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Generalized radiating stellar models with cosmological constant and electric charge

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Generalized radiating stellar models with cosmological constant and electric charge

by

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As the candidate's supervisors, I have approved this thesis for submission.

Signed: Prof Sunil D Maharaj November 2019

Signed: Dr Riven Narain November 2019

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Publications

The content contained within the pages of this thesis is based upon the following research papers (published and submitted)

Publication 1:

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Abstract

A general matter distribution, with the addition of the cosmological constant and electric charge, for the interior spacetime of a spherically symmetric radiating star undergoing gravitational collapse is considered in this investigation. The matching of the metric potentials and extrinsic curvature for the interior spacetime to the Vaidya exterior spacetime leads to the junction condition that relates the radial pressure to the heat flux. The presence of the cosmological constant and electric charge changes the nature of the problem significantly. Using Einstein-Maxwell field equations we express the junction condition as a Riccati equation in one of the metric potentials. In general this Riccati equation is not integrable. Special cases for particular matter distributions result in new classes of exact solutions to the Riccati equation. Previous results are also regained in this process. A transformation, called the horizon function, is then introduced to transform the Riccati equation into a simpler form. Several new classes of exact solutions are also found for the transformed Riccati equation. A new transformation called the generalized horizon function is introduced. This transformation preserves the form of the Riccati equation. The generalized horizon function leads to a transformed generalized Riccati equation. It is also possible to obtain earlier models by making assumptions on certain parameters. New models arise by restricting the values of parameters. The classes of solutions found can be given both implicitly and explicitly. The horizon function, and its generalization, can be obtained explicitly for all models.

Contents

Chapter 1

Introduction

Albert Einstein's theory of general relativity is the most profound and widely accepted theory of gravity in present times. It supersedes the theory of Newtonian gravity, which was the first consistent theory of gravity, since Einstein considered gravity to be a consequence of the curvature of spacetime. The mathematical structure for the formulation of general relativity is Riemannian geometry. It was formulated over a period of almost 10 years as an extension on the work of special relativity which Einstein published in 1905. General relativity describes the behaviour of gravitational fields and its effects on the interactions of astrophysical bodies in the universe. The predictions of general relativity have been shown to be consistent with observational data in the fields of relativistic astrophysics and cosmology.

Differential geometry, with the understanding of tensor calculus, is used as the mathematical basis for developing the framework of general relativity because it is able to represent spacetime as a manifold. Einstein expressed the metric tensor, Ricci scalar, Ricci tensor, stress energy momentum tensor and the cosmological constant in a tensorial form called the gravitational field equations. These equations tell us that matter and energy warps spacetime while the warping of spacetime forces matter and energy to move along geodesics. The Einstein field equations are governed by a coupled system of highly nonlinear partial differential equations which are very difficult to solve in general without making assumptions. The very first solutions to the Einstein field equations were obtained by Schwarzchild (1916a, 1916b), for the exterior and interior spacetimes for neutral matter, and by Reissner-Nordstrom (Reissner 1916, Nordstrom 1918) for charged matter. Vaidya (1951) introduced a nonstatic metric, for spherically symmetric bodies with outgoing null radiation, that is used to describe the exterior spacetime. By matching the exterior and interior spacetimes, we obtain the junction conditions to help with the modelling of radiating bodies. Santos (1985) formulated the boundary conditions by relating the heat flux and pressure. This model was first studied without the effects of shear and the cosmological constant. This work was extended by Glass (1989) and Maharaj and Govender (2000) in the presence of shear. Raychaudhuri (1955) pointed out that the slowest possible gravitational collapse arises in shear-free spacetimes.

The cosmological constant is an important component that is featured in the study of relativistic astrophysics and cosmology. Although scientists initially assumed it to be zero or fairly close to zero, an experimental discovery in 1998 proved otherwise. This discovery proved that the expanding universe is, in actual fact, accelerating. This implied that the cosmological constant could be nonzero and would have a deeper meaning in understanding the evolution of the universe. Physically it may be used to describe the energy density of space. It is also interpreted as a source of dark energy in the universe. The cosmological constant has been used in various studies for spherically symmetric radiating models. It mathematically and physically changes the nature of these stellar models. Also the effect of the cosmological constant in cosmology has been investigated by Lahav *et al* (1991), Balaguera-Antolinez *et al* (2006) and Debnath *et al* (2006). Riess *et al* (1998) provided measurements of type Ia supernovae which suggests that our universe may have a positive nonzero cosmological constant. Another analysis by Zehavi and Dekel (1999) observed the motion of low redshift galaxies which seems to give further evidence for a finite cosmological constant. The cosmological constant was also shown to play a diminishing role as the gravitational collapse proceeds in Oppenheimer-Snyder-deSitter spacetime by Markovic and Shapiro (2000). Sharif and Ahmad (2007) also observed that the cosmological constant slows down gravitational collapse and limits the size of a black hole. An interesting approach by Baier *et al* (2015) showed that it is possible to study black hole formation in four dimensional anti-de Sitter space with a negative cosmological constant.

Another important component in the study of gravitational collapse is the presence of the electromagnetic field. It appears in several investigations which gives a deeper understanding to our universe. The dynamics of the electromagnetic field, in a curved spacetime, is governed by Maxwell's equations. It is incorporated within the Einstein field equations by means of the electromagnetic energy momentum tensor. Stellar models are influenced by the presence of charge. In an earlier investigation Krori and Barua (1974) presented a study showing the complete exterior solution for a charged radiating sphere. Sharif and Azam (2012) studied the dynamical instability of expansion-free gravitational collapse due to the effects of the electromagnetic field. A recent study by Sharif and Abbas (2009) included the effects of the electromagnetic field and the cosmological constant, for spherically symmetric radiating bodies, which resulted in a faster collapse due to a reduction in the bound of cosmological constant. Another recent investigation by Ivanov (2019a), with the electromagnetic field, resulted in a generalization of dissipative collapse by making use of the horizon function.

The generalization of the junction conditions, described by the generalized Vaidya metric, was completed by Maharaj *et al* (2012) for spherically symmetric radiating models. Several exact solutions have been found for the Santos junction conditions, and some exact solutions are known for the generalized junction conditions. Thirukkanesh and Maharaj (2010) introduced a transformation that lead to a simplification of the boundary condition which resulted in exact solutions in terms of elementary functions. Thirukkanesh *et al* (2012a) studied a geodesic model which lead to several new exact solutions to the field equations by using various techniques. Ivanov (2016b) also introduced a transformation called the horizon function, which captures the physical importance of horizon formation. The horizon function was first used, in the absence of acceleration, to find new classes of exact solutions. The horizon function has proven to be useful in various studies of radiating stars including Ivanov (2019a, 2019b) and Mohanlal et al (2017). Another method used to investigate the field equations was the Lie method of infinitesimal generators. The Lie analysis was exploited by Abebe *et al* (2013, 2014a) and Mohanlal *et al* (2016). The geometrical approach of Lie symmetries gave rise to new classes of exact solutions for radiating stars with acceleration, expansion and shear.

It is important to find explicit solutions to the Einstein-Maxwell field equations for astrophysical applications such as a radiating star. These models are helpful in studying dissipative processes in stellar structures, particle production at the surface of stars, temperature profiles of stars and gravitational collapse. In addition to solving the Einstein-Maxwell equations we need to solve the junction conditions at the stellar surface. The main condition indicates that the radial pressure is proportional to the magnitude of the heat flux which is a nonlinear partial differential equation. This is the main objective of this thesis. The matter field is generalized to include the cosmological constant and the electromagnetic field.

The outline of this thesis is in terms of the following:

- Chapter 1: In this chapter we give a brief introduction to general relativity.
- Chapter 2: In this chapter we consider the generalized geodesic model. We study the general case including the effects of gravity, the cosmological constant and the electromagnetic field. The boundary condition is shown to be a Riccati equation in general. A

transformation reduces the boundary condition to a simpler equation. Several families of new exact solutions are found, both explicitly and implicitly. The exact solutions can be written in terms of elementary functions, elliptic integrals and Gaussian hypergeometric functions. We find that the cosmological constant and charge affects the gravitational behaviour of the model. We identify earlier models as special cases in this analysis.

- Chapter 3: In this chapter we study a geodesic radiating stellar model in which the Einstein field equations contain a cosmological constant and electric charge. Firstly, we study the boundary condition by introducing a generating function that is directly related to the horizon function. This generating function transforms the boundary condition into an algebraic equation which is solvable. New classes of exact solutions can be obtained by specifying a form for the generating function. It is possible to express physical quantities such as the mass and compactness factor in terms of the generating function. We also regain earlier results with only electric charge in this process. Secondly, we transform the boundary condition into a second order differential equation in terms of the generating function. By specifying certain parameters we solve the boundary condition by direct integration and obtain a new special class of exact solutions in terms of confluent hypergeometric functions.

- Chapter 4: In this chapter the cosmological constant and electric charge are incorporated in the Einstein-Maxwell field equations. Two approaches are used to investigate the problem. Firstly, the boundary condition is expressed as a generalized Riccati equation in one of the gravitational potentials. New classes of exact solutions are found which contain previous results in the absence of the cosmological constant and charge. Secondly, it is possible to preserve the form of the generalized Riccati equation by introducing a transformation called the horizon function. This transformation simplifes the generalized Riccati equation. We generate new solutions to the transformed Riccati equation when one of the metric function serves as a generating function. We also obtain other families of new classes of exact solutions where the horizon function serves as a generating function. Interestingly new uncharged solutions, not contained in previous studies, arise as special cases of the inhomogeneous Riccati equation in both approaches.

- Chapter 5: In this chapter we consider a spherically symmetric radiating model containing cosmological constant and electric charge. We propose a new transformation for the stellar boundary called the generalized horizon function. This transformation preserves the structure of the Riccati equation which governs the evolution of the boundary. Several new families of exact solutions, both implicit and explicit, to the transformed boundary condition are found. It is important to observe that the horizon function can be given explicitly in all cases. Results of previous studies arise as special cases in this general analysis.

- Chapter 6: In this chapter we conclude by discussing the findings of the previous chapters in a short summary.

Chapter 2

Generalized geodesic radiating models

2.1 Introduction

Models of radiating stars in general relativity have been studied in much detail in recent times since the completion of the junction conditions at the stellar surface by Santos (1985). In general the interior of the star includes an anisotropic energy momentum tensor and a spacetime manifold that is expanding, accelerating and shearing. Exact solutions to the field equations and boundary conditions in the general case were first generated by Thirukkanesh *et al* (2012a). Several families of exact solutions to this problem have been recently investigated by utilising the Lie symmetry generators in a group theoretical analysis. These include generalised Euclidean stars by Abebe *et al* (2014b), radiating bounded structures by Maharaj *et al* (2016), Mohanlal *et al* (2017) and radiating stars with exponential Lie symmetries by Mohanlal *et al* (2016). These studies reveal interesting geometrical features, and corresponding models can be used to study physical phenomena such as dissipation in gravitational collapse, see for example the perturbative study of Reddy *et al* (2015).

A useful approach in simplifying the model is to assume that the particle trajectories are geodesic. This makes the underlying boundary condition more tractable and helps in finding exact solutions. This assumption also helps in studying the physical behaviour associated with the rate of collapse, surface luminosity effects and changes in the temperature in the stellar interior. In particular Govender *et al* (1998) showed that the geodesic assumption makes it possible to find the temperature profiles in both causal thermodynamics and the Eckart theory. Dissipation and radiating bodies with neutrino flux were studied by Kolassis *et al* (1988) and Grammenos and Kolassis (1992), respectively. An exact solution where the model has a Friedmann-like limit was found by Naidu *et al* (2006). Families of exact solutions with geodesic particle trajectories were investigated by Thirukkanesh and Maharaj (2009, 2010) who reduced the problem to solutions of the standard differential equations of mathematical physics such as the Riccati equation. It is interesting to observe that self-similar solutions and travelling wave solutions are possible and arise in the group theoretical approach with Lie symmetries as shown by Abebe *et al* (2014a). Ivanov (2016a) studied spherical gravitational collapse in the presence of shear and heat flux, and he introduced generating functions to find new solutions to the boundary condition. Recently Tiwari and Maharaj (2017) found new geodesic radiating structures by generalising the potentials of earlier treatments.

The cosmological constant is often included in studies of the accelerated universe as it may be interpreted as a source of dark energy. Recently the influence of the cosmological constant has been considered in the collapse of a star in general relativity. Nayak *et al* (2015) studied the effect on a compact star with an equation of state provided by quark-meson coupling, and showed that the maximum mass of the star is affected by a nonzero cosmological constant. Exact solutions to the Einstein field equations and the boundary condition at the stellar surface have been found with a cosmological constant. Govender and Thirukkanesh (2009) found a geodesic model for a dissipating star, and showed that larger temperature gradients are produced at the core of the star in causal thermodynamics. Thirukkanesh *et al*

(2012b) generated conformally flat radiating stars, and showed that the cosmological constant enhances the temperature in the core region.

de Oliveira *et al* (1987) were the first to consider the dissipative effects of the electromagnetic field in gravitational collapse for a radiating star in general relativity. The presence of charge is important for the description of astrophysical processes, investigating the growth of inhomogeneities, describing structure formation, appearance of black holes and irregularities, and studying gravitational collapse phenomena. Sharif and Iftikhar (2015), Shah and Abbas (2018), Thirukkanesh and Govender (2013), and Nyonyi *et al* (2013, 2014) have included charge in their studies of gravitating fluids with heat flow. Ivanov (2019a) generated exact solutions with the geodesic assumption with shear and heat radiation. For a complete analysis we have included both the cosmological constant and the electromagnetic field in our treatment. In the absence of the cosmological constant and charge, we obtain the results for standard general relativity.

Thirukkanesh and Maharaj (2010) introduced a transformation for geodesic stars, acting under gravity only, that reduced the boundary condition to a separable equation. We show that this transformation may be extended to other matter fields which allows for a more complete model. For appropriate values of parameters, we regain the Thirukkanesh-Maharaj equations and other results. Our transformation for generalised matter fields allows for new exact solutions which we identify. We have structured our presentation to include the cosmological constant and charge. At relevant points we identify the cases and exact solutions for vanishing cosmological constant or neutral matter.

In section 2.2, we present the general shearing and expanding model of a radiating star, the corresponding Einstein field equations with the cosmological constant and charge, and the boundary condition at the surface of the star. Section 2.3 introduces the transformation from Thirukkanesh and Maharaj (2010). We show how the transformation in Thirukkanesh and Maharaj (2010) may be extended to include the cosmological constant and charge. In section 2.4, we study the transformed boundary condition for four cases with cosmological constant and charge. Several new families of exact solution to the boundary condition are found. Section 2.5 focusses on the outcome of this chapter and concluding remarks are made.

2.2 The model

We consider the physical situation where fluid particles are travelling in geodesic motion. Then the line element for a spherically symmetric model that describes the interior geometry of a radiating star is given by

$$ds^2 = -dt^2 + B^2 dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.1}$$

where $B = B(r,t)$ and $Y = Y(r,t)$ are the gravitational potentials. The fluid four-velocity $\mathbf{u}$ is comoving and can be expressed as $u^a = \delta^a_0$ so that $u^a u_a = -1$. The energy momentum tensor for the model in the interior has the form

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \epsilon_{ab} + E_{ab}, \tag{2.2}$$

where μ is the density, p is the isotropic pressure, q_a is the heat flux, ϵ_{ab} is the anisotropic stress and E_{ab} is the electromagnetic energy. The stress tensor is given by

$$\epsilon_{ab} = \left(p_{\parallel} - p_{\perp}\right)\left[n_a n_b - \frac{1}{3}h_{ab}\right], \tag{2.3}$$

where $p_{\parallel}$ is the radial pressure, $p_{\perp}$ is the tangential pressure, $h_{ab} = u_a u_b + g_{ab}$ is the projection tensor and $\mathbf{n}$ is a unit radial vector given by $n^a = \frac{1}{B}\delta^a_1$. The radial and tangential pressures can be expressed in terms of the isotropic pressure by $p = \frac{1}{3}\left(p_{\parallel} + 2p_{\perp}\right)$. The electromagnetic energy tensor is given by

$$E_{ab} = \frac{1}{4\pi}\left[F_a{}^c F_{bc} - \frac{1}{4}F^{cd}F_{cd}g_{ab}\right]. \tag{2.4}$$

Since the heat flow acts in the radial direction, we can write the heat flow vector $\mathbf{q}$ as

$$q^a = (0, Bq, 0, 0), \tag{2.5}$$

and $q^a u_a = 0$. The kinematical quantities are given as

$$\begin{aligned} \dot{u}^a &= (0,0,0,0)\,, && (2.6a)\\ \Theta &= \left(2\frac{Y_t}{Y} + \frac{B_t}{B}\right), && (2.6b)\\ \sigma &= \frac{1}{3}\left(\frac{Y_t}{Y} - \frac{B_t}{B}\right), && (2.6c) \end{aligned}$$

so that the acceleration $\dot{u}^a$ is a vanishing quantity. The expansion scalar Θ and the magnitude of the shear σ may be nonzero. Subscripts denote differentiation with respect to the coordinates r and t.

The Einstein-Maxwell equations with cosmological constant λ, in the stellar interior, are given by

$$\begin{aligned} G_{ab} + \lambda g_{ab} &= T_{ab}, && (2.7a)\\ F_{[ab;c]} &= 0, && (2.7b)\\ F^{ab}{}_{;b} &= \frac{1}{4\pi} J^a, && (2.7c) \end{aligned}$$

where Faraday's tensor $F_{ab} = \nu_{b;a} - \nu_{a;b}$ and the four-current $J^a = \kappa u^a$. Note that κ is the proper charge density and ν_a is the four-potential which has a simple form given by

$$\nu_a = (\Psi(r,t), 0, 0, 0), \tag{2.8}$$

so that $F_{01} = -F_{10} = -\Psi_r$. We can use (2.7a) and (2.7b) to get

$$\begin{aligned} \Psi_{rr} + \left(2\frac{Y_r}{Y} - \frac{B_r}{B}\right)\Psi_r &= \kappa B^2, && (2.9a)\\ \left(\frac{1}{B^2}\Psi_r\right)_t + \left(\frac{B_t}{B^3}\right)\Psi_r + \left(\frac{2Y_2}{B^2 Y}\right)\Psi_r &= 0. && (2.9b) \end{aligned}$$

Integrating (2.9) yields

$$
\begin{aligned}
\Psi_r &= \frac{Bl}{Y^2}, && (2.10a)\\
l(r) &= 4\pi \int^r \kappa B Y^2 dr, && (2.10b)
\end{aligned}
$$

where $l = l(r)$ is the total charge contained in the sphere up to radius r.

We can now write system (2.7) explicitly for the geodesic metric (2.1), the energy momentum tensor (2.2) and the four-potential (2.8) as

$$
\begin{aligned}
8\pi\mu + \frac{l^2}{Y^4} &= \left(2\frac{B_t}{B}\frac{Y_t}{Y} + \frac{{Y_t}^2}{Y^2}\right) - \frac{1}{B^2}\left(2\frac{Y_{rr}}{Y} + \frac{Y_r^2}{Y^2} - 2\frac{B_r}{B}\frac{Y_r}{Y} - \frac{B^2}{Y^2}\right) - \lambda, && (2.11a)\\
8\pi p_{\|} - \frac{l^2}{Y^4} &= -\left(2\frac{Y_{tt}}{Y} + \frac{{Y_t}^2}{Y^2}\right) + \frac{1}{B^2}\left(\frac{Y_r^2}{Y^2} - \frac{B^2}{Y^2}\right) + \lambda, && (2.11b)\\
8\pi p_{\perp} + \frac{l^2}{Y^4} &= -\left(\frac{B_{tt}}{B} + \frac{B_t}{B}\frac{Y_t}{Y} + \frac{Y_{tt}}{Y}\right) + \frac{1}{B^2}\left(\frac{Y_{rr}}{Y} - \frac{B_r}{B}\frac{Y_r}{Y}\right) + \lambda, && (2.11c)\\
8\pi q &= -\frac{2}{B}\left(\frac{B_t}{B}\frac{Y_r}{Y} - \frac{Y_{rt}}{Y}\right), && (2.11d)\\
4\pi\kappa &= \frac{l_r}{BY^2}. && (2.11e)
\end{aligned}
$$

The system (2.11) is expressed as a nonlinear system of partial differential equations which describes the gravitational and electromagnetic interactions in the interior of a spherically symmetric radiating body. When $\lambda = 0$ and $l = 0$ we regain the equations of Thirukkanesh and Maharaj (2010). The presence of the cosmological constant $\lambda \neq 0$ and the electromagnetic charge $l \neq 0$ changes the nature of the system, and the solutions that arise will be qualitatively different from the results in earlier investigations.

The surface of a spherically symmetric radiating star is the stellar boundary separating the interior and exterior spacetimes. The geodesic interior spacetime (2.1) must match, at the surface of the star, to the exterior radiating spacetime. We take the exterior spacetime to be the charged Vaidya metric

$$ds^2 = -\left(1 - \frac{2m(v)}{R} + \frac{Q^2}{R^2} - \frac{1}{3}\lambda R^2\right) dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.12}$$

where $m(v)$ is the mass of the star and Q is the total charge measured by an observer at infinity. The matching of the metrics (2.1) and (2.12), and the extrinsic curvature at the surface of the star, implies the junction condition

$$(p_{\|})_{\Sigma} \;\;=\;\; (q)_{\Sigma}, \tag{2.13a}$$

at the hypersurface Σ. Clearly the radial pressure $p_{\|}$ is nonzero at the surface Σ. Taking equation (2.13a), with equations (2.11b) and (2.11d), leads to the boundary condition

$$B_t \;\;=\;\; \left(\frac{Y_{tt}}{Y_r} + \frac{{Y_t}^2}{2YY_r} + \frac{1}{2YY_r} - \frac{\lambda Y}{2Y_r} - \frac{l^2}{2Y^3Y_r}\right) B^2 + \left(\frac{Y_{rt}}{Y_r}\right) B - \left(\frac{Y_r}{2Y}\right). \tag{2.14}$$

Note that this equation is given in the form of a Riccati equation in B. The presence of $\lambda \neq 0$ and $l \neq 0$ introduces the new terms $-\frac{\lambda Y}{2Y_r}$ and $-\frac{l^2}{2Y^3Y_r}$, respectively, which substantially changes the form of the boundary condition $p_{\|} = q$ on Σ.

2.3 Transformation

In an attempt to solve (2.14), with $\lambda = 0$ and $l = 0$, Thirukkanesh and Maharaj (2010) introduced the transformation

$$Z = \frac{B}{Y_r}. \tag{2.15}$$

This transformation also leads to a simplification of the more general equation (2.14) which allows for separable solutions. The transformation (2.15) reduces (2.14) to

$$Z_t = \frac{1}{2Y}\left[\tilde{F}Z^2 - 1\right], \tag{2.16}$$

where we have set

$$\tilde{F} = 2YY_{tt} + {Y_t}^2 - \lambda Y^2 - \frac{l^2}{Y^2} + 1. \tag{2.17}$$

Our objective in this chapter is to integrate (2.16) and obtain new solutions to the boundary condition.

If we set

$$\lambda = l = 0,$$

then (2.17) reduces to

$$F = 2YY_{tt} + Y_t^2 + 1. \tag{2.18}$$

The special case (2.18) was considered by Thirukkanesh and Maharaj (2010) and Ivanov (2016a).

By taking

$$\tilde{F} = 1 + m, \tag{2.19}$$

(2.17) becomes

$$2YY_{tt} + Y_t^2 - \lambda Y^2 - \frac{l^2}{Y^2} = m, \tag{2.20}$$

where m can be an arbitrary constant or a function of r. By expressing $Y_{tt} = \frac{1}{2}\frac{dY_t{}^2}{dY}$, (2.20) becomes a linear equation in $Y_t{}^2$. Integrating yields

$$Y_t{}^2 = \frac{\lambda Y^2}{3} - \frac{l^2}{Y^2} + \frac{n(r)}{Y} + m(r), \tag{2.21}$$

where $n(r)$ is an integration constant. Since (2.21) is a complicated equation to solve in general, we study equation (2.21) for four cases involving $\lambda = 0$, $\lambda \neq 0$, $l = 0$ and $l \neq 0$.

2.4 New classes of Solutions

Equations (2.16) and (2.21) need to be integrated to obtain exact solutions for the boundary condition. In this section we present models containing the cosmological constant ($\lambda = 0$, $\lambda \neq 0$) and electric charge ($l = 0$, $l \neq 0$). We give details only of the new solutions that we have been obtained. In the comprehensive tables we list all exact solutions, both new and previously discovered. In Table 2.1 we show the exact solutions that correspond to $\lambda = 0$ and $l = 0$. In Table 2.2 we list the exact solutions that correspond to $\lambda = 0$ and $l \neq 0$. In Table 2.3 we show the exact solutions that correspond to $\lambda \neq 0$ and $l = 0$. In Table 2.4 we list the exact solutions that correspond to $\lambda \neq 0$ and $l \neq 0$.

2.4.1 Case 1: $\lambda = 0$ and $l = 0$

We note that a rather simple class of new solutions, not contained in Thirukkanesh and Maharaj (2010), corresponds to $n = m = 0$. Then (2.21) gives

$$Y = f(r), \tag{2.22}$$

where $f(r)$ is the function of integration. Then (2.16) gives

$$Z = -\tanh\left[\frac{t}{2f(r)} + g(r)\right], \tag{2.23}$$

where $g(r)$ is another function of integration. From (2.15) we get

$$B = f_r \left(\frac{1 + h(r)\exp\left[\frac{t}{f(r)}\right]}{1 - h(r)\exp\left[\frac{t}{f(r)}\right]} \right), \tag{2.24}$$

where $h(r)$ is defined in terms of $g(r)$.

It is also possible to find another new set of solutions when $n \neq 0$ and $m \neq -1, 0$. Then (2.21) becomes

$$\frac{\sqrt{Y}}{\sqrt{mY + n}} dY = dt. \tag{2.25}$$

Two subcases arise: $m > 0$ and $m < -1$. Firstly, for $m > 0$ we obtain from (2.25),

$$\frac{\sqrt{Y}\sqrt{mY+n}}{m} - \frac{n}{m^{\frac{3}{2}}} \ln\left[\sqrt{Ym} + \sqrt{mY+n}\right] = t + f_1(r), \tag{2.26}$$

where $f_1(r)$ is the function of integration. Then (2.16) yields

$$Z = -\frac{1}{\sqrt{1+m}} \tanh\left[\frac{\sqrt{1+m}}{2}\int \frac{dt}{Y} + g_1(r)\right], \tag{2.27}$$

where $g_1(r)$ is another function of integration. The potential B becomes

$$B = -\frac{Y_r}{\sqrt{1+m}} \tanh\left[\frac{\sqrt{1+m}}{2}\int \frac{dt}{Y} + g_1(r)\right]. \tag{2.28}$$

Secondly, for $m < -1$ and $n > 0$ we get from (2.25),

$$\frac{\sqrt{Y}\sqrt{mY+n}}{m} - \frac{n}{m\sqrt{-m}} \arcsin\sqrt{-\frac{mY}{n}} = t + f_2(r), \tag{2.29}$$

where $f_2(r)$ is the function of integration. Again (2.16) yields

$$Z = -\frac{1}{\sqrt{-(1+m)}} \tan\left[\frac{\sqrt{-(1+m)}}{2}\int \frac{dt}{Y} + g_2(r)\right], \tag{2.30}$$

where $g_2(r)$ is the function of integration. The potential B becomes

$$B = -\frac{Y_r}{\sqrt{-(1+m)}} \tan\left[\frac{\sqrt{-(1+m)}}{2}\int \frac{dt}{Y} + g_2(r)\right]. \tag{2.31}$$

These solutions are real as $m < -1$. The new solutions (2.28) and (2.31) are given implicitly. The function Y has to satisfy (2.26) and (2.29) respectively.

Cases that have been studied previously arise when we let $n \neq 0$ and $m \neq 0$, for which the results of Thirukkanesh and Maharaj (2010) are regained (refer to their equations (13)-(16)). Also observe that if $n = 0$ and $m \neq -1, 0$ then the equations of Thirukkanesh and Maharaj (2010) are regained (refer to their equations (21)-(24)), and a special case of Zitha and Maharaj (2019) (refer to their equations (46)-(49c)). It is interesting to observe that new solutions can still be obtained in this simplest case considered in this section.

2.4.2 Case 2: $\lambda = 0$ and $l \neq 0$

A new class of solutions are obtained when we set $n = m = 0$. These results are given in terms of complex solutions which we omit in our presentation.

We can consider another new class of solutions when we set $n = 0$ and $m \neq -1, 0$. Then (2.21) gives

$$\frac{YdY}{\sqrt{mY^2 - l^2}} = dt. \tag{2.32}$$

which is a separable equation. By integrating (2.32) we get two subcases for $m > 0$ and $m < -1$, with

$$Y = \sqrt{m(t + f(r))^2 + \frac{l^2}{m}}, \tag{2.33}$$

where $f(r)$ is the integration function. Firstly, for $m > 0$, (2.16) yields

$$Z = -\frac{1}{\sqrt{1+m}} \tanh\left[\sqrt{\frac{1+m}{4m}} \ln\left[m(t+f) + \sqrt{m^2(t+f)^2 + l^2}\right] + g_1(r)\right], \tag{2.34}$$

where $g_1(r)$ is the function of integration. Then (2.15) becomes

$$\begin{aligned} B \;=\; & -\frac{1}{\sqrt{1+m}} \tanh\left[\sqrt{\frac{1+m}{4m}} \ln\left[m(t+f) + \sqrt{m^2(t+f)^2 + l^2}\right]\right. \\ & \left. + g_1(r)\right] \left(\frac{m^2(t+f)f_r + ll_r}{m\sqrt{m(t+f)^2 + \frac{l^2}{m}}}\right). \end{aligned} \tag{2.35}$$

Secondly, for $m < -1$, (2.33) becomes a complex function and is therefore neglected.

If we now consider the case when $n \neq 0$ and $m = 0$, then (2.21) becomes

$$\frac{YdY}{\sqrt{nY - l^2}} = dt. \tag{2.36}$$

Integrating (2.36) yields

$$Y = \frac{2l^4}{n^2}(x)^{-\frac{1}{3}} + \frac{1}{2}(x)^{\frac{1}{3}} - \frac{l^2}{n}, \tag{2.37}$$

where we have introduced

$$x=\frac{8l^6}{n^3}+9n(t+f(r))^2+\frac{3(t+f(r))}{n}\sqrt{16l^6+9n^4(t+f(r))^2}, \tag{2.38}$$

and $f(r)$ is the function of integration. Then (2.16) becomes

$$Z=-\tanh\left[\frac{1}{2}\int\frac{dt}{Y}+g(r)\right], \tag{2.39}$$

where $g(r)$ is another function of integration. Hence, the potential B is given by

$$B=-Y_r\tanh\left[\frac{1}{2}\int\frac{dt}{Y}+g(r)\right]. \tag{2.40}$$

This is an explicit solution as the t-dependence in (2.37) for Y, and therefore the potential B, is known.

We can now study the more general case in this section by letting $n\neq 0$ and $m\neq -1,0$. Then (2.21) gives

$$\frac{YdY}{\sqrt{mY^2+nY-l^2}}=dt. \tag{2.41}$$

We can solve (2.41) implicitly, which gives

$$\frac{\sqrt{mY^2+nY-l^2}}{m}-\frac{n}{2\sqrt{m^3}}\ln\left[n+2mY+2\sqrt{m}\sqrt{mY^2+nY-l^2}\right]=t+f(r), \tag{2.42}$$

where $f(r)$ is the function of integration. This gives us two solutions due to the restriction on m. Firstly, for $m>0$, (2.16) yields

$$Z=-\frac{1}{\sqrt{1+m}}\tanh\left[\frac{\sqrt{1+m}}{2}\int\frac{dt}{Y}+g_1(r)\right], \tag{2.43}$$

where $g_1(r)$ is the function of integration. The potential B becomes

$$B=-\frac{Y_r}{\sqrt{1+m}}\tanh\left[\frac{\sqrt{1+m}}{2}\int\frac{dt}{Y}+g_1(r)\right]. \tag{2.44}$$

Secondly, for $m < -1$ (2.16) yields

$$Z = -\frac{1}{\sqrt{-(1+m)}} \tan\left[\frac{\sqrt{-(1+m)}}{2}\int \frac{dt}{Y} + g_2(r)\right], \tag{2.45}$$

where $g_2(r)$ is the function of integration. Using (2.15), we have

$$B = -\frac{Y_r}{\sqrt{-(1+m)}} \tan\left[\frac{\sqrt{-(1+m)}}{2}\int \frac{dt}{Y} + g_2(r)\right]. \tag{2.46}$$

In the implicit solutions (2.44) and (2.46) the function Y has to satisfy (2.42).

2.4.3 Case 3: $\lambda \neq 0$ and $l = 0$

A new class of solutions is obtained when we take $n = m = 0$. Then (2.21) gives

$$Y = f(r)e^{\sqrt{\frac{\lambda}{3}}t}, \tag{2.47}$$

where $f(r)$ is the function of integration. From (2.16), we get

$$Z = \tanh\left[\frac{\sqrt{3}e^{-\sqrt{\frac{\lambda}{3}}t}}{2\sqrt{\lambda}f(r)} + g(r)\right], \tag{2.48}$$

where $g(r)$ is the function of integration. Therefore, the potential B becomes

$$B = e^{\sqrt{\frac{\lambda}{3}}t} \tanh\left[\frac{\sqrt{3}e^{-\sqrt{\frac{\lambda}{3}}t}}{2\sqrt{\lambda}f(r)} + g(r)\right](f_r). \tag{2.49}$$

The metric potentials B and Y are given in terms of elementary functions.

Another new set of solutions occurs when we let $n = 0$ and $m \neq -1, 0$. Then (2.21) is given by

$$\frac{dY}{\sqrt{\frac{\lambda}{3}Y^2 + m}} = dt, \tag{2.50}$$

which is a separable equation that yields three solutions due to restrictions on λ and m.

Firstly, for $\lambda > 0$ and $m > 0$, (2.50) becomes

$$Y = \frac{1}{2\lambda}\left[f(r)e^{\sqrt{\frac{\lambda}{3}}t} - \frac{3m\lambda}{f(r)e^{\sqrt{\frac{\lambda}{3}}t}}\right], \tag{2.51}$$

where $f(r)$ is the function of integration. Therefore (2.16) yields

$$Z = \frac{1}{\sqrt{1+m}} \tanh\left[\sqrt{\frac{1+m}{m}} \,\mathrm{arctanh}\left[\frac{f(r)}{\sqrt{3m\lambda}}e^{\sqrt{\frac{\lambda}{3}}t}\right] + g(r)\right]. \tag{2.52}$$

Using (2.15), the potential B is given by

$$B = \frac{f_r\left(e^{\sqrt{\frac{\lambda}{3}}t} + \frac{3m\lambda}{f(r)^2 e^{\sqrt{\frac{\lambda}{3}}t}}\right)}{2\lambda\sqrt{(1+m)}} \tanh\left[\sqrt{\frac{1+m}{m}} \,\mathrm{arctanh}\left[\frac{f(r)}{\sqrt{3m\lambda}}e^{\sqrt{\frac{\lambda}{3}}t}\right] + g(r)\right], \tag{2.53}$$

where $g(r)$ is the function of integration.

Secondly, for $\lambda > 0$ and $m < -1$, (2.50) becomes

$$Y = \frac{1}{2\lambda}\left[f(r)e^{\sqrt{\frac{\lambda}{3}}t} - \frac{3m\lambda}{f(r)e^{\sqrt{\frac{\lambda}{3}}t}}\right], \tag{2.54}$$

where $f(r)$ is the function of integration. Then (2.16) yields

$$Z = \frac{1}{\sqrt{-(1+m)}} \tan\left[\sqrt{\frac{1+m}{m}} \,\arctan\left[\frac{f(r)}{\sqrt{-3m\lambda}}e^{\sqrt{\frac{\lambda}{3}}t}\right] + g(r)\right]. \tag{2.55}$$

Then the potential B becomes

$$B = \frac{f_r\left(e^{\sqrt{\frac{\lambda}{3}}t} + \frac{3m\lambda}{f(r)^2 e^{\sqrt{\frac{\lambda}{3}}t}}\right)}{2\lambda\sqrt{-(1+m)}} \tan\left[\sqrt{\frac{1+m}{m}} \,\arctan\left[\frac{f(r)}{\sqrt{-3m\lambda}}e^{\sqrt{\frac{\lambda}{3}}t}\right] + g(r)\right], \tag{2.56}$$

where $g(r)$ is the function of integration.

Thirdly, for $\lambda < 0$ and $m > 0$, (2.21) yields

$$Y = \sqrt{-\frac{3m}{\lambda}} \sin\left[\sqrt{-\frac{\lambda}{3}}(t + f(r))\right], \tag{2.57}$$

where $f(r)$ is the function of integration. By integrating (2.16), we get

$$Z = -\frac{1}{\sqrt{1+m}} \tanh\left[2\sqrt{\frac{1+m}{m}}\left(x \,\arcsin x + \sqrt{1-x^2} + g(r)\right)\right], \tag{2.58}$$

where $x = \sqrt{-\frac{\lambda}{3}}(t + f(r))$. Hence, the potential B becomes

$$B = -\frac{(f_r)\sqrt{m}}{\sqrt{1+m}} \cos x \, \tanh\left[2\sqrt{\frac{1+m}{m}}\left(x \, \arcsin x + \sqrt{1-x^2} + g(r)\right)\right], \tag{2.59}$$

where $g(r)$ is the function of integration. Note that the three new solutions (2.53), (2.56) and (2.59) are given in terms of elementary functions.

We can also consider a general case in this section when $n \neq 0$ and $m \neq -1, 0$. Then (2.21) is given by

$$\sqrt{\frac{Y}{\frac{\lambda}{3}Y^3 + mY + n}}dY = dt. \tag{2.60}$$

Note that (2.60) involves elliptical integrals which are very complicated to solve in general. We can make progress by studying the cubic term in the denominator and finding its roots. Two cases arise when $\Delta = 0$ and $\Delta > 0$ where $\Delta = -\frac{4}{3}\lambda m^3 - 3\lambda^2 n^2$.

When $\Delta = 0$, we have $m < -1$ and $n = \sqrt{-\frac{4m^3}{9\lambda}}$ for which two subcases are analysed. Firstly, the cubic equation will not admit a single root (one triple root), since there is no quadratic term in the cubic equation. Secondly, the cubic equation admits one single root and one double root given by

$$Y_{0,1} = -\frac{3n}{2m} \quad \text{and} \quad Y_2 = \frac{3n}{m}.$$

Then equation (2.60) becomes

$$\sqrt{\frac{Y}{(Y + \frac{3n}{2m})^2(Y - \frac{3n}{m})}}dY = dt. \tag{2.61}$$

Equation (2.61) is integrated to give

$$2\ln\left[m\sqrt{Y} + \sqrt{m}\sqrt{mY - 3n}\right] - \frac{2\sqrt{3}}{3} \text{arctanh}\left[\frac{\sqrt{3mY}}{\sqrt{mY - 3n}}\right] = t + f(r), \tag{2.62}$$

where $f(r)$ is the function of integration. Then (2.16) yields

$$Z = -\frac{1}{\sqrt{-(1+m)}} \tan\left[\frac{\sqrt{-(1+m)}}{2}\int\frac{dt}{Y} + g(r)\right], \tag{2.63}$$

where $g(r)$ is another function of integration. So the potential B is given by

$$B = -\frac{Y_r}{\sqrt{-(1+m)}} \tan\left[\frac{\sqrt{-(1+m)}}{2}\int\frac{dt}{Y} + g(r)\right]. \tag{2.64}$$

where Y is given implicitly by (2.62).

The next case arises when we let $\Delta > 0$, then by definition $m < -1$ and $n > \sqrt{-\frac{4m^3}{9\lambda}}$. Thus, we can use a geometric representation to find three distinct real roots given by

$$\begin{aligned}
Y_0 &= 2\sqrt{-\frac{m}{\lambda}}\cos\left[\frac{1}{3}\arccos\left(\frac{3n}{2m}\sqrt{-\frac{\lambda}{m}}\right)\right], \\
Y_1 &= 2\sqrt{-\frac{m}{\lambda}}\cos\left[\frac{1}{3}\arccos\left(\frac{3n}{2m}\sqrt{-\frac{\lambda}{m}}\right) - \frac{2\pi}{3}\right], \\
Y_2 &= 2\sqrt{-\frac{m}{\lambda}}\cos\left[\frac{1}{3}\arccos\left(\frac{3n}{2m}\sqrt{-\frac{\lambda}{m}}\right) - \frac{4\pi}{3}\right],
\end{aligned}$$

where $-1 \leq \left(\frac{3n}{2m}\sqrt{-\frac{\lambda}{m}}\right) \leq 1$. We can then write equation (2.60) in the form

$$\sqrt{\frac{Y}{(Y-Y_0)(Y-Y_1)(Y-Y_2)}}dY = dt. \tag{2.65}$$

Equation (2.65) is integrated to yield

$$2(Y_0 - Y_2)\Pi(\phi, x, w) - 2Y_0 F(\phi, w) = \sqrt{Y_2(Y_0 - Y_1)}(t + f(r)), \tag{2.66}$$

where

$$\begin{aligned}
F(\phi, w) &= \int_0^\phi \left[1 - w^2\sin^2\theta\right]^{-1/2} d\theta, \\
\Pi(\phi, x, w) &= \int_0^\phi \left(1 - x\sin^2\theta\right)^{-1}\left[1 - w^2\sin^2\theta\right]^{-1/2} d\theta,
\end{aligned}$$

and we have set

$$x = \frac{Y_2 - Y_1}{Y_0 - Y_1}, \qquad w = x\frac{Y_0}{Y_2}, \qquad \sin\phi = \sqrt{\frac{(Y - Y_2)}{x(Y - Y_0)}}, \tag{2.67}$$

and $f(r)$ is the function of integration. Note that in the above we have followed the notation in Gradshteyn (2007) (refer to section 8.1 Elliptic Integrals and Functions) where $F(\phi, w)$ is the elliptic integral of the first kind, and $\Pi(\phi, x, w)$ is the elliptic integral of the third kind. Now (2.16) gives

$$Z = -\frac{1}{\sqrt{-(1+m)}} \tan\left[\frac{\sqrt{-(1+m)}}{2}\int \frac{dt}{Y} + g(r)\right], \tag{2.68}$$

where $g(r)$ is the function of integration. Using (2.15), we have

$$B = -\frac{Y_r}{\sqrt{-(1+m)}} \tan\left[\frac{\sqrt{-(1+m)}}{2}\int \frac{dt}{Y} + g(r)\right], \tag{2.69}$$

for which Y is expressed implicitly by (2.66). This class of solution is given implicitly in terms of special functions.

Note that when we let $n \neq 0$ and $m = 0$ in (2.60), then we regain the special case considered by Zitha and Maharaj (2019) (refer to their equations (26)-(30)). Their solution is presented in Table 2.3.

2.4.4 Case 4: $\lambda \neq 0$ and $l \neq 0$

Here we take the simplest subcase and we let $n = m = 0$. Then (2.21) is given by

$$\frac{YdY}{\sqrt{\frac{\lambda}{3}Y^4 - l^2}} = dt, \tag{2.70}$$

which is a separable equation. Integration gives

$$Y = \frac{f(r)e^{-t\sqrt{\frac{\lambda}{3}}}}{\sqrt{2\lambda}}\sqrt{f(r)e^{4t\sqrt{\frac{\lambda}{3}}} + 3\lambda l^2}, \tag{2.71}$$

where $f(r)$ is the function of integration. Then (2.16) gives

$$Z = -\tanh\left[\frac{\sqrt{2}fe^{\sqrt{\frac{\lambda}{3}}t}{}_2F_1\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{f}{3\lambda l^2}e^{4t\sqrt{\frac{\lambda}{3}}}\right)\sqrt{3+\frac{fe^{4t\sqrt{\frac{\lambda}{3}}}}{\lambda l^2}}}{\sqrt{fe^{4t\sqrt{\frac{\lambda}{3}}}+3\lambda l^2}}+g(r)\right], \tag{2.72}$$

where

$${}_2F_1\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{f}{3\lambda l^2}e^{4t\sqrt{\frac{\lambda}{3}}}\right) = \sum_{u=0}^{\infty}\frac{(\frac{1}{4})_u(\frac{1}{2})_u}{u!(\frac{5}{4})_u}\left[-\frac{f}{3\lambda l^2}e^{4t\sqrt{\frac{\lambda}{3}}}\right]^u, \tag{2.73}$$

is the Gaussian hypergeometric function. Therefore (2.15) expresses the potential B as

$$\begin{aligned} B &= -\tanh\left[\frac{\sqrt{2}fe^{\sqrt{\frac{\lambda}{3}}t}{}_2F_1\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{f}{3\lambda l^2}e^{4t\sqrt{\frac{\lambda}{3}}}\right)\sqrt{3+\frac{fe^{4t\sqrt{\frac{\lambda}{3}}}}{\lambda l^2}}}{\sqrt{fe^{4t\sqrt{\frac{\lambda}{3}}}+3\lambda l^2}}+g(r)\right] \\ &\quad\times\left(\frac{e^{-t\sqrt{\frac{\lambda}{3}}}\left[3e^{4t\sqrt{\frac{\lambda}{3}}}ff_r+6\lambda l(lf_r+fl_r)\right]}{2\sqrt{2}\sqrt{fe^{4t\sqrt{\frac{\lambda}{3}}}+3\lambda l^2}}\right), \end{aligned} \tag{2.74}$$

where $g(r)$ is the function of integration. This class of solution is given explicitly in terms of elementary and special functions.

We consider another class of solutions by letting $n = 0$ and $m \neq -1, 0$. Then (2.21) is given by

$$\frac{YdY}{\sqrt{\frac{\lambda}{3}Y^4+mY^2-l^2}} = dt, \tag{2.75}$$

which implies

$$Y = \frac{f(r)e^{-t\sqrt{\frac{\lambda}{3}}}}{2\sqrt{\lambda}}\sqrt{\left(f(r)e^{2t\sqrt{\frac{\lambda}{3}}}-3m\right)^2+12\lambda l^2}, \tag{2.76}$$

where $f(r)$ is the function of integration. We have two solutions for $m > 0$ and $m < -1$. Firstly, for $m > 0$, (2.16) yields

$$Z = -\frac{1}{\sqrt{1+m}}\tanh\left[\frac{\sqrt{1+m}}{2}\int\frac{dt}{Y}+g(r)\right]. \tag{2.77}$$

Therefore, the potential B becomes

$$\begin{aligned} B &= -\frac{1}{\sqrt{1+m}}\tanh\left[\frac{\sqrt{1+m}}{2}\int\frac{dt}{Y}+g(r)\right] \\ &\times\left(\frac{e^{-\sqrt{\frac{\lambda}{3}}}\left[f_r\left(9m^2+9mfe^{2t\sqrt{\frac{\lambda}{3}}}+2f^2e^{4t\sqrt{\frac{\lambda}{3}}}+12\lambda l^2\right)+12\lambda fll_r\right]}{2\sqrt{2}\sqrt{\left(f(r)e^{2t\sqrt{\frac{\lambda}{3}}}-3m\right)^2+12\lambda l^2}}\right), \end{aligned} \tag{2.78}$$

where $g(r)$ is the function of integration. Secondly, for $m<-1$, we can express (2.16) by

$$Z=-\frac{1}{\sqrt{-(1+m)}}\tan\left[\frac{\sqrt{-(1+m)}}{2}\int\frac{dt}{Y}+g(r)\right]. \tag{2.79}$$

Then the potential B is given by

$$\begin{aligned} B &= -\frac{1}{\sqrt{-(1+m)}}\tan\left[\frac{\sqrt{-(1+m)}}{2}\int\frac{dt}{Y}+g(r)\right] \\ &\times\left(\frac{e^{-\sqrt{\frac{\lambda}{3}}}\left[f_r\left(9m^2+9mfe^{2t\sqrt{\frac{\lambda}{3}}}+2f^2e^{4t\sqrt{\frac{\lambda}{3}}}+12\lambda l^2\right)+12\lambda fll_r\right]}{2\sqrt{2}\sqrt{\left(f(r)e^{2t\sqrt{\frac{\lambda}{3}}}-3m\right)^2+12\lambda l^2}}\right), \end{aligned} \tag{2.80}$$

where $g(r)$ is the function of integration. This is an explicit solution since Y is given in terms of the coordinate t in equation (2.76).

Another case arises when we let $n\neq 0$ and $m=0$. Then (2.21) is given by

$$\frac{Y\,dY}{\sqrt{\frac{\lambda}{3}Y^4+nY-l^2}}=dt. \tag{2.81}$$

Then (2.16) gives

$$Z=-\tanh\left[\frac{1}{2}\int\frac{dt}{Y}+g(r)\right], \tag{2.82}$$

where $g(r)$ is the function of integration. Using (2.15), the potential B becomes

$$B=-Y_r\tanh\left[\frac{1}{2}\int\frac{dt}{Y}+g(r)\right], \tag{2.83}$$

for which Y satisfies (2.81). However it is difficult to complete the integration in (2.81) for this case.

The more general case when studying equations (2.16) and (2.21) arises when we take $n \neq 0$ and $m \neq -1, 0$. Then (2.21) is expressed as

$$\frac{YdY}{\sqrt{\frac{\lambda}{3}Y^4 + mY^2 + nY - l^2}} = dt. \tag{2.84}$$

Equation (2.84) is too complex to integrate in general, but we can express Z and B as quadratures in some cases. We can find two solutions due to the restriction on m. Firstly, for $m > 0$, (2.16) yields

$$Z = -\frac{1}{\sqrt{1+m}} \tanh\left[\frac{\sqrt{1+m}}{2}\int\frac{dt}{Y} + g(r)\right], \tag{2.85}$$

where $g(r)$ is the function of integration. Using (2.15), B is given by

$$B = -\frac{Y_r}{\sqrt{1+m}} \tanh\left[\frac{\sqrt{1+m}}{2}\int\frac{dt}{Y} + g(r)\right], \tag{2.86}$$

for which Y satisfies (2.84). Secondly, for $m < -1$, (2.16) gives

$$Z = -\frac{1}{\sqrt{-(1+m)}} \tan\left[\frac{\sqrt{-(1+m)}}{2}\int\frac{dt}{Y} + g(r)\right], \tag{2.87}$$

where $g(r)$ is the function of integration. Then the potential B becomes

$$B = -\frac{Y_r}{\sqrt{-(1+m)}} \tan\left[\frac{\sqrt{-(1+m)}}{2}\int\frac{dt}{Y} + g(r)\right], \tag{2.88}$$

for which Y satisfies (2.84). Again the integration in (2.84) needs to be completed.

2.5 Discussion

In this work we have considered a generalized model of a geodesic radiating star with shear and expansion. We have extended earlier treatments by including the effects of cosmological constant and charge, when the fluid particles are in geodesic motion. It was necessary to solve the junction condition $(p_{\|})_\Sigma = (q)_\Sigma$ which is the master equation. The earlier transformation of Thirukkanesh and Maharaj (2010) was applied, and this led to a great simplification of the master equation. Several classes of exact solutions to the boundary condition were found. Particular classes of solutions were given explicitly, and other classes of solutions have an implicit form. The metric potentials are expressed in terms of elementary functions and special functions, involving elliptic integrals and Gaussian hypergeometric functions. We found that the presence of the cosmological constant and charge qualitatively changes the boundary condition, and allows for a wider category of solutions. Our analysis has yielded new solutions to the boundary condition and regained known solutions. Analytic expressions for the metric potentials were presented in Table 2.1 ($\lambda = 0$, $l = 0$), Table 2.2 ($\lambda = 0$, $l \neq 0$), Table 2.3 ($\lambda \neq 0$, $l = 0$) and Table 2.4 ($\lambda \neq 0$, $l \neq 0$). The earlier solutions of Thirukkanesh and Maharaj (2010), with $\lambda = 0$, $l = 0$, and Zitha *et al* (2019), with $\lambda \neq 0$, $l = 0$, arise as special cases in our treatment. It is interesting to note that the Riccati equation (2.14) can be solved even when $\lambda \neq 0$ and $l \neq 0$ indicating that the cosmological constant and charge may be included in these radiating stellar models.

2.6 Tables of results

Table 2.1: **New and known exact solutions for $\lambda = 0$ and $l = 0$**

Parameters	Gravitational potentials	Class
$n = 0$	$Y = f(r)$	New model
$m = 0$	$B = (f_r)\left(\frac{1+h(r)\exp\left[\frac{t}{f(r)}\right]}{1-h(r)\exp\left[\frac{t}{f(r)}\right]}\right)$	Explicit solution
$n \neq 0$	$\frac{\sqrt{Y}\sqrt{mY+n}}{m} - \frac{n}{m^{3/2}}\ln\left[\sqrt{Ym}+\sqrt{mY+n}\right] = t + f_1(r)$	New model
$m > 0$	$B = -\frac{Y_r}{\sqrt{1+m}}\ \tanh\left[\frac{\sqrt{1+m}}{2}\int\frac{dt}{Y} + g_1(r)\right]$	Implicit solution
$n > 0$	$\frac{\sqrt{Y}\sqrt{mY+n}}{m} - \frac{n}{m\sqrt{-m}}\ \arcsin\sqrt{-\frac{mY}{n}} = t + f_2(r)$	New model
$m < -1$	$B = -\frac{Y_r}{\sqrt{-(1+m)}}\ \tan\left[\frac{\sqrt{-(1+m)}}{2}\int\frac{dt}{Y} + g_2(r)\right]$	Implicit solution
$n \neq 0$	$Y = \left[R_1(r)t + R_2(r)\right]^{2/3}$	Thirukkanesh and Maharaj (2010)
$m = 0$	$B = \frac{2}{3}\left(\frac{1+f(r)\exp\left[3(R_1t+R_2)^{1/3}/R_1\right]}{1-f(r)\exp\left[3(R_1t+R_2)^{1/3}/R_1\right]}\right)\frac{(t(R_1)_r+(R_2)_r)}{(R_1+R_2)^{1/3}}$	Explicit solution
$n = 0$	$Y = R_1(r)t + R_2(r)$	Thirukkanesh and Maharaj (2010)
$m \neq -1, 0$	$B = \frac{t(R_1)_r+(R_2)_r}{\sqrt{{R_1}^2+1}}\left(\frac{1+g(r)(R_1t+R_2)^{\sqrt{{R_1}^2+1}/R_1}}{1-g(r)(R_1t+R_2)^{\sqrt{{R_1}^2+1}/R_1}}\right)$	Explicit solution

Table 2.2: **New exact solutions for $\lambda = 0$ and $l \neq 0$**

Parameters	Gravitational potentials	Class
$n = 0$	$Y = \sqrt{m(t+f(r))^2 + \frac{l^2}{m}}$	New model
$m > 0$	$B = -\frac{1}{\sqrt{1+m}} \tanh \left[\sqrt{\frac{1+m}{4m}} \ln \left[m(t+f) + \sqrt{m^2(t+f)^2 + l^2} \right] + g(r) \right] \left(\frac{m^2(t+f)f_r + ll_r}{m\sqrt{m(t+f)^2 + \frac{l^2}{m}}} \right)$	Explicit solution
$n \neq 0$	$Y = \frac{2l^4}{n^2}(x)^{-\frac{1}{3}} + \frac{1}{2}(x)^{\frac{1}{3}} - \frac{l^2}{n}$	New model
$m = 0$	$B = -Y_r \tanh \left[\frac{1}{2} \int \frac{dt}{Y} + g(r) \right]$ where $x = \frac{8l^6}{n^3} + 9n(t + f(r))^2 + \frac{3(t+f(r))}{n}\sqrt{16l^6 + 9n^4(t+f(r))^2}$	Explicit solution
$n \neq 0$	$\frac{\sqrt{mY^2+nY-l^2}}{m} - \frac{n}{2\sqrt{m^3}} \ln \left[n + 2mY + 2\sqrt{m}\sqrt{mY^2+nY-l^2} \right] = t + f(r)$	New model
$m > 0$	$B = -\frac{Y_r}{\sqrt{1+m}} \tanh \left[\frac{\sqrt{1+m}}{2} \int \frac{dt}{Y} + g(r) \right]$	Implicit solution
$n \neq 0$	$\frac{\sqrt{mY^2+nY-l^2}}{m} - \frac{n}{2\sqrt{m^3}} \ln \left[n + 2mY + 2\sqrt{m}\sqrt{mY^2+nY-l^2} \right] = t + f(r)$	New model
$m < -1$	$B = -\frac{Y_r}{\sqrt{-(1+m)}} \tan \left[\frac{\sqrt{-(1+m)}}{2} \int \frac{dt}{Y} + g(r) \right]$	Implicit solution

Table 2.3: **New and known exact solutions for $\lambda \neq 0$ and $l = 0$**

Parameters	Gravitational potentials	Class
$n = 0$	$Y = f(r)e^{\sqrt{\frac{\lambda}{3}}t}$	New model
$m = 0$	$B = e^{\sqrt{\frac{\lambda}{3}}t} \tanh\left[\sqrt{\frac{3}{\lambda}}\frac{e^{-\sqrt{\frac{\lambda}{3}}t}}{2f(r)} + g(r)\right](f_r)$	Explicit solution
$n \neq 0$	$Y = \left[\sqrt{\frac{3c(r)}{\lambda}} \sinh\left[\frac{\sqrt{3\lambda}}{2}(t + C)\right]\right]^{2/3}$	Zitha *et al* (2019)
$m = 0$	$B = Y_r \left(\frac{1+C_3 e^{\int 1/Y dt}}{1-C_3 e^{\int 1/Y dt}}\right)$	Explicit solution
$\lambda < 0$	$Y = \sqrt{-\frac{3m}{\lambda}} \sin\left[\sqrt{-\frac{\lambda}{3}}(t + f(r))\right]$	New model
$n = 0$	$B = -\frac{(f_r)\sqrt{m}}{\sqrt{1+m}} \cos x \tanh\Big[2\sqrt{\frac{1+m}{m}}\Big(x \arcsin x + \sqrt{1-x^2}$	Explicit solution
	$+ g(r)\Big)\Big]$	
$m > 0$	where $x = \sqrt{-\frac{\lambda}{3}}(t + f(r))$	
$\lambda > 0$	$Y = \frac{1}{2\lambda}\left[f(r)e^{\sqrt{\frac{\lambda}{3}}t} - \frac{3m\lambda}{f(r)e^{\sqrt{\frac{\lambda}{3}}t}}\right]$	New model
$n = 0$		
$m > 0$	$B = \frac{f_r}{2\lambda\sqrt{(1+m)}} \tanh\left[\sqrt{\frac{1+m}{m}} \operatorname{arctanh}\left[\frac{f(r)}{\sqrt{3m\lambda}}e^{\sqrt{\frac{\lambda}{3}}t}\right]\right.$	Explicit solution
	$\left.+g(r)\right]\left(e^{\sqrt{\frac{\lambda}{3}}t} + \frac{3m\lambda}{f(r)^2 e^{\sqrt{\frac{\lambda}{3}}t}}\right)$	
$\lambda > 0$	$Y = \frac{1}{2\lambda}\left[f(r)e^{\sqrt{\frac{\lambda}{3}}t} - \frac{3m\lambda}{f(r)e^{\sqrt{\frac{\lambda}{3}}t}}\right]$	New model
$n = 0$		
$m < -1$	$B = \frac{f_r}{2\lambda\sqrt{-(1+m)}} \tan\left[\sqrt{\frac{1+m}{m}} \arctan\left[\frac{f(r)}{\sqrt{-3m\lambda}}e^{\sqrt{\frac{\lambda}{3}}t}\right]\right.$	Explicit solution
	$\left.+g(r)\right]\left(e^{\sqrt{\frac{\lambda}{3}}t} + \frac{3m\lambda}{f(r)^2 e^{\sqrt{\frac{\lambda}{3}}t}}\right)$	

Continued on next page

Table 2.3: *Continued*

Parameters	Gravitational potentials	Class
$\Delta = 0$	$2\ln\left[m\sqrt{Y} + \sqrt{m}\sqrt{mY-3n}\right] + \frac{2\sqrt{3}}{3}\,\text{arctanh}\left[\frac{\sqrt{3mY}}{\sqrt{mY-3n}}\right]$	New model
$n \neq 0$	$= t + f(r)$	
$m < -1$	$B = -\frac{Y_r}{\sqrt{-(1+m)}}\,\tan\left[\frac{\sqrt{-(1+m)}}{2}\int\frac{dt}{Y} + g(r)\right]$	Implicit solution
$\Delta > 0$	$2(Y_0 - Y_2)\Pi(\phi, x, w) - 2Y_0F(\phi, w) = \sqrt{Y_2(Y_0 - Y_1)}(t + f(r))$	New model
$n > \sqrt{-\frac{4m^3}{9\lambda}}$	$B(r,t) = -\frac{Y_r}{\sqrt{-(1+m)}}\,\tan\left[\frac{\sqrt{-(1+m)}}{2}\int\frac{dt}{Y} + g(r)\right]$	Implicit solution
$m < -1$	where $x = \frac{Y_2 - Y_1}{Y_0 - Y_1}, \quad w = x\frac{Y_0}{Y_2}, \quad \sin\phi = \sqrt{\frac{(Y-Y_2)}{x(Y-Y_0)}}$	

Table 2.4: **New exact solutions for $\lambda \neq 0$ and $l \neq 0$**

Parameters	Gravitational potentials	Class
$n = 0$ $m = 0$	$Y = \frac{f(r)e^{-t\sqrt{\frac{\lambda}{3}}}}{\sqrt{2\lambda}}\sqrt{f(r)e^{4t\sqrt{\frac{\lambda}{3}}} + 3\lambda l^2}$ $B = \left(\frac{e^{-t\sqrt{\frac{\lambda}{3}}}\left[3e^{4t\sqrt{\frac{\lambda}{3}}}ff_r+6\lambda l(lf_r+fl_r)\right]}{2\sqrt{2}\sqrt{fe^{4t\sqrt{\frac{\lambda}{3}}}+3\lambda l^2}}\right)$ $\times\left(-\tanh\left[\frac{\sqrt{2}fe^{\sqrt{\frac{\lambda}{3}}t}\,{}_2F_1\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{f}{3\lambda l^2}e^{4t\sqrt{\frac{\lambda}{3}}}\right)\sqrt{3+\frac{fe^{4t\sqrt{\frac{\lambda}{3}}}}{\lambda l^2}}}{\sqrt{fe^{4t\sqrt{\frac{\lambda}{3}}}+3\lambda l^2}} + g(r)\right]\right)$	New model Explicit solution
$n = 0$ $m > 0$	$Y = \frac{f(r)e^{-t\sqrt{\frac{\lambda}{3}}}}{2\sqrt{\lambda}}\sqrt{\left(f(r)e^{2t\sqrt{\frac{\lambda}{3}}} - 3m\right)^2 + 12\lambda l^2}$ $B = -\frac{1}{\sqrt{1+m}}\tanh\left[\frac{\sqrt{1+m}}{2}\int\frac{dt}{Y} + g(r)\right]$ $\times\left(\frac{e^{-\sqrt{\frac{\lambda}{3}}}\left[f_r\left(9m^2+9mfe^{2t\sqrt{\frac{\lambda}{3}}}+2f^2e^{4t\sqrt{\frac{\lambda}{3}}}+12\lambda l^2\right)+12\lambda fll_r\right]}{2\sqrt{2}\sqrt{\left(f(r)e^{2t\sqrt{\frac{\lambda}{3}}}-3m\right)^2+12\lambda l^2}}\right)$	New model Explicit solution
$n = 0$ $m < -1$	$Y = \frac{f(r)e^{-t\sqrt{\frac{\lambda}{3}}}}{2\sqrt{\lambda}}\sqrt{\left(f(r)e^{2t\sqrt{\frac{\lambda}{3}}} - 3m\right)^2 + 12\lambda l^2}$ $B = -\frac{1}{\sqrt{-(1+m)}}\tan\left[\frac{\sqrt{-(1+m)}}{2}\int\frac{dt}{Y} + g(r)\right]$ $\times\left(\frac{e^{-\sqrt{\frac{\lambda}{3}}}\left[f_r\left(9m^2+9mfe^{2t\sqrt{\frac{\lambda}{3}}}+2f^2e^{4t\sqrt{\frac{\lambda}{3}}}+12\lambda l^2\right)+12\lambda fll_r\right]}{2\sqrt{2}\sqrt{\left(f(r)e^{2t\sqrt{\frac{\lambda}{3}}}-3m\right)^2+12\lambda l^2}}\right)$	New model Explicit solution
$n \neq 0$ $m = 0$	$YY_t = \sqrt{\frac{\lambda}{3}Y^4 + nY - l^2}$ $B(r,t) = -Y_r\tanh\left[\frac{1}{2}\int\frac{dt}{Y} + g(r)\right]$	New model Incomplete integration
$n \neq 0$ $m > 0$	$YY_t = \sqrt{\frac{\lambda}{3}Y^4 + mY^2 + nY - l^2}$ $B(r,t) = -\frac{Y_r}{\sqrt{1+m}}\tanh\left[\frac{\sqrt{1+m}}{2}\int\frac{dt}{Y} + g(r)\right]$	New model Incomplete integration
$n \neq 0$ $m < -1$	$YY_t = \sqrt{\frac{\lambda}{3}Y^4 + mY^2 + nY - l^2}$ $B(r,t) = -\frac{Y_r}{\sqrt{-(1+m)}}\tan\left[\frac{\sqrt{-(1+m)}}{2}\int\frac{dt}{Y} + g(r)\right]$	New model Incomplete integration

Chapter 3

A generating function and new exact solutions for geodesic matter

3.1 Introduction

Relativistic models of radiating stars have been studied in great detail to understand astrophysical phenomena such as temperature profiles, superdense matter formations and luminosities by applying restrictions on the acceleration, expansion and shear. Of particular interest is the process of gravitational collapse. Matching the exterior Vaidya spacetime (1951) with the interior spacetime of a radiating star leads to the junction conditions which were first obtained and studied by Santos (1985). Thirukkanesh and Maharaj (2010) was the first to find a transformation to solve the junction conditions to generate exact solutions in the absence of acceleration. Kolassis *et al* (1988) had earlier found an exact radiating model that regains, in the absence of heat flow, the Friedmann dust model using an adhoc approach. Particular models by Maharaj *et al* (2012), Thirukkanesh and Maharaj (2009) and Govender *et al* (2015) have also been studied with shear-free collapse, while Naidu *et al* (2006) were the first to obtain an exact solution with shear. De Oliveira *et al* (1985) studied gravitational collapse finding an exact model producing the slowest collapse of a radiating star in the absence of shear. These models generate much interest because they may be used to study the gravitational behaviour of radiating stars in general relativity.

The absence of acceleration in the model simplifies the equations and this is an assumption that is often used. This assumption does lead to models that are physically significant. Models that featured particles having geodesic motion were the earliest types studied. These models were obtained in the absence of shear, cosmological constant and electric charge. Acceleration-free motion in the presence of shear also leads to exact solutions, e.g. the shearing model of Naidu *et al* (2006). Many stellar models help to generate exact solutions with physical significance as shown by Thirukkanesh and Maharaj (2009, 2010), Pinheiro and Chan (2013) and Abebe *et al* (2014b). The Lie analysis of differential equations has been particularly useful in analysing these models as shown in Abebe *et al* (2014a) and Mohanlal *et al* (2016, 2017). The profile of the temperature in the stellar interior can be used to assist in determining the physical nature of the radiating sphere. Temperature profiles in both causal thermodynamics and the Eckart theory were found by Govender *et al* (1998).

The cosmological constant is used widely in general relativity as it contributes to the accelerated expansion of the universe and will affect the collapse of a radiating star. Exact solutions to the Einstein field equations and boundary condition at the stellar surface have been found which will be affected by the presence of the cosmological constant. The physical features of the collapsing star will be influenced by the presence of this additional parameter. Thirukkanesh *et al* (2012b) showed that the cosmological constant enhances the temperature in the core region of a radiating conformally flat star. We have therefore used the cosmological constant in our analysis to obtain new solutions. The electromagnetic field must also be considered in the treatment of gravitational collapse since it also influences the gravitational field. Its presence provides deeper physical insights to new exact solutions. The Einstein-Maxwell equations have been explored by Thirukkanesh and Maharaj (2008) for static fields. A general treatment for a dissipating star, in the presence of the electromagnetic field, was provided by Ivanov (2019a). Charge increases the mass, radius, compactness and surface red shift of the radiating star.

Our objective is to investigate the Einstein field equations with cosmological constant and the electric charge in geodesic motion. This extends earlier work where only charge was present. Although some previous studies involved the method of Lie symmetries, as shown by Abebe *et al* (2014a), we can perform simpler calculations to obtain results. The master equation is expressed as a Riccati equation in one of the potentials. This equation can be treated as an algebraic equation in one of the potentials; we can obtain the other potentials in a manner described as a solution generating function. This generating function was first obtained by Ivanov (2016a) without the presence of the cosmological constant. We extend the approach carried out by Ivanov to find a generalized solution generating function. Also a new class of solutions are found in terms of special functions by transforming the boundary condition to a second order differential equation. In section 3.2, we present the geodesic model of a radiating star, the corresponding Einstein field equations with the cosmological constant and electric charge. The boundary condition is written as a Riccati equation. Section 3.3 introduces two cases, with respect to the cosmological constant, to obtain solution generating algorithms that generalize Ivanov (2019a). In section 3.4 we find a new solution to the boundary condition by direct integration. Section 3.5 outlines the physical nature of a radiating body in the presence of the cosmological constant and electric charge by making use of the generating function. Some concluding remarks are made in section 3.6.

3.2 The model

We study fluid particles travelling in geodesic motion in a radiating star. Then the invariant line element for the interior geometry in a radiating model has the form

$$ds^2 = -dt^2 + B^2 dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.1}$$

where $B = B(r,t)$ and $Y = Y(r,t)$ are arbitrary functions. The spacetime (3.1) is spherically symmetric. The fluid four-velocity $\mathbf{u}$ has the form $u^a = \delta^a_0$ and is subsequently comoving so that $u^a u_a = -1$. The line element (3.1) generates

$$\dot{u}^a = (0,0,0,0), \tag{3.2a}$$

$$\Theta = \left(2\frac{Y_t}{Y} + \frac{B_t}{B}\right), \tag{3.2b}$$

$$\sigma = \frac{1}{3}\left(\frac{Y_t}{Y} - \frac{B_t}{B}\right). \tag{3.2c}$$

The energy momentum tensor describing the matter content has the general form

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \epsilon_{ab} + E_{ab}, \tag{3.3}$$

where μ is the energy density, p is the isotropic pressure, q_a is the heat flux, ϵ_{ab} is the anisotropic stress tensor and E_{ab} is the electromagnetic energy. The anisotropic stress tensor is defined by

$$\epsilon_{ab} = \left(p_{\parallel} - p_{\perp}\right)\left(n_a n_b - \frac{1}{3}h_{ab}\right), \tag{3.4}$$

where $p_{\parallel}$ is the radial pressure and $p_{\perp}$ is the tangential pressure. The stress tensor has been decomposed in terms of $h_{ab} = u_a u_b + g_{ab}$ which is the projection tensor and $\mathbf{n}$, a unit radial vector given by $n^a = \frac{1}{B}\delta^a_1$. The radial and tangential pressures are related to isotropic pressure by the equation $p = \frac{1}{3}\left(p_{\parallel} + 2p_{\perp}\right)$. The electromagnetic energy tensor has the form

$$E_{ab} = \frac{1}{4\pi}\left[F_a{}^c F_{bc} - \frac{1}{4}F^{cd}F_{cd}g_{ab}\right], \tag{3.5}$$

where F_{ab} is the Faraday tensor.

The Einstein-Maxwell equations with cosmological constant λ can be expressed in the form

$$G_{ab} + \lambda g_{ab} = T_{ab}, \tag{3.6a}$$

$$F_{[ab;c]} = 0, \tag{3.6b}$$

$$F^{ab}{}_{;c} = \frac{1}{4\pi} J^a, \tag{3.6c}$$

with the four-current $J^a = \kappa u^a$ and κ is the proper charge density. If we introduce the four-potential ν_a of the form

$$\nu_a = (\Psi(r,t), 0, 0, 0), \tag{3.7}$$

then the Faraday tensor $F_{ab} = \nu_{b;a} - \nu_{a;b}$ has two vanishing components $F_{01} = -F_{10} = -\Psi_r$. Maxwell's equations in (3.6) yield the conditions

$$\Psi_{rr} + \left(2\frac{Y_r}{Y} - \frac{B_r}{B}\right)\Psi_r = \kappa B^2, \tag{3.8a}$$

$$\left(\frac{1}{B^2}\Psi_r\right)_t + \left(\frac{B_t}{B^3}\right)\Psi_r + \left(\frac{2Y_2}{B^2 Y}\right)\Psi_r = 0. \tag{3.8b}$$

Integrating (3.8) gives

$$\Psi_r = \frac{Bl}{Y^2}, \tag{3.9a}$$

$$l = 4\pi \int^r \kappa B Y^2 dr, \tag{3.9b}$$

where $l = l(r)$ is a constant of integration. It represents the total charge contained in the sphere with radius r.

As we are modelling a radiating star we can take the heat to flow in the radial direction. Consequently the heat flow vector $\mathbf{q}$ must be of the form

$$q^a = (0, Bq, 0, 0), \tag{3.10}$$

with $q^a u_a = 0$. We assume the presence of geodesic metric (3.1), the charged energy momentum tensor (3.3) and the particular four-potential (3.7). Then the Einstein-Maxwell system (3.6) can be written explicitly as

$$8\pi\mu + \frac{l^2}{Y^4} = \left(2\frac{B_t}{B}\frac{Y_t}{Y} + \frac{{Y_t}^2}{Y^2}\right) - \frac{1}{B^2}\left(2\frac{Y_{rr}}{Y} + \frac{Y_r^2}{Y^2} - 2\frac{B_r}{B}\frac{Y_r}{Y} - \frac{B^2}{Y^2}\right) - \lambda, \tag{3.11a}$$

$$8\pi p_\parallel - \frac{l^2}{Y^4} = -\left(2\frac{Y_{tt}}{Y} + \frac{{Y_t}^2}{Y^2}\right) + \frac{1}{B^2}\left(\frac{Y_r^2}{Y^2} - \frac{B^2}{Y^2}\right) + \lambda, \tag{3.11b}$$

$$8\pi p_\perp + \frac{l^2}{Y^4} = -\left(\frac{B_{tt}}{B} + \frac{B_t}{B}\frac{Y_t}{Y} + \frac{Y_{tt}}{Y}\right) + \frac{1}{B^2}\left(\frac{Y_{rr}}{Y} - \frac{B_r}{B}\frac{Y_r}{Y}\right) + \lambda, \tag{3.11c}$$

$$8\pi q = -\frac{2}{B}\left(\frac{B_t}{B}\frac{Y_r}{Y} - \frac{Y_{rt}}{Y}\right), \tag{3.11d}$$

$$\kappa = \frac{l_r}{BY^2}. \tag{3.11e}$$

The system (3.11) comprises a system of coupled partial differential equations. This system is highly nonlinear and includes gravitational and electrical interactions for particles travelling on geodesics in the stellar interior. The influence of the cosmological constant and charge have also been included. When $\lambda = 0$ and $l = 0$ we regain earlier studies, in particular the equations of Thirukkanesh and Maharaj (2010). It is important to note that the cosmological constant ($\lambda \neq 0$) and electromagnetic field ($l \neq 0$) change the nature of the nonlinear system (3.11). The method of integration and the general solutions of (3.11) will necessarily be different from earlier investigations in the absence of λ and l. Consequently the physical behaviour of the geodesic radiating star will be qualitatively different.

The surface of the relativistic radiating star acts as the stellar boundary and separates the interior and exterior spacetimes. For consistency, the interior spacetime matches at the boundary, to the exterior spacetime. In general relativity we take the exterior spacetime to be

$$ds^2 = -\left(1 - \frac{2m(v)}{R} + \frac{Q^2}{R^2} - \frac{1}{3}\lambda R^2\right) dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.12}$$

which is the Vaidya metric with charge and cosmological constant. In (3.12), we note that $m(v)$ is the mass of the star and Q is the total charge measured by an observer at infinity. The matching of the metrics and the extrinsic curvature, at the boundary Σ, generates the junction conditions. These imply the restriction

$$(p_{\parallel})_{\Sigma} = (q)_{\Sigma}, \tag{3.13}$$

at the hypersurface Σ. Consequently the radial pressure $p_{\parallel}$ is proportional to the heat flux q at the surface Σ. Equation (3.13) can be written explicitly as

$$B_t = \left(\frac{Y_{tt}}{Y_r} + \frac{Y_t^2}{2YY_r} + \frac{1}{2YY_r} - \frac{\lambda Y}{2Y_r} - \frac{l^2}{2Y^3Y_r}\right) B^2 + \left(\frac{Y_{rt}}{Y_r}\right) B - \left(\frac{Y_r}{2Y}\right), \tag{3.14}$$

in terms of the potentials B and Y only. Note that we have written (3.14) as a Riccati equation in B. The presence of a cosmological constant and electric charge substantially changes the form and nature of the boundary condition. Thirukkanesh and Maharaj (2010) and Tiwari and Maharaj (2017) found particular classes of exact solutions for $\lambda = 0$ and $l = 0$. Ivanov (2016a, 2019a) studied (3.14) using the horizon function to obtain a generating function without the presence of the cosmological constant. Mahomed *et al* (2019a) used a special transformation to solve (3.14) with $\lambda \neq 0$. For our investigation, we show that general classes of solutions to (3.14) exist with $\lambda \neq 0$ and $l \neq 0$.

3.3 A solution generating algorithm

Ivanov (2016a) found a generating function for anisotropic spherical collapse with shear and null radiation. We show in this section that his approach may be extended to include a cosmological constant and charge. The horizon function, first introduced by Ivanov (2016b), has physical significance and is related to physical variables important for description of the collapse process. The horizon function, for particles travelling in geodesic motion, is defined by

$$H = \frac{Y_r}{B} + Y_t. \tag{3.15}$$

We may interpret the horizon function as replacing the potential B which is now given by

$$B = \frac{Y_r}{H - Y_t}, \tag{3.16}$$

where $H = H(r,t)$. Now substituting (3.16) in the junction condition (3.14) gives

$$2\left(H_t Y + H Y_t\right) = H^2 + \lambda Y^2 + \frac{l^2}{Y^2} - 1. \tag{3.17}$$

which is a Riccati equation in H. At this point it is convenient to introduce the new variable

$$D = HY, \tag{3.18}$$

where $D = D(r,t)$, in which case (3.17) has the equivalent form

$$\lambda Y^4 - \left(2D_t + 1\right) Y^2 + \left(D^2 + l^2\right) = 0, \tag{3.19}$$

Two cases arise in the analysis of equation (3.19). We consider each of these cases in turn.

3.3.1 Case 1: $\lambda = 0$

For this case (3.19) becomes

$$\left(2D_t + 1\right) Y^2 - \left(D^2 + l^2\right) = 0. \tag{3.20}$$

Equation (3.20) can be solved in terms of Y in general, and we find

$$Y = \sqrt{\frac{D^2 + l^2}{2D_t + 1}}. \tag{3.21}$$

Ivanov (2019a) was the first to obtain this result for charged matter. If we let $l = 0$ in (3.21) then we regain an earlier result of Ivanov (2016a) for uncharged matter.

We can write the potential B as

$$B = \frac{ll_r(2D_t+1) - l^2 D_{rt} + D[D_r(2D_t+1) - DD_{rt}]}{D(2{D_t}^2 + 3D_t + 1) + D_{tt}(D^2 + l^2)}, \tag{3.22}$$

in terms of the generating function D where we have used equations (3.16), (3.18) and (3.21). Specifying a form for D immediately leads to expressions for the potentials Y and B.

3.3.2 Case 2: $\lambda \neq 0$

This case has not been considered before. The boundary condition is highly nonlinear but it is possible to make progress. On closer inspection we realise that (3.19) may be interpreted as a fourth order polynomial in the variable Y. It is possible to solve (3.19) algebraically and we obtain

$$Y = \sqrt{\frac{(2D_t+1) \pm \sqrt{(2D_t+1)^2 - 4\lambda\left(D^2+l^2\right)}}{2\lambda}}, \tag{3.23}$$

for which $D_t > \sqrt{\lambda\left(D^2+l^2\right)} - \frac{1}{2}$ and $(D^2+l^2) > 0$. Note that we cannot regain the simpler solution in Case 1 from (3.23) since $\lambda \neq 0$ for this category of solutions. It is possible that $l = 0$ in (3.23) so that uncharged matter is regained. Equation (3.23) is a new solution to the boundary condition (3.14) in the presence of charge l and the cosmological constant λ.

It is interesting to observe that we can write the potential B in terms of the generating function D. Using equation (3.23) together with (3.16) and (3.18) we obtain

$$B = \frac{D_{rt}\left[\Delta \pm (2D_t+1)\right] \mp 2\lambda(DD_r + ll_r)}{2\lambda D\left[\Delta \pm D_t\right] \mp D_{tt}\left[(2D_t+1) \pm \Delta\right]}, \tag{3.24}$$

where $\Delta = \sqrt{(2D_t+1)^2 - 4\lambda\left(D^2+l^2\right)}$. This transparently shows that a choice for the generating function D yields expressions for the potentials Y and B.

We interpret the role of D as a generating function for both Case 1 and Case 2. A particular choice of D leads to a new solution. This approach is essentially an extension of a solution generating algorithm which was first proposed by Ivanov (2016a).

3.4 New travelling wave solutions

An interesting family of exact solutions to equation (3.19) was found by Ivanov (2019a). A similar approach to Ivanov in the presence of λ and l leads to another family of exact solution. We use the new variable

$$x = \int \frac{dr}{f(r)} - \frac{t}{a}, \tag{3.25}$$

which is related to the Lie group generator of Abebe *et al* (2014a). Here a is a constant, $f(r)$ is an arbitrary function and we take $D = D(x)$. We have a travelling wave if we let $f(r) = 1$ with a speed of $\frac{1}{a}$.

We can use equations (3.18) and (3.23) to show that $Y = Y(x)$ and $H = H(x)$ when $l(r)$ is a constant of integration. Tiwari and Maharaj (2017) suggested that the stellar radius has the form

$$Y(x) = (\beta x + \gamma)^{\epsilon}, \tag{3.26}$$

where β, γ, and ϵ are constants, which generated new models for different values of ϵ.

In our case we use equations (3.26) and (3.19) to find a new B. By setting

$$z = \beta x + \gamma, \tag{3.27}$$

we find that

$$D_t = -\frac{1}{a} D_x = -\frac{\beta}{a} D_z. \tag{3.28}$$

Then equation (3.19) becomes

$$D_z = -\frac{a}{2\beta} z^{-2\epsilon} D^2 - \frac{a}{2\beta}\left[\lambda z^{2\epsilon} + l^2 z^{-2\epsilon} - 1\right]. \tag{3.29}$$

which is nonlinear in D.

We introduce the transformation

$$\xi = z^{1-2\epsilon}, \tag{3.30}$$

so that equation (3.29) becomes

$$D_\xi = -\frac{a}{2\beta(1-2\epsilon)}D^2 - \frac{a}{2\beta(1-2\epsilon)}\left[\lambda\xi^{\frac{4\epsilon}{1-2\epsilon}} - \xi^{\frac{2\epsilon}{1-2\epsilon}} + l^2\right], \tag{3.31}$$

which is a Riccati equation in D. This equation is very difficult to solve in general. However, particular solutions exist. Ivanov (2019a) considered the case when $\epsilon \neq \frac{1}{2}$, $\lambda = l = 0$, and found a family of solutions in terms of Bessel functions.

It is more difficult to find exact solutions when $\lambda \neq 0$ and $l \neq 0$. However we can demonstrate the existence of a family of solutions for particular parameter values. We consider a special case of equation (3.31) by choosing a particular value of ϵ with $\lambda \neq 0$ and $l \neq 0$. If we let $\epsilon = \frac{1}{4}$ and $\beta = -a$, equation (3.31) becomes

$$D_\xi = D^2 + \lambda\xi^2 - \xi + l^2. \tag{3.32}$$

Equation (3.32) is a nonlinear first order equation. We can transform (3.32) to a second order linear equation by making the substitution

$$D = -\frac{U_\xi}{U}. \tag{3.33}$$

This leads to an equation of the form

$$U_{\xi\xi} + \left(\lambda\xi^2 - \xi + l^2\right)U = 0. \tag{3.34}$$

We now make the substitution

$$U = Ve^{s\xi^2}, \tag{3.35}$$

where s is a root of the quadratic equation $4s^2 + \lambda = 0$. This leads to an equation of the form

$$V_{\xi\xi} + (4s\xi)\,V_\xi + \left(2s + l^2 - \xi\right)V = 0. \tag{3.36}$$

The solution to equation (3.36) is given by

$$\begin{aligned} V &= e^{\xi/4s} W(x), &\quad (3.37a)\\ W(x) &= J\left(\frac{\frac{1}{16s^2}+2s+l^2}{8s}, \frac{1}{2}, -2sx^2\right), &\quad (3.37b)\end{aligned}$$

where $x = \xi + \frac{1}{8s^2}$. The quantity $J\left(\frac{\frac{1}{16s^2}+2s+l^2}{8s}, \frac{1}{2}, -2sx^2\right)$ is an arbitrary solution of the degenerate hypergeometric equation (see Example 2.1.2-70 in Polyanin (1995) for details). Hence the function D in (3.33) can easily be obtained with the help of (3.35) and (3.37). The solution will be given in terms of special functions. Hence existence of exact solutions to equation (3.31) with $\lambda \neq 0$ and $l \neq 0$ has been established. Note for this class of solutions $\lambda = -4s^2$ so that the cosmological constant has to be negative. Negative λ in stellar models has shown to be important in studying quantization of scalar fields in curved backgrounds, interactions with gravity, and research related to global hyperbolicity in manifolds. Astefanesei and Radu (2003) have studied boson stars with negative cosmological constant and several nontrivial solutions were identified. Other recent studies in stellar models involving negative λ in anti-de Sitter spacetimes include Buchel *et al* (2013) and Liebling and Palenzuela (2017). We also include an Appendix which indicates that other solutions arise as special cases of equations (3.29) and (3.31).

3.5 Physical Quantities

It is important to observe that important physical variables such as the mass and compactness factor can written in terms of the generating function. When $\lambda = 0$ Ivanov (2019a) expressed physically relevant fluid characteristics in terms of D and l. The derivative of (3.21) is given by

$$Y_t = \frac{(2D_t+1)DD_t - D^2 D_{tt}}{(2D_t+1)^{3/2}(D^2+l^2)^{1/2}}. \quad (3.38)$$

Then equation (3.21) with (3.18) gives

$$H_t = \frac{D^3 D_{tt} + [D_t(2D_t+1) + DD_{tt}]l^2}{(2D_t+1)^{3/2}(D^2+l^2)^{1/2}}. \tag{3.39}$$

The mass becomes

$$M = -\frac{D(D^2+l^2)^{1/2}D_{tt}}{(2D_t+1)^{3/2}} + \frac{(D_t+1)l^2}{(D^2+l^2)^{1/2}(2D_t+1)^{1/2}}, \tag{3.40}$$

and the compactness parameter yields

$$\frac{2M}{Y} = -\frac{2DD_{tt}}{2D_t+1} + \frac{2(D_t+1)l^2}{(D^2+l^2)^{1/2}}. \tag{3.41}$$

The above expressions take on a simpler form for uncharged matter when $l = 0$.

When $\lambda \neq 0$ it is also possible to express important stellar characteristics in terms of l, λ and D. To achieve this we need to find derivatives of Y and H. For our calculations, we set

$$\Delta = \pm\sqrt{(2D_t+1)^2 - 4\lambda\,(D^2+l^2)}. \tag{3.42}$$

Differentiating equation (3.23), we have

$$Y_t = \frac{1}{2\sqrt{2\lambda}\sqrt{(2D_t+1)\pm\Delta}}\left[2D_{tt} \pm \frac{2(2D_t+1)D_{tt} - 4\lambda DD_t}{\Delta}\right]. \tag{3.43}$$

Using equation (3.18) and (3.23), we obtain

$$H_t = \frac{\sqrt{2\lambda}D}{\sqrt{(2D_t+1)\pm\Delta}} - \frac{\sqrt{\lambda}D}{\sqrt{2}\left[(2D_t+1)\pm\Delta\right]^{3/2}}\left[2D_{tt} \pm \frac{2(2D_t+1)D_{tt} - 4\lambda DD_t}{\Delta}\right]. \tag{3.44}$$

Equations (3.42)-(3.44) apply when $\lambda \neq 0$. We have expressed Y_t and H_t in terms of the generating function D.

The mass of the sphere is given by

$$M = \frac{Y}{2}\left(1 + Y_t^2 - \frac{{Y_r}^2}{B^2}\right) + \frac{l^2}{2Y} - \frac{\lambda Y^3}{6}. \tag{3.45}$$

By applying the horizon function H to equation (3.45) gives

$$\frac{2M}{Y} = 1 - H^2 + 2Y_t H + \frac{l^2}{Y^2} - \frac{\lambda Y^2}{3}. \tag{3.46}$$

Then substituting equation (3.17) in (3.46) gives

$$M=-Y^2H_t+\frac{l^2}{Y}+\frac{1}{3}\lambda Y^2. \tag{3.47}$$

When $\lambda=0$, we observe that (3.47) becomes

$$M=-Y^2H_t+\frac{l^2}{Y}, \tag{3.48}$$

which was first found by Ivanov (2019a). In our case the charge l and cosmological constant λ contribute to the mass in (3.47).

Substituting equations (3.23) and (3.44) in (3.47) gives

$$\begin{aligned} M &= \frac{4\lambda l^2\left[\Delta\pm(D_t-1)\right]\mp 2\lambda D^2\left[(D_t+2)\pm\Delta\right]}{3\sqrt{\lambda}\sqrt{2(2D_t+1)^2-8\lambda(D^2+l^2)}\sqrt{(2D_t+1)\pm\Delta}} \\ &\quad+\frac{3DD_{tt}\left[\Delta\pm(2D_t+1)\right]-(D_t-1)(2D_t+1)\left[\Delta\pm(2D_t+1)\right]}{3\sqrt{\lambda}\sqrt{2(2D_t+1)^2-8\lambda(D^2+l^2)}\sqrt{(2D_t+1)\pm\Delta}}. \end{aligned} \tag{3.49}$$

Hence the mass is given solely in terms of the generating function D.

The compactness parameter $\frac{2M}{Y}$ can be expressed explicitly in terms of D, λ and l. When $\Delta=+\sqrt{(2D_t+1)^2-4\lambda\left(D^2+l^2\right)}$, the compactness parameter is given by

$$\begin{aligned} \frac{2M}{Y} &= \frac{8\lambda l^2\left[(D_t-1)+\Delta\right]-4\lambda D^2\left[(D_t+2)+\Delta\right]}{3\Delta\left[(2D_t+1)+\Delta\right]} \\ &\quad+\frac{6DD_{tt}\left[(2D_t+1)+\Delta\right]-2(D_t-1)(2D_t+1)\left[(2D_t+1)+\Delta\right]}{3\Delta\left[(2D_t+1)+\Delta\right]}. \end{aligned} \tag{3.50}$$

When $\Delta=-\sqrt{(2D_t+1)^2-4\lambda\left(D^2+l^2\right)}$, the compactness parameter is given by

$$\begin{aligned} \frac{2M}{Y} &= \frac{32\lambda^2l^4-16\lambda^2D^4-4(D_t-1)(2D_t+1)^2\left[\Delta-(2D_t+1)\right]}{3\Delta\left[\Delta-(2D_t+1)\right]^2} \\ &\quad+\frac{4\lambda D^2\left[5-3\Delta+D_t(2D_t-3\Delta+11)\right]+12DD_{tt}(2D_t+1)\left[\Delta-(2D_t+1)\right]}{3\Delta\left[\Delta-(2D_t+1)\right]^2} \\ &\quad+\frac{24\lambda D^3D_{tt}+8\lambda l^2\left[1+2\lambda D^2+D_t(3\Delta-8D_t-2)+3DD_{tt}\right]}{3\Delta\left[\Delta-(2D_t+1)\right]^2}. \end{aligned} \tag{3.51}$$

The special cases of uncharged matter with $l=0$ can be regained from (3.50) and (3.51).

When $\lambda \neq 0$, equations (3.49)-(3.51) are written in terms of D, the generating function. A choice of D will generate an exact solution to the boundary condition and produce functional forms for the mass M and the compactness factor $\frac{2M}{Y}$. Clearly the choice for D must be made such that the radiating star satisfies criteria for physical acceptability. Note also that we can easily write M and $\frac{2M}{Y}$ in terms of the horizon function H using $D = HY$ from equation (3.18). The horizon function is directly related to the redshift, formation of the horizon and black holes at the end of the collapse process. The role of H has been emphasized in the works of Ivanov (2016a, 2019a, 2019b).

3.6 Discussion

The particular constraint of zero acceleration was investigated in this chapter to study a collapsing star in the presence of the electromagnetic field l and the cosmological constant λ. Both the cosmological constant and electric charge have a great impact on the nature of the boundary condition. We explored two methods of finding new classes of exact solutions to the boundary condition. Firstly, we introduced a generating function D, by making use of the horizon function. This allowed us to extend the earlier work of Ivanov (2019a). By using the equation $D = HY$, we were able to transform the boundary condition into an algebraic equation in the potential Y. The algebraic equation was solved for two cases; when $\lambda = 0$ which regained the results obtained by Ivanov (2019a), and for $\lambda \neq 0$ which was a new result. All the gravitational potentials could then be expressed in terms of the generating function. By specifying a form for the generating function, we can find new classes of exact solutions. We showed that it is possible to express physical quantities such as the mass function M and the compactness parameter $\frac{2M}{Y}$, explicitly, in terms of the generating function D, the cosmological constant λ and the electric charge l. This is similar to the study of Ivanov

(2019a) who expressed important physical quantities in terms of D. Secondly, we expressed the boundary condition as a Riccati equation in terms of the generating function D using the Lie group generator of Abebe *et al* (2014a). It is not possible to solve this Riccati equation in general. By placing a constraint on a parameter we were able to transform the Riccati equation into a second order differential equation. This second order differential equation was solved exactly to find a new class of travelling wave solutions which are expressed in terms of confluent hypergeometric functions.

3.7 Appendix

In section 3.4 we presented an exact solution which was expressed in terms of special functions. We can also find other solutions to (3.29) and (3.31). We consider some examples here.

Example 1: $\lambda = 0,\ l \neq 0$

By setting $\lambda = 0$ and $l \neq 0$, (3.31) becomes

$$D_\xi = -\frac{a}{2\beta(1-2\epsilon)}D^2 + \frac{a}{2\beta(1-2\epsilon)}\xi^{\frac{2\epsilon}{1-2\epsilon}} - \frac{a}{2\beta(1-2\epsilon)}l^2. \tag{3.52}$$

To use the solution in Polyanin (1995) (see Example 1.2.2-1), we set $\epsilon = \frac{1}{4}$. Then (3.52) becomes

$$D_\xi = -\frac{a}{\beta}D^2 + \frac{a}{\beta}\xi - \frac{a}{\beta}l^2. \tag{3.53}$$

For $\frac{a}{\beta} \neq 0$, the substitution of

$$\frac{a}{\beta}u = \frac{a}{\beta}\xi - \frac{a}{\beta}l^2, \tag{3.54}$$

where $u = u(\xi)$, transforms (3.53) into an equation of the form

$$D_u = -\frac{a}{\beta}D^2 + \frac{a}{\beta}u. \tag{3.55}$$

Then the solution to (3.55) is given by

$$D = \frac{\beta}{a}\frac{V_u}{V}, \tag{3.56}$$

where

$$V(u) = \sqrt{u}\left[C_1 J_{\frac{1}{3}}\left(\frac{2ia}{3\beta}u^{\frac{3}{2}}\right) + C_2 Y_{\frac{1}{3}}\left(\frac{2ia}{3\beta}u^{\frac{3}{2}}\right)\right], \tag{3.57}$$

and $J_{\frac{1}{3}}$ and $Y_{\frac{1}{3}}$ are Bessel functions.

Example 2: $\lambda \neq 0$, $l \neq 0$

We use the software package Mathematica to find a new solution. By setting $\epsilon = \frac{1}{2}$ in equation (3.29), the solution is given by

$$D = \frac{i\left[S_1 U(v_0, v_1, w) + S_2 U(v_2, v_3, w) + S_3 L_{n_1}^{v_4}(w) - S_4 L_{n_2}^{v_1}(w)\right]}{2\beta\left[CU(v_0, v_1, w) + L_{n_1}^{v_4}(w)\right]}, \tag{3.58}$$

where

$$\begin{aligned}
S_1 &= 2\beta C(l - z\sqrt{\lambda}), & S_2 &= zC(ia(1 - 2l\sqrt{\lambda}) - 2\beta\sqrt{\lambda}),\\
S_3 &= 2\beta(l - z\sqrt{\lambda}), & S_4 &= 2\beta(2z\sqrt{\lambda}),\\
v_0 &= \frac{2ial + 2\beta - \frac{ia}{\sqrt{\lambda}}}{4\beta}, & v_1 &= 1 + \frac{ial}{\beta},\\
v_2 &= \frac{2ial + 6\beta - \frac{ia}{\sqrt{\lambda}}}{4\beta}, & v_3 &= 2 + \frac{ial}{\beta},\\
v_4 &= \frac{ial}{\beta}, & w &= \frac{iaz\sqrt{\lambda}}{\beta},\\
n_1 &= -v_0, & n_2 &= -v_2.
\end{aligned}$$

In the above $L_{n_i}^{v_j}(w)$ is the generalized Laguerre polynomial, $U(v_i, v_j, w)$ is the confluent hypergeometric function and C is the integration constant.

Chapter 4

A family of Riccati equations for radiating matter

4.1 Introduction

Radiating stellar models have been studied in many different physical situations over the years. The junction conditions at the stellar surface were completed by Santos (1985) who showed that the presence of heat flow must be taken into account so that the interior can match to the exterior Vaidya metric (1951) at the boundary. Particular solutions of the boundary condition have been studied in astrophysical settings including recent investigations of Naidu et el (2018), Govender *et al* (2017), Sharma *et al* (2015) and Pretel and da Silva (2019). A systematic approach to investigating this problem is to use the Lie analysis of infinitesimal symmetry generators. Abebe *et al* (2013, 2014a) used the Lie approach to study conformally flat and geodesic stars respectively. Stars with shear-free matter distributions were analysed by Abebe *et al* (2015). The interesting physical case of Euclidean stars, containing Newtonian stars in the limit, using the Lie analysis was studied by Govinder and Govender (2012) and

Abebe *et al* (2014b). The results for matter which are expanding, accelerating and shearing are contained in the works of Mohanlal *et al* (2016) using the Lie method of symmetry generators.

A second approach to solving the boundary condition at the stellar surface is to exploit physical properties such as the formation of horizons. This approach was first suggested by Ivanov (2016a) for anisotropic spherical collapse for geodesic particles with shear. Later Ivanov (2016b) extended this method to accelerating matter with shear and radiation. Recently generating functions in the presence of the electromagnetic field, with the horizon function, were considered by Ivanov (2019a, 2019b). Maharaj *et al* (2016) and Mohanlal *et al* (2017) showed that Riccati equations arise in the presence of the horizon function using the Lie approach. Several new families of exact solutions were obtained with the horizon function.

In this chapter we consider the boundary condition at the stellar surface for a general spherically symmetric line element. The horizon function of Ivanov (2016a) is utilized to reduce the Einstein-Maxwell system to generate a single evolution equation. The matter field is generalized to include the cosmological constant and the electromagnetic field. Our intention is to investigate the qualitative differences that arise by these additions. The boundary condition that arises is a generalization of a Riccati equation first derived by Thirukkanesh *et al* (2012a). We find that the addition of charge and cosmological constant to the Einstein field equations leads to several families of exact solutions to the Riccati equations that arise. Our analysis indicates that Riccati equations are the fundamental equations that determine the evolution of the boundary for a bounded radiating system such as a star. The presence of charge and the cosmological constant lead to several new Riccati equations.

In section 4.2, we present the geodesic model of a radiating star, the corresponding Einstein field equations with the cosmological constant, electric charge, and the boundary conditions at the surface. The boundary condition is expressed as a Riccati equation in one of the metric potentials. Section 4.3 introduces three cases for the Riccati equation. Various methods are employed so that the Riccati equation can be integrated to yield new classes of exact solutions. In section 4.4, we introduce the horizon function and the transformed form of the Riccati equation. Various techniques are used to find new classes of exact solutions corresponding to the transformed Riccati equation. Concluding remarks are given in section 4.5. Two tables showing new gravitational potentials and previous results are also provided.

4.2 The model

The metric for a general spherically symmetric matter distribution, in our case describing a radiating star, can be written as

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.1}$$

where $A = A(r,t)$, $B = B(r,t)$ and $Y = Y(r,t)$ are functions that describe gravity. The energy momentum tensor for the matter distribution is

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \epsilon_{ab} + E_{ab}, \tag{4.2}$$

where μ is the energy density, p denotes isotropic pressure, q_a is the heat flow, ϵ_{ab} is the anisotropic stress tensor and E_{ab} denotes the electromagnetic field. The radial and tangential pressures, $p_\parallel$ and $p_\perp$ respectively, can be written in terms of the isotropic pressure by $p = \frac{1}{3}\left(p_\parallel + 2p_\perp\right)$. The electromagnetic energy tensor is defined as

$$E_{ab} = \frac{1}{4\pi}\left[F_a{}^c F_{bc} - \frac{1}{4}F^{cd}F_{cd}g_{ab}\right], \tag{4.3}$$

where F_{ab} is the Faraday tensor. For charged matter with a cosmological constant λ the Einstein-Maxwell equations are given by

$$\begin{aligned} G_{ab} + \lambda g_{ab} &= T_{ab}, && (4.4a) \\ F_{ab;c} + F_{bc;a} + F_{ca;b} &= 0, && (4.4b) \\ F^{ab}{}_{;c} &= \frac{1}{4\pi} J^a, && (4.4c) \end{aligned}$$

where $J^a = \kappa u^a$ is the four-current and κ is the proper charge density.

We can write the Einstein-Maxwell equations as a system of coupled equations for the metric (4.1) and the energy momentum tensor (4.2). The field equations (4.4) then reduce to

$$\begin{aligned} 8\pi\mu + \frac{l^2}{Y^4} &= \frac{1}{A^2}\left(2\frac{B_t}{B}\frac{Y_t}{Y} + \frac{{Y_t}^2}{Y^2}\right) - \frac{1}{B^2}\left(2\frac{Y_{rr}}{Y} + \frac{Y_t^2}{Y^2} - 2\frac{B_r}{B}\frac{Y_r}{Y} - \frac{B^2}{Y^2}\right) - \lambda, && (4.5a) \\ 8\pi p_{\parallel} - \frac{l^2}{Y^4} &= \frac{1}{A^2}\left(-2\frac{Y_{tt}}{Y} - \frac{Y_t^2}{Y^2} + 2\frac{A_t}{A}\frac{Y_t}{Y}\right) + \frac{1}{B^2}\left(\frac{Y_r^2}{Y^2} + 2\frac{A_r}{A}\frac{Y_r}{Y}\right) - \frac{1}{Y^2} + \lambda, && (4.5b) \\ 8\pi p_{\perp} + \frac{l^2}{Y^4} &= -\frac{1}{A^2}\left(\frac{B_{tt}}{B} - \frac{A_t}{A}\frac{B_t}{B} + \frac{B_t}{B}\frac{Y_t}{Y} - \frac{A_t}{A}\frac{Y_t}{Y} + \frac{Y_{tt}}{Y}\right) \\ &\quad + \frac{1}{B^2}\left(\frac{A_{rr}}{A} - \frac{A_r}{A}\frac{B_r}{B} + \frac{A_r}{A}\frac{Y_r}{Y} - \frac{B_r}{B}\frac{Y_r}{Y} + \frac{Y_{rr}}{Y}\right) + \lambda, && (4.5c) \\ 8\pi q &= -\frac{2}{AB}\left(-\frac{Y_{rt}}{Y} + \frac{B_t}{B}\frac{Y_r}{Y} + \frac{A_r}{A}\frac{Y_t}{Y}\right), && (4.5d) \\ \kappa &= \frac{l_r}{BY^2}, && (4.5e) \end{aligned}$$

where $l = l(r)$ is the total charge within the stellar boundary. When $l = 0$ and $\lambda = 0$ we regain the equations of Thirukkanesh *et al* (2012a). The presence of cosmological constant $\lambda \neq 0$ and the electromagnetic charge $l \neq 0$ affects the dynamics of the gravitating model. Exact solutions of the system (4.5) will be very different from the results in earlier investigations.

The surface of spherically bound radiating matter is the boundary that separates the interior matter from the exterior spacetime. The spherically symmetric spacetime (4.1) must match, at the boundary, to the exterior radiating metric. We take the exterior metric to be

$$ds^2 = -\left(1 - \frac{2m(v)}{R} + \frac{Q^2}{R^2} - \frac{1}{3}\lambda R^2\right) dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.6}$$

where $m(v)$ denotes the mass of the star and Q is the total charge measured by an observer at infinity. The metric (4.1) is the Vaidya spacetime (1951) generalized to include the cosmological constant λ and charge l. We must match the line elements (4.1) and (4.6), and also the extrinsic curvature at the surface Σ of the star. This gives the junction conditions

$$Adt = \left[1 - 2\frac{m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv} + \frac{Q^2}{{R_\Sigma}^2} - \frac{1}{3}\lambda {R_\Sigma}^2\right]^{1/2} dv, \tag{4.7a}$$

$$(Y)_\Sigma = R_\Sigma(v), \tag{4.7b}$$

$$l_\Sigma = Q, \tag{4.7c}$$

$$m(v) = \left[\frac{Y}{2}\left(1 + \frac{{Y_t}^2}{A^2} - \frac{{Y_r}^2}{B^2}\right) + \frac{l^2}{2Y} - \frac{\lambda Y^3}{6}\right]_\Sigma, \tag{4.7d}$$

$$(p_\|)_\Sigma = (q)_\Sigma, \tag{4.7e}$$

at the hypersurface Σ. Note the important condition that the radial pressure $p_\|$ is nonvanishing at the surface Σ. The boundary condition (4.7e) can be written as the nonlinear differential equation

$$\begin{aligned} B_t = & \left(\frac{Y_{tt}}{AY_r} + \frac{{Y_t}^2}{2AYY_r} + \frac{A}{2YY_r} - \frac{A_tY_t}{A^2Y_r} - \frac{\lambda AY}{2Y_r} - \frac{l^2A}{2Y^3Y_r}\right)B^2 \\ & + \left(\frac{Y_{rt}}{Y_r} - \frac{Y_tA_r}{Y_rA}\right)B - \frac{A}{2}\left(\frac{2A_r}{A} + \frac{Y_r}{Y}\right). \end{aligned} \tag{4.8}$$

We can interpret (4.8) as a Riccati equation in B. The presence of cosmological constant λ and electric charge l substantially changes the form and nature of the boundary condition $p_\| = q$ on Σ. Thirukkanesh *et al* (2012a) found new classes of exact solutions for $\lambda = 0$ and $l = 0$ by transforming (4.8) to simpler forms. Ivanov (2016a, 2019a) also studied (4.8) using a particular transformation related to the formation of horizons, the so-called horizon function, to obtain a generating function without the presence of the cosmological constant. For our investigation, we will consider (4.8) in general with $\lambda \neq 0$ and $l \neq 0$.

4.3 New solutions: Riccati equation

Equation (4.8) has been studied before without the presence of the cosmological constant and electric charge. In our investigation, we consider how these parameters change the nature of the boundary condition by comparing previous results with new classes of exact solutions. We find that the parameters l and λ change the nature of the families of exact solutions that are possible. These parameters affect the gravitational dynamics of the collapsing star. The various solutions to the Riccati equation found are listed in Table 4.1.

4.3.1 Linear equation

We start by setting the coefficient of B^2 in (4.8) to zero which gives

$$A_t - \left[\frac{Y_{tt}}{Y_t} + \frac{Y_t}{2Y}\right] A = \left[\frac{1}{2YY_t} - \frac{\lambda Y}{2Y_t} - \frac{l^2}{2Y^3 Y_t}\right] A^3, \tag{4.9}$$

which is a Bernoulli equation in A. Solving (4.9) gives

$$A = \frac{\sqrt{3} Y Y_t}{\sqrt{\lambda Y^4 - 3l^2 - 3Y^2 + 3Y f(r)}}, \tag{4.10}$$

where $f(r)$ is the function of integration. Substituting (4.10) into (4.8) yields

$$B_t - \left(\frac{Y_{rt}}{Y_r} - \frac{Y_t A_r}{Y_r A}\right) B + \frac{A}{2}\left(\frac{2A_r}{A} + \frac{Y_r}{Y}\right) = 0, \tag{4.11}$$

which makes this a linear equation in B. The solution to (4.11) is given by

$$B = Y_r \exp\left[-\int \frac{Y_t A_r}{Y_r A} dt\right] \left(g(r) - \int \left(\frac{A_r}{Y_r} + \frac{A}{2Y}\right) \exp\left[\int \frac{Y_t A_r}{Y_r A} dt\right] dt\right), \tag{4.12}$$

where $g(r)$ is the integration constant, and we have

$$Y = Y(r,t), \tag{4.13}$$

as an arbitrary function. When $l = \lambda = 0$ we regain the results of Thirukkanesh *et al* (2012a). A particular choice for the potential Y will lead to explicit forms for the potential A and B.

4.3.2 Bernoulli equation

By setting the inhomogeneous term in (4.8) to zero we get

$$\frac{A}{2}\left(\frac{2A_r}{A} + \frac{Y_r}{Y}\right) = 0, \tag{4.14}$$

which yields

$$Y = \frac{f(t)}{A^2}, \tag{4.15}$$

where $f(t)$ is the integration constant. With the help of (4.15), equation (4.8) becomes

$$\begin{aligned} & B_t - \left[\frac{3f_t}{2f} - \frac{4A_t}{A} + \frac{A_{rt}}{A_r}\right] B \\ = & \left[\frac{\lambda A^2}{4A_r} + \frac{A^{10}l^2}{4f^4A_r} + \frac{7f_tA_t}{2AA_rf} - \frac{5{A_t}^2}{A^2A_r} - \frac{f_{tt}}{2fAr} + \frac{A_{tt}}{AA_r} - \left(\frac{{f_t}^2 + A^6}{4f^2A_r}\right)\right] B^2. \end{aligned} \tag{4.16}$$

Equation (4.16) is a Bernoulli equation in B which can be solved to give

$$B = \frac{f^{3/2}A_r}{A^4\left[\int Qdt + g(r)\right]}, \tag{4.17}$$

where

$$Q = -\frac{\lambda f^{3/2}}{4A^2} - \frac{A^6l^2}{4f^{5/2}} - \frac{7\sqrt{f}f_tA_t}{2A^5} + \frac{5f^{3/2}{A_t}^2}{A^6} + \frac{\sqrt{f}f_{tt}}{2A^4} - \frac{f^{3/2}A_{tt}}{A^5} + \frac{{f_t}^2}{4A^4\sqrt{f}} + \frac{A^2}{4\sqrt{f}},$$

and the constant of integration is $g(r)$. For this class of solutions we have

$$A = A(r,t), \tag{4.18}$$

as an arbitrary function. When $l = \lambda = 0$, we regain the results of Thirukkanesh *et al* (2012a) as a special case. If A is specified then we can obtain explicit forms for the potentials B and Y.

4.3.3 Inhomogeneous Riccati equation

Particular solutions to the Riccati equation (4.8) do exist but the presence of the parameters l and λ lead to complications. Setting the coefficient of B to zero in (4.8) gives

$$\frac{Y_{rt}}{Y_r} - \frac{Y_t A_r}{Y_r A} = 0, \tag{4.19}$$

which can be integrated to give

$$A = f(t) Y_t, \tag{4.20}$$

where $f(t)$ is the constant of integration. By substituting (4.20) into (4.8) we obtain

$$B_t = \left[\frac{Y_t}{2fYY_r} + \frac{fY_t}{2YY_r} - \frac{fl^2 Y_t}{2Y^3 Y_r} - \frac{\lambda f Y Y_t}{2Y_r} - \frac{f_t}{f^2 Y_r} \right] B^2 - \left[\frac{fY_r Y_t}{2Y} + fY_{rt} \right]. \tag{4.21}$$

Equation (4.21) is not integrable as it is presented. If we set $\lambda = l = 0$ in equation (4.21), we can regain the equations of Thirukkanesh *et al* (2012a). They presented an exact solution to equation (4.21) by assuming that the potential $Y(r,t)$ is a separable function. Unfortunately this approach does not work in (4.21). It is not possible to integrate (4.21) in general; particular solutions exist under certain assumptions. We demonstrate this below.

An interesting possibility arises if $Y_t = 0$. Then (4.19) is identically satisfied but the relation in (4.20) does not hold. If we let $Y_t = 0$ in equation (4.8) then the boundary condition becomes

$$B_t = \frac{A}{2Y_r} \left[\frac{1}{Y} - \lambda Y - \frac{l^2}{Y^3} \right] B^2 - \frac{A}{2} \left[\frac{2A_r}{A} + \frac{Y_r}{Y} \right], \tag{4.22}$$

which is a simpler Riccati equation in B. We observe that on setting $A = A(r)$ in equation (4.22), we can rewrite (4.22) as

$$B_t = f(r) B^2 + g(r), \tag{4.23}$$

where

$$f(r) = \frac{A}{2Y_r}\left[\frac{1}{Y} - \lambda Y - \frac{l^2}{Y^3}\right], \tag{4.24a}$$

$$g(r) = -\frac{A}{2}\left[\frac{2A_r}{A} + \frac{Y_r}{Y}\right]. \tag{4.24b}$$

An explicit solution to (4.23) can be found if we let

$$\alpha f(r) = g(r), \tag{4.25}$$

where α is the proportionality constant. Equation (4.25) implies

$$A_r = \left(\frac{\alpha}{2Y_r}\left[\lambda Y + \frac{l^2}{Y^3} - \frac{1}{Y}\right] - \frac{Y_r}{2Y}\right) A, \tag{4.26}$$

which is a separable equation in A. Integrating (4.26) gives

$$A = C_1 \exp\left[\int \left(\frac{\alpha}{2Y_r}\left[\lambda Y + \frac{l^2}{Y^3} - \frac{1}{Y}\right] - \frac{Y_r}{2Y}\right) dr\right], \tag{4.27}$$

where C_1 is the integration constant. By using equation (4.25), we can integrate (4.23) which yields two solutions for B. The first solution is given by

$$B = \sqrt{\alpha}\tan[\sqrt{\alpha}(f(r)t + C_2(r))], \qquad \alpha > 0, \tag{4.28}$$

where $C_2(r)$ is the constant of integration. The second solution is

$$B = \frac{\alpha}{\sqrt{-\alpha}}\left[\frac{C_3(r)\exp[2\sqrt{-\alpha}f(r)t] - 1}{C_3(r)\exp[2\sqrt{-\alpha}f(r)t] + 1}\right], \qquad \alpha < 0, \tag{4.29}$$

where $C_3(r)$ is the constant of integration. For this class of solutions

$$Y = Y(r), \tag{4.30}$$

which is an arbitrary function.

In the above solution the potential A is given in (4.27); the potential B contains A and Y through the function $f(r)$ and $g(r)$, and $Y(r)$ is arbitrary. This class of models obtained by integrating the Riccati equation is new. Observe that if $\lambda = l = 0$ the the potential A becomes

$$A = C_1 \exp\left[-\int \left(\frac{\alpha}{2YY_r} + \frac{Y_r}{2Y}\right) dr\right]. \tag{4.31}$$

We observe that this uncharged model is also a new solution to the Riccati equation and is not contained in earlier treatments.

4.4 New solutions: Transformed Riccati equation

Particular transformations reduce the boundary condition (4.8) to simpler forms. The horizon function was introduced by Ivanov (2016a) which gives a relation between the gravitational potentials and horizon formation. The transformation is given by

$$H = \frac{Y_r}{B} + \frac{Y_t}{A}, \tag{4.32}$$

where $H = H(r,t)$. The horizon function is used as a transformation to simplify equation (4.8) which becomes

$$H_t = \left[\frac{A}{2Y} + \frac{A_r}{Y_r}\right] H^2 - \left[\frac{A_r Y_t}{AY_r} + \frac{Y_t}{Y}\right] H + \frac{A}{2Y^3}\left[l^2 + \lambda Y^4 - Y^2\right]. \tag{4.33}$$

This is also a Riccati equation in H. We employ a similar method used to solve (4.8) in the previous section to integrate (4.33) for which we find that three cases arise. The various solutions are listed in Table 4.2.

4.4.1 Linear equation

We start by setting the coefficient of H^2 to zero so that

$$\frac{A}{2Y} + \frac{A_r}{Y_r} = 0. \tag{4.34}$$

This can be solved to give

$$A = \frac{f(t)}{\sqrt{Y}}, \tag{4.35}$$

where $f(t)$ is constant of integration. By substituting (4.35) into (4.33) we get

$$H_t + \left[\frac{Y_t}{2Y}\right] H = \frac{f}{2Y^{7/2}} \left[l^2 + \lambda Y^4 - Y^2\right], \tag{4.36}$$

which is a linear equation in H. We can solve this easily to give

$$H = \frac{1}{\sqrt{Y}} \left[\int \frac{f}{2Y^3} \left(l^2 + \lambda Y^4 - Y^2\right) dt + g(r)\right], \tag{4.37}$$

where $g(r)$ is the integration constant. By using equation (4.32) we can find the metric function B to be

$$B = \frac{f\sqrt{Y} Y_r}{f\left[\int \frac{f}{2Y^3} \left(l^2 + \lambda Y^4 - Y^2\right) dt + g(r)\right] - Y Y_t}. \tag{4.38}$$

For this class of solutions

$$Y = Y(r,t), \tag{4.39}$$

is arbitrary. Note that this class of exact solutions is different from the linear solutions given in section 4.3. The horizon function transformation (4.32) leads to a new class of exact models to the boundary condition (4.8).

4.4.2 Bernoulli equation

We can transform (4.33) into a Bernoulli equation in H. The condition for obtaining a Bernoulli equation is that

$$\lambda Y^4 - Y^2 + l^2 = 0, \tag{4.40}$$

which is a quadratic equation in Y. This condition is satisfied only if $Y = Y(r)$. Unfortunately this condition is inconsistent with the transformation (4.32) which requires $Y_t \neq 0$. Hence it is not possible to generate Bernoulli equations with the horizon function. This is in contrast with the results of section 4.3. We conclude that the transformation (4.32) restricts the number of solutions that are admissible.

4.4.3 Inhomogeneous Riccati equation

The Riccati equation (4.33) in H cannot be solved in general. By setting the coefficient of H to zero gives

$$\frac{A_r Y_t}{A Y_r} + \frac{Y_t}{Y} = 0. \tag{4.41}$$

This is a separable equation that can be solved to yield

$$A = \frac{f(t)}{Y}, \tag{4.42}$$

where $f(t)$ is the integration constant. Substituting (4.42) into (4.33), we get

$$H_t = -\left[\frac{f}{2Y^2}\right] H^2 + \frac{f}{2Y^4}\left[l^2 + \lambda Y^4 - Y^2\right], \tag{4.43}$$

which is a simpler Riccati equation in H. It is difficult to solve (4.43) to obtain an expression for H. However it is possible to express equation (4.43) in the form

$$(f\lambda - 2H_t)Y^4 - f(H^2+1)Y^2 + fl^2 = 0. \tag{4.44}$$

We can treat (4.44) as an algebraic equation in the variable Y. Four cases arise in the solution of (4.44). We consider these in turn. The various cases are listed in Table 4.2.

4.4.3.1 Case 1: $\lambda \neq 0$, $l \neq 0$

In this case we can solve (4.44) to get

$$Y = \sqrt{\frac{f(H^2+1) \pm \sqrt{f^2(H^2+1)^2 - 4(f\lambda - 2H_t)fl^2}}{2(f\lambda - 2H_t)}}, \tag{4.45}$$

where $H > \left[2l\sqrt{\frac{f\lambda-2H_t}{f}} - 1\right]^{1/2}$ and $H_t < \frac{1}{2}f\lambda$. Hence we have a new family of exact solutions to the Riccati equation (4.43). In this class we observe that the potentials A and Y are given in terms of the horizon function H. The function $H(r,t)$ is arbitrary. Here the quantity H plays the role of a generating function; a particular choice of H will lead to analytical forms for A and Y.

In spite of the complicated form of the potential Y in (4.45) it is possible to regain the potential B explicitly. By substituting (4.45) into (4.42) and (4.32), we obtain

$$B = \frac{2H_{rt}(f(H^2+1) \pm K_4) + (f\lambda - 2H_t)K_1}{2\sqrt{2(f\lambda-2H_t)[f(H^2+1)\pm K_4]}\left[H(f\lambda-2H_t) - \frac{1}{4f}K_2 \pm \frac{1}{4f(f\lambda-2H_t)}K_3\right]}, \tag{4.46}$$

where

$$\begin{aligned}
K_1 &= 2fHH_r \pm \frac{2f^2[(H+H^3)H_r - 2\lambda l l_r] + 4fl[2l_rH_t + lH_{rt}]}{K_4}, \\
K_2 &= (H^2+1)f_t + 2fHH_t \pm \frac{1}{K_4}\left(2f^2HH_t(H^2+1) + 4l^2f_tH_t + ff_t[(H^2+1)^2 - 4\lambda l^2] + 4fl^2H_{tt}\right), \\
K_3 &= \left[K_4 \pm f(H^2+1)\right](\lambda f_t - 2H_{tt}), \\
K_4 &= \sqrt{f^2(H^2+1)^2 - 4(f\lambda - 2H_t)fl^2}.
\end{aligned}$$

In these expressions the role of the horizon function H as a generating function is highlighted.

4.4.3.2 Case 2: $\lambda = 0$, $l \neq 0$

Solving (4.44) gives two solutions for

$$Y = \sqrt{\frac{f(H^2+1) \pm \sqrt{f^2(H^2+1)^2 + 8fl^2H_t}}{-4H_t}}, \tag{4.47}$$

where $H > \left[2l\sqrt{\frac{-2H_t}{f}} - 1\right]^{1/2}$ and $H_t < 0$. By substituting (4.47) into (4.42) and (4.32) the first solution, due to a positive inner root, is given by

$$B = \frac{2f^2 {H_t}^2 \left[4l^2 H_t H_{rt} - 8ll_r {H_t}^2 + \tilde{K_1}\right]}{\tilde{K_3}\sqrt{-{H_t}^3[f(H^2+1) + \tilde{K_3}]}\left[8f{H_t}^2 H - H_{tt}(f(H^2+1) + \tilde{K_3}) + H_t\tilde{K_2}\right]}, \tag{4.48}$$

where

$$\begin{aligned}
\tilde{K_1} &= (\tilde{K_3} + f(H^2+1))\left[H_{rt}(H^2+1) - 2HH_tH_r\right],\\
\tilde{K_2} &= f_t(H^2+1) + 2fHH_t + \frac{1}{\tilde{K_3}}\left(2f^2HH_t(H^2+1) + 4l^2f_tH_t + ff_t(H^2+1)^2 + 4fl^2H_{tt}\right),\\
\tilde{K_3} &= \sqrt{f^2(H^2+1)^2 + 8fl^2H_t}.
\end{aligned}$$

The second solution, due to a negative inner root, is given by

$$B = \frac{16f^2ll_r\sqrt{{H_t}^5} - 8f^2l^2\sqrt{{H_t}^3}H_{rt} + \overline{K_1}}{\sqrt{\overline{K_4} - f(H^2+1)}\left[f_tH_t(H^2+1)\overline{K_4} - 4l^2f_t{H_t}^2 + \overline{K_2} + \overline{K_3}\right]}, \tag{4.49}$$

where

$$\begin{aligned}
\overline{K_1} &= 2f^2\sqrt{H_t}(\overline{K_4} - f(H^2+1))\left[(H^2+1)H_{rt} - 2HH_tH_r\right],\\
\overline{K_2} &= f^2(H^2+1)\left[(H^2+1)H_{tt} - 2H{H_t}^2\right],\\
\overline{K_3} &= 10fH{H_t}^2\overline{K_4} - f\overline{K_4}H_{tt}(H^2+1) + fH_t\left[4l^2H_{tt} - f_t(H^2+1)^2\right],\\
\overline{K_4} &= \sqrt{f^2(H^2+1)^2 + 8fl^2H_t}.
\end{aligned}$$

The variable H can be used as a generating function for both solutions in (4.48) and (4.49).

4.4.3.3 Case 3: $\lambda \neq 0$, $l = 0$

Another case occurs when we set $l = 0$ and $\lambda \neq 0$ in equation (4.44). Then solving (4.44) gives

$$Y = \sqrt{\frac{f(H^2+1)}{f\lambda - 2H_t}}. \tag{4.50}$$

where $f(H^2+1) > 0$ and $H_t < \frac{f\lambda}{2}$. On substituting (4.50) into (4.42) and (4.32) yields

$$B = \frac{f^2\sqrt{f\lambda - 2H_t}\left[HH_r(f\lambda - 2H_t) + H_{rt}(H^2+1)\right]}{\sqrt{f(H^2+1)}\left[\lambda^2 f^3 H - 5\lambda f^2 H H_t + f_t H_t(H^2+1) + 6fHH_t{}^2 - fH_{tt}(H^2+1)\right]}. \quad (4.51)$$

For this class of solutions, we can take H to be a generating function.

4.4.3.4 Case 4: $\lambda = 0$, $l = 0$

For this case we obtain the solution

$$Y = \sqrt{-\frac{f(H^2+1)}{2H_t}}, \quad (4.52a)$$

$$B = \frac{\sqrt{2H_t}f^2\left[H_{rt}(H^2+1) - 2HH_rH_t\right]}{\sqrt{-f(H^2+1)}\left[fH_{tt}(H^2+1) - H_t(f_t(H^2+1) + 6fHH_t)\right]}, \quad (4.52b)$$

where $f(H^2+1) < 0$ and $H_t \neq 0$. This solution corresponds to an uncharged model and there is no cosmological constant. This is also a new exact solution to the boundary condition (4.33). A simple exact explicit solution to the Riccati equation in (4.8) was found by Thirukkanesh *et al* (2012a). In their model the potentials A and Y are separable functions. In our case the Ivanov transformation (4.32) leads to another family of solutions and the potentials $A = \frac{f(t)}{Y}$, B and Y have different functional forms. The quantity H serves as a generating function in our case.

4.5 Discussion

This chapter consists of two important approaches that discuss the boundary condition for a radiating star in general relativity, with cosmological constant and electric charge. We first studied the boundary condition as a Riccati equation in the potential B. This Riccati equation is very difficult to solve in general. However we obtained exact solutions for three cases: linear, Bernoulli and inhomogeneous Riccati. The linear and Bernoulli cases are extensions to the work of Thirukkanesh *et al* (2012a). The inhomogeneous Riccati case is a special case that allows the equation to become a separable equation after making certain assumptions for the potentials A and B. This allows for a simple new class of exact solutions. We next applied a transformation to the boundary condition, which was first introduced by Ivanov (2016a), called the horizon function. This transformation resulted in a simplification of the boundary condition which was then expressed as a Riccati equation in the variable H. Three cases arose when we solved the transformed Riccati equation: linear, Bernoulli and inhomogeneous Riccati. The linear case yields new classes of exact solutions in which the potential Y can be treated as a generating function. The Bernoulli case is disregarded because the horizon function breaks down due to the potential Y being a function of r only. The inhomogeneous Riccati case is characterized by four subcases when $\lambda \neq 0$, $\lambda = 0$, $l \neq 0$ and $l = 0$. Each subcase can be solved explicitly to yield a family of new exact solutions. These solutions are expressed in terms of the potential H which acts as a generating function. Tables 4.1 and 4.2 express the results collectively for both the Riccati and transformed Riccati equations. The solutions obtained in this chapter from the various Riccati equations can be used to study the physical features of the astrophysical model. It is possible to express stellar quantities such as the mass of the star from the junction condition (4.7d) as

$$2M = Y\left(\frac{1}{3} - H^2\right) + \frac{1}{3}\lambda Y^3 + \frac{5}{3}\frac{l^2}{Y}. \tag{4.53}$$

Note that (4.53) becomes equation (38) in Ivanov (2019a) when $\lambda = l = 0$. We also find the time derivative to be

$$2M_t = Y_t\left(\frac{1}{3} - 3H^2\right) + Y_t\left(\lambda Y^2 - \frac{5}{3}\frac{l^2}{Y^2}\right). \tag{4.54}$$

These expressions show how the cosmological constant and electric charge play a role in the mass function and its derivative. The compactness parameter $\frac{2M}{Y}$ can also be obtained using (4.53). These physical quantities are given in terms of the potential H which serves as a generating function. Specific choices of the generating function will permit a detailed physical analysis of the evolution of the star. This is the object of further research.

4.6 Tables of results

Table 4.1: **Exact solutions for the Riccati equation**

Cases	Gravitational potentials	Features
$\lambda \neq 0$ $l \neq 0$	$A = \frac{\sqrt{3} Y Y_t}{\sqrt{\lambda Y^4 - 3l^2 - 3Y^2 + 3Y f(r)}}$ $B = Y_r \exp\left[-\int \frac{Y_t A_r}{Y_r A} dt\right] \left(g(r) - \int \left(\frac{A_r}{Y_r} + \frac{A}{2Y}\right) \exp\left[\int \frac{Y_t A_r}{Y_r A} dt\right] dt\right)$ $Y = Y(r,t)$	New model Explicit solution
$\lambda \neq 0$ $l = 0$	$A = A(r,t)$ $B = \frac{f^{3/2} A_r}{A^4\left[\int Q dt + g(r)\right]}$ $Y = \frac{f(t)}{A^2}$ where $Q = -\frac{\lambda f^{3/2}}{4A^2} - \frac{A^6 l^2}{4 f^{5/2}} - \frac{7\sqrt{f} f_t A_t}{2A^5}$ $+\frac{5 f^{3/2} {A_t}^2}{A^6} + \frac{\sqrt{f} f_{tt}}{2A^4} - \frac{f^{3/2} A_{tt}}{A^5} + \frac{{f_t}^2}{4A^4\sqrt{f}} + \frac{A^2}{4\sqrt{f}}$	New model Explicit solution
$\lambda = 0$ $l \neq 0$	$A = C_1 \exp\left[\int \left(\frac{\alpha}{2Y_r}\left[\lambda Y + \frac{l^2}{Y^3} - \frac{1}{Y}\right] - \frac{Y_r}{2Y}\right) dr\right]$ $B = \sqrt{\alpha} \tan[\sqrt{\alpha}(f(r)t + C_2(r))]$ $Y = Y(r)$	New model Explicit solution
	$A = C_1 \exp\left[\int \left(\frac{\alpha}{2Y_r}\left[\lambda Y + \frac{l^2}{Y^3} - \frac{1}{Y}\right] - \frac{Y_r}{2Y}\right) dr\right]$ $B = \frac{\alpha}{\sqrt{-\alpha}}\left[\frac{C_3(r)\exp[2\sqrt{-\alpha} f(r) t] - 1}{C_3(r)\exp[2\sqrt{-\alpha} f(r) t] + 1}\right]$ $Y = Y(r)$	New model Explicit solution

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Table 4.1: *Continued*

Cases	Gravitational potentials	Features
	$A=\sqrt{\frac{YY_t{}^2}{h(r)-Y}}$	Known model
	$B=Y_r\exp\left(-\int\frac{A_rY_t}{AY_r}dt\right)\left(k(r)-\int\left[\left(\frac{A_r}{Y_r}+\frac{A}{2Y}\right)\exp\left(\int\frac{A_rY_t}{AY_r}dt\right)\right]dt\right)$	Explicit solution
	$Y=Y(r,t)$	
$\lambda=0$		
$l=0$	$A=A(r,t)$	Known model
	$B=\frac{A_rC_1{}^{3/2}}{A^4\left[\int Idt+g(r)\right]}$	Explicit solution
	$Y=\frac{C_1}{A^2}$	
	where $I=-\frac{7C_1{}^{1/2}(C_1)_tA_t}{2A^5}+\frac{5C_1{}^{3/2}A_t{}^2}{A^6}+\frac{C_1{}^{1/2}(C_1)_{tt}}{2A^4}$	
	$-\frac{C_1{}^{3/2}A_{tt}}{A^5}+\frac{C_1{}^2}{4A^4C_1{}^{1/2}}+\frac{A^2}{4C_1{}^{1/2}}$	

Table 4.2: **Exact solutions for the transformed Riccati equation**

Cases	Gravitational potentials	Features
$\lambda \neq 0$, $l \neq 0$	$A = \frac{f(t)}{\sqrt{Y}}$	New model
	$B = \frac{f\sqrt{Y}Y_r}{f\left[\int \frac{f}{2Y^3}(l^2+\lambda Y^4-Y^2)dt+g(r)\right]-YY_t}$	Explicit solution
	$H = \frac{1}{\sqrt{Y}}\left[\int \frac{f}{2Y^3}\left(l^2+\lambda Y^4-Y^2\right)dt+g(r)\right]$	Generating function Y
	$Y = Y(r,t)$	
	$A = \frac{f(t)}{Y}$	New model
	$B = \frac{2H_{rt}(f(H^2+1)\pm K_4)+(f\lambda-2H_t)K_1}{2\sqrt{2(f\lambda-2H_t)[f(H^2+1)\pm K_4]}\left[H(f\lambda-2H_t)-\frac{1}{4f}K_2\pm\frac{1}{4f(f\lambda-2H_t)}K_3\right]}$	Explicit solution
	$Y = \sqrt{\frac{f(H^2+1)\pm\sqrt{f^2(H^2+1)^2-4(f\lambda-2H_t)fl^2}}{2(f\lambda-2H_t)}}$	Generating function H
	$H = H(r,t)$	
$\lambda = 0$, $l \neq 0$	$A = \frac{f(t)}{Y}$	New model
	$B = \frac{2f^2{H_t}^2\left[4l^2H_tH_{rt}-8ll_r{H_t}^2+\tilde{K_1}\right]}{\tilde{K_3}\sqrt{-{H_t}^3[f(H^2+1)+\tilde{K_3}]}\left[8f{H_t}^2H-H_{tt}(f(H^2+1)+\tilde{K_3})+H_t\tilde{K_2}\right]}$	Explicit solution
	$Y = \sqrt{\frac{f(H^2+1)\pm\sqrt{f^2(H^2+1)^2+8fl^2H_t}}{-4H_t}}$	Generating function H
	$H = H(r,t)$	
	$A = \frac{f(t)}{Y}$	New model
	$B = \frac{16f^2ll_r\sqrt{{H_t}^5}-8f^2l^2\sqrt{{H_t}^3}H_{rt}+\bar{K_1}}{\sqrt{\bar{K_4}-f(H^2+1)}\left[f_tH_t(H^2+1)\bar{K_4}-4l^2f_t{H_t}^2+\bar{K_2}+\bar{K_3}\right]}$	Explicit solution
	$Y = \sqrt{\frac{f(H^2+1)\pm\sqrt{f^2(H^2+1)^2+8fl^2H_t}}{-4H_t}}$	Generating function H
	$H = H(r,t)$	

Continued on next page

Table 4.2: *Continued*

Cases	Gravitational potentials	Features
	$A = \frac{f(t)}{Y}$	New model
$\lambda \neq 0$	$B = \frac{f^2\sqrt{f\lambda-2H_t}\left[HH_r(f\lambda-2H_t)+H_{rt}(H^2+1)\right]}{\sqrt{f(H^2+1)}\left[\lambda^2 f^3 H-5\lambda f^2 HH_t+f_t H_t(H^2+1)+6fHH_t{}^2-fH_{tt}(H^2+1)\right]}$	Explicit solution
$l = 0$	$Y = \sqrt{\frac{f(H^2+1)}{f\lambda-2H_t}}$	Generating function H
	$H = H(r,t)$	
	$A = \frac{f(t)}{Y}$	New model
$\lambda = 0$	$B = \frac{\sqrt{2H_t}f^2\left[H_{rt}(H^2+1)-2HH_rH_t\right]}{\sqrt{-f(H^2+1)}[fH_{tt}(H^2+1)-H_t(f_t(H^2+1)+6fHH_t)]}$	Explicit solution
$l = 0$	$Y = \sqrt{-\frac{f(H^2+1)}{2H_t}}$	Generating function H
	$H = H(r,t)$	

Chapter 5

Generalized horizon functions and radiating stars

5.1 Introduction

The boundary condition for a radiating star in general relativity is a fundamental equation to describe the evolution of a radiating star. This equation was first derived by Santos (1985) by matching a shear-free matter distribution to the Vaidya exterior spacetime (1951). The junction condition was later generalized to include the effects of shearing matter and charged matter by de Oliveira and Santos (1987), Banerjee and Choudhary (1989), Tikekar and Patel (1992) and Maharaj and Govender (2000). Exact solutions have since been sought to the boundary condition to obtain insight in to the physical behaviour of gravitating matter and to study the temporal evolution of a collapsing body. Thirukkanesh *et al* (2012a) were the first to observe that the boundary condition can always be expressed as a Riccati equation in one of the potentials irrespective of the matter distribution for a general spherically symmetric metric in the interior of the star. Solutions of the Riccati equation are applicable to geodesic stars (see Abebe *et al* (2014a)), Euclidean stars (see Govinder and Govender (2012)) and shear-free stars (see Abebe *et al* (2015)), amongst others. These investigations relied on the Lie symmetry approach with infinitesimal generators. It is interesting to note that the study of Mohanlal *et al* (2016) with the Lie analysis gives rise to exponential functions in the symmetry generators which increases the complexity of the model.

A deep insight into the geometrical structure of the problem was obtained by realizing that the boundary condition could be simplified by introducing a new transformation. The transformation reduces the complexity of the boundary differential equation and can be related to physical features related to black holes and horizons. Consequently the transformations that arise are called horizon functions. This approach was first proposed by Ivanov (2016a) for a geodesic model in the presence of shear. This work was extended by Ivanov (2016b) with the addition of acceleration. Ivanov (2019a) generalized his earlier works by including the effects of the electromagnetic field with acceleration, shear and expansion. Particular solutions of the differential equation for the boundary condition have led to explicit forms for the horizon function. By using the Lie analysis Mohanlal *et al* (2016) showed how the horizon function preserves the form of the Riccati equation. The use of the horizon function leads to various classes of exact solutions.

The Ivanov horizon function has provided new insights into the behaviour of a collapsing star. This transformation maps the original Riccati equation into a simpler Riccati equation. A natural question is whether the Ivanov transformation is a special case of a more general transformation which leaves the structure of the Riccati equation intact. We show in this investigation that such a transformation exists. This leads to several different Riccati equations, for particular parameter values, and new families of exact solutions. The spacetime geometry is determined by a general spherically symmetric line element. The matter field is generalized to include the effects of charge and the cosmological constant. The cosmological constant is an important component in the matter distribution in general relativity. Several observations involving baryon acoustic oscillations, high redshifts and supernovae favour a nonzero value for the cosmological constant. Other studies involving stellar structures and gravitational collapse have been considered by Andreasson and Bohmer (2009), Rindler and Ishak (2007) and Bohmer and Harko (2005). The electromagnetic field is also an important factor that affects the behaviour of these models. For example, Pinheiro and Chan (2013) showed that the presence of an electric charge can delay black hole formation and prevent gravitational collapse.

In this chapter we study the Einstein-Maxwell field equations with cosmological constant and electric charge due to a generalized horizon function. The boundary condition is generalized to a new form which contains the original form due to Ivanov (2016a). Section 5.2 gives Einstein-Maxwell equations and the junction conditions for a spherical distribution. The boundary condition is expressed as a Riccati equation in one of the metric potentials. Section 5.3 introduces the generalized horizon function which maps the boundary condition from a Riccati equation into a new transformed Riccati equation. This Riccati equation, in terms of the generalized horizon function, becomes the master equation for our analysis. In section 5.4 we solve the master equation for several cases using conditions on the parameters. Several families of new exact solutions are found for both the metric potentials and the horizon function. Concluding remarks are made in section 5.5. We have also included tables for the various models that list known and new solutions.

5.2 The model

We are modelling the evolution of a radiating body in general relativity. The metric for the general spherically symmetric interior geometry is given by

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.1}$$

where $A = A(r,t)$, $B = B(r,t)$ and $Y = Y(r,t)$ are the metric potentials. The fluid four-velocity $\mathbf{u}$ is comoving and can be expressed as $u^a = \frac{1}{A}\delta^a_0$ so that $u^a u_a = -1$. The energy momentum tensor for the body in the interior of the boundary has the form

$$T_{ab} = (\mu + p)u_a u_b + pg_{ab} + q_a u_b + q_b u_a + \epsilon_{ab} + E_{ab}, \tag{5.2}$$

where μ is the energy density, p is the isotropic pressure, q_a is the heat flux vector, ϵ_{ab} is the anisotropic stress tensor and E_{ab} is the electromagnetic field. The radial and tangential pressures, $p_{\|}$ and $p_{\perp}$ respectively, can be expressed in terms of the isotropic pressure by $p = \frac{1}{3}\left(p_{\|} + 2p_{\perp}\right)$. The kinematical quantities $\dot{u}^a = \left(0, \frac{A_r}{AB^2}, 0, 0\right)$, $\Theta = \frac{1}{A}\left(2\frac{Y_t}{Y} + \frac{B_t}{B}\right)$ and $\sigma = \frac{1}{3A}\left(\frac{Y_t}{Y} - \frac{B_t}{B}\right)$, correspond to nonvanishing acceleration $\dot{u}^a$, the expansion scalar Θ and the magnitude of the shear σ, respectively. Subscripts denote differentiation with respect to r and t.

Since the heat flow acts in the radial direction for a radiating body we can write

$$q^a = (0, Bq, 0, 0), \tag{5.3}$$

for the heat flow vector $\mathbf{q}$ and $q^a u_a = 0$. The electromagnetic energy tensor is defined by

$$E_{ab} = \frac{1}{4\pi}\left[F_a{}^c F_{bc} - \frac{1}{4}F^{cd}F_{cd}g_{ab}\right]. \tag{5.4}$$

The Einstein-Maxwell equations are given by

$$\begin{aligned} G_{ab} + \lambda g_{ab} &= T_{ab}, && (5.5a) \\ F_{[ab;c]} &= 0, && (5.5b) \\ F^{ab}{}_{;c} &= \frac{1}{4\pi} J^a, && (5.5c) \end{aligned}$$

where Faraday's tensor $F_{ab} = \nu_{b;a} - \nu_{a;b}$ and the four-current $J^a = \kappa u^a$ in the presence of the cosmological constant λ. Note that κ is the proper charge density and ν_a is the four-potential which is of the chosen form

$$\nu_a = (\Psi(r,t), 0, 0, 0), \tag{5.6}$$

so that $F_{01} = -F_{10} = -\Psi_r$. For this form of ν_a we can reduce (5.5) to the system

$$\begin{aligned} \Psi_{rr} + \left(2\frac{Y_r}{Y} - \frac{B_r}{B} - \frac{A_r}{A}\right)\Psi_r &= \kappa A B^2, && (5.7a) \\ \left(\frac{1}{A^2B^2}\Psi_r\right)_t + \frac{1}{A^2B^2}\left(\frac{A_t}{A} + \frac{B_t}{B}\right)\Psi_r + \left(\frac{2Y_t}{A^2B^2Y}\right)\Psi_r &= 0, && (5.7b) \end{aligned}$$

for the metric (5.1). Integrating (5.7) yields

$$\begin{aligned} \Psi_r &= \frac{ABl}{Y^2}, && (5.8a) \\ l(r) &= 4\pi \int^r \kappa B Y^2 dr, && (5.8b) \end{aligned}$$

where $l = l(r)$ is the total charge contained in the interior of the radiating body.

We can express the field equations (5.5a)-(5.5c), for the metric (5.1) and the energy momentum tensor (5.2), as the coupled system

$$
\begin{aligned}
8\pi\mu + \frac{l^2}{Y^4} &= \frac{1}{A^2}\left(2\frac{B_t}{B}\frac{Y_t}{Y} + \frac{Y_t^2}{Y^2}\right) + \frac{1}{Y^2} - \frac{1}{B^2}\left(2\frac{Y_{rr}}{Y} + \frac{Y_t^2}{Y^2} - 2\frac{B_r}{B}\frac{Y_r}{Y}\right) - \lambda, &(5.9a)\\
8\pi p_{\|} - \frac{l^2}{Y^4} &= \frac{1}{A^2}\left(-2\frac{Y_{tt}}{Y} - \frac{Y_t^2}{Y^2} + 2\frac{A_t}{A}\frac{Y_t}{Y}\right) + \frac{1}{B^2}\left(\frac{Y_r^2}{Y^2} + 2\frac{A_r}{A}\frac{Y_r}{Y}\right) - \frac{1}{Y^2} + \lambda, &(5.9b)\\
8\pi p_{\perp} + \frac{l^2}{Y^4} &= -\frac{1}{A^2}\left(\frac{B_{tt}}{B} - \frac{A_t}{A}\frac{B_t}{B} + \frac{B_t}{B}\frac{Y_t}{Y} - \frac{A_t}{A}\frac{Y_t}{Y} + \frac{Y_{tt}}{Y}\right)\\
&\quad + \frac{1}{B^2}\left(\frac{A_{rr}}{A} - \frac{A_r}{A}\frac{B_r}{B} + \frac{A_r}{A}\frac{Y_r}{Y} - \frac{B_r}{B}\frac{Y_r}{Y} + \frac{Y_{rr}}{Y}\right) + \lambda, &(5.9c)\\
8\pi q &= -\frac{2}{AB}\left(-\frac{Y_{rt}}{Y} + \frac{B_t}{B}\frac{Y_r}{Y} + \frac{A_r}{A}\frac{Y_t}{Y}\right), &(5.9d)\\
\kappa &= \frac{l_r}{BY^2}. &(5.9e)
\end{aligned}
$$

Equations (5.9a)-(5.9e) are expressed as a nonlinear system of partial differential equations. This system is the most general for an expanding, accelerating and shearing gravitating relativistic fluid which includes the effects of the cosmological constant λ and the electromagnetic charge l.

The surface of a spherically symmetric radiating body is the stellar boundary separating the interior and exterior metrics. We take the exterior spacetime to be the Vaidya metric

$$ds^2 = -\left(1 - \frac{2m(v)}{R} + \frac{Q^2}{R^2} - \frac{1}{3}\lambda R^2\right)dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.10}$$

where $m(v)$ is the mass of the star and Q is the total charge measured by an observer at infinity. The metrics (5.1) and (5.10), and the extrinsic curvature, must match at the surface

Σ of the star. This matching generates the junction conditions

$$
\begin{aligned}
Adt &= \left[1 - 2\frac{m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv} + \frac{Q^2}{{R_\Sigma}^2} - \frac{1}{3}\lambda {R_\Sigma}^2\right]^{\frac{1}{2}} dv, &\text{(5.11a)}\\
(Y)_\Sigma &= R_\Sigma(v), &\text{(5.11b)}\\
l_\Sigma &= Q, &\text{(5.11c)}\\
m(v) &= \left[\frac{Y}{2}\left(1 + \frac{{Y_t}^2}{A^2} - \frac{{Y_r}^2}{B^2}\right) + \frac{l^2}{2Y} - \frac{\lambda Y^3}{6}\right]_\Sigma, &\text{(5.11d)}\\
(p_\parallel)_\Sigma &= (q)_\Sigma, &\text{(5.11e)}
\end{aligned}
$$

at the hypersurface Σ. Since the body is radiating the radial pressure $p_\parallel$ is nonzero at the surface Σ. Equation (5.11e), with equations (5.9b) and (5.9d), leads to the differential equation

$$
\begin{aligned}
B_t &= \left(\frac{Y_{tt}}{AY_r} + \frac{{Y_t}^2}{2AYY_r} + \frac{A}{2YY_r} - \frac{A_tY_t}{A^2Y_r} - \frac{\lambda AY}{2Y_r} - \frac{l^2A}{2Y^3Y_r}\right)B^2 \\
&\quad + \left(\frac{Y_{rt}}{Y_r} - \frac{Y_tA_r}{Y_rA}\right)B - \frac{A}{2}\left(\frac{2A_r}{A} + \frac{Y_r}{Y}\right). \qquad (5.12)
\end{aligned}
$$

Observe that this result is given in the form of a Riccati equation in the metric potential B. The presence of $\lambda \neq 0$ and $l \neq 0$ introduces the new terms $-\frac{\lambda AY}{2Y_r}$ and $-\frac{Al^2}{2Y^3Y_r}$, respectively. Therefore the cosmological constant and charge substantially changes the form of the boundary condition $p_\parallel = q$ on Σ.

5.3 Generalized model

We have written (5.12) as a Riccati equation in the variable B. Exact solutions to the nonlinear differential equation (5.12) have been found using various physical constraints. It is also possible to find solutions by transforming (5.12) to a different form using an appropriate change of variables for particular matter distributions. Thirukkanesh and Maharaj (2010) introduced the variable

$$\frac{1}{Z} = \frac{Y_r}{B}. \tag{5.13}$$

Ivanov (2016a) introduced another variable

$$H = \frac{Y_r}{B} + \frac{Y_t}{A}. \tag{5.14}$$

Both (5.13) and (5.14) transform (5.12) to simpler forms leading to new classes of exact solutions. Note that (5.13) and (5.14) lead to different families of solutions; transformation (5.13) is not contained in (5.14) except for the special case $Y = Y(r)$. A natural question is whether a generalized transformation exists which contains (5.13) and (5.14) as special cases. We need to find such a generalized transformation which preserves the form of (5.12) as a Riccati equation. Such transformations that preserve the form of the Riccati equation are difficult to find. We propose the new transformation

$$H = \alpha\frac{Y_r}{B} + \beta\frac{Y_t}{A} + \gamma, \tag{5.15}$$

where α, β, and γ are constants. Using (5.15) we can write (5.12) in the form

$$\begin{aligned} \alpha H_t \;=\; & \left[\frac{A}{2Y} + \frac{A_r}{Y_r}\right] H^2 - \left[\frac{A}{Y}\left(\frac{\beta Y_t}{A} + \gamma\right) + \frac{A_r}{Y_r}\left(2\gamma - \frac{(\alpha - 2\beta)Y_t}{A}\right)\right] H \\ & - \frac{A}{2Y}\left[\alpha^2\left(1 - \lambda Y^2 - \frac{l^2}{Y^2}\right) - \gamma^2\right] + \frac{\gamma}{Y_r}\left[\frac{\beta Y_r Y_t}{Y} + \gamma A_r - \frac{(\alpha - 2\beta)A_r Y_t}{A}\right] \\ & + \frac{(\alpha - \beta)}{A^2}\left[\alpha A_t Y_t - \alpha A Y_{tt} - \frac{\beta A_r Y_t^2}{Y_r} - \frac{A Y_t^2}{2Y}(\alpha + \beta)\right]. \end{aligned} \tag{5.16}$$

Remarkably (5.16) remains a Riccati equation in the variable H. Clearly the transformation (5.15) contains both (5.13) and (5.14) and thereby will regain previously found solutions, in addition to generating new solutions. There are several physically interesting cases contained in (5.16) that we will consider.

Firstly, we consider particles travelling along geodesic trajectories. In this case $A = 1$ and (5.16) becomes

$$\begin{aligned}
\alpha H_t &= \left[\frac{1}{2Y}\right] H^2 - \left[\frac{\beta Y_t}{Y} + \frac{\gamma}{Y}\right] H - (\alpha - \beta) \left[\alpha Y_{tt} + \frac{Y_t^2}{2Y}(\alpha + \beta)\right] \\
&\quad - \frac{1}{2Y}\left[\alpha^2 \left(1 - \lambda Y^2 - \frac{l^2}{Y^2}\right) - \gamma\left(2\beta Y_t + \gamma\right)\right], \qquad (5.17)
\end{aligned}$$

which is the most general evolution equation for the stellar boundary in the presence of charge l and the cosmological constant λ. If we let $\alpha = 1$, $\beta = 0$, $\gamma = 0$ then (5.17) reduces to

$$H_t = \left[\frac{1}{2Y}\right] H^2 - \frac{1}{2Y}\left[2YY_{tt} + Y_t^2 - \lambda Y^2 - \frac{l^2}{Y^2} + 1\right]. \qquad (5.18)$$

Now if we utilize the transformation (5.13) so that $H = \frac{1}{Z} = \frac{Y_r}{B}$ we find that (5.18) becomes

$$Z = \frac{1}{2Y}\left[FZ^2 - 1\right], \qquad (5.19a)$$

$$F = 2YY_{tt} + Y_t^2 - \lambda Y^2 - \frac{l^2}{Y^2} + 1. \qquad (5.19b)$$

These equations correspond to geodesic motion with charge l and the cosmological constant λ. When $l = 0$ and $\lambda = 0$, equations (5.19) become

$$Z = \frac{1}{2Y}\left[FZ^2 - 1\right], \qquad (5.20a)$$

$$F = 2YY_{tt} + Y_t^2 + 1, \qquad (5.20b)$$

which was the special case first considered by Thirukkanesh and Maharaj (2010).

Secondly, we consider the generalized Ivanov model for which (5.14) holds. This transformation is related to the appearance of horizons and black holes during collapse. If we let $\alpha = 1$, $\beta = 1$, $\gamma = 0$ then (5.16) has the form

$$H_t = \left[\frac{A}{2Y} + \frac{A_r}{Y_r}\right] H^2 - \left[\frac{A}{Y} + \frac{A_r}{Y_r}\right]\frac{Y_t}{A} H - \frac{A}{2Y}\left[1 - \lambda Y^2 - \frac{l^2}{Y^2}\right], \qquad (5.21)$$

which is the general Ivanov equation for the stellar boundary with charge l and cosmological constant λ. With $l = 0$, $\lambda = 0$ then (5.21) reduces to

$$H_t = \left[\frac{A}{2Y} + \frac{A_r}{Y_r}\right] H^2 - \left[\frac{A}{Y} + \frac{A_r}{Y_r}\right] \frac{Y_t}{A} H - \frac{A}{2Y}, \tag{5.22}$$

which was first generated by Ivanov (2016a). Mahomed *et al* (2019c) showed that (5.21) leads to a variety of new solutions.

5.4 Exact solutions

We now focus on equation (5.16) which holds because of the new transformation (5.15). This is a nonlinear equation in general of a rather complicated form. It is not possible to integrate this Riccati equation (in the variable H) in general. However we can demonstrate several families of exact solutions under particular conditions. The various exact solutions are listed in Table 5.1 for geodesic matter. The exact models that are integrable are provided in Table 5.2 for accelerating matter. The horizon function H has functional representations that can be given explicitly for both geodesic and accelerating particles; these are listed in Table 5.3.

5.4.1 Linear equation

We impose the condition

$$\frac{A}{2Y} + \frac{A_r}{Y_r} = 0. \tag{5.23}$$

Integrating (5.23) gives

$$A = \frac{f(t)}{\sqrt{Y}}. \tag{5.24}$$

Using (5.24) we find that (5.16) becomes

$$H_t + \left[\frac{Y_t}{2Y}\right] H = \frac{1}{2f^2Y^{7/2}}\Big[\alpha f^3\left(l^2 - Y^2 + \lambda Y^4\right) + \gamma f^2 Y^{5/2} Y_t$$
$$+2(\alpha-\beta)[Y^4 f_t Y_t - fY^3(Y_t^2 + YY_{tt})]\Big], \tag{5.25}$$

which is linear in the variable H. On integrating (5.25) we obtain

$$H = \frac{1}{\sqrt{Y}}\left[\int I dt + g(r)\right], \tag{5.26}$$

where

$$I = \frac{1}{2f^2Y^3}\Big[\alpha f^3\left(l^2 - Y^2 + \lambda Y^4\right) + \gamma f^2 Y^{5/2} Y_t + 2(\alpha-\beta)[Y^4 f_t Y_t - fY^3(Y_t^2 + YY_{tt})]\Big],$$

and $g(r)$ is the integration constant. Using equations (5.26), (5.24) and (5.15) gives the potential

$$B = \frac{\alpha f Y_r \sqrt{Y}}{f\left[\int I dt + g(r)\right] - \beta Y_t Y - \gamma f\sqrt{Y}}. \tag{5.27}$$

The metric functions A and B (including the horizon function H) are written in terms of the function Y which is arbitrary, $Y = Y(r,t)$. This is a new solution of the boundary condition (5.16) which contains l and λ. This result generalizes the work of Mahomed *et al* (2019c).

5.4.2 Bernoulli equation

It is possible to obtain Bernoulli equations with restrictions on the parameters α, β and γ. By imposing the condition

$$-\frac{A}{2Y}\left[\alpha^2\left(1 - \lambda Y^2 - \frac{l^2}{Y^2}\right) - \gamma^2\right] + \frac{\gamma A_r Y_t}{Y_r}\left[\frac{\beta Y_r}{YA_r} + \frac{\gamma}{Y_t} - \frac{(\alpha - 2\beta)}{A}\right]$$
$$+\frac{(\alpha-\beta)}{A^2}\left[\alpha A_t Y_t - \alpha A Y_{tt} - \frac{\beta A_r Y_t^2}{Y_r} - \frac{AY_t^2}{2Y}(\alpha+\beta)\right] = 0, \tag{5.28}$$

equation (5.16) takes the homogeneous form

$$H_t + \left[\frac{Y_t}{\alpha Y}\left(\beta + \frac{\gamma A}{Y_t}\right) + \frac{A_r Y_t}{\alpha A Y_r}\left(\frac{2\gamma A}{Y_t} - (\alpha - 2\beta)\right)\right] H = \left[\frac{A}{2\alpha Y} + \frac{A_r}{\alpha Y_r}\right] H^2, \tag{5.29}$$

which is clearly a Bernoulli equation in H. We can solve (5.29) to get

$$H = \exp\left[-\int J dt\right]\left(f(r) - \int \exp\left[-\int J dt\right] K dt\right)^{-1}, \tag{5.30}$$

where

$$\begin{aligned} J &= \frac{Y_t}{\alpha Y}\left(\beta + \frac{\gamma A}{Y_t}\right) + \frac{A_r Y_t}{\alpha A Y_r}\left(\frac{2\gamma A}{Y_t} - (\alpha - 2\beta)\right), \\ K &= \frac{A}{2\alpha Y} + \frac{A_r}{\alpha Y_r}, \end{aligned}$$

and $f(r)$ is the constant of integration. Using this solution in (5.30), we can find the potential

$$B = \frac{\alpha A Y_r}{A \exp\left[\int J dt\right]\left(-\int \exp\left[-\int J dt\right] K dt + f(r)\right)^{-1} - \beta Y_t - A\gamma}, \tag{5.31}$$

which is subject to the condition in (5.28).

By placing restrictions on α, β and γ, we can solve the consistency condition in (5.28) for specific cases given below.

5.4.2.1 Special case 1: Non-geodesic

In this example the fluid particles are accelerating. A special case is studied when α is arbitrary and $\beta = 0$, $\gamma = 0$. The consistency condition in (5.28) becomes

$$-\frac{\alpha A}{2Y}\left[1 - \lambda Y^2 - \frac{l^2}{Y^2}\right] + \frac{\alpha}{A^2}\left[A_t Y_t - A Y_{tt} - \frac{A Y_t^2}{2Y}\right] = 0. \tag{5.32}$$

We can rewrite (5.32) as

$$A_t - A\left[\frac{Y_t}{2Y} + \frac{Y_{tt}}{Y_t}\right] = \frac{A^3}{2YY_t}\left[1 - \lambda Y^2 - \frac{l^2}{Y^2}\right]. \tag{5.33}$$

It is clear that equation (5.33) is a Bernoulli equation in the metric function A. We can integrate (5.33) to give

$$A = \frac{\sqrt{3}YY_t}{\sqrt{\lambda Y^4 - 3l^2 - 3Y^2 + 3YC(r)}}, \tag{5.34}$$

where $C(r)$ is the constant of integration. Equation (5.30) gives

$$H = \exp\left[\int \frac{A_r Y_t}{AY_r} dt\right] \left(\int M dt + J(r)\right)^{-1}, \tag{5.35}$$

where

$$M = \exp\left[\int \frac{A_r Y_t}{AY_r} dt\right] \left(-\frac{1}{\alpha}\left[\frac{A}{2Y} + \frac{A_r}{Y_r}\right]\right),$$

and $J(r)$ is the integration constant. Then (5.31) implies

$$B = \frac{\alpha Y_r}{\exp\left[\int \frac{A_r Y_t}{AY_r} dt\right]} \left(\int M dt + J(r)\right). \tag{5.36}$$

We observe that the potentials A and B (including the horizon function H) are expressed in terms of $Y = Y(r,t)$ which is arbitrary. By specifying a choice of Y, we are able to find new classes of exact solutions. This class of models generalizes solutions of Thirukkanesh *et al* (2012a) and Mahomed *et al* (2019a) when we have $A \neq 1$, α is arbitrary, $l \neq 0$ and $\lambda \neq 0$.

5.4.2.2 Special case 2: Non-geodesic

Another special case arises for accelerated motion. This case is made possible by setting $Y_t = 0$ and $\gamma = 0$. These restrictions allow us to write (5.28) as

$$\lambda Y^4 - Y^2 + l^2 = 0, \tag{5.37}$$

which is a quadratic equation in Y. In this case $Y = Y(r)$. The solution to this algebraic equation is given by

$$Y = \sqrt{\frac{1 + \sqrt{1 - 4\lambda l^2}}{2\lambda}}. \tag{5.38}$$

where $l < \frac{1}{2\sqrt{\lambda}}$. We can substitute Y into (5.30) to give

$$H = -\frac{1}{\int Q dt + g(r)} \tag{5.39}$$

where

$$Q = \frac{\lambda A l l_r - A_r(1 - 4\lambda l^2 + \sqrt{1 - 4\lambda l^2})}{\alpha\sqrt{2\lambda} l l_r \sqrt{1 + \sqrt{1 - 4\lambda l^2}}},$$

and $g(r)$ is the constant of integration. By using equation (5.15) we can find B to be

$$B = \frac{\alpha\sqrt{2\lambda} l l_r \left[\int Q dt + g(r)\right]}{\sqrt{1 - 4\lambda l^2 + (1 - 4\lambda l^2)^{3/2}}}. \tag{5.40}$$

For this class of solutions

$$A = A(r,t), \tag{5.41}$$

is an arbitrary function. In this result, the the functions $A(r,t)$ and $l(r)$ serve as generating functions.

5.4.2.3 Special case 3: Geodesic

Another case of the consistency condition (5.28) can be studied for particles in geodesic motion. If we let $A = 1$, $\alpha = \beta$ and γ is arbitrary then (5.28) becomes

$$2\alpha\gamma Y^2 Y_t + \alpha^2 \lambda Y^4 + \left(\gamma^2 - \alpha^2\right) Y^2 + \alpha^2 l^2 = 0. \tag{5.42}$$

Equation (5.42) is nonlinear but can be integrated exactly. Solving equation (5.42) gives

$$\arctan\left[\frac{\sqrt{2\lambda}\alpha Y}{\sqrt{\gamma^2 - \alpha^2 + \chi}}\right]\sqrt{\gamma^2 - \alpha^2 + \chi} - \arctan\left[\frac{\sqrt{2\lambda}\alpha Y}{\sqrt{\gamma^2 - \alpha^2 - \chi}}\right]\sqrt{\gamma^2 - \alpha^2 - \chi}$$
$$= \sqrt{2\lambda}\alpha\chi\left(C(r) - \frac{t}{2\alpha\gamma}\right), \tag{5.43}$$

where

$$\chi = \sqrt{(\alpha^2 - \gamma^2)^2 - 4\alpha^2\lambda l^2},$$

and $C(r)$ is the integration constant. Equation (5.43) is an implicit solution in the potential Y. From equation (5.30) we have

$$H = \frac{1}{Y} \exp\left[-\int \frac{\gamma}{\alpha Y} dt\right] \left(\int K dt + g(r)\right)^{-1}. \tag{5.44}$$

where

$$K = -\frac{1}{2\alpha Y^2} \exp\left[-\int \frac{\gamma}{\alpha Y} dt\right],$$

and $g(r)$ is the constant of integration. Using equation (5.44) and (5.15), we can find the potential B in (5.31) to give

$$B = \frac{\alpha Y Y_r \left[\int K dt + g(r)\right]}{\exp\left[-\int \frac{\gamma}{\alpha Y} dt\right] - (\alpha Y Y_t + \gamma Y)\left[\int K dt + g(r)\right]}. \tag{5.45}$$

In this class of exact solutions the potential Y is given implicitly by (5.43).

5.4.2.4 Special case 4: Geodesic

Another case of geodesic motion can be considered if we let $A = 1$ and $\alpha = \beta = \gamma$. Then the consistency condition (5.28) becomes

$$2Y^2 Y_t + \lambda Y^4 + l^2 = 0. \tag{5.46}$$

Equation (5.46) is a first order nonlinear equation in Y. It can be solved to yield

$$\begin{aligned} 4\sqrt{2l}\lambda^{3/4}\left(C(r) - \frac{t}{2}\right) &= 2\arctan\left[1 + \frac{\sqrt{2}\lambda^{1/4} Y}{\sqrt{l}}\right] - 2\arctan\left[1 - \frac{\sqrt{2}\lambda^{1/4} Y}{\sqrt{l}}\right] \\ &+ \ln\left[l - \sqrt{2l}\lambda^{1/4} Y + \sqrt{\lambda} Y^2\right] - \ln\left[l + \sqrt{2l}\lambda^{1/4} Y + \sqrt{\lambda} Y^2\right], \end{aligned} \tag{5.47}$$

where $C(r)$ is the constant of integration. Equation (5.30) becomes

$$H = \frac{1}{Y} \exp\left[-\int \frac{1}{Y} dt\right] \left(f(r) - \int \frac{1}{2\alpha Y^2} \exp\left[-\int \frac{1}{Y} dt\right]\right)^{-1}, \tag{5.48}$$

where $f(r)$ is the constant of integration. Using equation (5.31) yields the metric function

$$B = \frac{\alpha Y Y_r \left(f(r) - \int \frac{1}{2\alpha Y^2} \exp\left[-\int \frac{1}{Y} dt\right]\right)}{\exp\left[-\int \frac{1}{Y} dt\right] - \alpha\left(YY_t + Y\right)\left(f(r) - \int \frac{1}{2\alpha Y^2} \exp\left[-\int \frac{1}{Y} dt\right]\right)}. \tag{5.49}$$

This model requires that the potential Y satisfies (5.47) implicitly.

5.4.3 Riccati equation

In general (5.16) is a Riccati equation in the variable H. It is not possible to obtain a general solution in this form as it is a very complicated equation. We consider some special cases for which equation (5.16) is integrable. To solve (5.16) we can impose the condition $Y_t = 0$. This yields the simpler equation

$$\alpha H_t = \left[\frac{A}{2Y} + \frac{A_r}{Y_r}\right] H^2 - \left[\frac{\gamma A}{Y} + \frac{2\gamma A_r}{Y_r}\right] H - \frac{A}{2Y}\left[\alpha^2\left(1 - \lambda Y^2 - \frac{l^2}{Y^2}\right) - \gamma^2\right] + \frac{\gamma^2 A_r}{Y_r}. \tag{5.50}$$

The reduced equation remains a Riccati equation containing the parameters α and γ. We show that equation (5.50) can be solved for the following subcases which we present below.

5.4.3.1 Solution 1

Equation (5.50) can be written in the form

$$\left[\frac{H^2}{Y_r} - \frac{2\gamma H}{Y_r} + \frac{\gamma^2}{Y_r}\right] A_r + \left[\frac{H^2}{2Y} - \frac{\gamma H}{Y} + \frac{\gamma^2}{2Y} - \frac{\alpha^2}{2Y}\left(1 - \lambda Y^2 - \frac{l^2}{Y^2}\right)\right] A = \alpha H_t. \tag{5.51}$$

In this representation we observe that it is a linear equation in A. We can solve (5.51) to give

$$A = \exp\left[-\int \frac{Q}{P} dr\right]\left(\int \frac{\alpha H_t}{P} \exp\left[\int \frac{Q}{P} dr\right] dr + f(t)\right), \tag{5.52}$$

where

$$\begin{aligned} P &= \left[\frac{H^2}{Y_r} - \frac{2\gamma H}{Y_r} + \frac{\gamma^2}{Y_r}\right], \\ Q &= \left[\frac{H^2}{2Y} - \frac{\gamma H}{Y} + \frac{\gamma^2}{2Y} - \frac{\alpha^2}{2Y}\left(1 - \lambda Y^2 - \frac{l^2}{Y^2}\right)\right], \end{aligned}$$

and $f(t)$ is the constant of integration. By using equation (5.15), we have

$$B = \frac{\alpha Y_r}{H - \gamma}. \tag{5.53}$$

In this category of solutions the potentials $H = H(r,t)$ and $Y = Y(r)$ are arbitrary functions. A particular choice will lead to a functional form for the potentials A and B. The quantities Y and H serve as generating functions.

5.4.3.2 Solution 2

We now make the assumption

$$\left[\frac{\gamma^2}{Y_r}\right] A_r - \left[\frac{\alpha^2}{2Y}\left(1 - \lambda Y^2 - \frac{l^2}{Y^2}\right) - \frac{\gamma^2}{2Y}\right] A = 0. \tag{5.54}$$

When $\gamma = 0$ we regain the condition in equation (5.37) for the Special case 2 in section 5.4.2. When $\gamma \neq 0$ in (5.54) another class of exact solutions is possible. Equation (5.54) is a separable equation in A which can be solved to give

$$A = f(t) \exp\left[\int K dr\right], \tag{5.55}$$

where

$$K = \frac{Y_r}{2\gamma^2 Y}\left[\alpha^2\left(1 - \lambda Y^2 - \frac{l^2}{Y^2}\right) - \gamma^2\right].$$

Then equation (5.50) becomes

$$H_t + \left[\frac{\gamma A}{\alpha Y} + \frac{2\gamma A_r}{\alpha Y_r}\right] H = \left[\frac{A}{2\alpha Y} + \frac{A_r}{\alpha Y_r}\right] H^2, \tag{5.56}$$

which is a Bernoulli equation in H. Solving (5.56) yields

$$H = \exp\left[-\int\left(\frac{\gamma A}{\alpha Y} + \frac{2\gamma A_r}{\alpha Y_r}\right) dt\right]\left(\int M dt + g(r)\right)^{-1}, \tag{5.57}$$

where

$$M = -\left(\frac{A}{2\alpha Y} + \frac{A_r}{\alpha Y_r}\right) \exp\left[-\int\left(\frac{\gamma A}{\alpha Y} + \frac{2\gamma A_r}{\alpha Y_r}\right) dt\right].$$

Then equation (5.15) gives

$$B = \frac{\alpha Y_r \left(\int M dt + g(r) \right)}{\exp \left[-\int \left(\frac{\gamma A}{\alpha Y} + \frac{2\gamma A_r}{\alpha Y_r} \right) dt \right] - \gamma \left(\int M dt + g(r) \right)}, \tag{5.58}$$

and $Y = Y(r)$ is an arbitrary potential. This category of models lead to functional forms for the potentials A and B. This automatically leads to a form for the horizon function H in (5.57).

5.5 Discussion

In this chapter we considered the Einstein-Maxwell system with cosmological constant and electric charge. The boundary condition was expressed as a Riccati equation in the gravitational potential B. Ivanov (2016a) had earlier introduced a transformation that simplified the boundary condition. In our present study we introduced a new transformation that generalizes that of Ivanov (2016a). This new transformation transforms the boundary condition in B to another Riccati equation in the variable H. This becomes the most general form of the boundary condition in terms of the horizon function H for a spherically symmetric radiating model. Since the boundary condition in the variable H is quite complex to solve in general, particular cases were investigated. Firstly we investigated the linear case by imposing a condition on one of the coefficients of the differential equation. Secondly we studied the Bernoulli case by setting the inhomogeneous term in the boundary condition to zero. Thirdly we showed that the transformed Riccati equation can be solved by restricting the potential Y in particular cases. In all three approaches the transformed Riccati equation was solved and explicit and implicit forms of the gravitational potentials were obtained. The horizon function H was found explicitly for each category of solutions. The various functional representations of the gravitational potentials A, B and Y and the horizon function H are given in Tables 5.1-5.3. The tables contain several new families of exact solutions arising from the transformed Riccati equation. We also identify known exact solutions that were found earlier. It is interesting to note that we can regain the models of Ivanov (2016b), Thirukkanesh and Maharaj (2010) and Mahomed *et al* (2019a, 2019b, 2019c).

5.6 Tables of results

Table 5.1: **Exact solutions for geodesic motion**

Parameters	Gravitational potentials	Features
$\alpha=\beta$ $\gamma\neq 0$	$\sqrt{2\lambda}\alpha\chi\left(C(r)-\frac{t}{2\alpha\gamma}\right)$ $=\arctan\left[\frac{\sqrt{2\lambda}\alpha Y}{\sqrt{\gamma^2-\alpha^2+\chi}}\right]\sqrt{\gamma^2-\alpha^2+\chi}$ $-\arctan\left[\frac{\sqrt{2\lambda}\alpha Y}{\sqrt{\gamma^2-\alpha^2-\chi}}\right]\sqrt{\gamma^2-\alpha^2-\chi}$ $B=\frac{\alpha YY_r\left[\int Kdt+g(r)\right]}{\exp\left[-\int\frac{\gamma}{\alpha Y}dt\right]-(\alpha YY_t+\gamma Y)\left[\int Kdt+g(r)\right]}$ where $K=-\frac{1}{2\alpha Y^2}\exp\left[-\int\frac{\gamma}{\alpha Y}dt\right]$	New model. Implicit solution.
$\alpha=\beta=\gamma$	$C(r)-\frac{t}{2}=\frac{1}{4\sqrt{2l}\lambda^{3/4}}\left(2\arctan\left[1+\frac{\sqrt{2}\lambda^{1/4}Y}{\sqrt{l}}\right]\right.$ $-2\arctan\left[1-\frac{\sqrt{2}\lambda^{1/4}Y}{\sqrt{l}}\right]+\ln\left[l-\sqrt{2l}\lambda^{1/4}Y+\sqrt{\lambda}Y^2\right]$ $\left.-\ln\left[l+\sqrt{2l}\lambda^{1/4}Y+\sqrt{\lambda}Y^2\right]\right)$ $B=\frac{\alpha YY_r\left(f(r)-\int\frac{1}{2\alpha Y^2}\exp\left[-\int\frac{1}{Y}dt\right]\right)}{\exp\left[-\int\frac{1}{Y}dt\right]-\alpha(YY_t+Y)\left(f(r)-\int\frac{1}{2\alpha Y^2}\exp\left[-\int\frac{1}{Y}dt\right]\right)}$	New model. Implicit solution.
$\alpha=\beta=1$ $\gamma=0$	$Y=\sqrt{\frac{D^2+l^2}{2D_t+1}}$ $B=\frac{ll_r(2D_t+1)-l^2D_{rt}+D[D_r(2D_t+1)-DD_{rt}]}{D(2{D_t}^2+3D_t+1)+D_{tt}(D^2+l^2)}$	Ivanov (2019a). $D=HY$ is a generating function. $\lambda=0$.
	$Y=\sqrt{\frac{(2D_t+1)\pm\sqrt{(2D_t+1)^2-4\lambda(D^2+l^2)}}{2\lambda}}$ $B=\frac{D_{rt}[\Delta\pm(2D_t+1)]\mp 2\lambda(DD_r+ll_r)}{2\lambda D[\Delta\pm D_t]\mp D_{tt}[(2D_t+1)\pm\Delta]}$ where $\Delta=\sqrt{(2D_t+1)^2-4\lambda\left(D^2+l^2\right)}$	Mahomed *et al* (2019b). $D=HY$ is a generating function.

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Table 5.1: *Continued*

Parameters	Gravitational potentials	Features
$\alpha = 1$ $\beta = \gamma = 0$	$Y = [R_1(r)t + R_2(r)]^{2/3}$ $B = \frac{2}{3}\left(\frac{1+f(r)\exp\left[3(R_1t+R_2)^{1/3}/R_1\right]}{1-f(r)\exp\left[3(R_1t+R_2)^{1/3}/R_1\right]}\right)\frac{(t(R_1)_r+(R_2)_r)}{(R_1+R_2)^{1/3}}$	Thirukkanesh and Maharaj (2010). $\lambda = l = 0$.
	$Y = R_1(r)t + R_2(r)$ $B = \frac{t(R_1)_r+(R_2)_r}{\sqrt{{R_1}^2+1}}\left(\frac{1+g(r)(R_1t+R_2)^{\sqrt{{R_1}^2+1}/R_1}}{1-g(r)(R_1t+R_2)^{\sqrt{{R_1}^2+1}/R_1}}\right)$	Thirukkanesh and Maharaj (2010). $\lambda = l = 0$.
$\alpha = 1$ $\beta = \gamma = 0$	Particular explicit forms for B and Y have been found with λ and l.	Mahomed *et al* (2019a).

Table 5.2: **Exact solutions for accelerated motion**

Parameters	Gravitational potentials	Features
$\alpha \neq 0$, $\beta \neq 0$ $\gamma \neq 0$	$A = \frac{f(t)}{\sqrt{Y}}$ $B = \frac{\alpha f Y_r \sqrt{Y}}{f\left[\int I dt + g(r)\right] - \beta Y_t Y - \gamma f \sqrt{Y}}$ $Y = Y(r,t)$ where $I = \frac{1}{2f^2Y^3}\Big[\alpha f^3\left(l^2 - Y^2 + \lambda Y^4\right) + \gamma f^2 Y^{5/2} Y_t$ $+2(\alpha - \beta)[Y^4 f_t Y_t - fY^3(Y_t^2 + YY_{tt})]\Big]$	New model. Explicit solution.
$\alpha \neq 0$, $\beta = 0$ $\gamma = 0$	$A = \frac{\sqrt{3}YY_t}{\sqrt{\lambda Y^4 - 3l^2 - 3Y^2 + 3Yf(r)}}$ $B = \frac{\alpha Y_r}{\exp\left[\int \frac{A_r Y_t}{AY_r} dt\right]}\left(\int M dt + J(r)\right)$ $Y = Y(r,t)$ where $M = \exp[\int \frac{A_r Y_t}{AY_r} dt]\left(-\frac{1}{\alpha}\left[\frac{A}{2Y} + \frac{A_r}{Y_r}\right]\right)$	New model. Explicit solution.
$\gamma = 0$	$A = A(r,t)$ $B = \frac{\alpha\sqrt{2\lambda} l l_r \left[\int Q dt + g(r)\right]}{\sqrt{1 - 4\lambda l^2 + (1 - 4\lambda l^2)^{3/2}}}$ $Y = \sqrt{\frac{1 + \sqrt{1 - 4\lambda l^2}}{2\lambda}}$ where $Q = \frac{\lambda A l l_r - A_r(1 - 4\lambda l^2 + \sqrt{1 - 4\lambda l^2})}{\alpha\sqrt{2\lambda} l l_r \sqrt{1 + \sqrt{1 - 4\lambda l^2}}}$	New model. Explicit solution. $A(r,t)$ and $l(r)$ are generating functions.
$\alpha = \beta = 1$ $\gamma = 0$	Particular explicit forms for A, B and Y have been found with λ and l.	Mahomed *et al* (2019c).

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Table 5.2: *Continued*

Parameters	Gravitational potentials	Features
	$A = \exp\left[-\int \frac{Q}{P}dr\right]\left(\int \frac{\alpha H_t}{P}\exp\left[\int \frac{Q}{P}dr\right]dr + f(t)\right)$	New model.
	$B = \frac{\alpha Y_r}{H-\gamma}$	Explicit solution.
	$Y = Y(r)$	$H(r,t)$ and $Y(r)$ are
	where $P = \left[\frac{H^2}{Y_r} - \frac{2\gamma H}{Y_r} + \frac{\gamma^2}{Y_r}\right]$	generating functions.
$\alpha \neq 0$	$Q = \left[\frac{H^2}{2Y} - \frac{\gamma H}{Y} + \frac{\gamma^2}{2Y} - \frac{\alpha^2}{2Y}\left(1 - \lambda Y^2 - \frac{l^2}{Y^2}\right)\right]$	
$\gamma \neq 0$	$A = f(t)\exp\left[\int \frac{Y_r}{2\gamma^2 Y}\left[\alpha^2\left(1 - \lambda Y^2 - \frac{l^2}{Y^2}\right) - \gamma^2\right]dr\right]$	New model.
	$B = \frac{\alpha Y_r\left(\int M dt + g(r)\right)}{\exp\left[-\int\left(\frac{\gamma A}{\alpha Y} + \frac{2\gamma A_r}{\alpha Y_r}\right)dt\right] - \gamma\left(\int M dt + g(r)\right)}$	Explicit solution.
	$Y = Y(r)$	
	where $M = -\left(\frac{A}{2\alpha Y} + \frac{A_r}{\alpha Y_r}\right)\exp\left[-\int\left(\frac{\gamma A}{\alpha Y} + \frac{2\gamma A_r}{\alpha Y_r}\right)dt\right]$	

Table 5.3: **Exact solutions for the horizon function H**

Parameters	Gravitational potentials	Features
$\alpha \neq 0$, $\beta \neq 0$	$H = \frac{1}{\sqrt{Y}}\left[\int I dt + g(r)\right]$	New model.
$\gamma \neq 0$	where $I = \frac{1}{2f^2Y^3}\Big[\alpha f^3\left(l^2 - Y^2 + \lambda Y^4\right) + \gamma f^2 Y^{5/2} Y_t$	Explicit solution.
	$+2(\alpha-\beta)[Y^4 f_t Y_t - fY^3({Y_t}^2 + YY_{tt})]\Big]$	
$\alpha \neq 0$, $\beta = 0$	$H = \exp\left[\int \frac{A_r Y_t}{AY_r} dt\right]\left(\int M dt + J(r)\right)^{-1}$	New model.
$\gamma = 0$	where $M = \exp[\int \frac{A_r Y_t}{AY_r} dt]\left(-\frac{1}{\alpha}\left[\frac{A}{2Y} + \frac{A_r}{Y_r}\right]\right)$	Explicit solution.
$\gamma = 0$	$H = -\frac{1}{\int Q dt + g(r)}$	New model.
	where $Q = \frac{\lambda All_r - A_r(1-4\lambda l^2 + \sqrt{1-4\lambda l^2})}{\alpha\sqrt{2\lambda} ll_r \sqrt{1+\sqrt{1-4\lambda l^2}}}$	Explicit solution.
	$H = \alpha\frac{Y_r}{B} + \gamma$	New model.
$\alpha \neq 0$		Explicit solution.
$\gamma \neq 0$	$H = \exp\left[-\int\left(\frac{\gamma A}{\alpha Y} + \frac{2\gamma A_r}{\alpha Y_r}\right)dt\right]\left(\int M dt + g(r)\right)^{-1}$	New model.
	where $M = -\left(\frac{A}{2\alpha Y} + \frac{A_r}{\alpha Y_r}\right)\exp\left[-\int\left(\frac{\gamma A}{\alpha Y} + \frac{2\gamma A_r}{\alpha Y_r}\right)dt\right]$	Explicit solution.
	$H = D\sqrt{\frac{2D_t+1}{D^2+Q^2}}$	Ivanov (2019a).
$\alpha = \beta = 1$	$H = D\sqrt{\frac{2\lambda}{(2D_t+1)\pm\sqrt{(2D_t+1)^2 - 4\lambda(D^2+l^2)}}}$	Mahomed *et al* (2019b).
$\gamma = 0$	Particular forms for H have been found with λ and l.	Mahomed *et al* (2019c).
$\alpha = 1$	$H = \frac{1-f(r)\exp\left[3(R_1 t + R_2)^{1/3}/R_1\right]}{1+f(r)\exp\left[3(R_1 t + R_2)^{1/3}/R_1\right]}$	Thirukkanesh and Maharaj (2010). $\lambda = l = 0$.
$\beta = \gamma = 0$	$H = \sqrt{{R_1}^2+1}\left(\frac{1-g(r)(R_1 t + R_2)^{\sqrt{{R_1}^2+1}/R_1}}{1+g(r)(R_1 t + R_2)^{\sqrt{{R_1}^2+1}/R_1}}\right)$	Thirukkanesh and Maharaj (2010). $\lambda = l = 0$.

Chapter 6

Conclusion

In this thesis we investigated spherically symmetric radiating models for particles in geodesic and accelerated motion with expansion and shear. We have also highlighted the effects and importance of both the cosmological constant and the electromagnetic field. The addition of these quantities changes the nature of previous spherically symmetric radiating models. The junction condition at the stellar surface was studied for matter that is geodesic and accelerating. In general the junction condition is a Riccati equation. As is well known the general Riccati equation is not integrable. We show that under particular assumptions exact solutions exist. We find new solutions to the Riccati equation and regain known solutions. We find that the horizon function of Ivanov (2016a) is a useful quantity to simplify the boundary condition.

In Chapter 2 we studied the boundary condition for particles travelling in geodesic motion. The boundary condition is expressed a Riccati equation. We introduced a transformation, from Thirukkanesh and Maharaj (2010), that leads to the simplification of this Riccati equation. We studied this simplified Riccati equation for several cases which includes the cosmological constant λ, the electric charge l and the potential Y. Several new classes of exact solutions were obtained by making various assumptions. Known solutions were regained. The results were summarized in four tables.

In Chapter 3 we studied a geodesic model in which the boundary condition was expressed as a Riccati equation in one of the metric potentials. In this analysis we obtained a solution generating algorithm and new classes of travelling wave solutions. We made use of an important transformation, which was first introduced by Ivanov (2016a), called the horizon function. Then we introduced a new variable called the generating function. Both the horizon function and the generating function allowed us to express the Riccati equation as an algebraic equation in the gravitational potential Y. Two cases were studied when $\lambda = 0$ and $\lambda \neq 0$. The potentials can be expressed in terms of a generating function. New exact solutions are possible when specifying a choice for the generating function. It was also possible to find new classes of travelling wave solutions by using a Lie generator of Abebe *et al* (2014a), the stellar radius found by Tiwari and Maharaj (2017). By making certain assumptions on the parameters we found new classes of exact solutions that were expressed as special functions. We also showed the important role of the generating function in important stellar quantities such as the mass and compactness parameter.

In Chapter 4 we consider a generalized spherically symmetric model. This model is expressed as a Riccati equation in one of the potentials. We use various techniques to study three cases pertaining to the generalized Riccati equation: linear, Bernoulli and inhomogeneous Riccati. We obtain new classes of exact solutions in the presence of the cosmological constant and charge. Previous results can also be obtained from these solutions. We then introduce a transformation, first formulated by Ivanov (2016a), called the horizon function. This transformation simplifies the generalized Riccati equation into a new form called the transformed Riccati equation. Several cases were investigated involving simplification of the generalized Riccati equation to such as: linear, Bernoulli and inhomogeneous Riccati types.

Various techniques were employed to yield new classes of exact solutions to the linear and Bernoulli cases. The inhomogeneous Riccati case was studied for four subcases: $\lambda \neq 0$, $\lambda = 0$, $l \neq 0$, and $l = 0$. We obtained new classes of exact solutions for radiating stars due to these four subcases in which the horizon function serves as a generating function. The tables of exact solutions give a collective overview of new and known results.

In Chapter 5 we investigated the effects of the cosmological constant and electric charge for generalized spherically symmetric radiating bodies. The boundary condition for this model is expressed in the form of a Riccati equation. We introduced a new transformation called the generalized horizon function given by

$$H = \alpha \frac{Y_r}{B} + \beta \frac{Y_t}{A} + \gamma, \tag{6.1}$$

where α, β, and γ are constants. This transformation can be used to obtain earlier transformations and models due to Ivanov (2016a) and Thirukkanesh and Maharaj (2010) for particular values of α, β and γ. The generalized horizon function preserves the form of the Riccati equation and transforms it into a generalized Riccati equation. The new Riccati equation is given by

$$\begin{aligned} \alpha H_t \;=\; & \left[\frac{A}{2Y} + \frac{A_r}{Y_r}\right] H^2 - \left[\frac{A}{Y}\left(\frac{\beta Y_t}{A} + \gamma\right) + \frac{A_r}{Y_r}\left(2\gamma - \frac{(\alpha - 2\beta) Y_t}{A}\right)\right] H \\ & - \frac{A}{2Y}\left[\alpha^2 \left(1 - \lambda Y^2 - \frac{l^2}{Y^2}\right) - \gamma^2\right] + \frac{\gamma}{Y_r}\left[\frac{\beta Y_r Y_t}{Y} + \gamma A_r - \frac{(\alpha - 2\beta) A_r Y_t}{A}\right] \\ & + \frac{(\alpha - \beta)}{A^2}\left[\alpha A_t Y_t - \alpha A Y_{tt} - \frac{\beta A_r Y_t^2}{Y_r} - \frac{A Y_t^2}{2Y}(\alpha + \beta)\right]. \end{aligned} \tag{6.2}$$

It is remarkable that the transformation (6.1) leads to (6.2) which remains a Riccati equation. Such assumptions are rare. It is very difficult to solve this generalized Riccati equation in general. However we used various methods to solve this equation for particular cases. The linear case was the most general case that yields new classes of exact solutions. The Bernoulli case was broken up into four subcases. By solving the consistency condition for each case we obtain new classes of exact solutions. The general Riccati case was explored via two subcases when we set $Y_t = 0$. This assumption yields special classes of new exact solutions. The tables of exact solutions combine these results for geodesic and nongeodesic motion. The horizon function can be found explicitly for the various classes of models.

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