PARTIAL EXCHANGEABILITY AND RELATED TOPICS

by

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PREFACE

The research described in this thesis was carried out in the Department of Mathematical Statistics, University of Natal, Durban, from February, 1984 to December, 1985 and from February, 1987 to December, 1991, under the supervision of Doctor A.I. Dale.

These studies represent original work by the authoress and have not been submitted in any form to another University. Where use was made of the work of others it has been duly acknowledged in the text.
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ABSTRACT

Partial exchangeability is the fundamental building block in the subjective approach to the probability of multi-type sequences which replaces the independence concept of the objective theory.

The aim of this thesis is to present some theory for partially exchangeable sequences of random variables based on well-known results for exchangeable sequences.

The reader is introduced to the concepts of partially exchangeable events, partially exchangeable sequences of random variables and partially exchangeable σ-fields, followed by some properties of partially exchangeable sequences of random variables.

Extending de Finetti's representation theorem for exchangeable random variables to hold for multi-type sequences, we obtain the following result to be used throughout the thesis:

There exists a σ-field, conditional upon which, an infinite partially exchangeable sequence of random variables behaves like an independent sequence of random variables, identically distributed within types.

Posing (i) a stronger requirement (spherical symmetry) and (ii) a weaker requirement (the selection property) than partial exchangeability on the infinite multi-type sequence of random variables, we obtain results related to de Finetti's representation theorem for partially exchangeable sequences of random variables.

Regarding partially exchangeable sequences as mixtures of independent and identically distributed (within types) sequences, we (i) give three possible expressions for the directed random measures of the partially exchangeable sequence and (ii) look at three possible expressions for the σ-field mentioned in de Finetti's representation theorem.

By manipulating random measures and using de Finetti's representation
theorem, we point out some concrete ways of constructing partially exchangeable sequences.

The main result of this thesis follows by extending de Finetti's representation theorem in conjunction with the Chatterji principle to obtain the following result:

Given any a.s. limit theorem for multi-type sequences of independent random variables, identically distributed within types, there exists an analogous theorem satisfied by all partially exchangeable sequences and by all sub-subsequences of some subsequence of an arbitrary dependent infinite multi-type sequence of random variables, tightly distributed within types.

We finally give some limit theorems for partially exchangeable sequences of random variables, some of which follow from the above mentioned result.
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CHAPTER 1
INTRODUCTION

The phrase events are independent with unknown probability $p$ is of no value in the subjective approach to probability, since learning through experience is of prime importance. To the objectivist independence is of fundamental importance. Clearly the subjectivist needed a basic concept to replace the independence concept of the objective approach.

Should independence be relaxed the next simplest thing is to continue to regard the order of the events as irrelevant, i.e. exchangeable (also called symmetric, permutable) events. Exchangeability, from a probabilistic point of view, was first introduced by J. Haag (1924a, 1928). He gave a slight indication (1924b, 1928) of the well-known representation theorem for exchangeable events (see Chapter 3). A precise statement and rigorous proof of this theorem was given by de Finetti (1932), following a paper he published in 1931 characterizing all stochastic processes which could be regarded as mixtures of coin tossing processes. This result was generalized by de Finetti (1937) so as to characterize mixtures of sequences of independent, identically distributed random variables. Not until the publication of this paper was the importance of the concept of exchangeability in the subjective approach to probability first noted. (A translation of this paper appears in Kyburg & Smokler (1964).)

Exchangeability is the fundamental building block in the subjective approach to probability which replaces the independence concept of the objective theory. Good background reading can be found in Link (1980), Hamaker (1977) and Goldstein (1986).

In 1938 de Finetti introduced the idea of partial exchangeability (see Chapter 2), often more appropriate than exchangeability. Partial exchangeability has not, to date, received anywhere near as much attention as exchangeability has. This thesis aims to provide some theory for partially exchangeable sequences of random variables, leading to limit theorems by extending well-known results for exchangeable sequences.
In Chapter 2 examples, definitions and properties of partially exchangeable sequences will be given.

Chapter 3 deals with de Finetti's theorem, extended to hold for partially exchangeable sequences. The proof, limitations and implications of this important theorem are thoroughly discussed.

Viewing partially exchangeable sequences as mixtures in Chapter 4 we obtain many more properties. Towards the end of this chapter, we construct some partially exchangeable sequences.

The definition of statutes in Chapter 5 leads to a whole new approach to partially exchangeable sequences. In this chapter we obtain an extension of Chatterji's principle (to allow multi-type sequences) and then give some limit theorems for partially exchangeable sequences.

**TERMINOLOGY AND NOTATION**

The following terminology and notation will be used throughout this thesis:

\[
\begin{align*}
\mathbb{R} &= \{\text{reals}\} \\
\mathbb{N} &= \{1, 2, 3, \ldots\} \\
|k| &= \{1, 2, 3, \ldots, k\} \\
B(\mathbb{R}) &= \text{denotes the class of Borel sets in } \mathbb{R}.
\end{align*}
\]

Unless otherwise indicated $S$ will denote a separable metrisable space. Topological spaces will be equipped with the $\sigma$-field generated by the open sets and product spaces will be given product topologies and product $\sigma$-fields.

Let $C(S)$ ($L^\infty(S)$) be the set of continuous (measurable), bounded, real valued functions on $S$.

$P(S)$ will denote the space of probability measures on $S$, equipped with the weak topology

\[
\lambda_n \Rightarrow \lambda
\]
if and only if

$$\lim_{n} \int f \, d\lambda_n = \int f \, d\lambda \quad \forall f \in C(S). \quad (1.1)$$

Note that $P(S)$ is itself a metrisable space (Parthasarathy (1967), Theorem 6.2). For measurable $A \subset \mathbb{R}$, the map $\lambda \rightarrow \lambda(A)$ from $P(\mathbb{R}) \rightarrow \mathbb{R}$ is measurable. Conversely (Jagers (1974)), if $\mathbb{R}_0$ is a dense subset of $\mathbb{R}$, then the collection of maps

$$\lambda \rightarrow \lambda([(-\infty, x]]) \quad x \in \mathbb{R}$$

generates the $\sigma$-field on $P(\mathbb{R})$.

Throughout this thesis $(\Omega, A, P)$ will denote a probability space. A measurable function $T : \Omega \rightarrow S$ will be called a random measure when $S = P(\mathbb{R})$; a random variable when $S = \mathbb{R}$ and in general a random map.

We shall denote $\{\omega; X(\omega) \in A\}$ by $[X \in A]$ for arbitrary random variable $X$ and Borel set $A$.

Let $L(T)$ denote the joint distribution of a countable sequence of random maps $T = \{T_{ij}; \ i \in [g], \ j \in \mathbb{N}\}$, for some finite $g \in \mathbb{N}$. Write $T_{in} \overset{D}{\rightarrow} T_i$ for $L(T_{in}) \Rightarrow L(T_i), \ \forall i \in [g]$. Write $\mathcal{F}(T)$ for the $\sigma$-field generated by $T$ and $I(A)$ for the indicator function of event $A$. For $s \in S$, write $\delta_s$ for the measure $\delta_s(A) = I(s \in A)$.

Let $g \in \mathbb{N}$ be an arbitrary, finite number. By the term $g$-fold infinite sequence we shall mean a sequence which contains an infinite number of items of each of $g$ types. A $g$-IID sequence shall mean a $g$-fold infinite sequence of independent random variables, identically distributed within types.

Vector $X$ and matrix $A$ will be denoted by $X$ and $A$ respectively.

Statements of theorems and various other results referred to in the thesis will be given in Appendix A1. Conditional independence is discussed in Appendix A2.
CHAPTER 2
PARTIALLY EXCHANGEABLE SEQUENCES

The aim of this chapter is to introduce the concept of partially exchangeable sequences for use in further chapters. This is done by giving definitions and examples of sequences of exchangeable events, partially exchangeable events, partially exchangeable random variables, the partially exchangeable $\sigma$-field, proving properties of partially exchangeable sequences of random variables and looking at sequences related to partially exchangeable sequences of random variables.

2.1 DEFINITIONS AND EXAMPLES

The following example gives an idea of what is meant by exchangeable events, an idea first introduced by J. Haag (1924a, 1928).

Consider a sequence of tosses of a single coin. There are many cases in which probabilities within each frequency group are equal. Suppose, for example, the coin is tossed 9 times and $HTHHTHTTH$ is obtained ($H =$ head, $T =$ tail). The judgement of the probability of the event head on the $10^{th}$ toss is likely to be affected by the frequency of heads and tails in the previous 9 tosses and not by the particular order in which the heads and tails were obtained. We are thus saying that all different sequences of 5 heads and 4 tails have the same probability and will all result in the same influence on the $10^{th}$ toss. This leads to the definition of exchangeable events.

Definition 2.1 (subjective viewpoint)
A sequence of events is said to be exchangeable if the events are symmetric in relation to our judgement of probability, i.e. the probability that we assign to a particular $n$ of these events occurring depends only on $n$ and not on the particular events chosen.

The corresponding definition in the objective theory is:

Definition 2.2 (objective viewpoint)
A sequence of events is exchangeable if the probability that any $n$ of these events occur depends only on $n$ and not on the particular events considered.
Definitions, theorems, etc. will be considered in an objective context, the corresponding subjective results follow by the obvious changes as demonstrated by Definitions 2.1 and 2.2.

Now suppose $g$ coins to be tossed. There are three different cases to be considered:

(i) If the coins are perfectly equal, then an exchangeable sequence of events will be generated.

(ii) The extreme opposite case is that in which the $g$ coins are all completely different. Each of the $g$ coins will generate a sequence of exchangeable events, with complete independence between the $g$ sequences.

(iii) Between these two cases lies an intermediate case: The outcomes of trials with one coin will influence the probability with respect to trials with other coins, but this influence is in a less direct manner than in (i), i.e. we have $g$ exchangeable sequences as in (ii) but with some interdependence between the sequences.

Case (iii) above leads to the definition of a partially exchangeable sequence of events.

**Definition 2.3**
A sequence of events is said to be $g$-fold partially exchangeable if the sequence of events splits into $g$ types (i.e. a $g$-fold sequence of events) and events of the same type are exchangeable (permutable, symmetric) in relation to probabilities, i.e. the joint probability of the occurrence of a particular $n_i$ events of type $i$, $i \in [g]$, depends only on the $n_i$'s, and not on the particular events chosen.

**Remark**
Note that a $g$-fold partially exchangeable sequence reduces to an exchangeable sequence in the $g = 1$ case.
Definition 2.4
A g-fold sequence of random variables, \(Z = \{Z_{ij}; i \in [g], j \in \mathbb{N}\}\), is said to be \(g\)-fold partially exchangeable if the joint distribution of any \(n_i\) random variables of type \(i, i \in [g]\), depends only on the \(n_i\)'s, and not on the particular random variables chosen.

Note that the term "g-fold" may be dropped in future discussions.

Attention will be restricted to a study of partially exchangeable sequences of random variables rather than partially exchangeable events since partially exchangeable events can be found as a special case of partially exchangeable random variables, by making use of indicator functions as demonstrated below:

Suppose that \(E = \{E_{ij}; i \in [g], j \in \mathbb{N}\}\) is an infinite \(g\)-fold sequence of events. Let \(Z_{ij} = I(E_{ij})\), \(\forall i \in [g], \forall j \in \mathbb{N}\).

Then \(E\) is a partially exchangeable sequence of events if the corresponding \(g\)-fold infinite sequence of random variables, \(Z = \{Z_{ij}; i \in [g], j \in \mathbb{N}\}\), is partially exchangeable.

Now let us give a detailed definition of a \(g\)-fold partially exchangeable sequence of random variables for the finite as well as the infinite case.

Definition 2.5
\(Z = \{Z_{ij}; j \in [n_i], i \in [g]\}\) is said to be a \((g\text{-fold})\) finite partially exchangeable sequence of random variables if

\[
L(Z_\pi) = L(Z)
\]

where

\[
Z_\pi = \{Z_{i\pi(j)}; j \in [n_i], i \in [g]\}
\]

\(n_i < \infty, n_i \in \mathbb{N} \quad \forall i \in [g]\)

\(\pi_i\) is any permutation of \(\{1, 2, \ldots, n_i\}\) \(\forall i \in [g]\).

In order to get the corresponding result for an infinite sequence of random variables we need the following definition.
Definition 2.6
A finite permutation of \( \{1, 2, \ldots \} \) is a map \( \pi : \mathbb{N} \to \mathbb{N} \) such that
\[
\# \{ i; \pi(i) \neq i \} < \infty
\]
i.e. \( \pi \) only permutes finitely many elements of the sequences, infinitely many retaining their original positions.

Definition 2.7
\( Z = \{ Z_{ij} ; i \in [g], j \in \mathbb{N} \} \) is said to be a \((g\text{-fold})\) infinite partially exchangeable sequence of random variables if
\[
L(Z) = L(Z_{x})
\]
where
\[
Z_{x} = \{ Z_{i\pi(j)} ; i \in [g], j \in \mathbb{N} \}
\]
for each finite permutation \( \pi \) of \( \{1, 2, \ldots \} \), \( i \in [g] \).

Suppose that there are \( g \) urns, urn \( i \) containing \( n_{i} \) balls, labelled \( z_{1}, z_{2}, \ldots, z_{n_{i}} \), \( \forall i \in [g] \). Let \( Z_{ij} \) denote the result of the \( j \)th draw from urn \( i \), then

(i) an infinite partially exchangeable sequence of random variables is formed by \( \{ Z_{ij} ; i \in [g], j \in \mathbb{N} \} \) when drawing with replacement from the urns.

(ii) a finite partially exchangeable sequence of random variables is formed by \( \{ Z_{ij} ; j \in [n_{i}], i \in [g] \} \) when drawing without replacement from the urns.

Case (i) follows immediately from Definition 2.7. In order to obtain the desired result for case (ii) note that \( Z = \{ Z_{ij} ; j \in [n_{i}], i \in [g] \} \) can have any of \( \prod_{i=1}^{g} n_{i}! \) outcomes, assigning a different \( z = \{ z_{ij} ; j \in [n_{i}], i \in [g] \} \) to \( Z \). Now let \( \Pi^{*}_{i} \) be a variable, denoting a permutation of \( \{1, 2, \ldots, n_{i}\} \) \( \forall i \in [g] \), so that \( \Pi^{*} = \prod_{i=1}^{g} \Pi^{*}_{i} \) denotes a random variable which has \( \prod_{i=1}^{g} n_{i}! \) outcomes, all equally probable. We may thus write
\[
Z = \{ z_{i\pi^{*}(j)} ; j \in [n_{i}], i \in [g] \}
\]
and have indexing by means of a variable.

Now let \( \pi_i \) be any of the \( n_i! \) permutations of \( \{1, 2, \ldots, n_i\}, i \in [g] \), then

\[
L(Z) = L \left( \left\{ z_{i\pi_i(j)}; j \in [n_i], i \in [g] \right\} \right)
\]

\[
= L \left( \left\{ z_{i\pi_i(j)}; j \in [n_i], i \in [g] \right\} \right)
\]

\[
= L \left( \{Z_i\} \right)
\]

where \( Z_i \) is as in Definition 2.5. Hence the desired result follows from Definition 2.5.

For the time being attention will be restricted to infinite partially exchangeable sequences of random variables; asymptotic results will be discussed in Chapter 5.

On extending the notion of an exchangeable \( \sigma \)-field (Aldous (1985) and Taylor, Daffer and Patterson (1985)) a definition for a partially exchangeable \( \sigma \)-field is obtained for use in further chapters. The following preliminary definition is required:

**Definition 2.8**

Call a subset \( B \) of \( \prod_{i=1}^{g} \mathbb{R} \) **partially exchangeable** if

\[
x = \{x_{ij}; i \in [g], j \in \mathbb{N}\} \in B \Rightarrow x_\pi = \{x_{i\pi_i(j)}; i \in [g], j \in \mathbb{N}\} \in B
\]

for each finite permutation \( \pi_i \) of \( \{1, 2, \ldots\}, i \in [g] \).

The corresponding \( B \)-**partially exchangeable event** is then

\[
|X \in B| = \{\omega; X(\omega) \in B\} = \{\omega; \{X_{ij}(\omega); i \in [g], j \in \mathbb{N}\} \in B\}
\]

**Definition 2.9**

Given a \( g \)-fold infinite sequence of random variables, \( X = \{X_{ij}; i \in [g], j \in \mathbb{N}\} \) the **partially exchangeable \( \sigma \)-field**, \( E_X \), is the set of all \( B \)-partially exchangeable events.
Remark
The exchangeable σ-field for a sequence of random variables \( X = \{ X_i; i \in \mathbb{N} \} \) may be found from the partially exchangeable σ-field by taking the special case \( g = 1 \).

2.2 SOME PROPERTIES OF PARTIALLY EXCHANGEABLE RANDOM VARIABLES

In this section some basic properties of partially exchangeable sequences are presented for use in further chapters. To simplify the notation we consider the \( g = 2 \) case only, the general theory follows by the obvious adjustments to the results. Throughout this section \( Z = \{ Z_{ij}; i \in [2], j \in \mathbb{N} \} \) will denote a 2-fold infinite sequence of random variables.

Theorem 2.1
If \( Z \) is partially exchangeable then \( Z \) is identically distributed within types.

Proof
\[
\forall i \in [2], Z_i = \{ Z_{ij}; j \in \mathbb{N} \} \text{ is exchangeable, and hence } \forall a, b \in \mathbb{R}, \forall j \in \mathbb{N},
\]
\[
P(Z_{i1} \leq a) = \lim_{b \to -\infty} P(Z_{i1} \leq a, Z_{i1} \leq b)
\]
\[
= \lim_{b \to -\infty} P(Z_{ij} \leq a, Z_{i1} \leq b)
\]
\[
= P(Z_{ij} \leq a)
\]

The following counter example demonstrates that a partially exchangeable sequence need not be independent. The example is numbered for future reference.

Example 2.1
Let \( Y = \{ Y_{ij}; i \in [2], j \in \mathbb{N} \} \) be a 2-IID sequence of random variables, and let \( Y_1 \) and \( Y_2 \) be two random variables, independent of \( Y \) and also of each other. Let
\[
Z_{ij} = Y_{ij} + Y_i
\]
\[\text{(2.1)}\]
where \( E(Y_{ij}) = 0 = E(Y_i) \) \( \forall i \in [2], \forall j \in \mathbb{N} \).
Then \( Z = \{Z_{ij}; i \in [2], j \in \mathbb{N}\} \) is a partially exchangeable sequence of random variables, and hence (Theorem 2.1) is identically distributed within types, so that, \( \forall i \in [2], \forall j \in \mathbb{N}, \)

\[
\var (Z_{ij}) = \var (Y_{i1}) + \var (Y_i)
\]

\[
\cov (Z_{i1}, Z_{i2}) = E (Z_{i1}Z_{i2}) - E^2 (Z_{i1})
\]

Using (2.1) and (2.2) we thus obtain

\[
\cov (Z_{i1}, Z_{i2}) = E (Y_i^2) - E^2 (Y_i)
\]

\[
= \var (Y_i)
\]

From this last result we clearly only have independence between \( Z \) variables of the same type if \( \var (Y_i) = 0 \) \( \forall i \in [2] \), which from the construction of the partially exchangeable sequence, \( Z \) in (2.1) need not generally be true. We are thus able to construct a partially exchangeable sequence which does not have independent random variables since the random variables need not even be independently distributed within types.

**Remark**

It follows immediately from the basic definitions that a 2-IID sequence is partially exchangeable: that the reverse need not hold follows from Theorem 2.1 and Example 2.1. The identical distribution within types of a partially exchangeable sequence follows from Theorem 2.1 but the independence of the random variables cannot be obtained in general. There is, however, a partial solution to this problem as will be seen in §3.1.

Partially exchangeable sequences do however partake of some of the properties of 2-IID sequences as the following example shows.

Let \( \{Z_{ij}; i \in [2], j \in \mathbb{N}\} \) be a partially exchangeable sequence and let

\[
Z = \{Z_{ij}; j \in [n_i], i \in [2]\}
\]

\[
Z_\pi = \{Z_{i\pi(i)}; j \in [n_i], i \in [2]\}
\]

where \( \forall i \in [2], n_i \in \mathbb{N}, \pi_i \) denotes any one of the \( n_i! \) permutations of \( \{1, 2, \ldots, n_i\} \).
Suppose that $X = \phi(Z)$ is a function of $\{Z_{ij}; j \in [n_i], i \in [2]\}$. Then clearly

$$L(X) = L(X_\pi)$$

where

$$X_\pi = \phi(Z_\pi).$$

We thus obtain

$$E(X) = (n_1!n_2!)^{-1} \sum_{\pi} E(X_\pi) = E[\psi(Z)]$$

where $\psi(Z) = (n_1!n_2!)^{-1} \sum \phi(Z_\pi)$ and the summation extends over all $n_1!n_2!$ possible permutations $\Pi_i$ of $\{1, 2, \ldots, n_i\}, i \in [2]$.

It immediately follows that if $\psi(Z) = 0$ then $E[\phi(Z)] = 0$, a result which is well-known when $Z$ is a 2-IID sequence. The relationship between partially exchangeable and 2-IID sequences will receive much more attention in further chapters, especially in §3.3.

Another interesting result which may be used to generalize the results of the following chapters is given in the next theorem.

**Theorem 2.2**

Let $Z = \{Z_{ij}; i \in [2], j \in N\}$ be a partially exchangeable sequence of random variables. If $h_i : R \rightarrow R$, $i \in [2]$, is a Borel measurable function, then $\{h_i(Z_{ij}); i \in [2], j \in N\}$ is a partially exchangeable sequence of random variables.

**Proof**

For any finite permutation $\Pi_i$ of $\{1, 2, \ldots\}, i \in [2]$ and Borel subsets $\{B_{ij}; i \in [2], j \in N\}$,
Remark
Note that by taking $h_i = h$ for $i \in [2]$ in Theorem 2.2 we see that a Borel function of a partially exchangeable sequence preserves the partial exchangeability.

2.3 SEQUENCES RELATED TO PARTIALLY EXCHANGEABLE SEQUENCES OF RANDOM VARIABLES

A brief discussion of the selection property, spherical symmetry and partially exchangeable arrays of random variables now follows.

The selection property for a sequence of random variables (Kingman (1978)) is extended below to permit 2-fold sequences.

Definition 2.10
A 2-fold sequence of random variables, $X = \{X_{ij}; i \in [2], j \in \mathbb{N}\}$, has the selection property if, for all integers

$$1 \leq m_1 < m_2 < \cdots < m_{k_i}, \quad \forall i \in [2], k_i \in \mathbb{N}.$$  

$$L \{X_{im_j}; j \in [k_i], i \in [2]\} = L \{X_{ij}; j \in [k_i], i \in [2]\}$$

where it is to be understood that the values of $m_j$ in $X_{1m_j}$ and $X_{2m_j}$ might differ, for any $j \in \text{minimum \{k_1, k_2\}}$.

The following theorem presents the relationship between partial exchangeability and the selection property.
Theorem 2.3
Suppose \( X \) to be a sequence of random variables which splits into two types.

(i) If \( X \) is a finite sequence, then \( X \) has the selection property if \( X \) is partially exchangeable. The reverse need not hold.

(ii) If \( X \) is an infinite sequence (infinitely many of each type, i.e. a 2-fold infinite sequence), then \( X \) has the selection property if and only if \( X \) is partially exchangeable.

Proof
It immediately follows from Definitions 2.5, 2.7 and 2.10 that \( X \) has the selection property if it is partially exchangeable (for finite and infinite sequences \( X \)). It thus remains to be shown that

(a) for \( X \) a finite sequence, \( X \) need not be a partially exchangeable if it has the selection property

(b) for \( X \) an infinite sequence, \( X \) is partially exchangeable if it has the selection property.

Consider case (a), i.e. let \( X = \{X_{ij}; j \in [n_i], n_i \in \mathbb{N}, i \in [2]\} \) have the selection property. For \( m_i < n_i \ \forall i \in [2] \), let \( \Pi_i \) be any permutation of \( \{1, 2, \ldots, n_i\} \) where

\[
1 \leq \Pi_i(1) < \Pi_i(2) < \cdots < \Pi_i(m_i) \quad \forall i \in [2]
\]

does not hold. Then \( \{X_{ij}; j \in [m_i], i \in [2]\} \) need not have the same distribution as \( \{X_{i\tau(j)}; j \in [m_i], i \in [2]\} \), i.e. \( X \) need not be partially exchangeable.

Case (b) will be proved in Chapter 3 (Theorem 3.4) since it follows on from an extension of the representation theorem as proved by Kingman (1978).

Remark
Comparing Definitions 2.7 and 2.10 it would seem that the requirement for \( X \), a 2-fold sequence of random variables, to have the selection property is a weaker requirement than partial exchangeability. This is in fact true if \( X \) is finite (Theorem 2.3 (i)), but if \( X \) has infinitely many variables of
each type then extending the concept of spherical symmetry (discussed by Kingman (1972 and 1978)) to permit 2-fold sequences, we obtain a stronger requirement than partial exchangeability.

Let $X$ and $Y$ be two-fold infinite sequences of random variables and let $X_m(Y_m)$ denote finite sequences of random variables, containing $m_i$ random variables of type $i$, $m_i \in \mathbb{N}, i \in [2]$, from $X(Y)$, i.e.

$$X_m = \{X_{ij}; j \in [m_i], i \in [2]\}$$

$$Y_m = \{Y_{ij}; j \in [m_i], i \in [2]\}.$$ 

Suppose $A = \{a_{ij}; i \in [m_1 + m_2], j \in [m_1 + m_2], a_{ij} \in \mathbb{R}\}$ to be an array. If

$$Y_m = X_m \widetilde{A}$$

where

$$\widetilde{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

for $A_1 = \{a_{ij}; i \in [m_1], j \in [m_1]\}$

$$A_2 = \{a_{ij}; i \in [m_2], j \in [m_2]\}$$

then $Y_m$ is a linear transformation of $X_m$.

Assume the above terminology until further notice.

**Definition 2.11**
In the above notation, let $Y_m$ be a linear transformation of $X_m$. If $\widetilde{A}$ is orthogonal with determinant 1 then $\widetilde{A}$ is a rotation array and $Y_m$ is a rotation of $X_m$.

**Definition 2.12**
$X_m$ is said to have (2-fold) spherical symmetry if the joint distribution of $X_m$ is the same as the joint distribution of $Y_m$ whenever $Y_m$ is a rotation of $X_m$. 
Note that the term 2-fold may be dropped in further discussions.

\( \mathbf{X} \) is said to have spherical symmetry if \( \mathbf{X}_m \) has spherical symmetry for all \( m_i \in \mathbb{N}, i \in [2] \).

**Remark**

It immediately follows from Definition 2.12 that if \( \mathbf{X} \) has spherical symmetry, then \( \mathbf{X} \) also has spherical symmetry within types, i.e. \( \{X_{ij}; j \in \mathbb{N}\} \) has spherical symmetry for all \( i \in [2] \).

The following theorem shows that spherical symmetry implies partial exchangeability.

**Theorem 2.4**

If \( \mathbf{X} \) has spherical symmetry, then \( \mathbf{X} \) is partially exchangeable.

**Proof:**

Let \( \Pi_i \) be any permutation of \( \{1, 2, \ldots, m_i\}, m_i \in \mathbb{N}, i \in [2] \). Then the linear mapping, \( \Pi \), defined by

\[
\Pi : \mathbb{R}^{m_1 + m_2} \rightarrow \mathbb{R}^{m_1 + m_2}; \quad \Pi(X_{ij}) = X_{i\pi_i(j)} \quad \forall j \in [m_i], \forall i \in [2]
\]

may be viewed as a linear transformation with a rotation matrix and hence \( \{X_{ij}; j \in [m_i], i \in [2]\} \) has the same joint distribution as \( \{X_{i\pi_i(j)}; j \in [m_i], i \in [2]\} \).

Since this holds for any \( m_i \in \mathbb{N} \), we thus have \( \mathbf{X} \) to be partially exchangeable.

In Chapter 3 (Theorem 3.5) a representation theorem for spherically symmetric sequences is given.

Aldous (1981) dealt with partially exchangeable arrays of random variables. Even though it is beyond the scope of this thesis to deal with arrays, the definitions below are given as interesting extensions of partial exchangeability.
Definition 2.13
For the random array $\mathbf{X} = \{X_{ij}; i \in \mathbb{N}, j \in \mathbb{N}\}$, say $\mathbf{X}$ is

(i) **row-exchangeable**, if the joint distribution of $\mathbf{X}$ is unaltered by permutations within rows

(ii) **column-exchangeable**, if the joint distribution of $\mathbf{X}$ is unaltered by permutations within columns

(iii) **row-and-column-exchangeable** if the joint distribution of $\mathbf{X}$ is unaltered by permutations within rows and columns.

Remark
Partial exchangeability clearly follows as a special case of row-exchangeability where only finitely many rows are allowed (the $i \in [2]$ case is considered in this thesis).

Next we define triangular arrays of exchangeable random variables, a very interesting type of exchangeability.

Definition 2.14
An array of random variables $\mathbf{X} = \{X_{ij}; i \in [i], j \in \mathbb{N}\}$ is a **triangular array of exchangeable random variables** if each row of $\mathbf{X}$ forms an exchangeable sequence.

Remark
CHAPTER 3

A REPRESENTATION THEOREM FOR PARTIALLY EXCHANGEABLE RANDOM VARIABLES

This chapter is devoted to proving de Finetti's representation theorem for partially exchangeable sequences which gives a relationship between a partially exchangeable sequence, $Z = \{Z_{ij}; i \in [2], j \in N\}$, and a 2-IID sequence of random variables. Conditions under which the representation theorem holds are discussed and finally further results concerning the tie-up between partially exchangeable and 2-IID sequences of random variables are given.

3.1 DE FINETTI'S REPRESENTATION THEOREM FOR MULTI-TYPE SEQUENCES OF RANDOM VARIABLES

Let $X_1, X_2, \ldots$ be a sequence of random variables taking values in $\{0, 1\}$. De Finetti's representation theorem (de Finetti (1932)) shows that $\{X_1, X_2, \ldots\}$ is exchangeable iff $\{X_1, X_2, \ldots\}$ is a mixture of coin tossing processes, i.e. $\forall n \in N$, and all strings $x_1, x_2, \ldots, x_n$ of 0's and 1's,

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \int p^t(1-p)^{n-t} \mu(dp)$$

where $t = \sum_{i=1}^{n} x_i$, $p$ is the probability of obtaining a 1 and $\mu$ is a probability on $[0, 1]$, uniquely determined by $P$.

This theorem has been generalized in several directions, the most famous of which undoubtedly is de Finetti's representation theorem for a partially exchangeable sequence of events (de Finetti (1937)) (see Appendix A1), which leads to the following statement of de Finetti's representation theorem for partially exchangeable random variables:

an infinite partially exchangeable sequence of random variables, $\{Z_{ij}; i \in [g], j \in N\}$, is a mixture of $g$-IID sequences.

Precisely what is meant by this mixture will be discussed in Chapter 4, but loosely, it means that there exists a $\sigma$-field conditional upon which the $g$-fold partially exchangeable sequence behaves like a $g$-IID sequence.
A precise statement and proof of this fundamental theorem for the \( g = 2 \) case follows in Theorem 3.1 (results reduce to work by Kingman (1978) for the \( g = 1 \) case).

Throughout this chapter let \( Z = \{Z_{ij}; i \in [2], j \in \mathbb{N}\} \) be an infinite 2-fold sequence of random variables.

**Definition 3.1**
A random variable is \((n, m)\)-symmetric, for \( n, m \in \mathbb{N} \), if it is a function of \( Z \) which is unchanged if the first \( n \) variables of type 1 and the first \( m \) variables of type 2 are permuted in any way (within types).

**Remark:**
Let \( \phi \) be a function of \( Z \) and let \( n, m \in \mathbb{N} \). From Definition 3.1 it immediately follows that if \( \phi \) is \((n, m)\)-symmetric, then it is also \((n - k, m - l)\)-symmetric for any \( k \in \{0, 1, \ldots, n - 1\} \) and any \( l \in \{0, 1, \ldots, m - 1\} \). Similarly, if \( \phi \) is not \((n, m)\)-symmetric, then it is not \((n + k, m + l)\)-symmetric, for any \( k, l \in \{0, 1, 2, \ldots\} \), \( k \) and \( l \) not both zero.

**Example**

(i) \[ \psi = (Z_{11} + Z_{12} + Z_{13}) Z_{21} + Z_{14}Z_{22} \]

is \((3,1)\)-symmetric but not \((4,1)\)- or \((3,2)\)-symmetric.

(ii) \[ \phi = Z_{11}Z_{12} (Z_{21} + Z_{22} + Z_{23} + Z_{24}) - Z_{25}Z_{13} \]

is \((2,4)\)-symmetric but not \((3,1)\)- or \((2,5)\)-symmetric.

**Definition 3.2**
Let \( \mathcal{F}_{nm} \) denote the \( \sigma \)-field generated by all \((n, m)\)-symmetric random variables and let

\[
\mathcal{F}_\infty = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{F}_{nm}. \tag{3.1}
\]

Note that (i) \( \mathcal{F}_{n+1,m} \subseteq \mathcal{F}_{nm} \) and \( \mathcal{F}_{n,m+1} \subseteq \mathcal{F}_{nm} \) \( \forall n, m \in \mathbb{N} \) \( \tag{3.2} \)

(ii) \( \mathcal{F}_\infty = \lim_{n \to \infty} \lim_{m \to \infty} \mathcal{F}_{nm}. \tag{3.3} \)
The following lemma presents a strong law of large numbers which will be used to prove a representation theorem for partially exchangeable random variables.

**Lemma 3.1**

Let $Z$ be an infinite sequence of partially exchangeable random variables and let $f$ be a measurable function for which $E|f(Z_{11})| < \infty$, $\forall i \in [2]$. Then

$$
\lim_{n \to \infty} n^{-1} \sum_{j=1}^{n} f(Z_{1j}) = E[f(Z_{11})/\mathcal{F}_{nm}] \text{ a.s. } \forall m \in \mathbb{N}
$$

and

$$
\lim_{m \to \infty} m^{-1} \sum_{j=1}^{m} f(Z_{2j}) = E[f(Z_{21})/\mathcal{F}_{nm}] \text{ a.s. } \forall n \in \mathbb{N}
$$

where

$$
\mathcal{F}_{nm} = \lim_{n \to \infty} \mathcal{F}_{nm} = \bigcap_{n=1}^{\infty} \mathcal{F}_{nm} \quad \forall m \in \mathbb{N}
$$

$$
\mathcal{F}_{nm} = \lim_{m \to \infty} \mathcal{F}_{nm} = \bigcap_{m=1}^{\infty} \mathcal{F}_{nm} \quad \forall n \in \mathbb{N}
$$

**Proof:**

Let $Y = g(Z_{11}, Z_{12}, \ldots, Z_{21}, Z_{22}, \ldots)$ be a bounded function which is $(n, m)$-symmetric for some $n, m \in \mathbb{N}$. Now $\{Z_{1j}; j \in \mathbb{N}\}$ is exchangeable and hence, $\forall j \leq n$,

$$
E[f(Z_{1j})Y] = E[f(Z_{1j})g(Z_{11}, Z_{12}, \ldots, Z_{21}, Z_{22}, \ldots)]
$$

$$
= E[f(Z_{11})g(Z_{1j}, Z_{12}, \ldots, Z_{1j-1}, Z_{21}, Z_{22}, \ldots, Z_{21}, Z_{22}, \ldots)]
$$

$$
= E[f(Z_{11})Y] .
$$

Hence

$$
\sum_{j=1}^{n} E[f(Z_{1j})Y] = nE[f(Z_{11})Y] .
$$

Now take $Y$ to be the indicator function of $A \in \mathcal{F}_{nm}$, so that

$$
\int_{A} n^{-1} \sum_{j=1}^{n} f(Z_{1j})dP = \int_{A} f(Z_{11})dP
$$
and hence

\[ E\left[ n^{-1} \sum_{j=1}^{n} f(Z_{1j}) / \mathcal{F}_{nm} \right] = E\left[ f(Z_{11}) / \mathcal{F}_{nm} \right] \text{ a.s. } \]

Since \( \sum_{j=1}^{n} f(Z_{1j}) \) is measurable w.r.t. \( \mathcal{F}_{nm} \),

\[ n^{-1} \sum_{j=1}^{n} f(Z_{1j}) = E\left[ f(Z_{11}) / \mathcal{F}_{nm} \right] \text{ a.s. } \]

For any \( m \in \mathbb{N} \), \( \{ \mathcal{F}_{nm}; n \in \mathbb{N} \} \) is a decreasing sequence of \( \sigma \)-fields, so that an elementary martingale convergence theorem (Appendix A1) yields

\[ \lim_{n \to \infty} n^{-1} \sum_{j=1}^{n} f(Z_{1j}) = E\left[ f(Z_{11}) / \mathcal{F}_{m} \right] \text{ a.s. } \]

which is the desired result, (3.4). Using variables of type 2, (3.5) follows similarly.

We are now in a position to prove a representation theorem for partially exchangeable random variables by following the line of thought adopted by Kingman (1978) but adapting his work to allow two types of random variables.

**Theorem 3.1**

Let \( Z = \{ Z_{ij}; i \in [2], j \in \mathbb{N} \} \) be a partially exchangeable sequence of random variables. Then

\[ P\left[ \cap_{i=1}^{2} \cap_{j=1}^{m_{i}} [Z_{ij} \leq z_{ij}] / \mathcal{F}_{\infty} \right] = \prod_{i=1}^{2} \prod_{j=1}^{m_{i}} P[Z_{ij} \leq z_{ij} / \mathcal{F}_{\infty}] \text{ a.s. } (3.7) \]

where \( m_{i} \in \mathbb{N}, z_{ij} \in \mathbb{R}, \forall i \in [2], \forall j \in [m_{i}] \).
Proof:
Let $f$ be a bounded measurable function on $\mathbb{R}^{m_1+m_2}$ and let $Y = g(Z)$ be $(n_1, n_2)$-symmetric for $n_i \geq m_i$, $n_i, m_i \in \mathbb{N}$, $\forall i \in [2]$. Now choose any $m_i$ variables from the first $n_i$ variables of type $i$, $i \in [2]$. Denote the chosen variables by $\{Z_{1\theta(1)}, \ldots, Z_{1\theta(m_1)}, Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)}\}$. There are clearly $\binom{n_1}{m_1} \binom{n_2}{m_2}$ ways of choosing the variables, the summation over all the possible choices will be denoted by $\Sigma_C$.

Due to the partial exchangeability of $Z$,

$$E \left[ f(Z_{1\theta(1)}, \ldots, Z_{1\theta(m_1)}, Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)}) g(Z) \right]$$

$$= E \left[ f(Z_{11}, \ldots, Z_{1m_1}, Z_{21}, \ldots, Z_{2m_2}) g'(Z) \right]$$

(3.8)

where $g'(Z)$ is just $g(Z)$ altered by interchanging $Z_{1\theta(k)}$ and $Z_{1k}$ $\forall k \in [m_1]$ and $Z_{2\alpha(k)}$ and $Z_{2k}$ $\forall k \in [m_2]$.

Since $g(Z)$ is $(n_1, n_2)$-symmetric, we thus obtain

$$E \left[ f(Z_{1\theta(1)}, \ldots, Z_{1\theta(m_1)}, Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)}) Y \right]$$

$$= E \left[ f(Z_{11}, \ldots, Z_{1m_1}, Z_{21}, \ldots, Z_{2m_2}) Y \right]$$

(3.9)

Any one of the $\binom{n_1}{m_1} \binom{n_2}{m_2}$ ways of choosing the variables would clearly satisfy (3.9). Now take $Y$ to be the indicator function of $A \in \mathcal{F}_{n_1, n_2}$, so that,

$$\int_A f(Z_{11}, \ldots, Z_{1m_1}, Z_{21}, \ldots, Z_{2m_2}) \, dP = \int_A \left[ \binom{n_1}{m_1} \binom{n_2}{m_2} \right]^{-1} \sum_C f(Z_{1\theta(1)}, \ldots, Z_{1\theta(m_1)}, Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)}) \, dP$$

and immediately then

$$E \left[ f(Z_{11}, \ldots, Z_{1m_1}, Z_{21}, \ldots, Z_{2m_2}) / \mathcal{F}_{n_1, n_2} \right]$$

$$= E \left\{ \left[ \binom{n_1}{m_1} \binom{n_2}{m_2} \right]^{-1} \sum_C f(Z_{1\theta(1)}, \ldots, Z_{1\theta(m_1)}, Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)}) / \mathcal{F}_{n_1, n_2} \right\} a.s.$$  

$$= \left[ \binom{n_1}{m_1} \binom{n_2}{m_2} \right]^{-1} \sum_C f(Z_{1\theta(1)}, \ldots, Z_{1\theta(m_1)}, Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)}) a.s. \, (3.10)$$
Using a martingale convergence theorem (Appendix A1), we get

\[\lim_{n_1, n_2 \to \infty} E \left[ f \left( Z_{11}, \ldots, Z_{1m_1}, Z_{21}, \ldots, Z_{2m_2} \right) \right] \]

\[= \lim_{n_1} \left\{ E \left[ f \left( Z_{11}, \ldots, Z_{1m_1}, Z_{21}, \ldots, Z_{2m_2} \right) \right] \bigg/ \mathcal{F}_{n_1} \right\} \text{ a.s.} \]

\[= E \left[ f \left( Z_{11}, \ldots, Z_{1m_1}, Z_{21}, \ldots, Z_{2m_2} \right) \right] / \mathcal{F}_\infty \text{ a.s.} \]

and hence, from (3.10),

\[\lim_{n_1, n_2 \to \infty} \left[ \left( \begin{array}{c} n_1 \\ m_1 \\ \end{array} \right) \left( \begin{array}{c} n_2 \\ m_2 \\ \end{array} \right) \right]^{-1} \sum_{C} f \left( Z_{\theta(1)}, \ldots, Z_{\theta(m_1), Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)} \bigg/ \mathcal{F}_\infty \right] \text{ a.s.} \quad (3.11)\]

If the \( m_i \) variables chosen from \( n_i \) for type \( i, i \in [2] \), allow repeats, i.e. need not all be distinct, then (3.11) becomes

\[\lim_{n_1, n_2 \to \infty} n_1^{-m_1} n_2^{-m_2} \sum_{D} f \left( Z_{\theta(1)}, \ldots, Z_{\theta(m_1), Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)} \bigg/ \mathcal{F}_\infty \right] \text{ a.s.} \quad (3.12)\]

where

\[\sum_{D} = \sum_{\theta(1)=1}^{n_1} \ldots \sum_{\theta(m_1)=1}^{n_1} \sum_{\alpha(1)=1}^{n_2} \ldots \sum_{\alpha(m_2)=1}^{n_2} \]

In particular if

\[f \left( Z_{11}, \ldots, Z_{1m_1}, Z_{21}, \ldots, Z_{2m_2} \right) = \prod_{i=1}^{2} \prod_{j=1}^{m_i} f_{ij}(z_{ij}) \quad ,\]

where

\[f_{ij}(z) = I \left[ Z_{ij} \leq z \right] \quad \forall z \in \mathbb{R}, \quad \forall j \in [m_i], \quad \forall i \in [2] \quad ,\]
then using (3.12),

\[
P \left[ \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} [Z_{ij} \leq z_{ij}] / \mathcal{F}_\infty \right]
\]

\[= E \left[ I \left( \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} [Z_{ij} \leq z_{ij}] \right) / \mathcal{F}_\infty \right] \text{ a.s.}
\]

\[= \lim_{n_1, n_2 \to \infty} n_1^{-m_1} n_2^{-m_2} \sum_D I \left( \left( \prod_{j=1}^{m_1} [Z_{1\theta(j)} \leq z_{1j}] \right) \cap \left( \prod_{j=1}^{m_2} [Z_{2\alpha(j)} \leq z_{2j}] \right) \right) \text{ a.s.}
\]

\[= \lim_{n_1, n_2 \to \infty} \left( \prod_{j=1}^{m_1} \left( n_1^{-1} \sum_{\theta(j)=1}^{n_1} I [Z_{1\theta(j)} \leq z_{1j}] \right) \right) \prod_{j=1}^{m_2} \left( n_2^{-1} \sum_{\alpha(j)=1}^{n_2} I [Z_{2\alpha(j)} \leq z_{2j}] \right) \text{ a.s. (3.13)}
\]

Using Lemma 3.1, we thus see that the right hand side of (3.13) may be replaced by

\[\prod_{j=1}^{m_1} \left\{ \lim_{n_2 \to \infty} E \left[ I (Z_{11} \leq z_{1j}) / \mathcal{F}_{n_2} \right] \right\} \prod_{j=1}^{m_2} \left\{ \lim_{n_1 \to \infty} E \left[ I (Z_{21} \leq z_{2j}) / \mathcal{F}_{n_1} \right] \right\}
\]

with probability one, and hence, using the martingale convergence theorem again (Appendix A1), we find that (3.13) becomes

\[
P \left[ Z_{11} \leq z_{11}, \ldots, Z_{1m_1} \leq z_{1m_1}, Z_{21} \leq z_{21}, \ldots, Z_{2m_2} \leq z_{2m_2} / \mathcal{F}_\infty \right]
\]

\[= \prod_{j=1}^{m_1} E \left[ I (Z_{11} \leq z_{1j}) / \mathcal{F}_\infty \right] \prod_{j=1}^{m_2} E \left[ I (Z_{21} \leq z_{2j}) / \mathcal{F}_\infty \right] \text{ a.s.}
\]

\[= \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} P \left[ Z_{ij} \leq z_{ij} / \mathcal{F}_\infty \right] \text{ a.s. .}
\]

**Remark**

Note that Theorem 3.1 shows that a partially exchangeable sequence \( Z = \{Z_{ij} : i \in [2], j \in \mathbb{N} \} \) is conditionally 2-IID given \( \mathcal{F}_\infty \), i.e. conditional on \( \mathcal{F}_\infty \) the partially exchangeable sequence behaves like a 2-IID sequence.
Corollary 3.1
Let \( Z = \{Z_{ij}; i \in [2], j \in \mathbb{N}\} \) be a partially exchangeable sequence with random distribution functions
\[
F_i(x) = P[Z_{ij} \leq x / \mathcal{F}_\infty] \quad x \in \mathbb{R}, i \in [2], j \in \mathbb{N}.
\]
If \( \mathcal{G} \) is any sub-\(\sigma\)-field of \( \mathcal{F}_\infty \) with respect to which \( F_1(x) \) and \( F_2(x) \) are measurable \( \forall x \in \mathbb{R} \), then
\[
P\left( \bigcap_{i=1}^{2} \bigcap_{j=1}^{m_i} [Z_{ij} \leq z_{ij}] / \mathcal{G} \right) = \prod_{i=1}^{2} \prod_{j=1}^{m_i} F_i(z_{ij})
\]
\( \forall m_i \in \mathbb{N}, \forall z_{ij} \in \mathbb{R}, \forall j \in \mathbb{N}, \forall i \in [2] \).

Proof:
Using Theorem 3.1,
\[
E \left\{ P \left( \bigcap_{i=1}^{2} \bigcap_{j=1}^{m_i} [Z_{ij} \leq z_{ij}] / \mathcal{F}_\infty \right) / \mathcal{G} \right\}
= E \left[ \prod_{i=1}^{2} \prod_{j=1}^{m_i} F_i(z_{ij}) / \mathcal{G} \right]
= \prod_{i=1}^{2} \prod_{j=1}^{m_i} F_i(z_{ij}).
\]

Remark
Taking \( \mathcal{G} = \mathcal{F}(F_1, F_2) \) in Corollary 3.1 we obtain
\[
P\left( \bigcap_{i=1}^{2} \bigcap_{j=1}^{m_i} [Z_{ij} \leq z_{ij}] / F_1, F_2 \right) = \prod_{i=1}^{2} \prod_{j=1}^{m_i} F_i(z_{ij})
\]
\( \forall m_i \in \mathbb{N}, \forall z_{ij} \in \mathbb{R}, \forall j \in \{m_i\}, \forall i \in [2] \), a form of de Finetti's representation theorem for partially exchangeable random variables which will be referred to in Chapter 4.

Using (3.16) we have a general method for constructing partially exchangeable sequences. First construct a sequence of \(2\)-IID random variables having common distribution functions within types \( F_1 \) and \( F_2 \). Then allow \( F_1 \) and
to vary randomly. The randomisation destroys the independence while preserving the partial exchangeability (see §4.3). Every infinite partially exchangeable sequence can be constructed in this way (see §4.3).

We now present a lemma and definition which we shall use in order to obtain an alternative representation theorem for partially exchangeable random variables.

**Lemma 3.2**  
Let $Y$ be a random variable and let $\mathcal{F}, \mathcal{G}$ be two $\sigma$-fields such that $\mathcal{F} \subset \mathcal{G}$. If  
\[
E[E(Y|\mathcal{G})]^2 = E[E(Y|\mathcal{F})]^2
\]
then  
\[
E[Y|\mathcal{G}] = E[Y|\mathcal{F}] \text{ a.s. .}
\]

**Proof**  
\[
E[E(Y|\mathcal{G}) - E(Y|\mathcal{F})]^2
\]
\[
= E[E(Y|\mathcal{G})]^2 - 2E[E(Y|\mathcal{G})E(Y|\mathcal{F})] + E[E(Y|\mathcal{F})]^2
\]
\[
= 0
\]
since  
\[
E[E(Y|\mathcal{G})E(Y|\mathcal{F})] = E\{E[E(Y|\mathcal{G})E(Y|\mathcal{F})/\mathcal{F}]\}
\]
\[
= E\{E(Y|\mathcal{F})E[E(Y|\mathcal{G})/\mathcal{F}]\}
\]
\[
= E[E(Y|\mathcal{F})E(Y|\mathcal{F})]
\]
\[
= E[E(Y|\mathcal{F})]^2 .
\]

**Definition 3.3**  
Let $Z$ be a 2-fold infinite sequence of random variables. The tail $\sigma$-field of $Z$ is defined by  
\[
\tau = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{F}_{nm}^\infty
\]
where
\[ \mathcal{F}_{nm}^\infty = \mathcal{F}(Z_{1,n+1}, Z_{1,n+2}, \ldots, Z_{2,m+1}, Z_{2,m+2}, \ldots) \quad \forall n, m \in \mathbb{N}. \] (3.17)

**Theorem 3.2**
A partially exchangeable sequence of random variables, \( Z \), is conditionally 2-IID given \( \tau \).

**Proof**
Let
\[ \mathcal{F}_{n}^\infty = \lim_{m} \mathcal{F}_{nm}^\infty \quad \forall n \in \mathbb{N} \]
and
\[ \mathcal{F}_{m}^\infty = \lim_{n} \mathcal{F}_{nm}^\infty \quad \forall m \in \mathbb{N} \]
Using Theorem 2.3 (ii) we see that, for \( m, n \in \mathbb{N} \),
\[ L(Z_{11}, Z_{12}, Z_{13}, \ldots, Z_{2m+1}, Z_{2m+2}, \ldots) = L(Z_{11}, Z_{1n+1}, Z_{1n+2}, \ldots, Z_{1m+1}, Z_{1m+2}, \ldots) \] (3.18)
and hence
\[ L[E(Z_{11}/\mathcal{F}_{1m}^\infty)] = L[E(Z_{11}/\mathcal{F}_{nm}^\infty)] . \] (3.19)
Using the martingale convergence theorem (Appendix A1) and noting that (3.19) holds \( \forall m, n \in \mathbb{N} \), we obtain the following
\[ L[E(Z_{11}/\mathcal{F}_{1m}^\infty)] = L[E(Z_{11}/\tau)] \] (3.20)
by taking the limit over all \( m \in \mathbb{N} \) and then over all \( n \in \mathbb{N} \). Since \( \tau \) is contained in \( \mathcal{F}_{1}^\infty \) we may use Lemma 3.2 to obtain
\[ E(Z_{11}/\mathcal{F}_{1}^\infty) = E(Z_{11}/\tau) \text{ a.s.} \] (3.21)
and thus
\[ E(Z_{11}/\tau, \mathcal{F}_{1}^\infty) = E(Z_{11}/\tau) \text{ a.s.} . \]
From this last result we conclude (Appendix A2, point A4 plus the remark below it) that \( Z_{11} \) and \( \mathcal{F}_{1}^\infty \) are conditionally independent given \( \tau \).

Applying the same argument to \( Z_{1s}, Z_{1s+1}, \) in place of \( Z_{11}, Z_{12}, \ldots \) one obtains, \( \forall s \in \mathbb{N} \),
\( Z_{1s} \text{ and } \mathcal{F}_s^\infty \text{ are conditionally independent given } \tau. \) \hspace{1cm} (3.22)

Using similar arguments we see that \( \forall t \in \mathbb{N} \),

\( Z_{2t} \text{ and } \mathcal{F}_t^\infty \text{ are conditionally independent given } \tau. \) \hspace{1cm} (3.23)

Let \( s \in \mathbb{N} \). Then

\[ \mathcal{F}_s^\infty \supseteq \mathcal{F}_{s+1}^\infty \hspace{1cm} \forall m \in \mathbb{N} \] \hspace{1cm} (3.24)

so that

\( Z_{1s} \text{ and } \mathcal{F}_s^\infty \text{ are conditionally independent given } \tau \hspace{1cm} \forall m \in \mathbb{N} \) \hspace{1cm} (3.25)

follows from (3.22). Similarly, from (3.23) we see that, for \( t \in \mathbb{N} \),

\( Z_{2t} \text{ and } \mathcal{F}_t^\infty \text{ are conditionally independent given } \tau \hspace{1cm} \forall n \in \mathbb{N}. \) \hspace{1cm} (3.26)

From (3.25), (3.26) and a property of conditional independence (Appendix A2, point A5) we see that \( Z \) is conditionally independent given \( \tau \).

We shall now show that \( Z \) is identically distributed within types given \( \tau \). Due to partial exchangeability, \( \forall m, n \in \mathbb{N} \),

\[ L(\{Z_{11}, Z_{1m+1}, Z_{1m+2}, \ldots Z_{2n}, Z_{2n+1}, Z_{2n+2}, \ldots\}) = L(\{Z_{1m}, Z_{1m+1}, Z_{1m+2}, \ldots Z_{2n}, Z_{2n+1}, Z_{2n+2}, \ldots\}) \] \hspace{1cm} (3.27)

which leads to

\[ E[\phi(Z_{11})/\mathcal{F}_m^\infty] = E[\phi(Z_{1m})/\mathcal{F}_m^\infty] \text{ a.s.} \] \hspace{1cm} (3.28)

for any bounded, measurable function, \( \phi \).

Now

\[ E\{E[\phi(Z_{11})/\mathcal{F}_m^\infty] /\tau\} = E\{E[\phi(Z_{1m})/\mathcal{F}_m^\infty] /\tau\} \text{ a.s.} \] \hspace{1cm} (3.29)

and since \( \tau \subset \mathcal{F}_m^\infty \), (3.29) becomes

\[ E[\phi(Z_{11})/\tau] = E[\phi(Z_{1m})/\tau] \text{ a.s. } \forall m \in \mathbb{N}. \] \hspace{1cm} (3.30)
Similarly
\[ E[\phi(Z_{21})/\tau] = E[\phi(Z_{2n})/\tau] \quad \text{a.s.} \quad \forall n \in \mathbb{N} \tag{3.31} \]
and hence, taking \( \phi \) to be an indicator function, (3.30) and (3.31) show that \( Z \) is identically distributed within types after conditioning on \( \tau \). Together with the conditional independence already shown, we thus get \( Z \) to be 2-IID conditional on \( \tau \).

**Remark**

Theorems 3.1 and 3.2 are both representation theorems for partially exchangeable random variables and are extensions of de Finetti's representation theorem for exchangeable random variables as found in Kingman (1978) and Aldous (1985) respectively, the difference between these two statements of the same theorem lying in the \( \sigma \)-field upon which the space is conditioned. An extension of de Finetti's representation theorem for exchangeable sequences to be applicable to a 2-fold partially exchangeable sequence would state the existence of a \( \sigma \)-field conditional upon which the partially exchangeable sequence behaves like a 2-IIID sequence, whether the conditioning \( \sigma \)-field is \( \mathcal{F}_\infty \) (as in Theorem 3.1) or \( \tau \) (as in Theorem 3.2) being irrelevant at this stage but to receive much attention in Chapter 4.

It is important to note that Theorems 3.1 and 3.2 cannot be applied to finite sequences of partially exchangeable random variables. It is beyond the scope of this thesis to consider finite sequences of random variables but for the interested reader a finite form of de Finetti's representation theorem for exchangeable random variables is to be found in Kendall (1967) and Heath and Sudderth (1976).

A more general result than Theorems 3.1 and 3.2 is the following:

**Theorem 3.3** de Finetti's Theorem

Let \( Z = \{Z_{ij}; i \in [2], j \in \mathbb{N} \} \) be a partially exchangeable sequence of random variables. Then there exists a \( \sigma \)-field conditional upon which \( Z \) behaves like a 2-IIID sequence.

The above result follows immediately from either Theorem 3.1 or Theorem 3.2.
Remarks

1) Note that this theorem is an extension of de Finetti's representation theorem for exchangeable sequences of random variables and thus not really de Finetti's theorem, but it will be referred to as such since the $g = 1$ case is commonly referred to as de Finetti's theorem.

2) Looking at Corollary 3.1 we find that there are many forms, other than Theorems 3.1 and 3.2, of de Finetti's representation theorem for partially exchangeable random variables.

3) The obvious extension of Theorem 3.3 to multi-type sequences shows that for a partially exchangeable sequence $Z = \{Z_{ij}, i \in [g], j \in N\}$, there exists a $\sigma$-field conditional upon which $Z$ behaves like a $g$-IID sequence or, as will be discussed in Chapter 4, "$Z$ is a mixture of $g$-IID sequences".

3.2 SOME LIMITATIONS AND IMPLICATIONS OF DE FINETTI'S THEOREM

De Finetti's theorem (Theorem 3.3) is a very useful result as it shows how the possible lack of independence in a partially exchangeable sequence of random variables (see Chapter 2, Example 2.1) can be partially overcome. The corresponding fundamental result for the $g = 1$ case has attracted a variety of generalizations, extensions and analogues as will be discussed here and in further chapters.

De Finetti's theorem holds in compact metric spaces, $S$, but breaks down in the absence of strong enough topological assumptions on $S$. In this regard Dubins and Freedman (1979) gave an example of a separable space $S$ where de Finetti's theorem for exchangeable random variables cannot be applied. It is thus possible to find a separable space $S$ where Theorem 3.3 does not hold.

Further work on the exchangeable case was done by Freedman (1980) and Dubins (1983).

Dacunha-Castelle (1974) and Kingman (1978) discussed the sufficiency of
weaker requirements than exchangeability of $Z$ to obtain de Finetti's theorem. Improving on these results, Aldous (1982) gave a minimal requirement.

We now prove de Finetti's theorem (Theorem 3.3) for a 2-fold infinite sequence which has the selection property (see Definition 2.10 in Chapter 2) rather than partial exchangeability. The following result is given in Kingman [(1978), p.188] and will be used in Theorem 3.4.

**Lemma 3.3**

Let $k \in [n]$ for any $n \in \mathbb{N}$. If, $\forall r \leq k$, the bounded sequence of variables \{\(a_r(j), j \geq 1\)} has Cesàro limit

$$\alpha_r = \lim_{n \to \infty} n^{-1} \sum_{j=1}^{n} a_r(j)$$

then

$$\lim_{n \to \infty} \binom{n}{k}^{-1} \sum_{M} \prod_{s=1}^{k} a_s(j_s) = \prod_{s=1}^{k} \alpha_s$$

where

$$M = \{(j_1, j_2, \ldots, j_k) ; j_1 < j_2 < \ldots < j_k \leq n\}.$$  

**Theorem 3.4**

Let $Z = \{Z_{ij}; i \in [2], j \in \mathbb{N}\}$ be a 2-fold infinite sequence of random variables, with $E(Z_{ii}) < \infty \forall i \in [2]$, which has the selection property (Definition 2.10). Then there exists a $\sigma$-field conditional upon which $Z$ behaves like a 2-IID sequence.

**Proof:**

It is sufficient to show that if $Z$ has the selection property then $Z$ is partially exchangeable as the desired result then follows immediately from Theorem 3.3.

For all $i \in [2]$ let

$$C_i = \bigcap_{n=1}^{\infty} C_{in}$$

where

$$C_{in} = \mathcal{F}(Z_{in}, Z_{i, n+1}, Z_{i, n+2}, \ldots) \quad \forall n \in \mathbb{N}.$$
Since $Z$ has the selection property, $Z_i = \{Z_{ij} ; j \in \mathbb{N}\}$ is stationary (Loève (1978), Vol. II, p.83) $\forall i \in [2]$, and hence by the Birkhoff ergodic theorem (Appendix A1) we have

$$\lim_{n \to \infty} n^{-1} \sum_{j=1}^{n} Z_{ij} = E(Z_{1i} / C_i) \text{ a.s., } \forall i \in [2].$$

(3.32)

Taking $f$ to be the indicator function of $[-\infty, x]$ for some $x \in \mathbb{R}$, one finds that $\{f(Z_{ij}); i \in [2], j \in \mathbb{N}\}$ has the selection property, so that

$$\lim_{n \to \infty} n^{-1} \# \{ j \leq n; Z_{ij} \leq x \} = F_i(x) \text{ a.s.}$$

(3.33)

where

$$F_i(x) = E(I[Z_{1i} \leq x] / C_i)$$

$$= P[Z_{1i} \leq x / C_i] \quad \forall i \in [2], x \in \mathbb{R}$$

follows from (3.32).

Now, $\forall r \in \mathbb{N}, \forall j \in \mathbb{N}$ let $a_r(j) = f_r(Z_{1j})$, $f_r(Z_{1j}) = I(Z_{1j} \leq x_r)$, then (3.33) and Lemma 3.3 with $\alpha_i = F_1(x_i) = P(Z_{11} \leq x_i)$ show that

$$\lim_{n \to \infty} \binom{n}{k}^{-1} \sum_{M} \prod_{s=1}^{k} f_r(Z_{1s}) = \prod_{s=1}^{k} F_1(x_s)$$

(3.34)

where

$$M = \{(j_1, j_2, \ldots, j_k); j_1 < j_2 < \cdots < j_k \leq n\}.$$

Similarly, for any $m \in \mathbb{N}$ and any $l \in [m]$,

$$\lim_{m' \to \infty} \binom{m}{l}^{-1} \sum_{M'} \prod_{s=1}^{l} I[Z_{2is} \leq x_i] = \prod_{s=1}^{l} F_2(x_s)$$

where

$$M' = \{(i_1, i_2, \ldots, i_l); i_1 < i_2 < \cdots < i_l \leq m\}.$$

Combining (3.34) and this previous result we obtain

$$\lim_{m' \to \infty} \frac{\binom{n}{k} \binom{m}{l}^{-1}}{\binom{m'}{k}} \sum_{M} \sum_{M'} f \left( \bigcap_{s=1}^{k} [Z_{1js} \leq x_s] \cap \bigcap_{s=1}^{l} [Z_{2is} \leq x_is] \right)$$

$$= \prod_{s=1}^{k} F_1(x_s) \prod_{r=1}^{l} F_2(x_s).$$

(3.35)
There are \( \binom{m}{k} \binom{n}{l} \) ways of choosing \( i_1 < i_2 < \cdots < i_l \leq m \) and \\
\( j_1 < j_2 < \cdots < j_k \leq n \) from \((1, 2, \ldots, m)\) and \((1, 2, \ldots, n)\) respectively. By 
the selection property of \( Z \), these choices will all result in 

\[
P \left\{ \left( \bigcap_{s=1}^{k} [Z_{1s} \leq x_{1s}] \right) \cap \left( \bigcap_{s=1}^{l} [Z_{2s} \leq x_{2s}] \right) \right\}
\]

being equal to 

\[
P \left\{ \left( \bigcap_{s=1}^{k} [Z_{1s} \leq x_{1s}] \right) \cap \left( \bigcap_{s=1}^{l} [Z_{2s} \leq x_{2s}] \right) \right\}
\]

so that, using (3.35) and noting that \( l \) and \( k \) depend on \( m \) and \( n \) respectively,

\[
\lim_{n \to \infty} \lim_{m \to \infty} P \left\{ \left( \bigcap_{s=1}^{k} [Z_{1s} \leq x_{1s}] \right) \cap \left( \bigcap_{s=1}^{l} [Z_{2s} \leq x_{2s}] \right) \right\} 
= E \left[ \prod_{i=1}^{k} F_1(x_{1i}) \prod_{r=1}^{l} F_2(x_{2r}) \right].
\]

The symmetry (permutability) of this last expression shows that \( Z \) is partially exchangeable.

Following the line of thought adopted by Kingman (1972) in the \( g = 1 \) case we now present a theorem that demonstrates how a more specific result than de Finetti's representation theorem may be obtained by posing a stronger condition than partial exchangeability on the infinite 2-fold sequence of random variables.

**Theorem 3.5**

Let \( Z = \{Z_{ij}; i \in [2], j \in \mathbb{N}\} \) be an infinite 2-fold sequence of random variables such that for all \( n_i \in \mathbb{N}, i \in [2], \) the distribution of the sequence of random variables \((Z_{11}, Z_{12}, \ldots, Z_{1n_1}, Z_{21}, Z_{22}, \ldots, Z_{2n_2})\) has spherical symmetry (Definition 2.12). Then there exist random variables \( V_1 \) and \( V_2 \), real and non-negative, such that, conditional on \( \mathcal{F}(V_1, V_2) \), \( Z \) is independent and identically distributed, within types as \( N(0, V_i) \).
Proof

\( Z \) is partially exchangeable (Theorem 2.4) and hence conditionally 2-IID given \( \mathcal{F}_\infty \) (Theorem 3.1), so that we may define a joint random distribution function for one variable of each type, as follows:

\[
F(x, y) = P[Z_{1k} \leq x, Z_{2j} \leq y/\mathcal{F}_\infty] \quad \forall k, j \in \mathbb{N}, \ x, y \in \mathbb{R}
\]

(\( \mathcal{F}_\infty \) as in Definition 3.2). Now define the corresponding joint characteristic function

\[
\phi(s, t) = \int \int \exp[i(sx + ty)]dF(x, y)
\]

\[
= E\{\exp[i(sZ_{1k} + tZ_{2j})]/\mathcal{F}_\infty\}
\]

\( \forall k, j \in \mathbb{N} \) and \( s, t \in \mathbb{R} \).

Now \( \phi(s, t) \) is thus a random, \( \mathcal{F}_\infty \)-measurable, continuous function. Using the conditional independence of \( Z \) (Theorem 3.1) and (3.36) it follows that

\[
\forall s_j, t_j \in \mathbb{R}, \ \forall n \in \mathbb{N}, \ j \in [n],
\]

\[
E\left[\exp i \sum_{j=1}^{n} (s_j Z_{1j} + t_j Z_{2j})/\mathcal{F}_\infty\right] = \prod_{j=1}^{n} \phi(s_j, t_j)
\]

and hence

\[
E\left[\prod_{j=1}^{n} \phi(s_j, t_j)\right] = E\left\{E\left[\exp i \sum_{i=1}^{n} (s_i Z_{1i} + t_i Z_{2i})/\mathcal{F}_\infty\right]\right\}
\]

\[
= E\left[\exp i \sum_{j=1}^{n} (s_j Z_{1j} + t_j Z_{2j})\right].
\]

Using a result from Lord (1954) (see Appendix A1) it follows that the right hand side of (3.38) is a function of \( \sum_{i=1}^{n} (t_i^2 + s_i^2) \) only.

For any \( u_i, v_i \in \mathbb{R}, \ i \in [2] \), define

\[
s = \left(u_1^2 + v_1^2\right)^{\frac{1}{2}} \quad t = \left(u_2^2 + v_2^2\right)^{\frac{1}{2}}.
\]
Then
\[ E \left[ |\phi(s, t) - \phi(u_1, u_2) \phi(v_1, v_2)|^2 \right] \]
\[ = E \left\{ |\phi(s, t) - \phi(u_1, u_2)\phi(v_1, v_2)| \left[ \phi(s, t) - \phi(u_1, u_2)\phi(v_1, v_2) \right] \right\} \]
\[ = E \left[ \phi(s, t)\phi(-s, -t) \right] - E \left[ \phi(s, t)\phi(-u_1, -u_2)\phi(-v_1, -v_2) \right] - E \left[ \phi(u_1, u_2)\phi(v_1, v_2)\phi(-s, -t) \right] + E \left[ \phi(u_1, u_2)\phi(v_1, v_2)\phi(-u_1, -u_2)\phi(-v_1, -v_2) \right]. \]

Using (3.39) and applying the result from Lord (1954) to each of the four terms in this last expression, it follows that they are all equal, so that
\[ E \left[ |\phi(s, t) - \phi(u_1, u_2) \phi(v_1, v_2)|^2 \right] = 0 \]
and hence
\[ \phi \left( \left( u_1^2 + v_1^2 \right)^{\frac{1}{2}}, \left( u_2^2 + v_2^2 \right)^{\frac{1}{2}} \right) = \phi(u_1, u_2) \phi(v_1, v_2) \quad \text{a.s.} \] (3.40)
follows from (3.39).

Now let \( u_1 = v_1 \) and \( u_2 = v_2 \) in (3.40). Then
\[ \phi(s, t) = \phi^2 \left( 2^{-\frac{1}{2}} s, 2^{-\frac{1}{2}} t \right) \] (3.41)

Now let
\[ \psi(s, t) = \ln \phi(s, t). \] (3.42)

Then (3.41) and (3.42) show that
\[ \psi(s, t) = 2\psi \left( 2^{-\frac{1}{2}} s, 2^{-\frac{1}{2}} t \right) \]
\[ = 2^n \psi \left( 2^{-\frac{n}{2}} s, 2^{-\frac{n}{2}} t \right) \]
so that
\[ \psi(s, t)(s^2 + t^2)^{-1} = \psi \left( 2^{-\frac{n}{2}} s, 2^{-\frac{n}{2}} t \right) \left( 2^{-\frac{n}{2}} s \right)^2 + (2^{-\frac{n}{2}} t)^2 \right)^{-1}. \] (3.43)
Now let
\[ m_i = E(Z_{ij}/\mathcal{F}_\infty) \]
\[ V_i = E(Z_{ij}/\mathcal{F}_\infty)^2 - m_i^2 \]
\[ Y_{ij} = (Z_{ij} - m_i)(V_i)^{-1/2} \quad \forall i \in [2], \forall j \in \mathbb{N}. \quad (3.44) \]

Since \( Z \) behaves like a 2-IID sequence once conditioned on \( \mathcal{F}_\infty \) (Theorems 2.4 and 3.1), it follows that
\[ E(Y_{ij}/\mathcal{F}_\infty) = 0 \quad \text{and} \quad \text{Var}(Y_{ij}/\mathcal{F}_\infty) = 1 \quad \forall i \in [2], \forall j \in \mathbb{N}. \quad (3.45) \]

Now let
\[ \phi_{Y_{1k}Y_{2j}}(s, t) = E[\exp(i(sY_{1k} + tY_{2j})/\mathcal{F}_\infty)] \quad (3.46) \]
\( s, t \in \mathbb{R}, \forall k, j \in \mathbb{N}. \)

Using (3.37) and (3.44), we have
\[ \phi(s, t) = \exp(i(sm_1 + tm_2)\phi_{Y_{1k}Y_{2j}}(sV_1^{1/2}, tV_2^{1/2})) \quad (3.47) \]
\( \forall s, t \in \mathbb{R}, \forall k, j \in \mathbb{N}. \) Now let
\[ \psi_{Y_{1k}Y_{2j}}(s, t) = \ln \phi_{Y_{1k}Y_{2j}}(s, t) \quad . \quad (3.48) \]

Replacing \( Z \) by \( Y = \{Y_{ij}; i \in [2], j \in \mathbb{N}\} \) [as in (3.44)] in (3.43) and using L'Hôpital's rule, (3.45) and (3.48), it follows that
\[ \psi_{Y_{1k}Y_{2j}}(s, t)(s^2 + t^2)^{-1} = -\frac{1}{2} \]
and hence
\[ \phi_{Y_{1k}Y_{2j}}(s, t) = \exp\left[ -\frac{1}{2}(s^2 + t^2) \right] \]
so that, from (3.47),
\[ \phi(s, t) = \exp(i(sm_1 + tm_2)\exp\left[ -\frac{1}{2}(s^2V_1 + t^2V_2) \right]) \quad . \quad (3.49) \]

Since \( Z \) has spherical symmetry the joint distribution of \((Z_{1k}, Z_{2j})\) is identical to the joint distribution of \((-Z_{1k}, -Z_{2j})\) \( \forall j, k \in \mathbb{N}, \) and hence from (3.36) \( \forall s, t \in \mathbb{R}, \phi(s, t) = \phi(-s, -t), \) so that
\[ \phi(s, t) = \frac{1}{2} [\phi(s, t) + \phi(-s, -t)] \quad . \]
From (3.49),
\[
\exp[i(sm_1 + tm_2)] \exp[-\frac{1}{2}(s^2V_1 + t^2V_2)]
\]
\[
= \frac{1}{2} \exp[-\frac{1}{2}(s^2V_1 + t^2V_2)] \{\exp[i(sm_1 + tm_2)] + \exp[i(-sm_1, -tm_2)]\}.
\]
This last equation only holds \(\forall s, t \in \mathbb{R}\), if \(m_1 = m_2 = 0\), and hence from (3.44), conditional on \(\mathcal{F}_\infty\), \(Z\) elements have zero mean. Using this last result and (3.49),
\[
\phi(s, t) = \exp[-\frac{1}{2}(s^2V_1 + t^2V_2)]
\]
(3.50)
Since \(\phi(s, t) = \phi(-s, -t)\), (3.50) yields
\[
\exp[-\frac{1}{2}(s^2V_1 + t^2V_2)] = \exp[-\frac{1}{2}(s^2V_1 + t^2V_2)].
\]
We thus conclude that \(V_1\) and \(V_2\) are real valued. Since \(|\phi(s, t)| \leq 1\) it follows from (3.50) that \(V_1\) and \(V_2\) are non-negative.

From (3.50),
\[
\phi(1, 0) = \exp\left(-\frac{1}{2}V_1^2\right)
\]
so that
\[
V_1 = -2\ln\phi(1, 0)
\]
and similarly
\[
V_2 = -2\ln\phi(0, 1)
\]
and hence (using (3.36)) it follows that \(V_1\) and \(V_2\) are \(\mathcal{F}_\infty\)-measurable random variables.

For any random variable \(X\),
\[
E(X/V_1, V_2) = E\left[E(X/\mathcal{F}_\infty)/V_1, V_2\right].
\]
(3.51)
Substituting \(X = \exp\left[i \sum_{r=1}^{\infty} (s_r Z_{1r} + t_r Z_{2r})\right]\) into (3.51) and using (3.50) and (3.37), one has
\[
E \left\{ \exp \left[ i \sum_{r=1}^{n} (s_r Z_{1r} + t_r Z_{2r}) \right] / V_1, V_2 \right\} \\
= E \left\{ E \left\{ \exp \left[ i \sum_{r=1}^{n} (s_r Z_{1r} + t_r Z_{2r}) \right] / \mathcal{F}_\infty \right\} / V_1, V_2 \right\} \\
= E \left\{ \prod_{r=1}^{n} \phi(s_r, t_r)/ V_1, V_2 \right\} \\
= \prod_{r=1}^{n} \exp \left[ -\frac{1}{2} (V_1 s_r^2 + V_2 t_r^2) \right].
\]
For any \( s_r, t_r \in \mathbb{R} \), \( \exp \left[ -\frac{1}{2} (V_1 s_r^2 + V_2 t_r^2) \right] \) is the characteristic function of a \( N \left( \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{cc} V_1 & 0 \\ 0 & V_2 \end{array} \right) \right) \) distribution, so that, conditional on \( \mathcal{F}_\infty \), \( Z \) is 2-IID as \( N(0, V_i) \) for variables of type \( i \), \( i \in [2] \).

Remarks:

1. Theorem 3.5 has a close resemblance, on the one hand to de Finetti's theorem (Theorem 3.3) and on the other hand to Maxwell's theorem (Appendix A1) which gives a characterization of the normal distribution.

2. Theorem 3.3 ensures the existence of some \( \sigma \)-field, conditional upon which the partially exchangeable sequence behaves like a 2-IID sequence, no mention being made of the particular type of distribution. Theorem 3.5 shows that posing the stronger requirement of spherical symmetry on the random variables enables us to know the exact distribution of the 2-IID sequence mentioned in de Finetti's theorem.

3. Ressel (1985) looked at the necessary and sufficient conditions required for an exchangeable sequence of random variables to behave like an IID sequence of a particular type (normal, exponential, Poisson, gamma) after conditioning on a \( \sigma \)-field. Freedman (1962) also gave many interesting examples in this regard.
3.3 ERGODICITY AND THE RELATIONSHIP BETWEEN PARTIALLY EXCHANGEABLE AND 2-IID SEQUENCES OF RANDOM VARIABLES

We have established that even though partially exchangeable sequences need not be 2-IID (Example 2.1), it is possible (Theorem 3.3) to regard the partially exchangeable sequence as a 2-IID sequence by making use of conditional probabilities. It is thus not surprising that the relationship between partial exchangeability and 2-IID leads to further results. In Chapter 5 we shall prove that all a.s. limit theorems for 2-IID random variables have an analogue for partially exchangeable random variables.

The class of partially exchangeable sequences can be viewed as the class of distributions invariant under certain transformations. Based on this idea of invariant distributions and using ergodic theory we prove an extension of the Hewitt-Savage zero-one law (Appendix A1) which gives a further connection between the concepts of partial exchangeability and independent identical distributions within types.

Assume the following general setting throughout this section:

Let $S$ be a Polish space, let $K$ denote a countable group of measurable maps $T : S \rightarrow S$ and let $P(S)$ be the set of probability measures on $S$.

**Definition 3.4**
For arbitrary map $T$ define the *induced map* $\tilde{T}$ as follows:

(i) If $T \in K$ then $\tilde{T} : P(S) \rightarrow P(S)$ where

$$\tilde{T}[L(X)] = L[T(X)] .$$

(ii) If $T$ is any measurable map from $S \times S$ to $S$, then $\tilde{T} : S \times P(S) \rightarrow P(S)$ where

$$\tilde{T}[s, L(X)] = L[T(s, X)]$$

for any random element $X$ and any $s \in S$. 

\[3.52\]
\[3.53\]
$X$, a random element of $S$, is \textit{invariant} if
\[ L[T(X)] = L(X) \quad \forall T \in K \quad (3.54) \]

$\mu$, a distribution on $S$, is an \textit{invariant distribution} if
\[ \hat{T}(\mu) = \mu \quad \forall T \in K. \quad (3.55) \]

Let $M$ denote the set of invariant distributions and suppose that $M$ is non-empty. A subset $A \subset S$ is an \textit{invariant subset} if
\[ T(A) = A \quad \forall T \in K. \quad (3.56) \]

$J$ is the \textit{invariant $\sigma$-field} made up by the family of invariant subsets.

An invariant distribution $\mu$ is \textit{ergodic} if
\[ \mu(A) = 0 \text{ or } 1 \quad \forall A \in J. \]

\textbf{Definition 3.5}

For $X = \{X_{ij}; i \in [2], j \in N\}$ an infinite two-fold sequence of random variables, $\{\omega; X(\omega) \in B\}$ is a \textit{partially exchangeable event} if $B$ is a subset of $R^\infty \times R^\infty$ such that

\[ (x_{11}, x_{12}, \ldots, x_{21}, x_{22}, \ldots) \in B \Rightarrow (x_{1,\pi_i(1)}, x_{1,\pi_i(2)}, \ldots, x_{2,\pi_i(1)}, x_{2,\pi_i(2)}, \ldots) \in B \]

where $\pi_i$ is any finite permutation of $\{1, 2, \ldots\}, \forall i \in [2].$

$E_X$, the \textit{partially exchangeable $\sigma$-field} is the set of all partially exchangeable events.

\textbf{Remark}

In Chapter 4 (Theorem 4.9) we shall show that for a partially exchangeable sequence $X$ the tail $\sigma$-field $\tau_X$ and $E_X$ coincide, i.e. we may condition on $E_X$ in Theorem 3.2.

If $S = R^\infty \times R^\infty, K = \{T_x; T_x \text{ finitely permutes } x_{ij} \text{ to } x_{i\pi(x)(j)}, i \in [2], j \in N\}$

then $X = \{X_{ij}; i \in [2], j \in N\}$ is partially exchangeable if and only if $X$ is invariant under $K$. The \textit{invariant $\sigma$-field} is the partially exchangeable
\( \sigma \)-field \( E_X \) and the next theorem shows that the ergodic processes are 2-IID sequences.

We now present a result based on the Hewitt-Savage zero-one law (Appendix A1) but differing from the latter by allowing multi-type sequences.

**Theorem 3.6**

Let \( X = \{X_{ij}; \ i \in [2], \ j \in \mathbb{N}\} \) be a 2-IID sequence of random variables. Then \( P(A) = 0 \) or \( 1 \) for all \( A \in E_X \).

**Proof**

Let \( \pi_i \) be any finite permutation of \( \{1, 2, \ldots\} \), \( i \in [2] \). Let

\[
X_\pi = \{X_{i\pi_i(j)}; i \in [2], \ j \in \mathbb{N}\}
\]

(3.57)

For \( B \in B(\mathbb{R}^\infty \times \mathbb{R}^\infty) \), let

\[
A = [X \in B]
\]

(3.58)

\[
\pi(A) = [X_\pi \in B] .
\]

(3.59)

For \( n \in \mathbb{N} \), \( m \in \mathbb{N} \) and

\[
B_{nm} \in B(\mathbb{R}^n \times \mathbb{R}^m)
\]

(3.60)

let

\[
A_{nm} = \{(X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, X_{22}, \ldots, X_{2m}) \in B_{nm}\} .
\]

(3.61)

For \( \pi_1 \) (\( \pi_2 \)) any permutation of \( [n] \) ([\( m \)]) respectively, let

\[
\pi_{nm}(X) = \{X_{1\pi_1(1)}, X_{1\pi_1(2)}, \ldots, X_{1\pi_1(n)}, X_{1n+1}, \ldots, X_{2\pi_2(1)}, X_{2\pi_2(2)}, \ldots,
X_{2m+1}, \ldots\}
\]

(3.62)

\[
\pi_{nm}(A) = [\pi_{nm}(X) \in B] .
\]

(3.63)

Suppose that \( A \in E_X \). Then \( A \) is a partially exchangeable event and hence

\[
\pi(A) = A
\]

(3.64)
for all \( \pi = \{\pi_i(j); i \in [2], j \in \mathbb{N} \} \) where \( \pi_i \) is a finite permutation of \( \{1, 2, \ldots\}, \forall i \in [2] \).

For \( n, m \in \mathbb{N} \), choose sets \( B_{nm} \in B(\mathbb{R}^n \times \mathbb{R}^m) \) such that
\[
\lim_{n,m \to \infty} P(A \Delta A_{nm}) = 0 .
\]

(3.65)

For \( B \in B(\mathbb{R}^\infty \times \mathbb{R}^\infty) \), let
\[
P_X(B) = P[X \in B]
\]

(3.66)

and for \( n, m \in \mathbb{N} \), let
\[
P_{\pi_{nm}}(X)(B) = P[\pi_{nm}(X) \in B].
\]

(3.67)

Since \( X \) is 2-IID, for any \( B \in B(\mathbb{R}^\infty \times \mathbb{R}^\infty) \), any \( n \in \mathbb{N}, m \in \mathbb{N} \),
\[
P_X(B) = P_{\pi_{nm}}(X)(B).
\]

(3.68)

From (3.58) and (3.61),
\[
P(A \Delta A_{nm}) = P\{[X \in B] \Delta \{X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, X_{22}, \ldots, X_{2m}\} \in B_{nm}\}
\]
\[
= P\{[X \in B \Delta B_{nm}]\}
\]
\[
= P_X(B \Delta B_{nm}).
\]

(3.69)

Using (3.68) we thus have
\[
P(A \Delta A_{nm}) = P_{\pi_{nm}}(X)(B \Delta B_{nm}).
\]

(3.70)

Since \( A \) is a partially exchangeable event,
\[
A = \pi_{nm}(A)
\]
i.e.
\[
[X \in B] = [\pi_{nm}(X) \in B]
\]
so that (3.70) becomes

\[ P(A \Delta A_{nm}) = P(\pi_{nm}(X) \in B \Delta B_{nm}) \]

\[ = P\{[\pi_{nm}(X) \in B] \Delta [\pi_{nm}(X) \in B_{nm}]\} \]

\[ = P\{[X \in B] \Delta \pi_{nm}(A_{nm})\} \]

\[ = P\{A \Delta \pi_{nm}(A_{nm})\} . \]

Using (3.65) we thus have that

\[ \lim_{n,m \to \infty} P[A \Delta \pi_{nm}(A_{nm})] = 0 . \]

(3.71)

Now

\[ P\{A \Delta [A_{nm} \cap \pi_{nm}(A_{nm})]\} \]

\[ = P\{(A \Delta A_{nm}) \cap [A \Delta \pi_{nm}(A_{nm})]\} + P\{(A \cap A_{nm}) \Delta [A \cap \pi_{nm}(A_{nm})]\} \]

so that (3.65) and (3.71) show that

\[ \lim_{n,m \to \infty} P\{A \Delta [A_{nm} \cap \pi_{nm}(A_{nm})]\} = 0 \]

and hence

\[ \lim_{n,m \to \infty} P[A_{nm} \cap \pi_{nm}(A_{nm})] = P(A) . \]

(3.72)

Since \( X \) is 2-IID,

\[ P[A_{nm} \cap \pi_{nm}(A_{nm})] \]

\[ = P\{[(X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, \ldots, X_{2m}) \in B_{nm}]
\]

\[ \cap [(X_{1\pi_{1}(1)}, X_{1\pi_{1}(n)}, X_{2\pi_{2}(1)}, \ldots, X_{2\pi_{2}(m)}) \in B_{nm}]\} \]

\[ = P\{[(X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, \ldots, X_{2m}) \in B_{nm}]
\]

\[ \cap [(X_{1n+1}, X_{1n+2}, \ldots, X_{2n+1}, X_{2m+1}, \ldots, X_{2m+2}) \in B_{nm}]\} \]

\[ = P\{[(X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, \ldots, X_{2m}) \in B_{nm}]
\]

\[ P\{[(X_{1n+1}, X_{1n+2}, \ldots, X_{1n+2}, X_{2n+1}, X_{2m+1}, \ldots, X_{2m+2}) \in B_{nm}]\} \]

\[ = P(A_{nm})P[\pi_{nm}(A_{nm})] . \]

(3.73)
From (3.71) and (3.65) respectively

\[ \lim_{n,m \to \infty} P[\pi_{nm}(A_{nm})] = P(A) \]

and

\[ \lim_{n,m \to \infty} P(A_{nm}) = P(A) \]

follows, so that (3.72) and (3.73) shows that

\[ P(A) = |P(A)|^2. \quad (3.74) \]

The desired result follows immediately since only \( P(A) = 0 \) or \( P(A) = 1 \) satisfies (3.74).

**Remark**

Considering \( \{X_i, i \in N\} \) (i.e. \( g = 1 \) case) Aldous and Pitman (1979) gave a weaker assumption than IID of \( X \) which is sufficient for the exchangeable \( \sigma \)-field to be trivial (i.e. all events in this \( \sigma \)-field have measure zero or one). Sendler (1975) also published work on this topic but the Aldous and Pitman results are more powerful.
CHAPTER 4

ANALYSIS OF PARTIALLY EXCHANGEABLE SEQUENCES

De Finetti's representation theorem for exchangeable random variables is commonly stated as:

\[
\text{an infinite exchangeable sequence is a mixture of IID sequences.}
\]

In this chapter we shall investigate exactly what is meant by a mixture; use this theory to draw conclusions about a 2-fold partially exchangeable sequence (since it turns out to be a mixture of 2-IID sequences) and finally construct partially exchangeable sequences.

4.1 MIXTURES AND DIRECTION

Definition 4.1
Given a family \( \{\mu_{\gamma}; \gamma \in \Gamma\} \) of distributions on a space \( S \), a distribution \( \nu \) is said to be a mixture of \( \{\mu_{\gamma}; \gamma \in \Gamma\} \) if

\[
\nu(\cdot) = \int_{\Gamma} \mu_{\gamma}(\cdot) \Theta(\,d\gamma) 
\]

for some distribution \( \Theta \) on \( \Gamma \).

A special case of the above definition describes the distribution of a sequence which is a mixture of 2-IID sequences:

Definition 4.2
Let \( Y = \{Y_{ij}; i \in [2], j \in \mathbb{N}\} \) be an infinite 2-fold sequence of random variables. Suppose that \( \Theta \) is a distribution on \( P(\mathbb{R}) \times P(\mathbb{R}) \). Then the distribution of \( Y \) is a mixture of 2-IID sequences if

\[
P(Y \in A) = \int_{P(\mathbb{R}) \times P(\mathbb{R})} \theta^\infty(A) \Theta(\,d\theta) 
\quad \forall A \in B(\mathbb{R}^\infty \times \mathbb{R}^\infty) \quad (4.1)
\]
where

\[ \theta = \theta_1 \times \theta_2 \]

\[ A = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} A_{ij} \quad \text{if } i,j \in [2], \forall i \in [2], \forall j \in \mathbb{N} \]

\[ \theta^\infty = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \theta_i, \]

the distribution on \(\mathbb{R}^\infty \times \mathbb{R}^\infty\) of a 2-IID sequence, variables of type \(i\) having distribution \(\theta_i, \forall i \in [2]\).

\[ \theta^\infty(A) = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \theta_i(A_{ij}). \]

**Remark**

Definition 4.2 merely gives the Bayesian idea that \(Y\) is 2-IID, variables of type \(i\) having common distribution \(\theta_i, \forall i \in [2]\), where \(\theta = \theta_1 \times \theta_2\) has prior distribution \(\Theta\).

We now discuss random measures and regular conditional distributions: this will enable us to deal with mixtures of random variables rather than mixtures of distributions.

**Definition 4.3**

A random measure \(\alpha\), is a \(P(\mathbb{R})\)-valued random variable, viewed as a function of two variables as follows:

\[ \alpha(\omega, \cdot) : B(\mathbb{R}) \to \mathbb{R}, \text{ is a probability measure for fixed } \omega \in \Omega \quad (4.2) \]

\[ \alpha(\cdot, A) : \Omega \to \mathbb{R}, \text{ is a random variable for fixed } A \in B(\mathbb{R}). \quad (4.3) \]

Random measures \(\alpha_1\) and \(\alpha_2\) are a.s. equivalent if they are a.s. equivalent when viewed as random variables in \(P(\mathbb{R})\), i.e.

\[ \alpha_1(\cdot, A) = \alpha_2(\cdot, A) \quad \text{a.s. } \forall A \in B(\mathbb{R}) \quad (4.4) \]

or equivalently,

\[ P\{\omega; \alpha_1(\omega, A) = \alpha_2(\omega, A)\} = 1 \quad \forall A \in B(\mathbb{R}). \]
Definition 4.4
A regular conditional distribution (r.c.d.) for a random variable $Y$ given a $\sigma$-field $\mathcal{F}$ is a random measure $\alpha$ such that
\[
\alpha(\cdot, A) = P(Y \in A/\mathcal{F}) \text{ a.s.} \quad \forall A \in B(\mathbb{R}) .
\] (4.5)

Remark
An r.c.d. is thus a special type of random measure, one which is obtained by conditioning on a $\sigma$-field (as the name suggests).

It is well known that r.c.d.'s exist and satisfy the fundamental property stated in the following lemma. This property will frequently be used to enable versions of conditional expectations to be computed.

Lemma 4.1 (Shirayayev (1984), page 231, exercise 3)
Let $X$ and $Y$ be random variables and suppose that $\alpha$ is an r.c.d. for $Y$ given a $\sigma$-field $\mathcal{F}$. If $X$ is $\mathcal{F}$-measurable and $E|g(X,Y)| < \infty$, then
\[
E[g(X,Y)/\mathcal{F}] = \int g(X,y) \alpha(\omega, dy) \text{ a.s.} .
\] (4.6)

We are now in a position to define a sequence which is a mixture of 2-IID random variables.

Definition 4.5
Let $\alpha_1$ and $\alpha_2$ be random measures and let $Y = \{Y_{ij}; i \in [2], j \in \mathbb{N}\}$ be an infinite 2-fold sequence of random variables. Then $Y$ is a mixture of 2-IID random variables, directed by $\alpha_1$ and $\alpha_2$ if
\[
\prod_{i=1}^{2} \prod_{\infty}^{\infty} \alpha_{i} \text{ is an r.c.d. for } Y \text{ given } \mathcal{F}(\alpha_1, \alpha_2)
\] (4.7)
or equivalently, if
\[
P(Y \in A/\alpha_1, \alpha_2) = \prod_{i=1}^{2} \prod_{1}^{\infty} \alpha_i(\cdot, A_{ij}) \text{ a.s.}
\] (4.8)

where $A = \prod_{i=1}^{2} \prod_{1}^{\infty} A_{ij}$, $A \in B(\mathbb{R}^\infty \times \mathbb{R}^\infty)$. 

Remark
Definition 4.2 differs from Definition 4.5 in that it defines the distribution of a sequence which is a mixture of 2-IID sequences.

Definition 4.6
Let $Y = \{Y_{ij} ; i \in [2], j \in \mathbb{N}\}$ be an infinite 2-fold sequence of random variables and let $\mathcal{F}$ be a $\sigma$-field. Then $Y$ is conditionally independent given $\mathcal{F}$ if

$$P[Y \in A/\mathcal{F}] = \prod_{i=1}^{2} \prod_{j=1}^{\infty} P[Y_{ij} \in A_{ij}/\mathcal{F}]$$

(4.9)

and $Y$ is conditionally identically distributed within types given $\mathcal{F}$ if,

$$P[Y_{ij} \in A_{ij}/\mathcal{F}] = P[Y_{ik} \in A_{ij}/\mathcal{F}] \quad \forall k, j \in \mathbb{N}, i \in [2].$$

(4.10)

where $A = \bigotimes_{i=1}^{2} \bigotimes_{j=1}^{\infty} A_{ij}$, $A \in B(\mathbb{R}^\infty \times \mathbb{R}^\infty)$.

$Y$ is conditionally 2-IID given $\mathcal{F}$ if both (4.9) and (4.10) hold.

Remark
Using Appendix A2 we may give various results, equivalent to (4.9), which would result in $Y$ being conditionally independent given $\mathcal{F}$.

Remark 4.1 (numbered for future reference)
Definition 4.5 may be re-stated as:

$Y$ is a mixture of 2-IID random variables directed by $\alpha_1$ and $\alpha_2$ if $Y$ is conditionally 2-IID given $\mathcal{F}(\alpha_1, \alpha_2)$ and

$$P(Y_{i1} \in A_{ij}/\alpha_1, \alpha_2) = \alpha_i(\cdot, A_{ij}) \quad \forall j \in \mathbb{N}, \forall i \in [2], \forall A_{ij} \in B(\mathbb{R}).$$

Theorem 4.1
Let $Y = \{Y_{ij} ; i \in [2], j \in \mathbb{N}\}$ be an infinite 2-fold sequence of random variables and let $\mathcal{F}$ be an arbitrary $\sigma$-field. Suppose that $Y$ is conditionally 2-IID given $\mathcal{F}$ and let $\alpha_i$ be an r.c.d. for $Y_{i1}$ given $\mathcal{F}$ $\forall i \in [2]$. Then

(i) $Y$ is a mixture of 2-IID sequences of random variables directed by $\alpha_1$ and $\alpha_2$. 
(ii) $Y$ and $\mathcal{F}$ are conditionally independent given $\mathcal{F}(\alpha_1, \alpha_2)$.

**Proof**

Let $A = \bigotimes_{i=1}^{2} \bigotimes_{j=1}^{\infty} A_{ij}$ for $A \in \mathcal{B}(\mathbb{R}^\infty \times \mathbb{R}^\infty)$. Clearly

$$\alpha_i(\cdot, A_{ij}) = P(Y_{i1} \in A_{ij} / \mathcal{F}) \quad \forall i \in [2], \forall j \in \mathbb{N} \quad (4.11)$$

and hence, using (4.9), (4.10) and (4.11),

$$P[Y \in A / \mathcal{F}] = \prod_{i=1}^{2} \prod_{j=1}^{\infty} P[Y_{ij} \in A_{ij} / \mathcal{F}]$$

$$= \prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}). \quad (4.12)$$

From (4.11) we see that $\alpha_i$ is $\mathcal{F}$-measurable $\forall i \in [2]$ and hence $\mathcal{F}(\alpha_i) \subset \mathcal{F}$ $\forall i \in [2]$. Using this and

$$\mathcal{F}(\alpha_1, \alpha_2) = \mathcal{F}[\mathcal{F}(\alpha_1) \cup \mathcal{F}(\alpha_2)] \quad (4.13)$$

we see that

$$\mathcal{F}(\alpha_1, \alpha_2) \subset \mathcal{F}. \quad (4.14)$$

From (4.12) and (4.14),

$$P[Y \in A / \alpha_1, \alpha_2] = E[P(Y \in A / \mathcal{F}) / \alpha_1, \alpha_2]$$

$$= \prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}) \quad (4.15)$$

which proves (i).

For any $A \in \mathcal{B}(\mathbb{R}^\infty \times \mathbb{R}^\infty)$, (4.12), (4.14) and (4.15) yield

$$P[Y \in A / \mathcal{F}, \mathcal{F}(\alpha_1, \alpha_2)] = P[Y \in A / \mathcal{F}]$$

$$= \prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij})$$

$$= P[Y \in A / \alpha_1, \alpha_2]$$
which is the desired result for (ii).

Suppose that \( Y = \{Y_{ij}; i \in [2], j \in \mathbb{N} \} \) is a mixture of 2-IID sequences for some unspecified random measures \( \alpha_1 \) and \( \alpha_2 \). The next theorem shows that we can determine the directed random measures, but we first need to prove the following Lemma.

**Lemma 4.2**

Let \( Y = \{Y_{ij}; i \in [2], j \in \mathbb{N} \} \) be a 2-IID sequence. Suppose, \( \forall i \in [2], \) that random variables of type \( i \) have distribution function \( F_i \). Then there exist empirical distribution functions \( \{F_{in}; n_i \in \mathbb{N} \} \), based on \( n_i \) samples from \( F_i \), such that

\[
\lim_{n_i \to \infty} F_{in}(\cdot, \omega) = F_i(\cdot) \quad \text{a.s. .}
\]

**Proof**

Let \( i \in [2] \). Fix \( n_i \in \mathbb{N} \) and for each \( \omega \in \Omega \), let the \( n_i \) real numbers \( \{Y_{ij}(\omega); j \in [n_i] \} \) be arranged in non-decreasing order, denoted by \( \{x_{ij}; j \in [n_i] \} \) where

\[
X_{i1}(\omega) \leq X_{i2}(\omega) \leq \ldots \leq X_{in_i}(\omega) .
\]

Now define a distribution function \( F_{in_i}(\cdot, \omega) \) as follows:

For \( x \in \mathbb{R} \),

\[
F_{in_i}(x, \omega) = \begin{cases} 
0 & \text{if } x < X_{i1}(\omega) \\
\frac{k}{n_i} & \text{if } X_{ik}(\omega) \leq x < X_{ik+1}(\omega) \quad \text{for } 1 \leq k < n_i \\
1 & \text{if } x \geq X_{in_i}
\end{cases}
\]

or equivalently

\[
F_{in_i}(x, \omega) = \frac{1}{n_i} \sum_{j=1}^{n_i} \xi_{ij}(x, \omega)
\]

where

\[
\xi_{ij}(x, \omega) = I \{Y_{ij}(\omega) \leq x \} .
\]
Call \( F_{n_i}(\cdot, \omega) \) the empirical distribution, based on \( n_i \) samples from \( F_i \). From the Glivenko-Cantelli theorem (Appendix A1) we thus have

\[
\lim_{n_i \to \infty} F_{n_i}(\cdot, \omega) = F_i(\cdot) \quad \text{a.s.}
\]

**Theorem 4.2**

Let \( Y = \{Y_{ij}; i \in [2], j \in \mathbb{N}\} \) be an infinite two-fold sequence of random variables. If \( Y \) is a mixture of 2-IID sequences, then \( Y \) is directed by \( \alpha_1 \) and \( \alpha_2 \), where

\[
\alpha_i = \Lambda_i\left(\{Y_{ij}; j \in \mathbb{N}\}\right) \quad \text{a.s.} \quad \forall i \in [2] \quad (4.16)
\]

for some function \( \Lambda_i \) from \( \mathbb{R}^\infty \) to \( P(\mathbb{R}) \).

**Proof:**

Fix \( n_i \in \mathbb{N} \quad \forall i \in [2] \) and define the following empirical distribution function for \( x \in \mathbb{R}, \omega \in \Omega \):

\[
\Lambda_{n_i} : \mathbb{R}^n \to P(\mathbb{R})
\]

by

\[
\Lambda_{n_i} [Y_{i1}(\omega), \ldots, Y_{in_i}(\omega)] = n_i^{-1} \sum_{j=1}^{n_i} \xi_{Y_{ij}}(\omega) \quad (4.17)
\]

where

\[
\xi_{Y_{ij}}(\omega)(A) = I \{Y_{ij}(\omega) \in A\}
\]

for \( A \subset \mathbb{R}, \forall j \in [n_i] \). Also define

\[
\Lambda_i : \mathbb{R}^\infty \to P(\mathbb{R})
\]

by

\[
\Lambda_i(\{Y_{ij}(\omega); j \in \mathbb{N}\}) = \lim_{n_i \to \infty} \Lambda_{n_i} (Y_{i1}(\omega), \ldots, Y_{in_i}(\omega)) \quad (4.18)
\]

Now let

\[
A \in B(\mathbb{R}^\infty \times \mathbb{R}^\infty), A = \bigotimes_{i=1}^{2} \bigotimes_{j=1}^{\infty} A_{ij}, A_{ij} \subset \mathbb{R} \quad \forall i \in [2], \forall j \in \mathbb{N}.
\]

Then, from (4.8),

\[
P\left(Y \in A/\alpha_1, \alpha_2\right) = \prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}) \quad \text{a.s.} \quad (4.19)
\]
Now let \( \alpha^\infty(\cdot, A) = \prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}) \).

From (4.19) and Lemma 4.1,
\[
\prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}) = E\{ I[Y(\omega) \in A]/\alpha_1, \alpha_2 \} \text{ a.s.}
= \int I[Y(\omega) \in A] \alpha^\infty(\omega, dY(\omega)) \text{ a.s.} \quad (4.20)
\]

From Lemma 4.2, (4.17), (4.18) and (4.20) we thus have
\[
\alpha_i = \Lambda_i \left\{ \{ Y_{ij}; j \in N \} \right\} \text{ a.s.} \forall i \in [2].
\]

Remark
If \( Y = \{ Y_{ij}; i \in [2], j \in N \} \) is a mixture of 2-IID sequences then we have an expression for the directing random measures.

So we may speak of the set of directing random measures as Theorem 4.2 gives an expression for the directing random measures which holds with probability one.

Corollary 4.1
Let \( Y = \{ Y_{ij}; i \in [2], j \in N \} \) be a mixture of 2-IID random variables. Then the directing random measures \( \alpha_1 \) and \( \alpha_2 \) are \( \tau \)-measurable, where \( \tau \) is the tail \( \sigma \)-field of \( Y \).

Proof
From Theorem 4.2 we have
\[
\alpha_i = \Lambda_i \left\{ \{ Y_{ij}; j \in N \} \right\} \text{ a.s.} \forall i \in [2]
\]
where \( \Lambda_i : \mathbb{R}^\infty \to P(\mathbb{R}) \) is as defined in Theorem 4.2. Let \( i \in [2] \). Changing a finite number of random variables in \( \{ Y_{ij}; j \in N \} \) will not change \( \Lambda_i \left\{ \{ Y_{ij}, j \in N \} \right\} \), hence \( \alpha_i \) is measurable w.r.t. \( \tau \), since
\[
\tau = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{F}(Y_{1n+1}, Y_{1n+2}, \ldots, Y_{2m+1}, Y_{2m+2}, \ldots)
\]

The following very useful result will often be referred to.
Theorem 4.3
Let \( Y = \{Y_{ij}, i \in [2], j \in \mathbb{N}\} \) be a mixture of 2-IID sequences and let \( \alpha_1 \) and \( \alpha_2 \) be the directing random measures. Let \( \tau \) denote the tail \( \sigma \)-field of \( Y \). Then the following are all a.s. equal:

(i) \( P \left[ Y_{ij} \in A_{ij}; j \in [n_i], \forall i \in [2]/Y_{1m_1, Y_{1m_1+1}, \ldots, Y_{2m_2, Y_{2m_2+1}, \ldots} \right] m_i > n_i, \forall i \in [2] \)

(ii) \( P \left[ Y_{ij} \in A_{ij}; j \in [n_i], \forall i \in [2]/Y_{1m_1, Y_{1m_1+1}, \ldots, Y_{2m_2, Y_{2m_2+2}, \ldots, \alpha_1, \alpha_2} \right] m_i > n_i, \forall i \in [2] \)

(iii) \( P \left[ Y_{ij} \in A_{ij}; j \in [n_i], \forall i \in [2]/\tau \right] \)

(iv) \( P \left[ Y_{ij} \in A_{ij}; j \in [n_i], \forall i \in [2]/\alpha_1, \alpha_2 \right] \)

(v) \( \prod_{i=1}^{2} \prod_{j=1}^{n_i} \alpha_i(\cdot, A_{ij}) \)

where

\( A_{ij} \in B(\mathbb{R}), \forall i \in [2], \forall j \in \mathbb{N}, m_i \in \mathbb{N}, n_i \in \mathbb{N}, \forall i \in [2] \).

Proof:
From the definition of the tail \( \sigma \)-field of \( Y \) (Definition 3.3) and Corollary 4.1 it follows that (i) and (ii) are equal.

Remark (4.1) shows that \( Y \) is conditionally independent given \( \mathcal{F} (\alpha_1, \alpha_2) \), so that \( \{Y_{ij}; j \in [n_i], \forall i \in [2]\} \) and \( \{Y_{ij}; j \in \{m_i + 1, m_i + 2, \ldots, \}, \forall i \in [2]\} \) are conditionally independent given \( \mathcal{F} (\alpha_1, \alpha_2) \) (i.e. once conditioned on \( \mathcal{F} (\alpha_1, \alpha_2) \), any random variable from the first sequence is independent of any one from the second sequence). By a standard property of conditional independence (Appendix A2, property A3) we see that (ii) = (iv). Also (iv) = (v) follows immediately from Definition 4.5.

Now

\( \mathcal{F} (\alpha_1, \alpha_2) = \mathcal{F} [\mathcal{F} (\alpha_1) \cup \mathcal{F} (\alpha_2)] \)

so that

\( \mathcal{F} (\alpha_1, \alpha_2) \subset \tau \) (4.21)
since $\alpha_1$ and $\alpha_2$ are both $\tau$-measurable (see Corollary 4.1). From the definition of $\tau$ it follows that

$$\tau \subset \mathcal{F} \left( Y_{1m_1}, Y_{1m_1+1}, \ldots, Y_{2m_2}, Y_{2m_2+1}, \ldots, \alpha_1, \alpha_2 \right), m_i \in \mathbb{N}, \ \forall i \in [2]. \quad (4.22)$$

Using (4.21), (4.22) and the fact that (ii) = (iv),

$$P \left[ Y_{ij} \in A_{ij}, j \in [n_i], \ \forall i \in [2]/\tau \right]$$

$$= E \left\{ E[I[Y_{ij} \in A_{ij}, j \in [n_i], \ \forall i \in [2]]/Y_{1m_1}, Y_{1m_1+1}, \ldots, Y_{2m_2}, Y_{2m_2+2}, \ldots, \alpha_1, \alpha_2]/\tau \right\}$$

$$= E \left\{ E[I[Y_{ij} \in A_{ij}, j \in [n_i], \ \forall i \in [2]]/\alpha_1, \alpha_2]/\tau \right\}$$

$$= E \left\{ I[Y_{ij} \in A_{ij}, j \in [n_i], \ \forall i \in [2]]/\alpha_1, \alpha_2 \right\}$$

so that (iii) = (iv).

**Theorem 4.4**

Let $Y = \{Y_{ij}; \ i \in [2], j \in \mathbb{N}\}$ be an infinite 2-fold sequence of random variables. If $Y$ is a mixture of 2-IID random variables then $Y$ is conditionally 2-IID given $\tau$, the tail $\sigma$-field of $Y$.

**Proof:**

Let $A_{ij} \in B(\mathbb{R}) \ \forall j \in [n_i], \ \text{for} \ n_i \in \mathbb{N}, \ \forall i \in [2]$. Then from Definition 4.5,

$$P \left[ Y_{ij} \in A_{ij}, j \in [n_i], \ \forall i \in [2]/\alpha_1, \alpha_2 \right] = \prod_{i=1}^{n_i} \prod_{j=1}^{n_i} \alpha_i(\cdot, A_{ij}). \quad (4.23)$$

Let $j \in [n_i], n_i \in \mathbb{N}, \ \forall i \in [2]$. Then from Remark 4.1 and Theorem 4.3 ((iii) = (iv)), $\forall A_{ij} \in B(\mathbb{R}),$

$$\alpha_i(\cdot, A_{ij}) = P \left( Y_{ij} \in A_{ij}/\alpha_1, \alpha_2 \right)$$

$$= P \left( Y_{ij} \in A_{ij}/\alpha_1, \alpha_2 \right)$$

$$= P \left( Y_{ij} \in A_{ij}/\tau \right) \quad \forall j \in [n_i], \ \forall i \in [2]. \quad (4.24)$$
So that
\[ \prod_{i=1}^{2} \prod_{j=1}^{n_i} \alpha_i(\cdot, A_{ij}) = \prod_{i=1}^{2} \prod_{j=1}^{n_i} P[Y_{ij} \in A_{ij}/\tau] \text{ a.s. .} \] (4.25)

Now consider the right hand side of (4.23). Using (iii) = (iv) in Theorem 4.3 we obtain
\[ P[Y_{ij} \in A_{ij}; j \in [n_i], \forall i \in [2]/\alpha_1, \alpha_2] = P[Y_{ij} \in A_{ij}; j \in [n_i], \forall i \in [2]/\tau] \]
and substituting this last result and (4.25) with (4.23) we see that
\[ P[Y_{ij} \in A_{ij}; j \in [n_i], \forall i \in [2]/\tau] = \prod_{i=1}^{2} \prod_{j=1}^{n_i} P[Y_{ij} \in A_{ij}/\tau] . \]

So that \( Y \) is conditionally independent given \( \tau \). The identical distributions within types given \( \tau \) comes from (4.24) since
\[ P[Y_{ij} \in A_{ij}/\tau] = P[Y_{ij} \in A_{ij}/\alpha_1, \alpha_2] \text{ a.s.} \]
and the right hand side of this equation is independent of \( j \) as far as \( Y_{ij} \) is concerned (see Remark 4.1).

**Corollary 4.2**
Let \( Y = \{Y_{ij}; i \in [2], j \in N\} \) be an infinite 2-fold sequence of random variables. If \( Y \) is a mixture of 2-IID random variables, then it is directed by random measures \( \alpha_1 \) and \( \alpha_2 \), and we have three possible expressions for them

(i) \( \alpha_i = \Lambda_i (\{Y_{ij}; j \in N\}) \text{ a.s., where } \Lambda_i \text{ is as in Theorem 4.2.} \)

(ii) \( \alpha_i \) is an r.c.d. for \( Y_{i1} \) given \( \tau \).

(iii) \( \alpha_i \) is an r.c.d. for \( Y_{i1} \) given \( \mathcal{F} (Y_{1m_1}, Y_{1m_1+1}, \ldots, Y_{2m_2}, Y_{2m_2+1}, \ldots) \)
\[ m_i \in N , \forall i \in [2] . \]

**Proof:**
Expression (i) for the d.r.m.'s comes from Theorem 4.2. From (4.24),
\[ \alpha_i (\cdot, A_{ij}) = P[Y_{ij} \in A_{ij}/\tau] \quad \forall j \in [n_i] \]
so that
\[ \alpha_i(.; A_{ij}) = P[Y_{i1} \in A_{ij}/\tau] \] 
(4.26)

and hence (ii) follows.

Taking \( n_i = 1 \), \( \forall i \in [2] \), using Theorem 4.3 (i) = (iii) and (4.26) gives (iii).

Remarks
Corollary 4.2 provides three expressions for the d.r.m.'s, they are all a.s. equal (as the d.r.m.'s are a.s. unique). Each of the expressions is useful in some circumstances.

4.2 PARTIALLY EXCHANGEABLE RANDOM VARIABLES, MIXTURES AND DIRECTION

In this section we obtain results for partially exchangeable sequences by applying the theory of §4.1.

Throughout this section \( Z = \{Z_{ij}; i \in [2], j \in \mathbb{N}\} \) will denote an infinite (2-fold) sequence of random variables.

De Finetti's theorem for 2-fold partially exchangeable random variables (Theorem 3.3) may be restated as follows:

Theorem 4.5
If \( Z \) is partially exchangeable then it is a mixture of 2-IID sequences, directed by \( \alpha_1 \) and \( \alpha_2 \), where
\[ \alpha_i(.; A_{ij}) = P[Z_{i1} \in A_{ij}/\tau] \quad \forall i \in [2], \forall j \in \mathbb{N}, \forall A_{ij} \in B(\mathbb{R}) \] 
(4.27)
and \( \tau \) denotes the tail \( \sigma \)-field of \( Z \).

Proof:
The desired result follows immediately from Theorems 3.2 and 4.1 (i).

Remark
Using Theorems 3.1 and 4.1 we could similarly show that a partially ex-
changeable sequence $Z$ is a mixture of 2-IID sequences, directed by

$$\alpha_i(\cdot, A_{ij}) = P \left[ Z_{i1} \in A_{ij} / \mathcal{F}_\infty \right], \quad i \in [2]$$  \hspace{1cm} (4.28)

where $\mathcal{F}_\infty$ is as in Theorem 3.1.

**Corollary 4.3**

$Z$ is a mixture of 2-IID sequences directed by $\alpha_1$ and $\alpha_2$ (as in (4.27) or (4.28)) if and only if $Z$ is partially exchangeable.

**Proof:**

Suppose that $Z$ is a mixture of 2-IID sequences, directed by $\alpha_1$ and $\alpha_2$ and let

$$\mathcal{F}_\infty(\cdot, A) = \prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij})$$  \hspace{1cm} (4.29)

where

$$A = \prod_{i=1}^{2} \prod_{j=1}^{\infty} A_{ij}, \text{ i.e. } A \in B(\mathbb{R}^\infty \times \mathbb{R}^\infty).$$

Then, using Lemma 4.1 and Definition 4.5,

$$\prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}) = \int I_{\{Z(\omega) \in A\}} F^\infty(\omega, dZ(\omega)).$$  \hspace{1cm} (4.30)

The right hand side of (4.30) shows that $Z$ is partially exchangeable. The reverse implication follows from Theorem 4.5.

**Remark**

Partially exchangeable sequences are usually defined in terms of the invariance of their joint distribution under permutations of the variables (see Definition 2.7). In view of Corollary 4.3 however, Definition 2.7 may be replaced by the following:

$Z$ is (2-fold) partially exchangeable if there exist random measures $\alpha_1$ and $\alpha_2$ such that $Z$ is a mixture of 2-IID sequences, directed by $\alpha_1$ and $\alpha_2$.

Aldous ([1977], page 61) defines an exchangeable sequence in this manner.
Theorem 4.6
If $\mathcal{F}$ is an arbitrary $\sigma$-field such that $Z$ is conditionally 2-IID given $\mathcal{F}$ then

(i) $Z$ is partially exchangeable

(ii) $Z$ and $\mathcal{F}$ are conditionally independent given $\mathcal{F}(\alpha_1, \alpha_2)$ where

$$\alpha_i(\cdot, A) = P[Z_{i1} \in A | \mathcal{F}] \quad \forall i \in [2] \quad \forall A \in B(\mathbb{R}). \quad (4.31)$$

Proof:
The desired result follows immediately from Theorem 4.1 and Corollary 4.3.

Remark 4.2 (numbered for future reference)
If there exists a $\sigma$-field $\mathcal{F}$ such that $Z$ is conditionally 2-IID given $\mathcal{F}$, then we may refer to $Z$ as being partially exchangeable without particular reference to the two d.r.m.'s (bearing in mind that they are of the form given in (4.31)).

Similarly, de Finetti’s theorem can be loosely stated as:

**Partially exchangeable sequences are mixtures of 2-IID sequences**

i.e. no reference to the two d.r.m.’s.

The next theorem however, gives various expressions for the d.r.m.’s of partially exchangeable sequences.

Theorem 4.7
If $Z$ is a partially exchangeable sequence, then it is directed by random measures $\alpha_1$ and $\alpha_2$ and we have three possible expressions for them.

(i) $\alpha_i = \Lambda_i \left( \{Z_{ij}; j \in \mathbb{N}\} \right)$ a.s. where $\Lambda_i$ is as in Theorem 4.2.

(ii) $\alpha_i$ is an r.c.d. for $Z_{i1}$ given $r$ the tail $\sigma$-field of $Z$.

(iii) $\alpha_i$ is an r.c.d. for $Z_{i1}$ given $\mathcal{F}(Z_{1m_1}, Z_{1m_1+1}, \ldots, Z_{2m_2}, Z_{2m_2+1}, \ldots)$

$$\forall i \in [2], \quad m_1 \in \mathbb{N}, \quad m_2 \in \mathbb{N}.$$
Proof: The desired result follows immediately from Theorem 4.5 and Corollary 4.2.

Theorem 4.4 leads us to the following very useful result:

**Theorem 4.8**
If \( Z \) is partially exchangeable, then \( Z \) is conditionally 2-IID given \( \tau \), the tail \( \sigma \)-field of \( Z \).

**Proof:**
If \( Z \) is partially exchangeable then it is a mixture of 2-IID sequences (Remark 4.2) and the result follows on application of Theorem 4.4.

The following definitions will be used to obtain other \( \sigma \)-fields which can replace the tail \( \sigma \)-field in Theorem 4.8.

**Definition 4.7**
Let \( Z \) be a partially exchangeable sequence of random variables and let \( V \) be any random variable. Then \( Z \) is *partially exchangeable over \( V \)* if

\[
L(V, Z) = L(V, Z_\pi)
\]

where

\[
Z_\pi = \{ Z_{i\pi(j)} ; \ i \in [2], \ j \in \mathbb{N} \}
\]

for \( \pi \), a finite permutation of \( \{1, 2, \ldots \} \), \( i \in [2] \).

**Definition 4.8**
Let \( Z \) be a partially exchangeable sequence of random variables and let \( \mathcal{G} \) be any \( \sigma \)-field. Then \( Z \) is *partially exchangeable over \( \mathcal{G} \)* if (4.32) holds for each random variable \( V \) which is measurable with respect to \( \mathcal{G} \).

**Lemma 4.3**
Let \( Z \) be a partially exchangeable sequence of random variables and let \( V \) be a random variable such that \( Z \) is partially exchangeable over \( V \). Then

(i) \( Z \) is conditionally 2-IID given \( (V, \alpha_1, \alpha_2) \) where \( \alpha_1 \) and \( \alpha_2 \) are the d.r.m.'s for \( Z \).
(ii) \(Z\) and \(V\) are conditionally independent given \(\mathcal{F}(\alpha_1, \alpha_2)\).

**Proof:**

Define a new 2-fold infinite sequence of random variables as follows:

\[
\hat{Z} = \{ \hat{Z}_{ij}; i \in [2], j \in \mathbb{N} \}
\]

where

\[
\hat{Z}_{ij} = (V, Z_{ij}) \quad \forall i \in [2], \forall j \in \mathbb{N}.
\]

Then \(\hat{Z}\) is clearly partially exchangeable. If we denote the d.r.m.'s of \(\hat{Z}\) by \(\hat{\alpha}_1\) and \(\hat{\alpha}_2\) then \(\hat{Z}\) is conditionally 2-IID given \(\mathcal{F}(\hat{\alpha}_1, \hat{\alpha}_2)\) (see (4.28)). In particular \(Z\) is conditionally 2-IID given \(\mathcal{F}(\hat{\alpha}_1, \hat{\alpha}_2)\).

Applying Theorem 4.7 (i) to \(\hat{Z}\) we see that for each \(i \in [2]\)

\[
\hat{\alpha}_i = A_i \left( \{ \hat{Z}_{ij}; j \in \mathbb{N} \} \right)
\]

\[
= \lim_{n_i \to \infty} n_i^{-1} \sum_{j=1}^{n_i} \delta_{\hat{Z}_{ij}}
\]

(4.33)

where, \(\forall A \in B(\mathbb{R}^2),\)

\[
\delta_{\hat{Z}_{ij}}(A) = \begin{cases} 
1 & \hat{Z}_{ij} = (V, Z_{ij}) \in A \\
0 & \text{elsewhere.}
\end{cases}
\]

(4.34)

If \(A = A_1 \times A_2\) for \(A_i \in B(\mathbb{R}) \forall i \in [2]\) then

\[
\delta_{\hat{Z}_{ij}}(A) = \delta_{Z_{ij}}(A_1) \delta_V(A_2) \quad \forall j \in \mathbb{N}
\]

where

\[
\delta_{Z_{ij}}(A_1) = I[Z_{ij} \in A_1] \quad \forall j \in \mathbb{N} \text{ and } \delta_V(A_2) = I[V \in A_2]
\]

so that (4.33) becomes

\[
\hat{\alpha}_i = \lim_{n_i \to \infty} n_i^{-1} \sum_{j=1}^{n_i} \delta_{Z_{ij}} \delta_V
\]

\[
= \delta_V \lim_{n_i \to \infty} n_i^{-1} \sum_{j=1}^{n_i} \delta_{Z_{ij}}
\]
and hence, from Theorem 4.7 (i),

$$\hat{\alpha}_i = \delta V \alpha_i .$$

(4.35)

We thus have a partially exchangeable sequence $\hat{Z}$ with d.r.m.'s $\delta V \alpha_1$ and $\delta V \alpha_2$. It thus follows that $\mathcal{F}(\hat{\alpha}_i) = \mathcal{F}(V, \alpha_i) \forall i \in [2]$, and hence

$$\mathcal{F}(\hat{\alpha}_1, \hat{\alpha}_2) = \mathcal{F}(V, \alpha_1, \alpha_2) .$$

We have thus shown (i) since $Z$ is conditionally 2-IID given $\mathcal{F}(\hat{\alpha}_1, \hat{\alpha}_2)$.

From (i) and Theorem 4.6 we see that $Z$ and $\mathcal{F}(V, \alpha_1, \alpha_2)$ are conditionally independent given $\mathcal{F}(\alpha_1, \alpha_2)$, i.e.

$$P[Z \in A/\mathcal{F}(V, \alpha_1, \alpha_2), \mathcal{F}(\alpha_1, \alpha_2)] = P[Z \in A/\mathcal{F}(\alpha_1, \alpha_2)]$$

so that

$$P[Z \in A/\mathcal{F}(V), \mathcal{F}(\alpha_1, \alpha_2)] = P[Z \in A/\mathcal{F}(\alpha_1, \alpha_2)]$$

and hence $Z$ and $V$ are conditionally independent given $\mathcal{F}$ (see Appendix A2, point A3) so that we have shown (ii).

We are now in a position to prove a very important result.

**Theorem 4.9**

Let $Z$ be a partially exchangeable sequence of random variables, directed by $\alpha_1$ and $\alpha_2$. Then

$$E_Z = \tau_Z = \mathcal{F}(\alpha_1, \alpha_2) \text{ a.s.}$$

(4.36)

where $E_Z$ and $\tau_Z$ are the partially exchangeable and tail $\sigma$-fields of $Z$ respectively, (see Definitions 3.3 and 3.5).

**Proof:**

Suppose that $A \in \tau_Z$. Then $A \in \mathcal{F}_{nm}^\infty$ for infinitely many values of $n, m \in \mathbb{N}$, where

$$\mathcal{F}_{nm}^\infty = \mathcal{F}(Z_{1n+1}, Z_{1n+2}, \ldots, Z_{2m+1}, Z_{2m+2}, \ldots)$$

and hence $A \in E_Z$ so that

$$\tau_Z \subset E_Z .$$

(4.37)
For \( B \in E_Z \), define random variable

\[
I_B(\omega) = \begin{cases} 1 & \omega \in B \\ 0 & \omega \notin B \end{cases}
\]

(4.38)

Then \( B \) is a subset of \( \Omega \) which \( Z \) maps to \( \mathbb{R}^\infty \times \mathbb{R}^\infty \) such that \( \forall \omega \in B \),

\[
Z(\omega) = Z_*(\omega)
\]

where

\[
Z_* = \{ Z_{i\pi_i(j)} ; i \in [2], j \in \mathbb{N} \}
\]

and \( \pi_i \) is any finite permutation of \( \{1, 2, \ldots \} \), \( \forall i \in [2] \).

Clearly then, since \( Z \) is partially exchangeable we obtain

\[
L(I_B, Z) = L(I_B, Z_*)
\]

Definition 4.7 shows that \( Z \) is partially exchangeable over \( I_B \) for \( B \in E_Z \). Lemma 4.3 then yields the following:

\( Z \) and \( I_B \) are conditionally independent given \( \mathcal{F}(\alpha_1, \alpha_2) \).

Using Definition 4.8, we thus have the following:

\( Z \) and \( E_Z \) are conditionally independent given \( \mathcal{F}(\alpha_1, \alpha_2) \).

Clearly \( E_Z \subset \mathcal{F}(Z) \) so that \( E_Z \) is conditionally independent of itself given \( \mathcal{F}(\alpha_1, \alpha_2) \). Hence

\[
E_Z \subset \mathcal{F}(\alpha_1, \alpha_2) \quad \text{a.s.}
\]

(4.39)

follows from A6 in Appendix A2. From Corollary 4.1 and Theorem 4.5,

\[
\mathcal{F}(\alpha_1, \alpha_2) \subset \tau_Z \quad \text{a.s.}
\]

(4.40)

Equations (4.39) and (4.40) thus yield

\[
E_Z \subset \mathcal{F}(\alpha_1, \alpha_2) \subset \tau_Z \quad \text{a.s.}
\]

(4.41)

Combining this with (4.37) we have

\[
E_Z = \mathcal{F}(\alpha_1, \alpha_2) = \tau(Z) \quad \text{a.s.}
\]

as required.
Remarks
From Theorem 4.8, Corollary 4.3 and Theorem 4.9 we see that $Z$ is partially exchangeable, directed by $\alpha_1$ and $\alpha_2$, if and only if $Z$ is conditionally 2-IID given either

(i) $\tau_Z$, the tail $\sigma$-field of $Z$

or

(ii) $E_Z$, the partially exchangeable $\sigma$-field of $Z$

or

(iii) $\mathcal{F}(\alpha_1, \alpha_2)$, where $\alpha_1$ and $\alpha_2$ are the d.r.m.'s of $Z$.

Theorem 4.6 thus gives the following three possible expressions for d.r.m.'s $\alpha_i$, $i \in [2]$.

Let $A \in B(\mathbb{R})$, then $\alpha_i(\cdot, A)$, the d.r.m., is any one of the following three expressions:

(i) $P(Z_{i1} \in A/\tau_Z)$ (see Theorem 4.7 (ii))

(ii) $P(Z_{i1} \in A/E_Z)$

(iii) $P(Z_{i1} \in A/\alpha_1, \alpha_2)$.

Remark
The three expressions for the set of directed random variables are a.s. equal as can be seen from Theorem 4.9. We may thus speak of the set of directed random measures.

4.3 CONSTRUCTION OF PARTIALLY EXCHANGEABLE RANDOM VARIABLES

The purpose of this section is to point out some concrete ways of constructing partially exchangeable sequences. Most of the results are direct consequences of de Finetti's theorem (Theorem 3.3) and show how to manipulate random measures.
Throughout this section $Z = \{Z_{ij}; i \in [2], j \in \mathbb{N}\}$ will denote an infinite 2-fold sequence of random variables. If $Z$ is partially exchangeable, then we denote the d.r.m.'s by $\alpha_i, i \in [2]$.

The next two examples may be considered to be degenerate partially exchangeable sequences:

(i) If $Z$ is 2-IID, variables of type $i$ having distribution $\theta_i, \forall i \in [2]$, then $Z$ may be viewed as a partially exchangeable sequence directed by $\theta_i, \forall i \in [2]$.

(ii) If $Z$ has $Z_{i1} = Z_{i2} = \cdots$ a.s., $\forall i \in [2]$, then $Z$ may be viewed as a partially exchangeable sequence with $\alpha_i(\cdot, A) = I(X_i \in A)$ $\forall A \in B(\mathbb{R})$ for some fixed random variable $X_i, \forall i \in [2]$.

A natural way to construct a partially exchangeable sequence (this was hinted at after Corollary 3.1) is to take a parametric family of distributions, choose the parameters randomly and then take a 2-IID sequence which has a distribution with these random parameters. The next example demonstrates this method.

Example
Denote the Normal $(\theta, \sigma^2)$ distribution by $\mu_{\theta, \sigma}$. Let $X = \{X_{ij}; i \in [2], j \in \mathbb{N}\}$ be a 2-IID sequence, $X_{ij} \sim N(0,1), \forall i \in [2], \forall j \in \mathbb{N}$.

If $Z$ is defined by

$$Z_{ij} = \Theta_i + S_i X_{ij}, \forall i \in [2], \forall j \in \mathbb{N}$$

where $\Theta_i$ and $S_i$ are random variables denoting the mean and standard deviation of $Z_{ij}$, then $Z$ is a partially exchangeable sequence of random variables, directed by $\alpha_i = \mu_{\Theta_i, S_i}, \forall i \in [2]$ (conditioning on $\Theta_i = \theta_i$, $S_i = \sigma_i, \forall i \in [2]$, clearly gives a 2-IID sequence).

In general however, $Z$ is a more complicated mixture of parametric families as the following examples demonstrate.
Example
Let \( Y = \{Y_{ij}; \ i \in [2], \ j \in \mathbb{N}\} \) and \( X = \{X_{ij}; \ i \in [2], \ j \in \mathbb{N}\} \) be 2-fold infinite sequences of random variables. Suppose that \( X \) is a 2-IID sequence, independent of \( Y \), such that \( \{X_{ij}; \ j \in \mathbb{N}\} \) takes distinct values from \( \{1, 2, \ldots\} \), \( \forall i \in [2] \). If \( Z \) is defined by
\[
Z_{ij} = Y_{i}X_{ij}, \quad \forall i \in [2], \forall j \in \mathbb{N}.
\]
then \( Z \) is partially exchangeable.

Using Theorem 4.7 (i) we next find expressions for the d.r.m.'s.

Let \( i \in [2] \),
\[
\alpha_i = \Lambda_i (\{Z_{ij}; \ j \in \mathbb{N}\})
\]
\[
= \lim_{n_i \to \infty} \sum_{j=1}^{n_i} \delta_{Y_{ij}} P(X_{i1} = j)
\]
\[
= \lim_{n_i \to \infty} \sum_{j=1}^{n_i} \delta_{Y_{ij}} P(X_{i1} = j)
\]
\[
= \sum_{j=1}^{\infty} \delta_{Y_{ij}} P_{ij}
\]
where
\[
P_{ij} = P(X_{i1} = j) \quad \forall j \in \mathbb{N}.
\]
The following simple method of construction also yields a partially exchangeable sequence.

Example 4.1 (numbered for future reference)
Let \( X = \{X_{ij}; \ i \in [2], \ j \in \mathbb{N}\} \) be a 2-fold infinite sequence of random variables. Suppose that \( X \) is a 2-IID sequence, variables of type \( i \) having distribution \( \theta_i \), \( \forall i \in [2] \). Let \( Y \) be an arbitrary random variable, which is independent of \( X \), with distribution \( \phi \). If \( Z \) is defined by
\[
Z_{ij} = f(Y, X_{ij}), \quad \forall i \in [2], \forall j \in \mathbb{N}
\]
for \( f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) an arbitrary Borel-measurable function, then \( Z \) is partially exchangeable. In order to find the d.r.m.'s of this sequence we recall the induced map as defined in Definition 3.4 for the special case \( S = \mathbb{R} \).
If \( f \) has induced map \( \tilde{f} : \mathbb{R} \times P(\mathbb{R}) \rightarrow P(\mathbb{R}) \) (see Definition 3.4 (ii)) then Corollary 4.2 (i) shows that \( Z \) is directed by \( f(Y, \theta_1) \) and \( f(Y, \theta_2) \).

**Remark**

Let \( Z \) be a partially exchangeable sequence of random variables directed by \( \alpha_1 \) and \( \alpha_2 \) and suppose that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is an arbitrary Borel measurable function. If \( f(Z) = \{f(Z_{ij}); i \in [2], j \in \mathbb{N}\} \) then \( f(Z) \) is a partially exchangeable sequence of random variables (see Theorem 2.2) and using Corollary 4.2 (i) we see that \( f(Z) \) is directed by \( \tilde{f}(\alpha_1) \) and \( \tilde{f}(\alpha_2) \).
CHAPTER 5
LIMIT THEOREMS

To conclude this thesis we present some limit theorems for partially exchangeable random variables. These limit theorems are immediate consequences of the main result of this chapter:

Given any a.s. limit theorem for 2-IID sequences of random variables, there exists an analogous theorem satisfied by all (2-fold) partially exchangeable sequences of random variables and by all sub-subsequences of some infinite subsequence of an arbitrary dependent 2-fold infinite sequence of random variables, tightly distributed within types.

Section 5.1 is devoted to notation, definitions, results and a brief discussion of the above mentioned theorem. This shows the curious link between some of the limit theorems of partially exchangeable sequences and those of sub-subsequences of arbitrary 2-fold infinite sequences.

The proof of the above mentioned main result appears in §5.2 while §5.3 is devoted to limit theorems for partially exchangeable sequences which follow from the main result.

5.1 NOTATION, DEFINITIONS AND DISCUSSION OF RESULTS

Komlós (1967) showed that if \( \{X_i, \ i \in \mathbb{N}\} \) is a sequence of random variables such that \( \sup E|X_i| < \infty \) then there exists an increasing sequence of integers \( \{n_1, n_2, \ldots\} \) and a random variable \( V \) such that

\[
\lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} X_{n_i} = V \quad \text{a.s.}
\]

This result was the prototype for the well-known Chatterji principle (Chatterji (1974a, 1974b, 1985 and 1986)):

For every a.s. limit theorem for IID random variables under certain moment conditions, there exists an analogous theorem such
that an arbitrarily-dependent sequence (under the same moment conditions) always contains a subsequence satisfying this analogous theorem.

Note that Komlós' result demonstrates Chatterji's principle where the limit theorem in question is the strong law of large numbers.

Chatterji's principle in conjunction with the relationship between IID and exchangeable random variables (see Chapter 3) led to the following conjecture of Kingman (1978):

Every limit property enjoyed by all exchangeable sequences is shared by some subsequence of every tight sequence.

Aldous provided a counterexample to the above conjecture (see Kingman (1978), p.190) and proved the following result in place of it:

For every a.s. limit theorem for IID random variables there exists an analogous theorem satisfied by all exchangeable sequences and by all sub-subsequences of some subsequence of an arbitrarily dependent tight sequence of random variables.

Following what Kingman calls "a brilliant display" by Aldous to prove the previous result (see Aldous (1977)) we extend these results in §5.2 to an infinite 2-fold sequence of random variables, a very powerful result which is an extension of Chatterji's principle and forms the main result of this chapter.

Unless otherwise indicated the following notation will be used here and in §5.2:

For \( \lambda \in P(\mathbb{R}) \), let \( \lambda^* \in P(\mathbb{R}^\infty) \) be the infinite product measure \( \lambda \times \lambda \times \cdots \). For a random measure \( \mu \), let \( \mu^* \) be the random map into \( P(\mathbb{R}^\infty) \) such that \( \mu^*(\omega) = \mu(\omega) \times \mu(\omega) \times \cdots \).

We shall be dealing with 2-fold infinite sequences and we shall use \( \mu(\nu) \) to denote the random measure associated with random variables of type 1 (2).

**Remark**

Clearly \( X = \{X_{ij}; i \in [2], j \in \mathbb{N}\} \) is a 2-fold infinite sequence of random
variables with \( L(X) = \lambda_1^* \times \lambda_2^* \), \( \lambda_i \in P(\mathbb{R}) \) \( \forall i \in [2] \), is a concise way of saying that \( X \) is a 2-IID sequence of random variables.

Introduce a special notation for subsequences of 2-fold sequences. Write \( n = (n_1, n_2) \) and \( m = (m_1, m_2) \) where \( n_1 = (n_{11}, n_{12}, n_{13}, \ldots) \) and \( m_1 = (m_{11}, m_{12}, m_{13}, \ldots) \) and these sequences denote strictly increasing sequences of positive integers (similarly \( n_2 \) and \( m_2 \)). Write \( m \subseteq n \) to mean that \( m_i \) is a subsequence of \( n_i \) \( \forall i \in [2] \). Write \( X_n \) for the subsequence \( \{X_{1n_{11}}, X_{1n_{12}}, \ldots, X_{2n_{21}}, X_{2n_{22}}, \ldots\} \) of \( X \), a 2-fold infinite sequence of random variables.

The following result will often be referred to in \$5.2\$.

**Theorem 5.1**

A 2-fold infinite sequence of random variables, \( Z = \{Z_{ij}; i \in [2], j \in \mathbb{N}\} \), is partially exchangeable if there exist random measures \( \mu^* \) and \( \nu^* \) such that

\[
\mu^* \times \nu^* \text{ is a r.c.d. for } Z \text{ given } f(\mu, \nu).
\]

**Proof**

The result follows immediately from Corollary 4.3.

**Remark**

Call the pair \((\mu, \nu)\) of the above theorem the 2-fold canonical random measure (2-c.r.m.) associated with the partially exchangeable sequence \( Z \).

**Remark 5.1**

Let \( X = \{X_i; i \in \mathbb{N}\} \) be a sequence of random variables. Then \( X_i \rightarrow X \sigma(L', L^\infty) \) for random variable \( X \) means that

\[
\lim_i E(X_i/B) = E(X/B)
\]

for each \( B \in B(\mathbb{R}) \), \( P(B) > 0 \).

**Remark 5.2**

It is well known (Meyer (1966), T23) that for \( X = \{X_i, i \in \mathbb{N}\} \) to be \( \sigma(L', L^\infty) \) relatively sequentially compact it is necessary and sufficient that \( X \) be uniformly integrable, and in particular it is sufficient that \( X \) be \( L^p \)-bounded for some \( p > 1 \).
Definition 5.1
For \( \lambda \in P(\mathbb{R}) \), write

\[
|\lambda|^p = \int_{-\infty}^{\infty} |x|^p \lambda(dx) \quad 0 < p < \infty \quad (5.2)
\]

\[
|\lambda|_1 = \int_{-\infty}^{\infty} x \lambda(dx)
\]

\[
|\lambda|_2 = \int_{-\infty}^{\infty} x^2 \lambda(dx) - (|\lambda|_1)^2.
\]

We refer to \( |\lambda|_1 \) and \( |\lambda|_2 \) as the mean and the variance of \( \lambda \) respectively. For definiteness write \( |\lambda|_1 = \infty \) if \( |\lambda|^1 = \infty \) and \( |\lambda|_2 = \infty \) if \( |\lambda|^2 = \infty \).

Remark
Tjur (1980) discusses distributions of random measures on the metrisable space (Parthasarathy (1967), T6.2) \( P(\mathbb{R}) \). Billingsley ((1968), p.238) defines moments of measures which are consistent with the definitions for \( |\lambda|_1 \) and \( |\lambda|_2 \) as given in Definition 5.1.

For the remainder of this section and §5.2 (unless otherwise indicated) assume \( X = \{X_{ij}; i \in [2], j \in \mathbb{N}\} \) to be a 2-fold infinite sequence of random variables such that \( \{L(X_{ij}); j \in \mathbb{N}\} \) is tight \( \forall i \in [2] \) (see Appendix A1).

The next two lemmas are extensions of work done by Aldous ((1977), pp.61-62, Lemma 2 and equations (3.2) and (3.3)). We re-state his result for each type of random variable separately, making the obvious adjustments.

Lemma 5.1

\[
E|\mu(\omega)|^p \leq \limsup_j E|X_{1j}|^p, \quad 0 < p < \infty
\]

\[
E|\nu(\omega)|^p \leq \limsup_j E|X_{2j}|^p, \quad 0 < p < \infty
\]

where

\[
E|X_{ij}|^p = \int \omega |X_{ij}(\omega)|^p dP(\omega), \quad i \in [2], \quad 0 < p < \infty.
\]
Lemma 5.2

If
\[ \alpha_1(\omega) = |\mu(\omega)|_1, \quad \alpha_2(\omega) = |\nu(\omega)|_1 \]
\[ \beta_1(\omega) = |\mu(\omega)|_2, \quad \beta_2(\omega) = |\nu(\omega)|_2 \]
then
\[ \sup_j E|X_{1j}| < \infty \quad \text{implies} \quad \alpha_1(\omega) < \infty \quad \text{a.s.} \]
\[ \sup_j E|X_{2j}| < \infty \quad \text{implies} \quad \alpha_2(\omega) < \infty \quad \text{a.s.} \]
\[ \sup_j E|X_{1j}|^2 < \infty \quad \text{implies} \quad \beta_1(\omega) < \infty \quad \text{a.s.} \]
\[ \sup_j E|X_{2j}|^2 < \infty \quad \text{implies} \quad \beta_2(\omega) < \infty \quad \text{a.s.}. \]

Remark
The previous two lemmas provide us with some technical results which random measures \( \mu \) and \( \nu \), which we construct in §5.2, satisfy.

Results in §5.2 are stated in terms of \( X, \mu \) and \( \nu \). When applying these results to specific theorems \( \alpha_i \) and \( \beta_i, i \in [2] \) play the role of means and variances in the 2-iid case.

Consider the following special property \( \beta \) which \( X \) might possess:

Definition 5.2

\( X \) possesses property \( \beta \) if there exists a (2-fold) partially exchangeable sequence of random variables \( Z = \{Z_{ij}; i \in [2], j \in \mathbb{N}\} \) such that

(i) \( (\mu, \nu) \) is the 2-c.r.m. for \( Z \)

(ii) \[ \sum_{i=1}^{2} \sum_{j=1}^{\infty} |X_{im,ij} - Z_{ij}| < \infty \quad \text{a.s. for some m.} \]

Remark
In §5.2 the results will be formulated so as to be almost obvious if \( X \) possesses property \( \beta \). However, \( X \) need not generally possess property \( \beta \) (see
Aldous (1977), p.81 for the trivial case where only one type of random variable is considered).

In the introduction to Chapter 5 mention was made of the main result of §5.2 which concerns *a.s. limit theorems for 2-IID random variables*. Exactly what is meant by this statement will be made clear by the following definition.

**Definition 5.3**

A *statute* $A$ is a measurable subset of $P(\mathbb{R}) \times P(\mathbb{R}) \times \mathbb{R}^\infty \times \mathbb{R}^\infty$ such that for each $(\lambda_1, \lambda_2) \in P(\mathbb{R}) \times P(\mathbb{R}),$

$$ (\lambda_1, \lambda_2, X(\omega)) \in A \text{ a.s. when } L(X) = \lambda_1^* \times \lambda_2^*. \quad (5.3) $$

This is equivalent to

$$ (\lambda_1^* \times \lambda_2^*) \{X(\omega) \in \mathbb{R}^\infty \times \mathbb{R}^\infty; (\lambda_1, \lambda_2, X(\omega)) \in A\} = 1 \quad (5.4) $$

or

$$ P \{\omega; (\lambda_1, \lambda_2, X(\omega)) \in A\} = 1 \quad (5.5) $$

when $X$ is a 2-IID sequence of random variables, $\lambda_i$ being the common distribution of random variables of type $i$, $\forall i \in [2]$.

The following examples of statutes representing some well-known a.s. limit theorems for IID random variables (i.e. for the $g = 1$ case where only one type of random variable is considered) appear in Aldous (1977):

**Example 5.1**

Let $X = \{X_i; i \in \mathbb{N}\}$ be an infinite sequence of IID random variables, then Kolmogorov's SLLN and the Law of the Iterated Logarithm (Shirayayev (1984), pp.366, 372 respectively) may be represented by the following statutes:

For each $\lambda \in P(\mathbb{R}),$

$$ A_1 = \left\{ (\lambda, X(\omega)); \lim_{n \to \infty} n^{-1} \sum_{i=1}^{n} X_i(\omega) = |\lambda|_1 \right\} \cup \left\{ (\lambda, X(\omega)); |\lambda|_1 = \infty \right\} \quad (5.6) $$

$$ A_2 = \left\{ (\lambda, X(\omega)); \limsup_{N \to \infty} \frac{1}{2N \log \log N} \sum_{i=1}^{N} X_i(\omega) - N|\lambda|_1 = (|\lambda|_2)^{\frac{1}{2}} \right\} \cup \left\{ (\lambda, X(\omega)); |\lambda|_2 = \infty \right\}. \quad (5.7) $$
Remark
The above two statutes will be extended in §5.3 to allow two types of random variables, i.e. an analogue of Kolmogorov’s SLLN and the Law of the Iterated Logarithm as a.s. limit theorems for 2-IID sequences will be presented in statute form.

Definition 5.4
A statute $A$ is said to be a limit statute if

$$\left(\lambda_1, \lambda_2, X(\omega) \right) \in A \text{ and } \sum_{i=1}^{2} \sum_{j=1}^{\infty} |X_{ij}(\omega) - X_{ij}(\omega)| < \infty$$

implies

$$\left(\lambda_1, \lambda_2, X'(\omega) \right) \in A.$$

5.2 PARTIAL EXCHANGEABILITY AND SUBSEQUENCES OF ARBITRARY DEPENDENT 2-FOLD INFINITE SEQUENCES OF RANDOM VARIABLES

As mentioned in the introduction to this chapter and in §5.1, this section contains a very powerful result which shows the link between a.s. limit theorems for 2-IID sequences of random variables and those of (i) partially exchangeable sequences and (ii) sub-subsequences of some subsequence of an arbitrary 2-fold infinite sequence of random variables which have a tight distribution (Appendix A1) for each type of random variable.

Any a.s. limit theorem for 2-IID random variables may be represented in statute form (see §5.3) and it thus remains to be shown that partially exchangeable sequences and the particular subsequences mentioned may be used in any statute.

For the remainder of this section, unless otherwise indicated, assume $X = \{X_{ij}; i \in [2], j \in \mathbb{N}\}$ to be a 2-fold infinite sequence of random variables, tightly distributed (Appendix A1) within types. To briefly describe the technique used we extract a subsequence $Y = \{Y_{ij}; i \in [2], j \in \mathbb{N}\}$ from $X$ and associate with it a partially exchangeable sequence $Z = \{Z_{ij}; i \in [2], j \in \mathbb{N}\}$.
We will then show that certain types of properties of $\mathbf{Z}$ are shared by $\mathbf{Y}$ (in particular some a.s. limit theorems).

**Remark**
Here follows a brief survey of related work for the $g = 1$ case:

1. Given a sequence $\mathbf{X} = \{X_i; i \in \mathbb{N}\}$ of random variables with a tight distribution, we can find a subsequence $\mathbf{Y} = \{Y_i; i \in \mathbb{N}\}$ which is asymptotically exchangeable, i.e. for some exchangeable sequence of random variables $\mathbf{Z} = \{Z_i; i \in \mathbb{N}\}$,

$$
\lim_{j} L(Y_{j+1}, Y_{j+2}, \ldots) = L(Z_1, Z_2, \ldots).
$$

This result was given independently by Dacunha-Castelle (1974) and Figiel and Sucheston (1976).

2. Berkes (1982) gave a survey of results concerning the structure of subsequences of random variables. He showed that the strongest exchangeability property that can be guaranteed for suitable subsequences of general sequences of random variables is a certain form of asymptotic exchangeability which he calls strong exchangeability at infinity.

**Theorem 5.2**
Let $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbb{N}\}$ be a partially exchangeable sequence of random variables. If $\mathbf{Z}$ has 2-c.r.m. $(\mu, \nu)$ then

$$
(\mu(\omega), \nu(\omega), \mathbf{Z}(\omega)) \in A \quad \text{a.s.} \quad (5.10)
$$

for any statute $A$.

**Proof**
From Lemma 4.1, using $\mathbf{z}$ for $\mathbf{Z}(\omega)$,

$$
\int_{\mathbb{R}^\infty \times \mathbb{R}^\infty} I[(\mu(\omega), \nu(\omega), \mathbf{z}) \in A] \mu^* \times \nu^*(\omega, dz)
$$

$$
= E \{I[(\mu, \nu, \mathbf{Z}) \in A] / \mathcal{F}(\mu, \nu)\} \quad \text{a.s.}. \quad (5.11)
$$
Also, from (5.4),
\[
\int_{\mathbb{R}^\infty \times \mathbb{R}^\infty} I \left( \{ (\mu(\omega), \nu(\omega), z) \in A \} \right) \mu^* \times \nu^*(\omega, dz) = 1 .
\] (5.12)

From (5.11) and (5.12) we thus have
\[
E \left( E \left( I \left( \{ (\mu, \nu, Z) \in A \} \right) \right) \right) = 1
\]
and hence
\[
E \left( I \left( (\mu, \nu, Z) \in A \right) \right) = 1
\]
so that
\[
(\mu(\omega), \nu(\omega), Z(\omega)) \in A \quad \text{a.s.}
\] (5.13)

Remark 5.3
Theorem 5.2 shows that a.s. limit theorems for 2-IID sequences of random variables extend immediately to (2-fold) partially exchangeable random variables, a result which is not surprising in view of our earlier work in Chapter 3 (in particular Theorem 3.3).

The following theorem shows that a.s. limit theorems for 2-IID sequences of random variables, which can be represented by a limit statute, extend immediately to all sub-subsequences of some particular subsequence of \( X \).

Theorem 5.3
Let \( A \) be any limit statute. Then there exists \( m \) such that for each \( n \subset m \),
\[
(\mu(\omega), \nu(\omega), X_n(\omega)) \in A \quad \text{a.s.}
\] (5.13)

Proof
The proof of this theorem will be given towards the end of this chapter as it relies on many results that still need to be presented.
Corollary 5.1
Theorem 5.3 is immediate if $X$ has property $\beta$.

Proof
From Definition 5.2 it follows that there exists a partially exchangeable sequence of random variables $Z = \{Z_{ij}; i \in [2], j \in \mathbb{N}\}$ with 2-c.r.m. $(\mu, \nu)$ and $m$ such that
\[
\sum_{i=1}^{2} \sum_{j=1}^{\infty} |X_{im_{ij}} - Z_{ij}| < \infty \quad a.s.
\]
(5.14)

Theorem 5.2, (5.14) and Definition 5.4 show that
\[
(\mu(\omega), \nu(\omega), X_{m}(\omega)) \in A \quad a.s.
\]
(5.15)
for any limit statute $A$.

Now suppose that $n \leq m$ and choose $Z'$ to be a subsequence of $Z$ indexed by the same subscripts as subsequence $X_n$ is from $X_m$.

From (5.14),
\[
\sum_{i=1}^{2} \sum_{j=1}^{\infty} |X_{im_{ij}} - Z'_{ij}| \leq \sum_{i=1}^{2} \sum_{j=1}^{\infty} |X_{im_{ij}} - Z_{ij}| < \infty \quad a.s.
\]
and hence
\[
(\mu(\omega), \nu(\omega), X_{n}(\omega)) \in A \quad a.s.
\]
follows by using the same argument as was used to obtain (5.15) (note that $Z'$ is partially exchangeable if $Z$ is).

Combining the results from Theorem 5.2 and Theorem 5.3 we thus have the main result of this section:

*Given any a.s. limit theorem for 2-IID random variables, there exists an analogous theorem satisfied by (2-fold) partially exchangeable sequences and by all sub-subsequences of some subsequence of an arbitrarily dependent 2-fold infinite sequence of random variables, tightly distributed within types.*
In order to prove Theorem 5.3 we need to construct the random measures \( \mu \) and \( \nu \) for variables of type 1 and 2 respectively. The method of construction reduces to a construction in Révész (1967, Theorem 6.1.1) when only one type of random variable is considered (i.e. the \( g = 1 \) case).

Using the Relative Compactness Criterion (Appendix A1) we see that \( \{L(X_{ij}); j \in \mathbb{N}\} \) is relatively compact \( \forall i \in [2] \), and hence

\[
L(X_{ij}) \Rightarrow \gamma_i \quad \forall i \in [2] \tag{5.16}
\]

where \( \gamma_i \in \mathcal{P}(\mathbb{R}), \ \forall i \in [2] \).

The following notation will be used throughout this section:

Results will be stated for random variables of type \( i \) where \( i \in [2] \). Let \( D_{i\infty} \) be a countably dense set of continuity points of \( \gamma_i \) (see (5.16)). For each \( k \in \mathbb{N} \) choose a finite set \( D_{ik} = \{x_{ikj}; j \in [q_k + 1]\} \) such that

\[
x_{ikj} < x_{ikj+1} < x_{ikj} + 2^{-k} \quad j \leq q_k \tag{5.17}
\]

\[
P \left[ x_{ik1} < x_{iin} < x_{ikq_k} \right] \geq 1 - 2^{-k} \quad n \in \mathbb{N} \tag{5.18}
\]

\[
D_{ik} \subset D_{ik+1} \tag{5.19}
\]

\[
D_{i\infty} = \bigcup_{k=1}^{\infty} D_{ik} \tag{5.20}
\]

Let \( J_{ik} \) be the set of intervals \((-\infty, x_{ik1}], (x_{ik1}, x_{ik2}], \ldots, (x_{ikq_k}, \infty)\) and let

\[
J_i = \bigcup_{k} J_{ik} \tag{5.21}
\]

Define \( \mathcal{P}_{ik}: \mathbb{R} \rightarrow D_{ik} \) by

\[
\mathcal{P}_{ik}(x) = \begin{cases} 
\inf \{x_{ikj} \in D_{ik}; x_{ikj} \geq x\} & \text{for } x \leq x_{ikq_k} \\
\infty & \text{else}
\end{cases}
\tag{5.22}
\]

for \( x \in \mathbb{R} \).
Note that \( P_{ik} \) is a constant on each \( J \in J_k, i \in [2], k \in \mathbb{N} \). For future reference note that (5.17) and (5.18) imply both the next two results:

\[
P \left[ \left| P_{ik}(X_{in}) - X_{in} \right| \geq 2^{-k} \right] \leq 2^{-k} \quad k, n \in \mathbb{N}, i \in [2] \tag{5.23}
\]

and for any \( V = \{ V_{ij}; i \in [2], j \in \mathbb{N} \} \) with \( L(V_{ij}) = \gamma_i, \gamma_i \in P(\mathbb{R}) \), \( j \in \mathbb{N}, i \in [2] \),

\[
P \left[ \left| P_{ik}(V_{ij}) - V_{ij} \right| \geq 2^{-k} \right] \leq 2^{-k} \quad \forall j \in \mathbb{N}, k \in \mathbb{N}, i \in [2] \tag{5.24}
\]

The following result follows immediately from work published by Aldous ((1977), p.69) by applying Aldous' results to each type of random variable separately.

**Theorem 5.4**

There exists a subsequence \( Y = X_n \) and random measures \( \mu \) and \( \nu \) such that

for each \( J \in J_1 \), \( \lim_{k \to \infty} E \left[ I(Y_{1k} \in J) / F_{1k-1} \right] (\omega) = \mu(\omega, J) \tag{5.25} \)

for each \( J \in J_2 \), \( \lim_{k \to \infty} E \left[ I(Y_{2k} \in J) / F_{2k-1} \right] (\omega) = \nu(\omega, J) \tag{5.26} \)

where \( \mu \) is \( F_1 \)-measurable and \( \nu \) is \( F_2 \)-measurable. For

\[
F_{k-1} = F(P_{i1}(Y_{i1}), P_{i2}(Y_{i2}), \ldots, P_{i(k-1)}(Y_{ik-1})) \quad i \in [2], k \in \mathbb{N} \tag{5.27}
\]

\[
F_i = \bigcup_k F_{ik} \quad i \in [2]. \tag{5.28}
\]

We now construct a partially exchangeable sequence \( Z \) whose 2-c.r.m. is \((\mu, \nu)\) and then compare properties of \( Y \) and \( Z \) (where \( Y, \mu, \nu \) are as in Theorem 5.4).

From (5.27) and (5.28) we note that

\[
F_i \subset F(Y) \quad i \in [2]. \tag{5.29}
\]

For the remainder of this section let \( Z = \{ Z_{ij}; i \in [2], j \in \mathbb{N} \} \) be a sequence of random variables such that

\[
\mu^* \times \nu^* \text{ is a r.c.d. for } Z \text{ given } F(Y, \mu, \nu). \tag{5.30}
\]
Since $\mu$ is $\mathcal{F}_1$-measurable and $\nu$ is $\mathcal{F}_2$-measurable (see Theorem 5.4), it follows that (5.29) and (5.30) yield
\[
\mu^* \times \nu^* \text{ is a r.c.d. for } Z \text{ given } \mathcal{F}(Y).
\] (5.31)

Using (5.30) and (5.31) we see that
\[
\mu^* \times \nu^* \text{ is a r.c.d. for } Z \text{ given } \mathcal{F}(\mu, \nu)
\]
so that $(\mu, \nu)$ is the 2-c.r.m. for the (2-fold) partially exchangeable sequence, $Z$.

**Theorem 5.5**

Let $V$ be an $\mathcal{F}(Y)$-measurable map into some separable, metrisable space, $S$. For all $f \in N$,

\begin{align*}
(i) \quad & \lim_{n \to \infty} L(V, Y_{1m}, Y_{2n}) = L(V, Z_{1j}, Y_{2n}) & \forall n \in N \tag{5.32} \\
(ii) \quad & \lim_{m \to \infty} L(V, Y_{1m}, Y_{2n}) = L(V, Y_{1m}, Z_{2j}) & \forall m \in N. \tag{5.33}
\end{align*}

Moreover suppose that $f \in L^\infty(S \times \mathbb{R} \times \mathbb{R})$ is such that, for each $s \in S$,

\[f(s, \cdot, Y_{2n})\text{ is a constant on each } J \in \mathcal{J}_1,\]

then

\begin{align*}
(iii) \quad & \lim_{m \to \infty} E[f(V, Y_{1m}, Y_{2n})] = E[f(V, Z_{1j}, Y_{2n})] & \forall j \in N \tag{5.34} \\
& \text{and if } f(s, Y_{1m}, \cdot) \text{ is a constant on each } J \in \mathcal{J}_2, \text{ then} & \\
(iv) \quad & \lim_{n \to \infty} E[f(V, Y_{1m}, Y_{2n})] = E[f(V, Y_{1m}, Z_{2j})]. & \forall j \in N. \tag{5.35}
\end{align*}

**Proof**

The proof of this theorem follows from Aldous’ related result (Aldous (1977), Lemma 12, p.72) for one type of random variable by making the obvious changes to his proof.

**Corollary 5.2**

$V$ and $Z_i = \{Z_{ij}; j \in N\}$ are conditionally independent given $\mathcal{F}_i$ \quad $\forall i \in [2]$.

**Proof**

For $i \in [2]$, let $J \in \mathcal{J}_i$ and let $A$ be a measurable subset of $S$. Then, using (5.29), we see that
\[ P [V \in A, Z_{ij} \in J/\mathcal{F}_i] \]
\[ = E \{ I(V \in A, Z_{ij} \in J)/\mathcal{F}_i \} \]
\[ = E \{ E [I(V \in A, Z_{ij} \in J)/Y]/\mathcal{F}_i \} \]
\[ = E \{ I(V \in A) E [I(Z_{ij} \in J)/Y]/\mathcal{F}_i \} \cdot \]  \hspace{1cm} (5.36)

Consider the \( i = 1 \) case and recall that \( \mu \) is \( \mathcal{F}_i \)-measurable (Theorem 5.4).

From (5.31) and (5.36) then, \( \forall j \in \mathbb{N} \),
\[ P [V \in A, Z_{ij} \in J/\mathcal{F}_i] \]
\[ = E \{ I[V \in A] \mu(\omega, J)/\mathcal{F}_i \} \]
\[ = \mu(\omega, J) E [I[V \in A]/\mathcal{F}_i] \]
\[ = E [\mu(\omega, J)/\mathcal{F}_i] E [I[V \in A]/\mathcal{F}_i] \]
\[ = E [E (I[Z_{ij} \in J]/Y)/\mathcal{F}_i] \cdot E [I[V \in A]/\mathcal{F}_i]. \]

From (5.29) then,
\[ P [V \in A, Z_{ij} \in J/\mathcal{F}_i] = P[Z_{ij} \in J/\mathcal{F}_i] P [V \in A/\mathcal{F}_i] \]

the \( i = 2 \) case follows similarly.

Remark
Since \( Y \) is an infinite subsequence of \( X \), (5.16) shows that for \( i \in [2] \),
\[ L(Y_{ij}) \Rightarrow \gamma_i \quad \forall j \in \mathbb{N} \cdot \]

Hence, from Theorem 5.5 we obtain, for \( i \in [2] \),
\[ L(Z_{ij}) = \gamma_i \quad \forall j \in \mathbb{N}. \]  \hspace{1cm} (5.37)

Consider a function \( g : P(\mathbb{R}) \times P(\mathbb{R}) \times \mathbb{R}^\infty \times \mathbb{R}^\infty \rightarrow \mathbb{R} \). We aim to use the asymptotic conditional independence property given by Theorem 5.5 to
show that $E[g(\mu, \nu, Y_n)]$ is close to $E[g(\mu, \nu, Z)]$ whenever $n$ increases sufficiently rapidly (exactly what is meant by this will be stated below). The situation is simplest when $g$ is continuous, but the alternative conditions below will sometimes be needed.

For each $x \in \mathbb{R}^\infty \times \mathbb{R}^\infty, y \in \mathbb{R}^\infty \times \mathbb{R}^\infty$ and $\lambda_i \in P(\mathbb{R}) \ i \in [2],$

$$g(\lambda_1, \lambda_2, x) \leq \liminf_{i_1, i_2} g(\lambda_1, \lambda_2, x_{1i_1}, \ldots, x_{2i_1}, x_{2i_1}, \ldots, y_{1i_1+1}, y_{1i_1+2}, \ldots, y_{2i_2+1}, y_{2i_2+2}, \ldots) \forall i_j \in \mathbb{N}, j \in [2]. \ (5.38)$$

If

$$P_{ij}(x_{ij}) = P_{ij}(y_{ij}) \forall i \in [2], \forall j \in \mathbb{N} \ (5.39)$$

then

$$g(\lambda_1, \lambda_2, x) = g(\lambda_1, \lambda_2, y) \ (5.39)$$

where $P_{ij}$ is as in (5.22).

**Definition 5.5**

Let $Q$ denote an assertion applicable to increasing sequences (within types) $n = \{n_{ij}; i \in [2], j \in \mathbb{N}\}$ of positive integers (i.e. $n_i = \{n_{ij}; j \in \mathbb{N}\}$ is increasing $\forall i \in [2]$) and let $Q'$ denote the set of $n \in \mathbb{N}^\infty \times \mathbb{N}^\infty$ for which $Q$ is true. If there exist functions $L_1$ and $L_2$, 

$$L_i : \{\text{finite sequences in } \mathbb{N}\} \to \mathbb{N} \forall i \in [2] \text{ such that, } \forall j \in \mathbb{N}, i \in [2],$$

$$n_{ij} \geq L_i(n_{i1}, n_{i2}, \ldots, n_{i-1}) \text{ implies } n \in Q', \quad (5.40)$$

then we say that $Q$ holds for all $n$ increasing sufficiently rapidly. (Note that we require the sequence to be increasing sufficiently rapidly for each type of variable, separately).

**Theorem 5.6**

Let $P^{(1)}_0(\mathbb{R})$ and $P^{(2)}_0(\mathbb{R})$ be two measurable subsets of $P(\mathbb{R})$ such that

$$\mu(\omega) \in P^{(1)}_0(\mathbb{R}) \ a.s. \text{ and } \nu(\omega) \in P^{(2)}_0(\mathbb{R}) \ a.s. \quad (5.41)$$

Let $P^{(i)}_0(\mathbb{R})$ be equipped with the separable, metrisable topology, $\tau_i$ such that

$$\lambda_{ij} \xrightarrow{\tau_i} \lambda_i \text{ implies } \lambda_{ij} \Rightarrow \lambda_i \quad \forall i \in [2]. \quad (5.42)$$
The weak topology and \( \tau_i \) generate the same \( \sigma \)-field \( \forall i \in [2] \). \hspace{1cm} (5.43)

Let \( \xi > 0 \) be given. Let \( g : P_0^{(1)}(\mathbb{R}) \times P_0^{(2)}(\mathbb{R}) \times \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R} \) satisfy either

(i) \( g \) is bounded and continuous

or

(ii) \( g \) is bounded, measurable and satisfies (5.38) and (5.39).

Then

\[
E[g(\mu, \nu, Y_n)] \leq E[g(\mu, \nu, Z)] + \xi
\]

for all \( n \) increasing sufficiently rapidly.

**Proof**

Suppose, inductively, that we have chosen the first \( k_i \) variables of type \( i, i \in [2] \), i.e. \( n_{i1} < n_{i2} < \ldots < n_{ik_i} \) have been specified \( \forall i \in [2] \).

Define random variables

\[
G_{00} = g(\mu, \nu, Z)
\]

\[
G_{ij} = g(\mu, \nu, Y_{1n_{i1}}, Y_{1n_{i2}}, \ldots, Y_{1n_{i1}}, Y_{2n_{i1}}, Y_{2n_{i2}}, \ldots, Y_{2n_{i1}})
\]

\[
Z_{1i+1}, Z_{1i+2}, \ldots, Z_{2j+1}, Z_{2j+2}, \ldots, i \in [k_1], j \in [k_2], k_i \in \mathbb{N}
\]

\[
\forall i \in [2]. \hspace{1cm} (5.46)
\]

For \( m_i > n_{ik_i} \), \( \forall i \in [2] \), define

\[
E_{m_1,m_2} = g(\mu, \nu, Y_{1n_{i1}}, Y_{1n_{i2}}, \ldots, Y_{1n_{i1}}, Y_{2n_{i1}}, Y_{2n_{i2}}, \ldots, Y_{2n_{ik_2}})
\]

\[
Y_{1m_1}, Y_{2m_2}, Z_{1k_1+2}, Z_{1k_1+3}, \ldots, Z_{2k_2+2}, Z_{2k_2+3}, \ldots.
\]

\[
\hspace{1cm} (5.47)
\]

We shall prove

\[
\lim_{m_1,m_2} E(B_{m_1,m_2}) = E(G_{k_1,k_2})
\]

in order to prove the theorem. We now proceed to show that if (5.48) holds then the desired result follows. We thus accept (5.48) for the time being.

From (5.46), (5.47) and (5.48) we see that there exist functions \( L_1 \) and \( L_2 \),

\[
L_i : \{\text{finite sequences in } \mathbb{N}\} \to \mathbb{N} \hspace{1cm} \forall i \in [2], \text{ such that}
\]
if 
\[ n_{ik_{i}+1} \geq L_i(n_{ik_1}, n_{ik_2}, \ldots, n_{ik_i}) \quad \forall i \in [2] \]
then \( E(G_{k_1+k_2}) \) and \( E(G_{k_1+k_2+1}) \) can be brought as close to \( E(G_{k_1+k_2}) \) as desired, i.e. from Definition 5.5, we may say that for all \( n \) increasing sufficiently rapidly, there exist \( \xi_i > 0, \forall i \in [2] \), such that
\[ E(G_{k_1+k_2}) \leq E(G_{k_1+k_2}) + \xi_i 2^{-k_i-1} \quad (5.49) \]
and
\[ E(G_{k_1+k_2+1}) \leq E(G_{k_1+k_2}) + \xi_i 2^{-k_i-1}. \quad (5.50) \]
Now suppose that \( n \) increases sufficiently rapidly, from (5.49) and (5.50) there exist \( \xi > 0 \) such that
\[ E(G_{k_1+k_2}) \leq E(G_{k_1+k_2}) + \xi_1 2^{-k_1} \]
\[ \vdots \]
\[ \leq E(G_{0k_2}) + \xi_1 \left( 2^{-k_1} + 2^{-k_1+1} + \ldots + 2^{-1} \right) \]
\[ \leq E(G_{0k_2}) + \xi_1 \]
\[ \leq E(G_{0k_2-1}) + \xi_2 2^{-k_2} + \xi_1 \]
\[ \vdots \]
\[ \leq E(G_{0,0}) + \xi \quad (5.51) \]
where \( \xi = \xi_1 + \xi_2 \). Since (5.51) holds \( \forall k_i \in \mathbb{N}, \forall i \in [2] \), we may use (5.45) to obtain the following:
If \( n \) increases sufficiently rapidly then there exists \( \xi > 0 \) such that
\[ \sup_{k_1, k_2} E(G_{k_1+k_2}) \leq E[g(\mu, \nu, Z)] + \xi. \quad (5.52) \]
At the outset of the theorem it was assumed that \( g \) either satisfies (i) or (ii). Assume that \( g \) satisfies (ii). From (5.38) with \( x = Y_n = \{ Y_{1i_{11}}, Y_{1i_{12}}, \ldots, Y_{2n_2}, Y_{2n_2}, \ldots \} \), \( y = Z = \{ Z_{11}, Z_{12}, \ldots, Z_{21}, Z_{22}, \ldots \} \), \( i = k_1, j = k_2 \), \( \lambda_1 = \mu \) and \( \lambda_2 = \nu \), we see that
\[ g(\mu, \nu, Y_n) \leq \lim_{k_1, k_2} \inf g(\mu, \nu, Y_{1i_{11}}, Y_{1i_{12}}, \ldots, Y_{1n_{1k_1}}, Y_{2n_1}, Y_{2n_2}, \ldots, Y_{2n_{2k_2}}, \]
\[ Z_{1k_1+1}, Z_{1k_1+2}, \ldots, Z_{2k_2+1}, Z_{2k_2+2}, \ldots) \)
and hence, from (5.46),
\[ g(\mu, \nu, Y_n) \leq \liminf_{k_1, k_2} G_{k_1, k_2}. \]  (5.53)

Since (5.53) follows immediately if \( g \) is continuous (which is assumed if \( g \) satisfies (i)) we thus have \( g \) to satisfy (5.53).

From (5.53) and Fatou's lemma (Appendix A1),
\[ E[g(\mu, \nu, Y_n)] \leq E\left(\liminf_{k_1, k_2} G_{k_1, k_2}\right) \leq \liminf_{k_1, k_2} E(G_{k_1, k_2}). \]  (5.54)

Using (5.52) and (5.54) we thus see that for all \( n \) increasing sufficiently rapidly, there exists \( \xi > 0 \) such that (5.44) follows, since
\[ E[g(\mu, \nu, Y_n)] \leq \liminf_{k_1, k_2} E(G_{k_1, k_2}) \leq \sup_{k_1, k_2} E(G_{k_1, k_2}) \leq E[g(\mu, \nu, Z)] + \xi. \]

The proof will thus be complete if (5.48) is shown to hold. For all \( i, j \in \mathbb{N} \), define functions \( g_{ij} : P_0^{(1)}(R) \times P_0^{(2)}(R) \times R^i \times R^j \to R \) by
\[ g_{ij} = (\lambda_1, \lambda_2, y_{11}, y_{12}, \ldots, y_{1i}, y_{21}, y_{22}, \ldots, y_{2j}) \]
\[ = \int g(\lambda_1, \lambda_2, y_{11}, y_{12}, \ldots, y_{1i}, y_{21}, y_{22}, \ldots, y_{2j}, x_{1i+1}, x_{1i+2}, \ldots, x_{2j+1}, x_{2j+2}, \ldots) \quad \lambda_1^\ast \times \lambda_2^\ast d(x) \]  (5.55)

where \( R = \left(\begin{array}{c} \infty \\ i+1 \end{array}\right) \times \left(\begin{array}{c} \infty \\ j+1 \end{array}\right). \)

From (5.55) and the assumptions made on \( g \) it immediately follows that \( g_{ij} \) is bounded and measurable \( \forall i, j \in \mathbb{N} \).

Define a random map \( V \) into \( P_0^{(1)}(R) \times P_0^{(2)}(R) \times R^{k_1} \times R^{k_2} \) for \( k_i \in \mathbb{N} \),
by

\[ V = (\mu, \nu, Y_{1m_1}, Y_{1m_2}, \ldots, Y_{1n_1}, Y_{2n_1}, Y_{2n_2}, \ldots, Y_{2n_2}) \]  

(5.56)

From (5.29) and the fact that \( \mu \) is \( F_1 \)-measurable we conclude that

\[ \mu \text{ is } F(Y) \text{-measurable.} \]  

(5.57)

Similarly for \( \nu \), so that (5.56) shows that

\[ V \text{ is } F(Y) \text{-measurable.} \]  

(5.58)

For random variables \( W_1 \) and \( W_2 \) and \( k_i \in \mathbb{N} \), \( i \in [2] \), we may regard \((V, W_1, W_2)\) as a random map into \( P_0^{(1)}(\mathbb{R}) \times P_0^{(2)}(\mathbb{R}) \times \mathbb{R}^{k_1+1} \times \mathbb{R}^{k_2+1} \). Using Lemma 4.1, the definitions of the functions involved and (5.31), we see that

\[ E(G_{k_1k_2}/Y) = g_{k_1k_2}(V) \text{ a.s.} \]  

(5.59)

\[ E(B_{m_1m_2}/Y) = g_{k_1+k_2+1}(V, Y_{1m_1}, Y_{2m_2}) \text{ a.s.} \]  

(5.60)

\[ E[g_{k_1+k_2+1}(V, Z_{1k_1+1}, Z_{2k_2+1}/Y)] = g_{k_1k_2}(V) \text{ a.s.} \]  

(5.61)

We now show that if

\[ \lim_{m_1, m_2} E[g_{k_1+k_2+1}(V, Y_{1m_1}, Y_{2m_2})] = E[g_{k_1+k_2+1}(V, Z_{1k_1+1}, Z_{2k_2+1})] \text{ a.s.} \]  

(5.62)

holds, then (5.48) follows.

From (5.60) and (5.62),

\[ \lim_{m_1, m_2} E[E(B_{m_1m_2}/Y)] \]

\[ = \lim_{m_1, m_2} E[g_{k_1+k_2+1}(V, Y_{1m_1}, Y_{2m_2})] \text{ a.s.} \]

\[ = E[g_{k_1+k_2+1}(V, Z_{1k_1+1}, Z_{2k_2+1})] \text{ a.s.} \]  

(5.63)

From a basic property of conditional expectations, (5.59), (5.61) and (5.63),
it follows that (5.48) holds, since
\[
\lim_{m_1,m_2} E(B_{m_1,m_2}) = E \left[ g_{k_1+k_2+1}(V, Z_{1k_1+1}, Z_{2k_2+1}) \right] \quad \text{a.s.}
\]
\[
= E \left\{ E \left[ g_{k_1+k_2+1}(V, Z_{1k_1+1}, Z_{2k_2+1}) \bigg/ Y \right] \right\} \quad \text{a.s.}
\]
\[
= E \left[ g_{k_1,k_2}(V) \right] \quad \text{a.s.}
\]
\[
= E \left[ E(G_{k_1,k_2}/Y) \right] \quad \text{a.s.}
\]
\[
= E \left( G_{k_1,k_2} \right) \quad \text{a.s.}
\]

The proof of the theorem is thus complete if (5.62) can be shown to hold. Once again, we consider the two cases separately.

Suppose that \( g \) satisfies (ii). Let \( k_1, k_2 \in \mathbb{N} \). From (5.55), for arbitrary \( m, n \in \mathbb{N} \),
\[
g_{k_1+k_2+1} (\lambda_1, \lambda_2, y_{11}, y_{12}, \ldots, y_{1k_1}, y_{21}, y_{22}, \ldots, y_{2k_2}, y_{1m}, y_{2n})
\]
\[
= \int_R g (\lambda_1, \lambda_2, y_{11}, y_{12}, \ldots, y_{1k_1+1}, y_{21}, y_{22}, \ldots, y_{2k_2+1}, x_{1k_1+2}, x_{1k_1+3}, \ldots, x_{2k_2+2}, x_{2k_2+3}, \ldots) 
\quad \lambda_1^1 \times \lambda_2^2 (dx)
\]
\[
(5.64)
\]
where \( R = \left( \mathbb{X}_{k_1+2} \times \mathbb{X}_{k_2+2} \right) \).

Now \( F_k^{k_{i+1}} \) is a constant on each \( J \in J_{k_{i+1}} \forall i \in [2] \) (this was noted just below the definition of \( F_k(x) \), see (5.22)), so that (5.39) and (5.55) immediately show that if we fix \( (\lambda_1, \lambda_2, y_{11}, \ldots, y_{1k_1}, y_{21}, \ldots, y_{2k_2}, y_{2n}) \) and consider \( g_{k_1+k_2+1} (\lambda_1, \lambda_2, y_{11}, y_{12}, \ldots, y_{1k_1}, y_{21}, y_{22}, \ldots, y_{2k_2}, y_{1m}, y_{2n}) \) as a function of \( y_{1m} \) only, then the right hand side of (5.64) is a constant on each \( J_1 \in J_{k_{1+1}} \).

We may thus use Theorem 5.5 (iii) to obtain
\[
\lim_{m \to \infty} E \left[ g_{k_1+k_2+1} (\lambda_1, \lambda_2, Y_{11}, \ldots, Y_{1k_1}, Y_{21}, \ldots, Y_{2k_2}, Y_{1m}, Y_{2n}) \right]
\]
\[
= \quad E \left[ g_{k_1+k_2+1} (\lambda_1, \lambda_2, Y_{11}, \ldots, Y_{1k_1}, Y_{21}, \ldots, Y_{2k_2}, Z_{1i}, Y_{2n}) \right] \quad \forall i \in \mathbb{N}.
\]

Using similar arguments we thus see that, \( \forall i, j \in \mathbb{N} \),
\[
\lim_{m,n \to \infty} E[g_{k_1 + k_2 + 1} (\lambda_1, \lambda_2, Y_{11}, \ldots, Y_{1k_1}, Y_{21}, \ldots, Y_{2k_2}, Y_{1m}, Y_{1n})]
\]
\[
= E[g_{k_1 + k_2 + 1} (\lambda_1, \lambda_2, Y_{11}, \ldots, Y_{1k_1}, Y_{21}, \ldots, Y_{2k_2}, Z_{1i}, Z_{2j})] \tag{5.65}
\]
so that (5.62) holds as it is just a special case of (5.65).

We now show that (5.62) holds if \( g \) satisfies (i). For \( V \) as in (5.56), \( V \) is an \( \mathcal{F}(Y) \)-measurable map into \( S = P_0^{(1)}(R) \times P_0^{(2)}(R) \times R^{k_1} \times R^{k_2} \), and we may thus use Theorem 5.5 (i) to see that

\[
\lim_{n \to \infty} L(V, Y_{1m}, Y_{2n}) = L(V, Z_{1i} Y_{2n}) \quad \forall i \in \mathbb{N} \tag{5.66}
\]
and

\[
\lim_{n \to \infty} L(V, Y_{1m}, Y_{2n}) = L(V, Y_{1m}, Z_{2j}) \quad \forall j \in \mathbb{N} \tag{5.67}
\]
so that (5.66) and (5.67) yield

\[
\lim_{m,n \to \infty} L(V, Y_{1m}, Y_{2n}) = L(V, Z_{1i}, Z_{2j}). \quad \forall i, j \in \mathbb{N} \tag{5.68}
\]

The continuity of \( g_{k_1 + k_2 + 1} \forall k_i \in \mathbb{N} \), \( i \in [2] \), follows from the continuity of \( g \) and hence (5.62) is obtained from (5.68). The proof of Theorem 5.6 is thus complete.

**Remark**

Theorem 5.6 reduces to a result by Aldous ((1977), Proposition 13, p.74) when only variables of one type are considered.

We are almost in a position to prove Theorem 5.3 but we first need establish the following lemmas:

**Lemma 5.3**

Let \( \{Q_{(j_1,j_2)(k_1,k_2)}; 1 \leq k_i \leq q_{j_i}, i \in [2], (j_1,j_2) \in \mathbb{N} \times \mathbb{N} \} \) be a collection of properties, each of which holds for all \( n \) increasing sufficiently rapidly (see Definition 5.5). Then there exists \( m \) which satisfies the following:

For each \( n \subset m \) and each \( (j_1,j_2) \in \mathbb{N} \times \mathbb{N} \), there exists

\[
\mathbb{N}^{(j_1,j_2)} = \{n^{j_1}_{11}, n^{j_1}_{12}, \ldots, n^{j_2}_{21}, n^{j_2}_{22}, \ldots \}
\]

such that
\( n^{(i,j)} \) satisfies properties \( Q_{(j_1,j_2)(k_1,k_2)} \), \( 1 \leq k_1 \leq q_{j_1}, i \in [2] \) (5.69)
\[ n^{j_1}_{ik} = n^{j_2}_{ik} \quad \text{for all } k > j_1, i \in [2]. \] (5.70)

**Proof**

The desired result follows from Aldous' related result (Aldous (1977), Lemma 14, p.76) since \( n \) increasing sufficiently rapidly requires \( n_i = \{n_{ij}; j \in N\} \) to increase sufficiently rapidly for all \( i \in [2] \) (see Definition 5.5).

**Lemma 5.4**

Let \( A \) be any limit statute, with \( A' \) as its complement in \( P(R) \times P(R) \times R^\infty \times R^\infty \). Let \( j_i \in N \) for all \( i \in [2] \) be given. Then
\[ P[(\mu, \nu, Y_n) \in A'] \leq 2^{-j_1-j_2} \] (5.71)
for all \( n \) increasing sufficiently rapidly.

**Proof**

We start off by showing that \( A' \) is a limit statute.

In this regard suppose that
\[ (\lambda_1, \lambda_2, x, x') \in A' \quad \text{and} \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |x_{ij} - x_{ij}'| < \infty \] (5.72)
for some \( x, x' \in R^\infty \times R^\infty \) and \( \lambda_i \in P(R) \), \( \forall i \in [2] \).

Suppose that \((\lambda_1, \lambda_2, x')\) is not in \( A' \). Then \((\lambda_1, \lambda_2, x) \in A\), and from the second part of (5.72), since \( A \) is a limit statute, we conclude that \((\lambda_1, \lambda_2, x) \in A\). Clearly then \((\lambda_1, \lambda_2, x) \notin A' \), which contradicts the first part of (5.72). We may thus conclude that \( A' \) is a limit statute if \( A \) is.

We cannot apply Theorem 5.6 directly to \( I(A') \) since (5.39) is not satisfied, so we need a more sophisticated approach.

Consider the functions \( p_{ij} \) as defined in (5.22). From (5.37) and (5.24), \forall j \in N \), we have that
\[ P \left[ |P_{ij}(Z_{ij}) - Z_{ij}| \geq 2^{-k} \right] \leq 2^{-k} \quad \forall k \in N, \forall i \in [2] \]
and hence, \( \forall j \in \mathbb{N}, \)
\[
P \left[ |Z_{ij} - P_{ij}(Z_{ij})| \geq 2^{-j} \right] \leq 2^{-j} \quad \forall j \in \mathbb{N}, \quad \forall i \in [2] \quad (5.73)
\]
so that
\[
2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |Z_{ij} - P_{ij}(Z_{ij})| < \infty \quad \text{a.s.} \quad (5.74)
\]
From Theorem 5.5, it follows that, \( \forall n \in \mathbb{N}^\infty \times \mathbb{N}^\infty, \forall i \in [2], \forall j \in \mathbb{N}, \)
\[
P \left[ |Y_{inij} - P_{ij}(Y_{inij})| \geq 2^{-j} \right] = P \left[ |Z_{ij} - P_{ij}(Z_{ij})| \geq 2^{-j} \right]
\]
so that
\[
2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |Y_{inij} - P_{ij}(Y_{inij})| < \infty \quad \text{a.s.} \quad (5.75)
\]
follows from (5.73) in the same way that (5.74) did.

Now define \( P : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty \times \mathbb{R}^\infty \) by
\[
P(x) = (P_{11}(x_{11}), P_{12}(x_{12}), \ldots, P_{21}(x_{21}), P_{22}(x_{22}), \ldots). \quad (5.76)
\]
Due to the symmetry of (5.74) and the fact that \( A' \) is a limit statute, we conclude that
\[
(\mu, \nu, Z(\omega)) \in A' \text{ if and only if } (\mu, \nu, P(Z(\omega))) \in A'
\]
for \( Z \) a partially exchangeable sequence with 2-c.r.m. \((\mu, \nu),\) so that
\[
P \left[ (\mu, \nu, Z) \in A' \right] = P \left[ (\mu, \nu, P(Z)) \in A' \right]. \quad (5.77)
\]
Similarly, from (5.75),
\[
P \left[ (\mu, \nu, Y_n) \in A' \right] = P \left[ (\mu, \nu, P(Y_n)) \in A' \right]. \quad (5.78)
\]
However, from Theorem 5.2 and (5.77),
\[
P \left[ (\mu, \nu, P(Z)) \in A \right] = 1
\]
so that
\[
P \left[ (\mu, \nu, P(Z)) \in A' \right] = 0. \quad (5.79)
\]
Since a probability measure on a metric space is regular, there exists an open set \( G \supset A' \) such that (5.79) leads us to

\[
P \left[ (\mu, \nu, P(Z)) \in G \right] \leq 2^{-j_1 - j_2 - 1} \quad \forall j_i \in \mathbb{N}, \ i \in [2]. \tag{5.80}
\]

Now let

\[
H = \{ (\lambda_1, \lambda_2, x); (\lambda_1, \lambda_2, P(x)) \in G \text{ for } \lambda_i \in P(\mathbb{R}) \quad \forall i \in [2], x \in \mathbb{R}^\infty \times \mathbb{R}^\infty \}.
\tag{5.81}
\]

For \( \lambda_i \in P(\mathbb{R}) \ \forall i \in [2], x \in \mathbb{R}^\infty \times \mathbb{R}^\infty \), let

\[
g(\lambda_1, \lambda_2, x) = I(H) = \begin{cases} 
1 & \text{if } (\lambda_1, \lambda_2, P(x)) \in G \\
0 & \text{if } (\lambda_1, \lambda_2, P(x)) \notin G.
\end{cases}
\tag{5.82}
\]

From (5.82) it follows that \( g(\lambda_1, \lambda_2, x) \) is bounded and measurable. Since \( g(\lambda_1, \lambda_2, x) \) satisfies (5.38) and (5.39) we may use Theorem 5.6 (ii) to see that, for all \( n \) increasing sufficiently rapidly,

\[
E [g(\mu, \nu, Y_n)] \leq E [g(\mu, \nu, Z)] + \xi.
\]

From (5.82) we thus have,

\[
E \{ I \left[ (\mu, \nu, P(Y_n)) \in G \right] \} \leq E \{ I \left[ (\mu, \nu, P(Z)) \in G \right] \} + \xi
\]

and hence

\[
P \left[ (\mu, \nu, P(Y_n)) \in G \right] \leq P \left[ (\mu, \nu, P(Z)) \in G \right] + \xi. \tag{5.83}
\]

From (5.80) and (5.83) we thus have that for all \( n \) increasing sufficiently rapidly,

\[
P \left[ (\mu, \nu, P(Y_n)) \in G \right] \leq 2^{-j_1 - j_2}. \tag{5.84}
\]

The lemma thus follows immediately from (5.78), (5.84) and the fact that \( A' \subset G \).

We are finally in a position to prove Theorem 5.3.
Proof of Theorem 5.3
Let $A$ be any limit statute. From Lemma 5.3 we know that $\exists m \in \mathbb{N}^\infty \times \mathbb{N}^\infty$, such that, for all $n \subset m$ and each $(j_1 \times j_2) \in \mathbb{N} \times \mathbb{N}$, we can find a $n^{(j_1,j_2)}$, as in (5.70), which satisfies a particular set of properties for sequences which increase sufficiently rapidly. From Lemma 5.4 then,

$$P \left[ (\mu, \nu, Y_{n^{(j_1,j_2)}}) \in A' \right] \leq 2^{-j_1-j_2} \quad (5.85)$$

for $j_i \in \mathbb{N}$, $\forall i \in [2]$.

From the construction of $n^{(j_1,j_2)}$ it follows that

$$\sum_{i=1}^{2} \sum_{k=1}^{\infty} \left| Y_{i_{ik}} - Y_{i_{ik}} \right| < \infty . \quad (5.86)$$

From the symmetry of (5.86) and the fact that $A'$ is a limit statute it immediately follows that

$$(\mu, \nu, Y_n) \in A'$ if and only if $(\mu, \nu, Y_{n^{(j_1,j_2)}}) \in A'$

so that

$$P \left[ (\mu, \nu, Y_n) \in A' \right] = P \left[ (\mu, \nu, Y_{n^{(j_1,j_2)}}) \in A' \right] . \quad (5.87)$$

From (5.85) and (5.87) then, for all $j_i \in \mathbb{N}$, $\forall i \in [2]$,

$$P \left[ (\mu, \nu, Y_n) \in A' \right] \leq 2^{-j_1-j_2}$$

so that

$$P \left[ (\mu, \nu, Y_n) \in A \right] = 1$$

and hence the desired result follows.

5.3 LIMIT THEOREMS FOR PARTIALLY EXCHANGEABLE SEQUENCES

We start off by extending some well-known limit theorems for IID random variables to hold for 2-IID sequences. Limit theorems for partially exchangeable sequences follow immediately (see Remark 5.3).

Let $X = \{X_{ij}; i \in [2], j \in \mathbb{N}\}$ be a 2-IID sequence of random variables,
and let \( S = \left\{ S_{n_1n_2} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} X_{ij}; n_i \in \mathbb{N}, \forall i \in [2] \right\} \).

**Definition 5.6**

A *tail event* on \( S \) is an event which is independent of \( \{S_{kl}; 1 \leq k \leq n_1, 1 \leq l \leq n_2\} \) for any finite \( n_i \in \mathbb{N}, i \in [2] \).

The following corollary to the Hewitt-Savage zero-one law for 2-IID sequences (Theorem 3.6) reduces to a result in Breiman (1968, p.64) when only variables of one type are considered.

**Corollary 5.3**

Let \( X = \{X_{ij}; i \in [2], j \in \mathbb{N}\} \) be a 2-IID sequence and \( S \) as defined above. Then every tail event on \( S \) has probability zero or one.

**Proof**

Let \( A \) be a tail event on \( S \). Then \( A \) is unaffected by finite permutations (within types) of \( X \), and hence \( A \in E_X \) (see Definition 3.5). The desired result follows from Theorem 3.6.

**Example**

Let \( X \) and \( S \) be as in Corollary 5.3, and let \( I_{n_1n_2} \in B(\mathbb{R}^2) \forall n_i \in \mathbb{N}, i \in [2] \).

Then

\[
A = [S_{n_1n_2} \in I_{n_1n_2} \text{ for infinitely many } n_1 \text{ and } n_2 \text{ from } \mathbb{N}]
\]

\[
B = \left[ \lim_{n_1, n_2} S_{n_1n_2} < \infty \right]
\]

are both tail events on \( S \) and hence, by Corollary 5.3, have measure zero or one.

The following limit theorem for 2-IID sequences is an extension of Kolmogorov's inequality (Appendix A1).

**Theorem 5.7**

Let \( X = \{X_{ij}; i \in [2], j \in \mathbb{N}\} \) be a 2-IID sequence of random variables with

\[
E(X_{ij}) = 0, \quad E(X_{ij}^2) < \infty \quad \forall i \in [2], \forall j \in \mathbb{N}
\]
Let 
\[ E(X_{ij}^2) = V_i \qquad \forall j \in \mathbb{N}, \ i \in [2]. \]

Then, \( \forall \xi > 0, \)
\[
P \left[ \max_{i \in [2]} \left| \sum_{i=1}^{k_i} \sum_{j=1}^{k_1} X_{ij} \right| \geq \xi \right] \leq 4\xi^{-2} \sum_{i=1}^{2} n_i V_i.
\]

**Proof**

For all \( k_i \in [n_i], \ i \in [2] \)
\[
\left| \sum_{i=1}^{2} \sum_{j=1}^{k_i} X_{ij} \right| \leq \left| \sum_{j=1}^{k_1} X_{1j} \right| + \left| \sum_{j=1}^{k_2} X_{2j} \right|
\]

and hence
\[
\max_{i \in [2]} \left| \sum_{i=1}^{k_i} \sum_{j=1}^{k_1} X_{ij} \right| \geq \xi
\]

\[
\Rightarrow \max_{1 \leq k_1 \leq n_1} \sum_{j=1}^{k_1} X_{1j} + \max_{1 \leq k_2 \leq n_2} \sum_{j=1}^{k_2} X_{2j} \geq \xi
\]

so that
\[
P \left[ \max_{i \in [2]} \left| \sum_{i=1}^{k_i} \sum_{j=1}^{k_1} X_{ij} \right| \geq \xi \right]
\]
\[
\leq P \left( \left\{ \max_{1 \leq k_1 \leq n_1} \left| \sum_{j=1}^{k_1} X_{1j} \right| \geq \xi/2 \right\} \cup \left\{ \max_{1 \leq k_2 \leq n_2} \left| \sum_{j=1}^{k_2} X_{2j} \right| \geq \xi/2 \right\} \right)
\]
\[
\leq P \left( \max_{1 \leq k_1 \leq n_1} \left| \sum_{j=1}^{k_1} X_{1j} \right| \geq \xi/2 \right) + P \left( \max_{1 \leq k_2 \leq n_2} \left| \sum_{j=1}^{k_2} X_{2j} \right| \geq \xi/2 \right). \quad (5.88)
\]

Now applying Kolmogorov's inequality (Appendix A1) to each term on the right hand side of (5.88), we see that
\[
P \left[ \max_{i \in [2]} \left| \sum_{i=1}^{k_i} \sum_{j=1}^{k_1} X_{ij} \right| \geq \xi \right] \leq 4\xi^{-2} (n_1 V_1 + n_2 V_2).
Lemma 5.5
Let $X$ be a 2-IID sequence with zero means, finite second moments, where $\sum_{i=1}^{2} \sum_{j=1}^{\infty} V_{ij}$ is convergent ($V_{ij} = \text{Var}(X_{ij})$). Then $\sum_{i=1}^{2} \sum_{j=1}^{\infty} X_{ij}$ is a.s. convergent.

Proof:
Since $\sum_{i=1}^{2} \sum_{j=1}^{\infty} V_{ij} < \infty$, it clearly follows that
$$\sum_{j=1}^{\infty} V_{ij} < \infty \quad \forall i \in [2],$$
and hence, from Clarke (1975, Lemma 3, p.179) we see that $\sum_{j=1}^{\infty} X_{ij}$ is a.s. convergent $\forall i \in [2]$. The desired result now follows immediately.

An idea obtained in a paper by Aldous (1981, p.590) led to the following extension of Kolmogorov's SLLN (see Appendix A1):

Theorem 5.8
Let $X$ be a 2-IID sequence of random variables with $E|X_{i1}| < \infty$, $\forall i \in [2]$ and let $E(X_{i1}) = \mu_i$, $\forall i \in [2]$. Then
$$\lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} X_{1k}X_{2k} = \mu_1\mu_2 \quad \text{a.s.}$$

Proof
Clearly $\{X_{1k}X_{2k} ; k \in \mathbb{N}\}$ is an IID sequence. Now
$$E|X_{1k}X_{2k}| \leq E|X_{1k}|E|X_{2k}| < \infty$$
so that Kolmogorov's above mentioned result yields
$$\lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} X_{1k}X_{2k} = \mu_1\mu_2 \quad \text{a.s.}$$

Using the notation of §5.1 and §5.2, in particular Definition 5.3, we now present the previous theorem in statute form.
$A_3 = \left\{ (\lambda_1, \lambda_2, x); \lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} x_{1k} x_{2k} = |\lambda_1| \cdot |\lambda_2| \right\} \cup$

$\left\{ (\lambda_1, \lambda_2, x); |\lambda_1| = \infty \text{ or } |\lambda_2| = \infty \right\}$

Similarly a statute representing an extension of the Law of the Iterated Logarithm (see Example 5.1) allowing 2-type sequences follows from statute $A_2$ in Example 5.1 in exactly the same way as $A_3$ above generalizes $A_1$:

$A_4 = \left\{ (\lambda_1, \lambda_2, x); \limsup_{N \to \infty} \frac{\left( \sum_{k=1}^{N} x_{1k} x_{2k} - N |\lambda_1| \cdot |\lambda_2| \right)}{(2N \log \log N)^{\frac{1}{2}}} \right\}$

$= \left\{ |\lambda_1| \cdot |\lambda_2| \cdot (|\lambda_1|)^2 |\lambda_2| + (|\lambda_1|)^2 |\lambda_2|^{\frac{1}{2}} \right\} \cup$

$\left\{ (\lambda_1, \lambda_2, x); |\lambda_1| \cdot |\lambda_2| \cdot (|\lambda_1|)^2 |\lambda_2| + (|\lambda_2|)^2 |\lambda_1| = \infty \right\}$,

where we have used the fact (see Clarke (1975), p.107) that, for any two random variables $X$ and $Y$,

$$\text{Var}(XY) = \text{Var}(X)\text{Var}(Y) + (E(X))^2\text{Var}(Y) + (E(Y))^2\text{Var}(X).$$

**Remark**

Theorems 5.2 and 5.3 and statutes $A_3$ and $A_4$ above demonstrate how a.s. limit theorems for 2-IID sequences of random variables extend to partially exchangeable sequences and to sub-subsequences of an arbitrary tight sequence of random variables.
REFERENCES


APPENDIX A1

De Finetti's representation theorem for partially exchangeable events (Dale (1984), p.234)
For every infinite sequence of $g$-fold partially exchangeable events, there
 corresponds a (unique) $g$-dimensional distribution function $F$ on $G \equiv [0, 1]^g$
such that

$$m_{r_1,r_2,\ldots,r_g}^{(n_1,n_2,\ldots,n_g)} = \int_G \prod_{i=1}^g \left( \binom{n_i}{r_i} \frac{r_i}{n_i} (1 - x_i)^{s_i} dF(x_1, x_2, \ldots, x_g)$$

where $r_i + s_i = n_i \quad \forall i \in [g]$

$m_{r_1,r_2,\ldots,r_g}^{(n_1,n_2,\ldots,n_g)}$ denotes the probability of $r_1, r_2, \ldots, r_g$ occurrences
in the $n_1, n_2, \ldots, n_g$ events respectively.

$x_i$ denotes the probability of an occurrence of event of type $i$.

Birkhoff ergodic theorem for stationary processes (Loève (1978), p.76)
Let $X = \{X_i, i \in \mathbb{N}\}$ be a stationary sequence of random variables. If $E(X_1)$ exists, then

$$\frac{\sum_{i=1}^n X_i}{n} \xrightarrow{\text{a.s.}} E(X_1/C) \quad a.s.$$ 

where $C = \bigcap_{n=1}^\infty C_n$

$C_n = \mathcal{F} \{X_n, X_{n+1}, \ldots\} \quad \forall n \in \mathbb{N}$. 
A result from Lord ((1954) p.45, result 2.2).
If \( X = \{X_i, i \in \mathbb{N}\} \) has spherical symmetry, then

\[
\phi(T) = \mathbb{E} \left[ \exp \left( i \sum_{j=1}^{n} t_j X_j \right) \right]
\]

is a function of \( \rho \) only, where \( \rho = |T| = |(t_1, t_2, \ldots, t_n)| \) (\(| \cdot | \) denotes modulus).

An independent spherically symmetric sequence, \( \{X_i; i \in \mathbb{N}\} \), has a \( N(0, \sigma^2) \) distribution for some \( \sigma^2 > 0 \).

Exchangeable events have a probability of 0 or 1 and exchangeable functions are degenerate on a sequence of IID random variables.

Glivenko-Cantelli theorem (Chung (1974), p.132)
Let \( \{X_n, n \in \mathbb{N}\} \) be a sequence of IID random variables, with common distribution function \( F \). Take \( \{X_j(\omega); j \in [n]\} \) for some \( n \in \mathbb{N} \) and arrange them in non-decreasing order, denoted by

\[
Y_{n_1}(\omega) \leq Y_{n_2}(\omega) \leq \cdots \leq Y_{n_n}(\omega).
\]

Now define a discrete distribution function \( F_n(\cdot, \omega) \) as follows:
For \( x \in \mathbb{R} \),

\[
F_n(x, \omega) = \begin{cases} 
0 & \text{if } x < Y_{n_1}(\omega) \\
\frac{k}{n} & \text{if } Y_{n_k}(\omega) \leq x < Y_{n_{k+1}}(\omega) \quad \forall k \in [n-1] \\
1 & \text{if } x \geq Y_{n_n}(\omega).
\end{cases}
\]

Call \( F_n(\cdot, \omega) \) the empiric distribution function based on \( n \) samples from \( F \).

Now introduce indicator random variables,

\[
\xi_j(x, \omega) = I[X_j(\omega) \leq x]
\]

Then

\[
F_n(x, \omega) = \frac{1}{n} \sum_{j=1}^{n} \xi_j(x, \omega)
\]
and hence, from the SLLN (Chung (1974), Theorem 5.1.2),

\[
\lim_{n \to \infty} F_n(x, \omega) = F(x) \quad \text{a.s.}
\]

**Martingale convergence theorem**
(Doob (1953), Chapter 7, Theorem 4.3)
Let \( Z \) be a random variable with \( E|Z| < \infty \), and let \( \cdots \supset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \) be Borel fields of measurable \( \omega \) sets. Let \( \mathcal{F}_\infty = \bigcap_n \mathcal{F}_n \) and let \( \mathcal{F}_\infty \) be the smallest Borel field of sets with \( \mathcal{F}_\infty \supset \bigcup_n \mathcal{F}_n \). Then

\[
\lim_{n \to \infty} E(Z/\mathcal{F}_n) = E(Z/\mathcal{F}_\infty) \quad \text{a.s.}
\]

\[
\lim_{n \to \infty} E(Z/\mathcal{F}_\infty) = E(Z/\mathcal{F}_\infty) \quad \text{a.s.}
\]

**Tight distributions of sequences of random variables**
A family \( \mathcal{P} \) of probabilities on \( \mathcal{A} \) is **tight** if \( \forall \xi > 0, \exists \) compact \( K_\xi \) such that

\[
P(K_\xi^c) < \xi \quad \forall \mathcal{P} \in \mathcal{P}, \text{ where } \mathcal{A} = \mathcal{F} \text{ (open sets of } S) \text{ and } S \text{ is a metric space.}
\]

**Relative Compactness**
Let \( S \) be a separable, complete, metric space with Borel field \( S \). A family \( \mathcal{P} \) of probabilities on \( S \) is **relatively compact** if every sequence of members of \( \mathcal{P} \) contains a subsequence which converges weakly to a probability on \( S \).

**Relative Compactness Criterion**
For the notation above, \( \mathcal{P} \) is relatively compact if and only if \( \mathcal{P} \) is tight. In fact the "if" part holds for general metric spaces \( S \).

**Fatou’s Lemma**
(Feller (1971), Vol. II, p.110)
Suppose \( \{U_n, n \in \mathbb{N}\} \) to be a sequence of non-negative, integrable functions. Then

\[
E \left( \liminf_{n \to \infty} U_n \right) \leq \liminf_{n \to \infty} E(U_n).
\]
Kolmogorov's inequality (Clarke (1975), p.177)
Suppose that \( \{X_i; i \in [n], n \in \mathbb{N}\} \) are independent random variables with zero means and finite second moments, and let
\[
V_k = \text{variance of } X_k, \quad k \in [n].
\]
Then, for any \( \xi < 0 \),
\[
P \left[ \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_i \right| \geq \xi \right] \leq \xi^{-2} \sum_{k=1}^{n} V_k.
\]

Kolmogorov's SLLN (Shirayayev (1984), p.366)
For \( \{X_i; i \in \mathbb{N}\} \) an IID sequence of random variables with \( E|X_i| < \infty \) \( \forall i \in \mathbb{N} \),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i = E(X_1) \quad a.s.
\]

APPENDIX A2

Listed below are properties of conditional independence taken from Aldous (1985), p.183.

An excellent elementary verification of these properties is given by Pfeiffer (1979) and a measure theoretic account is given by Chow and Teicher (1978).

In what follows sets are measurable and functions \( \phi_i \) are bounded and measurable \( \forall i \in [2] \).

Measurable, real valued random variables \( X \) and \( Y \) are conditionally independent given \( \sigma \)-field \( \mathcal{F} \) if

A1 \[
P[X \in A, Y \in B/\mathcal{F}] = P[X \in A/\mathcal{F}]P[Y \in B/\mathcal{F}] \quad \forall A, B \in \mathcal{B}(\mathbb{R})
\]

Each of the following is equivalent to A1

A2 \[
E[\phi_1(X)\phi_2(Y)/\mathcal{F}] = E[\phi_1(X)/\mathcal{F}]E[\phi_2(Y)/\mathcal{F}], \quad \forall \phi_1, \phi_2
\]
A3 \[ P [X \in A_1, Y] = P[X \in A_1] \]

A4 \[ E [\phi_i(X)/\mathcal{F}, Y] = E [\phi_i(X)/\mathcal{F}], \quad \forall i \in [2]. \]

In these definitions we can replace a random variable \( X \) by a \( \sigma \)-field \( \mathcal{G} \), by replacing events \([X \in A]\) with events \([G \in \mathcal{G}]\), and replacing functions \( \phi(X) \) with bounded random variables \( V \in \mathcal{G} \).

A sequence of random variables, \( \{X_i; i \in \mathbb{N}\} \) is conditionally independent given \( \sigma \)-field \( \mathcal{F} \) if the product form in A1 holds for each finite subset of \( \{X_i, i \in \mathbb{N}\} \).

Here are some properties. Let \( \mathcal{F} \) and \( \mathcal{C} \) denote arbitrary \( \sigma \)-fields.

A5 Suppose that for each \( j \geq 1, X_j \) and \( \mathcal{F} \{X_i; i > j\} \) are conditionally independent given \( \mathcal{F} \). Then \( \{X_i; i \in \mathbb{N}\} \) are conditionally independent given \( \mathcal{F} \).

A6 If \( X \) is conditionally independent of itself given \( \mathcal{F} \) then \( X \in \mathcal{F} \) a.s.

A7 Suppose \( X \) and \( \mathcal{F} \) are conditionally independent given \( \mathcal{C} \) and suppose that \( X \) and \( \mathcal{C} \) are conditionally independent given \( H \) where \( H \subset \mathcal{C} \), then \( X \) and \( \mathcal{F} \) are conditionally independent given \( H \).