and (3.3.4) that

$$\begin{aligned}
\|z_n - p\|^2 &= \|y_n - p\|^2 + \lambda_n^2 \|f y_n - f w_n\|^2 - 2\lambda_n \langle y_n - p, f y_n - f w_n \rangle \\
&\leq \|w_n - p\|^2 - \|y_n - w_n\|^2 - 2\lambda_n \langle y_n - p, f w_n \rangle + {\lambda_n}^2 \|f y_n - f w_n\|^2 \\
&- 2\lambda_n \langle y_n - p, f y_n - f w_n \rangle \\
&\leq \|w_n - p\|^2 - \|y_n - w_n\|^2 + \mu^2 \|y_n - w_n\|^2 - 2\lambda_n \langle y_n - p, f y_n \rangle.
\end{aligned} \tag{3.3.5}$$

Since $p \in \Gamma$ and $y_n \in C$, we have that $\langle fp, y_n - p\rangle \geq 0$. Hence, by the pseudo-monotonicity of f, we obtain that $\langle fy_n, y_n - p\rangle \geq 0$. Thus, we obtain from (3.3.5) that

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - (1-\mu^2)\|y_n - w_n\|^2, \tag{3.3.6}$$

which completes the proof of (i).

(ii) From Lemma 3.2.1 (ii),(iii),(iv), Lemma 3.2.2 and Lemma 3.2.6, we obtain that $P_C(I - \tau_n A^*(I - T_\lambda)A)$ is $\frac{1+\tau_n\|A\|^2}{2}$-average. That is, $P_C(I - \tau_n A^*(I - T_\lambda)A) = (1-\beta_n)I + \beta_n T_n, \ \ \forall n \geq 1$, where $\beta_n = \frac{1+\tau_n\|A\|^2}{2}$ and T_n is nonexpansive for all $n \geq 1$. Therefore, we can rewrite w_n from (3.3.1) as

$$w_n = (1-\beta_n)x_n + \beta_n T_n x_n, \ \ n \geq 1. \tag{3.3.7}$$

Again, let $p \in \Gamma$, then from (3.3.7), we obtain that

$$\begin{aligned} \|w_n - p\|^2 &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T_n x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - T_n x_n\|^2 \\ &\leq \|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - T_n x_n\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{3.3.8}$$

Also, we obtain from (3.3.3), (3.3.6), (3.3.8) and the convexity of $||.||^2$ that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \alpha_n\|g(x_n) - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &\leq \alpha_n\|g(x_n) - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 - (1-\alpha_n)(1-\mu^2)\|y_n - w_n\|^2 \\ &\leq \alpha_n\|g(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\quad -(1-\alpha_n)(1-\mu^2)\|y_n - w_n\|^2. \end{aligned} \tag{3.3.9}$$

□

Lemma 3.3.2. *Let $\{x_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by Algorithm 3.1, then the sequences $\{x_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{g(x_n)\}$ are bounded. Furthermore, if in addition, $\lim_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0$.*

Proof. By (3.3.3), (3.3.8) and Lemma 3.3.1, we obtain that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\|g(x_n) - p\| + (1-\alpha_n)\|z_n - p\| \\ &\leq \alpha_n\rho\|x_n - p\| + \alpha_n\|g(p) - p\| + (1-\alpha_n)\|z_n - p\| \\ &\leq \alpha_n\rho\|x_n - p\| + (1-\alpha_n)\|w_n - p\| + \alpha_n\|g(p) - p\| \\ &\leq \alpha_n\rho\|x_n - p\| + (1-\alpha_n)\|x_n - p\| + \alpha_n\|g(p) - p\| \\ &= (1-\alpha_n(1-\rho))\|x_n - p\| + \alpha_n\|g(p) - p\| \\ &\leq \max\left\{\|x_n - p\|, \frac{\|g(p) - p\|}{1-\rho}\right\} \\ &\ \ \vdots \\ &\leq \max\left\{\|x_1 - p\|, \frac{\|g(p) - p\|}{1-\rho}\right\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{g(x_n)\}$ are all bounded. Furthermore, since $\lim_{n\to\infty} \alpha_n = 0$, we obtain from (3.3.3) that

$$\|x_{n+1} - z_n\| = \alpha_n \|g(x_n) - z_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{3.3.10}$$

□

Lemma 3.3.3. *Let $\{x_n\}$, $\{w_n\}$ and $\{y_n\}$ be sequences generated by Algorithm 3.1 such that $\lim_{n\to\infty} \|w_n - y_n\| = 0 = \lim_{n\to\infty} \|w_n - x_n\|$. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some $v \in H$, then $v \in \Gamma$.*

Proof. Suppose that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some $v \in H$. Then, by our hypothesis, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$, both converging weakly to $v \in H$. We may also assume without loss of generality that, the subsequence $\{\tau_{n_k}\}$ of $\{\tau_n\}$ converges to a point say $\overline{\tau} \in \left(0, \frac{1}{\|A\|^2}\right)$. Also, by Lemma 3.2.2, $A^*(I - T_\beta)A$ is an inverse strongly monotone operator. Therefore, $\{A^*(I - T_\beta)Au_{n_k}\}$ is bounded. Hence, by the nonexpansivity of P_C, we obtain that

$$\begin{aligned}\|P_C(I - \tau_{n_k} A^*(I - T_\beta)A)x_{n_k} - P_C(I - \overline{\tau} A^*(I - T_\beta)A)x_{n_k}\| \\ \leq |\tau_{n_k} - \overline{\tau}| \|A^*(I - T_\beta)Ax_{n_k}\| \to 0, \quad \text{as} \quad k \to \infty.\end{aligned}$$

That is,

$$\lim_{k\to\infty} \|w_{n_k} - P_C(I - \overline{\tau} A^*(I - T_\beta)A)x_{n_k}\| = 0,$$

which implies from our hypothesis that

$$\lim_{k\to\infty} \|x_{n_k} - P_C(I - \overline{\tau} A^*(I - T_\beta)A)x_{n_k}\| = 0. \tag{3.3.11}$$

Thus, by Lemma 3.2.4, we obtain that $v \in F(P_C(I - \overline{\tau} A^*(I - T_\beta)A)$. It then follows from Lemma 3.2.3 that $v \in C \cap A^{-1}F(T_\beta)$, which together with Lemma 3.2.6 implies that

$$Av \in F(T_\beta) = F(T). \tag{3.3.12}$$

Now, by the characteristic property of P_C, we obtain for all $x \in C$ that

$$\begin{aligned} 0 &\leq \langle y_{n_k} - w_{n_k} + \lambda_{n_k} f w_{n_k}, x - y_{n_k} \rangle \\ &= \langle y_{n_k} - w_{n_k}, x - y_{n_k} \rangle + \lambda_{n_k} \langle f w_{n_k}, w_{n_k} - y_{n_k} \rangle + \lambda_{n_k} \langle f w_{n_k}, x - w_{n_k} \rangle \\ &\leq \|y_{n_k} - w_{n_k}\| \|x - y_{n_k}\| + \lambda_{n_k} \|f w_{n_k}\| \|w_{n_k} - y_{n_k}\| + \lambda_{n_k} \langle f w_{n_k}, x - w_{n_k} \rangle. \end{aligned} \tag{3.3.13}$$

Since $\lim_{k\to\infty} ||w_{n_k} - y_{n_k}|| = 0$, and by Lemma 3.2.8, $\min\{\gamma, \frac{\mu l}{L}\} \leq \lambda_n \leq \gamma$, we obtain by passing limit as $k \to \infty$ in (3.3.13) that

$$\liminf_{k\to\infty} \langle f x_{n_k}, x - x_{n_k} \rangle \geq 0 \;\; \forall x \in C. \tag{3.3.14}$$

Now, choose a sequence $\{\delta_k\}$ of positive numbers such that $\delta_{k+1} \leq \delta_k, \ \forall \ k \geq 1$ and $\delta_k \to 0$ as $k \to \infty$. Then, for each δ_k, we denote by N_k (which exists as a result of (3.3.14)) the smallest positive integer such that

$$\langle fx_{n_j}, x - x_{n_j} \rangle + \delta_k \geq 0 \ \ \forall j \geq N_k. \tag{3.3.15}$$

Since $\{\delta_k\}$ is decreasing, we have that $\{N_k\}$ is increasing. Furthermore, we set for each $k \geq 1$, $m_{N_k} = \frac{fx_{N_k}}{\|fx_{N_k}\|^2}$, provided $fx_{N_k} \neq 0$. Then it is easy to see that $\langle fx_{N_k}, m_{N_k} \rangle = 1$ for each $k \geq 1$. Thus, by (3.3.15), we have that

$$\langle fx_{N_k}, x + \delta_k m_{N_k} - x_{N_k} \rangle \geq 0,$$

which implies by the pseudo-monotonicity of f that

$$\langle f(x + \delta_k m_{N_k}), x + \delta_k m_{N_k} - x_{N_k} \rangle \geq 0. \tag{3.3.16}$$

Now, by the sequentially weakly continuity of f, we have that $\{fx_{n_k}\}$ converges weakly to fv. If $fv = 0$, then $v \in VI(C, f)$. On the other hand, if we suppose that $fv \neq 0$, then by the weakly lower semicontinuity of $\|\cdot\|$, we obtain that

$$0 < \|fv\| \leq \liminf_{k\to\infty} \|fx_{n_k}\|.$$

Since $\{x_{N_k}\} \subset \{x_{n_k}\}$, we obtain that

$$\begin{aligned} 0 \leq \limsup_{k\to\infty} \|\delta_k m_{N_k}\| &= \limsup_{k\to\infty} \left(\frac{\delta_k}{\|fx_{n_k}\|} \right) \\ &\leq \frac{\limsup_{k\to\infty} \delta_k}{\liminf_{k\to\infty} \|fx_{n_k}\|} \\ &\leq \frac{0}{\|fp\|} = 0. \end{aligned}$$

Therefore, $\lim_{k\to\infty} \|\delta_k m_{N_k}\| = 0$. Thus, letting $k \to \infty$ in (3.3.16) yields

$$\langle fx, x - v \rangle \geq 0 \ \ \forall x \in C, \tag{3.3.17}$$

which implies by Lemma 3.2.9 that $v \in VI(C, f)$. This together with (3.3.12) gives that $v \in \Gamma$. ☐

We now prove the main theorem of this paper.

Theorem 3.3.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Assume that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $z = P_\Gamma g(z)$.*

Proof. We consider two cases for our proof.

Case 1: Let $z = P_\Gamma g(z)$. Suppose that $\{\|x_n - z\|^2\}$ is monotone decreasing, then $\{\|x_n - z\|^2\}$ is convergent. Thus,

$$\lim_{n\to\infty} \|x_n - z\|^2 = \lim_{n\to\infty} \|x_{n+1} - z\|^2. \tag{3.3.18}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, we obtain from (3.3.9) and (3.3.18) that

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{3.3.19}$$

Again, from Algorithm 3.1, we obtain that

$$\|z_n - y_n\| = \lambda_n \|fy_n - fw_n\| \leq \mu \|w_n - y_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{3.3.20}$$

From (3.3.10), (3.3.19) and (3.3.20), we obtain that

$$\|z_n - w_n\| \to 0 \quad \text{and} \quad \|x_{n+1} - w_n\| \to 0, \text{ as } n \to \infty. \tag{3.3.21}$$

From (3.3.8), (3.3.9) and (3.3.18), we obtain that

$$\begin{aligned}\beta_n(1-\beta_n)\|x_n - T_n x_n\|^2 &\leq \|x_n - z\|^2 - \|w_n - z\|^2 \\ &\leq \|x_n - z\|^2 + \alpha_n \|g(x_n) - z\|^2 - \|x_{n+1} - z\|^2 \to 0, \quad \text{as} \quad n \to \infty,\end{aligned}$$

which implies from the definition of β_n that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.3.22}$$

Again, from (3.3.7) and (3.3.22), we obtain that

$$\|w_n - x_n\| = \beta_n \|x_n - T_n x_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

That is,

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.3.23}$$

From (3.3.21) and (3.3.23), we obtain that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.3.24}$$

By Lemma 3.3.2, we have that $\{x_n\}$ is bounded. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to some $v \in H$ and

$$\begin{aligned}\limsup_{n\to\infty} \langle g(z) - z, x_n - z\rangle &= \lim_{k\to\infty} \langle g(z) - z, x_{n_k} - z\rangle \\ &= \langle g(z) - z, v - z\rangle.\end{aligned} \tag{3.3.25}$$

Also, we obtain from (3.3.19), (3.3.23) and Lemma 3.3.3 that $v \in \Gamma$.

Thus, since $z = P_\Gamma g(z)$, we obtain from (3.3.25) that

$$\limsup_{n\to\infty} \langle g(z) - z, x_n - z\rangle \leq 0,$$

which implies from (3.3.24) that

$$\limsup_{n\to\infty} \langle g(z) - z, x_{n+1} - z\rangle = \limsup_{n\to\infty} \left(\langle g(z) - z, x_{n+1} - x_n\rangle + \langle g(z) - z, x_n - z\rangle\right) \leq 0. \tag{3.3.26}$$

Thus, from (3.3.3) and Lemma 2.1.6 (ii), we obtain that

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1-\alpha_n)^2\|z_n - z\|^2 + 2\alpha_n\langle g(x_n) - z, x_{n+1} - z\rangle \\ &= (1-\alpha_n)^2\|z_n - z\|^2 + 2\alpha_n\left(\langle g(x_n) - g(z), x_{n+1} - z\rangle + \langle g(z) - z, x_{n+1} - z\rangle\right) \\ &\leq (1-\alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\rho\|x_n - z\|\|x_{n+1} - z\| + 2\alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &\leq (1-2\alpha_n(1-\rho))\|x_n - z\|^2 + \alpha_n^2\|x_n - z\|^2 + 2\alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &= (1-2\alpha_n(1-\rho))\|x_n - z\|^2 \qquad (3.3.27) \\ &+ 2\alpha_n(1-\rho)\left[\frac{\alpha_n\|x_n - z\|^2}{2(1-\rho)} + \frac{\langle g(z) - z, x_{n+1} - z\rangle}{1-\rho}\right]. \end{aligned}$$

Using (3.3.26) and Lemma 3.2.5, we obtain that $\lim_{n\to\infty}\|x_n - z\|^2 = 0$. Hence, $\{x_n\}$ converges strongly to $z = P_\Gamma g(z)$.

Case 2: Suppose that $\{\|x_n - z\|^2\}$ is not monotone decreasing, then there exists a subsequence $\{\|x_{n_i} - z\|^2\}$ of $\{\|x_n - z\|^2\}$ such that $\|x_{n_i} - z\|^2 < \|x_{n_{i+1}} - z\|^2 \ \ \forall i \in \mathbb{N}$. Thus, by Lemma 3.2.7, there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $k \to \infty$ and the following holds

$$\|x_{m_k} - z\|^2 \leq \|x_{m_{k+1}} - z\|^2 \ \text{ and } \ \|x_k - z\|^2 \leq \|x_{m_k} - z\|^2. \tag{3.3.28}$$

Thus, we obtain from (3.3.9) that

$$(1-\alpha_{m_k})(1-\mu^2)\|y_{m_k} - w_{m_k}\|^2 \leq \|x_{m_k} - z\|^2 - \|x_{m_{k+1}} - z\|^2 + \alpha_{m_k}\|g(x_{m_k}) - z\|^2 \to 0,$$

which implies that

$$\lim_{k\to\infty}\|y_{m_k} - w_{m_k}\| = 0.$$

By similar arguments as in **Case 1**, we can show that

$$\lim_{k\to\infty}\|x_{m_k} - w_{m_k}\| = 0 = \lim_{k\to\infty}\|x_{m_k} - x_{m_{k+1}}\| = 0$$

and

$$\limsup_{k\to\infty}\langle g(z) - z, x_{m_{k+1}} - z\rangle \leq 0.$$

Now, for all $k \geq k_0$, we obtain from (3.3.27) that

$$\begin{aligned} \|x_{m_{k+1}} - z\|^2 &\leq (1-2\alpha_{m_k}(1-\rho))\|x_{m_k} - z\|^2 \\ &+ 2\alpha_{m_k}(1-\rho)\left[\frac{\alpha_{m_k}\|x_{m_k} - z\|^2}{2(1-\rho)} + \frac{\langle g(z) - z, x_{m_{k+1}} - z\rangle}{1-\rho}\right], \end{aligned}$$

which implies from (3.3.28) that

$$\|x_k - z\|^2 \leq \frac{\alpha_{m_k}\|x_{m_k} - z\|^2}{2(1-\rho)} + \frac{\langle g(z) - z, x_{m_{k+1}} - z\rangle}{1-\rho}.$$

Therefore, we obtain that $\limsup_{k\to\infty}\|x_k - z\| \leq 0$. Hence, $\{x_k\}$ converges strongly to z, where $z = P_\Gamma g(z)$. □

Now, by setting $H_1 = H_2$ and $T = I = A$ in Algorithm 3.1, we obtain the following result as a corollary of Theorem 3.3.4.

Corollary 3.3.5. *Let $\gamma > 0$, $l, \mu \in (0,1)$ and $x_1 \in H$ be given arbitrary. Then calculate x_{n+1} as follows:*

Step 1. *Compute*

$$y_n = P_C(x_n - \lambda_n f x_n),$$

where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda_n\|f x_n - f y_n\| \leq \mu\|x_n - y_n\|.$$

Step 2. *Compute*

$$x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)z_n,$$

where $z_n = y_n - \lambda_n(f y_n - f x_n)$.

Assume that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to $z = P_\Gamma g(z)$.

Remark 3.3.6. *Notice that our algorithm (Algorithm 3.1) is of viscosity-type. The motivation for using the viscosity-type algorithm over the Halpern-type (which also converges strongly) stems from the fact that viscosity-type algorithms have higher rate of convergence than the Halpern-type. More so, it was established in [75] that Halpern-type convergence theorems imply viscosity convergence theorems. In fact, by setting $g(x) = u$ for arbitrary but fixed $u \in H_1$ and for all $x \in H_1$ in Algorithm 3.1, we obtain the following result (with respect to an Halpern-type algorithm) as a corollary of Theorem 3.3.4.*

Corollary 3.3.7. *Let $\gamma > 0$, $l, \mu \in (0,1)$ and $u, x_1 \in H$ be given arbitrary. Then calculate x_{n+1} as follows:*

Step 1. *Compute*

$$w_n = P_C(x_n - \tau_n A^*(I - T_\beta)Ax_n) \text{ and } y_n = P_C(w_n - \lambda_n f w_n),$$

where T_β is as defined in Lemma 3.2.6 and λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda_n\|f w_n - f y_n\| \leq \mu\|w_n - y_n\|.$$

Step 2. *Compute*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)z_n,$$

where $z_n = y_n - \lambda_n(fy_n - fw_n)$.

Assume that $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $z = P_\Gamma u$.

Remark 3.3.8. *If we replace the pseudo-monotonicity assumption on* f *with monotonicity assumption, then we obtain another corollary of Theorem 3.3.4. In this case, the sequential weakly continuity assumption on* f *can be dispensed with. That is, we can obtain* (3.3.17) *from* (3.3.13) *without the extra assumption that* f *is sequentially weakly continuous.*

3.4 Numerical example

In this section, we give a numerical example of our algorithm in comparison with Algorithm (2.6.9) of Tian and Jiang [85] in an infinite dimensional Hilbert space. For the sake of comparison, we shall consider a monotone operator for our numerical experiment (since Algorithm (2.6.9) may not be applicable when f is pseudo-monotone). Let $H_1 = H_2 = L_2([0,1])$ be endowed with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt \ \ \forall\ x, y \in L_2([0,1])$$

and norm

$$||x|| := \Big(\int_0^1 |x(t)|^2 dt\Big)^{\frac{1}{2}} \ \ \forall\ x, y \in L_2([0,1]).$$

Let $C = \{x \in L_2([0,1]) : \langle y, x\rangle \le a\}$, where $y = t^2+1$ and $a = 2$. Then,

$$P_C(x) = \begin{cases} \frac{a-\langle y,x\rangle}{||y||^2_{L_2}} y + x, & \text{if } \ \langle y, x\rangle > a, \\ x, & \text{if } \ \langle y, x\rangle \le a. \end{cases}$$

Now, define the operator $f : L_2([0,1]) \to L_2([0,1])$ by

$$fx(t) = \int_0^1 \Big(x(t) - \Big(\frac{2tse^{t+s}}{e\sqrt{e^2-1}}\Big)\cos x(s)\Big)ds + \frac{2te^t}{e\sqrt{e^2-1}}, \quad x \in L_2([0,1]), \quad t \in [0,1].$$

Then f is 2-Lipschitz continuous and monotone on $L_2([0,1])$ (see [35]). Let $A, g, T : L_2([0,1]) \to L_2([0,1])$ be defined by $Ax(t) = \frac{2x(t)}{5}$, $gx(t) = \frac{2x(t)}{7}$ and $Tx(t) = -4x(t)$. Then, A is a bounded linear operator with adjoint $A^*x(t) = \frac{2x(t)}{5}$, g is a contraction with coefficient $\rho = \frac{2}{7}$ and T is $\frac{3}{5}$-strictly pseudocontractive. Thus, we can choose $\beta = \frac{3}{5}$, so that $T_\beta x(t) = -x(t)$. Take $\mu = \frac{1}{2} = l$, $\gamma = 1$ and $\alpha_n = \frac{1}{n+1}$ for all $n \ge 1$, then the conditions in Theorem 3.3.4 are satisfied. Now, consider the following cases.

Case 1: Take $x_1(t) = 2t$.

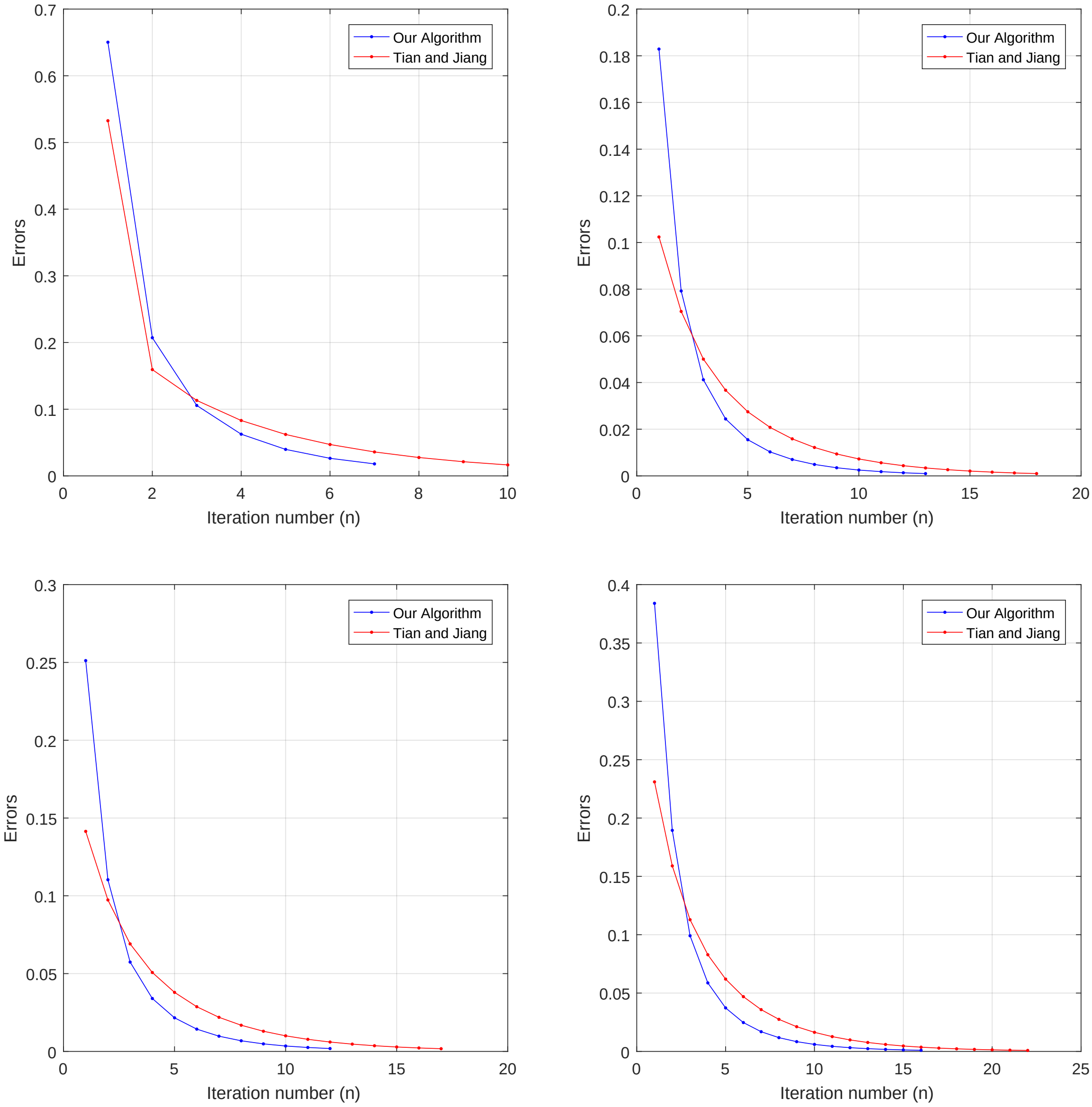

Figure 3.1: Errors vs Iteration numbers(n): **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

Case 2: Take $x_1(t) = t^3$.

Case 3: Take $x_1(t) = \sin t$.

Case 4: Take $x_1(t) = \cos t$.

By using these cases (**Case 1-Case 4** above), we compared Algorithms 3.1 (studied in this paper) with Algorithm (2.6.9) of Tian and Jiang [85] as shown in Figure 1. The graphs show that our algorithm converges faster than Algorithms (2.6.9) of Tian and Jiang [85]. This shows that our algorithm works well and have competitive advantages over the algorithm of Tian and Jiang [85].

Chapter 4

A Viscosity Iterative Algorithm for a Family of Monotone Inclusion Problems in an Hadamard Space

4.1 Introduction

In this chapter, we introduce a viscosity-type PPA which comprises of a finite sum of resolvents of monotone operators, and a generalized asymptotically nonexpansive mapping. We prove that the algorithm converges strongly to a common zero of a finite family of monotone operators, which is also a fixed point of a generalized asymptotically nonexpansive mapping in an Hadamard space. Furthermore, we give two numerical examples of our algorithm in finite dimensional spaces of real numbers and one numerical example in a non-Hilbert space setting, in order to show the applicability of our results.

More presicely, we study the following problem: Let X be an Hadamard space and X^* be its dual space. Let $T : X \to X$ be a uniformly asymptotically regular and uniformly L-Lipschitzian generalized asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Let $A_i : X \to 2^{X^*}$ be a finite family of multivalued monotone mappings which satisfy the range condition and g be a contraction mapping on X with coefficient $\tau \in (0, 1)$. Our interest is to find $\bar{v} \in F(T)$ such that

$$0 \in \cap_{i=1}^{N} A(\bar{v}). \tag{4.1.1}$$

Equation (4.1.1) can also be written as

$$\bar{v} \in \cap_{i=1}^{N} A^{-1}(0). \tag{4.1.2}$$

We shall denote the set of solutions of problem (4.1.1) by $\Gamma := F(T) \cap (\cap_{i=1}^{N} A_i^{-1}(0))$.

4.2 Preliminaries

In this section, we recall some important results that will be needed in our main results.

Remark 4.2.1. *(see [27, Remark 3.6]). If X is an Hadamard space and $A : X \to 2^{X^*}$ is a multivalued monotone mapping, then we have that*

$$d(J_\lambda^A x, J_\mu^A x) \leq \left(\sqrt{1 - \frac{\lambda}{\mu}}\right) d(x, J_\mu^A x), \quad \forall \quad x \in X, \quad 0 < \lambda \leq \mu.$$

Lemma 4.2.2. *[38] Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \cdots, N$ be a finite family of multivalued monotone operators. Then, for $\beta_i \in (0, 1)$ with $\sum_{i=0}^{N} \beta_i = 1$, the mapping $S_\lambda : X \to X$ defined by $S_\lambda x := \beta_0 x \oplus \beta_1 J_\lambda^{A_1} x \oplus \beta_2 J_\lambda^{A_2} x \oplus \cdots \oplus \beta_N J_\lambda^{A_N} x$ is nonexpansive with $F(S_\mu) \subseteq \cap_{i=1}^N F(J_\lambda^{A_i})$ for all $x \in X$, $0 < \lambda \leq \mu$.*

Remark 4.2.3. *[84] For a CAT(0) space X, if $\{x_i, \quad i = 1, 2, \cdots, N\} \subset X$, and $\alpha_i \in [0, 1]$, $1 = 1, 2, \cdots, N$. Then by induction , we can write*

$$\bigoplus_{i=1}^{N} \alpha_i x_i := (1 - \alpha_N) \bigoplus_{i=1}^{N} \frac{\alpha_i}{1 - \alpha_N} x_i \oplus \alpha_N x_N.$$

Lemma 4.2.4. *[27, 38] Let X be a CAT(0) space, $\{x_i, \quad i = 1, 2, \cdots, N\} \subset X$, $\{y_i, \quad i = 1, 2, \cdots, N\} \subset X$ and $\alpha_i \in [0, 1]$ for each $i = 1, 2, \cdots, N$ such that $\sum_{i=1}^{N} \alpha_i = 1$. Then*

$$d\left(\bigoplus_{i=1}^{N} \alpha_i x_i, \bigoplus_{i=1}^{N} \alpha_i y_i\right) \leq \sum_{i=1}^{N} \alpha_i d(x_i, y_i).$$

Definition 4.2.5. *Let $\{x_n\}$ be a bounded sequence in X and $r(., \{x_n\}) : X \to [0, \infty)$ be a continuous functional defined by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) := \inf\{r(x, \{x_n\}) : x \in X\}$, while the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$.*
It is well known that in an Hadamard space X, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be $\triangle$-convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\triangle$-$\lim_{n\to\infty} x_n = x$.

Lemma 4.2.6. *[29] Every bounded sequence in an Hadamard space always has a $\triangle$-convergent subsequence.*

Lemma 4.2.7. *[42] Let X be an Hadamard space, $\{x_n\}$ a sequence in X and $x \in X$. Then, $\{x_n\}$ $\triangle$-converges to x if and only if $\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0$ for all $y \in C$.*

Lemma 4.2.8. *[22] Let C be a closed convex subset of an Hadamard space X and $T : C \to X$ be a uniformly $L-$Lipschitizian and generalized asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\{x_n\}$ $\triangle$-converges to v and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, $Tv = v$.*

Lemma 4.2.9. *[77] (see also [40]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a metric space of hyperbolic type X and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $\liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = \beta_n x_n \oplus (1 - \beta_n) y_n$ for all $n \geq 0$ and $\limsup_{n\to\infty} (d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \leq 0$. Then $\lim_{n\to\infty} d(y_n, x_n) = 0$.*

4.3 Main results

Lemma 4.3.1. *Let X be an Hadamard space and X^* be its dual space. Let $T: X \to X$ be a generalized asymptotically nonexpansive mapping with the sequences $\{u_n\}$, $\{v_n\} \subset [0,\infty)$. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \dots, N$ be a finite family of multivalued monotone mappings which satisfy the range condition and g be a contraction mapping on X with coefficient $\tau \in (0,1)$. Suppose that $\Gamma := F(T) \cap \left(\cap_{i=1}^N A_i^{-1}(0)\right) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}_{n=1}^\infty$ is generated by*

$$\begin{cases} y_n = S_{\lambda_n} x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n, \\ w_n = \frac{\alpha_n}{1-\beta_n} g(x_n) \oplus \frac{\gamma_n}{1-\beta_n} T^n y_n, \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) w_n \quad n \geq 1, \end{cases} \tag{4.3.1}$$

where $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ are sequences in $(0,1)$ and $\{\lambda_n\}$ is a sequence of positive numbers such that,

(a) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(c) $\sum_{n=1}^{\infty} u_n < \infty, \quad \sum_{n=1}^{\infty} v_n < \infty,$

(d) $\beta_i \in (0,1)$ *with* $\sum_{i=0}^{N} \beta_i = 1.$

Then $\{x_n\}_{n=1}^\infty$ is bounded.

Proof. Let $v \in \Gamma$, then from (4.3.1) and Lemma 2.2.6, we have

$$\begin{aligned} d(x_{n+1}, v) &\leq \beta_n d(x_n, v) + (1-\beta_n) d(w_n, v) \\ &= \beta_n d(x_n, v) + (1-\beta_n)\left[d\left(\frac{\alpha_n}{1-\beta_n} g(x_n) \oplus \frac{\gamma_n}{1-\beta_n} T^n y_n, v\right)\right] \\ &\leq \beta_n d(x_n, v) + \alpha_n d(g(x_n), v) + \gamma_n d(T^n y_n, v) \\ &\leq \beta_n d(x_n, v) + \alpha_n \tau d(x_n, v) + \alpha_n d(g(v), v) + \gamma_n \left[(1+u_n) d(y_n, v) + v_n\right] \\ &\leq \beta_n d(x_n, v) + \alpha_n \tau d(x_n, v) + \alpha_n d(g(v), v) + \gamma_n \left[(1+u_n) d(x_n, v) + v_n\right] \\ &= (1 - \alpha_n + \alpha_n \tau + \gamma_n u_n) d(x_n, v) + \alpha_n d(g(v), v) + \gamma_n v_n \\ &\leq (1 - \alpha_n + \alpha_n \tau + u_n) d(x_n, v) + d(g(v), v) + v_n \\ &= \left[(1+u_n) - \alpha_n(1-\tau)\right] d(x_n, v) + d(g(v), v) + v_n \\ &\leq (1+u_n) d(x_n, v) + d(g(v), v) + v_n \end{aligned} \tag{4.3.2}$$

By the same method as above, we obtain

$$d(x_n, v) \leq (1+u_{n-1}) d(x_{n-1}, v) + d(g(v), v) + v_{n-1}$$

Thus (4.3.2) becomes

$$\begin{aligned}
d(x_{n+1},v) &\le (1+u_n)(1+u_{n-1})d(x_{n-1},v) + (1+u_n)\left[d(g(v),v)+v_{n-1}\right] + d(g(v),v) + v_n \\
&= \Pi_{i=0}^{1} d(x_{n-1},v) + (1+u_n)\left[d(g(v),v)+v_{n-1}\right] + d(g(v),v) + v_n \\
&\le (1+u_n)(1+u_{n-1})(1-u_{n-2})d(x_{n-2},v) + (1+u_n)(1+u_{n-1})\left[d(g(v),v),v_{n-2}\right] \\
&+ (1+u_n)\left[d(g(v),v)+v_{n-1}\right] + d(g(v),v) + v_n \\
&= \Pi_{i=0}^{2}(1+u_{n-i})d(x_{n-2},v) + \Pi_{i=0}^{1}(1+u_{n-i})\left[d(g(v),v)+v_{n-2}\right] \\
&+ (1+u_n)\left[d(g(v),v)+v_{n-1}\right] + d(g(v),v) + v_n \\
&= \Pi_{i=0}^{3}(1+u_{n-i})d(x_{n-3},v) + \Pi_{i=0}^{2}(1+u_{n-i})\left[d(g(v),v)+v_{n-3}\right] \\
&+ \Pi_{i=0}^{1}(1+u_{n-i})\left[d(g(v),v)+v_{n-2}\right] + (1+u_n)\left[d(g(v),v)+v_{n-1}\right] + d(g(v),v) + v_n \\
&\vdots \\
&\le \Pi_{i=0}^{n-1}(1+u_{n-i})d(x_1,v) + \Pi_{i=0}^{n-2}(1+u_{n-i})\left[d(g(v),v)+v_1\right] + \cdots \\
&+ \Pi_{i=0}^{2}(1+u_{n-i})\left[d(g(v),v)+v_{n-3}\right] + \Pi_{i=0}^{1}(1+u_{n-i})\left[d(g(v),v)+v_{n-2}\right] \\
&+ (1+u_n)\left[d(g(v),v)+v_{n-1}\right] + \left[d(g(v),v)+v_n\right].
\end{aligned}$$

Since $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, we obtain that $\{x_n\}$ is bounded and consequently $\{w_n\}$, $\{g(x_n)\}$, $\{T^n y_n\}$ and $\{y_n\}$ are all bounded. □

Theorem 4.3.2. *Let X be an Hadamard space and X^* be its dual space. Let $T : X \to X$ be a uniformly asymptotically regular and uniformly L-Lipschitzian generalized asymptotically nonexpansive mapping with the sequences $\{u_n\}$, $\{v_n\} \subset [0,\infty)$. Let $A_i : X \to 2^{X^*}$, $i = 1,2,\ldots,N$ be a finite family of multivalued monotone mappings which satisfy the range condition and g be a contraction mapping on X with coefficient $\tau \in (0,1)$. Suppose that $\Gamma := F(T) \cap \left(\cap_{i=1}^{N} A_i^{-1}(0)\right) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by (4.3.1), where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $(0,1)$ and $\{\lambda_n\}$ is a sequence of positive numbers such that*

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1, \alpha_n + \beta_n + \gamma_n = 1 \ \forall\ n \ \ge 1$,

(c) $\sum_{n=1}^{\infty} u_n < \infty$, $\sum_{n=1}^{\infty} v_n < \infty$,

(d) $\beta_i \in (0,1)$ *with* $\sum_{i=0}^{N} \beta_i = 1$,

(e) $0 < \lambda \le \lambda_n \quad \forall n \ge 1$.

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\bar{v} \in \Gamma$.

Proof. **Step 1:** We show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. From (4.3.1) and Lemma 2.2.6, we have

$$\begin{aligned}
d(w_{n+1}, w_n) &= d\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_{n+1}) \oplus \frac{\gamma_{n+1}}{1-\beta_{n+1}}T^{n+1}y_{n+1}, \frac{\alpha_n}{1-\beta_n}g(x_n) \oplus \frac{\gamma_n}{1-\beta_n}T^n y_n\right)\\
&\leq d\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_{n+1}) \oplus \frac{\gamma_{n+1}}{1-\beta_{n+1}}T^{n+1}y_{n+1}, \frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_{n+1}) \oplus \frac{\gamma_{n+1}}{1-\beta_{n+1}}T^{n+1}y_n\right)\\
&+ d\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_{n+1}) \oplus \frac{\gamma_{n+1}}{1-\beta_{n+1}}T^{n+1}y_n, \frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_n) \oplus \frac{\gamma_{n+1}}{1-\beta_{n+1}}T^n y_n\right)\\
&+ d\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_n) \oplus \frac{\gamma_{n+1}}{1-\beta_{n+1}}T^n y_n, \frac{\alpha_n}{1-\beta_n}g(x_n) \oplus \frac{\gamma_n}{1-\beta_n}T^n y_n\right)\\
&= d\Bigg(\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_{n+1}) \oplus \left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)T^{n+1}y_{n+1},\\
&\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_{n+1}) \oplus \left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)T^{n+1}y_n\Bigg)\\
&+ d\Bigg(\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_{n+1}) \oplus \left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)T^{n+1}y_n,\\
&\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_n) \oplus \left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)T^n y_n\Bigg)\\
&+ d\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}g(x_n) \oplus \left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)T^n y_n, \frac{\alpha_n}{1-\beta_n}g(x_n) \oplus \left(1-\frac{\alpha_n}{1-\beta_n}\right)T^n y_n\right)\\
&\leq \left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)d(T^{n+1}y_{n+1}, T^{n+1}y_n) + \frac{\alpha_{n+1}}{1-\beta_{n+1}}d(g(x_{n+1}), g(x_n))\\
&+ \left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)d(T^{n+1}y_n, T^n y_n) + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| d(T^n y_n, g(x_n))\\
&\leq \left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)[(1+u_{n+1})d(y_{n+1}, y_n) + v_{n+1}] + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)\tau d(x_{n+1}, x_n)\\
&+ \left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)d(T^{n+1}y_n, T^n y_n) + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| d(T^n y_n, g(x_n)).
\end{aligned} \tag{4.3.3}$$

Now, without loss of generality, we may assume that $0 < \lambda_{n+1} \leq \lambda_n, \ \forall\, n \geq 1$. Thus, from

(2.2.6), Lemma 4.2.4, condition (d) and Remark 4.2.1, we have

$$
\begin{aligned}
d(y_{n+1}, y_n) &= d\left(\beta_0 x_{n+1} \oplus \beta_1 J^{A_1}_{\lambda_{n+1}} x_{n+1} \oplus \cdots \oplus \beta_N J^{A_N}_{\lambda_{n+1}} x_{n+1}, \quad \beta_0 x_n \oplus \beta_1 J^{A_1}_{\lambda_n} x_n \oplus \cdots \oplus \beta_N J^{A_N}_{\lambda_n} x_n\right)\\
&\leq \beta_0 d(x_{n+1}, x_n) + \sum_{i=1}^{N} \beta_i d(J^{A_i}_{\lambda_{n+1}} x_{n+1}, J^{A_i}_{\lambda_n} x_n)\\
&\leq \beta_0 d(x_{n+1}, x_n) + \sum_{i=1}^{N} \beta_i d(J^{A_i}_{\lambda_{n+1}} x_{n+1}, J^{A_i}_{\lambda_{n+1}} x_n) + \sum_{i=1}^{N} \beta_i d(J^{A_i}_{\lambda_{n+1}} x_n, J^{A_i}_{\lambda_n} x_n)\\
&\leq d(x_{n+1}, x_n) + \left(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\right) \sum_{i=1}^{N} \beta_i d(J^{A_i}_{\lambda_n} x_n, x_n)\\
&\leq d(x_{n+1}, x_n) + \left(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\right) M, \qquad (4.3.4)
\end{aligned}
$$

where $M = \sup_{n\geq 1} \left\{ \sum_{i=1}^{N} \beta_i d(J^{A_i}_{\lambda_n} x_n, x_n) \right\}$.
Substituting equation (4.3.4) into equation (4.3.3), we obtain

$$
\begin{aligned}
d(w_{n+1}, w_n) &\leq \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right) \left[(1+u_{n+1}) \left[d(x_{n+1}, x_n) + \left(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\right) M\right] + v_{n+1}\right]\\
&+ \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| d(T^n y_n, g(x_n)) + \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right) d(T^{n+1} y_n, T^n y_n)\\
&+ \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right) \tau d(x_{n+1}, x_n)\\
&= \left[\frac{\alpha_{n+1}}{1-\beta_{n+1}} \tau + \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)(1+u_{n+1})\right] d(x_{n+1}, x_n)\\
&+ \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)(1+u_{n+1}) \left(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\right) M + \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right) v_{n+1}\\
&+ \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| d(T^n y_n, g(x_n)) + \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right) d(T^{n+1} y_n, T^n y_n)\\
&= \left[(1+u_{n+1}) - \frac{\alpha_{n+1}}{1-\beta_{n+1}}(1+u_{n+1}-\tau)\right] d(x_{n+1}, x_n) + \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right) v_{n+1}\\
&+ \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)(1+u_{n+1}) \left(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\right) M\\
&+ \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right) d(T^{n+1} y_n, T^n y_n) + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| d(T^n y_n, g(x_n)),
\end{aligned}
$$

which implies

$$\begin{aligned} d(w_{n+1}, w_n) - d(x_{n+1}, x_n) \leq & \left(u_{n+1} - \frac{\alpha_{n+1}}{1-\beta_{n+1}}(1+u_{n+1}-\tau)\right) d(x_{n+1}, x_n) \\ & + \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)(1+u_{n+1})\left(\sqrt{1-\frac{\lambda_{n+1}}{\lambda_n}}\right) M \\ & + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| d(T^n y_n, g(x_n)) \\ & + \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right) v_{n+1} + \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right) d(T^{n+1}y_n, T^n y_n). \end{aligned}$$

Since $\{\lambda_n\}$ is monotone nonincreasing and bounded below, we have that $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \lambda_{n+1} = c$, where $0 < c \leq \lambda$. Thus, we obtain from conditions (a) and (c), and by the uniform asymptotic regularity of T that

$$\limsup_{n\to\infty}(d(w_{n+1}, w_n) - d(x_{n+1}, x_n)) \leq 0.$$

Therefore, it follows from Lemma 4.2.9 and condition (b) that,

$$\lim_{n\to\infty}(w_n, x_n) = 0. \tag{4.3.5}$$

From (2.2.1), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\beta_n x_n \oplus (1-\beta_n)w_n, x_n) \\ &= (1-\beta_n)d(w_n, x_n) \to \quad 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{4.3.6}$$

Step 2: We next show that $\lim_{n\to\infty} d(y_n, Ty_n) = 0 = \lim_{n\to\infty} d(w_n, T(S_{\lambda_n})w_n)$.
Since $J_{\lambda_n}^{A_i}$ is firmly nonexpansive for each $i = 1, 2, \cdots, N$, we obtain from Lemma 2.5.2 that

$$d^2(J_{\lambda_n}^{A_i} x_n, v) \leq d^2(v, x_n) - d^2(J_{\lambda_n}^{A_i} x_n, x_n). \tag{4.3.7}$$

Also, from Lemma 2.2.6 (b) and Lemma 4.2.2, we have

$$\begin{aligned} d^2(w_n, v) \leq & \frac{\alpha_n}{1-\beta_n} d^2(g(x_n), v) + \frac{\gamma_n}{1-\beta_n} d^2(T^n y_n, v) \\ \leq & \frac{1}{1-\beta_n}\left[\alpha_n d^2(g(x_n), v) + \gamma_n[(1+u_n)d(y_n, v) + v_n]^2\right] \\ = & \frac{1}{1-\beta_n}\left[\alpha_n d^2(g(x_n), v) + \gamma_n(1+u_n)^2 d^2(y_n, v) + 2\gamma_n(1+u_n)v_n d(y_n, v) + v_n^2\gamma_n\right] \\ \leq & \frac{1}{1-\beta_n}\left[\alpha_n d^2(g(x_n), v) + 2\gamma_n(1+u_n)v_n d(y_n, v) + v_n^2\gamma_n\right] \\ & + \frac{1}{1-\beta_n}\left[\gamma_n(1+u_n)^2\left(\sum_{i=0}^{N} \beta_i d^2(J_{\lambda_n}^{A_i}(x_n), v)\right)\right], \end{aligned}$$

where $J_{\lambda_n}^{A_0} x_n = x_n$. Thus, it follows from (4.3.7) that

$$d^2(w_n, v) \leq \frac{1}{1-\beta_n}\big[\alpha_n d^2(g(x_n), v) + 2\gamma_n(1+u_n)v_n d(y_n, v) + v_n^2\gamma_n$$
$$+ \gamma_n(1+u_n)^2 d^2(v, x_n) - \gamma_n(1+u_n)^2 \sum_{i=1}^{N} \beta_i d^2(J_{\lambda_n}^{A_i} x_n, x_n)\big]$$
$$\leq \frac{1}{1-\beta_n}\left[\alpha_n d^2(g(x_n), v) + 2\gamma_n(1+u_n)v_n d(y_n, v) + v_n^2\gamma_n\right]$$
$$+ \left(1 - \frac{\alpha_n}{1-\beta_n}\right)\left[(1+u_n)^2 d^2(v, x_n) - (1+u_n)^2 \sum_{i=1}^{N} \beta_i d^2(J_{\lambda_n}^{A_i} x_n, x_n)\right],$$

which implies from (4.3.5) and conditions (a) and (c) that

$$(1+u_n)^2 \sum_{i=1}^{N} d^2(J_{\lambda_n}^{A_i} x_n, x_n) \to 0, \quad \text{as} \quad n \to \infty.$$

Hence, by condition (c), we obtain that

$$\sum_{i=1}^{N} d^2(J_{\lambda_n}^{A_i} x_n, x_n) \to 0, \quad \text{as} \quad n \to \infty. \tag{4.3.8}$$

Therefore, we obtain from Lemma 4.2.2 that

$$d(S_{\lambda_n} x_n, x_n) \leq \sum_{i=0}^{N} \beta_i d(J_{\lambda_n}^{A_i} x_n, x_n) \to 0, \quad \text{as} \quad n \to \infty.$$

That is,

$$\lim_{n\to\infty} d(S_{\lambda_n} x_n, x_n) = \lim_{n\to\infty} d(y_n, x_n) = 0. \tag{4.3.9}$$

It follows from Remark 4.2.3 that (4.3.1) is equivalent to:

$$\begin{cases} y_n = S_{\lambda_n} x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n, \\ x_{n+1} = \alpha_n g(x_n) \oplus (1-\alpha_n)\frac{\beta_n x_n \oplus \gamma_n T^n y_n}{(1-\alpha_n)} \quad n \geq 1. \end{cases} \tag{4.3.10}$$

Thus, from (2.2.1), we have

$$d\left(x_{n+1}, \frac{\beta_n x_n \oplus \gamma_n T^n y_n}{(1-\alpha_n)}\right) = \alpha_n d\left(g(x_n), \frac{\beta_n x_n \oplus \gamma_n T^n y_n}{(1-\alpha_n)}\right) \to 0, \quad \text{as} \quad n \to \infty. \tag{4.3.11}$$

Also, from (2.2.1), we have that

$$d\left(x_n, \frac{\beta_n x_n \oplus \gamma_n T^n y_n}{(1-\alpha_n)}\right) = \frac{\gamma_n}{1-\alpha_n} d(x_n, T^n y_n),$$

which implies from (4.3.6) and (4.3.11) that

$$\frac{\gamma_n}{1-\alpha_n}d(x_n, T^n y_n) = d\left(x_n, \frac{\beta_n x_n \oplus \gamma_n T^n y_n}{(1-\alpha_n)}\right)$$
$$\leq d(x_n, x_{n+1}) + d\left(x_{n+1}, \frac{\beta_n x_n \oplus \gamma_n T^n y_n}{1-\alpha_n}\right) \to 0, \quad \text{as} \quad n \to \infty.$$

This implies that

$$\lim_{n\to\infty} d(x_n, T^n y_n) = \lim_{n\to\infty} d(x_n, T^n S_{\lambda_n} x_n) = 0. \tag{4.3.12}$$

By the uniform asympototic regularity of T, (4.3.9) and (4.3.12), we obtain that

$$\begin{aligned} d(y_n, Ty_n) &\leq d(y_n, T^n y_n) + d(T^n y_n, T^{n+1} y_n) + d(T^{n+1} y_n, Ty_n) \\ &\leq (1+L)d(y_n, T^n y_n) + d(T^n y_n, T^{n+1} y_n) \\ &\leq (1+L)[d(y_n, x_n) + d(x_n, T^n y_n)] + d(T^n y_n, T^{n+1} y_n) \to 0, \quad (4.3.11) \\ & \text{as } n \to \infty. \end{aligned}$$

as $n \to \infty$. Since $\{x_n\}$ is bounded and X is complete, we obtain from Lemma 4.2.6 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\triangle\text{-}\lim_{k\to\infty} x_{n_k} = \bar{v}$. Again, since S_{λ_n} is nonexpansive and every nonexpansive mapping is demiclosed, it follows from (4.3.9), condition (e) and Lemma 4.2.2 that $\bar{v} \in F(S_{\lambda_n}) \subseteq \cap_{i=1}^N F(J_\lambda^{A_i})$.
Also, we obtain from Lemma 4.2.8, (4.3.9) and (4.3.13) that $\bar{v} \in F(T)$. Hence, we obtain that $\bar{v} \in \Gamma$.

Now, using (4.3.5) and (4.3.12), we have

$$\begin{aligned} d(w_n, T^n(S_{\lambda_n})w_n) &\leq d(w_n, x_n) + d(x_n, T^n(S_{\lambda_n})x_n) + d(T^n(S_{\lambda_n})x_n, T^n(S_{\lambda_n})w_n) \\ &\leq d(w_n, x_n) + d(x_n, T^n(S_{\lambda_n})x_n) + Ld(x_n, w_n) \\ &= (1+L)d(w_n, x_n) + d(x_n, T^n(S_{\lambda_n})x_n) \to 0, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} d(w_n, T^n(S_{\lambda_n})w_n) = 0. \tag{4.3.11}$$

Step 3: Next, we show that $\limsup_{n\to\infty} \langle \overrightarrow{g(\bar{v})\bar{v}}, \overrightarrow{x_n\bar{v}} \rangle \leq 0$.
Let $T_m^h x := \beta_m x \oplus (1-\beta_m)w$, where $w = \frac{\alpha_m}{1-\beta_m} g(x_n) \oplus \frac{\gamma_m}{1-\beta_m} T^m(S_{\lambda_m})x$. Then, we have that T_m^h is a contraction for each $m \geq 1$. By Banach contraction principle, there exists a unique fixed point z_m of T_m^h $\forall\ m \geq 1$. That is $z_m = \beta_m z_m \oplus (1-\beta_m)w_m$, where $w_m = \frac{\alpha_m}{1-\beta_m} g(z_m) \oplus \frac{\gamma_m}{1-\beta_m} T^m(S_{\lambda_m} z_m)$. Moreover, $\lim_{n\to\infty} z_m = \bar{v} \in \Gamma$ (see [72]). Thus, we have

$$\begin{aligned} d(z_m, w_n) &= d(\beta_m z_m \oplus (1-\beta_m)w_m, w_n) \\ &\leq \beta_m d(z_m, w_n) + (1-\beta_m)d(w_m, w_n), \end{aligned}$$

which implies that

$$(1-\beta_m)d(z_m, w_n) \leq (1-\beta_m)d(w_m, w_n).$$

This further implies that

$$d(z_m, w_n) \leq d(w_m, w_n). \tag{4.3.12}$$

We have from Lemma 2.2.6 (e) that

$$\begin{aligned}
d^2(w_m, w_n) &= \langle \overrightarrow{w_m w_n}, \overrightarrow{w_m w_n} \rangle \\
&= \left\langle \overrightarrow{w_m T^m(S_{\lambda_m} z_m)}, \overrightarrow{w_m w_n} \right\rangle + \left\langle \overrightarrow{T^m(S_{\lambda_m} z_m) w_n}, \overrightarrow{w_m w_n} \right\rangle \\
&\leq \frac{\alpha_m}{1-\beta_m} \left\langle \overrightarrow{g(z_m) T^m(S_{\lambda_m} z_m)}, \overrightarrow{w_m w_n} \right\rangle + \left\langle \overrightarrow{T^m(S_{\lambda_m} z_m) w_n}, \overrightarrow{w_m w_n} \right\rangle \\
&= \frac{\alpha_m}{1-\beta_m} \left\langle \overrightarrow{g(z_m) T^m(S_{\lambda_m} z_m)}, \overrightarrow{w_m z_m} \right\rangle + \frac{\alpha_m}{1-\beta_m} \left\langle \overrightarrow{g(z_m) T^m(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_n} \right\rangle \\
&+ \left\langle \overrightarrow{T^m(S_{\lambda_m} z_m) T^m(S_{\lambda_m} w_n)}, \overrightarrow{w_m w_n} \right\rangle + \left\langle \overrightarrow{T^m(S_{\lambda_m} w_n) w_n}, \overrightarrow{w_m w_n} \right\rangle \\
&\leq \frac{\alpha_m}{1-\beta_m} \left\langle \overrightarrow{g(z_m) T^m(S_{\lambda_m} z_m)}, \overrightarrow{w_m z_m} \right\rangle + \frac{\alpha_m}{1-\beta_m} \left\langle \overrightarrow{g(z_m) w_n}, \overrightarrow{z_m w_n} \right\rangle \\
&+ \frac{\alpha_m}{1-\beta_m} \left\langle \overrightarrow{w_n T^m(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_n} \right\rangle + \left\langle \overrightarrow{T^m(S_{\lambda_m} z_m) T^m(S_{\lambda_m} w_n)}, \overrightarrow{w_m w_n} \right\rangle \\
&+ \left\langle \overrightarrow{T^m(S_{\lambda_m} w_n) w_n}, \overrightarrow{w_m w_n} \right\rangle \\
&\leq \frac{\alpha_m}{1-\beta_m} d(g(z_m), T^m(S_{\lambda_m} z_m)) d(w_m, z_m) + \frac{\alpha_m}{1-\beta_m} \left\langle \overrightarrow{g(z_m) z_m}, \overrightarrow{z_m w_n} \right\rangle \\
&+ \frac{\alpha_m}{1-\beta_m} \left\langle \overrightarrow{z_m T^m(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_n} \right\rangle + d(T^m(S_{\lambda_m} z_m), T^m(S_{\lambda_m} w_n)) d(w_m, w_n) \\
&+ d(T^m(S_{\lambda_m} w_n), w_n) d(w_m, w_n) \\
&\leq \frac{\alpha_m}{1-\beta_m} d(g(z_m), T^m(S_{\lambda_m} z_m)) d(w_m, z_m) + \frac{\alpha_m}{1-\beta_m} \left\langle \overrightarrow{g(z_m) z_m}, \overrightarrow{z_m w_n} \right\rangle \\
&+ \frac{\alpha_m}{1-\beta_m} d(z_m, T^m(S_{\lambda_m} z_m)) d(z_m, w_n) + \left[(1+u_m) d(z_m, w_n) + v_m\right] d(w_m, w_n) \\
&+ d(T^m(S_{\lambda_m} w_n), w_n) d(w_m, w_n),
\end{aligned}$$

which implies from (4.3.12) that

$$\begin{aligned}
d^2(w_m, w_n) &\leq \frac{\alpha_m}{1-\beta_m} d(g(z_m), T^m(S_{\lambda_m} z_m)) d(w_m, z_m) + \frac{\alpha_m}{1-\beta_m} \left\langle \overrightarrow{g(z_m) z_m}, \overrightarrow{z_m w_n} \right\rangle \\
&+ \frac{\alpha_m}{1-\beta_m} d(z_m, T^m(S_{\lambda_m} z_m)) d(w_m, w_n) + \left[(1+u_m) d(w_m, w_n) + v_m\right] d(w_m, w_n) \\
&+ d(T^m(S_{\lambda_m} w_n), w_n) d(w_m, w_n) \\
&= \frac{\alpha_m}{1-\beta_m} d(g(z_m), T^m(S_{\lambda_m} z_m)) d(w_m, z_m) + \frac{\alpha_m}{1-\beta_m} \left\langle \overrightarrow{g(z_m) z_m}, \overrightarrow{z_m w_n} \right\rangle \\
&+ \frac{\alpha_m}{1-\beta_m} d(z_m, T^m(S_{\lambda_m} z_m)) d(w_m, w_n) + (1+u_m) d^2(w_m, w_n) + v_m d(w_m, w_n) \\
&+ d(T^m(S_{\lambda_m} w_n), w_n) d(w_m, w_n).
\end{aligned}$$

Therefore,

$$\left\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{w_n z_m} \right\rangle \leq d(g(z_m), T^m(S_{\lambda_m} z_m))d(w_m, z_m) + d(z_m, T^m(S_{\lambda_m} z_m))d(w_m, w_n)$$
$$+ u_m \left(\frac{1-\beta_m}{\alpha_m} \right) d^2(w_m, w_n)$$
$$+ \left(\frac{1-\beta_m}{\alpha_m} \right) v_m d(w_m, w_n) + \left(\frac{1-\beta_m}{\alpha_m} \right) d(T^m(S_{\lambda_m} w_n), w_n)d(w_m, w_n).$$

Thus, taking lim sup as $n \to \infty$ first, then as $m \to \infty$, it follows from (4.3.12) and (4.3.5) that

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \left\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{w_n z_m} \right\rangle \leq 0. \tag{4.3.13}$$

Furthermore, we have

$$\left\langle \overrightarrow{g(\bar{v})\bar{v}}, \overrightarrow{x_n \bar{v}} \right\rangle = \left\langle \overrightarrow{g(\bar{v})g(z_m)}, \overrightarrow{x_n \bar{v}} \right\rangle + \left\langle \overrightarrow{g(z_m)\bar{v}}, \overrightarrow{x_n \bar{v}} \right\rangle$$
$$\leq \left\langle \overrightarrow{g(\bar{v})g(z_m)}, \overrightarrow{x_n \bar{v}} \right\rangle + \left\langle \overrightarrow{z_m \bar{v}}, \overrightarrow{x_n \bar{v}} \right\rangle + \left\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{x_n w_n} \right\rangle$$
$$+ \left\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{w_n z_m} \right\rangle + \left\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{z_m \bar{v}} \right\rangle$$
$$\leq d(g(\bar{v}), g(z_m))d(x_n, \bar{v}) + d(z_m, \bar{v})d(x_n, \bar{v}) + d(g(z_m), z_m)d(x_n, w_n)$$
$$+ \left\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{w_n z_m} \right\rangle + d(g(z_m), z_m)d(z_m, \bar{v})$$
$$\leq (1+\tau)d(z_m, \bar{v})d(x_n, \bar{v}) + \left\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{w_n z_m} \right\rangle + [d(x_n, w_n) + d(z_m, \bar{v})]d(g(z_m), z_m).$$

Thus, from (4.3.5), (4.3.13) and the fact that $\lim_{m\to\infty} z_m = \bar{v}$, we obtain

$$\limsup_{n\to\infty} \left\langle \overrightarrow{g(\bar{v})\bar{v}}, \overrightarrow{x_n \bar{v}} \right\rangle = \limsup_{m\to\infty} \limsup_{n\to\infty} \left\langle \overrightarrow{g(\bar{v})\bar{v}}, \overrightarrow{x_n \bar{v}} \right\rangle$$
$$\leq \limsup_{m\to\infty} \limsup_{n\to\infty} \left\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{w_n z_m} \right\rangle \leq 0. \tag{4.3.13}$$

Step 4: Finally, we show that $\{x_n\}$ converges strongly to $\bar{v} \in \Gamma$.
From Lemma 2.2.7 , we obtain

$$\langle \overrightarrow{w_n \bar{v}}, \overrightarrow{x_n \bar{v}} \rangle \leq \frac{\alpha_n}{1-\beta_n} \left\langle \overrightarrow{g(x_n)v}, \overrightarrow{x_n \bar{v}} \right\rangle + \frac{\gamma_n}{1-\beta_n} \left\langle \overrightarrow{T^n(S_{\lambda_n} x_n)v}, \overrightarrow{x_n \bar{v}} \right\rangle$$
$$\leq \frac{\alpha_n}{1-\beta_n} \left\langle \overrightarrow{g(x_n)g(v)}, \overrightarrow{x_n \bar{v}} \right\rangle + \frac{\alpha_n}{1-\beta_n} \left\langle \overrightarrow{g(v)v}, \overrightarrow{x_n \bar{v}} \right\rangle + \frac{\gamma_n}{1-\beta_n} d(T^n(S_{\lambda_n} x_n), v)d(x_n, v)$$
$$\leq \frac{\alpha_n}{1-\beta_n} \tau d^2(x_n, v) + \frac{\alpha_n}{1-\beta_n} \left\langle \overrightarrow{g(v)v}, \overrightarrow{x_n \bar{v}} \right\rangle + \frac{\gamma_n}{1-\beta_n} \left[(1+u_n)d(x_n, v) + v_n \right] d(x_n, v)$$
$$= \left[\frac{\alpha_n}{1-\beta_n} \tau + \left(\frac{\gamma_n}{1-\beta_n} \right) (1+u_n) \right] d^2(x_n, v)$$
$$+ \left(\frac{\gamma_n}{1-\beta_n} \right) v_n d(x_n, v) + \frac{\alpha_n}{1-\beta_n} \left\langle \overrightarrow{g(v)v}, \overrightarrow{x_n \bar{v}} \right\rangle.$$

Thus, we obtain from Lemma 2.2.6 that

$$\begin{aligned} d^2(x_{n+1},\bar{v}) &\leq \beta_n d^2(x_n,\bar{v}) + (1-\beta_n)d^2(w_n,\bar{v}) \\ &= \beta_n d^2(x_n,\bar{v}) + (1-\beta_n)\left\langle \overrightarrow{w_n\bar{v}}, \overrightarrow{w_n\bar{v}}\right\rangle \\ &= \beta_n d^2(x_n,\bar{v}) + (1-\beta_n)\left[\left\langle \overrightarrow{w_n\bar{v}}, \overrightarrow{w_nx_n}\right\rangle + \left\langle \overrightarrow{x_n\bar{v}}, \overrightarrow{w_n\bar{v}}\right\rangle\right] \\ &\leq [\beta_n + \alpha_n\tau + \gamma_n + \gamma_n u_n]d^2(x_n,\bar{v}) + \gamma_n v_n d(x_n,\bar{v}) + \alpha_n\left\langle \overrightarrow{g(\bar{v})\bar{v}}, \overrightarrow{x_n\bar{v}}\right\rangle \\ &\quad + (1-\beta_n)d(w_n,\bar{v})d(w_n,x_n) \\ &= [1-\alpha_n(1-\tau)]d^2(x_n,\bar{v}) + \gamma_n u_n d^2(x_n,\bar{v}) + \gamma_n v_n d(x_n,\bar{v}) \\ &\quad + \alpha_n(1-\tau)\left[\frac{1}{(1-\tau)}\left\langle \overrightarrow{g(\bar{v})\bar{v}}, \overrightarrow{x_n\bar{v}}\right\rangle\right] + (1-\beta_n)d(w_n,\bar{v})d(w_n,x_n). \end{aligned}$$

That is,

$$d^2(x_{n+1},\bar{v}) \leq (1-\sigma_n)d^2(x_n,\bar{v}) + \sigma_n\delta_n + \theta_n,$$

where $\sigma_n := \alpha_n(1-\tau)$, $\delta_n := \left[\frac{1}{(1-\tau)}\left\langle \overrightarrow{g(\bar{v})\bar{v}}, \overrightarrow{x_n\bar{v}}\right\rangle\right]$ and $\theta_n = \gamma_n u_n d^2(x_n,\bar{v}) + \gamma_n v_n d(x_n,\bar{v}) + (1-\beta_n)d(w_n,\bar{v})d(w_n,x_n)$.
It follows from conditions (a), (c), (4.3.14) and Lemma 3.2.5 that $d(x_n,\bar{v}) \to 0$, as $n \to \infty$. Hence, we conclude that $\{x_n\}$ converges strongly to $\bar{v} \in \Gamma$. □

If T is a nonexpansive mapping in Theorem 4.3.2, then we obtain the following corollary.

Corollary 4.3.3. *Let X be an Hadamard space and X^* be its dual space. Let $T : X \to X$ be a nonexpansive mapping and g be a contraction mapping on X with coefficient $\tau \in (0,1)$. Let $A_i : X \to 2^{X^*}$ be a finite family of multivalued monotone mappings which satisfy the range condition. Suppose that $\Gamma := F(T) \cap \left(\cap_{i=1}^N A_i^{-1}(0)\right) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}_{n=1}^\infty$ is generated by*

$$\begin{cases} y_n = S_{\lambda_n}x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1}x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N}x_n, \\ w_n = \frac{\alpha_n}{1-\beta_n}g(x_n) \oplus \frac{\gamma_n}{1-\beta_n}Ty_n, \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n)w_n \quad n \geq 1, \end{cases} \tag{4.3.4}$$

where $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ are sequences in $(0,1)$ and $\{\lambda_n\}$ is a sequence of positive numbers such that,

(a) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(b) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1, \alpha_n + \beta_n + \gamma_n = 1\ \forall\ n \geq 1$,

(c) $\beta_i \in (0,1)$ *with* $\sum_{i=0}^{N}\beta_i = 1$,

(d) $0 < \lambda \leq \lambda_n\ \ \forall n \geq 1$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $\bar{v} \in \Gamma$.

By setting $g(x_n) = u$ for all $n \geq 1$, $u \in X$ fixed in Theorem 4.3.2, we obtain the following corollary with the Halpern-type algorithm.

Corollary 4.3.4. *Let* X *be an Hadamard space and* X^* *be its dual space. Let* $T : X \to X$ *be uniformly asymptotically regular and uniformly L-Lipschitzian generalized asymptotically nonexpansive mapping with the sequences* $\{u_n\}$, $\{v_n\} \subset [0, \infty)$. $A_i : X \to 2^{X^*}$ *be a finite family of multivalued monotone mappings which satisfy the range condition. Suppose that* $\Gamma := F(T) \cap \left(\cap_{i=1}^{N} A_i^{-1}(0)\right) \neq \emptyset$ *and for arbitrary* $u, x_1 \in X$, *the sequence* $\{x_n\}_{n=1}^{\infty}$ *is generated by*

$$\begin{cases} y_n = S_{\lambda_n} x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n, \\ w_n = \frac{\alpha_n}{1-\beta_n} u \oplus \frac{\gamma_n}{1-\beta_n} T^n y_n, \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) w_n \quad n \geq 1, \end{cases} \tag{4.3.5}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, *and* $\{\gamma_n\}_{n=1}^{\infty}$ *are sequences in* $(0, 1)$ *and* $\{\lambda_n\}$ *is a sequence of positive numbers such that,*

(a) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1, \alpha_n + \beta_n + \gamma_n = 1 \ \forall\ n \ \geq 1,$

(c) $\sum_{n=1}^{\infty} u_n < \infty, \quad \sum_{n=1}^{\infty} v_n < \infty,$

(d) $0 < \lambda \leq \lambda_n \quad \forall n \geq 1,$

(e) $\beta_i \in (0, 1)$ *with* $\sum_{i=0}^{N} \beta_i = 1.$

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $\bar{v} \in \Gamma$.

By setting $T \equiv I$ (where I is the identity mapping on X) in Theorem 4.3.2, we obtain the following corollary.

Corollary 4.3.5. *Let* X *be an Hadamard space and* X^* *be its dual space. Let* g *be a contraction mapping on* X *with coefficient* $\tau \in (0, 1)$ *and* $A_i : X \to 2^{X^*}$ *be a finite family of multivalued monotone mappings which satisfy the range condition. Suppose that* $\Gamma := \cap_{i=1}^{N} A_i^{-1}(0) \neq \emptyset$ *and for arbitrary* $x_1 \in X$, *the sequence* $\{x_n\}_{n=1}^{\infty}$ *is generated by*

$$\begin{cases} y_n = S_{\lambda_n} x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n, \\ w_n = \frac{\alpha_n}{1-\beta_n} g(x_n) \oplus \frac{\gamma_n}{1-\beta_n} y_n, \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) w_n \quad n \geq 1, \end{cases} \tag{4.3.6}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *and* $\{\gamma_n\}_{n=1}^{\infty}$ *are sequences in* $(0, 1)$ *and* $\{\lambda_n\}$ *is a sequence of positive numbers such that,*

(a) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1, \alpha_n + \beta_n + \gamma_n = 1 \ \forall \ n \ \geq 1,$

(c) $\beta_i \in (0,1)$ *with* $\sum_{i=0}^{N} \beta_i = 1,$

(d) $0 < \lambda \leq \lambda_n \quad \forall n \geq 1.$

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $\bar{v} \in \Gamma$.

4.4 Numerical examples

In this section, we first present a numerical example of Algorithm (4.3.1) in real line to show its efficiency. We also present two numerical examples of Algorithm (4.3.6) in 2-dimensional space of real numbers and in an Hadamard space (non-Hilbert space), to show its applicability.

Throughout this section, we shall take $\alpha_n = \frac{1}{2n+4}$ and $\gamma_n = \frac{n+2}{2n+4}$, $\forall n \geq 1$ and $\beta_n = \frac{n+1}{2n+4}$, $\forall \ n$. Furthermore, we shall take $\beta_i = \frac{1}{4}$, $i = 0,1,2,3$ and $g(x) = \frac{2}{5}x \ \forall x \in X$. Thus, the conditions of Corollary 4.3.5 are satisfied. Hence, for $x_1 \in X$, Algorithm (4.3.1) becomes:

$$\begin{cases} y_n = \frac{1}{4}\left(x_n + J_{\lambda_n}^{A_1} x_n + J_{\lambda_n}^{A_2} x_n + J_{\lambda_n}^{A_3} x_n\right), \\ w_n = \frac{2n+4}{n+3}\left[\frac{2}{10n+20}(x_n) + \frac{n+2}{2n+4}(T^n y_n)\right], \\ x_{n+1} = \frac{n+1}{2n+4}(x_n) + \frac{n+3}{2n+4}(w_n) \quad n \geq 1, \end{cases} \tag{4.4.1}$$

while Algorithm (4.3.6) becomes:

$$\begin{cases} y_n = \frac{1}{4}\left(x_n + J_{\lambda_n}^{A_1} x_n + J_{\lambda_n}^{A_2} x_n + J_{\lambda_n}^{A_3} x_n\right), \\ w_n = \frac{2n+4}{n+3}\left[\frac{2}{10n+20}(x_n) + \frac{n+2}{2n+4}(y_n)\right], \\ x_{n+1} = \frac{n+1}{2n+4}(x_n) + \frac{n+3}{2n+4}(w_n) \quad n \geq 1. \end{cases} \tag{4.4.2}$$

We now compute the monotone operator A and its resolvent $J_{\lambda_n}^{A}$ in the following examples.

Example 4.4.1. *Let* $X = \mathbb{R}$ *be endowed with the usual metric. Define* $T : \mathbb{R} \to \mathbb{R}$ *by*

$$Tx = \begin{cases} x, & x \in (-\infty, 0), \\ kx, & x \in [0, \frac{1}{2}], \\ \sin x, & x \in (\frac{1}{2}, \infty), \end{cases} \tag{4.4.3}$$

where $k \in (0,1)$. *Then, it follows by similar argument as in [95] that* T *is a generalized asymptotically nonexpansive mapping with* $u_n = 2k^n$ *and* $v_n = k^n$. *Also,* T *satisfies the assumptions in Theorem 4.3.2, and* $\sum_{n=1}^{\infty} u_n < \infty$ *and* $\sum_{n=1}^{\infty} v_n < \infty$. *Thus, condition (c) of Theorem 4.3.2 is satisfied.*

Now, define $A_i(x) = 3ix$. Then, A_i is monotone for each $i = 1, 2, 3$. Recall that $[t\overrightarrow{ab}] \equiv t(b-a)$, for all $t \in \mathbb{R}$ and $a, b \in \mathbb{R}$ (see [43]). Thus, for each $x \in \mathbb{R}$, we have that

$$J_{\lambda_n}^{A_i}(x) = z \iff \frac{1}{\lambda_n}(x - z) \in A_i z \iff z = (I + \lambda_n A_i)^{-1}x.$$

We now consider the following cases for our initial points $x_1 \in \mathbb{R}$.

Case 1: *Take $x_1 = 3$.* **Case 2:** *Take $x_1 = -5$.*

Case 3: *Take $x_1 = 0.8$.* **Case 4:** *Take $x_1 = 0.5$.*

Example 4.4.2. *Let $X = \mathbb{R}^2$ be endowed with the Euclidean norm $||.||_2$. Then for $i = 1$, we define $A_1 : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$A_1(x) = (x^{(1)} - 2x^{(2)},\ 2x^{(1)} + x^{(2)}).$$

Clearly, A_1 is a monotone operator.

Hence, we compute the resolvent of A_1 as follows:

$$\begin{aligned} J_{\lambda_n}^{A_1}(x) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_n & -2\lambda_n \\ 2\lambda_n & \lambda_n \end{bmatrix} \right)^{-1} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \\ &= \frac{1}{1 + 2\lambda_n + 5\lambda_n^2} \begin{bmatrix} 1 + \lambda_n & 2\lambda_n \\ -2\lambda_n & 1 + \lambda_n \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}, \end{aligned}$$

which implies that

$$J_{\lambda_n}^{A_1}(x) = \left(\frac{(1 + \lambda_n)x^{(1)} + 2\lambda_n x^{(2)}}{1 + 2\lambda_n + 5\lambda_n^2},\ \frac{(1 + \lambda_n)x^{(2)} - 2\lambda_n x^{(1)}}{1 + 2\lambda_n + 5\lambda_n^2} \right).$$

Now, for $i = 2, 3$, we define $A_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$A_2(x) = (x^{(1)} - x^{(2)},\ x^{(1)} + x^{(2)}), \quad A_3(x) = (x^{(2)},\ -x^{(1)}).$$

Thus, by similar argument as above, we obtain that

$$J_{\lambda_n}^{A_2}(x) = \left(\frac{(1 + \lambda_n)x^{(1)} + \lambda_n x^{(2)}}{1 + 2\lambda_n + 2\lambda_n^2},\ \frac{(1 + \lambda_n)x^{(2)} - \lambda_n x^{(1)}}{1 + 2\lambda_n + 2\lambda_n^2} \right),$$

$$J_{\lambda_n}^{A_3}(x) = \left(\frac{x^{(1)} - \lambda_n x^{(2)}}{1 + \lambda_n^2},\ \frac{x^{(2)} + \lambda_n x^{(1)}}{1 + \lambda_n^2} \right).$$

We consider the next example in a non-Hilbert space setting.

Example 4.4.3. *[27] Let $Y = \mathbb{R}^2$ be an $\mathbb{R}$-tree with the radial metric d_r, where $d_r(x, y) = d(x, y)$ if x and y are situated on a Euclidean straight line passing through the origin and $d_r(x, y) = d(x, \mathbf{0}) + d(y, \mathbf{0}) := \|x\| + \|y\|$ otherwise. Let $p = (1, 0)$ and $X = B \cup C$, where*

$$B = \{(h, 0) : h \in [0, 1]\} \ \text{ and } \ C = \{(h, k) : h + k = 1, h \in [0, 1)\}.$$

Then X is an Hadamard space. Thus, for each $[\overrightarrow{tab}] \in X^*$, we obtain that

$$[\overrightarrow{tab}] = \begin{cases} \{\overrightarrow{scd} : c, d \in B, s \in \mathbb{R},\ t(\|b\| - \|a\|) = s(\|d\| - \|c\|)\} & a, b \in B, \\ \{\overrightarrow{scd} : c, d \in C \cup \{\boldsymbol{0}\}, s \in \mathbb{R},\ t(\|b\| - \|a\|) = s(\|d\| - \|c\|)\} & a, b \in C \cup \{\boldsymbol{0}\}, \\ \{\overrightarrow{tab}\} & a \in B,\ b \in C. \end{cases}$$

Now, define $A : X \to 2^{X^*}$ by

$$A(x) = \begin{cases} \{[\overrightarrow{\boldsymbol{0}p}]\} & x \in B, \\ \\ \{[\overrightarrow{\boldsymbol{0}p}], [\overrightarrow{\boldsymbol{0}x}]\} & x \in C. \end{cases}$$

Then, A is monotone and its resolvent $J^A_{\lambda_n}$ is defined by

$$J^A_{\lambda_n}(x) = \begin{cases} \{z = (h - \lambda_n, 0)\} & x = (h, 0) \in B, \\ \\ \{z = (h', k') \in C : (1 + \lambda_n)^2(h'^2 + k'^2) = h^2 + k^2\} & x = (h, k) \in C. \end{cases}$$

We now consider the following cases for our initial vectors $x_1 \in \mathbb{R}^2$.

Case I: Take $x_1 = (3,\ -0.5)^T$. **Case II:** Take $x_1 = (-1,\ 0.5)^T$.

Case III: Take $x_1 = (-3,\ -4)^T$. **Case IV:** Take $x_1 = (0.9,\ 10)^T$.

Remark 4.4.4. *For each of the cases above, we compared the convergence rate of the different parameter λ_n as shown in the figures below, that is, $\lambda_n 1 = \frac{n+2}{2n+5}$, $\lambda_n 2 = \frac{n+7}{n+1}$, $\lambda_n 3 = \frac{100n+1}{n+2}$ for* **Case 1-4** *and $\lambda_n 1 = \frac{n+1}{10n+3}$, $\lambda_n 2 = \frac{2n}{n+4}$, $\lambda_n 3 = \frac{15n+3}{n+5}$ for* **Case I-IV**.

We observe from the numerical results that the sequence $\{\lambda_n\}$ converging to a number far away from 0 has a better convergence rate than that converging to a number closer to 0. This validates the condition on λ_n in our main results.

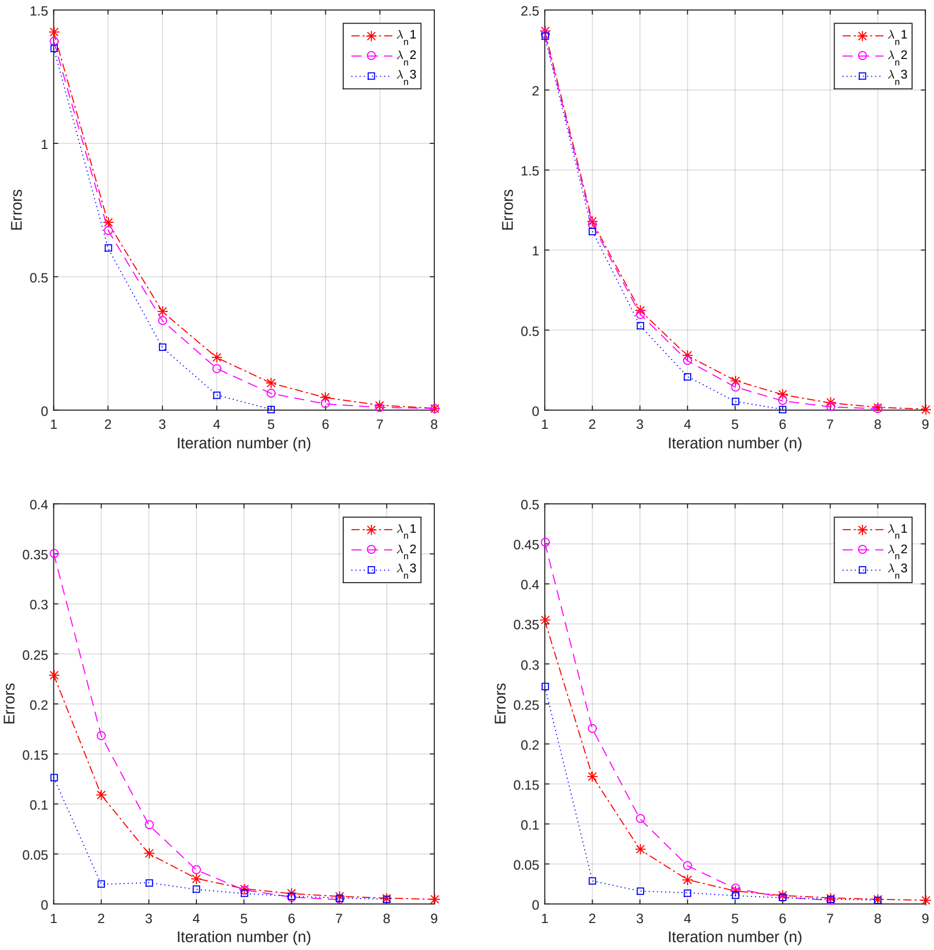

Figure 4.1: Errors vs Iteration numbers for **Example 4.1**: **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

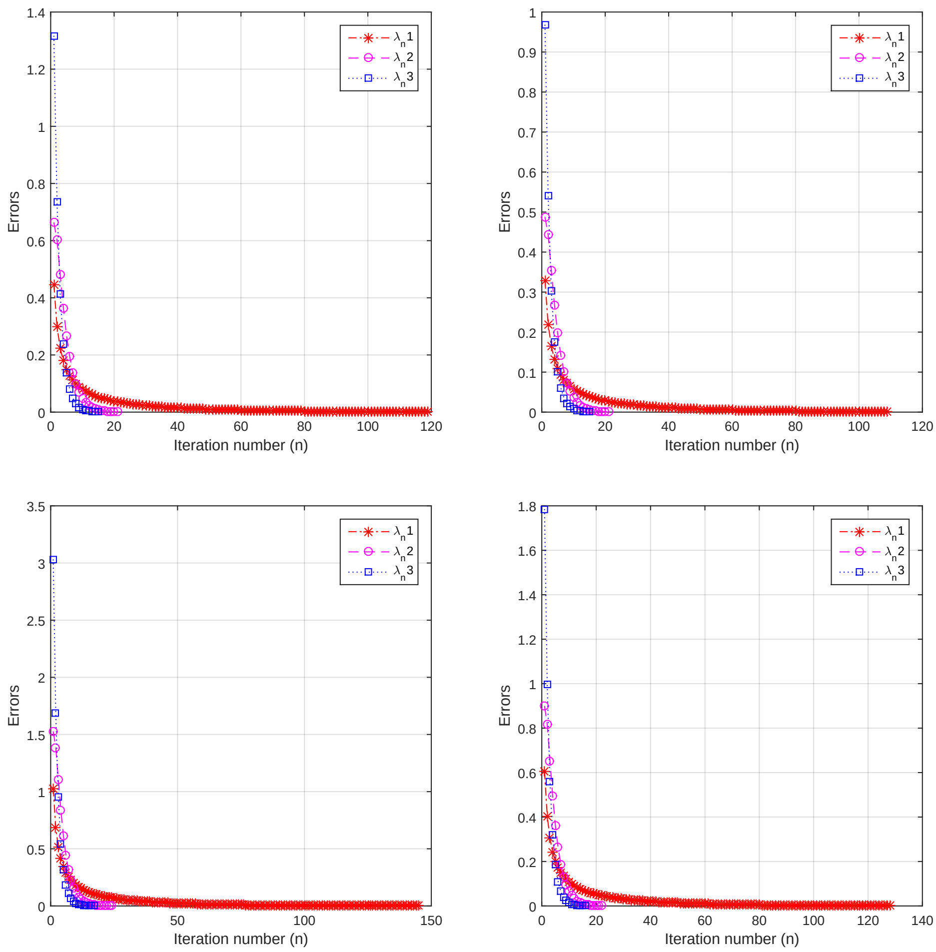

Figure 4.2: Errors vs Iteration numbers for **Example 4.2**: **Case I** (top left); **Case II** (top right); **Case III** (bottom left); **Case Iv** (bottom right).

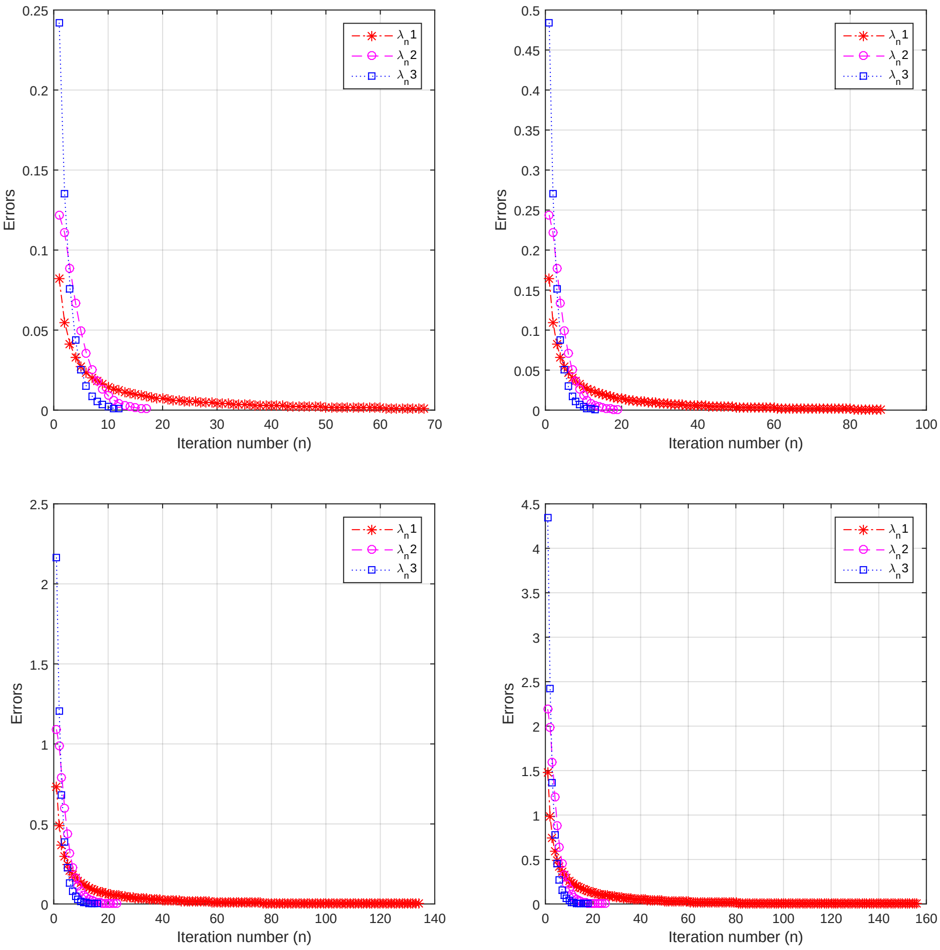

Figure 4.3: Errors vs Iteration numbers for **Example 4.3**: **Case I** (top left); **Case II** (top right); **Case III** (bottom left); **Case IV** (bottom right).

Chapter 5

On Generalized Demimetric Mappings and Monotone Operators in Hadamard spaces

5.1 Introduction

Recently, Takahashi [83] introduced a new class of nonlinear mappings in a real Hilbert space which he defined as follows:
Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. A mapping $T : C \to H$ is called k-demimetric, if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$ such that for any $x \in C$ and $y \in F(T)$, we have

$$\langle x - y, x - Tx \rangle \geq \frac{1-k}{2}\|x - Tx\|^2.$$

The class of demimetric mappings is of central importance in optimization since it contains many common types of operators emanating from optimization. For instance, the class of k-demimetric mappings with $k \in (-\infty, 1)$ is known to cover the class of k-demicontractive mappings with $k \in [0, 1)$, generalized hybrid mappings, the metric projections and the resolvents of maximal monotone operators (which are known as useful tools for solving optimization problems) in Hilbert spaces (see [4, 83] and the references therein). Thus, many authors have studied this class of mappings in both Hilbert and Banach spaces (see [52, 81, 82, 83]). This was recently extended to Hadamard spaces by Aremu et al. [4]. They defined demimetric mappings in an Hadamard space as follows: Let X be a CAT(0) space and C be a nonempty, closed and convex subset of X. A mapping $T : C \to X$ is said to be k-demimetric if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$ such that

$$\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle \geq \frac{1-k}{2} d^2(x, Tx) \tag{5.1.1}$$

for all $x \in C$, $\quad y \in F(T)$.
Furthermore, they gave an example of a demimetric mapping and established some fixed point results for this class of mappings and proved a strong convergence theorem for

approximating a common solution of finite family of minimization problems and fixed point problems for this class of mappings in Hadamard spaces.

In 2018, Kawasaki and Takahashi [45] generalized the class of demimetric mappings as follows: Let C be a nonempty, closed and convex subset of a smooth Banach space E and θ be a real number with $\theta \neq 0$. A mapping $T : C \to E$ with $F(T) \neq \emptyset$ is called generalized demimetric if

$$\theta\langle x - y, J(x - Tx)\rangle \geq \|x - Tx\|^2 \tag{5.1.2}$$

for all $x \in C$ and $y \in F(T)$, where J is a duality mapping on E. This class of mappings has also been studied in Banach spaces by Takahashi [79].

Motivated by the above results, we introduce and study the class of generalized demimetric mappings in Hadamard spaces. We also propose a Halpern-type PPA comprising of this class of mappings and a finite composition of resolvents of monotone operators, and prove that it converges strongly to a common zero of a finite family of monotone operators which is also a fixed point of a generalized demimetric mapping in an Hadamard space. We apply our results to solve a finite family of convex minimization problems, VIPs and convex feasibility problems in Hadamard spaces.

Let C be a nonempty closed and convex subset of an Hadamard space and $T : C \to C$ be a θ-generalized demimetric mapping with $\theta \neq 0$ and $F(T) \neq \emptyset$. Let $A_i : X \to 2^{X^*}, i = 1, 2, \cdots, N$ be a multivalued monotone mappings that satisfy the range condition. Our major interest in this chapter, is to study the problem: Find $z \in F(T)$ such that

$$0 \in \cap_{i=1}^{N} A(z). \tag{5.1.3}$$

5.2 Preliminaries

In this section, we recall some important results that will be needed in our study.

Definition 5.2.1. *Let C be a nonempty closed and convex subset of an Hadamard space X. A mapping $T : C \to C$ is said to be $\triangle$-demiclosed, if for any bounded sequence $\{x_n\}$ in X such that $\triangle\text{-}\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x = Tx$.*

Lemma 5.2.2. *[30] Let X be an Hadamard space and $T : X \to X$ be a nonexpansive mapping. Then T is $\triangle$-demiclosed.*

Lemma 5.2.3. *[58] Let $\{a_n\}$ be a sequence of non-negative numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n T_n,$$

where $\{T_n\}$ is a sequence of real numbers bounded from above and $\{\alpha_n\} \subset [0, 1]$ satisfies $\sum \alpha_n = \infty$. Then it holds that

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} T_n.$$

Lemma 5.2.4. *[88] Let X be an Hadamard space, for any $t \in [0, 1]$ and $u, v \in X$, let $u_t = tu \oplus (1 - t)v$. Then for all $x, y \in X$, we have*

$$\langle\overrightarrow{u_t x}, \overrightarrow{uy}\rangle \leq t\langle\overrightarrow{ux}, \overrightarrow{uy}\rangle + (1 - t)\langle\overrightarrow{vx}, \overrightarrow{uy}\rangle.$$

5.3 Main results

Following the idea of (5.1.1) and (5.1.2), we define θ-generalized demimetric mappings in CAT(0) spaces as follows:

Definition 5.3.1. *Let X be a CAT(0) space and $T : X \to X$ be a nonlinear mapping. T is called θ-generalized demimetric, if $F(T) \neq \emptyset$ and there exists $\theta \neq 0$ such that*

$$\theta\langle \overrightarrow{xy}, \overrightarrow{xTx}\rangle \geq d^2(x, Tx), \tag{5.3.1}$$

for all $x \in X$ and $y \in F(T)$.

Remark 5.3.2. *The following are examples of generalized demimetric mappings in CAT(0) spaces.*

1. *If $T : X \to X$ is a k- strictly pseudocontractive mapping with $k \in [0, 1)$ and $F(T) \neq \emptyset$, then T is $\left(\frac{2}{1-k}\right)$-generalized demimetric mapping. This follows from the same argument as in [4].*

2. *If $T : X \to X$ is a generalized hybrid mapping with $F(T) \neq \emptyset$, then T is a 2-generalized demimetric mapping. Indeed, for $x \in F(T)$ and $y \in X$, we obtain from (2.2.5) that*

$$d^2(x, Ty) \leq d^2(x, y). \tag{5.3.2}$$

   Also, from (2.2.3), we have that

$$2\langle \overrightarrow{yx}, \overrightarrow{yTy}\rangle = d^2(y, Ty) + d^2(x, y) - d^2(x, Ty),$$

   which implies from (5.3.2) that

$$\begin{aligned} 2\langle \overrightarrow{yx}, \overrightarrow{yTy}\rangle &\geq d^2(y, Ty) + d^2(x, y) - d^2(x, y) \\ &= d^2(y, Ty). \end{aligned}$$

   Hence, T is a 2-generalized demimetric mapping. Therefore, nonexpansive, nonspreading and hybrid mappings are examples of generalized demimetric mappings.

3. *If $T : X \to X$ is a k-demicontractive mapping, then T is a $\left(\frac{2}{1-k}\right)$-generalized demimetric.*

We now study some properties of θ-generalized demimetric mappings in Hadamard spaces.

Proposition 5.3.3. *Let X be an Hadamard space and $T : X \to X$ be a θ-generalized demimetric mapping with $\theta > 0$. Then T is a $(1 - \frac{2}{\theta})$-demimetric.*

Proof. It follows from the definition of demimetric mapping and θ-generalized demimetric mapping. □

Proposition 5.3.4. *Let X be an Hadamard space and $T : X \to X$ be a θ-generalized demimetric mapping with $\theta \neq 0$. Then $F(T)$ is closed and convex.*

Proof. We first show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $\{x_n\}$ converges to x^*. Then from the definition of θ-generalized demimetric mappings, we have

$$\theta\langle\overrightarrow{x^*x_n}, \overrightarrow{x^*Tx^*}\rangle \geq d^2(x^*, Tx^*), \tag{5.3.3}$$

which implies from Cauchy Schwartz inequality that

$$\theta d(x^*, x_n)d(x^*, Tx^*) \geq d^2(x^*, Tx^*).$$

Taking the limit of both sides, we obtain $0 \geq d^2(x^*, Tx^*)$, which implies that $x^* = Tx^*$. Thus, $x^* \in F(T)$. Therefore, $F(T)$ is closed. Next, we show that $F(T)$ is convex. For this, let $x, y \in F(T)$. Then, it suffices to show that $(tx \oplus (1-t)y) \in F(T)$, for $t \in [0, 1]$. Set $z = tx \oplus (1-t)y, \quad t \in [0, 1]$, then we obtain from Lemma 5.2.4 that

$$\begin{aligned}
d^2(z, Tz) &= \langle\overrightarrow{zTz}, \overrightarrow{zTz}\rangle \\
&= \langle\overrightarrow{(tx \oplus (1-t)y)Tz}, \overrightarrow{zTz}\rangle \\
&\leq t\langle\overrightarrow{xTz}, \overrightarrow{zTz}\rangle + (1-t)\langle\overrightarrow{yTz}, \overrightarrow{zTz}\rangle \\
&= t\left[\langle\overrightarrow{xz}, \overrightarrow{zTz}\rangle + \langle\overrightarrow{zTz}, \overrightarrow{zTz}\rangle\right] + (1-t)\left[\langle\overrightarrow{yz}, \overrightarrow{zTz}\rangle + \langle\overrightarrow{zTz}, \overrightarrow{zTz}\rangle\right] \\
&\leq \frac{-t}{\theta}d^2(z, Tz) + td^2(z, Tz) - \frac{(1-t)}{\theta}d^2(z, Tz) + (1-t)d^2(z, Tz) \\
&= \frac{-1}{\theta}d^2(z, Tz) + d^2(z, Tz),
\end{aligned}$$

which implies that $\frac{1}{\theta}d^2(z, Tz) \leq 0$. Since $\theta \neq 0$, we obtain that $z \in F(T)$ as required. □

Lemma 5.3.5. *Let X be a CAT(0) space and $T : X \to X$ be a θ-generalized demimetric mapping with $\theta \neq 0$. Suppose that $S_\lambda x = \lambda x \oplus (1-\lambda)Tx$ with $\theta \leq \frac{2}{1-\lambda}$ and $\lambda \in (0, 1)$, then S_λ is quasi-nonexpansive and $F(S_\lambda) = F(T)$.*

Proof. Let $x \in X$ and $z \in F(T)$, then since T is θ-generalized demimetric, we obtain from Lemma 5.2.4 that

$$\begin{aligned}
\langle\overrightarrow{zx}, \overrightarrow{xS_\lambda x}\rangle &= \langle\overrightarrow{(\lambda x \oplus (1-\lambda)Tx)x}, \overrightarrow{xz}\rangle \\
&\leq \lambda\langle\overrightarrow{xx}, \overrightarrow{xz}\rangle + (1-\lambda)\langle\overrightarrow{Txx}, \overrightarrow{xz}\rangle \\
&= (1-\lambda)\langle\overrightarrow{Txx}, \overrightarrow{xz}\rangle \\
&\leq \frac{-(1-\lambda)^2}{\theta(1-\lambda)}d^2(x, Tx).
\end{aligned} \tag{5.3.4}$$

Now, from (2.2.1), we obtain that $d^2(x, S_\lambda x) = (1-\lambda)^2 d^2(x, Tx)$. Substituting into (5.3.4), we obtain

$$\langle\overrightarrow{zx}, \overrightarrow{xS_\lambda x}\rangle \leq \frac{-1}{\theta(1-\lambda)}d^2(x, S_\lambda x),$$

which implies that

$$\langle \overrightarrow{xz}, \overrightarrow{xS_\lambda x}\rangle \geq \frac{1}{\theta(1-\lambda)} d^2(x, S_\lambda x)$$
$$\geq \frac{1}{2} d^2(x, S_\lambda x).$$

Thus, we obtain that

$$d^2(x, S_\lambda x) + d^2(z, x) - d^2(z, S_\lambda x) \geq d^2(x, S_\lambda x),$$

which implies that

$$d^2(z, S_\lambda x) \leq d^2(z, x).$$

Hence, S_λ is quasi-nonexpansive.
Next, we show that $F(S_\lambda) = F(T)$. From (2.2.1), we obtain that

$$d(x, S_\lambda x) = (1-\lambda) d(x, Tx).$$

This implies that $S_\lambda x = x$ if and only if $Tx = x$. Therefore, $F(S_\lambda) = F(T)$. ☐

Theorem 5.3.6. *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \cdots, N$ be a finite family of multivalued monotone mappings satisfying the range condition and $T : X \to X$ be a θ-generalized demimetric mapping with $\theta \neq 0$. Suppose that $\Gamma := F(T) \cap (\cap_{i=1}^N A_i^{-1}(0)) \neq \emptyset$ and for arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ is defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n S_\mu (J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n), \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \quad n \geq 1, \end{cases} \tag{5.3.5}$$

where $S_\mu x := \mu x \oplus (1-\mu)Tx$ such that S_μ is Δ-demiclosed, with $\theta \leq \frac{2}{1-\mu}$, $\mu \in (0,1)$, $\lambda \in (0,\infty)$ and $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ are in $(0,1)$ satisfying the following:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^\infty \alpha_n = \infty,$

(C2) $0 < a \leq \beta_n, \ \gamma_n \leq b < 1.$

Then $\{x_n\}$ converges strongly to an element of Γ.

Proof. We first show that $\{x_n\}$ is bounded.
Let $p \in \Gamma$, from (5.3.5), Lemma 2.2.6 and Lemma 5.3.5, we have

$$\begin{aligned} d^2(z_n, p) &= d^2((1-\gamma_n)y_n \oplus \gamma_n S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n), p) \\ &\leq (1-\gamma_n)d^2(y_n, p) + \gamma_n d^2(S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n), p) \\ &- \gamma_n(1-\gamma_n)d^2(y_n, S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n)) \\ &\leq (1-\gamma_n)d^2(y_n, p) + \gamma_n d^2(y_n, p) - \gamma_n(1-\gamma_n)d^2(y_n, S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n)) \quad (5.3.6) \\ &\leq d^2(y_n, p). \end{aligned}$$

We also have from (5.3.5) and (2.2.1) that

$$d(x_{n+1}, y_n) = \beta_n d(z_n, y_n), \tag{5.3.7}$$

which implies that

$$d^2(z_n, y_n) = \frac{\alpha_n}{\beta_n}\left(\frac{d^2(x_{n+1}, y_n)}{\alpha_n\beta_n}\right). \tag{5.3.8}$$

From (5.3.5), (5.3.6), (5.3.8) and Lemma 2.2.6, we obtain

$$\begin{aligned} d^2(x_{n+1}, p) &\leq (1-\beta_n)d^2(y_n, p) + \beta_n d^2(z_n, p) - \beta_n(1-\beta_n)d^2(y_n, z_n) && (5.3.9)\\ &\leq d^2(y_n, p) - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n) && (5.3.10)\\ &\leq d^2(y_n, p). \end{aligned}$$

Thus, we obtain from Lemma 2.2.6 that

$$\begin{aligned} d(x_{n+1}, p) &\leq d(y_n, p)\\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(u, p)\\ &\leq \max\ \{d(x_n, p), d(u, p)\}\\ &\vdots\\ &\leq \max\ \{d(x_1, p), d(u, p)\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded and consequently, $\{y_n\}$ and $\{z_n\}$ are all bounded.
Next, we show that

$$\lim_{n\to\infty} d(y_n, S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n)) = 0.$$

From (5.3.5),(5.3.9) and Lemma 2.2.6, we have that

$$\begin{aligned} d^2(x_{n+1}, p) &\leq d^2(y_n, p) - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n)\\ &= d^2((1-\alpha_n)x_n \oplus \alpha_n u, p) - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n)\\ &\leq (1-\alpha_n)^2 d^2(x_n, p) + \alpha_n^2 d^2(u, p) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{x_n p}, \overrightarrow{up}\rangle - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n)\\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n^2 d^2(u, p) - 2\alpha_n(1-\alpha_n)\langle\overrightarrow{x_n p}, \overrightarrow{pu}\rangle - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n)\\ &= (1-\alpha_n)d^2(x_n, p) + \alpha_n(-d_n), && (5.3.11) \end{aligned}$$

where

$$d_n = \left[2(1-\alpha_n)\langle\overrightarrow{x_n p}, \overrightarrow{pu}\rangle - \alpha_n d^2(u, p) + \frac{1}{\beta_n\alpha_n}(1-\beta_n)d^2(x_{n+1}, y_n)\right]. \tag{5.3.12}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, they are bounded below. Thus, $\{d_n\}$ is bounded below, which implies that $\{-d_n\}$ is bounded above.
Therefore, we obtain from Lemma 5.2.3 and condition $C1$ of Theorem 5.3.6 that

$$\begin{aligned}\limsup_{n\to\infty} d^2(x_n,p) &\leq \limsup_{n\to\infty}(-d_n)\\ &= -\liminf d_n,\end{aligned} \tag{5.3.13}$$

which implies that $\liminf(d_n) \leq -\limsup d^2(x_n,p)$. Thus, we conclude that $\liminf\limits_{n\to\infty} d_n$ exists.
Hence, we obtain from (5.3.12) and condition $C1$ of Theorem 5.3.6 that

$$\liminf_{n\to\infty} d_n = \liminf_{n\to\infty}\left[2\langle\overrightarrow{x_np},\overrightarrow{pu}\rangle + \frac{1}{\beta_n\alpha_n}(1-\beta_n)d^2(x_{n+1},y_n)\right].$$

Since $\{x_n\}$ is bounded and X is complete, we obtain from Lemma 4.2.6 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\triangle$-$\lim\limits_{k\to\infty} x_{n_k} = z \in X$, and

$$\liminf_{n\to\infty} d_n = \lim_{k\to\infty}\left[2\langle\overrightarrow{x_{n_k}p},\overrightarrow{pu}\rangle + \frac{1}{\beta_{n_k}\alpha_{n_k}}(1-\beta_{n_k})d^2(x_{n_k+1},y_{n_k})\right], \tag{5.3.14}$$

for some subsequences $\{y_{n_k}\}$, $\{\beta_{n_k}\}$ and $\{\alpha_{n_k}\}$ of $\{y_n\}, \{\beta_n\}$ and $\{\alpha_n\}$ respectively.
Using the fact that $\{x_n\}$ is bounded and $\liminf\limits_{n\to\infty} d_n$ exists, we get that
$\left\{\frac{1}{\beta_{n_k}\alpha_{n_k}}(1-\beta_{n_k})d^2(x_{n_k+1},y_{n_k})\right\}$ is bounded. Also, by condition $C2$, we obtain that
$\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k}) \geq \frac{1}{\alpha_{n_k}\beta_{n_k}}(1-b) > 0$. Thus $\left\{\frac{1}{\beta_{n_k}\alpha_{n_k}}d^2(x_{n_k+1},y_{n_k})\right\}$ is bounded.
Again, from C1 and C2, we obtain that $0 < \frac{\alpha_{n_k}}{\beta_{n_k}} \leq \frac{\alpha_{n_k}}{a} \to 0$, as $k \to \infty$. Thus, $\frac{\alpha_{n_k}}{\beta_{n_k}} \to 0$ as $k \to \infty$.
Therefore, we obtain from (5.3.8) that

$$\lim_{k\to\infty} d(z_{n_k},y_{n_k}) = 0. \tag{5.3.15}$$

From (5.3.7), (5.3.15) and condition C2, we obtain that

$$\lim_{k\to\infty} d(x_{n_k+1},y_{n_k}) = 0. \tag{5.3.16}$$

Also, from (5.3.6) and (5.3.15), we have

$$\begin{aligned}\gamma_{n_k}(1-\gamma_{n_k})d^2(y_{n_k},S_\mu(J_\lambda^N\circ J_\lambda^{N-1}\circ\cdots J_\lambda^2\circ J_\lambda^1 y_{n_k})) &\leq d^2(y_{n_k},p) - d^2(z_{n_k},p)\\ &\leq d^2(y_{n_k},z_{n_k}) + 2d(y_{n_k},z_{n_k})d(z_{n_k},p)\\ &+ d^2(z_{n_k},p) - d^2(z_{n_k},p) \to 0,\\ &\text{ as } k\to\infty.\end{aligned}$$

Thus, from Condition $C2$, we have that

$$\lim_{k\to\infty}(y_{n_k},S_\mu(J_\lambda^N\circ J_\lambda^{N-1}\circ\cdots J_\lambda^2\circ J_\lambda^1 y_{n_k})) = 0. \tag{5.3.17}$$

Next, we show that $\lim_{k\to\infty} d(v_{n_k}, S_\mu v_{n_k}) = 0$.
Let $v_{n_k} = \Phi_\lambda^N y_{n_k}$, where $\Phi_\lambda^N = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots J_\lambda^2 \circ J_\lambda^1$ with $\Phi_\lambda^0 = 1$. Since J_λ^N is firmly nonexpansive, we obtain from Remark 2.5.3 and (5.3.17) that

$$\begin{aligned} d^2(v_{n_k}, \Phi_\lambda^{N-1} y_{n_k}) &\leq d^2(p, \Phi_\lambda^{N-1} y_{n_k}) - d^2(p, v_{n_k}) \\ &\leq d^2(p, y_{n_k}) - d^2(p, S_\mu v_{n_k}) \\ &\leq d^2(p, S_\mu v_{n_k}) + 2d(p, S_\mu v_{n_k}) d(S_\mu v_{n_k}, y_{n_k}) \\ &+ d^2(S_\mu v_{n_k}, y_{n_k}) - d^2(p, S_\mu v_{n_k}) \to 0 \ \text{ as } \ k \to \infty. \end{aligned} \tag{5.3.18}$$

Similarly, since J_λ^{N-1} is firmly nonexpansive, we obtain that

$$\begin{aligned} d^2(\Phi_\lambda^{N-1} y_{n_k}, \Phi_\lambda^{N-2} y_{n_k}) &\leq d^2(p, \Phi_\lambda^{N-2} y_{n_k}) - d^2(p, \Phi_\lambda^{N-1} y_{n_k}) \\ &\leq d^2(p, y_{n_k}) - d^2(p, v_{n_k}) \\ &\leq d^2(p, y_{n_k}) - d^2(p, S_\mu v_{n_k}) \\ &\leq d^2(p, S_\mu v_{n_k}) + 2d(p, S_\mu v_{n_k}) d(S_\mu v_{n_k}, y_{n_k}) \\ &+ d^2(S_\mu v_{n_k}, y_{n_k}) - d^2(p, S_\mu v_{n_k}) \to 0 \ \text{ as } \ k \to \infty. \end{aligned} \tag{5.3.19}$$

In the same manner, we can show that

$$\lim_{k\to\infty} d^2(\Phi_\lambda^{N-2} y_{n_k}, \Phi_\lambda^{N-3} y_{n_k}) = \lim_{k\to\infty} d^2(\Phi_\lambda^{N-3} y_{n_k}, \Phi_\lambda^{N-4} y_{n_k}) = \cdots = \lim_{k\to\infty} d^2(\Phi_\lambda^1 y_{n_k}, y_{n_k}) = 0. \tag{5.3.20}$$

Thus,

$$d(v_{n_k}, y_{n_k}) \leq d(\Phi_\lambda^N y_{n_k}, \Phi_\lambda^{N-1} y_{n_k}) + d(\Phi_\lambda^{N-1} y_{n_k}, \Phi_\lambda^{N-2} y_{n_k}) + \cdots + d(\Phi_\lambda^1 y_{n_k}, y_{n_k}).$$

This implies from (5.3.18), (5.3.19) and (5.3.20) that

$$\lim_{k\to\infty} d(v_{n_k}, y_{n_k}) = \lim_{k\to\infty} d(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots J_\lambda^2 \circ J_\lambda^1 y_{n_k}, y_{n_k}) = 0. \tag{5.3.21}$$

Furthermore, from (5.3.17) and (5.3.21), we obtain

$$\lim_{k\to\infty} d(v_{n_k}, S_\mu v_{n_k}) = 0. \tag{5.3.22}$$

Lastly, we show that $\{x_n\}$ converges strongly to $z \in \Gamma$.
From (5.3.5) and condition $C1$, we obtain

$$\begin{aligned} d(y_{n_k}, x_{n_k}) &= d((1-\alpha_{n_k}) x_{n_k} \oplus \alpha_{n_k} u, x_{n_k}) \\ &= \alpha_{n_k} d(u, x_{n_k}) \to 0 \ \text{ as } \ k \to \infty. \end{aligned} \tag{5.3.23}$$

Since $\triangle\text{-}\lim_{k\to\infty} x_{n_k} = z$, we obtain from (5.3.23) that $\triangle\text{-}\lim_{k\to\infty} y_{n_k} = z$, and from (5.3.21) that $\triangle\text{-}\lim_{k\to\infty} v_{n_k} = z$. By the demicloseness of S_μ, (5.3.22) and Lemma 5.3.5, we obtain that $z \in F(S_\mu) = F(T)$. Since $J_\lambda^i, \ \ i = 1, 2, \cdots, N$ are nonexpansive mappings and the composition of nonexpansive mappings is nonexpansive, we obtain from (5.3.21), Lemma 5.2.2 and Lemma 2.5.4 that $z \in F(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1) = F(J_\lambda^N) \cap F(J_\lambda^N) \cap F(J_\lambda^{N-1}) \cap$

$\cdots \cap F(J_\lambda^2) \cap F(J_\lambda^1)$. Hence $z \in \Gamma$.
Furthermore, we obtain from Lemma 4.2.7 that

$$\limsup_{n\to\infty} \langle \overrightarrow{zu}, \overrightarrow{x_{n_k} z} \rangle \geq 0.$$

Thus, we obtain from (5.3.14) and (5.3.16) that

$$\liminf_{n\to\infty} d_n = 2 \lim_{k\to\infty} \langle \overrightarrow{zu}, \overrightarrow{x_{n_k} z} \rangle \geq 0.$$

Hence from (5.3.13), we have

$$\limsup_{n\to\infty} d^2(x_n, z) \leq -\liminf_{n\to\infty} d_n \leq 0.$$

Therefore, $\lim_{n\to\infty} d(x_n, z) = 0$ and this implies that $\{x_n\}$ converges strongly to $z \in \Gamma$. □

Setting $T \equiv I$ (where I is the identity mapping) in Theorem 5.3.6, we have the following result:

Corollary 5.3.7. *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \cdots, N$ be a finite family of multivalued monotone mappings satisfying the range condition. Suppose that $\Gamma := \cap_{i=1}^N A_i^{-1}(0) \neq \emptyset$ and for arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ is defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n, \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \quad n \geq 1, \end{cases} \tag{5.3.24}$$

where $\lambda \in (0, \infty)$ and $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ are in $(0, 1)$ satisfying the following:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^\infty \alpha_n = \infty,$

(C2) $0 < a \leq \beta_n, \ \gamma_n \leq b < 1.$

Then $\{x_n\}$ converges strongly to an element of Γ.

Setting $N = 1$ in Theorem 5.3.6, we have the following result:

Corollary 5.3.8. *Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^X$ be a multivalued monotone mapping that satisfies the range condition and $T : X \to X$ be a θ-generalized demimetric mapping with $\theta \neq 0$. Suppose that $\Gamma := F(T) \cap A^{-1}(0) \neq \emptyset$ and for arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ is defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n S_\mu(J_\lambda^A y_n), \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \quad n \geq 1, \end{cases} \tag{5.3.25}$$

where $S_\mu x := \mu x \oplus (1-\mu)Tx$ such that S_μ is Δ-demiclosed, with $\theta \leq \frac{2}{1-\mu}$, $\mu \in (0, 1)$, $\lambda \in (0, \infty)$ and $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ are in $(0, 1)$ satisfying the following:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(C2) $0 < a \leq \beta_n, \; \gamma_n \leq b < 1.$

Then $\{x_n\}$ *converges strongly to an element of* Γ.

If T is nonexpansive in Corollary (5.3.8), we obtain the following result.

Corollary 5.3.9. *Let* X *be an Hadamard space and* X^* *be its dual space. Let* $A : X \to 2^{X^*}$ *be a multivalued monotone mapping satisfying the range condition and* $T : X \to X$ *be a nonexpansive mapping. Suppose that* $\Gamma := F(T) \cap A^{-1}(0) \neq \emptyset$ *and for arbitrary* $u, x_1 \in X$, *the sequence* $\{x_n\}$ *is defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n T(J_\lambda^A y_n), \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \quad n \geq 1, \end{cases} \tag{5.3.26}$$

with $\lambda \in (0,\infty)$ *and* $\{\alpha_n\}_{n=1}^{\infty}, \; \{\beta_n\}_{n=1}^{\infty}, \; \{\gamma_n\}_{n=1}^{\infty}$ *are in* $(0,1)$ *satisfying the following:*

(C1) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(C2) $0 < a \leq \beta_n, \; \gamma_n \leq b < 1.$

Then $\{x_n\}$ *converges strongly to an element of* Γ.

5.4 Application to some optimization problems

In this section, we apply our results to solve some optimization problems.

Definition 5.4.1. *Let* X *be an Hadamard space and* $f : X \to (-\infty, \infty]$ *be a proper, convex and lower semicontinuous function with domain* $D(f) := \{x \in X : f(x) < +\infty\}$. *The function* $f : X \to (-\infty, \infty]$ *is called*

(i) proper, if $D(f) \neq \emptyset$,

(ii) convex, if

$$f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \quad \forall \;\; x, y \in X \quad \text{and} \quad \lambda \in (0,1),$$

(iii) lower semicontinuous at a point $x \in D(f)$, *if*

$$f(x) \leq \liminf_{n\to\infty} f(x_n),$$

for each sequence $\{x_n\}$ *in* $D(f)$ *such that* $\lim_{n\to\infty} x_n = x$.

(iv) f is lower semicontinuous on $D(f)$, if it is lower semicontinuous at any point in $D(f)$.

Definition 5.4.2. *[43] Let X be an Hadamard space and X^* be its dual space. The subdifferential of f is the multivalued function $\partial f : X \to 2^{X^*}$ defined by*

$$\partial f(x) = \begin{cases} \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz} \rangle \quad \forall \quad z \in X\}, & \text{if } x \in D(f), \\ \emptyset, & \text{otherwise.} \end{cases} \tag{5.4.1}$$

Theorem 5.4.3. *[43] Let $f : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function on an Hadamard space X with dual X^*, then*

(i) f attains its minimum at $x \in X$ if and only if $0 \in \partial f(x)$,

(ii) $\partial f : X \to 2^{X^*}$ is a monotone operator,

(iii) for any $y \in X$ and $\alpha > 0$, there exists a unique point $x \in X$ such that $[\alpha \overrightarrow{xy}] \in \partial f(x)$, that is $D(J_\lambda^{\partial f}) = X$, for all $\lambda > 0$.

Definition 5.4.4. *Let C be a nonempty, closed and convex subset of X. The indicator function $\delta_C : X \to \mathbb{R}$ defined by*

$$\delta_C x = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.4.2}$$

It is generally known that δ_C is a proper convex. Thus, by Theorem 5.4.3 (ii)(iii), we have that the subdifferential of δ_C, given by

$$\partial \delta_C(x) = \begin{cases} \{x^* \in X^* : \langle x^*, \overrightarrow{xz} \rangle \quad \leq 0 \quad \forall \quad z \in C\} \quad \text{if} \quad x \in C \\ \emptyset \ \text{otherwise,} \end{cases} \tag{5.4.3}$$

is a monotone operator which satisfies the range condition.

5.4.1 Variational inequality problem

Recently, Khatibzabdeh and Rajbar [49] formulated a VIP associated with a nonexpansive mapping in an Hadamard space as follows: Find $x \in C$ such that

$$\langle \overrightarrow{Txx}, \overrightarrow{xy} \rangle \geq 0 \quad \forall \quad y \in C. \tag{5.4.4}$$

Recall that the metric projection $P_C : X \to C$ is defined for $x \in X$ by $d(x, P_C x) = \inf_{y \in C} d(x, y)$ and is characterized by $z = P_C x$ if and only if $\langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \leq 0, \quad \forall \quad y \in C$ (see [49]). Using the characterization of P_C, we obtain that

$$x = P_C T x \quad \text{if and only if} \quad \langle \overrightarrow{Txx}, \overrightarrow{xy} \rangle \geq 0 \quad \forall \quad y \in C.$$

Thus, we have that $x \in F(P_C \circ T)$ if and only if x solves (5.4.4). From (2.5.1), we have that

$$z = J_\lambda^{\partial \delta_C} x \iff \left[\frac{1}{\lambda}\overrightarrow{zx}\right] \in \partial \delta_C z \iff \langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \leq 0, \quad \forall \; y \in C \iff z = P_C x. \quad (5.4.5)$$

Letting $z = x$, we obtain that $x = P_C x$ if and only if $x \in (\partial \delta_C)^{-1}(0)$. Thus, $x \in (\partial \delta_C)^{-1}(0) \cap F(T) \implies x \in F(P_C) \cap F(T) \implies x \in F(P_C \circ T)$.
Suppose the solution set of Problem (5.4.4) is Υ. Setting $A = \partial \delta_C$ in Corollary 5.3.9, we apply Corollary 5.3.9 to obtain the following result for approximating solutions of VIP in Hadamard spaces.

Theorem 5.4.5. *Let C be a nonempty closed and convex subset of an Hadamard space X and X^* be its dual space. Let $T : X \to X$ be a nonexpansive mapping. Suppose that $\Upsilon \neq \emptyset$ and for arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ is defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n T(J_\lambda^{\partial \delta_C} y_n), \quad n \geq 1, \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \quad n \geq 1, \end{cases} \quad (5.4.6)$$

with $\lambda \in (0, \infty)$ and $\{\alpha_n\}_{n=1}^\infty, \; \{\beta_n\}_{n=1}^\infty, \; \{\gamma_n\}_{n=1}^\infty \subset (0,1)$ satisfying the following:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^\infty \alpha_n = \infty,$

(C2) $0 < a \leq \beta_n, \; \gamma_n \leq b < 1.$

Then $\{x_n\}$ converges strongly to an element of Υ.

5.4.2 Convex feasibility problem

The convex feasibility problem is defined as follows: Find $x \in C$ such that

$$x \in \bigcap_{i=1}^N C_i, \quad (5.4.7)$$

where C is a nonempty closed and convex subset of X and $C_i, \quad i = 1, 2, \cdots, N$ is a finite family of nonempty closed and convex subsets of C such that $\bigcap_{i=1}^N C_i \neq \emptyset$.

From (5.4.5), we have that $x = J_\lambda^{\partial \delta_{C_i}} x$ if and only if $x = P_{C_i} x, \quad i = 1, 2, \cdots, N$. Setting $A_i = \partial \delta_{C_i}$ in Corollary (5.3.7) and $J_\lambda^i = P_{C_i}, \quad i = 1, 2, \cdots, N$ in Algorithm 5.3.24, we can apply Corollary 5.3.7 to approximate solutions of (5.4.7).

5.4.3 Convex minimization problem

The minimization problem is to find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y). \tag{5.4.8}$$

Observe from Theorem 5.4.3 (i) that (5.4.8) can be written as: Find $x \in X$ such that

$$0 \in \partial f(x). \tag{5.4.9}$$

Thus, by setting $A = \partial f$ in Theorem 5.3.6, we obtain the following result.

Theorem 5.4.6. *Let X be an Hadamard space and X^* be its dual space. Let $f_i : X \to (-\infty, \infty]$, $\quad i = 1, 2, \cdots, N$ be a finite family of proper, convex and lower semicontinuous functions and $T : X \to X$ be a θ-generalized demimetric mapping with $\theta \neq 0$. Suppose that $\Upsilon := F(T) \cap (\cap_{i=1}^{N} \partial f_i^{-1}(0)) \neq \emptyset$ and for arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ is defined by*

$$\begin{cases} y_n = (1 - \alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1 - \gamma_n)y_n \oplus \gamma_n S_\mu(J_\lambda^{\partial f_N} \circ J_\lambda^{\partial f_{N-1}} \circ \cdots \circ J_\lambda^{\partial f_2} \circ J_\lambda^{\partial f_1} y_n), \\ x_{n+1} = (1 - \beta_n)y_n \oplus \beta_n z_n, \quad n \geq 1, \end{cases} \tag{5.4.10}$$

where $S_\mu x := \mu x \oplus (1-\mu)Tx$ such that S_μ is Δ-demiclosed, with $\theta \leq \frac{2}{1-\mu}$, $\quad \mu \in (0, 1)$, $\quad \lambda \in (0, \infty)$ and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty} \subset (0, 1)$, satisfying

(C1) $\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(C2) $0 < a \leq \beta_n, \ \gamma_n \leq b < 1.$

Then $\{x_n\}$ converges strongly to an element of Υ.

Chapter 6

Conclusion, Contribution to knowledge and Future Research

6.1 Conclusion

In this dissertation, we proposed some iterative methods for approximating solutions of SVIPs and MIPs in Hilbert spaces and Hadamard spaces respectively. In Chapter 3, we established a strong convergence theorem for approximating solutions of split pseudo-monotone variational inequality problems in Hilbert spaces without the prior knowledge of Lipschitz constant of the pseudo-monotone, and with minimized number of projections as compared with other results in the literature. We also gave numerical examples in this chapter in a real Hilbert space. In Chapter 4, we established a strong convergence theorem for approximating a common solution of a finite family of MIPs and fixed point problem for a generalized asymptotically nonexpansive mapping in Hadamard spaces. Several numerical examples of our established theorem are given in support of the theorem. In Chapter 5, we introduced the class of generalized demimetric mappings in Hadamard spaces and prove several fixed point results concerning this class of mappings. We further obtained a strong convergence result of the sequence generated by our iterative scheme, and applied the established results to solve other optimization problems like the VIPs, convex feasibility problems and convex minimization problems.

6.2 Contribution to knowledge

The contribution of this work are as follows:

(i) In [21, 85], Censor *et al.*, Tian and Jian proposed algorithms for approximating the solution of the SVIPs and proved that the sequence generated by the algorithms converges weakly to a solution of the spilt variational ineqaulity problem when the operator is inverse strongly monotone and monotone (with a nonexpansive mapping) respectively. Our work in Chapter 3 generalizes these results since we obtained a strong convergence results when the underlying operator is pseudo-monotone and the mapping is strictly pseudocontractive. Also, our algorithm is designed in such a way that it does not depend on the knowledge of the Lipschitz constant of the underlying operator unlike the algorithms considered in [21, 85]. Furthermore, our algorithm has minimized number of projections compared to that of Tian and Jian [85], and our numerical experiments in Section 3.4 show that our algorithm performs better than that considered by Tian and Jian [85].

(ii) Ranijbar and Khatibzabeh [69] proposed the Mann and Halpern-type PPAs in Hdamard spaces for approximating solutions of MIPs. In Chapter 4 of this dissertation, we introduced a viscosity PPA for approximating a common solution of a finite family of MIPs and fixed point problem for generalized asymptotic nonexpansive mapping. Besides the fact that the problem considered in this chapter (Chapter 4) generalizes the problem studied by Ranijbar and Khatibzabeh [69], the Mann and Halpern algorithms of Ranijbar and Khatibzabeh [69] are special cases of the viscosity algorithm proposed in this chapter. Moreover, it has been established that viscosity type algorithms have higher convergence rates than Halpern iterations (see [67] for more advantages of viscosity algorithms over Halpern algorithms).

(iii) In Chapter 5, we introduced and studied the class of generalized demimetric mappings in Hadamard spaces. We proposed an iterative scheme which converges strongly to a common zero of a finite family of monotone operators which is also a fixed point of the newly introduced generalized demimetric mappings in an Hadamard space. The results obtained in this chapter generalize the results obtained in [4].

6.3 Future Research

In this section, we discuss some possible areas of future research.

The Armijo-like search rule (3.3.2) can be seen as a local approximation of the Lipschitz constant L of the pseudo-monotone and Lipschitz continuous operator f. Thus, the Lipschitz constant need not to be known. Hence, the stepsize $\{\lambda_n\}$ is given self-adaptively in our algorithm, which implies that the $\{\lambda_n\}$ in our algorithm does not depend on the knowledge of L. However, as observed in [73, Remark 5.3], Armijo-like search rules involve some evaluations of the operator f in the inner iteration and additional projections, in order to determine whether a certain "candidate" predictor stepsize satisfies the required inequality (i.e., inequality (3.3.2)). This may affect the efficiency of algorithms with Armijo-like search rules. To avoid this, one may relax the Lipschitz continuity assumption on the operator f. In our future research, we shall replace the Lipschitz continuity assumption on f with uniform continuity and try to obtain similar results as in Chapter 3. When this is achieved, we shall generalize the obtained results from Hilbert space settings to other Banach spaces more general than Hilbert spaces.

In Hadamard spaces, we are aware of only two results on VIPs (see [2, 49]), for which the underlying operators are nonexpansive and inverse strongly monotone operators. Thus, the theory of VIPs in Hadamard spaces are still in the developing stage. Part of our future research would be to study VIPs in Hadamard spaces.

It is generally known that Hilbert spaces are the only Banach spaces which are examples of Hadamard spaces. To further generalize established results in other Banach spaces like p-uniformly convex Banach spaces, Noar and Silberman [64] introduced the notion of p-uniformly convex metric spaces in 2011, which are natural generalization of the classical notion of p-uniformly convex Banach spaces. The notion of p-uniformly convex metric space is defined as follows: Let $1 < p < \infty$, a metric space X is called p-uniformly convex with parameter $c > 0$ if X is a geodesic space and for all $x, y, z \in X$ and $t \in [0, 1]$, we have

$$d(z, (1-t)x \oplus ty)^p \leq (1-t)d(z, x)^p + td(z, y)^p - \frac{c}{2}t(1-t)d(x, y)^p.$$

In our future research, we shall try to generalize the results in Chapter 4 and 5 to the framework of p-uniformly convex metric spaces.

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