ASPECTS OF DISTANCE MEASURES IN GRAPHS

by

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To my family

Preface

The study described in this thesis was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban, during the period May 2008 to December 2011. This thesis was completed under the supervision of Professor P. A. Dankelmann and Dr. S. Mukwembi.

This study represents original work by the author and has not been submitted in any form to another university nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

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Declaration 2-Publications

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and/or include research presented in this thesis (include publications in preparation, submitted, *in press* and published and give details of the contributions of each author to the experimental work and writing of each publication.)

Publication 1.
P. Ali, P. Dankelmann, S. Mukwembi, Upper bounds on the Steiner diameter of a Graph. (accepted for publication in Discrete Appl. Math.)

Publication 2.
P. Ali, P. Dankelmann, S. Mukwembi, The radius of k-Connected Planar graphs with Bounded Faces. (submitted)

Publication 3.
P. Ali, P. Dankelmann, S. Mukwembi, Steiner diameter of maximal planar graphs. (in preparation)

signed:

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Abstract

In this thesis we investigate bounds on distance measures, namely, Steiner diameter and radius, in terms of other graph parameters.

The thesis consists of four chapters. In Chapter 1, we define the most significant terms used throughout the thesis, provide an underlying motivation for our research and give background in relevant results.

Let G be a connected graph of order p and S a nonempty set of vertices of G. Then the Steiner distance $d(S)$ of S is the minimum size of a connected subgraph of G whose vertex set contains S. If n is an integer, $2 \leq n \leq p$, the Steiner n-diameter, $diam_n(G)$, of G is the maximum Steiner distance of any n-subset of vertices of G. In Chapter 2, we give a bound on $diam_n(G)$ for a graph G in terms of the order of G and the minimum degree of G. Our result implies a bound on the ordinary diameter by Erdös, Pach, Pollack and Tuza. We obtain improved bounds on $diam_n(G)$ for K_3-free graphs and C_4-free graphs.

In Chapter 3, we prove that, if G is a 3-connected plane graph of order p and maximum face length l then the radius of G does not exceed $\frac{p}{6} + \frac{5l}{6} + \frac{5}{6}$. For constant l, our bound improves on a bound by Harant. Furthermore we extend these results to 4- and 5-connected planar graphs.

Finally, we complete our study in Chapter 4 by providing an upper bound on $diam_n(G)$ for a maximal planar graph G.

Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction

This chapter aims to define the most significant terms that will be used in this thesis, provide underlying motivation for our study and present relevant background. We will define terms that have not been defined in this chapter in subsequent chapters, as the need arises.

1.2 Graph Theory Terminology

A *graph*, $G = (V, E)$, graph consists of a finite nonempty set V of elements called *vertices* and a (possible empty) set E of 2-element subsets of V called *edges*. For concepts not defined here we refer the reader to [117]. The number of elements in V is called the *order* and usually denoted by $p(G)$, or simply by p if the graph is understood and the number of elements in E is called the *size* of G and usually denoted by $q(G)$, or simply by q if the graph is understood. If G has only one vertex, then we say G is *trivial*; otherwise G is *nontrivial*. Let $e = \{u, v\} \in E(G)$. Then we say that e *connects* u and v. We also say u and v are *adjacent*, while e is *incident* with u and v. We simply write $e = uv$ instead of $e = \{u, v\}$. If all the vertices of G are pairwise adjacent, then G is *complete*. A complete graph on p vertices is denoted by K_p. A K_3 is referred to as a triangle.

A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is a *spanning subgraph* of G. For any set $S \subseteq V(G)$, the *induced subgraph* of G is the maximal subgraph of G with vertex set S and is denoted by $G[S]$.

The *degree*, $deg(v)$, of a vertex of G is the number of edges incident with v. A vertex of degree 1 is called an *end-vertex*. The minimum degree, $\delta(G)$, is the minimum of the degrees of vertices in G. The *neighbourhood*, $N_G(v)$, of a vertex $v \in V$ is the set of all vertices adjacent to v in G; while the *closed neighbourhood*, $N_G[v]$, is the union of $\{v\}$ and its neighbourhood. For a nonempty proper subset $A \subset V(G)$, $N[A]$ is the set of all vertices x of G of distance at most one to some $a \in A$, i.e., $N[A] := \{x \in V(G) : d_G(x,a) \leq 1 \text{ for some } a \in A\}$ and $N^2[A]$ is the set of all vertices x of G of distance at most two to some $a \in A$, i.e., $N^2[A] := \{x \in V(G) : d_G(x,a) \leq 2 \text{ for some } a \in A\}$.

A *walk* W in a graph G is an alternating sequence

$$W : v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$$

of vertices and edges such that $e_i = v_{i-1}v_i$ for $i = 1, 2, \ldots, k$. Since the vertices that appear in a walk determine the edges in the walk, we can omit edges in the description of a walk, and denote the walk W by $v_0, v_1, v_2, \ldots, v_{k-1}, v_k$. Since W starts at v_0 and ends at v_k, it is said to join v_0 and v_k; it is also referred to as a v_0-v_k walk. We call k the *length* of W. If all v_i are distinct, then W is called a *path*. A path $v_0v_1v_2, \ldots, v_k$ that begins at vertex v_0 and ends at vertex v_k is called a *v_0-v_k path*. A *closed walk* in G is a walk of the form $v_0v_1v_2, \ldots, v_k$ where $v_0 = v_k$. If all the vertices except v_0 of a closed walk $v_0v_1v_2, \ldots, v_k$ are distinct and $p \geq 3$, then the closed walk is called a *cycle* of *length* k or simple an *k-cycle* and is usually denoted by C_k. We say G is *connected* if every pair of vertices is connected by a path. The components of a graph are the maximal connected subgraphs of the graph. A *tree* is a connected graph with no cycles. A graph without cycles is called a *forest*. So each component of a forest is a tree.

For $S \subseteq V(G)$, $G - S$ is the graph obtained from G by deleting every vertex in S and all edges incident with it. If $S = \{v\}$, we sometimes write $G - v$ instead of $G - \{v\}$. A *separating set* of G is a set of vertices of G whose removal increases the number of components of G. A separating set with only one vertex is called a *cut vertex*. A *vertex cutset* of G is a separating set of G. The *vertex-connectivity*, $\kappa(G)$, of G is the minimum number of vertices whose deletion from G results in a disconnected or trivial graph. We say G is *k-vertex-connected* or simply *k-connected* if G is connected and $\kappa(G) \geq k$. .

Let G_1 and G_2 be two vertex disjoint graphs. The *union* $G_1 \cup G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *join* $G_1 + G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$.

A graph G is *planar* if it can be embedded into the plane with no crossing edges. A *plane graph* is a planar graph together with an embedding into the plane. A plane graph divides the plane into *faces*. The union of the vertices and edges of G incident with a face f of G is called the *boundary* of f. Two vertices u and v share a face if they are on the boundary of a common face. The length of a face in a plane graph G is the length of the walk in G that bounds it. A planar graph in which every face is a triangle is called a maximal planar graph.

1.3 Distance Concepts

All graphs considered here and in the sequel are connected and nontrivial, unless otherwise specified. Let G be a graph of order p. The *distance*, $d_G(u, v)$, between two vertices u, v of G is the length of a shortest u-v path in G. The *eccentricity*, $ex(v)$, of a vertex $v \in V(G)$ is the maximum distance between v and any other vertex in G. The maximum eccentricity of vertices of G is the *diameter* of G, denoted by $diam(G)$. The minimum eccentricity of vertices of G is the *radius* of G, denoted by $rad(G)$. The center of a graph is the set of all vertices of G of minimum eccentricity. Every vertex of G of minimum eccentricity is a *centre vertex* of G. The median of a graph is the set of all vertices of G of minimum distance.

The *distance* of u in G is defined as

$$\sigma(u, G) = \sum_{w \in V(G)} d_G(u, w),$$

and the *distance* of G as

$$\sigma(G) = \sum_{\{u,w\} \subseteq V(G)} d_G(u, w) = \frac{1}{2} \sum_{u \in V(G)} \sigma(u, G).$$

The *average distance*, $\mu(G)$, is defined as the average of the distances between all unordered pairs of vertices, that is,

$$\mu(G) = \binom{p}{2}^{-1} \sum_{\{u,w\} \subseteq V} d_G(u, w).$$

Let S be a set of vertices of G. Let H be a connected subgraph of G of minimum size which contains S. Then H is a tree, known as a *Steiner tree* for S, and the size of H is the *Steiner distance* of S in G, denoted by $d_G(S)$. If $|S| = 2$, then the Steiner distance of S is the (ordinary) distance between the two vertices of S, so the Steiner distance generalises the ordinary distance between two vertices. Let n be an integer such that $2 \leq n \leq p$. The *n-eccentricity*, $ex_n(v)$, of a vertex $v \in V(G)$ is the maximum Steiner distance of any *n-subset* of vertices of G containing v. The maximum n-eccentricity among the vertices of G is the *n-diameter* of G, denoted by $diam_n(G)$, while the minimum n-eccentricity of G is the *n-radius* of G, denoted by $rad_n(G)$. The *Steiner n-distance* of a vertex v of G is the sum of the Steiner distances of all sets of n vertices that contain v. The *Steiner n-centre*, $C_n(G)$, G is the subgraph induced by the vertices of minimum n-eccentricity in G. The *Steiner n-median*, $M_n(G)$, of G is the subgraph induced by the vertices of minimum Steiner n-distance.

The set $S_G(S)$ consists of all vertices in G that lie on some Steiner tree for S. If $S_G(S) = V(G)$, then S is called a Steiner set of G. The *Steiner number*, $s(G)$, of G is the minimum cardinality among the Steiner sets of G.

The *i-th distance layer*, $N_i(v)$, of a vertex $v \in V(G)$ is the set of vertices at distance i from v for each $i = 0, 1, 2, \ldots, ex_G(v)$. We define the sets $N_{\leq i}(v) = \bigcup_{0\leq j \leq i} N_j(v)$ and $N_{\geq i}(v) = \bigcup_{i\leq j \leq ex_G(v)} N_j(v)$.

For a nonempty subset $A \subseteq V(G)$ and a vertex v, the distance $d(v, A)$ between v and A is defined as $\min_{a\in A} d(v, a)$. For nonempty subsets $A, B \subseteq V(G)$, $d(A, B) = min\ \{d(a, b)|a \in A, b \in B\}$ and for nonempty subsets $X, Y \subseteq E(G)$, $d(X, Y) = d(V(X), V(Y))$ where $V(X)$ and $V(Y)$ are the set of vertices incident with edges in X and Y, respectively. If e, f are edges of G, then we write $d(e, f)$ instead of $d(\{e\}, \{f\})$.

A subgraph H of G is said to be distance preserving from a vertex v in G if $d_H(v, w) = d_G(v, w)$ for all $w \in V(H)$.

The *k-th power* of G, denoted by G^k, is the graph with the same vertex set as G, in which two vertices $a, b \in V(G)$ are adjacent if $d_G(a, b) \leq k$. For a positive integer k, a *k-packing* of G is a subset $A \subset V(G)$ with $d_G(a, b) > k$ for all $a, b \in A$.

1.4 Motivation for Distance and Steiner Distance

This section presents underlying motivation for our research.

In [59], authors commented on the importance of distances. Distances play a central role in the study of graphs. Hence, research on distance concepts in graphs has attracted much attention in the literature. A major impetus for research on distance concepts in graphs has certainly been its wide applications. Its applications range from facility loacation problems and network design in operations research to prediction of properties of chemical compounds in chemistry, from measuring closeness of groups of individuals in sociology to identifying important role players, in, for example, the internet

The Steiner tree problem for a graph is a discrete analogue of the well-known geometric Steiner problem:

In an Euclidean space (usually an Euclidean plane) find the shortest possible network of line segments interconnecting a set of given points.

The geometric Steiner problem dates back from the 17th century when Pierre Fermat proposed the following problem: Find in the Euclidean plane a point to minimize the distance from this point to three given points [72]. Gauss (1777-1855) generalized the problem. Surveys on the geometric Steiner problem have been given by Winter [118], Gilbert and Pollak [51] and Hwang and Richards [68]. The problem was named in honour of Jacob Steiner, a professor at the University of Berlin who hugely contributed to mathematics.

The Steiner problem for graphs was originally formulated by Hakimi [60] in 1971. The Steiner distance in graphs was introduced by Chartrand, Oellermann, Tian and Zou [21]. Since then, the problem has received considerable attention in the literature. It can be formulated as follows:

For a given connected graph G and subset S of the vertex set $V(G)$ of G find a connected subgraph with the minimum number of edges that contains S.

A solution of the Steiner problem is necessarily a tree which is called a Steiner tree for S. Steiner trees have many applications, such as design of communication and computer networks, design of circuits (Very Large Scale Integration) and analysis of biological networks. In multiprocessor computer networks, for example, it may be desirable to connect a certain set of processors with a subnetwork that uses the least number of communication links. A Steiner tree for vertices, corresponding to the processors that need to be connected, corresponds to such a desired network.

There are many types of communication networks and among them are telecommunications networks or the internet [88]. Two or more people can communicate simultaneously during so-called teleconferences or during chatroom sessions on the internet. For example, it may be desirable to connect such a group of people with a subnetwork that uses fewest number of communication links. A Steiner tree for vertices, corresponding to the people that need to be connected, corresponds to such a desired network.

When designing VLSI circuits, sets of pins are placed on a chip [81]. Each set of pins share electrical signal. For example, it may be desired to connect such a set with a subnetwork that uses the least number of communication links. A Steiner tree for vertices, corresponding to the pins that need to be connected, corresponds to such a desired network.

Cells of living organisms fulfill many functions. The function of a single cell depends on the interplay between proteins, genes and other biochemical components. Sometimes, it might be necessary to find biological relationships between a set of proteins or genes [100]. For example, it may be desirable to connect such a set of proteins or genes with a subnetwork that uses the least number of communication links. A Steiner tree for vertices, corresponding to the set of relevant proteins or genes that need to be connected, corresponds to such a desired network.

1.5 Survey of Results on Radius and Diameter

The diameter and radius are the most common of the classical distance parameters in graph theory.

The following relationship between the radius and the diameter follows directly from the definition of the radius and from the triangle inequality:

$$\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G).$$

For trees we have a much stronger relationship between the radius and the diameter as shown by the following classical theorem which is essentially due to Jordan [73].

Theorem 1.5.1. *Let T be a tree of order $p \geq 2$.*
(a) The centre of T consist of a single vertex or of two adjacent vertices.
(b) If the centre of T consists of a single vertex then $\mathrm{diam}(T) = 2\mathrm{rad}(T)$*, and*

if the centre of T *consists of two adjacent vertices then* $\mathrm{diam}(T) = 2\mathrm{rad}(T) - 1$.

Hedetniemi (see[15]) proved the following folklore result on centres.

Theorem 1.5.2. *Let* H *be a graph. Then* H *is the centre of some graph* G.

Proof. The graph G can be constructed from H by adding two new vertices, u and v, which are adjacent to every vertex of H but not to one another, and then adding two further vertices u' and v' which are adjacent only to u and v, respectively. It is easy to verify that H is the central subgraph of G since only the vertices of H have eccentricity 2 in G. □

The following result on the upper bound on the radius in terms of order is well known.

Proposition 1.5.3. *Let* c *be any central vertex of a connected graph* G*, and let* T_c *be a spanning tree of* G *which is distance-preserving from* c*. Then* $c \in C(T_c)$*, and* $rad(T_c) = rad(G)$.

Proof. Since T_c is distance-preserving from c, $rad(T_c) \leq e_{T_c}(c) = e_G(c) = rad(G)$. Since removing edges cannot decrease the eccentricity of any vertex, it follows that $rad(T_c) = rad(G)$ and that $c \in C(T_c)$. □

Proposition 1.5.4. *For any connected graph* G *of order* p *and radius* r,

$$r \leq \lfloor \frac{1}{2} p \rfloor.$$

Proof. Let c be any central vertex of G, and let T_c be a spanning tree of G which is distance-preserving from c. By Proposition 1.5.3, $\mathrm{rad}(T_c) = r$, and hence $\mathrm{diam}(T_c) = 2r$ or $2r - 1$. Now let P be any diametral path of T_c, and note that P has $\mathrm{diam}(T_c) + 1 \geq 2r$ vertices. It follows that $p \geq 2r$, and hence that $r \leq \lfloor \frac{1}{2} p \rfloor$. □

It is tedious but not difficult to show that equality holds if and only if
(1) G is a path or cycle, or
(2) p is odd and G consists of a path or cycle of order $2r$, a vertex w, and

one, two, or three edges joining w to vertices which are at most distance 2 apart in $G - w$.

Several upper bounds on the diameter and radius in terms of other graph parameters are known. Erdös, Pach, Pollack, and Tuza [44] proved the following results.

Theorem 1.5.5. *[44] Let G be a connected graph of order p and minimum degree $\delta \geq 2$. Then*

(i) $diam(G) \leq \frac{3p}{\delta+1} - 1$,

(ii) $rad(G) \leq \frac{3p}{2(\delta+1)} + 5$.

Furthermore, (i) and (ii) are tight apart from the exact value of the additive constants, and for every $\delta \geq 5$ equality holds in (i) for infintely many values of p.

Proof. (i) Denote $\mathrm{diam}(G)$ by d and let v be a vertex of G such that $ex_G(v) = d$. Every vertex in $N_i(v)$ has neighbours only in $N_{i-1} \cup N_i \cup N_{i+1}$. By the condition on minimum degree, $|N_{i-1}(v)| + |N_i(v)| + |N_{i+1}(v)| \geq \delta + 1$ for all integers i with $0 \leq i \leq d$ where $N_{-1}(v) = \emptyset = N_{d+1}(v)$. Define the integer k by $d = 3k + r$, $r \in \{0, 1, 2\}$. Hence,

$$p \geq \sum_{i=1}^{k}(|N_{3i-1}(v)| + |N_{3i}(v)| + |N_{3i+1}(v)|) \geq (k+1)(\delta+1)$$

Rearranging and using $k = \frac{d-r}{3} \geq \frac{d-2}{3}$ yields the result.

(ii) The proof is based on the observation that, given a centre vertex v, there exists vertices w_i at distance r or $r-1$ from v, $i = 1, 2$, and shortest paths P_i from v to w_i, with the following property: no vertex u_1 of P_1 shares a neighbour with a vertex u_2 of P_2, unless u_1 or u_2 are very close to one of the vertices v, w_1 or w_2. Given P_1 and P_2 one can find approximately $2\mathrm{rad}(G)/3$ vertices with disjoint neighbourhoods by choosing every third vertex on P_1 and P_2. This yields, approximately $p \geq \frac{2}{3}\mathrm{rad}(G)(\delta + 1)$, and so bound (ii) follows.

To show that (i) is tight apart from the exact value of the additive constant, consider the following graph. Given integers p, k, δ with $k > 1, \delta > 5$

and $p = k(\delta+1)+2$, let $G_{p,\delta} = G_0 + G_1 + \cdots + G_{3k-1}$, where

$$G_i = \begin{cases} K_1 & \text{if } i \equiv 0 \bmod 3 \text{ or } i \equiv 2 \bmod 3, \\ K_\delta & i = 1,\, 3k-2, \\ K_{\delta-1} & \text{otherwise.} \end{cases}$$

Clearly, $G_{p,\delta}$ has minimum degree δ, p vertices, $\text{diam}(G_{p,\delta}) = 3\frac{p-2}{\delta+1} - 1$ and $\text{rad}(G_{p,\delta}) = \lceil \frac{3(p-2)}{2(\delta+1)} - \frac{1}{2} \rceil$ □

Erdös, Pach, Pollack, and Tuza [44] further gave the following improved bounds on the diameter for triangle-free and C_4-free graphs.

Theorem 1.5.6. *[44] Let G be a connected graph of order p and minimum degree $\delta \geq 2$. Then*

(i) for triangle free graphs G, diam$(G) \leq 4\lceil \frac{p-\delta-1}{2\delta} \rceil$,

and

(ii) for C_4-free graphs G, diam$(G) \leq \frac{5p}{\delta^2 - 2[\delta/2]+1}$.

Furthermore the bounds are tight apart from the exact value of the additive constants, and for every $\delta \geq 5$ equality holds for infintely many values of p.

They also gave improved bounds on the radius for triangle-free and C_4-free graphs. Using different methods, Dankelmann, Dlamini and Swart [29, 34] proved similar bounds on the radius for $K_{2,t}$-free graphs and $K_{3,3}$-free graphs. The following is a well-known bound on the diameter in terms of order and vertex connectivity.

Proposition 1.5.7. *Let G be a κ-connected graph of order p. Then*

diam$(G) \leq \lfloor \frac{p+\kappa-2}{\kappa} \rfloor$.

Furthermore, the bound is sharp apart from an additive constant.

Proof. Write $\text{diam}(G) = d$ and let v be a vertex with $ex_G(v) = d$. Since G is κ-connected, for $i = 1, \ldots, d-1$ we have $|N_i| \geq \kappa$ and $|N_d| \geq 1$. Then

$$p = \sum_{i=0}^{d} (|N_i|) \geq 1 + (d-1)\kappa + 1$$

Therefore, $d \leq \lfloor \frac{p+\kappa-2}{\kappa} \rfloor$. □

To show that the bound in Proposition 1.5.7 is sharp, consider the following graph. Given integers p, κ, k with $p = 2 + (k-1)\kappa$ an $k \geq 2$, let

$$G_{p,\kappa} = G_0 + G_1 + \cdots + G_k$$

where $G_0 = K_1 = G_k$ and $G_i = K_\kappa$ for $i = 1, \ldots, k-1$. Then $G_{p,\kappa}$ is κ-connected of order p and has $\text{diam}(G_{p,\kappa}) = k = \frac{p+\kappa-2}{\kappa}$, as desired. In [63], Harant and Walther gave bounds on the radius in terms of order and vertex-connectivity. For even $\kappa(G)$, the bound in Proposition 1.5.7 on the diameter is also sharp for the radius as shown by the composition $G = C_{\frac{2p}{\kappa}}[K_{\frac{\kappa}{2}}]$, where $2p \equiv 0 \bmod \kappa$. For odd $\kappa(G)$, Harant and Walther [63] proved that

$$\text{rad}(G) \leq \frac{p}{\kappa(G)+1} + O(\log p),$$

and conjectured that $\text{rad}(G) \leq \frac{p}{\kappa(G)+1} + C$ for some constant C. Harant [61] showed that for $\kappa(G) = 3$, the $O(\log p)$ term can be replaced by 8. Using different methods, Mukwembi [84] proved that for odd $\kappa(G) \geq 3$, the $O(\log p)$ term can be replaced by $1 + \frac{16}{\kappa(G)+1}$. It has, however, been shown by Egawa and Inoue [42] that for odd $\kappa(G) \geq 3$, the $O(\log p)$ term can be replaced by $1 + \frac{9}{2\kappa(G)}$. On the other hand, Iida and Kobayashi [70] obtained a slightly better bound by showing that if $\kappa(G) \geq 3$, $\kappa(G)$ odd, then the $O(\log p)$ term can be replaced by $1 + \frac{1}{\kappa(G)}$.

Vizing [112] determined the following result on the maximum size of a graph of given order and radius, which yields a bound on the radius in terms of order and size.

Theorem 1.5.8. *[112] The maximum number of edges in a graph on p vertices with radius r is*

$$\begin{cases} \frac{p(p-1)}{2}, & r = 1; \\ \lfloor \frac{p(p-2)}{2} \rfloor & r = 2; \\ \frac{p^2-4pr+5p+4r^2-6r}{2} & r \geq 3. \end{cases}$$

Extremal graphs for the three possibilities are K_p for radius 1, and $K_p - \{a\ 1\text{-}factor\}$ when p is even and radius is 2. When radius is 2 and p is odd, an extremal graph is obtained from the complete graphs $K_{\frac{p-1}{2}}$ and $K_{\frac{p+1}{2}}$, by *joining each vertex of* $K_{\frac{p+1}{2}}$ *with* $\frac{p-3}{2}$ *vertices in such a way that no vertex of* $K_{\frac{p-1}{2}}$ *achieves degree* $p-1$*. An extremal graph for radius at least 3 is shown in Figure 1.1. This graph consists of a complete graph* K_{p-2r} *all of whose vertices are joined to three consecutive vertices on a cycle* C_{2r}*.*

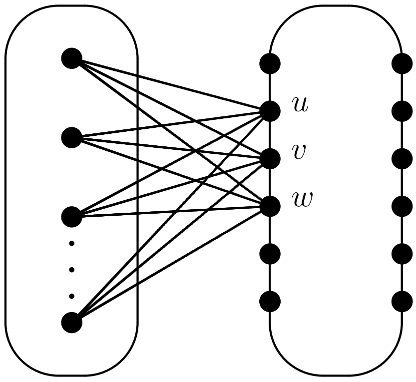

Figure 1.1: An extremal graph on p vertices with radius r.

The following similar result for bipartite graphs is due to Dankelmann, Swart and van den Berg [34].

Theorem 1.5.9. *For natural numbers* p *and* r *such that* $p \geq 2r \geq 2$*, the maximum number of edges in a bipartite graph of order* p *and radius at least* r *is* $b(p,r)$*, where*

a) $b(p,1) = p-1$,

b) $b(p,2) = \lfloor \frac{p^2}{4} \rfloor$,

c) $b(p,3) = \lfloor \frac{p^2}{4} \rfloor - \lfloor \frac{p}{2} \rfloor$

d) $b(p,3) = \lfloor \frac{p^2}{4} \rfloor - pr + p^2 + 2(p-r)$ *for* $p \geq 2r \geq 8$.

The bipartite graph with radius 1 and the maximum number of edges is the star $K_{1,p-1}$*. The bipartite graph with radius 2 and the maximum number of edges is the complete bipartite graph* $K_{\lceil \frac{p}{2} \rceil, \lfloor \frac{p}{2} \rfloor}$*. The bipartite graph with radius*

3 and the maximum number of edges is obtained from the complete graph $K_{\lceil \frac{p}{2} \rceil, \lfloor \frac{p}{2} \rfloor}$, *by the removal of a maximum edge cover. If G is a bipartite graph with radius 4 and the maximum number of edges, then $G \in B(p,r)$. The set $B(p,r)$ consists of all graphs G obtained from C_{2r} with three consecutive vertices replaced by aK_1, bK_1, cK_1, where* $a+c = \lceil \frac{p-2r+3}{2} \rceil, b = \lfloor \frac{p-2r+3}{2} \rfloor$, *or* $a+c = \lfloor \frac{p-2r+3}{2} \rfloor, b = \lceil \frac{p-2r+3}{2} \rceil$.

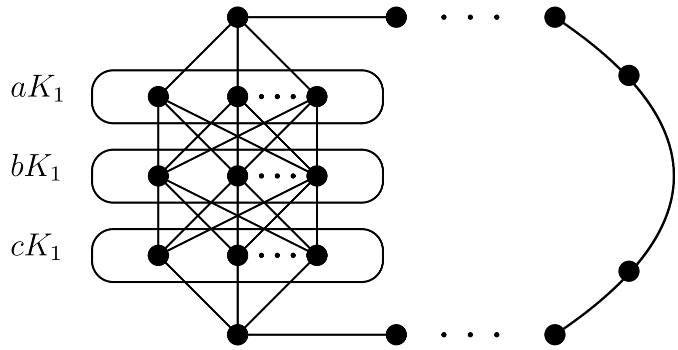

Figure 1.2: An example of a graph in B.

A lower bound on the radius of a graph was given by Kim and West [78], who showed that the radius of a triangle-free planar graph with no vertex of degree 1 or 2 is at least 3. We note that if $\kappa = 3$ in Proposition 1.5.7, then $\text{diam}(G) \leq \lfloor \frac{p+1}{3} \rfloor$. So the diameter of maximal planar graphs does not exceed $\frac{p+1}{3}$ since maximal planar graphs are 3-connected. Fulek, Morić, and Pritchard [48] proved that for every connected planar graph G of order p and size m,

$$\text{diam}(G) \leq \frac{4(p-1)-m}{3}. \tag{1.1}$$

Since for maximal planar graphs $m = 3p-6$, the bound in (1.1) becomes

$$\text{diam}(G) \leq \frac{p+2}{3}.$$

For 3-connected planar graphs, Harant [62] proved the following upper bound on the radius in terms of order and maximum face length.

Theorem 1.5.10. *If G is a 3-connected planar graph of order p and whose faces have length at most l, then*

$$\operatorname{rad}(G) \leq \frac{p}{6} + l + \frac{3}{2}.$$

However, no graphs which attain the bound were constructed.

In this thesis we improve on the bound by Harant [62] and extend these results to 4- and 5-connected planar graphs. In addition we demonstrate that the bounds are sharp. In Chapter 3 we strengthen this bound by Harant [62] to

$$\operatorname{rad}(G) \leq \frac{p}{6} + \frac{5l}{6} + \frac{5}{6}.$$

We also prove that for 4-connected planar graphs of order p, maximum face length l and radius $\operatorname{rad}(G)$ the bound

$$\operatorname{rad}(G) \leq \frac{p}{8} + \frac{5l}{4} + 1$$

holds and for 5-connected planar graphs of order p, maximum face length l and radius $\operatorname{rad}(G)$ the bound

$$\operatorname{rad}(G) \leq \frac{p}{10} + \frac{8l}{5} + 1$$

holds. We furthermore show that for large p and constant l our bounds are sharp, apart from an additive constant.

1.6 Survey of Results involving Steiner Distance

The problem of determining the Steiner distance can be solved in polynomial time in the following cases: If $S = V(G)$, then we have the minimum spanning tree problem, and an optimal solution can be found using Kruskal's algorithm. If there are exactly two vertices, say, $S = \{a, b\}$, then we have the shortest $a - b$ path problem, and an optimal solution can be found using Dijkstra's algorithm. There are classes of graphs, for example Halin graphs, interval graphs, partial 2-trees and distance hereditary graphs for which the problem of determining the Steiner distance has polynomial solutions [5, 113, 118, 119]. Since the problem of determining the Steiner distances for other classes of graphs is known to be NP-hard [50], it is desirable to have good bounds.

Chartrand, Oellermann, Tian and Zou [21] introduced the Steiner distance in graphs. Moreover, the ordinary radius and diameter were generalised. They showed the following result.

Proposition 1.6.1. *Let $n \geq 3$ be an integer and suppose that G is a tree with at least n end-vertices. If v is a vertex of G with $ex_n(v) = \mathrm{rad}_n(G)$, then there exists a set S of $n-1$ end-vertices of G such that $d(S \cup \{v\}) = ex_n(v)$ and $v \in V(T_S)$.*

They also established the following relationship between the n-diameter and $(n-1)$-diameter of a tree.

Proposition 1.6.2. *Let $n \geq 3$ and G be a tree of order at least $p \geq n$. Then*

$$\mathrm{diam}_n(G) \leq \frac{n}{n-1}\mathrm{diam}_{n-1}(G).$$

The following results establish a relationship between the n-diameter and n-radius of a tree.

Proposition 1.6.3. *[21] Let $n \geq 3$ be an integer and G be a tree of order $p \geq n$. Then*

$$diam_{n-1}(G) = rad_n(G).$$

Theorem 1.6.4. *[21] If $n \geq 2$ is an integer and G is a tree of order $p \geq n$, then*

$$\mathrm{rad}_n(G) \leq \mathrm{diam}_n(G) \leq \frac{n}{n-1}\mathrm{rad}_n(G).$$

Proof. For $n = 2$ the result is well-known. Let $n \geq 3$. By Proposition 1.6.2 and Proposition 1.6.3,

$$\mathrm{rad}_n(G) = \mathrm{diam}_{n-1}(G) \leq \mathrm{diam}_n(G)$$

which provides the first inequality. Now by Proposition 1.6.2 and Proposition 1.6.3,

$$\begin{aligned} \mathrm{diam}_n(G) &\le \frac{n}{n-1}\mathrm{diam}_{n-1}(G) \\ &= \frac{n}{n-1}\mathrm{rad}_n(G) \end{aligned}$$

which establishes the second inequality. □

The star $K_{1,n}$ or any tree obtained from the star by subdividing each edge k times for some $k \ge 1$ shows that the upper bound given in Theorem 1.6.4 is sharp [89]. It was conjectured in [21] that Theorem 1.6.4 can be extended to all connected graphs. This conjecture was disproved by Henning, Oellermann and Swart [66] who showed the following result.

Theorem 1.6.5. *If $n \ge 3$ is an integer, then there exists a graph G_n such that*

$$\mathrm{diam}_n(G_n) = \frac{2(n+1)}{2n-1}\mathrm{rad}_n(G_n).$$

Moreover, they established the following upper bounds for the 3-diameter and 4-diameter in terms of 3-radius and 4-radius, respectively.

Theorem 1.6.6. *If G is a connected graph of order at least 3, then*

$$\mathrm{diam}_3(G) \le \frac{8}{5}\mathrm{rad}_3(G).$$

Theorem 1.6.7. *If G is a connected graph of order at least 3, then*

$$\mathrm{diam}_4(G) \le \frac{10}{7}\mathrm{rad}_4(G).$$

The graphs G_n in Theorem 1.6.5 show that the bounds given in Theorems 1.6.6 and 1.6.7 are sharp. Henning, Oellermann and Swart [66] also conjectured that for all $n \ge 3$ every connected graph G of order at least n has $\mathrm{diam}_n(G_n) = \frac{2(n+1)}{2n-1}\mathrm{rad}_n(G_n)$. So far the conjecture has been settled for cases $n = 2, 3, 4$. If it were true for all values of n, then the graphs they constructed would show that this bound is sharp. Henning, Oellermann and Swart [66] also demonstrated that the relationship given in Proposition 1.6.2 does not hold for all graphs in general. However in [67] the following result was established.

Theorem 1.6.8. *For a connected graph G and integer $n \geq 3$, then* $\operatorname{diam}_n(G) \leq \frac{n+1}{n-1}\operatorname{diam}_{n-1}(G)$.

For each $n \geq 4$ the graph obtained from the complete bipartite graph $K_{n,n}$ by deleting a 1-factor shows that the bound in Theorem 1.6.8 is sharp. For $n = 3$, the graph G of Figure 1.3 shows sharpness of the bound in Theorem 1.6.8 [89].

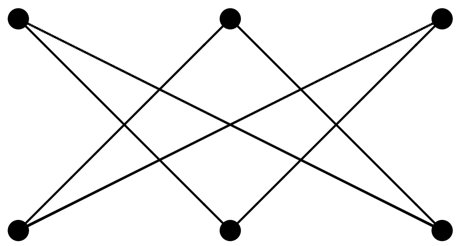

Figure 1.3: The graph G.

Bounds on the n-diameter of a graph in terms of other graph parameters have not been fully investigated. This can be attributed to the complexity of the problems. The following result by Dankelmann, Swart and Oellermann [33] is the only one known.

Theorem 1.6.9. *Let G be a connected graph of order p and minimum degree δ and let $2 \leq n \leq p$. Then*

$$\operatorname{diam}_n(G) \leq \frac{3p}{\delta + 1} + 3n.$$

Theorem 1.6.9 extends the result for all connected graphs in Theorem 1.5.5 (i).

The Steiner centre is a measure of centrality with a basis from the Steiner distance. The result in Theorem 1.5.2 was extended by Oellermann and Tian [92] as follows.

Theorem 1.6.10. *Let $n \geq 2$ be an integer and H a graph. Then H is the n-centre of some graph G.*

They also showed that the n-centre of a tree is connected. They further obtained the following characterization of n-centres of trees.

Theorem 1.6.11. *A tree H is the n-centre of some tree if and only if*

(1) $n \geq 3$ *and H has at most $n-1$ end-vertices, or*

(2) $n \leq 2$ *and H is isomorphic to K_1 or K_2.*

The proof of Theorem 1.6.11 gives a linear algorithm for finding these centres.

The Steiner distance and the median of a graph are the basis of yet another type of centre in a graph called the Steiner n-median. Since the problem of finding the Steiner n-distance is known to be NP-hard, one expects the problem of finding the Steiner n-median to be difficult. Beineke, Oellermann and Pippert [5] obtained efficient algorithms to the two problems for the class of trees. They showed that the n-median of any tree is connected. They also characterized trees that are n-medians of trees as follows.

Theorem 1.6.12. *A tree H of order p is the n-median of some tree if and only if one of the following holds.*

(a) *H is K_1,*

(b) *H is K_2,*

(c) $p = n$,

(d) *H has at most $n - p + 1$ end-vertices.*

They further obtained algorithms for finding the Steiner n-median of a tree and the Steiner n-distances of all vertices in a tree. They established sharp upper and lower bounds for the n-median values of trees of given order.

Oellermann [90] observed that $M_n(T) \subset C_n(T)$ if T is a tree of order p, $2 \leq n \leq p \leq 2n-2$. The distance between the $C_n(T)$ and $M_n(T)$ is 0 in this case. However, it was shown in the following theorem that the distance between the n-centre and n-median of a tree of order $p \geq 2n-1$ can be arbitrarily large.

Theorem 1.6.13. *Let T_1 be any tree with at most $n-1$ end vertices and let T_2 be a tree isomorphic to K_1 or K_2. Let $d \geq 1$ be an integer. Then there exists a tree T with $C_n(T) = T_1$, $M_n(T) = T_2$ and $d(T_1, T_2) = d$.*

Chartrand and Zhang [23] defined the Steiner number and showed that if G is a connected graph of order p, then $s(G) \leq p - \kappa(G)$. They showed that the nontrivial paths have the smallest possible Steiner number 2 and the complete graph, K_p has the largest possible Steiner number p as follows.

Theorem 1.6.14. *Every nontrivial tree with exactly k end-vertices has Steiner number k.*

Theorem 1.6.15. *If G is a connected graph of order $p \geq 2$, then $s(G) = p$ if and only if $G = K_p$.*

They characterized connected graphs of order p having Steiner number $p-1$ as follows.

Theorem 1.6.16. *Let G be a connected graph of order $p \geq 3$. Then $s(G) = p-1$ if and only if G contains a cut-vertex of degree $p-1$.*

They further proved the following realization results.

Theorem 1.6.17. *For every pair k, p of integers with $2 \leq k \leq p$ there exists a connected graph G of order p such that $s(G) = k$.*

Theorem 1.6.18. *For positive integers r, d, and $k \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with rad$(G) = r$, diam$(G) = d$ and $s(G) = k$.*

Let k, l, s and m be nonnegative integers with $m \geq s \geq 2$ and k and l not both zero. In [57] a connected graph G was defined to be *k-vertex* *l-edge* (s,m)*-Steiner distance stable*, if every set S of s vertices of G with $d_G(S) = m$, and every set A consisting of at most k vertices of $G - S$ and at most l edges of G, $d_{G-A}(S) = d_G(S)$. It was shown that if G is k-vertex l-edge (s,m)-Steiner distance stable, then G is k-vertex l-edge $(s-1,m)$ Steiner distance stable for $s \geq 3$ and k-vertex l-edge $(s,m+1)$-Steiner distance stable for any s. It was also shown that the converse of neither of these two results hold. If G is a connected graph and S an independent set of s vertices of G such that $d_G(S) = m$, then S is called an $I(s,m)$-set. Goddard, Oellermann and Swart [57] defined a connected graph to be *k-vertex $l-edge$ $I(s,m)$-steiner distance stable*, if for every $I(s,m)$-set S of G, and every set A consisting of at most k vertices of $G - S$ and at most l edges of G, $d_{G-A}(S) = d_G(S)$. They showed that every k-vertex l-edge $I(3,m)$-Steiner distance stable graph, $m \geq 4$, is k-vertex l-edge $I(3,m+1)$-Steiner distance stable.

Let G be a k-connected graph of order p where $k \geq 1$, and let S be a set of vertices of G. Chartrand and Tian [22] defined the *Steiner $i - distance$*, $d_i(S)$ of S as the minimum size among all i-connected subgraphs containing S for $1 \leq i \leq k$. They defined the (i,n)-eccentricity of a vertex v of G as the maximum Steiner i-distance , $d_i(S)$, of a set S containing v with $|S| = n$ for $1 \leq i \leq k$ and $1 \leq n \leq p$. The (i,n)-centre, $C_{i,n}(G)$, of G was defined as the subgraph induced by those vertices with minimum (i,n)-eccentricity. An i-connected graph G of order p is called an (i,n,p)*-graph*, where $2 \leq i < n \leq p$, if $d_i(S) = \lceil \frac{i|S|}{2} \rceil$ for every set S of n vertices of G. They proved that every (i,n,p)-graph, $2 \leq i \leq n \leq p$, is $(p-n+i)$-connected. They established upper and lower bounds for the circumference of nonhamiltonian $(2,n,p)$-graphs. They also proved that for every graph H and integers $i,n \geq 2$, there exists an i-connected graph such that $C_{i,n}(G) \cong H$.

The average Steiner distance of a graph marries the concepts of Steiner distance and average distance of a graph. Dankelmann, Swart and Oellermann [35] defined the average Steiner distance $\mu_n(G)$, or *average n-distance* as the average of the Steiner distances of all n-subsets of $V(G)$, i.e.

$$\mu_n(G) = \binom{p}{n}^{-1} \sum_{S\subset V,\, |S|=n} d_G(S).$$

They showed that $\mu_n(G) \leq \mu_k(G) + \mu_{n+1-k}(G)$ for $2 \leq k \leq n-1$, and that the range of the Steiner k-distance of a graph is given by:

Theorem 1.6.19. *If G is a connected graph of order p, then*

$$n - 1 \leq \mu_n(G) \leq \frac{n-1}{n+1}(p+1)$$

with equality on the left if and only if G is $(p+1-n)$-connected or $p = n$ and equality on the right if and only if G is a path or $p = n$.

They also showed that $\mu_n(T) \leq \frac{n}{k}\mu_k(T)$ for $2 \leq k \leq n-1$, and that the range of the Steiner k-distance of a tree T is given by:

Theorem 1.6.20. *If T is a tree of order $p \geq n \geq 2$, then*

$$n(1 - \frac{1}{p}) \leq \mu_n(T) \leq \frac{n-1}{n+1}(p+1)$$

equality holds if and only if T is a star or path, respectively, or in either case if $n = p$.

Moreover, they outlined a polynomial algorithm that finds the average Steiner k-distance of a tree. Bounds on μ_n for 2-connected graphs and for k-connected graphs were given in [36] as follows.

Theorem 1.6.21. *Let G be a 2-connected graph of order p and let $2 \leq n \leq p$. Then*

$$\mu_n(G) \leq \mu_n(C_p).$$

Equality holds if and only if $G = C_p$ or $n \geq p - 1$.

Let $H_{p,k}$ be the graph obtained from the complete graph K_k and a path of order $p - k$ with end vertices v_1 and v_1' by joining v_1' to one vertex of K_k for $k < p$. For $k = p$, let $H_{p,k}$ be the complete graph K_p and let v_1 be a vertex of K_p.

Theorem 1.6.22. *Let G be a connected graph of order $p \geq n \geq 2$ and chromatic number k and let v be a vertex of G. Then*

(i) $d_n(v_1, G) \leq d_n(v_1, H_{p,k})$

and

(ii) $\mu_n(G) \leq \mu_n(H_{p,k})$,

with equality if and only if $v = v_1$ and $G = H_{p,k}$, respectively.

In this thesis we prove upper bounds on the n-diameter of all connected graphs, triangle-free graphs and C_4-free graphs in terms of order and minimum degree. We further prove upper bounds on the n-diameter of maximal planar graphs in terms of order. In Chapter 2 we show that for all connected graphs G with minimum degree δ and order p,

$$\text{diam}_n(G) \leq \frac{3p}{\delta + 1} + 2n - 3,$$

in a sense, this generalises Theorem 1.5.5 (i) by Erdös, Pach, Pollack, and Tuza [44]. We also show that for triangle-free graphs G with minimum degree δ and order p,

$$\text{diam}_n(G) \leq \frac{2p}{\delta} + 3n - 3,$$

and for C_4-free graphs G with minimum degree δ and order p,

$$\text{diam}_n(G) \leq \frac{5p}{\delta^2 - 2\lfloor \delta/2 \rfloor + 1} + 4n - 5,$$

in a sense, these generalise Theorem 1.5.6 by Erdös, Pach, Pollack, and Tuza [44]. Moreover we construct graphs to show that these bounds are close to best possible. In Chapter 4 we show that for maximal planar graphs G of order p,

$$\text{diam}_n(G) \leq \frac{p}{3} + \frac{8n}{3} - 5,$$

in a sense, this generalises Proposition 1.5.7. Further, we demonstrate that, apart from the additive constant, the given bound is best possible.

Chapter 2

Upper Bounds on the Steiner Diameter of a Graph

2.1 Introduction

In this chapter we give upper bounds on the Steiner n-diameter, $diam_n(G)$, for all connected graphs, K_3-free graphs and C_4-free graphs. In addition, we construct graphs to show that the bounds are asymptotically best possible.

We will make frequent use of the following lemma.

Lemma 2.1.1. *Let G be a connected graph, $S \subseteq V(G)$ and j be a positive integer. Then*

$$d_G(S) \leq j \cdot d_{G^j}(S).$$

Proof. Recall that $G^j[S]$ is the subgraph of G^j induced by S. Let T_j be a Steiner tree of S in G^j with vertex set $W \cup S$, where $W := \{x | x \in V(T_j) - S\}$. Let $|W| = k$. Then T_j has $(|S| + k - 1)$ edges, $e_1, e_2, \ldots, e_{|S|+k-1}$.

Let $e_i = a_i b_i$ for $i = 1, 2, \ldots, |S| + k - 1$ and let P_i be a shortest $a_i - b_i$

path in G. Let $V_i = V(P_i) - \{a_i, b_i\}$. Then $|V_i| \leq j - 1$. Now

$$\begin{aligned}
|\bigcup_{i=1}^{|S|+k-1} V(P_i)| &= |S \cup W \cup \bigcup_{i=1}^{|S|+k-1} V_i| \\
&\leq |S| + |W| + \sum_{i=1}^{|S|+k-1} |V_i| \\
&\leq |S| + k + (j-1)(|S| + k - 1) \\
&= j|S| + jk - j + 1.
\end{aligned}$$

Since the graph induced by $\bigcup_{i=1}^{|S|+k-1} V(P_i)$ in G is connected and contains S, it follows that

$$d_G(S) \leq j|S| + jk - j = j \cdot d_{G^j}(S). \qquad \square$$

2.2 Steiner diameter of all connected graphs, triangle-free and C_4-free graphs

Theorem 2.2.1. *Let G be a connected graph of order p and minimum degree $\delta \geq 2$. If $2 \leq n \leq p$, then*

$$\text{diam}_n(G) \leq \frac{3p}{\delta + 1} + 2n - 5. \tag{2.1}$$

Proof. Let $S = \{f_1, f_2, \ldots, f_n\} \subseteq V(G)$ be a set of n vertices such that $d_G(S) = \text{diam}_n(G)$. Construct a maximal 2-packing A of G using the following procedure: Let $a_1 = f_1$ and $A = \{a_1\}$. If $A = \{a_1, a_2, \ldots, a_{i-1}\}$ and if there exists a vertex a_i in G with $d_G(a_i, A) = 3$, add a_i to A. Continue adding vertices a_i with $d_G(a_i, A) = 3$ until each vertex not in A is within distance 2 of A.

Let $T_1 \leq G$ be the forest with vertex set $N_G[A]$, and whose edge set consists of all edges of G incident with a vertex in A. By our construction of A, there exist $|A| - 1$ edges in G, each of them joining two neighbours of distinct vertices of A, whose addition to T_1 yields a tree $T_2 \leq G$.

Now each vertex $b \in V(G) - V(T_2)$ is adjacent to some vertex $b' \in V(T_2)$. Let T be the spanning tree of G with edge set $E(T_2) \cup \{bb' | b \in V(G) - V(T_2)\}$.

We show that

$$T^3[A] \text{ is connected.} \tag{2.2}$$

To prove (2.2) it suffices to show that for every vertex a_i there exists a path from a_i to a_1 in $T^3[A]$. We use induction on i. For $i = 1$, there is a walk from a_1 to a_1 of length 0. For $i > 1$, by our construction of A, there is a vertex a_j at distance 3 of a_i in T such that $j < i$. By the induction hypothesis, there is a path from a_j to a_1 in $T^3[A]$. This path together with the edge $a_i a_j$ in $T^3[A]$ yields a path from a_i to a_1 in $T^3[A]$. This proves (2.2).

We now prove that

$$d_T(S) \leq \frac{3p}{\delta + 1} + 2n - 5. \tag{2.3}$$

First let $b_1, b_2, \ldots, b_n \in A$ be such that b_i is a vertex in A closest to f_i in T for $i = 1, 2, \ldots, n$. Note that $b_1 = a_1$. Since $T^3[A]$ is connected, we have

$$d_{T^3[A]}\left(\{b_1, b_2, \ldots, b_n\}\right) \leq |A| - 1,$$

and so by Lemma 2.1.1,

$$d_T\left(\{b_1, b_2, \ldots, b_n\}\right) \leq 3(|A| - 1).$$

By our construction of A and T, we have $d_T(b_1, f_1) = 0$ and $d_T(b_j, f_j) \leq 2$ for $j = 2, \ldots, n$, so that $\sum_{j=1}^{n} d_T(b_j, f_j) \leq 2n - 2$. Hence

$$\begin{aligned} d_T(S) &\leq d_T\left(\{b_1, b_2, \ldots, b_n\}\right) + \sum_{j=1}^{n} d_T(b_j, f_j) \\ &\leq 3(|A| - 1) + 2n - 2 \\ &= 3|A| + 2n - 5. \end{aligned} \tag{2.4}$$

Since all $a_i \in A$ have at least δ neighbours and since the closed neighbourhoods of the a_i are disjoint, we have

$$|A|(\delta + 1) \leq p,$$

which implies that $|A| \leq \frac{p}{\delta+1}$. Hence, by (2.4),

$$d_T(S) \leq \frac{3p}{\delta+1} + 2n - 5.$$

This proves (2.3). The theorem now follows since $\mathrm{diam}_n(G) = d_G(S) \leq d_T(S)$. □

The following graphs show that, for constant δ and n, the bound in Theorem 2.2.1 is best possible, apart from the value of the additive constant. For given integers $\delta, k > 0$, let $G_1, G_2, \ldots, G_k$ be disjoint copies of the complete graph $K_{\delta+1}$, and let $a_ib_i \in E(G_i)$. Let $G_{k,\delta}$ be the graph obtained from the union of $G_1, G_2, \ldots, G_k$ by deleting the edges a_ib_i for $i = 2, 3, \ldots, k-1$ and adding the edges $a_{i+1}b_i$ for $i = 1, 2, \ldots, k-1$. Clearly, $p(G_{k,\delta}) = k(\delta+1)$, so $k = \frac{p(G_{k,\delta})}{\delta+1}$. If $2 \leq n \leq 2\delta$ then, by a simple calculation, $\mathrm{diam}_n(G_{k,\delta}) = 3k + n - 5$ and so $\mathrm{diam}_n(G_{k,\delta}) = 3\frac{p(G_{k,\delta})}{\delta+1} + n - 5$. In this case the difference between $\mathrm{diam}_n(G_{k,\delta})$ and the bound in Theorem 2.2.1 is at most n. For $n > 2\delta$ we use the estimate $\mathrm{diam}_n(G_{k,\delta}) \geq \mathrm{diam}_2(G_{k,\delta}) = 3\frac{p(G_{k,\delta})}{\delta+1} - 3$. In this case, the difference between the Steiner n-diameter of $G_{k,\delta}$ and the bound in Theorem 2.2.1 is bounded by the additive constant $2n-2$.

We note that a very slight modification of the proof of Theorem 2.2.1 yields that the n-diameter of the tree T constructed in this proof is at most $\frac{3p}{\delta+1} + 2n - 3$. Hence we obtain the following corollary:

Corollary 2.2.1. *Let G be a connected graph of order p and minimum degree $\delta \geq 2$. If $2 \leq n \leq p$, then G has a spanning tree T with*

$$\mathrm{diam}_n(T) \leq \frac{3p}{\delta+1} + 2n - 3.$$

Theorem 2.2.2. *Let G be a connected triangle-free graph of order p and minimum degree $\delta \geq 2$. If $2 \leq n \leq p$, then*

$$\mathrm{diam}_n(G) \leq \frac{2p}{\delta} + 3n - 6. \tag{2.5}$$

Proof. Let $S = \{v_1, v_2, \ldots, v_n\} \subseteq V(G)$ be a set of n vertices such that $d_G(S) = \mathrm{diam}_n(G)$. First we find a matching M of G using the following procedure: Choose $e_1 \in E(G)$ incident with v_1 and let $M = \{e_1\}$. Let $M = \{e_1, e_2, \ldots, e_{i-1}\}$. If there exists an edge e_i in G with $d_G(e_i, V(M)) = 3$, add e_i to M. Repeat this process until each edge not in M is within distance 2 of M in G.

Let $T_1 \leq G$ be the forest with vertex set $N_G[V(M)]$, and whose edge set consists of all edges incident with a vertex in $V(M)$. By our construction of M, there exist $|M| - 1$ edges in G, each of them joining two distinct components of T_1, whose addition to T_1 yields a tree $T_2 \leq G$.

Now each vertex in $V(G) - V(T_2)$ is within distance 3 of some vertex of T_2. Let $T \geq T_2$ be a spanning tree of G in which $d_T(b, V(M)) = d_G(b, V(M))$ for every vertex $b \in V(G)$. Let $T_m \leq T$ be the smallest subtree of T containing M. It follows that

$$|E(T_m)| \leq 4|M| - 3.$$

Now $v_1 \in V(T_m)$ and for each $v_i \in S - \{v_1\}$, we have $d_T(v_i, V(T_m)) \leq 3$. Hence

$$\begin{aligned} d_T(S) &\leq |E(T_m)| + 3(n-1) \\ &\leq 4|M| + 3n - 6. \end{aligned} \tag{2.6}$$

Let $uv \in M$. Since G is triangle-free u and v do not have any neighbours in common. Since u and v have at least δ neighbours each and since the open neighbourhoods of u and v are disjoint, we obtain

$$2\delta|M| \leq p.$$

Isolating $|M|$ yields $|M| \leq \dfrac{p}{2\delta}$ and hence, by (2.6),

$$d_T(S) \leq \frac{2p}{\delta} + 3n - 6.$$

The theorem now follows since $\mathrm{diam}_n(G) = d_G(S) \leq d_T(S)$. □

The following graphs show that, for constant δ and n, the bound in Theorem 2.2.2 is best possible, apart from an additive constant. Given integers

$\delta, k > 0$, let $G_1^{'}, G_2^{'}, \dots, G_k^{'}$ be disjoint copies of the complete bipartite graph $K_{\delta,\delta}$ and let $a_i b_i \in E(G_i^{'})$. Let $G_{k,\delta}^{'}$ be the graph obtained from the union of $G_1^{'}, G_2^{'}, \dots, G_k^{'}$ by deleting the edges $a_i b_i$ for $i = 2, 3, \dots, k-1$ and adding edges $a_{i+1} b_i$ for $i = 1, 2, \dots, k-1$. Note that $p(G_{k,\delta}^{'}) = 2\delta k$ and, thus, $k = \frac{p(G_{k,\delta}^{'})}{2\delta}$. If $n \leq 2\delta - 2$ then, by a simple calculation, $\operatorname{diam}_n(G_{k,\delta}^{'}) = 4k + n - 7$ and so $\operatorname{diam}_n(G_{k,\delta}^{'}) = 2\frac{p(G_{k,\delta}^{'})}{\delta} + n - 7$. In this case the difference between $\operatorname{diam}_n(G_{k,\delta}^{'})$ and the bound in Theorem 2.2.2 is at most $2n+1$. For $n > 2\delta - 2$ we use the estimate $\operatorname{diam}_n(G_{k,\delta}^{'}) \geq \operatorname{diam}_2(G_{k,\delta}^{'}) = 2\frac{p(G_{k,\delta}^{'})}{\delta} - 5$. In this case, the difference between the Steiner n-diameter of $G_{k,\delta}^{'}$ and the bound in Theorem 2.2.2 is bounded by the additive constant $3n - 1$.

We note that a very slight modification of the proof of Theorem 2.2.2 yields that the n-diameter of the tree T constructed in this proof is at most $\frac{2p}{\delta} + 3n - 3$. Hence we obtain the following corollary:

Corollary 2.2.2. *Let G be a connected triangle-free graph of order p and minimum degree $\delta \geq 2$. If $2 \leq n \leq p$, then G has a spanning tree T with*

$$\operatorname{diam}_n(T) \leq \frac{2p}{\delta} + 3n - 3.$$

Theorem 2.2.3. *(i) Let G be a connected C_4-free graph of order p and minimum degree $\delta \geq 2$. If $2 \leq n \leq p$, then*

$$\operatorname{diam}_n(G) \leq \frac{5p}{\delta^2 - 2\lfloor \delta/2 \rfloor + 1} + 4n - 9.$$

(ii) If δ is an integer such that $\delta = q - 1$ for some prime power q, then there exists an infinite number of C_4-free graphs G with minimum degree $\delta \geq 2$ such that,

$$\operatorname{diam}_n(G) \geq \frac{5p(G)}{\delta^2 + 3\delta + 2} + n - 3.$$

Proof. (i) Let $S = \{f_1, f_2, \dots, f_n\} \subseteq V(G)$ be a set of n vertices such that $d_G(S) = \operatorname{diam}_n(G)$. First we construct a maximal 4-packing A of

G using the following procedure: Let $a_1 = f_1$ and $A = \{a_1\}$. If $A = \{a_1, a_2, \ldots, a_{i-1}\}$ and if there exists a vertex a_i in G with $d_G(a_i, A) = 5$, add a_i to A. Add vertices with $d_G(a_i, A) = 5$ to A until each of the vertices not in A is within distance 4 of A.

For $a_i \in A$, let $T_1(a_i)$ be a tree with vertex set $N_G^2[a_i]$, which is distance preserving to a_i. Then $T_1 = \bigcup_{a_i \in A} T_1(a_i)$ is a subforest of G. By our construction of A, there exist $|A| - 1$ edges in G, each of them joining two components of T_1, whose addition to T_1 yields a tree $T_2 \leq G$.

Now each vertex in $V(G) - V(T_2)$ is within distance 4 of some vertex of T_2. Let $T \geq T_2$ be a spanning tree of G in which $d_T(b, A) = d_G(b, A)$ for each $b \in V(G)$.

As in the proof of (2.2),

$$T^5[A] \text{ is connected.}$$

We now prove that

$$d_T(S) \leq \frac{5p}{\delta^2 - 2\lfloor \delta/2 \rfloor + 1} + 4n - 9. \tag{2.7}$$

First let $b_1, b_2, \ldots, b_n \in A$ be such that b_i is a vertex in A closest to f_i in T for $i = 1, 2, , \ldots, n$. Note that $b_1 = a_1$. Since $T^5[A]$ is connected, we have

$$d_{T^5}(\{b_1, b_2, \ldots, b_n\}) \leq |A| - 1,$$

and by Lemma 2.1.1,

$$d_T(\{b_1, b_2, \ldots, b_n\}) \leq 5(|A| - 1).$$

By our construction of A and T, we have $d_T(b_1, f_1) = 0$ and $d_T(b_j, f_j) \leq 4$ for $j = 2, \ldots, n$, so that $\sum_{j=1}^{n} d(b_j, f_j) \leq 4n - 4$. Hence

$$\begin{aligned} d_T(S) &\leq d_T(\{b_1, b_2, \ldots, b_n\}) + \sum_{j=1}^{n} d(b_j, f_j) \\ &\leq 5(|A| - 1) + 4n - 4 \\ &= 5|A| + 4n - 9. \end{aligned} \tag{2.8}$$

Since G is C_4-free, no two neighbours of $a_i \in A$ have a common neighbour apart from a_i. Since all $a_i \in A$ have at least δ neighbours and since the closed neighbourhoods of the a_i are disjoint, we have

$$|N_G^2[a_i]| \geq \delta^2 - 2\lfloor \delta/2 \rfloor + 1.$$

Hence

$$(\delta^2 - 2\lfloor \delta/2 \rfloor + 1)|A| \leq p$$

which implies that $|A| \leq \dfrac{p}{\delta^2 - 2\lfloor \delta/2 \rfloor + 1}$. Hence, by (2.8),

$$d_T(S) \leq \frac{5p}{\delta^2 - 2\lfloor \delta/2 \rfloor + 1} + 4n - 9.$$

This proves (2.7). The theorem now follows since $\mathrm{diam}_n(G) = d_G(S) \leq d_T(S)$.

(ii) To prove the second part of the theorem, consider the following graph $G''_{k,\delta}$, first described in [44]. Let q be a prime power and let $GF(q)^3$ be a 3-dimensional vector space over $GF(q)$, the finite field of order q. Let H be the graph whose vertices are the 1-dimensional subspaces of $GF(q)^3$. Let $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ be adjacent in H if $\underline{x} \cdot \underline{y} = 0$.

Claim 2.2.4. If S is a subset of $GF(q)^n$ then the orthogonal complement of S, denoted by $S^{\perp}$, is a subspace of $GF(q)^n$.

Proof. It suffices to show that vector addition and scalar multiplication are satisfied. Let $\underline{x} \in S$ and let $\underline{y}, \underline{z} \in S^{\perp}$. Since $\underline{y} \cdot \underline{x} + \underline{z} \cdot \underline{x} = 0$ we have $(\underline{y} + \underline{z}) \cdot \underline{x} = 0$. Also if $c \in GF(q)$ then $c\,(\underline{y} \cdot \underline{x}) = (c\underline{y}) \cdot \underline{x} = 0$. Therefore $(\underline{y} + \underline{z}), c\underline{y} \in S^{\perp}$. This completes the proof of the claim. □

Claim 2.2.5. Let $\langle \underline{x} \rangle$, $\langle \underline{y} \rangle$ be 1-dimensional subspaces of $GF(q)^3$. Then the following hold:

(a) if $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ are nonadjacent in H, then they have exactly one common neighbour,

(b) if $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ are adjacent, then at most one of $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ is self-orthogonal,

(c) if $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ are adjacent and one of $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ is self-orthogonal, then $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ have no common neighbour in H,

(d) if $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ are adjacent and both are not self-orthogonal, then $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ have exactly one common neighbour.

Proof. Let $\langle \underline{x}, \underline{y} \rangle^{\perp} = \langle \underline{z} \rangle$ for some $\underline{z} \in GF(q)^3$, $\underline{z} \neq \underline{0}$. Furthermore,

(i) if $\langle \underline{z} \rangle \neq \langle \underline{x} \rangle$ and $\langle \underline{z} \rangle \neq \underline{y}$ then $\langle \underline{z} \rangle$ is the unique neighbour of $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ and

(ii) if $\langle \underline{z} \rangle = \langle \underline{x} \rangle$ or$\langle \underline{z} \rangle = \langle \underline{y} \rangle$ then $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ have no common neighbour.

Consider the following cases:

Case 1. Let $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ be nonadjacent. Then $\langle \underline{x} \rangle, \langle \underline{y} \rangle \nsubseteq \langle \underline{x}, \underline{y} \rangle^{\perp}$ and so $\langle \underline{z} \rangle \neq \langle \underline{x} \rangle$ and $\langle \underline{z} \rangle \neq \langle \underline{y} \rangle$. Hence $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ have exactly one common neighbour.

Case 2. Let $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ be adjacent. Suppose that both are self-orthogonal. Then $\langle \underline{x} \rangle, \langle \underline{y} \rangle \subseteq \langle \underline{x}, \underline{y} \rangle^{\perp}$. Therefore $\langle \underline{x}, \underline{y} \rangle^{\perp}$ is a 2-dimensional subspace of $GF(q)^3$. This yields a contradiction. Hence the assertion holds.

Case 3. Let $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ be adjacent and let one of $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ be self-orthogonal, say, $\langle \underline{x} \rangle$ is self-orthogonal. Then $\langle \underline{x} \rangle \subseteq \langle \underline{x}, \underline{y} \rangle^{\perp}$ and thus $\langle \underline{z} \rangle = \langle \underline{x} \rangle$. Hence $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ have no common neighbour.

Case 4. Let $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ be adjacent and both be not self-orthogonal. Then $\langle \underline{x} \rangle, \langle \underline{y} \rangle \nsubseteq \langle \underline{x}, \underline{y} \rangle^{\perp}$. Therefore $\langle \underline{z} \rangle \neq \langle \underline{x} \rangle$ and $\langle \underline{z} \rangle \neq \langle \underline{y} \rangle$. Thus $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ have exactly one common neighbour. □

Remark 2.2.6. It follows that any two vertices of H have no more than one common neighbour. Hence H is C_4-free.

Note that the number of nonzero vectors in $GF(q)^3$ is $q^3 - 1$. Since $GF(q)$ has q elements, every 1-dimensional subspace has $q - 1$ nonzero vectors. Each of these $q-1$ nonzero vectors spans the same 1-dimensional subspace. Hence each 1-dimensional subspace is spanned by $q-1$ different vectors. Consequently, the number of 1-dimensional subspaces of $GF(q)^3$ is $(q^3-1)/(q-1) = q^2+q+1$. Hence H has q^2+q+1 vertices. Now let $\langle \underline{x} \rangle$ be a 1-dimensional subspace of $GF(q)^3$. Then $\langle \underline{x} \rangle^\perp$ is a 2-dimensional subspace of $GF(q)^3$ and thus isomorphic to $GF(q)^2$. Also note that $GF(q)^2$ has q^2-1 nonzero vectors. Thus the number of 1-dimensional subspaces of $GF(q)^2$ is $(q^2-1)/(q-1) = q+1$. Hence each vertex $\langle \underline{x} \rangle$ of H has degree q if $\langle \underline{x} \rangle$ is self orthogonal or has degree $q+1$ otherwise.

Let $\langle \underline{u} \rangle, \langle \underline{v} \rangle, \langle \underline{z} \rangle \in V(H)$ be fixed vertices satisfying $\underline{u} \cdot \underline{z} = \underline{v} \cdot \underline{z} = \underline{z} \cdot \underline{z} = 0$. Since z is self-orthogonal by Claim 2.2.5 (b), $\langle \underline{u} \rangle$ and $\langle \underline{v} \rangle$ cannot be self-orthogonal.

Claim 2.2.7. There exists a self-orthogonal vector $\underline{z}$ in $GF(q)^3$.

Proof. Let $H^* = GF(q) \setminus \{0\}$ be the multiplicative group of $GF(q)$. Let $\phi : H^* \to H^*$ be the homomorphism $\phi(x) = x^2$. Then $\phi(1) = 1^2 = 1$ and also $\phi(-1) = (-1)^2 = 1$. Thus $\{1, -1\} \subseteq \ker \phi$. Now to show that $\ker \phi \subseteq \{1, -1\}$, let $x \in \ker \phi$. So $\phi(x) = x^2 = 1$ or, equivalently, $x^2 - 1 = (x+1)(x-1) = 0$. Thus $x = 1$ or -1 and so $x \in \{1, -1\}$. Thus $\ker \phi \subseteq \{1, -1\}$. Hence $\ker \phi = \{1, -1\}$.

Consider the following cases:

Case 1. q is even. Then $\ker \phi = \{1, -1\} = \{1\}$. Hence

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 + 1 + 0 = 0.$$

Case 2. q is odd. Then $\ker \phi = \{1, -1\}$. Let $Q = \operatorname{im} \phi$.
Case 2(*a*). $-1 \in Q$. Then $-1 = a^2$ for some $a \in GF(q)$. Then

$$\begin{pmatrix} 1 \\ a \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ a \\ 0 \end{pmatrix} = 0.$$

Case 2(*b*). $-1 \notin Q$. Note that $\{0\} \cup Q$ is not closed under addition because if $\{0\} \cup Q$ was closed under addition then since $\{0\} \cup Q$ is closed under multiplication, $\{0\} \cup Q$ would be a subfield with $\frac{(q+1)}{2}$ elements and this is impossible. Hence, there exist $a, b \in Q$ such that $a + b \notin Q$. Since $H^* = Q \cup (-1)Q$, and since $a + b \neq 0$, we have $a + b \in (-1)Q$, that is, $a + b = (-1) \cdot c$ for some $c \in Q$. Since $a, b, c \in Q$, there exist x, y, z such that $a = x^2$, $b = y^2$, $c = z^2$. For these x, y, z we have $x^2 + y^2 = -z^2$, that is,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

and this completes the proof. □

Claim 2.2.8. If $\langle \underline{z} \rangle$ is self-orthogonal then no two neighbours of $\langle \underline{z} \rangle$ are adjacent.

Proof. Let $\langle \underline{u} \rangle, \langle \underline{v} \rangle \in N(\langle \underline{z} \rangle)$. Clearly, $\underline{z}, \underline{v}, \underline{u} \in \langle \underline{z} \rangle^{\perp}$ and $\dim\langle \underline{u}, \underline{v}, \underline{z} \rangle \leq \dim\langle \underline{z} \rangle^{\perp} = 2$. Since $\underline{u}, \underline{v}$ and $\underline{z}$ are linearly dependent then without loss of generality, $\langle \underline{u} \rangle$ is a linear combination of $\langle \underline{v} \rangle$ and $\langle \underline{z} \rangle$. Hence there exists $\alpha, \beta \in GF(q)$ with

$$\underline{u} = \alpha \underline{z} + \beta \underline{v}, \;\; \beta \neq 0. \tag{2.9}$$

Now applying dot product to (3.2), we obtain

$$\begin{aligned} \underline{u} \cdot \underline{v} &= (\alpha \underline{z} + \beta \underline{v}) \cdot \underline{v} \\ &= \alpha \underline{z} \cdot \underline{v} + \beta \underline{v} \cdot \underline{v} \\ &= \beta \cdot (\underline{v} . \underline{v}) \\ &\neq 0, \end{aligned}$$

since $\beta \neq 0$ and $\underline{v}$ is not self-orthogonal. It follows that $\langle \underline{v} \rangle$ and $\langle \underline{u} \rangle$ are nonadjacent. □

Let $\langle \underline{u}_0 \rangle = \langle \underline{z} \rangle, \langle \underline{u}_1 \rangle, \langle \underline{u}_2 \rangle, \ldots, \langle \underline{u}_q \rangle$ and let $\langle \underline{v}_0 \rangle = \langle \underline{z} \rangle, \langle \underline{v}_1 \rangle, \langle \underline{v}_2 \rangle, \ldots, \langle \underline{v}_q \rangle$ denote the neighbours of $\langle \underline{u} \rangle$ and $\langle \underline{v} \rangle$, respectively.

Claim 2.2.9. (a) For every i with $1 \leq i \leq q$, there exists a unique $j(i)$ with $1 \leq j(i) \leq q$ such that $\langle \underline{u}_i \rangle \langle \underline{v}_{j(i)} \rangle \in E(H)$.
Renumber $\underline{v}_1, \ldots, \underline{v}_q$ such that $j(i) = i$ for all i.

(b) For every i, $1 \leq i \leq q$,

(i) no $\langle \underline{u}_i \rangle$ or $\langle \underline{v}_i \rangle$ is adjacent to $\langle \underline{z} \rangle$ in H.

(ii) no $\langle \underline{u}_i \rangle$ is adjacent to $\langle \underline{v} \rangle$ and no $\langle \underline{v}_i \rangle$ is adjacent to $\langle \underline{u} \rangle$ in H.

Proof. (a) For each i, $1 \leq i \leq q$, $\langle \underline{u}_i \rangle$ and $\langle \underline{v} \rangle$ have exactly one common neighbour since $\langle \underline{u}_i \rangle$ and $\langle \underline{v} \rangle$ are nonadjacent and by Claim 2.2.5 (a). The common neighbour is $\langle \underline{v}_i \rangle$, $i \neq 0$ since if we suppose $i = 0$ and so $\langle \underline{v}_0 \rangle = \langle \underline{z} \rangle$ and by Claim 2.2.8 this is impossible. Proofs for $\langle \underline{v}_i \rangle$ and $\langle \underline{u} \rangle$ are similar. This completes the proof of part (a) of the claim.

(b) Suppose $\langle \underline{u}_i \rangle$ is adjacent to $\langle \underline{z} \rangle$. By Claim 2.2.8, this is impossible. Therefore, $\langle \underline{u}_i \rangle$ and $\langle \underline{z} \rangle$ are nonadjacent. Proof for $\langle \underline{v}_i \rangle$ is similar. This completes the proof of the first part. The second part follows from Remark 2.2.6. This concludes the proof of part (b) of the claim. □

Let $H^{'}$ denote the graph obtained from H by removing the vertex $\langle \underline{z} \rangle$ and all edges of the form $\underline{u}_i \underline{v}_j$, $1 \leq i \leq q$. Clearly, every vertex of $H^{'}$ has degree at least $q - 1 = \delta$. By Claim 2.2.5 (a), $\langle \underline{u}_i \rangle$ and $\langle \underline{v}_j \rangle$, $i \neq j$, have exactly one common neighbour since they are nonadjacent. Hence $d_{H'}(\langle \underline{u} \rangle, \langle \underline{v} \rangle) = 4$.

Note that the number of vertices of $H^{'}$ is $q^2 + q$. For $p \geq 5k + n - 2$ a multiple of $q^2 + q = \delta^2 + 3\delta + 2$, let $G^{''}_{k,\delta}$ so that $k \geq n - 2$ be the graph obtained from the union of $k = \dfrac{p}{\delta^2 + 3\delta + 2}$ disjoint copies $H^{'}_1, H^{'}_2, \ldots, H^{'}_k$ of $H^{'}$ by adding the edges $\underline{u}^t\ \underline{v}^{t+1}$ for $1 \leq t \leq k-1$, where $\langle \underline{u}^t \rangle$ and $\langle \underline{v}^t \rangle$ are the vertices in $H^{'}_t$ corresponding to $\langle \underline{u} \rangle$ and $\langle \underline{v} \rangle$ in $H^{'}$. Let $\langle \underline{c} \rangle$ be a vertex in H such that $\langle \underline{c} \rangle$ is adjacent to $\langle \underline{z} \rangle$ and common neighbours of $\underline{u}_1$ and $\underline{v}_2$. Since $\langle \underline{z} \rangle$ and the common neighbours of $\underline{u}_1$ and $\underline{v}_2$ are nonadjacent, and by Claim 2.2.5 (a), such a vertex exists. The Steiner tree containing $\langle \underline{u} \rangle$, $\langle \underline{v} \rangle$ and $\langle \underline{c} \rangle$ has more than four edges since $\langle \underline{z} \rangle$ is self-orthogonal and by Claim 2.2.8 $\langle \underline{c} \rangle$ cannot be adjacent to $\langle \underline{u} \rangle$ and $\langle \underline{v} \rangle$, and since by Remark 2.2.6 $\langle \underline{c} \rangle$ cannot be adjacent to $\langle \underline{u}_1 \rangle$ and $\langle \underline{v}_2 \rangle$. Let $\langle \underline{c}^t \rangle$ be the copy of $\langle \underline{c} \rangle$ in $H_t^{'}$. Thus $d_{H_t^{'}}(\langle \underline{v}^t \rangle, \langle \underline{c}^t \rangle, \langle \underline{u}^t \rangle) = 5$, $1 \leq t \leq k$. Now let $S = \{\langle \underline{v}^1 \rangle, \langle \underline{u}^k \rangle, \langle \underline{c}^1 \rangle, \langle \underline{c}^2 \rangle, \ldots \langle \underline{c}^{n-2} \rangle\}$, $1 \leq k \leq t$. Hence $d(S) = 5(k-1) + 4 + n - 2 = 5k + n - 3 \leq \mathrm{diam}_n(G_{k,\delta}^{''})$ and so $\mathrm{diam}_n(G) \geq \frac{5p}{\delta^2+3\delta+2} + n - 3$, which concludes the proof of the second part of the theorem. □

As above, a very slight modification of the proof of Theorem 2.2.3 yields the following result.

Corollary 2.2.3. *Let G be a connected C_4-free graph of order p and minimum degree $\delta \geq 2$. If $2 \leq n \leq p$, then G has a spanning tree T with*

$$\mathrm{diam}_n(T) \leq \frac{5p}{\delta^2 - 2\lfloor \delta/2 \rfloor + 1} + 4n - 5.$$

Chapter 3

The Radius of k-Connected Planar Graphs with Bounded Faces

3.1 Introduction

This chapter discusses yet another graph invariant, the radius of a graph. In particular, it presents an upper bound on the radius of a connected plane graph in terms of order and maximum face length. We prove that if G is a 3-connected plane graph of order p, maximum face length l and radius $\text{rad}(G)$, then the bound

$$\text{rad}(G) \leq \frac{p}{6} + \frac{5l}{6} + \frac{5}{6}$$

holds. For constant l, our bound is shown to be asymptotically sharp and improves on a bound by Harant [62]. Furthermore we extend these results to 4- and 5-connected planar graphs.

The following definition will be useful throughout this chapter.

Let G be a connected plane graph of order p. From now on let z be a fixed, not necessarily central, vertex of G and let $\text{ex}(z) = r$. For each

$i = 0, 1, \ldots, r$ let

$$N_i := \{x \in V(G) | d_G(x, z) = i\}.$$

A vertex $x \in N_i$ is *active* if $i \leq r - 1$ and x has a neighbour in N_{i+1}. We denote by A_i the set of active vertices in N_i.

The following lemma is important for the main results of this chapter.

Lemma 3.1.1. *Let G, and A_i be as above and let $1 \leq i \leq r - 1$.*

(a) If G is 3-connected and $u \in A_i$, then there exist distinct vertices $v, w \in A_i - \{u\}$ such that u and v share a face, and u and w share a face.

(b) If G is 4-connected and $u, v, w \in A_i$, then at least one of u, v, w shares a face with a vertex in $A_i - \{u, v, w\}$.

Proof. (a) Since u is a vertex of A_i, it has neighours in N_{i-1} and in N_{i+1}. Number the neighbours of u as $x_0, x_1, \ldots, x_l$ such that the edges ux_i appear in clockwise order, x_0 is in N_{i-1} and, say, x_k is in N_{i+1}. Denote the face containing u, x_j and x_{j+1} by f_j for $j = 0, 1, \ldots, l$ where subscripts are taken modulo $l + 1$. Let P_j be the x_j-x_{j+1} path of the vertices on the boundary of f_j except u in clockwise order.

We show that there exist a $j_1 \in \{0, 1, \ldots, k - 1\}$ such that the boundary of f_{j_1} contains a vertex $v \in A_i - \{u\}$. Consider the walk, $W := x_0\overrightarrow{P_0}x_1\overrightarrow{P_1}x_2 \ldots x_{k-1}\overrightarrow{P_{k-1}}x_k$ i.e., the $x_0 - x_k$ walk that traverses the vertices of P_0 then $P_1, P_2, \ldots, P_{k-1}$. Let b be the first vertex of W in N_{i+1} and let v be the predecessor of b in W. Then v is in N_i. Since v has a neighbour in N_{i+1}, we have $v \in A_i$. Furthermore, v is on the boundary of f_{j_1} for some $j_1 \in \{0, 1, \ldots, k - 1\}$. Similarly we can show that there exists a $j_2 \in \{k, k + 1, \ldots, l\}$ such that the boundary of f_{j_2} contains a vertex $w \in A_i - \{u\}$. It remains to show that $v \neq w$. Suppose $v = w$. Join u and v by an edge that goes through face f_{j_1}, and another edge through face f_{j_2}, thus creating a plane multigraph. The new edges form a 2-cycle, C_2. Since the last x_j that precedes v on W and the first x_j that succeeds v on W are on different sides of C_2, the inside and the outside of C_2 both contain vertices. So any path between vertices inside C_2 and those vertices outside has to pass through u or v, and hence u and v form a cutset, a contradiction to the 3-connectedness of G.

(b) Suppose that none of u, v, w shares a face with a vertex in $A_i - \{u, v, w\}$. By the proof and notation of Lemma 3.1.1(a), we have v on the boundary of f_{j_1} for some $j_1 \in \{0, 1, \ldots, k-1\}$ and w on the boundary of f_{j_2} for some $j_2 \in \{k, k+1, \ldots, l\}$. Also $v \neq w$. By Lemma 3.1.1(a) and the assumption that none of u, v, w shares a face with a vertex in $A_i - \{u, v, w\}$, we conclude that u and v share a face, u and w share a face, and v and w share a face. So we can add new edges between u and v through face f_{j_1}, between u and w through face f_{j_2} and between v and w, thus creating a plane multigraph. The three new edges form a 3-cycle, C_3. Since the last x_j that precedes v on W and the first x_j that succeeds v on W are on different sides of C_3, the inside and the outside of C_3 both contain vertices. Thus any path between vertices inside C_3 and those vertices outside has to pass through u or v or w, and hence u, v and w form a cutset, a contradiction to the 4-connectedness of G. □

Definition 3.1.2. Let G, z and A_i be as above. For $i \in \mathbb{N}$ and $1 \leq i \leq r-1$ we define $\hat{H}_i$ to be the graph with vertex set A_i, where two vertices are adjacent in $\hat{H}_i$ if and only if they share a face in G.

Lemma 3.1.3. *Let $\hat{H}_i$ be as above and let u be a vertex of $\hat{H}_i$. If G is 5-connected, then u has two neighbours v and w that have no common neighbour other than u.*

Proof. By the proof and notation of Lemma 3.1.1, we have v on the boundary of f_{j_1} for some $j_1 \in \{0, 1, \ldots, k-1\}$ and w on the boundary of f_{j_2} for some $j_2 \in \{k, k+1, \ldots, l\}$. Also $v \neq w$. Suppose that v and w share a neighbour $a \neq u$ in $\hat{H}_i$, so v and a share a face f' and w and a share a face f''. As above we can add edges to G: between u and v through face f_{j_1}, between u and w through face f_{j_2}, between v and a through face f' and between w and a through face f'', thus creating a plane multigraph. Now the four edges uv, uw, va and wa form a 4-cycle, C_4. Since the last x_j that precedes v on W

and the first x_j that succeeds v on W are on different sides of C_4, the inside and the outside of C_4 both contain vertices. Thus any path between vertices inside C_4 and those vertices outside has to pass through u, v, w or a, and hence u, v, w and a form a cutset, a contradiction to the 5-connectedness of G. □

Lemma 3.1.4. *Let $\hat{H}_i$ be as above. If G is 3-connected, then $\delta(\hat{H}_i) \geq 2$. Moreover,*

a) each component of $\hat{H}_i$ has at least three vertices.

b) if G is 4-connected, then each component of $\hat{H}_i$ has at least four vertices.

c) if G is 5-connected, then each component of $\hat{H}_i$ has at least five vertices.

Proof. a) Let G be 3-connected. By Lemma 3.1.1 (a), $\hat{H}_i$ has minimum degree at least two. Hence each component of $\hat{H}_i$ has at least three vertices.

b) Let G be 4-connected. By a), each component of $\hat{H}_i$ has at least three vertices. Suppose that $\hat{H}_i$ has a component with three vertices, say, u, v, w. But by Lemma 3.1.1 (b) at least one of u, v, w shares a face in G, and is thus adjacent in $\hat{H}_i$, with a fourth vertex, a contradiction.

c) Let G be 5-connected. By Lemma 3.1.3, each component of $\hat{H}_i$ has a vertex u with two neighbours v and w that have no common neighbour other than u. By $\delta(\hat{H}_i) \geq 2$, both v and w have a neighbour other than u. Hence the component has at least five vertices. □

Lemma 3.1.5. *Let G be* 3*-connected and z as above. Let $i \in \{1, 2, \ldots, r-1\}$.*

(a) If $|A_i| = 3$ then there exists a vertex $z_i \in A_i$ with $d_G(z_i, v) \leq \lfloor \frac{l}{2} \rfloor$ for all $v \in A_i$.

(b) If $4 \leq |A_i| \leq 5$ then there exists a vertex $z_i \in A_i$ with $d_G(z_i, v) \leq l$ for all $v \in A_i$.

Proof. (a) Since $\hat{H}_i$ has minimum degree two and exactly three vertices, $\hat{H}_i$ is connected. Fix a vertex z_i of $\hat{H}_i$ and let $v \in A_i$ be arbitrary. Since any two vertices that are adjacent in $\hat{H}_i$ are joined by a path of length at most $\lfloor \frac{l}{2} \rfloor$ in G, the $z_i - v$ path in $\hat{H}_i$ yields a $z_i - v$ path in G of length at most $\lfloor \frac{l}{2} \rfloor$. Hence $d_G(z_i, v) \leq \lfloor \frac{l}{2} \rfloor$, as desired.

(b) Since $\hat{H}_i$ has minimum degree at least two and at most five vertices, $\hat{H}_i$ is connected and has a vertex z_i of eccentricity at most two. As in (a), this implies that $d_G(z_i, v) \leq l$ for all $v \in A_i$. □

Lemma 3.1.6. *Let G be 4-connected and z as above. Let $i \in \{1, 2, \ldots, r-1\}$. If $6 \leq |A_i| \leq 7$ then there exists a vertex $z_i \in A_i$ with $d(z_i, v) \leq \lfloor \frac{3l}{2} \rfloor$ for all $v \in A_i$.*

Proof. Since $\hat{H}_i$ has at most seven vertices, it follows by Lemma 3.1.4 that $\hat{H}_i$ is connected. By Lemma 3.1.1 a), $\hat{H}_i$ has minimum degree at least two. Hence $\hat{H}_i$ has a vertex z_i of eccentricity at most three. As in Lemma 3.1.5, this implies that $d_G(z_i, v) \leq \lfloor \frac{3l}{2} \rfloor$ for all $v \in A_i$. □

Lemma 3.1.7. *Let G be 5-connected and z as above. Let $i \in \{1, 2, \ldots, r-1\}$. If $8 \leq |A_i| \leq 9$ then there exists a vertex $z_i \in A_i$ with $d(z_i, v) \leq 2l$ for all $v \in A_i$.*

Proof. Since $\hat{H}_i$ has at most nine vertices, it follows by Lemma 3.1.4 c) that $\hat{H}_i$ is connected. Also $\hat{H}_i$ has minimum degree at least two by Lemma 3.1.1 a). So $\hat{H}_i$ has a vertex z_i of eccentricity at most four. As in Lemma 3.1.5, this implies that $d_G(z_i, v) \leq 2l$ for all $v \in A_i$. □

From now on let z be a central vertex of G, i.e., a vertex of eccentricity $r = \text{rad}(G)$. We employ the notation

$$N_{\leq i} = \bigcup_{0 \leq j \leq i} N_j \quad \textit{and} \quad N_{\geq i} = \bigcup_{i \leq j \leq r} N_j.$$

Form a spanning tree T of G that is distance preserving from z. For a vertex $y \in V(G)$, denote by $T(z, y)$, the set of vertices on the path connecting z and y in T.

3.2 Upper Bounds on the Radius of 3-,4- and 5-Connected Planar Graphs with Bounded Faces

Theorem 3.2.1. *Let G be a 3-connected plane graph of order p, maximum face length l and radius r. Then*

$$r \leq \frac{p}{6} + \frac{5}{6}l + \frac{5}{6}. \tag{3.1}$$

Proof. We first bound the cardinalities of the N_i from below. The following claim immediately follows from the 3-connectedness of G:

Claim 1. Let $i \in \{1, 2, \ldots, r-1\}$. Then $|N_i| \geq 3$.

This bound can be improved if i is not too close to 0 or r.

Claim 2. Let $i \in \{\lfloor \frac{l}{2} \rfloor + 1, \lfloor \frac{l}{2} \rfloor + 2, \ldots, r - \lfloor \frac{l}{2} \rfloor - 1\}$. Then $|N_i| \geq 4$.

Proof of Claim 2. By way of contradiction suppose $|N_i| = 3$ for some $i \in \{\lfloor \frac{l}{2} \rfloor + 1, \lfloor \frac{l}{2} \rfloor + 2, \ldots, r - \lfloor \frac{l}{2} \rfloor - 1\}$. Let $z_i \in A_i$ be as in Lemma 3.1.5. Let x_1 denote the unique vertex of $T(z, z_i)$ which belongs to $N_{\lfloor \frac{l}{2} \rfloor + 1}$. We show that $ex(x_1) \leq r - 1$. First let $y \in N_{\leq i-1}$. Then

$$\begin{aligned} d(x_1, y) &\leq d(x_1, z) + d(z, y) \\ &\leq \lfloor \frac{l}{2} \rfloor + 1 + i - 1 \\ &\leq \lfloor \frac{l}{2} \rfloor + 1 + r - \lfloor \frac{l}{2} \rfloor - 1 - 1 \\ &= r - 1. \end{aligned}$$

Now let $y \in N_{\geq i}$. Let $y_i \in T(z, y) \cap N_i$ so that $d(x_1, z_i) = i - \lfloor \frac{l}{2} \rfloor - 1$. By Lemma 3.1.5 we have $d(z_i, y_i) \leq \lfloor \frac{l}{2} \rfloor$. Also $d(y_i, y) \leq r - i$. It follows that

$$\begin{aligned} d(x_1, y) &\leq d(x_1, z_i) + d(z_i, y_i) + d(y_i, y) \\ &\leq i - \lfloor \frac{l}{2} \rfloor - 1 + \lfloor \frac{l}{2} \rfloor + r - i \\ &= r - 1. \end{aligned}$$

Therefore, $ex(x_1) \leq r-1$, contradicting the fact that r is the radius of G. □

Claim 3. Let $i \in \{l+1, l+2, \ldots, r-l-1\}$. Then $|N_i| \geq 6$.

Proof of Claim 3. Suppose to the contrary that $|N_i| \leq 5$ for some $i \in \{l+1, l+2, \ldots, r-l-1\}$. Let $z_i \in A_i$ be as in Lemma 3.1.5. Let x_2 denote the unique vertex of $T(z, z_i)$ which belongs to N_{l+1}. We show that $ex(x_2) \leq r-1$. First let $y \in N_{\leq i-1}$. Then

$$\begin{aligned} d(x_2, y) &\leq d(x_2, z) + d(z, y) \\ &\leq l + 1 + i - 1 \\ &\leq l + 1 + r - l - 1 - 1 \\ &= r - 1. \end{aligned}$$

Now let $y \in N_{\geq i}$. Let $y_i \in T(z, y) \cap N_i$ so that $d(x_2, z_i) = i - l - 1$. By Lemma 3.1.5 we have $d(z_i, y_i) \leq l$. Also $d(y_i, y) \leq r - i$. It follows that

$$\begin{aligned} d(x_2, y) &\leq d(x_2, z_i) + d(z_i, y_i) + d(y_i, y) \\ &\leq i - l - 1 + l + r - i \\ &= r - 1. \end{aligned}$$

Therefore, $ex(x_2) \leq r-1$, contradicting the fact that r is the radius of G. □

We now complete the proof of the theorem. By Claim 1, Claim 2 and Claim 3, we have

$$\begin{aligned} p &= |N_0| + (|N_1| + \cdots + |N_{\lfloor \frac{l}{2} \rfloor}|) + (|N_{\lfloor \frac{l}{2} \rfloor + 1}| + \cdots + |N_l|) + (|N_{l+1}| + \cdots + \\ &\quad |N_{r-l-1}|) + (|N_{r-l}| + \cdots + |N_{r - \lfloor \frac{l}{2} \rfloor - 1}|) + (|N_{r - \lfloor \frac{l}{2} \rfloor}| + \cdots + |N_{r-1}|) + |N_r| \\ &\geq 1 + 3\lfloor \frac{l}{2} \rfloor + 4(l - \lfloor \frac{l}{2} \rfloor) + 6(r - 2l - 1) + 4(-\lfloor \frac{l}{2} \rfloor + l) + 3\lfloor \frac{l}{2} \rfloor + 1 \\ &= -4 - 4l - 2\lfloor \frac{l}{2} \rfloor + 6r, \end{aligned}$$

and (3.1) follows. □

Corollary 3.2.1. *Let G be a 3-connected maximal planar graph of order p. Then*

$$r \leq \frac{p}{6} + \frac{10}{3}.$$

The following graphs show that for fixed l the bound in Theorem 3.2.1 is best possible, apart from the value of the additive constant. For an even integer $k \geq 4$, let $G_1, G_2, \ldots, G_k$ be disjoint copies of the cycle C_3, and let $a_i, b_i, c_i \in V(G_i)$. Let G'_k be the graph obtained from the union of $G_1, G_2, \ldots, G_k$ by adding the edges $a_{i+1}a_i, b_{i+1}b_i, c_{i+1}c_i, a_{i+1}b_i, c_{i+1}b_i, a_{i+1}c_i$ for $i = 1, 2, \ldots, k-1$. Furthermore let C_l be a cycle with vertices $j_1, j_2, \ldots, j_l$. Now join the graphs C_l and G'_k by adding the edges $j_1a_1, j_2a_1, j_2b_1, j_la_1$ and j_ic_1 for $i = 2, 3, \ldots, l$ thus obtaining a planar graph H_k. Clearly, $p(H_k) = 3k + l$ so that $k = \frac{p(H_k)-l}{3}$. By a simple calculation, $\mathrm{rad}(H_k) = \frac{k}{2}$ and so $\mathrm{rad}(H_k) = \frac{p(H_k)}{6} - \frac{l}{6}$. Figure 3.1 shows the graph H_k.

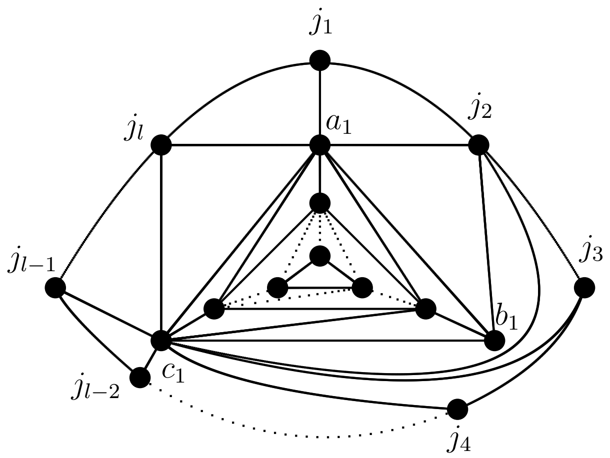

Figure 3.1: The graph H_k.

Theorem 3.2.2. *Let G be a 4-connected plane graph of order p, maximum face length l and radius r. Then*

$$r \leq \frac{p}{8} + \frac{5}{4}l + 1. \tag{3.2}$$

Proof. Recall that z is a central vertex of G. We first bound the cardinalities of the N_i from below. The following claim immediately follows from the 4-connectedness of G:

Claim 1. Let $i \in \{1, 2, \ldots, r-1\}$. Then $|N_i| \geq 4$.

This bound can be improved if i is not too close to 0 or r.

Claim 2. Let $i \in \{l+1, l+2, \ldots, r-l-1\}$. Then $|N_i| \geq 6$.

Proof of Claim 2. Suppose to the contrary that $|N_i| \leq 5$ for some $i \in \{l+1, l+2, \ldots, r-l-1\}$. Let $z_i \in A_i$ be as in Lemma 3.1.5. Let x_1 denote the unique vertex of $T(z, z_i)$ which belongs to N_{l+1}. We show that $ex(x_1) \leq r-1$. First let $y \in N_{\leq i-1}$. Then

$$\begin{aligned} d(x_1, y) &\leq d(x_1, z) + d(z, y) \\ &\leq l + 1 + i - 1 \\ &\leq l + 1 + r - l - 1 - 1 \\ &= r - 1. \end{aligned}$$

Now let $y \in N_{\geq i}$. Let $y_i \in T(z, y) \cap N_i$ so that $d(x_1, z_i) = i - l - 1$. By Lemma 3.1.5 we have $d(z_i, y_i) \leq l$. Also $d(y_i, y) \leq r - i$. It follows that

$$\begin{aligned} d(x_1, y) &\leq d(x_1, z_i) + d(z_i, y_i) + d(y_i, y) \\ &\leq i - l - 1 + l + r - i \\ &= r - 1. \end{aligned}$$

Therefore, $ex(x_1) \leq r-1$, contradicting the fact that r is the radius of G. □

Claim 3. Let $i \in \{\lfloor \frac{3l}{2} \rfloor + 1, \lfloor \frac{3l}{2} \rfloor + 2, \ldots, r - \lfloor \frac{3l}{2} \rfloor - 1\}$. Then $|N_i| \geq 8$.

Proof of Claim 3. Suppose to the contrary that $|N_i| \leq 7$ for some $i \in \{\lfloor \frac{3l}{2} \rfloor + 1, \lfloor \frac{3l}{2} \rfloor + 2, \ldots, r - \lfloor \frac{3l}{2} \rfloor - 1\}$. Let $z_i \in A_i$ be as in Lemma 3.1.6. Let x_2 denote the unique vertex of $T(z, z_i)$ which belongs to $N_{\lfloor \frac{3l}{2} \rfloor + 1}$. We show that $ex(x_2) \leq r-1$. First let $y \in N_{\leq i-1}$. Then

$$\begin{aligned} d(x_2, y) &\leq d(x_2, z) + d(z, y) \\ &\leq \lfloor \frac{3l}{2} \rfloor + 1 + i - 1 \\ &\leq \lfloor \frac{3l}{2} \rfloor + 1 + r - \lfloor \frac{3l}{2} \rfloor - 1 - 1 \\ &= r - 1. \end{aligned}$$

Now let $y \in N_{\geq i}$. Let $y_i \in T(z, y) \cap N_i$ so that $d(x_2, z_i) = i - \lfloor \frac{3l}{2} \rfloor - 1$. By Lemma 3.1.6 we have $d(z_i, y_i) \leq \lfloor \frac{3l}{2} \rfloor$. Also $d(y_i, y) \leq r - i$. It follows that

$$\begin{aligned} d(x_2, y) &\leq d(x_2, z_i) + d(z_i, y_i) + d(y_i, y) \\ &\leq i - \lfloor \frac{3l}{2} \rfloor - 1 + \lfloor \frac{3l}{2} \rfloor + r - i \\ &= r - 1. \end{aligned}$$

Therefore, $ex(x_2) \leq r-1$, contradicting the fact that r is the radius of G. □

We now complete the proof of the theorem. By Claim 1, Claim 2 and Claim 3, we have

$$\begin{aligned} p &= |N_0| + (|N_1| + \cdots + |N_l|) + (|N_{l+1}| + \cdots + |N_{\lfloor \frac{3l}{2} \rfloor}|) \\ &\quad +(|N_{\lfloor \frac{3l}{2} \rfloor+1}| + \cdots + |N_{r-\lfloor \frac{3l}{2} \rfloor-1}|) + (|N_{r-\lfloor \frac{3l}{2} \rfloor}| + \cdots + |N_{r-l-1}|) \\ &\quad +(|N_{r-l}| + \cdots + |N_{r-1}|) + |N_r| \\ &\geq 1 + 4l + 6(\lfloor \frac{3l}{2} \rfloor - l) + 8(r - 2\lfloor \frac{3l}{2} \rfloor - 1) + 6(-l + \lfloor \frac{3l}{2} \rfloor) + 4l + 1 \\ &= -8 - 10l + 8r, \end{aligned}$$

and (3.2) follows. □

The following graphs show that for fixed l the bound in Theorem 3.2.2 is best possible, apart from the value of the additive constant. For an even integer $k \geq 6$, let $G_1, G_2, \ldots, G_k$ be disjoint copies of the 4-cycle C_4, and let $a_i, b_i, c_i, d_i \in V(G_i)$. Let G''_k be the graph obtained from the union of $G_1, G_2, \ldots, G_k$ by adding the edges $a_{i+1}a_i, b_{i+1}b_i, c_{i+1}c_i, d_{i+1}d_i, a_{i+1}d_i, b_{i+1}a_i$, $c_{i+1}b_i, d_{i+1}c_i$, for $i = 1, 2, \ldots, k-1$ and $a_k c_k$. Furthermore let C_l be a cycle with vertices $j_1, j_2, \ldots, j_l$. Now join the graphs C_l and G''_k by adding the edges $j_1a_1, j_la_1, j_1b_1, j_2b_1, j_3b_1, j_3c_1$ and j_id_1 for $i = 3, 4, \ldots, l$ thus obtaining a planar graph H'_k. Clearly, $p(H'_k) = 4k + l$ so that $k = \frac{p(H'_k)-l}{4}$. By a simple calculation, $\text{rad}(H'_k) = \frac{k}{2}$ and so $\text{rad}(H'_k) = \frac{p(H'_k)}{8} - \frac{l}{8}$. Figure 3.2 shows the graph H'_k.

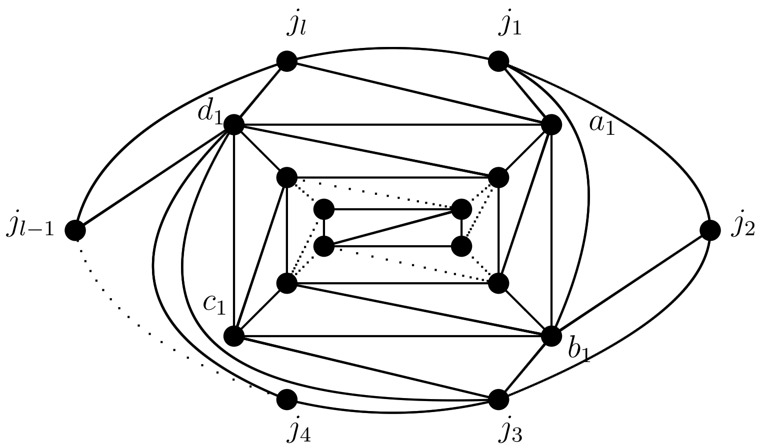

Figure 3.2: The graph H'_k.

Theorem 3.2.3. *Let G be a 5-connected plane graph of order p, maximum face length l and radius r. Then*

$$r \leq \frac{p}{10} + \frac{8}{5}l + 1. \tag{3.3}$$

Proof. Recall that z is a central vertex of G. We first bound the cardinalities of the N_i from below. The following claim immediately follows from the 5-connectedness of G:

Claim 1. Let $i \in \{1, 2, \ldots, r-1\}$. Then $|N_i| \geq 5$.

This bound can be improved if i is not too close to 0 or r.

Claim 2. Let $i \in \{l+1, l+2, \ldots, r-l-1\}$. Then $|N_i| \geq 6$.

Proof of Claim 2. Suppose to the contrary that $|N_i| \leq 5$ for some $i \in \{l+1, l+2, \ldots, r-l-1\}$. Let $z_i \in A_i$ be as in Lemma 3.1.5. Let x_1 denote the unique vertex of $T(z, z_i)$ which belongs to N_{l+1}. We show that

$ex(x_1) \leq r-1$. First let $y \in N_{\leq i-1}$. Then

$$\begin{aligned} d(x_1,y) &\leq d(x_1,z)+d(z,y) \\ &\leq l+1+i-1 \\ &\leq l+1+r-l-1-1 \\ &= r-1. \end{aligned}$$

Now let $y \in N_{\geq i}$. Let $y_i \in T(z,y) \cap N_i$ so that $d(x_1,z_i) = i-l-1$. By Lemma 3.1.5 we have $d(z_i,y_i) \leq l$. Also $d(y_i,y) \leq r-i$. It follows that

$$\begin{aligned} d(x_1,y) &\leq d(x_1,z_i)+d(z_i,y_i)+d(y_i,y) \\ &\leq i-l-1+l+r-i \\ &= r-1. \end{aligned}$$

Therefore, $ex(x_1) \leq r-1$, contradicting the fact that r is the radius of G. □

Claim 3. Let $i \in \{\lfloor \frac{3l}{2} \rfloor + 1, \lfloor \frac{3l}{2} \rfloor + 2, \ldots, r - \lfloor \frac{3l}{2} \rfloor - 1\}$. Then $|N_i| \geq 8$.

Proof of Claim 3. Suppose to the contrary that $|N_i| \leq 7$ for some $i \in \{\lfloor \frac{3l}{2} \rfloor + 1, \lfloor \frac{3l}{2} \rfloor + 2, \ldots, r - \lfloor \frac{3l}{2} \rfloor - 1\}$. Let $z_i \in A_i$ be as in Lemma 3.1.6. Let x_2 denote the unique vertex of $T(z,z_i)$ which belongs to $N_{\lfloor \frac{3l}{2} \rfloor + 1}$. We show that $ex(x_2) \leq r-1$. First let $y \in N_{\leq i-1}$. Then

$$\begin{aligned} d(x_2,y) &\leq d(x_2,z)+d(z,y) \\ &\leq \lfloor \frac{3l}{2} \rfloor + 1 + i - 1 \\ &\leq \lfloor \frac{3l}{2} \rfloor + 1 + r - \lfloor \frac{3l}{2} \rfloor - 1 - 1 \\ &= r-1. \end{aligned}$$

Now let $y \in N_{\geq i}$. Let $y_i \in T(z,y) \cap N_i$ so that $d(x_2,z_i) = i - \lfloor \frac{3l}{2} \rfloor - 1$. By Lemma 3.1.6 we have $d(z_i,y_i) \leq \lfloor \frac{3l}{2} \rfloor$. Also $d(y_i,y) \leq r-i$. It follows that

$$\begin{aligned} d(x_2,y) &\leq d(x_2,z_i)+d(z_i,y_i)+d(y_i,y) \\ &\leq i - \lfloor \frac{3l}{2} \rfloor - 1 + \lfloor \frac{3l}{2} \rfloor + r - i \\ &= r-1. \end{aligned}$$

Therefore, $ex(x_2) \leq r-1$, contradicting the fact that r is the radius of G. □

Claim 4. Let $i \in \{2l+1, 2l+2, \ldots, r-2l-1\}$. Then $|N_i| \geq 10$.

Proof of Claim 4. Suppose to the contrary that $|N_i| \leq 9$ for some $i \in \{2l+1, 2l+2, \ldots, r-2l-1\}$. Let $z_i \in A_i$ be as in Lemma 3.1.7. Let x_2 denote the unique vertex of $T(z, z_i)$ which belongs to N_{2l+1}. We show that $ex(x_2) \leq r-1$. First let $y \in N_{\leq i-1}$. Then

$$\begin{aligned} d(x_2, y) &\leq d(x_2, z) + d(z, y) \\ &\leq 2l + 1 + i - 1 \\ &\leq 2l + 1 + r - 2l - 1 - 1 \\ &= r - 1. \end{aligned}$$

Now let $y \in N_{\geq i}$. Let $y_i \in T(z, y) \cap N_i$ so that $d(x_2, z_i) = i - 2l - 1$. By Lemma 3.1.7 we have $d(z_i, y_i) \leq 2l$. Also $d(y_i, y) \leq r - i$. It follows that

$$\begin{aligned} d(x_2, y) &\leq d(x_2, z_i) + d(z_i, y_i) + d(y_i, y) \\ &\leq i - 2l - 1 + 2l + r - i \\ &= r - 1. \end{aligned}$$

Therefore, $ex(x_2) \leq r-1$, contradicting the fact that r is the radius of G. □

We now complete the proof of the theorem. By Claim 1, Claim 2, Claim 3 and Claim 4, we have

$$\begin{aligned} p &= |N_0| + (|N_1| + \cdots + |N_l|) + (|N_{l+1}| + \cdots + |N_{\lfloor \frac{3l}{2} \rfloor}|) \\ &\quad + (|N_{\lfloor \frac{3l}{2} \rfloor + 1}| + \cdots + |N_{2l}|) + (|N_{2l+1}| + \cdots + |N_{r-2l-1}|) \\ &\quad + (|N_{r-2l}| + \cdots + |N_{r-\lfloor \frac{3l}{2} \rfloor - 1}|) + (|N_{r-\lfloor \frac{3l}{2} \rfloor}| + \cdots + |N_{r-l-1}|) \\ &\quad + (|N_{r-l}| + \cdots + |N_{r-1}|) + |N_r| \\ &\geq 1 + 5l + 6(\lfloor \frac{3l}{2} \rfloor - l) + 8(2l - \lfloor \frac{3l}{2} \rfloor) + 10(r - 4l - 1) + 8(2l - \lfloor \frac{3l}{2} \rfloor) \\ &\quad + 6(\lfloor \frac{3l}{2} \rfloor - l) + 5l + 1 \\ &= -10 - 16l + 10r, \end{aligned}$$

and (3.3) follows. □

The following graphs show that for fixed l the bound in Theorem 3.2.3 is best possible, apart from the value of the additive constant. For an even integer $k \geq 10$, let $G_1, G_2, \ldots, G_k$ be disjoint copies of the 5-cycle, C_5, and let $a_i, b_i, c_i, d_i, w_i \in V(G_i)$. Let G_k''' be the graph obtained from the union of $G_1, G_2, \ldots, G_k$ by adding the edges $a_{i+1}a_i, b_{i+1}b_i, c_{i+1}c_i, d_{i+1}d_i, w_{i+1}w_i, a_{i+1}w_i, b_{i+1}a_i, c_{i+1}b_i, d_{i+1}c_i, w_{i+1}d_i$ for $i = 1, 2, \ldots, k-1$ and a new vertex v_k adjacent to a_k, b_k, c_k, d_k and w_k. Furthermore let C_l be a cycle with vertices $j_1, j_2, \ldots, j_l$. Now join the graphs C_l and G_k''' by adding the edges $j_1w_1, j_lw_1, j_1a_1, j_2a_1, j_2b_1, j_3b_1, j_3c_1$, and j_id_1 for $i = 3, 4, \ldots, l$ thus obtaining a planar graph H_k''. Clearly, $p(H_k'') = 5k + l$ so that $k = \frac{p(H_k'')-l}{5}$. By a simple calculation, $\text{rad}(H_k'') = \frac{k}{2}$ and so $\text{rad}(H_k'') = \frac{p(H_k'')}{10} - \frac{l}{10}$. Figure 3.3 shows the graph H_k''.

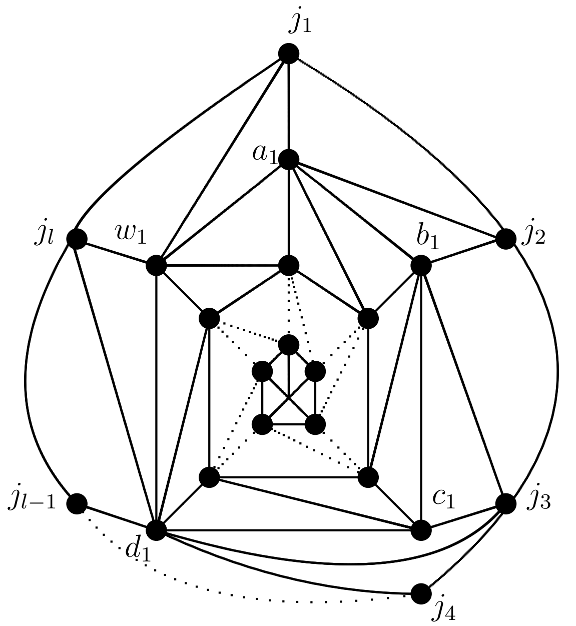

Figure 3.3: The graph H_k''.

Chapter 4

Steiner Diameter of Maximal Planar Graphs

4.1 Introduction

In this chapter we continue with the discussion on Steiner n-diameter started in Chapter 2. We consider maximal planar graphs. In particular, we give an upper bound on the Steiner n-diameter of a maximum planar graph of given order. Moreover, we construct graphs to show that the bounds are asymptotically sharp.

Let G be a maximal planar graph of order p. For vertices $y, z \in V(G)$, denote by $P(z, y)$, a z-y shortest path in G. If G is rooted at a vertex, say a_0, and $i \in \mathbb{N}_0$, then the *i-th* distance layer is the set

$$N_i := \{x \in V(G) | d_G(x, a_0) = i\}.$$

A vertex $x \in N_i$ is *active* if x has a neighbour in N_{i+1}. We denote by A_i the set of active vertices in N_i.

The following result, proved in Chapter 3, will help us prove the next corollary:

Lemma 4.1.1. *Let G be a 3-connected planar graph, and A_i be as above and let $1 \leq i \leq r-1$. If G is 3-connected and $u \in A_i$, then there exist distinct* *vertices $v, w \in A_i - \{u\}$ such that u and v share a face, and u and w share a face.*

Corollary 4.1.2. *Let G be a maximal planar graph, and A_i be as above and let $i \in \mathbb{N}_0$. If G is maximal planar and $u \in A_i$, then there exist two distinct vertices in $A_i - \{u\}$ both of which are adjacent to u.*

Proof. By Lemma 4.1.1, there exist two distinct vertices $v, w \in A_i - \{u\}$ such that u and v, and u and w, share a face. Since every face in a maximal planar graph is a triangle, u is adjacent to v and to w. $\square$

4.2 An Upper Bound on the Steiner Diameter of Maximal Planar Graphs

In this section we give an upper bound on the Steiner n-diameter in terms of order p.

Theorem 4.2.1. *Let G be a maximal planar graph of order p. If $2 \le n \le p$, then*

$$\operatorname{diam}_n(G) \le \frac{p}{3} + \frac{8n}{3} - 5. \tag{4.1}$$

Proof. Let $S = \{a_0, a_1, \ldots, a_{n-1}\}$ be a set of n vertices in G such that $d_n(S) = diam_n(G)$. Root G at a_0.

The idea of the proof is as follows: We construct a sequence $T_1 \le T_2 \le \ldots \le T_{n-1}$ of subtrees of G such that the vertices $a_0, a_1, \ldots, a_i$ are contained in T_i for $i \in \{1, 2, \ldots, n-1\}$. We further construct sets $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_{n-1}$ with $B_i \subseteq V(T_i)$ for $i = 1, 2, \ldots, n-1$ such that for each $v \in B_i$ there exists a set A_v, consisting of v and two neighbours of v not in T_i. We show that the sets A_v are pairwise disjoint, and that almost all vertices of T_i are in B_i, which implies that $|V(T_{n-1})|$ is only slightly greater than $\frac{p}{3}$, which in turn implies our bound on $\operatorname{diam}_n(G)$. We also construct sets $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_{n-1}$ with $C_i \subseteq V(T_i)$ and $C_i \cap B_i = \emptyset$ for $i = 1, 2, \ldots, n-1$.

We first consider a_1. Let T_1 be the tree $P(a_0, a_1)$. Let $B_1 = V(T_1) \backslash \{a_0, a_1\}$ and $C_1 = V(T_1) \backslash (B_1 \cup \{a_0, a_1\})$ so that $C_1 = \emptyset$. Assume that $a_0, a_1, \ldots, a_{i-1}$, have been considered and a tree T_{i-1} and sets B_{i-1} and C_{i-1} have been found. To incorporate a_i into T_i, we consider $P(a_0, a_i)$. Let $j \geq 1$ be the largest integer such that there exist $y \in V(P(a_0, a_i)) \cap N_j$, and $x \in V(T_{i-1})$ with $d_G(x, y) \leq 2$. We may assume that $P(y, a_i)$ is the y-a_i section of the path $P(a_0, a_i)$. We define T_i, B_i and C_i as follows:

i) $T_i = T_{i-1} \cup P(x, y) \cup P(y, a_i)$,

ii) $B_i = B_{i-1} \cup (V(P(y, a_i)) \backslash \{y, a_i\})$,

iii) $C_i = C_{i-1} \cup (V(P(x, y)) \backslash \{x\})$.

Note that for each $i = 1, 2, \ldots, n-1$,

$$|V(T_i)| \leq |B_i| + |C_i \cup \{a_0, a_1, \ldots, a_i\}|$$

which implies that

$$|B_i| \geq |V(T_i)| - |C_i \cup \{a_0, a_1, \ldots, a_i\}|.$$

Now the set C_1 is empty, and C_i has at most two vertices more than C_{i-1}. Hence the set C_i contains at most $2(i-1)$ vertices, and we have

$$\begin{aligned} |B_i| &\geq |V(T_i)| - (2i - 2 + i + 1) \\ &= |V(T_i)| - 3i + 1. \end{aligned} \tag{4.2}$$

Finally, let T be the tree T_{n-1}. From (4.2) we get

$$\begin{aligned} |B_{n-1}| &\geq |V(T_{n-1})| - 3(n-1) + 1 \\ &= |V(T)| - 3n + 4. \end{aligned}$$

Now each vertex in B_{n-1} is active. By Corollary 4.1.2, each vertex $v \in B_{n-1}$ is adjacent to two vertices v' and v'' in the same distance layer. Define $A_v := \{v, v', v''\}$. By our construction of T_{n-1}, the sets A_v for $v \in B_{n-1}$ are pairwise disjoint, and do not contain any vertex of S. Hence $\sum_{v \in B_{n-1}} |A_v| + |S| \leq p$ and so

$$3(|V(T)| - 3n + 4) + n \leq p$$

which implies that

$$|V(T)| \leq \frac{p}{3} + \frac{8n}{3} - 4.$$

The theorem now follows since $d_n(S) \leq |V(T)| - 1$.

The following graphs show that, for constant n, the bound in Theorem 4.2.1 is best possible, apart from the value of the additive constant. For an integer $k \geq 4$, let $G_1, G_2, \ldots, G_k$ be disjoint copies of the cycle C_3, and let $a_i, b_i, c_i \in V(G_i)$. Let G'_k be the graph obtained from the union of $G_1, G_2, \ldots, G_k$ by adding the edges $a_{i+1}a_i, b_{i+1}b_i, c_{i+1}c_i, a_{i+1}b_i, c_{i+1}b_i, a_{i+1}c_i$ for $i = 1, 2, \ldots, k-1$. Clearly, $p(G'_k) = 3k$ so that $k = \frac{p(G'_k)}{3}$. By a simple calculation, $\text{diam}_n(G'_k) = k + n - 3$ and so $\text{diam}_n(G'_k) = \frac{p(G'_k)}{3} + n - 3$. Figure 4.1 shows the graph G'_k. □

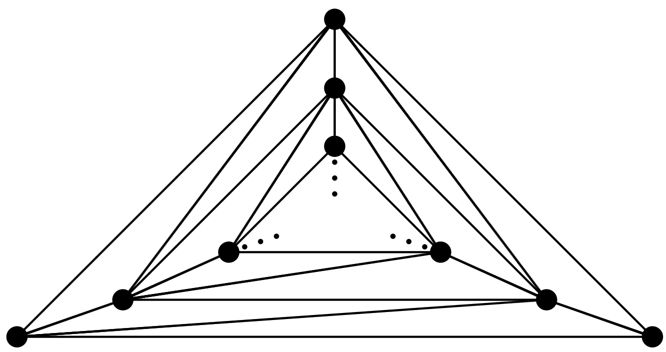

Figure 4.1: The graph G'_k.

The following example shows that in Theorem 4.2.1 it is essential that G is maximal planar. Let H be the cartesian product of K_2 and a cycle $C_{\frac{p}{2}}$, where p is even, i.e., let $V(H) = \{a_0, a_1, \ldots, a_{\frac{p}{2}-1}, b_0, b_1, \ldots, b_{\frac{p}{2}-1}\}$ and $E(H) = \{a_i a_{i+1} | i = 0, 1, \ldots, \frac{p}{2} - 1\} \cup \{b_i b_{i+1} | i = 0, 1, \ldots, \frac{p}{2} - 1\} \cup \{a_i b_i | i = 0, 1, \ldots, \frac{p}{2} - 1\}$ where subscripts are taken modulo $\frac{p}{2}$. If n divides $\frac{p}{2}$, then the set $S_n = \{a_i | i \in \{0, \frac{p}{2n}, 2\frac{p}{2n}, 3\frac{p}{2n}, (n-1)\frac{p}{2n}\}\}$ has n vertices and $d(S_n) = \frac{n-1}{n} \cdot \frac{p}{2}$. Hence $\text{diam}_n(H) \geq \frac{n-1}{2n} p$ which for constant $n \geq 4$ and large p is greater than

$\frac{p}{3}+\frac{8}{3}n-4$. Since H is planar and 3-connected, this shows that Theorem 4.2.1 does not hold for 3-connected planar graphs. Figure 4.2 shows the graph H and the set S_n for $p = 16$ and $n = 4$.

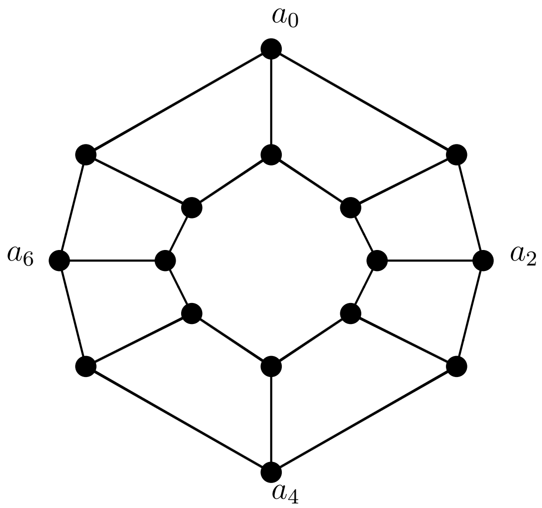

Figure 4.2: A 3-connected planar graph for which Theorem 4.2.1 does not hold.

Chapter 5

Conclusion

In this thesis we have determined upper bounds on the n-diameter and radius of a graph in terms of order, minimum degree and maximum face length. In Chapter 2 we established upper bounds, which are asymptotically sharp, on the n-diameter of all connected graphs, triangle-free and C_4-free graphs, in a sense, these generalise results by Erdös, Pach, Pollack, and Tuza [44]. In Chapter 3 we established upper bounds on the radius of a graph in terms of order and maximum face length. In particular, we found upper bounds, which can be attained, on the radius of k-connected planar graphs in terms of order and maximum face length thus improving results by Harant [62]. In Chapter 4 we provided an upper bound, which is asymptotically sharp, on the n-diameter of maximal planar graphs in terms of order.

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