FISCHER MATRICES AND CHARACTER TABLES OF GROUP EXTENSIONS

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Abstract

In this dissertation, we study the method of Fischer matrices for constructing the character tables of group extensions. We apply this method to calculate the character tables of all the maximal subgroups of the Janko group J_1 and one maximal subgroup of the Janko group J_2. Many of these maximal subgroups have the form $\overline{G} = N.G$ where N is a normal subgroup of $\overline{G}$ and $\overline{G}/N \cong G$. ($\overline{G}$ is an extension of N by G.) If the extension is split, the character table of $\overline{G}$ can be determined by constructing a matrix corresponding to each conjugacy class of G. The character table of G can then be determined from these matrices and the character tables of certain subgroups of G, called the inertia groups. We have described this method and used it to calculate the character tables of the maximal subgroups of J_1. We have also shown how the Fischer matrix method can be used to calculate the character table of any group extension, by considering projective characters, and used this more general method to determine the character table of the maximal subgroup of J_2 of the form $3{\cdot}PGL_2(9)$, a non-split extension of the cyclic group of order 3 by $PGL_2(9)$.

PREFACE

The work described in this dissertation was carried out in the Department of Mathematics and Applied Mathematics, University of Natal, Pietermaritzburg, from January to August 1993, under the supervision of Professor Jamshid Moori.

These studies represent original work of the author and have not otherwise been submitted in any form for any degree or diploma to any University. Where use has been made of the work of others it is duly acknowledged in the text.

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Chapter 1

INTRODUCTION

Since the classification of all finite simple groups, more recent work in group theory has involved methods of calculating character tables of finite groups. In particular, the character tables of all maximal subgroups of the sporadic simple groups have not yet been determined. Many of these maximal subgroups are extensions of elementary abelian groups so methods have been developed for the calculation of character tables of extensions of elementary abelian groups. If $\overline{G}$ is an extension of N by G, then Fischer showed how the character table of $\overline{G}$ can be determined by constructing a matrix corresponding to each conjugacy class of G. The character table of $\overline{G}$ can then be determined from these matrices and the character tables of certain subgroups of G, called the inertia groups. This method applies not only to extensions of elementary abelian groups, but also to extensions of any normal subgroup of N with the property that each character of N can be extended to its inertia group. In particular, List has used this method to determine the characters of groups of the form $2^{n-\epsilon}.S_n$ [19] , and List and Mahmoud have determined the characters of wreath products [20]. More recently, Fischer has extended these methods and shown how these matrices (which he calls Clifford matrices) can be constructed for extensions of any p-group where p is a prime [7].

In this dissertation, we describe this method of Fischer matrices and apply it to determine the character tables of the maximal subgroups of the Janko group J_1. Chapters 2 and 3 provide a review of basic definitions and results on character theory and group extensions which are then applied in chapter 4 to describe the Fischer matrix methods. After giving some examples of the use of these methods in chapter 5, we apply the methods to determine the character tables of all maximal subgroups of J_1.

To calculate the character tables of the maximal subgroups of J_1, we were able to use the basic Fischer matrix methods as they were used by List [19], List and Mahmoud [20] and Salleh [28], since all the group extensions were of elementary abelian groups. However, these methods cannot be used for certain non-split extensions. In particular, the maximal subgroup of the Janko group J_2 of the form $3 \cdot PGL_2(9)$ is nonsplit and its character table cannot be calculated in the same way. In an attempt to generalize these methods to such groups, it is necessary to consider projective representations and characters. We have given some results on projective representations and characters in chapter 7 and shown how they can be used to construct Fischer matrices for any group extension.

In order to apply these methods, the projective characters of the inertia groups must be known and these can be difficult to determine for some groups, so this method is not easily applicable to any group extension. But we have used it to determine the character table of the maximal subgroup of J_2, and thus demonstrated how to determine Fischer matrices for non-split extensions.

Chapter 2

THEORY OF CHARACTERS

In this chapter we give preliminary results on group characters that will be needed to develop the theory in later chapters. Definitions and basic properties of group representations and characters are given in the first section; in sections 2.2 and 2.3 we show how the characters of factor groups and direct products of groups can be determined, and then consider the relationship between characters of a group and those of its subgroups in 2.4. Finally we give some results on permutation characters that will be used in later calculations.

In the first section most proofs have been omitted but we give references to the book by Feit [5] which has a complete treatment of the results. Following Feit, we use the classical approach of matrix representations, as opposed to considering modules over rings and algebras. The module approach does allow for greater simplicity in some proofs but we are concerned with the properties of characters which can be derived through matrix representations without developing the theory of rings and modules. Isaacs [15] and Lederman [18] provide further references for the results of this chapter and Curtis and Reiner [4] give an extensive treatment of representation theory through the module-theoretic approach. Throughout, G denotes a group and F denotes a field. We write 1, rather than 1_G for the identity element of G.

2.1 Representations and Characters

Definition 2.1.1 Let G be a finite group and F a field. An *F-representation* of G is a homomorphism $T : G \to GL_n(F)$ for some integer n (where $GL_n(F)$, the general linear group, is the multiplicative group of all non-singular $n \times n$ matrices over F).

The homomorphism T is said to have *degree* n. Two F-representations T_1 and T_2 of G are *equivalent* if there exists $P \in GL_n(F)$ such that $T_2(g) = P^{-1}T_1(g)P$ for all $g \in G$. An F-representation T of G is *reducible* if it is equivalent to a representation U where

$$U(g) = \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix}$$

for all $g \in G$. If T is not reducible, it is said to be *irreducible*. T is defined to be *fully reducible* if it is equivalent to a representation U where

$$U(g) = \begin{pmatrix} S_1(g) & 0 \\ 0 & S_2(g) \end{pmatrix}$$

for all $g \in G$. T is *completely reducible* of it is equivalent to one of the form

$$g \mapsto \begin{pmatrix} S_1(g) & & & \\ & S_2(g) & & \\ & & \ddots & \\ & & & S_r(g) \end{pmatrix},$$

where each S_i is an irreducible F-representation of G. Then $S_1, S_2, \ldots, S_r$ are called *constituents* of T.

Theorem 2.1.1 (Mashke's theorem) *Let G be a finite group. If F is a field of characteristic zero, or whose characteristic does not divide $|G|$, then every F-representation of G is completely reducible.*

Proof: See [5, (1.1)]. □

Theorem 2.1.2 (Schur's lemma) *Let T_1 and T_2 be irreducible F-representations of G and suppose S is a non-zero matrix over F such that $T_1(g)S = ST_2(g)$ for all $g \in G$. Then S is nonsingular and T_1 is equivalent to T_2.*

Proof: See [5, (1.2)]. □

Corollary 2.1.3 *Let F be an algebraically closed field, and T an irreducible F-represen tation of G. Then the only matrices that commute with every $T(g)$ $(g \in G)$ are the scalar matrices.*

Proof: See [5, (1.4)]. □

Definition 2.1.2 If T is an F-representation of G, then the *character* afforded by T is the function $\chi_T : G \to F$ defined by $\chi_T(g) = \text{trace}(T(g))$ for $g \in G$. The *degree* of χ_T is the degree of T. The *trivial character* is the character 1_G defined by $1_G(g) = 1_F$ for all $g \in G$. An *irreducible character* is a character afforded by an irreducible representation.

Lemma 2.1.4 *The following properties hold.*

1. *A character of G is constant on the conjugacy classes of G.*
2. *Equivalent representations afford the same character.*
3. *For any character χ, $\chi(1)$ is the degree of χ.*
4. *The sum of any two characters of G is again a character of G.*

Proof: Parts 1 and 2 follow from the fact that for matrices A and P, $\text{trace}(P^{-1}AP) = \text{trace}(A)$.

3. Let χ have degree n. Then $\chi(1) = \text{trace}(I_n) = n$.
4. Let χ_T and χ_U be characters of G, afforded by the representations T and U respectively. Define the function S on G by $S(g) = \begin{pmatrix} T(g) & 0 \\ 0 & U(g) \end{pmatrix}$. Then S is a representation of G with $\chi_S = \chi_T + \chi_U$.

□

From now on, we will consider representations and characters of a finite group G over the complex field $\mathbb{C}$.

Theorem 2.1.5 *The following properties hold.*

1. *Two representations of G have the same character if and only if they are equivalent.*
2. *The number of irreducible characters of G is equal to the number of conjugacy classes of G.*

3. *Any character of G can be written as a sum of irreducible characters.*

Proof:

1. See [5, (2.6)]
2. See [5, (2.16)]
3. This follows from Mashke's Theorem (Theorem 2.1.1).

□

Lemma 2.1.6 *Let χ be a character of G afforded by a representation T of degree n. Then for $g \in G$, $T(g)$ is similar to a diagonal matrix $diag(\epsilon_1, \ldots, \epsilon_n)$ where each ϵ_i is a complex root of unity. Then $\chi(g) = \epsilon_1 + \cdots + \epsilon_n$ and $\chi(g^{-1}) = \overline{\chi(g)}$, where $\overline{x}$ denotes the complex conjugate of x.*

Proof: See [15, (2.15)]. □

Note 1 We will denote the set of all irreducible characters of G by Irr(G). These irreducible characters are presented in a table, called the *character table* of G. In this table, the columns correspond to the conjugacy classes of G and the rows to the irreducible characters, with entry a_{ij} being the value of the i^{th} irreducible character on an element of the j^{th} conjugacy class. This character table satisfies certain orthogonality relations, which we give in the next theorem.

Definition 2.1.3 The *inner product* of two characters χ_1 and χ_2 of G is defined by

$$<\chi_1, \chi_2>_G = |G|^{-1} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)}.$$

Theorem 2.1.7 (Orthogonality relations) *Let $Irr(G) = \{\chi_1, \ldots, \chi_r\}$ and let $\{g_1, \ldots, g_r\}$ be a set of representatives of the conjugacy classes of G. Then*

1. $|G|^{-1} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \delta_{ij}$, *that is,* $<\chi_i, \chi_j> = \delta_{ij}$
2. $\sum_{s=1}^{k} \chi_s(g_i)\overline{\chi_s(g_j)} = \delta_{ij}|C_G(g_i)|$.

Proof: For part 1 see [5, (2.9)] and for part 2 see [5, (2.14)]. □

Theorem 2.1.8 *Let* $Irr(G) = \{\chi_1, \ldots, \chi_r\}$ *and let* χ *be any character of* G. *Then*

1. χ *can be expressed uniquely as* $\chi = \sum_{i=1}^{r} a_i\chi_i$ *where the* a_i *are nonnegative integers.*

2. *If* $\chi = \sum_{i=1}^{r} a_i\chi_i$ *then* $<\chi, \chi> = \sum_{i=1}^{r} a_i^2$.

3. χ *is irreducible if and only if* $<\chi, \chi> = 1$.

Proof:

1. By theorem 2.1.5(3), $\chi = \sum_{i=1}^{r} a_i\chi_i$ for nonnegative integers a_i. For each i, $<\chi, \chi_i> = <\sum_{j=1}^{r} a_j\chi_j, \chi_i> = a_i<\chi_i, \chi_i> = a_i$ by the orthogonality relation (2.1.7(1)), so the a_i 's are unique.

2. Follows from the orthoganality relation (2.1.7(1)).

3. Follows from parts 1 and 2.

□

Note 2 If ϕ is any class function on G (that is, a function that is constant on the conjugacy classes of G), then ϕ can be uniquely expressed in the form $\phi = \sum_{i=1}^{r} a_i\chi_i$ where $a_i \in \mathbb{C}$ and $\text{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$. Furthermore, ϕ is a character if and only if all the a_i are nonnegative integers and $\phi \neq 0$. (See [15, (2.8)]).

Note 3 If $\chi = \sum_{i=1}^{r} a_i\chi_i$, as in the above theorem, then those χ_i with $a_i > 0$ are called the *irreducible constituents* of χ. We also say that χ contains a_i copies of the irreducible character χ_i.

2.2 Normal Subgroups

Lemma 2.2.1 *Let* χ *be a character of* G *afforded by the representation* T. *Then* $g \in ker(T)$ *if and only if* $\chi(g) = \chi(1)$.

Proof: Let $n = \chi(1)$, so n is the degree of T. If $g \in \ker(T)$ then $T(g) = I_n = T(1)$, where I_n is the $n \times n$ identity matrix, so $\chi(g) = n = \chi(1)$. Conversely, assume $\chi(g) = \chi(1) = n$. By lemma 2.1.6, $\chi(g) = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$ where each ϵ_i is a complex root of unity. Therefore, $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = n$. But $|\epsilon_i| = 1$ for all i, so we must have $\epsilon_i = 1$ for all i. Hence $T(g)$ is similar to $\mathrm{diag}(\epsilon_1, \ldots, \epsilon_n) = I_n$, so $g \in \ker(T)$. □

Definition 2.2.1 Let χ be a character of G. We define

$$ker(\chi) = \{g \in G : \chi(g) = \chi(1)\}.$$

Note 1 By the previous lemma, $\ker(\chi)$ is a normal subgroup of G (since it is the kernel of some group homomorphism). Also, if N is any normal subgroup of G then it is the intersection of some of the $\ker(\chi_i)$, where $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$. (See [15, p 23].)

Now the next result shows that the character table of G/N (where $N \trianglelefteq G$) can be obtained from that of G. Here $N \trianglelefteq G$ indicates that N is a normal subgroup of G.

Theorem 2.2.2 *Let $N \trianglelefteq G$.*

1. *If χ is a character of G with $N \subset ker(\chi)$ then $\hat{\chi}$ defined by $\hat{\chi}(gN) = \chi(g)$ is a character of G/N.*

2. *If $\hat{\chi}$ is a character of G/N then the function χ defined by $\chi(g) = \hat{\chi}(gN)$ is a character of G.*

3. *In both of the above, $\chi \in Irr(G)$ if and only if $\hat{\chi} \in Irr(G/N)$.*

Proof:

1. Let T be a representation that affords χ. Then $N \subset \ker(T)$ so $\hat{T}$ defined on G/N by $\hat{T}(gN) = T(g)$ is well-defined and it is a representation of G/N that affords $\hat{\chi}$.

2. As above, if $\hat{T}$ affords $\hat{\chi}$ then T affords χ.

3. We have

$$\begin{aligned} <\chi,\chi>_G = |G|^{-1}\sum_{g\in G}|\chi(g)|^2 &= |G|^{-1}\sum_{g\in G}|\hat{\chi}(gN)|^2 \\ &= |G|^{-1}|N|\sum_{gN\in G/N}|\hat{\chi}(gN)|^2 \\ &= |G/N|^{-1}\sum_{gN\in G/N}|\hat{\chi}(gN)|^2 \\ &= <\hat{\chi},\hat{\chi}>_{G/N}. \end{aligned}$$

Now by theorem 2.1.8(3), $\chi \in \text{Irr}(G)$ iff $<\chi,\chi>_G = 1$ iff $<\hat{\chi},\hat{\chi}>_{G/N} = 1$ iff $\hat{\chi} \in \text{Irr}(G/N)$.

□

Note 2 In the notation of the previous theorem, we say that $\chi \in \text{Irr}(G)$ has been lifted from $\hat{\chi} \in \text{Irr}(G/N)$. By identifying the characters χ and $\hat{\chi}$, we may say that $\text{Irr}(G/N) = \{\chi \in \text{Irr}(G) : N \subset \ker(\chi)\}$.

2.3 Products of Characters

We showed in Lemma 2.1.4 that the sum of any two characters is again a character. We now show that the product $\chi\psi$ of characters χ and ψ defined by $\chi\psi(g) = \chi(g)\psi(g)$ is also a character. We will then show how the character table of a direct product of two groups can be easily constructed from tbe character tables of its factor groups.

First, we define the tensor product of two matrices.

Definition 2.3.1 Let $P = (p_{ij})_{m\times m}$ and $Q = (q_{ij})_{n\times n}$ be square matrices. Define the $mn \times mn$ matrix $P \otimes Q$ by

$$P \otimes Q = (p_{ij}Q) = \begin{pmatrix} p_{11}Q & p_{12}Q & \cdots & p_{1m}Q \\ p_{21}Q & p_{22}Q & \cdots & p_{2m}Q \\ \vdots & \vdots & & \vdots \\ p_{m1}Q & p_{m2}Q & \cdots & p_{mm}Q \end{pmatrix}.$$

Then

$$\begin{aligned} \text{trace}(P \otimes Q) &= p_{11}\text{trace}(Q) + p_{22}\text{trace}(Q) + \cdots + p_{mm}\text{trace}(Q) \\ &= \text{trace}(P)\text{trace}(Q). \end{aligned}$$

Definition 2.3.2 If T and U are representations of G, then the *tensor product* $T \otimes U$ is defined by $(T \otimes U)(g) = T(g) \otimes U(g)$.

The tensor product $T \otimes U$ is a representation of G with $\chi_{T\otimes U} = \chi_T\chi_U$. Thus the product of two characters is again a character of G.

Now let $G = H \times K$ be the direct product of H and K. Let T be a representation of H of degree m with character χ_T, and let U be a representation of K of degree n with character χ_U.

Definition 2.3.3 With notation as above, we define the direct product of T and U, $T \otimes U$, as follows: Each $g \in G$ can be written uniquely as $g = hk$ for $h \in H$ and $k \in K$, and we define $(T \otimes U)(g) = T(h) \otimes U(k)$, where $\otimes$ on the right hand side is the tensor product of Definition 2.3.1.

For $G = H \times K$ the $T \otimes U$ defined by Definition 2.3.3 is a representation of G of degree mn, and

$$\chi_{T \otimes U}(g) = \chi_T(h)\chi_U(k)$$

where $g = hk$.

With this definition, the product of a character of H and a character of K is a character of G and all the characters of G can be constructed in this way, according to the following theorem.

Theorem 2.3.1 *Let $G = H \times K$ be the direct product of the groups H and K. Then the product of any irreducible character of H and any irreducible character of K is an irreducible character of G. Moreover, every irreducible character of G can be constructed in this way.*

Proof: Let $\chi_T \in \mathrm{Irr}(H)$ and $\chi_U \in \mathrm{Irr}(K)$, with $\chi = \chi_T\chi_U = \chi_{T \otimes U}$ as defined above. Then χ is a character of G. We now show that χ is irreducible, by showing that $<\chi, \chi> = 1$. Let $g \in G$ be written as $g = hk$, $h \in H$, $k \in K$. Then

$$\begin{aligned}
\sum_{g \in G} |\chi(g)|^2 &= \sum_{h \in H} \sum_{k \in K} |\chi_T(h)\chi_U(k)|^2 \\
&= \sum_{h \in H} \sum_{k \in K} |\chi_T(h)|^2 |\chi_U(k)|^2 \\
&= (\sum_{h \in H} |\chi_T(h)|^2)(\sum_{k \in K} |\chi_U(k)|^2) \\
&= |H||K|
\end{aligned}$$

since χ_T and χ_U are irreducible characters of H and K respectively.
Therefore $|G|^{-1} \sum_{g \in G} |\chi(g)|^2 = 1$, as required.

If $|\mathrm{Irr}(H)| = r$ and $|\mathrm{Irr}(K)| = s$, then we obtain rs irrednc ible characters of G in this way. These are all the irreducible characters of G, since G has rs conjugacy classes. Notice that hk and $h'k'$ are conjugate of G if and only if h and h' are conjugate in H and k and k' are conjugate in K, so the conjugacy classes of G are of the form $C_1C_2 = \{h.k : h \in C_1, k \in C_2\}$, where C_1 is a conjugacy class of H and C_2 a conjugacy class of K. □

2.4 Induced Characters

Let H be a subgroup of G. If θ is a character of G then it can be restricted to H to give the character $\theta|_H$ of H. We now show how a character of H can be induced to G, to give a character of G.

Definition 2.4.1 Let $H \leq G$ and let ϕ be a class function of H. Then ϕ^G, the *induced class function* on G is defined by

$$\phi^G(g) = |H|^{-1} \sum_{x \in G} \phi^0(xgx^{-1})$$

where ϕ^0 is defined on G by

$$\begin{cases} \phi^0(y) = \phi(y) & \text{if } y \in H, \\ \phi^0(y) = 0 & \text{if } y \notin H. \end{cases}$$

Then ϕ^G is a class function of G, and $\phi^G(1) = [G : H]\, \phi(1)$.

Theorem 2.4.1 *If ϕ is a character of H where $H \leq G$, then ϕ^G is a character of G.*

Proof: Let T be a representation of H that affords ϕ, say of degree n. Now in the following we define the induced representation T^* on G: Let $\{x_1, \ldots, x_r\}$ be a set of representatives for the right cosets of H in G (a transversal for H in G), where $r = [G : H]$. Extend T to all of G by defining $T(g)$ to be the zero matrix for $g \in G - H$. Now for $g \in G$, define $T^*(g) = (T(x_i g x_j^{-1}))_{i,j=1}^r$, where each $T(x_i g x_j^{-1})$ is a submatrix of degree n, so $T^*(g)$ is a matrix of degree rn. We show that $T^*(g)T^*(h) = T^*(gh)$ for $g, h \in G$. this is equivalent to showing that for all fixed $i, j \in \{1, \ldots, r\}$,

$$\sum_{k=1}^{r} T(x_i g x_k^{-1}) T(x_k h x_j^{-1}) = T(x_i g h x_j^{-1}) \tag{2.1}$$

If $x_i g h x_j^{-1} \notin H$, then the right-hand side of (2.1) is zero. But in this case we must have $x_i g x_k^{-1} \notin H$ or $x_k h x_j^{-1} \notin H$ for each $k \in \{1, \ldots, r\}$, so the left-hand side of (2.1) is also zero.

Now assume $v = x_i g h x_j^{-1} \in H$. The element $x_i g$ belongs to exactly one right coset, say $x_i g \in H x_s$, so $u = x_i g x_s^{-1} \in H$. If $k \neq s$, then $x_i g x_k^{-1} \notin H$. Therefore the sum on the left-hand side of (2.1) reduces to one term, with $k = s$. Then (2.1) reduces to $T(u)T(u^{-1}v) = T(v)$ which is true since $u, v \in H$.

Now T^* is a representation of G so it affords a character θ, say, of G with $\theta(g) = \sum_{i=1}^{r} \phi^0(x_i g x_i^{-1})$ (since T affords ϕ). We claim that $\theta = \phi^G$.

Since ϕ is a class function on H, $\phi(h x_i g x_i^{-1} h^{-1}) = \phi(x_i g x_i^{-1})$ for $h \in H$. Thus

$$\begin{aligned} |H|.\theta(g) &= \textstyle\sum_{h\in H} \sum_{i=1}^{r} \phi^0(h x_i g x_i^{-1} h^{-1}) \\ &= \textstyle\sum_{x\in G} \phi^0(x g x^{-1}) \\ &= |H|.\phi^G(g) \end{aligned}$$

□

Note 1 Note that from the proof of the above theorem, we get an alternative formula for the induced character: Let T be a set of representatives for the right cosets of H in G. Then

$$\phi^G(g) = \sum_{t\in T} \phi^0(t g t^{-1}).$$

Induction and restriction of characters are related by the following result.

Theorem 2.4.2 (Frobenius reciprocity theorem) *Let $H \leq G$ and suppose ϕ is a character of H, and θ a character of G. Then*

$$<\phi, \theta|_H>_H = <\phi^G, \theta>_G.$$

Proof: We have $<\phi^G, \theta>_G = \frac{1}{|G|} \sum_{g\in G} \phi^G(g)\overline{\theta(g)} = \frac{1}{|G|}\frac{1}{|H|} \sum_{g\in G} \sum_{x\in G} \phi^0(x g x^{-1})\overline{\theta(g)}$.

Now for a fixed $x \in G$, as g runs through G, so does $x g x^{-1} = y$, and $\theta(y) = \theta(g)$, since θ is a class function on G. Therefore

$$\begin{aligned} <\phi^G, \theta>_G &= \frac{1}{|G|}\frac{1}{|H|} \sum_{x\in G} \sum_{y\in G} \phi^0(y)\overline{\theta(y)} \\ &= \frac{1}{|H|} \sum_{y\in G} \phi^0(y)\overline{\theta(y)} \end{aligned}$$

$$= \frac{1}{|H|}\sum_{y\in H}\phi(y)\overline{\theta(y)}$$
$$= <\phi, \theta|_H>_H.$$

□

Corollary 2.4.3 *Let* $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}, \mathrm{Irr}(H) = \{\psi_1, \ldots, \psi_s\}$ *where* $H \le G$. *Suppose* $\chi_i|_H = \sum_{j=1}^{s} a_{ij}\psi_j$ *and* $\psi_j^G = \sum_{i=1}^{r} b_{ij}\chi_i$. *Then* $a_{ij} = b_{ij}$ *for all* i, j.

Proof: By Theorem 2.4.2 we have $a_{ij} = <\chi_i|_H, \psi_j> = <\chi_i, \psi_j^G> = b_{ij}$. □

To compute the value of an induced character we will use the following lemma.

Lemma 2.4.4 *Assume* $H \le G$, ϕ *is a character of* H *and* $g \in G$. *Let* $[g]$ *denote the conjugacy class of* G *containing* g. *If* $H \cap [g]$ *is empty, then* $\phi^G(g) = 0$. *Otherwise, choose representatives* $x_1, \ldots, x_m$ *for the classes of* H *that fuse to* $[g]$. *Then*

$$\phi^G(g) = |C_G(g)|\sum_{i=1}^{m}\frac{\phi(x_i)}{|C_H(x_i)|}.$$

Proof: By definition, $\phi^G(g) = \frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1})$. If $H \cap [g] = \Phi$, then $xgx^{-1} \notin H$ for all $x \in G$, so $\phi^0(xgx^{-1}) = 0$ for all $x \in G$ and $\phi^G(g) = 0$.

Now we assume $H \cap [g] \ne \Phi$. As x runs over G, xgx^{-1} covers $[g]$ exactly $|C_G(g)|$ times, so $\phi^G(g) = \frac{|C_G(g)|}{|H|}\sum_{y\in[g]}\phi^0(y)$. Now $\phi^0(y) = 0$ if $y \notin H$, and $[g] \cap H$ contains $[H : C_H(x_i)]$ conjugates of each x_i. Therefore $\phi^G(g) = |C_G(g)|\sum_{i=1}^{m}\frac{\phi(x_i)}{|C_H(x_i)|}$. □

2.5 Permutation Characters

In this section, we will describe an important type of character, a permutation character. Knowledge of the permutation characters of a group leads to information about the subgroup structure of the group.

First, we give definitions of the permutation action of G, where, as before, G is a finite group.

Definition 2.5.1 G *acts* on a finite set Ω if for each $g \in G$ and $\alpha \in \Omega$, there is an element α^g in Ω such that $\alpha^1 = \alpha$ and $(\alpha^g)^h = \alpha^{gh}$ for all $\alpha \in \Omega$ and $g, h \in G$.

Equivalently, G acts on Ω if there is a homomorphism $\rho : G \to S_\Omega$, where S_Ω is the set of all permutations of Ω (the symmetric group on Ω).

Now let Ω denote a finite set.

Definition 2.5.2 Let $\alpha \in \Omega$, where G acts on Ω. The *orbit* of G on Ω containing α is $\alpha^G = \{\alpha^g : g \in G\}$. The *stabilizer* of α in G is $G_\alpha = \{g \in G : \alpha^g = \alpha\}$.

The action of G on Ω is said to be *transitive* if G has only one orbit on Ω.

Lemma 2.5.1 *For G acting on Ω and $\alpha \in \Omega$, we have*

1. *G_α is a subgroup of G.*
2. $|\alpha^G| = [G : G_\alpha]$

Proof:

1. Since $1 \in G_\alpha$, $G_\alpha \neq \Phi$. Now let $g, h \in G_\alpha$. Then $\alpha^h = \alpha$ implies that $\alpha = (\alpha^h)^{h^{-1}} = \alpha^{h^{-1}}$. Now since $\alpha^g = \alpha^{h^{-1}} = \alpha$, we have $\alpha^{gh^{-1}} = (\alpha^g)^{h^{-1}} = \alpha^{h^{-1}} = \alpha$. Hence $gh^{-1} \in G_\alpha$. Therefore $G_\alpha \leq G$.
2. We produce a one-one correspondence between α^G and G/G_α, the set of all left cosets of G_α in G:

   Define $\phi : \alpha^G \to G/G_\alpha$ by $\phi(\alpha^g) = gG_\alpha$. This is a well-defined one-one function, since $\alpha^g = \alpha^h \iff \alpha^{gh^{-1}} = \alpha \iff gh^{-1} \in G_\alpha \iff gG_\alpha = hG_\alpha$. The function is clearly onto, so this proves the result.

□

Corollary 2.5.2 *The length of any orbit of G on Ω divides the order of G.*

Proof: Follows from Lemma 2.5.1. □

If G acts on Ω, this action defines a representation of G: Let $\Omega = \{\alpha_1, \ldots, \alpha_n\}$ and for each $g \in G$ define the $n \times n$ matrix π_g by $\pi_g = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } \alpha_i^g = \alpha_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then π_g is the permutation matrix of the action of g, and $A : G \to \mathrm{GL}_n(\mathbb{C})$ given by $A(g) = \pi_g$ is a representation of G.

The character ϕ afforded by this representation is called a *permutation character*, and $\phi(g) = |\{\alpha \in \Omega : \alpha^g = \alpha\}|$, that is, $\phi(g)$ is the number of points of Ω fixed by g. The degree of this permutation character is $|\Omega|$.

Note 1 Let $H \leq G$, then G acts on the set of all right cosets of H in G, by $(Ha)^g = Hag$. This action is transitive and gives rise to a permutation character of degree $[G : H]$.

This permutation character is in fact the trivial character 1_H of H induced to G. If we denote this permutation character by χ, then $\chi(g)$ is the number of points of $\omega = \{Ha_1, \ldots, Ha_r\}$ fixed by g, where $\{a_1, \ldots, a_r\}$ is a transversal for H in G. Now $(Ha_i)^g = Ha_i$ if and only if $Ha_ig = Ha_i$ if and only if $a_iga_i^{-1} \in H$, so

$$\chi(g) = \sum_{i=1}^{r} \phi^0(a_iga_i^{-1}), \text{ where } \phi^0(y) = \begin{cases} 1 & \text{if } y \in H \\ 0 & \text{if } y \notin H. \end{cases}$$

Thus $\chi = (1_H)^G$.

Conversely, if G acts transitively on any set, then the associated permutation character is induced from the trivial character of some subgroup of G, according to the following theorem.

Theorem 2.5.3 *Let G act transitively on Ω. Let $\alpha \in \Omega$ and let $H = G_\alpha$. Then $(1_H)^G$ is the permutation character of the action, where 1_H is the trivial character of H.*

Proof: Since G acts transitively, $\alpha^G = \Omega$. Therefore, by Lemma 2.5.1 there is a one-one correspondence between Ω and the set of right cosets of H in G, given by $\alpha^k \mapsto Hk$ for $k \in G$.

Let $g \in G$. Then $(\alpha^k)^g = \alpha^k \iff \alpha^{kgk^{-1}} = \alpha \iff kgk^{-1} \in H \iff Hk = Hkg \iff Hk = (Hk)^g$, where G acts on the right cosets of H as in Note 1 above. Therefore the permutation character of the action of G on Ω is the same as the permutation character of the action of G on the right cosets of H in G, which is $(1_H)^G$. □

Corollary 2.5.4 *If G acts on Ω with permutation character χ and has k orbits on Ω, then $<\chi, 1_G> = k$.*

Proof: Write $\Omega = \bigcup_{i=1}^{k} \Theta_i$, where Θ_i are the orbits of G on Ω. Let χ_i be the permutation character of G on Θ_i, so $\chi = \sum_{i=1}^{k} \chi_i$. For $\alpha \in \Theta_i$, we have $\chi_i = (1_{G_{\alpha_i}})^G$

by Theorem 2.5.3 ,so $<\chi_i, 1_G> = <1_{G_{\alpha_i}}, 1_{G_{\alpha_i}}> = 1$ by Frobenius reciprocity (Theorem 2.4.2). Thus $<\chi, 1_G> = k$. □

Every subgroup of G gives rise to a permutation character, as shown by the previous results. Conversely, we can show the existence of a subgroup H if we can identify the character $(1_H)^G$. Because this character is a transitive permutation character, it must satisfy certain necessary conditions. We give these conditions in Theorem 2.5.6, but first prove a lemma.

Lemma 2.5.5 *If G acts transitively on Ω, then all subgroups G_α of G (for $\alpha \in \Omega$) are conjugate in G.*

Proof: Let $\alpha, \beta \in \Omega$. We show that G_α and G_β are conjugate in G, that is, we show that there is an $h \in G$ with $G_\alpha = (G_\beta)^h = hG_\beta h^{-1}$.

Since G acts transitively on Ω, there is some $h \in G$ such that $\alpha^h = \beta$. Now $g \in G_\alpha \iff \alpha^g = \alpha \iff \beta^{h^{-1}g} = \beta^{h^{-1}} \iff \beta^{h^{-1}gh} = \beta \iff h^{-1}gh \in G_\beta \iff g \in (G_\beta)^h$, so $G_\alpha = (G_\beta)^h$ as required. □

Theorem 2.5.6 *Let $H \leq G$ and $\chi = (1_H)^G$. Then*

1. *$\chi(1)$ divides the order of G.*
2. *$<\chi, \psi> \leq \psi(1)$ for all $\psi \in \mathrm{Irr}(G)$.*
3. *$<\chi, 1_G> = 1$.*
4. *$\chi(g)$ is a nonnegative integer for all $g \in G$.*
5. *$\chi(g) \leq \chi(g^m)$ for all $g \in G$ and m a nonnegative integer.*
6. *$\chi(g) = 0$ if the order of g does not divide $\frac{|G|}{\chi(1)}$.*
7. *$\chi(g)\frac{|[g]|}{\chi(1)}$ is an integer for all $g \in G$.*

Proof: Let Ω be the set of all right cosets of H in G, so χ is the permutation character of G on Ω.

1. This is clear, since $\chi(1) = [G : H]$.
2. By Frobenius reciprocity, $<\chi, \psi> = <(1_H)^G, \psi> = <1_H, \psi|_H> \leq \psi(1)$.

3. This follows from Corollary 2.5.4, since χ is a transitive permutation character.

4. $\chi(g)$ is the number of points of Ω fixed by g, so must be a nonnegative integer.

5. Each point of Ω fixed by g is fixed by g^m, so the number of points fixed by g cannot exceed the number of points fixed by g^m.

6. We know that $\frac{|G|}{\chi(1)} = |H|$ so if the order of g does not divide $|H|$ then no conjugate of g lies in H, hence $(1_H)^G(g) = 0$.

7. Let $\mathcal{S} = \{(\alpha, x) : \alpha \in \Omega, x \in [g], \alpha^x = \alpha\}$. Since χ is constant on $[g]$, we have $|[g]|\chi(g) = |\mathcal{S}| = \sum_{\alpha\in\Omega} |[g] \cap G_\alpha|$. By Lemma 2.5.5, all subgroups G_α are conjugate in G, so $|[g] \cap G_\alpha| = m$ is independent of α, and $\chi(g)|[g]| = m|\Omega| = m\chi(1)$.

□

The following result will be used in later calculations to determine the conjugacy class fusions of subgroups of G.

Theorem 2.5.7 *Let $H \leq G$, with $\chi = (1_H)^G$. Let $g \in G$ and let $x_1, \ldots, x_m$ be representatives of the conjugacy classes of H that fuse to $[g]$. Then*

$$\chi(g) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_H(x_i)|}.$$

(If $H \cap [g] = \Phi$, then $\chi(g) = 0$).

Proof: This follows from Lemma 2.4.4. □

Chapter 3

GROUP EXTENSIONS

We now go on to consider group extensions and their characters. We first give definitions and basic results on group extensions and introduce notation. We have used the books by Rotman [27] and Gorenstein [10] as references for the first section; there are also many other books on group theory which cover the material. In section 3.2 we describe a method that can be used to determine the conjugacy classes of group extensions, although we restrict ourselves to extensions of abelian groups. These methods were used by Moori [22, 23] and Salleh [28] to determine the conjugacy classes of extensions of elementary abelian groups. We then consider the characters of group extensions in section 3.3. This theory is known as Clifford theory as it is based on an important result by Clifford [2] (Theorem 3.3.1). We used Isaacs [15] and Curtis and Reiner [4] as references for this section.

3.1 Definitions, Notation and Basic Results

Definition 3.1.1 If N and G are groups, an *extension* of N by G is a group $\overline{G}$ that satisfies the following properties

1. $N \trianglelefteq \overline{G}$
2. $\overline{G}/N \cong G$.

We say that $\overline{G}$ is a *split extension* of N by G if $\overline{G}$ contains subgroups N and G_1 with $G_1 \cong G$ such that

1. $N \trianglelefteq \overline{G}$
2. $NG_1 = \overline{G}$
3. $N \cap G_1 = 1$.

In this case $\overline{G}$ is also called a *semi-direct product* of N and G, and we identify G_1 and G.

Note 1 If $\overline{G}$ is a semi-direct product of N and G then every $\overline{g} \in \overline{G}$ has a unique expression of the form $\overline{g} = ng$ where $n \in N$ and $g \in G$. Multiplication in $\overline{G}$ satisfies $(n_1g_1)(n_2g_2) = n_1n_2^{g_1}g_1g_2$, where n^g denotes gng^{-1}.

Definition 3.1.2 The *automorphism group* of a group G, denoted by $\text{Aut}(G)$, is the set of all automorphisms of G under the binary operation of composition.

If $\overline{G}$ is a split extension of N by G, then there is a homomorphism $\theta : G \rightarrow \text{Aut}(N)$ given by $\theta_g(n) = gng^{-1} = n^g$ $(n \in N, g \in G)$, where we denote $\theta(g)$ by θ_g. Thus G acts on N, and we say that the extension $\overline{G}$ realizes θ.

Conversely, given any groups N and G, and $\theta : G \rightarrow \text{Aut}(N)$, we can define a semi-direct product of N by G that realizes θ as follows. Let $\overline{G}$ be the set of ordered pairs (n, g) $(n \in N, g \in G)$ with multiplication given by $(n_1, g_1)(n_2, g_2) = (n_1\theta_{g_1}(n_2), g_1g_2)$. Then $\overline{G}$ is a semi-direct product of N by G.

Hence a split extension of N by G is completely described by the map $\theta : G \rightarrow \text{Aut}(N)$, that is to say, it is described by the way G acts on N.

Following ATLAS [3], we denote an arbitrary extension of N by G by $N.G$. A split extension is denoted by $N : G$ or $N :^{\theta} G$ where $\theta : G \rightarrow \text{Aut}(N)$ determines the extension. A case of $N.G$ that is not split is denoted by $N \cdot G$.

If $\overline{G}$ is a split extension of N by G, then $\overline{G} = NG = \bigcup_{g \in G} Ng$, so G may be regarded as a right transversal for N in $\overline{G}$ (that is, a complete set of right coset representatives of N in $\overline{G}$). Now suppose $\overline{G}$ is any extension of N by G, not necessarily split, Then, since $\overline{G}/N \cong G$, there is an onto homomorphism $\lambda : \overline{G} \rightarrow G$ with kernel N. For $g \in G$ define a lifting of g to be an element $\overline{g} \in \overline{G}$ such that $\lambda(\overline{g}) = g$. Then choosing a lifting of each element of G, we get the set $\{\overline{g} : g \in G\}$ which is a transversal for N in $\overline{G}$.

We now show that even for a non-split extension of N by G, if N is abelian, G acts on N. This lemma and its proof were obtained from Rotman [27, 7.17].

Lemma 3.1.1 *Let $\overline{G}$ be an extension of N by G, with N abelian, Then there is a homomorphism $\theta : G \rightarrow Aut(N)$ such that $\theta_g(n) = \overline{g}n\overline{g}^{-1}$ $(n \in N)$, and θ is independent of the choice of liftings $\{\overline{g} : g \in G\}$.*

Proof: For $a \in \overline{G}$, denote conjugation by a by γ_a. Since N is normal in $\overline{G}$, $\gamma_a|_N$ is an automorphism of N and the function $\mu : \overline{G} \to \mathrm{Aut}(N)$ defined by $\mu(a) = \gamma_a|_N$ is a homomorphism.

If $a \in N$, then $\mu(a) = 1_N$, since N is abelian. Therefore there is a homomorphism $\mu^* : \overline{G}/N \to \mathrm{Aut}(N)$ defined by $\mu^*(Na) = \mu(a)$.

Now $G \cong \overline{G}/N$ and for any lifting $\{\overline{g} : g \in G\}$, the map $\phi : G \to \overline{G}/N$ defined by $\phi(g) = N\overline{g}$ is an isomorphism. If $\{\overline{g}_1 : g \in G\}$ is another choice of liftings, then $\overline{g}\overline{g}_1^{-1} \in N$ for every $g \in G$ so that $N\overline{g} = N\overline{g}_1$. Therefore the isomorphism ϕ is independent of the choice of liftings. Now let $\theta : G \to \mathrm{Aut}(N)$ be the composite $\mu^* \circ \phi$. If $g \in G$ and $\overline{g}$ is a lifting, then $\theta(g) = \mu^*(\phi(g)) = \mu^*(N\overline{g}) = \mu(\overline{g}) \in \mathrm{Aut}(N)$, so for $n \in N$, $\theta_g(n) = \mu(\overline{g})(n) = \overline{g}n\overline{g}^{-1}$, as required. □

Note 2 Let $\overline{G}$ be an extension of an abelian group N by G. For each $g \in G$ we choose a lifting $\overline{g} \in \overline{G}$, and for convenience we take $\overline{1} = 1$. We identify G with $\overline{G}/N$ under the isomorphism $g \mapsto N\overline{g}$. Now $\{\overline{g} : g \in G\}$ is a right transversal for N in $\overline{G}$ so every element $h \in \overline{G}$ has a unique expression of the form $h = n\overline{g}$ $(n \in N, g \in G)$, and we have the following relations.

1. $\overline{g}n = n^g\overline{g}$, where $n \in N$ and $g \in G$

2. $\overline{g}\overline{h} = f(g,h)\overline{gh}$ for some $f(g,h) \in N$, where $g, h \in G$.

Here we use n^g to denote $\theta_g(n)$ as given in the previous lemma.

3.2 Conjugacy Classes of $\overline{G} = N.G$ (N abelian)

In this section we assume that N is abelian, so the preceding lemma and Note 2 above apply.

To determine the conjugacy classes of $\overline{G}$ we analyse the cosets $N\overline{g}$, where $\overline{G} = \bigcup_{g \in G} N\overline{g}$. It is only necessary to consider one coset $N\overline{g}$ for each conjugacy class of G with representative g, and the corresponding classes of $\overline{G}$ are determined by the action (by conjugation) of C_g, the set stabilizer in $\overline{G}$ of $N\overline{g}$.

Now $N \subseteq C_g$, since for $n \in N$ and $n_1\overline{g} \in N\overline{g}$, $n(n_1\overline{g})n^{-1} = nn_1(n^{-1})^g\overline{g} \in N\overline{g}$ by the relations in Note 2 in Section 3.1.

Therefore $N \trianglelefteq C_g$ and we have $C_g/N = C_{\overline{G}/N}(N\overline{g})$ because

$$
\begin{aligned}
Nh \in C_{\overline{G}/N}(N\overline{g}) &\iff NhN\overline{g}(Nh)^{-1} = N\overline{g} \\
&\iff NhNn\overline{g}h^{-1} = N\overline{g},\ \forall n \in N \\
&\iff Nhn\overline{g}h^{-1} = N\overline{g},\ \forall n \in N \\
&\iff hn\overline{g}h^{-1} \in N\overline{g}\ ,\forall n \in N \\
&\iff h \in C_g \\
&\iff Nh \in C_g/N.
\end{aligned}
$$

Therefore C_g is an extension of N by $C_G(g)$, identifying $C_{\overline{G}/N}(N\overline{g})$ and $C_G(g)$.

Now we determine the orbits of $C_g = N.C_G(g)$ on $N\overline{g}$. Let $h \in N\overline{g}$ and let $C_N(h)$ be the stabilizer in N of h. Then for any $nh \in N\overline{g}$ $(n \in N)$, $(nh)^x = n^x h^x = nh$ for $x \in C_N(h)$, since N is abelian. Therefore $C_N(h)$ fixes each element in $N\overline{g}$. Let $k = |C_N(h)|$. Then under conjugation by N each element of $N\overline{g}$ is conjugate to $\frac{|N|}{k}$ elements of $N\overline{g}$, so $N\overline{g}$ splits into k blocks with $\frac{|N|}{k}$ elements in each block. Denote these blocks by $Q_1, \ldots, Q_k$.

The orbits of C_g (that is, the conjugacy classes of $N\overline{g}$) are unions of these blocks which fuse together by the action of C_g. Since $C_g = N.C_G(g)$, this fusion is completely determined by the action of $\{\overline{h} : h \in C_G(g)\}$. For suppose Q_i and Q_j fuse $(i \neq j)$. Then there exist $n_1\overline{g} \in Q_i$, $n_2\overline{g} \in Q_j$ such that $(n_1\overline{g})^k = n_2\overline{g}$ for some $k \in C_g$. But $k \in C_g$ implies that $k = n\overline{h}$ for some $n \in N$, $h \in C_G(g)$. So $(n_1\overline{g})^{n\overline{h}} = n_2\overline{g}$ implies that$((n_1\overline{g})^n)^{\overline{h}} = n_2\overline{g}$. Now $(n_1\overline{g})^n \in Q_i$, so by the action of $\overline{h}$, Q_i and Q_j have fused.

Suppose f blocks fuse to form an orbit Ω of C_g. Then $|\Omega| = f\frac{|N|}{k}$. Let $x \in \Omega$. Then the stabilizer in C_g of x is $C_{\overline{G}}(x)$, so $|\Omega| = \frac{|C_g|}{|C_{\overline{G}}(x)|} = \frac{|N||C_G(g)|}{|C_{\overline{G}}(x)|}$ (by Lemma 2.5.1). Therefore $|C_{\overline{G}}(x)| = \frac{k|C_G(g)|}{f}$.

So to calculate the conjugacy classes of $\overline{G}$ we need to find the values of k and f for each conjugacy class of G. Note that the values of k can be determined from the action of G on N (given in Lemma 3.1.1):

Consider a class representative g of G. For this class, k is the number of elements of N that fix h, for $h \in N\overline{g}$. Take $h = \overline{g}$. Now for $n \in N$,

$$n \text{ fixes } \overline{g} \iff n\overline{g}n^{-1} = \overline{g} \iff \overline{g}n\overline{g}^{-1} = n \iff n^g = n.$$

Therefore k is the number of elements of N fixed by g, which equals $\chi(g)$ where χ is the permutation character of the action of G on N.

3.3 Clifford Theory

We now consider the characters of $\overline{G}$, an extension of N by G. Here N is any group, not necessarily abelian.

Let $\theta \in \mathrm{Irr}(N)$, where $N \trianglelefteq \overline{G}$. Then θ^g defined by $\theta^g(n) = \theta(gng^{-1})$, where $g \in \overline{G}$ and $n \in N$, is a character of N, and is said to be *conjugate* to θ in $\overline{G}$. $\overline{G}$ permutes $\mathrm{Irr}(N)$ by $g : \theta \mapsto \theta^g$. Since N acts trivially on $\mathrm{Irr}(N)$, $\mathrm{Irr}(N)$ is permuted by $\overline{G}/N$, by $gN : \theta \mapsto \theta^g$.

All of our work in this section and the next chapter is dependent on the next result. This result is due to Clifford [2] and is thus known as Clifford's theorem, but we give a proof from Isaacs [15].

Theorem 3.3.1 *Let $N \trianglelefteq \overline{G}$ and $\chi \in \mathrm{Irr}(\overline{G})$. Let θ be an irreducible constituent of $\chi|_N$ and suppose that $\theta = \theta_1, \theta_2, \ldots, \theta_t$ are the distinct conjugates of θ in $\overline{G}$. Then $\chi|_N = e\sum_{i=1}^{t}\theta_i$ where $e = <\chi|_N, \theta>$.*

Proof: We compute $\theta^{\overline{G}}|_N$. Define θ^0 on $\overline{G}$ by

$$\theta^0(x) = \begin{cases} \theta(x) & \text{if } x \in N \\ 0 & \text{if } x \notin N. \end{cases}$$

For $n \in N$, we have $\theta^{\overline{G}}(n) = |N|^{-1}\sum_{x\in\overline{G}}\theta^0(xnx^{-1})$. Since $xnx^{-1} \in N \ \forall x \in \overline{G}$ we have $\theta^{\overline{G}}(n) = |N|^{-1}\sum_{x\in\overline{G}}\theta^x(n)$. Therefore $|N|\theta^{\overline{G}}|_N = \sum_{x\in\overline{G}}\theta^x$, and if $\phi \in \mathrm{Irr}(N)$ and $\phi \notin \{\theta_i : 1 \leq i \leq t\}$ then $0 = <\sum_{x\in\overline{G}}\theta^x, \phi>$, so $<\theta^{\overline{G}}|_N, \phi> = 0$. Since χ is an irreducible constituent of $\theta^{\overline{G}}$ by Frobenius reciprocity, it follows that $<\chi|_N, \phi> = 0$. Thus all the irreducible constituents of $\chi|N$ are among the θ_i, so $\chi|_N = \sum_{i=1}^{t}<\chi|_N, \theta_i>\theta_i$. But $<\chi|_N, \theta_i> = <\chi|_N, \theta>$ since θ_i and θ are conjugate, so the proof is complete. □

Definition 3.3.1 Let $N \trianglelefteq \overline{G}$ and $\theta \in \mathrm{Irr}(N)$. Then $I_{\overline{G}}(\theta) = \{g \in \overline{G} : \theta^g = \theta\}$ is the *inertia group* of θ in $\overline{G}$.

Since $I_{\overline{G}}(\theta)$ is the stabilizer of θ in the action of $\overline{G}$ on $\mathrm{Irr}(N)$, we have that $I_{\overline{G}}(\theta)$ is a subgroup of $\overline{G}$ and $N \subseteq I_{\overline{G}}(\theta)$. Also, $[\overline{G} : I_{\overline{G}}(\theta)]$ is the size of the orbit containing θ, so in the formula $\chi|_N = e\sum_{i=1}^{t}\theta_i$, we have $t = [\overline{G} : I_{\overline{G}}(\theta)]$.

As a consequence of Clifford's theorem, we have the following theorem.

Theorem 3.3.2 *Let $N \trianglelefteq \overline{G}$, $\theta \in \mathrm{Irr}(N)$ and $\overline{H} = I_{\overline{G}}(\theta)$. Then induction to $\overline{G}$ maps the irreducible characters of $\overline{H}$ that contain θ in their restriction to N faithfully onto the irreducible characters of $\overline{G}$ which contain θ in their restriction to N.*

Proof: Let $\mathcal{A} = \{\psi \in \mathrm{Irr}(\overline{H}) : <\psi|_N, \theta> \neq 0\}$, $\mathcal{B} = \{\chi \in \mathrm{Irr}(\overline{G}) : <\chi|_N, \theta> \neq 0\}$. We will show that the map $\psi \mapsto \psi^{\overline{G}}$ maps $\mathcal{A}$ faithfully onto $\mathcal{B}$, and, furthermore, for $\psi \in \mathcal{A}$, $<\psi|_N, \theta> = <\psi^{\overline{G}}|_N, \theta>$.

Let $\psi \in \mathcal{A}$. We first show that $\psi^{\overline{G}} \in \mathcal{B}$. Let χ be any irreducible constituent of $\psi^{\overline{G}}$. Then by Frobenius reciprocity, ψ is an irreducible constituent of $\chi|_{\overline{H}}$, and since θ is a constituent of $\psi|_N$, we have that θ is a constituent of $\chi|_N$, so $\chi \in \mathcal{B}$. We now show that $\chi = \psi^{\overline{G}}$. Let $\theta = \theta_1, \ldots, \theta_t$ be the $\overline{G}$-conjugates of θ, so that $t = [\overline{G} : \overline{H}]$ and $\chi|_N = e\sum_{i=1}^{t} \theta_i$. Since θ is the only $\overline{H}$-conjugate of θ (because $\overline{H} = I_{\overline{G}}(\theta)$), we have that $\psi|_N = f\theta$ for some f. But ψ is a constituent of $\chi|_{\overline{H}}$, so $f \leq e$. Therefore, by counting degrees,

$$e.t.\theta(1) = \chi(1) \leq \psi^{\overline{G}}(1) = f.t.\theta(1) \leq e.t.\theta(1) \tag{3.1}$$

Equality must hold throughout (3.1), so $\chi(1) = \psi^{\overline{G}}(1)$ and therefore $\chi = \psi^{\overline{G}}$, as required.

Now we show that the map is onto. Let $\chi \in \mathcal{B}$. Since θ is a constituent of $\chi|_N$, there must be some irreducible constituent ψ of $\chi|_{\overline{H}}$ with $<\psi|_N, \theta> \neq 0$. Then $\psi \in \mathcal{A}$ and χ is a constituent of $\psi^{\overline{G}}$ (as above). Note that by (3.1), $<\chi|_N, \theta> = e = f = <\psi|_N, \theta>$.

To show that the map is one-one we need to show that for $\psi \in \mathcal{A}$, ψ is the unique irreducible constituent of $\psi^{\overline{G}}|_{\overline{H}}$ which lies in $\mathcal{A}$. Suppose $\psi_1 \in \mathcal{A}$ such that ψ_1 is a constituent of $\psi^{\overline{G}}|_{\overline{H}} = \chi|_{\overline{H}}$ and $\psi_1 \neq \psi$. Then

$$<\chi|_N, \theta> \geq <(\psi + \psi_1)|_N, \theta> = <\psi|_N, \theta> + <\psi_1|_N, \theta> > <\psi|_N, \theta>,$$

a contradiction. This completes the proof. □

The above theorem shows that to find the irreducible characters of $\overline{G}$ that contain θ in their restriction to N, it suffices to find the irreducible characters of $\overline{H} = I_{\overline{G}}(\theta)$ that contain θ in their restriction. If θ can be extended to an irreducible character ψ of $\overline{H}$ (that is $\psi \in \mathrm{Irr}(\overline{H})$ with $\psi|_N = \theta$), then the relevant characters of $\overline{H}$ can be obtained by using the following theorem.

Theorem 3.3.3 (Gallagher [8]) *With $N, \overline{G}, \theta$ and $\overline{H}$ as above, if θ extends to a character $\psi \in \mathrm{Irr}(\overline{H})$ then as β ranges over all irreducible characters of $\overline{H}$ that contain N in their kernel, $\beta\psi$ ranges over all irreducible characters of $\overline{H}$ that contain θ in their restriction.*

Proof: By definition of $\overline{H}$, θ is the only $\overline{H}$-conjugate of θ, so by Clifford's theorem $\theta^{\overline{H}}|_N = f\theta$ for some integer f. Comparing degrees, $\theta^{\overline{H}}|_N = [\overline{H} : N]\theta$, so $<\theta^{\overline{H}}, \theta^{\overline{H}}> = <\theta, \theta^{\overline{H}}|_N> = [\overline{H} : N]$.

Now we claim that $\theta^{\overline{H}} = \sum_\beta \beta(1).\beta\psi$, where β runs over all irreducible characters of $\overline{H}$ that contain N in their kernel, or, equivalently, over all irreducible characters of $\overline{H}/N$. Both $\theta^{\overline{H}}$ and $\sum_\beta \beta(1)\beta\psi$ are zero off N because for $g \notin N$, $\theta^{\overline{H}}(g) = 0$ since $xgx^{-1} \notin N$ $\forall x \in \overline{G}$, and $\sum_\beta \beta(1)(\beta\psi)(g) = \sum_\beta(\beta(1)\beta(g))\psi(g) = 0$ (column orthogonality for character table of $\overline{H}/N$, since $g \notin N$).

Also $\theta^{\overline{H}}|_N = [\overline{H} : N]\theta = (\sum_\beta \beta(1)\beta\psi)|_N$ because for $g \in N$, $\sum_\beta \beta(1)\beta(g)\psi(g) = \sum_\beta(\beta(1))^2.\psi(g) = [\overline{H} : N]\psi(g) = [\overline{H} : N]\theta(g)$.

Therefore $\theta^{\overline{H}} = \sum_\beta \beta(1)\beta\psi$ as claimed.

Now $[\overline{H} : N] = <\theta^{\overline{H}}, \theta^{\overline{H}}> = <\sum_\beta \beta(1)\beta\psi, \sum_\gamma \gamma(1)\gamma\psi> = \sum_{\beta,\gamma} \beta(1)\gamma(1)<\beta\psi, \gamma\psi>$. The diagonal terms contribute at least $\sum \beta(1)^2 = [\overline{H} : N]$, so the $\beta\psi$ are irreducible and distinct. These $\beta\psi$ are all the irreducible constituents of $\theta^{\overline{H}}$, so are all the irreducible characters of $\overline{H}$ that contain θ in their restriction, since for $\phi \in \mathrm{Irr}(\overline{H})$, $<\phi|_N, \theta> = <\phi, \phi^{\overline{H}}>$.

□

Note 1 Now suppose $\overline{G}$ is an extension of N by G. If every irreducible character of N can be extended to its inertia group in $\overline{G}$, then by application of Theorems 3.3.2 and 3.3.3, the characters of $\overline{G}$ can be obtained as follows:

Let $\theta_1, \ldots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $\mathrm{Irr}(N)$. For each i, let $\overline{H}_i = I_{\overline{G}}(\theta_i)$ and let $\psi_i \in \mathrm{Irr}(\overline{H}_i)$ with $\psi_i|N = \theta_i$. Now each irreducible character of $\overline{G}$ contains some θ_i in its restriction to N by Clifford's theorem, so by Theorems 3.3.2 and 3.3.3 we have

$$\mathrm{Irr}(\overline{G}) = \bigcup_{i=1}^{t} \{(\beta\psi_i)^{\overline{G}} : \beta \in \mathrm{Irr}(\overline{H}_i), N \subset \ker(\beta)\}.$$

Hence the characters of $\overline{G}$ fall into blocks, with each block corresponding to an inertia group.

We now quote some results which give sufficient conditions for the irreducible characters of N to be extendible to their respective inertia groups, so that the above method can be used to calculate the characters of $\overline{G}$.

The following result and proof was obtained from Curtis and Reiner ([4, page 353]).

Theorem 3.3.4 (Mackey's theorem) *Suppose that N is a normal subgroup of $\overline{H}$ such that N is abelian and $\overline{H}$ is a semi-direct product of N and H for some $H \leq \overline{H}$. If $\theta \in \mathrm{Irr}(N)$ is invariant in $\overline{H}$ (that is, $\theta^h = \theta$, $\forall h \in \overline{H}$) then θ can be extended to a linear character of $\overline{H}$.*

Proof: Since $\overline{H}$ is a semi-direct product, any $h \in \overline{H}$ can be written uniquely as $h = nk$, $n \in N$, $k \in H$. Define χ on $\overline{H}$ by $\chi(nk) = \theta(n)$. Since N is abelian, θ has degree 1 so is linear, and the fact that $\theta = \theta^h$ for all $h \in \overline{H}$ implies that $\theta(n) = \theta(hnh^{-1})$ for all $h \in \overline{H}$. Then if $h_1 = n_1k_1$, $h_2 = n_2k_2$, we have $\chi(h_1h_2) = \chi(n_1k_1n_2k_2) = \chi(n_1n_2^{k_1}k_1k_2) = \theta(n_1n_2^{k_1}) = \theta(n_1)\theta(n_2^{k_1}) = \theta(n_1)\theta(n_2) = \chi(h_1)\chi(h_2)$. Therefore χ is a linear character of $\overline{H}$, and $\chi|_N = \theta$. □

In most cases that we will consider, N is abelian and the extension is split, so Mackey's theorem will apply.

Mackey's theorem is a corollary of a more general result by Karpilovsky which we state without proof.

Theorem 3.3.5 *[17] Let the group $\overline{H}$ contain a subgroup H of order n such that $\overline{H} = NH$ for N normal in $\overline{H}$ and let $\chi \in \mathrm{Irr}(N)$ be invariant in $\overline{H}$. Then χ extends to an irreducible character of $\overline{H}$ if the following conditions hold:*

1. *$(m, n) = 1$ where $m = \chi(1)$,*

2. *$N \cap H \leq N'$ where N' is the derived subgroup of N.*

Another extension theorem is the following:

Theorem 3.3.6 *[9] If N is a normal subgroup of $\overline{H}$ and θ is an irreducible character of N that is invariant in $\overline{H}$, then θ is extendable to an irreducible character of $\overline{H}$ if $([\overline{H} : N], \frac{|N|}{\theta(1)}) = 1$.*

Chapter 4

FISCHER MATRICES

Let $\overline{G}$ be an extension of N by G, with the property that every irreducible character of N can be extended to its inertia group. With the notation of the previous chapter we have that

$$\mathrm{Irr}(\overline{G}) = \bigcup_{i=1}^{t} \{(\beta\psi_i)^{\overline{G}} : \beta \in \mathrm{Irr}(\overline{H}_i) \text{ with } N \subset \ker(\beta)\}.$$

Now we show how the character table of $\overline{G}$ can be constructed using this result. We construct a matrix for each conjugacy class of G (the Fischer matrices). Then the character table of $\overline{G}$ can be constructed using these matrices and the character tables of factor groups of the inertia groups. These constructions of Fischer matrices have been discussed and used by Salleh [28], List [19] and List and Mahmoud [20].

4.1 Definitions

As previously, let $\theta_1, \ldots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $\mathrm{Irr}(N)$, and let $\overline{H}_i = I_{\overline{G}}(\theta_i)$ and $H_i = \overline{H}_i/N$. Let ψ_i be an extension of θ_i to $\overline{H}_i$. We take $\theta_1 = 1_N$, so $\overline{H}_1 = \overline{G}$ and $H_1 = G$.

We consider a conjugacy class $[g]$ of G with representative g.

Let $X(g) = \{x_1, \ldots, x_{c(g)}\}$ be representatives of $\overline{G}$-conjugacy classes of elements of the coset $N\overline{g}$. Take $x_1 = \overline{g}$.

Let $R(g)$ be a set of pairs (i, y) where $i \in \{1, \ldots, t\}$ such that H_i contains an element of $[g]$, and y ranges over representatives of the conjugacy classes of H_i that fuse to $[g]$. Corresponding to this $y \in H_i$, let $\{y_{l_k}\}$ be representatives of conjugacy classes of $\overline{H}_i$ that contain liftings of y.

If $\beta \in \mathrm{Irr}(\overline{H}_i)$ with $N \subset \ker(\beta)$, then β has been lifted from some $\hat{\beta} \in \mathrm{Irr}(H_i)$, with $\hat{\beta}(y) = \beta(y_{l_k})$ for any lifting y_{l_k} of y. For convenience we write $\beta(y)$ for $\hat{\beta}(y)$.

Now, using the formula for induced characters given in Lemma 2.4.4, we have

$$\begin{aligned}
(\psi_i\beta)^{\overline{G}}(x_j) &= \sum_{y:(i,y)\in R(g)} \sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}(\psi_i\beta)(y_{l_k}) \\
&= \sum_{y:(i,y)\in R(g)} \sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}\psi_i(y_{l_k})\hat{\beta}(y) \\
&= \sum_{y:(i,y)\in R(g)} \left(\sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}\psi_i(y_{l_k})\right)\beta(y)
\end{aligned}$$

By $\sum_k{}'$ we mean that we sum over those k for which y_{l_k} is conjugate to x_j in $\overline{G}$.

Now we define the Fischer matrix $M(g) = \left(a^j_{(i,y)}\right)$ with columns indexed by $X(g)$ and rows indexed by $R(g)$ by

$$a^j_{(i,y)} = \sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}\psi_i(y_{l_k}). \tag{4.1}$$

Then

$$(\psi_i\beta)^{\overline{G}}(x_j) = \sum_{y:(i,y)\in R(g)} a^j_{(i,y)}\beta(y). \tag{4.2}$$

The rows of $M(g)$ can be divided into blocks, each block corresponding to an inertia group. Denote the submatrix corresponding to H_i by $M_i(g)$, and let $C_i(g)$ be the fragment of the character table of H_i consisting of the columns corresponding to classes that fuse to $[g]$. Then, by relation (4.2), the characters of $\overline{G}$ at the classes represented by $X(g)$ obtained from inducing characters of $\overline{H}_i$ are given by the matrix product $C_i(g).M_i(g)$.

4.2 Properties of Fischer Matrices

In this section we will give some properties of the Fischer matrices which help in their computation. First we state a result of Brauer and prove a lemma which will be needed later.

Lemma 4.2.1 (Brauer) *Let A be a group of automorphisms of a group K. Then A also acts on $\mathrm{Irr}(K)$ and the number of orbits of A on $\mathrm{Irr}(K)$ is the same as that on the conjugacy classes of K.*

Proof: See [10, 4.5.2]. □

Lemma 4.2.2 *Let A be a group of automorphisms of a group K, so A acts on $\mathrm{Irr}(K)$ and on the conjugacy classes of K with the same number of orbits on each by the previous lemma. Suppose we have the following matrix describing these actions:*

$$\begin{array}{c} \\ s_1 \\ s_2 \\ \vdots \\ s_i \\ \vdots \\ s_t \end{array}
\begin{array}{c} \begin{array}{ccccccc} 1 = l_1 & l_2 & \cdots & l_j & \cdots & l_t \end{array} \\
\left(\begin{array}{ccccccc}
1 & 1 & \cdots & 1 & \cdots & 1 \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2t} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{it} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{t1} & a_{t2} & \cdots & a_{tj} & \cdots & a_{tt}
\end{array} \right) \end{array}$$

where $a_{1j} = 1$ for $j = 1, \ldots, t$,

l_j's are lengths of orbits of A on the conjugacy classes of K,

s_i's are lengths of orbits of A on $\mathrm{Irr}(K)$,

a_{ij} is the sum of s_i irreducible characters of K on the element x_j, where x_j is an element of the orbit of length l_j.

Then the following relation holds for $i, i' \in \{1, \ldots, t\}$:

$$\sum_{j=1}^{t} a_{ij}\overline{a_{i'j}}l_j = |K|s_i\delta_{ii'}.$$

Proof: Let $\underline{s_i}$ denote the sum of s_i irreducible characters of K, so $\underline{s_i}(x_j) = a_{ij}$. Then $<\underline{s_i}, \underline{s_{i'}}> = |K|^{-1}\sum_{j=1}^{t} l_j\underline{s_i}(x_j)\overline{\underline{s_{i'}}(x_j)} = |K|^{-1}\sum_{j=1}^{t} l_j a_{ij}\overline{a_{i'j}}$. But by orthogonality of irreducible characters, $<\underline{s_i}, \underline{s_{i'}}> = \delta_{ii'}s_i$, so $\sum_{j=1}^{t} l_j a_{ij}\overline{a_{i'j}} = |K|s_i\delta_{ii'}$.

□

Now let $M(g) = (a^{j}_{(i,y)})$ be the Fischer matrix for $\overline{G} = N.G$ at $g \in G$. We present $M(g)$ with corresponding "weights" for columns and rows as follows:

$$
\begin{array}{l|cccc|}
 & |C_{\overline{G}}(x_1)| & |C_{\overline{G}}(x_2)| & \cdots & |C_{\overline{G}}(x_{c(g)})| \\
|C_{H_1}(g)| & 1 & 1 & \cdots & 1 \\
\hline
|C_{H_2}(y)| & a^1_{(2,y)} & a^2_{(2,y)} & \cdots & \\
|C_{H_2}(y')| & a^1_{(2,y')} & a^2_{(2,y')} & \cdots & \\
\vdots & \vdots & & & \\
\hline
|C_{H_i}(y)| & a^1_{(i,y)} & a^2_{(i,y)} & \cdots & \\
\vdots & \vdots & & & \\
\hline
|C_{H_t}(y)| & a^1_{(t,y)} & a^2_{(t,y)} & \cdots & \\
\vdots & \vdots & & &
\end{array}
$$

The matrix $M(g)$ is divided into blocks (separated by horizontal lines), each corresponding to an inertia group. Note that $a^j_{(1,g)} = 1$ for all $j \in \{1, \ldots, c(g)\}$.

Fischer has shown that $M(g)$ is square and nonsingular (see [20]). In the following propositions and note we give further properties of Fischer matrices.

Proposition 4.2.3 (column orthogonality)

$$\sum_{(i,y)\in R(g)} |C_{H_i}(y)| a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} = \delta_{jj'} |C_{\overline{G}}(x_j)|.$$

Proof: The partial character table of $\overline{G}$ at classes $x_1, \ldots, x_{c(g)}$ is

$$\begin{bmatrix} C_1(g)M_1(g) \\ \vdots \\ C_t(g)M_t(g) \end{bmatrix}$$

where $C_i(g)$, $M_i(g)$ are as defined in section 4.1.

By column orthogonality of the character table of $\overline{G}$, we have

$$|C_{\overline{G}}(x_j)|\delta_{jj'} = \sum_{i=1}^{t} \sum_{\beta_i \in \mathrm{Irr}(H_i)} \left(\sum_{y:(i,y)\in R(g)} a^j_{(i,y)} \beta_i(y)) \overline{(\sum_{y':(i,y')\in R(g)} a^{j'}_{(i,y')} \beta_i(y')} \right)$$

$$= \sum_{i=1}^{t} \sum_{\beta_i \in \mathrm{Irr}(H_i)} \left(\sum_{y} a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} \beta_i(y) \overline{\beta_i(y)} + \sum_{y} \sum_{y' \neq y} a^j_{(i,y)} \overline{a^{j'}_{(i,y')}} \beta_i(y) \overline{\beta_i(y')} \right)$$
$$= \sum_{i=1}^{t} \left(\sum_{y} a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} \sum_{\beta_i \in \mathrm{Irr}(H_i)} \beta_i(y) \overline{\beta_i(y)} + \sum_{y} \sum_{y' \neq y} a^j_{(i,y)} \overline{a^{j'}_{(i,y')}} \sum_{\beta_i \in \mathrm{Irr}(H_i)} \beta_i(y) \overline{\beta_i(y')} \right)$$
$$= \sum_{i=1}^{t} \left(\sum_{y} a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} |C_{H_i}(y)| + 0 \right)$$
$$= \sum_{(i,y) \in R(g)} a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} |C_{H_i}(y)|.$$

□

Proposition 4.2.4 (List [19]) *At the identity of G, the matrix $M(1)$ is the matrix with rows equal to orbit sums of the action of $\overline{G}$ on* Irr(N) *with duplicate columns discarded.*

For this matrix we have $a^j_{(i,1)} = [G : H_i]$, and an orthogonality relation for rows:

$$\sum_{j=1}^{t} a^j_{(i,1)} a^j_{(i',1)} \cdot |C_{\overline{G}}(x_j)|^{-1} = \delta_{ii'} |C_{H_i}(1)|^{-1} = \delta_{ii'} |H_i|^{-1}.$$

Proof: The $(i,1), j^{\text{th}}$ entry of $M(1)$ is

$$a^j_{(i,1)} = \sum_{k} \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|} \psi_i(y_{l_k})$$

where we sum over representatives of conjugacy classes of $\overline{H}_i$ that fuse to $[x_j]$ in $\overline{G}$. Therefore $a^j_{(i,1)} = \psi_i^{\overline{G}}(x_j)$. By Theorem 3.3.2 $\psi_i^{\overline{G}}$ is an irreducible character of $\overline{G}$, and $<\psi_i^{\overline{G}}|_N, \theta_i> = <\psi_i|_N, \theta_i> = 1$. Therefore, by Clifford's theorem (Theorem 3.3.1), $\psi_i^{\overline{G}}|_N = \sum_\alpha \chi_\alpha$, where we sum over all $\chi_\alpha \in \mathrm{Irr}(N)$ in the orbit containing θ_i. Now $x_j \in N$, and $a^j_{(i,1)} = \sum_\alpha \chi_\alpha(x_j)$. The orthogonality relation follows by Lemma 4.2.2.
□

Note 1 If N is an elementary abelian group (which is the case for our calculations), then List [19] has also shown the following for $M(g)$, where $g \neq 1$:

If $\overline{G}$ is a split extension of N by G, then $M(g)$ is the matrix of orbit sums of C_g (as defined in section 3.2) acting on the rows of the character table of a certain factor group of N with duplicate columns discarded.

If the extension is not split, $M(g)$ is the matrix of orbit sums of C_g acting on the rows of the character table with duplicate columns discarded and with each row multiplied by a p^{th} root of unity where $|N| = p^n$ for some n. It may be that the root of unity for each row is 1.

For these matrices (N elementary abelian, any extension) $a^1_{(i,y)} = \frac{|C_G(g)|}{|C_{H_i}(y)|}$, and we have an orthogonality relation for rows (as a consequence of Lemma 4.2.2):

$$\begin{aligned}\sum_{j=1}^{c(g)} m_j a^j_{(i,y)} \overline{a^j_{(i',y')}} &= \delta_{(i,y)(i',y')} |C_G(g)| |C_{H_i}(y)|^{-1} |N| \\ &= \delta_{(i,y)(i',y')} a^1_{(i,y)} |N|\end{aligned}$$

where $m_j = [C_g : C_{\overline{G}}(x_j)]$.

(In the notation of section 3.2, m_j is the length of the orbit Ω of C_g, so $m_j = \frac{l \cdot |N|}{k}$.)

The relations given in the above propositions and note will be used later in our calculations of Fischer matrices, so for convenience we list them in a theorem.

Theorem 4.2.5 *For a Fischer matrix $M(g) = (a^j_{(i,y)})$ of $\overline{G} = N.G$ we have the following relations.*

1. $a^j_{(1,g)} = 1$ *for all* $j \in \{1, \ldots, c(g)\}$.
2. $\sum_{(i,y) \in R(g)} |C_{H_i}(y)| a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} = \delta_{jj'} |C_{\overline{G}}(x_j)|$.
   If N is elementary abelian, then
3. $a^1_{(i,y)} = \frac{|C_G(g)|}{|C_{H_i}(y)|}$, *and*
4. $\sum_{j=1}^{c(g)} m_j a^j_{(i,y)} \overline{a^j_{(i',y')}} = \delta_{(i,y)(i',y')} a^1_{(i,y)} |N|$.

Chapter 5

EXAMPLES

We will give in this chapter examples of the use of the methods discussed in the previous two chapters (to calculate conjugacy classes and character tables of extension groups).

5.1 The group $2^3 : GL_3(2)$

Let N be an elementary abelian group of order 8, so $N \cong V_3(2)$, the vector space of dimension three over a field of two elements. Let $G \cong GL_3(2)$. We determine the character table of $\overline{G} = N : G$, where G acts naturally on N. From ATLAS [3], we have the character table of G, which we give in Table 5.1.

Let N be generated by $\{e_1, e_2, e_3\}$ with $e_i^2 = 1$ for $1 \leq i \leq 3$, so

$$N = \{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}.$$

To determine the conjugacy classes of $\overline{G}$ we analyse the cosets Ng where g is a representative of a class of G. (Note that the extension is split, so $\overline{G} = \bigcup_{g \in G} Ng$). We use the notation of section 3.2, so $|C_{\overline{G}}(x)| = \frac{k \cdot |C_G(g)|}{f}$, where f of the k blocks of the coset Ng have fused to give a class of $\overline{G}$ containing x.

- $g = 1$: For g the identity of G, g fixes all elements of N, so $k = 8$. Then under the action of $C_G(g) = G$ we have two orbits with $f = 1$ and $f = 7$, so this coset gives two classes of $\overline{G}$:

$$x = 1, \quad \text{class } (1), \quad |C_{\overline{G}}(x)| = 8 \times 168 = 1344;$$
$$x = e_1, \quad \text{class } (2_1), \quad |C_{\overline{G}}(x)| = \tfrac{8 \times 168}{7} = 192.$$

class	$(1A)$	$(2A)$	$(3A)$	$(4A)$	$(7A)$	$(7B)$
centralizer	168	8	3	4	7	7
χ_1	1	1	1	1	1	1
χ_2	3	-1	0	1	a	$\overline{a}$
χ_3	3	-1	0	1	$\overline{a}$	a
χ_4	6	2	0	0	-1	-1
χ_5	7	-1	1	-1	0	0
χ_6	8	0	-1	0	1	1

$$a = \tfrac{1}{2}(-1+\sqrt{7}i)$$

Table 5.1: Character table of $GL_3(2)$

- $g \in (2A)$: We take $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ with $|C_G(g)| = 8$ The action of g on N is represented by the cycle structure $(1)(e_1)(e_1e_2e_3)(e_2e_3)(e_2\ e_3)(e_1e_2\ e_1e_3)$, so $k = 4$.

  The four orbits of N on Ng are $\{g, e_2e_3g\}, \{e_1g, e_1e_2e_3g\}, \{e_2g, e_3g\}$ and $\{e_1e_2g, e_1e_3g\}$.

  Now we act $C_G(g) = \left\langle \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\rangle$ on these orbits.

  For $eg \in Ng$, $h \in C_G(g)$, $(eg)^h = e^hg^h = e^hg$ so we obtain the following orbits: $\{g, e_2e_3g\}^{C_G(g)} = \{g, e_2e_3g\}$, $\{e_1g, e_1e_2e_3g\}^{C_G(g)} = \{e_1g, e_1e_2e_3g\}$, $\{e_2g, e_3g\}^{C_G(g)} = \{e_2g, e_3g, e_1e_2g, e_1e_3g\}$.

  Therefore we get three classes of $\overline{G}$:

$$\begin{array}{llll} f=1, & x=g, & \text{class } (2_2), & |C_{\overline{G}}(x)| = 4\times 8 = 32; \\ f=1, & x=e_1g, & \text{class } (2_3), & |C_{\overline{G}}(x)| = 32; \\ f=2, & x=e_2g, & \text{class } (4_1), & |C_{\overline{G}}(x)| = \frac{4\times 8}{2} = 16. \end{array}$$

- $g \in (3A)$: We take $g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ with $|C_G(g)| = 3$. The action of g on N is represented by $(1)(e_1e_2e_3)(e_1\ e_2\ e_3)(e_1e_2\ e_1e_3\ e_2e_3)$.

  Hence $k = 2$, so we must have just two blocks. These cannot fuse together under $C_G(g)$, since $g^{C_G(g)} = \{g\}$. Therefore we have two classes of $\overline{G}$, each with $f = 1$:

$$\begin{array}{lll} x = g, & \text{class } (3_1), & |C_{\overline{G}}(x)| = 2 \times 3 = 6; \\ x = e_1g, & \text{class } (6_1), & |C_{\overline{G}}(x)| = 6. \end{array}$$

- $g \in (4A)$: Again we get two classes of $\overline{G}$:

$$\begin{array}{lll} x = g, & \text{class } (4_2), & |C_{\overline{G}}(x)| = 8; \\ x = e_1g, & \text{class } (4_3), & |C_{\overline{G}}(x)| = 8. \end{array}$$

For classes $(7A)$ and $(7B)$ we have $k = 1$, so each coset has just one class in $\overline{G}$. These are classes (7_1) and (7_2) of $\overline{G}$, each with centralizer of order 7.

Thus the conjugacy classes of $\overline{G}$ are as follows:

class of G	$(1A)$		$(2A)$			$(3A)$		$(4A)$		$(7A)$	$(7B)$
class of $\overline{G}$	(1)	(2_1)	(2_2)	(2_3)	(4_1)	(3_1)	(6_1)	(4_2)	(4_3)	(7_1)	(7_2)
centralizer	1344	192	32	32	16	6	6	8	8	7	7

Now we determine the Fischer matrices: G has two orbits on N, hence two orbits on $\mathrm{Irr}(N)$. These must have lengths 1 and 7. The inertia groups are $\overline{H}_1 = \overline{G}$ and $\overline{H}_2$, where $[\overline{G} : \overline{H}_2] = 7$. Let $H_2 = \overline{H}_2/N$, then H_2 is a subgroup of G with $[G : H_2] = 7$. Therefore $H_2 \cong S_4$ (by considering the maximal subgroups of G given in ATLAS [3]). The character table of H_2 is given in Table 5.2, and the class fusions of H_2 in G in Table 5.3.

Now to calculate the Fischer matrices we will use the relations of Theorem 4.2.5. Note that all the relations hold, since N is elementary abelian.

Corresponding to the identity of G, we have

$$M(1) = \begin{array}{c} \\ 168 \\ 24 \end{array} \begin{array}{c} \begin{array}{cc} 1344 & 192 \end{array} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array},$$

class	(1A)	(2A)	(2B)	(3A)	(4A)
centralizer	24	4	8	3	4
χ_1	1	1	1	1	1
χ_2	1	−1	1	1	−1
χ_3	2	0	2	−1	0
χ_4	3	1	−1	0	−1
χ_5	3	−1	−1	0	1

Table 5.2: Character Table of $H_2 = S_4$

class of H_2	class of G
(1A)	(1A)
(2A)	(2A)
(2B)	(2A)
(3A)	(3A)
(4A)	(4A)

Table 5.3: Fusion of H_2 in G

say. By the relation 4.2.5(1), $a = b = 1$ and by the relation 4.2.5(3), $c = \frac{168}{24} = 7$. Now by the relation 4.2.5(2), we have $168 \times 1 \times 1 + 24 \times 7 \times d = 0$, so $d = -1$. Therefore $M(1) = \begin{pmatrix} 1 & 1 \\ 7 & -1 \end{pmatrix}$.

Now suppose $g \in (2A)$. Then $M(g)$ is a 3×3 matrix since Ng has three $\overline{G}$-conjugacy classes. Let

$$M(g) = \begin{matrix} & 32 & 32 & 16 \\ 8 & 1 & 1 & 1 \\ 4 & 2 & a & b \\ 8 & 1 & c & d \end{matrix}.$$

The entries of the first row and column follow from relations 4.2.5(1) and 4.2.5(3). To calculate a, b, c and d we will use the column orthogonality relation 4.2.5(2). For the second column, $8 + 4|a|^2 + 8|c|^2 = 32 \Rightarrow |a|^2 + 2|c|^2 = 6 \Rightarrow |a| = 1$ and $|c| = 1$. But by orthogonality of columns 1 and 2, we have $8 + 8a + 8c = 0$, so $a + c = -1$. Therefore $a = -2$ and $c = 1$. Similarly, $b = 0$ and $d = -1$.

The other matrices are determined similarly, and all the Fischer matrices of $\overline{G}$ are given below.

$[g]$	$M(g)$
$(1A)$	$\begin{matrix} & 1344 & 192 \\ 168 & 1 & 1 \\ 24 & 7 & -1 \end{matrix}$
$(2A)$	$\begin{matrix} & 32 & 32 & 16 \\ 8 & 1 & 1 & 1 \\ 4 & 2 & -2 & 0 \\ 8 & 1 & 1 & -1 \end{matrix}$
$(3A)$	$\begin{matrix} & 6 & 6 \\ 3 & 1 & 1 \\ 3 & 1 & -1 \end{matrix}$
$(4A)$	$\begin{matrix} & 8 & 8 \\ 4 & 1 & 1 \\ 4 & 1 & -1 \end{matrix}$

$$(7A) \qquad \begin{array}{c} \quad 7 \\ 7 \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

$$(7B) \qquad \begin{array}{c} \quad 7 \\ 7 \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

For each matrix $M(g)$, we write $|C_{\overline{G}}(x_j)|$ above column j and on the left of row (i,y) we write $|C_{H_i}(y)|$.

Now we can calculate the characters of $\overline{G}$, which fall into two blocks with inertia groups $\overline{G}$ and $\overline{H}_2$, from these matrices and the character tables of G and H_2, by multiplying rows of $M(g)$ witb sections of the character tables corresponding to g.

At the identity of G we have $M(g) = \begin{pmatrix} 1 & 1 \\ 7 & -1 \end{pmatrix}$. Now we multiply each row by columns of tables 5.1 and 5.2 respectively to get the value of the characters of $\overline{G}$ on $\overline{G}$-classes (1) and (2_1) as follows;

$$\begin{bmatrix} 1 \\ 3 \\ 3 \\ 6 \\ 7 \\ 8 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \\ 3 & 3 \\ 6 & 6 \\ 7 & 7 \\ 8 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} \begin{bmatrix} 7 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 7 & -1 \\ 14 & -2 \\ 21 & -3 \\ 21 & -3 \end{bmatrix}.$$

Similarly, the characters corresponding to class $(2A)$ of G are

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 2 \\ 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ 2 & 2 & -2 \\ 1 & -3 & 1 \\ -3 & 1 & 1 \end{bmatrix}$$

which give the values of the characters of $\overline{G}$ on $\overline{G}$-classes $(2_2), (2_3)$ and (4_1). Similarly for other classes of G, so we get the character table of $\overline{G}$ given in Table 5.4. It is divided into two blocks corresponding to the two inertia groups.

class	(1)	(2_1)	(2_2)	(2_3)	(4_1)	(3_1)	(6_1)	(4_2)	(4_3)	(7_1)	(7_2)
centralizer	1344	192	32	32	16	6	6	8	8	7	7
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	3	−1	−1	−1	0	0	1	1	a	$\overline{a}$
χ_3	3	3	−1	−1	−1	0	0	1	1	$\overline{a}$	a
χ_4	6	6	2	2	2	0	0	0	0	−1	−1
χ_5	7	7	−1	−1	−1	1	1	−1	−1	0	0
χ_6	8	8	0	0	0	−1	−1	0	0	1	1
χ_7	7	−1	3	−1	−1	1	−1	1	−1	0	0
χ_8	7	−1	−1	3	−1	1	−1	−1	1	0	0
χ_9	14	−2	2	2	−2	−1	1	0	0	0	0
χ_{10}	21	−3	1	−3	1	0	0	−1	1	0	0
χ_{11}	21	−3	−3	1	1	0	0	1	−1	0	0

$$a = \tfrac{1}{2}(-1 + \sqrt{7}i)$$

Table 5.4: Character Table of $\overline{G} = 2^3 : GL_3(2)$

5.2 A group of the form $2^4.S_6$

As a second example, we determine the Fischer matrices and hence the character table of a group of the form $2^4.S_6$, a subgroup of the holomorph $2^4.A_8$ and of $\overline{M}_{22}$, the automorphism group of the Mathieu group M_{22} (see Moori [22]). This character table has been determined by Moori by different methods, but we are concerned here with using the methods of Fischer matrices.

Let $\overline{G} = N.G$ be the group mentioned above where N is an elementary abelian group of order 16, so $N \cong V_4(2)$ and $G \cong S_6$. We shall calculate the conjugacy classes of $\overline{G}$ using two different constructions of $\overline{G}$. In the first we regard G as a group of linear transformations, and in the second method we consider $\overline{G}$ as a subgroup of $\overline{M}_{22}$

Conjugacy Classes of $\overline{G}$ (Method 1)

S_6 is a maximal subgroup of $A_8 \cong GL_4(2)$. In fact, S_6 is isomorphic to $SP_4(2)$, the set of all 4×4 matrices over a field of two elements that preserve a non-singular symplectic form. The isomorphism is given by Huppert [14, II.9.21]. Now $G \cong SP_4(2)$ acts naturally on $N \cong V_4(2)$. We can thus determine exactly how G acts on N, and use the methods of section 3.2 to determine the conjugacy classes of $\overline{G}$. The computations were done using CAYLEY [1].

In Table 5.5 we give the conjugacy classes of G and the number of points of N fixed by each class representative g, which we denote by k.

$[g]$	(1)	(2A)	(2B)	(2C)	(3A)	(3B)	(4A)	(4B)	(5A)	(6A)	(6B)
$\|C_G(g)\|$	720	48	48	16	18	18	8	8	5	6	6
k	16	8	4	4	4	1	2	2	1	2	1

Table 5.5: Conjugacy Classes of $G \cong S_6$

Now we analyse the cosets Ng for class representatives g of G. The coset falls into k blocks under the action of N, then we determine how these fuse under the action of $C_G(g)$. In each case, the action of $C_G(g)$ was calculated using CAYLEY.

- $g = 1$: In this case $k = 16$ and under the action of $C_G(g)$ on N we have two orbits of lengths 1 and 15, so we get two classes of $\overline{G}$, the identity class and a class of involutions (2_1).

- $\underline{g \in (2A)}$: We have $|C_G(g)| = 48$. Let $g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ which is an element of this class of G. The orbits of $C_G(g)$ on N are $\{(1,1,1,0)\}$, $\{(0,0,0,0)\}$, $\{(1,1,1,1),(0,0,1,0),(0,1,1,1),(1,1,0,0),(1,0,1,0),(0,1,0,0),(0,0,0,1),(1,0,0,1)\}$, and $\{(1,0,1,1),(0,1,1,0),(0,0,1,1),(1,1,0,1),(0,1,0,1),(1,0,0,0)\}$. From Table 5.5, $k = 8$, so Ng splits into eight blocks $Q_1, \ldots, Q_8$ of length 2 under the action of N. The block Q_1 containing g is fixed by $C_G(g)$, so we have a class (2_2) of $\overline{G}$ with $f = 1$. We also have blocks $Q_2 = \{(1,1,1,1)g,(0,0,0,1)g\}$, $Q_3 = \{(0,0,1,0)g,(1,1,0,0)g\}$, $Q_4 = \{(1,0,0,1)g,(0,1,1,1)g\}$ and $Q_5 = \{(1,0,1,0)g,(0,1,0,0)g\}$. Now we act $C_G(g)$ on these blocks. Note that $(vg)^h = v^h g$ for $h \in C_G(g)$, so the action of $C_G(g)$ on Ng is determined by the action of $C_G(g)$ on N. Thus, from the action of $C_G(g)$ on N we can see that Q_2, Q_3, Q_4 and Q_5 fuse together to give a class of $\overline{G}$ with $f = 4$, so $|C_{\overline{G}}(x)| = \frac{8\times48}{4} = 2^5.3$ where $x = ug$, say $u = (1,1,1,1)$. Now $x^2 = ugug = uu^g = (1,1,1,1) + (0,0,0,1) = (1,1,1,0)$. Therefore x has order 4 and we represent this class of $\overline{G}$ by (4_1).

  $C_G(g)$ also has an orbit of length 6 on N, so the remaining three blocks must fuse together to give a class of $\overline{G}$ with representative $x = ug$, $u = (1,0,1,1)$. Then $x^2 = ugug = uu^g = (1,0,1,1) + (1,0,1,1) = \underline{0}$, so x has order 2. Thus we have class (2_3) of $\overline{G}$ with $|C_{\overline{G}}(x)| = \frac{8\times48}{3} = 2^7$.

- $\underline{g \in (2B)}$: We have $|C_G(g)| = 48$ and $k = 4$, so the coset Ng has four blocks of length 4. $C_G(g)$ acting on N has three orbits of lengths $12, 3$ and 1. So when $C_G(g)$ acts on the blocks of Ng there will be fusions $f = 1$ and $f = 3$, giving classes (2_4) and (4_2) of $\overline{G}$.

Similarly, we have the actions of $C_G(g)$ on N for all remaining classes $[g]$ of G, so we get the conjugacy classes of $\overline{G}$, as in Table 5.6.

Conjugacy Classes of $\overline{G}$ (Method 2)

We now determine the conjugacy classes of $\overline{G}$ by an alternative method, by regarding $\overline{G}$ as a subgroup of $\overline{M}_{22}$.

G acting on N fixes one point and acts transitively on the remaining 15. From the character table of S_6 (Table 5.11) the permutation character of S_6 acting on 15 points

class of G	f	class of $\overline{G}$	$C_{\overline{G}}(x)$
(1)	1	(1)	$2^8.3^2.5$
	15	(2_1)	$2^8.3$
$(2A)$	1	(2_2)	$2^7.3$
	4	(4_1)	$2^5.3$
	3	(2_3)	2^7
$(2B)$	1	(2_4)	$2^6.3$
	3	(4_2)	2^6
$(2C)$	1	(2_5)	2^6
	2	(4_3)	2^5
	1	(4_4)	2^6
$(3A)$	1	(3_1)	$2^3.3^2$
	3	(6_1)	$2^3.3$
$(3B)$	1	(3_2)	2.3^2
$(4A)$	1	(4_5)	2^4
	1	(8_1)	2^4
$(4B)$	1	(4_6)	2^4
	1	(8_2)	2^4
$(5A)$	1	(5_1)	5
$(6A)$	1	(6_2)	$2^2.3$
	1	(12_1)	$2^2.3$
$(6B)$	1	(6_3)	2.3

Table 5.6: Conjugacy Classes of $\overline{G} = 2^4.S_6$

is $\chi = \chi_1 + \chi_3 + \chi_7$. (This was obtained by using Theorem 2.5.6). Then for a class representative g, the number k of points of N fixed by g is $a = \chi(g)$. These values are given in Table 5.7. In Table 5.7 we label each conjugacy class according to its cycle structure.

$[g]$	1^6	$1^4 2$	$1^3 3$	$1^2 4$	$1^2 2^2$	123	15	6	24	2^3	3^2
$\lvert C_G(g)\rvert$	$2^4.3^2.5$	$2^4.3$	2.3^2	2^3	2^4	2.3	5	2.3	2^3	$2^4.3$	2.3^2
k	16	4	1	2	4	1	1	2	2	8	4

Table 5.7: Conjugacy Classes of $G = S_6$

The coset Ng splits into k blocks, and we now determine the values of f, the fusion of these blocks under $C_G(g)$. For the identity coset we have values $f = 1$ and $f = 15$, so classes of $\overline{G}$ are (1) and (2a) with centralizer $\frac{16\times 720}{15} = 2^8.3$. Also, for each coset Ng we will have one class of $\overline{G}$ formed from the identity block, with $f = 1$.

To determine the remaining f values, we consider $\overline{G}$ as a subgroup of $\overline{M_{22}}$. The permutation character $\phi = (1_{\overline{G}})^{\overline{M_{22}}}$ is given in [22]. Referring to the character table of $\overline{M_{22}}$ in [22] we have $\phi = \underline{1} + \underline{21'} + \underline{55}$. The values of ϕ on the conjugacy classes of $\overline{M_{22}}$ are given in Table 5.8 (considering only the classes of $\overline{M_{22}}$ that contain an element of $\overline{G}$). We use the ATLAS [3] notation for $\overline{M_{22}}$-classes, and label them + or - if they lie inside or outside M_{22}, respectively.

Now for a representative y of a class of $\overline{M_{22}}$, we have by Theorem 2.5.7 that $\phi(y) = \sum_x |C_{\overline{M_{22}}}(y)|/|C_{\overline{G}}(x)|$, where x runs over representatives of conjugacy-classes of $\overline{G}$ that fuse to $[y]$ in $\overline{M_{22}}$.

Classes of elements of orders 2, 4, 8

In Table 5.9 we give values of $\frac{|C_{\overline{M_{22}}}(y)|}{|C_{\overline{G}}(x)|}$ for classes of elements of orders 2, 4 and 8 and using the above expression we determine the fusion of elements to $\overline{M_{22}}$ and the conjugacy classes of $\overline{G}$.

For each class of G we have one class of $\overline{G}$ with $f = 1$, giving us the classes (1), (2b), (4b), (2c), (2d) and (4f). Also we have class (2a) from the identity coset. From the entries in Table 5.9 corresponding to these classes we see that (2a) and (2c) must fuse to (2A) in $\overline{M_{22}}$ and no other class fuses to (2A). Therefore (2b) and (2d) must fuse to (2B) and there is no other fusion to (2B). Also (4b) and (4f) must each fuse to one of (4D) and (4C).

Now we consider each class of G (given in Table 5.7) and the possible f values.

class $[y]$ of $\overline{M}_{22}$	$\lvert C_{\overline{M}_{22}}(y)\rvert$	$\phi(y)$
(1)	$2^8.3^2.5.7.11$	77
$(2A)^+$	$2^8.3$	13
$(2B)^-$	$2^7.3.7$	21
$(2C)^-$	$2^7.5$	5
$(4B)^+$	2^5	1
$(4A)^+$	2^6	5
$(4D)^-$	2^6	5
$(4C)$	$2^5.3$	1
$(8A)^+$	2^4	1
$(8B)^-$	2^4	1
$(3A)^+$	$2^3.3^2$	5
$(6A)^+$	$2^3.3$	1
$(6B)^-$	$2^2.3$	3
$(12A)^-$	$2^2.3$	1

Table 5.8: Conjugacy Classes of $\overline{M}_{22}$

- Class 1^42: In this case $k = 4$, so we have four blocks. We have $f = 1$ for one block so other f values are 1,2 or 3, to give a class of $\overline{G}$ containing x with $|C_{\overline{G}}(x)| = \frac{4\times|C_G(g)|}{f} = \frac{2^6\times 3}{f}$. Since x has order 2 or 4, it can fuse to one of the following classes of $\overline{M_{22}}$: $(2C),(4B),(4A),(4D)$ or $(4C)$. If $f = 1$, $|C_{\overline{G}}(x)| = 2^6.3$ but this does not divide $|C_{\overline{M}_{22}}(y)|$ for any of the above classes, so $f \neq 1$. Hence we also cannot have $f = 2$, so we must have $f = 3$ and $|C_{\overline{G}}(x)| = 2^6$. Therefore x fuses to $(4A)$ or $(4D)$, and we get class $(4a)$ of $\overline{G}$. But $(2b)$ fuses to $(2B)^-$ in $\overline{M}_{22}$ so lies outside M_{22}. Therefore elements of $(4a)$ which are products of an element of N and $g \in (2b)$ must also lie outside M_{22}. Therefore $(4a)$ fuses to $(4D)^-$.

- Class 1^24: Here $k = 2$ and $|C_G(g)| = 8$, so besides $(4b)$ we have another class with $f = 1$ and $|C_{\overline{G}}(x)| = 2^4$. From the values of $\frac{|C_{\overline{M}_{22}}(y)|}{|C_{\overline{G}}(x)|}$ we see that this class cannot fuse to any class of elements of order 4, so must have order 8 and fuse to $(8A)$ or $(8B)$.

- Class 1^22^2: For this class, $k = 4$ and $|C_G(g)| = 2^4$. We have $\overline{G}$-class $(2c)$ with $f = 1$, and other classes of $\overline{G}$ must have $f \in \{1,2,3\}$ and $|C_{\overline{G}}(x)| = \frac{2^6}{f}$. Therefore $f = 3$ is not possible, so there is a class with $f = 1$ and $|C_{\overline{G}}(x)| = 2^6$. This class

			$\overline{M}_{22}$ − class $[y]$	1	$2A$	$2B$	$2C$	$4B$	$4A$	$4D$	$4C$	$8A$	$8B$
			$\lvert C_{\overline{M}_{22}}(y)\rvert$	$2^8.3^2.5.7.11$	$2^8.3$	$2^7.3.7$	$2^7.5$	2^5	2^6	2^6	$2^5.3$	2^4	2^4
			$\phi(y)$	77	13	21	5	1	5	5	1	1	1
$[g]$	f	$[x]$	$\lvert C_{\overline{G}}(x)\rvert$										
1^6	1	(1)	$2^8.3^2.5$	77									
	15	$(2a)$	$2^8.3$		1								
1^42	1	$(2b)$	$2^6.3$		4	14							
	3	$(4a)$	2^6				10		1	1			
1^24	1	$(4b)$	2^4					2	4	4	6		
	1	$(8a)$	2^4					2	4	4	6	1	1
1^22^2	1	$(2c)$	2^6		12	42	10						
	1	$(4c)$	2^6				10		1	1			
	2	$(4d)$	2^5				20	1			3		
2^3	1	$(2d)$	$2^7.3$		2	7							
	4	$(4e)$	$2^5.3$								1		
	1	$(2e)$	2^7				5						
24	1	$(4f)$	2^4					2	4	4	10		
	1	$(8b)$	2^4									1	1

Table 5.9:

must then fuse to $(4A)$, we label it $(4c)$. If there is another class with $f = 1$ it will have $|C_{\overline{G}}(x)| = 2^6$ and there is no class of $\overline{M}_{22}$ that it can fuse to, so this is not possible. Hence we must have $f = 2$ and $|C_{\overline{G}}(x)| = 2^5$ and this class must fuse to $(4B)$. We label it $(4d)$.

- Class 2^4: Here $k = 2$ so there are two classes each with $f = 1$. The second class must fuse to a class of $\overline{M}_{22}$ of elements of order 8.

- Class 2^3: We have $k = 8$, and $|C_G(g)| = 2^4.3$. This coset must give rise to a class $(4e)$ that fuses to $(4C)$ with $|C_{\overline{G}}(x)| = 2^5.3$, so has $f = 4$. Now for the remaining blocks, $f \in \{1, 2, 3\}$. If $f = 1$, $|C_{\overline{G}}(x)| = 2^7.3$ and if $f = 2$, $|C_{\overline{G}}(x)| = 2^6.3$. Neither of these is possible since the class must fuse to $(4C)$, so $f = 3$.

Classes of elements of orders 3, 6, 12

In Table 5.10 we give values of $\frac{|C_{\overline{M}_{22}}(y)|}{|C_{\overline{G}}(x)|}$ for classes of elements of orders 3, 6, and 12.

			$\overline{M}_{22}$ − class $[y]$	$(3A)$	$(6A)$	$(6B)$	$(12A)$
			$\lvert C_{\overline{M}_{22}}(y)\rvert$	$2^3.3^2$	$2^3.3$	$2^2.3$	$2^2.3$
			$\phi(y)$	5	1	3	1
G − class	f	$\overline{G}$ − class	$\lvert C_{\overline{G}}(x)\rvert$				
1^33	1	$(3a)$	2.3^2	4			
3^2	1	$(3b)$	$2^3.3^2$	1			
	3	$(6a)$	$2^3.3$		1		
123	1	$(6b)$	2.3			2	2
6	1	$(6c)$	$2^2.3$			1	
	1	$(12a)$	$2^2.3$				1

Table 5.10:

- Class 1^33: Here $k = 1$, so we have one class $(3a)$ of $\overline{G}$, with $f = 1$ and $|C_{\overline{G}}(x)| = 2.3^2$. This must fuse to $(3A)$ in $\overline{M}_{22}$.

- Class 3^2: This class has $k = 4$. There is one class with $f = 1$ and $|C_{\overline{G}}(x)| = 4.2.3^2 = 2^3.3^2$. This must fuse to $(3A)$ in $\overline{M}_{22}$. Now other classes of $\overline{G}$ can have $f = 1, 2$ or 3. If $f = 1$, $|C_{\overline{G}}(x)| = 2^3.3^2$, and this does not divide $|C_{\overline{M}_{22}}(y)|$ for any possible class of $\overline{M}_{22}$. Hence we must have $f = 3$, and a class $(6a)$ of $\overline{G}$ with $|C_{\overline{G}}(x)| = 2^3.3$. This class fuses to the $\overline{M}_{22}$-class $(6A)$.

- Classes 123 and 6: These have $k = 1$ and $k = 2$ respectively so we get classes $(6b)$, $(6c)$ and $(12a)$ of $\overline{G}$, as in Table 5.10. these fuse to $\overline{M_{22}}$-classes $(6B), (6B)$ and $(12A)$ respectively.

Fischer Matrices of G

S_6 acting on N has two orbits, so has two orbits on $\mathrm{Irr}(N)$. These must have lengths 1 and 15. Thus the inertia groups are $\overline{H}_1 = \overline{G}$ and $\overline{H}_2$ where $[\overline{G} : \overline{H}_2] = 15$. If $H_2 = \overline{H}_2/N$ then $H_2 \leq S_6$ with $[S_6 : H_2] = 15$. Thus H_2 is a subgroup of S_6 of order 48, and its character table is given in Table 5.12. (See [21]). The class labels in Table 5.12 indicate the fusion of H_2 in $G = S_6$. Now using the conjugacy classes of $\overline{G}$ from Tables 5.9 and 5.10 and the fusion of H_2 in G, we get the Fischer matrices $M(g)$ for class representatives g of G, given below. The entries were calculated from the relations in Theorem 4.2.5.

$[g]$	$M(g)$
1^6	$\begin{array}{c} \\ 720 \\ 48 \end{array}\begin{array}{cc} 2^8.3^2.5 & 2^8.3 \\ \left(\begin{array}{c} 1 \\ 15 \end{array}\right. & \left.\begin{array}{c} 1 \\ -1 \end{array}\right) \end{array}$
1^42	$\begin{array}{c} \\ 48 \\ 16 \end{array}\begin{array}{cc} 2^6.3 & 2^6 \\ \left(\begin{array}{c} 1 \\ 3 \end{array}\right. & \left.\begin{array}{c} 1 \\ -1 \end{array}\right) \end{array}$
1^22^2	$\begin{array}{c} \\ 16 \\ 16 \\ 8 \end{array}\begin{array}{ccc} 2^6 & 2^5 & 2^6 \\ \left(\begin{array}{c} 1 \\ 1 \\ 2 \end{array}\right. & \begin{array}{c} 1 \\ -1 \\ 0 \end{array} & \left.\begin{array}{c} 1 \\ 1 \\ -2 \end{array}\right) \end{array}$
2^3	$\begin{array}{c} \\ 48 \\ 8 \\ 48 \end{array}\begin{array}{ccc} 2^7.3 & 2^7 & 2^5.3 \\ \left(\begin{array}{c} 1 \\ 6 \\ 1 \end{array}\right. & \begin{array}{c} 1 \\ -2 \\ 1 \end{array} & \left.\begin{array}{c} 1 \\ 0 \\ -1 \end{array}\right) \end{array}$
1^24	$\begin{array}{c} \\ 8 \\ 8 \end{array}\begin{array}{cc} 2^4 & 2^4 \\ \left(\begin{array}{c} 1 \\ 1 \end{array}\right. & \left.\begin{array}{c} 1 \\ -1 \end{array}\right) \end{array}$

$$24 \qquad \begin{array}{c} \\ 8 \\ 8 \end{array} \begin{array}{c} 2^4 \quad 2^4 \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$$

$$3^2 \qquad \begin{array}{c} \\ 18 \\ 6 \end{array} \begin{array}{c} 2^3.3^2 \quad 2^3.3 \\ \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \end{array}$$

$$1^3 3 \qquad \begin{array}{c} \\ 18 \end{array} \begin{array}{c} 2.3^2 \\ \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

$$123 \qquad \begin{array}{c} \\ 6 \end{array} \begin{array}{c} 2.3 \\ \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

$$6 \qquad \begin{array}{c} \\ 6 \\ 6 \end{array} \begin{array}{c} 2^2.3 \quad 2^2.3 \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$$

$$15 \qquad \begin{array}{c} \\ 5 \end{array} \begin{array}{c} 5 \\ \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

Now by multiplication of the relevant columns of the character tables of G and H_2 (Tables 5.11 and 5.12) and the rows of the Fischer matrices, we get the character table of $\overline{G}$, given in Table 5.13. The characters are divided into blocks, corresponding to the two inertia groups.

class	1^6	1^42	1^22^2	2^3	1^24	24	3^2	1^33	123	6	15
centralizer	720	48	16	48	8	8	18	18	6	6	5
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	5	3	1	−1	1	−1	−1	2	0	−1	0
χ_3	9	3	1	3	−1	1	0	0	0	0	−1
χ_4	10	2	−2	−2	0	0	1	1	−1	1	0
χ_5	5	1	1	−3	−1	−1	2	−1	1	0	0
χ_6	16	0	0	0	0	0	−2	−2	0	0	1
χ_7	5	−1	1	3	1	−1	2	−1	−1	0	1
χ_8	10	−2	−2	2	0	0	1	1	1	−1	1
χ_9	9	−3	1	−3	1	1	0	0	0	0	1
χ_{10}	5	−3	1	1	−1	−1	−1	2	0	1	0
χ_{11}	1	−1	1	−1	−1	1	1	1	−1	−1	1

Table 5.11: Character Table of $G = S_6$

class	1^6	1^42	$(1^22^2)_1$	$(1^22^2)_2$	$(2^3)_1$	$(2^3)_2$	1^24	24	3^2	6
centralizer	48	16	16	8	8	48	8	8	6	6
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	−1	1	1	−1	−1	−1	1	1	−1
χ_3	1	1	1	−1	−1	1	−1	−1	1	1
χ_4	1	−1	1	−1	1	−1	1	−1	1	−1
χ_5	2	2	2	0	0	2	0	0	−1	−1
χ_6	2	−2	2	0	0	−2	0	0	−1	1
χ_7	3	1	−1	1	−1	−3	1	−1	0	0
χ_8	3	−1	−1	1	1	3	−1	−1	0	0
χ_9	3	1	−1	−1	1	−3	−1	1	0	0
χ_{10}	3	−1	−1	−1	−1	3	1	1	0	0

Table 5.12: Character Table of H_2

class of G	1^6		$1^4 2$		$1^2 2^2$			2^3			$1^2 4$	
centralizer	$2^8.3^2.5$	$2^8.3$	$2^6.3$	2^6	2^6	2^5	2^6	$2^7.3$	2^7	$2^5.3$	2^4	2^4
class of $\overline{G}$	(1)	(2*a*)	(2*b*)	(4*a*)	(2*c*)	(4*d*)	(4*c*)	(2*d*)	(2*e*)	(4*e*)	(4*b*)	(8*a*)
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	5	5	3	3	1	1	1	−1	−1	−1	1	1
χ_3	9	9	3	3	1	1	1	3	3	3	−1	−1
χ_4	10	10	2	2	−2	−2	−2	−2	−2	−2	0	0
χ_5	5	5	1	1	1	1	1	−3	−3	−3	−1	−1
χ_6	16	16	0	0	0	0	0	0	0	0	0	0
χ_7	5	5	−1	−1	1	1	1	3	3	3	1	1
χ_8	10	10	−2	−2	−2	−2	−2	2	2	2	0	0
χ_9	9	9	−3	−3	1	1	1	−3	−3	−3	1	1
χ_{10}	5	5	−3	−3	1	1	1	1	1	1	−1	−1
χ_{11}	1	1	−1	−1	1	1	1	−1	−1	−1	−1	−1
χ_{12}	15	−1	3	−1	3	−1	−1	7	−1	−1	1	−1
χ_{13}	15	−1	−3	1	3	−1	−1	−7	1	1	−1	1
χ_{14}	15	−1	3	−1	−1	−1	3	−5	3	−1	−1	1
χ_{15}	15	−1	−3	1	−1	−1	3	5	−3	1	1	−1
χ_{16}	30	−2	6	−2	2	−2	2	2	2	−2	0	0
χ_{17}	30	−2	−6	2	2	−2	2	2	2	−2	0	0
χ_{18}	45	−3	3	−1	1	1	−3	−9	−1	3	1	−1
χ_{19}	45	−3	−3	1	1	1	−3	9	1	−3	−1	1
χ_{20}	45	−3	3	−1	−3	1	1	3	−5	3	−1	1
χ_{21}	45	−3	−3	1	−3	1	1	−3	5	−3	1	−1

Table 5.13: Character Table of $\overline{G} = 2^4.S^6$
(continued on next page)

class of G	2 4		3^2		$1^3 3$	1 2 3	6		1 5
centralizer	2^4	2^4	$2^3.3^2$	$2^3.3$	2.3^2	2.3	$2^2.3$	$2^2.3$	5
class of $\overline{G}$	$(4f)$	$(8b)$	$(3b)$	$(6a)$	$(3a)$	$(6b)$	$(6c)$	$(12a)$	$(5a)$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	−1	−1	−1	−1	2	0	−1	−1	0
χ_3	1	1	0	0	0	0	0	0	−1
χ_4	0	0	1	1	1	−1	1	1	0
χ_5	−1	−1	2	2	−1	1	0	0	0
χ_6	0	0	−2	−2	−2	0	0	0	1
χ_7	−1	−1	2	2	−1	−1	0	0	0
χ_8	0	0	1	1	1	1	−1	−1	0
χ_9	1	1	0	0	0	0	0	0	−1
χ_{10}	−1	−1	−1	−1	2	0	1	1	0
χ_{11}	1	1	1	1	1	−1	−1	−1	1
χ_{12}	1	−1	3	−1	0	0	1	−1	0
χ_{13}	1	−1	3	−1	0	0	−1	1	0
χ_{14}	−1	1	3	−1	0	0	1	−1	0
χ_{15}	−1	1	3	−1	0	0	−1	1	0
χ_{16}	0	0	−3	1	0	0	−1	1	0
χ_{17}	0	0	−3	1	0	0	1	−1	0
χ_{18}	−1	1	0	0	0	0	0	0	0
χ_{19}	−1	1	0	0	0	0	0	0	0
χ_{20}	1	−1	0	0	0	0	0	0	0
χ_{21}	1	−1	0	0	0	0	0	0	0

Character Table of $\overline{G} = 2^4.S^6$

(cont.)

5.3 Holomorph of C_p

Definition 5.3.1 The *holomorph* of a group N is $N : \mathrm{Aut}(N)$, where $\mathrm{Aut}(N)$ acts naturally on N.

Lemma 5.3.1 *If C_p is the cyclic group of order p (p prime), then $Aut(C_p) \cong C_{p-1}$.*

Proof: Let $C_p = <x>$. Each $\alpha \in \mathrm{Aut}(C_p)$ is determined by $\alpha(x)$, so that $\mathrm{Aut}(C_p) = \{\alpha_1, \ldots, \alpha_{p-1}\}$ where α_i is defined by $\alpha_i(x) = x^i$ for $i = 1, \ldots, p-1$. Now let Z_p^* be the multiplicative group of nonzero elements of $Z_p \cong Z/pZ$, and define $\phi : \mathrm{Aut}(C_p) \to Z_p^*$ by $\phi(\alpha_i) = \bar{i}$. Then ϕ is an automorphism so that $\mathrm{Aut}(C_p) \cong Z_p^*$ But (for example, see Rotman [27, 2.16]) the group of nonzero elements of a finite field is cyclic, so $\mathrm{Aut}(C_p) \cong C_{p-1}$. □

Now we construct the Fischer matrices and character table of the holomorph of C_p, which is $C_p : C_{p-1}$. Let $\overline{G} = N : G$ where $N \cong C_p$, $G \cong C_{p-1}$. If $N = <x>$ then each element of $G = \mathrm{Aut}(N)$ maps x onto a different non-identity element of N. Therefore the orbits of G on N have lengths 1 and $p-1$.

To find the conjugacy classes of $\overline{G}$ we analyse the cosets Ng for each $g \in G$ and find the values of k (the order of the stabilizer in N of g). Since k divides $|N|$ and $|N| = p$, we must have $k = p$ or $k = 1$. If $k = p$ then $n^g = n$ for all $n \in N$, so $g = e$, the identity of $\mathrm{Aut}(N) = G$. Hence for non-identity element g we have $k = 1$.

Now the classes of $\overline{G}$ are as follows: For $g = e$, $k = p$ and there are fusions $f = 1$ ad $f = p-1$. For $f = 1$, we have the identity class of $\overline{G}$. For $f = p-1$, we have a class of $\overline{G}$ containing x of order p with $|C_{\overline{G}}(x)| = \frac{p(p-1)}{p-1} = p$. We denote this class by (p). Corresponding to the cosets Ng where $g \neq e$, we have $k = 1$ so there is one class of $\overline{G}$ containing x for each non-identity x in G, with $|C_{\overline{G}}(x)| = p-1$.

Since G has two orbits on N, it has two orbits on $\mathrm{Irr}(N)$ and these must have lengths 1 and $p-1$. Therefore the inertia groups are $\overline{H}_1 = \overline{G}$ and $\overline{H}_2 = N$ with $H_1 = G$ and $H_2 = \{e\}$ respectively.

For $g = e$, the Fischer matrix is

$$M(e) = \begin{array}{c} \\ p-1 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} p(p-1) & p \end{array} \\ \begin{pmatrix} 1 & 1 \\ p-1 & -1 \end{pmatrix} \end{array}.$$

For $g \neq e$, $M(g) = (\,1\,)$, since H_2 does not fuse to any non-identity class of G.

Now the characters of $\overline{G}$ are determined from the matrices $M(g)$ and the character tables of G and H_2. At conjugacy classes of $\overline{G}$ corresponding to $g = e$, the character values in the $\overline{G}$-block are

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix},$$

and in the $\overline{H}_2$-block,

$$\begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} p-1 & -1 \end{bmatrix} = \begin{bmatrix} p-1 & -1 \end{bmatrix}.$$

Corresponding to $g \neq e$, we have the character table of G for the $\overline{G}$-block, and zero in the $\overline{H}_2$-block, so we have the character table of $\overline{G}$ as in Table 5.14.

class	(1)	(p)	Cl_1	Cl_2	$\cdots$	Cl_{p-2}
centralizer	$p(p-1)$	p	$p-1$	$p-1$	$\cdots$	$p-1$
χ_1	1					
χ_2	1					
$\vdots$	$\vdots$		X			
χ_{p-1}	1					
χ_p	$p-1$	-1	0	0	$\cdots$	0

$Cl_1, \ldots, Cl_{p-2}$ are the non-identity classes (elements) of C_{p-1}
X denotes the character table of C_{p-1}

Table 5.14: Character Table of $C_p : C_{p-1}$

For example, considering the case $p = 7$ we can determine the character table of $C_7 : C_6$ which we give in Table 5.15.

class	(1)	(7)	(2)	(3_1)	(3_2)	(6_1)	(6_2)
centralizer	42	7	6	6	6	6	6
χ_1	1	1	1	1	1	1	1
χ_2	1	1	-1	$-\overline{a}$	$-a$	a	$\overline{a}$
χ_3	1	1	1	$-a$	$-\overline{a}$	$-\overline{a}$	$-a$
χ_4	1	1	-1	1	1	-1	-1
χ_5	1	1	1	$-\overline{a}$	$-a$	$-a$	$-\overline{a}$
χ_6	1	1	-1	$-a$	$-\overline{a}$	$\overline{a}$	a
χ_7	6	-1	0	0	0	0	0

$$a = \frac{1}{2}(1+\sqrt{3}i)$$

Table 5.15: Character Table of $C_7 : C_6$

Chapter 6

MAXIMAL SUBGROUPS OF J_1

J_1, the smallest Janko group, is a sporadic simple group of order 175560. Janko [16] constructed it as a subgroup of $GL_7(11)$, and it is characterized by the following properties:

1. Sylow 2-subgroups of J_1 are abelian,
2. J_1 has no subgroup of index 2, and
3. J_1 contains an involution t such that $C_{J_1}(t) = \langle t \rangle \times F$, where $F \cong A_5$.

From ATLAS [3], we bave the character table of J_1 and a list of its maximal subgroups. We give the character table in the Appendix, and list the maximal subgroups in Table 6.1.

The character tables of these maximal subgroups have been calculated and are available through GAP (see [29]); our aim here is to show how the conjugacy classes and character tables of these groups can be calculated using the theory and methods discussed in chapters 3 and 4. We also give the class fusions of these maximal subgroups to J_1 and their permutation characters. We follow the ATLAS notation in writing the irreducible constituents of these characters where we refer to each irreducible character by its degree and distinguish different characters of the same degree by $a, b, c, \ldots$etc. So $76b$ denotes the second irreducible character of degree 76, and we abbreviate $76a + 76b$ to $76ab$, for example. The permutation characters which we determine in this chapter (with the exception of $L_2(11)$) are not listed in ATLAS.

Order	Index	Structure	Specification
660	266	$L_2(11)$	
168	1045	$2^3:7:3$	Sylow 2-normalizer
120	1463	$2\times A_5$	
114	1540	$19:6$	Sylow 19-normalizer
110	1596	$11:10$	Sylow 11-normalizer
60	2926	$D_6\times D_{10}$	Sylow 3, Sylow 5-normalizer
42	4180	$7:6$	Sylow 7-normalizer

Table 6.1: Maximal subgroups of J_1

6.1 $L_2(11)$

This is the general linear group of degree 11 over a field of two elements factored by its centre and its character table is given in ATLAS. In Table 6.2 we give its conjugacy classes with fusion to J_1. The permutation character is $1_{L_2(11)}^{J_1} = 1a + 56ab + 76a + 77a$.

$[g]$	$\lvert C_{L_2(11)}(g)\rvert$	$\to J_1$	power maps π^2	π^3
$(1A)$	660	$(1A)$		
$(2A)$	12	$(2A)$		
$(3A)$	6	$(3A)$		
$(5A)$	5	$(5A)$	$(5B)$	
$(5B)$	5	$(5B)$	$(5A)$	
$(6A)$	6	$(6A)$	$(3A)$	$(2A)$
$(11A)$	11	$(11A)$		
$(11B)$	11	$(11A)$		

Table 6.2: Conjugacy Classes of $L_2(11)$

6.2 $2^3 : 7 : 3$

The normalizer of a Sylow 2-subgroup in J_1 is a split extension of an elementary abelian group of order 8 by a non-abelian group of order 21, which is a split extension of C_7 by C_3. First, we construct the character table of $G = K : Q$ where $K = <\alpha> \cong C_7$ and $Q = <\beta> \cong C_3$. The group Q acts on K, with the action determined by the action of β. Since β has order 3, it must act as $(e)(\alpha\alpha^2\alpha^4)(\alpha^3\alpha^6\alpha^5)$. Then the conjugacy classes of G are as follows: For the coset Kq where q is the identity of Q, we have $k = 7$ and $f = 1, 3, 3$. So we get the identity class of G and two classes of elements of order 7. Now for the cosets $K\beta$ and $K\beta^{-1}$ we must have $k = 1$ (since k divides 7 but $k \neq 7$ for if $k = 7$ then K and Q commute and G is abelian, a contradiction). Thus we get two classes of elements of order 3. Table 6.3 gives the conjugacy classes of G.

class of Q		(1)		(3_1)	(3_2)
class of G	(1)	(7_1)	(7_2)	(3_1)	(3_2)
centralizer	21	7	7	3	3

Table 6.3: Conjugacy Classes of $G = 7 : 3$

Since Q has three orbits on K it has three orbits on $\mathrm{Irr}(K)$, and these must have lengths 1, 3, 3 (since the length of any orbit must divide $|Q| = 3$). Referring to the character table of K (Table 6.4), the orbits of Q on K are $\{e\}$, $\{\alpha, \alpha^2, \alpha^4\}$ and $\{\alpha^3, \alpha^6, \alpha^5\}$. Hence we find the orbits on $\mathrm{Irr}(K)$: Since $\chi_2^\beta(\alpha) = \chi_2(\alpha^\beta) = \chi_2(\alpha^2) = \chi_3(\alpha)$, we have $\chi_2^\beta = \chi_3$. Similarly, $\chi_2^{\beta^{-1}} = \chi_5$. Therefore the orbits of Q on $\mathrm{Irr}(K)$ are $\{\chi_1\}$, $\{\chi_2, \chi_3, \chi_5\}$ and $\{\chi_4, \chi_6, \chi_7\}$.

Now the rows of $M(e)$, the Fischer matrix corresponding to the identity of Q, are orbit sums of the action of Q on $\mathrm{Irr}(K)$ with duplicate columns discarded (Proposition 4.2.4), so $M(e) = \begin{pmatrix} 1 & 1 & 1 \\ 3 & b & c \\ 3 & c & b \end{pmatrix}$, where

$$b = e^{\frac{2\pi i}{7}} + e^{\frac{4\pi i}{7}} + e^{\frac{8\pi i}{7}} = \frac{1}{2}(-1 + \sqrt{7}i),$$

$$c = e^{\frac{12\pi i}{7}} + e^{\frac{10\pi i}{7}} + e^{\frac{6\pi i}{7}} = \frac{1}{2}(-1 - \sqrt{7}i) = \bar{b}.$$

Each row of $M(e)$ corresponds to an inertia group $\overline{H}_i$ where $\overline{H}_1 = \overline{G}$ and $\overline{H}_2 = \overline{H}_3 = K$, so H_2 and H_3 are trivial (where $H_i = \overline{H}/K$). The remaining Fischer matrices are

element of C_7	e	α	α^2	α^3	α^4	α^5	α^6
χ_1	1	1	1	1	1	1	1
χ_2	1	a	a^2	a^3	a^4	a^5	a^6
χ_3	1	a^2	a^4	a^6	a	a^3	a^5
χ_4	1	a^3	a^6	a^2	a^5	a	a^4
χ_5	1	a^4	a	a^5	a^2	a^6	a^3
χ_6	1	a^5	a^3	a	a^6	a^4	a^2
χ_7	1	a^6	a^5	a^4	a^3	a^2	a

$$a = e^{\frac{2\pi i}{7}}$$

Table 6.4: Character Table of $C_7 =< \alpha >$

$M(\beta) = M(\beta^{-1}) = (1)$. Now, from the matrix $M(e)$ and the character table of C_3, we get the character table of $G \cong 7:3$, given in Table 6.5.

Now let $\overline{G} = N : G$, where N is an elementary abelian group of order 8. Let $N = \{0, e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3\}$. The action of G on N is determined by the actions of α and β of orders 7 and 3 respectively. These actions are as follows.

$$\alpha : (0)(e_1 \ \ e_2 \ \ e_3 \ \ e_1 + e_2 \ \ e_2 + e_3 \ \ e_1 + e_2 + e_3 \ \ e_1 + e_3)$$

$$\beta : (0)(e_1)(e_2 \ \ e_3 \ \ e_2 + e_3)(e_1 + e_2 \ \ e_1 + e_3 \ \ e_1 + e_2 + e_3)$$

Thus G has orbits of lengths 1 and 7. Now with the action of G on N, the methods of section 3.2 give the conjugacy classes of $\overline{G}$, given in Table 6.6. The fusion to J_1 is obvious, and with this fusion known we determine the permutation character by Theorem 2.5.7. It is

$$1_{\overline{G}}^{J_1} = 1a + 56ab + 76a + 77bc + 120abc + 133a + 209a.$$

Fischer Matrices of $\overline{G}$

G has two orbits on $\mathrm{Irr}(N)$ of lengths 1 and 7, so the inertia groups are $\overline{H_1} = \overline{G}$ with $H_1 = G$ and $\overline{H_2} = N : H_2$ where $H_2 \leq G$ with $[G : H_2] = 7$. Hence $|H_2| = 3$, so $H_2 \cong C_3$. Now we get the Fischer matrices, from the conjugacy classes of $\overline{G}$ and the relations in Theorem 4.2.5. The Fischer matrices are given in Table 6.7.

$[g]$	(1)	(7_1)	(7_2)	(3_1)	(3_2)
$\lvert C_G(g)\rvert$	21	7	7	3	3
χ_1	1	1	1	1	1
χ_2	1	1	1	a	$\overline{a}$
χ_3	1	1	1	$\overline{a}$	a
χ_4	3	b	$\overline{b}$	0	0
χ_5	3	$\overline{b}$	b	0	0

$a = \frac{1}{2}(-1 + \sqrt{3}i)$, $b = \frac{1}{2}(-1 + \sqrt{7}i)$

Table 6.5: Character Table of $G = 7:3$

class of G	f	class of $\overline{G}$	centralizer	$\to J_1$	power maps π^2	π^3
(1)	1	(1)	168	1a		
	7	(2_1)	24	2a		
(7_1)	1	(7_1)	7	7a	(7_2)	
(7_2)	1	(7_2)	7	7a	(7_1)	
(3_1)	1	(3_1)	6	3a	(3_2)	
	1	(6_1)	6	6a	(3_2)	(2_1)
(3_2)	1	(3_2)	6	3a	(3_1)	
	1	(6_2)	6	6a	(3_1)	(2_1)

Table 6.6: Conjugacy Classes of $\overline{G} \cong 2^3 : 7 : 3$

$[g]$	$M(g)$
(1)	$\begin{array}{c} \\ 21 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} 168 & 24 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 7 & -1 \end{pmatrix} \end{array}$
(7_1)	$\begin{array}{c} \\ 7 \end{array}\begin{array}{c} 7 \\ \begin{pmatrix} 1 \end{pmatrix} \end{array}$
(7_2)	$\begin{array}{c} \\ 7 \end{array}\begin{array}{c} 7 \\ \begin{pmatrix} 1 \end{pmatrix} \end{array}$
(3_1)	$\begin{array}{c} \\ 3 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} 6 & 6 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$
(3_2)	$\begin{array}{c} \\ 3 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} 6 & 6 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$

Table 6.7: Fischer Matrices of $\overline{G} \cong 2^3 : 7 : 3$

class	(1)	(2_1)	(7_1)	(7_2)	(3_1)	(6_1)	(3_2)	(6_2)
centralizer	168	24	7	7	6	6	6	6
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	a	a	$\overline{a}$	$\overline{a}$
χ_3	1	1	1	1	$\overline{a}$	$\overline{a}$	a	a
χ_4	3	3	b	$\overline{b}$	0	0	0	0
χ_5	3	3	$\overline{b}$	b	0	0	0	0
χ_6	7	-1	0	0	1	-1	1	-1
χ_7	7	-1	0	0	a	$-a$	$\overline{a}$	$-\overline{a}$
χ_8	7	-1	0	0	$\overline{a}$	$-\overline{a}$	a	$-a$

$$a = \tfrac{1}{2}(-1 + \sqrt{3}i),\ b = \tfrac{1}{2}(-1 + \sqrt{7}i)$$

Table 6.8: Character Table of $\overline{G} \cong 2^3 : 7 : 3$

Now by multiplication of the columns of the character tables of G (Table 6.5) and $H_2 \cong C_3$ with the rows of the Fischer matrices, we get the character table of $\overline{G}$ (Table 6.8).

class	(1)	(2_1)	(3_1)	(5_1)	(5_2)
centralizer	60	4	3	5	5
power π^2			(3_1)	(5_2)	(5_1)
maps π^3				(5_2)	(5_1)
χ_1	1	1	1	1	1
χ_2	3	-1	0	a	b
χ_3	3	-1	0	b	a
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$a = \frac{1}{2}(1+\sqrt{5}),\ b = \frac{1}{2}(1-\sqrt{5})$

Table 6.9: Character Table of A_5

6.3 $2 \times A_5$ and $D_6 \times D_{10}$

By Theorem 2.3.1 each character of $H \times K$ is a product of a character of H and a character of K. So from the character tables of C_2 and A_5 (Table 6.9), we get the character table of $2 \times A_5$ (Table 6.10). The fusion to J_1 is determined by the power maps, and we can then calculate the permutation character as

$$1_{2\times A_5}^{J_1} = 1a + 56ab + 76a + 77aa + 120abc + 133aa + 209aa.$$

Similarly, from the character tables of D_6 and D_{10} (Tables 6.11 and 6.12) we get the character table of $D_6 \times D_{10}$ (Table 6.13), and

$$1_{D_6\times D_{10}}^{J_1} = 1a + 56ab + 76aaa + 77aaa + 120aabbcc + 133aaaabc + 209aaaa.$$

class	(1)	(2_1)	(3_1)	(5_1)	(5_2)	(2_2)	(2_3)	(6_1)	(10_1)	(10_2)
centralizer	120	8	6	10	10	120	8	6	10	10
$\rightarrow J_1$	1a	2a	3a	5a	5b	2a	2a	6a	10a	10b
power maps π^2				(5_2)	(5_1)			(3_1)	(5_2)	(5_1)
π^3				(5_2)	(5_1)			(2_1)		
π^5									(2_1)	(2_1)
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	−1	−1	−1	−1	−1
χ_3	3	−1	0	a	b	3	−1	0	a	b
χ_4	3	−1	0	a	b	−3	1	0	$-a$	$-b$
χ_5	3	−1	0	b	a	3	−1	0	b	a
χ_6	3	−1	0	b	a	−3	1	0	$-b$	$-a$
χ_7	4	0	1	−1	−1	4	0	1	−1	−1
χ_8	4	0	1	−1	−1	−4	0	−1	1	1
χ_9	5	1	−1	0	0	5	1	−1	0	0
χ_{10}	5	1	−1	0	0	−5	−1	1	0	0

$a = \frac{1}{2}(1+\sqrt{5}),\ b = \frac{1}{2}(1-\sqrt{5})$

Table 6.10: Character Table of $2 \times A_5$

$[h]$	(1)	(2_1)	(3_1)
$\lvert C_{D_6}(h)\rvert$	6	2	3
χ_1	1	1	1
χ_2	1	−1	1
χ_3	2	0	−1

Table 6.11: Character Table of $D_6 \cong S_3$

$[k]$	(1)	(2_1)	(5_1)	(5_2)
$\lvert C_{D_{10}}(k)\rvert$	10	2	5	5
χ_1	1	1	1	1
χ_2	1	-1	1	1
χ_3	2	0	a	b
χ_4	2	0	b	a

$a = \frac{1}{2}(-1+\sqrt{5})$, $b = \frac{1}{2}(-1-\sqrt{5})$

Table 6.12: Character Table of D_{10}

class	(1)	(2_1)	(5_1)	(5_2)	(2_2)	(2_3)	(10_1)	(10_2)	(3_1)	(6_1)	(15_1)	(15_2)
centralizer	60	12	30	30	20	4	10	10	30	6	15	15
$\rightarrow J_1$	1a	2a	5a	5b	2a	2a	10a	10b	3a	6a	15a	15b
power maps π^2			(5_2)	(5_1)			(5_2)	(5_1)		(3_1)		
π^3										(2_1)	(5_2)	(5_1)
π^5							(2_2)	(2_2)			(3_1)	(3_1)
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	−1	1	1	1	−1	1	1	1	−1	1	1
χ_3	2	0	a	b	2	0	a	b	2	0	a	b
χ_4	2	0	b	a	2	0	b	a	2	0	b	a
χ_5	1	1	1	1	−1	−1	−1	−1	1	1	1	1
χ_6	1	−1	1	1	−1	1	−1	−1	1	−1	1	1
χ_7	2	0	a	b	−2	0	$-a$	$-b$	2	0	a	b
χ_8	2	0	b	a	−2	0	$-b$	$-a$	2	0	b	a
χ_9	2	2	2	2	0	0	0	0	−1	−1	−1	−1
χ_{10}	2	−2	2	2	0	0	0	0	−1	1	−1	−1
χ_{11}	4	0	$2a$	$2b$	0	0	0	0	−2	0	$-a$	$-b$
χ_{12}	4	0	$2b$	$2a$	0	0	0	0	−2	0	$-b$	$-a$

$$a = \tfrac{1}{2}(-1+\sqrt{5}),\ b = \tfrac{1}{2}(-1-\sqrt{5})$$

Table 6.13: Character Table of $D_6 \times D_{10}$

6.4 Sylow 19-normalizer

Definition 6.4.1 A *Frobenius group* is a finite group $\overline{G}$ that contains a nontrivial normal subgroup N such that if x is a non-identity element of N then $C_{\overline{G}}(x) \subset N$.

Now let $\overline{G}$ be a Sylow 19-normalizer in J_1. Then by Janko [16], $\overline{G}$ is a Frobenius group with structure 19:6. Let $\overline{G} = N : G$ where $N \cong C_{19}$ and $G \cong C_6$. Then we have the following lemma.

Lemma 6.4.1 *When G acts on N, it fixes one point and has three orbits of length 6.*

Proof: G fixes the identity of N. Now if x is a non-identity element of N, then $C_{\overline{G}}(x) \subset N$, since $\overline{G}$ is a Frobenius group. Therefore, for $e \neq g \in G$, $g \notin C_{\overline{G}}(x)$. (Because $G \cap N = \{e\}$). So no nonidentity element of G fixes x. Now we consider the action of β on N, where $G = <\beta>$. Since β has order 6, its nontrivial orbits on N can have lengths 2, 3 or 6. But if β has an orbit of length 2 or 3 then β^2 or β^3 (respectively) fixes a non-identity element of N which is not possible. Therefore β (and hence G) has three orbits of length 6. □

Now let $N = <\alpha>$, $G = <\beta>$. Then $\overline{G}$ is a subgroup of J_1 and J_1 has three conjugacy classes of elements of order 19, namely $19a, 19b = (19a)^2$ and $19c = (19a)^4$. Therefore α, α^2 and α^4 are all in different classes of J_1, so must be in different classes of $\overline{G}$. Therefore the three orbits of length 6 of G on N have representatives α, α^2 and α^4 respectively.

Conjugacy Classes of $\overline{G}$

For each $g \in G$, we find $k = |C_N(g)|$ and f, the block fusions. for $g = e$, $k = 19$ and (by action of G on N) $f = 1, 6, 6, 6$. Thus we have the identity class of $\overline{G}$ and three classes of elements of order 19, each with centralizer $\frac{19 \times 6}{6} = 19$. For $g \neq e$, $k = 1$ (since $k|19$ and $k \neq 19$), so $f = 1$. The conjugacy classes of $\overline{G} = 19:6$ are given in Table 6.15.

Fischer Matrices of $\overline{G}$

G has four orbits on N, so has four orbits on Irr(N). One orbit is trivial and the others must have lengths that divide $|G|$, so there are three orbits of length 6. Thus the inertia groups are $\overline{H}_1 = \overline{G}$ with $H_1 = G$ and $\overline{H}_2 = \overline{H}_3 = \overline{H}_4 = N$, so $H_2 = H_3 = H_4 = \{e\}$.

The Fischer matrix $M(e)$ has rows that are orbit sums of the action of G on $\mathrm{Irr}(N)$. Since the orbits of G on N are $\{\alpha, \alpha^8, \alpha^7, \alpha^{-1}, \alpha^{-8}, \alpha^{-7}\}$, $\{\alpha^2, \alpha^{-3}, \alpha^{-5}, \alpha^{-2}, \alpha^3, \alpha^5\}$ and $\{\alpha^4, \alpha^{-6}, \alpha^9, \alpha^{-4}, \alpha^6, \alpha^{-9}\}$, the orbit sums of G on $\mathrm{Irr}(N)$ are

$$c = x + x^8 + x^7 + x^{18} + x^{11} + x^{12},$$

$$d = x^2 + x^{16} + x^{14} + x^{17} + x^3 + x^5$$

and

$$e = x^4 + x^{13} + x^9 + x^{15} + x^6 + x^{10},$$

where $x = e^{\frac{2\pi i}{19}}$, a primitive 19^{th} root of unity.

Therefore $M(e) = \begin{array}{c} \\ 6 \\ 1 \\ 1 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 114 & 19 & 19 & 19 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 6 & c & d & e \\ 6 & d & e & c \\ 6 & e & c & d \end{pmatrix} \end{array}$, where c, d, e are given by the above relations.

For $g \neq e$, $M(g) = (\,1\,)$, since H_2, H_3 and H_4 do not fuse to nonidentity classes of G.

Now from the character table of G (Table 6.14) and the Fischer matrices, we get the character table of $\overline{G}$ (Table 6.15). We can also determine the permutation character $1_{19:6}^{J_1}$ from the fusion to J_1 (which is given in Table 6.15) and obtain

$$1_{19:6}^{J_1} = 1a + 56ab + 76aa + 77abc + 120abc + 133aa + 209aa.$$

class	(1)	(6_1)	(3_1)	(2)	(3_2)	(6_2)
χ_1	1	1	1	1	1	1
χ_2	1	a	$-b$	-1	$-a$	b
χ_3	1	$-b$	$-a$	1	$-b$	$-a$
χ_4	1	-1	1	-1	1	-1
χ_5	1	$-a$	$-b$	1	$-a$	$-b$
χ_6	1	b	$-a$	-1	$-b$	a

$a = \frac{1}{2}(1 + \sqrt{3}i)$, $b = \frac{1}{2}(1 - \sqrt{3}i) = \overline{a}$

Table 6.14: Character Table of C_6

class / centralizer		(1) 114	(19_1) 19	(19_2) 19	(19_3) 19	(2) 6	(3_1) 6	(3_2) 6	(6_1) 6	(6_2) 6
	$\to J_1$	1*a*	19*a*	19*b*	19*c*	2*a*	3*a*	3*a*	6*a*	6*a*
power map	π^2		19_2	19_3	19_1				3_1	3_2
	χ_1	1	1	1	1	1	1	1	1	1
	χ_2	1	1	1	1	-1	$-\bar{a}$	$-a$	a	$\bar{a}$
	χ_3	1	1	1	1	1	$-a$	$-\bar{a}$	$-\bar{a}$	$-a$
	χ_4	1	1	1	1	-1	1	1	-1	-1
	χ_5	1	1	1	1	1	$-\bar{a}$	$-a$	$-a$	$-\bar{a}$
	χ_6	1	1	1	1	-1	$-a$	$-\bar{a}$	$\bar{a}$	a
	χ_7	6	c	d	e	0	0	0	0	0
	χ_8	6	d	e	c	0	0	0	0	0
	χ_9	6	e	c	d	0	0	0	0	0

$$a = \tfrac{1}{2}(1+\sqrt{3}i)$$
$$c = x + x^8 + x^7 + x^{18} + x^{11} + x^{12}$$
$$d = x^2 + x^{16} + x^{14} + x^{17} + x^3 + x^5$$
$$e = x^4 + x^{13} + x^9 + x^{15} + x^6 + x^{10}$$
$$\text{where } x = e^{\frac{2\pi i}{19}}$$

Table 6.15: Character Table of 19:6

6.5 Sylow 11 and 7-normalizers

The Sylow 11 and 7- normalizers of J_1 are Frobenius groups with structures $11:10$ and $7:6$ respectively. Let $\overline{G} = N:G$ be a Sylow 11-normalizer with $N \cong C_{11}$ and $G \cong C_{10}$.The group G acts on N by conjugation so that each nontrivial element of G fixes only the identity of N (since $\overline{G}$ is a Frobenius group). Thus the orbits of G on N have lengths 1 and 10. This group $\overline{G}$ is the holomorph of G, so its character table follows from section 5.3 and the character table of C_{10}. We give the character table of $\overline{G}$ in Table 6.16. Using the fusion of $\overline{G}$ to J_1 given in Table 6.16, the permutation character is

$$1_{11:10}^{J_1} = 1a + 56ab + 76ab + 77aa + 120abc + 133abc + 209aa$$

Similarly, a Sylow 7-normalizer of J_1 is the holomorph of C_7. Its character table is given in section 5.3. In Table 6.17 we give the character table of $7:6$ with fusions to J_1 and we have

$$1_{7:6}^{J_1} = 1a + 56aabb + 76aaab + 77aabbcc + 120aaabbbccc + 133aaaabbcc + 209aaaaa$$

class	(1)	(11)	(10_1)	(5_1)	(10_2)	(5_2)	(2)	(5_3)	(10_3)	(5_4)	(10_4)
centralizer	110	11	10	10	10	10	10	10	10	10	10
$\rightarrow J_1$	1*a*	11*a*	10*b*	5*a*	10*a*	5*b*	2*a*	5*b*	10*a*	5*a*	10*b*
power map π^2			(5_1)	(5_2)	(5_3)	(5_4)		(5_1)	(5_2)	(5_3)	(5_4)
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	a	b	$-\bar{b}$	$-\bar{a}$	-1	$-a$	$-b$	$\bar{b}$	$\bar{a}$
χ_3	1	1	b	$-\bar{a}$	$-a$	$\bar{b}$	1	b	$-\bar{a}$	$-a$	$\bar{b}$
χ_4	1	1	$-\bar{b}$	$-a$	$\bar{a}$	b	-1	$\bar{b}$	a	$-\bar{a}$	$-b$
χ_5	1	1	$-\bar{a}$	$\bar{b}$	b	$-a$	1	$-\bar{a}$	$\bar{b}$	b	$-a$
χ_6	1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_7	1	1	$-a$	b	$\bar{b}$	$-\bar{a}$	1	$-a$	b	$\bar{b}$	$-\bar{a}$
χ_8	1	1	$-b$	$-\bar{a}$	a	$\bar{b}$	-1	b	$\bar{a}$	$-a$	$-\bar{b}$
χ_9	1	1	$\bar{b}$	$-a$	$-\bar{a}$	b	1	$\bar{b}$	$-a$	$-\bar{a}$	b
χ_{10}	1	1	$\bar{a}$	$\bar{b}$	$-b$	$-a$	-1	$-\bar{a}$	$-\bar{b}$	b	a
χ_{11}	10	-1	0	0	0	0	0	0	0	0	0

$$a = -e^{6\pi i/5},\ b = e^{2\pi i/5}$$

Table 6.16: Character Table of 11 : 10

class		(1)	(7₁)	(2)	(3₁)	(3₂)	(6₁)	(6₂
centralizer		42	7	6	6	6	6	6
	$\to J_1$	1a	7a	2a	3a	3a	6a	6a
power map	π^2						(3₁)	(3₂)
	χ_1	1	1	1	1	1	1	1
	χ_2	1	1	−1	$-\overline{a}$	$-a$	a	$\overline{a}$
	χ_3	1	1	1	$-a$	$-\overline{a}$	$-\overline{a}$	$-a$
	χ_4	1	1	−1	1	1	−1	−1
	χ_5	1	1	1	$-\overline{a}$	$-a$	$-a$	$-\overline{a}$
	χ_6	1	1	−1	$-a$	$-\overline{a}$	$\overline{a}$	a
	χ_7	6	−1	0	0	0	0	0

$$a = \frac{1}{2}(1 + \sqrt{3}i)$$

Table 6.17: Character Table of 7:6

Chapter 7

PROJECTIVE CHARACTERS

In chapter 4, we showed how Fischer matrices could be used to determine the characters of a group $\overline{G}$ with a normal subgroup N such that every irreducible character of N can be extended to its inertia group. Now, in order to generalize these methods to other group extensions, we need to define and discuss projective representations and characters. In the first section we define projective representations and show how they are related to ordinary representations. (In this chapter we refer to the group representations and characters that we defined in chapter 2 as ordinary representations and characters). We then go on to define and give some properties of projective characters in section 7.2. In section 7.3 we relate projective representations and characters to Clifford theory and hence generalize the Fischer matrix methods.

7.1 Projective Representations

Definition 7.1.1 Let G be a group and F a field. A *projective F-representation* of G of degree n is a mapping $P : G \to GL_n(F)$ such that for every $g, h \in G$ there exists a scalar $\alpha(g,h) \in F$ such that $P(g)P(h) = \alpha(g,h)P(gh)$. The function $\alpha : G \times G \to F$ is the associated *factor set* of P. (From the definition it is clear that $\alpha(g,h) \neq 0_F$ for all $g, h \in G$, so $\alpha : G \times G \to F^*$).

Note 1 The projective general linear group is the factor group

$$PGL_n(F) = GL_n(F)/Z(GL_n(F))$$

where $Z(GL_n(F))$ is the centre of $GL_n(F)$ which consists of all nonzero scalar matrices. If P is a projective F-representation of G then the composition of P with the natural homomorphism $GL_n(F) \to PGL_n(F)$ is a homomorphism $G \to PGL_n(F)$. Conversely, if $\pi : G \to PGL_n(F)$ is any homomorphism, a projective representation P of G can be defined by setting $P(g)$ equal to any element of the coset $\pi(g)$ of $Z(GL_n(F))$ in $GL_n(F)$. Hence the projective F-representations of G can be identified with the homomorphisms of G into the projective general linear group.

Before giving further results on projective representations, we need to consider their associated factor sets.

Lemma 7.1.1 *Let α be the associated factor set of a projective representation P of G. Then α satisfies*

$$\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z)$$

for all $x, y, z \in G$.

Proof: By associativity we have

$$P(x)P(y)P(z) = \alpha(x, y)P(xy)P(z) = \alpha(x, y)\alpha(xy, z)P(xyz)$$

and

$$P(x)P(y)P(z) = \alpha(y, z)P(x)P(yz) = \alpha(y, z)\alpha(x, yz)P(xyz).$$

Now the result follows since $P(xyz)$ is invertible. □

Any function $\alpha : G \times G \to F^*$ that satisfies $\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$ is called an F^*-factor set of G. By Lemma 7.1.1, the associated factor set of any projective F-representation of G is an F^*-factor set of G. Conversely, every F^*-factor set is associated with a projective representation (see [15, 11.6]). We will consider projective representations and factor sets over the complex field $\mathbb{C}$ from now on.

Definition 7.1.2 Two factor sets α and α' are said to be *equivalent* if there exists a function $\rho : G \to \mathbb{C}^*$ such that $\alpha'(x, y) = \rho(x)\rho(y)\left(\rho(xy)\right)^{-1}\alpha(x, y)$ for all $x, y \in G$. This is an equivalence relation, and we denote the equivalence class of the factor set α by $[\alpha]$.

For factor sets α and α', let $\alpha\alpha'$ denote the function defined by $(\alpha\alpha')(x, y) = \alpha(x, y)\alpha'(x, y)$ for $x, y \in G$. Then $\alpha\alpha'$ is a factor set, as is α^{-1} defined by $\alpha^{-1}(x, y) = \left(\alpha(x, y)\right)^{-1}$.

Definition 7.1.3 The set of equivalence classes of factor sets forms an abelian group M by defining $[\alpha][\alpha'] = [\alpha\alpha']$. The identity of M is $[1]$ where 1 is the factor set $1(x,y) = 1$ for all $x, y \in G$, and $[\alpha]^{-1} = [\alpha^{-1}]$. The group M is called the *multiplier* of G.

As with ordinary representations, we define equivalence and irreducibility of projective representations.

Definition 7.1.4 Two projective representations P_1 and P_2 of G are *equivalent* if there is a non-singular matrix T such that for all $g \in G$,

$$P_1(g) = c(g)TP_2(g)T^{-1} \text{ for some } c(g) \in \mathbb{C}^*.$$

If $c(g) = 1$ for all $g \in G$ then P_1 and P_2 are *linearly equivalent*. A projective representation P is *irreducible* if it is not linearly equivalent to a projective representation of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Lemma 7.1.2 *If two projective representations are equivalent then they have equivalent factor sets; if they are linearly equivalent they have equal factor sets.*

Proof: Let P_1 and P_2 be equivalent projective representations with factor sets α_1 and α_2 respectively. Suppose T is a non-singular matrix and $c : G \to \mathbb{C}^*$ such that

$$P_1(g) = c(g)TP_2(g)T^{-1} \text{ for all } g \in G.$$

Now for $g, h \in G$,

$$\begin{aligned}
\alpha_1(g,h) &= P_1(g)P_1(h)(P_1(gh))^{-1} \\
&= c(g)TP_2(g)T^{-1}c(h)TP_2(h)T^{-1}(c(gh))^{-1}T(P_2(gh))^{-1}T^{-1} \\
&= c(g)c(h)(c(gh))^{-1}TP_2(g)P_2(h)(P_2(gh))^{-1}T^{-1} \\
&= c(g)c(h)(c(gh))^{-1}\alpha_2(g,h),
\end{aligned}$$

so α_1 and α_2 are equivalent. If P_1 and P_2 are linearly equivalent, then $c(g) = 1$ for all $g \in G$ in the above expressions, so $\alpha_1 = \alpha_2$. □

Now we show how the projective representations of a group G can be determined from the ordinary representations of a so-called representation group of G. We follow Isaacs [15] in developing the following theory.

Definition 7.1.5 A *central extension* of G is a group H together with a homomorphism π of H onto G such that $\ker \pi$ lies in the centre of H.

Lemma 7.1.3 *Let (H, π) be a central extension of G with $A = \ker \pi$. Let X be a set of coset representatives for A in H, and write $X = \{x_g : g \in G\}$, where $\pi(x_g) = g$. Define $\alpha : G \times G \to A$ by $x_g x_h = \alpha(g,h) x_{gh}$. Then α is an A-factor set of G and the equivalence class of α is independent of the choice of X.*

Proof: The fact that α is a factor set follows from associativity in H. If $Y = \{y_g : g \in G\}$ is another set of coset representatives then $y_g = \mu(g) x_g$ for some $\mu(g) \in A$, for each $g \in G$. Now

$$\begin{aligned} y_g y_h = \mu(g)\mu(h) x_g x_h &= \mu(g)\mu(h)\alpha(g,h) x_{gh} \\ &= \mu(g)\mu(h)(\mu(gh))^{-1}\alpha(g,h) y_{gh}, \end{aligned}$$

so the factor set given by Y is equivalent to α, as required. □

Corollary 7.1.4 *Let H be a central extension of G with A,X and α as in the previous lemma. Let T be an ordinary representation of H such that the restriction $T|_A$ is the scalar representation λI for some $\lambda \in Hom(A, \mathbb{C}^*)$, that is*

$$T(a) = \begin{pmatrix} \lambda(a) & & & \\ & \lambda(a) & & \\ & & \ddots & \\ & & & \lambda(a) \end{pmatrix}_{n \times n} \quad \forall a \in A,$$

where $n = \deg T$. Define $P(g) = T(x_g)$ for $g \in G$. Then P is a projective representation of G with factor set $\lambda(\alpha)$, where $\lambda(\alpha)(g,h) = \lambda(\alpha(g,h))$. Furthermore, P is irreducible if and only if T is and the equivalence class of P is independent of the choice of coset representatives X.

Proof: We have

$$P(g)P(h) = T(x_g)T(x_h) = T(x_g x_h) = T(\alpha(g,h)x_{gh}) = \lambda(\alpha(g,h))P(gh),$$

so P is a projective representation with factor set $\lambda(\alpha)$. Now if $y \in H$, then $y = a x_g$ where $g = \pi(y)$ and $a \in A$. Thus

$$T(y) = P(g)\lambda(a) = P(\pi(y))\lambda(y x_{\pi(y)}^{-1}) = P(\pi(y))\mu(y),$$

where $\mu : H \to \mathbb{C}^*$ is defined by $\mu(y) = \lambda(yx_{\pi(y)}^{-1})$. Therefore $T(H)$ and $P(G)$ span the same vector space of matrices over $\mathbb{C}$, so the result about irreducibility follows. Also, if P_1 is the projective representation determined by another choice of coset representatives, then $P_1(\pi(y)) = T(y)\mu_1(y)^{-1} = P(\pi(y))\mu(y)\mu_1(y)^{-1}$, so P_1 and P are equivalent, since for $g \in G$, $g = \pi(y)$ for some $y \in H$. Let $c(g) = \mu(y)\mu_1(y)^{-1}$, then $P_1(g) = c(g)P(g)$. □

Note tbat if T is an ordinary irreducible representation of H then the condition that $T|_A$ be a scalar representation is satisfied by Schur's Lemma (Corollary 2.1.2), since A lies in the centre of H.

Definition 7.1.6 A projective representation of G that can be constructed from an ordinary representation of a central extension H of G as in Corollary 7.1.4 is said to be *lifted* to H. A *representation group* of G is a central extension H of G such tbat every projective representation of G can be lifted to H.

Every group has a representation group by the following result known as Schur's theorem, which we state without proof.

Theorem 7.1.5 *Let G be a finite group of order n. Then G has at least one representation group H of order mn where $m = |M|$ and the kernel of the extension is isomorphic to the multiplier M of G.*

Proof: See, for example, [15, 11.17] □

7.2 Projective Characters

Definition 7.2.1 If P is a projective representation of G, then the *projective character* ξ of P is defined by

$$\xi(g) = \text{trace}(P(g)) \quad \text{for all } g \in G.$$

We say ξ is *irreducible* if P is, and ξ has factor set α, where α is the factor set of P.

The projective characters of G can be determined from the the ordinary characters of a representation group (H, π) of G. Let $\pi : H \to G$ define the extension H of G, and let $\{x_g : g \in G\}$ be a set of coset representatives for $\ker(\pi)$ in H. If P is a projective representation of G with projective character ξ then there is an ordinary representation T of H such that $P(g) = T(x_g)$ for $g \in G$. Let χ be the character of H afforded by T, then $\xi(g) = \chi(x_g)$ for all $g \in G$.

Definition 7.2.2 Given a factor set α of G, an element $g \in G$ is said to be *α-regular* if $\alpha(g,x) = \alpha(x,g)$ for all $x \in C_G(g)$.

If g is α-regular, so is every conjugate of g, and an element g is α-regular if and only if g is β-regular for every factor set β equivalent to α. So we can define a conjugacy class of G to be α-regular if each of its elements is α-regular.

Now we have analogues of results for ordinary characters.

Theorem 7.2.1
1. *The number of irreducible projective characters of G with factor set α is equal to the number of α-regular conjugacy classes of G.*

2. *Let $\xi_1, \ldots, \xi_t$ be the projective characters of G with factor set α, and let $C_1, \ldots, C_t$ be the α-regular conjugacy classes of G with g_i a representative of C_i for $i = 1, \ldots, t$. Then*

$$\sum_{i=1}^{t} \xi_i(g_j)\overline{\xi_i(g_k)} = \delta_{jk}|C_G(g_j)| \text{ for } j,k \in \{1,\ldots,t\}$$

3. *An element g of G is α-regular if and only if there is an irreducible projective character ξ of G with factor set α such that $\xi(g) \neq 0$.*

Proof: See [11] □

We have shown that the projective characters of G can be determined from the ordinary characters of a representation group H of G. Haggarty and Humphreys [11] show that it is possible to determine the projective characters of G with a given factor set without the full representation group of G: Suppose α is a factor set of G, with $[\alpha]$ having order e in the multiplier M. Let ω be an e^{th} root of unity and let β be a representative of $[\alpha]$ whose values are powers of ω. For $g, h \in G$ define $a(g,h)$ by $\beta(g,h) = \omega^{a(g,h)}$. Let L be the group generated by an element x of order e and elements x_g ($g \in G$) with multiplication $x^i x_g x^j x_h = x^{i+j} x^{a(g,h)} x_{gh}$. Then L is a quotient of the representation group H and any projective representation of G with factor set α can be lifted to an ordinary representation of L. Thus the projective characters of G with factor set α can be determined from the ordinary character table of L.

7.3 Projective Representations and Clifford Theory

We will now show how projective representations can be used to generalize our results of section 3.3 and hence the Fischer matrix methods of chapter 4.

Definition 7.3.1 Let $N \trianglelefteq \overline{G}$. If Y is an irreducible (ordinary) representation of N then for $g \in \overline{G}$, Y^g defined by $Y^g(n) = Y(gng^{-1})$, $n \in N$, is a representation of N, called a *conjugate* of Y. The *inertial group* of Y, $T(Y)$, is the set of all $g \in G$ such that Y is equivalent to Y^g. Note that $T(Y) = I_{\overline{G}}(\theta)$ where θ is the character of N afforded by Y.

Now let Y be an irreducible representation of N, where $N \trianglelefteq \overline{G}$ and let $\overline{H} = T(Y)$, so Y is equivalent to all its conjugates in $\overline{H}$. The following theorem shows that Y can always be extended to a projective representation of $\overline{H}$ and gives a necessary and sufficient condition for Y to be extendable to an ordinary representation of $\overline{H}$. This theorem and the next one are originally due to Clifford [2]; we state them without proof and then restate the results in the form in which we will use them, in terms of projective and ordinary characters.

Theorem 7.3.1 *Let $N, \overline{G}, Y$ and $\overline{H}$ be as above. Then Y extends to a projective representation X of $\overline{H}$ with factor set $\overline{\alpha}$ such that $\overline{\alpha}$ is constant on cosets of N in $\overline{H}$. Therefore $\overline{\alpha}$ can be regarded as a factor set α of $H = \overline{H}/N$ defined by $\alpha(Nh, Nk) = \overline{\alpha}(h, k)$. Also, α satisfies $\alpha^{d|N|} \sim 1$ where d is the degree of Y. Furthermore, Y extends to a linear representation of $\overline{H}$ if and only if $\alpha \sim 1$.*

Proof: See [25, 3.5.7]. □

Theorem 7.3.2 *Let $N \trianglelefteq \overline{G}$, Y an irreducible representation of N with $\overline{H} = T(Y)$ and $H = \overline{H}/N$. Extend Y to a projective representation X of $\overline{H}$ as in Theorem 7. 3. 1with factor set $\overline{\alpha}$. Then*

1. *If W is an irreducible representation of H that has Y as one of its irreducible constituents in its restriction to N then there exists an irreducible projective representation Z of H with factor set α^{-1} such that W is equivalent to the representation $\overline{Z} \otimes X$ of $\overline{H}$, where α is the factor set of H obtained from $\overline{\alpha}$, and $\overline{Z}$ is the representation of $\overline{H}$ obtained naturally from Z.*

2. *If, conversely, Z is any irreducible projective representation of H with factor set α^{-1}, then $\overline{Z} \otimes X$ is an irreducible representation of $\overline{H}$ which is equivalent to some representation that contains Y in its restriction to N.*

Proof: See [25, 3.5.8]. □

Corollary 7.3.3 *Let $N \trianglelefteq \overline{G}$, $\theta \in Irr(N)$ and $\overline{H} = I_{\overline{G}}(\theta)$. Then θ extends to a projective character ξ of $\overline{H}$ with factor set $\overline{\alpha}$ that is constant on cosets of N, so $\overline{\alpha}$ can be regarded as a factor set α of $H = \overline{H}/N$. Also, α satisfies $\alpha^{d|N|} \sim 1$ where d is the degree of θ. Now as η runs over all irreducible projective characters of H with factor set α^{-1}, $\xi\overline{\eta}$ runs over all irreducible characters of $\overline{H}$ that contain θ in their restriction to N where $\overline{\eta}$ is the projective character of $\overline{H}$ obtained naturally from η.*

Proof: This follows from Theorems 7.3.1 and 7.3.2 by considering the character of each representation (projective or ordinary) where the product of characters corresponds to tensor product of representations. □

Note that in the above corollary, if θ extends to an ordinary character of $\overline{H}$ then by the last statement of Theorem 7.3.1, $\alpha \sim 1$ so the relevant projective characters of H have trivial factor set. These are the ordinary irreducible characters of H so this special case is the result given in Theorem 3.3.3.

Now by Theorem 3.3.2 and Corollary 7.3.3, the characters of $\overline{G} = N.G$ can be obtained as follows: Let $\theta_1, \ldots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $\mathrm{Irr}(N)$. Let $\overline{H}_i = I_{\overline{G}}(\theta_i)$ and let ξ_i be a projective character of $\overline{H}_i$ with factor set $\overline{\alpha}_i$ such that $\xi|_N = \theta_i$. Then

$$\mathrm{Irr}(\overline{G}) = \bigcup_{i=1}^{t} \{(\xi_i\overline{\eta})^{\overline{G}} : \eta \text{ is an irreducible projective character of } H_i \text{ with factor set } \alpha_i^{-1}\},$$

where α_i is obtained from $\overline{\alpha}_i$ and $\overline{\eta}$ from η as in Corollary 7.3.3.

7.4 Fischer Matrices

With the notation of the previous section, consider conjugacy class $[g]$ of G. Let $X(g) = \{x_1, \ldots, x_{c(g)}\}$ be representatives of $\overline{G}$-conjugacy-classes of elements of the coset $N\overline{g}$. Take $x_1 = \overline{g}$, a lifting of g. Let $R(g)$ be a set of pairs (i, y) where $i \in \{1, \ldots, t\}$ such that H_i contains an element of $[g]$, and y ranges over representatives of the α_i^{-1}-regular classes of H_i that fuse to $[g]$. Corresponding to this $y \in H_i$, let $\{y_{l_k}\}$ be representatives of conjugacy classes of $\overline{H}_i$ that contain liftings of y. Let $y_{l_1} = \overline{y}$.

Now, as in 4.1,we have

$$(\xi_i\overline{\eta})^{\overline{G}}(x_j) = \sum_{y:(i,y)\in R(g)} \left(\sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|} \xi_i(y_{l_k}) \right) \eta(y)$$

(summing over those k for which y_{l_k} is conjugate to x_j in $\overline{G}$)

We let

$$a^j_{(i,y)} = \sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|} \xi_i(y_{l_k}),$$

so

$$(\xi_i\overline{\eta})^{\overline{G}}(x_j) = \sum_{y:(i,y)\in R(g)} a^j_{(i,y)}\eta(y).$$

Again we denote tbe matrix $(a^j_{(i,y)})$ by $M(g)$. This is the Fischer matrix for $\overline{G}$ at g, and we obtain the characters of $\overline{G}$ by multiplying the relevant columns of the projective characters of H_i with factor set α_i^{-1} by rows of $M(g)$.

Lemma 7.4.1 *The Fischer matrix $M(g)$ as defined above satisfies*

1. $a^j_{(1,g)} = 1$ *for all* $j \in \{1, \ldots, c(g)\}$.
2. *(column orthogonality)* $\sum_{(i,y)\in R(g)} a^j_{(i,y)}\overline{a^{j'}_{(i,y)}}|C_{H_i}(y)| = \delta_{jj'}|C_{\overline{G}}(x_j)|$

Proof:

1. Follows from the definition.
2. As in Proposition 4.2.3, using the projective character orthogonality (Theorem 7.2.1(2)) for H_i.

□

Chapter 8

THE GROUP $3 \cdot PGL_2(9)$, A MAXIMAL SUBGROUP OF J_2

The Janko group J_2 is a sporadic simple group of order 604800 discovered by Hall and Wales [12]. Its character table is given in ATLAS [3], we give this table in the Appendix. J_2 has nine conjugacy classes of maximal subgroups, determined by Finkelstein and Rudvalis [6]. In this chapter we will determine the conjugacy classes and character table of one of these maximal subgroups, the normalizer in J_2 of the subgroup generated by an element of class $(3a)$. This group $\overline{G}$ is a nonsplit extension of $N \cong C_3$ by $G \cong PGL_2(9)$. The group G is isomorphic to $A_6.2_2$ (see ATLAS). For this group $\overline{G}$ it is not the case that every irreducible character of N can be extended to its inertia group so we cannot use the same results on Fischer matrices that we used for our other examples; in this case we will use the Fischer matrices that discussed in chapter 7.

8.1 Conjugacy Classes of $\overline{G}$

We will use the methods of section 3.2 to determine the conjugacy classes of $\overline{G}$. For $g \in G$, we denote by $\overline{g}$ a lifting of g in $\overline{G}$, so $\lambda(\overline{g}) = g$ where $\lambda : \overline{G} \to G$ is the natural homomorphism. By Lemma 3.1.1, G acts on N such that $n^g = \overline{g}n\overline{g}^{-1}$. We consider a coset $N\overline{g}$ for each class representative g of G, and the conjugacy classes are determined by first acting N, then acting $\{\overline{h} : h \in C_G(g)\}$ on the orbits of N. If N has k orbits on $N\overline{g}$ and f of these fuse to give a class of $\overline{G}$ with element x then $|C_{\overline{G}}(x)| = \frac{k \times |C_G(g)|}{f}$.

In Table 8.1 we give the character table of G. Referring to the character table of

class $[g]$		(1)	(2_1)	(3_1)	(4_1)	(5_1)	(5_2)	(2_2)	(8_1)	(8_2)	(10_1)	(10_2)
$\|[g]\|$		1	45	80	90	72	72	36	90	90	72	72
$\|C_G(g)\|$		720	16	9	8	10	10	20	8	8	10	10
power	π^2			(3_1)	(2_1)	(5_2)	(5_1)		(4_1)	(4_1)	(5_2)	(5_1)
maps	π^3					(5_2)	(5_1)		(8_2)	(8_1)	(10_2)	(10_1)
	π^5										(2_2)	(2_1)
	χ_1	1	1	1	1	1	1	1	1	1	1	1
	χ_2	1	1	1	1	1	1	-1	-1	-1	-1	-1
	χ_3	10	2	1	-2	0	0	0	0	0	0	0
	χ_4	8	0	-1	0	$-a$	$-b$	2	0	0	a	b
	χ_5	8	0	-1	0	$-a$	$-b$	-2	0	0	$-a$	$-b$
	χ_6	8	0	-1	0	$-b$	$-a$	2	0	0	b	a
	χ_7	8	0	-1	0	$-b$	$-a$	-2	0	0	$-b$	$-a$
	χ_8	9	1	0	1	-1	-1	-1	1	1	-1	-1
	χ_9	9	1	0	1	-1	-1	1	-1	-1	1	1
	χ_{10}	10	-2	1	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	0	0
	χ_{11}	10	-2	1	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	0	0

$$a = -\tfrac{1}{2} + \tfrac{\sqrt{5}}{2},\ b = -\tfrac{1}{2} - \tfrac{\sqrt{5}}{2}$$

Table 8.1: Character Table of $PGL_2(9)$

J_2 in the Appendix, ATLAS gives the permutation character

$$\phi = 1_{\overline{G}}^{J_2} = 1a + 63a + 90a + 126a.$$

In Table 8.2 we give the values of ϕ on the classes of J_2.

Let $N = \{e, n, n^{-1}\}$. Since n and n^{-1} are conjugate in J_2, $n^x = n^{-1}$ for some $x \in J_2$. Then $N^x = N$, so x is an element of the normalizer in J_2 of N which is $\overline{G}$. Therefore n and n^{-1} are conjugate in $\overline{G}$, so $n^g = n^{-1}$ for some $g \in G$. Therefore G fixes e and acts transitively on the two points n and n^{-1}. The permutation character of this transitive action is $\chi_1 + \chi_2$, so for each $g \in G$, the number k of points of N fixed by g is $1 + (\chi_1 + \chi_2)(g)$. These values, using Table 8.1 for values of χ_1 and χ_2, are given below.

class of J_2	centralizer	$\phi = 1_G^{J_2}$
$(1a)$	604800	280
$(2a)$	1920	40
$(2b)$	240	12
$(3a)$	1080	1
$(3b)$	36	4
$(4a)$	96	4
$(5a)$	300	10
$(5b)$	300	10
$(6a)$	24	1
$(8a)$	8	2
$(10a)$	20	2
$(10b)$	20	2
$(12a)$	12	1
$(15a)$	15	1
$(15b)$	15	1

Table 8.2:

class $[g]$of G	(1_1)	(2_1)	(3_1)	(4_1)	(5_1)	(5_2)	(2_2)	(8_1)	(8_2)	(10_1)	(10_2)
k	3	3	3	3	3	3	1	1	1	1	1

Now we consider the cosets $N\overline{g}$.

- $g = e$: Here $\overline{g} = e$ and $N\overline{g} = N$; G has orbits of lengths 1 and 2 on N, so we have classes (1) and (3_1) of $\overline{G}$. Class (3_1) contains the element n, and $|C_{\overline{G}}(n)| = \frac{3\times 720}{2} = 1080$. Now (3_1) fuses to $(3a)$ in J_2 and $\frac{|C_{J_2}(n)|}{|C_{\overline{G}}(n)|} = \frac{1080}{1080} = 1 = \phi(n)$, so no other classes of $\overline{G}$ fuse to $(3a)$.

- $g \in (2_1)$: We have $|C_G(g)| = 16$ and $k = 3$. Therefore g fixes all elements of N so $\overline{g}$ and n commute. Now $\lambda(\overline{g}^2) = (\lambda(\overline{g}))^2 = g^2 = e$, so $\overline{g}^2 \in N = \{e, n, n^{-1}\}$. If $\overline{g}^2 = e$, then $\overline{g} = e$ or $\overline{g}$ has order 2. But $\overline{g} = e$ is not possible, for then $\lambda(\overline{g}) = g$ implies $g = e$, a contradiction. If $\overline{g}^2 = n$ or n^{-1} then $(\overline{g})^6 = e$ so 6 divides the order of $\overline{g}$. Therefore $o(\overline{g}) \in \{1, 2, 3, 6\}$. But $o(\overline{g}) \neq 1$ by the above. If $o(\overline{g}) = 3$ then $(\overline{g})^2 = n$ or n^{-1} so $\overline{g} = n^{-1}$ or n. Then $\overline{g} \in N$ and $\lambda(\overline{g}) = e$, a contradiction. Therefore $\overline{g}$ has order 2 or 6. Suppose $\overline{g}$ has order 6. Then $n\overline{g}$ also has order 6, so $N\overline{g}$ has three elements of order 6. Since $\overline{g}^2 \in N$, $\overline{g}^2 \in (3a)$ in J_2, so $\overline{g} \in (6a)$ and $|C_{J_2}(\overline{g})| = 24$. Therefore $|C_{\overline{G}}(\overline{g})|$ divides 24, where $|C_{\overline{G}}(\overline{g})| = \frac{3\times 16}{f}$ for $f \in \{1, 2, 3\}$. The only possibility is $f = 2$. Then there is another class with $f = 1$, and hence centralizer 48. But this is not possible (it cannot fuse to J_2). Therefore $o(\overline{g}) \neq 6$, so $o(\overline{g}) = 2$ and $n\overline{g}$ and $n^{-1}\overline{g}$ have order 6. Thus we have class (2_1) of $\overline{G}$ with centralizer 24.

- $g \in (3_1)$: Then $|C_G(g)| = 9$ and again $k = 3$, so $\overline{g}$ commutes with n. Since g has order 3, $\overline{g}$ has order 3 or 9. But J_2 has no elements of order 9, so $\overline{g}$ must have order 3, and hence $n\overline{g}$ and $n^{-1}\overline{g}$ also have order 3. These must all be in class $(3b)$ of J_2 by the comments made in the first case above and Table 8.2; with $|C_{J_2}(\overline{g})| = 36$. Now $|C_{\overline{G}}(\overline{g})| = \frac{3\times 9}{f}$ for $f \in \{1, 2, 3\}$, such that $|C_{\overline{G}}(\overline{g})|$ divides 36. The only possibility is $f = 3$, so we get class (3_2) of $\overline{G}$ with $|C_{\overline{G}}(\overline{g})| = 9$.

- Classes (5_1) and (5_2): Similarly, by considering fusions to J_2, we get classes (5_1) and (15_1) of $\overline{G}$ from class (5_1) of G, and classes (5_2) and (15_2) from class (5_2) of G.

- Classes $(2_2), (8_1), (8_2), (10_1), (10_2)$: These classes all have $k = 1$, so each corresponds to one class of $\overline{G}$ with $f = 1$.

class of G	class $[x]$ of $\overline{G}$	$\lvert C_{\overline{G}}(x)\rvert$	$\to J_2$
(1_1)	(1)	2160	$(1a)$
	(3_1)	1080	$(3a)$
(2_1)	(2_1)	48	$(2a)$
	(6_1)	24	$(6a)$
(3_1)	(3_2)	9	$(3b)$
(4_1)	(4_1)	24	$(4a)$
	(12_1)	12	$(12a)$
(5_1)	(5_1)	30	$(5a)$
	(15_1)	15	$(15b)$
(5_2)	(5_2)	30	$(5b)$
	(15_2)	15	$(15a)$
(2_2)	(2_2)	20	$(2b)$
(8_1)	(8_1)	8	$(8a)$
(8_2)	(8_2)	8	$(8a)$
(10_1)	(10_1)	10	$(10a)$
(10_2)	(10_2)	10	$(10b)$

Table 8.3: Conjugacy Classes of $\overline{G} = 3 \cdot PGL_2(9)$

- $g \in (4_1)$: Here $|C_G(g)| = 8$ and $k = 3$ so $\overline{g}$ commutes with n. By the value of $\phi = 1_{\overline{G}}^{J_2}$ on class $(4a)$ of J_2, we see that $\overline{G}$ must have a class of elements of order 4 that fuses to $(4a)$. Therefore $\overline{g}$ has order 4 and $n\overline{g}, n^{-1}\overline{g}$ have order 12. So we must have class (4_1) of $\overline{G}$ with $|C_{\overline{G}}(\overline{g})| = \frac{3\times 8}{1} = 24$. Now $|C_{\overline{G}}(n\overline{g})| = \frac{3\times 8}{f}$ where $f \in \{1, 2\}$. But $|C_{\overline{G}}(n\overline{g})| = 12$, so we must have $f = 2$.

We list all the conjugacy classes of $\overline{G}$ in Table 8.3.

8.2 Fischer Matrices of $\overline{G}$

$\overline{G}$ has two orbits on N, hence two orbits on $\mathrm{Irr}(N)$, so they must have lengths 1 and 2. So the inertia groups are $\overline{H}_1 = \overline{G}$ with $H_1 = G$ and $\overline{H}_2 = N.H_2$ where H_2 is a subgroup of G of index 2. Since $[G : H_2] = 2$, $H_2 \trianglelefteq G$ so H_2 is a union of conjugacy classes

class	(1_1)	(2_1)	(3_1)	(3_2)	(4_1)	(5_1)	(5_2)
centralizer	360	8	9	9	4	5	5
η_1	3	-1	0	0	1	a	b
η_2	3	-1	0	0	1	b	a
η_3	6	2	0	0	0	1	1
η_4	9	1	0	0	1	-1	-1
η_5	15	-1	0	0	-1	0	0

$$a = \frac{1}{2} - \frac{\sqrt{5}}{2}, \; b = \frac{1}{2} + \frac{\sqrt{5}}{2}$$

Table 8.4: Projective Characters of $H_2 = A_6$ with factor set α^{-1}

of G. Since $|H_2| = 360$, H_2 contains elements of order 3 and 5. From Table 8.1, H_2 must contain the (3_1)-class of G and it contains both (5_1) and (5_2) since (5_2) contains squares of elements of (5_1). Now there are a further 135 elements of H_2, some of which must have order 2. If H_2 contains the class (2_2) of G, then there another $135 - 36 = 99$ elements which is impossible. Therefore H_2 contains the (2_1) class of G and this leaves another 90 elements, so $(4_1) \subset H_2$. Therefore $H_2 = (1) \cup (2_1) \cup (3_1) \cup (4_1) \cup (5_1) \cup (5_2)$, so $H_2 \cong A_6$. Then $\overline{H}_2$ is a nonsplit extension of C_3 by A_6, so is isomorphic to $3.A_6$, the threefold proper covering of A_6 [6].

If θ_1 and θ_2 are representatives of the orbits of $\overline{G}$ on $\mathrm{Irr}(N)$, then $\theta_1 = 1_N$ and θ_2 is a nontrivial character of N, necessarily of degree 1. Now θ_1 extends to the trivial character of $\overline{G}$, but θ_2 does not extend to an irreducible character of $\overline{H}_2 = 3.A_6$ since $3.A_6$ has no nontrivial characters of degree 1. (Considering the character table of $3.A_6$ in ATLAS). Therefore we need to consider the results of chapter 7, and by Corollary 7.3.3 θ_2 extends to a projective character of $\overline{H}_2$ with factor set $\overline{\alpha}$. Then we get the corresponding factor set α of H_2 such that $\alpha^3 \sim 1$. Since θ_2 does not extend to an ordinary character of $\overline{H}_2$ it is not the case that $\alpha \sim 1$, so $[\alpha]$ has order 3 and therefore $[\alpha^{-1}]$ has order 3. So the projective characters of H_2 with factor set α^{-1} can be obtained from the ordinary characters of $3.H_2$. Thus, from the ATLAS table of $3.A_6$ we have the projective characters of H_2 with factor set α^{-1}, which are given in Table 8.4. This table indicates that classes (3_1) and (3_2) of H_2 are not α^{-1}-regular by Theorem 7.2.1(3) so H_2 has five α^{-1}-regular classes and five characters with factor set α^{-1}, as required by Theorem 7.2.1(1).

Next we construct the Fischer matrices $M(g)$ for class representatives g of G which are given in Table 8.5. The entries were obtained from the relations in Lemma 7.4.1 and the fact that all entries must be real (since the characters of $\overline{G}$ are all real - every element of $\overline{G}$ is conjugate to its inverse). For classes $[g] = (1_1), (2_1), (4_1), (5_1), (5_2)$ of G there is one α_i^{-1}-regular class that fuses to $[g]$ so these classes have 2×2 Fischer matrices. For the remaining classes there is no fusion from H_2 so these matrices are trivial.

We construct the character table of $\overline{G}$ from the Fischer matrices, the character table of G (Table 8.1) and the projective character table of H_2 (Table 8.4). We have the conjugacy classes of $\overline{G}$ from Table 8.3.

Corresponding to the identity class of G (classes (1) and (3_1) of $\overline{G}$), the characters in the $\overline{G}$ are obtained by multiplying the first column of the character table of G by the first row of $M(g)$, ie

$$\begin{bmatrix} 1 \\ 1 \\ 10 \\ 8 \\ 8 \\ 8 \\ 8 \\ 9 \\ 9 \\ 10 \\ 10 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 10 & 10 \\ 8 & 8 \\ 8 & 8 \\ 8 & 8 \\ 8 & 8 \\ 9 & 9 \\ 9 & 9 \\ 10 & 10 \\ 10 & 10 \end{bmatrix}$$

The characters in the $\overline{H}_2$-block are obtained by multiplying the first column of Table 8.4 by the second row of $M(g)$, ie

$$\begin{bmatrix} 3 \\ 3 \\ 6 \\ 9 \\ 15 \end{bmatrix} \begin{bmatrix} 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 6 & -3 \\ 12 & -6 \\ 18 & -9 \\ 30 & -15 \end{bmatrix}$$

Similarly for all other classes of G, and we get the character table of $\overline{G}$ (Table 8.2).

$[g]$	$M(g)$
(1_1)	$\begin{array}{c} \\ 720 \\ 360 \end{array}\begin{array}{c} \begin{array}{cc} 2160 & 1080 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \end{array}$
(2_1)	$\begin{array}{c} \\ 16 \\ 8 \end{array}\begin{array}{c} \begin{array}{cc} 48 & 24 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \end{array}$
(3_1)	1
(4_1)	$\begin{array}{c} \\ 8 \\ 4 \end{array}\begin{array}{c} \begin{array}{cc} 24 & 12 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \end{array}$
(5_1)	$\begin{array}{c} \\ 10 \\ 5 \end{array}\begin{array}{c} \begin{array}{cc} 30 & 15 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \end{array}$
(5_2)	$\begin{array}{c} \\ 10 \\ 5 \end{array}\begin{array}{c} \begin{array}{cc} 30 & 15 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \end{array}$
(2_2)	1
(8_1)	1
(8_2)	1
(10_1)	1
(10_2)	1

Table 8.5: Fischer Matrices of $\overline{G}$

class	(1)	(3_1)	(2_1)	(6_1)	(3_2)	(4_1)	(12_1)	(5_1)	(15_1)	(5_2)	(15_2)
centralizer	2160	1080	48	24	9	24	12	30	15	30	15
$\rightarrow J_2$	$1a$	$3a$	$2a$	$6a$	$3b$	$4a$	$12a$	$5a$	$15b$	$5b$	$15a$
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	1	1
χ_3	10	10	2	2	1	-2	-2	0	0	0	0
χ_4	8	8	0	0	-1	0	0	a	a	b	b
χ_5	8	8	0	0	-1	0	0	a	a	b	b
χ_6	8	8	0	0	-1	0	0	b	b	a	a
χ_7	8	8	0	0	-1	0	0	b	b	a	a
χ_8	9	9	1	1	0	1	1	-1	-1	-1	-1
χ_9	9	9	1	1	0	1	1	-1	-1	-1	-1
χ_{10}	10	10	-2	-2	1	0	0	0	0	0	0
χ_{11}	10	10	-2	-2	1	0	0	0	0	0	0
χ_{12}	6	-3	-2	1	0	2	-1	$2a$	$-a$	$2b$	$-b$
χ_{13}	6	-3	-2	1	0	2	-1	$2b$	$-b$	$2a$	$-a$
χ_{14}	12	-6	4	-2	0	0	0	2	-1	2	-1
χ_{15}	18	-9	2	-1	0	2	-1	-2	1	-2	1
χ_{16}	30	-15	-2	1	0	-2	1	0	0	0	0

$a = \frac{1}{2} - \frac{\sqrt{5}}{2}$, $b = \frac{1}{2} + \frac{\sqrt{5}}{2}$

Table 8.6: Character Table of $\overline{G} = 3 \cdot PGL_2(9)$
(continued on next page)

class	(2_3)	(8_1)	(8_2)	(10_1)	(10_2)
centralizer	20	8	8	10	10
$\rightarrow J_2$	$2b$	$8a$	$8a$	$10a$	$10b$
χ_1	1	1	1	1	1
χ_2	-1	-1	-1	-1	-1
χ_3	0	0	0	0	0
χ_4	2	0	0	$-a$	$-b$
χ_5	-2	0	0	a	b
χ_6	2	0	0	$-b$	$-a$
χ_7	-2	0	0	b	a
χ_8	-1	1	1	-1	-1
χ_9	1	-1	-1	1	1
χ_{10}	0	$\sqrt{2}$	$-\sqrt{2}$	0	0
χ_{11}	0	$-\sqrt{2}$	$\sqrt{2}$	0	0
χ_{12}	0	0	0	0	0
χ_{13}	0	0	0	0	0
χ_{14}	0	0	0	0	0
χ_{15}	0	0	0	0	0
χ_{16}	0	0	0	0	0

$a = \frac{1}{2} - \frac{\sqrt{5}}{2}$, $b = \frac{1}{2} + \frac{\sqrt{5}}{2}$

Character Table of $\overline{G} = 3 \cdot PGL_2(9)$
(cont.)

APPENDIX

class	1a	2a	3a	5a	5b	6a	7a	10a	10b	11a	15a	15b	19a	19b	19c
centralizer	175560	120	30	30	30	6	7	10	10	11	15	15	19	19	19
power maps π^2				5b	5a			5b	5b		15b	15a	19b	19c	19a
π^3											5b	5a			
π^5								2a	2a		3a	3a			
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	56	0	2	$-2a$	$-2b$	0	0	0	0	1	a	b	−1	−1	−1
χ_3	56	0	2	$-2b$	$-2a$	0	0	0	0	1	b	a	−1	−1	−1
χ_4	76	4	1	1	1	1	−1	−1	−1	−1	1	1	0	0	0
χ_5	76	−4	1	1	1	−1	−1	1	1	−1	1	1	0	0	0
χ_6	77	5	−1	2	2	−1	0	0	0	0	−1	−1	1	1	1
χ_7	77	−3	2	a	b	0	0	a	b	0	a	b	1	1	1
χ_8	77	−3	2	b	a	0	0	b	a	0	b	a	1	1	1
χ_9	120	0	0	0	0	0	1	0	0	−1	0	0	c	d	e
χ_{10}	120	0	0	0	0	0	1	0	0	−1	0	0	e	c	d
χ_{11}	120	0	0	0	0	0	1	0	0	−1	0	0	d	e	c
χ_{12}	133	5	1	−2	−2	−1	0	0	0	1	1	1	0	0	0
χ_{13}	133	−3	−2	$-a$	$-b$	0	0	a	b	1	$-a$	$-b$	0	0	0
χ_{14}	133	−3	−2	$-b$	$-a$	0	0	b	a	1	$-b$	$-a$	0	0	0
χ_{15}	209	1	−1	−1	−1	1	−1	1	1	0	−1	−1	0	0	0

$$a = \tfrac{1}{2}(-1+\sqrt{5}),\quad b = \tfrac{1}{2}(-1-\sqrt{5}),\quad c = z + z^7 + z^8 + z^{11} + z^{12} + z^{18},$$
$$d = z^2 + z^3 + z^5 + z^{14} + z^{16} + z^{17},$$
$$e = z^4 + z^6 + z^9 + z^{10} + z^{13} + z^{15}$$
$$(z = e^{2\pi i/19})$$

Character Table of J_1

class	$1a$	$2a$	$2b$	$3a$	$3b$	$4a$	$5a$	$5b$	$5c$	$5d$
centralizer	604800	1920	240	1080	36	96	300	300	50	50
power maps π^2				$3a$	$3b$	$2a$	$5b$	$5a$	$5d$	$5c$
π^3							$5b$	$5a$	$5d$	$5c$
π^5										
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	14	-2	2	5	-1	2	$-3a$	$-3b$	$a+2$	$b+2$
χ_3	14	-2	2	5	-1	2	$-3b$	$-3a$	$b+2$	$a+2$
χ_4	21	5	-3	3	0	1	$a+4$	$b+4$	$-2a$	$-2b$
χ_5	21	5	-3	3	0	1	$b+4$	$a+4$	$-2b$	$-2a$
χ_6	36	4	0	9	0	4	-4	-4	1	1
χ_7	63	15	-1	0	3	3	3	3	-2	-2
χ_8	70	-10	-2	7	1	2	$-5a$	$-5b$	0	0
χ_9	70	-10	-2	7	1	2	$-5b$	$-5a$	0	0
χ_{10}	90	10	6	9	0	-2	1	1	0	0
χ_{11}	126	14	6	-9	0	2	1	1	1	1
χ_{12}	160	0	4	16	1	0	-5	-5	0	0
χ_{13}	175	15	-5	-5	1	-1	0	0	0	0
χ_{14}	189	-3	-3	0	0	-3	$-3a$	$-3b$	$a+2$	$b+2$
χ_{15}	189	-3	-3	0	0	-3	$-3b$	$-3a$	$b+2$	$a+2$
χ_{16}	224	0	-4	8	-1	0	c	d	$2a$	$2b$
χ_{17}	224	0	-4	8	-1	0	d	c	$2b$	$2a$
χ_{18}	225	-15	5	0	3	-3	0	0	0	0
χ_{19}	288	0	4	0	-3	0	3	3	-2	-2
χ_{20}	300	-20	0	-15	0	4	0	0	0	0
χ_{21}	336	16	0	-6	0	0	-4	-4	1	1

$a = \frac{1}{2}(-1+\sqrt{5}),\ b = \frac{1}{2}(-1-\sqrt{5}),\ c = 2\sqrt{5}-1,\ d = -2\sqrt{5}-1$

Character Table of J_2

(continued on next page)

class	6a	6b	7a	8a	10a	10b	10c	10d	12a	15a	15b
centralizer	24	12	7	8	20	20	10	10	12	15	15
power maps π^2	3a	3b	7a	4a	5b	5a	5d	5c	6a	15b	15a
π^3	2a	2b			10b	10a	10d	10c	4a	5b	5a
π^5					2b	2b	2a	2a		3a	3a
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	−1	0	0	a	b	$-a$	$-b$	−1	0	0
χ_3	1	−1	0	0	b	a	$-b$	$-a$	−1	0	0
χ_4	−1	0	0	−1	a	b	0	0	1	$-b$	$-a$
χ_5	−1	0	0	−1	b	a	0	0	1	$-a$	$-b$
χ_6	1	0	1	0	0	0	−1	−1	1	−1	−1
χ_7	0	−1	0	1	−1	−1	0	0	0	0	0
χ_8	−1	1	0	0	$-a$	$-b$	0	0	−1	a	b
χ_9	−1	1	0	0	$-b$	$-a$	0	0	−1	b	a
χ_{10}	1	0	−1	0	1	1	0	0	1	−1	−1
χ_{11}	−1	0	0	0	1	1	−1	−1	−1	1	1
χ_{12}	0	1	−1	0	−1	−1	0	0	0	1	1
χ_{13}	3	1	0	−1	0	0	0	0	−1	0	0
χ_{14}	0	0	0	1	a	b	a	b	0	0	0
χ_{15}	0	0	0	1	b	a	b	a	0	0	0
χ_{16}	0	−1	0	0	1	1	0	0	0	$-b$	$-a$
χ_{17}	0	−1	0	0	1	1	0	0	0	$-a$	$-b$
χ_{18}	0	−1	1	−1	0	0	0	0	0	0	0
χ_{19}	0	1	1	0	−1	−1	0	0	0	0	0
χ_{20}	1	0	−1	0	0	0	0	0	1	0	0
χ_{21}	−2	0	0	0	0	0	1	1	0	−1	−1

$a = \frac{1}{2}(-1+\sqrt{5}),\ b = \frac{1}{2}(-1-\sqrt{5}),\ c = 2\sqrt{5}-1,\ d = -2\sqrt{5}-1$

Character Table of J_2
(cont.)

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