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RELATIVISTIC THERMODYNAMICS

OF RADIATING STARS

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Relativistic Thermodynamics of Radiating Stars

by

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As the candidate's supervisor and co-supervisor, we have approved this dissertation for submission.

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Abstract

In this research, on the topic of relativistic thermodynamics of radiating stars, the following three case studies are investigated:

Gravitational collapse in spacially isotropic coordinates – The nature of a dissipative collapse process of a spherically symmetric star which has been perturbed into a dynamical state from an initial static configuration is studied. The collapse process involves dissipation of energy in the form of a radial heat flux. The perturbation in the density and pressure profiles are such that the star is always close to hydrostatic equilibrium. The temperature profiles are studied using a causal heat transport equation.

Radiating collapse in the presence of anisotropic stresses – The effect of anisotropic stresses are investigated for a collapsing fluid sphere dissipating energy in the form of a radial flux. The collapse process starts from an initial static sphere described by the Bowers and Liang solution, and then proceeds until the time of formation of the horizon. We find that the surface redshift increases as the stellar fluid moves away from isotropy. The evolution of the temperature profiles is analysed by employing a causal heat transport equation of the Maxwell-Cattaneo form. Both the Eckart and causal temperatures are enhanced by anisotropy at each interior point of the stellar configuration.

The influence of an equation of state during radiative collapse – A linear equation of state is imposed on a static configuration which undergoes radiative gravitational collapse. Various values of the equation of state parameter allow descriptions of different matter content from classical stars to dust and also dark energy stars. The physical parameters are shown to behave in a meaningful and realistic manner.

This thesis is dedicated to

My grandfather, R.O. Atkinson, who worked hard so that I could study.

Look deep into nature, and then you will understand everything better.
There are only two ways to live your life. One is as though nothing is a miracle. The other is as though everything is a miracle. – A. Einstein

On the Sun too, there are spots – Russian saying meaning that nobody is perfect, and so perhaps no theory is or ever will be perfect.

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DECLARATION 1 - PLAGIARISM

I, Robert Sacha Bogadi, student number: 213572912, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. The following thesis has not been submitted for any degree or examination at any other university.

3. The following thesis does not contain other persons data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

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DECLARATION 2 - PUBLICATIONS

Details of contributions to publications that form part of the research presented in this thesis.

Publication 1

Govender M., Bogadi R.S., Sharma R., Das S., Gravitational collapse in spatially isotropic coordinates, *Astrophys. Space Sci.* **361**, 33 (2016).

Publication 2

Govender M., Bogadi R.S., Lortan D.B., Maharaj S.D., Radiating collapse in the presence of anisotropic stresses, *Int. J. Mod. Phys. D*, **25**, 1650037 (2016).

Publication 3

Govender M., Bogadi R.S., Maharaj S.D., The influence of an equation of state during radiative collapse, *Int. J. Mod. Phys. D*, in press (2017).

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Contents

Chapter 1

Introduction

Stars have been a source of fascination and intrigue since ancient times. They have been both a source of mysticism, even in present times, and also of scientific interest and value in guiding sailors across vast oceans and travellers across deserts. For most of the history of mankind, their physical nature had been largely unknown until the arrival of Galileo in the 16th century. Through the efforts of Galileo Galilei ($1564 - 1642$), considered the father of astronomy with his re-invention of the telescope, the stage had been set for the scientific investigation of celestial bodies. Although forced to retract statements about the nature of our solar system on the grounds of religious conformity, his ideas and discoveries nevertheless remained alive to be re-investigated and confirmed by his successors. Not long thereafter, one of the greatest physicists of all time, Isaac Newton ($1642 - 1726$), formulated a theory of gravitation which helped to explain and calculate the motion of bodies within our planetary system. Newton's theory worked well and the force between two bodies due to their mutual property of mass could be calculated using an inverse square law which at the time seemed to be very accurate, subject only to the inaccuracies in the measurements of mass, distance and Newton's gravitational constant. Moreover, gravity and the gravitational force were taken to be one and the same thing, thus highlighting the fact that the real underlying physical nature of gravity had been hidden and thus unknown. Then, in

the early part of the 20th century, Albert Einstein came up with revolutionary ideas that completely changed mankind's view on gravity. No longer was gravity seen to be some mysterious force of attraction between two or more masses, but that it was rather the mass of an object itself that caused changes in the properties of a so called four-dimensional 'spacetime', thereby altering the preferred paths of motion of the other bodies. This is the subject of the general theory of relativity which was made known to the scientific community by Einstein (1916).

The theory very soon proved to be a success after accounting for the anomalous motion of the planet Mercury which was the first approximate solution of Einstein's equations as presented by him in 1915. Mercury is the planet closest to the Sun and displays an anomalous precession which is not accountable for in Newton's theory. This was first recognized in 1859 by Urbain Le Verrier. Although other planets are also responsible for Mercury's precession, there was always a disagreement between observation and Newton's theory that could not be reconciled. This was finally accounted for by the curvature of spacetime near the Sun according to Einstein's theory. A few years later, accurate measurements of the degree of deflection of rays of light which passed near the Sun were made during a solar eclipse, and Einstein's calculations agreed with the results of this experiment. This was a triumph for the theory of general relativity, however a problem did arise not long thereafter in connection with the stability of the universe, and for this reason, Einstein introduced a cosmological constant into his field equations. Later measurements, notably by Hubble, showed the universe to be expanding thus making the cosmological constant unnecessary. In recent times, this research has led to the theory of dark matter and dark energy. Apart from the uneasiness as to whether Einstein's equations should incorporate a cosmological constant or not, the theory still enjoys continued success even until very recently (February 2016) with the detection of gravitational waves, a desired consequence of the theory of general relativity.

An interesting area to which general relativity is commonly applied is the nature

of compact objects. By compact, we mean matter densities which are greater than $(10^{15}g/cm^3)$. Such high matter densities belong to objects such as neutron stars, gravastars, and in the extreme case where the density becomes infinite, to black holes. These objects have a marked effect on the structure of spacetime, and their study has produced novel insights into the nature of gravity itself. This research aims to gain further insight into the gravitational collapse process leading up to such compact objects through the study of relativistic thermodynamics.

1.1   Stars and general relativity

Stars, in all states, are the most massive objects to be found throughout the cosmos. They are the best candidates for studying and testing the general theory of relativity (GR). The definitive experiment undertaken to test GR involved measuring the deflection of light from stars by the Sun during a solar eclipse. In May 1919, Arthur Eddington and his co-workers took photographs of the stars whose light passed near the Sun during a total solar eclipse. The experiment demanded highly accurate and precise measurements. The measurements made confirmed Einstein's theory and thereafter made his theory world famous.

In more recent times, the physics of neutron stars and black holes have been studied using general relativity (Ghezzi 2005; Goswami and Joshi 2004b). The principal pioneers of this research were Karl Schwarzschild, Subrahmanyan Chandrasekhar, David Finkelstein, Joseph Taylor and Russell Hulse, Roger Penrose and Stephen Hawking. However, it is interesting to note that the idea of an object being so massive that not even light can escape from it goes back to 1783 when John Michell wrote to Henry Cavendish of the Royal Society:

"If the semi-diameter of a sphere of the same density as the Sun were to exceed that of the Sun in the proportion of 500 to 1, a body falling from an infinite height towards it would have acquired at its surface greater velocity than that of light, and

consequently supposing light to be attracted by the same force in proportion to its vis inertiae, with other bodies, all light emitted from such a body would be made to return towards it by its own proper gravity." – John Michell

John Wheeler was one of the first scientists to introduce the term 'black hole' in 1967. Initially, the term encountered opposition in strict scientific circles but then became acceptable, entering into the scientific terminologies of astrophysics and cosmology.

1.2 Gravitational collapse

A star, during its lifetime, supports its weight by fusing hydrogen into helium, but it is inevitable that the hydrogen available will eventually be depleted. As the hydrogen supply diminishes, the chemical composition within the core of the star changes and thus the star evolves with time. As fusion proceeds, the number of atomic nuclei decreases and this causes a reduction in pressure, thereby affecting the gravitational stability of the star. This results in the core being squeezed more tightly so that the contraction increases the density and temperature of the core. The rate of nuclear reactions then increases allowing additional energy to flow outwards causing outer layers to expand so that the star appears larger and brighter. However, the onset of helium fusion then causes some contraction of the star up until the helium has been almost exhausted whereafter the star again expands into a giant. Finally, after all possible nuclear reactions have reached completion and the star has lost some of its mass during times of expansion, gravity finally dominates and the star reduces to a certain type of remnant depending on its mass.

Particularly massive stars do not merely become giants during the final stages of reaching their end states, but instead display violent and highly luminous events called supernova explosions. The subsequent gravitational collapse thereafter is much stronger and can lead to a neutron star or black hole remnant. The material of a

neutron star is highly dense and has been described by physicists as a fluid of neutrons or one giant nucleus. Such objects of about 1 solar mass may be typically about 20km in diameter. If the core of a star is more than 3 solar masses when it collapses, the gravitational collapse is so powerful that a neutron star end state is no longer sufficient to support the weight of the star. It effectively collapses to zero spatial dimension although some physicists argue that quantum mechanics prevents this singularity. The properties of black holes are ideal for applying and studying the theory of general relativity since the curvature of spacetime becomes infinite thus exemplifying the direct relationship between gravity and spacetime curvature. Even under such extreme conditions, the Einstein field equations may be solved in order to discover and study some of the properties of black holes such as event horizons. The first solution was found by Karl Schwarzschild and is known as the Schwarzschild black hole.

In 1915, Albert Einstein formulated a theory of gravity which was independent of an object's physical properties, and dependent only on the geometry of space and time. This linked space and time into what is called a four-dimensional spacetime. The theory, known as general relativity, makes use of differential geometry and its connection with the physical properties of an object, namely its energy and momentum, to yield a set of nonlinear coupled differential equations which link gravitational acceleration with the curvature of spacetime. In this way, gravity may be thought of independently from an object's mass, and associated only with the curvature of spacetime. This is further justified by the equivalence principle in which gravitational acceleration and gravitational effects due to an accelerated body are indistinguishable.

In this chapter, the mathematics needed for working with general relativity will be firstly considered, leading to the formulation of the Einstein field equations. These equations are used to describe the gravitational collapse of stars and they predict an end state for this collapse which may typically be a white dwarf star, a neutron star, or a black hole in the case of sufficiently massive stars.

The concepts of vectors, covectors (1-forms) and tensors are assumed which are

part of the mathematical tools required for describing physical quantities in GR. The four-dimensional geometry of spacetime is introduced and applied to the curvature of spacetime, leading to the Einstein curvature tensor. A basic overview of the energy momentum tensor which describes an object's state of being within the phase space is given. Then the components of the Einstein tensor for a spherically symmetric spacetime are derived. A general energy momentum tensor with associated kinematical quantities is also used for which the Einstein field equations are derived for an isotropic spherical polar coordinate system. Finally, Vaidya's solution, which is a solution of the Einstein field equations for an exterior spacetime filled null radiation, is presented.

1.3 Differential geometry

The general theory of relativity requires the representation of space and time as a four–dimensional manifold which is continuous and differentiable at all points. This representation incorporates a branch of mathematics known as differential geometry and is well documented in standard texts on general relativity (Schutz 2009). Upon the four–dimensional manifold is imposed a metric which is represented by a symmetric tensor **g** with signature $(-+++)$. Any event or point on this spacetime manifold is represented by the vector $x^a = (x^0, x^1, x^2, x^3)$, where x^0 is the temporal component and x^1, x^2 and x^3 are the spatial components. The metric tensor, which is locally Lorentzian in nature, may be used to measure invariant distances between events as given in infinitesimal form by

$$ds^2 = g_{ab}dx^a dx^b, \tag{1.3.1}$$

where ds^2 is the square of the infinitesimal distance, not to be confused with an area. It is commonly known as the *line element.* The metric connection coefficients $\Gamma^a{}_{bc}$, also known as the Christoffel symbols, are expressed in terms of the metric tensor and its derivatives by

$$\Gamma^a{}_{bc} = \tfrac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}), \tag{1.3.2}$$

where the commas denote partial differentiation with respect to the variable proceeding the comma. In terms of a graphical representation, $\Gamma^a{}_{bc}$ represents the ath component of the change in basis vectors $\vec{e}_b$ as a consequence of parallel transport of a vector along the basis vectors $\vec{e}_c$. The connection coefficients preserve inner products under parallel transport and take into account the variation in the basis vectors from one event to another in the spacetime. This is one of the fundamental theorems of Riemannian (also known as elliptic) geometry.

The Riemann curvature tensor, which is essentially a mathematical construct used to measure the spacetime curvature, is obtained from combinations of the derivatives and products of connection coefficients (1.3.2) and is given by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}. \tag{1.3.3}$$

Contraction of the Riemann tensor (1.3.3), with the use of the metric tensor g_{ab}, results in the Ricci tensor R_{ab} and is given by

$$R_{ab} = g_{ea}g^{cd}R^e{}_{bcd} = \Gamma^d{}_{ab,d} - \Gamma^d{}_{ad,b} + \Gamma^e{}_{ab}\Gamma^d{}_{ed} - \Gamma^e{}_{ad}\Gamma^d{}_{eb}. \tag{1.3.4}$$

Further contraction of the Ricci tensor gives the simplest curvature invariant of a Riemannian manifold, namely the Ricci scalar R, and is given by

$$R = g^{ab}R_{ab} = R^a{}_a. \tag{1.3.5}$$

The Einstein curvature tensor, which is a rank two symmetric tensor composed of the Ricci tensor and Ricci scalar, is given by

$$G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab}, \tag{1.3.6}$$

and is in fact a measure of the rate of change of volume of a collection of test particles initially at rest which fall freely in spacetime. An important property of the Einstein curvature tensor is that its covariant derivative vanishes, namely

$$G^{ab}{}_{;b} = 0, \tag{1.3.7}$$

where the semicolon denotes covariant differentiation. In general relativity, this is commonly known as the contracted Bianchi identity and shows that the Einstein tensor is divergence–free, a requirement for the local conservation of energy and momentum.

Einstein then made the connection between the curvature tensor and the energy momentum tensor, thereby formulating the Einstein field equations which are the fundamental equations of the general theory of relativity. According to Einstein, "*Space acts on matter, telling it how to move. In turn, matter reacts back on space, telling it how to curve*" (Misner *et al* 1973). This is somewhat reminiscent of Newton's third law of motion which states that for every force or action, there is an equal but opposite force or reaction. The relationship, expressed in geometrized units, is given as

$$G_{ab} = T_{ab}, \tag{1.3.8}$$

where T_{ab} is the energy momentum tensor. Geometrised units are made use of in this research in which certain constants or combinations thereof are scaled to unity for ease of calculation. These include the speed of light $c \to 1$, and Einstein's coupling coefficient $\kappa = 8\pi G/c^4 \to 1$. It is also noted that equation (1.3.8) is a compact representation of 16 nonlinear coupled differential equations.

It is also noted that the Einstein field equations may be obtained through variation of the Einstein-Hilbert action given by

$$S = \int d^n x \sqrt{-g} \left(\frac{1}{\kappa} \left(R - 2\Lambda \right) + \mathcal{L}_{matter} \right), \tag{1.3.9}$$

where Λ is a cosmological constant and $\mathcal{L}_{matter}$ is the matter field Lagrangian. Such an action integral equation is the starting point for the extension of general relativity to higher dimensions (Carloni and Dunsby 2007).

1.4 Spherically symmetric spacetimes

1.4.1 The energy momentum tensor for an imperfect fluid

The energy momentum tensor determines the amount of mass–energy which is contained in a unit volume. The mass–energy produces the spacetime curvature which is quantified by the Einstein tensor G_{ab}. A perfect fluid (an ideal gas model) can be described as a collection of noninteracting particles travelling through spacetime with a four–velocity $\mathbf{u}$, having an energy density ρ with pressure p remaining isotropic. The various contributions to the energy density ρ, which is a total energy, include the rest mass energy, the kinetic energy, and the various potentials. The perfect fluid model no longer applies if any of the effects of bulk viscosity, shear stress, heat flow, anisotropic pressure and free–streaming radiation are involved. The energy momentum tensor for an imperfect fluid, according to Herrera $et\ al$ (2009), is given by

$$T_{ab} = (\rho + p_t + \Pi)u_a u_b + (p_t + \Pi)g_{ab} + (p_r - p_t)\chi_a\chi_b + q_a u_b + q_b u_a + \epsilon l_a l_b + \pi_{ab}, \quad (1.4.1)$$

where ρ is the energy density, p_r is the radial pressure, p_t is the tangential pressure, Π is the bulk viscosity, u^a is the comoving timelike fluid four–velocity, χ_a is a radial four–vector, q^a is the heat flux vector, ϵ is the null radiation density, l_a is the null four–vector and π_{ab} is the shear viscosity tensor. These quantities have the following properties:

$$u^a u_a = -1, \qquad u^a q_a = 0, \qquad l^a u_a = -1, \qquad l^a l_a = 0, \qquad (1.4.2a)$$

$$\chi^a \chi_a = 1, \qquad \chi^a V_a = 0, \qquad \pi_{ab} V^b = 0, \qquad \pi_{[ab]} = 0, \qquad (1.4.2b)$$

$$\pi^a_a = 0, \qquad (1.4.2c)$$

where $[ab]$ implies anti-symmetrisation with respect to indices a, b. For the comoving metric where

$$ds^2 = -A^2(r,t)dt^2 + B^2(r,t)dr^2 + Y^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (1.4.3)$$

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the above nonzero quantities become

$$u^a = (A^{-1}, 0, 0, 0), \qquad\qquad q^a = (0, q_1, 0, 0), \qquad\qquad (1.4.4\text{a})$$

$$l_a = (A^{-1}, B^{-1}, 0, 0), \qquad\qquad \chi^a = (0, B^{-1}, 0, 0). \qquad\qquad (1.4.4\text{b})$$

In cases which are shear–free as is the case in this research, the condition

$$\frac{\dot{B}}{B} = \frac{\dot{Y}}{Y}$$

is obtained where $Y = rB$. The shear–free line element is then

$$ds^2 = -A^2 dt^2 + B^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \qquad\qquad (1.4.5)$$

where A, B are both functions of t and r. We note that the line element given by (1.4.5) is isotropic and comoving.

1.4.2 The exterior field: Vaidya's solution for a null radiation filled exterior spacetime

Oppenheimer and Snyder (1939) solved the Einstein field equations for a star undergoing gravitational collapse. They used a spherically symmetric dust cloud model with a void exterior, known as a Schwarzschild exterior spacetime. This work lay the foundation for the study of gravitational collapse of massive stars and led to the Vaidya solution (Vaidya 1953) which incorporates radiant energy into a description of the exterior spacetime. The Vaidya solution is a unique solution of the Einstein field equations for a spherically symmetric star which radiates energy in the form of null radiation. The Vaidya metric is given by

$$ds^2 = -\left(1 - \frac{2m(v)}{\mathcal{R}} \right) dv^2 - 2dvd\mathcal{R} + \mathcal{R}^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right), \qquad\qquad (1.4.6)$$

with coordinates given by $(x^a) = (v, \mathcal{R}, \theta, \phi)$. The quantity $m(v)$ is just the Newtonian mass of the self–gravitating object as measured by an observer, located at infinity. The

energy momentum tensor for the exterior matter distribution, which is just the null radiation, is given by

$$T_{ab} = \mu l_a l_b, \tag{1.4.7}$$

where $l_a = (1, 0, 0, 0)$ is a radial null four–vector as defined for the Vaidya solution which satisfies the condition $l^a l_a = 0$. The Ricci tensor has only one nonvanishing component, namely

$$R_{00} = -\frac{2}{\mathcal{R}^2} \frac{dm}{dv}. \tag{1.4.8}$$

The Ricci scalar (1.3.5) for the line element given by (1.4.3) is $R = 0$. Thus the only nonzero component of the Einstein tensor (1.3.6) is

$$G_{00} = -\frac{2}{\mathcal{R}^2} \frac{dm}{dv}. \tag{1.4.9}$$

It can be shown that the only surviving component of T_{ab} is given by

$$T_{00} = \mu.$$

From the Einstein field equations $G_{ab} = T_{ab}$, it follows that

$$\mu = -\frac{2}{\mathcal{R}^2} \frac{dm}{dv}. \tag{1.4.10}$$

The radiation energy density (ϵ) should be nonnegative, and so it follows that

$$\frac{dm}{dv} \leq 0.$$

It is thus seen that the mass function $m(v)$ decreases with proper time v which is consistent with a loss in energy in the form of radiation.

1.5   Junction conditions

In solving a system of differential equations, as is the case with Einstein's field equations, one obtains solutions which include constants of integration, a natural consequence of the process of integration. In many physical problems, these constants are

assigned values based on predetermined boundary conditions, having been set according to physical intuition. In the case of general relativity, these are known as the junction conditions and their presence is always highlighted in articles on relativity. Junction conditions arise from the physical nature of a star separating spacetime into two main regions, namely, spacetime containing the matter medium of a star, and spacetime extending from the surface of the star outwards which ideally contains only radiation emanating from the star. Solutions to Einstein's field equations for the interior of a star were found by Oppenheimer and Snyder (1939), for the non-adiabatic collapse of a spherically symmetric matter distribution in the form of a dust cloud having a Schwarzschild exterior, and by Misner and Sharp (1964) for an interior matter distribution modelled as a perfect fluid with static exterior. The solution to Einstein's field equations for an exterior which is void, is the Schwarzschild (1916) solution, whereas Vaidya's (1953) solution describes the exterior of a star containing null radiation. The solutions to Einstein's field equations for both the interior and the exterior spacetime must show agreement at the interior–exterior interface in order to make sense of the collapse process. This is known as the matching of interior and exterior spacetimes of the star. The criteria for matching involve the continuity of intrinsic and extrinsic curvature over the interior and exterior solutions as given by Darmois (1927), Lichnerowicz (1955) and O'Brien and Synge (1952). The junction conditions given by Darmois and O'Brien and Synge are equivalent to those of Bonnor and Vickers (1981).

Initial work (Misner and Sharp 1964) on adiabatic collapse showed that the pressure vanishes at the boundary of a static star, which was regarded as physically realistic for some time. A full set of shear–free junction conditions for non-adiabatic, null radiation producing collapse was then introduced by Santos (1985) through an approach similar to that of Israel (1966). A consequence thereof is that the pressure at the boundary of a star is proportional to the flow of heat. Applying these junction conditions to solutions of Einstein's equations provides temporal behaviour (Santos 1985, Herrera *et al* 2008). Santos shows that the pressure at the boundary is non–vanishing in general and that

the boundary conditions are due to the *"local conservation of momentum"* across the hypersurface.

In leading up to the first junction condition, the following equations arise, namely

$$A(t, r_\Sigma)\dot{t} \ = \ 1, \tag{1.5.1a}$$

$$Y(t, r_\Sigma) \ = \ \mathcal{Y}(\eta), \tag{1.5.1b}$$

$$\mathcal{R}_\Sigma(v) \ = \ \mathcal{Y}(\eta), \tag{1.5.1c}$$

$$\left(1 - \frac{2m(v)}{\mathcal{R}} + 2\frac{d\mathcal{R}}{dv}\right)_\Sigma \ = \ \left(\frac{1}{\dot{v}^2}\right)_\Sigma. \tag{1.5.1d}$$

Upon eliminating η from the above equations we are able to obtain the necessary and sufficient conditions required by the spacetime geometry for the first junction condition to be satisfied. In summary, the results of the first junction condition can be collectively written as

$$A(t, r_\Sigma)dt \ = \ \left(1 - \frac{2m(v)}{\mathcal{R}_\Sigma} + 2\frac{d\mathcal{R}_\Sigma}{dv}\right)^{\frac{1}{2}} dv, \tag{1.5.2a}$$

$$Y(t, r_\Sigma) \ = \ \mathcal{R}_\Sigma(v). \tag{1.5.2b}$$

Similarly, a summary of the second juction condition which involves matching the extrinsic curvatures is now given. The necessary and sufficient conditions required of the spacetimes in order to satisfy the second junction condition are

$$m(v) \ = \ \left[\frac{Y}{2}\left(1 + \frac{\dot{Y}^2}{A^2} - \frac{Y'^2}{B^2}\right)\right]_\Sigma, \tag{1.5.3a}$$

$$\left(p_r + \Pi + \frac{2}{3}\Omega + \epsilon\right)_\Sigma \ = \ \left(q_1 B + \epsilon\right)_\Sigma, \tag{1.5.3b}$$

where the shear viscosity $\Omega = \frac{3}{2}\pi_1^1$. It is worthwhile noting that the junction conditions (1.5.3a) and (1.5.3b) are independent of any particular form for the metric functions

A, B or Y, but rather has been established as a general result for spherically symmetric, shearing spacetimes. Since the radiation energy density ϵ appears on both sides of equation (1.5.3b), we can rewrite it as

$$\left(p_r + \Pi + \frac{2}{3}\Omega\right)_\Sigma = \left(q_1 B\right)_\Sigma. \tag{1.5.4}$$

According to Herrera *et al* (2009), the left hand side of (1.5.4) can be regarded as an "effective" radial pressure. If we eliminate the bulk viscosity but not the shear viscosity, we obtain the result (for shearing spacetime) of Naidu *et al* (2006), namely $\left(p_r\right)_\Sigma = \left(q_1 B\right)_\Sigma$. The most general matching conditions for the spherically symmetric shearing spacetimes $\mathcal{Z}^+$ and $\mathcal{Z}^-$ are given by equations (1.5.2) and (1.5.3). Equation (1.5.3b) simply tells us that the effective radial pressure comprising of the radial pressure, the bulk viscous pressure and the shear stress, is proportional to the magnitude of the heat flow $q_1 B$ which is nonvanishing in general as shown by Herrera *et al* (2009). Thus the radial pressure $\left(p_r\right)_\Sigma$ on the boundary can only be zero when the heat flow $\left(q_1 B\right)_\Sigma$, the bulk viscosity Π and the shear viscosity Ω are all zero. When this happens, no heat is radiated to the exterior spacetime and thus the exterior spacetime is no longer the Vaidya spacetime but becomes the Schwarzschild exterior spacetime. In disregarding bulk and shear viscosities, the conditions as obtained by Santos in the shear–free limit with isotropic pressure are given by

$$m(v) = \left(\frac{r^3 B}{2A^2}B_t{}^2 - r^2 B_r - \frac{r^3}{2B}B_r{}^2\right)_\Sigma, \tag{1.5.5a}$$

$$p_\Sigma = (q_1 B)_\Sigma. \tag{1.5.5b}$$

There is a physical interpretation of the second junction condition which can be arrived at by taking into account the momentum flux across the boundary in the radial direction. With the knowledge that (1.5.3a) represents the total energy for a sphere of radius r lying within the hypersurface Σ we are able to express the mass as a function

of radial and time coordinates, namely

$$m(t, r) = \left[\frac{Y}{2} \left(1 + \frac{\dot{Y}^2}{A^2} - \frac{Y'^2}{B^2} \right) \right]_\Sigma. \tag{1.5.6}$$

Partial differentiation of $m(t, r)$ with respect to t yields

$$\left(\frac{\partial m}{\partial t} \right)_\Sigma = \left[\dot{Y} \left(\frac{\ddot{Y}Y}{A^2} + \frac{\dot{Y}^2}{2A^2} - \frac{Y'^2}{2B^2} - \frac{\dot{A}\dot{Y}Y}{A^3} + \frac{1}{2} \right) \right.$$

$$\left. - \frac{Y'\dot{Y}'Y}{B^2} + \frac{\dot{B}Y'^2Y}{B^3} \right]_\Sigma.$$

By using the Einstein field equations for radial pressure and heat flux, the expression may be rewritten as

$$\left(\frac{\partial m}{\partial t} \right)_\Sigma = \left[-\frac{Y^2}{2} \left(\dot{Y} \left(p_r + \Pi + \epsilon + \frac{2}{3}\Omega \right) + \frac{AY'}{B} (q_1 B + \epsilon) \right) \right]_\Sigma. \tag{1.5.7}$$

Substitution of (1.5.3b) in (1.5.7) yields

$$\left(\frac{\partial m}{\partial t} \right)_\Sigma = \left[-\frac{AY^2}{2} \left(\frac{\dot{Y}}{A} + \frac{Y'}{B} \right) \left(p_r + \Pi + \epsilon + \frac{2}{3}\Omega \right) \right]_\Sigma. \tag{1.5.8}$$

The fact that the radial coordinate is comoving with respect to the hypersurface Σ enables us to write

$$\left(\frac{\partial m}{\partial t} \right)_\Sigma = \left(\frac{dm}{dt} \right)_\Sigma = \left(\frac{\dot{v}}{\dot{t}} \frac{dm}{dv} \right)_\Sigma. \tag{1.5.9}$$

If we consider

$$A(t, r_\Sigma)\dot{t} = 1, \tag{1.5.10a}$$

$$Y(t, r_\Sigma) = \mathcal{Y}(\eta), \tag{1.5.10b}$$

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in addition to (1.5.2b), (1.5.8) and (1.5.9) it can be shown that

$$\left(-\frac{2}{\mathcal{R}^2}\frac{dm}{dv}\dot{v}^2\right)_\Sigma = \left(p_r + \Pi + \epsilon + \frac{2}{3}\Omega\right)_\Sigma.$$
(1.5.11)

As a result of the initiative due to Lindquist *et al* (1965) we know that the energy density of the radiation detected by an observer situated on the hypersurface Σ (having a four–velocity v^a) is expressed as

$$\epsilon = v^a v^b T_{ab},$$
(1.5.12)

where the four–velocity expressed in component form is

$$v^a = (\dot{v}, \dot{\mathcal{R}}, 0, 0).$$
(1.5.13)

The Einstein tensor for the metric (1.4.6) is given by

$$G_{ab} = -\frac{2}{\mathcal{R}^2}\frac{dm}{dv}\delta_a^0\delta_b^0 = T_{ab}.$$
(1.5.14)

Employing (1.5.14) and (1.5.13) in equation (1.5.12), we obtain the radiation energy density that an observer situated on the hypersurface Σ measures, i.e.

$$\epsilon = \left(-\frac{2}{\mathcal{R}^2}\frac{dm}{dv}\dot{v}^2\right)_\Sigma.$$
(1.5.15)

If we consider the radial momentum flux in exterior spacetime (Z^+) with an energy density given by (1.5.15), it becomes apparent that equation (1.5.11) implies the local conservation of momentum which takes into account bulk viscous effects, shear viscosity and null radiation (Santos 1985).

1.6 Thermodynamics

In studying the evolution of a system such as a massive star undergoing gravitational collapse, it is necessary to determine the temperature and luminosity profiles of the star. The first order Eckart formalism was initially used to study temperatures where heat conduction is governed by Fourier's law, namely,

$$q^a = -\kappa h^{ab} \left(T_{,b} + T\dot{u}_b \right) \tag{1.6.1}$$

where κ is the thermal conductivity, $\dot{u}_b = u_{b;c}u^c$ is the four–acceleration and $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor. This equation has been used previously in the study of radiating stars (de Oliveira *et al* 1985, Govender *et al* 1998). The temperatures calculated in this way provide a reasonable approximation when the fluid is close to static equilibrium, however at later stages of the collapse process, the fluid is far from quasi–stationary equilibrium and the Eckart description is no longer accurate. This is largely due to the noncausal nature of the description which allows for superluminous propagation velocities. Furthermore, in the classical formalism, the constitutive equations are assumed to be algebraic which is not in accordance with experiments in plasma physics and high frequency currents. The constitutive equations should contain time-derivative terms to account for relaxational effects. The noncausal theory also predicts unstable equilibrium states. Experiments on neutron scattering in liquids and low temperature investigations of phonon propagation in solids indicate that the theory is unsatisfactory at high frequencies and short wavelengths.

In order to make improvements in the theory of irreversible thermodynamics, second order effects in the dissipative fluxes have to be included (Maartens 1996). The entropy flux vector S^a is defined by

$$S^a = Snu^a + \frac{\mathcal{R}^a}{T}, \tag{1.6.2}$$

where S is the specific entropy, n is the particle number density and $\mathcal{R}^a$ represents the dissipation. In the Eckart theory, $\mathcal{R}^a$ is an algebraic function of the particle four–current $n^a = nu^a$ and the energy momentum tensor. At equilibrium, $\mathcal{R}^a$ vanishes.

In order to restore causality and stability, the algebraic form of $\mathcal{R}^a$ must at least be second order in the dissipative fluxes in extended irreversible thermodynamics. This generates a system of transport equations which govern the behaviour of the dissipative fluxes and their associated quantities. If it is assumed that there is no viscous–type heat coupling, then we have the following relationship for the temperature

$$\tau h^{ab} u^c q_{b;c} + q^a = -\kappa h^{ab} \left(T_{,b} + T \dot{u}_b \right). \qquad (1.6.3)$$

This is known as the covariant relativistic Maxwell–Cattaneo heat transport equation (Maartens 1996) and is used in this research as the starting point for then proceeding to calculate temperature profiles for the models under investigation. It is one of the transport equations that appears in the truncated Israel–Stewart theory (Israel and Stewart 1979). In the above equation, τ is the relaxation time and when $\tau = 0$ we regain the Fourier equation.

Gravitational collapse is a highly dissipative process and any reasonable model of gravitational collapse has to include dissipation and relaxation time (Herrera and Santos 1997a). Martínez (1996) clearly highlights the effect of relaxation time especially for early and late stages of the collapse process. Di Prisco *et al* (1997, 2007) have shown that relaxation time has a direct impact on the final mass, compactness and luminosity profile of a radiating star. Govender and co–workers have shown that relaxational effects can lead to higher core temperatures and enhanced cooling at the surface of the collapsing body (Govender *et al* 1998, 2003, 2010, 2013; Govinder and Govender 2012). In order to explore the contributions due to relaxational effects we employ the causal heat transport equation of the Maxwell-Cattaneo form. The Maxwell–Cattaneo form has many desirable features as pointed out by Joseph and Preziosi (1989) since it satisfies the relativistic causality requirement and rate–type equations like it are "*extensively used in the theory of viscoelastic fluids and in relaxing gas dynamics*".

1.6.1　General case

The truncated form of the Israel–Stewart causal heat transport equation is the Maxwell–Cattaneo equation which is expressed as

$$\tau_r h_a{}^b \dot{q}_b + q_a = -\kappa\left(h_a{}^b \nabla_b \mathcal{T} + \mathcal{T} \dot{u}_a\right), \tag{1.6.4}$$

where $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor, $\mathcal{T}(t,r)$ is the local equilibrium temperature, $\kappa (\geq 0)$ is the thermal conductivity, and $\tau_r (\geq 0)$ is the relaxation time scale with which causal, stable behaviour is achieved. The noncausal Eckart heat transport equation is obtained by setting the relaxation time $\tau_r = 0$ in (1.6.4). With the aid of the metric (1.4.3), equation (1.6.4), with $\tau_r \neq 0$, becomes

$$\tau_r (qB)\dot{} + A(qB) = -\kappa\frac{(A\mathcal{T})'}{B}. \tag{1.6.5}$$

The thermodynamic coefficients associated with radiative transfer are well motivated by Govender *et al* (1998, 1999) and Martínez (1996). Following Martínez, we take the thermal conductivity to be

$$\kappa = \gamma \mathcal{T}^3 \tau_c, \tag{1.6.6}$$

where $\gamma = \frac{4}{3}b$ ($b = \frac{7}{8}a$ for neutrinos, with a being a radiation constant) and τ_c is the mean collision time. Martínez finds the temperature dependance of the collision time, by assuming that the neutrinos generated as a consequence of thermal emission, to be $\tau_c \propto T^{-\frac{3}{2}}$. Based on his findings, we assume a power law relationship of the form

$$\tau_c = \left(\frac{\xi}{\gamma}\right)\mathcal{T}^{-\omega}, \tag{1.6.7}$$

where ξ and ω are positive constants, with $\omega = \frac{3}{2}$ for thermally generated neutrinos. It is easy to see that the case of $\omega = 0$ corresponds to constant collision time, which holds true for a limited temperature range. Martínez assumes that the speed of the thermal and viscous signal to be comparable to the speed of sound in the fluid medium, which implies that it is reasonable to say that the relaxation time is proportional to the mean

collision time. This is expressed as

$$\tau_r = \left(\frac{\psi\gamma}{\alpha}\right)\tau_c, \tag{1.6.8}$$

where ψ (≥ 0) is a constant. Employing the definitions for τ_r and κ, it can be shown that equation (1.6.5) takes the form

$$\psi(qB)\dot{\mathcal{T}}^{-\omega} + A(qB) = -\xi\frac{\mathcal{T}^{3-\omega}(A\mathcal{T})'}{B}, \tag{1.6.9}$$

where ψ can be considered to be a 'causality index', that enables us to quantify the impact of relaxation effects on the system.

1.6.2 Luminosity and surface redshift

The total luminosity detected or measured by an observer at rest at an infinite distance away from the surface of a star is given by

$$L_\infty(v) = -\frac{dm}{dv} = \lim_{\mathcal{R}\to\infty}\frac{\mathcal{R}^2}{2}\epsilon\frac{1}{\dot{v}^2}, \tag{1.6.10}$$

where the mass $m(v)$ is a function of the retarded time v (Lindquist *et al* 1965). Furthermore, it is noted that $\frac{dm}{dv} \leq 0$ for the luminosity L_∞ to be positive. The luminosity measured by an observer situated on Σ is given by

$$L_\Sigma = \left(\frac{\mathcal{R}^2\epsilon}{2}\right)_\Sigma = \left(\frac{Y^2\epsilon}{2}\right)_\Sigma. \tag{1.6.11}$$

The surface redshift z_Σ, which is in actual fact the change in the frequency of the radiation emitted from the hyper-surface Σ of the star is expressed as

$$1 + z_\Sigma = \frac{dv}{d\eta} = \left(\frac{\dot{Y}}{A} + \frac{Y'}{B}\right)_\Sigma^{-1}. \tag{1.6.12}$$

Note that as the star undergoes continued gravitational collapse, it will at some point reach the event horizon which is also referred to as the Schwarzschild radius, i.e. $\mathcal{R} = 2m(v)$. When the star's radius is less than Schwarzschild radius, signals take an

infinitely long time to reach an observer on the outside. Thus, we can determine the time it takes for an event horizon to form. The above expressions allow us to write

$$L_\infty = -\frac{dm}{dv} = \left(\frac{\mathcal{R}^2}{2}\epsilon\frac{1}{\dot{v}^2}\right)_\Sigma.$$
(1.6.13)

We can rewrite (1.6.13), evaluated on the hypersurface Σ as

$$L_\infty = -\frac{dm}{dv} = \left\{\frac{Y^2}{2}\left(p_r + \Pi + \epsilon + \frac{2}{3}\Omega\right)\left[\frac{Y'}{B} + \frac{\dot{Y}}{A}\right]^2\right\}_\Sigma.$$
(1.6.14)

With the aid of (1.6.11), (1.6.12) and (1.5.2b), relation (1.6.13) can be recast in the form

$$\frac{L_\Sigma}{L_\infty} = \left(1 + z_\Sigma\right)^2,$$
(1.6.15)

which expresses the ratio of the luminosity on the hypersurface L_Σ to the luminosity an infinite distance away L_∞ in terms of the surface redshift.

1.6.3 Noncausal solutions

The noncausal solutions of (1.6.9) are obtained by setting $\psi = 0$. These solutions are due to Govinder and Govender (2001) for shear–free radiating collapse, and are given by

$$(AT)^{4-\omega} = \frac{\omega - 4}{\alpha}\int A^{4-\omega}qB^2dr + F(t) \ , \omega \neq 4$$
(1.6.16)

$$\ln(AT) = -\frac{1}{\xi}\int qB^2dr + F(t) \ , \omega = 4$$
(1.6.17)

where $F(t)$ is an arbitrary integration function. The quantity $F(t)$ is obtained by applying the surface temperature boundary condition

$$\left(T^4\right)_\Sigma = \left(\frac{1}{r^2B^2}\right)_\Sigma\left(\frac{L_\infty}{4\pi\delta}\right),$$
(1.6.18)

where L_∞ represents the total luminosity (1.6.14) observed at a very far distance and δ is a positive constant. Causal solutions of (1.6.9) have been obtained by Govinder and Govender (2001) for constant and non-constant collision times. These are given below.

1.6.4 Causal solutions: $\omega = 0$ and $\omega = 4$

The case of $\omega = 0$ in (1.6.7) corresponds to constant mean collision time. Upon integrating the causal transport equation (1.6.9) with $\omega = 0$, we obtain the following temperature profile

$$(AT)^4 = -\frac{4}{\alpha}\left[\psi \int A^3 B(qB)\dot{} dr + \int A^4 qB^2 dr\right] + F(t). \qquad (1.6.19)$$

Setting $\omega = 4$ in (1.6.7) produces a model that is valid for a specific range of temperatures as pointed out by Govinder and Govender (2001). The resulting equation has the form of a Bernoulli equation in AT and the solution to this equation has the form

$$\begin{aligned}
(AT)^4 &= -\frac{4\psi}{\alpha}\exp\left(-\int \frac{4qB^2}{\alpha}dr\right)\int A^3 B(qB)\dot{}\exp\left(\int \frac{4qB^2}{\alpha}dr\right) \\
&\quad + F(t)\exp\left(-\int \frac{4qB^2}{\xi}dr\right).
\end{aligned} \qquad (1.6.20)$$

Temperature profiles (causal and noncausal) for (1.6.9) have been obtained for various models such as shear–free (horizon–free) collapse (Naidu and Govender 2007), radiating anisotropic collapse with shear (Naidu *et al* 2006), Euclidean star models (Govender *et al* 2010, 2012), dissipative collapse with cosmological constant (Thirukkanesh *et al* 2012a), collapse involving first–order perturbations (Maharaj *et al* 2011), shearing, radiative collapse with expansion and acceleration (Thirukkanesh *et al* 2012b), models highlighting the effect of shear (Govender *et al* 2014), as well as the role of anisotropy on the perturbed temperature profiles (Reddy *et al* 2015).

Chapter 2

Gravitational collapse in spatially isotropic coordinates

2.1 Introduction

Looking for exact solutions to Einstein's field equations which are capable of describing realistic astrophysical systems has been an area of active research since the discovery of the Schwarzschild solution in 1916 (Schwarzschild 1916). There are many solutions to the field equations however many of these are not physically viable (Delgaty 1998). For instance, solutions might return negative values for energy density, heat dissipation and temperature which are unrealistic and thus unacceptable. Thus various techniques and assumptions based on particle physics, hydrostatic equilibrium and physical observations have been used to generate physically meaningful models of stellar objects. Since exact solutions are not easy to come by, various ad hoc approaches have also been used to simplify the nonlinearity of the field equations so as to generate solutions which are well behaved and which can be utilized to describe realistic physical processes. A comprehensive study of exact static solutions of the Einstein field equations, which are physically reasonable, yields only a small class of solutions that satisfy all of the conditions for hydrostatic equilibrium and causality.

Nevertheless, in the absence of any reliable information about the physics of matter at extremely high densities, a geometrical approach has been found to be a worthwhile technique in the study of compact stellar objects, which are the best candidates for understanding particle interactions under extreme conditions. For example, the Tikekar superdense stellar model which describes the gravitational field of a highly compact spherically symmetric star, was shown to exhibit a reasonable equation of state for neutron stars (Karmakar *et al* 2007; Sarwe and Tikekar 2010). This then prompted many researchers to look for exact solutions capable of describing a large variety of astrophysical systems where relativistic effects cannot be ignored, thereby expanding the Tikekar model to include charge (Maharaj and Govender 2000), pressure anisotropy (Sharma and Tikekar 2012a), quark matter (Sharma and Mukherjee 2001), scalar fields and higher dimensional systems (Patel and Singh 2001). Within a set of realistic models, the static stellar model proposed by Pant and Sah (1985) is of particular interest. The Pant and Sah model describes a spherically symmetric compact star in spatially isotropic coordinates whose solution regains the well known Buchdahl polytrope solution of index 5. The physical viability of the Pant and Sah model was recently studied in detail by Deb *et al* (2012) and it has been shown that this model can be utilized to describe a wide variety of compact stellar objects including strange stars.

In this research, we have incorporated dynamical effects into the Pant and Sah model by allowing certain model parameters to evolve with time. This has allowed us to investigate the non-adiabatic collapse of a star in a spatially isotropic spacetime. Work on non-adiabatic gravitational collapse was previously done by de Oliveira and Santos (1988). In our time-dependent model, we have assumed that the star begins its collapse from an initial static configuration by dissipating energy in the form of a radial heat flux. The collapse proceeds in such a manner so as to ensure that the mass loss is small and that the stellar body is in quasi-static equilibrium. This corresponds to the final stage of a star, just before the formation of the compact object. We regain the Pant and Sah model as the static limit of the dynamical collapse process.

Although the issue of gravitational collapse was first taken up by Oppenheimer and Snyder (1939), the study of a more realistic collapse scenario in the presence of dissipative processes came about through the incorporation of Vaidya's metric, corresponding to the exterior gravitational field of a radiating star. A formal treatment of the junction conditions required the smooth matching of the collapsing core to the exterior non-empty spacetime and was set down by Santos (1985). These junction conditions spurred an interest in studying dissipative collapse, with much of the early work having been done by Herrera and co-workers (Herrera *et al* 1997). For a radiating collapsing star, the pressure at the boundary is proportional to the magnitude of the heat flux and hence gives rise to a temporal evolution equation for the metric functions. As in the case of a static model, various solutions for radiating stars have been based on physics, dynamical stability and ad hoc assumptions. The interior spacetime of these models has been generalised to include (apart from heat flow) pressure anisotropy, bulk viscosity, shear–stresses and electromagnetic fields (Thirukkanesh and Govender 2013). A comprehensive review of various approaches and analyses involving gravitationally collapsing systems has been done by Joshi and Malafarina (2011). An interesting approach was adopted by Kramer (1992) in which the interior Schwarzschild solution was written in isotropic coordinates and the mass parameter was allowed to become time-dependent. The radiating Schwarzschild-like solution had as its source term, a perfect fluid with heat flow. Since the interior was radiating energy, the exterior was non-empty and was described by Vaidya's outgoing metric. Kramer provided a first integral of the boundary condition required for the matching of the interior to the Vaidya solution. Maharaj and Govender (1997) presented the full temporal behaviour of the Kramer model in terms of Li integrals. The complicated form of the analytical solution for the temporal behaviour did not warrant a full study of the physics of the model. The present work takes up the initiative to use the Kramer algorithm to provide a full descriptive model of dissipative gravitational collapse. As the collapse process begins in a massive star, after exhausting all of its thermonuclear fuel, prediction of the final

stage of the collapsing star becomes very much speculative in nature (Chandrasekhar 1934). In fact, one of the the most outstanding challenges in general relativity has been the prediction of the end state of a gravitationally bounded system. In the context of the Cosmic Censorship Conjecture, the general relativistic prediction is that such a collapse must terminate in a black hole; though there are several counter–examples where it has been shown that a naked singularity is more likely to be formed (Joshi and Malafarina 2011). The nature of singularities is a topic for continued research (Sharif and Siddiqa 2010). In our dynamical model, we show that the star begins its collapse from an initial static configuration with acceptable physical conditions which are always close to hydrostatic equilibrium.

2.2 Interior and exterior spacetimes

We write the interior spacetime of a spherically symmetric shear–free collapsing star in spatially isotropic coordinates as

$$ds_-^2 = -A^2(r,t)dt^2 + B^2(r,t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \qquad (2.2.1)$$

We assume that the material composition filling the interior of the collapsing object is an imperfect fluid with dissipation in the form of a radial heat flux and accordingly we express the energy momentum tensor in the form

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a, \qquad (2.2.2)$$

where ρ is the energy density, p is the isotropic fluid pressure, $u^a = (1/A)\delta_0^a$ is the timelike four–velocity of the fluid and $q^a = (0, q, 0, 0)$ is the heat flux vector which is orthogonal to the velocity vector so that $q^a u_a = 0$. The Einstein field equations

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describing the dynamics of the system are then obtained as

$$\rho = 3\frac{1}{A^2}\frac{B_t{}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B_{rr}}{B} - \frac{B_r{}^2}{B^2} + \frac{4}{r}\frac{B_r}{B}\right), \qquad (2.2.3a)$$

$$p_r = \frac{1}{A^2}\left(-2\frac{B_{tt}}{B} - \frac{B_t{}^2}{B^2} + 2\frac{A_t}{A}\frac{B_t}{B}\right)$$

$$+\frac{1}{B^2}\left(\frac{B_r{}^2}{B^2} + 2\frac{A_r}{A}\frac{B_r}{B} + \frac{2}{r}\frac{A_r}{A} + \frac{2}{r}\frac{B_r}{B}\right), \qquad (2.2.3b)$$

$$p_t = -2\frac{1}{A^2}\frac{B_{tt}}{B} + 2\frac{A_t}{A^3}\frac{B_t}{B} - \frac{1}{A^2}\frac{B_t{}^2}{B^2} + \frac{1}{r}\frac{A_r}{A}\frac{1}{B^2}$$

$$+\frac{1}{r}\frac{B_r}{B^3} + \frac{A_{rr}}{A}\frac{1}{B^2} - \frac{B_r{}^2}{B^4} + \frac{B_{rr}}{B^3}, \qquad (2.2.3c)$$

$$q = -\frac{2}{AB}\left(-\frac{B_{rt}}{B} + \frac{B_r B_t}{B^2} + \frac{A_r}{A}\frac{B_t}{B}\right). \qquad (2.2.3d)$$

Combining equations (2.2.3b) and (2.2.3c), we obtain

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} - \left(2\frac{B_r}{B} + \frac{1}{r}\right)\left(\frac{A_r}{A} + \frac{B_r}{B}\right) = 0, \qquad (2.2.4)$$

which is the pressure isotropy equation. We note that this equation does not contain time derivatives of the metric functions. If we take a static solution $(A_0(r), B_0(r))$ of the pressure isotropy equation and allow the constants to become time–dependent, then this solution will satisfy the metric (2.2.1).

The exterior spacetime, in the presence of an outgoing radiation flux around the spherically symmetric collapsing matter source, is described by the Vaidya metric (Vaidya 1953)

$$ds_+^2 = -\left(1 - \frac{2m(v)}{r}\right)dv^2 - 2dvdr + r^2\left(d\theta^2 + r^2\sin^2\theta d\phi^2\right), \qquad (2.2.5)$$

where the total mass $m(v)$ is a function of the retarded time v. Assuming that Σ divides the spacetimes into two distinct regions, the junction conditions which join smoothly the interior spacetime (2.2.1) and the exterior spacetime (2.2.5) across Σ, forming the boundary of the star, are

$$p_\Sigma = (qB)_\Sigma, \qquad (2.2.6)$$

$$m_\Sigma = \left[\frac{r^3}{2}\left(\frac{BB_t^2}{A^2} - \frac{B_r^2}{B}\right) - r^2 B_r\right]_\Sigma, \qquad (2.2.7)$$

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where m_Σ is the total mass within a sphere of radius r_Σ. This is according to Santos (1985).

2.3   Generating dynamical solutions

Note that in the metric (2.2.1), the metric potentials $A(r,t)$ and $B(r,t)$ are yet to be specified. To generate a viable dynamical model, let us assume that the system begins its collapse from an initial static configuration $(A_0(r), B_0(r))$. For the initial static configuration, we choose the Pant and Sah solution (Pant and Sah 1985) which describes the interior spacetime of a static spherically symmetric star in isotropic coordinates. In our construction, we generalize the Pant and Sah solution so as to develop a model of a collapsing star which satisfies the time dependence of the metric (2.2.1). We note that (2.2.4) admits a solution

$$A(r,t) \;=\; \frac{a(1-\alpha(r)k(t))}{(1+\alpha(r)k(t))}, \tag{2.3.1}$$

$$B(r,t) \;=\; \frac{(1+\alpha(r)k(t))^2}{(1+r^2/R^2)}, \tag{2.3.2}$$

for an arbitrary $k(t)$, where

$$\alpha(r) = \sqrt{\frac{1+r^2/R^2}{1+\lambda r^2/R^2}}. \tag{2.3.3}$$

Obviously the static limit of the model is obtained by setting $k(t) = K$, a constant (i.e., $\dot{k} = 0$). For an evolving system, we need to determine $k(t)$ which can be obtained by solving the junction condition (2.2.6). The resultant 'surface equation' in this construction turns out to be highly nonlinear in nature and extremely difficult to solve. However, it is possible to generate an approximate solution of the equation by setting $k(t) = K + \epsilon h(t)$, with $0 < \epsilon << 1$. Neglecting terms $\mathcal{O}(\epsilon^2)$ and noting that pressure at the boundary of the initial static star vanishes, the surface equation then assumes a simple form

$$\mu\ddot{h} + \nu\dot{h} + \eta h = 0, \tag{2.3.4}$$

where μ, ν and η are constants evaluated at the boundary Σ. They are given by

$$\mu = -\left[\frac{\alpha(1+K\alpha)^5}{a^2(K\alpha-1)^2}\right]_{\Sigma}, \tag{2.3.5a}$$

$$\nu = \left[\frac{\alpha(1+K\alpha)^2\sqrt{(1-\alpha^2)(\alpha^2\lambda-1)}}{aR(K\alpha-1)^2}\right]_{\Sigma}, \tag{2.3.5b}$$

$$\eta = \left[\frac{4\alpha - 6K\alpha^2 + 2\lambda K\alpha^6(1+2K\alpha(K\alpha-1))}{R^2(k^2\alpha^2-1)^2}\right]_{\Sigma}. \tag{2.3.5c}$$

Equation (2.3.4) is easily solvable and the most general solution of the equation can be written as

$$h(t) = Ce^{\frac{t}{2\mu}\left(-\nu-\sqrt{\nu^2-4\mu\eta}\right)} + De^{\frac{t}{2\mu}\left(-\nu+\sqrt{\nu^2-4\mu\eta}\right)}, \tag{2.3.6}$$

where C and D are integration constants.

We assume that the collapse begins in the remote past $(t \to -\infty)$ from an initial static configuration as the star loses its equilibrium. This implies that we must have $k(t \to -\infty) = K$, where K is a constant as described in the static Pant and Sah model. For a collapsing (contracting) sphere, without any loss of generality, we set $D = 0$ which ensures that $\dot{k}$, $\dot{h} < 0$. In addition to this, for real values of $h(t)$, we need to fix the model parameters so that the condition $\nu^2 \geq 4\mu\eta$ is satisfied. Subsequently, from equations (2.2.3a)–(2.2.3d), the energy density, pressure and heat flux density are obtained as

$$\rho = 12\left(\frac{1}{R^2(1+K\alpha)^5} + \frac{\epsilon\alpha^2\dot{h}^2(t)}{a^2(1-K\alpha)^2}\right), \tag{2.3.7a}$$

$$p = \frac{-4}{a^2(1-K\alpha)^3}\left[\left(\frac{(a(1-K\alpha))^2}{R^2(1+K\alpha)^5}\right) + 4\alpha^2 + 2\alpha\left(r^2+R^2\right)\left(1-K^2\alpha^2\right) \right. $$
$$\left. -\epsilon\left(\frac{\alpha^2}{r^2+R^2}\right)K\dot{h}^2(t) + \epsilon^2\alpha^2\left(2K+h(t)\right)h(t)\ddot{h}(t))\right], \tag{2.3.7b}$$

$$q = \frac{4r\epsilon\alpha\dot{h}(t)}{aR^2\left(1-K^2\alpha^2\right)^2}. \tag{2.3.7c}$$

We see from equations (2.3.7a) and (2.3.7b) that for the initial static configuration,

$$\rho_s(r) = \frac{12}{R^2(1+K\alpha)^5}, \tag{2.3.8}$$

$$p_s(r) = \frac{-4}{R^2(1-K\alpha)(1+K\alpha)^5} - \frac{16\alpha^2}{a^2(1-K\alpha)^3}. \tag{2.3.9}$$

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From (2.2.7), the total mass $m(r,t)$ within a radius $r \leq r_\Sigma$, at any instant t, is obtained as

$$m(r,t) = -\frac{2r^5 R^2}{(r^2 + R^2)^3} + \frac{2r^3 R^2 \left(1 + \alpha \left(K + \epsilon h(t)\right)\right)}{(r^2 + R^2)^2} +$$
$$\frac{2r^3 \left(1 + \alpha \left(K + \epsilon h(t)\right)\right)^6 \epsilon \dot{h}^2(t)}{a^2 \left(1 + r^2/R^2\right)^2 \left(1 + \lambda r^2/R^2\right) \left(1 - \alpha \left(K + \epsilon h(t)\right)\right)^2}. \qquad (2.3.10)$$

Then the mass of the initial static configuration is

$$m_s(r) = 2r^3 R^2 \left(\frac{\left(r^2 + R^2\right)\left(1 + K\alpha\right) - r^2}{(r^2 + R^2)^3} \right). \qquad (2.3.11)$$

Let b be the radial distance where the pressure of the initial static star vanishes, i.e., $p_s(r = b) = 0$. The physical radius b_0 of the initial static star can then be obtained from the relation (Deb *et al* 2012)

$$b_0 = b \left(1 + \frac{m_s(b_0)}{2b} \right)^2. \qquad (2.3.12)$$

At the boundary we match the static interior solution to that of the Schwarzschild exterior, and obtain

$$\frac{(1 + K\alpha_b)^2}{(1 + b^2/R^2)} = \left(1 + \frac{m_s(b)}{2b} \right)^2, \qquad (2.3.13\text{a})$$

$$K = \frac{1}{\sqrt{\lambda \alpha_b^3}}, \qquad (2.3.13\text{b})$$

which yields the condition

$$[(1 + a)^4 + K^4(1 - a)^4 - 8(1 + a)^2 + 16$$

$$-2K^2(1 - a^2)^2 - 8K^2(1 - a)^2]$$

$$+[2\lambda(1 + a)^4 - 16\lambda(1 + a)^2$$

$$-8(1 + a)^2 + 32(1 + \lambda) - 2K^2(1 + \lambda)(1 - a^2)^2$$

$$-8K^2(2 + \lambda)(1 - a)^2 + 2K^4(1 - a)^4]y^2$$

$$+[\lambda^2(1 + a)^4 - 8\lambda^2(1 + a)^2$$

$$-8\lambda(1 + a)^2(1 + 4\lambda + \lambda^2)$$

$$-2\lambda K^2(1 - a^2)^2 - 8K^2(1 - a)^2(1 + 2\lambda)$$

$$+K^4(1 - a)^4]y^4 - [8\lambda^2(1 + a)^2 - 32(1 + \lambda)$$

$$-8\lambda K^2(1 - a)^2]y^6 + 16\lambda^2 y^8 = 0, \tag{2.3.14}$$

where

$$y = \frac{b}{R} \quad \text{and} \quad \alpha_b = \sqrt{\frac{1 + y^2}{1 + \lambda y^2}}. \tag{2.3.15}$$

Equations (2.3.12)–(2.3.14) can be utilized to fix the model parameters for a specific choice of mass and radius of the initial static star.

2.4 Physical analysis

In order to analyze the physical behaviour of the collapsing model, let us start with one of the static models as given by Deb *et al* (2012) in which the parameters were set as follows: $A = 4$, $\lambda = 0.1211$ and $K = 2.2$. These parameters characterise a star of mass $M = 2.44\ M_\odot$ with a radius of $r_0 = 8.197$ km and with $R = 1.819$ km. These parameters characterise the X-ray pulsar, 4U 1700-37, as studied by Deb and co-workers.

In our time-dependent model, we set $C = -1$ and $\epsilon = 0.01$. This choice allows us to obtain physically reasonable temporal behaviour with the value of ϵ well within our

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perturbative framework. By setting $r = 16$km, we observe a calculated mass of 2.44 $M_\odot$ which varies negligibly from past to present times. This is evident in Figure 2.1. The radiated mass is dissipated to the exterior spacetime via a radial heat flux. Figure 2.2 shows that the energy dissipated throughout the collapse process can be viewed as a weak heat flux approximation. Such a collapse process was previously investigated by Lemos (1998) for a Friedmann-like dissipative sphere. In their scenario, the particles making up the stellar fluid exhibited geodesic motion. It is interesting to note that we obtain similar results even though the four–acceleration of the particles making up the stellar fluid in our model is nonzero. The corresponding dynamical nature of the pressure and energy density are depicted in Figure 2.3 and Figure 2.4 respectively. In the next section we turn our attention to the thermal behaviour of the collapsing model.
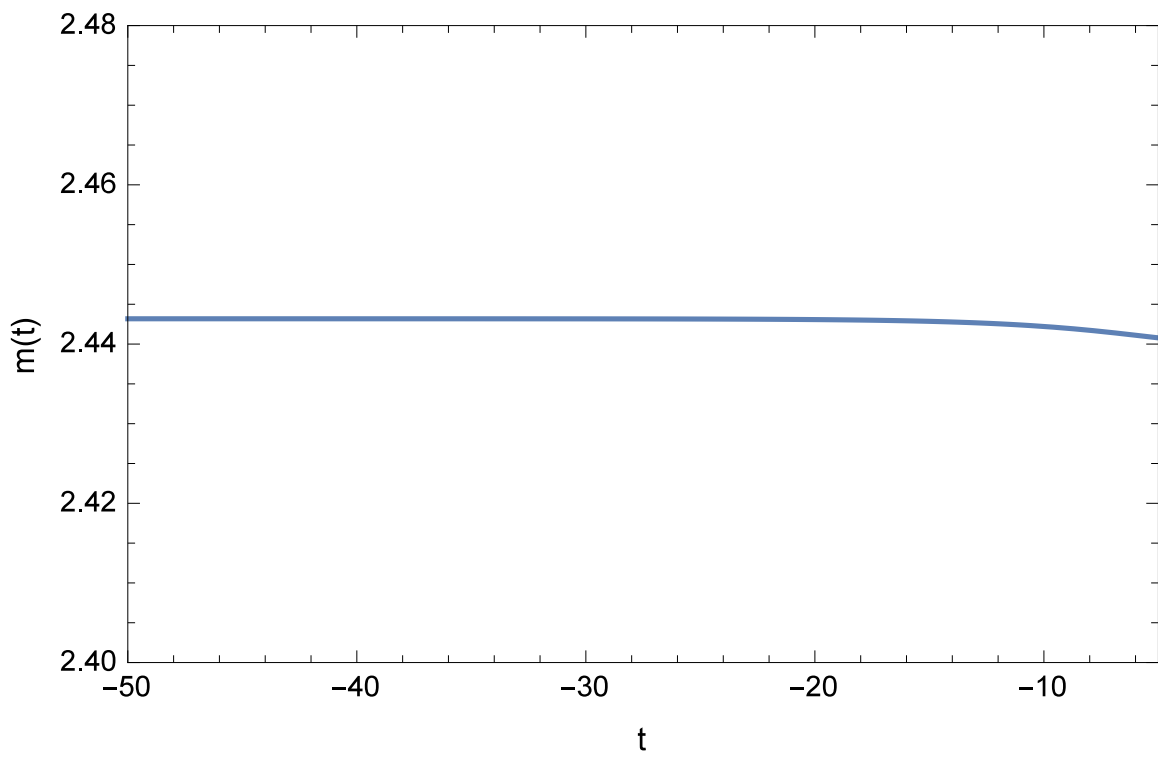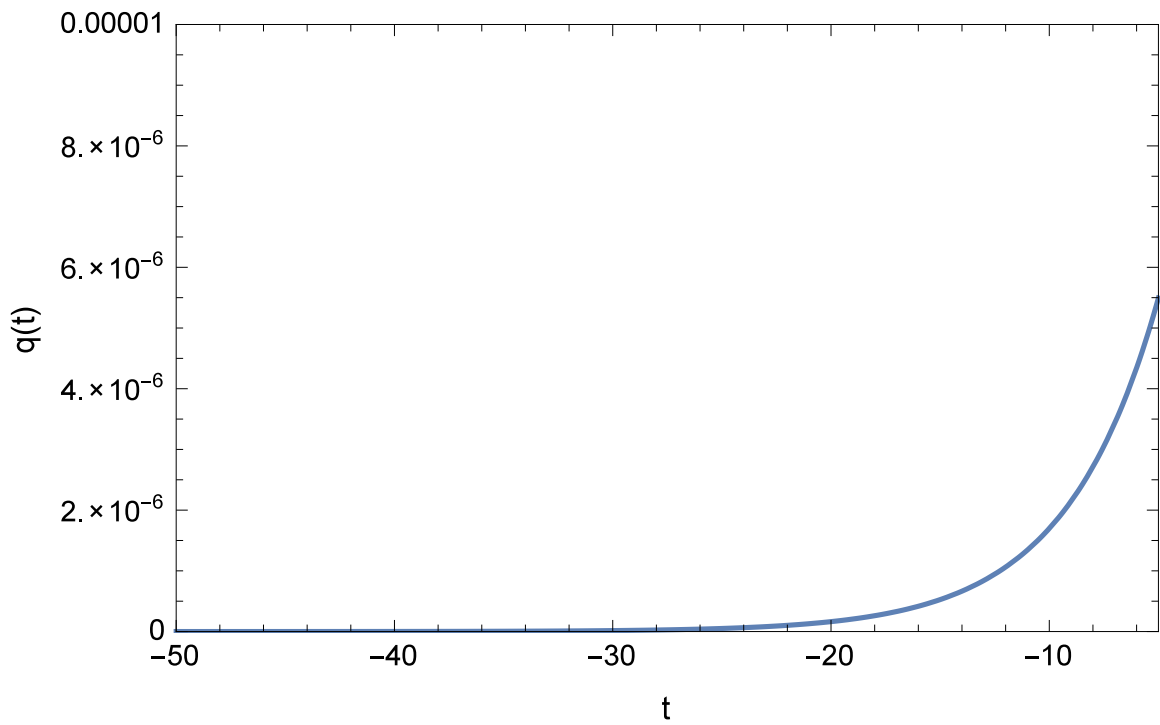
Figure 2.1: Evolution of mass $m(t)$.

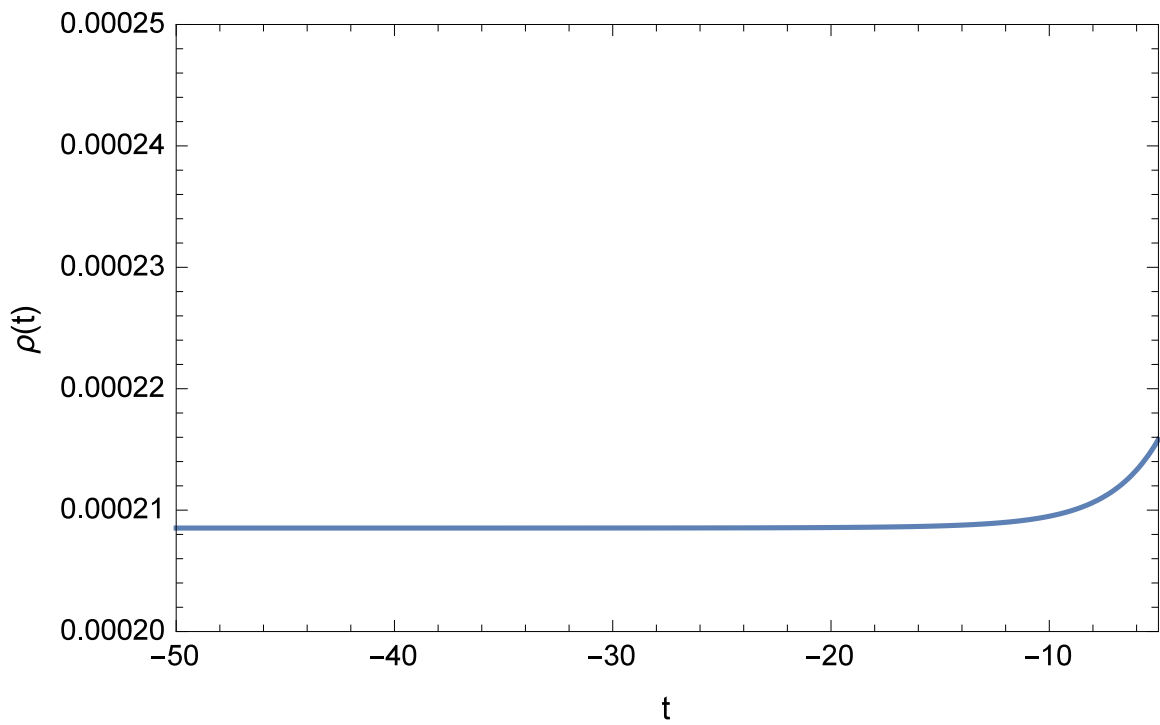Figure 2.2: Evolution of heat flux $q(t)$.

Figure 2.3: Evolution of energy density $\rho(t)$.
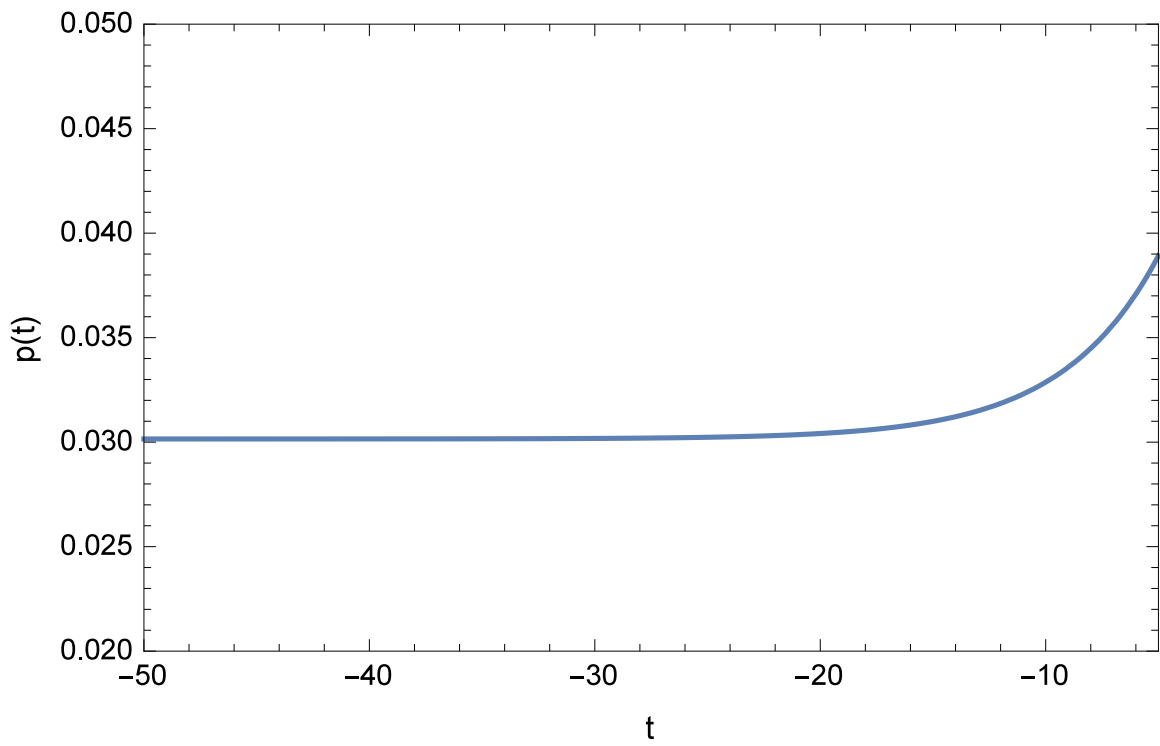
Figure 2.4: Evolution of pressure $p(t)$.

2.5   Thermal evolution

We now investigate the thermal evolution of the collapsing system generated in §2.4. It is well known that the Eckart formalism of thermodynamics has somewhat unphysical aspects towards it. In particular, this includes super-luminal propagation velocities for the dissipative fluxes as well as the prediction of unstable equilibrium states (Anile *et al* 1998). Gravitational collapse of a stellar object is dissipative in nature and is usually accompanied by heat generation via neutrino emission or free-streaming radiation. Various investigations have shown that relaxational effects predict higher core temperatures and significantly different luminosity profiles when compared to their noncausal counterparts (Govender *et al* 1998, 1999, 2013; Naidu *et al* 2006).

As already outlined in §1.6, in order to study the impact of the relaxational effects brought about by heat flow, it is necessary to employ a causal heat transport equation as given by the following truncated form (Thirukkanesh and Maharaj 2009),

$$\tau h_a{}^b \dot{q}_b + q_a = -\kappa(h_a{}^b \nabla_b T + T \dot{u}_a), \tag{2.5.1}$$

where κ is the thermal conductivity, $\dot{u}_b = u_{b;a}u^a$ is the four–acceleration, $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor and τ is the relaxation time. We obtain the Eckart temperature by setting $\tau = 0$ in (2.5.1). We assume that the neutrinos are thermally generated within the stellar core with energies of the order of $k_B T$. At neutron star densities, neutrino trapping takes place via electron–neutrino scattering and nucleon absorption. The mean collision time for thermally generated neutrinos is given by

$$\tau_{\rm c} \propto T^{-3/2}, \tag{2.5.2}$$

to good approximation (Martínez 1996). Following (2.5.2), we adopt a power law dependence for the thermal conductivity and relaxation time:

$$\kappa = \gamma T^3 \tau_{\rm c}, \qquad \tau_{\rm c} = \left(\frac{\alpha}{\gamma}\right) T^{-\sigma}, \tag{2.5.3}$$

where $\alpha \geq 0$, $\gamma \geq 0$ and $\sigma \geq 0$ are constants. We further assume that the relaxation

time is directly proportional to the mean collision time

$$\tau = \left(\frac{\beta\gamma}{\alpha}\right)\tau_{\mathrm{c}}, \tag{2.5.4}$$

where β (≥ 0) is a constant. The causal heat transport equation (2.5.1) for the above assumptions then reduces to

$$\beta(qB)\dot{T}^{-\sigma} + A(qB) = -\alpha\frac{T^{3-\sigma}(AT)'}{B}, \tag{2.5.5}$$

where the Eckart temperature T_0 is obtained by setting $\beta = 0$ in (2.5.5). In the case of constant collision time $(\sigma = 0)$ we are in a position to write down the solution to (2.5.5), namely

$$(AT)^4 = -\frac{4}{\alpha}\left[\beta\int A^3 B(qB)_{,t}\mathrm{d}r + \int A^4 qB^2\mathrm{d}r\right] + F(t), \tag{2.5.6}$$

where $F(t)$ is an arbitrary function of integration. The function $F(t)$ can be determined by making use of the effective surface temperature of a star as given by

$$(T^4)_\Sigma = \left(\frac{1}{r^2 B^2}\right)_\Sigma \left(\frac{L_\infty}{4\pi\delta}\right), \tag{2.5.7}$$

where L_∞ is the total luminosity at infinity and δ (> 0) is a constant.

Making use of the solution generated in §2.4 and for the particular case considered in §2.5, we have plotted the temperature within the collapsing stellar core as a function of time at $r = 1km$ in Figure 2.5. The plots show an increase in temperature with time which is to be expected from increases in pressure and energy density as already shown. The plot $\beta = 0$ represents the noncausal case. We see that a causal temperature, $\beta > 0$, is always greater than a noncausal temperature calculation. It is clear that relaxational effects contribute to the enhancement of the temperature for a collapsing star system as described in our work. It is interesting to note that even though our model is based on a weak heat flux approximation $(0 < \epsilon << 1)$, relaxational effects lead to very different outcomes for the temperature. The results confirm earlier findings by Govender and Govinder (2001).
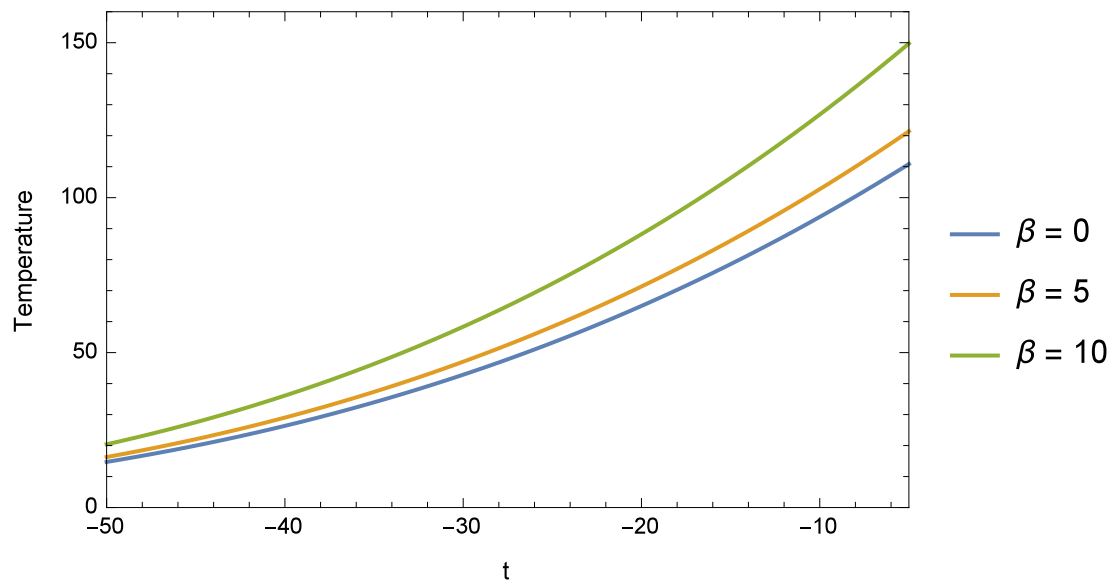
Figure 2.5: Temperature profiles at $r = 1km$

2.6 Discussion

In this chapter, we have generated a dynamical solution from the static stellar model given by Pant and Sah (1985) to investigate the nature of dissipative collapse. This has been achieved by allowing a constant parameter within the static model to evolve with time, still allowing the model to remain a solution of the Einstein field equations. The resulting dynamical model is a radiating collapsing star with heat conduction enveloped by a radiation atmosphere. In our construction, the star begins its collapse from an initial static configuration as described by the Pant and Sah model. Unlike many previous models describing collapse from an initial static configuration, the usual method of assuming metric separability in the variables r and t has not been adopted in our approach. Secondly, the background spacetime has been couched in spatially isotropic coordinates. The static model of Pant and Sah, as analyzed by Deb *et al* (2012), has the following key features: (1) $\rho > 0$, $p > 0$; (2) $\rho' < 0$, $p' < 0$; (3) $dp/d\rho < 1$. This implies that the collapse begins from a physically acceptable initial configuration which includes the fulfillment of (at least) the weak energy condition. The collapse is found to proceed without formation of an event horizon, with heat generated mostly due to the collapse process as the mass has been shown to be largely conserved. We have also studied the thermodynamics of the collapsing star within the framework of extended irreversible thermodynamics. Our results confirm earlier findings (through various different approaches) that relaxational effects can significantly alter the physical characteristics such as the temperature of the collapsing system. It must be pointed out that the spheroidal parameter k (in the static case) is usually chosen on an ad hoc basis to fit theoretical data to observed data related to neutron stars and strange stars. Our approach in allowing the spheroidal parameter to dynamically evolve with time gives snapshots of the collapse process, particularly during the latter stages just before the formation of the remnant.

Chapter 3

Radiating collapse in the presence of anisotropic stresses

3.1 Introduction

The end state of gravitational collapse processes is still a hotly debated area of research amongst astrophysicists, relativists and physicists who strongly adhere to quantum theory. Oppenheimer and Snyder (1939) determined the final outcome of the collapse of a spherical dust configuration in the absence of dissipation. The exterior of the collapsing dust sphere was taken to be the vacuum Schwarzschild solution. Although very idealised, the Oppenheimer–Snyder collapse model sheds new light on the collapse process, particularly towards the latter stages of the evolution of the collapsing system. Vaidya (1951) provided the unique solution which describes the exterior gravitational field of a radiating sphere dissipating energy in the form of null radiation. The Vaidya solution can be thought of as describing the atmosphere enveloping the collapsing body. The discovery of the Vaidya solution prompted researchers to consider more realistic effects in gravitational collapse, particularly the influence of dissipative fluxes when a star leaves hydrostatic equilibrium. A collapsing star which is radiating energy divides spacetime into two distinct regions: the interior spacetime representing

the collapsing stellar fluid and the exterior spacetime describing the atmosphere of the star. As mentioned in the previous chapter, the junction conditions required for the smooth matching of the interior spacetime to Vaidya's outgoing solution were provided by Santos (1985) by taking the interior of the star to be described by a spherically symmetric, shear–free line element in comoving isotropic coordinates. These junction conditions led to an abundance of radiating stellar models with the chief investigators being Herrera and co-workers (Herrera *et al* 1997, 1998, 2002, 2004). The roles played by shear and anisotropy during dissipative collapse have revealed exciting phenomena such as cracking, modification to the adiabatic index, cavity formation and horizon–free collapse.

The effect of dissipation on the evolution of collapsing spheres has been further investigated within the framework of causal thermodynamics (Govender *et al* 1998). Relaxational effects due to heat dissipation, bulk viscosity and shear were shown to affect the luminosity and temperature profiles of radiating stars (Govender *et al* 1999). At high core densities and temperatures, the relaxation time can be as large as one second. It has been shown that causal temperature profiles are higher than their noncausal counterparts throughout the interior of the collapsing star.

In this work, we consider the collapse of a spherical matter configuration within a model which is initially static. The static model is described by the Bowers and Liang solution (Bowers and Liang 1974) which is a generalisation of the Schwarzschild incompressible sphere to include pressure anisotropy. Our aim is to investigate the effect of pressure anisotropy on the subsequent dynamics of the collapsing system. The study is organised in the following way. Firstly, we present the Einstein field equations describing a spherically symmetric matter distribution with radial heat flux. Then we introduce the Vaidya solution which represents the gravitational field of the radiating star. In addition, we provide an outline of the junction conditions required for the smooth matching of the interior spacetime to the Vaidya spacetime. Then the Bowers and Liang solution representing the static core is introduced. The radiating model and

the associated physics is then discussed following which a conclusion, highlighting the main results of the work, is given.

3.2   Shear–free spacetimes

We investigate the gravitational collapse of a shear–free matter distribution with spherical symmetry, which is a reasonable assumption when modelling a relativistic radiating star. In this case, there exist coordinates for which the line element may be expressed in a form that is simultaneously isotropic and comoving. With the coordinates $(x^a) = (t, r, \theta, \phi)$, the line element for the interior spacetime of the stellar model takes the form

$$ds^2 = -A_0^2(r)dt^2 + f^2(t)\left[B_0^2(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],\qquad(3.2.1)$$

where the metric functions are yet to be determined.

In this work we consider a model which represents a spherically symmetric, shear–free fluid configuration with heat conduction. For our model, the energy momentum tensor for the stellar fluid becomes

$$T_{ab} = (\rho + p_t)u_au_b + p_tg_{ab} + (p_r - p_t)\chi_a\chi_b + q_au_b + q_bu_a,\qquad(3.2.2)$$

where ρ is the energy density, p_r and p_t are the radial and tangential stresses respectively, χ^a is a unit spacelike four–vector along the radial direction and $q^a = (0, q, 0, 0)$ is the heat flow vector which is assumed to flow in the radial direction because of spherical symmetry. The fluid four–velocity $\mathbf{u}$ is comoving and is given by

$$u^a = \frac{1}{A}\delta_0^a.\qquad(3.2.3)$$

The following relations also need to be satisfied, namely

$$u^au_a = -1,\qquad u^aq_a = 0,\qquad \chi^a\chi_a = 1,\qquad \chi^au_a = 0.$$

The fluid collapse rate $\Theta = u^a_{;a}$ of the stellar model is given by

$$\Theta = 3\frac{\dot{B}}{AB},\qquad(3.2.4)$$

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where dots represent differentiation with respect to t.

The nonzero components of the Einstein field equations for the line element (3.2.1) are

$$\rho = \frac{1}{f^2}\left[\frac{1}{r^2} - \frac{1}{r^2 B_0^2} + \frac{2B_0'}{rB_0^3}\right] + \frac{3\dot{f}^2}{A_0^2 f^2}, \tag{3.2.5a}$$

$$p_r = \frac{1}{f^2}\left[-\frac{1}{r^2} + \frac{1}{B_0^2 r^2} + \frac{2A_0'}{rA_0 B_0^2}\right] - \frac{1}{A_0^2}\left[2\frac{\ddot{f}}{f} + \frac{\dot{f}^2}{f^2}\right], \tag{3.2.5b}$$

$$p_t = \frac{1}{f^2}\left[\frac{A_0''}{A_0 B_0^2} + \frac{A_0'}{rA_0 B_0^2} - \frac{B_0,}{rB_0^3} - \frac{A_0' B_0'}{A_0 B_0^3}\right] - \frac{1}{A_0^2}\left[2\frac{\ddot{f}}{f} + \frac{\dot{f}^2}{f^2}\right], \tag{3.2.5c}$$

$$q = -\frac{2A_0'\dot{f}}{A_0^2 B_0^2 f^3}. \tag{3.2.5d}$$

We rewrite (3.2.5a)–(3.2.5c) in the form

$$\rho = \frac{\rho_s}{f^2} + \frac{3\dot{f}^2}{A_0^2 f^2}, \tag{3.2.6a}$$

$$p_r = \frac{(p_r)_s}{f^2} - \frac{1}{A_0^2}\left[2\frac{\ddot{f}}{f} + \frac{\dot{f}^2}{f^2}\right], \tag{3.2.6b}$$

$$p_t = \frac{(p_t)_s}{f^2} - \frac{1}{A_0^2}\left[2\frac{\ddot{f}}{f} + \frac{\dot{f}^2}{f^2}\right], \tag{3.2.6c}$$

where ρ_s, $(p_r)_s$ and $(p_t)_s$ denote the energy density, radial pressure and tangential pressure respectively of the initial static star. These are clearly given by

$$\rho_s = \left[\frac{1}{r^2} - \frac{1}{r^2 B_0^2} + \frac{2B_0'}{rB_0^3}\right], \tag{3.2.7a}$$

$$(p_r)_s = \left[-\frac{1}{r^2} + \frac{1}{B_0^2 r^2} + \frac{2A_0'}{rA_0 B_0^2}\right], \tag{3.2.7b}$$

$$(p_t)_s = \left[\frac{A_0''}{A_0 B_0^2} + \frac{A_0'}{rA_0 B_0^2} - \frac{B_0,}{rB_0^3} - \frac{A_0' B_0'}{A_0 B_0^3}\right]. \tag{3.2.7c}$$

The anisotropy parameter can be obtained from (3.2.5b) and (3.2.5c) according to a simple difference in pressures, namely

$$\Delta(r, t) = (p_r - p_t), \tag{3.2.8}$$

which is in accordance with 'Equation (7)' in Sharma and Das (2013). In order to construct a static model of the initial configuration, Sharma and Das assumed that the

anisotropy parameter Δ was separable in r and t, i.e.

$$\Delta\left(r,t\right) = \frac{\Delta_s(r)}{f^2(t)}. \tag{3.2.9}$$

Then equation (3.2.8) reduces to

$$\Delta_s(r) = \left[-\frac{A_0''}{A_0 B_0^2} + \frac{A_0'}{rA_0}B_0^2 + \frac{B_0'}{rB_0^3} + \frac{A_0'B_0'}{A_0 B_0^3} + \frac{1}{r^2 B_0^2} - \frac{1}{r^2} \right], \tag{3.2.10}$$

as given by Sharma and Das, which is clearly independent of t. They further utilised the Finch and Skea ansatz (Finch and Skea 1998) for B_0, i.e.

$$B_0 = \left(1 + \frac{r^2}{\mathcal{R}^2} \right)^{\frac{1}{2}}, \tag{3.2.11}$$

where $\mathcal{R}$ is the curvature parameter. The Finch and Skea ansatz has been successfully used to model compact stars (Hansraj and Maharaj 2006).

In order to solve equation (3.2.10), Sharma and Das assumed a particular profile for the anisotropy parameter based on physically reasonable behaviour. This equation reduces to a second order equation in A_0 for which there is a general solution. Hence the initial static configuration can now be fully specified. In this work, we adopt a completely different approach. We begin with an initial static configuration described by a Bowers and Liang model which is a generalisation of the Schwarzschild uniform density sphere to include anisotropic stress. The physical viability of the Bowers and Liang solution has been extensively investigated in a recent study by Reddy et al (2015). In their study, Reddy et al considered an initially static matter configuration described by the Bowers and Liang solution which starts to collapse and dissipate energy in the form of a radial heat flux. The collapse proceeds in such a manner so as to ensure that the star is always close to hydrostatic equilibrium. We present the essential features of the Bowers and Liang static model which we will use as the static core.

3.3 Junction conditions

Since the interior of the star is radiating energy, the exterior spacetime is appropriately described by Vaidya's outgoing solution (Vaidya 1951) given by

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right) dv^2 - 2dvdR + R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \qquad (3.3.1)$$

The quantity $m(v)$ represents the Newtonian mass of the gravitating body as measured by an observer at infinity. The metric (3.3.1) is the unique spherically symmetric solution of the Einstein field equations for radiation in the form of a null fluid.

The Einstein tensor for the line element (3.3.1) is given by

$$G_{ab} = -\frac{2}{R^2}\frac{dm}{dv}\delta_a^0\delta_b^0 . \qquad (3.3.2)$$

The energy momentum tensor for null radiation assumes the form

$$T_{ab} = \Phi w_a w_b , \qquad (3.3.3)$$

where the null four–vector is given by $w_a = (1, 0, 0, 0)$. Thus from (3.3.2) and (3.3.3), according to Einstein's theory, we have

$$\Phi = -\frac{2}{R^2}\frac{dm}{dv} , \qquad (3.3.4)$$

for the energy density of the null radiation. Since the star is radiating energy to the exterior spacetime, it is necessary to have $\frac{dm}{dv} \leq 0$.

The necessary conditions for the smooth matching of the interior spacetime to the exterior spacetime was first presented by Santos (1985) in his seminal paper. The junction conditions for the line elements (3.2.1) and (3.3.1) are thus given by

$$p_\Sigma = (qB)_\Sigma, \qquad (3.3.5)$$

$$m(v) = \left\{\frac{r^3}{2}\left(\frac{\dot{B}^2 B}{A^2} - \frac{B'^2}{B}\right) - r^2 B'\right\}_\Sigma . \qquad (3.3.6)$$

3.4 A Bowers–Liang static core

We use a Bowers and Liang model (Bowers and Liang 1974) with constant density in which the gravitational potentials are given by

$$A_0^2 = \left[\frac{3\left(1 - 2M/R\right)^{h/2} - \left(1 - 2m/r\right)^{h/2}}{2} \right]^{2/h}, \tag{3.4.1a}$$

$$B_0^2 = \left(1 - \frac{2m}{r}\right)^{-1}. \tag{3.4.1b}$$

Using these potentials in (3.2.7a) – (3.2.7c), the static energy density and pressures are given by

$$\rho_s = 6M/R^3, \tag{3.4.2a}$$

$$(p_r)_s = -\rho_s \left[\left(1 - \frac{2Mr^2}{R^3}\right)^{h/2} - \left(1 - \frac{2M}{R}\right)^{h/2} \right] \times \left[\left(1 - \frac{2Mr^2}{R^3}\right)^{h/2} \right.$$
$$\left. -3\left(1 - \frac{2M}{R}\right)^{h/2} \right]^{-1}, \tag{3.4.2b}$$

$$(p_t)_s = 2M\left(1 - \frac{2Mr^2}{R^3}\right)^{h/2} \left(2Mr^2\left[\left(1 - \frac{2Mr^2}{R^3}\right)^{h/2} - 3h\left(1 - \frac{2M}{R}\right)^{h/2}\right] \right.$$
$$\left. -\left[\left(1 - \frac{2Mr^2}{R^3}\right)^{h/2} - 3\left(1 - \frac{2M}{R}\right)^{h/2}\right] R^3 \right) \times \left(\left[\left(1 - \frac{2Mr^2}{R^3}\right)^{h/2} \right.\right.$$
$$\left.\left. -3\left(1 - \frac{2M}{R}\right)^{h/2} \right]^2 R^3 \left(R^3 - 2Mr^2\right) \right)^{-1}, \tag{3.4.2c}$$

respectively. The static anisotropy parameter, $\Delta_s = (p_t)_s - (p_r)_s$, is then expressed as

$$\Delta_s = -2M\left[2\left(1 - \frac{2Mr^2}{R^3}\right)^{h}\left(2Mr^2 - R^3\right) - 9\left(1 - \frac{2M}{R}\right)^{h}\left(R^3 - 2Mr^2\right) \right.$$
$$\left. +3\left(1 - \frac{2Mr^2}{R^3}\right)^{h/2}\left(1 - \frac{2M}{R}\right)^{h/2}\left(2(h-4)Mr^2 + 3R^3\right) \right] \times$$
$$\left(\left[\left(1 - \frac{2Mr^2}{R^3}\right)^{h/2} - 3\left(1 - \frac{2M}{R}\right)^{h/2} \right]^2 R^3\left(R^3 - 2Mr^2\right) \right)^{-1}. \tag{3.4.3}$$

According to Herrera and Santos (1997b), the anisotropy parameter may also be written as

$$\Delta = \frac{4}{3}\pi C r^2 \left(\rho_o + p_{ro}\right)\left(\rho_o + 3p_{ro}\right)\left(1 - \frac{2m}{r}\right)^{-1}, \tag{3.4.4}$$

where C, the anisotropic factor, measures the degree of anisotropy and is given by

$$h = 1 - 2C. \tag{3.4.5}$$

From (3.4.5) it is clear that $h = 1$ corresponds to $C = 0$ which is the isotropic case, while $h = 2$ and $h = 4$ correspond respectively to $C = -\frac{1}{2}$ and $C = -\frac{3}{2}$ which imply that the radial pressure dominates the tangential pressure. The model has a limiting case of $h = 0$ which corresponds to the Florides solution (Florides 1974) which is not considered in our study.

The critical value of $2M/R$ which results in infinite central pressure is given by

$$\left(\frac{2M}{R}\right)_{crit} = 1 - \left(\frac{1}{3}\right)^{2/h}, \tag{3.4.6}$$

and the associated critical mass by

$$M_{crit} = \left(\frac{3}{32\pi\rho_o}\right)^{1/2} \left[1 - \left(\frac{1}{3}\right)^{2/h}\right]^{3/2}. \tag{3.4.7}$$

The redshift z at the star's surface is expressed as

$$z = \left(1 - \frac{2M}{r_\Sigma}\right)^{-1/2} - 1, \tag{3.4.8}$$

and by considering equation (3.4.6), we obtain a simple expression for the critical value of the redshift, namely

$$z_{crit} = 3^{1/h} - 1. \tag{3.4.9}$$

The Bowers and Liang model is an extension of the interior Schwarzschild solution to include anisotropic pressures. It has been used extensively to investigate the role played by local anisotropy in highly dense matter distributions of the order of $10^{15} g.cm^{-3}$. The model augments or diminishes the equilibrium mass and redshift relative to isotropic values. In particular, the anisotropy allows for arbitrarily large surface redshifts as observed in quasars.

3.5 A radiating stellar model

By using equations (3.2.5d) and (3.2.6b) in (3.3.5), with $((p_r)_s)_\Sigma = 0$, we obtain

$$2f\ddot{f} + \dot{f}^2 - 2a\dot{f} = 0 \,, \tag{3.5.1}$$

where

$$a = \left(\frac{A_0'}{B_0}\right)_\Sigma \,.$$

The positivity of a is due to the requirement that the static solution space (A_0, B_0) must be matched to the Schwarzschild exterior solution. Since the star is collapsing we require $\dot{f} < 0$. A first integral of (3.5.1) is given by

$$\dot{f} = \frac{-2a}{\sqrt{f}} \left(1 - \sqrt{f}\right) . \tag{3.5.2}$$

This has also been obtained by Sharma and Das (2013). Then by using (3.4.6) and the fact that f must be positive, we have

$$0 \le f(t) \le 1. \tag{3.5.3}$$

Integrating (3.5.2) we obtain

$$t = \frac{1}{a} \left[\frac{f}{2} + \sqrt{f} + \ln\left(1 - \sqrt{f}\right)\right] , \tag{3.5.4}$$

where we have used the translation $t \to t + \alpha$. Note that $f(t)$ decreases monotonically from its value of unity at $t = -\infty$ to zero at $t = 0$ where a physical singularity is encountered. Physically this implies that the collapse starts at $t = -\infty$ from a static perfect fluid sphere with its interior solution described by the solution (A_0, B_0). The radial and temporal behaviour of our collapsing stellar model have now been fully specified.

We are in a position to investigate the physical viability of the collapsing sphere. As already mentioned, the use of extended irreversible thermodynamics has been well motivated and extensively used to determine the causal temperature profiles of radiating stars. The causal heat transport equation for the line element (3.2.1) is given

by

$$\tau(qB)^{\cdot} + AqB = -\frac{\kappa(AT)'}{B} \, ,$$
(3.5.5)

where

$$\kappa = \gamma T^3 \tau_{\rm c}, \qquad \tau_{\rm c} = \left(\frac{\alpha}{\gamma}\right) T^{-\sigma}, \qquad \tau = \left(\frac{\beta\gamma}{\alpha}\right)\tau_{\rm c}\, ,$$
(3.5.6)

are physically reasonable choices for the thermal conductivity κ, the mean collision time between massive and massless particles τ_c, and the relaxation time τ. The quantities $\alpha \geq 0$, $\beta \geq 0$ and $\sigma \geq 0$ are constants. With these assumptions, the causal heat transport equation (3.5.5) becomes

$$\beta(qB)^{\cdot}T^{-\sigma} + A(qB) = -\alpha\frac{T^{3-\sigma}(AT)'}{B} \, .$$
(3.5.7)

We are in a position to integrate (3.5.7) for the case $\sigma = 0$ which corresponds to constant collision time. Note that by setting $\beta = 0$ in (3.5.7) we obtain the noncausal heat transport equation. Figure 3.1 shows a comparison of the noncausal (Eckart) versus the causal temperature profiles at each interior point of the collapsing sphere. It is clear that the causal temperature is higher than the noncausal temperature throughout the stellar interior. These results confirm earlier findings by various researchers. An interesting phenomenon that is highlighted for the first time is the effect of anisotropy (h) on the temperature profiles. We observe that both causal and noncausal temperatures increase with an increase in anisotropy (larger h). Recall that larger h values correspond to $\Delta = p_t - p_r$ becoming more negative, i.e. the radial pressure dominates the tangential pressure. The effect of this is to slow down the collapse which means that less heat is being driven out into the exterior spacetime. The net effect is to produce a higher core temperature when the radial pressure dominates the tangential pressure. Figure 3.2 shows the variation of the surface redshift with the time connected parameter f. Recall that $f \to 1$ corresponds to early times, i.e. the model is static and is described by the Bowers and Liang solution. As the collapse proceeds ($f \to 0$) we note that the anisotropy plays a significant role in the behaviour of the surface redshift.

The surface redshift increases with an increase in anisotropy. This trend in the surface redshift has been pointed out in several static models, and we observe the same for our dynamical model. Figure 3.3 plots the time of formation of the horizon (f_{bh}) as a function of the anisotropy parameter. Recall that as h increases, the radial pressure dominates the tangential pressure. Furthermore, larger values of f correspond to earlier times of the collapse process. With this information, Figure 3.3 indicates that the horizon is reached sooner with an increase in anisotropy (radial pressure dominating the tangential pressure). This is an interesting feature of our model which directly links the anisotropy in the pressure to the dynamical nature of the collapse. It was pointed out by Joshi *et al* (2003) that shear or anisotropy affects the rate and direction of the collapse making the motion incoherent. Our model clearly indicates that the inclusion of pressure anisotropy during dissipative collapse can lead to very different outcomes when compared to homogeneous, isotropic collapse.
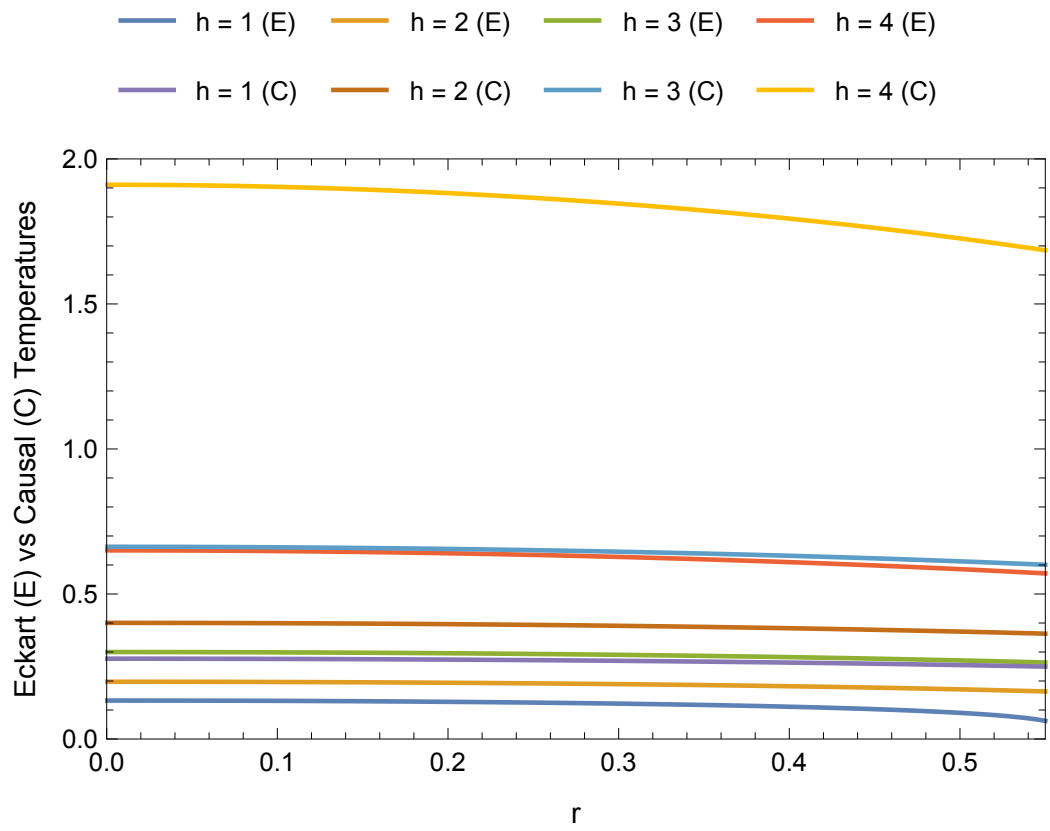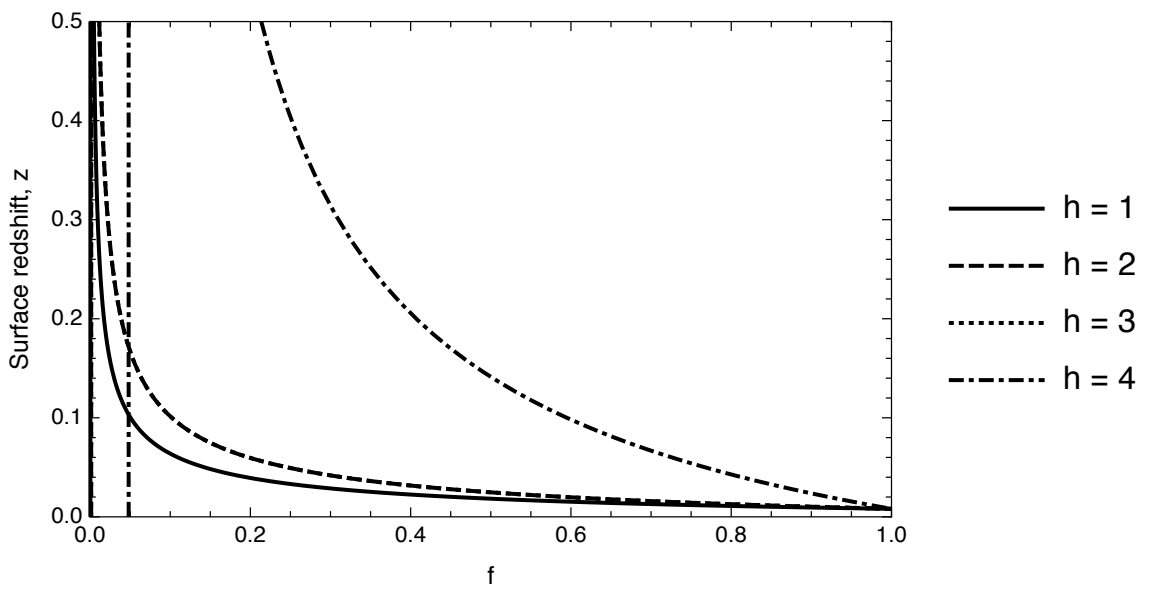
Figure 3.1: Temperature profiles.
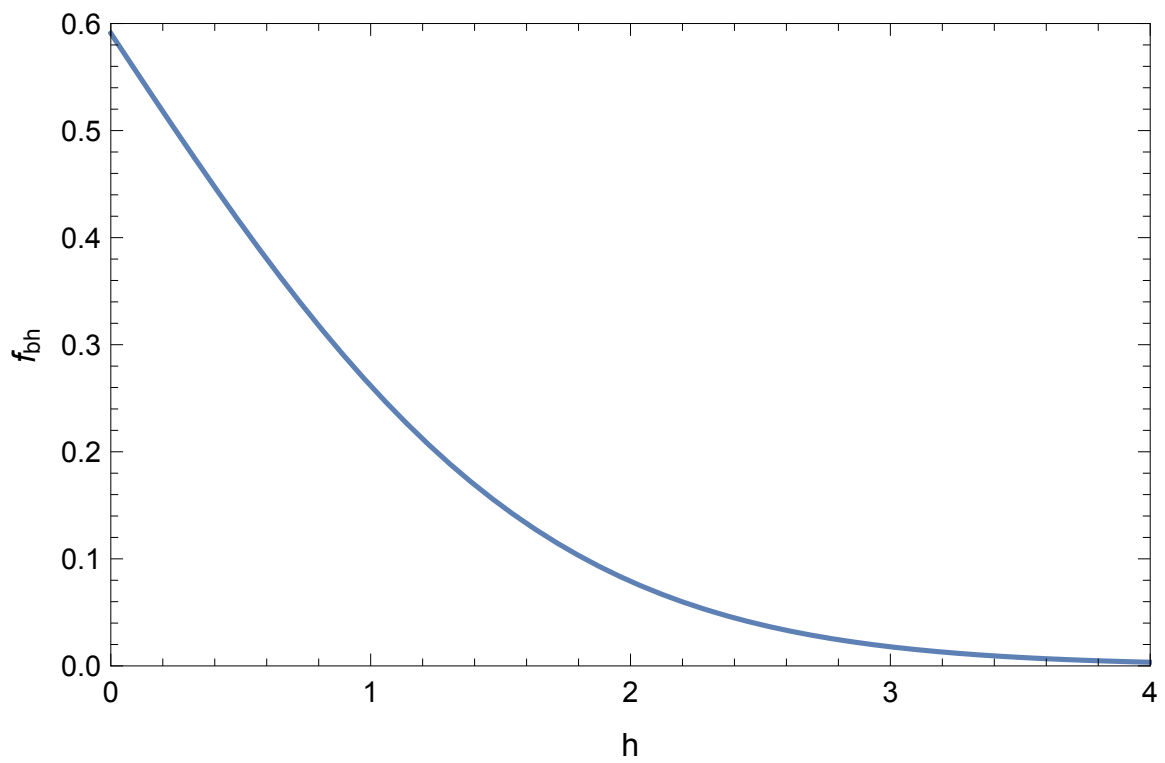
Figure 3.2: Surface redshift as a function of f.

Figure 3.3: Time of formation of horizon as a function of the anisotropy parameter.

Chapter 4

The influence of an equation of state during radiative collapse

4.1 Introduction

In astrophysics, the gravitational collapse of compact objects remains the key phenomenon for applying the Einstein field equations and investigating physically viable solutions. The first such study is the well-known gravitational collapse of a spherical distribution of dust-matter with a Schwarzschild exterior, studied by Oppenheimer and Snyder (1939). This idealised system was further restricted to being adiabatic and hence largely unphysical in nature since gravitational collapse is necessarily a dissipative process. Many improvements towards more realistic models such as the inclusion of shear, anisotropic stresses, electromagnetic field, extension to higher order gravity theories amongst others (Govender *et al* 2014; Govinder *et al* 1998; Chan 2000; Herrera and Santos 1997b; Sharma and Das 2013; Sharif and Abbas 2011a, 2011b; Carloni and Dunsby 2007), have been made, spurred by the replacement of the Schwarzschild exterior metric with that of Vaidya's radiating metric (Vaidya 1953; de Oliveira 1985). However, the increasing sophistication of the model reduces the possibility of finding exact solutions due to the highly nonlinear nature of the Einstein field equations. Nev-

ertheless, careful approximations and calculations often provide reasonable results and new insight into the collapse process. It is well known that the junction conditions which ensure the smooth matching of the interior spacetime to the exterior spacetime in the case of dissipative collapse lead to a temporal evolution equation (Govender *et al* 2003; Maharaj *et al* 2011). Various methods, ranging from Lie analysis (Msomi *et al* 2010) through to physically motivated assumptions such as vanishing of the Weyl stresses (Maharaj and Govender 2005), horizon-free condition and vanishing of shear, have been adopted to solve this boundary condition. The importance of initial conditions has also been studied (Sharma and Tikekar 2012b; Pinheiro and Chan 2011). In a recent paper by Naidu and Govender (2016), they showed that the pressure of the initial static configuration plays a vital role in the subsequent collapse of the star. In their study, they compared the outcome of dissipative collapse arising from a pressureless core with that of nonzero pressure. Studies on the importance of pressures in determining the end-state of gravitational collapse have also been done by Goswami and Joshi (2002).

In this study, we start off with a spherically symmetric static core obeying an equation of state of the form $p_r = \alpha\rho - \eta$ where $\eta = \alpha\rho_s$ (ρ_s being the surface density of the star). Linear equations of state have previously been used in modelling gravitational collapse (Goswami and Joshi 2004a). Hydrostatic equilibrium is lost and the star undergoes shear–free, dissipative collapse, radiating energy to the exterior Vaidya spacetime. We solve the junction conditions which ensures the continuity of the momentum flux across the boundary of the collapsing star. The solution of the boundary condition determines the temporal behaviour of our model. We show that the equation of state parameter plays a pivotal role in determining the behaviour of our dynamical model. Our analyses of various physically motivated models such as dust ($\alpha = 0$), pure radiation fluid ($\alpha = 1/3$), stiff fluid ($\alpha = 1$) and dark fluid ($\alpha = -1/3$) reveal that the associated thermodynamical quantities such as density, radial pressure and tangential pressure are all well behaved. We further employ a causal heat transport equation

to determine the role played by relaxational effects during collapse. Our study shows that the causal temperature is everywhere greater than the noncausal temperature. We also show for the first time the effect of the equation of state parameter α on the temperature profiles of our radiating models.

4.2   Radiating interior

The interior spacetime of a spherically symmetric radiating star can be modelled using the shear–free line element

$$ds_-^2 = -A_0^2(r)dt^2 + f^2(t)\left[B_0^2(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \qquad (4.2.1)$$

where (A_0, B_0) represent the initial static configuration potentials. The exterior is governed by the well known Vaidya metric (Vaidya 1951) for a matter-free, null radiation filled spacetime, given by

$$ds_+^2 = -\left(1 - \frac{2m(v)}{\mathcal{R}}\right)dv^2 - 2dvd\mathcal{R} + \mathcal{R}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (4.2.2)$$

where $m(v)$ represents the mass as perceived by an observer located at infinity.

In our model, the energy momentum tensor for the stellar fluid is given by

$$T_{ab} = (\rho + p_t)u_au_b + p_tg_{ab} + (p_r - p_t)\chi_a\chi_b + q_au_b + q_bu_a, \qquad (4.2.3)$$

where ρ is the energy density, p_r and p_t the radial and tangential pressures respectively and q^a is the heat flow vector assumed to flow in the radial direction because of spherical symmetry. The fluid four–velocity $\mathbf{u}$ is comoving and is given by

$$u^a = \frac{1}{A}\delta_0^a. \qquad (4.2.4)$$

The Einstein field equations for the line element (4.2.1) and energy momentum

tensor (4.2.3) are given by

$$8\pi\rho = \frac{1}{f^2}\left[\frac{1}{r^2} - \frac{1}{r^2 B_0^2} + \frac{2B_0'}{r B_0^3}\right] + \frac{3\dot{f}^2}{A_0^2 f^2}, \tag{4.2.5a}$$

$$8\pi p_r = \frac{1}{f^2}\left[-\frac{1}{r^2} + \frac{1}{r^2 B_0^2} + \frac{2A_0'}{r A_0 B_0^2}\right] - \frac{1}{A_0^2}\left[2\frac{\ddot{f}}{f} + \frac{\dot{f}^2}{f^2}\right], \tag{4.2.5b}$$

$$8\pi p_t = \frac{1}{f^2}\left[\frac{A_0''}{A_0 B_0^2} + \frac{A_0'}{r A_0 B_0^2} - \frac{B_0'}{r B_0^3} - \frac{A_0' B_0'}{A_0 B_0^3}\right] - \frac{1}{A_0^2}\left[2\frac{\ddot{f}}{f} + \frac{\dot{f}^2}{f^2}\right], \tag{4.2.5c}$$

$$8\pi q^1 = -\frac{2A_0'\dot{f}}{A_0^2 B_0^2 f^3}, \tag{4.2.5d}$$

which may be written more concisely as

$$\rho = \frac{\rho_s}{f^2} + \frac{3\dot{f}^2}{8\pi A_0^2 f^2}, \tag{4.2.6a}$$

$$p_r = \frac{(p_r)_s}{f^2} - \frac{1}{8\pi A_0^2}\left[2\frac{\ddot{f}}{f} + \frac{\dot{f}^2}{f^2}\right], \tag{4.2.6b}$$

$$p_t = \frac{(p_t)_s}{f^2} - \frac{1}{8\pi A_0^2}\left[2\frac{\ddot{f}}{f} + \frac{\dot{f}^2}{f^2}\right], \tag{4.2.6c}$$

$$q^1 = -\frac{2A_0'\dot{f}}{8\pi A_0^2 B_0^2 f^3}, \tag{4.2.6d}$$

where ρ_s, $(p_r)_s$ and $(p_t)_s$ denote energy density, radial pressure and tangential pressure for the static configuration respectively. These thermodynamical quantities for the static case are given by

$$\rho_s = \frac{1}{8\pi}\left[\frac{1}{r^2} - \frac{1}{r^2 B_0^2} + \frac{2B_0'}{r B_0^3}\right], \tag{4.2.7a}$$

$$(p_r)_s = \frac{1}{8\pi}\left[-\frac{1}{r^2} + \frac{1}{r^2 B_0^2} + \frac{2A_0'}{r A_0 B_0^2}\right], \tag{4.2.7b}$$

$$(p_t)_s = \frac{1}{8\pi}\left[\frac{A_0''}{A_0 B_0^2} + \frac{A_0'}{r A_0 B_0^2} - \frac{B_0'}{r B_0^3} - \frac{A_0' B_0'}{A_0 B_0^3}\right]. \tag{4.2.7c}$$

The anisotropy parameter which measures the difference between radial and tangential pressures is defined as

$$\Delta(r,t) = (p_r - p_t) = \frac{1}{8\pi f^2}\left[-\frac{A_0''}{A_0 B_0^2} + \frac{A_0'}{r A_0 B_0^2} + \frac{B_0'}{r B_0^3} + \frac{A_0' B_0'}{A_0 B_0^3} + \frac{1}{r^2 B_0^2} - \frac{1}{r^2}\right], \tag{4.2.8}$$

which is also written concisely as

$$\Delta(r,t) = (p_r - p_t) = \frac{1}{8\pi f^2}\Delta_s(r), \tag{4.2.9}$$

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where Δ_s refers to the anisotropy parameter for the static configuration.

Following Sharma and Das (2013), we again adopt the Finch and Skea ansatz (Finch and Skea 1998) for the metric potential B_0 :

$$B_0^2(r) = \left(1 + \frac{r^2}{R^2}\right). \tag{4.2.10}$$

This ansatz also has been used by other researchers (Hansraj and Maharaj 2006) in modelling compact objects in general relativity.

Our approach is different from that of Sharma in which a form for the anisotropy parameter Δ is chosen in order to obtain A_0. Here we impose an equation of state of the form

$$p_r = \alpha\rho - \eta, \tag{4.2.11}$$

as previously used by Govender and Thirukkanesh (2015). We assume that the static configuration obeys an equation of state of the form

$$(p_r)_s = \alpha\rho_s - \eta_s, \tag{4.2.12}$$

where $\eta_s = \alpha(\rho_s)_\Sigma$. Then, substituting our expressions for density and radial pressure given by (4.2.7a) and (4.2.7b) respectively, we obtain

$$-\frac{1}{r^2} + \frac{1}{r^2 B_0^2} + \frac{2}{r}\frac{A_0'}{A_0 B_0^2} = \alpha\left[\frac{1}{r^2} - \frac{1}{r^2 B_0^2} + \frac{2}{r}\frac{B_0'}{B_0^3}\right] - \eta_s, \tag{4.2.13}$$

which is easily rewritten as

$$\frac{A_0'}{A_0} = \frac{rB_0^2}{2}\left[\alpha\left(\frac{1}{r^2} - \frac{1}{r^2 B_0^2} + \frac{2}{r}\frac{B_0'}{B_0^3}\right) + \frac{1}{r^2} - \frac{1}{r^2 B_0^2} - \eta_s\right]. \tag{4.2.14}$$

Using the expression for B_0 in (4.2.10), we obtain

$$\frac{A_0'}{A_0} = \frac{r}{2}\left[\alpha\left(\frac{1}{R^2} + \frac{2}{R^2 + r^2}\right) + \frac{1}{R^2} - \eta_s\left(1 + \frac{r^2}{R^2}\right)\right], \tag{4.2.15}$$

which integrates to

$$\ln(A_0) = \frac{r^2}{4R^2}(1 + \alpha) + \frac{\alpha}{2}\ln(R^2 + r^2) - \frac{\eta_s r^2}{4}\left(1 + \frac{r^2}{2R^2}\right) + d, \tag{4.2.16}$$

where d is a constant of integration obtained by matching the interior spacetime to the exterior Schwarzschild solution in isotropic coordinates.

From the Einstein field equations (4.2.5a)–(4.2.5d), the initial static core is now completely specified in terms of (A_0, B_0).

4.3   Physical analysis

In order to obtain $f(t)$, we invoke the junction conditions due to Santos (1985). Conservation of momentum flux across the hypersurface Σ requires nonvanishing of the radial pressure at the surface, i.e.

$$p_\Sigma = (qB)_\Sigma. \tag{4.3.1}$$

For our model, this leads to a temporal evolution equation, namely

$$2\ddot{f}f + \dot{f}^2 - 2n\dot{f} = 0, \tag{4.3.2}$$

where

$$n = \left(\frac{A_0'}{B_0}\right)_\Sigma. \tag{4.3.3}$$

This equation easily integrates to

$$t = \frac{1}{n}\left[\frac{f}{2} + \sqrt{f} + \log\left(1 - \sqrt{f}\right)\right], \tag{4.3.4}$$

and has been used by various researchers to study dissipative gravitational collapse (Sharma and Das 2013). By making use of our expressions for A_0 and B_0, we now rewrite the solutions to the Einstein field equations as

$$\rho = \frac{(\rho)_s}{f^2} + \frac{3n^2(1 - \sqrt{f})^2}{2\pi A_0^2 f^3}, \tag{4.3.5a}$$

$$p_r = \frac{(p_r)_s}{f^2} + \frac{n^2(1 - \sqrt{f})}{2\pi A_0^2 \sqrt{f^5}}, \tag{4.3.5b}$$

$$p_t = \frac{(p_t)_s}{f^2} + \frac{n^2(1 - \sqrt{f})}{2\pi A_0^2 \sqrt{f^5}}, \tag{4.3.5c}$$

$$q = \frac{nA_0'(1 - \sqrt{f})}{2\pi A_0^2 B_0^2 \sqrt{f^7}}, \tag{4.3.5d}$$

where the static quantities are given by

$$8\pi(\rho)_s = \frac{R^2}{R^2 + r^2}\left(\frac{2}{R^2 + r^2} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{4.3.6a}$$

$$8\pi(p_r)_s = \frac{1}{R^2 + r^2}\left(X + \frac{R^2}{r^2}\right) - \frac{1}{r^2}, \tag{4.3.6b}$$

$$8\pi(p_t)_s = \frac{X}{2\left(R^2 + r^2\right)}\left(\frac{Xr^2}{2R^2} + \frac{R^2 - r^2}{R^2 + r^2}\right) + \frac{1 + \alpha - \eta\left(R^2 + 3r^2\right)}{2\left(R^2 + r^2\right)} + \frac{(\alpha - 1)R^2}{\left(R^2 + r^2\right)^2} - \frac{2\alpha R^2 r^2}{\left(R^2 + r^2\right)^3}, \tag{4.3.6c}$$

where

$$X = 1 + \alpha - \eta\left(R^2 + r^2\right) + \frac{2\alpha R^2}{R^2 + r^2}. \tag{4.3.7}$$

If we require that our dynamical model also obeys a linear equation of state of the form given in equation (4.2.11), then the Einstein field equations (4.2.6a) and (4.2.6b) lead to

$$\frac{(p_r)_s}{f^2} - \frac{1}{8\pi A_0^2}\left[\frac{2\ddot{f}}{f} + \frac{\dot{f}^2}{f^2}\right] = \frac{\alpha\rho_s}{f^2} + \frac{3\alpha\dot{f}^2}{8\pi A_0^2 f^2} - \eta. \tag{4.3.8}$$

By making use of equations (4.2.12) and (4.3.2), this leads to

$$\dot{f} = -\frac{2n}{3\alpha}. \tag{4.3.9}$$

Then substituting back into the boundary condition (4.3.2), we arrive at $\alpha = -1/3$. In order to proceed further in our attempt to model a realistic collapsing star, we require that the static core should obey an equation of state. This leads us to investigate different values of α which re-assumes a variable form in equations (4.3.6b) and (4.3.6c). It is clear that α is a key parameter in these static thermodynamical quantities as well as the dynamical quantities given by equations (4.2.6a)–(4.2.6d).

4.4 Thermodynamics

As previously emphasised, the use of extended irreversible thermodynamics has been well motivated and extensively used to determine the causal temperature profiles of

radiating stars (Govender *et al* 1998). The causal heat transport equation for the line element (4.2.1) is again given by

$$\tau(qB)\dot{} + AqB = -\frac{\kappa(AT)'}{B},$$ (4.4.1)

where

$$\kappa = \gamma T^3 \tau_{\mathrm{c}}, \qquad \tau_{\mathrm{c}} = \left(\frac{\psi}{\gamma}\right) T^{-\sigma}, \qquad \tau = \left(\frac{\beta\gamma}{\psi}\right) \tau_{\mathrm{c}},$$ (4.4.2)

are physically reasonable choices for the thermal conductivity κ, the mean collision time between massive and massless particles τ_{c}, and the relaxation time τ. The quantities $\psi \geq 0$, $\beta \geq 0$ and $\sigma \geq 0$ are constants. With these assumptions the causal heat transport equation (4.4.1) becomes

$$\beta(qB)\dot{}T^{-\sigma} + A(qB) = -\psi\frac{T^{3-\sigma}(AT)'}{B}.$$ (4.4.3)

The integration of (4.4.3) for constant and variable collision times has been provided by Govender and Govinder (2001). The resulting equation for constant mean collision times as given in §1.6.4 is

$$(AT)^4 = -\frac{4}{\alpha}\left[\psi \int A^3 B(qB)\dot{}dr + \int A^4 qB^2 dr\right] + F(t).$$ (4.4.4)

As already shown in §2.5 in more detail, the constant of integration $F(t)$ is obtained by equating the expression for luminosity with the fourth power of the temperature, in accordance with the well known Stefan–Boltzmann law for radiative heat.
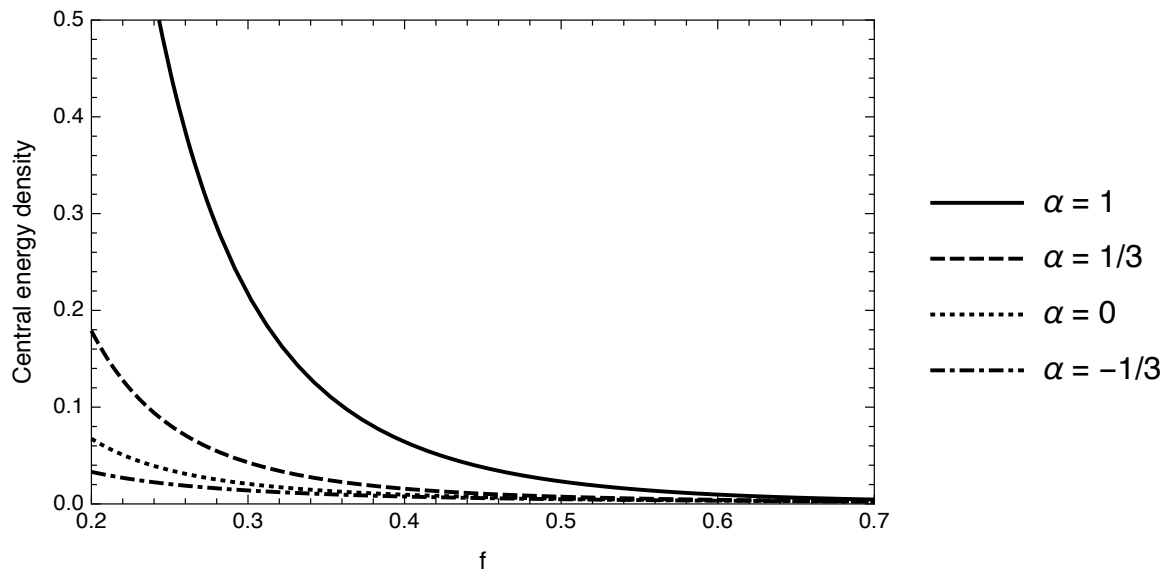
Figure 4.1: Evolution of central energy density $\rho(f)$
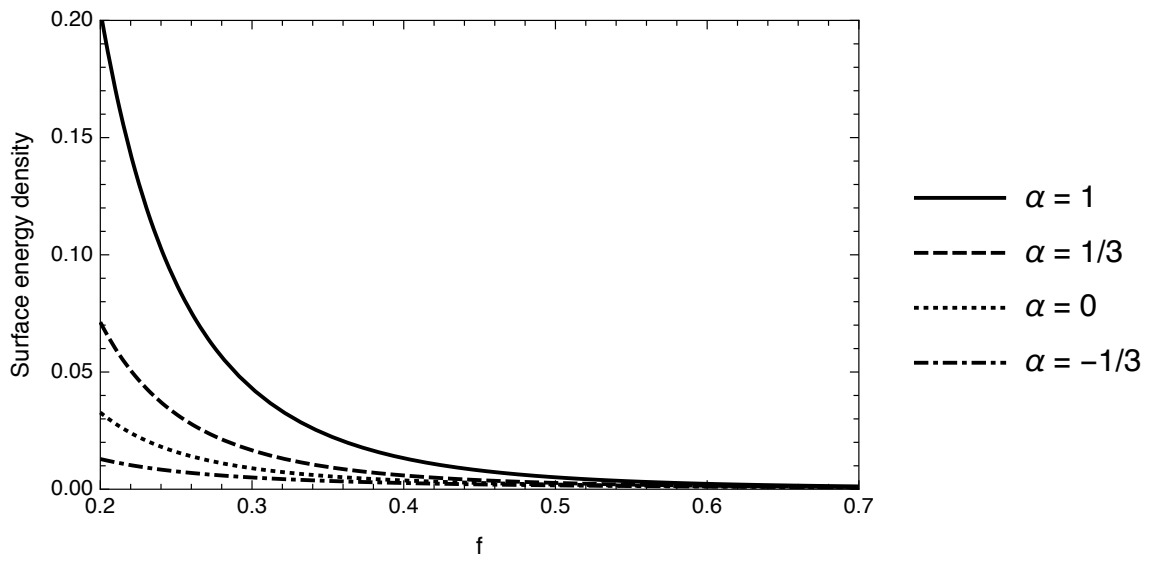
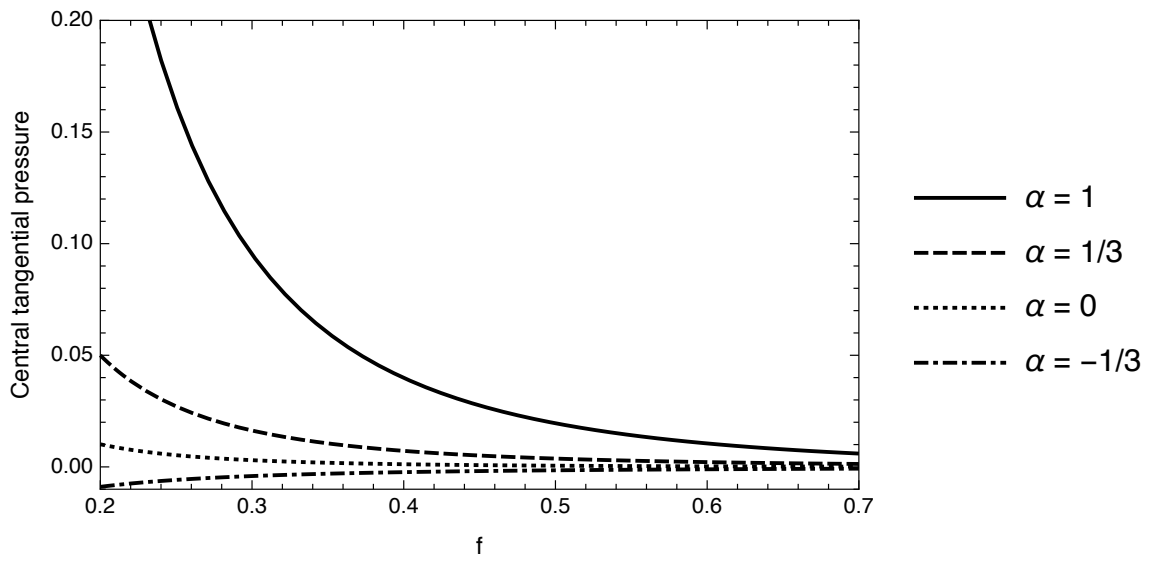Figure 4.2: Evolution of surface energy density $\rho(f)$

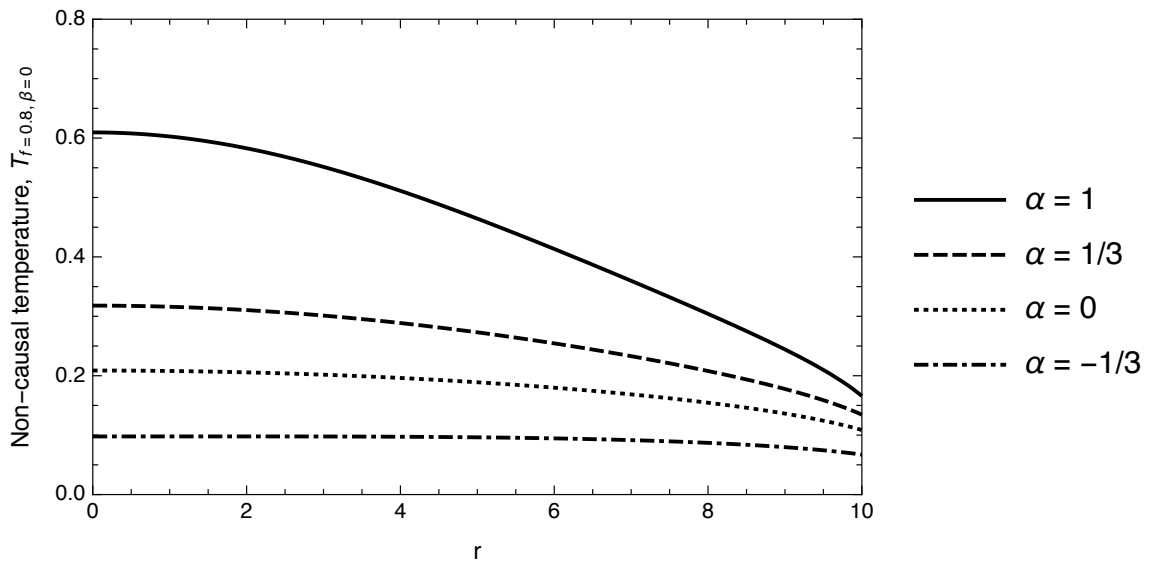Figure 4.3: Evolution of tangential pressure $p_t(f)$

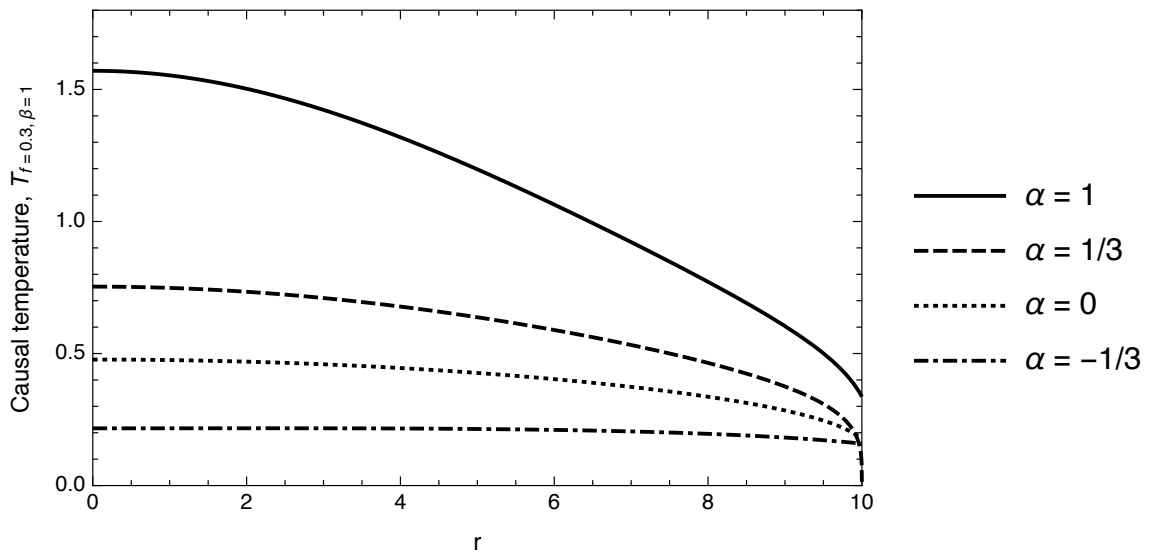Figure 4.4: Non-causal temperature ($\beta = 0$) profiles for various α

Figure 4.5: Causal temperature ($\beta = 1$) profiles for various α

4.5 Discussion

Figure 4.1 shows the evolution of the central density as a function of time $f(t)$. Recalling that smaller values of f correspond to later times, Figure 4.1 shows that the central density increases as the collapse proceeds as expected. It is also clear that α influences the density gradient and profile. Figure 4.2 illustrates that the surface density is always less than the central density as the collapse proceeds. We also note that there is a sharper drop-off in the surface density as compared to the central density. Figure 4.3 shows the behaviour of central pressure which increases rapidly as the collapse proceeds. We note a negative pressure for $\alpha < 0$ which is not abnormal within the regime of dark star models (Lobo 2006) and is clearly a possible outcome of the equation of state used. Figure 4.4 and Figure 4.5 both show that the noncausal and causal temperatures are well behaved at each interior point of the star. Our analyses clearly show that an increase in α leads to higher temperatures within the collapsing core. This can be understood in terms of the behaviour of the density and pressure. We have seen from Figure 4.1 and Figure 4.2 that an increase in α results in an increase in density and in effect, an increase in pressure ($p_r = \alpha\rho - \eta$). A more compact and dense core would imply a greater generation of heat, thus leading to higher temperatures. Our results clearly display the effect of the relaxation time as the star loses hydrostatic equilibrium. In addition, they confirm earlier findings that the causal temperature is higher everywhere as compared to its noncausal counterpart. We are also in a position to show that relaxational effects (Govender *et al* 1999) cannot be ignored even when we make the transition from baryonic matter ($\alpha > 0$) to the phantom regime ($-1 < \alpha < -1/3$).

Chapter 5

Conclusion

The research project undertaken was based primarily on the subject of Einstein's general theory of relativity. Gravitational collapse and the final state of stars which include white dwarfs, neutron stars, black holes (Shapiro and Teukolosky 1983) and more recently, dark energy stars (Lobo 2006), are suitable testing grounds for the theory of general relativity. In this research, the emphasis was primarily on very massive objects that would ultimately collapse to a black hole singularity or a gravastar. The theory was introduced in concise form through the mathematics of differential geometry which is the most natural way of describing spacetime and the motion of objects therein. As is the case with differential equations which describe physical systems, boundary conditions, known as junction conditions in general relativity, were reviewed as prescribed by Santos (1985). This provided us with the mathematical tools required for studying and analysing our astrophysical systems. In addition, the theory of causal thermodynamics with regard to heat transport was also incorporated in order to enhance our understanding of the physical processes taking place in our models of collapsing stars.

In the first study on gravitational collapse in spatially isotropic coordinates, a dynamic model was generated from an initially static configuration. The gravitational potentials were extended to being time dependent by allowing a parameter to vary.

This time dependence was introduced as a linear perturbation in much the same way as the Hamiltonian of an isolated hydrogen atom might be perturbed by an electric or magnetic field except that we have perturbed an already nonexact solution. Although the value in further approximating a system might be applicable in limited regions, our results reveal that it was still worthwhile and added insight into the collapse process. A detailed examination revealed reasonable behaviours in energy density, pressures, heat and temperature profiles with respect to time.

A second study was undertaken of a system undergoing radiative collapse in the presence of anisotropic stresses. A suitable interior metric in which a time-dependent function $f(t)$ assumes the position of the gravitational potential for the spatially varying part was used. The Vaidya metric was again used for the external spacetime. Then by using a Bowers–Liang static core model which is suitable for investigating systems with anisotropy, a radiating model was obtained by introducing the static potentials into the time-dependent field equations. Application of causal and noncasual thermodynamics then provided comparisons of both causal versus noncausal situations as well as the degree of anisotropy. It was found that anisotropy assisted in elevating the temperatures in both scenarios. The time of formation of the horizon was also investigated and found to decrease with anisotropy. From this study, it is thus concluded that pressure anisotropy provides an additional mechanism for the collapse process to proceed more rapidly towards the end state.

Lastly, a system undergoing radiative collapse subject to a linear equation of state was investigated. By substituting the solutions of the Einstein field equations into the equation of state, the intitial static configuration could be specified in terms of the static gravitational potentials. Then by substituting these static potentials back into the field equations containing the time-dependent parameter $f(t)$, a new radiating model was obtained. During this process, the importance of the equation of state parameter α was highlighed which was then identified as a parameter for investigation. Physically viable results were shown for both positive and negative values of α thus

proving the usefulness of this model in describing the radiative collapse of both regular and dark energy stars.

Finally, it is worth noting the recent experimental confirmation of gravitational waves (GW150914) which were also a prediction of general relativity. Gravitational wave physics opens up a new detection method for discovering and ultimately studying objects such as black holes and the so-called dark energy stars which do not emit electromagnetic or particle-like radiation.

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