Iterative schemes for approximating common solutions of certain optimization and fixed point problems in Hilbert spaces

by

Musa Adewale Olona

Dissertation submitted in fulfilment of the requirements for the degree
of
Master of Science

The University of KwaZulu-Natal

School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal, South Africa.

January, 2021.

Iterative schemes for approximating common solutions of certain optimization and fixed point problems in Hilbert spaces.

by

Musa Adewale Olona

As the candidate's supervisor, I have approved this dissertation for submission

Prof. O. T. Mewomo

. . . .

Dedication

This dissertation is dedicated to the Almighty Allah.

Acknowledgements

All thanks and adoration to God, the almighty for endowing me with guidance, strength and viable health to complete my MSc program at the prestigious University of KwaZulu-Natal, Durban, South Africa (UKZN).

I owe my heartfelt gratitude to my able, effective and efficient supervisor Prof. O.T. Mewomo for his kindness, support, encouragement, contributions and advice from the beginning to the end of this work. Most especially, when the ship of hope was sinking he came to my aid. I will forever be grateful to him. I am also grateful for his timely and careful proof-reading of this dissertation which greatly improved the quality of this dissertation.

My deep appreciation to my parents (Mr. and Mrs. Olona), wife (Qudrat Fakayode), son (Mustopha), siblings (Sefiat, Zainab, Luth, Idris and Qudus), in-law (Mr. and Mrs. Fakayode), friends and family (most especially, Uncle Dimeji Olona, Abdul-lateef Olona, Olona Hussain and Dr Mrs Tijani) for thier love, prayer, support, encouragement, advice and endurance during my MSc program in South Africa.

My profound gratitude to T.O. Alakoya and A.O.-E. Owolabi for their time, patience, academic support, productive discussion and contribution to the success of this work. I am sure it would have been an insurmountable task without their support, encouragement and contributions.

My profound gratitude goes to my senior colleagues; Dr. C. Izuchukwu, A. Taiwo, G.N. Ogwo, K.O. Aremu, O.K. Oyewole, A.A. Akindele, H.A. Abass and Emeka Godwin. I really appreciate their utmost concern and support throughout this work. I wish them all the best and successful completion of their doctoral study.

I am thankful to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and Science at the University of KwaZulu-Natal (UKZN), for giving me tuition fee remission, up-to-date facilities, and providing a conducive learning and research environment during my program.

I specially thank my mentor (Gafari Lukumon), my friend (Adegoke Yusuff, Ajeigbe Yusuff, Raheem Idowu) your contribution can not be quantified. May almighty God grant all your wishes.

I acknowledge and appreciate Engineer Ibrahim Taofeek and Duad Sulaimon for your unforgettable support, prayer and advice during the processing of my visa and my stay in South Africa. May the Almighty Allah continue to elevate you immensely.

I must appreciate the significant role of proprietors (especially Alhaji Ismail Folorunso), Principal (Mr O.I Oseni) and staff of SAF School, Iseyin. You are such a wonderful family.

I express my gratitude to Westville usrah committee, Durban Muslim community, Jamia KZN for their care and hospitality. May Allah reward you abundantly.

I appreciate my sisters Waheed Suliat, Azeez Khadijah, Olanrewaju Aisha. Thanks for your support and prayers.

I acknowledge the impact and support of Movement for Ekunle Development (Art. Ajayi

Akeem and Ayandare Lateef) and Progressive Minded coalition (Uthmon Sulaimon and Eyinafe Abdul-lateef).

My sincere thanks to 166 family and friends (Dr Adeyinka Gbadebo, Dr Rabiu Musa, Dr Seun, Mr Muhammad Buhari, Mr Kashamu Ibrahim and others). Your hospitality, encouragement and support worth more than I can explain.

Abstract

In this dissertation, we introduce a shrinking projection method of an inertial type with self-adaptive step size for finding a common element of the set of solutions of Split Generalized Equilibrium Problem (SGEP) and the set of common fixed points of a countable family of nonexpansive multivalued mappings in real Hilbert spaces. The self-adaptive step size incorporated helps to overcome the difficulty of having to compute the operator norm while the inertial term accelerates the rate of convergence of the propose algorithm. Under standard and mild conditions, we prove a strong convergence theorem for the sequence generated by the proposed algorithm and obtain some consequent results. We apply our result to solve Split Mixed Variational Inequality Problem (SMVIP) and Split Minimization Problem (SMP), and present numerical examples to illustrate the performance of our algorithm in comparison with other existing algorithms. Moreover, we investigate the problem of finding common solutions of Equilibrium Problem (EP), Variational Inclusion Problem (VIP)and Fixed Point Problem (FPP) for an infinite family of strict pseudo-contractive mappings. We propose an iterative scheme which combines inertial technique with viscosity method for approximating common solutions of these problems in Hilbert spaces. Under mild conditions, we prove a strong theorem for the proposed algorithm and apply our results to approximate the solutions of other optimization problems. Finally, we present a numerical example to demonstrate the efficiency of our algorithm in comparison with other existing methods in the literature. Our results improve and complement contemporary results in the literature in this direction.

Contents

Declaration

This dissertation has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used, proper reference has been made.

Musa Adewale Olona

. . . .

Contributed papers from the dissertation

Papers from the dissertation submitted and still in the refereeing process.

1. **M.A Olona,** T.O Alakoya, A.O-E Owolabi and O.T. Mewomo, Inertial shrinking projection algorithm with self-adaptive step size for split generalized equilibrium and fixed point problems for a countable family of nonexpansive multivalued mappings. Summited to Demonstratio Mathematica.

2. **M.A Olona,** T.O Alakoya, A.O-E Owolabi and O.T. Mewomo, Inertial algorithm for solving equilibrium, variational inclusion and fixed point problems for an infinite family of strict pseudocontractive mappings. Summited to Journal of Nonlinear Functional Analysis.

   The results presented in Chapter 3 of this dissertation are from Paper 1 while results presented in Chapter 4 of this dissertation are from Paper 2. Both papers are submitted to journals and we hope for positive outcome.

CHAPTER 1

Introduction

1.1 Background of study

Many authors have studied and proposed several iterative algorithms for solving optimization problems because of its key role in the area of research such as convex and nonlinear analysis.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $\phi : C \times C \to \mathbb{R}$, $F : C \times C \to \mathbb{R}$ be two bifunctions. The *Generalized Equilibrium Problem* (GEP) is to find a point $x^* \in C$ such that

$$F(x^*, y) + \phi(x^*, y) \geq 0, \quad \forall \quad y \in C. \tag{1.1.1}$$

The solution set of the GEP is denoted by $GEP(F, \phi)$. In particular, If we set $\phi = 0$ in (1.1.1), then the GEP reduces to the classical Equilibrium Problem (EP), which is to find a point $x^* \in C$ such that $F(x^*, y) \geq 0, \quad \forall \quad y \in C$. The solution set of EP is denoted by $EP(F)$, (see [5, 39, 75] and the references contained therein).

Suppose H_1 and H_2 are real Hilbert spaces and C, Q are non empty closed convex subsets of H_1 and H_2, respectively. Let $F_1, \phi_1 : C \times C \to \mathbb{R}$ and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ be bifunctions, and $A : H_1 \to H_2$ be a bounded linear operator. The *Split Generalized Equilibrium Problem* (SGEP) is defined as follows: Find $x^* \in C$ such that

$$F_1(x^*, x) + \phi_1(x^*, x) \geq 0, \quad \forall \quad x \in C, \tag{1.1.2}$$

and such that

$$y^* = Ax^* \in Q \ \text{ solves } \ F_2(y^*, y) + \phi_2(y^*, y) \geq 0, \quad \forall \quad y \in Q. \tag{1.1.3}$$

We denote the solution set of SGEP (1.1.2)-(1.1.3) by

$$SGEP(F_1, \phi_1, F_2, \phi_2) := \{x^* \in C : x^* \in GEP(F_1, \phi_1) \quad \text{and} \quad Ax^* \in GEP(F_2, \phi_2)\}.$$

Let $A : H \to H$ be a single-valued operator and $B : H \to 2^H$ be a multi-valued operator. The *Variational Inclusion Problem* (VIP) is defined as follows: Find a point $\hat{x} \in H$ such that

$$0 \in (A + B)\hat{x}. \tag{1.1.4}$$

The solution set of VIP (1.1.4) is denoted by $(A + B)^{-1}(0)$ and referred to as the set of zero points of $A+B$. The VIP (1.1.4) includes, as special cases, convex programming, split feasibility problems, variational inequalities and minimization problems. More precisely, some concrete problems in machine learning, image processing and linear inverse problems can be modeled mathematically as VIP (1.1.4), for example, see [24, 28, 33, 62]. There are several methods for solving VIP (1.1.4) with the most successful among the methods being the forward-backward splitting method introduced in [44, 59]. Specifically, the forward-backward splitting method is presented as follows:

$$x_{n+1} = (I + \lambda_n B)^{-1}(I - \lambda_n A)(x_n),$$

where λ_n is a positive parameter, the operator $(I - \lambda_n A)$ is the so-called forward operator and $(I+\lambda_n B)^{-1}$ is the resolvent operator, which was introduced in [51] and is often referred to as the backward operator. Recently, several authors have studied and extended the forward-backward splitting method, for example, see [3, 62, 86].
Let $S : H \to H$ be a nonlinear mapping, a point $\hat{x} \in H$ is called a fixed point of S if $S\hat{x} = \hat{x}$. We denote by $F(S)$, the set of all fixed points of S, i.e.

$$F(S) = \{\hat{x} \in H : S\hat{x} = \hat{x}\}. \tag{1.1.5}$$

If S is a multivalued mapping, i.e., $S : H \to 2^H$, then $x^* \in H$ is called a fixed point of S if

$$x^* \in Sx^*. \tag{1.1.6}$$

The fixed point theory for multivalued mappings can be utilized in various areas such as game theory, control theory, mathematical economics, etc. Fixed point is one of the most effective and successful methods for solving optimization problems such as equilibrium problem, variational inclusion problem and many more.

In this dissertation, our goal is to propose some iterative schemes for approximating the solutions of some important optimization problems in Hilbert spaces. We establish the strong convergence of the sequences generated by our iterative schemes and present some numerical experiments to illustrate the performance of our methods as well as compare them with some related methods in the literature.

1.2 Research motivation

In 2016, Suantai et al. [72] introduced the following iterative scheme for solving Split Equilibrium Problem and Fixed Point Problem of nonspreading multi-valued mapping in

Hilbert spaces:

$$\begin{cases} x_1 \in C \quad \text{arbitrarily}, \\ u_n = T_{r_n}^{F_1}\big(I - \gamma A^*(I - T_{r_n}^{F_2})A\big)x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Su_n, \end{cases} \tag{1.2.1}$$

for all $n \geq 1$, where C is a nonempty closed convex subset of a real Hilbert space H, $\{\alpha_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$, S is a nonspreading multivalued mapping, and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and A^* is the adjoint of the bounded linear operator A. Under the following conditions on the control sequences:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\liminf_{n\to\infty} r_n > 0$,

the authors proved that the sequence $\{x_n\}$ generated by (1.2.1) converges weakly to $p \in F(S) \cap SEP(F_1, F_2) \neq \emptyset$.

Bauschke and Combettes [9] pointed out that in solving Optimization Problems, strong convergence of iterative schemes are more desirable than their weak convergence counterparts. Hence, the need to construct iterative schemes that generate strong convergence sequence.

Takahashi *et al.* [85] introduced an iterative scheme known as the shrinking projection method for approximating the fixed point of nonexpansive single-valued mapping in Hilbert spaces. The shrinking projection method is a famous method, which plays a significant role in obtainning strong convergence for approximating fixed points of nonlinear mappings. The method has received much attention due to its applications, and it has been developed to solve many problems, such as, EPs, VIPs and FPPs in Hilbert spaces (see, for example [42]).

Very recently, Phuengrattana and Lerkchaiyaphum [60] introduced the following shrinking projection method for solving SGEP and FPP of a countable family of nonexpansive multivalued mappings: For $x_1 \in C$ and $C_1 = C$, then

$$\begin{cases} u_n = T_{r_n}^{(F_1,\phi_1)}(I - \gamma A^*(I - T_{r_n}^{(F_2,\phi_2)})A)x_n, \\ z_n = \alpha_n^{(0)} x_n + \alpha_n^{(1)} y_n^{(1)} + \ldots + \alpha_n^{(n)} y_n^{(n)}, \qquad y_n^{(i)} \in S_i u_n, \\ C_{n+1} = \{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \qquad n \in \mathbb{N}. \end{cases} \tag{1.2.2}$$

They proved that if

(i) $\liminf_{n\to\infty} r_n > 0$,

(ii) $\lim_{n\to\infty} \alpha_n^{(i)}$ exists for all $i \geq 0$,

then the sequence $\{x_n\}$ generated by (1.2.2) converges strongly to $P_\Gamma x_1$, where $\Gamma = \bigcap_{i=1}^\infty F(S_i) \cap SGEP(F_1, \phi_1, F_2, \phi_2) \neq \emptyset$ and S_i is a countable family of nonexpansive multivalued mappings.

It is important to point out at this point that the step size γ of the above algorithms plays an essential role in the convergence properties of iterative methods. The results obtained by the authors in [72] and [60], and several other related results in the literature involve step size that requires prior knowledge of the operator norm, $||A||$. One of the drawbacks with such algorithms is that they are usually not easy to implement because they require computation of the operator norm $||A||$, which is very difficult if not impossible to calculate or even estimate. Moreover, the step size defined by such algorithms are often very small and deteriorates the convergence rate of the algorithm. In practice, a larger stepsize can often be used to yield better numerical results.

Based on the heavy ball methods of a two-order time dynamical system, Polyak [61] first proposed an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. The inertial algorithm is a two-step iteration where the next iterate is defined by making use of the previous two iterates. Recently, several researchers have constructed some fast iterative algorithms by using inertial extrapolation (see, e.g., [2, 3, 6, 10, 21, 25, 37]).

Motivated by the above results and the ongoing research interest in this direction, we present a new self-adaptive inertial shrinking projection algorithm, which does not require any prior knowledge of the operator norm for finding a common element of the set of solutions of SGEP and the set of common fixed points of a countable family of nonexpansive multivalued mappings in Hilbert spaces. We prove strong convergence theorem for the proposed algorithm and obtain some consequent results. Moreover, we apply our results to solving Split Mixed Variational Inequality Problem (SMVIP) and Split Minimization Problem (SMP), and we provide numerical examples to illustrate the efficiency of the proposed algorithm in comparison with existing results in the current literature.

In [45], Liu introduced the following algorithm for finding a common element of the set of solutions of EP and set of fixed points of a k-strictly pseudocontractive mapping in the setting of real Hilbert spaces:

Algorithm 1.2.1.

$$
\begin{aligned}
F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \quad \forall y \in C, \\
y_n = \beta_n u_n + (I - \beta_n) S u_n, \\
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n D) y_n, \quad \forall\, n \in \mathbb{N},
\end{aligned}
$$

where $S : C \to H$ is a k-strictly pseudocontractive mapping, $f : H \to H$ is a contraction with constant $\rho \in (0, 1)$ and D is a strongly positive bounded linear operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\rho}$. Under some conditions on the control parameters, the author proved that the sequence generated by the algorithm converges strongly to an element in the solution set, which also solves certain variational inequality.

Wang in [91] proposed the following general composite iterative method for approximating a common solution of an infinite family of strict pseudo-contractions in Hilbert spaces:

Algorithm 1.2.2.

$$\begin{cases} x_1 \in C \\ y_n = \beta_n x_n + (1-\beta_n)W_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n D)y_n, \quad \forall \quad n \geq 1, \end{cases}$$

where W_n is a mapping defined by (4.2.1), f is a contraction with constant $\rho \in (0,1), D$ is a k-Lipschitzian and η - strongly monotone operator with $0 < \mu < 2\eta/k^2$. Under appropriate conditions on the control parameters, they proved that the sequence generated by Algorithm 1.2.2 converges strongly to a common element of the fixed points of an infinite family of strict pseudo-contractions, which is a also a unique solution of certain variational inequality problem.

In 2018, Cholamjiak *et al.* [19] introduced the following inertial forward-backward splitting algorithm, which combines Halpern and Mann iteration methods for solving inclusion problems in Hilbert spaces:

Algorithm 1.2.3.

$$\begin{aligned} y_n &= x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} &= \beta_n u + \xi_n y_n + \mu_n J^B_{\lambda_n}(y_n - \lambda_n A y_n), \quad n \geq 1, \end{aligned}$$

where $A : H \to H$ is a k-inverse strongly monotone operator and $B : H \to 2^H$ is a maximal monotone operator, $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}, 0 < \lambda_n \leq 2k$, $\{\alpha_n\} \subset [0, \alpha]$ with $\alpha \in [0, 1)$ and $\{\beta_n\}, \{\xi_n\}$ and $\{\mu_n\}$ are sequences in $[0, 1]$ with $\beta_n + \xi_n + \mu_n = 1$. Under the following conditions on the control parameters:

(1) $\sum_{n=1}^{\infty} \alpha_n||x_n - x_{n-1}|| < \infty$;

(2) $\lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty$;

(3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2k$;

(4) $\liminf_{n\to\infty} \mu_n > 0$,

they proved that the sequence generated by Algorithm 1.2.3 converges strongly to an element in the solution set.

However, authors have pointed out that the summability condition (1) of Algorithm 1.2.3 is a drawback in its implementation (see [52]).

More recently, Thong and Vinh [89], studied the problem of finding a common element of the set of solutions of variational inclusion problem and the fixed points set of a nonexpansive mapping. They introduced the following modified inertial forward-backward splitting algorithm combined with viscosity technique for finding a common solution of the problems in Hilbert spaces.

Algorithm 1.2.4.

Initialization: Select $x_0, x_1 \in H$ and set $n := 1$.

Step 1. Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}),$$
$$z_n = (I + \lambda B)^{-1}(I - \lambda A)w_n.$$

If $z_n = w_n$ then stop (z_n is a solution to (1.3.3)). Otherwise, go to **Step 2**.

Step 2. Compute

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)Tz_n.$$

Let $n := n + 1$ and return to **Step 1**.

Where $T : H \to H$ is a nonexpansive mapping, $f : H \to H$ is a contraction with constant $\rho \in [0, 1)$, $A : H \to H$ is a k-inverse strongly monotone operator, $B : H \to 2^H$ is a maximal monotone operator and $\lambda \in (0, 2k)$ is the step size of the algorithm. Under the following conditions on the control sequences:

(1) $\{\beta_n\} \subset (0, 1), \lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty, \lim_{n\to\infty} \frac{\beta_{n-1}}{\beta_n} = 1;$

(2) $\{\alpha_n\} \subset [0, \alpha), \alpha > 0, \lim_{n\to\infty} \frac{\alpha_n}{\beta_n}||x_n - x_{n-1}|| = 0,$

the authors proved that the sequence generated by Algorithm (1.2.4) converges strongly to an element in the solution set.

We observe that the summability condition in Algorithm 1.2.3 has been dispensed in Algorithm 1.2.4. However, we point out that the step size of Algorithm 1.2.4 is a constant and hence admits the same value for each iteration. Moreover, additional restriction was imposed on the control parameter β_n, that is, $\lim_{n\to\infty} \frac{\beta_{n-1}}{\beta_n} = 1$.

Motivated and inspired by the results in [19, 45, 89, 91], and the ongoing research in this direction, we study the problem of finding common solutions of Equilibrium Problem (EP), Variational Inclusion Problem (VIP)and Fixed Point Problem (FPP) for an infinite family of strict pseudocontractive mappings. We propose an iterative scheme which combines inertial technique with viscosity method for approximating common solutions of these problems in Hilbert spaces. Under mild conditions, we prove a strong theorem for the proposed algorithm and apply our results to approximate the solutions of other optimization problems. Finally, we present a numerical example to demonstrate the efficiency of our algorithm in comparison with other existing methods in the literature. Our results improve and complement contemporary results in the literature in this direction.

1.3 Statement of problems

This dissertation focus on the following problems:

- Split Generalized Equilibrium Problem (SGEP): Let $C \subseteq H_1$ and $Q \subseteq H_2$, where H_1 and H_2 are real Hilbert spaces. Let $F_1, \phi_1 : C \times C \to \mathbb{R}$ and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions, and $A : H_1 \to H_2$ be a bounded linear operator. The *SGEP* is defined as follows: Find $x^* \in C$ such that

$$F_1(x^*, x) + \phi_1(x^*, x) \geq 0, \quad \forall \quad x \in C, \tag{1.3.1}$$

and such that

$$y^* = Ax^* \in Q \ \text{ solves } \ F_2(y^*, y) + \phi_2(y^*, y) \geq 0, \quad \forall \quad y \in Q. \tag{1.3.2}$$

We denote the solution set of SGEP (1.3.1)-(1.3.2) by

$$SGEP(F_1, \phi_1, F_2, \phi_2) := \{x^* \in C : x^* \in GEP(F_1, \phi_1) \quad \text{and} \quad Ax^* \in GEP(F_2, \phi_2)\}.$$

- Variational Inclusion Problem: Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and let $A : H \to H$ be a single-valued operator and $B : H \to 2^H$ be a multi-valued operator. The *Variational Inclusion Problem* (VIP) is formulated as finding a point $\hat{x} \in H$ such that

$$0 \in (A + B)\hat{x}. \tag{1.3.3}$$

We propose some iterative schemes for approximating the solutions of these problems, establish the strong convergence of the sequences generated by these iterative schemes and present numerical experiments to illustrate the performance of these iterative schemes as well as compare them with some related iterative schemes in the literature. We apply these results to solve some other important optimization problems.

1.4 Objectives

The main objectives of this work are

(i) to review some essential results on Split Generalized Equilibruim Problem (SGEP) and Variational Inclusion Problem (VIP),

(ii) to propose some iterative schemes for approximating the solutions of SGEP, VIP and related optimization problems,

(iii) to establish the strong convergence of the sequences generated by the proposed algorithms and obtain some consequent results,

(iv) to provide numerical experiments to illustrate the performance of the proposed algorithms in comparison with some existing results in the current literature,

(v) to apply our results to study certain optimization problems.

1.5 Organization of the dissertation

The remaining chapters of this dissertation are organized as follows
Chapter 2: In this chapter, we recall some basic notions, definitions and preliminary results that are useful in establishing our main results.

Chapter 3: In this chapter, we introduce a shrinking projection method of inertial type with self-adaptive step size for finding a common element of the set of solutions of split generalized equilibrium problem and the set of common fixed points of a countable family of nonexpansive multivalued mappings in real Hilbert spaces. Also, we present numerical examples to illustrate the efficiency of the algorithm in comparison with other existing algorithms.

Chapter 4: In this chapter, we study the problem of finding common solutions of Equilibrium Problem (EP), Variational Inclusion Problem (VIP) and Fixed Point Problem (FPP) for an infinite family of strict pseudocontractive mappings. We propose an iterative scheme which combines inertial technique with viscosity method for approximating common solutions of these problems in Hilbert spaces. Under mild conditions, we prove a strong theorem for the proposed algorithm and apply our results to approximate the solutions of other optimization problems. Finally, we present a numerical example to demonstrate the efficiency of our algorithm in comparison with other existing methods in the literature.

Chapter 5: In this chapter, we present conclusion, highlight our contribution to knowledge and give some possible areas for future research work.

CHAPTER 2

Preliminaries

In this chapter, we recall some basic notions, definitions and preliminary results that will be employed in this study.

2.1 Some useful results in Hilbert space

In this dissertation, our study is carried out in the framework of Hilbert space. Thus, we give the definition of Hilbert space with examples and some basic results that are useful in establishing our main results.

Definition 2.1.1. Let H be a linear space over the scalar field $\mathbb{F}$, ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). An inner product on H is a mapping $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{F}$ satisfying the following conditions for all $x, y, z \in H, \mu, \lambda \in \mathbb{F}$:

(i) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$,

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$,

(ii) $\langle \mu x + \lambda y, z\rangle = \mu\langle x, z\rangle + \lambda\langle y, z\rangle$.

The pair $(H, \langle \cdot, \cdot \rangle)$ is called an inner product space.

Specifically, from (ii) and (iii), the following property can be deduced:

(iv) $\langle x, \mu y + \lambda z\rangle = \bar{\mu}\langle x, y\rangle + \bar{\lambda}\langle x, z\rangle$.

Definition 2.1.2. A Hilbert space is a complete inner product space, that is, an inner product space $(H, \langle \cdot, \cdot \rangle)$ in which every Cauchy sequence in H converges to a point in H.

The following are examples of Hilbert space:

Example 2.1.1. (i) The space $\mathbb{R}^n$ is a Hilbert space with the inner product defined as follows:

$$\langle \alpha, \beta \rangle = \alpha_1, \beta_1 + \alpha_2, \beta_2 + \cdots + \alpha_n, \beta_n = \sum_{i=1}^{n} \alpha_i \beta_i,$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ are in $\mathbb{R}^n$.

(ii) The space $l^2(\mathbb{C})$ is a Hilbert space with the inner product defined as follows:

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n},$$

where $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$ are in $l^2(\mathbb{C})$.

(iii) The space $L^2(\mathbb{R})$ of real valued functions such that

$$\int_{\mathbb{R}} |f(x)|^2 dx < \infty,$$

is a Hilbert space with the inner product defined as follows:

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) g(x) dx,$$

where f, g are in $L^2(\mathbb{R})$.

The following results will be needed in the sequel:

Lemma 2.1.2. *[78] In a real Hilbert space H, the following inequalities hold for all $x, y \in H$:*

(i) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle;$

(ii) $||x + y||^2 = ||x||^2 + 2\langle x, y \rangle + ||y||^2;$

(iii) $||x - y||^2 = ||x||^2 - 2\langle x, y \rangle + ||y||^2.$

Lemma 2.1.3. *[36] Let H be a Hilbert space, $\{x_n\}$ be a sequence in H, and $\alpha_1, \alpha_2, ...\alpha_N$ be real numbers such that $\sum_{i=1}^{N} \alpha_i = 1$. Then*

$$\left\| \sum_{i=1}^{N} \alpha_i x_i \right\|^2 = \sum_{i=1}^{N} \alpha_i \|x_i\|^2 - \sum_{1 \leq i,j \leq N} \alpha_i \alpha_j \|x_i - x_j\|^2. \quad (2.1.1)$$

Lemma 2.1.4. *[70] Let H be a Hilbert space, and let $\{x_n\}$ be a sequence in H. Let $u, v \in H$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ that converge weakly to u and v respectively, then $u = v$.*

Lemma 2.1.5. *[50] Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x, y, z \in H$and a real number α, the set $\{u \in C : \|y - u\|^2 \leq \|x - u\|^2 + \langle z, u \rangle + \alpha\}$ is closed and convex.*

2.2 Some useful operators and important results

The following are useful operators and fundamental functional analysis results needed in this study:

Definition 2.2.1. Let C be a nonempty closed convex subset of a real Hilbert space H. A single-valued mapping $S : C \to C$ is said to be

- *L-Lipschitz* if there exist $L > 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall \quad x, y \in C;$$

  if $L = 1$, then S is nonexpansive while S is called a contraction if $L \in (0, 1)$,

- *δ-inverse strongly monotone* if there exists a positive real number δ such that

$$\langle x - y, Sx - Sy\rangle \geq \delta||Sx - Sy||^2, \quad \forall\ x, y \in C;$$

- *monotone* if and only if

$$\langle y - x, Sy - Sx\rangle \geq 0, \quad \forall\ x, y \in C.$$

If S is δ-inverse strongly monotone, for each $\gamma \in (0, 2\delta]$, it is known that $I - \gamma S$ is a nonexpansive single-valued mapping.

A subset K of H is called *proximal* if for each $x \in H$, there exists $y \in K$ such that

$$||x - y|| = d(x, K).$$

We denote by $CB(C), CC(C), K(C)$ and $P(C)$ the families of all nonempty closed bounded subsets of C, nonempty closed convex subset of C, nonempty compact subsets of C, and nonempty proximal bounded subsets of C respectively. The Pompeiu-Hausdorff metric on $CB(C)$ is defined by

$$H(A, B) := \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\},$$

for all $A, B \in CB(C)$. Let $S : C \to 2^C$ be a multivalued mapping. We say that S satisfies the *endpoint condition* if $Sp = \{p\}$ for all $p \in F(S)$. For multivalued mappings $S_i : C \to 2^C$ $(i \in \mathbb{N})$ with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, we say that S_i satisfies the *common endpoint condition* if $S_i(p) = \{p\}$ for all $i \in \mathbb{N}, p \in \cap_{i=1}^{\infty} F(S_i)$.

We recall some basic and useful definitions on multivalued mappings.

Definition 2.2.2. A multivalued mapping $S : C \to CB(C)$ is said to be *nonexpansive* if

$$H(Sx, Sy) \leq ||x - y||, \quad \forall \quad x, y \in C.$$

The class of nonexpansive multivalued mappings contains the class of nonexpansive single-valued mappings. If S is a nonexpansive single-valued mapping on a closed convex subset of a Hilbert space, then $F(S)$ is closed and convex. The closedness of $F(S)$ can easily be extended to the multivalued case. However, the convexity of $F(S)$ cannot be extended (see, e.g., [?]). But, if S is a nonexpansive multivalued mapping which satisfies the endpoint condition, then $F(S)$ is always closed and convex as shown by the following result:

Lemma 2.2.1. *[20] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to CB(C)$ be a nonexpansive multivalued mapping with $F(S) \neq \emptyset$ and $Sp = \{p\}$ for each $p \in F(S)$. Then $F(S)$ is a closed and convex subset of C.*

The *best approximation operator* P_S for a multivalued mapping $S : C \to P(C)$ is defined by

$$P_S(x) := \{y \in Sx : ||x - y|| = d(x, Sx)\}.$$

It is known that $F(S) = F(P_S)$ and P_S satisfies the endpoint condition. Song and Cho [68] gave an example of a best approximation operator P_S which is nonexpansive, but where S is not necessarily nonexpansive.

Definition 2.2.3. Let $S : C \to CB(C)$ be a multivalued mapping. The multivalued mapping $I-S$ is said to be demiclosed at zero if for any sequence $\{x_n\} \subset C$ which converges weakly to q and the sequence $\{||x_n - u_n||\}$ converges strongly to 0, where $u_n \in Sx_n$, then $q \in F(S)$. If S is a multivalued nonexpansive mapping, then $I - S$ is demiclosed at zero.

Lemma 2.2.2. *[30, 54] Let C be a nonempty closed convex subset of a real Hilbert space H, and let $P_C : H \to C$ be the metric projection. Then*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall \ \ x \in H, y \in C.$$

The following are examples of metric projection:

Example 2.2.3. Let $C = [a, b]$ be a closed rectangle in $\mathbb{R}^n$, where $a = (a_1, a_2, \cdots, a_n)^T$ and $b = (b_1, b_2, \cdots, b_n)^T$. The metric projection with the i^{th} coordinate denoted by $(P_C x)_i$ is given by

$$(P_C x)_i = \begin{cases} a_i, & x_i < a_i, \\ x_i, & x_i \in [a_i, b_i], \\ b_i, & x_i > b_i, \end{cases}$$

for $1 \leq i \leq n$.

Example 2.2.4. Let $C = \{y \in H : \langle \eta, y \rangle = \beta\}$ be a hyperplane with $\eta \neq 0$, then the metric projection onto C is defined by

$$P_C x = x - \frac{\langle \eta, x \rangle - \beta}{\|\eta\|^2} \eta, \quad \forall \ \eta \in \mathbb{R}.$$

Assumption 2.2.5. *Let C be a nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ and $\phi_1 : C \times C \to \mathbb{R}$ be two bifunctions satisfying the following conditions:*

(A1) $F_1(x,x) = 0$ *for all* $x \in C$*,*

(A2) F_1 *is monotone, that is,* $F_1(x,y) + F_1(y,x) \leq 0$ *for all* $x, y \in C$*,*

(A3) F_1 *is upper hemicontinuous, that is, for all* $x, y, z \in C$*,* $\lim_{t \downarrow 0} F_1\big(tz + (1-t)x, y\big) \leq F_1(x,y)$*,*

(A4) for each $x \in C, y \mapsto F_1(x,y)$ *is convex and lower semicontinuous,*

(A5) $\phi_1(x,x) \geq 0$ *for all* $x \in C$*,*

(A6) for each $y \in C, x \mapsto \phi_1(x,y)$ *is upper semicontinuous,*

(A7) for each $x \in C, y \mapsto \phi_1(x,y)$ *is convex and lower semicontinuous,*

and assume that for fixed $r > 0$ *and* $z \in C$*, there exists a nonempty compact convex subset* K *of* H_1 *and* $x \in C \cap K$ *such that*

$$F_1(y,x) + \phi_1(y,x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \quad \forall y \in C \backslash K.$$

Lemma 2.2.6. *[48] Let* C *be a nonempty closed and convex subset of a Hilbert space* H_1*. Let* $F_1 : C \times C \to \mathbb{R}$ *and* $\phi_1 : C \times C \to \mathbb{R}$ *be two bifunctions satisfying Assumption (2.2.5). Assume that* ϕ_1 *is monotone. For* $r > 0$ *and* $x \in H_1$*, define a mapping* $T_r^{(F_1,\phi_1)} : H_1 \to C$ *as follows:*

$$T_r^{(F_1,\phi_1)}(x) = \left\{ z \in C : F_1(z,y) + \phi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}, \tag{2.2.1}$$

for all $x \in H_1$, Then

(i) for each $x \in H_1, T_r^{(F_1,\phi_1)}(x) \neq \emptyset$,

(ii) $T_r^{(F_1,\phi_1)}$ is single-valued,

(iii) $T_r^{(F_1,\phi_1)}$ is firmly nonexpansive, that is, for any $x, y \in H_1$,

$$\|T_r^{(F_1,\phi_1)}x - T_r^{(F_1,\phi_1)}y\|^2 \leq \langle T_r^{(F_1,\phi_1)}x - T_r^{(F_1,\phi_1)}y, x - y\rangle,$$

(iv) $F(T_r^{(F_1,\phi_1)}) = GEP(F_1, \phi_1)$,

(v) $GEP(F_1, \phi_1)$ is compact and convex.

Furthermore, assume that $F_2 : Q \times Q \to \mathbb{R}$ and $\phi_2 : Q \times Q \to \mathbb{R}$ satisfy *Assumption 2.2.5*, where Q is a nonempty closed and convex subset of a Hilbert space H_2. For all $s > 0$ and $w \in H_2$, define the mapping $T_s^{(F_2,\phi_2)} : H_2 \to Q$ by

$$T_s^{(F_2,\phi_2)}(v) = \left\{ w \in Q : F_2(w,d) + \phi_2(w,d) + \frac{1}{s}\langle d - w, w - v\rangle \geq 0, \quad \forall d \in Q \right\}. \tag{2.2.2}$$

Then we have:

(vi) For each $v \in H_2, T_s^{(F_2,\phi_2)}(v) \neq \emptyset$,

(vii) $T_s^{(F_2,\phi_2)}$ is single-valued,

(viii) $T_s^{(F_2,\phi_2)}$ is firmly nonexpansive,

(ix) $F(T_s^{(F_2,\phi_2)}) = GEP(F_2, \phi_2)$,

(x) $GEP(F_2, \phi_2)$ is closed and convex,

where $GEP(F_2, \phi_2)$ is the solution set of the following generalized equilibrium problem: find $y^* \in Q$ such that

$$F_2(y^*, y) + \phi_2(y^*, y) \geq 0 \quad \forall\, y \in Q.$$

Moreover, $SGEP(F_1, \phi_1, F_2, \phi_2)$ is a closed and convex set.

Lemma 2.2.7. *[22] Let C be a nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ and $\phi_1 : C \times C \to \mathbb{R}$ be two bifunctions satisfying Assumption 2.2.5, and let $T_r^{(F_1,\phi_1)}$ be defined as in Lemma 2.2.6 for $r > 0$. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then*

$$\|T_{r_2}^{(F_1,\phi_1)}y - T_{r_1}^{(F_1,\phi_1)}x\| \leq \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \|T_{r_2}^{(F_1,\phi_1)}y - y\|.$$

2.3 Some useful results on metric projection

Definition 2.3.1. Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The metric projection P_C is a map defined on H onto C which assigns to each $x \in H$, the unique point in C, denoted by $P_C x$ such that

$$||x - P_C x|| = \inf\{||x - y|| : \; y \in C\}.$$

It is well known that $P_C x$ is characterized by the inequality $\langle x - P_C x, z - P_C x\rangle \leq 0 \;\; \forall z \in C$ and P_C is a firmly nonexpansive mapping. Thus, P_C is nonexpansive. Moreover, P_C satisfies the following properties:

(i) $\langle x - y, P_C x - P_C y\rangle \geq ||P_C x - P_C y||^2$, for every $x, y \in C$;

(ii) for $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y\rangle \geq 0, \quad \forall y \in C; \tag{2.3.1}$$

(iii) for $x \in H$ and $y \in C$,

$$||y - P_C(x)||^2 + ||x - P_C(x)||^2 \leq ||x - y||^2. \tag{2.3.2}$$

CHAPTER 3

On Split Generalized Equilibrium and Fixed Point Problems

3.1 Introduction

In this chapter, we introduce a shrinking projection method of inertial type with self-adaptive step size for finding a common element of the set of solutions of split generalized equilibrium problem and the set of common fixed points of a countable family of nonexpansive multivalued mappings in real Hilbert spaces. The self-adaptive step size incorporated helps to overcome the difficulty of having to compute the operator norm while the inertial term accelerates the rate of convergence of the proposed algorithm. Under standard and mild conditions, we prove a strong convergence theorem for the problems under consideration and obtain some consequent results. Finally, we apply our result to solving split mixed variational inequality and split minimization problems, and we present numerical examples to illustrate the efficiency of our algorithm in comparison with other existing algorithms.
Precisely, we study the following problem: Let $C \subseteq H_1$ and $Q \subseteq H_2$, where H_1 and H_2 are real Hilbert spaces. Let $F_1, \phi_1 : C \times C \to \mathbb{R}$ and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions, and $A : H_1 \to H_2$ be a bounded linear operator. The $SGEP$ is defined as follows: Find $x^* \in C$ such that

$$F_1(x^*, x) + \phi_1(x^*, x) \geq 0, \quad \forall \quad x \in C, \tag{3.1.1}$$

and such that

$$y^* = Ax^* \in Q \;\; \text{solves} \;\; F_2(y^*, y) + \phi_2(y^*, y) \geq 0, \quad \forall \quad y \in Q. \tag{3.1.2}$$

We denote the solution set of SGEP (3.1.1)-(3.1.2) by

$$SGEP(F_1, \phi_1, F_2, \phi_2) := \{x^* \in C : x^* \in GEP(F_1, \phi_1) \quad \text{and} \quad Ax^* \in GEP(F_2, \phi_2)\}.$$

3.2 Main results

In this section, we state and prove our strong convergence theorem for finding a common element of the set of solutions of SGEP and the set of common fixed points of a countable family of nonexpansive multivalued mappings in real Hilbert spaces.

Theorem 3.2.1. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}$ be a countable family of nonexpansive multivalued mappings of C into $CB(C)$. Let $F_1, \phi_1 : C \times C \to \mathbb{R}$, $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.2.5. Let ϕ_1, ϕ_2 be monotone, ϕ_1 be upper hemicontinuous, and F_2 and ϕ_2 be upper semicontinuous in the first argument. Assume that $\Omega = \bigcap_{i=1}^{\infty} F(S_i) \cap SGEP(F_1, \phi_1, F_2, \phi_2) \neq \emptyset$ and $S_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{\infty} F(S_i)$. Let $x_0, x_1 \in C$ with $C_1 = C$, and let $\{x_n\}$ be a sequence generated as follows:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = T_{r_n}^{(F_1,\phi_1)}(I - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})A)w_n, \\ z_n = \alpha_{n,0}u_n + \sum_{i=1}^{n} \alpha_{n,i} y_{n,i}, \quad y_{n,i} \in S_i u_n, \\ C_{n+1} = \{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle + \theta_n^2\|x_{n-1} - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.2.1}$$

$$\gamma_n = \begin{cases} \frac{\tau_n\|(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2}{\|A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2} & \text{if} \quad Aw_n \neq T_{r_n}^{(F_2,\phi_2)}Aw_n, \\ \gamma & \text{otherwise } (\gamma \text{ being any nonnegative real number}), \end{cases}$$

where $0 < a \leq \tau_n \leq b < 1, \{\theta_n\} \subset \mathbb{R}$, $\{\alpha_{n,i}\} \subset (0,1)$, such that $\sum_{i=0}^{n} \alpha_{n,i} = 1$, and $\{r_n\} \subset (0,\infty)$. Suppose that the following conditions hold:

(C1) $\liminf_{n\to\infty} r_n > 0$,

(C2) The limits $\lim_{n\to\infty} \alpha_{n,i} \in (0,1)$ *exist for all* $i \geq 0$.

Then the sequence $\{x_n\}$ generated by (3.2.1), converges strongly to $P_\Omega x_1$.

Proof. We divide the proof into several steps as follows:
Step 1: First, we show that $\{x_n\}$ is well-defined for every $n \in \mathbb{N}$.
By Lemma 2.2.1 and Lemma 2.2.6, we have that $SGEP(F_1, \phi_1, F_2, \phi_2)$ and $\bigcap_{i=1}^{\infty} F(S_i)$ are closed and convex subsets of C. Therefore, the solution set Ω is a closed and convex subset of C. By Lemma 2.1.5, it then follows that C_{n+1} is closed and convex for each $n \in \mathbb{N}$. Let $p \in \Omega$, then we have $p = T_{r_n}^{F_1,\phi_1}p$ and $Ap = T_{r_n}^{(F_2,\phi_2)}(Ap)$. Since $T_{r_n}^{(F_1,\phi_1)}$ is nonexpansive, by Lemma 2.1.2 we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}^{(F_1,\phi_1)}(w_n - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n) - p\|^2 \\ &\leq \|w_n - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n - p\|^2 \\ &= \|w_n - p\|^2 + \gamma_n^2\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 - 2\gamma_n\langle w_n - p, A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle. \end{aligned} \tag{3.2.2}$$

By the firmly nonexpansivity of $I-T_{r_n}^{(F_2,\phi_2)}$, we get

$$\begin{aligned}\langle w_n-p, A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle &= \langle Aw_n-Ap, (I-T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle\\ &= \langle Aw_n-Ap, (I-T_{r_n}^{(f_2,\phi_2)})Aw_n-(I-T_{r_n}^{(F_2,\phi_2)})Ap\rangle\\ &\geq \|(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2. \qquad (3.2.3)\end{aligned}$$

By substituting (3.2.3) into (3.2.2), applying the definition of γ_n and the condition on τ_n, we obtain

$$\begin{aligned}\|u_n-p\|^2 &\leq \|w_n-p\|^2+\gamma_n^2\|A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2-2\gamma_n\|(I-T_{r_n}^{(F_1,\phi_1)})Aw_n\|^2\\ &= \|w_n-p\|^2-\gamma_n\big[2\|(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2-\gamma_n\|A^*(I-T_{r_n}^{(F_1,\phi_1)})Aw_n\|^2\big]\\ &= \|w_n-p\|^2-\gamma_n(2-\tau_n)\|(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 \qquad (3.2.4)\\ &\leq \|w_n-p\|^2. \qquad (3.2.5)\end{aligned}$$

Applying Lemma (2.1.3) and using (3.2.5), we have

$$\begin{aligned}\|z_n-p\|^2 &= \|\alpha_{n,0}u_n+\sum_{i=1}^{n}\alpha_{n,i}y_{n,i}-p\|^2\\ &\leq \alpha_{n,0}\|u_n-p\|^2+\sum_{i=1}^{n}\alpha_{n,i}\|y_{n,i}-p\|^2-\alpha_{n,0}\sum_{i=1}^{n}\alpha_{n,i}\|u_n-y_{n,1}\|^2\\ &= \alpha_{n,0}\|u_n-p\|^2+\sum_{i=1}^{n}\alpha_{n,i}d(y_{n,i},S_ip)^2-\alpha_{n,0}\sum_{i=1}^{n}\alpha_{n,i}\|u_n-y_{n,i}\|^2\\ &\leq \alpha_{n,0}\|u_n-p\|^2+\sum_{i=1}^{n}\alpha_{n,i}H(S_iu_n,S_ip)^2-\alpha_{n,0}\sum_{i=1}^{n}\alpha_{n,i}\|u_n-y_{n,i}\|^2\\ &\leq \alpha_{n,0}\|u_n-p\|^2+\sum_{i=1}^{n}\alpha_{n,i}\|u_n-p\|^2-\alpha_{n,0}\sum_{i=1}^{n}\alpha_{n,i}\|u_n-y_{n,i}\|^2\\ &\leq \|u_n-p\|^2-\alpha_{n,0}\sum_{i=1}^{n}\alpha_{n,i}\|u_n-y_{n,i}\|^2 \qquad (3.2.6)\\ &\leq \|u_n-p\|^2. \qquad (3.2.7)\end{aligned}$$

Also, by applying Lemma 2.1.2(iii), we get

$$\begin{aligned}\|w_n-p\|^2 &= \|\big(x_n-p-\theta_n(x_{n-1}-x_1)\big)\|^2,\\ &= \|x_n-p\|^2-2\theta_n\langle x_n-p, x_{n-1}-x_n\rangle+\theta_n^2\|x_{n-1}-x_n\|^2. \qquad (3.2.8)\end{aligned}$$

By using (3.2.5) and (3.2.8) in (3.2.7), we have

$$\|z_n-p\|^2 \leq \|x_n-p\|^2-2\theta_n\langle x_n-p, x_{n-1}-x_n\rangle+\theta_n^2\|x_{n-1}-x_n\|^2. \qquad (3.2.9)$$

This shows that $p\in C_{n+1}$, and it follows that $\Omega\subset C_{n+1}\subset C_n$. Therefore, $P_{C_{n+1}}x_1$ is well-defined for every $x_1\in C$ and the sequence $\{x_n\}$ is well defined.

Step 2: Next, we show that $\lim_{n\to\infty} x_n = q$ for some $q \in C$.
We know that Ω is a nonempty closed convex subset of H_1, then there exists a unique $w \in \Omega$ such that $w = P_\Omega x_1$. Since $x_n = P_{C_n} x_1$ and $x_{n+1} \in C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$, we have

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \quad \forall \ n \in \mathbb{N}. \tag{3.2.10}$$

Similarly, since $\Omega \subset C_n$, we have

$$\|x_n - x_1\| \leq \|w - x_1\|, \quad \forall \ n \in \mathbb{N}. \tag{3.2.11}$$

Therefore, by (3.2.10) and (3.2.11) $\{\|x_n - x_1\|\}$ is bounded and nondecreasing, and it follows that $\{x_n\}$ is bounded. Consequently, $\{w_n\}, \{u_n\}, \{z_n\}$ and $\{y_{n,i}\}$ are bounded. Hence, $\lim_{n\to\infty} \|x_n - x_1\|$ exists. From the construction of C_n, it is clear that $x_m = P_{C_m} x_1 \in C_m \subset C_n$ for $m > n \geq 1$. By Lemma (2.2.2), we have that

$$\|x_m - x_n\|^2 \leq \|x_m - x_1\|^2 - \|x_n - x_1\|^2 \to 0 \qquad \text{as} \quad m, n \to \infty. \tag{3.2.12}$$

Since $\lim_{n\to\infty} \|x_n - x_1\|$ exists, then it follows that $\{x_n\}$ is a Cauchy sequence. By the completeness of H_1 and the closedness of C, we have that there exists an element $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

Step 3: We next show that $\lim_{n\to\infty} \|y_{n,i} - u_n\| = 0$ for all $i \in \mathbb{N}$.
From (3.2.12). we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.2.13}$$

Since $x_{n+1} \in C_{n+1}$, then we have

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 - 2\theta_n \langle x_n - x_{n+1}, x_{n-1} - x_n \rangle + \theta_n^2 \|x_{n-1} - x_n\|^2.$$

By (3.2.13), we obtain

$$\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0. \tag{3.2.14}$$

By applying (3.2.13) and (3.2.14), we get

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, \quad n \to \infty. \tag{3.2.15}$$

Hence, $\lim_{n\to\infty} z_n = q$.
By the triangle inequality we have that

$$\begin{aligned}\|w_n - x_n\| &= \|x_n + \theta_n(x_n - x_{n-1}) - x_n\| \\ &\leq \|x_n - x_n\| + \theta_n \|x_n - x_{n-1}\|.\end{aligned}$$

By (3.2.13), we obtain

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.2.16}$$

Applying (3.2.15) and (3.2.16), we get

$$\|z_n - w_n\| \leq \|z_n - x_n\| + \|x_n - w_n\| \to 0, \quad n \to \infty. \tag{3.2.17}$$

From (3.2.5) and (3.2.6), we obtain

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \alpha_{n,0} \sum_{i=1}^{n} \alpha_{n,i} \|u_n - y_{n,i}\|^2,$$

which implies that

$$\begin{aligned}\alpha_{n,0}\alpha_{n,i}\|u_n - y_{n,i}\|^2 &\leq \alpha_{n,0} \sum_{i=1}^{n} \alpha_{n,i} \|u_n - y_{n,i}\|^2 \\ &\leq \|w_n - p\|^2 - \|z_n - p\|^2 \\ &\leq \|w_n - z_n\|(\|w_n - p\| + \|z_n - p\|).\end{aligned}$$

By the conditions on $\{\alpha_{n,i}\}$ and using (3.2.17), we get

$$\lim_{n\to\infty} \|u_n - y_{n,i}\| = 0, \quad \forall \quad i \in \mathbb{N}. \tag{3.2.18}$$

Step 4: We show that $\|u_n - x_n\| = 0$.
Substituting (3.2.4) into (3.2.7), we have

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \gamma_n(2 - \tau_n)\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2. \tag{3.2.19}$$

From this, we obtain

$$\begin{aligned}\gamma_n(2 - \tau_n)\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 &\leq \|w_n - p\|^2 - \|z_n - p\|^2 \\ &\leq \|w_n - z_n\|(\|w_n - p\| + \|z_n - p\|).\end{aligned}$$

By the definition of γ_n, condition on τ_n and (3.2.17), we get

$$\frac{\tau_n(2 - \tau_n)||(I - T_{r_n}^{(F_2,\phi_2)})Aw_n||^4}{||A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n||^2} \to 0, \quad n \to \infty,$$

which implies that

$$\frac{||(I - T_{r_n}^{(F_2,\phi_2)})Aw_n||^2}{||A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n||} \to 0, \quad n \to \infty.$$

Since $||A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n||$ is bounded, then it follows that

$$||(I - T_{r_n}^{(F_2,\phi_2)})Aw_n|| \to 0, \quad n \to \infty. \tag{3.2.20}$$

From this, we obtain

$$\|A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\| \leq \|A^*\|\|(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\| = \|A\|\|(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\| \to 0, \quad n \to \infty. \tag{3.2.21}$$

Since $T_{r_n}^{(F_1,\phi_1)}$ is firmly nonexpansive and $I-\gamma_n A^*(I-T_{r_n}^{(F_2,\phi_2)})A$ is nonexpansive by invoking Lemma 2.1.2(ii), we obtain

$$\begin{aligned}\|u_n-p\|^2 &= \|T_{r_n}^{(F_1,\phi_1)}(I-\gamma_n A^*(I-T_{r_n}^{(F_2,\phi_2)})A)w_n - T_{r_n}^{(F_1,\phi_1)}p\|^2\\ &\le \langle T_{r_n}^{(F_1,\phi_1)}(I-\gamma_n A^*(I-T_{r_n}^{(F_2,\phi_2)})A)w_n - T_{r_n}^{(F_1,\phi_1)}p,\\ &\quad (I-\gamma_n A^*(I-T_{r_n}^{(F_2,\phi_2)})A)w_n - p\rangle\\ &= \langle u_n-p, (I-\gamma_n A^*(I-T_{r_n}^{(F_2,\phi_2)})A)w_n-p\rangle\\ &= \frac{1}{2}\Big[\|u_n-p\|^2+\|(I-\gamma_n A^*(I-T_{r_n}^{(F_2,\phi_2)})A)w_n-p\|^2\\ &- \|u_n-w_n+\gamma_n A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2\Big]\\ &\le \frac{1}{2}\Big[\|u_n-p\|^2+\|w_n-p\|^2-\big(\|u_n-w_n\|^2+\gamma_n^2\|A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2\\ &+2\gamma_n\langle u_n-w_n, A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle\big)\Big],\end{aligned}$$

which implies that

$$\begin{aligned}\|u_n-p\|^2 &\le \|w_n-p\|^2-\|u_n-w_n\|^2-\gamma_n^2\|A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|\\ &+2\gamma_n\langle w_n-u_n, A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle\\ &\le \|w_n-p\|^2-\|u_n-w_n\|^2+2\gamma_n\|w_n-u_n\|\|A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|. \end{aligned} \tag{3.2.22}$$

Substituting (3.2.22) into (3.2.6), we have

$$\|z_n-p\|^2 \le \|w_n-p\|^2-\|u_n-w_n\|^2+2\gamma_n\|w_n-u_n\|\|A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|.$$

From this, we get

$$\begin{aligned}\|u_n-w_n\|^2 &\le \|w_n-p\|^2-\|z_n-p\|^2+2\gamma_n\|w_n-u_n\|\|A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|\\ &\le \|w_n-p\|^2-\|z_n-p\|^2+2\gamma_n M\|A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|\\ &\le \|w_n-z_n\|(\|w_n-p\|+\|z_n-p\|)+2\gamma_n M\|A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|, \end{aligned}\tag{3.2.23}$$

where $M=\sup\{\|w_n-u_n\| : n\in\mathbb{N}\}$.
By applying (3.2.17) and (3.2.21) to (3.2.23), we get

$$\lim_{n\to\infty}\|u_n-w_n\|=0. \tag{3.2.24}$$

Combining this together with (3.2.16) and (3.2.17), we have

$$\|u_n-z_n\|\le\|u_n-w_n\|+\|w_n-z_n\|\to 0,\quad n\to\infty \tag{3.2.25}$$

and

$$\|u_n-x_n\|\le\|u_n-w_n\|+\|w_n-x_n\|\to 0,\quad n\to\infty. \tag{3.2.26}$$

Step 5: Next, we show that $q\in\bigcap_{i=1}^{\infty}F(S_i)$.
By (3.2.18), for all $i\in\mathbb{N}$, we get that

$$\lim_{n\to\infty} d(u_n,S_iu_n)\le\lim_{n\to\infty}\|u_n-y_{n,i}\|=0. \tag{3.2.27}$$

For each $i \in \mathbb{N}$, we have

$$\begin{aligned} d(q, S_i q) &\le \|q - u_n\| + \|u_n - y_{n,i}\| + d(y_{n,i}, S_i q) \\ &\le \|q - u_n\| + d(u_n, S_i u_n) + H(S_i u_n, S_i q) \\ &\le 2\|q - u_n\| + d(u_n, S_i u_n). \end{aligned}$$

By (3.2.26), we have that $\lim_{n\to\infty} u_n = q$. Then it follows from (3.2.27) that

$$d(q, S_i q) = 0 \quad \forall\, i \in \mathbb{N}.$$

This show that $q \in S_i q$ for all $i \in \mathbb{N}$, which implies that $q \in \bigcap_{i=1}^{\infty} F(S_i)$.

Step 6: Next, we show that $q \in GEP(F_1, \phi_1, F_2, \phi_2)$.
First, we will show that $q \in GEP(F_1, \phi_1)$. Since $u_n = T_{r_n}^{(F_1,\phi_1)}\big(I - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})A\big)w_n$, then by Lemma 2.2.6, we obtain

$$F_1(u_n, y) + \phi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle \ge 0, \quad \forall\, y \in C,$$

which implies that

$$F_1(u_n, y) + \phi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle - \frac{1}{r_n}\langle y - u_n, \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle \ge 0, \quad \forall\, y \in C.$$

Since F_1 and ϕ_1 are monotone, we have

$$\frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle - \frac{1}{r_n}\langle y - u_n, \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle \ge F_1(y, u_n) + \phi_1(y, u_n), \quad \forall\, y \in C.$$

By (3.2.16) and (3.2.24), and $\lim_{n\to\infty} x_n = q$, we obtain $\lim_{n\to\infty} u_n = q$. Then by Condition (C1), (3.2.20), (3.2.24), Assumption 2.2.5, (A4) and (A7), it follows that

$$0 \ge F_1(y, q) + \phi_1(y, q) \quad \forall\, y \in C.$$

Let $y_t = ty + (1-t)q$ for all $t \in (0, 1]$ and $y \in C$. Then, $y_t \in C$, and thus $F_1(y_t, q) + \phi_1(y_t, q) \le 0$. Therefore, by Assumption 2.2.5, (A1)-(A7), we obtain

$$\begin{aligned} 0 &\le F_1(y_t, y_t) + \phi_1(y_t, y_t) \\ &\le t\big(F_1(y_t, y) + \phi_1(y_t, y)\big) + (1-t)\big(F_1(y_t, q) + \phi_1(y_t, q)\big) \\ &\le t\big(F_1(y_t, y) + \phi_1(y_t, y)\big) + (1-t)\big(F_1(q, y_t) + \phi_1(q, y_t)\big) \\ &\le F_1(y_t, y) + \phi_1(y_t, y). \end{aligned}$$

This implies that,

$$F_1(y_t, y) + \phi_1(y_t, y) \ge 0, \quad \forall \quad y \in C.$$

Letting $t \to 0$, and by using assumption together with the upper hemicontinuity of ϕ_1, we obtain

$$F_1(q, y) + \phi_1(q, y) \ge 0, \quad \forall y \in C.$$

This implies that $q \in GEP(F_1, \phi_1)$.

We next show that $Aq \in GEP(F_2, \phi_2)$. Since A is a bounded linear operator, then $Aw_n \to Aq$. Thus, from (3.2.20) we have

$$T_{r_n}^{(F_2,\phi_2)} Aw_n \to Aq. \tag{3.2.28}$$

By the definition of $T_{r_n}^{(F_2,\phi_2)} Aw_n$, we have

$$F_2\big(T_{r_n}^{(F_2,\phi_2)} Aw_n, y\big) + \phi_2(T_{r_n}^{(F_2,\phi_2)} Aw_n, y) + \frac{1}{r_n}\langle y - T_{r_n}^{(F_2,\phi_2)} Aw_n, T_{r_n}^{(F_2,\phi_2)} Aw_n - Aw_n\rangle \geq 0, \quad \forall y \in Q.$$

Since F_2 and ϕ_2 are upper semicontinuous in the first argument, then it follows from (3.2.28) that,

$$F_2(Aq, y) + \phi_2(Aq, y) \geq 0, \quad \forall\ y \in Q.$$

This implies that $Aq \in GEP(F_2, \phi_2)$. Hence, $q \in SGEP(F_1, \phi_1, F_2, \phi_2)$.

Step 7: Lastly, we show that $q = P_\Omega x_i$.
We know that $x_n = Pc_n x_1$ and $\Omega \subset C_n$, then it follows that $\langle x_1 - x_n, x_n - p\rangle \geq 0$ for all $p \in \Omega$. Hence, we have $\langle x_1 - q, q - p\rangle \geq 0$ for all $p \in \Omega$. This implies that $q = P_\Omega x_1$.

Consequently, we can conclude by Steps 1-7 that $\{x_n\}$ converges strongly to $q = P_\Omega x_1$ as required. □

If $\phi_1 = \phi_2 = 0$ in (3.1.1)-(3.1.2), then the SGEP reduces to the SEP. Hence, from Theorem 3.2.1, we obtain the following consequent result for approximating a common element of the set of solutions of SEP and the set of common fixed points of a countable family of nonexpansive multivalued mappings.

Corollary 3.2.2. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}$ be a countable family of nonexpansive multivalued mappings of C into $CB(C)$. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.2.5. Let F_2 be upper semicontinuous in the first argument. Assume that $\Omega = \bigcap_{i=1}^\infty F(S_i) \cap SEP(F_1, F_2) \neq \emptyset$ and $S_i p = \{p\}$ for each $p \in \bigcap_{i=1}^\infty F(S_i)$. Let $x_0, x_1 \in C$ with $C_1 = C$, and let $\{x_n\}$ be a sequence generated as follows:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = T_{r_n}^{F_1}(I - \gamma_n A^*(I - T_{r_n}^{F_2})A)w_n, \\ z_n = \alpha_{n,0}u_n + \sum_{i=1}^n \alpha_{n,i} y_{n,i}, \quad y_{n,i} \in S_i u_n, \\ C_{n+1} = \{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle + \theta_n^2\|x_{n-1} - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.2.29}$$

$$\gamma_n = \begin{cases} \frac{\tau_n\|(I - T_{r_n}^{F_2})Aw_n\|^2}{\|A^*(I - T_{r_n}^{F_2})Aw_n\|^2} & \text{if} \quad Aw_n \neq T_{r_n}^{F_2} Aw_n, \\ \gamma & \text{otherwise } (\gamma \text{ being any nonnegative real number}). \end{cases}$$

where $0 < a \leq \tau_n \leq b < 1, \{\theta_n\} \subset \mathbb{R}$, $\{\alpha_{n,i}\} \subset (0, 1)$, such that $\sum_{i=0}^n \alpha_{n,i} = 1$, and $\{r_n\} \subset (0, \infty)$. Suppose that the following conditions hold:

(C1) $\liminf_{n\to\infty} r_n > 0$,

(C2) The limits $\lim_{n\to\infty} \alpha_{n,i} \in (0,1)$ *exist for all* $i \geq 0$.

Then the sequence $\{x_n\}$ *generated by* (3.2.29)*, converges strongly to* $P_\Omega x_1$.

By the properties of the best approximation operator, we obtain the following consequent result.

Corollary 3.2.3. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}$ be a countable family of multivalued mappings of C into $P(C)$ such that P_{S_i} is nonexpansive and $I - S_i$ is demiclosed at zero for each $i \in \mathbb{N}$. Let $F_1, \phi_1 : C \times C \to \mathbb{R}$, $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.2.5. Let ϕ_1, ϕ_2 be monotone, ϕ_1 be upper hemicontinuous, and F_2 and ϕ_2 be upper semicontinuous in the first argument. Assume that $\Omega = \bigcap_{i=1}^{\infty} F(S_i) \cap SGEP(F_1, \phi_1, F_2, \phi_2) \neq \emptyset$. Let $x_0, x_1 \in C$ with $C_1 = C$, and let $\{x_n\}$ be a sequence generated as follows:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = T_{r_n}^{(F_1,\phi_1)}(I - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})A)w_n, \\ z_n = \alpha_{n,0}u_n + \sum_{i=1}^{n} \alpha_{n,i}y_{n,i}, \quad y_{n,i} \in P_{S_i}u_n, \\ C_{n+1} = \{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle + \theta_n^2\|x_{n-1} - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.2.30}$$

$$\gamma_n = \begin{cases} \frac{\tau_n\|(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2}{\|A^*(I-T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2} & \text{if} \quad Aw_n \neq T_{r_n}^{(F_2,\phi_2)}Aw_n, \\ \gamma & \text{otherwise } (\gamma \text{ being any nonnegative real number}). \end{cases}$$

where $0 < a \leq \tau_n \leq b < 1, \{\theta_n\} \subset \mathbb{R}$, $\{\alpha_{n,i}\} \subset (0,1)$, such that $\sum_{i=0}^{n} \alpha_{n,i} = 1$, and $\{r_n\} \subset (0,\infty)$. Suppose that the following conditions hold:

(C1) $\liminf_{n\to\infty} r_n > 0$,

(C2) The limits $\lim_{n\to\infty} \alpha_{n,i} \in (0,1)$ *exist for all* $i \geq 0$.

Then the sequence $\{x_n\}$ *generated by* (3.2.30)*, converges strongly to* $P_\Omega x_1$.

Proof. Since P_{S_i} satisfies the common endpoint condition and $F(S_i) = F(P_{S_i})$ for each $i \in \mathbb{N}$, then the result follows from Theorem 3.2.1. □

3.3 Applications

In this section, we apply our main result to approximate the solutions of some important optimization problems.

3.3.1 Split mixed variational inequality and fixed point problems

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $B : H \to H$ be a single-valued mapping and $\phi : C \times C \to \mathbb{R}$ be a bifunction. The *Mixed Variational Inequality Problem* (MVIP) is defined as follows:

$$\text{Find } \ x^* \in C \ \text{ such that } \ \langle y - x^*, Bx^* \rangle + \phi(x^*, y) \geq 0, \quad \forall\, y \in C. \tag{3.3.1}$$

We denote the set of solution of MVIP by $MVI(C, B, \phi)$. If we take $\phi = 0$ in (3.3.1), then the MVIP reduces to the *Variational Inequality Problem* (VIP), which is to find a point $x^* \in C$ such that $\langle y - x^*, Bx^* \rangle \geq 0, \quad \forall\, y \in C$. The solution set of the VIP is denoted by $VI(C, B)$. Variational inequality was first introduced independently by Fichera [26] and Stampacchia [69]. The VIP is a useful mathematical model that unifies many important concepts in applied mathematics, such as necessary optimality conditions, complementarity problems, network EPs, and systems of nonlinear equations (see [24, 27, 32, 34]). Several methods have been proposed and analyzed for solving VIP and related OPs, see [1, 4, 15, 29, 35, 41, 79, 81] and references therein.

Here, we apply our result to study the following *Split Mixed Variational Inequality Problem* (SMVIP):

$$\text{Find } \ x^* \in \bigcap_{i=1}^{\infty} F(S_i) \ \text{ such that } \ \langle x - x^*, B_1 x^* \rangle + \phi_1(x^*, x) \geq 0, \quad \forall\, x \in C, \tag{3.3.2}$$

and such that

$$y^* = Ax^* \in Q \ \text{ solves } \ \langle y - y^*, B_2 y^* \rangle + \phi_2(y^*, y) \geq 0, \quad \forall\, y \in Q, \tag{3.3.3}$$

where C and Q are nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, $\{S_i\}$ is a countable family of nonexpansive multivalued mappings of C into $CB(C), A : H_1 \to H_2$ is a bounded linear operator, $B_1 : C \to H_1, B_2 : Q \to H_2$ are monotone mappings, and $\phi_1 : C \times C \to \mathbb{R}, \phi_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying assumptions (A5)-(A7). Moreover, ϕ_1, ϕ_2 are monotone with ϕ_1 being upper hemicontinuous and ϕ_2 upper semicontinuous in the first argument. We denote the solution set of problem (3.3.2)-(3.3.3) by Ω and assume that $\Omega \neq \emptyset$. By taking $F_j(x, y) := \langle y - x, B_j x \rangle, j = 1, 2$, then the SMVIP (3.3.2)-(3.3.3) becomes the problem of finding a solution of the SGEP (3.1.1)-(3.1.2) which is also a solution of the countable family of nonexpansive multivalued mappings $\{S_i\}$. In addition, all the conditions of Theorem 3.2.1 are satisfied. Hence, Theorem 3.2.1 provides a strong convergence theorem for approximating a common solution of SMVIP and fixed point of a countable family of nonexpansive multivalued mappings.

3.3.2 Split minimization and fixed point problems

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $f : C \to \mathbb{R}, g : Q \to \mathbb{R}$ be two operators and $A : H_1 \to H_2$ be a bounded linear operator, then the *Split Minimization Problem* (SPM) is defined as follows:

$$\text{Find } \; x^* \in C \; \text{ such that } \; f(x^*) \leq f(x), \quad \forall\, x \in C, \tag{3.3.4}$$

and such that

$$y^* = Ax^* \in Q \; \text{ solves } \; g(y^*) \leq g(y), \quad \forall\, y \in Q. \tag{3.3.5}$$

We denote the solution set of SMP (3.3.4)-(3.3.5) by Φ and assume that $\Phi \neq \emptyset$. For some recent results on iterative algorithms for solving MP, (see [7, 8] and the references contained therein). Let $F_1(x,y) := f(y) - f(x)$ for all $x, y \in C$ and $F_2(u,v) := f(v) - f(u)$ for all $u, v \in Q$, and taking $\phi_1 = \phi_2 = 0$ in the SGEP (3.1.1)-(3.1.2). Then $F_1(x,y)$ and $F_2(u,v)$ satisfy assumptions (A1)-(A4) provided f and g are convex and lower semi-continuous on C and Q, respectively. Clearly, ϕ_1 and ϕ_2 satisfy assumptions (A5)-(A7). Therefore, from Theorem 3.2.1 we obtain a strong convergence theorem for approximating a common solution of SMP and fixed point problem for a countable family of nonexpansive multivalued mappings in real Hilbert spaces.

3.4 Numerical examples

In this section, we present some numerical experiments to illustrate the performance of our algorithm as well as comparing it with Algorithm 1.2.2 in the literature. All numerical computations were carried-out using Matlab version R2019(b).

We define the sequences $\{\alpha_{n,i}\}$ as follows for each $i \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$:

$$\alpha_{n,i} = \begin{cases} \frac{1}{b^{i+1}}\left(\frac{n}{n+1}\right), & n > i, \\ 1 - \frac{n}{n+1}\left(\sum_{k=1}^{n} \frac{1}{b^k}\right), & n = i, \\ 0, & n < i, \end{cases} \tag{3.4.1}$$

where $b > 1$.

Example 3.4.1. Let $H_1 = H_2 = \mathbb{R}$ and $C = Q = [0, 10]$. Let $A : H_1 \to H_2$ be defined by $Ax = \frac{x}{3}$ for all $x \in H_1$. Then, we have that $A^*y = \frac{y}{3}$ for all $y \in H_2$. For $x \in C, i \in \mathbb{N}$, we define the multivalued mappings $S_i : C \to CB(C)$ as follows:

$$S_i(x) = \left[0, \frac{x}{10i}\right], \quad \forall \;\; i \in \mathbb{N}. \tag{3.4.2}$$

It can easily be checked that S_i is nonexpansive for all $i \in \mathbb{N}$, $S_i(0) = \{0\}$, and $\bigcap_{i=1}^{\infty} F(S_i) = \{0\}$. We define the bifunctions $F_1, \phi_1 : C \times C \to \mathbb{R}$ by $F_1(x,y) = y^2 + 3xy - 4x^2$ and $\phi_1(x,y) = y^2 - x^2$ for $x, y \in C$, and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ by $F_2(w,v) = 2v^2 + wv - 3w^2$ and $\phi_2(w,v) = w - v$ for $w, v \in Q$. Choose $r_n = \frac{n-3}{n+2}, \theta_n = 0.8$, and $\tau_n = 0.7$. It can easily be verified that all the conditions of Theorem 3.2.1 are satisfied with $\Omega = \{0\}$. Now, we compute $T_r^{(F_1,\phi_1)}(x)$. We find $u \in C$ such that for all $z \in C$

$$\begin{aligned}
&0 \leq F_1(u,z) + \phi_1(u,z) + \frac{1}{r}\langle z-u, u-x\rangle \\
&\quad = 2z^2 + 3uz - 5u^2 + \frac{1}{r}\langle z-u, u-x\rangle \\
&\quad \Leftrightarrow \\
&0 \leq 2rz^2 + 3ruz - 5ru^2 + (z-u)(u-x) \\
&\quad = 2rz^2 + 3ruz - 5ru^2 + uz - xz - u^2 + ux \\
&\quad = 2rz^2 + (3ru + u - x)z + (-5ru^2 - u^2 + ux).
\end{aligned}$$

Let $h(z) = 2rz^2 + (3ru + u - x)z + (-5ru^2 - u^2 + ux)$. Then $h(z)$ is a quadratic function of z with coefficients $a = 2r, b = 3ru + u - x$, and $c = -5ru^2 - u^2 + ux$. We determine the discriminant $\triangle$ of $h(z)$ as follows:

$$\begin{aligned}
\triangle &= (3ru + u - x)^2 - 4(2r)(-5ru^2 - u^2 + ux) \\
&= 49r^2u^2 + 14ru^2 - 14rux + u^2 - 2ux + x^2 \\
&= ((7r+1)u - x)^2. \qquad (3.4.3)
\end{aligned}$$

By Lemma 2.2.6, $T_r^{(F_1,\phi_1)}$ is single-valued. Hence, it follows that $h(z)$ has at most one solution in $\mathbb{R}$. Therefore, from (3.4.3) we have that $u = \frac{x}{7r+1}$. This implies that $T_r^{(F_1,\phi_1)}(x) = \frac{x}{7r+1}$. Similarly, we compute $T_r^{(F_2,\phi_2)}(v)$. Find $w \in Q$ such that for all $d \in Q$

$$T_s^{(F_2,\phi_2)}(v) = \left\{ w \in Q : F_2(w,d) + \phi_2(w,d) + \frac{1}{s}\langle d-w, w-v\rangle \geq 0, \quad \forall\, d \in Q \right\}.$$

By following similar procedure as above, we obtain $w = \frac{v+s}{5s+1}$. This implies that $T_s^{(F_2,\phi_2)}(v) = \frac{v+s}{5s+1}$. We take $y_{n,i} = \frac{u_n}{10i}$ for all $i \in \mathbb{N}$. Then Algorithm (3.2.1) becomes

$$\begin{cases}
w_n = x_n + \theta_n(x_n - x_{n-1}), \\
u_n = \frac{w_n}{7r_n+1} - \gamma_n \frac{15w_nr_n + 2w_n - 3r_n}{9(7r_n+1)(5r_n+1)}, \\
z_n = \alpha_{n,0}u_n + \sum_{i=1}^{n} \alpha_{n,i}\frac{u_n}{10i}, \\
C_{n+1} = \left\{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle + \theta_n^2\|x_{n-1} - x_n\|^2\right\}, \\
x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N},
\end{cases}$$

where

$$\gamma_n = \begin{cases}
\frac{\tau_n\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2} & \text{if} \quad Aw_n \neq T_{r_n}^{(F_2,\phi_2)}Aw_n, \\
\gamma & \text{otherwise } (\gamma \text{ being any nonnegative real number}).
\end{cases}$$

In this example, we set the parameter b on $\{\alpha_{n,i}\}$ in (3.4.1) to be $b = 50$ and we choose different initial values as follows:
Case Ia: $x_0 = \frac{11}{2}, x_1 = \frac{2}{5}$;
Case Ib: $x_0 = 8, x_1 = 1$;
Case Ic: $x_0 = 5, x_1 = \frac{7}{10}$;

Case Id: $x_0 = 6, x_1 = \frac{4}{5}$.

We compare the performance of our Algorithm (3.2.1) with Algorithm (1.2.2). The stopping criterion used for our computation is $|x_{n+1} - x_n| < 10^{-4}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 3.1 and Table 3.1.

Table 3.1: **Numerical results for Example 3.4.1**

		Alg. 1.2.2	Alg. 3.2.1
Case Ia	CPU time (sec)	2.1794	0.1722
	No. of Iter.	13	3
Case Ib	CPU time (sec)	2.2136	0.1514
	No. of Iter.	14	3
Case Ic	CPU time (sec)	2.2338	0.1517
	No. of Iter.	14	3
Case Id	CPU time (sec)	2.1757	0.1495
	No. of Iter.	14	3

Example 3.4.2. Let $H_1 = H_2 = L_2([0,1])$ with the inner product defined as

$$\langle x, y \rangle = \int_0^1 x(t)y(t)dt, \quad \forall x, y \in L_2([0,1]).$$

Let

$$C := \{x \in H_1 : \langle a, x \rangle = d\},$$

where $a = 2t^2$ and $d \geq 0$. Here, we have

$$P_C(x) = x + \frac{d - \langle a, x \rangle}{||a||^2} a.$$

Also, let

$$Q := \{x \in H_2 : \langle c, x \rangle \leq e\},$$

where $c = \frac{t}{3}$ and $e = 1$, we get

$$P_Q(x) = x + \max\left\{0, \frac{e - \langle c, x \rangle}{||c||^2} c\right\}.$$

We define $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ by $F_1(x, y) = \langle L_1 x, y - x \rangle$ and $F_2(x, y) = \langle L_2 x, y - x \rangle$, where $L_1 x(t) = \frac{x(t)}{2}$ and $L_2 x(t) = \frac{x(t)}{5}$. It can easily be verified that F_1 and F_2 satisfy conditions (A1)-(A4). Also, take $\phi_1 = \phi_2 = 0$. Moreover, let $A : L_2([0,1]) \to L_2([0,1])$ be defined by $Ax(t) = \frac{x(t)}{2}$ and $A^* y(t) = \frac{y(t)}{2}$. Then, A is a bounded linear operator. We consider the case for which the countable family of nonexpansive

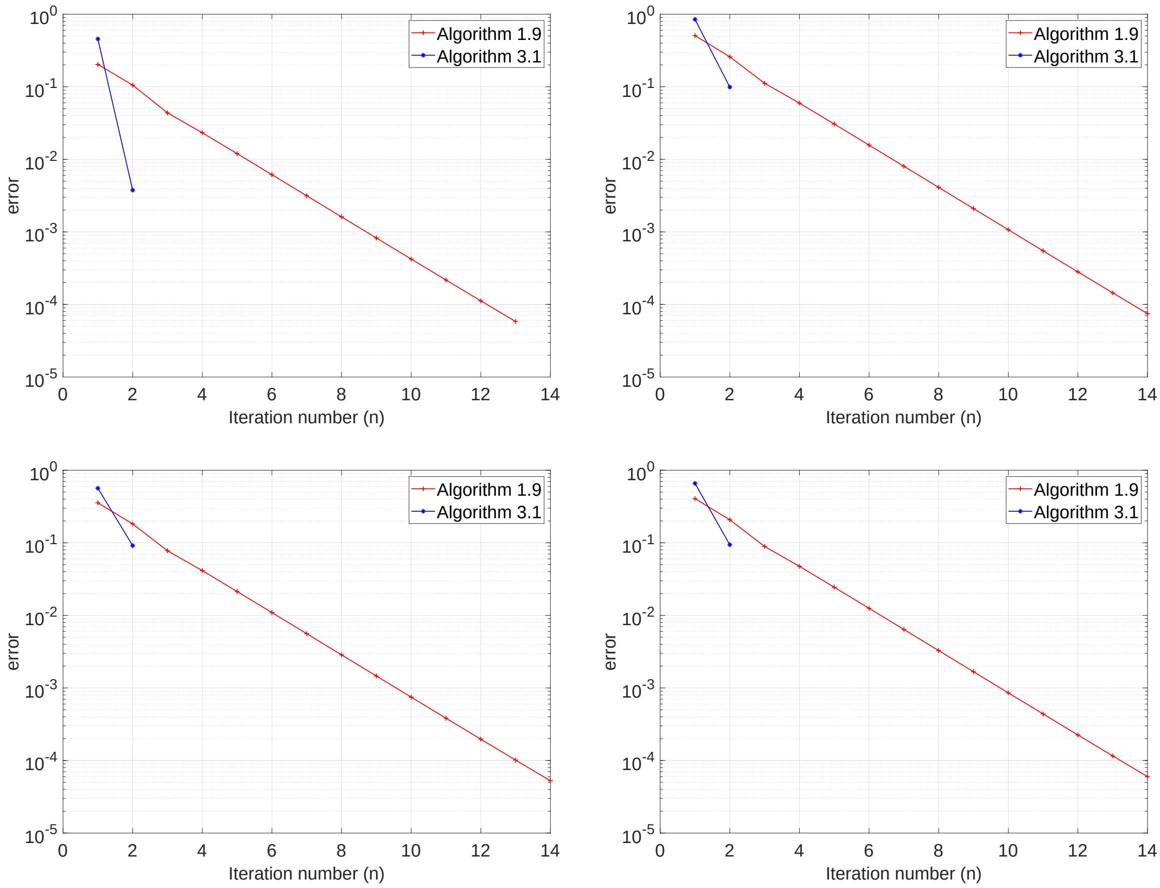

Figure 3.1: Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id.

multivalued mappings $\{S_i\}$ are singled-valued. Define a countable family of nonexpansive mappings $S_i : L^2([0,1]) \to L^2([0,1])$ by

$$(S_i x)(t) = \int_0^1 t^i x(s) ds \quad \text{for all } t \in [0,1].$$

Observe that S_i is nonexpansive for each $i \in \mathbb{N}$. Choose $\theta_n = 0.9, \tau_n = 0.8, r_n = \frac{n}{n+1}$. It can easily be checked that all the conditions on the control sequences in Theorem 3.2.1 are satisfied. Next, we compute $T_r^{(F_1,\phi_1)}(x)$. We find $z \in C$ such that for all $y \in C$

$$\begin{aligned}
&F_1(z,y) + \phi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0\\
&\Leftrightarrow \langle \frac{z}{2}, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0\\
&\Leftrightarrow \frac{z}{2}(y - z) + \frac{1}{r}(y - z)(z - x) \geq 0\\
&\Leftrightarrow (y - z)[rz + 2(z - x)] \geq 0\\
&\Leftrightarrow (y - z)[(r + 2)z - 2x] \geq 0. \qquad (3.4.4)
\end{aligned}$$

According to Lemma 2.2.6,

$$T_r^{(F_1,\phi_1)}(x) = \left\{ z \in C : F_1(z,y) + \phi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \ \forall\, y \in C \right\},$$

is single-valued for all $x \in H_1$. Hence, from (3.4.4) we have that $z = \frac{2x}{r+2}$. This implies that $T_r^{(F_1,\phi_1)}(x) = \frac{2x}{r+2}$. Similarly, we compute $T_r^{(F_2,\phi_2)}(v)$. We find $w \in Q$ such that for all $d \in Q$

$$T_s^{(F_2,\phi_2)}(v) = \left\{ w \in Q : F_2(w,d) + \phi_2(w,d) + \frac{1}{s}\langle d - w, w - v\rangle \geq 0, \quad \forall\, d \in Q \right\}.$$

Following similar procedure as above, we obtain $w = \frac{5v}{s+5}$. This implies that $T_s^{(F_2,\phi_2)}(v) = \frac{5v}{s+5}$. Then Algorithm (3.2.1) becomes

$$\begin{cases}
w_n = x_n + \theta_n(x_n - x_{n-1}),\\
u_n = \frac{2w_n}{r_n+2} - \gamma_n \frac{2r_n+5}{2(r_n+5)(r_n+2)} w_n,\\
z_n = \alpha_{n,0} u_n + \sum_{i=1}^{n} \alpha_{n,i} S_i u_n,\\
C_{n+1} = \{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle + \theta_n^2\|x_{n-1} - x_n\|^2\},\\
x_{n+1} = P_{C_{n+1}} x_1, \quad n \in \mathbb{N},
\end{cases}$$

where

$$\gamma_n = \begin{cases}
\frac{\tau_n\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2} & \text{if} \quad Aw_n \neq T_{r_n}^{(F_2,\phi_2)} Aw_n,\\
\gamma & \text{otherwise } (\gamma \text{ being any nonnegative real number}).
\end{cases}$$

Here, we set the parameter b on $\{\alpha_{n,i}\}$ in (3.4.1) to be $b = 2$ and we choose different initial values as follows:

Case Ia: $x_0 = t^3, x_1 = t^2 + t^4$;
Case Ib: $x_0 = t^2 + t^6 + t^8, x_1 = t^3$;
Case Ic: $x_0 = t^5 + t^9 + t^{11}, x_1 = t^5$;
Case Id: $x_0 = t + t^2 + t^4 + t^6, x_1 = t^2 + t^7$.

We compare the performance of our Algorithm (3.2.1) with Algorithm (1.2.2). The stopping criterion used for our computation is $||x_{n+1} - x_n|| < 10^{-4}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 3.2 and Table 3.2.

Table 3.2: **Numerical results for Example 3.4.2**

		Alg. 1.2.2	Alg. 3.2.1
Case Ia	CPU time (sec) No of Iter.	2.2241 23	1.3724 19
Case Ib	CPU time (sec) No. of Iter.	2.2247 23	1.2772 18
Case Ic	CPU time (sec) No of Iter.	2.1359 22	1.3056 18
Case Id	CPU time (sec) No of Iter.	2.3458 25	1.4506 20

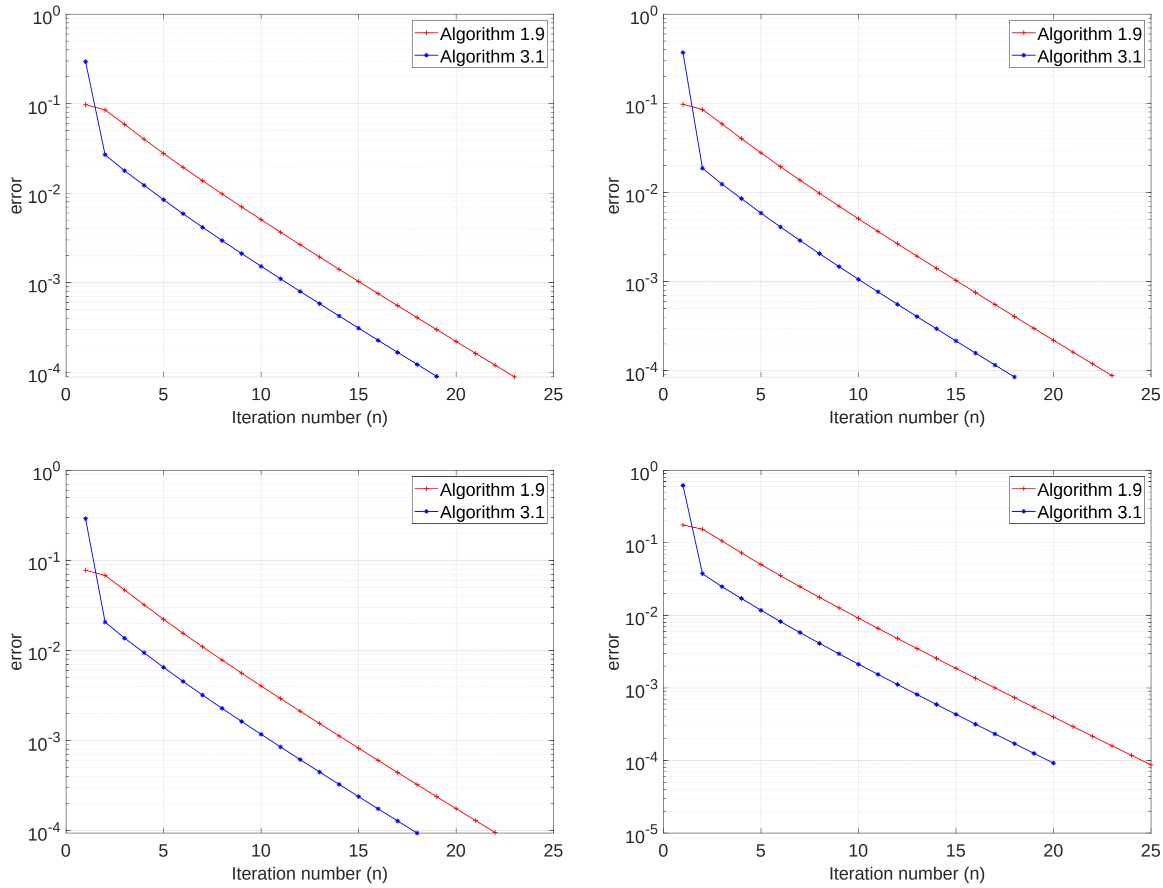

Figure 3.2: Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id.

CHAPTER 4

Inertial Algorithm for Solving Equilibrium, Variational Inclusion and Fixed Point Problems

4.1 Introduction

In this Chapter, we study the problem of finding common solutions of Equilibrium Problem (EP), Variational Inclusion Problem (VIP)and Fixed Point Problem (FPP) for an infinite family of strict pseudocontractive mappings. We propose an iterative scheme which combines inertial technique with viscosity method for approximating common solutions of these problems in Hilbert spaces. Under mild conditions, we prove a strong theorem for the proposed algorithm and apply our results to approximate the solutions of other optimization problems. Finally, we present a numerical example to demonstrate the efficiency of our algorithm in comparison with other existing methods in the literature. Our results improve and complement contemporary results in the literature in this direction.

More precisely, we study the following problem: Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and let $A : H \to H$ be a single-valued operator and $B : H \to 2^H$ be a multi-valued operator. The *Variational Inclusion Problem* (VIP) is formulated as finding a point $\hat{x} \in H$ such that

$$0 \in (A + B)\hat{x}.$$

Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction. The *Equilibrium Problem* (shortly, EP) in the sense of Blum and Oettli [11] is to find $\hat{x} \in C$ such that

$$F(\hat{x}, y) \geq 0, \quad \forall\, y \in C.$$

4.2 Preliminaries

In this section, we recall some useful definitions and lemmas required for establishing our main results.

Lemma 4.2.1. *[77, 78] Let H be a real Hilbert space, $\lambda \in (0,1)$, then $\forall x, y \in H$, we have*

(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$*;*

(ii) $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$*;*

(iii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$.

Lemma 4.2.2. *[76, 77] For each $x_1, \ldots, x_m \in H$ and $\alpha_1, \ldots, \alpha_m \in [0,1]$ with $\sum_{i=1}^m \alpha_i = 1$, the following holds:*

$$||\alpha_1 x_1 + \ldots + \alpha_m x_m||^2 = \sum_{i=1}^m \alpha_i ||x_i||^2 - \sum_{1\leq i<j\leq m} \alpha_i\alpha_j ||x_i - x_j||^2.$$

Lemma 4.2.3. *[?] Let $\{a_n\}, \{c_n\} \subset \mathbb{R}_+, \{\sigma_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that*

$$a_{n+1} \leq (1-\sigma_n)a_n + b_n + c_n \quad \text{for all } n \geq 0.$$

Assume $\sum_{n=0}^\infty |c_n| < \infty$. Then the following results hold:
(1) If $b_n \leq \beta\sigma_n$ for some $\beta \geq 0$, then $\{a_n\}$ is a bounded sequence.
(2) If we have

$$\sum_{n=0}^\infty \sigma_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{b_n}{\sigma_n} \leq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 4.2.4. *[2] Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^\infty \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n, \quad \text{for all } \; n \geq 1,$$

if $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.

Definition 4.2.1. Let H be a real Hilbert space H. A mapping $T : H \to H$ is said to be:

(1) *L-Lipschitz continuous*, where $L > 0$, if

$$||Tx - Ty|| \leq L||x - y||, \quad \forall \; x, y \in H;$$

if $L \in [0,1)$, then T is called a *contraction mapping;*

(2) *nonexpansive* if T is 1−Lipschitz continuous;

(3) *k-strictly pseudo-contractive* if there exists a constant $k \in [0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \quad \forall \ x, y \in H;$$

(4) *monotone* if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall \ x, y \in H;$$

(5) *k-inverse-strongly monotone* (k-ism), if there exists a constant $k > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq k\|Ax - Ay\|^2, \quad \forall \quad x, y \in H;$$

(6) *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \quad \forall \quad x, y \in H,$$

or equivalently,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall \quad x, y \in H.$$

Observe that the class of k-strict pseudo-contractive mappings properly contains the class of nonexpansive mappings. That is, T is nonexpansive if and only if T is 0-strict pseudo-contractive. It is known that if T is a k-strict pseudo-contractiion and $F(T) \neq \emptyset$, then $F(T)$ is a closed convex subset of H (see [92]). Strict pseudo-contractions have many applications, due to their ties with inverse strongly monotone operators. It is known that, if B is a strongly monotone operator, then $T = I - B$ is a strict pseudo-contraction, and so we can recast a problem of zeros for B as a fixed point problem for T, and vice versa (see e.g. [17, 67]).

Lemma 4.2.5. *[92] Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a k-strict pseudo-contractive mapping. Define a mapping $T : C \to C$ by $Tx = \alpha x + (1 - \alpha)Sx$ for all $x \in C$ and $\alpha \in [k, 1)$. Then T is a nonexpansive mapping such that $F(T) = F(S)$.*

Definition 4.2.2. [91] Let $\{S_n\}$ be a sequence of k_n-strict pseudo-contractions. Define $S'_n = t_n I + (1 - t_n)S_n, t_n \in [k_n, 1)$. Then, by Lemma 4.2.5, S'_n is nonexpansive. In this paper, we consider the mapping W_n defined by

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \zeta_n S'_n U_{n,n+1} + (1 - \zeta_n)I, \\ U_{n,n-1} = \zeta_{n-1} S'_{n-1} U_{n,n} + (1 - \zeta_{n-1})I, \\ \cdots, \\ U_{n,k} = \zeta_k S'_k U_{n,k+1} + (1 - \zeta_k)I, \\ U_{n,k-1} = \zeta_{k-1} S'_{k-1} U_{n,k} + (1 - \zeta_{k-1})I, \\ \cdots, \\ U_{n,2} = \zeta_2 S'_2 U_{n,3} + (1 - \zeta_2)I, \\ W_n = U_{n,1} = \zeta_1 S'_1 U_{n,2} + (1 - \zeta_1)I. \end{cases} \tag{4.2.1}$$

where $\{\zeta_i\}$ is a sequence of real numbers such that $0 \leq \zeta_i \leq 1$ for all $i \geq 1$. For each $n \geq 1$, such a mapping W_n is nonexpansive.

We have the following lemmas related to the mapping W_n, which are needed in proving our main results.

Lemma 4.2.6. *[66] Let $\{S'_i\}$ be an infinite family of nonexpansive mappings on a Hilbert space H such that $\bigcap_{i=1}^{\infty} F(S'_i) \neq \emptyset$ and $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq b < 1$ for all $i \geq 1$. Then we have the following:*

(1) W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(S'_i)$ for each $n \geq 1$;

(2) for each $x \in H$ and for each positive integer k, the $\lim_{n\to\infty} U_{n,k}x$ exists;

(3) the mapping W defined by

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x \quad \text{for all } x \in H \tag{4.2.2}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(S'_i)$, which is called the modified W-mapping generated by $S_1, S_2, \cdots, \zeta_1, \zeta_2, \cdots$ and $t_1, t_2, \cdots$.

By combining Lemma 4.2.5 and Lemma 4.2.6, it follows that $F(W) = \bigcap_{i=1}^{\infty} F(S'_i) = \bigcap_{i=1}^{\infty} F(S_i)$.

Lemma 4.2.7. *[16] Let $\{S'_i\}$ be an infinite family of nonexpansive mappings on a Hilbert space H such that $\bigcap_{i=1}^{\infty} F(S'_i) \neq \emptyset$ and $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq b < 1$ for all $i \geq 1$, where b is a positive real number. If K is any bounded subset of H, then*

$$\lim_{n\to\infty} \sup_{x\in K} ||Wx - W_n x|| = 0.$$

Lemma 4.2.8. *[57] Each Hilbert space H satisfies the Opial condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$ holds for every $y \in H$ with $y \neq x$.*

Lemma 4.2.9. *[92] If S is a k-strict pseudo-contraction on closed convex subset C of a real Hilbert space H, then $I - S$ is demiclosed at any point $y \in H$.*

Lemma 4.2.10. *[84] Let $A : H \to H$ be a k-inverse-strongly monotone mapping, then*

1. *A is $\frac{1}{k}$-Lipschitz continuous and monotone mapping;*
2. *if λ is any constant in $(0, 2]$, then the mapping $I - \lambda A$ is nonexpansive , where I is the identity mapping on H.*

Definition 4.2.3. Let $B : H \to 2^H$ be a multi-valued maximal monotone mapping. Then the resolvent mapping $J_\lambda^B : H \to H$ associated with B is defined by $J_\lambda^B(x) = (I + \lambda B)^{-1}(x) \quad \forall \quad x \in H$, for some $\lambda > 0$, where I is the identity operator on H.

It is well known that if $B : H \to 2^H$ is a multi-valued maximal monotone mapping and $\lambda > 0$, then $\mathrm{Dom}(J_\lambda^B) = H$, and J_λ^B is single-valued and firmly nonexpansive mapping (see [83] for more details on maximal monotone mapping).

Assumption 4.2.11. *For solving the EP, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:*

(A1) $F(x, x) = 0$ *for all* $x \in C$*;*

(A2) F *is monotone, that is,* $F(x, y) + F(y, x) \leq 0$ *for all* $x, y \in C$*;*

(A3) F *is upper hemicontinuous, that is, for all* $x, y, z \in C$, $\lim_{t\downarrow 0} F\big(tz + (1-t)x, y\big) \leq F(x, y)$*;*

(A4) for each $x \in C, y \mapsto F(x, y)$ *is convex and lower semicontinuous.*

Lemma 4.2.12. *[48] Let* C *be a nonempty closed convex subset of a Hilbert space* H *and* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying Assumption 4.2.11. For* $r > 0$ *and* $x \in H$*, define a mapping* $T_r^F : H \to C$ *as follows:*

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall\ y \in C\}. \tag{4.2.3}$$

Then T_r^F *is well defined and the following hold:*

(1) for each $x \in H, T_r^F(x) \neq \emptyset$*;*

(2) T_r^F *is single-valued;*

(3) T_r^F *is firmly nonexpansive, that is, for any* $x, y \in H$*,*

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle;$$

(4) $F(T_r^F) = EP(F)$*;*

(5) $EP(F)$ *is closed and convex.*

Lemma 4.2.13. *[46] Let* E *be a real Banach space. Let* $B : E \to 2^E$ *maximal monotone operator and* $A : E \to E$ *be a k-inverse strongly monotone mapping on* E*. Define* $T_\lambda = (I + \lambda B)^{-1}(I - \lambda A)$, $\lambda > 0$*. Then we have*

(i) $F(T_\lambda) = (A + B)^{-1}(0)$*;*

(ii) for $0 < s \leq \lambda$ *and* $x \in E$, $\|x - T_s x\| \leq 2\|x - T_\lambda x\|$*.*

Lemma 4.2.14. *[83] Let* $B : H \to 2^H$ *be a set-valued maximal monotone mapping and* $\lambda > 0$*. Then* J_λ^B *is a single-valued and firmly nonexapansive mapping.*

4.3 Main results

In this section, we present the proposed algorithm and investigate its convergence. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : H \to H$ be a $k-$ism and $B : H \to 2^H$ be a maximal monotone mapping. Let $f : H \to H$ be a contraction mapping with coefficient $\rho \in (0, 1)$. Let $\{W_n\}$ be a sequence defined by (4.2.1) and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 4.2.11. Suppose that the solution set denoted by $\Gamma = (A + B)^{-1}(0) \cap EP(F) \cap \bigcap_{i=1}^{\infty} F(S_i)$ is nonempty, where $S_i : H \to H$ is an infinite family of k_i-strict pseudo-contractions. We establish the convergence of the algorithm under the following conditions on the control parameters:

(C1) Let $\{\delta_n\}, \{\xi_n\}, \{\mu_n\} \subset (0, 1), \{\beta_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$;

(C2) Let $\alpha > 0, \{\theta_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\theta_n}{\beta_n} = 0$;

(C3) $0 < \liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 2k, \{r_n\} \subset (0, \infty)$ such that $\liminf_{n\to\infty} r_n > 0$.

Now, the proposed algorithm is presented as follows:

Algorithm 4.3.1.

Step 0 : Select initial data $x_0, x_1 \in H$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose α_n such that $0 \le \alpha_n \le \hat{\alpha}_n$ with $\hat{\alpha}_n$ defined by

$$\hat{\alpha}_n = \begin{cases} \min\left\{\alpha, \ \frac{\theta_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases} \tag{4.3.1}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Step 3: Compute

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \ge 0, \quad \forall \quad y \in H.$$

Step 4: Compute

$$v_n = \delta_n w_n + (1 - \delta_n)u_n.$$

Step 5: Compute

$$z_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)v_n.$$

Step 6: Compute

$$x_{n+1} = \beta_n f(x_n) + \xi_n x_n + \mu_n W_n z_n.$$

Set n := n + 1 and return to **Step 1.**

Remark 4.3.2. By conditions (C1) and (C2), one can easily verify from (4.3.1) that

$$\lim_{n\to\infty} \alpha_n||x_n - x_{n-1}|| = 0 \text{ and } \lim_{n\to\infty} \frac{\alpha_n}{\beta_n}||x_n - x_{n-1}|| = 0. \tag{4.3.2}$$

Now, we state the strong convergence theorem as follows:

Theorem 4.3.3. *Suppose that $\{x_n\}$ is a sequence generated by Algorithm 4.3.1 such that conditions (C1)-(C3) are satisfied and $\Gamma \neq \emptyset$. Then the sequence $\{x_n\}$ converges strongly to an element $\hat{x} \in \Gamma$, where $\hat{x} = P_\Gamma \circ f(\hat{x})$.*

First, we prove some lemmas which will be employed in establishing Theorem 4.3.3.

Lemma 4.3.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.3.1, then $\{x_n\}$ is bounded.*

Proof. Let $q \in \Gamma$, then by Lemma 4.2.14 and the conditions on the control parameters, we have

$$\begin{aligned} \|z_n - q\|^2 =& \|(I + \lambda_n B)^{-1}(I - \lambda_n A)v_n - (I - \lambda_n B)^{-1}(I - \lambda_n A)q\|^2 \\ &\leq \|v_n - q - \lambda_n(Av_n - Aq)\|^2 \\ &= \|v_n - q\|^2 - 2\lambda_n\langle Av_n - Aq, v_n - q\rangle + \lambda_n^2\|Av_n - Aq\|^2 \\ &\leq \|v_n - q\|^2 - 2\lambda_n k\|Av_n - Aq\|^2 + \lambda_n^2\|Av_n - Aq\| \\ &= \|v_n - q\|^2 - (2k - \lambda_n)\lambda_n\|Av_n - Aq\|^2 & (4.3.3) \\ &\leq \|v_n - q\|^2. & (4.3.4) \end{aligned}$$

Thus, from (4.3.4), we have

$$\|z_n - q\| \leq \|v_n - q\|. \tag{4.3.5}$$

From the definition of w_n, we have

$$\begin{aligned} \|w_n - q\| &= \|x_n + \alpha_n(x_n - x_{n-1}) - q\| \\ &\leq \|x_n - q\| + \alpha_n\|x_n - x_{n-1}\| \\ &= \|x_n - q\| + \beta_n\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|. \end{aligned} \tag{4.3.6}$$

From Remark 4.3.2, it is known that $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0$. Then there exists a constant $L_1 > 0$ such that $\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \leq L_1$, for all $n \geq 1$. Thus from (4.3.6), we obtain

$$\|w_n - q\| \leq \|x_n - q\| + \beta_n L_1. \tag{4.3.7}$$

Let $T_{r_n}^F w_n = \{u_n \in C : F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0\}$. This implies that $u_n = T_{r_n}^F w_n$. Since $q \in \Gamma$, then $T_{r_n}^F q = q$. By the nonexpansiveness of $T_{r_n}^F$, we have

$$||u_n - q|| = ||T_{r_n}^F w_n - q|| \leq ||w_n - q||. \tag{4.3.8}$$

From the definition of v_n and by applying (4.3.8), we have

$$\begin{aligned} \|v_n - q\| &= \|\delta_n w_n + (1 - \delta_n)u_n - q\| \\ &\leq \delta_n\|w_n - q\| + (1 - \delta_n)\|u_n - q\| \\ &\leq \delta_n\|w_n - q\| + (1 - \delta_n)\|w_n - q\| \\ &= \|w_n - q\|. \end{aligned} \tag{4.3.9}$$

By combining (4.3.5), (4.3.7) and (4.3.9), we have

$$\|z_n - q\| \le \|x_n - q\| + \beta_n L_1. \tag{4.3.10}$$

Now, by applying (4.3.5), (4.3.6) and (4.3.9), we obtain

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\beta_n f(x_n) + \xi_n x_n + \mu_n W_n z_n - q\| \\
&= ||\beta_n(f(x_n) - f(q)) + \beta_n(f(q) - q) + \xi_n(x_n - q) + \mu_n(W_n z_n - q)|| \\
&\le \beta_n \rho ||x_n - q|| + \beta_n ||fq - q|| + \xi_n ||x_n - q|| + \mu_n ||z_n - q|| \\
&\le \beta_n \rho ||x_n - q|| + \beta_n ||fq - q|| + \xi_n ||x_n - q|| + \mu_n(||x_n - q|| + \beta_n L_1) \\
&= (1 - \beta_n(1-\rho))||x_n - q|| + \beta_n ||fq - q|| + \mu_n \beta_n L_1 \\
&= (1 - \beta_n(1-\rho))||x_n - q|| + \beta_n(1-\rho)\frac{||fq - q|| + \mu_n L_1}{1-\rho} \\
&\le (1 - \beta_n(1-\rho))||x_n - q|| + \beta_n(1-\rho)M^*,
\end{aligned}$$

where $M^* := \sup_{n\in N}\{\frac{||fq-q||+\mu_n L_1}{1-\rho}\}$. Setting $a_n := ||x_n - q||,\ \ b_n := \beta_n(1-\rho)M^*,\ \ c_n := 0,$ and $\sigma_n := \beta_n(1-\rho)$. By Lemma 4.2.3(1) and the assumptions on the control parameters, it follows that $\{||x_n - q||\}$ is bounded and thus $\{x_n\}$ is bounded. Consequently, $\{w_n\}, \{u_n\}, \{v_n\}, \{z_n\}, \{f(x_n)\}$ are all bounded. □

Lemma 4.3.5. *The following inequality holds for all $q \in \Gamma$ and $n \in \mathbb{N}$:*

$$\begin{aligned}
||x_{n+1} - q||^2 \le& \Big(1 - \frac{2\beta_n(1-\rho)}{(1-\beta_n\rho)}\Big)||x_n - q||^2 + \frac{2\beta_n(1-\rho)}{(1-\beta_n\rho)}\Big\{\frac{\beta_n}{2(1-\rho)}L_3 \\
&+ \frac{3L_2\mu_n(1-\beta_n)}{2(1-\rho)}\frac{\alpha_n}{\beta_n}||x_n - x_{n-1}|| + \frac{1}{(1-\rho)}\langle f(q) - q, x_{n+1} - q\rangle\Big\} \\
&- \frac{\mu_n(1-\beta_n)}{(1-\beta_n\rho)}\Big\{(2k - \lambda_n)\lambda_n||Av_n - Aq||^2 + \delta_n(1-\delta_n)||w_n - u_n||^2\Big\}.
\end{aligned}$$

Proof. Let $q \in \Gamma$, then by applying the Cauchy-Schwartz inequality and Lemma 4.2.1(i), we have

$$\begin{aligned}
||w_n - q||^2 &= ||x_n + \alpha_n(x_n - x_{n-1}) - q||^2 \\
&= ||x_n - q||^2 + \alpha_n^2||x_n - x_{n-1}||^2 + 2\alpha_n\langle x_n - q, x_n - x_{n-1}\rangle \\
&\le ||x_n - q||^2 + \alpha_n^2||x_n - x_{n-1}||^2 + 2\alpha_n||x_n - x_{n-1}||||x_n - q|| \\
&= ||x_n - q||^2 + \alpha_n||x_n - x_{n-1}||(\alpha_n||x_n - x_{n-1}|| + 2||x_n - q||) \\
&\le ||x_n - q||^2 + 3L_2\alpha_n||x_n - x_{n-1}|| \\
&= ||x_n - q||^2 + 3L_2\beta_n\frac{\alpha_n}{\beta_n}||x_n - x_{n-1}||,
\end{aligned} \tag{4.3.11}$$

where $L_2 := \sup_{n\in\mathbb{N}}\{||x_n - q||, \alpha_n||x_n - x_{n-1}||\} > 0$.

Also, by applying Lemma 4.2.2 and (4.3.8), we obtain

$$\begin{aligned}||v_n - q||^2 &= ||\delta_n w_n + (1-\delta_n)u_n - q||^2 \\ &= \delta_n||w_n - q||^2 + (1-\delta_n)||u_n - q||^2 - \delta_n(1-\delta_n)||w_n - u_n||^2 \\ &\le \delta_n||w_n - q||^2 + (1-\delta_n)||w_n - q||^2 - \delta_n(1-\delta_n)||w_n - u_n||^2 \\ &= ||w_n - q||^2 - \delta_n(1-\delta_n)||w_n - u_n||^2. \end{aligned} \tag{4.3.12}$$

By invoking Lemma 4.2.1, and using (4.3.3), (4.3.11) and (4.3.12), we have

$$\begin{aligned}||x_{n+1} - q||^2 &= ||\beta_n f(x_n) + \xi_n x_n + \mu_n W_n z_n - q||^2 \\ &= ||\beta_n(f(x_n) - q) + \xi_n(x_n - q) + \mu_n(W_n z_n - q)||^2 \\ &\le ||\xi_n(x_n - q) + \mu_n(W_n z_n - q)||^2 + 2\beta_n\langle f(x_n) - q, x_{n+1} - q\rangle \\ &\le \xi_n^2||x_n - q||^2 + \mu_n^2||W_n z_n - q||^2 + 2\xi_n\mu_n||x_n - q||||W_n z_n - q|| \\ &+ 2\beta_n\langle f(x_n) - q, x_{n+1} - q\rangle \\ &\le \xi_n^2||x_n - q||^2 + \mu_n^2||z_n - q||^2 + 2\xi_n\mu_n||x_n - q||||z_n - q|| \\ &+ 2\beta_n\langle f(x_n) - q, x_{n+1} - q\rangle \\ &\le \xi_n^2||x_n - q||^2 + \mu_n^2||z_n - q||^2 + \xi_n\mu_n(||x_n - q||^2 + ||z_n - q||^2) \\ &+ 2\beta_n\langle f(x_n) - q, x_{n+1} - q\rangle \\ &= \xi_n(\xi_n + \mu_n)||x_n - q||^2 + \mu_n(\mu_n + \xi_n)||z_n - q||^2 + 2\beta_n\langle f(x_n) - q, x_{n+1} - q\rangle \\ &\le \xi_n(1-\beta_n)||x_n - q||^2 + \mu_n(1-\beta_n)\Big\{||x_n - q||^2 + 3L_2\beta_n\frac{\alpha_n}{\beta_n}||x_n - x_{n-1}|| \\ &- (2k - \lambda_n)\lambda_n||Av_n - Aq||^2 - \delta_n(1-\delta_n)||w_n - u_n||^2\Big\} \\ &+ 2\beta_n\langle f(x_n) - f(q), x_{n+1} - q\rangle + 2\beta_n\langle f(q) - q, x_{n+1} - q\rangle \\ &\le \xi_n(1-\beta_n)||x_n - q||^2 + \mu_n(1-\beta_n)\Big\{||x_n - q||^2 + 3L_2\beta_n\frac{\alpha_n}{\beta_n}||x_n - x_{n-1}|| \\ &- (2k - \lambda_n)\lambda_n||Av_n - Aq||^2 - \delta_n(1-\delta_n)||w_n - u_n||^2\Big\} + 2\beta_n\rho||x_n - q||||x_{n+1} - q|| \\ &+ 2\beta_n\langle f(q) - q, x_{n+1} - q\rangle \end{aligned}$$

$$\begin{aligned}&\le \xi_n(1-\beta_n)||x_n - q||^2 + \mu_n(1-\beta_n)\Big\{||x_n - q||^2 + 3L_2\beta_n\frac{\alpha_n}{\beta_n}||x_n - x_{n-1}|| \\ &- (2k - \lambda_n)\lambda_n||Av_n - Aq||^2 - \delta_n(1-\delta_n)||w_n - u_n||^2\Big\} + \beta_n\rho(||x_n - q||^2 \\ &+ ||x_{n+1} - q||^2) + 2\beta_n\langle f(q) - q, x_{n+1} - q\rangle \\ &= \big((1-\beta_n)^2 + \beta_n\rho\big)||x_n - q||^2 + \beta_n\rho||x_{n+1} - q||^2 \\ &+ 3L_2\mu_n(1-\beta_n)\beta_n\frac{\alpha_n}{\beta_n}||x_n - x_{n-1}|| \\ &+ 2\beta_n\langle f(q) - q, x_{n+1} - q\rangle - \mu_n(1-\beta_n)\Big\{(2k - \lambda_n)\lambda_n||Av_n - Aq||^2 \\ &+ \delta_n(1-\delta_n)||w_n - u_n||^2\Big\}. \end{aligned}$$

This implies that

$$\begin{aligned}
||x_{n+1}-q||^2 \le& \frac{(1-2\beta_n+\beta_n^2+\beta_n\rho)}{(1-\beta_n\rho)}||x_n-q||^2+\frac{\beta_n}{(1-\beta_n\rho)}\Big\{3L_2\mu_n(1-\beta_n)\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||\\
&+2\langle f(q)-q,x_{n+1}-q\rangle\Big\}-\frac{\mu_n(1-\beta_n)}{(1-\beta_n\rho)}\Big\{(2k-\lambda_n)\lambda_n||Av_n-Aq||^2\\
&+\delta_n(1-\delta_n)||w_n-u_n||^2\Big\}\\
\le& \frac{(1-2\beta_n++\beta_n\rho)}{(1-\beta_n\rho)}||x_n-q||^2+\frac{\beta_n^2}{(1-\beta_n\rho)}||x_n-p||^2\\
&+\frac{\beta_n}{(1-\beta_n\rho)}\Big\{3L_2\mu_n(1-\beta_n)\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||+2\langle f(q)-q,x_{n+1}-q\rangle\Big\}\\
&-\frac{\mu_n(1-\beta_n)}{(1-\beta_n\rho)}\Big\{(2k-\lambda_n)\lambda_n||Av_n-Aq||^2+\delta_n(1-\delta_n)||w_n-u_n||^2\Big\}\\
\le& \Big(1-\frac{2\beta_n(1-\rho)}{(1-\beta_n\rho)}\Big)||x_n-q||^2+\frac{2\beta_n(1-\rho)}{(1-\beta_n\rho)}\Big\{\frac{\beta_n}{2(1-\rho)}L_3\\
&+\frac{3L_2\mu_n(1-\beta_n)}{2(1-\rho)}\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||+\frac{1}{(1-\rho)}\langle f(q)-q,x_{n+1}-q\rangle\Big\}\\
&-\frac{\mu_n(1-\beta_n)}{(1-\beta_n\rho)}\Big\{(2k-\lambda_n)\lambda_n||Av_n-Aq||^2+\delta_n(1-\delta_n)||w_n-u_n||^2\Big\},
\end{aligned}$$

where $L_3 := \sup\{||x_n-q||^2 : n\in\mathbb{N}\}$. This completes the proof. □

Lemma 4.3.6. *The following inequality holds for all $q\in\Gamma$ and $n\in\mathbb{N}$:*

$$\begin{aligned}
||x_{n+1}-q||^2 \le& (1-\beta_n)||x_n-q||^2+\beta_n\Big\{||f(x_n)-q||^2+3L_2\mu_n\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||\Big\}\\
&+2L_4||Av_n-Aq||-\mu_n||v_n-z_n||^2-\xi_n\mu_n||W_nz_n-x_n||^2.
\end{aligned}$$

Proof. Applying the fact that $(I+\lambda_nB)^{-1}$ is firmly nonexpansive and $I-\lambda_nA$ is nonexpansive, we have

$$\begin{aligned}
\|z_n-q\|^2 =& \|(I+\lambda_nB)^{-1}(I-\lambda_nA)v_n-\|(I+\lambda_nB)^{-1}(I-\lambda_nA)q\|^2\\
\le& \langle z_n-q,(I-\lambda_nA)v_n-(I-\lambda_nA)q\rangle\\
=& \frac{1}{2}\|(I-\lambda_nA)v_n-(I-\lambda_nA)q\|^2+\frac{1}{2}\|z_n-q\|^2-\frac{1}{2}\|(I-\lambda_nA)v_n\\
&-(I-\lambda_nA)q-(z_n-q)\|^2\\
\le& \frac{1}{2}\|v_n-q\|^2+\frac{1}{2}\|z_n-q\|^2-\frac{1}{2}\|v_n-z_n-\lambda_n(Av_n-Aq)\|^2\\
\le& \frac{1}{2}\|v_n-q\|^2+\frac{1}{2}\|z_n-q\|^2-\frac{1}{2}\|v_n-z_n\|^2-\frac{1}{2}\lambda_n^2\|Av_n-Aq\|^2\\
&+\lambda_n\|v_n-z_n\|\|Av_n-Aq\|.
\end{aligned}$$

This implies that

$$\|z_n-q\|^2 \le \|v_n-q\|^2-\|v_n-z_n\|^2+2\lambda_n\|v_n-z_n\|\|Av_n-Aq\|. \tag{4.3.13}$$

By applying Lemma 4.2.2 and using (4.3.9), (4.3.11) and (4.3.13) we have

$$
\begin{aligned}
||x_{n+1}-q||^2 &= ||\beta_n f(x_n)+\xi_n x_n+\mu_n W_n z_n - q||^2\\
&= \beta_n||f(x_n)-q||^2+\xi_n||x_n-q||^2+\mu_n||W_n z_n-q||^2-\xi_n\mu_n||W_n z_n-x_n||^2\\
&\le \beta_n||f(x_n)-q||^2+\xi_n||x_n-q||^2+\mu_n||z_n-q||^2-\xi_n\mu_n||W_n z_n-x_n||^2\\
&\le \beta_n||f(x_n)-q||^2+\xi_n||x_n-q||^2+\mu_n\Big\{||x_n-q||^2+3L_2\beta_n\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||\\
&- \|v_n-z_n\|^2+2\lambda_n\|v_n-z_n\|\|Av_n-Aq\|\Big\}-\xi_n\mu_n||W_n z_n-x_n||^2\\
&= (1-\beta_n)||x_n-q||^2+\beta_n\Big\{||f(x_n)-q||^2+3L_2\mu_n\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||\Big\}\\
&+2\mu_n\lambda_n||v_n-z_n||||Av_n-Aq||-\mu_n||v_n-z_n||^2-\xi_n\mu_n||W_n z_n-x_n||^2\\
&\le (1-\beta_n)||x_n-q||^2+\beta_n\Big\{||f(x_n)-q||^2+3L_2\mu_n\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||\Big\}\\
&+2L_4||Av_n-Aq||-\mu_n||v_n-z_n||^2-\xi_n\mu_n||W_n z_n-x_n||^2,
\end{aligned}
$$

where $L_4 := \sup_{n\in\mathbb{N}}\{\mu_n\lambda_n||v_n-z_n||\}$. Hence, the desired result. □

Lemma 4.3.7. *Let $q\in\Gamma$. Suppose $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that* $\liminf_{k\to\infty}(||x_{n_k+1}-q||-||x_{n_k}-q||)\ge 0$. *Then $x_{n_k}\rightharpoonup x^*\in\Gamma$, i.e. $w_\omega(x_n)\subset\Gamma$.*

Proof. Suppose $q\in\Gamma$. Then, from Lemma 4.3.5 we obtain

$$
\begin{aligned}
&\frac{\mu_{n_k}(1-\beta_{n_k})}{(1-\beta_{n_k}\rho)}\delta_{n_k}(1-\delta_{n_k})||w_{n_k}-u_{n_k}||^2\\
&\le \Big(1-\frac{2\beta_{n_k}(1-\rho)}{(1-\beta_{n_k}\rho)}\Big)||x_{n_k}-q||^2-||x_{n_k+1}-q||^2+\frac{2\beta_{n_k}(1-\rho)}{(1-\beta_{n_k}\rho)}\Big\{\frac{\beta_{n_k}}{2(1-\rho)}L_3\\
&+\frac{3L_2\mu_{n_k}(1-\beta_{n_k})}{2(1-\rho)}\frac{\alpha_{n_k}}{\beta_{n_k}}||x_{n_k}-x_{n_k-1}||+\frac{1}{(1-\rho)}\langle f(q)-q,x_{n_k+1}-q\rangle\Big\}.
\end{aligned}
$$

By the hypothesis of Lemma 4.3.7 together with the fact that $\lim_{k\to\infty}\beta_{n_k}=0$, we have

$$\frac{\mu_{n_k}(1-\beta_{n_k})}{(1-\beta_{n_k}\rho)}\delta_{n_k}(1-\delta_{n_k})||w_{n_k}-u_{n_k}||^2\to 0,\quad k\to\infty.$$

Therefore, we have

$$||w_{n_k}-u_{n_k}||\to 0,\quad k\to\infty. \tag{4.3.14}$$

By following similar argument, we get from Lemma 4.3.5 that

$$(2k-\lambda_{n_k})\lambda_{n_k}||Av_{n_k}-Aq||^2\to 0,\quad k\to\infty.$$

By the conditions on k and λ_n, it follows that

$$||Av_{n_k}-Aq||\to 0,\quad k\to\infty. \tag{4.3.15}$$

Also, from Lemma 4.3.6, we have

$$
\begin{aligned}
\mu_{n_k}||v_{n_k}-z_{n_k}||^2 &\le (1-\beta_{n_k})||x_{n_k}-q||^2-||x_{n_k+1}-q||^2+\beta_{n_k}\Big\{||f(x_{n_k})-q||^2\\
&+3L_2\mu_{n_k}\frac{\alpha_{n_k}}{\beta_{n_k}}||x_{n_k}-x_{n_k-1}||\Big\}+2L_4||Av_{n_k}-Aq||.
\end{aligned}
$$

By the hypothesis of Lemma 4.3.7 and using (4.3.15) together with the condition of β_n, we have

$$\mu_{n_k}||v_{n_k} - z_{n_k}||^2 \to 0, \quad k \to \infty.$$

By the condition on μ_n, it follows that

$$||v_{n_k} - z_{n_k}|| \to 0, \quad k \to \infty. \tag{4.3.16}$$

Following similar argument, from Lemma 4.3.6, we obtain

$$||W_{n_k} z_{n_k} - x_{n_k}|| \to 0, \quad k \to \infty. \tag{4.3.17}$$

By Remark 4.3.2, we have

$$||w_{n_k} - x_{n_k}|| = \alpha_{n_k}||x_{n_k} - x_{n_k-1}|| \to 0, \quad k \to \infty. \tag{4.3.18}$$

By the definition of v_n and using (4.3.14), we obtain

$$\begin{aligned} ||v_{n_k} - w_{n_k}|| &= ||\delta_{n_k} w_{n_k} + (1 - \delta_{n_k})u_{n_k} - w_{n_k}|| \\ &\leq \delta_{n_k}||w_{n_k} - w_{n_k}|| + (1 - \delta_{n_k})||u_{n_k} - w_{n_k}|| \to 0, \quad k \to \infty. \end{aligned} \tag{4.3.19}$$

By applying (4.3.16), (4.3.17), (4.3.18) and (4.3.19), we obtain

$$||W_{n_k} z_{n_k} - z_{n_k}|| \to 0, \quad k \to \infty. \tag{4.3.20}$$

Combining (4.3.17) and (4.3.20) we have

$$||x_{n_k} - z_{n_k}|| \to 0, \quad k \to \infty. \tag{4.3.21}$$

Also, using (4.3.14), (4.3.16), (4.3.18) and (4.3.21) we get

$$||x_{n_k} - u_{n_k}|| \to 0, \qquad ||x_{n_k} - v_{n_k}|| \to 0, \quad k \to \infty. \tag{4.3.22}$$

By applying (4.3.17) and the condition on β_n, we get

$$\begin{aligned} ||x_{n_k+1} - x_{n_k}|| &= ||\beta_{n_k} f(x_{n_k}) + \xi_{n_k} x_{n_k} + \mu_{n_k} W_{n_k} z_{n_k} - x_{n_k}|| \\ &\leq \beta_{n_k}||f(x_{n_k}) - x_{n_k}|| + \xi_{n_k}||x_{n_k} - x_{n_k}|| + \mu_{n_k}||W_{n_k} z_{n_k} - x_{n_k}|| \to 0, \quad k \to \infty. \end{aligned} \tag{4.3.23}$$

We next show that $w_\omega(x_n) \subset \cap_{i=1}^{\infty} F(S_i) = F(W)$. Let $x^* \in w_\omega(x_n)$ and suppose that $x^* \notin F(W)$, that is, $Wx^* \neq x^*$. From (4.3.21), we have that $w_\omega(x_n) = w_\omega(z_n)$. By Lemma 4.2.8, we have

$$\begin{aligned} \liminf_{k\to\infty} ||z_{n_k} - x^*|| &< \liminf_{k\to\infty} |z_{n_k} - Wx^*||| \\ &\leq \liminf_{k\to\infty} \{||z_{n_k} - Wz_{n_k}|| + ||Wz_{n_k} - Wx^*||\} \\ &\leq \liminf_{k\to\infty} \{||z_{n_k} - Wz_{n_k}|| + ||z_{n_k} - x^*||\}. \end{aligned} \tag{4.3.24}$$

Since $x_{n_k} \in K$ for all $k \geq 1$ and $\lim_{k\to\infty} ||x_{n_k} - z_{n_k}|| = 0$, we obtain

$$\begin{aligned} ||Wz_{n_k} - z_{n_k}|| &\leq ||Wz_{n_k} - W_{n_k}z_{n_k}|| + ||W_{n_k}z_{n_k} - z_{n_k}|| \\ &\leq \sup_{x\in K} ||Wx - W_{n_k}x|| + ||W_{n_k}z_{n_k} - z_{n_k}||. \end{aligned}$$

By applying Lemma 4.2.7 and (4.3.20), we have $\lim_{k\to\infty} ||Wz_{n_k} - z_{n_k}|| = 0$. Combining this with (4.3.24) yields

$$\liminf_{k\to\infty} ||z_{n_k} - x^*|| < \liminf_{k\to\infty} ||z_{n_k} - x^*||,$$

which is a contradiction. Hence, we have

$$x^* \in F(W) = \cap_{i=1}^{\infty} F(S_i), \quad \text{i.e.,} \quad w_\omega(x_n) \subset F(W) = \bigcap_{i=1}^{\infty} F(S_i). \tag{4.3.25}$$

Next, we show that $x^* \in EP(F)$. From the definition of $T^F_{r_{n_k}} w_{n_k}$, we have that

$$F(u_{n_k}, y) + \frac{1}{r_{n_k}} \langle y - u_{n_k}, u_{n_k} - w_{n_k} \rangle \geq 0, \quad \forall y \in C. \tag{4.3.26}$$

By the monotonicity of F, we have

$$\frac{1}{r_{n_k}} \langle y - u_{n_k}, u_{n_k} - w_{n_k} \rangle \geq F(y, u_{n_k}), \quad \forall\ y \in C.$$

Since $x_{n_k} \rightharpoonup x^*$, then by (4.3.22) it follows that $u_{n_k} \rightharpoonup x^*$. By combining (4.3.18) and (4.3.22), and applying condition (A4) together with the fact that $\liminf_{k\to\infty} r_{n_k} > 0$, we obtain

$$F(y, x^*) \leq 0, \quad \forall\ y \in C. \tag{4.3.27}$$

Let $y_t = ty + (1-t)x^*, \quad \forall\ t \in (0, 1]$ and $y \in C$. This implies that $y_t \in C$, and it follows from (4.3.27) that $F(y_t, x^*) \leq 0$. So, by applying conditions (A1)-(A4), we have

$$\begin{aligned} 0 &= F(y_t, y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, x^*) \\ &\leq tF(y_t, y). \end{aligned}$$

Hence, we have

$$F(y_t, y) \geq 0, \quad \forall\ y \in C. \tag{4.3.28}$$

Letting $t \to 0$, by condition (A3), we get

$$F(x^*, y) \geq 0, \quad \forall\ y \in C.$$

This implies that

$$x^* \in EP(F). \tag{4.3.29}$$

Finally, we show that $x^* \in (A+B)^{-1}(0)$. Let $T_{n_k} = (I + \lambda_{n_k} B)^{-1}(I - \lambda_{n_k} A)$, then from the definition of z_n and by applying (4.3.16) we have

$$\lim_{k\to\infty} \|T_{n_k} v_{n_k} - v_{n_k}\| = \lim_{k\to\infty} \|z_{n_k} - v_{n_k}\| = 0.$$

Since $\liminf_{k\to\infty} \lambda_{n_k} > 0$, there exists $\delta > 0$ such that $\lambda_{n_k} \geq \delta$ for all $k \geq 1$. By Lemma 4.2.13(ii), we have

$$\lim_{k\to\infty} \|T_\delta v_{n_k} - v_{n_k}\| \leq 2 \lim_{k\to\infty} \|T_{n_k} v_{n_k} - v_{n_k}\| = 0.$$

By Lemma 4.2(ii) and Lemma 4.2.14, we have that T_δ is nonexpansive and $v_{n_k} \rightharpoonup x^*$. By the demiclosedness of $I - T_\delta$, we have that $x^* \in F(T_\delta)$. By Lemma 4.2.13(i) we obtain

$$x^* \in (A+B)^{-1}(0). \tag{4.3.30}$$

Hence, by combining (4.3.25), (4.3.29) and (4.3.30) we have that $w_\omega(x_n) \subset \Gamma$ as required. □

Now, we prove the strong convergence result Theorem 4.3.3.

Proof. **Proof of Theorem 4.3.3.**

Let $\hat{x} = P_\Gamma \circ f(\hat{x})$. Then it follows from Lemma 4.3.5 that

$$\begin{aligned} ||x_{n+1} - \hat{x}||^2 \leq \Big(1 - \frac{2\beta_n(1-\rho)}{(1-\beta_n\rho)}\Big)||x_n - \hat{x}||^2 + \frac{2\beta_n(1-\rho)}{(1-\beta_n\rho)}\Big\{\frac{\beta_n}{2(1-\rho)}L_3 \\ + \frac{3L_2\mu_n(1-\beta_n)}{2(1-\rho)}\frac{\alpha_n}{\beta_n}||x_n - x_{n-1}|| + \frac{1}{(1-\rho)}\langle f(\hat{x}) - \hat{x}, x_{n+1} - \hat{x}\rangle\Big\}. \end{aligned} \tag{4.3.31}$$

Now, we claim that the sequence $\{||x_n - \hat{x}||\}$ converges to zero. In order to establish this, by Lemma 4.2.4, it suffices to show that $\limsup_{k\to\infty}\langle f(\hat{x}) - \hat{x}, x_{n_k+1} - \hat{x}\rangle \leq 0$ for every subsequence $\{||x_{n_k} - \hat{x}||\}$ of $\{||x_n - \hat{x}||\}$ satisfying

$$\liminf_{k\to\infty}(||x_{n_k+1} - \hat{x}|| - ||x_{n_k} - \hat{x}||) \geq 0.$$

Suppose that $\{||x_{n_k} - \hat{x}||\}$ is a subsequence of $\{||x_n - \hat{x}||\}$ such that

$$\liminf_{k\to\infty}(||x_{n_k+1} - \hat{x}|| - ||x_{n_k} - \hat{x}||) \geq 0.$$

Then, by Lemma 4.3.7, we have that $w_\omega\{x_n\} \subset \Gamma$. It also follows from (4.3.21) that $w_\omega\{z_n\} = w_\omega\{x_n\}$. By the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup x^\dagger$ and

$$\lim_{j\to\infty}\langle f(\hat{x}) - \hat{x}, x_{n_{k_j}} - \hat{x}\rangle = \limsup_{k\to\infty}\langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\rangle = \limsup_{k\to\infty}\langle f(\hat{x}) - \hat{x}, z_{n_k} - \hat{x}\rangle. \tag{4.3.32}$$

Since $\hat{x} = P_\Gamma \circ f(\hat{x})$, it follows from (4.3.32) that

$$\limsup_{k\to\infty}\langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\rangle = \lim_{j\to\infty}\langle f(\hat{x}) - \hat{x}, x_{n_{k_j}} - \hat{x}\rangle = \langle f(\hat{x}) - \hat{x}, x^\dagger - \hat{x}\rangle \leq 0. \tag{4.3.33}$$

Hence, by (4.3.23) and (4.3.33), we have

$$\begin{aligned} \limsup_{k\to\infty}\langle f(\hat{x}) - \hat{x}, x_{n_k+1} - \hat{x}\rangle &\leq \limsup_{k\to\infty}\langle f(\hat{x}) - \hat{x}, x_{n_k+1} - x_{n_k}\rangle + \limsup_{k\to\infty}\langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\rangle \\ &= \langle f(\hat{x}) - \hat{x}, x^\dagger - \hat{x}\rangle \leq 0. \end{aligned} \tag{4.3.34}$$

Applying Lemma 4.2.4 to (4.3.31), and using (4.3.34) together with Remark 4.3.2 and the condition on β_n, we deduce that $\lim_{n\to\infty}||x_n - \hat{x}|| = 0$ as required. Hence, that completes the proof. □

Taking $S_n = S$ for all $n \geq 1$ in Theorem 4.3.3, then we obtain the following consequent result.

Corollary 4.3.8. *Let C be a nonempty closed convex subset of a real Hilbert space H and $f : H \to H$ be a contraction mapping with coefficient $\rho \in (0,1)$. Let $\{W_n\}$ be a sequence defined by (4.2.1) and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 4.2.11. Suppose that the solution set denoted by $\Gamma \neq \emptyset$ and let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 4.3.9.

Step 0 : Select initial data $x_0, x_1 \in H$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose α_n such that $0 \leq \alpha_n \leq \hat{\alpha}_n$ with $\hat{\alpha}_n$ defined by

$$\hat{\alpha}_n = \begin{cases} \min\left\{\alpha, \ \frac{\theta_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases} \tag{4.3.35}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Step 3: Compute

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0, \quad \forall \quad y \in H.$$

Step 4: Compute

$$v_n = \delta_n w_n + (1 - \delta_n)u_n.$$

Step 5: Compute

$$z_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)v_n.$$

Step 6: Compute

$$x_{n+1} = \beta_n f(x_n) + \xi_n x_n + \mu_n S z_n.$$

Set n := n + 1 and return to **Step 1.**

Suppose that conditions (C1)-(C3) *are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 4.3.9 converges strongly to a point $\hat{x} \in \Gamma$, where $\hat{x} = P_\Gamma \circ f(\hat{x})$.*

4.4 Applications

In this section, we present some applications of our main result to approximate the solutions of related optimization problems.

4.4.1 Variational inequality problem

Here, we apply our result to approximating common solutions of variational inclusion, variational inequality and fixed point problems.

Let C be a nonempty closed convex subset of a real Hilbert space H, and $P : H \to H$ be a single-valued mapping. The *Variational Inequality Problem* (VIP) is defined as follows:

$$\text{Find } \ x^* \in C \ \text{ such that } \ \langle y - x^*, Px^* \rangle \geq 0, \quad \forall \, y \in C. \tag{4.4.1}$$

The solution set of the VIP is denoted by $VI(C, P)$. Variational inequality was first introduced independently by Fichera [26] and Stampacchia [69]. The VIP is a useful mathematical model that unifies many important concepts in applied mathematics, such as necessary optimality conditions, complementarity problems, network equilibrium problems, and systems of nonlinear equations. Several methods have been proposed and analyzed by authors for solving VIP and related optimization problems, see [1, 15, 41] and references therein.

If we take $F(x, y) := \langle y - x, Px \rangle$, then the VIP (4.4.1) becomes the EP (4.1). Moreover, all the conditions of Theorem 4.3.3 are satisfied. Hence, Theorem 4.3.3 provides a strong convergence theorem for approximating common solutions of variational inclusion, variational inequality and fixed point problems for an infinite family of strict pseudocontractions.

4.4.2 Split feasibility and fixed point problems

In this subsection, we derive a scheme for approximating common solutions of split feasibility problem, equilibrium problem and fixed point problem from Algorithm 4.3.1.

Let H_1 and H_2 be two real Hilbert spaces and let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The Split Feasibility Problem (SFP) is defined as follows:

$$\text{Find } \ x^* \in C \ \text{ such that } \ Ax^* \in Q, \tag{4.4.2}$$

where $A : H_1 \to H_2$ is a bounded linear operator. Let the solution set of SFP (4.4.2) be denoted by Ω. In 1994, the Split Feasibility Problem (SFP) was introduced by Censor and Elfving [14] in finite dimensional Hilbert spaces for modelling inverse problems which arise from phase retrievals and in medical image reconstruction [12]. Furthermore, the problem (4.4.2) is also useful in various disciplines such as computer tomography, image restoration, and radiation therapy treatment planning [13, 23]. The problem has been studied by numerous researchers, see [12, 33]. Let f be a proper, lower semi-continuous convex function of H into $(-\infty, \infty)$. Then the *subdifferential* ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) - f(y) \leq \langle z, x - y \rangle, \forall y \in H, \} \quad \forall x \in H.$$

Let C be a nonempty closed convex subset of a real Hilbert space H and i_c be the indicator function on C, that is

$$i_c(x) = \begin{cases} 0 & \text{if } \ x \in C; \\ \infty & \text{if } \ x \notin C. \end{cases}$$

Moreover, we define the *normal cone* $N_C u$ of C at $u \in C$ as follows:

$$N_C u = \{z \in H : \langle z, v - u \rangle \leq 0, \forall v \in C\}.$$

It is known that i_C is a proper, lower semi-continuous and convex function on H. Hence, the subdifferential ∂i_C of i_C is a maximal monotone operator. Therefore, we define the resolvent $J_r^{\partial i_C}$ of ∂i_C, $\forall r > 0$ as follows:

$$J_r^{\partial i_C} x = (I + r\partial i_C)^{-1} x, \forall x \in H.$$

Moreover, for each $x \in C$, we have

$$\begin{aligned} \partial i_C x &= \{z \in H : i_C x + \langle z, u - x \rangle \leq i_C u, \forall u \in H\} \\ &= \{z \in H : \langle z, u - x \rangle \leq 0, \forall u \in C\} \\ &= N_C x. \end{aligned}$$

Hence, for all $\alpha > 0$, we derive

$$\begin{aligned} u = \partial i_C x &\iff x \in u + r\partial i_C u \\ &\iff x - u \in r\partial i_C u \\ &\iff u = P_C x. \end{aligned}$$

It is known that $A^*(I - P_Q)A$ is $1/||A||^2$-inverse strongly monotone [12]. Hence, by applying Theorem 4.3.3, we obtain the following strong convergence theorem for approximating common solutions of SFP, EP and FPP for an infinite family of strict pseudocontractive mappings.

Theorem 4.4.1. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $f : H_1 \to H_1$ be a contraction mapping with coefficient $\rho \in (0, 1)$. Let $\{W_n\}$ be a sequence defined by (4.2.1) and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 4.2.11. Suppose that the solution set denoted by $\Gamma = \Omega \cap EP(F) \cap \bigcap_{i=1}^{\infty} F(S_i)$ is nonempty and let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 4.4.2.

Step 0 : Select initial data $x_0, x_1 \in H$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose α_n such that $0 \leq \alpha_n \leq \hat{\alpha}_n$ with $\hat{\alpha}_n$ defined by

$$\hat{\alpha}_n = \begin{cases} \min\left\{\alpha, \ \frac{\theta_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases} \tag{4.4.3}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Step 3: Compute

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0, \quad \forall \quad y \in H.$$

Step 4: Compute

$$v_n = \delta_n w_n + (1 - \delta_n)u_n.$$

Step 5: Compute

$$z_n = P_C\big[v_n - \lambda_n A^*(I - P_Q)Av_n\big].$$

Step 6: Compute

$$x_{n+1} = \beta_n f(x_n) + \xi_n x_n + \mu_n W_n z_n.$$

Set n := n + 1 and return to **Step 1.**

Suppose that conditions (C1)-(C3) *are satisfied. Then the sequence* $\{x_n\}$ *generated by Algorithm 4.4.2 converges strongly to a point* $\hat{x} \in \Gamma$*, where* $\hat{x} = P_\Gamma \circ f(\hat{x})$.

4.5 Numerical example

In this section, we provide numerical example to illustrate the efficiency of our algorithm in comparison with Algorithm 1.2.3 and Algorithm 1.2.4 in the literature.

Example 4.5.1. Let $H = (l_2(\mathbb{R}), \|\cdot\|_2)$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_n, \ldots), x_j \in \mathbb{R} : \sum_{j=1}^{\infty} |x_j|^2 < \infty\}, ||x||_2 = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}}$ for all $x \in l_2(\mathbb{R})$. Let $A : H \to H$ be defined by $Ax = \frac{x}{2}$ for all $x \in H$, and let $B : H \to H$ be defined by $Bx = \frac{3}{2}x$. Define the bifunction F by $F(x, y) = x(y - x)$. It can be verified that

$$T_r^F x = \frac{x}{1 + r} \quad \text{for all} \quad x \in H.$$

Define an infinite family of mappings $S_n : H \to H$ by

$$S_n x := -\frac{2}{n}x \quad \text{for all } x \in H.$$

It can easily be verified that S_n is k_n-strict pseudo-contractive for each $n \in \mathbb{N}$. Define $S'_n = t_n I + (1 - t_n)S_n, t_n \in [k_n, 1)$. Let $\{\zeta_n\}$ be a sequence of nonnegative real numbers defined by $\zeta_n = \{\frac{n}{3n-1}\}$ for all $n \in \mathbb{N}$ and W_n be generated by $\{S_n\}, \{\zeta_n\}$ and $\{t_n\}$. Let $f(x) = \frac{1}{3}x$, then $\rho = \frac{1}{3}$ is the Lipschitz constant for f. Choose $\alpha = 0.8, \beta_n = \frac{1}{n+2}, \xi_n = \mu_n = \frac{n+1}{2(n+2)}, \theta_n = \frac{1}{(n+2)^2}, \delta_n = \frac{n}{2n+1}, \lambda_n = \frac{n+1}{2n+3}, r_n = \frac{n}{2n+3}, t_n = \frac{1}{n+3}$ in Algorithm 4.3.1 and we take $\alpha_n = \frac{1}{10n+1}, u = (1, -\frac{1}{2}, \frac{1}{4}, \cdots)$ in Algorithm 1.2.3 and $\lambda = 0.01$ in Algorithm 1.2.4. It can easily be verified that all the conditions of Theorem 4.3.3 are satisfied.

We choose different initial values as follows:
Case IIa: $x_0 = (-2, 1, -\frac{1}{2}, \cdots)$, $x_1 = (\frac{1}{5}, -\frac{1}{10}, \frac{1}{20}, \cdots)$,
Case IIb: $x_0 = (-4, 1, -\frac{1}{4}, \cdots)$, $x_1 = (1, \frac{1}{5}, \frac{1}{25}, \cdots)$,

Case IIc: $x_0 = (-\frac{5}{2}, \frac{5}{4}, -\frac{5}{8}, \cdots)$, $x_1 = (\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \cdots)$,
Case IId: $x_0 = (-5, 1, -\frac{1}{5}, \cdots)$, $x_1 = (1, -0.1, 0.01, \cdots)$.

Using MATLAB 2019(b), we compare the performance of Algorithm 4.3.1 with Algorithm 1.2.3 and Algorithm 1.2.4. The stopping criterion used for our computation is $||x_{n+1} - x_n|| < 10^{-4}$. We plot the graphs of errors against the number of iterations in each case. The numerical result is reported in Figure 4.1 and Table 4.1.

Table 4.1: **Numerical results for Example 4.5.1**

		Alg. 1.2.3	Alg. 1.2.4	Alg. 4.3.1
Case I	CPU time (sec)	0.0518	0.0216	0.0536
	No. of Iter.	200	105	14
Case II	CPU time (sec)	0.0320	0.0172	0.0266
	No. of Iter.	200	130	13
Case III	CPU time (sec)	0.0303	0.0151	0.0259
	No. of Iter.	200	119	14
Case IV	CPU time (sec)	0.0468	0.0198	0.0236
	No. of Iter.	200	155	13

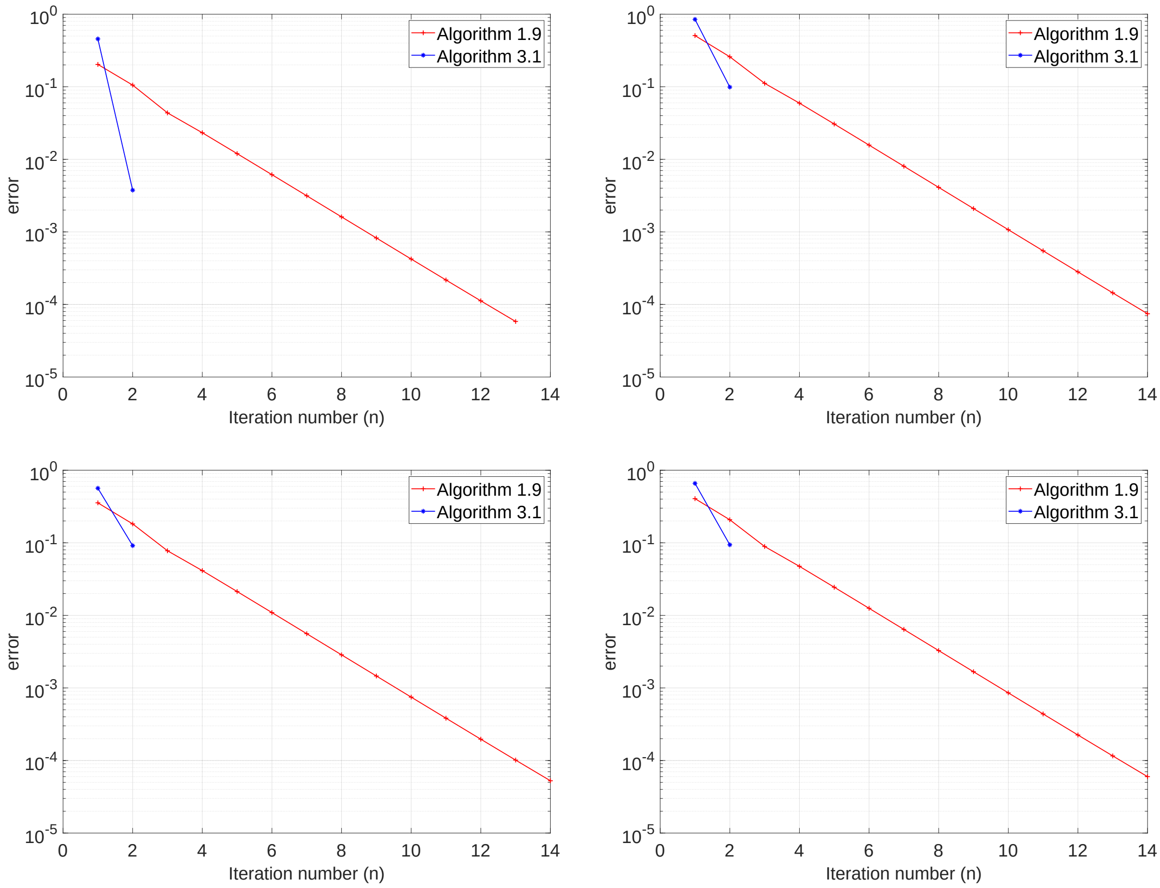

Figure 4.1: Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

CHAPTER 5

Conclusion, Contribution to Knowledge and Future Research

5.1 Conclusion

In this dissertation, we studied and introduced iterative schemes for approximating common solutions of Split Generalized Equilibrium, Variational Inclusion Problem and Fixed Point Problem in real Hilbert spaces. In Chapter 3 we proved a strong convergence theorem for the problem of finding common solutions of split generalized equilibrium and fixed point problems for a countable family of nonexpansive multivalued mappings and obtained some consequent results. We applied our result to solving split mixed variational inequality and split minimization problems, and we also presented numerical examples to illustrate the efficiency of our algorithm in comparison with other existing algorithms. Our results complement and generalize several other results in this direction in the current literature. In Chapter 4, we introduced an iterative scheme which combines inertial technique with viscosity method for approximating common solutions of variational inclusion problem, equilibrium problem and fixed point problem for an infinite family of strict-pseudocontractive mappings in Hilbert spaces. Under mild conditions, we proved a strong theorem for the proposed algorithm and apply our results to approximate the solutions of other optimization problems. Finally, we present a numerical example to demonstrate the efficiency of our algorithm in comparison with other existing methods in the literature. Our results improve and complement contemporary results in the literature in this direction.

5.2 Contribution to knowledge

As earlier pointed out, our results in this study generalize and improve some recent results in the literature. The following contributions are made in this study:

(1) In [72], Suantai et al. introduced an iterative scheme for approximating common solution of Split Equilibrium Problem and Fixed Point Problem of nonspreading multivalued mapping in Hilbert spaces and proved a weak convergence theorem for the proposed algorithm. On the other hand, Phuengrattana and Lerkchaiyaphum [60] introduced a shrinking projection method for approximating common solutions of Split Generalized Equilibrium Problem and Fixed Point Problem for a countable family of nonexpansive multivalued mappings in Hilbert spaces and proved a strong convergence theorem for the proposed algorithm. The results obtained by the authors in [72] and [60] require the prior knowledge of the operator norm, which is often very difficult to estimate. Hence, this is a major drawback in the implementation of the proposed methods. However, in Chapter 3 we introduced a new self-adaptive inertial shrinking projection algorithm, which does not require any prior knowledge of the operator norm for finding a common element of the set of solutions of Split Generalized Equilibrium Problem and the set of common fixed points of a countable family of nonexpansive multivalued mappings in Hilbert spaces. We proved strong convergence theorem for the proposed algorithm and obtained some consequent results. Hence, our results in Chapter 3 generalize and improve the results obtained by the authors in [72] and [60].

(2) In [45], Liu introduced an algorithm for finding a common element of the set of solutions of Equilibrium Problem and set of fixed points of a k-strictly pseudocontractive mapping in the setting of real Hilbert spaces and obtained a strong result. Wang in [91] proposed an iterative method for approximating a common solution of an infinite family of strict pseudocontractions in Hilbert spaces. On the other hand, Cholamjiak *et al.* in [19] introduced an inertial forward-backward splitting algorithm, which combines Halpern and Mann iteration methods for solving inclusion problems in Hilbert spaces and proved a strong convergence theorem for the proposed algorithm. Meanwhile, Thong and Vinh [89], studied the problem of finding a common element of the set of solutions of variational inclusion problem and the fixed points set of a nonexpansive mapping. The authors introduced a modified inertial forward-backward splitting algorithm combined with viscosity technique for finding a common solution of the problems in Hilbert spaces and obtained a strong convergence result. However, the authors in [19] and [89] established their results under some stringent conditions on the control parameters. In Chapter 4, we studied the problem of finding common solutions of Equilibrium Problem, Variational Inclusion Problem and Fixed Point Problem for an infinite family of strict pseudocontractive mappings. We proposed an iterative scheme which combines inertial technique with viscosity method for approximating common solutions of these problems in Hilbert spaces. Under relaxed conditions on the control parameters, we proved a strong theorem for the proposed algorithm and apply our results to approximate the solutions of other optimization problems. Therefore, our results in Chapter 4 extend, improve and generalize the results obtained by the authors in [45], [91], [19] and [89].

5.3 Future research

Our results in this dissertation were obtained in Hilbert space settings. In our future research, we will like to extend the results obtained in this dissertation to Banach and Hadamard spaces which are more general spaces than Hilbert space.

Bibliography

[1] Alakoya, T.O., Jolaoso, L.O., Mewomo, O.T., A self adaptive inertial algorithm for solving split variational inclusion and fixed point problems with applications, *J. Ind. Manag. Optim.,* 2020, DOI:10.3934/jimo.2020152.

[2] Alakoya, T.O., Jolaoso, L.O., Mewomo, O.T., Modified inertial subgradient extragradient method with self adaptive stepsize for solving monotone variational inequality and fixed point problems, *Optimization,* 2020, DOI:10.1080/02331934.2020.1723586.

[3] Alakoya, T.O., Jolaoso, L.O., Mewomo, O.T., Two modifications of the inertial Tseng extragradient method with self-adaptive step size for solving monotone variational inequality problems, *Demonstr. Math.,* 2020, doi.org/10.1515/dema-2020-0013.

[4] Alakoya, T.O., Taiwo, A., Mewomo, O.T., Cho, Y.J., An iterative algorithm for solving variational inequality, generalized mixed equilibrium, convex minimization and zeros problems for a class of nonexpansive-type mappings, *Ann. Univ. Ferrara Sez. VII Sci. Mat.,* DOI: 10.1007/s11565-020-00354-2

[5] Alakoya, T.o., Taiwo, A., Mewomo,O.T., On system of split generalised mixed equilibrium and fixed point problems for multivalued mappings with no prior knowledge of operator norm, *Fixed Point Theory,* 2020, (accepted, to appear).

[6] Alakoya, T.O., Jolaoso, L.O., Mewomo, O.T. Two modifications of the inertial Tseng extragradient method with self-adaptive step size for solving monotone variational inequality problems, *Demonstr. Math.* 2020, doi.org/10.1515/dema-2020-0013.

[7] Aremu, K.O., Abass, H.A., Izuchukwu, C., Mewomo, O.T., A viscosity-type algorithm for an infinitely countable family of (f, g)-generalized k-strictly pseudononspreading mappings in CAT(0) spaces, *Analysis*, 2020, **40** (1), 19-37.

[8] Aremu,K.O., Izuchukwu, C., Ogwo, G.N., Mewomo,O.T., Multi-step Iterative algorithm for minimization and fixed point problems in p-uniformly convex metric spaces, *J. Ind. Manag. Optim.*, 2020, doi:10.3934/jimo.2020063.

[9] Bauschke, H.H., Combettes, P.L., A weak-to-strong convergence principle for Fejer-monotone methods in Hilbert spaces, *Math. Oper. Res.* 2001, **26**(2), 248-264.

[10] Beck A., Teboulle M., A fast iterative shrinkage-thresholding algorithm for linear inverse problem, *SIAM J. Imaging Sci.* **2** (1), 2009, 183-202.

[11] Blum, E., Oettli, W., From optimization and variational inequalities to equilibrium problems, *The Mathematics Student*, 1994, **63**(1-4), 123-145.

[12] Byrne, C., A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Probl.* **20**, 2004, 103-120.

[13] Censor, Y., Bortfeld, T., Martin, B., Trofimov, A., A unified approach for inversion problems in intensity modulated radiation therapy, *Phys. Med. Biol.* **51**, 2006, 2353-2365.

[14] Censor Y, Elfving T., A multi projection algorithm using Bregman projection in a product space. *Numerical Algorithms.*, 1994; **8**: 221–239.

[15] Censor, Y., Gibali, A., Reich, S., The subgradient extragradient method for solving variational inequalities in Hilbert space, *J. Optim. Theory Appl.* 2011, **148**(2), 318-335.

[16] Chang S.S,Joseph Lee H.W, and Chan C.K, "A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization," *Nonlinear Analysis*, 2009, vol. **70**, no. 9, pp. 3307–3319.

[17] Chen, R., Yao, Y., Strong convergence theorems for strict pseudo-contractions in Hilbert spaces, *J. Appl. Math. Comput.*, 2010, **32**, 69-82.

[18] Cholamjiak, P., Suantai, S., Iterative methods for solving equilibrium problems, variational inequalities and
fixed points of nonexpansive semigroups, *J. Glob. Optim.*, 2013, **57**, 1277-1297.

[19] Cholamjiak W, Cholamjiak P, Suantai S. An inertial forward-backward splitting method for solving inclusion problems in Hilbert spaces. *J Fixed Point TheoryAppl.* (2018);20:42. doi:10.1007/s11784-018-0526-5.

[20] Cholamjiak, W., Suantai, S., A hybrid method for a countable family of multivalued maps, equilibrium problems, and variational inequality problems, *Discrete Dyn. Nat. Soc.* 2010, (2020) Art. ID 349158.

[21] Cholamjiak, P., Shehu, A., Inertial forward-backward splitting method in Banach spaces with application to compressed sensing, *Appl. Math.,* 2019 **64**, 409-435.

[22] Cianciaruso, F., Marino, G., Muglia, L., Yao, Y., A hybrid projection algorithm for finding solutions of mixed equilibrium problem and variational inequality problem, *Fixed Point Theory Appl.* 2010, **2010**, Art. ID 383740.

[23] Combettes, P.L The convex feasibility problem in image recovery. In: Hawkes, P. (ed.) Advances in Imaging and Electron Physics, Academic Press, New York, 1996, pp. 155-270.

[24] Combettes, P.L., Wajs, V.R. Signal recovery by proximal forward-backward splitting. *Multiscale Model. Simul.* (2005), **4**, 1168–1200. [CrossRef].

[25] Dong,Q., Jiang, D., Cholamjiak, P., Shehu, A., strong convergence result involving an inertial forward–backward algorithm for monotone inclusions, *J. Fixed Point Theory Appl.,* 2017 **19**, 3097-3118.

[26] Fichera, G., Sul problema elastostatico di Signorini con ambigue condizioni al contorno, *Atti Accad. Naz. Lincei VIII. Ser. Rend. Cl. Sci. Fis. Mat. Nat.,* 1963, **34**, 138-142.

[27] Gibali, A., Reich, S., Zalas, R., Outer approximation methods for solving variational inequalities in Hilbert space, *Optimization,* 2017, **66**, 417-437.

[28] Gibali, A., Jolaoso,L.O., Mewomo,O.T., Taiwo,A., Fast and simple Bregman projection methods for solving variational inequalities and related problems in Banach spaces, *Results Math.*,2020, **75**, Art. No. 179, 36 pp.

[29] Godwin, E.C. Izuchukwu, Mewomo, O.T. An inertial extrapolation method for solving generalized split feasibility problems in real Hilbert spaces, *Boll. Unione Mat. Ital.,* 2020, DOI 10.1007/s40574-020-00.

[30] Goebel, K., Reich, S., Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, *Dekker, New York,* 1984.

[31] Hieu, D.V., Muu, L.D., Anh, P.K., Parallel hybrid extragradient methods for pseudomotone equilibrium problems and nonexpansive mappings, *Numer. Algorithms,* 2016, **73**, 197-217.

[32] C. Izuchukwu, A.A. Mebawondu, O.T. Mewomo, A New Method for Solving Split Variational Inequality Problems without Co-coerciveness, *J. Fixed Point Theory Appl.,* 2020, **22** (4), Art. No. 98, 23 pp.

[33] C. Izuchukwu, G.N. Ogwo, O.T. Mewomo, An Inertial Method for solving Generalized Split Feasibility Problems over the solution set of Monotone Variational Inclusions, *Optimization,* 2020, DOI:10.1080/02331934.2020.1808648.

[34] C. Izuchukwu, C. C. Okeke, O. T. Mewomo, Systems of Variational Inequalities and multiple-set split equality fixed point problems for countable families of multivalued type-one demicontractive-type mappings, *Ukraïn. Mat. Zh.*, 2019, **71** (11), 1480-1501.

[35] C. Izuchukwu, G.C. Ugwunnadi, O.T. Mewomo, A.R. Khan and M. Abbas, Proximal-type algorithms for split minimization problem in p-uniformly convex metric space, *Numerical Algorithms*, 2019, **82**, 909 - 935.

[36] Jolaoso, L.O., Alakoya, T.O., Taiwo, A., Mewomo, O.T., A parallel combination extragradient method with Armijo line searching for finding common solution of finite families of equilibrium and fixed point problems, *Rend. Circ. Mat. Palermo II,* 2020, **69**(2), 475-495.

[37] Jolaoso, L.O., Alakoya, T.O., Taiwo, A., Mewomo, O.T., Inertial extragradient method via viscosity approximation approach for solving Equilibrium problem in Hilbert space, *Optimization,* 2020, DOI:10.1080/02331934.2020.1716752.

[38] Jolaoso, L.O., Taiwo, A., Alakoya, T.O., Mewomo, O.T., A unified algorithm for solving variational inequality and fixed point problems with application to the split equality problem, *Comput. Appl. Math.,* 2020, **39**(1), Art. 38. 2.211-232.

[39] L.O. Jolaoso, K.O. Oyewole, K.O. Aremu, O.T. Mewomo, A new efficient algorithm for finding common fixed points of multivalued demicontractive mappings and solutions of split generalized equilibrium problems in Hilbert spaces, *Int. J. Comput. Math.,* 2020, https://doi.org/10.1080/00207160.2020.1856823.

[40] Kazmi, K.R, Rizvi, S.H., Iterative approximation of a common solution of a split generalized equilibrium problem and a fixed point problem for nonexpansive semigroup, *Math. Sci. (Springer),* 2013, **7**, Art. 1, 10 pp.

[41] Khan, S.H., Alakoya, T.O., Mewomo, O.T., Relaxed projection methods with self-adaptive step size for solving variational inequality and fixed point problems for an infinite family of multivalued relatively nonexpansive mappings in Banach spaces *Math. Comput. Appl.,* 2020, **25**, Art. 54.

[42] Kimura, Y., Convergence of a sequence of sets in a Hadamard space and the shrinking projection method for a real Hilbert ball.*Abstr. Appl. Anal.,* 2010, **2010**, Art. ID 582475.

[43] Kumam, P and Jaiboon, C. Approximation of common solutions to system of mixed equilibrium problems, variational inequality problem, and strict pseudo-contractive mappings. *Fixed Point Theory and Applications.*, 2011, vol.2011, Article ID 347204, 30 pages.

[44] Lions, P.L., Mercier, B., Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.*, 1979, **16**, 964-979.

[45] Liu Y., A general iterative method for equilibrium problems and strict pseudo-contractions in Hilbert spaces, *Nonlinear Analysis*, 2009, **71**(10), pp. 4852-4861.

[46] López G, MárquezMV, Wang F, et al. Forward-backward splitting methods for accretive operators in Banach spaces. *Abstr Appl Anal.* **2010**(2012). Article ID 109236.

[47] Luo, C., Ji, H., Li, Y., Utility-based multi-service bandwidth allocation in the 4G heterogeneous wireless networks, *IEEE Wireless Communication and Networking Conference.* 2009, DOI:10.1109/WCNC.2009.4918017.

[48] Ma, Z., Wang, L., Chang, S.S., Duan, W., Convergence theorems for split equality mixed equilibrium problems with applications, *Fixed Point Theory Appl.* 2015, **31**.

[49] Mann, W. Mean value methods in iterations, *Proc. Amer. Math. Soc.,* **4** (1953), 506-510.

[50] Martinez-Yanesa, C., Xu, H.K., Strong convergence of the CQ method for fixed point iteration processes, *Nonlinear Anal.,* 2006, **64**, 2400–2411.

[51] Moreau JJ. Proximité et dualité dans un espace Hilbertien. *Bull Soc Math France.* 1965;**93**:273–299.

[52] Moudafi A, Oliny M. Convergence of a splitting inertial proximal method for monotone operators. *J. Comput. Appl. Math.* (2003) **155**:447–454.

[53] Moudafi, A. Viscosity approximation methods for fixed-point problems, *J. Math. anal. appl.,* **241**(1), (2000) 46-55.

[54] Nakajo, K., Takahashi, W., Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.,* 2003, **279**, 372–379.

[55] Ogwo,G.N., Izuchukwu,C., Aremu,K.O., Mewomo,O.T.,, A viscosity iterative algorithm for a family of monotone inclusion problems in an Hadamard space, *Bull. Belg. Math. Soc. Simon Stevin*, 2020, **27**, 127-152.

[56] Okeke, C.C., Izuchukwu, C., Mewomo, O.T., Strong convergence results for convex minimization and monotone variational inclusion problems in Hilbert spaces, *Rend. Circ. Mat. Palermo II,* 2020, **69** (2), 675-693.

[57] Opial, Z., Weak convergence of the sequence of successive approximation for nonexpansive mappings, *Bull. Am. Math. Soc.,* 1967, **73**, 591-597.

[58] Oyewole, O.K., Abass, H.A., Mewomo, O.T., A Strong convergence algorithm for a fixed point constrainted split null point problem, *Rend. Circ. Mat. Palermo II*, 2020, DOI:10.1007/s12215-020-00505-6.

[59] Passty, G.B., Ergodic convergence to a zero of the sum of monotone operators in Hilbert space, *J. Math. Anal. Appl.*, 1979, **72**, 383-390.

[60] Phuengrattana, W., Lerkchaiyaphum, K., On solving the split generalized equilibrium problem and the fixed point problem for a countable family of nonexpansive multivalued mappings, *Fixed Point Theory Appl.,* 2018, Art. 6.

[61] Polyak, B.T., Some methods of speeding up the convergence of iterative methods, *Zh. Vychisl. Mat. Mat. Fiz.,* 1964, **4**, 1-17.

[62] Raguet H, Fadili J, Peyré G. A generalized forward-backward splitting. *SIAM J Imaging Sci.* (2013), **6** 1199–1226.

[63] Rehman, H., Kumam, P., Cho, Y.J., Yordsorn, P. Weak convergence of explicit extragradient algorithms for solving equilibirum problems. *J. Inequal. Appl.* (2019), **2019**, 1–25. [CrossRef]

[64] Rockafellar RT. On the maximal monotonicity of subdifferential mappings. *Pac J Math.* 1970;**33** 209–216.

[65] Schaefer, H. *ber die Methods Sukzessiver Approximation, (German),*jber. Deutsch. Math. Verein, **59** (1957), Abt 1, 131-140.

[66] Shimoji, K., Takahashi, W. Strong convergence to common fixed points of infinite nonexpansive mappings and applications,*Taiwanese J. Math.,* 2001, **5**(2), 387-404.

[67] Shahazad, N., Zegeye, H., Approximating of common point of fixed points of a pseudo-contractive mapping and zeros of sum of monotone mappings, *Fixed Point Theory Appl.*, 2014, **85**.

[68] Song, Y., Cho, Y. J., Some note on Ishikawa iteration for multivalued mappings, *Bull. Korean Math. Soc.,* 2011, **48**(3), 575-584.

[69] Stampacchia, G., Formes bilineaires coercitives sur les ensembles convexes, *C. R. Acad. Sci. Paris,* 1964, **258**, 4413-4416.

[70] Suantai, S.,, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.,* 2005, **311**, 506–517.

[71] Suantai, S., Cholamjiak, P., Algorithms for solving generalize equilibrium problems and fixed points of nonexpansive semigroups in Hilbert spaces, *Optimization*, 2014, **63**, 799-815.

[72] Suantai, S., Cholamjiak, P., Cho, Y.J., Cholamjiak, W., On solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces, *Fixed Point Theory and Appl.* 2016, **2016**, Art. 35.

[73] Suantai S, Pholasa N, Cholamjiak P. The modified inertial relaxed CQ algorithm for solving the split feasibility problems. *J Ind Manag Optim.* (2018) **14**:1595–1615.

[74] Suparatulatorn, R., Khemphet. A. Tseng type methods for inclusion and fixed point problems with applications. *Mathematics* 2019, **7**, 1175. [CrossRef]

[75] Taiwo, A., Alakoya, T.O., Mewomo, O.T., Strong convergence theorem for solving equilibrium problem and fixed point of relatively nonexpansive multi-valued mappings in a Banach space with applications, *Asian-Eur. J. Math.,* DOI:10.1142/S1793557121501370.

[76] Taiwo,A., Alakoya,T.O., Mewomo, O.T., Halpern-type iterative process for solving split common fixed point and monotone variational inclusion problem between Banach spaces, *Numer. Algorithms*, 2020, DOI: 10.1007/s11075-020-00937-2.

[77] Taiwo, A., Jolaoso, L.O., Mewomo, O.T., General alternative regularization method for solving Split Equality Common Fixed Point Problem for quasi-pseudocontractive mappings in Hilbert spaces, *Ric. Mat.* 2020, **69** (1), 235–259.

[78] Taiwo, A., Jolaoso, L.O., and Mewomo, O.T., Viscosity approximation method for solving the multiple-set split equality common fixed-point problems for quasi-pseudocontractive mappings in Hilbert Spaces, *J. Ind. Manag. Optim.,* 2020, doi:10.3934/jimo.2020092.

[79] Taiwo, A., Jolaoso, L.O., Mewomo, O.T., Inertial-type algorithm for solving split common fixed-point problem in Banach spaces, *J. Sci. Comput.*, 2020, DOI: 10.1007/s10915-020-01385-9.

[80] Taiwo, A., Jolaoso,L.O., Mewomo,O.T., Gibali,A., On generalized mixed equilibrium problem with α-β-μ bifunction and μ-τ monotone mapping, *J. Nonlinear Convex Anal.,* 2020, **21** (6), 1381-1401.

[81] Taiwo, A., Owolabi, A. O.-E., Jolaoso, L.O., Mewomo, O.T., Gibali, A. A new approximation scheme for solving various split inverse problems, Afr. Mat., 2020, DOI:https://doi.org/10.1007/s13370-020-00832-y.

[82] Takahashi, W., Introduction to Nonlinear and Convex Analysis, *Yokohama Publishers, Yokohama* (2009).

[83] Takahashi W. Nonlinear functional analysis-fixed point theory and its applications. *Yokohama: Yokohama Publishers* (2000).

[84] Takahashi W., Toyoda M. Weak convergence theorems for nonexpansive mappings and monotone mappings. *J Optim Theory Appl.* (2003),**118**, 417–428.

[85] Takahashi, W., Takeuchi, Y., Kubota, R., Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.,* 2008, **341**, 276–286.

[86] Thong, D.V., Cholamjiak, P. Strong convergence of a forward–backward splitting method with a new step size for solving monotone inclusions. *Comput. Appl. Math.* (2019),**38**(94). [CrossRef]

[87] Tian, M. A general iterative algorithm for nonexpansive mappings in Hilbert spaces. *Nonlinear Analysis* 2010, **73**, no. 3, pp. 689–694.

[88] Tseng P. A modified forward-backward splitting method for maximal monotone mappings. *SIAM J Control Optim.* (2000), **38**, 431–446.

[89] Viet Thonga V.D. and Vinh, N.T Inertial methods for fixed point problems and zero point problems of the sum of two monotone mappings. *Optimization* 2019, **68**, **5**, 1037–1072.

[90] Wang, A. Viscosity Approximation Methods for Equilibrium Problems, Variational Inequality Problems of Infinitely Strict Pseudocontractions in Hilbert Spaces. *Journal of Applied Mathematics.* Volume 2012, Article ID 150145, 20 pages.

[91] Wang, S A general iterative method for obtaining an infinite family of strictly pseudo-contractive mappings in Hilbert spaces. *Applied Mathematics Letters.*, **24**, no. 6, pp. 901–907, 2011.

[92] Zhou, Y., Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert spaces, *Nonlinear Anal.*, 2008, **69**, 456-462.