Energy Of Graphs: The Eigen-Complete Difference Ratio With Its Asymptotic, Domination And Area Aspects

By

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Completed in fulfilment of the academic requirements for the degree of Masters of science in Mathematics

in the

School of Mathematics, Statistics and Computer Science, University of kwaZulu-Natal Durban/Pietermaritzburg

December, 2015

Abstract

This dissertation brings together two important concepts in graph theory the energy of a graph and the complete graph. The energy of a graph is the sum of the absolute values its eigenvalues, and originated from the determination of the sum of π -electron energy in a molecule represented by a molecular graph- i.e. a graph where the vertices represent atoms and the edges bonds between atoms. Important theorems, such as the Lovazs and Lollipop theorems, are used to find eigenvalues of classes of graphs while analytic methods are used to determine simplified expressions of the energy of classes of graphs.
As a result of the investigation, in the literature, of the difference of the energy of two graphs G and H, on the same number n of vertices, we adapted this idea by making one of the graphs the complete graph. This premise is based on the fact that the complete graph is a very important and well-studied class of graphs. Since the complete graph does not have the largest energy of all graphs, it is customary to see how the energy of other classes of graphs compare to that of the complete graph, and graphs with energy less than that of the complete graph are referred to as hypoenergenic. It would therefore be significant to compare the energy of a complete graph, with the energy of any other graph G, in terms of how close their energies are, and how the energy of G compares with the energy of the complete graph, where a large number of vertices are involved. This comparison provided for the original definition of the eigen-complete difference ratio of a graph, which is the main thrust of this dissertation.
Considering a graph as a molecular graph, the complete graph translates to that of a molecule with all possible bonds between atoms i.e. a strongly bonded molecule. The eigen-complete difference ratio allowed for the investigation of the domination effect of the energy of graphs on the energy of the complete graph, when a large number of vertices are involved. We found that a strongly regular graph dominated in the largest negative way, while the star graph with rays of length one had a domination effect of one- the largest possible positive domination effect. The lollipop graph with base the complete graph had domination effect of zero, indicating its behavior as similar to the complete graph. Cycles, paths and wheels are shown to have same asymptotic convergence of their eigen-complete ratio,identical to $\frac{\pi-2}{\pi}$; providing a significant link to the π -electron molecule. We attached the average degree to the Riemann integral of this eigen-complete difference area to determine eigen-complete difference area associated with classes of graphs similar to that of other graph theoretical ratios in the literature.

Declaration

I, Samson Ogagaoghene Ojako, declare that

(i) The research reported in this thesis, except where otherwise indicated, is my original research.

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Publications

The publications on which the research presented in this thesis is based are reproduced in Appendix [112].Details of the contributions to these publications by this author are:

1 Winter P. A. and Ojako S. O. The Eigen-complete Difference Ratio of classes of Graphs- Domination, Asymptotes and Area. Journal of Advances in Mathematics, 10(9), 3791-3802. (2015).

Also, the publication formed part of a book authored by Dr Paul August Winter

Titled:
Graph Theory and Calculus: Ratios of Classes of Graphs. Publisher: LAP Lambert Academic Publishing (May 6, 2015).

S.O. Ojako — 28 /04 /2016

Student Signature — Date

Dr. Paul A. Winter — 28 / 04 / 2016

Supervisor(Name and Signature) — Date

Contents

Preface

This research work described in this dissertation was carried out in the school of Mathematics, University of KwaZulu Natal, Durban, from August 2013 to December 2015, under the supervision of Dr Paul August Winter.

This dissertation represent original work by the author and has not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

Acknowledgements

I praise God,the almighty for providing me this opportunity and granting me the capability to proceed successfully. This thesis appears in its current form due to the assistance and guidance of several people. I would therefore like to offer my sincere thanks to all of them.
Firstly, I would like to express my sincere gratitude to my supervisor Dr. PAUL AUGUST WINTER for the continuous support of my MSc study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better supervisor/advisor and mentor for my MSc study. Besides my supervisor, I would like to thank the rest of the staff and my colleagues of the School of Mathematics, Computer Science and Statistics: Prof. Simon Mukwembi, Dr Gareth Amery, Dr L.Scribani of blessed memory, Dr. Moopanar, Dr Aruna K, Dr Hansraj Sudan, Dr Brian Chilabwe, Dr O. Mewomo, Dr Mhlongo, Dr G. Govender, Dr. N. Proscovia, Dr Collins Okeke, Dr Usiholo, Ms Princess Bavuyile Nhlangulela, and a host of others, for their insightful comments and encouragement, but also for the hard questions which propelled me to widen my research from various perspectives. My sincere thanks also go to Dr. Uyi Osariyekemwen, Dr. Ireyuwa Igbinosa, and Dr. Festus Osayamwen, Dr Efe Orumwense, who provided me an opportunity to join their team as intern, and who gave me access to the University of KwaZulu Natal and research facilities. Without their precious support and advice it would not be possible to conduct this research.
I thank my fellow researchmates; Mr.Langalihle Mazibuko, Mr. Lindizwi Dlamini, Dr.Tendai Shumai (my LaTeX Instructor), Dr Godwin Achama, Mr Alex Somto (LaTeX Guru), Mr. Philip Smith, Mr. Ojolo Tolulope Lembola, Dr. Qudus Wale and friends in the following institution, University of KwaZulu Natal (PMB Campus). In particular, I am grateful to Dr. Paul A.Winter for enlightening me the first glance of the research.
Last but not the least, I would like to thank my family: my parents and to my brothers and sister for supporting me spiritually throughout writing this thesis and my life in general. My Son, Daniel Oruaroghene Samson, Miss Elohor Samson and my lovely wife, Miss Onome Iburho, for their patience and love.
I would like to express my special appreciation and thanks to my spiritual advisers, Mr Sanele Mkwanazi, Pastor Oket, Pastor Gbendro, Pastor Olugbemi, Miss Lindiwe Mthethwa and a host of others.
GOD ALMIGHTY BLESS YOU ALL. AMEN.

Chapter 1

Introduction

1.1 Introduction

This dissertation brings together two important concepts in graph theory: the energy of a graph and the complete graph. The energy of a graph is the sum of the absolute values its eigenvalues (defined explicitly in chapter 4) and originated (see [33]) from the determination of the sum of π-electron energy in a molecule represented by a molecular graph - i.e. a graph (formally defined in chapter 2) where the vertices represent atoms and the edges bonds between atoms.
Brualdi [13], introduced the difference of the energy of two graphs G and H on the same number n of vertices - we adopt this idea and making one of the graphs the complete graph (a graph where every pair of vertices is joined by an edge defined formally in the next chapter). This premise is based on the fact that the complete graph is a very important and well-studied class of graphs; for example, because of the fullness of its edges, it has a high degree of connectivity and robustness. The complete graph is often used to apply a new graph theoretical concept, and to verify known concepts, such as radius, diameter and chromatic number (defined in chapter 2). Given that the complete graph does not have the largest energy of all graphs, it is customary, (see hypoenergetic graphs in Koolen, Moulton, Gutman, and Vidovic [65]) to see how the energies of other classes of graphs compare to that of the energy of the complete graph.

It would therefore be significant to compare the energy of a complete graph with the energy of any other graph G, in terms of how close their energies are, and how the energy of G compares with the energy of the complete graph, when a large number of vertices are involved. This provided for the definition of *the eigen-complete difference ratio of a graph*, see Winter and Ojako[112], which we present in chapter 5.
Model atoms to vertices and bonds to edges, then the complete graph translated to that of a molecule with all possible bonds between atom i.e. a strongly bonded molecule. The eigen-complete difference ratio allowed for the investigation of the domination effect of the energy of graphs versus the energy of the complete graph, when a large number of vertices are involved. For cycles, paths and wheels (formally defined in chapter 3), the domination effect is $\frac{\pi-2}{\pi}$, which provides a mathematical link to the idea of the π-electron energy associated with such molecules.

Chapter 2 presents the basic graph theoretical definitions needed for subsequent chapters, and it is here where the different classes of graphs are defined and discussed. These classes form the basis for the investigation of eigen-complete difference ratios, asymptotes and areas in chapter 5.
The eigenvalues of graphs arise from the algebraic definition involving matrices and vectors, and in chapter 3 we define the eigenvalues of a graph in terms of the adjacency matrix of a graph. In this chapter we present different important theorems used to find the eigenvalues of graphs such as the Lovaszs theorem and the lollipop theorem. This chapter also illustrates the different methods used to find eigenvalues, such as difference equations and matrix manipulation.
Once we have determined the eigenvalues of classes of graphs, it is possible to determine the energies of classes of graphs. The complete graph on n vertices has an integral energy of $(2n-2)$, but for other classes of graphs, such as cycles and paths and wheels, the energy expression is not simple, and involves the sum of cosine of a rational number. However, analytical methods based on Winter and Jessop [101] are presented to obtain simplified expressions of the energy of path and cycles in terms of the cotangent and cosecant. We show that for large n, paths, cycles and wheels on n vertices, have the same energy of $\frac{4n}{\pi}$, confirming their hypoenergenic aspects i.e. they have energy less than that of the complete graph. Once we have found the energies of classes of graphs, we present a new ratio, the eigen-complete difference ratio of classes of graphs, in chapter 5 this was puplished in Winter and Ojako[112]. We used the idea of the energy difference between two graphs, and the significance of the complete graph, to formulate this eigen-complete difference ratio, which allowed for the investigation of the domination effect that the energy of graphs have, with respect to the complete graph, when a large number of vertices are involved. This idea can be translated to molecular structures where the energy of a heavily bonded molecule is significant. The cycles, paths and wheels on n vertices are also shown to have the same eigen-complete domination effect of $\frac{\pi-2}{\pi}$ which is equivalent to the solution $y = f(\pi)$, of the separable differential equation:

$$y(x) + \left(\frac{1}{x}\right)\frac{dy}{dx} = \frac{1}{x} = \frac{1}{x}; y(3) = \frac{1}{3}.$$

We attached the average degree to the Riemann integral of this eigen-complete difference ratio to determine eigen-complete difference areas associated with classes of graphs similar to that of: the eigen-pair ratio of classes of graphs, [113], the tree-cover ratio of graphs,[110] , the eigen-energy formation ratio see [102], the t-compete sequence ratio see [101], the chromatic-cover ratio, [108], the tree-3-cover ratio [110], the chromatic-complete difference ratio [108], graph theory and calculus: ratios of classes of graphs [106],the eigen-cover ratio [112] and the eigen-3-cover ratio [113]. We applied the above definitions also to the complements of classes of graphs.

Chapter 2

Definition of Graph Theoretical Terms

2.1 Introduction

Graph theory is an expanding area in mathematical research, and has range of specialized usage of terms related to "graph" (defined below). Some authors use the same word with different meanings while some use different words to mean in same thing. This chapter attempts to describe the majority of current usage of graph basic terms. One of the earliest people to experiment with graph theory was a man by the name of Euler (pronounced "Oiler") (1707-1783) [95]. He tried to solve the problem of crossing seven bridges onto an island without using any of them more than once. Since then, the study of graphs has been applied to a large number of real world problems. Today, graphs are everywhere and are used in many diverse industries, from computer networking (such as the internet), urban planning, to shipping lanes, etc.

2.2 Graph Theory

Graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. We shall adopt the graph theoretical notation of Harris, Hirst, and Mossinghoff (see [45]).
Algebraic graph theory is the branch of mathematics that studies graphs by using algebraic properties of associated matrices. We discuss this in chapter 3.

2.2.1 Graphs

A *graph* G consists of an ordered pair $G = (V, E)$ of 2 disjoint sets V, E such that $V \cap E = \emptyset$ with $V = \{v_1, v_2, \cdots, v_n\}$; $E = \{e_1, e_2, \cdots, e_m\}; n \geq 1;\ m \geq 0$. The elements of V are called *vertices*, the elements of E called *edges*, which are 2-element subsets of V (the endpoints of the edge).

2.2.2 Vertex of Graphs

A *vertex* (plural vertices) also known as *node*, is the fundamental unit of which graphs are formed: an *undirected graph* consists of a set of *vertices* and a set of edges (unordered pairs of vertices) formally defined below. In a diagram of a graph, a vertex is usually represented by a circle with a label and an edge is represented by a line. A *directed graph* consists of a set of *vertices* and a set of *arcs* (ordered pairs of vertices with arrow showing direction in the graph diagram). From the point of view of graph theory, vertices are treated as featureless and indivisible objects, although they may have additional structure depending on the application from which the graph arises. The two vertices forming an edge are said to be the endpoints of this edge, and the edge is said to be incident to the vertices. A vertex v is said to be *adjacent* to another vertex w if the graph contains an edge v, w. A *simple graph* is an undirected graph without *loops* (an edge with endpoints coinciding) or *multiple edges* (more than one edge, each with the same endpoints).
The *neighbourhood* of a vertex v are all the vertices adjacent to v denoted by $N(v)$.
The *degree of a vertex* in a graph is the number of edges incident to it, denoted by $d(v)$.
The *maximum degree* or more explicitly, *maximum vertex degree* of a graph G is the largest vertex degree of G denoted $\triangle(G)$.
The *minimum degree* of a graph G, denoted by $\delta(G)$, is the smallest vertex degree of G. The degree sum formula states that: given a graph, $G = (V, E)$;

$$\sum_{v \in \in V} d(v) = 2\,|E|\,.$$

The formula implies that in any graph G, the number of vertices with odd degree is even. This statement (as well as the degree sum formula) is known as the handshaking lemma. The handshaking lemma (corollary 2.2.1 below) states that every finite undirected graph has an even number of vertices with odd degree.

Theorem 2.2.1. *The graph* $G = (V, E)$ *where* $V = \{v_1, v_2, \cdots, v_n\}$ *and* $E = \{e_1, e_2, \cdots, e_n\}$ *satisfies*

$$\sum_{i=1}^{n} d(v_i) = 2m.$$

Proof. Since each edge of graph G is incident with exactly two vertices, therefore if we sum the degrees of the vertices of graph G, we count each edge twice. □

corollary 2.2.1. *Every graph has an even number of vertices of odd degree.*

Proof. If the vertices $v_1, v_2, \cdots, v_k$ have odd degrees and the vertices $v_{k+1}, v_{k+2}, \cdots, v_n$ have even degrees, then,

$$d(v_1) + d(v_2) + \cdots + d(v_k) = 2m - d(v_{k+1}) + d(v_{k+2}) + \cdots + d(v_n)$$

is even number. Therefore k is even. □

In more colloquial terms, in a party of people some of whom shake hands, an even number of people must have shaken an odd number of other people's hands.

The handshaking lemma (corollary 2.2.1) is a consequence of the degree sum formula (Theorem 2.2.1) for a graph. Both results were proved by*Leonhard Euler (1736)* in his famous paper on the Seven Bridges of Königsberg that began the study of graph theory [29]. The vertices of odd degree in a graph are sometimes called odd nodes or odd vertices; in this terminology, the handshaking lemma can be restated as the statement that every graph has an even number of odd nodes. An *isolated vertex* is a vertex with degree zero; that is, a vertex that is not an endpoint of any edge. A *leaf vertex* or *leaf* (also *pendant vertex* or end vertex) is a vertex with degree one.

2.2.3 Edge of a Graph

An *edge* is drawn as a line connecting two vertices, called *endpoints* or *end vertices* or *end-vertices*. An *edge* with end-vertices x and y is denoted by xy (without any symbol in between).

The edge set of G is denoted by $E(G)$ or E, when there is no danger of confusion. An edge xy is called *incident* to a vertex when this vertex is one of the endpoints x or y. The *size* of a graph is the number of its edges, i.e. $[E(G)]$.

2.2.4 Walk of a Graph

A *walk* of a graph G is a sequence of vertices and edges, of G, $v_1, e_1, v_1, \cdots, v_n$ which start and end with a vertex, where each edge's endpoints are the preceding and following vertices in the sequence.
A walk is *closed* if its first and last vertices are the same, and *open* if they are different.

2.2.5 Length of Walk

The *length* l of a walk is the number of edges that it uses. For an open (if its first and last vertices are not the same) walk, $l = n - 1$ where n is the number of vertices visited (a vertex is counted each time it is visited). For a closed walk, $l = n$ (the start/end vertex is listed twice but is not counted twice).

2.2.6 Trail of a Graph

A *trail* is walk in which all the edges are distinct (that is, no edge is repeated). A closed trail has been called a *tour*.

2.2.7 Isomorphic Graphs

If two graphs are *isomorphic*, they must have the same number of vertices, the same number of edges, the same degrees for corresponding vertices, the same number of connected components, the same number of loops and the same number of parallel edges.
In other words, an isomorphism of graphs G and H is a bijection (Gross and Yellen [38]) between the vertex sets of G and H; $f : v(G) \to v(H)$; such that any two

vertices u and v of G are adjacent in H if and only if $f(u)$ and $f(v)$ are adjacent in H. This kind of bijection is generally called *"edge-preserving bijection"*, in accordance with the general notion of isomorphism being a structure-preserving bijection. If an isomorphism exists between two graphs H and G, then the graphs are isomorphic and we write: $G \approx H$.

2.2.8 Path And Cycle of a Graph

A *path* of a graph is a walk where the vertices are distinct. A *cycle* of a graph may be defined either as a closed walk with no repetitions of vertices and edges allowed, other than the repetition of the starting and ending vertex, or a closed path with only start and end vertex being the same. A graph is *acyclic* if it has no cycles.

2.2.9 A Connected Graph

A graph is *connected* if there is a path between every pair of vertices. A graph that is not connected is disconnected. A graph with one vertex is connected. An edgeless graph with two or more vertices is disconnected. In an undirected graph G, two vertices u and v are called connected if G contains a path from u to v. If the two vertices are additionally connected by a path of length 1, i.e. by a single edge, the vertices are called *adjacent*. A graph is therefore said to be connected if every pair of vertices in the graph is connected.

2.2.10 Cut Vertex And Vertex Separator

A *cut vertex* is a vertex, whose removal from a connected graph, would disconnect the graph. A *vertex separator* is a collection of vertices, whose removal from a connected graph, would disconnect the graph.

2.2.11 Subgraph, Supergraph, Induced Subgraph And Component of A Graph

A *subgraph* S of a graph G is a graph whose set of vertices and set of edges are all subsets of G. (Since every set is a subset of itself, every graph is a subgraph of itself.) In the other direction, a *supergraph* of a graph H is a graph G in which H is a subgraph. We say a graph G contains a *subgraph* H if some subgraph of G is H or is isomorphic to H (see definition of isomorphm of graphs in section 2.2.8).

A subgraph H is a *spanning* subgraph, or *factor* of a graph G if it has the same vertex set as G. We say H spans G.

A subgraph H of a graph G is said to be *induced* (or *full*) if, for any pair of vertices x and y of H, xy is an edge of H if and only if xy is an edge of G. In other words, H is an induced subgraph of G if it has exactly the edges that appear in G over the same vertex set. If the vertex set of the graph H is the subset S of $V(G)$ then H can be written as $G[S]$, and is said to be *induced* by S.

A *component* of a graph (or a connected component) of an undirected graph is a (proper) subgraph in which any two vertices are connected, and which is connected to no additional vertices in the supergraph. A vertex with no incident edges is itself a connected component.

2.2.12 Distance, Radius, Diameter And Eccentricity of Graphs

The *distance* between two vertices in a graph G, is the number of edges in a shortest path connecting them. This is also known as the *geodesic distance* of G. Note that there may be more than one shortest path between two vertices. If there is no path connecting the two vertices, i.e., if they belong to different components, then conventionally the distance is defined as infinite.

The *eccentricity* $e(v)$ of a vertex v in a connected graph G, with vertex set V, is the maximum graph distance between v and any other vertex u of graph G for $u, v \in V$.
The *radius* r of a graph G with vertex set V is the *minimum eccentricity* over all vertices of G. It is written as: $V_{min} \in [e(V)]$.
To find the *diameter of a graph*, first find the shortest path between each pair of vertices. The greatest length of any of these paths is the diameter of the graph. To find the radius of a graph, first find the shortest path between each pair of vertices, then, the least length of any of these paths will give the radius of the graph. The diameter, R is $R = V_{max} \in [e(V)]$.
For a disconnected graph, all vertices are defined to have infinite eccentricity.

2.2.13 Caterpillar Graph

Caterpillar graph is a tree in which all the vertices are within distance 1 of a central "path" (defined in section 2.2.8). They were first studied in a series of papers by Harary and Schwenk. The name was suggested by Hobbs, A, J Harary and Schwenk (see [47]). They colorfully write, "a caterpillar is a tree which metamorphoses into a path when its cocoon of endpoints is removed". We define regular caterpillar graphs in section 2.2.22 in terms of a remnant path (fragment graph) when all the end vertices are removed.

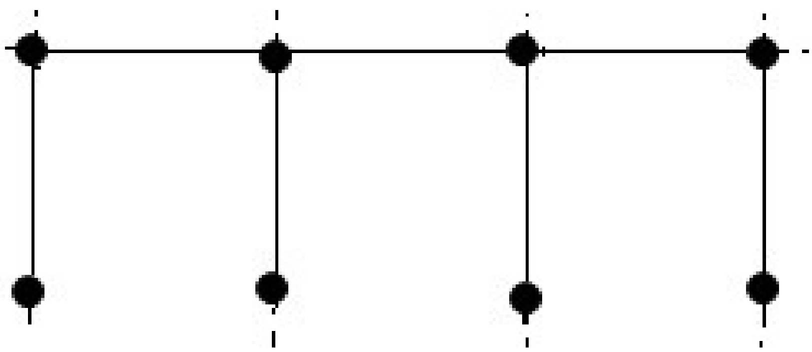

Figure 2.1: Caterpillar graph $CT(k, l)$

2.2.14 Independent Set

An *independent set* is a set of vertices in a graph, such that no two vertices of the graph $G(V, E)$ are adjacent (i.e. there is no edge connecting the two vertices). In other words, it is a set $I \subseteq V$ such that for every two vertices in I, there is no edge connecting the two. Equivalently, each edge in the graph has at most one endpoint in I.

2.2.15 Clique of a graph

A c*lique*, C, in an undirected graph $G = (V, E)$ is a subset of the vertices, $C \subseteq V$, such that every two distinct vertices are adjacent. This is equivalent to the condition that the subgraph of G induced by C is complete. The term clique may also refer to the subgraph directly in some cases.
A *maximal clique* is a clique that cannot be extended by including one more adjacent vertex, that is, a clique which does not exist exclusively within the vertex set of a larger clique.
A *maximum clique* of a graph, G, is a clique, such that there is no clique with more vertices.
The clique number $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G.
The *intersection number* of G is the smallest number of cliques that together cover all edges of G.
The *clique cover number* of a graph G is the smallest number of cliques of G whose union covers $V(G)$. The opposite of a clique is an independent set, in the sense that every clique corresponds to an independent set in the complement graph. The clique cover problem concerns finding as few cliques as possible that include every vertex in the graph.

2.2.16 Complement of A Graph

The *complement* $\overline{G}$ of a graph G is the graph with the same vertex set as G and: $e \in E(\overline{G}) \Leftrightarrow e \notin E(G)$.

2.2.17 The Conjugate Pair

A *conjugate pair* is a pair of the form $a \pm \sqrt{b}$ where $a, b \in \Re,\ b > 0$.
We shall show later that if a graph has the associated eigenvalue $\frac{(a+\sqrt{b})}{2}$(defined in section 3.2.1), then it has an eigenvalue $\frac{(a-\sqrt{b})}{2}$ and vice versa. Adding the pair of conjugates $\frac{(a+\sqrt{b})}{2}$ and $\frac{(a-\sqrt{b})}{2}$, we obtain the integer a. Their product is $\frac{(a^2-b)}{4}$ which is an integer, provided the numerator is a multiple of 4.

2.2.18 The Path As A Graph

A *path graph* $P_n = P$ is a simple connected graph with $|V_p| = |E_p| + 1$ that can be drawn so that all of its vertices and edges lie on a single straight line. An n vertex path graph is denoted P_n.
The length of a path P_n is the number of edges in P_n and is $n - 1$. A path always has a first vertex called its *start vertex* and a last vertex called its *end vertex*. Both of them are called *end or terminal vertices* of the path. The other vertices in the path are internal vertices.

2.2.19 Cycle As A Graph

A cycle graph is a simple connected graph C with $|V(C)| = |E(C)|$ that can be drawn so that all of its vertices and edges lie on a circle. An n vertex cycle graph is denoted by C_n . The length l of a cycle is the number of edges in it which is n . A cycle is not allowed to have a length zero.

2.2.20 Tree As A Graph

A *tree* is an undirected graph in which any two vertices are connected by exactly one simple path. In other words, any connected graph without cycles is a tree - see Theorem 2.2.2 below.

Theorem 2.2.2. *Let G be a connected graph, then G is a tree iff G has no cycles.*

Proof. Suppose T is a tree, and let $u, v \in E(T)$. Since T is connected, we know there exists a unique u, v *path* in T between any pair uv of vertices. Suppose now that there is a cycle in G with vertices: $u = u_1, u_2, \cdots, v$ and edges $u_1u_2, u_2u_3, \cdots, u_{k-1}v, vu_1$. Then there are two paths between u and v, namely, u_1v and $u_1, u_2, \cdots, v$. This is a contradiction with our assumption that between every two vertices there is a unique path connecting them. Hence, G contains no cycles. □

The forest as a disjoint union of trees

A forest is an acyclic graph (i.e., a graph without any graph cycles). Forests consist only of (possibly disconnected) trees, hence the name "forest." Examples of forests include the singleton graph, empty graphs, and all trees. A forest with components and nodes has graph edges.
A tree is called a *rooted tree* if one vertex has been designated the root (start vertex), in which case the edges have a natural orientation, towards or away from the root, (Diestel, Reinhard [27]). *Rooted trees,* often with additional structure such as ordering of the neighbours at each vertex, are a key data structure in computer science. In a context where trees are supposed to have a root, a tree without any designated root is called a *free tree* [97]. The term "tree" was coined in 1857 by the British mathematician Arthur Cayley [68].

Also, a tree can be seen as an undirected simple graph G that satisfies any of the following equivalent conditions:

- G is connected and has no cycles.

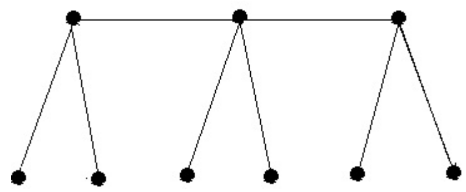

Figure 2.2: A tree with 9 vertices and 8 edges

- G has no cycles, and a simple cycle is formed if any edge is added to G.
- G is connected, but is disconnected if any single edge, or non-end vertex, is removed from G.
- Any two vertices in G are connected by a unique path.

2.2.21 The Complete Graph

The *complete graph* is denoted by K_n and defined as a graph on n vertices such that there is an edge between any pair of vertices. In other words, a complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A drawing of a complete graph, with its vertices placed on a regular polygon, is sometimes referred to as a mystic rose.

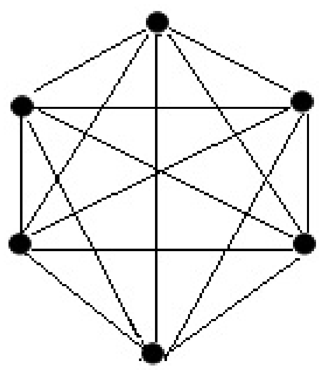

Figure 2.3: A complete graph on 6 vertices K_6

2.2.22 The Star Graph

A *star graph* on n vertices with rays of length 1, denoted by $S_{n,1}$, is the tree of order n with diameter 2; in which case a star, with $n > 2$, has $n - 1$ leaves (vertices of degree one or end vertices). Stars may also be described as the only connected graphs in which at most one vertex has degree greater than one. A star graph on n

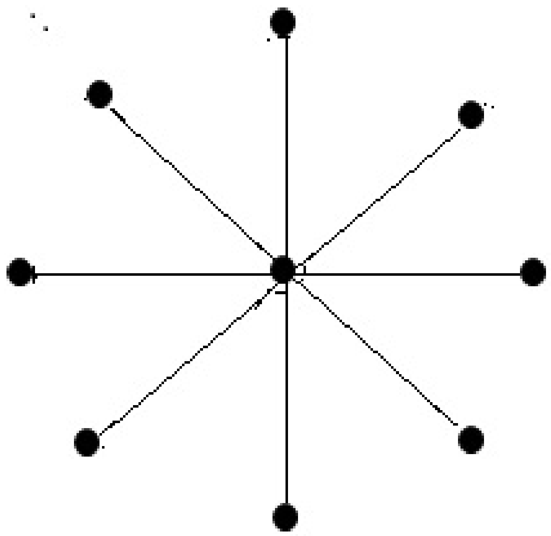

Figure 2.4: Star graph $S_{8,1}$

vertices with k rays of length 2 is obtained from the star graph with rays of length 1 by inserting a vertex in each edge-i.e. by subdividing each edge with a vertex. The number of vertices is therefore $n = 2k + 1$ so that $k = \frac{(n-1)}{2}$. Let $S_{n,k(2)}$ denote the star graph on n vertices with k rays of length 2

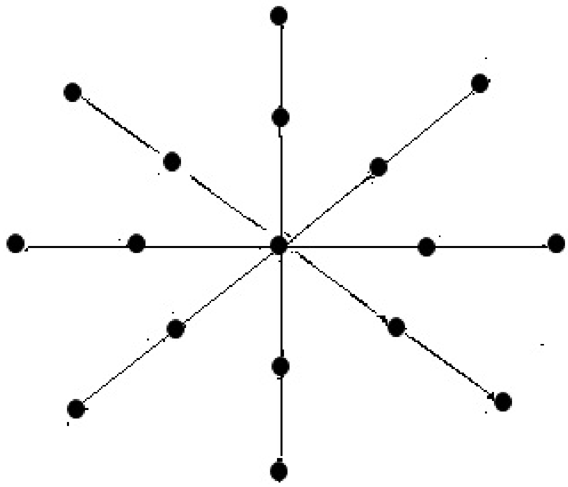

Figure 2.5: Star graph $S_{17,8(2)}$

2.2.23 l-Regular Caterpillar Graph $CT(k,l)$

An l-regular caterpillar graph is obtained by attaching l pendant edges(An edge of a graph in which one of its vertices is a pendant vertex) to each vertex of the path P_k. It is denoted by $CT(k,l)$ where k and l denote the number of vertices on the path and the number of pendant edges respectively. This graph will have $n = k(l+1)$ vertices. A vertex v of G is said to be a pendant vertex if and only if it has degree 1. The caterpillar graph is a planar graph. (see section 2.2.24).

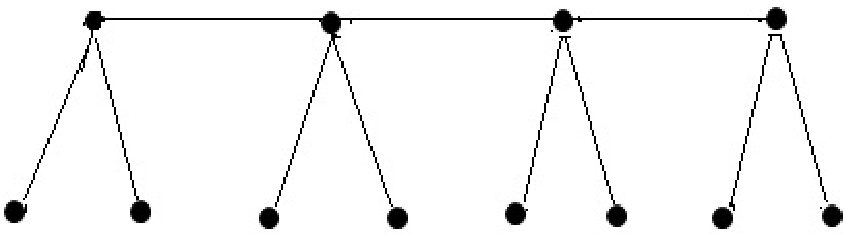

Figure 2.6: Caterpillar $CT(4,2)$

2.2.24 Bipartite Graphs: the Complete Split-Bipartite Graph

A *bipartite graph* G is a graph whose vertex set V can be partitioned into two subsets A and B such that each edge of G has one endpoint in A and one endpoint in B. The pair A, B is called the *vertex bipartition* of G, and A and B are called the *bipartition subsets*. The two sets A and B may be thought of as a colouring of the graph with two colours: if one colours all vertices in A blue, and all vertices in B green, each edge has endpoints of differing colours, as is required in the graph colouring problem [56]. In contrast, such a colouring is impossible in the case of a non-bipartite graph, such as a triangle: after one node is coloured blue and another green, the third vertex of the triangle is connected to vertices of both colours, preventing it from being assigned either colour.

Theorem 2.2.3. *A bipartite graph is a graph that does not contain any odd-length cycles.*

Proof. Let $G = (V, E)$ be bipartite given that A and B are the partitioned subsets of V So, let $V = A \cup B$ (number of vertices in a bipartite graph. see definition above) and $A \cap B = \phi$. Then, all edges $e \in E$ are such that e is of the form $\{a, b\}$ where $a \in A$ and $b \in B$.
Suppose G has (at least) one odd cycle C_n with length of C_n is n. Let $C_n = (v_1, v_2, v_3, \cdots, v_n, v_1)$ Also, let $v_1 \in A$. It follows that $V_2 \in B - v$, and hence $v_3 \in A$ and so on. Hence we see that for all $k \in \{1, 2, 3, \cdots, \cdots, n\}$, we have: $v_k \in \{A : B : odd\ k : even\ k : \cdots\}$. But as n is odd, $v_n \in A$.
Also, $v_1 \in A$ and $v_n v_1 \in C_n$. So $v_n v_1 \in E$ contradicts the assumption that G is bipartite. Hence, if G is bipartite, it has no odd cycles. □

A *complete bipartite graph* is a simple bipartite graph such that every vertex in one of the bipartition subsets is joined to every vertex in the other bipartition subset. Any complete bipartite graph that has m vertices in one of its bipartition subsets and n vertices in the other is denoted by $K_{m,n}$; it has mn edges, joining every vertex of one partition to every vertex of the other partition. The *complete split-bipartite graph* is when we split the vertex-set of the complete bipartite graph into identical parts of size $\frac{n}{2}$.

Examples of Bipartite Graph

- Every tree is bipartite.
- Cycle graphs with an even number of vertices are bipartite.
- Every planar (defined in section 2.2.24 below) graph whose faces all have even length is bipartite. A face of G is a closed walk, and an odd closed walk contains an odd cycle. So the bipartite plane graph has no face of odd length.

Bipartite graphs may be characterized in several different ways

A graph is bipartite if and only if it is 2-colorable, (i.e. its chromatic number is less than or equal to 2: the chromatic number of a graph is the least number of colours require to colour the vertices of the graph so that adjacent vertices do not have same colour) [56], [26].

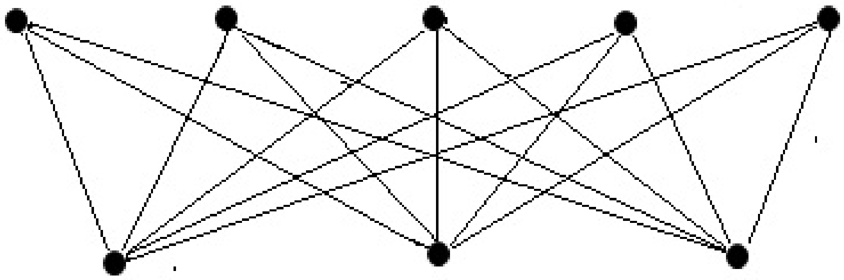

Figure 2.7: The complete bipartite graph $K_{5,3}$

2.2.25 Planar Graph

A *planar graph* is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. Such a drawing is called a plane graph or planar embedding of the graph. A plane graph can be defined as a planar graph with a mapping from every node to a point on a plane, and from every edge to a plane curve on that plane, such that the extreme points of each curve are the points mapped from its end nodes, and all curves are disjoint except on their extreme points. Every graph that can be drawn on a plane can be drawn on the sphere as well, and vice versa [12]. In other words, a graph G_p is said to be planar if there exists some geometric representation of G_p which can be drawn on a plane such that no two of its edges intersect. A planar representation of a graph divides the plane into regions also called *faces*. Such a region is characterized by the set of edges (or the set of vertices) forming its boundary.

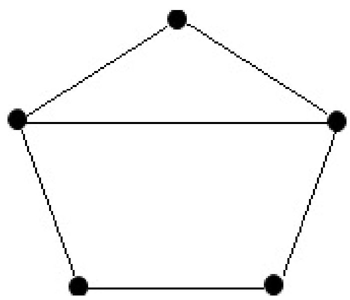

Figure 2.8: A planar graph

Theorem 2.2.4 (Kuratowski's theorem)**.** *Kuratowski's theorem (Kuratowski and Kazimierz [60]) states that a graph is not planar if and only if it contains* K_5 *or* $K_{3,3}$*. They are the "atoms" of nonplanar graphs; at least one of them is contained in every nonplanar graph.*

Examples of non-planar graph

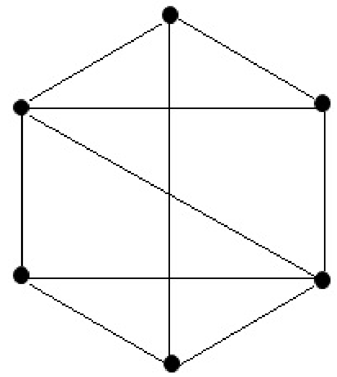

Figure 2.9: A non planar graph

The complete split-bipartite graph on 6 vertices: $K_{3,3}$

The Utilities Graph: The complete split-bipartite graph on 6 vertices is called the utility graph. We may imagine we have 3 utilities (gas=G, water=W, electricity=E) and houses(A, B, C) and every house needs to be connected to every utility. It's called bipartite because there are two disjoint subsets of vertices, and all the vertices in one subset are connected to all the vertices in the second subset (and vice versa).

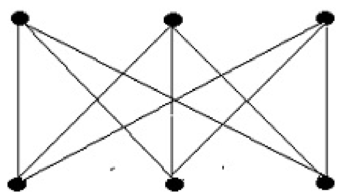

Figure 2.10: The utilities graph

A simple proof of the non-planarity of the utility graph can be effected by noting that the graph consists of a graph cycle $G - A - W - B - E - C$, to which the three edges must be added. Now, for each of the edges, we have to choose whether to draw the edge inside or outside the graph cycle, and so for two of the edges, we must make the same choice. But two lines can't be drawn on the same side without crossing, hence the utility graph is not a planar (Buckley [21]).

2.2.26 The Wheel Graph

The *wheel graph* on n vertices is composed of a cycle on $n - 1$ vertices with a vertex v (the *center* of the wheel) joined by a single edge to each vertex of the cycle.

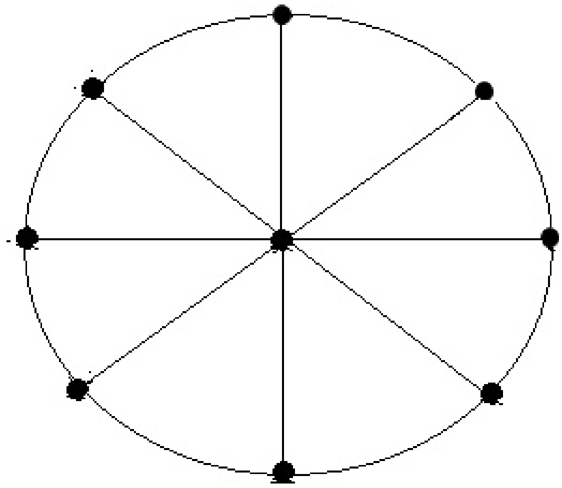

Figure 2.11: wheel graph W_9

2.2.27 The Dual-Star Graph

A *dual star* DuS_n is defined as two star graphs with m rays of length 1 each on $\frac{n}{2}$ vertices joined by an edge (its *center edge*) connecting their centers. Graph duality is a topological generalization of the geometric concepts of dual polyhedra and is in turn generalized algebraically by the concept of a dual matroid. However, the notion described in this page is different from the edge-to-vertex dual (line graph) of a graph and should not be confused with it. The term "dual" is used because this property is symmetric, meaning that if H is a dual of G, then G is a dual of H (if G is connected). When discussing the dual of a graph G, the graph G itself may be referred to as the "primal graph".

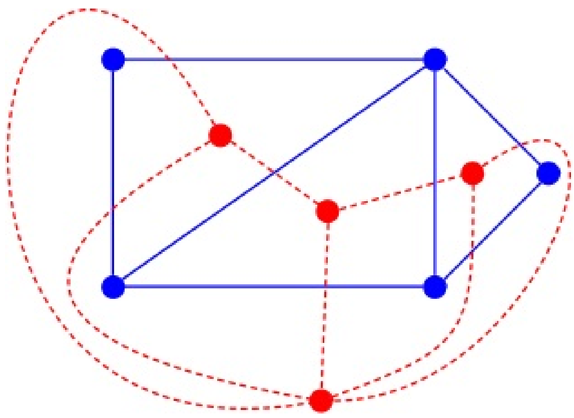

Figure 2.12: The Dual-Star graph

2.2.28 The Line Graph of G ($i.e., L(G)$)

The *line graph* of a graph G, denoted by $L(G)$, is defined as a graph that has the following properties:

- there is a vertex in $L(G)$ for every edge of G;
- two vertices of $L(G)$ are adjacent if and only if they correspond to two edges of G with a common end vertex.

The name line graph comes from a paper by Harary and Norman [48]. Both Whitney [104] and Angela [100] [64] used the construction before this. Other terms used for the line graph include the *covering graph*, the derivative, the edge-to-vertex dual, the conjugate, the representative graph, the edge graph, the interchange graph, the adjoint graph, and the derived graph.

Whitney[104] proved that, with one exceptional case,the structure of a connected graph G can be recovered completely from its line graph. Many other properties of line graphs follow by translating the properties of the underlying graph from vertices into edges, and by Whitney's theorem (see Whitney [104]) the same translation can also be done in the other direction. Line graphs are claw-free(as is the complement of any triangle-free graph), and the line graphs of bipartite graphs are perfect(a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph) [23].

Line graphs can be characterized by nine forbidden subgraphs, and can be recognized in linear time.

The line graph of a complete graph $L(K_n)$ has $p = \frac{n(n-1)}{2}$ vertices, and also the number of edges q of $L(K_n)$ is half the sum of the squares of the degrees of the vertices of K_n minus the number of edges of K_n. Thus:

$$q = \frac{(n-1)^2}{2} - \frac{n(n-1)}{2}$$

$$q = \frac{n(n-1)(n-2)}{2}.$$

2.2.29 Strongly Regular Graph

A *strongly regular graph* is defined as follows. Let $G = (V, E)$ be a regular graph with v vertices and degree k. Then G is said to be strongly regular if there are also integers λ and μ such that [10]:

- Every two adjacent vertices have λ common neighbours.
- Every two non-adjacent vertices have μ common neighbours.

A graph of this kind is sometimes denoted by $srg(v, k, \lambda, \mu)$.

2.2.30 The Lollipop Graph

The *lollipop graph*, denoted by $H_{n,p}$ is obtained by appending a cycle C_n, or a complete graph K_n, to a pendant vertex with a bridge or edge. We will only deal with the latter case, where the complete graph on $(n-1)$ vertices has an end vertex appended to any of its vertices, and will be denoted by LP_n (see [118].

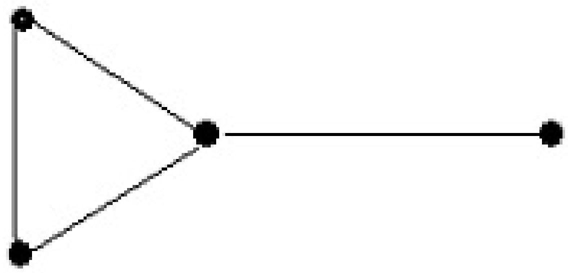

Figure 2.13: Lollipop graph LP_4

2.3 Conclusion

In our review of graph theoretical terms, we observed that a graph can take on many forms which vary from undirected to directed. The edges and vertices form an integral part of any graph, while the edges contribute to its "connectivity". We introduced many classes of graphs, such as bipartite, star, wheel, path, cycles etc. The complete graph K_n is a "well connected" graph in which every pair of vertices is adjacent and with graph theoretical properties, such that the diameter and the radius ($radius = 1, diameter = 2$) that can easily be determined. The importance of the complete graph will be used in conjunction with the energy of other classes of graphs, which involves algebraic graph theory which we discuss in the next chapter.

Chapter 3

Linear Algebra And Algebraic Graph Theory

3.1 Introduction

The *Linear algebra* of graphs is the branch of mathematics that studies graphs by using algebraic properties of associated matrices. Moreover, the theory of association schemes and coherent configurations studies the algebra generated by associated matrices. In this chapter, algebraic methods are applied to problems about graphs. This is in contrast to geometric, combinatoric, or algorithmic approaches. There are three main branches of algebraic graph theory, involving the use of linear algebra, the use of group theory, and the study of graph invariants.
Linear algebraic graph theory involves the study of graphs in connection with linear algebra. Especially, it studies the collection of eigenvalues (defined in Subsection 3.2.1) of the adjacency and Laplacian matrices (defined in section 3.2.3) of a graph (this part of algebraic graph theory is also called spectral graph theory which will be studied in the subsequent chapters). Several theorems relate properties of the eigenvalues of special graphs to other graph properties. As a simple example, a connected graph with diameter d (see Subsection 2.2.12) will have at least $d + 1$ distinct values in the collection of its eigenvalues.
The relationship between a graph and the eigenvalues of its adjacency and Laplacian matrix is explained in detail in section 3.2 below and studied explicitly in spectra graph theory.
The focus of this chapter is placed on various families, or classes, of graphs based on their adjacency and Laplacian matrices from which their eigenvalues are derived. Such graphs include Complete, Bipartite, Star graphs with rays of length 1- and star graphs with rays of length 2, Path, cycle, Lollipop, strongly regular graphs, etc. Such classes of graphs were introduced in chapter two.
Two basic theorems (the Lovasz and the Lollipop) are discussed with proofs and application. The Lovasz theorem is used to determine the eigenvalues of trees while the Lollipop theorem is used to find the eigenvalues of graphs with an end vertex connected to a subgraph whose eigenvalues are already known. Different methods were applied to find the eigenvalues of some selected graphs mentioned above. We need the eigenvalues of classes of graphs so that we can determine their energies

which we address in the next chapter.

3.2 The Adjacency Matrix, Laplacian Matrix and their Eigenvalues

3.2.1 Eigenvalues And Eigenvectors

Eigenvalues are a special set of scalars associated with a linear system of equations (i.e. a matrix equation) that are sometimes also known as characteristic roots, characteristic values [27], proper values, or latent roots [78]. Eigenvalues have their greatest importance in dynamic problems. To explain eigenvalues, we first explain eigenvectors. *Eigenvectors* are a special set of vectors associated with a linear system of equations (i.e., a matrix equation) that are sometimes also known as characteristic vectors, proper vectors, or latent vectors [78]. An eigenvector or characteristic vector of a square matrix is a vector that points in a direction which is invariant under the associated linear transformation. In other words, if $\underline{v}$ is a nonzero vector, then it is an eigenvector of a $n \times n$ square matrix A if Av is a scalar multiple of $\underline{v}$. This condition could be written as the equation $A\underline{v} = \lambda\underline{v}$.
Almost all vectors change direction, when they are multiplied by A, so eigenvectors are exceptional vectors $\underline{x}$ which are in the same direction as $A\underline{x}$. The value λ is said to be an eigenvalue of A. The eigenvalue tells whether the special vector $\underline{x}$ is stretched or shrunk or reversed or left unchanged when it is multiplied by A. The eigenvalue could be zero. Then, $A\underline{x} = 0\underline{x}$ means that this eigenvector $\underline{x}$ is the value of $\underline{x}$ that solve the equation $A\underline{x} = 0$.
If A is the identity matrix (an $n \times n$ matrix with $1s$ down the main diagonal and 0 everywhere else-defined formally below), then every vector $\underline{x}$ satisfies $A\underline{x} = I\underline{x}$. All vectors are eigenvectors of I. All eigenvalues are $\lambda = 1$.

3.2.2 Adjacency Matrix of a Graph

An *adjacency matrix* is seen as a means of representing which vertices of a graph have common edges (are adjacent). The adjacency matrix of a finite graph G on n vertices, defined by A_G or $A(G)$, is the $n \times n$ matrix where the non-diagonal entry a_{ij} is the number of edges from vertex i to vertex j, and the diagonal entry a_{ii} depending on the convention, is either zero, or the number of edges (loops) from vertex i to itself. The *adjacency matrix* can also be called the connection matrix. For a simple labeled graph G is a matrix with rows and columns labeled by graph vertices, with a 1 or 0 in position v_i, v_j according to whether v_i and v_j are adjacent or not. For a simple graph with no self-loops, the adjacency matrix must have zeros on the diagonal. For an undirected graph, the adjacency matrix is symmetric. There exists a unique adjacency matrix for each isomorphic class of graphs (Subsection 2.2.7) and it is not the adjacency matrix of any other isomorphic class of graphs.

3.2.3 Degree Matrix

The *degree matrix* is a diagonal matrix which contains information about the number of edges incidence to each vertex in a graph G. It is used together with the adjacency matrix to construct the Laplacian matrix of a graph G.
Given a graph $G = (V, E)$ with $|v| = n$, the degree matrix D_G is the $n \times n$ diagonal matrix defined as:

$$d_{ij} = \begin{Bmatrix} deg(v_i) & if & i = j \\ 0 & & otherwise \end{Bmatrix}$$

where $deg(v_i)$ is the number of times an edge terminates at the vertex.

3.2.4 The Laplacian Matrix

The *Laplacian matrix*, sometimes called the admittance matrix, is a matrix representation of a graph. It can be used to calculate the number of spanning subgraphs that are trees. The Laplacian matrix can be used to find many other properties of the graph. For example,it approximates the sparest cut of a graph through the second eigenvalue of its Laplacian. Given a simple graph G with n vertices, its Laplacian matrix LM_G, is $LM_G = D_G - A_G$, where D is the degree matrix $(n \times n)$ matrix with the degree of the ith vertex in the i, i entry: formally defined above) and A is the adjacency matrix of the graph defined above. From the definition it follows that:

$$L_{i,j} = \begin{Bmatrix} deg(v_i) & if & i = j & & & & \\ -1 & if & i \neq j & v_i & adjacent & to & v_j \\ 0 & & otherwise & & & & \end{Bmatrix}$$

$deg(v_i)$ is degree of the vertex i.

3.2.5 Identity Matrix

The *identity matrix* or unit matrix of size n is the $n \times n$ square matrix with $1s$ on the main diagonal and zeros elsewhere. It is denoted by I_n or simply by I if the size is immaterial or can be trivially determined by the context. See the matrices below.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

3.2.6 Adjacency Matrix of L-Regualar Caterpillar Graph

The adjacency matrix of the l-regular caterpillar graph (defined in section 2.2.4) is $n \times n$ and takes the general form:

$$A(CT(k,l)) = \begin{bmatrix} A(P_k) & I_{k,k} & I_{k,k} & \cdots & I_{k,k} \\ I_{k,k} & 0_{k,k} & 0_{k,k} & \cdots & 0_{k,k} \\ I_{k,k} & 0_{k,k} & 0_{k,k} & \cdots & 0_{k,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{k,k} & 0_{k,k} & 0_{k,k} & \cdots & 0_{k,k} \end{bmatrix}$$

where $I_{k,k}$ is repeated l times horizontally and l times vertically, and $A(P_k)$ is written once and is the first entry in the diagonal. For $l = 1$ the 1-caterpillar graph $CT(k,l)$ has $n = 2k$ *vertices*, and an adjacency matrix of the form

$$A(CT(k,l)) = \begin{bmatrix} A(P_k) & I_{k,k} \\ I_{k,k} & 0_{k,k} \end{bmatrix}$$

. Figure 3.1 depicts $CT(4,1)$

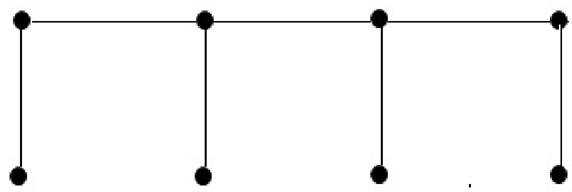

Figure 3.1: A caterpillar graph $CT(4,1)$

For $l = 2$ the 1-caterpillar graph $CT(k, 2)$ has $3k$ vertices, and an adjacency matrix of the form

$$A(CT(k,2)) = \begin{bmatrix} A(P_k) & I_{k,k} & I_{k,k} \\ I_{k,k} & 0_{k,k} & 0_{k,k} \\ I_{k,k} & 0_{k,k} & 0_{k,k} \end{bmatrix}$$

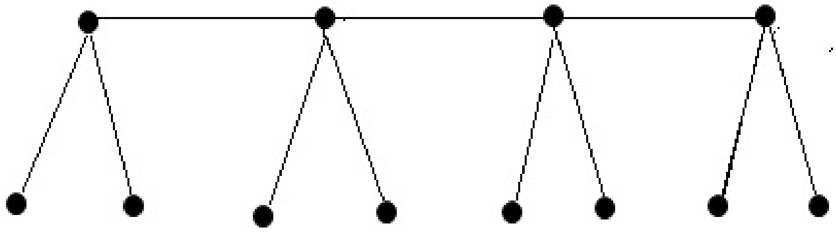

Figure 3.2: Caterpillar graph $CT(4, 2)$

3.2.7 Adjacency Matrix of Star Graph With Rays of Length 1

A star graph, S_n, is a connected graph on n vertices where one vertex has degree $n-1$ and the other $n-1$ vertices have degree 1 (see section 2.2.22). A star graph is a special case of a complete bipartite graph in which one set has 1 vertex and the other set has $n-1$ vertices:$S_n = K_{1,n-1}$.
Below is a star graph of 5 vertices, $S_5 = K_{1,4}$.

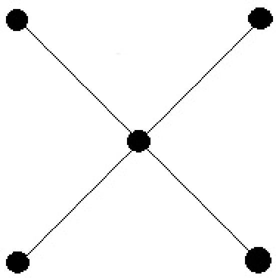

Figure 3.3: Star graph $S_5 = K_{1,4}$

The adjacency matrix of a star graph S_n is:

$$A(S_n) = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

In otherwords,

$$a_{i,j} = \begin{Bmatrix} 1 \quad if \quad i \quad or \quad j = 1 \quad but \quad & i \neq j \\ 0 & otherwise \end{Bmatrix}.$$

3.3 The Characteristic Polynomial

The *characteristic polynomial* (a determinant expression) of a square matrix is a polynomial which has the eigenvalues (λ defined above) as roots. Given a square matrix A we want to find a polynomial whose zeros are the eigenvalues of A . For a diagonal matrix A, the characteristic polynomial is easy to define. If the diagonal entries are $a_1, a_2, a_3, \cdots$, then the characteristic polynomial will be:$(t-a_1)(t-a_2)(t-a_3)\cdots$ This works because the diagonal entries are also the eigenvalues of this matrix. For a general matrix A , one can proceed as follows. A scalar λ is an eigenvalue of A if and only if there is an eigenvector $\underline{v} \neq 0$ such that $A\underline{v} = \lambda\underline{v}$ or $[\lambda I - A]\underline{v} = 0$ (where I is the identity matrix).
Since $\underline{v}$ is non-zero, this means that the matrix (*the characteristic matrix*) $[\lambda I - A]$ is singular (non-invertible), which in turn means that its determinant is 0. Thus the roots of the function $det(\lambda I - A)$ are the eigenvalues of A, and it is clear that this determinant is a polynomial (*the characteristic polynomial*) in λ. The *characteristic equation* is the equation obtained by equating to zero the characteristic polynomial. The characteristic polynomial of a graph is the characteristic polynomial of its adjacency or Laplacian matrix. It is a graph invariant (a graph property that depends only on the abstract structure not on graph representations), though it is not complete: the smallest pair of non-isomorphic graphs with the same characteristic polynomial has five vertices [53]. We define $det(\lambda I - A(G))$ as $P_\lambda^{A(G)}$ where $A(G)$ is the adjacency matrix of the graph G. We define $det(\mu I - LM(G))$ as $P_\mu^{LM(G)}$ where $LM(G)$ is the Laplacian matrix of the graph G

3.4 Theorems Used To Find Eigenvalues

In finding the eigenvalues of some special graphs, we applied two basic theorems which include the Lovasz theorem and the Lollipop theorem.

Example

[1] Given a tree T with seven 7 vertices. We show that:

$$det(\lambda I - A_G) = det(\lambda I - A_{G-e}) - det(\lambda I - A_{G-\{e\}})$$

Where e is the edge uv and (See [94]), $\{e\}$ is the edge uv together with the vertices u and v. See diagram below. The adjacency matrix of T is

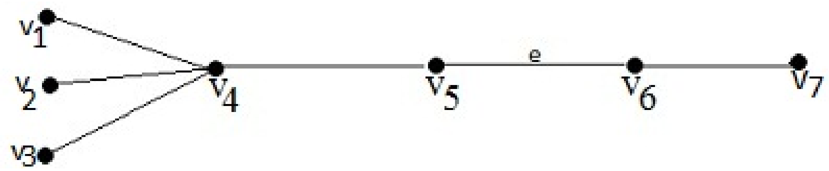

Figure 3.4: The tree T

$$A_G = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristics matrix will be:

$$(\lambda I - A_G) = \begin{bmatrix} \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & \lambda & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \lambda & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & \lambda \end{bmatrix}.$$

The expansion along the $5th$ column gives:

$$det(\lambda I - A_G) = (-1)^{4+5}(-1)|M_{4,5}| + (-1)^{5+5}\lambda|M_{5,5}| + (-1)^{6+5}(-1)|M_{6,5}| + 0$$

$$det(\lambda I - A_G) = |M_{4,5}| + \lambda|M_{5,5}| + |M_{6,5}| \tag{3.1}$$

$$|M_{4,5}| = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{vmatrix} = B;$$

$$\lambda\,|M_{5,5}| = \lambda \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 & 0 \\ -1 & -1 & -1 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & -1 & \lambda \end{vmatrix} = C;$$

$$|M_{6,5}| = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 & 0 \\ -1 & -1 & -1 & \lambda & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{vmatrix} = D.$$

We must work only with B , C , and D. Now we consider graphs with edge removed: Now, if the edge e between the vertices (5) and (6) are removed from the tree T, we have the disconnected tree $T-e$ as below:

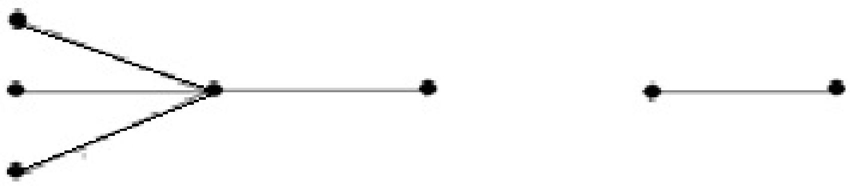

Figure 3.5: The tree $T-e$

T with edge e removed is $G-e$, and the adjacency matrix of $(G-e)$ is:

$$A_{G-e} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial will be:

$$(\lambda I - A_{G-e}) = \begin{bmatrix} \lambda & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & \lambda & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & \lambda \end{bmatrix}.$$

Expanding along 5th column:

$$|\lambda I - A_{G-e}| = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & \lambda & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & & 0 & -1 & \lambda \end{vmatrix}$$

$$|\lambda I - A_{G-e}| = (-1)^{4+5}(-1)|M_{4,5}| - (-1)^{5+5}\lambda|M_{5,5}| = |M_{4,5}| + \lambda|M_{5,5}|.$$

$$= \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & -1 & \lambda \end{vmatrix} + \lambda \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 & 0 \\ -1 & -1 & -1 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & -1 & \lambda \end{vmatrix} = B' + C$$

Note $det(\lambda I - A_{G-e})$ equals matrix C plus a matrix B' similar to matrix B with the 4,5 position different.

$$det(\lambda I - A_{G-e}) = B' + C = M_{4,5} + \lambda M_{5,5} \tag{3.2}$$

Now we must work with D to get $det(\lambda I - A_{G-\{e\}})$

Figure 3.6: The graph $T - \{e\}$

$$det(\lambda I - A_{G-\{e\}}) = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 & 0 \\ -1 & -1 & -1 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{vmatrix},$$

$$|M_{6,5}| = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 & 0 \\ -1 & -1 & -1 & \lambda & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & \lambda \end{vmatrix} = D.$$

We work with D :
Expanding along fifth column:

$$D = (-1)^{5+5}(-1)|N_{5,5}| - (-1)^{6+5}|N_{6,5}| = -|N_{5,5}| + |N_{6,5}|$$

this implies

$$D = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 & 0 \\ -1 & -1 & -1 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{vmatrix} + \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 & 0 \\ -1 & -1 & -1 & \lambda & 0 \\ 0 & 0 & 0 & -1 & 0 \end{vmatrix},$$

where

$$E = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 & 0 \\ -1 & -1 & -1 & \lambda & 0 \\ 0 & 0 & 0 & -1 & 0 \end{vmatrix} = 0.$$

Therefore,

$$D = det(\lambda I - A_{G-\{e\}}) + E = -det(\lambda I - A_{G-\{e\}}) + 0$$

$$D = -det(\lambda I - A_{G-\{e\}}) \tag{3.3}$$

Now, recall that

$$M_{4,5} = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & -1 & \lambda \end{vmatrix} = B; \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & -1 & \lambda \end{vmatrix} = B'.$$

Expansion along the $6th$ row of B will give:

$$B = (-1)^{6+5}(-1)|N_{6,5}| - (-1)^{6+6}\lambda|N_{6,6}| = |N_{6,5}| - \lambda|N_{6,6}|$$

$$B = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} - \lambda \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & \lambda \end{vmatrix}.$$

Expansion along the $5th$ row of the $2nd$ Minor gives:

$$B = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} - \lambda^2 \begin{vmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix}.$$

Also, the expansion along the $6th$ row of B' will give:

$$B' = (-1)^{6+5}(-1)|N_{6,5}| - (-1)^{6+6}\lambda|N_{6,6}| = |N_{6,5}| - \lambda|N_{6,6}|$$

$$B' = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} - \lambda \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{vmatrix}.$$

Expansion along the $5th$ row of the $2nd$ minor gives:

$$B' = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} - \lambda^2 \begin{vmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix},$$

and,

$$B = \begin{vmatrix} \lambda & 0 & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} - \lambda^2 \begin{vmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix}.$$

Therefore,

$$B = B'.$$

Now, to show:

$$det(\lambda I - A_G) = det(\lambda I - A_{G-e}) - det(\lambda I - A_{G-\{e\}})$$

From (3.1) above, we have:

$$det(\lambda I - A_G) = |M_{4,5}| + \lambda|M_{5,5}| + |M_{6,5}|.$$

This implies that:

$$det(\lambda I - A_G) = B + C - D. \tag{3.4}$$

Also, from (3.2)

$$det(\lambda I - A_{G-e}) = |M_{4,5}| + \lambda |M_{5,5}|,$$

we have

$$det(\lambda I - A_{G-e}) = B' + C.$$

We've proved that $B = B'$
Therefore,

$$det(\lambda I - A_{G-e}) = B + C. \tag{3.5}$$

From (3.3), we have:

$$-det(\lambda I - A_{G-\{e\}}) = |M_{6,5}|$$

implying that

$$-det(\lambda I - A_{G-\{e\}}) = D. \tag{3.6}$$

Adding equations 3.4, 3.5 and 3.6, we have

$$det(\lambda I - A_G) = B + C + D = det(\lambda I - A_{G-e}) - det(\lambda I - A_{G-\{e\}}) \tag{3.7}$$

Theorem 3.4.1 (Lovasz's Theorem)**.** *This theorem states that, If G is a tree and $e \in E(G)$, then,*

$$det(\lambda I - A_G) = det(\lambda I - A_{G-e}) - det(\lambda I - A_{G-\{e\}})$$

Where e is the edge uv and $\{e\}$ is the edge e and its vertices u and v .

Proof. As G is tree, we can label its vertices in such a way that e joins the points k and $k+1$ and there are no other edges between a point $i(1 \leq i \leq k)$ and a point $j(k+1 \leq j \leq n)$ so that e is a cut edge of G and none of the $v(t)$ are adjacent to $v(k)$ for $1 \leq t \leq (k-2)$.

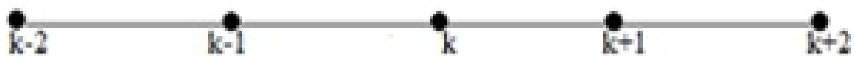

Figure 3.7: A path as a subgraph of a tree G

The adjacency matrix A_G is :

$$A_G = \begin{bmatrix} 0 & a_{1,2} & \cdots & a_{1,k-1} & a_{1,k} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & 0 & \cdots & a_{2,k-1} & a_{2,k} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \cdots & a_{3,k-1} & a_{3,k} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & 0 & a_{k-1,k} & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} & 0 & a_{k,k+1} & a_{k,k+2} & \cdots & a_{k,n-1} & a_{k,n} \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k-1} & a_{k+1,k} & 0 & a_{k+1,k+2} & \cdots & a_{k+1,n-1} & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & 0 & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,k-1} & a_{n,k} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & 0 \end{bmatrix}$$

The characteristic matrix is:

$$(\lambda I - A_G) = \begin{bmatrix} \lambda & a_{1,2} & \cdots & a_{1,k-1} & a_{1,k} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & \cdots & a_{2,k-1} & a_{2,k} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \cdots & a_{3,k-1} & a_{3,k} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & \lambda & a_{k-1,k} & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} & \lambda & a_{k,k+1} & a_{k,k+2} & \cdots & a_{k,n-1} & a_{k,n} \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k-1} & a_{k+1,k} & \lambda & a_{k+1,k+2} & \cdots & a_{k+1,n-1} & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,k-1} & a_{n,k} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{bmatrix}$$

Now, expansion of the determinant by its kth column gives:

$$det(\lambda I - A_G) = (-1)^{2k-1} a_{k-1,k}|M_{k-1,k}| + \lambda |M_{k,k}| + (-1)^{2k+1} a_{k+1,k}|M_{k+1,k}|$$

Since

$$det(\lambda I - A_G) = (-1)^{1+k} a_{1,k}|M_{1,k}| + (-1)^{2+k} a_{2,k}|M_{2,k}| + \cdots +$$
$$(-1)^{2k-1} a_{k-1,k}|M_{k-1,k}| + \lambda |M_{k,k}| + (-1)^{2k+1} a_{k+1,k}|M_{k+1,k}| + \cdots$$
$$+(-1)^{n-1,k} a_{n-1,k}|M_{n-1,k}| + (-1)^{n+k} a_{n,k}|M_{n,k}|$$

$$det(\lambda I - A_G) = |M_{k-1,k}| + \lambda |M_{k,k}| + |M_{k+1,k}| \tag{3.8}$$

$$|M_{k-1,k}| = \begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-1} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-1} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-2,k+1} & a_{k-2,k+2} & \cdots & a_{k-2,n-1} & a_{k-2,n} \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-1} & a_{k,k+1} & a_{k,k+2} & \cdots & a_{k,n-1} & a_{k,n} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,k-1} & \lambda & a_{k+1,k+2} & \cdots & a_{k+1,n-1} & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix} = B$$

$$|\lambda|M_{k,k}| = \begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-1} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-1} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & a_{k-2,k-1} & a_{k-2,k+1} & a_{k-2,k+2} & \cdots & a_{k-2,n-1} & a_{k-2,n} \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & \lambda & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,k-1} & \lambda & a_{k+1,k+2} & \cdots & a_{k+1,n-1} & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix} = C$$

$$||M_{k+1,k}| = \begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-1} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-1} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & a_{k-2,k-1} & a_{k-2,k+1} & a_{k-2,k+2} & \cdots & a_{k-2,n-1} & a_{k-2,n} \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & \lambda & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-1} & \lambda & a_{k,k+2} & \cdots & a_{k,n-1} & a_{k,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix} = D$$

$$det(\lambda I - A_G) = M_{k-1,k} + \lambda M_{k,k} + M_{k+1,k} = B + C + D \tag{3.9}$$

Now we must work with B; C $andD$

Now we look at graph with edge removed: Now, if the edge e between the vertices k and $k+1$ are removed from the tree G , we have the disconnected tree $G-e$ as below:

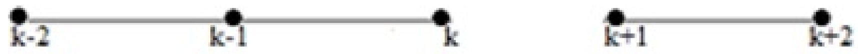

Figure 3.8: A tree G with edge e removed

The adjacency matrix A_{G-e} is:

$$A_{G-e} = \begin{bmatrix} 0 & a_{1,2} & \cdots & a_{1,k-1} & a_{1,k} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & 0 & \cdots & a_{2,k-1} & a_{2,k} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \cdots & a_{3,k-1} & a_{3,k} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & 0 & a_{k-1,k} & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} & 0 & 0 & a_{k,k+2} & \cdots & a_{k,n-1} & a_{k,n} \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k-1} & 0 & 0 & a_{k+1,k+2} & \cdots & a_{k+1,n-1} & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & 0 & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,k-1} & a_{n,k} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & 0 \end{bmatrix}$$

Then the characteristic matrix is:

$$(\lambda I - A_{G-e}) = \begin{bmatrix} \lambda & a_{1,2} & \cdots & a_{1,k-1} & a_{1,k} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & \cdots & a_{2,k-1} & a_{2,k} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \cdots & a_{3,k-1} & a_{3,k} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & \lambda & a_{k-1,k} & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} & \lambda & 0 & a_{k,k+2} & \cdots & a_{k,n-1} & a_{k,n} \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k-1} & 0 & \lambda & a_{k+1,k+2} & \cdots & a_{k+1,n-1} & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,k-1} & a_{n,k} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{bmatrix}$$

Now, the expansion of the determinant by its first k columns gives:

$$det(\lambda I - A_{G-e}) = (-1)^{2k-1} a_{k-1,k} M_{k-1,k} + \lambda M_{k,k} + (-1)^{2k+1} a_{k+1,k} M_{k+1,k}$$

Since

$$det(\lambda I - A_{G-e}) = (-1)^{1+k} a_{1,k} M_{1,k} + (-1)^{2+k} a_{2,k} M_{2,k} + \cdots +$$

$$(-1)^{2k-1}a_{k-1,k}M_{k-1,k} + \lambda M_{k,k} + (-1)^{2k+1}a_{k+1,k}M_{k+1,k} + \cdots$$
$$+(-1)^{n-1,k}a_{n-1,k}M_{n-1,k} + (-1)^{n+k}a_{n,k}M_{n,k}$$

Therefore,

$$det(\lambda I - A_{G-e}) = M_{k-1,k} + \lambda M_{k,k}$$

$$|M_{k-1,k}| = \begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-1} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-1} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-1} & 0 & a_{k,k+2} & \cdots & a_{k,n-1} & a_{k,n} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,k-1} & \lambda & a_{k+1,k+2} & \cdots & a_{k+1,n-1} & a_{k+1,n} \\ a_{k+2,1} & a_{k+2,2} & a_{k+2,3} & \cdots & a_{k+2,k-1} & a_{k+2,k+1} & \lambda & \cdots & a_{k+2,n-1} & a_{k+2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix} = B'$$

$$\lambda\,|M_{k,k}| = \lambda \begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-1} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-1} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & a_{k-2,k-1} & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & \lambda & 0 & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,k-1} & \lambda & a_{k+1,k+2} & \cdots & a_{k+1,n-1} & a_{k+1,n} \\ a_{k+2,1} & a_{k+2,2} & a_{k+2,3} & \cdots & a_{k+2,k-1} & a_{k+2,k+1} & \lambda & \cdots & a_{k+2,n-1} & a_{k+2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix} = C$$

Therefore,

$$det(\lambda I - A_{G-e}) = M_{k-1,k} + \lambda M_{k,k} = B' + C \tag{3.10}$$

where B' is similar to B Now, we must work with D to get: $det(\lambda I - A_{G-\{e\}})$

Now, removing the vertices k and $(k+1)$, we have the graph as below $G - \{e\}$

Figure 3.9: A tree G with vertices k and $k+1$ removed:$G - \{e\}$

$$det(\lambda I - A_{G-\{e\}}) = \begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-1} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-1} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & a_{k-2,k-1} & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & \lambda & 0 & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k+2,1} & a_{k+2,2} & a_{k+2,3} & \cdots & a_{k+2,k-1} & a_{k+2,k+1} & \lambda & \cdots & a_{k+2,n-1} & a_{k+2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix}$$

Now, we have :

$$\lambda\,|M_{k,k}| = \lambda \begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-1} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-1} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & a_{k-2,k-1} & a_{k-2,k+1} & a_{k-2,k+2} & \cdots & a_{k-2,n-1} & a_{k-2,n} \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & \lambda & 0 & a_{k,k+2} & \cdots & a_{k,n-1} & a_{k,n} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,k-1} & \lambda & a_{k+1,k+2} & \cdots & a_{k+1,n-1} & a_{k+1,n} \\ a_{k+2,1} & a_{k+2,2} & a_{k+2,3} & \cdots & a_{k+2,k-1} & a_{k+2,k+1} & \lambda & \cdots & a_{k+2,n-1} & a_{k+2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix} = C$$

Now, working with D,

$$|M_{k+1,k}| = \begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-1} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-1} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & a_{k-2,k-1} & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & \lambda & 0 & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-1} & \lambda & a_{k,k+2} & \cdots & a_{k,n-1} & a_{k+1,n} \\ a_{k+2,1} & a_{k+2,2} & a_{k+2,3} & \cdots & a_{k+2,k-1} & a_{k+2,k+1} & \lambda & \cdots & a_{k+2,n-1} & a_{k+2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix} = D$$

The expansion along $(k+1)th$ column will give:

$$D = (-1)^{2+k} a_{1,k+1} N_{1,k+1} + (-1)^{3+k} a_{2,k+1} N_{2,k+1} + (-1)^{4+k} a_{3,k+1} N_{3,k+1} + \cdots$$

$+$

$$(-1)^{2k-1} a_{k-2,k+1} N_{k-2,k+1} + (-1)^{2k} a_{k-1,k+1} N_{k-11,k+1} + (-1)^{2k+1} a_{k,k+1} N_{k,k+1} +$$

$$(-1)^{2k+3} a_{k+2,k+1} N_{k+2,k+1} + \cdots + (-1)^{n+k} a_{n-1,k+1} N_{n-1,k+1} + (-1)^{n+k+1} a_{n,k+1} N_{n,k+1}$$

Therefore,

$$D = -N_{k,k+1} + N_{k+2,k+1}$$

$$D = -\begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-2} & a_{1,k-1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-2} & a_{2,k-1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-2} & a_{3,k-1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-2,k-1} & a_{k-2,k+2} & \cdots & a_{k-2,n-1} & a_{k-1,n} \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,k-2} & \lambda & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k+2,1} & a_{k+2,2} & a_{k+2,3} & \cdots & a_{k+2,k-1} & a_{k+2,k+1} & \lambda & \cdots & a_{k+2,n-1} & a_{k+2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix} +$$

$$\begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-2} & a_{1,k-1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-2} & a_{2,k-1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-2} & a_{3,k-1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-2,k-1} & a_{k-2,k+2} & \cdots & a_{k-2,n-1} & a_{k-1,n} \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,k-2} & \lambda & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-1} & a_{k,k+1} & \lambda & \cdots & a_{k,n-1} & a_{k,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix}$$

The expansion along $(k+2)th$ column of the $2nd$ minor will give:

$$D = -\begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-2} & a_{1,k-1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-2} & a_{2,k-1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-2} & a_{3,k-1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-2,k-1} & a_{k-2,k+2} & \cdots & a_{k-2,n-1} & a_{k-1,n} \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,k-2} & \lambda & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k+2,1} & a_{k+2,2} & a_{k+2,3} & \cdots & a_{k+2,k-1} & a_{k+2,k+1} & \lambda & \cdots & a_{k+2,n-1} & a_{k+2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix} +$$

$$\begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-2} & a_{1,k-1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-2} & a_{2,k-1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-2} & a_{3,k-1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-2,k-1} & a_{k-2,k+2} & \cdots & a_{k-2,n-1} & a_{k-1,n} \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,k-2} & \lambda & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-1} & a_{k,k+1} & \lambda & \cdots & a_{k,n-1} & a_{k,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix} +$$

$$\begin{vmatrix}
\lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-2} & a_{1,k-1} & \cdots & a_{1,n-1} & a_{1,n} \\
a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-2} & a_{2,k-1} & \cdots & a_{2,n-1} & a_{2,n} \\
a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-2} & a_{3,k-1} & \cdots & a_{3,n-1} & a_{3,n} \\
\vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \cdots & \vdots & \vdots \\
a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-2,k-1} & \cdots & a_{k-2,n-1} & a_{k-1,n} \\
a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,k-2} & \lambda & \cdots & a_{k-1,n-1} & a_{k-1,n} \\
a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-2} & a_{k,k-1} & \cdots & a_{k,n-1} & a_{k,n} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-2} & a_{n-1,k-11} & \cdots & \lambda & a_{n-1,n} \\
a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-2} & a_{n,k-1} & \cdots & a_{n,n-1} & \lambda
\end{vmatrix}$$

$$D = -(\lambda I - A_{G-\{e\}}) + E + F$$

where $F = 0$

$$D = -\begin{vmatrix}
\lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-2} & a_{1,k-1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\
a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-2} & a_{2,k-1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\
a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-2} & a_{3,k-1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-2,k-1} & a_{k-2,k+2} & \cdots & a_{k-2,n-1} & a_{k-1,n} \\
a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,k-2} & \lambda & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\
a_{k+2,1} & a_{k+2,2} & a_{k+2,3} & \cdots & a_{k+2,k-1} & a_{k+2,k+1} & \lambda & \cdots & a_{k+2,n-1} & a_{k+2,n} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\
a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda
\end{vmatrix} +$$

$$\begin{vmatrix}
\lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-2} & a_{1,k-1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\
a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-2} & a_{2,k-1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\
a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-2} & a_{3,k-1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-2,k-1} & a_{k-2,k+2} & \cdots & a_{k-2,n-1} & a_{k-2,n} \\
a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,k-2} & \lambda & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\
a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-1} & a_{k,k+1} & \lambda & \cdots & a_{k,n-1} & a_{k,n} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\
a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda
\end{vmatrix}$$

Expansion along the $(n-1)th$ column of the 2^{nd} Minor will give:

$$D = -(\lambda I - A_{G-\{e\}}) + E$$

$$E = \begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-2} & a_{1,k-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-2} & a_{2,k-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-2} & a_{3,k-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-2,k-1} & a_{k-2,n} \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,k-2} & \lambda & a_{k-1,n} \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-2} & a_{k,k-1} & a_{k,n} \end{vmatrix} = 0$$

$$D = -\begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-2} & a_{1,k-1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-2} & a_{2,k-1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-2} & a_{3,k-1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-2,k-1} & a_{k-2,k+2} & \cdots & a_{k-2,n-1} & a_{k-1,n} \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,k-2} & \lambda & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k+2,1} & a_{k+2,2} & a_{k+2,3} & \cdots & a_{k+2,k-1} & a_{k+2,k+1} & \lambda & \cdots & a_{k+2,n-1} & a_{k+2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix} -$$

$$\begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-2} & a_{1,k-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-2} & a_{2,k-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-2} & a_{3,k-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-2,k-1} & a_{k-2,n} \\ a_{k-1,1} & a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,k-2} & \lambda & a_{k-1,n} \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-2} & a_{k,k-1} & a_{k,n} \end{vmatrix}$$

Therefore,

$$D = -(\lambda I - A_{G-\{e\}}) \tag{3.11}$$

Now for Minors B and B'

$$|M_{k-1,k}| = \begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-1} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-1} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-1} & a_{k,k+1} & a_{k,k+2} & \cdots & a_{k,n-1} & a_{k,n} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,k-1} & \lambda & a_{k+1,k+2} & \cdots & a_{k+1,n-1} & a_{k+1,n} \\ a_{k+2,1} & a_{k+2,2} & a_{k+2,3} & \cdots & a_{k+2,k-1} & a_{k+2,k+1} & \lambda & \cdots & a_{k+2,n-1} & a_{k+2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix}_{edge\{e\}remo}$$

and

$$|M_{k-1,k}| = \begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-1} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-1} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-1} & 0 & a_{k,k+2} & \cdots & a_{k,n-1} & a_{k,n} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,k-1} & \lambda & a_{k+1,k+2} & \cdots & a_{k+1,n-1} & a_{k+1,n} \\ a_{k+2,1} & a_{k+2,2} & a_{k+2,3} & \cdots & a_{k+2,k-1} & a_{k+2,k+1} & \lambda & \cdots & a_{k+2,n-1} & a_{k+2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix}_{edge\{e\}remo}$$

Using column operations to get rid of any non-zero entries in the last row between $k+2$ and $(n-1)$, the following steps are applicable.
B and B' differ in the $(k,k+1)^{th}$ entry after we have removed the k^{th} column from G and $G-e$ So that in $B, a_{k,k+1} \neq 0$ and in $B', a_{k,k+1} = 0$

$$|M_{k-1,k}| = \begin{vmatrix} \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & \lambda & a_{2,3} & \cdots & a_{2,k-1} & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & \lambda & \cdots & a_{3,k-1} & a_{3,k+1} & a_{3,k+2} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{k-2,1} & a_{k-2,2} & a_{k-2,3} & \cdots & \lambda & a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n-1} & a_{k-1,n} \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,k-1} & a_{k,k+1} & a_{k,k+2} & \cdots & a_{k,n-1} & a_{k,n} \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,k-1} & \lambda & a_{k+1,k+2} & \cdots & a_{k+1,n-1} & a_{k+1,n} \\ a_{k+2,1} & a_{k+2,2} & a_{k+2,3} & \cdots & a_{k+2,k-1} & a_{k+2,k+1} & \lambda & \cdots & a_{k+2,n-1} & a_{k+2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & a_{n-1,k+2} & \cdots & \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,k-1} & a_{n,k+1} & a_{n,k+2} & \cdots & a_{n,n-1} & \lambda \end{vmatrix} = B$$

Replacing column I with $I + a_j$ where j is column and it does not change the determinant. This operation is done on matrices B and B' below. The problem is the different entries in $k, k+1$ for B and B' - note that in the nth row all entries from column 1 to $k-1$ are 0 (because there are no adjacent vertices)

Case 1, $\lambda \neq 0$
There is a ($\lambda \neq 0$) in the n, n position of the matrices B and B'. Let b_j be a nonzero entry in one of the column positions $(k+2)$ to $(n-1)$ in the nth row, i.e. j runs from $(k+2)$ to $(n-1)$. These columns are identical in B and B'. Let this position be I_j which contains b_j. Replacing column I_j with column $I_j - \left(\frac{b_j}{\lambda}\right)$ will convert all non-zero entries from $(k+2)$ to $(n-1)$ to 0. This operation will not affect the determinants of matrices B and B' when expanding along the nth row.

Case 2: $\lambda = 0$
We have $\lambda = 0$ in the n, n position. Let b_j be a nonzero entry in one of the column positions $(k+2)$ to $(n-1)$ in the nth row, i.e. j runs from $(k+2)$ to $(n-1)$. These columns are identical in B and B'. Let this position be I_j which contains b_j. Swop the nth column with this jth column so that we have a 1 in the n, n position.

Case 2.1
Proceed as in *case* 1 if there are any other non-zero positions in columns $(k+2)$ to $(n-1)$ using this non-zero 1 in the n, n position. Following **cases 1 and 2** for B and B' will keep their determinants the same when expanding along nth row. Now proceed to row $n-1$ where there is a λ in the $(n-1, n-1)$ position. Convert relevant non-zero entries in this row to 0 in columns from $k-2$ column using cases 1 and 2. Expanding along this $n-1th$ row will keep the determinant of B and B' the same. Continue until row $k-1$. When we expand the along the kth row, the different entries in the $k, k+1$ entry for B and B' will not be relevant so that: $B = B'$

From above, we have:

$$B = B' \tag{3.12}$$

we may proceed to calculate. from above, we have: $det(\lambda I - A_G) = B + C + D \ \cdots$ (3.8)

$$det(\lambda I - A_{G-e}) = M_{k-1,k} + \lambda M_{k,k} = B' + C \ \cdots \ (3.9)$$

$$det(\lambda I - A_{G-\{e\}}) \ \cdots \ (3.10)$$

Therefore, since we have (3.12)

$$det(\lambda I - A_G) = det(\lambda I - A_{G-e}) - det(\lambda I - A_{G-\{e\}})$$

□

Theorem 3.4.2 (Lollipop Theorem)**.** *Let x_i be a vertex of degree one in the graph G on n vertices, and let x_j be the vertex adjacent to x_i. Let G_1 be the sub-graph* *obtained from G by deleting the vertex x_i, and let G_2 be the sub-graph obtained from G by deleting both vertices x_i and x_j . Then*

$$P_{A(G)}(\lambda) = \lambda P_{A(G_1)}(\lambda) - P_{A(G_2)}(\lambda)$$

See [8]

Proof. Without loss of generality, let $i < j$, so row i comes before row j in

$$P_{A(G)}(\lambda) = det(\lambda I - A(G))$$

Then we have,

$$(\lambda I - A(G))_{i,i} = \lambda.$$

$$(\lambda I - A(G))_{i,j} = -1.$$

$$(\lambda I - A(G))_{i,k} = 0 \ for \ 1 \leq k \leq n \ and \ k \neq i \ and \ k \neq j.$$

Expand the determinant of $(\lambda I - A(G))$ along the ith row, where there are only two non-zero entries as defined above. Then,

$$P_{A(G)}(\lambda) = (-1)^{i+j}(\lambda)M_{i,i} + (-1)^{i+j}(-1)M_{i,j}$$

now,

$$M_{i,i} = det(\lambda I - A_{(G)})$$

so,

$$P_{A(G)}(\lambda) = \lambda P_{A(G_1)}(\lambda) + (-1)^{i+j+1}M_{i,j}$$

Now expand $M_{i,j}$ along the ith column, which only has one non-zero entry of -1 in the $(j-1)th$ row as x_i has degree one and is only adjacent to x So,

$$P_{A(G)}(\lambda) = \lambda P_{A(G_1)}(\lambda) + (-1)^{i+j+1}(-1)^{i+j-1}(-1)P_{A(G_2)}(\lambda)$$

$$= \lambda P_{A(G_1)}(\lambda) - P_{A(G_2)}(\lambda)$$

□

3.5 Eigenvalues of complete graphs, bipartite graphs, star graphs, cycle graphs, path graphs

3.5.1 Eigenvalues of Complete Graphs

A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. For any complete graph K_n, the eigenvalues are $n-1$ (multiplicity 1) and -1(with multiplicity $n-1$). See Jessop [59].

Lemma 3.5.1. *Let J denote the $n \times n$ matrix of all 1s then the eigenvalues of J are n (with multiplicity 1) and 0 (with multiplicity $(n-1)$.*

Proof. Using the eigenvector $\underline{v}$ with only 1s as its entries we see that

$$J\underline{v} = n\underline{v}.$$

so that n is an eigenvalue of J . Since

$$tr(J) = n,$$

and the trace is the sum of the eigenvalues,

$$Trace = \sum^{n} \lambda i = n + 0$$

follows that the remaining $n-1$ eigenvalues of J must all equal to 0. □

Theorem 3.5.1. *Any complete graph K_n has one eigenvalue $(n-1)^1$ and eigenvalues of $(-1)^{n-1}$.*

Proof. Generally, the complete graph K_n has an adjacency matrix A which is given below as:

$$A = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}.$$

where $a_{i,j} = 1$ when $i \neq j$; $a_{ij} = 0$ when $i = j$
Let J be matrix with all 1s and I is the identity matrix. The rank of matrix J is 1(that is, there is one nonzero eigenvalue equal to n ; with eigenvector $\underline{1} = (1, 1, \cdots, 1)$ while the remaining eigenvalues are 0. Let $\underline{x}$ be the $n \times 1$ vector of $1's$ only and A the adjacency matrix of the complete graph. Then, subtracting the identity matrix shifts all eigenvalues by (-1) .
i.e.

$$\begin{aligned} A\underline{x} &= (J - I)\underline{x} \\ &= J\underline{x} - I\underline{x} \\ &= n\underline{x} - \underline{x} = (n-1)\underline{x}, \end{aligned}$$

we have

$$A\underline{x} = (n-1)\underline{x}.$$

Therefore $n-1$ is an eigenvalue of the adjacency matrix A.
Since the complete graph is regular of degree $n-1$ it has exactly one eigenvalue equal to $n-1$(see spectra of graphs by Haemers [9]).
Since the complete graph has diameter 2, it has 2 distinct eigenvalues (see [9]).
Let the second eigenvalue be p, of which there are $n-1$. Then, since $tr(A) = 0$, and A is the adjacency matrix of the complete graph (see [9]), we have:

$$(n-1) + p(n-1) = 0$$

Which implies $p = -1$.

$$\therefore A\underline{x} = 0\underline{x} - 1\underline{x} \Rightarrow A\underline{x} = -I\underline{x}$$

$$\therefore A = -1.$$

Therefore, the eigenvalues of the complete graph K_n, are $(n-1)^1$ and $(-1)^{n-1}$. $\square$

3.5.2 Eigenvalues of Bipartite Graphs

Example: Below is a bipartite graph, $K_{x,y}$, on 6 vertices.

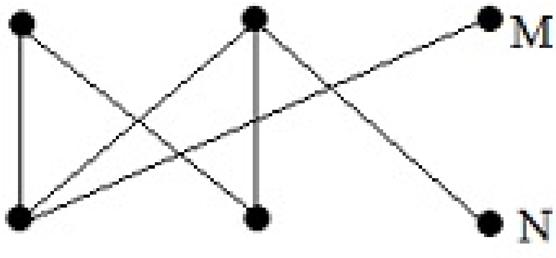

Figure 3.10: A bipartite graph on 6 vertices

Below is the complete bipartite graph $K_{3,3}$.

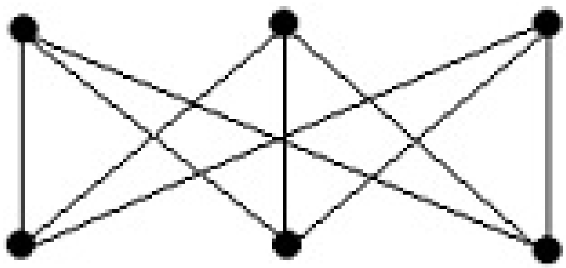

Figure 3.11: A bipartite graph on $K_{3,3}$

Theorem 3.5.2. *The adjacency matrix of a complete bipartite graph $K_{m,n}$ has eigenvalues: $(\sqrt{mn})^1$, $(-\sqrt{mn})^1$ and 0^{m+n-2}.*

Proof. The adjacency matrix of $K_{m,n}$ with order $(m+n)(m+n)$ is of the form:

$$A_{K_{m\times n}} = \begin{bmatrix} 0_{m\times m} & 1_{m\times n} \\ 1_{n\times m} & 0_{n\times n} \end{bmatrix}.$$

Now, let define matrices $B_{2\times 2}$ and $S_{(m+n)\times 2}$ to be:

$$B = \begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix}_{2\times 2}$$

and

$$S = \begin{bmatrix} \left(\frac{1}{m}\right)_{m\times 1} & 0_{m\times 1} \\ 0_{n\times 1} & \left(\frac{1}{n}\right)_{n\times 1} \end{bmatrix}_{(m+n)\times 2}$$

The idea that $m > 0$ and $n > 0$ allows us to construct S. We have:

$$AS = \begin{bmatrix} 0_{m\times m} & 1_{m\times n} \\ 1_{n\times m} & 0_{n\times n} \end{bmatrix} \begin{bmatrix} \left(\frac{1}{m}\right)_{m\times 1} & 0_{m\times 1} \\ 0_{n\times 1} & \left(\frac{1}{n}\right)_{n\times 1} \end{bmatrix}_{(m+n)\times 2} = \begin{bmatrix} 0_{m\times 1} & 1_{m\times 1} \\ 0_{n\times 1} & 0_{n\times 1} \end{bmatrix}_{(m+n)\times 2},$$

and

$$SB = \left[\begin{bmatrix} \left(\frac{1}{m}\right)_{m\times 1} & 0_{m\times 1} \\ 0_{n\times 1} & \left(\frac{1}{n}\right)_{n\times 1} \end{bmatrix}_{(m+n)\times 2} \right] \begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix} = \begin{bmatrix} 0_{m\times 1} & 1_{m\times 1} \\ 0_{n\times 1} & 0_{n\times 1} \end{bmatrix}_{(m+n)\times 2}.$$

From the above, it is obvious that: $AS = SB$. Here, if λ is an eigenvalue of B with eigenvector $\underline{v}$ then:

$$B\lambda = \lambda\underline{v}$$

$$AS = SB \Rightarrow AS\underline{v} = SB\underline{v} = S\lambda\underline{v} = \lambda S\underline{v}$$

This implies

$$AS\underline{v} = \lambda S\underline{v}.$$

So that λ is an eigenvalue of A with eigenvector $S\underline{v}$.

Now, matrix B has eigenvalues $\pm\sqrt{mn}$, which are the nonzero eigenvalues of $K_{m,n}$. But the rank of matrix A is 2, therefore the rest of the eigenvalues are 0. The method applied in the above proof is called the *equitable partition.* □

3.5.3 Eigenvalues of Star Graphs

We use induction and the Lovasz theorem to find the eigenvalues of star graphs.

Theorem 3.5.3. *The eigenvalues of a star graph with m rays of length 1 on n vertices are 0^{n-2} and $\pm\sqrt{n-1}$*

Proof. By induction: The theorem is true for a star graph with rays of length one on 2 and 3 vertices. See[59]
Assume:

$$Det\left(\lambda I - A_{S_{n-1}}\right) = \lambda^{n-3}\left[\lambda^2 - (n-2)\right]$$

where $\lambda^{n-3}\left[\lambda^2-(n-2)\right]$ is the characteristic polynomial of a star graph on $n-1$ vertices. The characteristic polynomial of the star of rays of length 1 on n vertices is:

$$(\lambda I - A_{S_n}) = \begin{bmatrix} \lambda & -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & \lambda & 0 & \cdots & 0 \\ -1 & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ -1 & 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

Using cofactor expansion along the second row, we have the determinant as:

$$det(\lambda I - A_{S_n}) = -1a_{21}C_{21} + a_{22}C_{22} - a_{23}C_{23} + a_{24}C_{24} - \cdots + (-1)^{i+j}a_{ij}C_{ij}$$

$$= -1C_{21} + \lambda C_{22} - 0C_{23} + 0C_{24} - \cdots + 0C_{ij}$$
$$= C_{21} + \lambda C_{22}$$

. Since a_{21} is negative.

$$= \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} + \lambda \begin{bmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda & 0 & \cdots & 0 \\ -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

It is obvious that C_{22} is the determinant of the matrix $(\lambda I - A_{S_{n-1}})$. Now let the matrix C_{21} be B. Then,

$$B = \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

Using cofactor expansion of B along the first column, we get:

$$detB = -1C_{11} = -1 \begin{vmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{vmatrix}.$$

But $C_{11} = det\lambda I_{n-2}$, where I_{n-2} is the $n-2$ identity matrix.
Therefore, $detB = -\lambda^{n-2}$
and

$$det(\lambda I - A_{S_n}) = -(-1)(-\lambda^{n-2}) + \lambda det(\lambda I - A_{S_{(n-1)}})$$

$$= \lambda det(\lambda I - A_{S_{(n-1)}}) - \lambda^{n-2}.$$

Using the inductive hypothesis we have:

$$\begin{aligned} det(\lambda I - A_{S_n}) &= \lambda\left(\lambda^{n-3}\left(\lambda^2 - (n-2)\right) - \lambda^{n-2}\right) \\ &= \lambda^{n-2}\left(\lambda^2 - (n-2))\right) - \lambda^{n-2} \\ &= \lambda^{n-2}\left(\lambda^2 - (n-2) - \right) \\ &= \lambda^{n-2}\left(\lambda^2 - (n-2)\right) \end{aligned}$$

Now, let

$$= det\left(\lambda I - A_{S_n}\right) = 0$$

This implies,

$$\lambda^{n-2}\left[\lambda^2 - (n-1)\right] = 0.$$

Thus, either $\lambda^{n-2} = 0$ or $[\lambda^2 - (n-1)] = 0$. Therefore, $\lambda = 0^{n-2}$ or

$$\lambda^2 = (n-1) \Rightarrow \lambda = \pm\sqrt{(n-1)}.$$

□

Thus by induction the eigenvalues, λ, of a star graph of rays of length 1 on n vertices are: 0^{n-2} and $\pm\sqrt{(n-1)}$.

Theorem 3.5.4. *For a star graphs $S_{m,2}$ with m rays of length 2 with $n = 2m+1$, the eigenvalues of $S_{m,2}$ are: 1 and -1, each of multiplicity $m-1 = \frac{n-3}{2}$, one eigenvalue 0, and two eigenvalues*

$$\lambda = \pm\sqrt{m+1} = \pm\sqrt{\frac{n+1}{2}},$$

Proof. Now, considering a star graph S_n of $m+1$ rays of length 2 where $n' = 2(m+1)+1$ and $e = uv$ where u is the center adjacent to v , we must show that:

$$det(\lambda I - A_{S_{n'}}) = \lambda(\lambda^2 - 1)^m[\lambda^2 - (m+2)]$$

Now, considering a star graph $S^{n'}$ of $(m+1)$ rays of length 2 where $n' = 2(m+1)+1$ and $e = uv$ where u centre adjacent to v. Given:

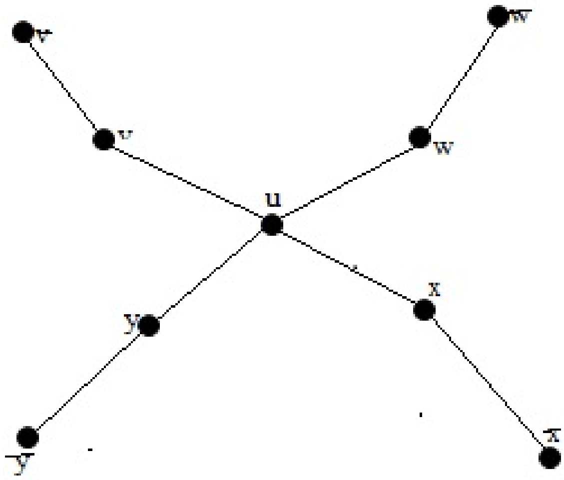

Figure 3.12: A star graph with 4 rays

Let find the eigenvalues as follows.
for $n' = 2(m+1)+1$ is of 4 rays of length 2
$n' = 2(m+1)+1$ vertices.
we must show:

$$det(\lambda I - A_{S_n}) = \lambda(\lambda^2 - 1)^m[\lambda^2 - (m+2)]$$

By removing the edge $e = uv$ from S_n , i.e from the star with $m+1$ rays of length 2

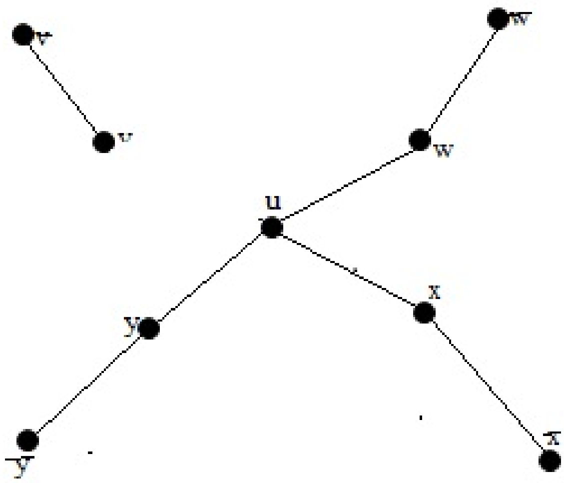

Figure 3.13: A disconnected star graph

$$G - e = S_n \cup K_2$$

Now removing the vertices $u, v \in V$ from S_n we obtain

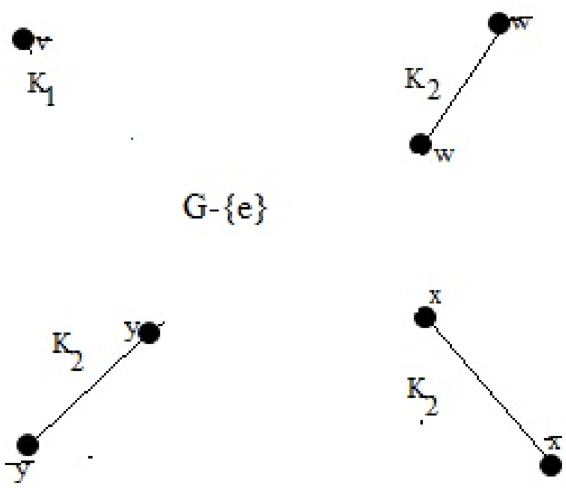

Figure 3.14: A star graph with central vertex removed

For the star with $m + 1$ rays of length 2, we have:

$$G - \{e\} = (K_2)^m \cup (K_1)$$

and ,

$$det(\lambda I - A_{S_n}) = \lambda(\lambda^2 - 1)^{m-1}[\lambda^2 - (m+1)].$$

$K_1 = \lambda$
Therefore,

$$det(\lambda I - A_{S_n}) = det(\lambda I - A_{G-e}) - det(\lambda I - A_{G-\{e\}})$$

(Lovasz Theorem).

$$= (\lambda I - A_{S_n}) \cup (\lambda I - A_{K_2}) - (\lambda I - A_{K_2})^m \cup (\lambda I - A_{K_1})$$
$$= det\,(\lambda I - A_{S_n})\,det\,(\lambda I - A_{K_2}) - [det\,(\lambda I - A_{K_2})]^m\,\lambda$$
$$= \lambda\left(\lambda^2 - 1\right)^{m-1}\left[\lambda^2 - (m+1)\right]\left(\lambda^2 - 1\right) - \lambda\left(\lambda^2 - 1\right)^m$$
$$= \lambda\left(\lambda^2 - 1\right)^{m-1+1}\left[\lambda^2 - (m+1)\right] - \lambda\left(\lambda^2 - 1\right)^m$$
$$= \lambda\left(\lambda^2 - 1\right)^m\left[\lambda^2 - (m+1)\right] - \lambda\left(\lambda^2 - 1\right)^m$$
$$= \lambda\left(\lambda^2 - 1\right)^m\left[\lambda^2 - (m+1)) - 1\right]$$
$$det\,(\lambda I - A_{S_n}) = \lambda\left(\lambda^2 - 1\right)^m\left[\lambda^2 - (m+2)\right]$$

Thus, the eigenvalues are:

$$\lambda = 0,\ \lambda = \pm(\sqrt{1})^m,\ \lambda = \pm(\sqrt{m+2})^1,$$

So that the theorem is proved by induction. □

3.5.4 Eigenvalues of The Path Graph

We need the following theorem and lemma:

Theorem 3.5.5 (Gershgorins Theorem)**.** *This theorem states that every eigenvalue λ of matrix A_{nn} satisfies:[21]*

$$|\lambda - a_{ii}| \leq \sum_{j \neq i}^{n} |a_{ij}| \ \ for \ i = \{1, 2, 3, \cdots, n\}.$$

Proof. Suppose that λ is an eigenvalue of the matrix A. The matrix $|\lambda I - A|$ is Strictly Diagonally Dominant (SDD) if:

$$|\lambda - a_{ii}| \leq \sum_{j \neq i}^{n} |a_{ij}| \ \ for \ i = \{1, 2, 3, \cdots, \}$$

If the Gershgorin's Theorem is not satisfied, then $|\lambda I - A|$ is SDD.
If $|\lambda I - A|$ is SDD then, it is non-singular and as a result,λ is not an eigenvalue.
If λ is to be an eigenvalue, then:

$$|\lambda - a_{ii}| \leq \sum_{j \neq i}^{n} |a_{ij}| \ \ for \ i = \{1, 2, 3, \cdots, n\}$$

□

Lemma 3.5.2. *Suppose α and β are the roots of $av_{i+1} - \lambda v_i + bv_{i-1} = 0. for \ a, b \neq 0$. Then α^i , β^i are solutions of $av_{i+1} - \lambda v_i + bv_{i-1} = 0$.*

Proof. Let α be a solution to the difference equation:$av_{i+1} - \lambda v_i + bv_{i-1} = 0$. Thus, if $v_i = \alpha^i$, we have

$$a\alpha^{1+1} - \lambda\alpha^1 + b\alpha^{1-1} = a\alpha^2 - \lambda\alpha + b = 0.$$

let $v_i = \alpha^i$, and also let a, b be nonzero constant. Then the difference equation will be:

$$av_{i+1} - \lambda v_i + bv_{i-1} = 0, \ a\alpha^{i+1} - \lambda\alpha + b\alpha^(i - 1).$$

By factorization, we have:

$$a\alpha^{i+1} - \lambda\alpha^i + b\alpha^{i-1}$$

Taking out α^{i-1} one is left with the quadratic as a factor. That is:

$$a\alpha^{i+1} - \lambda\alpha^i + b\alpha^{i-1} = \alpha^{i-1}\left(a\alpha^2 - \lambda\alpha + b\right) = 0;$$

$$\Rightarrow a\alpha^2 - \lambda\alpha + b = 0.$$

Similarly by putting $v_i = \beta^i$ we get the desired result as

$$\beta^{i-1}\left(a\beta^2 - \lambda\beta + b\right) = \beta a^{i-1}.0 = 0.$$

Assuming that β^i is also a solution of the equation

$$v_{i+1} - \lambda v_i + v_{i-1} = 0.$$

From the above, we have established that:
$v_i = \alpha^i$ and $v_i = \beta^i$ are solutions to the difference equation

$$v_{i+1} - \lambda v_i + v_{i-1} = 0.$$

Suppose $v_i = a\alpha^i + b\beta^i$ where a and b are constants, then,

$$av_{i+1} - \lambda v_i + bv_{i-1} = a(\alpha^{i+1} + \beta^{i+1}) - \lambda(\alpha^i + \beta^i) + b(\alpha^{i-1} + \beta^{i-j}).$$

$$= a\alpha^{i+1} + \lambda\alpha^i - b\alpha^{i-1} + a\beta^{i+1} + \lambda\beta^i - b\beta^{i-1}$$

$$= \alpha^i \left(a\alpha^2 + \lambda\alpha - b\right) + \beta^i \left(a\beta^2 + \lambda\beta - b\right)$$

$= \alpha^i.0 + \beta^i.0 = 0$ provided $a \neq 0$.
This implies that: $v_i = a\alpha^i + b\beta^i$ is a solution to the difference equation:

$$v_{i+1} - \lambda v_i + v_{i-1} = 0.$$

□

Theorem 3.5.6. *Let A_n be the adjacency matrix of the path graph P_n. The eigenvalues of A_n are: $2cos\frac{\pi j}{n+1}$ for $j = 1, 2, 3, \cdots, n$.*

Proof. The adjacency matrix A_n has the form:

$$A_n = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

We know that λ is an eigenvalue of A_n if and only if there exists a nonzero vector $\underline{v} = (v_1, v_2, v_3, \cdots, v_n)^t$ such that $A_n\underline{v} = \lambda\underline{v}$. Writing the condition $A_n\underline{v} = \lambda\underline{v}$ in coordinates, we obtain the system of equations:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n-3} \\ v_{n-2} \\ v_{n-1} \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n-3} \\ v_{n-2} \\ v_{n-1} \\ v_n \end{bmatrix}.$$

This implies that:

$$\begin{Bmatrix} v_2 = \lambda v_1 \\ v_1 + v_3 = \lambda v_2 \\ v_2 + v_4 = \lambda v_3 \\ \vdots \\ v_{n-1} = \lambda v_n \end{Bmatrix}.$$

From the above system of equation, it is seen that:$v_2 = \lambda v_1$.
This implies that: $v_n = \lambda v_{n-1}$
Also, $v_1 + v_3 = \lambda v_2 \Rightarrow v_3 = \lambda v_2 - v_1 \Rightarrow v_n = \lambda v_{n-1} - v_{n-2}$

and $v_2 + v_4 = \lambda v_3 \Rightarrow v_4 = \lambda v_3 - v_2 \Rightarrow v_n = \lambda v_{n-1} - v_{n-2}$

Therefore, $v_{n-1} = \lambda v_n \Rightarrow v_n = \lambda v_{n+1}$.
We have the initial conditions $v_0 = 0 = v_{n+1}$ since v_0 and v_{n+1} are respectively before and after the start vertex and the end vertex of the path P_n.

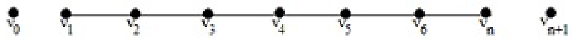

Figure 3.15: A path graph of n vertices

Then, the system of equation will become linear recurrence:
$v_{i+1} + v_{i-1} = \lambda v_i,\ 1 \leq i \leq n$
The system of equation implies that:
$v_{i+1} - \lambda v_i + v_{i-1} = 0$
(Homogeneous difference equation).
Now, given that E is an $n \times n$ matrix this will give: v_i is replaced with E. This is because, v_i is a vector (row or column matrix) while E is an $n \times n$ matrix where $n \neq 0$.

$$E^{i+1} - \lambda E^i + E^{i-1} = 0$$

$$E^i E - \lambda E^i + E^i E^{-1} = 0$$

$$E^i \left(E - \lambda + E^{-1}\right) = 0$$

Multiplying the terms in the bracket by E ,we have:

$$E^i \left(E^2 - \lambda E + 1\right) = 0$$

Hence, the linear recurrence can also be written as:

$$\left(E^2 - \lambda E + 1\right) v = 0.$$

its solution has the form $v_i = a\alpha^i + b\beta^i$ (see lemma 3.6.1 above) where a and b are constants). (unless $\alpha = \beta$ to make the equation linear), where α, β are the solutions of the characteristic equation:

$$E^2 - \lambda E + 1 = 0$$

which has α, β as a non-trivial solution of the characteristic equation:

$$x^2 - \lambda x + 1 = 0$$

In particular, $\alpha\beta = 1$ (roots of quadratic equation).
From the initial data $v_0 = 0 = v_{n+1}$, we deduce $\alpha^{n+1} = \beta^{n+1}$ as below.
The general form of second order difference equation is:

$$ay_{k+2} + by_{k+1} + cy_k = 0.$$

Which has a characteristic equation of:

$$ax^2 + bx + c = 0.$$

From above, we have the difference equation

$$v_{i+1} - \lambda v_i + v_{i-1} = 0$$

Initial value condition:

$$v_0 = 0 = v_{n+1}$$

For $i = 0$ implies

$$v_0 = a\alpha^0 + b\beta^0 \Rightarrow 0 = a + b,$$

from which $a = -b$. For,

$$\begin{aligned} 0 = v_{n+1} &= a\alpha^{n+1} + b\beta^{n+1} \\ &= -b\alpha^{n+1} + b\beta^{n+1} \\ &= b\left[-\alpha^{n+1} + \beta^{n+1}\right] \end{aligned}$$

For non-trivial solution,

$$b \neq 0 \Rightarrow -\alpha^{n+1} + \beta^{n+1} = 0 \Rightarrow \alpha^{n+1} = \beta^{n+1}$$

But,

$$\begin{aligned} \beta = \alpha^{-1} &\Rightarrow \alpha^{n+1} = \frac{1}{\alpha^{n+1}} \\ &\Rightarrow \alpha^{(2n+2)} = 1. \end{aligned}$$

This, along with the equation $\alpha\beta = 1$ (the product of the roots of quadratic equation) gives

$$\begin{Bmatrix} \alpha^{2n+2} & = 1 \\ \beta & = \frac{1}{\alpha} \end{Bmatrix}$$

That is, $\alpha = \sqrt[2n+2]{1}$
Hence α is some $(2n+2)$ root of unity.
By De Moivres Theorem (Borges 1998) [7]

$$\alpha = cis\left(\frac{0 + 2\pi j}{2n+2}\right); \; j = 0, 1, \cdots, 2n+1.$$

this implies that,

$$\alpha = cis\left(\frac{\pi j}{n+1}\right);\ j = 0, 1, 2, \cdots, (2n+1).$$

Therefore,

$$\beta = \frac{1}{\alpha} \Rightarrow \beta = \overline{\alpha}.$$

But $\lambda = \alpha + \beta$ (sum of the roots of quadratic equation). $= \alpha + \overline{\alpha} = 2Re(\alpha)$

$$= 2cos\left(\frac{\pi j}{n+1}\right),\ j = 0, 1, \cdots, (2n+1).$$

Consequently,

$$\lambda = \alpha + \beta = 2Re(\alpha) = 2cos\left(\frac{\pi j}{n+1}\right),\ j = 0, 1, 2, \cdots, (2n+1).$$

By periodicity of cosine and its evenness, we have
$cos(-x) = cos\ x$ and $cos\left(2\pi - \frac{\pi j}{n+1}\right) = cos\left(\frac{\pi(j)}{n+1}\right)$. Moreover,

$$cos\left(\frac{\pi j}{n+1}\right) = cos(\pi)\left(\frac{2n+2}{n+1} - \frac{j}{n+1}\right).$$

$$= cos(\pi)\left(\frac{2n+2-j}{n+1}\right),\ j = 0, 1, 2, \cdots, (2n+1).$$

Thus,

$$2cos\left(\frac{\pi j}{n+1}\right) = 2cos\left(\frac{\pi(2n+2-j)}{n+1}\right).$$

We need only consider the possibilities $j = 0, 1, 2, \cdots, (n+1)$.
This is because, if we go beyond $n+1$ then

$$\frac{2n+2-j}{n+1} \leq \frac{n}{n+1}.$$

That is, for $j = n+2$, we have

$$\frac{2n+2-j}{n+1} = \frac{2n+2-n-2}{n+1} = \frac{n}{n+1},$$

for $j = n+3$, we have

$$\frac{2n+2-j}{n+1} = \frac{2n+2-n-3}{n+1} = \frac{n-1}{n+1},$$

and so on.
Three (3) cases to be considered. (1) $j = 0$; (2) $j = n+1$; (3) all the other js.

CASE 1. if $j = 0$,

$$\lambda = 2cos\pi\left(\frac{2n+2-j}{n+1}\right),$$
$$= 2cos2\pi = 2.$$

Similarly, we can show that j can be $n+1$ as follows.

CASE 2 if $j = n+1$,

$$\lambda = 2cos\pi\left(\frac{2n+2-n-1}{n+1}\right) = 2cos\pi = -2.$$

To show that the eigenvalues of a path graph can't be two (2) and negative two (-2) we use theorem 3.5.5 above:
We had established that:

$$\lambda = cos\pi\left(\frac{2n+2-j}{n+1}\right), \; j = 0, 1, \cdots, (n+1)$$

Now,

$$(A_{P_n} - \lambda) = \begin{bmatrix} -\lambda & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -\lambda & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\lambda & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -\lambda & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda \end{bmatrix}.$$

Let $\lambda = 2$. Then, $Row1 : |-2| \leq |1| + |0| + |0| + |0| + \cdots + |0| + |0| + |0|$ not true
.
Also,let $\lambda = -2$ then, $Row1 : |2| \leq |1| + |0| + |0| + |0| + \cdots + |0| + |0| + |0|$ not true.

An $n \times n$ Matrix has exactly n eigenvalues, so we conclude that:

$$\lambda = 2cos\left(\frac{\pi j}{n+1}\right), \; j = 1, 2, \cdots, n$$

are indeed the eigenvalues of the adjacency matrix of the Path graph. □

3.5.5 Eigenvalues of The Cycle Graph

Theorem 3.5.7. *Let A_n be the adjacency matrix of the cycle graph C_n. The eigenvalues of A_{C_n} are:*

$$cos\left(\frac{2\pi k}{n}\right); \; for\; k = 0, 1, 2, 3, \cdots, (n-1).$$

Proof. The adjacency matrix A_n has the form:

$$A_{C_n} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Writing the condition $A_n\underline{v} = \lambda\underline{v}$ in coordinates, we obtain the system of equations:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix},$$

$$\Leftrightarrow \left\{ \begin{array}{ll} v_2 + v_n & = \lambda v_1 \\ v_1 + v_3 & = \lambda v_2 \\ v_2 + v_4 & = \lambda v_3 \\ \vdots & \\ v_1 + v_{n-1} & = \lambda v_n \end{array} \right\}$$

From the above system of equation, it is seen that:

$$v_2 + v_n = \lambda v_1 \Rightarrow v_n = \lambda v_1 - v_2,$$

Also,

$$v_1 + v_3 = \lambda v_2 \Rightarrow v_3 = \lambda v_2 - v_1,$$

$$v_2 + v_4 = \lambda v_3 \Rightarrow v_4 = \lambda v_3 - v_2.$$

We deduced that,

$$v_i = \lambda v_{i-1} - v_{i-2};\ 1 \leq i \leq n \tag{3.13}$$

$$v_1 + v_{n-1} = \lambda v_n \tag{3.14}$$

We have the initial conditions

$$v_0 = v_n \ and \ v_{n+1} = v_1$$

Since the start vertex and the end vertex are the same the system of equations becomes the linear recurrence.

$$v_i = \lambda v_{i-1} - v_{i-2};\ 1 \leq i \leq n\ \cdots, (3.13).$$

Equation (3.13) is a second order linear homogeneous difference equation (Homogeneous difference equation, [Mallik 2000]). □

Theorem 3.5.8. *The second order linear homogeneous difference equation:*

$$x_{k+2} + ax_{k+1} + bx_k = 0$$

has $x_k = x_0^k$ as a non-trivial solution if and only if x_0 is a solution of the characteristic equation: $x^2 + ax + b = 0$.

Proof. Considering the equation (3.13) above, we have:

$$v_i = \lambda v_{i-1} - v_{i-2};\ 1 \leq i \leq n.$$

On substituting $v_i = x^i$ as a solution, we find that the resulting characteristic equation will be

$$v_i - \lambda v_{i-1} + v_{i-2} = 0 \Rightarrow x^i - \lambda x^{i-1} + x^{i-2} = 0,$$

$$\Rightarrow x^i(1 - \lambda x^{-1} + x^{-2}) = 0.$$

This implies either $x^i = 0$ or $1 - \lambda x^{-1} + x^{-2} = 0$.
Multiply the second equation by x^2

$\Rightarrow x^2 - \lambda x + 1 = 0$. This is only true if $x^i \neq 0$. Since $x^i \in V$ and $0 \notin V$, its solution has the form $x_i = a\alpha^i + b\beta^i$ (unless $\alpha = \beta$), where a and b are arbitrary constants, α and β are the solutions of the equation: $x^2 - \lambda x + 1 = 0$.
In particular, $\alpha\beta = 1,\ \ \alpha + \beta = \lambda$.
From the initial data, $x_0 = x_n$ and $x_{n+1} = x_1,\ \ \alpha^n = 1$ and $\alpha^{n+1} = \alpha^n.\alpha^1 = \alpha$.
The second one implies $\alpha^n = 1$.
This, along with the equation $\alpha\beta = 1$ gives us: $\begin{Bmatrix} \alpha^n & = 1 \\ \beta & = \alpha^{-1} \end{Bmatrix}$.
Hence, α is some nth root of unity (i.e $\alpha = \sqrt[n]{1}$) and $(n + 1)th$ root of unity $(\alpha = \sqrt[n+1]{1})$.
Consequently

$$\lambda = \alpha + \beta = \alpha + \alpha^{-1} = 2Re(\alpha),$$

where $\lambda = \alpha + \beta$ (sum of the complex roots of a quadratic equation).
However, $\alpha = (1)^{\frac{1}{n}}$

$$= \left[cos\left(\frac{2k\pi}{n}\right) + isin\left(\frac{2k\pi}{n}\right)\right],$$

$$Re(\alpha) = cos\left(\frac{2k\pi}{n}\right).$$

Therefore,

$$\lambda = 2Re(\alpha) = 2cos\left(\frac{2k\pi}{n}\right),\ k = 1, 2, \cdots, (n-1).$$

□

3.5.6 Eigenvalues of The Wheel Graph

Let W_n be wheel graph on n vertices with $(n-1)$ spokes and with the central vertex labelled v_1 and the outer vertices $v_2, v_3, \cdots, v_n$. The adjacency matrix of W_n is

$$A_{W_n} = \overline{K_1} \bigoplus C_{n-1} = \begin{bmatrix} A & J_{1,n-1} \\ (J_{1,n-1})^T & B \end{bmatrix},$$

where A is 1×1 matrix of a single vertex, so $m(row) = 1$ and B is an $(n-1) \times (n-1)$ adjacency matrix of a cycle on (n-1) vertices.

Theorem 3.5.9. *Let*

$$U_k^T = \left[I\rho_{m,k}^1 \; I\rho_{m,k}^2 \cdots, I\rho_{m,k}^{(m-1)} \right],$$

and

$$V_j^T = \left[I\rho_{n,j}^1 \; I\rho_{n,j}^2 \cdots, I\rho_{n,j}^{(n-1)} \right],$$

where $\rho_{m,k} = e^{\frac{2\pi k}{m}}$ *for* $1 \leq k \leq (m-1)$ *and* $\rho_{n,j} = e^{\frac{2\pi j}{n}}$ *for* $1 \leq j \leq (n-1)$. *Let square matrices* $A = [a_1, a_2, \cdots, a_m]$ *and* $B = [b_1, b_2, \cdots, b_n]$ *be two circulant matrices. Then,*

$$1. \; A \oplus B = \begin{bmatrix} A & J_{m,n} \\ J_{n,m} & B \end{bmatrix}$$

$$2. \; CSET(A \oplus B) = \{W_1, W_2, \cdots, W_{m+n}\}$$

= set of vectors of $A \oplus B$
where $W_k^T = \left[0_{1,m}, V_k^T \right]$ *if* $1 \leq k \leq (m-1)$,
and

$$\left\{ W_n^T, W_{n+m}^T \right\} = \left\{ [J_{1,m}, \alpha J_{1,n}] \,|n\alpha^2 + \alpha(d_A - d_B) - m = 0 \right\},$$

where $d_A = a_1 + a_2 + \cdots + a_m$.
and $d_B = b_1 + b_2 + \cdots + b_n$.

3. The eigenvalues λ_k *of* $A \oplus B$ *are given by:*

$$\lambda_k = b_1 + b_2\rho_n^k + b_3\rho_n^{2k} + \cdots + b_n\rho_n^{(n-1)k},$$

for $1 \leq k \leq (n-1)$,

$$\lambda_{k+n} = a_1 + a_2\rho_m^k + a_3\rho_m^{2k} + \cdots + a_m\rho_m^{(m-1)k},$$

for $1 \leq k \leq (m-1)$,
and

$$\{\lambda_n, \lambda_{n+m}\} = \left\{ n\alpha + d_A | n\alpha^2 + \alpha(d_A - d_B) - m = 0 \right\}.$$

See Gross and Yellen [41].

Proof. To show that $W_k^T = \left[0_{1,m}, V_k^T\right]$ is not an eigenvector for $k = 0$ we let

$$C = A \oplus B = \begin{bmatrix} A & J_{m,n} \\ J_{n,m} & B \end{bmatrix}$$

so that,

$$CW_0^T = \begin{bmatrix} A & J_{m,n} \\ J_{n,m} & B \end{bmatrix} \left[0_{1,m}, V_0^T\right] = \begin{bmatrix} A & J_{m,n} \\ J_{n,m} & B \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} m \\ m \\ . \\ d_b \\ d_b \\ . \\ d_b \end{bmatrix} = \lambda \left[0_{1,m}, V_0^T\right] = \lambda \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix},$$

which means $m = 0$ is impossible.
To show that

$$W_j^T = \left[0_{1,m}, V_j^T\right] = \left[000 \cdots 001 \rho_{n,j}^1 \; \rho_{n,j}^2 \cdots \; \rho_{n,j}^{(n-1)}\right]^T ; \; 1 \leq j \leq (n-1)$$

is an eigenvector of C consider:

$$CW_j^T = \begin{bmatrix} A & J_{m,n} \\ J_{n,m} & B \end{bmatrix} \left[000 \cdots 001 \rho_{n,j}^1 \; \rho_{n,j}^2 \cdots \; \rho_{n,j}^{(n-1)}\right]^T .$$

The first m rows look like:

$$\sum_{k=0}^{n-1} \rho_{n,j}^k \; \sum_{k=0}^{n-1} \rho_{n,j}^k cdots \cdots \sum_{k=0}^{n-1} \rho_{n,j}^k .$$

Then, from Theorem 3.2

$$\sum_{k=0}^{n-1} \rho_{n,j}^k = 0,$$

so the first m rows of CW_j^T are 0.
The next n rows look like:

$$b_1 + b_2 \rho_{n,j}^1 + b_3 \rho_{n,j}^2 + \cdots + b_n \rho_{n,j}^{(n-1)},$$

for $1 \leq j \leq (n-1)$ which is from the eigenvalue corresponding to eigenvector

$$\left[1, \rho_{n,j}^1, \rho_{n,j}^2, \cdots, \rho_{n,j}^{(n-1)}\right], \; for \; 1 \leq j \leq (n-1).$$

Thus we have proved that:

$$W_j^T = \left[0_{1,m}, V_j^T\right] = \left[000 \cdots 001 \rho_{n,j}^1 \; \rho_{n,j}^2 \cdots \; \rho_{n,j}^{(n-1)}\right]^T ; \; 1 \leq j \leq (n-1)$$

is an eigenvector of $C = A \oplus B$.
Applying the same method, we can show that

$$W_{n+j}^T = \left[U_j^T, 0_{1,n}\right] \ for\ 1 \leq j \leq (m-1),$$

are eigenvectors of $C = A \oplus B$.

To determine the first set of eigenvalues of $A \oplus B$ we set $\underline{v} = \left(0_{1,m}, \underline{x}^T\right)^T$ where $\underline{x}^T = (x_1, x_2, \cdots, x_n)$ and solve for $(A \oplus B)\,\underline{v} = \lambda \underline{v}$. We specifically select $\underline{v}$ to be of this form as we understand the join of sub-graphs A and B, and this vector isolates the edges in B.

Solving $(A \oplus B)\,\underline{v} = \lambda \underline{v}$ we get

$$\begin{bmatrix} A_{m,m} & J_{m,n} \\ J_{n,m} & B_{n,n} \end{bmatrix} \begin{bmatrix} 0_{m,1} \\ x_{n,1} \end{bmatrix} = \lambda \begin{bmatrix} 0_{m,1} \\ x_{n,1} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} J_{m,m} & \underline{x}_{n,1} \\ B_{n,m} & \underline{x}_{n,1} \end{bmatrix} = \lambda \begin{bmatrix} 0_{n,1} \\ \underline{x}_{n,1} \end{bmatrix}$$

Solving $B\underline{x} = \lambda \underline{x}$ we obtain the eigenvalues of B which are, as per Theorem 2.2.1(see [59])

$$\lambda_k = b_1 + b_2\rho_j + b_3\rho_j^2 + \cdots + b_m\rho_j^{(n-1)},$$

for $1 \leq k \leq (n-1)$.

To determine the next set of eigenvalues of $A \oplus B$ we set $\underline{v} = \left(\underline{x}^T, 0_{1,n}\right)^T$ where $\underline{x}^T = (x_1, x_2, \cdots, x_m)$ and solve for $(A \oplus B)\,\underline{v} = \lambda \underline{v}$. We specifically select $\underline{v}$ to be of this form as we understand the join of sub-graphs A and B, and this vector isolates the edges in A.

Solving $(A \oplus B)\,\underline{v} = \lambda \underline{v}$ we get

$$\begin{bmatrix} A_{m,m} & J_{m,n} \\ J_{n,m} & B_{n,n} \end{bmatrix} \begin{bmatrix} 0_{m,1} \\ x_{n,1} \end{bmatrix} = \lambda \begin{bmatrix} 0_{m,1} \\ x_{n,1} \end{bmatrix}$$

$$\begin{bmatrix} A_{m,m} & \underline{x}_{m,1} \\ J_{n,m} & \underline{x}_{m,1} \end{bmatrix} = \lambda \begin{bmatrix} \underline{x}_{m,1} \\ 0_{n,1} \end{bmatrix}$$

Solving $A\underline{x} = \lambda \underline{x}$ we obtain the eigenvalues of A

$$\lambda_{n+k} = a_1 + a_2\rho_j + a_3\rho_j^2 + \cdots + a_m\rho_j^{(m-1)},$$

for $1 \leq k \leq (m-1)$.

To find the eigenvalues λ_n and λ_{n+m} of $(A \oplus B)$ we solve: $(A \oplus B)\,\underline{v} = \lambda \underline{v}$, where $\underline{v} = (J_{1,m}, \alpha J_{1,n})^T$.

The edges between the two graphs A and B, which form the join between the sub-graphs, are significant in the determination of the conjugate eigen-pair of the adjacency matrix of the resultant graph. We used the factor of α in the vector $\underline{v}$ to assist in obtaining the conjugate eigenvalues as follows:

$$\begin{bmatrix} A_{m,m} & J_{m,n} \\ J_{n,m} & A_{n,n} \end{bmatrix} \begin{bmatrix} J_{n,m} \\ B_{n,m} \end{bmatrix} = \lambda \begin{bmatrix} J_{m,1} \\ \alpha_{n,1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (d_A + n\alpha)_{m,1} \\ (m + \alpha d_B)_{n,1} \end{bmatrix} = \lambda \begin{bmatrix} J_{m,1} \\ \alpha J_{n,1} \end{bmatrix},$$

$$d_A + n\alpha = \lambda$$

$$m + \alpha d_B = \lambda\alpha.$$

Now, from theorem 2.1.1 and Corollary 2.1.1 of Jessop's thesis(see [59]), the wheel graph W_n has eigenvalues:

$$\lambda_k = b_1 + b_2\rho_n^k + b_3\rho_n^{2k} + b_4\rho_n^{3k} + \cdots + b_n\rho_n^{(n-1)k},$$

for $1 \leq k \leq (n-2)$,

$$\lambda_k = \rho_n^k + \rho^{(n-1)k},$$

where

$$\rho_n = exp\left(\frac{2\pi i}{n}\right).$$

Also,

$$\lambda_{n+k} = a_1 + a_2\rho_n^k + a_3\rho_n^{2k} + a_4\rho_n^{3k} + \cdots + a_n\rho_n^{(m-1)k},$$

for $1 \leq k \leq (m-1)$,

Which gives no eigenvalues for $m - 1 = 0$

$$\{\lambda_{n-1}, \lambda_n\} = \{(n-1)\alpha + d_A\},$$

where $(n-1)\alpha^2 + \alpha(d_A - d_B) - 1 = 0$ and $d_A = 0$, $d_B = 2$.

That is $(n-1)\alpha^2 - 2\alpha - 1 = 0$,

$$\Rightarrow \alpha = \frac{2 \pm \sqrt{4 - 4(n-1)}}{2(n-1)}.$$

Therefore,

$$\{\lambda_{n-1}, \lambda_n\} \{(n-1)\alpha + d_A\}$$

$$= \left\{ (n-1)\left(\frac{2 \pm \sqrt{4+4(n-1)}}{2(n-1)} \right) \right\} = 0.$$

$$\{\lambda_{n-1}, \lambda_n\} \{(n-1)\alpha + d_A\}$$

$$= \left\{ \frac{2 \pm \sqrt{4+4(n-1)}}{2} \right\} = \left\{ 1 \pm \sqrt{n} \right\}$$

□

3.5.7 Eigenvalues of The Lollipop Graph

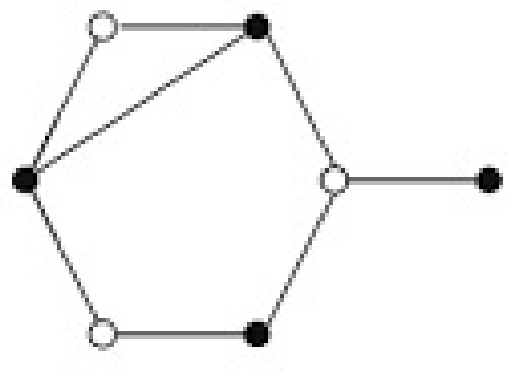

Figure 3.16: A Lollipop Graph

Theorem 3.5.10. *If $LP(G)$ is the complete graph on $n-1$ vertices (the base of the lollipop graph) joined to a single end vertex x_2 by an edge x_1x_2, and let G' be the subgraph of G induced by removing the vertex x_1, and let G'' be the subgraph of G induced by removing the vertex x_2. Then the eigenvalues of G are:* $\lambda = 0$; $\lambda = -1$ *(multiplicity $n-3$)*; $\lambda = \frac{(n-2)+\sqrt{n^2-4n+8}}{2}$;$\lambda = \frac{(n-2)-\sqrt{n^2-4n+8}}{2}$.

Proof. Using the Lollipop theorem (see theorem 3.4.2).

$$P_{A(G}(\lambda) = \lambda P_{A(G'}(\lambda) - P_{A(G''}(\lambda) = \lambda(\lambda+1)^{n-2}(\lambda-(n-2)) - \lambda(\lambda+1)^{n-3}(\lambda-(n-3))$$

$$= \lambda(\lambda+1)^{n-3}\left[(\lambda+1)(\lambda-(n-2)) - (\lambda-(n-3))\right]$$

$$= \lambda(\lambda+1)^{n-3}\left[\lambda^2 - \lambda(n-2) + \lambda - (n-2) - \lambda + (n-3)\right]$$

$$= \lambda(\lambda+1)^{n-3}\left[(\lambda^2 - \lambda(n-2) - 1\right].$$

The roots of quadratic are:

$$\lambda = \frac{(n-2) \pm \sqrt{n^2 - 4n + 4 + 4}}{2}.$$

we have the following roots
$\lambda = 0$; $\lambda = -1$ $(multiplicity\ n-3)$; $\lambda = \frac{(n-2)+\sqrt{n^2-4n+8}}{2}$;$\lambda = \frac{(n-2)-\sqrt{n^2-4n+8}}{2}$.

□

3.5.8 Eigenvalues of the Dual-Star Graph

A dual star graph DuS_n is defined as two star graphs with m rays of length 1 (each on $\frac{n}{2}$ vertices). It has 4 non-zero eigenvalues found as solutions of the following equation with small trees(an undirected graph in which any two vertices are connected by exactly one path):

$$x^4 - (2m+1)x^2 + m^2 = 0 \Rightarrow x^4 - (n-1)x^2 + \frac{(n-2)^2}{4} = 0,$$

$$\Rightarrow x^2 = \frac{(n-1) \pm \sqrt{(n-)^2 - (n-2)^2}}{2} = \frac{(n-1) \pm \sqrt{2n-3}}{2},$$

$$\Rightarrow x = \pm\sqrt{\frac{(n-1) + \sqrt{2n-3}}{2}}\ or\ \pm\sqrt{\frac{(n-1) - \sqrt{2n-3}}{2}}.$$

3.6 Eigenvalues of The Line Graph of the Complete Graph,the L-Regular Caterpillar Graph and the Caterpillar Graph

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3.6.1 Eigenvalues of the Line Graph of the complete graph K_n

The line graph $L(K_n)$ of K_n has $\frac{n(n-1)}{2}$ [13] vertices. The number q of edges is the sum of the square of the degrees minus the number of edges of K_n.
Thus,

$$q = \frac{n(n-1)^2}{2} - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}[n-1-1] = \frac{n(n-1)(n-2)}{2}.$$

Also,

$$2n^2 - 6n = \frac{4n(n-1)}{2} - 4n = 4p - 4n$$

$$n^2 - n - 2p = 0 \Longrightarrow n = \frac{1 \pm \sqrt{1+8p}}{2} = \frac{1 + \sqrt{1+8p}}{2}.$$

3.6.2 Eigenvalues of the L-Regular Caterpillar Graph and the caterpillar graph

To obtain the eigenvalues of the caterpillar graph we need the eigenvalues of the Laplacian of its line graph. Now, by using the format of $A(CT(k,2))$ as above, we have the following:

Line Graph of The Caterpillar Graph $L(CT(k,l))$

For the caterpillar graph $CT(k,l)$, its line graph is the sequence of k cliques(see section 2.2.15) $K_{l+1}, K_{l+2}, K_{l+3}, \cdots, K_{l+2}, K_{l+1}$ in this order, such that two consecutive cliques have exactly one vertex in common. This graph will have $n = kl + (k-1)$ vertices, and

$$m = \frac{k(k-l}{2}(k-2) + 2\frac{(k-1)(k-2)}{2} = \frac{(k+2)(k-1)(k-2)}{2}.$$

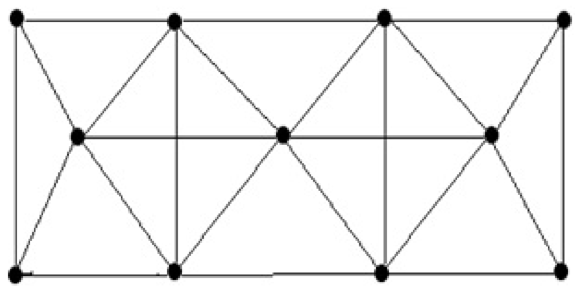

Figure 3.17: Line graph $L(CT(4,2))$ of Caterpillar $L(CT(4,2))$

Lemma 3.6.1. *If G is a bipartite graph and if λ is a nonzero Laplacian eigenvalue of G then $\lambda - 2$ is an eigenvalue of $L(G)$ [86].*

Proof. Now since $CT(k,l)$ is a bipartite graph, the eigenvalues of $L(CT(k,l))$ can be derived from the Laplacian eigenvalues of $CT(k,l)$ (see [86] for these eigenvalues), namely:
$\lambda_i = \lambda - 2 = 1 - 2 = -1$ with multiplicity $k(l-1)$,

$$\lambda_j = \lambda_i - 2 = \frac{1}{2}\left(l - 1 + \sigma_j - \sqrt{\sigma^2 + 2(l+1)\sigma_j + (l^2 + 6l}\right),$$

where

$$\sigma_j = 2cos\left(\frac{(k+1-j)\pi}{k}\right),$$

for $j = 2, 3, \cdots, k$ and

$$\mu_{k+j} = \lambda_j - 2 = \frac{1}{2}\left(l - 1 + \sigma_j + \sqrt{\sigma^2 + 2(l+1)\sigma_j + (l^2 + 6l}\right),$$

where

$$\sigma_j = 2cos\left(\frac{(k+1-j)\pi}{k}\right)$$

, for $j = 1, 2, 3, \cdots, k$. ∎

3.7 Conclusion

In this chapter, we presented different methods of finding the eigenvalues of some classes of graphs so that we can determine their energies in the next chapter. Some of the methods use the definition of the adjacency matrix while others used known results/theorems. Some of the proofs illustrate the combinatorial aspects of the associated theorems in determining the eigenvalues of adjacency matrices associated with graphs.
The complete graph K_n has $\frac{n(n-1)}{2}$ edges, and is a regular graph of degree $(n-1)$. All complete graphs are their own maximal cliques [1]. They are maximally connected(i.e. every pair of distinct vertices is connected by a pair of unique edges: one in each direction) as the only vertex cut which disconnects the graph is the complete set of vertices. It is very well centrally connected (a well-covered graph or an undirected graph in which every minimal vertex cover has the same size as every other minimal vertex cover) and has conjugate pair

$$\left[\frac{n-2}{2} \pm \sqrt{\frac{n^2}{4}}\right]$$

For the bipartite graph, its "bi-centrality" is easy to see (chromatic number is two and its maximum clique size is also two). It has the pair of conjugate eigenvalues: $[0 \pm \sqrt{mn}]$

The path graph contains only vertices of degree 2 and 1. In particular, it has two terminal vertices that have degree 1 , while all others (if any) have degree 2. The path and the cycle graphs don't have an explicit central vertex.

For the wheel graph, the connectivity of the central vertex to every other vertex gave rise to the formation of a vector that resulted in a conjugate pair of eigenvalues $(1 \pm \sqrt{n})$. There is a conjugate pair of eigenvalues of the adjacency matrix related to the star graphs (rays of length one; length two), which are: $\left[0 \pm \sqrt{(n-1)}\right]$; $\left[1 \pm \sqrt{\frac{n-1}{2}}\right]$. In the next chapter we use the eigenvalues found in this chapter to determine the energies of the graphs discussed so far.

Chapter 4

Determining The Energy of Classes of Graphs With Emphasis On Analytical Methods

4.1 Introduction

Definition 4.1.1. *The energy of a graph G is the sum of the absolute values of the eigenvalues associated with the adjacency matrix of G [65], [8]).*

We determine the energy of different classes of graphs using the eigenvalues determined in the previous chapter– the purpose of this will be explicated in the next chapter. Finding a simple expression for the energy of a class of graphs can be trivial, especially if the eigenvalues are integral (for the complete graph, for example) on it can be a challenge (for graphs for which the eigenvalues involve the cosine of a rational function of n) as we illustrate below. We express the energy of cycles, paths and wheels, on n vertices, in terms of simplified expressions involving the cotangent and the secant of either $\frac{\pi}{n}$; $\frac{\pi}{n-1}\frac{\pi}{2n}$ or $\frac{\pi}{2(n-1)}$. We also find that these classes of graphs have the same energy of $\frac{4n}{\pi}$, for large values of n.

4.2 Finding Eigenvalues of Lollipop Type Graphs With The Complete Graph

The lollipop graph, defined in section 2.2.28, and denoted by $LP(H_{n-1})$, is obtained by appending a known class of graphs H_{n-1} on $n-1$ vertices to a pendant vertex of a path of length 1. We shall only deal with the case $H_{n-1} = K_{n-1}$.

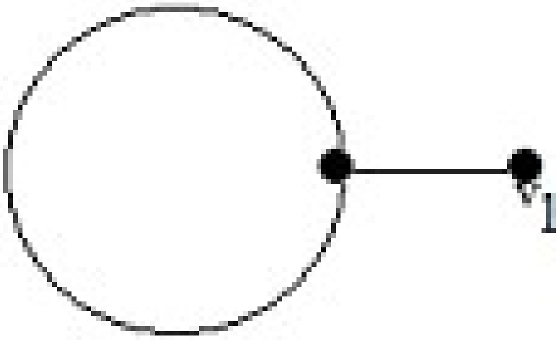

Figure 4.1: Lollipop graph 1

The eigenvalues of $LP(H_{n-1})$ are (see theorem 3.2): $\lambda = 0$; $\lambda = -1$ (*multiplicity* $n-3$); $\lambda = \frac{(n-2)+\sqrt{n^2-4n+8}}{2}$;$\lambda = \frac{(n-2)-\sqrt{n^2-4n+8}}{2}$.
Hence, the following theorem:

Theorem 4.2.1. *Energy of this $LP(H_{n-})$ is:*

$$0 + 1(n-3) + \frac{(n-2)+\sqrt{n^2-4n+8}}{2} + \frac{\sqrt{n^2-4n+8}-(n-2)}{2}$$

since $n \geq 4$, then

$$E(LP(H_{n-1})) = (n+3) + \sqrt{n^2-4n+8}$$

4.3 Dual-Star Graph

A dual star graph DuS_n is defined as two star graphs with m rays of length 1 (each on $\frac{n}{2}$ vertices) joined by an edge (its center edge) connecting their centers (see section 2.2.25). This graph has 4 non-zero eigenvalues found as solutions of the following equation see Haemers [54] with small trees:

$$\Rightarrow x = \pm\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}} \ or \ \pm\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}.$$

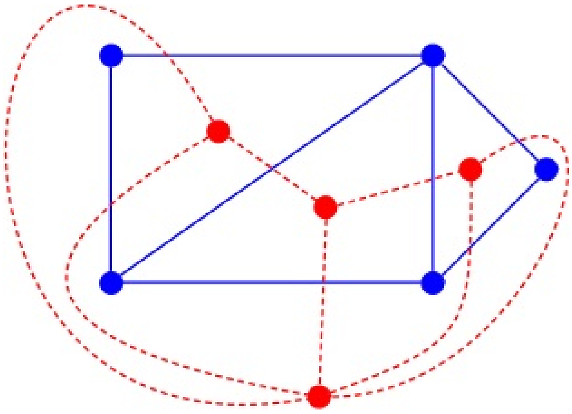

Figure 4.2: Dual star graph

The energy of this graph is therefore given by the following theorem:

Theorem 4.3.1.

$$E(DuS_n) = 2\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}} + 2\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}$$

4.4 Energy of Strongly Regular Graphs

Koolen and Moulton [62] have proved that the energy of a graph on n vertices is at most $\frac{n(n+\sqrt{n}}{2}$, and that equality holds if and only if the graph is strongly regular with parameters

$$\left[\frac{n(n+\sqrt{n})}{2}, \frac{(n+2\sqrt{n}}{4}, \frac{(n+2\sqrt{n}}{4}\right]$$

. Such graphs are equivalent to a certain type of *Hadamard matrices*. Here we survey constructions of these Hadamard matrices and the related strongly regular graphs [94]

4.5 Energy of The Line Graph of The Complete Graph

The line graph $L(K_p)$ of K_p on n vertices has energy $2p^2 - 6p$ [13]. Converting the energy to being a function of n we get (discussed in section 3.5):

$$E(L(K_p)) = 2p^2 - 6p = 4\frac{p(p-1)}{2} - 4p = 4n - 2 - 2\sqrt{1+8n}.$$

For large n this value is greater than the energy of the complete graph.

4.6 Planar Graphs

A graph G_p is said to be planar if there exists some geometric representation of G_p which can be drawn on a plane such that no two of its edges intersect (see section

2.2.24). A plane representation of a graph divides the plane into regions also called faces. A region is characterized by the set of edges (or the set of vertices) forming its boundary.

4.6.1 Bounds on The Energy of Planar Graphs

Let G_p be a planar graph with n vertices and m edges. Denote the energy of G_p by $E(G_p)$. Clearly,

$$E(G_p) \leq \sqrt{2mn} \tag{4.1}$$

$$m \leq 3n - 6 \tag{4.2}$$

A lower bound for the energy of a graph solely in terms of its number of vertices

$$E(G_p) \geq 2\sqrt{n-1} \tag{4.3}$$

Theorem 4.6.1. *For a planar graph G_p*

$$2\sqrt{\frac{m+3}{3}} \leq E(G_p) \leq \sqrt{6n(n-2)}.$$

Proof. If G_p is a planar graph with n vertices and m edges, then (4.2) holds. Using (4.2) in (4.1), we obtain

$$E(G_p) \leq \sqrt{2(3n-6)n} = \sqrt{6n(n-2)} \tag{4.4}$$

Using (4.2) in (4.3), we obtain

$$E(G_p) \geq 2\sqrt{\frac{m+6}{3} - 1} = 2\sqrt{\frac{m+3}{3}} \tag{4.5}$$

Combining (4.4) and (4.5), the result follows. □

Theorem 4.6.2. *If G_p is a connected planar graph, then*

$$2\sqrt{\frac{n+2}{3}} \leq E(G_p) \leq \sqrt{6n(n-1)}$$

Proof. For a connected planar graph, $m \geq n - 1$ and using this in (4.5), we get

$$E(G_p) \geq 2\sqrt{\frac{n-1+3}{3} - 1} = 2\sqrt{\frac{n+2}{3}}.$$

Together with (4.4), we obtain

$$2\sqrt{\frac{n+2}{3}} \leq E(G_p) \leq \sqrt{6n(n-1)}.$$

□

4.7 Polyhedron

A *polyhedron* is a solid bounded by surfaces, called faces, each of which is a plane. A polyhedron is said to be *convex* if any two of its interior points can be joined by a straight line lying entirely within the region. The vertices and edges of a polyhedron, which form a skeleton of the solid, give a simple graph in three dimensional space. For a convex polyhedron this graph is planar(see [87]. A simple connected plane graph G_p is called *polyhedral* if $d(v) \geq 3$ for each vertex v of G_p and $d(\phi) \geq 3$ for every face ϕ of G_p.

Theorem 4.7.1. *If G_p is a polyhedral graph, then*

$$\sqrt{2(n+2)} \leq E(G_p) \leq \sqrt{6n(n-1)}.$$

Proof. As G_p is polyhedral,$d(\phi) \geq 3$ for every vertex v of G_p. So,

$$2m = \sum d(\phi) \geq 3, \ or \ m \geq \frac{3n}{2}. \tag{4.6}$$

Now, using (4.6) in (4.5), we get

$$E(G_p) \geq 2\sqrt{\frac{3n+6}{6}} = \sqrt{2(n+2)}.$$

Together with (4.4), we get the required result. □

If G is a bipartite graph with $n > 2$ vertices, then [83]

$$E(G) \leq \frac{n}{\sqrt{8}}\left(\sqrt{n} + \sqrt{2}\right) \tag{4.7}$$

and [83])

$$E(G) \leq 2\left(\frac{2m}{n}\right) + \sqrt{(n-2)\left(2m - 2\left(\frac{2m}{n}\right)^2\right)} \tag{4.8}$$

Theorem 4.7.2. *If G_p is a planar bipartite graph, then*

$$E(G_p) \leq \frac{3m-4}{8}\left(\sqrt{3m-4} + 2\right).$$

Proof. Since G_p is planar bipartite, then

$$m \geq \frac{2}{3}(n+2) \ or \ n \leq \frac{3m-4}{2} \tag{4.9}$$

Using (4.9) in (4.7), the result follows. □

A polyhedron is called *regular* if it is convex and its faces are congruent regular polygons. We know that, the only regular polyhedra are the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. Balakrishnan and Ranganathan in [8], showed that if G is a regular graph of degree k , then

$$E(G) \leq k + \sqrt{k(n-1)(n-k)} \tag{4.10}$$

By using (4.10), we see that the tetrahedron, the cube and the dodecahedron are not hyperenergetic (having energy greater than the complete graph (see section 4.11 below), since they are regular graphs of degree 3.
This can also be verified by using, the result of [42] as shown below.
Show that for these three regular polyhedras, $m = \frac{3n}{2}$ and $m = \frac{3n}{2} = 2n - n - \frac{n-4}{2}$<2n-2 , and hence the argument. Now by using (4.10), $E(octahedron)$ and $E(icosahedron) \leq 5 + \sqrt{385}$. For the planar graph G_p , with n vertices, m edges and f faces, we have the well-known Euler's formula: $n - m + f = 2$ [29] Together with (4.2), we obtain

$$m \geq \frac{3f - 12}{2} \tag{4.11}$$

and

$$n \geq \frac{f + 4}{2} \tag{4.12}$$

With the help of (4.11) and (4.12), the bounds for the energy of a planar graph solely in terms of its faces f can be obtained as below.

$$E(G_p) \geq \left(\frac{f+4}{2}\right)\sqrt{\frac{f+2}{2}} \tag{4.13}$$

$$E(G_p) \geq \sqrt{\frac{3}{2}f^2 - 16} \tag{4.14}$$

$$E(G_p) \geq \sqrt{2(f+2)} \tag{4.15}$$

Furthermore, the smallest non-planar graphs are K_5 and $K_{3,3}$ and we see that both are not hyperenergetic (section 4.11) From (4.7),

$$E(K_{3,3}) \leq \frac{6}{\sqrt{8}}\left(\sqrt{6} + \sqrt{2}\right) < 2n - 2 = 10 \tag{4.16}$$

4.8 Analytical Methods To Obtain The Energy of Graphs

This section is based on the article by Winter and Jessop [111] .We show that the energy of cycles, paths and wheels are the same for large values of n, i.e.$\frac{4n}{\pi}$

4.8.1 Energy of The Path Graph

Let G be the path graph P_n on n vertices, and with $n - 1$ edges.

corollary 4.8.1. *The eigenvalues of the path P_n are $\lambda_i = 2cos\left(\frac{\pi j}{n+1}\right)$; $j = 1, \cdots, n$ (each with multiplicity 1) for $n \geq 2$ See [59].*

Theorem 4.8.1 (1). *For p even, then*

$$\sum_{1}^{\frac{p}{2}} 2cos\left(\frac{\pi j}{p}\right) = cot\left(\frac{\pi}{2p}\right) - 1$$

,
2. for p odd, then

$$\sum_{1}^{\frac{p-1}{2}} 2cos\left(\frac{\pi j}{p}\right) = cosec\left(\frac{\pi}{2p}\right) - 1$$

3.

$$\sum_{1}^{\frac{n-1}{4}} 2cos\left(\frac{2\pi j}{n}\right) = \sum_{1}^{\frac{n-3}{4}} 2cos\left(\frac{2\pi j}{n}\right) = \frac{1}{2}cosec\left(\frac{\pi}{2p}\right) - 1$$

Proof. The positive cosine eigenvalues are therefore:

$$2cos\left(\frac{\pi}{n}\right); 2cos\left(\frac{\pi 2}{n}\right); 2cos\left(\frac{\pi 3}{n}\right); \cdots ; 2cos\left(\frac{\pi}{n}\right)\left(\frac{k}{2}\right)$$

Let

$$C = \sum_{1}^{\frac{k}{2}} 2cos\left(\frac{\pi j}{n}\right);\ S = \sum_{1}^{\frac{k}{2}} 2sin\left(\frac{\pi j}{n}\right).$$

and

$$\gamma = cos\left(\frac{\pi}{n}\right) + isin\left(\frac{\pi}{n}\right).$$

so that

$$\gamma^{\frac{k}{2}} = \left[cos\left(\frac{\pi}{n}\right) + isin\left(\frac{\pi}{n}\right)\right]^{\frac{k}{2}} = cos\left(\frac{\pi}{n}\frac{k}{2}\right) + isin\left(\frac{\pi}{n}\frac{k}{2}\right).$$

Then,

$$C + iS = \left[2cos\left(\frac{\pi}{n}\right) + isin\left(\frac{\pi}{n}\right)\right] + \left[2cos\left(\frac{2\pi}{n}\right) + isin\left(\frac{2\pi}{n}\right)\right] + \cdots$$
$$+ \left[2cos\left(\frac{\pi}{n}\right)\left(\frac{k-2}{2}\right) + 2isin\left(\frac{\pi}{n}\right)\left(\frac{k}{2}\right)\right]$$

$$= 2\gamma + 2\gamma^2 + \cdots + 2\gamma^{\frac{k}{2}}$$

$$= 2\gamma\left(1 + \gamma + \cdots + \gamma^{\frac{k}{2}-1}\right)$$
$$= 2\gamma\frac{1-\gamma^{\frac{k}{2}}}{1-\gamma}$$

(this step was provided by Indulal and Vijayakumar [58]

$$= 2\gamma\frac{\left[1 - \left(cos\left(\frac{\pi}{n}\frac{k}{2}\right) + isin\left(\frac{\pi}{n}\frac{k}{2}\right)\right)\right]}{\left(1 - cos\left(\frac{\pi}{n}\right)\right) - isin\left(\frac{\pi}{n}\right)}$$

$$= 2\left(cos\frac{\pi}{n} + isin\frac{\pi}{n}\right)\frac{\left[1 - \left(cos\left(\frac{\pi}{n}\frac{k}{2}\right) + isin\left(\frac{\pi}{n}\frac{k}{2}\right)\right)\right]}{\left(1 - cos\left(\frac{\pi}{n}\right)\right) - isin\left(\frac{\pi}{n}\right)} \times \frac{\left(1 - cos\left(\frac{\pi}{n}\right)\right) + isin\left(\frac{\pi}{n}\right)}{\left(1 - cos\left(\frac{\pi}{n}\right)\right) + isin\left(\frac{\pi}{n}\right)}$$

$$= 2\frac{\left[cos\frac{\pi}{n} + isin\frac{\pi}{n}\right]\left[\left(1 - cos\frac{\pi}{n}\right) + isin\frac{\pi}{n}\right]\left[\left(1 - cos\left(\frac{\pi}{n}\frac{k}{2}\right)\right) - isin\left(\frac{\pi}{n}\frac{k}{2}\right)\right]}{\left(1 - cos\left(\frac{\pi}{n}\right)\right)^2 + isin^2\left(\frac{\pi}{n}\right)}$$

$$= 2\frac{\left[cos\frac{\pi}{n}\left(1 - cos\frac{\pi}{n}\right) + isin\frac{\pi}{n}\left(1 - cos\frac{\pi}{n}\right) + icos\frac{\pi}{n}sin\frac{\pi}{n} - sin^2\frac{\pi}{n}\right]\left[\left(1 - cos\left(\frac{\pi}{n}\frac{k}{2}\right)\right) - isin\left(\frac{\pi}{n}\frac{k}{n}\right)\right]}{1 - 2cos\frac{\pi}{n} + cos^2\frac{\pi}{n} + sin^2\frac{\pi}{n}}$$

$$= \frac{2\left[cos\frac{\pi}{n} - cos^2\frac{\pi}{n} + isin\frac{\pi}{n} - icos\frac{\pi}{n}sin\frac{\pi}{n} + icos\frac{\pi}{n}sin\frac{\pi}{n} - sin^2\frac{\pi}{n}\right]\left[\left(1 - cos\left(\frac{\pi}{n}\frac{k}{2}\right)\right) - isin\left(\frac{\pi}{n}\frac{k}{n}\right)\right]}{2 - 2cos\frac{\pi}{n}}$$

$$\frac{\left[\left(cos\frac{\pi}{n} - 1\right) + i\left(sin\frac{\pi}{n} - cos\frac{\pi}{n}sin\frac{\pi}{n} + cos\frac{\pi}{n}sin\frac{\pi}{n}\right)\right]\left[\left(1 - cos\left(\frac{\pi}{n}\frac{k}{2}\right)\right) - isin\left(\frac{\pi}{n}\frac{k}{2}\right)\right]}{1 - cos\frac{\pi}{n}}$$

$$= \frac{\left[\left(cos\frac{\pi}{n} - 1\right) + isin\frac{\pi}{n}\right]\left[\left(1 - cos\left(\frac{\pi}{n}\frac{k}{2}\right)\right) - isin\left(\frac{\pi}{n}\frac{k}{2}\right)\right]}{1 - \left(cos^2\frac{\pi}{n} - sin^2\frac{\pi}{n}\right)}$$

$$= \frac{\left[\left(cos\frac{\pi}{n} - 1\right) + isin\frac{\pi}{n}\right]\left[\left(1 - cos\left(\frac{\pi}{n}\frac{k}{2}\right)\right) - isin\left(\frac{\pi}{n}\frac{k}{2}\right)\right]}{\left(2sin^2\frac{\pi}{n}\right)}.$$

Taking the real parts:

$$C = \sum_{1}^{\frac{k}{2}} 2cos\left(\frac{\pi j}{n}\right)$$

$$= \frac{\left[\left(cos\frac{\pi}{n} - 1\right)\left(1 - cos\left(\frac{\pi}{n}\frac{k}{2}\right)\right) + sin\frac{\pi}{n}sin\left(\frac{\pi}{n}\frac{k}{2}\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

$$= \frac{\left[cos\frac{\pi}{n} - 1 - cos\frac{\pi}{n}cos\left(\frac{\pi}{n}\frac{k}{2}\right) + cos\left(\frac{\pi}{n}\frac{k}{2}\right) + sin\frac{\pi}{n}sin\left(\frac{\pi}{n}\frac{k}{2}\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

$$= \frac{\left[\left(cos\frac{\pi}{n} - 1 - \left(cos\frac{\pi}{n}cos\left(\frac{\pi}{n}\frac{k}{2}\right) - sin\frac{\pi}{n}sin\left(\frac{\pi}{n}\frac{k}{2}\right)\right) + cos\left(\frac{\pi}{n}\frac{k}{2}\right)\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

$$= \frac{\left[\left(cos\frac{\pi}{n} - 1 - (cos\left(\frac{\pi}{n} + \frac{\pi}{n}\frac{k}{2}\right) + cos\left(\frac{\pi}{n}\frac{k}{2}\right)\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

For $k = n$, we obtain

$$C = \sum_{1}^{\frac{n}{2}} 2cos\left(\frac{\pi j}{n}\right)$$

$$= \frac{\left[\left(cos\frac{\pi}{n} - 1 - (cos\left(\frac{\pi}{n} + \frac{\pi}{2}\right) + cos\left(\frac{\pi}{2}\right)\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

$$= \frac{\left[\left(cos^2\frac{\pi}{2n} - sin^2\frac{\pi}{2n} - 1 - cos\left(\frac{\pi}{n} + \frac{\pi}{2}\right)\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

$$= \frac{\left[\left(cos^2\frac{\pi}{2n} - 1 - sin^2\frac{\pi}{2n} - \left(cos\left(\frac{\pi}{n}\right)cos\left(\frac{\pi}{2}\right) - sin\left(\frac{\pi}{n}\right)sin\left(\frac{\pi}{2}\right)\right)\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

$$= \frac{\left[-2sin^2\left(\frac{\pi}{2n}\right) + sin\left(\frac{\pi}{n}\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

$$= \frac{\left[-2sin^2\left(\frac{\pi}{2n}\right) + 2sin\left(\frac{\pi}{2n}\right) cos\left(\frac{\pi}{2n}\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

$$= cot\left(\frac{\pi}{2n}\right) - 1.$$

this gives result [1] of theorem 4.8.1
Recall

$$C = \sum_{1}^{\frac{k}{2}} 2cos\left(\frac{\pi j}{n}\right) = \frac{\left[\left(cos\frac{\pi}{n} - 1 - (cos\left(\frac{\pi}{n} + \frac{\pi}{n}\frac{k}{2}\right) + cos\left(\frac{\pi}{n}\frac{k}{2}\right)\right))\right]}{\left(2sin^2\frac{\pi}{2n}\right)}.$$

for $k = n - 1$, we obtain

$$C = \sum_{1}^{\frac{n-1}{2}} 2cos\left(\frac{\pi j}{n}\right) = \frac{\left[\left(cos\frac{\pi}{n} - 1 - (cos\left(\frac{\pi}{n} + \frac{\pi}{n}\frac{n-1}{2}\right) + cos\left(\frac{\pi}{n}\frac{n-1}{2}\right)\right))\right]}{\left(2sin^2\frac{\pi}{2n}\right)}.$$

$$= \frac{\left[\left(cos\frac{\pi}{n} - 1 - (cos\left(\frac{\pi}{n}\frac{n+1}{2}\right) + cos\left(\frac{\pi}{n}\frac{n-1}{2}\right)\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

$$= \frac{\left[cos\frac{\pi}{n} - 1 - \left(cos\left(\frac{\pi}{n}\right)cos\left(\frac{\pi}{2n}\right) - sin\left(\frac{\pi}{2}\right)sin\left(\frac{\pi}{2n}\right)\right) + \left(cos\left(\frac{\pi}{2}\right)cos\left(\frac{\pi}{2n}\right) + sin\left(\frac{\pi}{2}\right)sin\left(\frac{\pi}{2n}\right)\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

$$= \frac{\left[\left(cos\frac{\pi}{n} - 1\right) - \left(-sin\left(\frac{\pi}{2n}\right)\right) + \left(sin\left(\frac{\pi}{2n}\right)\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

$$= \frac{\left[-2sin^2\frac{\pi}{2n} + 2sin\left(\frac{\pi}{2n}\right)\right]}{\left(2sin^2\frac{\pi}{2n}\right)}$$

$$= cosec\left(\frac{\pi}{2n}\right) - 1$$

this gives result 2 of theorem 4.7.1

Let

$$D = \sum_{1}^{\frac{k}{2}} 2cos\left(\frac{\pi j}{n}\right) = \frac{\left[\left(cos\frac{2\pi}{n} - 1\right) - \left(cos\left(\frac{2\pi}{n} + \frac{2\pi}{n}\frac{k}{2}\right) + cos\left(\frac{2\pi}{n}\frac{k}{2}\right)\right)\right]}{\left(2sin^2\frac{\pi}{n}\right)}$$

for $k = \frac{n-1}{2}$, we obtain

$$D = \sum_{1}^{\frac{n-1}{4}} 2cos\left(\frac{2\pi j}{n}\right) = \frac{\left[\left(cos\frac{2\pi}{n} - 1\right) - \left(cos\left(\frac{2\pi}{n} + \frac{2\pi}{n}\frac{n-1}{4}\right) + cos\left(\frac{2\pi}{n}\frac{n-1}{4}\right)\right)\right]}{\left(2sin^2\frac{\pi}{n}\right)}$$

$$= \frac{\left[\left(cos\frac{2\pi}{n} - 1\right) - \left(cos\left(\frac{2\pi}{n}\frac{n+3}{4}\right) + cos\left(\frac{2\pi}{n}\frac{n-1}{4}\right)\right)\right]}{\left(2sin^2\frac{\pi}{n}\right)}$$

$$= \frac{\left[\left(-2sin^2\frac{\pi}{n} - \left(cos\left(\frac{\pi}{2}\right)cos\left(\frac{3\pi}{2n}\right) - sin\left(\frac{\pi}{2}\right)sin\left(\frac{3\pi}{2n}\right)\right) + \left(cos\left(\frac{\pi}{2}\right)cos\left(\frac{\pi}{2n}\right) + sin\left(\frac{\pi}{2}\right)sin\left(\frac{\pi}{2n}\right)\right)\right)\right]}{\left(2sin^2\left(\frac{\pi}{n}\right)\right)}$$

$$= -1 + \frac{\left[sin\left(\frac{3\pi}{2n}\right) + sin\left(\frac{\pi}{2n}\right)\right]}{\left(2sin^2\left(\frac{\pi}{n}\right)\right)}$$

$$= -1 + \frac{\left[sin\left(\frac{\pi}{n}\right)cos\left(\frac{\pi}{2n}\right) + cos\left(\frac{\pi}{n}\right)sin\left(\frac{\pi}{2n}\right) + sin\left(\frac{\pi}{2n}\right)\right]}{\left(2sin^2\left(\frac{\pi}{n}\right)\right)}$$

$$= -1 + \frac{\left[2sin\left(\frac{\pi}{2n}\right)cos\left(\frac{\pi}{2n}\right)cos\left(\frac{\pi}{2n}\right) + sin\left(\frac{\pi}{2n}\right)\left(cos\left(\frac{\pi}{n}\right) + 1\right)\right]}{\left(2sin^2\left(\frac{\pi}{n}\right)\right)}$$

$$= -1 + \frac{\left[2sin\left(\frac{\pi}{2n}\right)cos\left(\frac{\pi}{2n}\right)cos\left(\frac{\pi}{2n}\right) + sin\left(\frac{\pi}{2n}\right)\left(cos^2\left(\frac{\pi}{2n}\right) - sin^2\left(\frac{\pi}{2n}\right) + 1\right)\right]}{\left(2sin^2\left(\frac{\pi}{n}\right)\right)}$$

$$= -1 + \frac{\left[2sin\left(\frac{\pi}{2n}\right)cos\left(\frac{\pi}{2n}\right)cos\left(\frac{\pi}{2n}\right) + sin\left(\frac{\pi}{2n}\right)\left(2cos^2\left(\frac{\pi}{2n}\right)\right)\right]}{\left(2sin^2\left(\frac{\pi}{n}\right)\right)}$$

$$= -1 + \frac{\left[2sin\left(\frac{\pi}{2n}\right)cos^2\left(\frac{\pi}{2n}\right)\right]}{\left[4sin^2\left(\frac{\pi}{2n}\right)cos^2\left(\frac{\pi}{2n}\right)\right]}$$

$$= -1 + \frac{1}{2}cosec\left(\frac{\pi}{2n}\right)$$

this gives result 3 of theorem 4.8.1
for $k = \frac{n-3}{2}$, we get

$$D = \sum_{1}^{\frac{n-3}{4}} 2cos\left(\frac{2\pi j}{n}\right)$$

$$= \frac{\left[\left(cos\frac{2\pi}{n} - 1 - cos\left(\frac{2\pi}{n} + \frac{2\pi}{n}\frac{n-3}{4}\right) + cos\left(\frac{2\pi}{n}\frac{n-3}{4}\right)\right)\right]}{\left[2sin^2\left(\frac{\pi}{n}\right)\right]}$$

$$= \frac{\left[\left(cos\frac{2\pi}{n} - 1 - cos\left(\frac{2\pi}{n}\frac{n+1}{4}\right) + cos\left(\frac{2\pi}{n}\frac{n-3}{4}\right)\right)\right]}{\left[2sin^2\left(\frac{\pi}{n}\right)\right]}$$

$$= \frac{\left[\left(-2sin^2\left(\frac{\pi}{n}\right) - \left(cos\left(\frac{\pi}{2}\right)cos\left(\frac{\pi}{2n}\right) - sin\left(\frac{\pi}{2}\right)sin\left(\frac{\pi}{2n}\right)\right) + \left(cos\left(\frac{\pi}{2}\right)cos\left(\frac{3\pi}{2n}\right) + sin\left(\frac{\pi}{2}\right)sin\left(\frac{3\pi}{2n}\right)\right)\right)\right]}{\left[2sin^2\left(\frac{\pi}{n}\right)\right]}$$

$$= -1 + \frac{\left[sin\left(\frac{\pi}{2n}\right) + sin\left(\frac{3\pi}{2n}\right)\right]}{\left[2sin^2\left(\frac{\pi}{n}\right)\right]}$$

$$= -1 + \frac{\left[sin\left(\frac{\pi}{2n}\right) + sin\left(\frac{\pi}{n}\right) cos\left(\frac{\pi}{2n}\right) + cos\left(\frac{\pi}{n}\right) sin\left(\frac{\pi}{2n}\right)\right]}{\left[2sin^2\left(\frac{\pi}{n}\right)\right]}$$

$$= -1 + \frac{\left[sin\left(\frac{\pi}{2n}\right) + 2sin\left(\frac{\pi}{2n}\right) cos^2\left(\frac{\pi}{2n}\right) + \left[cos^2\left(\frac{\pi}{2n}\right) - \left(sin^2\left(\frac{\pi}{2n}\right)\right)\right] sin\left(\frac{\pi}{2n}\right)\right]}{\left[2sin^2\left(\frac{\pi}{n}\right)\right]}$$

$$= -1 + \frac{\left[sin\left(\frac{\pi}{2n}\right) + 2sin\left(\frac{\pi}{2n}\right) cos^2\left(\frac{\pi}{2n}\right) + \left[cos^2\left(\frac{\pi}{2n}\right) - \left(1 - cos^2\left(\frac{\pi}{2n}\right)\right)\right] sin\left(\frac{\pi}{2n}\right)\right]}{\left[2sin^2\left(\frac{\pi}{n}\right)\right]}$$

$$= -1 + \frac{\left[sin\left(\frac{\pi}{2n}\right) + 2sin\left(\frac{\pi}{2n}\right) cos^2\left(\frac{\pi}{2n}\right) + \left[cos^2\left(\frac{\pi}{2n}\right) - 1\right] sin\left(\frac{\pi}{2n}\right)\right]}{\left[2sin^2\left(\frac{\pi}{n}\right)\right]}$$

$$= -1 + \frac{\left[4sin\left(\frac{\pi}{2n}\right) cos^2\left(\frac{\pi}{2n}\right)\right]}{8sin^2\frac{\pi}{2n}cos^2\frac{\pi}{2n}} = -1 + \frac{1}{2}cosec\frac{\pi}{2n}.$$

This gives result 4 of the theorem. □

Theorem 4.8.2. *The energy of the path* P_n *is:*
1. for n *even,*

$$E(P_n) = \sum_{j=1}^{n} \left|2cos\left(\frac{\pi j}{n+1}\right)\right| = 2\left[cosec\left(\frac{\pi}{2(n+1)}\right) - 1\right],$$

2. n *odd,*

$$E(P_n) = \sum_{j=1}^{n} \left|2cos\left(\frac{\pi j}{n+1}\right)\right| = 2\left[cot\left(\frac{\pi}{2(n+1)}\right) - 1\right].$$

Proof. As per Corollary 4.8.1 the eigenvalues of the path P_n are $2cos\left(\frac{\pi j}{n+1}\right)$; $j = 1, \cdots, n$, so

$$E(P_n) = \sum_{i=1}^{n} |\lambda_i| = E(P_n) = \sum_{j=1}^{n} \left|2cos\left(\frac{\pi j}{n+1}\right)\right|.$$

1. Now let n be even, ie. $n = 2t$. Then

$$E(P_n) = \sum_{j=1}^{n} \left|2cos\left(\frac{\pi j}{n+1}\right)\right| = \sum_{j=1}^{2t} \left|2cos\left(\frac{\pi j}{2t+1}\right)\right|$$

$$= \sum_{j=1}^{t} \left[\left| 2cos\left(\frac{\pi j}{2t+1}\right) \right| + \left| 2cos\left(\frac{\pi(2t+1-j)}{2t+1}\right) \right| \right]$$

$$= \sum_{j=1}^{t} \left| 2cos\left(\frac{\pi j}{2t+1}\right) \right| + \left| 2cos\left(\frac{\pi(2t+1)}{2t+1}\right) cos\left(\frac{\pi j}{2t+1}\right) + sin\left(\frac{\pi(2t+1)}{2t+1}\right) sin\left(\frac{\pi j}{2t+1}\right) \right|$$

$$= \sum_{j=1}^{t} \left| 2cos\left(\frac{\pi j}{2t+1}\right) \right| + \left| 2cos(\pi) cos\left(\frac{\pi j}{2t+1}\right) + sin(\pi) sin\left(\frac{\pi j}{2t+1}\right) \right|$$

$$= \sum_{j=1}^{t} \left| 2cos\left(\frac{\pi j}{2t+1}\right) \right| + \left| -2cos(\pi) cos\left(\frac{\pi j}{2t+1}\right) \right|.$$

$$= 2\sum_{j=1}^{t} \left| 2cos\left(\frac{\pi j}{2t+1}\right) \right|.$$

Now, $\frac{j}{2t+1} < 0.5$ for $1 \leq j \leq t$, so $cos\left(\frac{\pi j}{2t+1}\right) \geq 0$. Therefore,

$$E(P_n) = 2\sum_{j=1}^{t} 2cos\left(\frac{\pi j}{2t+1}\right) = 2\sum_{j=1}^{\frac{n}{2}} 2cos\left(\frac{\pi j}{n+1}\right).$$

Setting $k = n+1$, we get

$$E(P_n) = 2\sum_{j=1}^{\frac{k-1}{2}} 2cos\left(\frac{\pi j}{k}\right)$$

Now applying result 2 from Theorem 4.8.1, we obtain

$$E(P_n) = 2\left[cosec\left(\frac{\pi}{2k}\right) - 1\right] = 2\left[cosec\left(\frac{\pi}{2(n+1)}\right) - 1\right]$$

2. Now let n be odd, ie. $n = 2t + 1$. Then

$$E(P_n) = \sum_{j=1}^{n} \left|2cos\left(\frac{\pi j}{n+1}\right)\right| = \sum_{j=1}^{2t+1} \left|2cos\left(\frac{\pi j}{2t+2}\right)\right|$$

$$= \sum_{j=1}^{t} \left[\left|2cos\left(\frac{\pi j}{2t+2}\right)\right| + \left|2cos\left(\frac{\pi(2t+2-j)}{2t+2}\right)\right|\right] + \left|2cos\left(\frac{\pi(t+1)}{2t+2}\right)\right|$$

$$= \sum_{j=1}^{t} \left[\left|2cos\left(\frac{\pi j}{2t+2}\right)\right| + 2cos\left(\frac{\pi(2t+2)}{2t+2}\right)cos\left(\frac{\pi j}{2t+2}\right)\right] +$$
$$\left[sin\left(\frac{\pi(2t+2)}{2t+2}\right)sin\left(\frac{\pi j}{2t+2}\right)\right] + 2cos\left(\frac{\pi}{2}\right)$$

$$= \sum_{j=1}^{t} \left|2cos\left(\frac{\pi j}{2t+2}\right)\right| + \left|2cos(\pi)cos\left(\frac{\pi}{j}2t+2\right) + sin(\pi)sin\left(\frac{\pi j}{2t+2}\right)\right|$$

$$= \sum_{j=1}^{t} \left|2cos\left(\frac{\pi j}{2t+2}\right)\right| + \left|-2cos\left(\frac{\pi}{j}2t+2\right)\right|$$

$$= 2\sum_{j=1}^{t} \left|2cos\left(\frac{\pi j}{2t+2}\right)\right|$$

Now, $\frac{j}{2t+2} < 5$ for $1 \le j \le t$, so $2cos\left(\frac{\pi j}{2t+2}\right) \ge 0$. Therefore,

$$E(P_n) = 2\sum_{j=1}^{t} 2cos\left(\frac{\pi j}{2t+2}\right) = 2\sum_{j=1}^{\frac{n-1}{2}} 2cos\left(\frac{\pi j}{n+1}\right)$$

setting $k = n + 1$, we get

$$E(P_n) = 2\sum_{j=1}^{\frac{k-2}{2}} 2cos\left(\frac{\pi j}{k}\right)$$

$$= 2\sum_{j=1}^{\frac{k}{2}-1} 2cos\left(\frac{\pi j}{k}\right) + 2cos\left(\frac{\pi}{k}.\frac{k}{2}\right)$$

$$= 2\sum_{j=1}^{\frac{k}{2}} 2cos\left(\frac{\pi j}{k}\right)$$

Now applying result 1 from Theorem 4.7 we get

$$E(P_n) = 2\left[cot\left(\frac{\pi j}{2k}\right) - 1\right]$$

$$2\left[cot\left(\frac{\pi j}{2(n+1)}\right) - 1\right]$$

□

4.8.2 Energy of The Cycle Graph

Let G be the cycle graph C_n on n vertices, and with n edges.

corollary 4.8.2. *The eigenvalues of the cycle* C_n *are* $\lambda_j = 2cos\left(\frac{2\pi j}{n}\right); j = 0, \cdots, n-1$ *each with multiplicity* 1 *for* $n \geq 3$ *. See Jessop [58].*

Lemma 4.8.1.

$$2\left[cos\left(\frac{\pi}{2t+1}\right) + cos\left(\frac{3\pi}{2t+1}\right) + \cdots + cos\left(\frac{(2t-1)\pi}{2t+1}\right)\right]$$

$$= 2\sum_{r=1}^{t} cos\left[\frac{\pi(2t-2r+1)}{2t+1}\right] = 1;\ \ t = 1, 2, \cdots .$$

See Winter [110] From Lemma 4.1

$$\sum_{r=1}^{t} cos\left[\frac{\pi(2t-2r+1)}{2t+1}\right] = \frac{1}{2};\ \ t = 1, 2, \cdots .$$

Now $n = 2t + 1$ *so*

$$\sum_{r=1}^{t} cos\left(\frac{\pi(n-2r)}{n}\right) = \frac{1}{2};\ \ t = 1, 2, \cdots .$$

$$\Rightarrow cos\left(\frac{\pi(n-2)}{n}\right)+cos\left(\frac{\pi(n-4)}{n}\right)+\cdots+cos\left(\frac{\pi(n-2(t-1)))}{n}\right)+cos\left(\frac{\pi(n-2t)}{n}\right)=0.5$$

$$\Rightarrow cos\left(\frac{\pi(n-2)}{n}\right)+cos\left(\frac{\pi(n-4)}{n}\right)+\cdots+cos\left(\frac{3\pi}{n}\right)+cos\left(\frac{\pi}{n}\right)=0.5$$

$$\Rightarrow cos\left(\frac{\pi}{n}\right)+cos\left(\frac{3\pi}{n}\right)+\cdots+cos\left(\frac{\pi(n-4)}{n}\right)+cos\left(\frac{\pi(n-2)}{n}\right)=0.5$$

Lemma 4.8.2. *For n odd, and $n = 2t + 1$ where t is even and $t = 2q$, then*

$$\sum_{j=\frac{t}{2}+1}^{t}\left|cos\left(\frac{2\pi j)}{n}\right)\right|=\sum_{j=1}^{\frac{t}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right|+0.5.i.e.\sum_{j=q+1}^{2q}\left|cos\left(\frac{2\pi j}{n}\right)\right|=\sum_{j=1}^{q}\left|cos\left(\frac{2\pi j}{n}\right)\right|+0.5 \tag{4.17}$$

2. For n odd, and $n = 2t + 1$ where t is odd and $t = 2q$, then

$$\sum_{j=\frac{t-1}{2}+1}^{t}\left|cos\left(\frac{2\pi j)}{n}\right)\right|=\sum_{j=1}^{\frac{t-1}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right|+0.5i.e.\sum_{j=q+1}^{2q+1}\left|cos\left(\frac{2\pi j}{n}\right)\right|=\sum_{j=1}^{q}\left|cos\left(\frac{2\pi j}{n}\right)\right|+0.5 \tag{4.18}$$

1. From Lemma 4.8.1 we consider n odd i.e. $n = 2t + 1$ and t even i.e. $t = 2q$ and $n = 4q + 1$. for $q = 2, 3, 4, \cdots, n$, we have:

$$2\sum_{r=1}^{t} cos\left(\frac{\pi(2t-2r+1)}{2t+1}\right)=1;\ t=1,2,\cdots, \tag{4.19}$$

$$\Rightarrow 2\sum_{r=1}^{2q} cos\left(\frac{\pi(4q-2r+1)}{4q+1}\right)=0.5;\Rightarrow= A+B=0.5, \tag{4.20}$$

where

$$A = \left[cos\left(\frac{\pi}{4q+1}\right) + cos\left(\frac{3\pi}{4q+1}\right) + cos\left(\frac{5\pi}{4q+1}\right) + \cdots + cos\left(\frac{(2q-1)\pi}{4q+1}\right)(q\ terms)\right], \tag{4.21}$$

and

$$B = \left[cos\left(\frac{(2q+1)\pi}{4q+1}\right) + cos\left(\frac{(2q+3)\pi}{4q+1}\right) + cos\left(\frac{(2q+5)\pi}{4q+1}\right) + \cdots + cos\left(\frac{(4q-1)\pi}{4q+1}\right) 9q\ terms)\right]. \tag{4.22}$$

Note that all terms in A are positive. Now

$$A = cos\left(\frac{\pi}{4q+1}\right) + cos\left(\frac{3\pi}{4q+1}\right) + cos\left(\frac{5\pi}{4q+1}\right) + \cdots + cos\left(\frac{(2q-1)\pi}{4q+1}\right), \tag{4.23}$$

$$= cos\left(\frac{(4q+1)-4q\pi}{4q+1}\right) + cos\left(\frac{(4q+1)-(4q-2)\pi}{4q+1}\right) \tag{4.24}$$

$$+cos\left(\frac{(4q+1)-(4q-4)\pi}{4q+1}\right) + \cdots + cos\left(\frac{(4q+1)-(2q+2)\pi}{4q+1}\right), \tag{4.25}$$

$$= -cos\left(\frac{(4q)\pi}{4q+1}\right) + cos\left(\frac{(4q-2)\pi}{4q+1}\right) \tag{4.26}$$

$$+cos\left(\frac{(4q-4)\pi}{4q+1}\right) + \cdots + cos\left(\frac{(2q+2)\pi}{4q+1}\right) - \sum_{j=q+1}^{2q} cos\frac{2j\pi}{4q+1}. \tag{4.27}$$

Also,

$$B = cos\left(\frac{(2q+1)\pi}{4q+1}\right) + cos\left(\frac{(2q+3)\pi}{4q+1}\right) + cos\left(\frac{(2q+5)\pi}{4q+1}\right) + \cdots + cos\left(\frac{(4q-1)\pi}{4q+1}\right) \tag{4.28}$$

$$= cos\left(\frac{((4q+1)-(2q))\pi}{4q+1}\right) + cos\left(\frac{((4q+1)-(2q-2))\pi}{4q+1}\right) \tag{4.29}$$

$$+cos\left(\frac{((4q+1)-(2q-4))\pi}{4q+1}\right) + \cdots + cos\left(\frac{((4q+1)-(2))\pi}{4q+1}\right) \tag{4.30}$$

$$= -cos\left(\frac{(2q)\pi}{4q+1}\right) - cos\left(\frac{(2q-2)\pi}{4q+1}\right) - cos\left(\frac{(2q-4)\pi}{4q+1}\right) - \cdots - cos\left(\frac{(2)\pi}{4q+1}\right) = -C \tag{4.31}$$

where,

$$C = cos\left(\frac{(2q)\pi}{4q+1}\right) + cos\left(\frac{(2q-2)\pi}{4q+1}\right) + cos\left(\frac{(2q-4)\pi}{4q+1}\right) \tag{4.32}$$

$$+ \cdots + cos\left(\frac{(2)\pi}{4q+1}\right) = \sum_{j=1}^{q} cos\frac{(2j)\pi}{4q+1} \tag{4.33}$$

and all terms in C are positive.

Therefore:

$$A + B = 0.5 and B = -C, \Rightarrow A = C + 0.5 \Rightarrow -\sum_{j=q+1}^{2q} cos\frac{2\pi j}{4q+1} = \sum_{j=1}^{q} cos\frac{2\pi j}{4q+1} + 0.5 \tag{4.34}$$

Taking the absolute values of both sides, we obtain:

$$\sum_{j=q+1}^{2q} \left|cos\frac{2\pi j}{4q+1}\right| = \sum_{j=1}^{q} \left|cos\frac{2\pi j}{4q+1}\right| + 0.5 \tag{4.35}$$

$$\Rightarrow \sum_{j=\frac{t}{2}+1}^{t} \left|cos\frac{2\pi j}{n}\right| = \sum_{j=1}^{\frac{t}{2}} \left|cos\frac{2\pi j}{n}\right| + 0.5 \tag{4.36}$$

2. From Lemma 4.8.1, we have for n odd i.e. $n = 2t+1$ and t is odd i.e. $t = 2q+1$ and $n = 4q+3$, we then for $q = 2, 3, 4, \cdots$, we obtain

$$2\sum_{r=1}^{t} cos\left(\frac{\pi(2t-2r+1)}{2t+1}\right) = 1;\ t = 1, 2, \cdots \tag{4.37}$$

$$\Rightarrow 2\sum_{r=1}^{2q+1} cos\left(\frac{\pi(4q+2-2r+1)}{4q+3}\right) = 0.5. \tag{4.38}$$

$$\Rightarrow \left[cos\left(\frac{\pi}{4q+3}\right) + cos\left(\frac{3\pi}{4q+3}\right) + cos\left(\frac{5\pi}{4q+3}\right) + \cdots + cos\left(\frac{(2q+1)\pi}{4q+3}\right)\right] + \tag{4.39}$$

$$\left[cos\left(\frac{(2q+3)\pi}{4q+3}\right) + cos\left(\frac{(2q+5)\pi}{4q+3}\right) + cos\left(\frac{(2q+7)\pi}{4q+3}\right) + \cdots + cos\left(\frac{(4q+3)\pi}{4q+3}\right)\right] \tag{4.40}$$

$$= [A] + [B] = 0.5. \tag{4.41}$$

where

$$A = cos\left(\frac{\pi}{4q+3}\right) + cos\left(\frac{3\pi}{4q+3}\right) + cos\left(\frac{5\pi}{4q+3}\right) + \cdots + cos\left(\frac{(2q+1)\pi}{4q+3}\right); qterms. \tag{4.42}$$

and

$$B = cos\left(\frac{(2q+3)\pi}{4q+3}\right) + cos\left(\frac{(2q+5)\pi}{4q+3}\right) + cos\left(\frac{(2q+7)\pi}{4q+3}\right) \tag{4.43}$$

$$+ \cdots + cos\left(\frac{(4q+3)\pi}{4q+3}\right);\ q+1\ terms \tag{4.44}$$

Note that all terms in $[A]$ are positive. Now

$$A = cos\left(\frac{\pi}{4q+3}\right) + cos\left(\frac{3\pi}{4q+3}\right) + cos\left(\frac{5\pi}{4q+3}\right) + \cdots + cos\left(\frac{(2q+1)\pi}{4q+3}\right) \tag{4.45}$$

$$= cos\left(\frac{(4q+3-(4q+2))\pi}{4q+3}\right) + cos\left(\frac{(4q+3)-(4)\pi}{4q+3}\right) \tag{4.46}$$

$$+cos\left(\frac{(4q+3)-(4q-2)\pi}{4q+3}\right) + \cdots + cos\left(\frac{(4q+3)-(2q+2)\pi}{4q+3}\right) \tag{4.47}$$

$$= -cos\left(\frac{(4q+2)\pi}{4q+3}\right) - cos\left(\frac{(4q)\pi}{4q+3}\right) - cos\left(\frac{(4q-2)\pi}{4q+3}\right) - \cdots - cos\left(\frac{(2q+2)\pi}{4q+3}\right) \tag{4.48}$$

$$= -\sum_{j=q+1}^{2q+1}\left(cos\frac{2j\pi}{4q+3}\right) \tag{4.49}$$

$$B = cos\left(\frac{(2q+3)\pi}{4q+3}\right) + cos\left(\frac{(2q+5)\pi}{4q+3}\right) \tag{4.50}$$

$$+cos\left(\frac{(2q+7)\pi}{4q+3}\right) + \cdots + cos\left(\frac{(4q+3)\pi}{4q+3}\right) \tag{4.51}$$

$$= cos\left(\frac{((4q+3)-(2q))\pi}{4q+3}\right) + cos\left(\frac{((4q+3)-(2q-2))\pi}{4q+3}\right) \tag{4.52}$$

$$+cos\left(\frac{((4q+3)-(2q-4))\pi}{4q+3}\right) + \cdots + cos\left(\frac{((4q+3)-(2))\pi}{4q+3}\right) \tag{4.53}$$

$$= -cos\left(\frac{(2q)\pi}{4q+3}\right) - cos\left(\frac{(2q-2)\pi}{4q+3}\right) - cos\left(\frac{(2q-4)\pi}{4q+3}\right) - \cdots - cos\left(\frac{(2)\pi}{4q+3}\right) = -C \tag{4.54}$$

where,

$$C = cos\left(\frac{(2q)\pi}{4q+3}\right) + cos\left(\frac{(2q-2)\pi}{4q+3}\right) + cos\left(\frac{(2q-4)\pi}{4q+3}\right) + \cdots \tag{4.55}$$

$$+cos\left(\frac{(2)\pi}{4q+3}\right) = \sum_{j=1}^{q} cos\left(\frac{(2j)\pi}{4q+3}\right) \tag{4.56}$$

and all terms in C are positive.

Therefore

$$A + B = 0.5$$

$$\Rightarrow A = C + 0.5$$

$$\Rightarrow -\sum_{j=q+1}^{2q+1} cos\left(\frac{2\pi j}{4q+3}\right) = \sum_{j=1}^{q} cos\left(\frac{2\pi j}{4q+3}\right) + 0.5 \tag{4.57}$$

Taking absolute values of both sides, we get

$$\Rightarrow \sum_{j=q+1}^{2q+1} \left|cos\left(\frac{2\pi j}{4q+3}\right)\right| = \sum_{j=1}^{q} \left|cos\left(\frac{2\pi j}{4q+3}\right)\right| + 0.5 \tag{4.58}$$

$$\Rightarrow \sum_{j=\frac{t-1}{2}+1}^{t} \left|cos\left(\frac{2\pi j}{n}\right)\right| = \sum_{j=1}^{\frac{t-1}{2}} \left|cos\left(\frac{2\pi j}{n}\right)\right| + 0.5 \tag{4.59}$$

□

Theorem 4.8.3. *The energy of the cycle C_n is given by:*
1. For n even and $n = 2t$,
1.1 for t even, then

$$E(C_n) = \sum_{j=0}^{n-1} \left| 2cos\left(\frac{2\pi j}{n}\right)\right| = 4cot\left(\frac{\pi}{n}\right); \tag{4.60}$$

1.2 for t odd, then

$$E(C_n) = \sum_{j=0}^{n-1} \left| 2cos\left(\frac{2\pi j}{n}\right)\right| = 4cosec\left(\frac{\pi}{n}\right). \tag{4.61}$$

2. for n odd, $n = 2t + 1$;
2.1 for t even, then

$$E(C_n) = \sum_{j=0}^{n-1} \left| 2cos\left(\frac{2\pi j}{n}\right)\right| = 4\left[1 + \sum_{k=1}^{\frac{n-1}{4}} cos\left(\frac{2\pi k}{n}\right)\right] = 2cosec\left(\frac{\pi}{2n}\right) \tag{4.62}$$

2.2 for t odd, then

$$E(C_n) = \sum_{j=0}^{n-1} \left| 2cos\left(\frac{2\pi j}{n}\right)\right| = 4\left[1 + \sum_{k=1}^{\frac{n-3}{4}} cos\left(\frac{2\pi k}{n}\right)\right] = 2cosec\left(\frac{\pi}{2n}\right) \tag{4.63}$$

Proof. From Corollary 4.8.2, eigenvalues of the cycle C_n are:

$$2cos\left(\frac{2\pi j}{n}\right); j = 0, 1, \cdots, (n-1), \tag{4.64}$$

$$E(C_n) = \sum_{i=1}^{n} |\lambda_i| = \sum_{j=0}^{n-1} \left| 2cos\left(\frac{2\pi j}{n}\right)\right|. \tag{4.65}$$

1. Now let n be even, i.e $n = 2t$. Then

$$E(C_n) = \sum_{j=0}^{n-1} \left| 2cos\left(\frac{2\pi j}{n}\right)\right| = \sum_{j=0}^{2t-1} \left| 2cos\left(\frac{2\pi j}{n}\right)\right| \tag{4.66}$$

$$2\left[\left|cos\left(\frac{0}{n}\right)\right| + \left|cos\left(\frac{2\pi}{n}\right)\right| + \left|cos\left(\frac{4\pi}{n}\right)\right| + \cdots\right] \tag{4.67}$$

$$\left[\left|cos\left(\frac{2\pi t}{n}\right)\right| + \left|cos\left(\frac{2\pi(t+1)}{n}\right)\right|\right] + \cdots + \left[\left|cos\left(\frac{2\pi(2t-1)}{n}\right)\right|\right] \tag{4.68}$$

$$= 2\sum_{k=0}^{t-1}\left[\left|cos\left(\frac{2\pi k}{n}\right)\right| + \left|cos\left(\frac{2\pi(k+t)}{n}\right)\right|\right] \tag{4.69}$$

$$= 2\sum_{k=0}^{t-1}\left[\left|cos\left(\frac{2\pi k}{2t}\right)\right| + \left|cos\left(\frac{2\pi k}{2t} + \frac{2\pi t}{2t}\right)\right|\right] \tag{4.70}$$

$$= 2\sum_{k=0}^{t-1}\left[\left|cos\left(\frac{2\pi k}{2t}\right)\right| + \left|cos\left(\frac{2\pi k}{2t}\right) cos(\pi) - sin\left(\frac{2\pi k}{2t}\right) sin(\pi)\right|\right] \tag{4.71}$$

$$= 2\sum_{k=0}^{t-1}\left[\left|cos\left(\frac{2\pi k}{2t}\right)\right| + \left|-cos\left(\frac{2\pi k}{2t}\right)\right|\right] \tag{4.72}$$

$$= 2\sum_{k=0}^{t-1}\left|2cos\left(\frac{2\pi k}{2t}\right)\right| \tag{4.73}$$

Now,

$$E(C_n) = 2\sum_{k=0}^{2t-1}\left|2cos\left(\frac{2\pi j}{n}\right)\right| = 2\sum_{k=0}^{t-1}\left|2cos\left(\frac{2\pi k}{2t}\right)\right| = 2\sum_{k=0}^{t-1}\left|2cos\left(\frac{\pi k}{t}\right)\right| \tag{4.74}$$

$$= 2\left[|2cos(0)| + \sum_{k=1}^{t-1}\left|2cos\left(\frac{\pi k}{t}\right)\right|\right] \tag{4.75}$$

$$= 2\left[2 + \sum_{k=1}^{t-1}\left|2cos\left(\frac{\pi k}{t}\right)\right|\right] \tag{4.76}$$

setting $l = t - 1$, then

$$E(C_n) = 2\left[2 + \sum_{k=1}^{l}\left|2cos\left(\frac{\pi k}{l+1}\right)\right|\right] \tag{4.77}$$

1.1 For t even and l odd, we have from Theorem 4.8.1 result 2,

$$\sum_{k=1}^{l}\left|2cos\left(\frac{\pi k}{l+1}\right)\right| = 2\left[cot\left(\frac{\pi}{2(l+1)}\right) - 1\right] \tag{4.78}$$

So,

$$E(C_n) = 2\left[2 + \sum_{k=1}^{l}\left|2cos\left(\frac{\pi k}{l+1}\right)\right|\right] = 2\left[2 + 2\left(cot\left(\frac{\pi}{2(l+1)}\right) - 1\right)\right] \quad (4.79)$$

$$4\left(1 + cot\left(\frac{\pi}{2(t)}\right) - 1\right) = 4cot\left(\frac{\pi}{n}\right) \quad (4.80)$$

1.2 For t odd and l even, we have from Theorem 4.8.3 result 1,

$$\sum_{k=1}^{l}\left|2cos\left(\frac{\pi k}{l+1}\right)\right| = 2\left[cosec\left(\frac{\pi}{2(l+1)}\right) - 1\right] \quad (4.81)$$

So,

$$E(C_n) = 2\left[2 + \sum_{k=1}^{l}\left|2cos\left(\frac{\pi k}{l+1}\right)\right|\right] = 2\left[2 + 2\left(cosec\left(\frac{\pi}{2(l+1)}\right) - 1\right)\right] \quad (4.82)$$

$$= 4\left(1 + cosec\left(\frac{\pi}{2(t)}\right) - 1\right) = 4cosec\left(\frac{\pi}{n}\right) \quad (4.83)$$

2. Now let n be odd, ie. $n = 2t + 1$ Then

$$E(C_n) = \sum_{j=0}^{2t}\left|2cos\left(\frac{2\pi j}{n}\right)\right| = 2\left|cos\left(\frac{0}{n}\right)\right| + \sum_{j=1}^{2t}\left|2cos\left(\frac{2\pi j}{n}\right)\right| \quad (4.84)$$

$$= 2\left|cos\left(\frac{0}{n}\right)\right| + \sum_{j=1}^{t}\left[\left|2cos\left(\frac{2\pi j}{2t+1}\right)\right| + \left|2cos\left(\frac{2\pi(2t+1-j)}{2t+1}\right)\right|\right] \quad (4.85)$$

$$= 2 + \sum_{j=1}^{t}\left[\left|2cos\left(\frac{2\pi j}{2t+1}\right)\right| + \left|2cos\left(2\pi - \frac{2\pi j}{2t+1}\right)\right|\right] \quad (4.86)$$

$$= 2 + 2\sum_{j=1}^{t}\left[\left|2cos\left(\frac{2\pi j}{2t+1}\right)\right| + \left|cos(2\pi)cos\left(\frac{2\pi j}{2t+1}\right) - sin(2\pi)sin\left(\frac{2\pi j}{2t+1}\right)\right|\right] \quad (4.87)$$

$$= 2 + \sum_{j=1}^{t}\left[\left|cos\left(\frac{2\pi j}{2t+1}\right)\right| + \left|cos\left(\frac{2\pi j}{2t+1}\right)\right|\right] \quad (4.88)$$

$$= 2 + 4\sum_{j=1}^{t}\left[\left|cos\left(\frac{2\pi k}{n}\right)\right|\right] \tag{4.89}$$

2.1 Now for t even, we obtain

$$E(C_n) = 2 + 4\sum_{j=1}^{t}\left[\left|cos\left(\frac{2\pi k}{n}\right)\right|\right] \tag{4.90}$$

$$= 2 + 4\left[\sum_{j=1}^{\frac{t}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right| + \sum_{j=\frac{t}{2}+1}^{t}\left|cos\left(\frac{2\pi j}{n}\right)\right|\right] \tag{4.91}$$

Now from Lemma 4.8.2, for t even,

$$\sum_{j=\frac{t}{2}+1}^{t}\left|cos\left(\frac{2\pi j}{n}\right)\right| = \sum_{j=1}^{\frac{t}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right| + 0.5 \tag{4.92}$$

So,

$$E(C_n) = 2 + 4\left[\sum_{j=1}^{\frac{t}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right| + \sum_{j=1}^{\frac{t}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right| + 0.5\right] \tag{4.93}$$

$$= 4 + 8\left[\sum_{j=1}^{\frac{t}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right|\right] \tag{4.94}$$

$$= 4\left(1 + 2\left[\sum_{j=1}^{\frac{t}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right|\right]\right) \tag{4.95}$$

Now for $1 \leq j \leq \frac{t}{2}$, from theorem 4.8.1

$$2cos\left(\frac{2\pi(j)}{n}\right) \geq 0 \tag{4.96}$$

so,

$$E(C_n) = 4\left(1 + 2\left[\sum_{j=1}^{\frac{n-1}{4}} cos\left(\frac{2\pi j}{n}\right)\right]\right) = 4\left[1 + 2\sum_{j=1}^{\frac{n-1}{4}} cos\left(\frac{2\pi j}{n}\right)\right] = 2cosec\left(\frac{\pi}{2n}\right) \tag{4.97}$$

From theorem 4.8.3

2.2 Now for t odd, we get

$$E(C_n) = 2 + 4\sum_{j=1}^{t}\left|cos\left(\frac{2\pi j}{n}\right)\right| \tag{4.98}$$

$$= 2 + 4\left[\sum_{j=1}^{\frac{t-1}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right| + \sum_{j=\frac{t-1}{2}+1}^{t}\left|cos\left(\frac{2\pi j}{n}\right)\right|\right] \tag{4.99}$$

Now from Lemma 4.8.2, for t odd,

$$\sum_{j=\frac{t-1}{2}+1}^{t}\left|cos\left(\frac{2\pi j}{n}\right)\right| = \sum_{j=1}^{\frac{t-1}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right| + 0.5. \tag{4.100}$$

$$E(C_n) = 2 + 4\left[\sum_{j=1}^{\frac{t-1}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right| + \sum_{j=1}^{\frac{t-1}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right| + 0.5\right] \tag{4.101}$$

$$= 4 + 8\left[\sum_{j=1}^{\frac{t-1}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right|\right] \tag{4.102}$$

$$= 4\left(1 + 2\left[\sum_{j=1}^{\frac{t-1}{2}}\left|cos\left(\frac{2\pi j}{n}\right)\right|\right]\right) \tag{4.103}$$

Now for $1 \leq j \leq \frac{t}{2}$, so from theorem 4.8.1

$$E(C_n) = 4\left(1 + 2\sum_{j=1}^{\frac{t-1}{2}} 2cos\left(\frac{2\pi j}{n}\right)\right) = 4\left[1 + 2\sum_{j=1}^{\frac{n-3}{4}} cos\left(\frac{2\pi j}{n}\right)\right] = 2cosec\left(\frac{\pi}{2n}\right) \tag{4.104}$$

□

4.8.3 Energy of The Wheel Graph

Let G be the wheel graph W_n on n vertices, and with $n-1$ spokes, with $n \geq 4$.

corollary 4.8.3. *The eigenvalues of the wheel graph W_n are $\lambda = 1 \pm \sqrt{n}$ (each with multiplicity 1), and $\lambda_j = 2cos\left(\frac{2\pi j}{n-1}\right)$; $j = 1, 2, \cdots, (n-2)$(each with multiplicity 1). See Jessop [59]*

Theorem 4.8.4. *The energy of the wheel graph W_n For n even, is:*

$$E(W_n) = 2\sqrt{n} - 2 + 2cosec\left(\frac{\pi}{2(n-1)}\right) \tag{4.105}$$

for n odd, and $n = 2t + 1$
1.1 for t even, then

$$E(W_n) = 2\sqrt{n} - 2 + 4cot\left(\frac{\pi}{n-1}\right) \tag{4.106}$$

1.2 For t odd, then

$$E(W_n) = 2\sqrt{n} - 2 + 4cosec\left(\frac{\pi}{n-1}\right) \tag{4.107}$$

Proof. for $n \geq 4$, we have

$$E(W_n) = \sum_{1}^{n} |\lambda_i| \tag{4.108}$$

$$= \left|1 + \sqrt{n}\right| + \left|1 - \sqrt{n}\right| + \sum_{j=1}^{n-2} 2\left|cos\left(\frac{2\pi j}{n-1}\right)\right| \tag{4.109}$$

$$= 1 + \sqrt{n} - 1 + \sqrt{n} + \sum_{j=1}^{n-2} \left|2cos\left(\frac{2\pi j}{n-1}\right)\right| \tag{4.110}$$

$$= 2\sqrt{n} + \sum_{j=1}^{n-2} \left|2cos\left(\frac{2\pi j}{n-1}\right)\right| \tag{4.111}$$

setting $k = n - 1$, then

$$E(W_n) = 2\sqrt{k+1} + \sum_{j=1}^{k-1} \left|2cos\frac{2\pi j}{k}\right| \tag{4.112}$$

$$= 2\sqrt{k+1} + \sum_{j=0}^{k-1} \left|2cos\frac{2\pi j}{k}\right| - |2cos(0)| \tag{4.113}$$

$$= 2\sqrt{k+1} - 2 + \sum_{j=0}^{k-1} \left| 2cos \frac{2\pi j}{k} \right| \tag{4.114}$$

From Theorem 4.8.4 we get 1. For n even, k odd,

$$E(W_n) = 2\sqrt{k+1} - 2 + \sum_{j=0}^{k-1} \left| 2cos \frac{2\pi j}{k} \right| \tag{4.115}$$

$$= 2\sqrt{k+1} - 2 + 2cosec\left(\frac{\pi}{2k}\right) \tag{4.116}$$

$$2\sqrt{n} - 2 + 2cosec\left(\frac{\pi}{2(n-1)}\right) \tag{4.117}$$

This proves 1 of Theorem 4.8.4

2. For $n = 2t+1$, k even, and $k = 2t$,
2.1 For t even, then

$$E(W_n) = 2\sqrt{k+1} - 2 + 4cot\left(\frac{\pi}{k}\right) \tag{4.118}$$

Now $n = k+1 = 2t+1$, so n is odd. Therefore

$$E(W_n) = 2\sqrt{n} - 2 + 4cot\left(\frac{\pi}{n-1}\right)$$

This proves 2.1 of Thoerem 4.8.4
2.2 For t odd, then

$$E(W_n) = 2\sqrt{k+1} - 2 + 4cosec\left(\frac{\pi}{k}\right)$$

Now $n = k+1 = 2t+1$, so n is odd. Therefore

$$E(W_n) = 2\sqrt{n} - 2 + 4cosec\left(\frac{\pi}{n-1}\right)$$

This proves 2.2 of Thoerem 4.8.4 □

Lemma 4.8.3. *For large values of* n *the following expressions behave in the following way:*
1.

$$\lim_{n\longrightarrow\infty}\left[cosec\left(\frac{\pi}{n}\right)\right] \approx \frac{n}{\pi};\ n\ large$$

2.

$$\lim_{n \longrightarrow \infty} \left[cot \left(\frac{\pi}{n} \right) \right] \approx \frac{n}{\pi}; \; n \; large$$

Proof. We use the following results
(i) $\lim_{x \longrightarrow 0^+} sinx = x$
and
(ii) $\lim_{x \longrightarrow 0^+} cosx = 1.$

$$\lim_{n \longrightarrow \infty} \left[cosec \left(\frac{\pi}{n} \right) \right] = \lim_{n \longrightarrow \infty} \left[\frac{1}{sin \left(\frac{\pi}{n} \right)} \right] = \frac{1}{\frac{\pi}{n}} \approx \frac{n}{\pi} \tag{4.119}$$

$$2. \lim_{n \longrightarrow \infty} \left[cot \left(\frac{\pi}{n} \right) \right] = \lim_{n \longrightarrow \infty} \left[\frac{cos \left(\frac{\pi}{n} \right)}{sin \left(\frac{\pi}{n} \right)} \right] = \frac{1}{\frac{\pi}{n}} \approx \frac{n}{\pi} \tag{4.120}$$

□

Theorem 4.8.5. *For large n , the energy of cycles, paths and wheels (as classes denoted by $\Im$) is:*

$$\lim_{n \longrightarrow \infty} E(\Im) = \frac{4n}{\pi} \tag{4.121}$$

Proof. 1. For n even, $n = 2t$ and t even,

$$E(C_n) = \sum_{j=0}^{n-1} \left| 2cos \left(\frac{2\pi j}{n} \right) \right| = 4cot \left(\frac{\pi}{n} \right) \tag{4.122}$$

For large n the energy of the cycle C_n) will therefore be

$$\lim_{n \longrightarrow \infty} [E(C_n)] = \lim_{n \longrightarrow \infty} \left[4 \left(cot \frac{\pi}{n} \right) \right] \approx 4\frac{n}{\pi} = \frac{4n}{\pi} \tag{4.123}$$

2. For n even, $n = 2t$ and t odd,

$$E(C_n) = \sum_{j=0}^{n-1} \left| 2cos \left(\frac{2\pi j}{n} \right) \right| = 4cosec \left(\frac{\pi}{n} \right) \tag{4.124}$$

For large n the energy of the cycle C_n) will therefore be

$$\lim_{n \longrightarrow \infty} [E(C_n)] = \lim_{n \longrightarrow \infty} \left[4 \left(cosec \frac{\pi}{n} \right) \right] \approx 4\frac{n}{\pi} = \frac{4n}{\pi} \tag{4.125}$$

3. For n odd, $n = 2t + 1$ and all t,

$$E(C_n) = 2cosec\left(\frac{\pi}{2n}\right) \tag{4.126}$$

For large n the energy of the cycle C_n) will therefore be

$$\lim_{n\longrightarrow\infty}\left[2\left(cosec\frac{\pi}{2n}\right)\right] \approx 2\frac{2n}{\pi} = \frac{4n}{\pi} \tag{4.127}$$

Therefore, for large n, the energy of the cycle C_n is:

$$\lim_{n\longrightarrow\infty}[E(C_n)] = \frac{4n}{\pi} \tag{4.128}$$

The same can be applied to path and wheel graphs. □

4.9 Energy of The Line Graph of The Regular Caterpillar Graph

A caterpillar graph is a tree with the property that the removal of its end points leaves a path. An l regular caterpillar is obtained by attaching l pendant edges to each vertex of the path P_k (see section 2.2.18). A caterpillar graph is denoted by $CT(k, l)$ where k and l denote the number of vertices on the path and the number of pendant edges respectively. This graph will have $n = k(l + 1$ vertices and the adjacency matrix of the caterpillar graph is an $n \times n$ matrix and takes the general form:

$$A(CT(k,l)) = \begin{bmatrix} A(P_k) & I_{k,k} & I_{k,k} & \cdots & I_{k,k} \\ I_{k,k} & 0_{k,k} & 0_{k,k} & \cdots & 0_{k,k} \\ I_{k,k} & 0_{k,k} & 0_{k,k} & \cdots & 0_{k,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{k,k} & 0_{k,k} & 0_{k,k} & \cdots & 0_{k,k} \end{bmatrix}$$

where $I_{k,k}$ is repeated l times horizontally and l times vertically. For $l = 1$ the l-caterpillar graph $CT(k, l)$ has $n = 2k$ vertices, and an adjacency matrix of the form:

$$A(CT(k,1)) = \begin{bmatrix} A(P_k) & 1_{k,k} \\ 1_{k,k} & 0_{k,k} \end{bmatrix}$$

Now, by using the format of $A(CT(k, l))$ as above, we have the following by calculation:

For $l = 2$ the 1-caterpillar graph $CT(k, l)$ has $3k$ vertices, and an adjacency matrix of the form:

$$A(CT(k,2)) = \begin{bmatrix} A(P_k) & I_{k,k} & I_{k,k} \\ I_{k,k} & 0_{k,k} & 0_{k,k} \\ I_{k,k} & 0_{k,k} & 0_{k,k} \end{bmatrix}.$$

k	n	E(CT(k,l))
2	4	4.4721
3	6	6.8990
4	8	9.3317
5	10	11.7636
6	12	14.1957
7	14	16.6277
10	20	23.9237

Table 4.1: Energy of the Caterpillar graphs

Now, by using the format of $A(CT(k,2))$ as above, we have the following by calculation:

k	n	E(CT(k,2))
2	6	6
3	9	9.153
4	12	12.307
5	15	15.461
6	18	16.271
7	21	21.770
10	30	31.233

Table 4.2: Energy of the Caterpillar graphs

For the caterpillar graph $CT(k,l)$, its line graph is the sequence of k cliques $K_{l+1}, K_{l+2}, \cdots, K_{l+2}, K_{l+1}$ in this order, such that two consecutive cliques have exactly one vertex in common. This graph will have $n = kl + (k-1)$ vertices, and $m = \frac{k(k-1)}{2}(k-2) + 2\frac{(k-1)(k-2)}{2} = \frac{(k+2)(k-1)(k-2)}{2}$ edges.

Theorem 4.9.1. *The energy of $L(CT(k,l))$ is:*

$$E(L(CT(k,l))) = k(l-1) + \sum_{j=2}^{k} \left| \frac{1}{2} \left(l - 1 + \sigma_j - \sqrt{\sigma^2 + 2(l+1)\sigma_j + (l^2+6l+1)} \right) \right| \tag{4.129}$$

$$+ \sum_{j=1}^{k} \left| \frac{1}{2} \left(l - 1 + \sigma_j + \sqrt{\sigma^2 + 2(l+1)\sigma_j + (l^2+6l+1)} \right) \right| \tag{4.130}$$

Where

$$\sigma_j = 2cos\left(\frac{(k+1-j)\pi}{k}\right) \tag{4.131}$$

for $j = 1, \cdots, k$

Proof. The Laplacian eigenvalues of the caterpillar graph $CT(k,l)$ are given by subsection 3.5.7 and since $CT(k,l)$ is a bipartite graph, from section 3.6.2, the eigenvalues of $L(CT(k,l))$ can be derived from the Laplacian eigenvalues of $CT(k,l)$, namely

$$\mu = \lambda - 2 = 1 - 2 = -1,$$

with multiplicity $k(l-1)$.

$$\mu_j = \lambda_j - 2 = \left(12\left(l - 1 + \sigma_j - \sqrt{\sigma^2 + 2(l+1)\sigma_j + (l^2+6l+1)}\right)\right) \tag{4.132}$$

Where $\sigma_j = 2cos\left(\frac{(k+1-j)\pi}{k}\right)$, for $j = 2, \cdots, k$ and

$$\mu_{k+j} = \lambda_j - 2 = \left(\frac{1}{2}\left(l - 1 + \sigma_j + \sqrt{\sigma^2 + 2(l+1)\sigma_j + (l^2+6l+1)}\right)\right) \tag{4.133}$$

Where $\sigma_j = 2cos\left(\frac{(k+1-j)\pi}{k}\right)$, for $j = 1, \cdots, k$ and $\mu_{k+j} = \lambda_j - 2$
Therefore the energy of $L(CT(k,l))$ is:

$$E(L(CT(k,l))) = \sum_{1}^{n} |\lambda| = k(l-1) + \sum_{j=2}^{k} \left|\frac{1}{2}\left(l - 1 + \sigma_j - \sqrt{\sigma^2 + 2(l+1)\sigma_j + (l^2+6l+1)}\right)\right| \tag{4.134}$$

$$+ \sum_{j=1}^{k} \left|\frac{1}{2}\left(l - 1 + \sigma_j + \sqrt{\sigma^2 + 2(l+1)\sigma_j + (l^2+6l+1)}\right)\right| \tag{4.135}$$

Where

$$\sigma_j = 2cos\left(\frac{(k+1-j)\pi}{k}\right)$$

, for $j = 1, \cdots, k$. □

4.10 Energy of some other Graphs

4.10.1 Complete Graph

Let G be the complete graph K_n on n vertices, and with $\frac{n(n-1)}{2}$ edges. The eigenvalues of the complete graph K_n on n vertices are $\lambda = 1$ (with multiplicity $(n-1)$), and $\lambda - (n-1)$ with multiplicity 1(See Jessop [57] and section 3.5.1). So the energy of the complete graph is:

$$E(K_n) = 2(n-1) \tag{4.136}$$

.

4.10.2 Complete Split-Bipartite Graph $K_{\frac{n}{2},\frac{n}{2}}$

Let G be the complete split-bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ on vertices, and with $\frac{n^2}{4}$ edges. The eigenvalues of the complete split-bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ are $\lambda = 0$ (with multiplicity $n-2$) and $\lambda = \pm\frac{n}{2}$ (each with multiplicity 1), (See jessop2014matrices). So the energy of the complete split-bipartite graph is:

$$E\left(K_{\frac{n}{2},\frac{n}{2}}\right) = n \tag{4.137}$$

4.10.3 Star Graph, With Rays of Length 1

Let G be the star graph $S_{n-1,1}$ on n vertices, and with $n-1$ rays of length 1, $n \geq 2$. The eigenvalues of the star graph $S_{n-1,1}$ are $\lambda = 0$ (with multiplicity $n-2$) and $\lambda = \pm\sqrt{n-1}$ (each with multiplicity 1) (See jessop2014matrices).
So the energy of the star graph $S_{n-1,1}$ is:

$$E\left(S_{n-1,1}\right) = 2\sqrt{n-1} \tag{4.138}$$

4.10.4 Star Graph, With Rays of Length 2

Let G be the star graph $S_{\frac{n-1}{2},2}$ on n vertices, and with $\frac{n-1}{2}$ rays of length 2, $n \geq 3$. The eigenvalues of the star graph $S_{\frac{n-1}{2},2}$ are $\lambda = 0$ (with multiplicity 1), $\lambda = \pm 1$ (each with multiplicity $\frac{n-1}{2} - 1$) and $\lambda = \pm\sqrt{\frac{n-1}{2} + 1}$ (each with multiplicity 1)(see [59]).
So the energy of the star graph $S_{n-1,1}$ is:

$$E\left(S_{\frac{n-1}{2},1}\right) = n - 3 + \sqrt{2}\sqrt{n+1} \tag{4.139}$$

4.11 Hyperenergenic Graphs

A graph G having energy greater than the complete graph on the same number of vertices is called *hyperenergetic* i.e.

$$E(G) > 2(n-1) \tag{4.140}$$

(see Koolen Moulton [65]. Most of the classes of graphs all have the same energy of $\frac{4n}{\pi}$ for large values of n confirming their hypoenergetic nature.

4.12 Conclusion

In this chapter, we analyzed the energy of the path graph on n vertices, the energy of the cycle graph on n vertices, and the energy of the wheel graph on n vertices. We then used the relationship between the eigenvalues of the line graph of the caterpillar graph and the Laplacian eigenvalues of the caterpillar graph, in order to calculate the energy of the line graph of the caterpillar graph. We expressed the energy of cycles, paths and wheels in terms of simplified expressions using the cotangent or

the cosecant, and showed that these classes of graphs all have the same energy of $\frac{4n}{\pi}$ for large values of n confirming their hypoenergetic nature.
The expressions for the energy of some other types of graphs were then stated, based on the known eigenvalues of these graphs; namely the complete graph, the complete bi-partite graph, the star graph with rays of length 1, the star graph with rays of length 2, the lollipop graph, the dual star graph.
We also presented the energy of strongly regular graphs and the line graph of the complete graph.
The energies of the various types of graphs was calculated and we concluded, that, except for strongly regular graphs and the line graph of the complete graph, the graphs analyzed are hyperenergetic (see Koolen Moulton Gutman[63]). Thus the complete graph does not have the greatest energy on n vertices for the types of graphs analyzed.
However, the complete graph is significant in that it is mostly used when determining graph theoretical properties involving a definition such as diameter, radius etc. We address the relevance of this issue in the next chapter.

Chapter 5

The Eigen-Complete Difference Ratio Of Classes Of Graphs-Domination, Asymptotes And Area

5.1 Introduction

As discussed in the previous chapter the energy of a graph is the sum of the absolute values of the eigenvalues of the adjacency matrix of the graph in consideration. This quantity is studied in the context of spectral graph theory. In short, for an n -vertex graph G with adjacency matrix A having eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$, the energy $E(G)$ is defined as:

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \,. \tag{5.1}$$

It is related to the sum of π -electron energy in a [**?**] molecule represented by a molecular graph- i.e. a graph where the vertices represent atoms and the edges bonds between atoms. If we know some chemistry, then we might fully appreciate the origin of graph energy. In a private communication, Gutman (see [42]) claimed that the HMO (Huckel molecular orbital) theory is nowadays superseded by new theories that provide better explanations and which do not make unnecessary assumptions. Graph energy became a very popular topic of mathematical research; this is evident in the reviews and recent papers. In the paper Energy of Graphs by Brualdi [13] the difference of the energy of two graphs G and H on the same number n of vertices was presented- we adopt this idea by making one of the graphs the complete graph. Although the complete graph K_n does not have the maximum energy of all graphs (see [51]), it is a very important and well-studied class of graphs – for example it has a high degree of connectivity and robustness. Given that the complete graph does not have the largest energy of all graphs it is customary (see hypoenergetic graphs), i.e. graphs with energy less than the complete graph, in (Winter and Sarvate [102]) to see how the energy of other classes of graphs compare to that of the energy of the complete graph.

Thus, one would like to compare its energy with the energy of any other graph G in terms of how close their energies are, and how the energy of G compares with the energy of K_n where a large number of vertices are involved. This provided the motivation for the eigen-complete difference ratio of a graph (see Winter and Ojako [112]) which we present in this chapter, which is original in its entire.
This energy idea can be translated to that of molecules made up of atoms with bonds, where we map the atoms to vertices and bonds to edges, and allow one to investigate how other molecular energies compare with that of a molecule with all possible bonds between atoms –i.e. a strongly bonded molecule. The eigen-complete difference ratio thus allows for the investigation of the domination effect of the energy of graphs on the energy of the complete graph when a large number of vertices are involved. We found that this domination effect is the greatest negatively (positively) for a strongly regular graph (star graphs with rays of length one), and is zero for the lollipop graph. For paths, cycles and wheel graphs, the domination effect is $\frac{\pi-2}{\pi}$, which provides a mathematical link to the idea of π-electron energy associated with such molecules.

5.2 Ratios and graphs

Ratios have been an important aspect of graph theoretical definitions. Examples of ratios are: expanders, (see Alon and Spencer [3], the central ratio of a graph [14], eigen-pair ratio of classes of graphs [111], Independence and Hall ratios [90], tree-cover ratio of graphs [110], eigen-energy formation ratio [102] and t-compete sequence ratio [101], the chromatic-cover ratio [108], the tree-3-cover ratio [110], the chromatic-complete difference ratio [107], graph theory and calculus: ratios of classes of graphs [**?**],the eigen-cover ratio [113] and the eigen-3-cover ratio [**?**]. We now introduce the idea of ratio, asymptotes and areas involving energy difference between the complete graph and G.

5.2.1 Eigen-Complete Difference Ratio- Asymptotes, Domination Effect And Area

Let K_n be the complete graph on n vertices.

Definition 5.2.1. *The difference between the energy of K_n and a connected graph G on the same number of vertices n is given by:*

$$\langle D_n^G = E(K_n) - E(G) \tag{5.2}$$

This is called the eigen-complete difference associated with G. If the graph G in consideration belongs to a class $\Im$ of graphs of order n, then the complete-energy difference associated with $\Im$ is defined as:

$$\langle D_n^{\Im} = E(K_n) - E(G); G \in \Im. \tag{5.3}$$

Now, dividing the eigen complete difference by the energy of K_n will give an average of the eigen-complete difference with respect to G. This provides motivation for the following definition.

Definition 5.2.2. *The eigen-complete difference ratio with respect to $G(\Im)$, is defined as*

$$Rat\langle D_n^G = \frac{E(K_n) - E(G)}{E(K_n)}; Rat\langle D_n^{\Im} = \frac{E(K_n) - E(G)}{E(K_n)}; G \in \Im \tag{5.4}$$

*This ratio allows for the definition of the eigen-complete difference asymptote and the eigen-complete difference area, similar to that of : the eigen-pair ratio of classes of graphs [111], the tree-cover ratio of graphs [110], the eigen-energy formation ratio [102]), the t-compete sequence ratio [101], the chromatic-cover ratio [108], the tree-3-cover ratio [110], the chromatic-complete difference ratio [108], graph theory and calculus: ratios of classes of graphs ([**?**],the eigen-cover ratio ([113] and the eigen-3-cover ratio (see [**?**].*

Definition 5.2.3. *If the eigen-complete difference ratio is a function $f(n)$ of the order of $G \in \Im$, then its horizontal asymptote results in the eigen-complete difference asymptote:*

$$Asymrat\langle D_n^{\Im} = \lim_{n\to\infty} \left[\frac{E(K_n) - E(G)}{E(G)}\right]; G \in \Im \tag{5.5}$$

This asymptote allows for the investigation of the effect of the energy of a graph G on the complete graph when a large number of vertices are involved, referred to as the eigen-complete difference domination effect.

Definition 5.2.4. *Attaching the average degree of graph G with m' edges, to the Riemann integral of:*

$$RatD_n^{\Im} = \frac{E(K_n) - E(G)}{E(K_n)}; G \in \Im \tag{5.6}$$

we obtain the eigen-complete difference area:

$$Arat\langle D_n^{\Im} = \frac{2m'}{n} \left| \int \left[\frac{E(K_n) - E(G)}{E(K_n)}\right] dn \right| \tag{5.7}$$

; with $Arat\langle D_k^{\Im} = 0$.
The average degree is referred to as the "length of the area", while the integral part is the "height of the area".

Lemma 5.2.1. *The eigen-complete difference ratio can take on one of the following:*

$$E(G) < E(K_n) \Rightarrow Rat\langle D_n^{\Im} = \frac{E(K_n) - E(G)}{E(K_n)} = 1 - \frac{E(G)}{E(K_n)} > 0; G \in \Im \tag{5.8}$$

$$E(G) < E(K_n) \Rightarrow Rat\langle D_n^{\Im} = \frac{E(K_n) - E(G)}{E(K_n)} = 1 - \frac{E(G)}{E(K_n)} < 0; G \in \Im \tag{5.9}$$

$$E(G) < E(K_n) \Rightarrow Rat\langle D_n^{\Im} = \frac{E(K_n) - E(G)}{E(K_n)} = 0; G \in \Im \tag{5.10}$$

If $G_p = \Im$ is a polyhedral graph, then from Theorem 4.7.1

$$\sqrt{2(n+2)} \leq E(G_p) \leq \sqrt{6n(n-1)} \Rightarrow \frac{2n-2-\sqrt{6n(n-1)}}{2n-2} \leq \tag{5.11}$$

$$Rat\langle D_n^{\Im} \leq \frac{2n-2-\sqrt{2(n+1)}}{2n-2} \Rightarrow \frac{2-\sqrt{6}}{2} \leq Asymrat\langle D_n^{\Im} \leq 1 \tag{5.12}$$

5.3 Examples

5.3.1 The complete split-bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$

The energy of this graph is n (see section 4.10.2) and it has $\frac{n^2}{4}$ edges while that of the complete graph is $2n-2$ (see section 4.10.1) so that:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n) - E(K_{\frac{n}{2},\frac{n}{2}})}{2n-2}\right] = \frac{(2n-2)-n}{2n-2} = \frac{n-2}{2n-2} \tag{5.13}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n \longrightarrow \infty} \left[\frac{n-2}{2n-2}\right] = \frac{1}{2} \tag{5.14}$$

$$Arat\langle D_n^{\Im} = \frac{n}{2} \int \left[\frac{n-2}{2n-2}\right] dn = \frac{n}{4} \int \frac{n-2}{n-1} dn = \frac{n}{4}(n - ln(n-1) + c) \tag{5.15}$$

with smallest order 2 we have: $c = -2$

5.3.2 The Star Graph $K_{1,n-1}$ with $n-1$ Rays of Length 1

The energy of this star graph is $2\sqrt{n-1}$ (see section 4.10.3) so that:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n) - E(K_{1,n-1}}{2n-2}\right] = \frac{(2n-2)-2\sqrt{n-1}}{2(n-1)} = 1 - \frac{1}{\sqrt{n-1}} \tag{5.16}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n \longrightarrow \infty} \left[1 - \frac{1}{\sqrt{n-1}}\right] = 1 \tag{5.17}$$

$$AratD_n^{\Im} = \frac{2(n-1)}{n} \int \left[1 - \frac{1}{\sqrt{n-1}}\right] dn = \frac{2(n-1)}{n} \left[n - 2\sqrt{n-1} + c\right] \tag{5.18}$$

. With smallest star graph on 2 vertices we have: $c = 0$

5.3.3 Star Graphs $S_{r,2}$ With r Rays Of Length 2

The energy of this star graph with $r = n-1$ edges is (see section 4.10.4): $n-3+\sqrt{2}\sqrt{n+1}$ so that:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(K_{1,n-1}}{2n-2}\right] = \frac{(2n-2)-(n-3+\sqrt{2}\sqrt{n+1})}{2n-2} = \frac{n+1-\sqrt{2}\sqrt{n+1}}{2n-2} \tag{5.19}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n\longrightarrow\infty}\left[\frac{n+1-\sqrt{2}\sqrt{n+1}}{2n-2}\right] = \frac{1}{2} \tag{5.20}$$

$$Arat\langle D_n^{\Im} = \frac{2(n-1)}{n}\int\left[\frac{n+1-\sqrt{2}\sqrt{n+1}}{2n-2}\right]dn \tag{5.21}$$

$$= \frac{2(n-1)}{n}\int\left[\frac{n-1}{2(n-1)}+\frac{2}{2(n-1)}-\frac{1}{\sqrt{2}}\frac{\sqrt{n+1}}{\sqrt{n-1}}\right]dn \tag{5.22}$$

$$= \frac{2(n-1)}{n}\left[\frac{n}{2}+in(n-1)-\frac{1}{\sqrt{2}}.A+C\right] \tag{5.23}$$

$$A = \int\frac{\sqrt{n+1}}{\sqrt{n-1}}dn = \tag{5.24}$$

let $n = u^2+1 \Longrightarrow dn \Longrightarrow= 2udu$

$$\Longrightarrow A = \int\frac{\sqrt{u^2+2}}{u}2udu$$

put $u = \sqrt{2}tant$ so that

$$A = 4\int sec^3tdt = 4\left[\frac{secttant+in(sect+tant}{2}\right] \tag{5.25}$$

Thus:

$$A = 2\left[\frac{\sqrt{n-1}}{\sqrt{2}}\frac{\sqrt{n+1}}{\sqrt{2}}+in\left(\frac{\sqrt{n+1}}{\sqrt{2}}+\frac{\sqrt{n-1}}{\sqrt{2}}\right)\right] \tag{5.26}$$

$$= \sqrt{n^2-1}+2in\left(\frac{\sqrt{n+1}}{\sqrt{2}}+\frac{\sqrt{n-1}}{\sqrt{2}}\right) \tag{5.27}$$

The smallest such star graph is on 3 vertices so that

$$A = \sqrt{8}+2ln\left(\frac{\sqrt{4}+\sqrt{2}}{\sqrt{2}}\right) = \sqrt{8}+2ln\left(\sqrt{2}+1\right) \tag{5.28}$$

Thus;

$$c = 2 + \sqrt{2}ln\left(1+\sqrt{2}\right) - \frac{3}{2} - ln2 \tag{5.29}$$

And eigen-complete difference area is:

$$= \frac{2(n-1)}{n}\left[\frac{n}{2} + ln(n-1) - \frac{1}{\sqrt{2}}\left(\sqrt{n^2-1} + 2ln\left(\frac{\sqrt{n+1}}{\sqrt{2}} + \frac{\sqrt{n-1}}{\sqrt{2}}\right)\right)\right] \tag{5.30}$$

$$\left[+2 + \sqrt{2}ln(1+\sqrt{2} - \frac{3}{2} - ln2\right] \tag{5.31}$$

5.3.4 The Line Graph Of K_n

The line graph $L(K_n)$ of K_n has $p = \frac{n(n-1)}{2}$ vertices and energy $2n^2 - 6n$ (see section 4.5). The number q of edges is the sum of the square of the degrees minus the number of edges of K_n (see brualdi2006energy). Hence,

$$q = n\frac{(n-1)^2}{2} - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}[n-1-1] = \frac{n(n-1)(n-2)}{2} \tag{5.32}$$

$$= 2n^2 - 6n = 4\frac{n(n-1)}{2} - 4n = 4p - 4n \tag{5.33}$$

$$n^2 - n - 2p = 0 \Rightarrow n = \frac{1 \pm \sqrt{1+8p}}{2} = \frac{1+\sqrt{1+8p}}{2} \tag{5.34}$$

Thus:

$$E(L(K_n)) = 2n^2 - 6n = 4\frac{n(n-1}{2} - 4n = 4p - 2 - 2\sqrt{1+8p} \tag{5.35}$$

$$Rat\langle D_p^{\Im} = \left[\frac{E(K_p) - E(L(K_n))}{E(K_{p)}}\right] = \frac{2p-2-4p+2+\sqrt{1+8p}}{2p-2} = \frac{-2p+\sqrt{1+8p}}{2p-2} \tag{5.36}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n\to\infty}\left[\frac{-2p-\sqrt{1+8p}}{2p-2}\right] = -1 \tag{5.37}$$

$$Arat\langle D_n^{\Im} = \frac{2q}{p}\left|\int \frac{-2p+\sqrt{1+8p}}{2p-2}dp\right| = \frac{2q}{p}\int \frac{-2p+\sqrt{1+8p}}{2p-2}dp \tag{5.38}$$

$$n\frac{2q}{p}\int\left[-\frac{2p-2}{2p-2} + \frac{-2+\sqrt{1+8p}}{2p-2}\right]dp; u^2 = 1+8p \Longrightarrow dp = \frac{udu}{4}; p = \frac{u^2-1}{8} \tag{5.39}$$

$$= \frac{2q}{p}\left[(-p) + \int \frac{(-2+u)}{\left(\frac{u^2-1}{4} - 2\right)}\frac{udu}{4}\right] \tag{5.40}$$

$$= \frac{2q}{p}\left[(-p) + \int \frac{(-2+u)}{\frac{u^2-9}{4}}\frac{udu}{4}\right] = \frac{2q}{p}\left[(-p) + \int \frac{u^2-2}{u^2-9}du\right] \tag{5.41}$$

$$\frac{2q}{p}\left[(-p) + \int du + \int \frac{7}{u^2-9}du\right] = \frac{2q}{p}\left|(-p) + \int du + \int \frac{7}{u^2-9}du\right| \tag{5.42}$$

$$\frac{2q}{p}\left[p - \sqrt{1+8p} - \frac{7}{6}\left(ln(\sqrt{1+8p}-3)\right) + \frac{7}{6}ln(\sqrt{1+8p}+3)\right] + c \tag{5.43}$$

$p = 3$ yields

$$c = -3 + 5 + \frac{7}{6}ln(2) - \frac{7}{6}ln(8) = 2 + \frac{7}{6}ln(2) - \frac{7}{6}ln(8) \tag{5.44}$$

Thus, the eigen-complete area of the line graph of K_n on p vertices is:

$$\frac{2q}{p}\left[p - \sqrt{1+8p} - \frac{7}{6}ln(\sqrt{1+8p}-3) + \frac{7}{6}ln(\sqrt{1+8p}+3) + 2 + \frac{7}{6}ln(2) - \frac{7}{6}ln(8)\right] \tag{5.45}$$

5.3.5 Strongly Regular Graphs

Koolen and Moulton [65] have proved that the energy of a graph on n vertices is at most $\frac{n(1+\sqrt{n}}{2}$, and that equality holds if and only if the graph is strongly regular with parameters $\frac{n(1+\sqrt{n})}{2}$, $\frac{(n+2\sqrt{n}}{4}$, $\frac{(n+2\sqrt{n}}{4}$ (see section 4.4).
Its energy is $\frac{n(1+\sqrt{n}}{2}$ and to find the number of edges m' we use:

$$\sum_1^n d(v) = 2m' \Longrightarrow n\frac{(n+\sqrt{n})}{2} = 2m' \Rightarrow m' = \frac{n(n+\sqrt{n})}{4} \tag{5.46}$$

Thus,

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n - E(SR(G)}{2n-2}\right] = \frac{(2n-2) - \frac{n(1+\sqrt{n})}{2}}{2n-2} = \frac{3n - n\sqrt{n} - 4}{4n-4} \tag{5.47}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n\to\infty} \frac{3n - n\sqrt{n} - 4}{4n-4} = -\infty \tag{5.48}$$

$$Arat\langle D_n^{\Im} \frac{2m'}{n}\left|\int \left[\frac{3n - n\sqrt{n} - 4}{4n-4}\right] dn\right| \tag{5.49}$$

$$= \frac{(n+\sqrt{n})}{2}\left|\int\left[\frac{3n-n\sqrt{n}-4}{4n-4}\right]dn\right| \tag{5.50}$$

$$\frac{(n+\sqrt{n}}{2}\int\left[\frac{3}{4}\left(\frac{4n-4}{4n-4}\right)-\left(\frac{1+n\sqrt{n}}{4n-4}\right)\right]dn = \frac{(n+\sqrt{n}}{2}\left[\frac{3}{4}n-\int\frac{1+n\sqrt{n}}{4n-4}dn\right] \tag{5.51}$$

Now,

$$\int\frac{1=n\sqrt{n}}{4n-4}dn = \frac{1}{4}\ln(n-1)+\frac{1}{4}\int\frac{n\sqrt{n}}{n-1}dn \tag{5.52}$$

Setting $n = u^2$, we have

$$= \frac{1}{4}ln(n-1)+\frac{1}{4}\int\frac{u^2u}{u^2-1}2udu \tag{5.53}$$

But

$$\frac{1}{2}\int\frac{u^4}{u^2-1}du\frac{1}{2}\int u^2\frac{u^2-1}{u^2-1}+\frac{u^2}{u^2-1}du \tag{5.54}$$

$$= \frac{1}{6}u^3+\frac{1}{2}\int\frac{u^2-1}{u^2-1}+\frac{1}{u^2-1}du \tag{5.55}$$

$$= \frac{1}{6}u^3+\frac{1}{2}u+\frac{1}{2}ln\frac{u-1}{u+1} = \frac{1}{6}n^{\frac{3}{2}}+\frac{1}{2}\sqrt{n}+\frac{1}{2}ln\frac{\sqrt{n}-1}{\sqrt{n}+1} \tag{5.56}$$

So the eigen-complete area (without absolute sign) is:

$$\frac{(n+\sqrt{n}}{2}\left[\frac{3}{4}n-\frac{1}{4}ln(n-1)-\frac{1}{6}n^{\frac{3}{2}}-\frac{1}{2}\sqrt{n}-\frac{1}{2}ln\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)+c\right] \tag{5.57}$$

The term $n^{\frac{3}{2}}$ dominates for large n, so introducing absolute sign:

$$\frac{(n+\sqrt{n}}{2}\left[\frac{1}{6}n^{\frac{3}{2}}-\frac{3}{4}n+\frac{1}{4}ln(n-1)+\frac{1}{2}\sqrt{n}+\frac{1}{2}ln\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)+c\right] \tag{5.58}$$

5.3.6 Lollipop Graph

The energy of the lollipop graph with base the complete graph on $n-1$ vertices is (see section 4.2):

$$(n-3)+\sqrt{n^2-4n+8} \tag{5.59}$$

To find $\sum dv$ for this graph we have:

$$\sum dv = (n-2)(n-2)+(n-1)+1 = (n-2)(n-2)+n = n^2-3n+4 = 2m' \tag{5.60}$$

Thus,

$$RatD_n^{\Im} = \left[\frac{E(K_n) - E(LP(G))}{2n-2}\right] = \frac{(2n-2) - \left[(n-3) + \sqrt{n^2-4n+8}\right]}{2n-2} = \frac{n+1-\sqrt{n^2-4n+8}}{2n-2} \tag{5.61}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n \longrightarrow \infty} \left[\frac{n+1-\sqrt{n^2-4n+8}}{2n-2}\right] = \tag{5.62}$$

Multiply top and bottom of by $n+1+\sqrt{n^2-4n+8}$ to obtain

$$\frac{(n+1)^2 - (n^2-4n+8)}{(2n-2)(n+1+\sqrt{n^2-4n+8})} \tag{5.63}$$

For large n, the numerator is of order $6n$ and the denominator is of order $4n^2$ so that:

$$Asymrat\langle D_n^{\Im} = \lim_{n \longrightarrow \infty} \left[\frac{6n}{4n^2}\right] = 0 \tag{5.64}$$

$$Arat\langle D_n^{\Im} = \frac{2m'}{n} \int \left[\frac{n+1-\sqrt{n^2-4n+8}}{2n-2}\right] dn = \frac{m'}{n} \int \frac{n+1-\sqrt{n^2-4n+8}}{n-1} dn \tag{5.65}$$

$$= \frac{m'}{n} \int \left[\frac{n-1}{n-1} + \frac{2-\sqrt{n^2-4n+8}}{n-1}\right] dn = \frac{m'}{n} \left[n + \int \frac{2-\sqrt{(n-2)^2+4}}{n-1}\right] dn. \tag{5.66}$$

5.3.7 The Line Graph Of The L-Regular Caterpillar Graph

The energy of $L(CT(k,l))$ is (see section 4.9):

$$E(L(CT(k,l))) = k(l-1) + \sum_{j=2}^{k} \left| \frac{1}{2} \left(l - 1 + \sigma_j - \sqrt{\sigma_j^2 + 2(l+1)\sigma_j + (l^2+6l+1)}\right) \right| + \tag{5.67}$$

$$\sum_{j=2}^{k} \left| \frac{1}{2} \left(l - 1 + \sigma_j + \sqrt{\sigma_j^2 + 2(l+1)\sigma_j + (l^2+6l+1)}\right) \right| \tag{5.68}$$

Where

$$\sigma_j = 2cos\left(\frac{(k+1-j)\pi}{k}\right) \tag{5.69}$$

for $j = 1, 2, \cdots, k$
if we take $l = 2$ then $n = 3k$
so that:

$$E\left(L\left(CT\left(\frac{n}{3},1\right)\right)\right) = \frac{n}{3}(2-1) + \sum_{j=2}^{\frac{n}{3}} \left| \frac{1}{2}\left(2 - 1 + \sigma^j - \sqrt{\sigma^2 + 2(2+1)\sigma_j + (4+12+1}\right) \right| \quad (5.70)$$

$$+ \sum_{j=2}^{\frac{n}{3}} \left| \frac{1}{2}\left(2 - 1 + \sigma^j + \sqrt{\sigma^2 + 2(2+1)\sigma_j + (4+12+1}\right) \right| \quad (5.71)$$

where,

$$\sigma_j = 2cos\left(\frac{\left(\frac{n}{3} + 1 - j\right)\pi}{\frac{n}{3}}\right) \ for\ j = 1, \cdots, k \quad (5.72)$$

for large $n = n^L$ and

$$\sigma_j = 2cos\left(\frac{\left(\frac{n}{3} + 1 - j\right)\pi}{\frac{n}{3}}\right) \leq 2 \quad (5.73)$$

The energy behaves like:

$$E\left(L\left(CT\left(\frac{n}{3},1\right)\right)\right) \approx \frac{n^L}{3} + \sum_{j=2}^{\frac{n^L}{3}} \left| \frac{1}{2}\left(1 + \sigma_j - \sqrt{\sigma_j^2}\right) \right| \quad (5.74)$$

$$+ \sum_{j=2}^{\frac{n^L}{3}} \left| \frac{1}{2}\left(1 + \sigma_j + \sqrt{\sigma_j^2}\right) \right| \quad (5.75)$$

$$= \frac{n^L}{3} + \frac{1}{2}\left(\frac{n^L}{3} - 1\right) + \frac{1}{2}\left(\frac{n^L}{3} + 4\right) \approx \frac{2n^L}{3} \quad (5.76)$$

Thus for n large and $\Im = \left(L\left(CT\left(\frac{n}{3},1\right)\right)\right)$ we have

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n) - E\left(L\left(CT\left(\frac{n}{3},1\right)\right)\right)}{2n-2}\right] = \frac{(2n-2) - \frac{2n}{3}}{2n-2} \quad (5.77)$$

$$= \frac{\frac{4n}{3} - 2}{2n-2} = \frac{4n-6}{6n-6} = \frac{2n-3}{3n-3} \quad (5.78)$$

$$Asymrat\langle D_n^{\Im} = \lim_{n \longrightarrow \infty} \left[\frac{4n-6}{6n-6}\right] = \frac{2}{3} \quad (5.79)$$

$$Arat\langle D_n^{\Im} = \frac{2(n-1)}{n} \int \frac{1}{3}\left[\frac{2n-3}{n-1}\right] dn \quad (5.80)$$

$$= \frac{2(n-1)}{n} \int \left[\frac{2}{3}\frac{(n-1)}{n-1)} - \frac{1}{3(n-1)}\right] dn = \frac{2(n-1)}{n}\left[\frac{2}{3}n - \frac{1}{3}ln(n-1) + c\right]; c = -\frac{4}{3} \quad (5.81)$$

5.3.8 Dual Star Graphs

The energy of the dual star graph on n vertices is (Theorem 4.2):

$$E(DuS_n) = 2\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}} + 2\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}} \tag{5.82}$$

Thus the eigen-complete difference ratio, asymptote and area are:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(DuS_n)}{2n-2}\right] = \frac{(2n-2)-2\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}}+2\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}}{2n-2} \tag{5.83}$$

$$AsymratD_n^{\Im} = \lim_{n\longrightarrow\infty}\left[\frac{(2n-2)-\left[2\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}}+2\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}\right]}{2n-2}\right] = 1 \tag{5.84}$$

$$Arat\langle D_n^{\Im} = \frac{2(n-1)}{n}\int \frac{1}{3}\left[\frac{(2n-2)-\left[2\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}}+2\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}\right]}{2n-2}\right] dn \tag{5.85}$$

5.3.9 Paths, Cycles And Wheels On An Even Number Of Vertices

We illustrate how one can determine the eigen-complete difference ratio, and its associated aspects, of paths, cycles and wheels on an even number of vertices using the results found in chapter 4.
1. for n even,

$$E(P_n) = \sum_{j=1}^{n}\left|2cos\left(\frac{\pi j}{n+1}\right)\right| = 2\left[cosec\left(\frac{\pi}{2(n+1}-1\right)-1\right] \tag{5.86}$$

see section 4.8.3
2. for n even, n=2t, and t even,

$$E(C_n) = \sum_{j=0}^{n-1}\left|2cos\left(\frac{2\pi j}{n}\right)\right| = 4cot\left(\frac{\pi}{n}-1\right) \tag{5.87}$$

see section 4.8.5
3. for n even, $n=2t$ then

$$E(W_n) = 2\sqrt{n}-2+2cosec\left(\frac{\pi}{2(n-1}\right) \tag{5.88}$$

see section 4.8.7
The lemma 4.8.3, will simplify the eigen-complete difference ratio when a large number of vertices are involved.

The path graph

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n) - E(P_n)}{2n-2}\right] = \frac{(2n-2) - 2\left[cosec\left(\frac{\pi}{2(n+1)}\right) - 1\right]}{2n-2} \tag{5.89}$$

$$= \frac{(n-1) - \left[cosec\left(\frac{\pi}{2(n+1)}\right) - 1\right]}{n-1} \tag{5.90}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n\longrightarrow\infty} \frac{(n-1) - \left[cosec\left(\frac{\pi}{2(n+1)}\right) - 1\right]}{n-1} \tag{5.91}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n\longrightarrow\infty} \frac{(n-1) - \frac{2n}{\pi}}{n-1} \approx \frac{\pi-2}{\pi} \tag{5.92}$$

From Lemma (4.8.3)

$$Arat\langle D_n^{\Im} = \frac{2(n-1)}{n} \int \left[\frac{(n-1) - \left[cosec\left(\frac{\pi}{2(n+1)}\right) - 1\right]}{n-1}\right] dn \tag{5.93}$$

The cycle graph

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n) - E(C_n)}{2n-2}\right] = \frac{(2n-2) - 4cot\left(\frac{\pi}{n}\right)}{2n-2} \tag{5.94}$$

$$= \frac{(n-1) - 2cot\left(\frac{\pi}{n}\right)}{n-1} \tag{5.95}$$

$$Asyrat\langle D_n^{\Im} = \lim_{n\longrightarrow\infty} \frac{(n-1) - 2cot\left(\frac{\pi}{n}\right)}{n-1} \tag{5.96}$$

$$Asyrat\langle D_n^{\Im} = \lim_{n\longrightarrow\infty} \frac{(n-1) - 2cot\left(\frac{\pi}{n}\right)}{n-1} \approx \frac{(n-1) - 2\left(\frac{\pi}{n}\right)}{n-1} = \frac{\pi-2}{\pi} \tag{5.97}$$

by lemma 4.8.3.

$$Arat\langle D_n^{\Im} = 2\int \left[\frac{(n-1) - 4cot\left(\frac{\pi}{n}\right)}{n-1}\right] dn \tag{5.98}$$

The wheel graph

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n) - E(W_n)}{2n-2}\right] \tag{5.99}$$

$$= \frac{(2n-2) - \left[2\sqrt{n} + 2 + 8\sum_{j=1}^{\frac{n-4}{4}} cos\left(\frac{2\pi j}{n-1}\right)\right]}{2n-2} \tag{5.100}$$

$$\frac{(n-1) - \left[\sqrt{n} + 1 + 4\sum_{j=1}^{\frac{n-4}{4}} cos\left(\frac{2\pi j}{n-1}\right)\right]}{n-1} \tag{5.101}$$

$$Asymrat\langle D_n^{\Im} = \lim_{v \longrightarrow \infty} \frac{(n-1) - \left[\sqrt{n} + 1 + 4\sum_{j=1}^{\frac{n-4}{4}} cos\left(\frac{2\pi j}{n-1}\right)\right]}{n-1} \tag{5.102}$$

$$\approx \frac{(n-1-2\left(\frac{n}{\pi}\right)}{n-1} = \frac{\pi-2}{\pi} bylemma 4.8.3 \tag{5.103}$$

$$Arat\langle D_n^{\Im} = 2\left(\frac{2n-2}{n}\right)\int\left[\frac{(n-1) - \left[\sqrt{n} + 1 + 4\sum_{j=1}^{\frac{n-4}{4}} cos\left(\frac{2\pi j}{n-1}\right)\right]}{n-1}\right] dn \tag{5.104}$$

. The same eigen-complete difference ratio will be obtained for cycles, paths and wheels on an odd number of vertices.
Thus for paths, cycles, and wheels, we have shown that their eigen-complete difference ratio converges to: $\frac{\pi-2}{\pi} = 1 - \frac{2}{\pi}$.
We show, in the theorem below, that this value is equivalent to the solution $y = f(\pi)$ of a separable differential equation.

Theorem 5.3.1. *If $y = f(x)$ is a solution of the separable variable differential equation:*

$$y + \left(\frac{1}{x}\right)\frac{dy}{dx} = \frac{1}{x}; y(3) = \frac{1}{3} \tag{5.105}$$

then the asymptotic convergence, of the eigen-complete difference ratio of paths and cycles on an even number of vertices, is given by:

$$y = f(\pi) = \frac{\pi-2}{\pi}$$

.

Proof.

$$y + \left(\frac{1}{x}\right)\frac{dy}{dx} = \frac{1}{x};\ y(3) = \frac{1}{3}; \tag{5.106}$$

$$IF : e^{\int \frac{1}{x}dx} = x \Rightarrow \frac{d}{dx}[y, x] = x\left(\frac{1}{x}\right) \tag{5.107}$$

$$\Rightarrow \frac{d}{dx}[y.x] = 1 \Rightarrow yx = x + c; \Rightarrow y = 1 + \frac{c}{x}; y(3) = \frac{1}{3} \Rightarrow c = -2 \tag{5.108}$$

$$\Rightarrow y(x) = 1 - \frac{2}{x} \Rightarrow y(\pi) = 1 - \frac{2}{\pi} \tag{5.109}$$

□

We have also proved the following theorem:

Theorem 5.3.2.

$$Rat\langle D_n^{\Im}; \tag{5.110}$$

$$Asymrat\langle D_n^{\Im} \tag{5.111}$$

and

$$Arat\langle D_n^{\Im} \tag{5.112}$$

for the following classes of graphs are, respectively:

$$\Im = K_{\frac{n}{2},\frac{n}{2}} : \frac{n-2}{2n-2}; \frac{1}{2}; \frac{4}{n}[n - ln(n-1) - 2] \tag{5.113}$$

$$\Im = K_{i,n-1} : 1 - \frac{1}{\sqrt{n-1}}; 1; \frac{2(n-1)}{n}[n - 2\sqrt{n-1}] \tag{5.114}$$

$$\Im = K_{2,r} : \frac{n+1-\sqrt{2}\sqrt{n+1}}{2n-2}; \frac{1}{2} \tag{5.115}$$

$$\frac{2(n-1)}{n}\left[\frac{n}{2} + ln(n-1) - \frac{1}{\sqrt{2}}.\left[\sqrt{n^2-1} + 2ln\left(\frac{\sqrt{n+1}}{\sqrt{2}} + \frac{\sqrt{n-1}}{\sqrt{2}}\right)\right]\right] + \tag{5.116}$$

$$\left[2 + \sqrt{2}ln(1+\sqrt{2} - \frac{3}{2} - ln2\right] \tag{5.117}$$

$$\Im = L(K_n) : \frac{-2p+\sqrt{1+8p}}{2p-2}; -1; \tag{5.118}$$

$$\frac{2q}{p}\left[p-\sqrt{1+8p}+\frac{7}{6}ln(\sqrt{1+8p}-3)+\frac{7}{6}ln(\sqrt{1+8p}+3)+2+\frac{7}{6}ln(2)-\frac{7}{6}ln(8)\right] \tag{5.119}$$

where $q = \frac{n(n-1)(n-2)}{2}$

$$\Im = SR(G) : \frac{3n-n\sqrt{n}-4}{4n-4}; -\infty$$

$$\frac{(n+\sqrt{n})}{2}\left[\frac{1}{6}n^{\frac{3}{2}}-\frac{3}{4}n+\frac{1}{4}ln(n-1)+\frac{1}{2}\sqrt{n}+\frac{1}{2}ln\frac{\sqrt{n}-1}{\sqrt{n}+1}+c\right] \tag{5.120}$$

$$\Im = LP(G) : \frac{n+1-\sqrt{n^2-4n+8}}{2n-2}; 0 \tag{5.121}$$

$$Arat\langle D_n^{\Im} = 2\frac{m'}{n}\int\left[\frac{n+1-\sqrt{n^2-4n+8}}{2n-2}\right]dn \tag{5.122}$$

$$\Im = L(CT(k,2)) : \frac{2n-3}{3n-3}; \frac{2}{3}\frac{2(n-1)}{n}\left[\frac{2}{3}n-\frac{1}{3}ln(n-1)-\frac{4}{3}\right];\ n\ large \tag{5.123}$$

$$\Im = DuS_n : \frac{(2n-2)-\left[2\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}}+2\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}\right]}{2n-2}, 1 \tag{5.124}$$

$$Arat\langle D_n^{\Im} = \frac{2(n-1)}{n}\int\frac{1}{3}\left[\frac{(2n-2)-\left[2\sqrt{\frac{(n-1)+\sqrt{2n-3}}{2}}+2\sqrt{\frac{(n-1)-\sqrt{2n-3}}{2}}\right]}{2n-2}\right]dn \tag{5.125}$$

$$\Im = P_n; n = 2t : \frac{(n-1)-\left[cosec\left(\frac{\pi}{2(n+1)}\right)\right]}{n-1}; \frac{\pi-2}{\pi} \tag{5.126}$$

$$Arat\langle D_n^{\Im} = \frac{2(n-1)}{n}\int\left[\frac{(n-1)-\left[cosec\left(\frac{\pi}{2(n+1)}\right)\right]}{n-1}\right]dn \tag{5.127}$$

$$\Im = C_n; n = 2t = 4q : \frac{(n-1)-2cot\left(\frac{\pi}{n}\right)}{n-1}; \frac{\pi-2}{\pi} \tag{5.128}$$

$$Arat\langle D_n^{\Im} = 2\int\left[\frac{(n-1)-4cot\left(\frac{\pi}{n}\right)}{n-1}\right]dn \tag{5.129}$$

$$\Im = W_n; n = 2t : \frac{(n-1)-\left[\sqrt{n}+1+4\sum_{j=1}^{\frac{n-4}{4}} cos\left(\frac{2\pi j}{n-1}\right)\right]}{n-1}; Asymrat\langle D_n^{\Im} = \frac{\pi-2}{\pi} \tag{5.130}$$

$$Arat\langle D_n^{\Im} = 2\frac{2(n-1)}{n}\int\left[\frac{(n-1)-\left[\sqrt{n}+1+4\sum_{j=1}^{\frac{n-4}{4}} cos\left(\frac{2\pi j}{n-1}\right)\right]}{n-1}\right]dn \tag{5.131}$$

Lemma 5.3.1.

$$\frac{3n-n\sqrt{n}-4}{4n-4} \leq Rat\langle D_n^{\Im} \leq 1$$

Proof. Since $E(G) \geq 0$ for any graph G we get the right hand inequality:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(G)}{2n-2}\right] \leq \frac{E(K_n)}{2n-2} = 1 \tag{5.132}$$

And since

$$E(G) \leq E(SR(G)) \Rightarrow E(K_n) - E(SR(G)) \geq 2n-2-\frac{n(1+\sqrt{n})}{2} \tag{5.133}$$

we get the left hand inequality. □

Lemma 5.3.2. *The domination eigen-complete effect is at most one and is greatest negatively for the strongly regular graph examined in the above example.*

Proof. The strongly regular graph in the above example has the greatest energy of all graphs so that:

$$Asymrat\langle D_n^{\Im} = \lim_{n\to\infty}\left[\frac{3n-n\sqrt{n}-4}{4n-4}\right] = -\infty$$

□

The above lemmas can be used to verify the following theorem:

Theorem 5.3.3. $Asymrat\langle D_n^{\Im} \in (-\infty, 1]$ *with end points attained for the strongly regular graph and star graphs with rays of length* 1, *respectively. Moreover,* $Asymrat\langle D_n^{\Im} =$ 0 *for the lollipop graph.*

corollary 5.3.1. *The eigen-complete difference height of the strongly regular graph above is the greatest of all eigen-complete heights.*

Proof. Since $E(SR(G)) \geq E(G')$ for any graph $(G' \in \Im$, we have

$$\left|\int\left[\frac{E(K_n)-E(SR(G))}{E(K_n}\right]dn\right| = \int\left[\frac{E(SR(G))-E(K_n)}{E(K_n)}\right]dn \geq \int\left[\frac{E(G')-E(K_n)}{E(K_n)}\right]dn \tag{5.134}$$

□

Conjecture 1. *Except for strongly regular graphs, the eigen-complete difference asymptote lies on the interval* $[-1, 1]$.

5.4 Eigen-Complete Different Ratios Of Complements of Classes of Graphs

5.4.1 The Complete-Split Bipartite Graph

The complement of $K_{\frac{n}{2},\frac{n}{2}}$ consists of two disjoint copies of $K_{\frac{n}{2}}$. It energy is therefore: $2n-4$ so that:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(G)}{2n-2}\right] = \frac{2n-2-(2n-4)}{2n-2} = \frac{1}{n-1} \tag{5.135}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n\longrightarrow\infty}\left[\frac{1}{n-1}\right] = 0 \tag{5.136}$$

.

$$Arat\langle D_n^{\Im} = \frac{n-2}{2n}\int\left[\frac{1}{n-1}\right]dn = \frac{n-2}{n}\left[ln(n-1)+c\right] \tag{5.137}$$

Smallest such graph occurs for $n-4$ so that: $c=-ln(3)$
The eigen-complete difference ratio for the complement of the complete-split bipartite graph is:

$$f(n) = \frac{1}{n-1}.$$

The eigen-complete difference ratio of the original graph $K_{\frac{n}{2},\frac{n}{2}}$ is:

$$g(n) = Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(G)}{2n-2}\right] = \frac{2n-2-n}{2n-2} = \frac{n-2}{2n-2} = \frac{n}{2n-2} - \frac{2}{2n-2} \tag{5.138}$$

$$= \frac{n}{2}f(n) - f(n) = f(n)\left(\frac{n}{2}-1\right) \Rightarrow g'(n) = \frac{f(n)}{2} + \frac{nf'(n)}{2} - f'(n) = f'(n)\left[\frac{n}{2}-1\right] + \frac{f(n)}{2} \tag{5.139}$$

$$g'(n) = \frac{(2n-2)-2(n-2)}{(2n-2)^2} = \frac{2}{(2n-2)^2} = f'(n)\left[\frac{n-2}{2}\right] + \frac{f(n)}{2} \tag{5.140}$$

$$\Rightarrow f'(n) + \frac{1}{n-2}f(n) = \frac{2g'(n)}{(n-)} = \frac{4}{(n-2)(2n-2)^2} = \frac{1}{(n-2)(n-1)^2} \tag{5.141}$$

$$\frac{d(f(n))}{dn} + \frac{1}{n-2}f(n) = \frac{1}{(n-2)(n-1)^2} \tag{5.142}$$

$$IF = (n-2) \Rightarrow (n-2)f(n) = \int\left[\frac{1}{(n-1)^2}\right]dn \tag{5.143}$$

$$\Rightarrow (n-2)f(n) = -\frac{1}{n-1} + c \Rightarrow f(n) = -\frac{1}{(n-2)(n-1)} + \frac{c}{(n-c)} \tag{5.144}$$

$$\Rightarrow \frac{1}{n-1} = -\frac{1}{(n-2)(n-1)} + \frac{c}{(n-2)} \Rightarrow 1 = -\frac{1}{n-2} + \frac{c(n-1)}{n-2} \tag{5.145}$$

$$\Rightarrow n-2 = -1 + c(n-1) \Rightarrow c = 1 \tag{5.146}$$

Thus,

$$\Rightarrow f(n) = -\frac{1}{(n-2)(n-1) + \frac{1}{(n-2)}} \tag{5.147}$$

Giving a different way of expressing:$\frac{1}{n-1}$.
Thus with $f(n)$ and $g(n)$, we associate the quadratic expression:

$$h(n) = n^2 - 3n + 2$$

$$h(n+1) - h(n) = n^2 + 2n + 1 - 3n - 3 + 2 - n^2 + 3n - 2 = 2n - 2 \tag{5.148}$$

$$h(n+2) - h(n+1) = n^2 + 4n + 4 - 3n - 6 + 2 - (n^2 + 2n + 1 - 3n - 3 + 2) = 2n \tag{5.149}$$

$$h(n+3) - h(n+2) = n^2 + 6n + 9 - 3n - 9 + 2 - \tag{5.150}$$

$$(n^2 + 4n + 4 - 3n - 6 + 2) = 2n + 2 \tag{5.151}$$

The second difference involving (5.148), (5.149) and (5.150) is an arithmetic sequence with common difference 2. Thus we have the following theorem:

Theorem 5.4.1. *The eigen-difference ratio $g(n)$ of the complete-split bipartite and the eigen-difference ratio $f(n)$ its complement are related by the following equation:*

$$f'(n) + \frac{1}{n-2} f(n) = \frac{2g'(n)}{(n-2)} \tag{5.152}$$

Proof. The equation above results in the differential equation:

$$\frac{d(f(n))}{dn} + \frac{1}{n-2} f(n) = \frac{1}{(n-2)(n-1)^2} \tag{5.153}$$

With general solution:

$$\Rightarrow f(n) = -\frac{1}{(n-2)(n-1)} + \frac{c}{(n-2)} \tag{5.154}$$

□

corollary 5.4.1. *The equation in the above theorem yields the following quadratic sequence:* $0, 2, 6, \cdots, n^2 - 3n + 2, \cdots$ *With second difference sequence with common difference 2:* $2, 4, 6, \cdots, 2n - 2, \cdots$

5.4.2 Star Graphs With Rays Of Length 1

The compliment of the star graph with rays of length one (on at least three vertices) is a complete graph on $n-1$ vertices together with an isolated vertex. Its energy is therefore: $2n-4$ so that:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(G)}{2n-2}\right] = \frac{2n-2-2n+4}{2n-2} = \frac{2}{2n-2} = \frac{1}{n-1} \tag{5.155}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n\to\infty}\left[\frac{1}{n-1}\right] = 0 \tag{5.156}$$

$$Arat\langle D_n^{\Im} = \frac{m'}{n}\int\left[\frac{1}{n-1}\right]dn = \frac{(n-1)(n-2)}{2n}[ln(n-1)+c] \tag{5.157}$$

where, for $n=3$, we get $c=-ln(2)$.
The energy of the original star graph is:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(K_{1,n-1})}{2n-2}\right] = \frac{(2n-2)-2\sqrt{n-1}}{2(n-1)} = 1-\sqrt{\frac{1}{n-1}} = f(n), \tag{5.158}$$

While that of its complement is:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(G)}{2n-2}\right] = \frac{1}{n-1} = g(n). \tag{5.159}$$

Also, differentiating $f(n)$ yields

$$f'(n) = \frac{1}{2}(n-1)^{-\frac{3}{2}}. \tag{5.160}$$

Thus:

$$f(n) = 1-\sqrt{g(n)} \Longrightarrow g(n) = [1-f(n)]^2 \tag{5.161}$$

$$\Rightarrow f^2(n)-2f(n)+1-g(n) = 0. \tag{5.162}$$

5.4.3 The Lollipop Graph With Complete Graph On $n-1$ Vertices As Base

The compliment of the lollipop graph consists of a star graphs on $n-1$ vertices and an isolated vertex. Its energy is therefore $2\sqrt{n-2}$.

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(G)}{2n-2}\right] = \frac{2n-2-2\sqrt{n-2}}{2n-2} = \frac{n-1-\sqrt{n-2}}{n-1} = g(n) \tag{5.163}$$

The ratio of the original graph is:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(LP(G))}{2n-2}\right] = \frac{n+1-\sqrt{n^2-4n+8}}{2n-2} = f(n) \tag{5.164}$$

5.5 Conclusion

In this chapter we used the idea of energy difference between two graphs and the significance of the complete graph to formulate the eigen-complete difference ratio. This allowed for the investigation of the domination effect that the energy of graphs have with respect to the complete graph when a large number of vertices are involved. This idea can be applied to molecules with a large number of atoms, where one desires to examine molecules whose energy may dominate the molecule that is very well bonded. We found that a strongly regular graph dominated in the largest negative way, while the star graph with rays of length one had a domination effect of one- the largest possible positive domination effect. The lollipop graph with base the complete graph had domination effect of zero. Cycles, paths and wheels are shown to have the same eigen-complete domination effect of $\frac{\pi-2}{\pi}$ which is equivalent to the solution $y = f(\pi)$ of the separable differential equation:

$$y + \left(\frac{1}{x}\right)\frac{dy}{dx} = \frac{1}{x}, y(3) = \frac{1}{3}. \tag{5.165}$$

We attached the average degree to the Riemann integral of this eigen-complete difference ratio to determine eigen-complete difference areas associated with classes of graphs and applied the above ideas to the complement of classes of graphs. We showed that the eigen-complete difference ratios of the complete-split bipartite graph and its complement are related by a differential equation with an associated quadratic sequence with second difference being a sequence with common difference of two.

Chapter 6

Summary and Conclusion

6.1 Summary

This dissertation brings together two important concepts in graph theory the energy of a graph and the complete graph. The energy of a graph is the sum of the absolute values of its eigenvalues, and originated from the determination of the sum of π -electron energy in a molecule represented by a molecular graph (i.e. a graph where the vertices represent atoms and the edges bonds between atoms). Important theorems, such as the Lovasz and the Lollipop theorems, are used to find eigenvalues of classes of graphs, while analytic methods are used to determine simplified expressions for the energy of classes of graphs.
Considering a graph as a molecular graph, then the complete graph translates to that of a molecule with all possible bonds between atoms; i.e. a strongly bonded molecule. The eigen-complete difference ratio allowed for the investigation of the domination effect of the energy of graphs on the energy of the complete graph, when a large number of vertices are involved.
In chapter two, we defined some basic graph theorectical terms that were used during the course for this research. We adopted the graph theoretical notation of Harris, J. M., Hirst, J. L. and Mossinghoff, M.[45] with little addition from similar materials.

In Chapter three, we talked about the Linear algebra of graphs as the branch of mathematics that studies graphs by using the linear algebraic properties of associated matrices, the theory of association schemes and coherent configurations studies of the algebra generated by associated matrices. In this chapter, algebraic methods were applied to problems about graphs. This is in contrast to geometric, combinatoric, or algorithmic approaches. The relationship between a graph and the eigenvalues of its adjacency and Laplacian matrix were explained in detail. We applied the Lovasz theorem and the Lollipop theorems to the problem determining the eigenvalues of some graphs whose energies were considered in the subsequent chapters.
In chapter four, we discussed extensively the determination of the energies of classes of graphs with an emphasis on analytical methods. We determined the energy of different classes of graphs using the eigenvalues calculated in the previous chapters. We expressed the energy of cycles, paths and wheels, on n vertices, in terms of

simplified expressions involving the cotangent and the secant of either $\frac{\pi}{n}, \frac{\pi}{n-1}\frac{\pi}{2n}$ or $\frac{\pi}{2(n-1)}$ and showed that these classes of graphs have the same energy of $\frac{4n}{\pi}$ for large values of n.
In chapter five, the eigen-complete difference ratio of classes of graphs domination, asymptotes and area was discussed.

Although the complete graph K_n does not have the maximum energy of all graphs (see [51], it is a very important and well-studied class of graphs, for example it has a high degree of connectivity and robustness. Given that the complete graph does not have the largest energy of all graphs it is customary (see hypoenergetic graphs, i.e. graphs with energy less than the complete graph, to see how the energy of other classes of graphs compare to that of the energy of the complete graph.

Thus one would like to compare its energy with the energy of any other graph G in terms of how close their energies are, and how the energy of G compares with the energy of K_n where a large number of vertices are involved and this provided the motivation for the eigen-complete difference ratio of a graph (see Winter and Ojako[104]) which we present in chapter five, which is original in its entire.

6.2 Conclusion

This dissertation involved the merging of two important concepts in graph theory the complete graph and the energy of graphs. The significance of the complete graph in terms of its strong connectivity, and its practical usefulness in realizing graph theoretical concepts is well documents. The connection between π-electron energy of a molecule and the energy of a molecular graph is well researched. These two concepts were used to define a new ratio, the eigen-complete difference ratio of a graph G , belonging to a class of graphs $\Im$:

$$Rat\langle D_n^G = \left[\frac{E(K_n) - E(G)}{E(K_n)}\right]; \tag{6.1}$$

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n) - E(G)}{E(K_n)}\right]; \ G \in \Im \tag{6.2}$$

By considering the asymptotic convergence of the eigen-complete difference ratio, we were able to investigate the domination effect of the energy of graphs, on the energy of the complete graph, when a large number of vertices are involved. We found that this domination effect is the greatest negatively (positively) for a strongly regular graph (star graphs with rays of length one), and is zero for the lollipop graph. For paths, cycles and wheel graphs, the domination effect is $f(\pi) = \frac{\pi-2}{\pi}$. We showed that this value $f(\pi)$ is a solution of the separable variable differential equation:

$$y + \left(\frac{1}{x}\right)\frac{dy}{dx} = \frac{1}{x}, y(3) = \frac{1}{3} \tag{6.3}$$

We also claim that :$Asymrat\langle D_n^{\Im} \in (-\infty, 1]$ with end points attained for the strongly regular graph and star graphs with rays of length 1, respectively, and showed that

$$Asymrat\langle D_n^{\Im} = 0$$

for the lollipop graph.
Attaching the average degree of graph G, with m' edges, to the absolute value of the Riemann integral of this ratio we obtain the eigen-complete difference area:

$$Arat\langle D_n^{\Im} = \frac{2m'}{n} \left| \int \left[\frac{E(K_n) - E(G)}{E(K_n)} \right] dn \right| \tag{6.4}$$

with $Arat\langle D_k^{\Im} = 0$
where k is the smallest order of $G \in \Im$.
The average degree is referred to as the length of the area, while the integral part is the height of the area.
We determined the area, either exactly or in terms of an integral, of known classes of graphs such as the complete-split bipartite graph, path and cycle graphs, star graphs with rays of length one and two, dual star graphs, the wheel graph, lollipop graph with the complete graph as its base, strongly regular graphs with the maximum energy, the line graph of the complete graph and the regular caterpillar graph. The area of strongly regular graphs with maximum energy behaves, for large n, like $\frac{n^{\frac{5}{2}}}{12}$ which appears to provide the largest eigen-complete difference area of all classes of graphs, followed by the area of the complete-split bipartite graph with area, for large n, behaving as $\frac{n^2}{4}$. We also proved that the height of strongly regular graphs with maximum energy is the greatest of heights associated with all other classes of graphs. Future research will involve determining the eigen-complete difference ratio of other classes of graphs as well as using a different means source, such as the Laplacian energy.

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