SPHERICALLY SYMMETRIC CHARGED EINSTEIN–MAXWELL SOLUTIONS

by

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Abstract

In this thesis we study spherically symmetric spacetimes with a perfect fluid source which incorporates charge. We seek explicit solutions to the Einstein–Maxwell system of equations. For nonaccelerating spherically symmetric models a charged, dust solution is found. With constant pressure the equations reduce to quadratures. Particular solutions are also found, with no acceleration, with the equation of state $p = (\gamma - 1)\mu$. The Lie analysis is utilised to reduce the Einstein–Maxwell equations to a system of ordinary differential equations. The evolution of the model depends on a Riccati equation for this general class of accelerating, expanding and shearing spacetimes with charge. Also arbitrary choices for the gravitational potentials lead to explicit solutions in particular cases. With constant gravitational potential λ we generate a simple nonvacuum model. The analysis, in this case, enables us to reduce the solution to quadratures. With the value $\gamma = 2$, for a stiff equation of state, we find that the solution is expressable in terms of elementary functions. Throughout the thesis we have attempted to relate our results to previously published work, and to obtain the uncharged perfect fluid limit where appropriate.

To

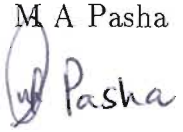
My family especially my mother Surryia Begum and all my friends for being pillars of support and encouragement.

Preface and Declaration

The study described in this thesis was carried out in the School of Mathematical and Statistical Sciences, University of Natal, Durban, during the period April 1996 to December 1999. This thesis was completed under the supervision of Professor S D Maharaj and Professor P G L Leach.

This study represents original work by the author and has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

M A Pasha



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Contents

1 Introduction

The general theory of relativity is a relativistic theory of gravitation. It has wide applications in relativistic astrophysics and cosmology. The theory of general relativity extends the special theory of relativity by incorporating gravitational effects. In general relativity the gravitational field of a body is contained in the curvature of the spacetime. The Riemann tensor describes the curvature of the spacetime manifold. Spacetime is taken to be a four–dimensional, differentiable manifold endowed with a symmetric, nondegenerate metric tensor field. In relativity the line element describes, not only the metric properties of the manifold, but also incorporates the gravitational field. The spacetime geometry of general relativity only locally resembles that of special relativity. However, globally the geometries differ in that the differentiable manifold is not flat. The geometry of spacetime is represented by the Einstein tensor which is defined in terms of the Ricci tensor, the Ricci scalar and metric tensor. The matter content and the electromagnetic contribution are described by the symmetric energy–momentum tensor, which is coupled to the gravitational field via the Einstein field equations. The electromagnetic field is subject to the Maxwell equations. The Einstein–Maxwell field equations are a system of highly nonlinear second order partial differential equations and difficult to solve.

There exist many solutions to the Einstein field equations in the literature.

Exact solutions to the field equations are very important because they facilitate the investigation of the physical properties of specific models. Although a large number of solutions are known today, many of these are not physically significant. For a comprehensive list of known exact solutions which are of physical relevance, the reader is refered to Kramer *et al* (1980), Krasinski (1997), Ryan and Shepley (1975) and Shapiro and Teukolsky (1983), amongst others. In this thesis we study spherically symmetric gravitational fields with a perfect fluid energy–momentum tensor and an electromagnetic field. In particular we seek exact solutions to the Einstein–Maxwell system by choosing a specific form for the electromagnetic gauge potential. Spherically symmetric models are physically significant and are extensively utilised in a variety of applications. In astrophysics the collapse of a star can be modelled by a spherically symmetric gravitational field as pointed out by Shapiro and Teukolsky (1983). In cosmology spherically symmetric spacetimes have been used to model the gravitational behaviour and evolution of the early universe (Krasinski 1997). The spherically symmetric models are an important generalisation of the Robertson–Walker models, the standard cosmology models which are both homogeneous and isotropic. To describe many situations of physical significance we need to incorporate the effects of anisotropies and inhomogeneities. Anisotropic cosmologies have been analysed by a number of authors to study the effect of deviations from the isotropic universe (Ryan and Shepley 1975). The motivation for studying inhomogeneous models is to analyse the deviation from homogeneity based on observational evidence (Krasinski 1997). Inhomogeneous, spherically symmetric models are proving to be highly useful in the study of gravitational collapse as observed by Joshi (1993).

Most of the exact solutions found have vanishing shear. The evolution of shear–free models can be reduced to an Emden–Fowler equation. These shear–free models are mathematically simpler to study than their counterparts with shear. A recent general treatment of the shear–free case is given by Maharaj *et al* (1996). Broad classes of shear–free solutions have also been presented by Srivastava (1987), Stephani (1983) and Sussman (1988a, 1988b). Kramer *et al* (1980) have listed and categorised most shear–free spherically symmetric solutions. These solutions are normally given in comoving coordinates which facilitate the physical interpretation of the models. In contrast there are few spherically symmetric solutions having nonzero shear. Some known solutions with shear, are those of Gutman and Bespalko (1967) and of Wesson (1978); these admit a stiff equation of state in comoving coordinates. McVittie and Wiltshire (1975, 1977), Szafron (1977), Szekeres (1975) and Vaidya (1968) also found shearing solutions but these were given in noncomoving coordinates and are more difficult to interpret. In recent attempts to obtain exact solutions a conformal symmetry requirement is imposed on the spacetime manifold, so that spacetime is invariant under the action of a group of conformal motions. A number of exact solutions have been found including those of Herrera and Ponce de Leon (1985a, 1985b), Herrera *et al* (1984), Maartens and Maharaj (1990), Maartens *et al* (1986, 1995, 1996), Mason and Maartens (1987), Saridakis and Tsamparlis (1991) and Tsamparlis and Mason (1985). Most of these solutions have been concerned with astrophysical applications. Spherically symmetric cosmological models, admitting a conformal Killing vector, have been studied by Dyer *et al* (1987) and Maharaj *et al* (1991), amongst others. Various spacetimes admitting an inheriting conformal Killing vector, a special conformal symmetry, have been analysed by Coley (1991) and Coley and Tupper (1989, 1990). The spherically symmetric perfect fluid solutions obeying a barotropic equation of state are related to those obtained by Gutman

and Bespalko (1967), Hajj-Boutros (1985), Lake (1983), Shaver and Lake (1988) and Wesson (1978). These are incorporated in the general class of Maharaj *et al* (1993) which obey an equation of state that generalises the stiff equation of state. Kitamura (1989, 1994, 1995a, 1995b), using the characteristic system method devised by Takeno (1966), derived exact solutions for a perfect fluid with shear, expansion and acceleration. The Kitamura class is related to the models of Gutman and Bespalko (1967), Sussman (1991), Van den Berg and Wils (1985) and Wesson (1978). All the solutions listed above are in comoving coordinates. Noncomoving coordinates are more to difficult to analyse physically as pointed out by Knutsen (1995).

In most cases the solutions given above apply to neutral matter. Our intention in this thesis is to incorporate the electromagnetic field which means that the Einstein field equations are supplemented with the Maxwell equations. We are therefore seeking solutions to the Einstein–Maxwell system in spherically symmetric spacetimes. Clearly this is more complicated than the case of uncharged matter with only the Einstein field equations. By assuming a particular form for the electromagnetic gauge potential we find shearing solutions under a variety of assumptions.

This thesis is organised as follow:

- Chapter 1: Introduction.
- Chapter 2: In this chapter we present an overview and background material necessary for later chapters.
- Chapter 3: We present solutions to the Einstein–Maxwell system of equations that are nonaccelerating. The solutions are separated into the categories $Y' = 0$

and $Y' \neq 0$. Nonexpanding solutions are also found. An equation of state is specified and exact solutions are demonstrated.

- Chapter 4: A Lie analysis of the Einstein–Maxwell system is performed. The system is reduced to a quadrature. We regain the uncharged solution of Govender (1997) in the appropriate limit.

- Chapter 5: Particular solutions to the Einstein–Maxwell system are found by making *ad hoc* choices for the gravitational potentials. We also utilise the Lie analysis and specify a stiff equation of state to obtain explicit solutions which are expanding, accelerating and shearing.

- Chapter 6: Conclusion

2 Spacetime Geometry and Field Equations

2.1 Introduction

In this chapter we discuss briefly those concepts in differential geometry, the general theory of relativity and electromagnetism relevant to this thesis. For a more detailed account of differential geometry and tensor calculus the reader is referred to Choquet-Bruhat *et al* (1977), de Felice and Clarke (1990), Hawking and Ellis (1973) and Misner *et al* (1973). In §2.2 we introduce the metric tensor field, the connection coefficients, the Riemann tensor, the Ricci tensor, Ricci scalar and the Einstein tensor. We define in §2.3, the spherically symmetric line element in comoving coordinates and give the kinematical quantities. The nonvanishing components of the connection coefficients, the Ricci tensor, the Ricci scalar and the Einstein tensor are explicitly calculated for spherically symmetric spacetimes. In §2.4 we derive the Einstein-Maxwell system of equations for spherically symmetric spacetimes containing a charged perfect fluid. These equations are obtained for a particular form of the electromagnetic gauge potential which is chosen on physical grounds. Exact solutions to the Einstein-Maxwell system are presented in subsequent chapters. The Einstein field equations for an uncharged fluid are regained in §2.5 as a special case of the results from §2.4.

2.2 Spacetime Geometry

In the general theory of relativity we take spacetime to be a four–dimensional differentiable manifold. The manifold is endowed with a symmetric, nonsingular metric field $\mathbf{g}$ with signature $(-+++)$ and locally has the structure of Euclidean space in that it may be covered by coordinate patches. The manifold supports a differentiable structure because the passage between the coordinate patches in the overlapping coordinate neighbourhoods is smooth. Note that the global structure of the manifold may be very different from that of Euclidean space. Points on the manifold are labelled by real coordinates (x^a), where x^0 is timelike and x^1, x^2, x^3 are spacelike. For convenience we use units in which the speed of light $c = 1$. For a rigorous definition of differentiable manifolds and for material on differential geometry the reader is referred to texts by Bishop and Goldberg (1968), de Felice and Clarke (1990), Hawking and Ellis (1973), Misner *et al* (1973) and Wald (1984).

The fundamental line element defining the invariant distance between neighbouring points on the manifold is given by

$$ds^2 = g_{ab}dx^a dx^b, \tag{2.1}$$

where $\mathbf{g}$ is the symmetric, nondegenerate metric tensor field. The metric connection $\mathbf{\Gamma}$ is defined in terms of the metric tensor field and its derivatives by

$$\Gamma^a{}_{bc} = \tfrac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}), \tag{2.2}$$

where a comma denotes partial differentiation. The fundamental theorem of Riemannian geometry implies the existence of a unique symmetric connection $\mathbf{\Gamma}$ that preserves inner products under parallel transport (do Carmo 1992).

Let $\mathbf{Z}$ be a type $(0,1)$ vector field defined on the manifold. Then the covariant derivative of $\mathbf{Z}$ is given by

$$Z_{a;b} = Z_{a,b} - \Gamma^{d}{}_{ab} Z_d,$$

where the semicolon denotes covariant differentiation. The covariant derivative is a generalisation of the partial derivative such that when operating on an (r,s) tensor field it produces an $(r,s+1)$ tensor field. On covariantly differentiating twice and forming the difference $Z_{a;bc} - Z_{a;cb}$ we obtain

$$Z_{a;bc} - Z_{a;cb} = (\Gamma^{d}{}_{ac,b} - \Gamma^{d}{}_{ab,c} + \Gamma^{e}{}_{ac}\Gamma^{d}{}_{eb} - \Gamma^{e}{}_{ab}\Gamma^{d}{}_{ec}) Z_d$$

$$= R^{d}{}_{abc} Z_d,$$

where $R^{d}{}_{abc}$ are the components of the Riemann tensor field $\mathbf{R}$. The Riemann tensor provides a measure of the curvature of a manifold; it gives a measure of deviation from the flatness of the Minkowski spacetime of the special theory of relativity. In a curved spacetime $\mathbf{R}$ does not vanish globally. The Riemann tensor is a type (1,3) tensor and is defined as

$$R^{a}{}_{bcd} = \Gamma^{a}{}_{bd,c} - \Gamma^{a}{}_{bc,d} + \Gamma^{a}{}_{ec}\Gamma^{e}{}_{bd} - \Gamma^{a}{}_{ed}\Gamma^{e}{}_{bc} \tag{2.3}$$

in terms of the connection coefficients (2.2). The Riemann tensor components R_{abcd} satisfy the following useful identities

$$R_{abcd} = -R_{bacd}$$

$$R_{abcd} = -R_{abdc}$$

$$R_{abcd} = R_{cdab}$$

$$R_{abcd} + R_{acdb} + R_{adbc} = 0$$

$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0.$$

These identities assist in calculations that involve the curvature of the manifold and are important in the formulation of the Einstein field equations.

On contraction of the Riemann tensor (2.3) we obtain the Ricci tensor

$$R_{ab} = R^{c}{}_{acb}$$

$$= \Gamma^{d}{}_{ab,d} - \Gamma^{d}{}_{ad,b} + \Gamma^{e}{}_{ab}\Gamma^{d}{}_{ed} - \Gamma^{e}{}_{ad}\Gamma^{d}{}_{eb}. \tag{2.4}$$

On contracting the Ricci tensor (2.4) we obtain the Ricci scalar or curvature scalar

$$R = g^{ab}R_{ab}$$

$$= R^{a}{}_{a}. \tag{2.5}$$

The Einstein tensor $\mathbf{G}$ is constructed in terms of the Ricci tensor (2.4) and the Ricci scalar (2.5) as follows

$$G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab}. \tag{2.6}$$

The Einstein field tensor $\mathbf{G}$ has zero divergence so that

$$G^{ab}{}_{;b} = 0, \tag{2.7}$$

which follows directly from the definition (2.6). This property of the Einstein tensor is sometimes called the Bianchi identity and generates the conservation of energy–momentum via the Einstein field equations.

2.3 Spherically Symmetric Spacetimes

We are principally concerned with the behaviour of the gravitational field in spherically symmetric spacetimes. The most general line element (2.1) in the case of spherically symmetric spacetimes in coordinates (x^a)=(t, r, θ, Φ) is given by

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + Y^2(t,r)\left[d\theta^2 + \sin^2\theta d\Phi^2\right], \tag{2.8}$$

where the gravitational potentials ν, λ and Y are functions only of the spacetime coordinates t and r. It should be noted that the coordinates used in the line element (2.8) are comoving and not isotropic. The four–velocity $\mathbf{u}$ has the form

$$u^a = \left(e^{-\nu}, 0, 0, 0\right). \tag{2.9}$$

In the comoving frame of reference (2.9), for the spherically symmetric metric (2.8), the various kinematical quantities are given by

$$\omega_{ab} = 0 \tag{2.10a}$$

$$\dot{u}^a = (0, \nu', 0, 0) \tag{2.10b}$$

$$\Theta = e^{-\nu}\left(\dot{\lambda} + \frac{2\dot{Y}}{Y}\right) \tag{2.10c}$$

$$\sigma_1^1 = \sigma_2^2 = -\tfrac{1}{2}\sigma_3^3 = \tfrac{1}{3}e^{-\nu}\left(\frac{\dot{Y}}{Y} - \dot{\lambda}\right), \tag{2.10d}$$

where dots and primes denotes partial differentiation with respect to t and r respectively. The kinematical quantities are the vorticity tensor ω_{ab}, the acceleration vector $\dot{u}^a$, the expansion scalar Θ (or the rate of expansion) and the magnitude of the shear σ (or the rate of shear). The vorticity vanishes since the spacetime is spherically symmetric. The acceleration, expansion and shear are nonzero in general. Note that most of the exact solutions listed in the literature, corresponding to the metric (2.8), are categorised in terms of the kinematical quantities as pointed out by Kramer *et al* (1980) and Krasinski (1997). In this thesis we are concerned with nonzero shear so that our solutions must satisfy the condition

$$\frac{\dot{Y}}{Y} - \dot{\lambda} \neq 0.$$

If the shear becomes zero then, after a suitable coordinate transformation, (2.8) assumes the form

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\Phi^2\right)\right], \tag{2.11}$$

where $\nu = \nu(t,r)$ and $\lambda = \lambda(t,r)$ are metric functions. In the form (2.11) the coordinates are simultaneously comoving and isotropic; the line element has this form only when the shear vanishes. A comprehensive analysis of the shear-free field equations corresponding to (2.11) was performed by Maharaj *et al* (1996). The governing equation for the evolution of the spacetime (2.11) is of the form

$$\frac{d^2z}{dx^2} = f(x)z^2,$$

which is a generalised Emden–Fowler equation of index two.

The nonzero connection coefficients (2.2) for the line element (2.8) are given by

$$\Gamma^0{}_{00} = \dot{\nu} \qquad \Gamma^0{}_{01} = \nu'$$

$$\Gamma^0{}_{11} = e^{2(\lambda-\nu)}\dot{\lambda} \qquad \Gamma^0{}_{22} = e^{-2\nu}Y\dot{Y}$$

$$\Gamma^0{}_{33} = \sin^2\theta\, e^{-2\nu}Y\dot{Y} \qquad \Gamma^1{}_{00} = e^{2(\nu-\lambda)}\nu'$$

$$\Gamma^1{}_{01} = \dot{\lambda} \qquad \Gamma^1{}_{11} = \lambda'$$

$$\Gamma^1{}_{22} = -e^{2\nu}YY' \qquad \Gamma^1{}_{33} = -\sin^2\theta\, e^{-2\lambda}YY'$$

$$\Gamma^2{}_{02} = \frac{\dot{Y}}{Y} \qquad \Gamma^2{}_{12} = \frac{Y'}{Y}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \qquad \Gamma^3{}_{03} = \frac{\dot{Y}}{Y}$$

$$\Gamma^3{}_{13} = \frac{Y'}{Y} \qquad \Gamma^3{}_{23} = \cot\theta.$$

The number of nonzero coefficients is higher when the shear is nonvanishing; this is related to the greater range of behaviour for the potentials. It is now possible to calculate the Ricci tensor (2.4) using the above connection coefficients.

The nonzero Ricci tensor components take the form

$$R_{00} = -\ddot{\lambda} - \dot{\lambda}^2 + \dot{\lambda}\dot{\nu} + 2\dot{\nu}\frac{\dot{Y}}{Y} - 2\frac{\ddot{Y}}{Y}$$

$$+ e^{2(\nu-\lambda)}\left(\nu'' + \nu'^2 - \nu'\lambda' + 2\nu'\frac{Y'}{Y}\right) \tag{2.12a}$$

$$R_{01} = 2\left(\dot{\lambda}\frac{Y'}{Y} + \nu'\frac{\dot{Y}}{Y} - \frac{\dot{Y}'}{Y}\right) \tag{2.12b}$$

$$R_{11} = -\nu'' - \nu'^2 + \lambda'\nu' + 2\lambda'\frac{Y'}{Y} - 2\frac{Y''}{Y}$$

$$+ e^{2(\lambda-\nu)}\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + 2\dot{\lambda}\frac{\dot{Y}}{Y}\right) \tag{2.12c}$$

$$R_{22} = e^{-2\nu}Y\dot{Y}\left(\dot{\lambda} - \dot{\nu} + \frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{\dot{Y}}\right)$$

$$+ e^{-2\lambda}YY'\left(\lambda' - \nu' - \frac{Y'}{Y} - \frac{Y''}{Y'}\right) + 1 \tag{2.12d}$$

$$R_{33} = \sin^2\theta\, R_{22}. \tag{2.12e}$$

With the help of the Ricci tensor components of (2.12) and the definition (2.5) we calculate the Ricci scalar

$$R = 2e^{-2\nu}\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + 2\dot{\lambda}\frac{\dot{Y}}{Y} - 2\dot{\nu}\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} + \frac{\ddot{Y}}{Y}\right)$$

$$- 2e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' - 2\lambda'\frac{Y'}{Y} + 2\nu'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + 2\frac{Y''}{Y}\right) + \frac{2}{Y^2}. \tag{2.13}$$

The Ricci tensor components (2.12) and the Ricci scalar (2.13) generate the nonzero components of the Einstein tensor (2.6). These components are

$$G_{00} = 2\dot{\lambda}\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} - e^{2(\nu-\lambda)}\left(-2\lambda'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + 2\frac{Y''}{Y}\right) + \frac{e^{2\nu}}{Y^2} \tag{2.14a}$$

$$G_{01} = 2\dot{\lambda}\frac{Y'}{Y} + 2\nu'\frac{\dot{Y}}{Y} - 2\frac{\dot{Y}'}{Y} \tag{2.14b}$$

$$G_{11} = 2\nu'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + e^{2(\lambda-\nu)}\left(2\dot{\nu}\frac{\dot{Y}}{Y} - \frac{\dot{Y}^{2^2}}{Y} - 2\frac{\ddot{Y}}{Y})\right) - \frac{e^{2\lambda}}{Y^2} \tag{2.14c}$$

$$G_{22} = -e^{-2\nu}\left[\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}\right)Y^2 + \left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y}\right)Y\right]$$

$$+ e^{-2\lambda}\left[\left(\nu'' + \nu'^2 - \nu'\lambda'\right)Y^2 + \left(\nu'Y' - \lambda'Y' + Y''\right)Y\right] \tag{2.14d}$$

$$G_{33} = \sin^2\theta\, G_{22}, \tag{2.14e}$$

for the line element (2.8).

2.4 The Einstein–Maxwell Field Equations

The Einstein–Maxwell field equations describe the coupling between the curvature of spacetime and the matter content $\mathbf{T}$ which also includes the electromagnetic field. In addition to the Einstein equations we also need to consider the Maxwell equations which govern the behaviour of the electromagnetic field. Charge is introduced

through the electromagnetic four–potential **A** which defines the electromagnetic field tensor **F**. The electromagnetic field tensor **F** is defined in terms of the four–potential A, and is given by

$$F_{ab} = A_{b;a} - A_{a;b}. \tag{2.15}$$

The electromagnetic contribution **E** to the energy–momentum tensor is given by the result

$$E_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}.$$

The uncharged matter contribution to **T** is

$$M_{ab} = (\mu + p)u_a u_b + pg_{ab},$$

which corresponds to a perfect fluid. Here μ is the energy density and p is the isotropic pressure. The total energy–momentum tensor is then given by

$$T_{ab} = M_{ab} + E_{ab},$$

for barotropic matter and charge respectively. Then the Einstein–Maxwell field equations can be expressed as the system

$$G_{ab} = T_{ab}$$

$$G_{ab} = M_{ab} + E_{ab} \tag{2.16a}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{2.16b}$$

$$F^{ab}{}_{;b} = J^a, \tag{2.16c}$$

where J^a is the four-current density. We write the four-current as

$$J^a = \epsilon u^a. \tag{2.17}$$

The quantity ϵ in (2.17) is the proper charge density.

We utilise the freedom in the gauge by choosing the four-potential $\mathbf{A}$. We make the choice

$$A_a = (\phi(t,r), 0, 0, 0), \tag{2.18}$$

which is consistent with spherical symmetry and has been widely used in the study of inhomogeneous cosmological models (Sussman 1987, Sussman 1988a, Sussman 1988b). The quantity $\phi(t,r)$ in (2.18) is called the electromagnetic gauge potential. Note that we have taken ϕ to be a function of both the radial and time coordinates t and r which is the same as the dependence of the metric potentials ν, λ and Y. The nonzero components of the electromagnetic field tensor are

$$F_{10} = -F_{01} = \phi'. \tag{2.19}$$

On using the components (2.19) we calculate the electromagnetic contribution to the energy-momentum tensor which is given by

$$E_{ab} = \text{diag}\left(\frac{1}{2}e^{-2\lambda}\phi'^2, -\frac{1}{2}e^{-2\nu}\phi'^2, \frac{1}{2}e^{-2(\nu+\lambda)}Y^2\phi'^2, \frac{1}{2}e^{-2(\nu+\lambda)}Y^2\phi'^2\sin^2\theta\right).$$

The uncharged matter contribution to the energy-momentum tensor $\mathbf{T}$ is

$$M_{ab} = \text{diag}\left(\mu e^{\nu}, pe^{2\lambda}, pY^2, p\sin^2\theta Y^2\right).$$

The nonzero components of the total energy-momentum tensor become

$$T_{ab} = \text{diag}\Big(\mu e^{2\nu} + \frac{1}{2}e^{-2\lambda}\phi'^2, pe^{2\lambda} - \frac{1}{2}e^{-2\nu}\phi'^2,$$
$$pY^2 + \frac{1}{2}e^{-2(\nu+\lambda)}Y^2\phi'^2, p\sin^2\theta Y^2 + \frac{1}{2}e^{-2(\nu+\lambda)}Y^2\phi'^2\sin^2\theta\Big). \tag{2.20}$$

From (2.20) and (2.14) we obtain the Einstein field equations (2.16a) as

$$\mu = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - \lambda' Y' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}e^{-2\nu}\left(\dot{\lambda}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right)$$

$$- \frac{1}{2}e^{-2(\lambda+\nu)}\phi'^2 \tag{2.21a}$$

$$p = -\frac{1}{Y^2} + \frac{2}{Ye^{2\lambda}}\left(\nu' Y' + \frac{Y'^2}{2Y}\right) - \frac{2}{Y}e^{-2\nu}\left(\ddot{Y} - \dot{\nu}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right)$$

$$+ \frac{1}{2}e^{-2(\lambda+\nu)}\phi'^2 \tag{2.21b}$$

$$p = e^{-2\lambda}\left[\left(\nu'' + \nu'^2 - \nu'\lambda'\right) + \frac{1}{Y}\left(\nu' Y' - \lambda' Y' + Y''\right)\right]$$

$$- e^{-2\nu}\left[\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}\right) + \frac{1}{Y}\left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y}\right)\right]$$

$$- \frac{1}{2}e^{-2(\lambda+\nu)}\phi'^2 \tag{2.21c}$$

$$0 = \dot{Y}' - \dot{Y}\nu' - Y'\dot{\lambda}, \tag{2.21d}$$

for a spherically symmetric model with charged matter.

Given the nonzero components of the electromagnetic field tensor (2.19) it is possible to generate the Maxwell equations. The first Maxwell equation (2.16b) is identically satisfied. The second Maxwell equation (2.16c) is identically satisfied for $a = 2, 3$. We generate the conditions

$$e^{-2\lambda-\nu}\phi'\left(\lambda'+\nu'-\frac{\phi''}{\phi'}-2\frac{Y'}{Y}\right)=\epsilon \tag{2.22a}$$

$$\phi'\left(\dot{\lambda}+\dot{\nu}-2\frac{\dot{Y}}{Y}\right)=\dot{\phi}', \tag{2.22b}$$

from (2.16c), (2.17) and (2.19) where we have set $a = 0$ and $a = 1$, respectively. We take (2.22a) as the definition of the proper charge density, given in terms of the gravitational potentials and the electromagnetic gauge potential. Equation (2.22b) can be immediately integrated to give the expression

$$\phi' = \frac{1}{Y^2}e^{\lambda+\nu}K(r)$$

where $K(r)$ is a function of integration. We can treat this expression as a definition for ϕ. Consequently we need consider only (2.21) to generate a solution and treat (2.22b) as a consistency condition. Note that (2.22b) does not arise for static gravitational field. The system of equations (2.21)–(2.22) comprise the Einstein–Maxwell equations for the spherically symmetric models corresponding to the four–potential (2.18).

For charged matter the conservation of energy–momentum is given by

$$T^{ab}{}_{;b} = (M^{ab} + E^{ab})_{;b} = 0,$$

where we have used the field equation (2.16a) and the Bianchi condition (2.7). With $a = 1$ and $a = 0$ the conservation equations become

$$p' + (\mu + p)\nu' + e^{-2\lambda-2\nu}\phi'^2\left(\lambda'+\nu'-\frac{\phi''}{\phi'}-2\frac{Y'}{Y}\right)=0 \tag{2.23a}$$

$$\dot{\mu} + (\mu + p)(\dot{\lambda} + \frac{2}{Y}\dot{Y}) + e^{-2\lambda - 2\nu}\phi'^2\left(-\dot{\lambda} - \dot{\nu} + \frac{\dot{\phi}'}{\phi'} + 2\frac{\dot{Y}}{Y}\right) = 0. \qquad (2.23b)$$

Of course the conservation equations (2.23) may be derived directly from the field equations.

Exact solutions in cosmology to the system (2.21)–(2.22) have been found for particular choices of the electromagnetic gauge potential ϕ. A comprehensive review of such solutions is provided by Krasinski (1997). Of particular note are the extensive analyses of Sussman (1987, 1988a, 1988b) and Srivastava (1992) for inhomogeneous cosmological models with an electromagnetic field. The recent class of solutions of Moodley *et al* (1999), regains in the uncharged limit, many familiar shearing models, including the Gutman–Bespalko (1967) solution with a stiff equation of state. Examples of solutions corresponding to charged relativistic stars are given by Humi and Mansour (1984), Pant and Sah (1979) and Patel and Mehta (1995). Exact solutions of (2.21)–(2.22), for charged stars with a conformal symmetry, were found by Herrera *et al* (1984), Herrera and Ponce de Leon (1985a, 1985b) and Maartens and Maharaj (1990), amongst others.

2.5 The Einstein Field Equations

For uncharged matter the energy momentum tensor $\mathbf{T}$ is given by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab}.$$

With $\phi = \phi(t)$ we observe that $\mathbf{F} = \mathbf{0}$ from (2.19). The Einstein–Maxwell equations then reduce to those corresponding to the above form of $\mathbf{T}$ for uncharged matter.

The Einstein equations become

$$\mu = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - \lambda' Y' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}e^{-2\nu}\left(\dot{\lambda}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right) \tag{2.24a}$$

$$p = -\frac{1}{Y^2} + \frac{2}{Ye^{2\lambda}}\left(\nu' Y' + \frac{Y'^2}{2Y}\right) - \frac{2}{Y}e^{-2\nu}\left(\ddot{Y} - \dot{\nu}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right) \tag{2.24b}$$

$$p = e^{-2\lambda}\left[\left(\nu'' + \nu'^2 - \nu'\lambda'\right) + \frac{1}{Y}\left(\nu' Y' - \lambda' Y' + Y''\right)\right]$$

$$- e^{-2\nu}\left[\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}\right) + \frac{1}{Y}\left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y}\right)\right] \tag{2.24c}$$

$$0 = \dot{Y}' - \dot{Y}\nu' - Y'\dot{\lambda}, \tag{2.24d}$$

from (2.21) for uncharged matter.

The Maxwell equations (2.22) are identically satisfied. The conservation equations (2.23) become

$$p' + (\mu + p)\nu' = 0, \tag{2.25a}$$

$$\dot{\mu} + (\mu + p)\left(\dot{\lambda} + \frac{2}{Y}\dot{Y}\right) = 0, \tag{2.25b}$$

corresponding to momentum conservation and energy conservation respectively. The equations (2.25) are first order differential equations which can also be derived directly from the field equations (2.24).

Previous attempts have succeeded, in some cases, in finding exact solutions to the system (2.24) for nonzero shear. Maharaj *et al* (1993) presented a class of

accelerating, expanding and shearing metrics which contain the solutions of Gutman and Bespalko (1967), Hajj-Boutros (1987), Shaver and Lake (1988) and Wesson (1978). The solutions of Maharaj *et al* (1993) admit a conformal Killing vector and contain a stiff equation of state (see also Maharaj and Maharaj 1994). Van den Berg and Wils (1985) found shearing solutions for nonaccelerating and accelerating fluids. Kitamura (1994) recently presented a class of shearing metrics using a method proposed by Takeno (1966); this class admits a conformal Killing vector acting in the radial direction (Kitamura 1995a, 1995b). A number of solutions have been found in noncomoving coordinates. These include the solutions of McVittie and Wiltshire (1975, 1977) and Vaidya (1968). The physical analysis of solutions in noncomoving coordinates has been performed by Bonnor and Knutsen (1993) and Knutsen (1992, 1995). Clearly few solutions are known for nonzero shear as the field equations are difficult to integrate; the shear–free case is easier to handle.

3 Nonaccelerating and Nonexpanding Solutions to the Field Equations

3.1 Introduction

In this chapter we consider nonaccelerating and nonexpanding solutions to the coupled Einstein-Maxwell system. A number of exact solutions are identified. In §3.2 we investigate solutions having both shear and expansion but the acceleration vanishes. This class of solutions is divided into two categories, according to whether $Y' = 0$ or $Y' \neq 0$. Firstly we consider the case $Y' = 0$ in §3.2.1 and a particular solution is given for charged dust. Then the case $Y' \neq 0$ is studied in §3.2.2. The solution of the field equations is reduced to quadratures by assuming that $p =$ constant. Three cases arise in the solution and we consider each case in turn. The uncharged solutions of Govender (1997) are regained by setting a constant to zero. Nonexpanding solutions are considered briefly in §3.3 and the behaviour of the potentials is reduced to a quadrature. In §3.4 we impose the equation of state $p = (\gamma - 1)\mu$. A solution valid for all γ is found by assuming that the potential λ is constant. With no restriction on λ and by setting $\gamma = 2$, a second solution is generated.

3.2 Nonaccelerating Solutions

In this section we find solutions having shear and expansion, but no acceleration so that $\dot{u}^a = 0$. Therefore from (2.10b) this condition implies that

$$\nu = \nu(t). \tag{3.1}$$

Kramer *et al* (1980) and Govender (1997), in their analysis of uncharged fluids, point out that the nonaccelerating models may be divided into two categories: $Y' = 0$ and $Y' \neq 0$. We analyse the case $Y' = 0$ in §3.2.1. Later in §3.2.2 we investigate the case $Y' \neq 0$.

3.2.1 The Case $Y' = 0$

If we consider the case $Y' = 0$, then we can take

$$Y = t, \tag{3.2}$$

without any loss of generality. Then the field equations (2.21) reduce to

$$\mu t^2 = 1 + e^{-2\nu}\left(2t\dot{\lambda} + 1\right) - \frac{\phi'^2 t^2}{2e^{2\nu+2\lambda}} \tag{3.3a}$$

$$pt^2 = -1 + e^{-2\nu}\left(2t\dot{\nu} - 1\right) + \frac{\phi'^2 t^2}{2e^{2\nu+2\lambda}} \tag{3.3b}$$

$$pt^2 = -e^{-2\nu}\left[t^2\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}\right) + t\left(\dot{\lambda} - \dot{\nu}\right)\right] - \frac{\phi'^2 t^2}{2e^{2\nu+2\lambda}}. \tag{3.3c}$$

The energy density and pressure are defined by (3.3a) and (3.3b) respectively. Equations (3.3b) and (3.3c) generate the condition of pressure isotropy

$$\frac{\phi'^2 t^2}{e^{2\nu+2\lambda}} - 1 + e^{-2\nu}(2\dot{\nu}t - 1) + e^{-2\nu}\left[t^2\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}\right) + t\left(\dot{\lambda} - \dot{\nu}\right)\right] = 0. \quad (3.4)$$

Thus a solution to (3.4) generates a solution to the Einstein–Maxwell system (3.3). Equation (3.4) contains three dependent functions ϕ, ν and λ; to obtain a solution we need to specify the form of two functions, and then integrate (3.4) to find the third function. We have reduced the evolution of the cosmological model, with $Y' = 0$, to the single equation (3.4).

Van den Berg and Wils (1985) and Govender (1997) have presented solutions to (3.3), for uncharged spherically symmetric fluid3, corresponding to radiation and st)ff matter. Here we demonstrate ! simple solution to the system (3.3). We make the assumptions

$$\nu = \nu_0 \quad (3.5a)$$

$$p = 0, \quad (3.5b)$$

where ν_0 is a constant. Then the energy density is

$$\mu = \frac{2}{t}e^{-2\nu_0}\dot{\lambda},$$

and the electromagnetic gauge potential is given by

$$\phi'^2 = \frac{2}{t^2}\left(1 + e^{-2\nu_0}\right)e^{2\nu_0+2\lambda}.$$

We observe that both μ and ϕ are given in terms of λ. It remains to obtain the gravitational potential λ: this follows from (3.3b) and (3.3c). The constraint on the

behaviour of λ is given by

$$t^2\left(\ddot{\lambda}+\dot{\lambda}^2\right)+t\dot{\lambda}+1+e^{2\nu_0}=0. \tag{3.6}$$

It is convenient to make the transformation

$$t=e^s$$

$$\lambda=\ln\sigma.$$

Then (3.6) is transformed to

$$\sigma_{ss}+\left(1+e^{2\nu_0}\right)\sigma=0.$$

This is a linear second order equation which has solution

$$\sigma=C_1\cos\left(\sqrt{1+2e^{\nu_0}}s\right)+C_2\sin\left(\sqrt{1+e^{2\nu_0}}s\right),$$

where C_1 and C_2 are constants. Finally the gravitational potential λ has the form

$$\lambda=\ln\left[C_1\cos\left(\sqrt{1+2e^{\nu_0}}\ln t\right)+C_2\sin\left(\sqrt{1+2e^{\nu_0}}\ln t\right)\right]. \tag{3.7}$$

Therefore we have demonstrated the explicit solution (3.7) to the nonlinear equation (3.6), and consequently the system (3.3). The line element is given by

$$ds^2=-e^{2\nu_0}dt^2+\left[C_1\cos\sqrt{1+2e^{\nu_0}}\ln t+C_2\sin\sqrt{1+2e^{\nu_0}}\ln t\right]^2dr^2$$

$$+t^2\left(d\theta^2+\sin^2\theta d\Phi^2\right).$$

This solution follows because of the simple choice in (3.5). Other choices will represent wider possibilities for the behaviour of the gravitational potentials. Our objective here was to demonstrate the existence of solutions. A general analysis on the system (3.3) will be the object of future investigations.

3.2.2 The Case $Y' \neq 0$

In this section we consider the second possibility $Y' \neq 0$, in the class of nonaccelerating solutions. If $Y' \neq 0$, then we can make the choice

$$\nu = 0$$

as pointed out by Kramer *et al* (1980) and Govender (1997). The field equations (2.21) then become

$$\mu = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - \lambda' Y' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}\left(\dot{\lambda}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right)$$

$$- \frac{\phi'^2}{2e^{2\lambda}} \tag{3.8a}$$

$$p = -\frac{1}{Y^2} + e^{-2\lambda}\left(\frac{Y'^2}{Y^2}\right) - \frac{2}{Y}\left(\ddot{Y} + \frac{\dot{Y}^2}{2Y}\right)$$

$$+ \frac{\phi'^2}{2e^{2\lambda}} \tag{3.8b}$$

$$p = \frac{e^{-2\lambda}}{Y}\left(Y'' - \lambda' Y'\right) - \left[\ddot{\lambda} + \dot{\lambda}^2 + \frac{1}{Y}\left(\dot{\lambda}\dot{Y} + \ddot{Y}\right)\right]$$

$$- \frac{\phi'^2}{2e^{2\lambda}} \tag{3.8c}$$

$$0 = \dot{Y}' - Y'\dot{\lambda}. \tag{3.8d}$$

It is possible to study the integration of the system (3.8) in general.

Equations (3.8d) has solution

$$e^{2\lambda} = \frac{Y'^2}{1 - \epsilon f^2(r)} \qquad \epsilon = 0, \pm 1, \tag{3.9}$$

where $f(r)$ is a function of integration. A combination of equations (3.8a), (3.8b) and (3.8c) yields

$$\mu + 3p = -2\left(\ddot{\lambda} + \dot{\lambda}^2\right) - 4\frac{\ddot{Y}}{Y} - \frac{\phi'^2}{e^{2\lambda}}.$$

With the help of (3.8d) and (3.9), we can eliminate the variable λ appearing in the last equation. Thus we obtain the energy density μ in terms of Y, ϕ, and p:

$$\mu = -3p - 4\frac{\ddot{Y}}{Y} - 2\frac{\ddot{Y}'}{Y'} - \frac{\phi'^2 Y'^2}{(1 - \epsilon f^2(r))}. \tag{3.10}$$

In order to obtain an expression for p in terms of Y, we substitute (3.9) into (3.8b) which results in

$$Y^2 p = -2Y\ddot{Y} - \dot{Y}^2 - \epsilon f^2(r) + \frac{\phi'^2 Y^2}{2e^{2\lambda}}. \tag{3.11}$$

It remains to find a solution to (3.11). The remaining field equation (3.8c) becomes a consistency condition; we can treat this equation as a definition for ϕ. Once (3.11) is integrated we can express the solution as the line element

$$ds^2 = -dt^2 + \frac{Y'^2}{1 - \epsilon f^2(r)}dr^2 + Y^2\left(d\theta^2 + \sin^2\theta d\Phi^2\right), \tag{3.12}$$

where $\epsilon = 0, \pm 1$. It remains to determine Y for the line element (3.12). To find Y we must specify p explicitly in (3.11). We make the particular choice

$$p = \text{constant},$$

which was also used by Govender (1997) for uncharged fluids. This choice enables us to solve (3.11) in terms of quadratures.

It seems that the explicit integration of (3.11) is not possible as Y depends on both t and r, and f is arbitrary. However, we will demonstrate that it is possible to make progress. In order to further facilitate the integration procedure in (3.11), it is convenient to introduce the new variables τ and y:

$$t = \frac{1}{\sqrt{p}}\tau$$

$$Y = \frac{f}{\sqrt{p}}y.$$

Then (3.11) can be written as

$$2yy_{\tau\tau} + {y_\tau}^2 + y^2 + \epsilon - \frac{\phi'^2 y^2}{2pe^{2\lambda}} = 0, \tag{3.13}$$

where the subscript refers to differentiation with respect to τ. Note that, even though $y = y(\tau, r)$, we may essentially treat (3.13) as an ordinary differential equation. We shall now present solutions to (3.13) for the three different values of ϵ.

Case 1 : $\epsilon = 0$

It is necessary to eliminate the electromagnetic gauge potential from (3.13). We make the choice

$$\phi'^2 = 2Cpe^{2\lambda},$$

for the electromagnetic gauge potential where C is constant. We let $\epsilon = 0$ and it is convenient to replace y by

$$y = u^{2/3}$$

Under this transformation (3.13) reduces to the simple differential equation

$$4u_{\tau\tau} + (3 - C)u = 0$$

in terms of u. The above equation is immediately integrated to yield

$$u = a_1 \sin \frac{\sqrt{(3-C)}}{2}\tau + a_2 \cos \frac{\sqrt{(3-C)}}{2}\tau,$$

where $a_1(r)$ and $a_2(r)$ are functions of integration. The solution of (3.13) then becomes

$$Y = \frac{f(r)}{\sqrt{p}}\left[a_1 \sin \frac{\sqrt{(3-C)p}}{2}t + a_2 \cos \frac{\sqrt{(3-C)p}}{2}t\right]^{2/3} \tag{3.14}$$

for the case $\epsilon = 0$. When $C = 0$, we regain the uncharged solution of Govender (1997) which is given by

$$Y = \frac{f(r)}{\sqrt{p}}\left[a_1 \sin \frac{\sqrt{3p}}{2}t + a_2 \cos \frac{\sqrt{3p}}{2}t\right]^{2/3}$$

from (3.14).

Case II : $\epsilon = -1$

In this case we make the choice

$$\phi'^2 = \frac{2C_1 p e^{2\lambda}}{y^2}$$

for the electromagnetic gauge potential and C_1 is a constant. It is easy to show that (3.13) becomes

$$2yy_{\tau\tau} + {y_\tau}^2 + y^2 - 1 - C_1 = 0. \tag{3.15}$$

Then it is easy to see that (3.15) admits the first integral

$$y{y_\tau}^2 = a_3 + y(C_1 + 1) - \frac{1}{3}y^3,$$

where $a_3(r)$ is an integration function. This is a first order equation and the variables separate. Then a second integration results in

$$\tau - \tau_0 = \int \frac{\sqrt{y}dy}{\left[a_3 + y(C_1+1) - \frac{1}{3}y^3\right]^{1/2}},$$

where $\tau_0(r)$ is a function of integration. At this stage we need to introduce a new variable

$$v = \frac{1}{y},$$

to bring the integral into standard form. The above integral can then be written as

$$\tau - \tau_0 = -\int \frac{dv}{v\left[a_3v^3 + v^2(C_1+1) - \frac{1}{3}\right]^{1/2}}. \tag{3.16}$$

Thus the differential equation (3.15) has been reduced to the quadrature (3.16) which can be evaluated in terms of elliptic integrals in general (Gradshteyn and Ryzhik 1994). Note that when $C_1 = 0$, then (3.16) reduces to the integral of Govender (1997) in his analysis of uncharged matter. We obtain his solution

$$\tau - \tau_0 = -\int \frac{dv}{v\left[a_3v^3 + v^2 - \frac{1}{3}\right]^{1/2}}$$

from (3.16).

Case III : $\epsilon = 1$

Here we make the choice

$$\phi'^2 = \frac{2pC_2e^{2\lambda}}{y^2}$$

for the electromagnetic gauge potential and C_2 is a constant. Then equation (3.13) becomes

$$2yy_{\tau\tau} + {y_\tau}^2 + y^2 + 1 - C_2 = 0.$$

It is easy to show that this second order equation can be integrated to give

$$y{y_\tau}^2 = a_4 - y(C_2 + 1) - \frac{1}{3}y^3,$$

where $a_4(r)$ is an integration function. This is first order differential equation which reduces to the quadrature

$$\tau - \tau_0 = \int \frac{\sqrt{y}dy}{\left[a_4 - y(C_2 + 1) - \frac{1}{3}y^3\right]^{1/2}}.$$

As for *Case II*, we utilise the transformation

$$v = \frac{1}{y},$$

so that the above integral can then be written as

$$\tau - \tau_0 = -\int \frac{dv}{v\left[a_4 v^3 - v^2(C_2 + 1) - \frac{1}{3}\right]^{1/2}}. \tag{3.17}$$

The cubic expression $a_3 v^3 - v^2(C_2+1) - \frac{1}{3}$ may be factorised and the integral expressed in terms of elliptic functions (Gradshteyn and Ryzhik 1994). Note that when $C_2 = 0$ we regain the uncharged solution of Govender (1997) which is given by

$$\tau - \tau_0 = -\int \frac{dv}{v\left[a_3 v^3 - v^2 - \frac{1}{3}\right]^{1/2}}$$

from (3.17).

3.3 Nonexpanding Solutions

In this section we briefly study solutions that are shearing and accelerating, but are expansion–free. If the cosmological model is nonexpanding, then it can be seen from

(2.10c) that the condition

$$\dot{\lambda} = -2\frac{\dot{Y}}{Y}$$

must hold, where $\dot{Y} \neq 0$. Integration of the above equation gives the metric potential

$$e^{2\lambda} = Y^{-4},$$

where we have eliminated the constant of integration by a coordinate transformation in the metric. In order to find an expression for the metric potential ν we substitute this form of $e^{2\lambda}$ into (2.21d), and then integrate the resulting equation to obtain

$$e^{2\nu} = f^2(t)Y^4\dot{Y}^2,$$

where $f(t)$ is a function of integration. We observe from (2.23b) and (2.22b) that the energy density μ (for nonexpanding solutions) is a function of r only. Equation (2.21a) can be written as

$$2Y^5Y'' + 5Y^4Y'^2 + 3f^{-2}(t)Y^{-4} + \mu(r)Y^2 - 1 + \frac{1}{2}\phi'^2Y^2\dot{Y}^{-2}f^{-2}(r) = 0, \qquad (3.18)$$

which relates the two quantities μ and Y. Equation (2.21b) can be used to express the pressure p in terms of Y:

$$p = -Y^{-2} + 2Y^3\dot{Y}'\dot{Y}^{-1}Y' + 5Y^2Y'^2 + 2f^{-3}(t)\dot{f}(t)Y^{-5}\dot{Y}^{-1} + 3f^{-2}(t)Y^{-6}$$

$$+\frac{1}{2}\phi'^2f^{-2}(r)\dot{Y}^{-2}. \qquad (3.19)$$

Thus we have reduced the solution of the field equations to (3.18) and (3.19). (Note that (2.21c) has to be satisfied in addition). The metric can now be written solely in terms of Y as

$$ds^2 = -f^2(t)Y^4\dot{Y}^2dt^2 + Y^{-4}dr^2 + Y^2\left(d\theta^2 + \sin^2\theta d\Phi^2\right).$$

To complete the solution we need to determine the gravitational potential Y.

If Y is known, and $\mu(r)$ and $f(t)$ prescribed, then p can be computed from (3.19). To find Y we must specify $\mu(r)$ explicitly in (3.18). We make the choice

$$\mu = \text{constant},$$

and assume that

$$\phi'^2 = \frac{2C_3}{Y^2\dot{Y}^{-2}f^{-2}(r)},$$

for the electromagnetic gauge potential and C_3 is constant. With this choice of μ (3.18) can be integrated to give

$$Y^5Y'^2 - f^{-2}(t)Y^{-3} + \frac{\mu}{3}Y^3 - Y(1 - C_3) = A(t),$$

where $A(t)$ is an integration constant. This differential equation may be reduced to quadratures. For a qualitative treatment see Skripkin (1960) for uncharged fluids. We can write the solution to the above differential equation as the quadrature

$$r - r_0 = \int \frac{Y^4 dY}{\left[\frac{1}{f^2} + AY^3 + Y^4(1 - C_3) - \frac{\mu}{3}Y^6\right]^{1/2}}. \tag{3.20}$$

We observe that, in the special case when $A = 0$, this integral may be expressed in terms of elliptic functions (Gradshteyn and Ryzhik 1994). When $C_3 = 0$, we regain as a special case the solution of Govender (1997) in his analysis of uncharged matter. We obtain his solution

$$r - r_0 = \int \frac{Y^4 dY}{\left[\frac{1}{f^2} + AY^3 + Y^4 - \frac{\mu}{3}Y^6\right]^{1/2}}.$$

from (3.20).

3.4 Equation of State

In this section we assume that the equation of state

$$p = (\gamma - 1)\mu$$

relates μ to p. The quantity γ is a constant. As in §3.2 we take

$$\nu = \nu(t)$$

so that solutions in this section are also nonaccelerating. Here we introduce the new independent variable

$$s = e^t.$$

Then the equations governing the behaviour of the gravitational potentials are

$$\mu = 1 + e^{-2\nu}(2\lambda_s + 1) - \frac{\phi'^2}{2e^{2\nu+2\lambda-2s}} \tag{3.21a}$$

$$(\gamma - 1)\mu = e^{-2\nu}(2\nu_s - 1) + \frac{\phi'^2}{2e^{2\nu+2\lambda-2s}} \tag{3.21b}$$

$$(\gamma - 1)\mu = -e^{-2\nu}\left(\lambda_{ss} + {\lambda_s}^2 - \lambda_s\nu_s - \nu_s\right) - \frac{\phi'^2}{2e^{2\nu+2\lambda-2s}} \tag{3.21c}$$

$$0 = \mu_s + \mu(\gamma\lambda_s + 2(\gamma - 1)) + \left(\frac{\phi'^2}{2e^{2\nu+2\lambda-2s}}\right)_s$$

$$+ 2\left(\frac{\phi'^2}{2e^{2\nu+2\lambda-2s}}\right), \tag{3.21d}$$

where subscripts denotes differentiation with respect to s. Note that equation (3.21a), (3.21b) and (3.21c) correspond to the field equations (2.21). We have also include the conservation equation (3.21d), which corresponds to (2.23b). In this section we find that the conservation equation simplifies the solution process.

If we differentiate (3.21a) with respect to s we obtain

$$\mu_s + \left(\frac{\phi'^2}{2e^{2\nu+2\lambda-2s}}\right)_s = e^{-2\nu}\left(2\lambda_{ss} - 4\nu_s\lambda_s - 2\nu_s\right). \tag{3.22}$$

From (3.21a) and (3.21b) we obtain

$$\gamma\mu = 2e^{-2\nu}\left(\nu_s + \lambda_s\right). \tag{3.23}$$

Substituting (3.22) and (3.21b) in equation (3.21d) we generate the result

$$\lambda_s\left(-2e^{-2\nu}\left(\nu_s + \lambda_s\right) + \gamma\mu\right) = 0. \tag{3.24}$$

We need only consider the case

$$\lambda_s = 0, \tag{3.25}$$

arising from (3.24). The second case of $-2e^{-2\nu}\left(\nu_s + \lambda_s\right) + \gamma\mu = 0$ is equivalent to (3.23). With the condition (3.25), the system (3.21) becomes

$$\mu = 1 + e^{-2\nu} - \frac{\phi'^2}{2e^{2\nu+2\lambda-2s}} \tag{3.26a}$$

$$(\gamma - 1)\,\mu = e^{-2\nu}\left(2\nu_s - 1\right) + \frac{\phi'^2}{2e^{2\nu+2\lambda-2s}} \tag{3.26b}$$

$$(\gamma - 1)\mu = e^{-2\nu}\nu_s - \frac{\phi'^2}{2e^{2\nu+2\lambda-2s}} \tag{3.26c}$$

$$0 = \mu_s + 2\mu(\gamma - 1) + \left(\frac{\phi'^2}{2e^{2\nu+2\lambda-2s}}\right)_s$$

$$+ 2\left(\frac{\phi'^2}{2e^{2\nu+2\lambda-2s}}\right), \tag{3.26d}$$

which we now attempt to integrate.

From equation (3.23) we have

$$\gamma\mu = 2e^{-2\nu}\nu_s. \tag{3.27}$$

Also adding equations (3.26b) and (3.26c) gives

$$2(\gamma - 1)\mu = -1 + e^{-2\nu}\left(3\nu_s - 1\right). \tag{3.28}$$

Eliminate μ from (3.27) and (3.28) to get the first order differential equation

$$\nu_s e^{-2\nu}\left(\gamma - 4\right) + e^{-2\nu}\gamma + \gamma = 0. \tag{3.29}$$

Note that (3.29) can be written as

$$-\frac{1}{2}\left(\gamma - 4\right)\left(e^{-2\nu}\right)_s + \gamma(e^{-2\nu}) + \gamma = 0$$

which has the solution

$$e^{-2\nu} = At^{(2\gamma)/(\gamma-4)} - 1, \tag{3.30}$$

where A is constant. From (3.27) and (3.30) we can write the energy density as a function of t:

$$\mu = \frac{-2A}{\gamma - 4}t^{(\gamma+4)/(\gamma-4)}. \tag{3.31}$$

Finally substituting (3.30) and (3.31) in (3.26a), we get the gravitational gauge potential ϕ:

$$\phi'^2 = 2At^{-2}e^{2\nu+2\lambda}\left(\frac{2}{\gamma - 4}t^{(\gamma+4)/(\gamma-4)} + t^{(2\gamma)/(\gamma-4)}\right).$$

The line element for this class of solutions has the form

$$ds^2 = -\left(At^{(2\gamma)/(\gamma-4)} - 1\right)^{-1} dt^2 + e^{2\lambda}dr^2 + t^2\left(d\theta^2 + \sin^2\theta d\Phi^2\right), \qquad (3.32)$$

where λ = constant. Hence we have found a simple class of solutions (3.30) to the Einstein–Maxwell equations by taking λ to be constant in the nonaccelerating case. The gravitational potential and the energy density are given explicitly in terms of the variable t. This is a simple class of solutions which does not constraint the range of γ. When $A = 0$, $\mu = 0$ and we regain vacuum. We believe that this class of models is new.

The solution (3.32) holds for constant λ. We now seek a solution to the system (3.26) with $\lambda \neq$ constant. From (3.21c), (3.23) and (3.21a), we can eliminate μ to obtain

$$2\left(\frac{\phi'^2}{2e^{2\nu+2\lambda-2s}}\right) - 1 = -e^{-2\nu}\left(\lambda_{ss} + {\lambda_s}^2 - \lambda_s\nu_s + \nu_s - 1\right). \qquad (3.33)$$

Similarly eliminating μ from (3.21a), (3.23) and (3.21d) gives

$$2\left(\frac{\phi'^2}{2e^{2\nu+2\lambda-2s}}\right) - 1 = -e^{-2\nu}\left(\lambda_{ss} + {\lambda_s}^2 - \lambda_s\nu_s + \nu_s - 1\right). \qquad (3.34)$$

Clearly (3.33) and (3.34) are same. Hence we need only consider three equations. We take the relevant equations to be

$$\mu = 1 - \frac{\phi'^2}{2e^{2\nu+2\lambda-2s}} + e^{-2\nu}\left(2\lambda_s + 1\right) \qquad (3.35a)$$

$$\gamma\mu = 2e^{-2\nu}\left(\nu_s + \lambda_s\right) \qquad (3.35b)$$

$$2\left(\frac{\phi'^2}{2e^{2\nu+2\lambda-2s}}\right) - 1 = -e^{-2\nu}\left(\lambda_{ss} + {\lambda_s}^2 - \lambda_s\nu_s + \nu_s - 1\right). \qquad (3.35c)$$

From (3.35a) and (3.35b) we have

$$2\left(\frac{\phi'^2}{2e^{2\nu+2\lambda-2s}}\right)-1=1+2e^{-2\nu}\left(2\lambda_s+1\right)-\frac{4}{\gamma}e^{-2\nu}\left(\nu_s+\lambda_s\right). \tag{3.36}$$

We can eliminate ϕ from (3.35c) and (3.36) to obtain

$$\nu_s\left(\lambda_s-1+\frac{4}{\gamma}\right)-e^{2\nu}=\lambda_{ss}+{\lambda_s}^2+4\lambda_s\left(1-\frac{1}{\gamma}\right)+1. \tag{3.37}$$

Let us now make the transformation

$$u=e^{\nu}$$

to simplify (3.37). Under this transformation (3.37) can be written as

$$u_s-\frac{2\left(\lambda_{ss}+{\lambda_s}^2+4\lambda_s\left(1-1/\gamma\right)+1\right)u}{\lambda_s-1+4/\gamma}-\frac{2u^2}{\lambda_s-1+4/\gamma}=0. \tag{3.38}$$

It is convenient to introduce the quantities

$$A=\frac{\lambda_{ss}+{\lambda_s}^2+4\lambda_s\left(1-1/\gamma\right)+1}{\lambda_s-1+4/\gamma}$$

$$B=\frac{2}{\lambda_s-1+4/\gamma}$$

Then (3.38) can be written as

$$\frac{u_s}{u^2}-\frac{2A}{u}-B=0.$$

This equation can be simplified further if we let

$$v=\frac{1}{u},$$

so that

$$v_s+2Av+B=0.$$

This linear equation has the solution

$$v = e^{-2\int A}\left[K - \int Be^{2\int A}\right] \tag{3.39}$$

in terms of v where K is constant.

It is not possible to evaluate the integrals in (3.39) for general γ. We can make progress with $\gamma = 2$ corresponding to the stiff equation of state. With this assumption (3.39) becomes

$$v = \frac{Ke^{-2(\lambda+s)} - 1}{(\lambda_s + 1)^2}.$$

Then using the transformation $v = \frac{1}{u}$ and $u = e^{2\nu}$ we get

$$e^{2\nu} = -\frac{(\lambda_s + 1)^2}{Ke^{-2(\lambda+s)} - 1}.$$

From equation (3.35c), we get the gravitational gauge potential ϕ as

$$\phi'^2 = e^{2\nu+2\lambda-2s}\left(1 - e^{-2\nu}\left(\lambda_{ss} + {\lambda_s}^2 - \lambda_s\nu_s + \nu_s - 1\right)\right).$$

From equation (3.35b), we get the energy density μ as

$$\mu = -\frac{(\lambda_s + 1)^2(\lambda_s + \nu_s)}{Ke^{-2(\lambda+s)} - 1}.$$

In terms of the variable t we have

$$e^{2\nu} = -\frac{(t\lambda_t + 1)^2}{Kt^{-2}e^{-2\lambda} - 1} \tag{3.40a}$$

$$\phi'^2 = t^{-2}e^{2\nu+2\lambda}\left(1 - e^{-2\nu}\left(t^2(\lambda_{tt} + {\lambda_t}^2 - \lambda_t\nu_t\right) + t(\nu_t + \lambda_t) - 1\right) \tag{3.40b}$$

$$\mu = -\frac{t(t\lambda_t + 1)^2(\lambda_t + \nu_t)}{Kt^{-2}e^{-2\lambda} - 1}. \tag{3.40c}$$

Thus for this case with $\gamma = 2$, for a stiff equation of state, the line element becomes

$$ds^2 = -\frac{(t\lambda_t + 1)^2}{Kt^{-2}e^{-2\lambda} - 1}dt^2 + e^{2\lambda}dr^2 + t^2\left(d\theta^2 + \sin^2\theta d\Phi^2\right) \tag{3.41}$$

We observe that the solution (3.41) is given in terms of λ. Note that the energy density μ and the electromagnetic gauge potential ϕ are also defined in terms of λ in (3.40): the quantity ν is given in terms of λ. The behaviour of λ is arbitrary in this class of solution. This is in contrast to the stiff equation of state solution presented in §5.4 when we specify initially that λ should be constant.

4 Self–Similar Solutions

4.1 Introduction

In this chapter we utilise the Lie analysis of differential equations to generate a solution to the Einstein–Maxwell system. In §4.2 we briefly introduce the basic concepts related to the technique of the Lie analysis. We need to choose an infinitesimal generator G such that its extension leaves the field equations invariant. This method is used to generate ordinary differential equations that arise in our relativistic charged model. A Lie analysis of the symmetry generators, for the appropriate equations, motivates a form for the self–similarity variable. In §4.3 we generate the self–similar form of the field equations. The Einstein–Maxwell field equations and conservation equations, which are highly nonlinear differential equations, are written in terms of this self–similar variable, and they are reduced to a system of ordinary differential equations. The field equations are further simplified by redefining one of the gravitational potentials. In §4.4 we determine the solutions to the field equations, in principle, by introducing a new independent variable η, and choosing a form for the electromagnetic gauge potential. The solution to the Einstein–Maxwell system is reduced to a second order, nonlinear equation in one of the gravitational potentials. This equation is transformed firstly to a first order Riccati equation, and then to a linear, second order differential equation. Two cases arise: $C = 0$ and $C \neq 0$. When

$C = 0$ we regain the uncharged solution of Govender (1997). When $C \neq 0$ we reduce the behaviour of the model to quadratures.

4.2 Lie Symmetries and the Similarity Generator

In this section we briefly discuss those concepts in the Lie analysis which help in finding solutions to the field equations. For a more detailed account of the Lie analysis of differential equations the reader is referred to texts by Bluman and Kumei (1989), Olver (1993), Kamke E (1983) and Stephani (1989). The Lie analysis of differential equations is an important area of research in mathematical physics and a detailed analysis is not possible here. In particular the Lie approach has proved to be a useful technique in finding exact solutions in general relativity. Different classes of solutions for spherically symmetric models have been found by Leach *et al* (1992), Maharaj *et al* (1996) and Stephani (1983). The analyses of Srivastava (1987) and Sussman (1986, 1987, 1988a, 1988b) are comprehensive reviews, in terms of invariance transformations of differential equations, for the same class of models for both neutral matter and charged matter.

We first define some important concepts concerning Lie symmetries of algebraic equations and then proceed to Lie symmetries of partial differential equations. Let $\mathbf{y} = (y^1, y^2, \ldots, y^n)$ lie in the region $D \subset \Re^n$ and ϵ is an element of $C \subset \Re$. Then

$$\mathbf{y}^* = G(\mathbf{y}; \epsilon) \tag{4.1}$$

represents a one–parameter (ϵ) Lie group of transformations. We define the infinites-

imal generator G as the operator

$$G = G(\mathbf{y}) = \xi(\mathbf{y}) \cdot \nabla = \sum_{i=1}^{p} \xi^i(\mathbf{y}) \frac{\partial}{\partial y^i},$$

where ∇ is the gradient operator

$$\nabla = \left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \cdots, \frac{\partial}{\partial y^p} \right)$$

and

$$\xi(\mathbf{y}) = \frac{\partial G}{\partial \epsilon}(\mathbf{y}; \epsilon) \mid_{\epsilon=0} .$$

The infinitely differentiable function $F(\mathbf{y})$ is said to be an invariant function of the Lie group of transfor-ations (4.1) if and only if

$$GF(\mathbf{y}) \equiv 0. \tag{4.2}$$

The associated characteristic system of equations

$$\frac{dy^1}{\xi^1(y)} = \frac{dy^2}{\xi^2(y)} = \cdots = \frac{dy^p}{\xi^p(y)}$$

follows from (4.2). The general solution of (4.2) can then be written as

$$F^1(y^1, \ldots, y^p) = c_1$$

$$F^2(y^1, \ldots, y^p) = c_2$$

$$\vdots$$

$$F^{p-1}(y^1, \ldots, y^p) = c_{p-1},$$

where the $c_1, \ldots, c_{p-1}$ are constants of integration.

Similar results may be presented for systems of partial differential equations. Consider a system of n–th order partial differential equations, with p independent variables $\mathbf{y} = (y^1, \ldots, y^p)$ and q dependent variables $\mathbf{w} = (w^1, \ldots, w^q)$ and

with derivatives of w with respect to y up to order n, given by

$$F_\beta\left(\mathbf{y}, \mathbf{w}^{(n)}\right) = 0. \tag{4.3}$$

We let $g_\epsilon = \exp(\epsilon G)$ be a one–parameter (ϵ) Lie group of the system with transformations given by

$$(\mathbf{y}^*, \mathbf{w}^*) = g_\epsilon \cdot (\mathbf{y}, \mathbf{w}) = (\Psi_\epsilon(\mathbf{y}, \mathbf{w}), \Phi_\epsilon(\mathbf{y}, \mathbf{w})). \tag{4.4}$$

Let us suppose that the generator given by

$$G = \sum_{i=1}^{p} \xi^i(\mathbf{y}, \mathbf{w}) \frac{\partial}{\partial y^i} + \sum_{\alpha=1}^{q} \phi_\alpha(\mathbf{y}, \mathbf{w}) \frac{\partial}{\partial w^\alpha}, \tag{4.5}$$

with

$$\xi^i(\mathbf{y}, \mathbf{w}) = \frac{d}{d\epsilon} \Psi^i_\epsilon(\mathbf{y}, \mathbf{w}) \mid_{\epsilon=0}, \qquad i = 1, \ldots, p$$

$$\phi_\alpha(\mathbf{y}, \mathbf{w}) = \frac{d}{d\epsilon} \Phi^\alpha_\epsilon(\mathbf{y}, \mathbf{w}) \mid_{\epsilon=0}, \qquad \alpha = 1, \ldots, q$$

is an infinitesimal generator of the system (4.3). Then $G^{[n]}$, the n–th extension of G given by (4.5), is defined as

$$G^{[n]} = G + \sum_{\alpha=1}^{q} \sum_{J} \left[\left(D_J \left(\phi_\alpha - \sum_{i=1}^{p} \xi^i w^\alpha_i \right) + \sum_{i=1}^{p} \xi^i w^\alpha_{J,i} \right) \frac{\partial}{\partial w^\alpha_J} \right], \tag{4.6}$$

where

$$w^\alpha_i = \frac{\partial w^\alpha}{\partial y^i},$$

$$w^\alpha_{J,i} = \frac{\partial w^\alpha_J}{\partial y^i}$$

and the total derivative D_J is defined by

$$D_J = D_{j_1} D_{j_2} \ldots D_{j_k}.$$

It should be noted that the summation is over all multi-indices $J = (j_1, \ldots, j_k)$ with $1 \leq j_k \leq p$, $1 \leq k \leq n$. The transformations defined by (4.4) leave the system of partial differential equations (4.3) invariant if and only if

$$G^{[n]} F_\beta|_{F_\beta = 0} = 0,$$

where $G^{[n]}$ is given by (4.6).

4.3 Self–Similar Form of the Field Equations

We now apply the theory of Lie symmetries to the field equations (2.21) to reduce the number of independent variables appearing in them. We have performed these calculations by hand but for more complicated systems it is advisable to use computer packages designed for this purpose. We need to choose an infinitesimal generator G such that its extension leaves the field equations invariant. We choose the infinitesimal generator G as

$$G = \mathcal{A}r\frac{\partial}{\partial r} + \mathcal{B}t\frac{\partial}{\partial t} + \mathcal{C}\mu\frac{\partial}{\partial \mu} + \mathcal{D}p\frac{\partial}{\partial p} + \mathcal{E}Y\frac{\partial}{\partial Y} + \mathcal{G}\phi\frac{\partial}{\partial \phi}$$

$$+\mathcal{H}\frac{\partial}{\partial \lambda} + \mathcal{I}\frac{\partial}{\partial \nu}, \tag{4.7}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{G}$, $\mathcal{H}$ and $\mathcal{I}$ are constants. Note that we cannot use $\mathcal{H}\lambda\frac{\partial}{\partial \lambda}$ and $\mathcal{I}\nu\frac{\partial}{\partial \nu}$ because this will introduce a multiplier of λ/ν in the exponential terms appearing in (2.21). The $1st$ and $2nd$ extensions of G follow from (4.7). With the help of (4.6) we obtain for the $1st$ extension

$$G^{[1]} = G + (\mathcal{C} - \mathcal{A})\mu'\frac{\partial}{\partial \mu'} + (\mathcal{C} - \mathcal{B})\dot{\mu}\frac{\partial}{\partial \dot{\mu}} + (\mathcal{D} - \mathcal{A})p'\frac{\partial}{\partial p'} + (\mathcal{D} - \mathcal{B})\dot{p}\frac{\partial}{\partial \dot{p}}$$

$$+(\mathcal{E}-\mathcal{A})Y'\frac{\partial}{\partial Y'}+(\mathcal{E}-\mathcal{B})\dot{Y}\frac{\partial}{\partial \dot{Y}}+(\mathcal{G}-\mathcal{A})\phi'\frac{\partial}{\partial \phi'}(\mathcal{G}-\mathcal{B})\dot{\phi}\frac{\partial}{\partial}\dot{\phi}$$

$$-\mathcal{A}\lambda'\frac{\partial}{\partial \lambda'}-\mathcal{B}\dot{\lambda}\frac{\partial}{\partial \dot{\lambda}}-\mathcal{A}\nu'\frac{\partial}{\partial \nu'}-\mathcal{B}\dot{\nu}\frac{\partial}{\partial \dot{\nu}} \tag{4.8}$$

and for the $2nd$ extension

$$G^{[2]}=G^{[1]}+(\mathcal{C}-2\mathcal{A})\mu''\frac{\partial}{\partial \mu''}+(\mathcal{C}-\mathcal{A}-\mathcal{B})\dot{\mu}'\frac{\partial}{\partial \dot{\mu}'}+(\mathcal{C}-2\mathcal{B})\ddot{\mu}\frac{\partial}{\partial \ddot{\mu}}$$

$$+(\mathcal{D}-2\mathcal{A})p''\frac{\partial}{\partial p''}+(\mathcal{D}-\mathcal{A}-\mathcal{B})\dot{p}'\frac{\partial}{\partial \dot{p}'}+(\mathcal{D}-2\mathcal{B})\ddot{p}\frac{\partial}{\partial \ddot{p}}$$

$$+(\mathcal{E}-2\mathcal{A})Y''\frac{\partial}{\partial Y''}+(\mathcal{E}-\mathcal{A}-\mathcal{B})\dot{Y}'\frac{\partial}{\partial \dot{Y}'}+(\mathcal{E}-2\mathcal{B})\ddot{Y}\frac{\partial}{\partial \ddot{Y}}$$

$$+(\mathcal{G}-2\mathcal{A})\phi''\frac{\partial}{\partial \phi''}+(\mathcal{G}-\mathcal{A}-\mathcal{B})\dot{\phi}'\frac{\partial}{\partial \dot{\phi}'}+(\mathcal{G}-2\mathcal{B})\ddot{\phi}\frac{\partial}{\partial \ddot{\phi}}$$

$$-2\mathcal{A}\lambda''\frac{\partial}{\partial \lambda''}-(\mathcal{A}+\mathcal{B})\dot{\lambda}'\frac{\partial}{\partial \dot{\lambda}'}-2\mathcal{B}\ddot{\lambda}\frac{\partial}{\partial \ddot{\lambda}}-2\mathcal{A}\nu''\frac{\partial}{\partial \nu''}$$

$$-(\mathcal{A}+\mathcal{B})\dot{\nu}'\frac{\partial}{\partial \dot{\nu}'}-2\mathcal{B}\ddot{\nu}\frac{\partial}{\partial \ddot{\nu}}. \tag{4.9}$$

In order to determine the constants $\mathcal{A},...,\mathcal{I}$ appearing in G we apply (4.7) and its extensions (4.8) and (4.9) to the system of field equations (2.21) and the conservation equations (2.23). This generates the following system of equations:

$$\mathcal{C}\mu=(-2\mathcal{E})\frac{1}{Y^2}-(-2\mathcal{H}-2\mathcal{A})\frac{2}{Y}e^{-2\lambda}\left(Y''-\lambda'Y'+\frac{Y'^2}{2Y}\right)$$

$$+(-2\mathcal{I}-2\mathcal{B})\frac{2}{Y}e^{-2\nu}\left(\dot{\lambda}\dot{Y}+\frac{\dot{Y}^2}{2Y}\right)$$

$$- \frac{\phi'^2}{e^{2\nu+2\lambda}} (\mathcal{G} - \mathcal{A} - \mathcal{H} - \mathcal{I})$$

$$\mathcal{D}p = (-2\mathcal{E})\frac{1}{Y^2} + (-2\mathcal{H} - 2\mathcal{A})\frac{2}{Y}e^{-2\lambda}\left(\nu' Y' + \frac{Y'^2}{2Y}\right)$$

$$- (-2\mathcal{I} - 2\mathcal{B})\frac{2}{Y}e^{-2\nu}\left(\ddot{Y} - \dot{\nu}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right)$$

$$+ \frac{\phi'^2}{e^{2\nu+2\lambda}} (\mathcal{G} - \mathcal{A} - \mathcal{H} - \mathcal{I})$$

$$(\mathcal{D} + \mathcal{E})pY = (-2\mathcal{H} + \mathcal{E} - 2\mathcal{A})e^{-2\lambda}\left[(\nu'' + \nu'^2 - \nu'\lambda') + \frac{1}{Y}(\nu' Y' - \lambda' Y' + Y'')\right]$$

$$- (-2\mathcal{I} + \mathcal{E} - 2\mathcal{B})e^{-2\nu}\left[(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}) + \frac{1}{Y}\left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y}\right)\right]$$

$$- \frac{\phi'^2}{e^{2\nu+2\lambda}} (\mathcal{G} - \mathcal{A} - \mathcal{H} - \mathcal{I} + \mathcal{E})$$

$$0 = (\mathcal{E} - \mathcal{B} - \mathcal{A})\dot{Y}' - (\mathcal{E} - \mathcal{B} - \mathcal{A})\dot{Y}\nu' - (\mathcal{E} - \mathcal{B} - \mathcal{A})Y'\dot{\lambda}$$

$$(\mathcal{D} - \mathcal{A})p' = -\left[(\mathcal{C} - \mathcal{A})\mu\nu' + (\mathcal{D} - \mathcal{A})p\nu'\right]$$

$$+ \frac{\phi'^2}{e^{2\nu+2\lambda}}\left[-\frac{\phi''}{\phi'} + \lambda' + \nu' - \frac{2Y'}{Y}\right][2(\mathcal{G} - \mathcal{A}) + 2\mathcal{I} + 4\mathcal{H} + \mathcal{A}]$$

$$(\mathcal{C} - \mathcal{B})\dot{\mu} = -\left[(\mathcal{C} - \mathcal{B})\mu\dot{\lambda} + (\mathcal{D} - \mathcal{B})p\dot{\lambda} + (\mathcal{C} - \mathcal{B})\mu\frac{2\dot{Y}}{Y} + (\mathcal{D} - \mathcal{B})\frac{2\dot{Y}}{Y}\right]$$

$$-\frac{\phi'^2}{e^{2\nu+2\lambda}}\left[\frac{2\dot{Y}}{Y}+\frac{\dot{\phi}'}{\phi'}-\dot{\nu}-\dot{\lambda}\right][2(\mathcal{G}-\mathcal{A})-4\mathcal{I}+2\mathcal{H}-\mathcal{B}].$$

The above equations will be satisfied if we set

$$\mathcal{A}=\mathcal{B}=\mathcal{E}$$

$$\mathcal{C}=\mathcal{D}=-2\mathcal{E}$$

$$\mathcal{G}=\mathcal{H}=\mathcal{I}=0.$$

The above set of conditions is consistent with the field equations (2.21) and the conservation equations (2.23). Without any loss in generality we set

$$\mathcal{E}=1.$$

Thus the infinitesimal generator (4.7) becomes

$$G=r\frac{\partial}{\partial r}+t\frac{\partial}{\partial t}-2\mu\frac{\partial}{\partial \mu}-2p\frac{\partial}{\partial p}+Y\frac{\partial}{\partial Y}, \tag{4.10}$$

where G is known as the self-similar infinitesimal generator. We observe that the form of G given in (4.10) is the same as that utilised by Govender (1997) in his investigation of uncharged inhomogeous cosmological models. It is remarkable that the presence of charge does not affect the form of the generator G.

To make progress we need to reduce the number of variables in the field equations. We introduce the new invariant $F(u,y,f,g)$ as pointed out in §4.2. The

associated characteristic system is written as

$$\frac{dr}{r} = \frac{dt}{t} = \frac{dY}{Y} = \frac{d\mu}{-2\mu} = \frac{dp}{-2p},$$

which upon integration yields

$$u = \frac{r}{t}$$

$$y = \frac{r}{Y}$$

$$f = \mu r^2$$

$$g = pr^2,$$

where u, y, f and g are characteristics. Thus the characteristic $u = r/t$ (also called the similarity variable) allows the introduction of the new independent variable u which simplifies the field equations. In this way the field equations are converted to a system of ordinary differential equations. The metric potentials, the electromagnetic gauge potential, energy density and pressure are expressed in terms of the independent variable $u = r/t$ as

$$\lambda = \lambda(u) \qquad \nu = \nu(u)$$

$$\phi = \phi(u) \qquad Y = ry(u)$$

$$\mu = \frac{f(u)}{r^2} \qquad p = \frac{g(u)}{r^2},$$

where the new functions y, f, g depend on u only.

It should be kept in mind that the infinitesimal generator G allows the new independent variable to reduce the field equations to a simpler form. The form of G has to be carefully chosen so that simplification does in fact occur. The similarity variable u allows us to express the field equations (2.21) as the following system

$$f = \frac{1}{y^2} - \frac{2}{e^{2\lambda}y}\left(2uy_u + u^2y_{uu} - (y+uy_u)u\lambda_u + \frac{(y+uy_u)^2}{2y}\right)$$

$$+ \frac{2u^4}{e^{2\nu}y}\left(y_u\lambda_u + \frac{y_u^2}{2y}\right) - \frac{u^2{\phi_u}^2}{2e^{2\nu+2\lambda}} \tag{4.11a}$$

$$g = -\frac{1}{y^2} + \frac{2}{e^{2\lambda}y}\left((y+uy_u)u\nu_u + \frac{(y+uy_u)^2}{2y}\right)$$

$$- \frac{2u^3}{e^{2\nu}y}\left(uy_{uu} + 2y_u - uy_u\nu_u + \frac{uy_u^2}{2y}\right) + \frac{u^2{\phi_u}^2}{2e^{2\nu+2\lambda}} \tag{4.11b}$$

$$yg = \frac{u}{e^{2\lambda}}\left(uy\left(\nu_{uu} + \nu_u^2 - \nu_u\lambda_u\right) + (y+uy_u)(\nu_u - \lambda_u) + 2y_u + uy_{uu}\right)$$

$$- \frac{u^3}{e^{2\nu}}\left(y\left(2\lambda_u + u\lambda_{uu} + u\lambda_u^2 - u\lambda_u\nu_u\right) + 2y_u + uy_{uu} + uy_u(\lambda_u - \nu_u)\right)$$

$$- \frac{u^2{\phi_u}^2}{2e^{2\nu+2\lambda}} \tag{4.11c}$$

$$0 = 2y_u + uy_{uu} + uy_u\nu_u - \lambda_u(y+uy_u), \tag{4.11d}$$

where the subscripts denote differentiation with respect to u. The conservation equations (2.23) transform to

$$ug_u - 2g = -u\nu_u(f+g)$$

$$-\frac{u^2{\phi_u}^2}{e^{2\nu+2\lambda}}\left(-\frac{u\phi_{uu}}{\phi_u}+1+u\lambda_u+u\nu_u-2(y+uy_u)\right) \tag{4.12a}$$

$$f_u = -(g+f)\left(\lambda_u+\frac{2y_u}{y}\right)$$

$$-\frac{u^2{\phi_u}^2}{e^{2\nu+2\lambda}}\left(\frac{2y_u}{y}-\frac{1}{u}+\frac{\phi_{uu}}{\phi_u}-\nu_u-\lambda_u\right). \tag{4.12b}$$

We apply the conservation laws (4.12) and the field equations (4.11) to obtain solutions with a self–similar variable. Note that we do not get new information from the conservation equations. However the conservation equations help in the integration process and are used in addition to the field equations.

It is possible to further simplify the form of the differential equations (4.11)–(4.12) by introducing a new independent variable η given by

$$u = e^{\eta}.$$

Then equations (4.11) and (4.12) transform to

$$f = \frac{1}{y^2}-\frac{2}{e^{2\lambda}y}\left(y_{\eta\eta}+y_\eta-(y+y_\eta)\lambda_\eta+\frac{(y+y_\eta)^2}{2y}\right)$$

$$+\frac{2}{e^{2(\nu-\eta)}y}\left(y_\eta\lambda_\eta+\frac{{y_\eta}^2}{2y}\right)-\frac{{\phi_\eta}^2}{2e^{2\nu+2\lambda}} \tag{4.13a}$$

$$g = -\frac{1}{y^2} + \frac{2}{e^{2\lambda}y}\left((y+y_\eta)\nu_\eta + \frac{(y+y_\eta)^2}{2y}\right)$$

$$-\frac{2}{e^{2(\nu-\eta)}y}\left(y_{\eta\eta} + y_\eta - y_\eta\nu_\eta + \frac{{y_\eta}^2}{2y}\right) + \frac{{\phi_\eta}^2}{2e^{2\nu+2\lambda}} \tag{4.13b}$$

$$yg = \frac{1}{e^{2\lambda}}\left((\nu_{\eta\eta} - \nu_\eta + {\nu_\eta}^2 - \nu_\eta\lambda_\eta)y + (y+y_\eta)(\nu_\eta - \lambda_\eta) + y_{\eta\eta} + y_\eta\right)$$

$$-\frac{1}{e^{2(\nu-\eta)}}\left((\lambda_{\eta\eta} + \lambda_\eta + {\lambda_\eta}^2 - \lambda_\eta\nu_\eta)y + y_\eta(\lambda_\eta - \nu_\eta) + y_{\eta\eta} + y_\eta\right)$$

$$-\frac{{\phi_\eta}^2}{2e^{2\nu+2\lambda}} \tag{4.13c}$$

$$y_{\eta\eta} + y_\eta = y_\eta\nu_\eta + (y+y_\eta)\lambda_\eta \tag{4.13d}$$

$$g_\eta - 2g = -\nu_\eta(f+g) - \frac{{\phi_\eta}^2}{e^{2\nu+2\lambda}}\left(\frac{\phi_{\eta\eta}}{\phi_\eta} + \lambda_\eta + \nu_\eta - 2(y+y_\eta)\right) \tag{4.13e}$$

$$f_\eta = -(f+g)\left(\lambda_\eta + \frac{2y_\eta}{y}\right)$$

$$+\frac{{\phi_\eta}^2}{e^{2\nu+2\lambda}}\left(\frac{2y_\eta}{y} - 1 + \frac{\phi_{\eta\eta}}{\phi_\eta} - \nu_\eta\lambda_\eta\right). \tag{4.13f}$$

The field equations (4.13) may be simplified further by introducing a new variable. This new potential σ is related to the old potential ν by

$$\sigma = \nu - \eta.$$

As a result we are in a position to express the field equations as the system

$$f = \frac{1}{y^2} - \frac{2}{e^{2\lambda}y}\left(y_{\eta\eta} + y_\eta - (y+y_\eta)\lambda_\eta + \frac{(y+y_\eta)^2}{2y}\right)$$

$$+\frac{2}{e^{2\sigma}y}\left(y_\eta\lambda_\eta + \frac{{y_\eta}^2}{2y}\right) - \frac{{\phi_\eta}^2}{2e^{2\nu+2\lambda}} \tag{4.14a}$$

$$g = -\frac{1}{y^2} + \frac{2}{e^{2\lambda}y}(y+y_\eta)(\sigma_\eta+1) + \frac{(y+y_\eta)^2}{2y}$$

$$-\frac{2}{e^{2\sigma}y}\left(y_{\eta\eta} - \sigma_\eta y_\eta + \frac{{y_\eta}^2}{2y}\right) + \frac{{\phi_\eta}^2}{2e^{2\nu+2\lambda}} \tag{4.14b}$$

$$yg = \frac{1}{e^{2\lambda}}\Big((\sigma_{\eta\eta} + \sigma_\eta + {\sigma_\eta}^2 - \lambda_\eta(\sigma_\eta+1))y + (y+y_\eta)(\sigma_\eta+1-\lambda_\eta) + y_{\eta\eta} + y_\eta\Big)$$

$$-\frac{1}{e^{2\sigma}}\Big((\lambda_{\eta\eta} + {\lambda_\eta}^2 - \lambda_\eta\sigma_\eta)y + y_\eta(\lambda_\eta - \sigma_\eta - 1) + y_{\eta\eta} + y_\eta\Big)$$

$$-\frac{{\phi_\eta}^2}{2e^{2\nu+2\lambda}} \tag{4.14c}$$

$$y_{\eta\eta} + y_\eta = y_\eta(\sigma_\eta+1) + (y+y_\eta)\lambda_\eta \tag{4.14d}$$

$$g_\eta - 2g = -(\sigma_\eta+1)(f+g)$$

$$-\frac{{\phi_\eta}^2}{e^{2\nu+2\lambda}}\left(\frac{\phi_{\eta\eta}}{\phi_\eta} + \lambda_\eta + (\sigma_\eta+1) - 2(y+y_\eta)\right) \tag{4.14e}$$

$$f_\eta = -(f+g)\left(\lambda_\eta + \frac{2y_\eta}{y}\right)$$

$$+\frac{{\phi_\eta}^2}{e^{2\nu+2\lambda}}\left(\frac{2y_\eta}{y} - \frac{\phi_{\eta\eta}}{\phi_\eta} - (\sigma - \eta + 1) - \lambda_\eta\right). \tag{4.14f}$$

The system of equations (4.14) is essentially equivalent to (4.13). The simplified system of field equations (4.14) governs the behaviour of the gravitational field and electromagnetic field with a self–similar variable for a charged spherically symmetric relativistic fluid.

4.4 The Gravitational Potential y

The system (4.14) is highly nonlinear and it seems unlikely that these equations can be solved in general. We can make progress for a particular case by assuming that

$$\sigma = \lambda.$$

Then the system of ordinary differential equations (4.14) reduces to the simpler form

$$f = \frac{1}{y^2} - \frac{2}{e^{2\sigma}y}\left(y_{\eta\eta} + 2y_\eta + \frac{y}{2} - (y + 2y_\eta)\sigma_\eta\right) - \frac{{\phi_\eta}^2}{2e^{2\nu+2\sigma}} \tag{4.15a}$$

$$g = -\frac{1}{y^2} + \frac{2}{e^{2\sigma}y}\left(-y_{\eta\eta} + 2\sigma_\eta y_\eta + 2y_\eta + \sigma_\eta y + \frac{3y}{2}\right)$$

$$+\frac{{\phi_\eta}^2}{2e^{2\nu+2\sigma}} \tag{4.15b}$$

$$yg = \frac{1}{e^{2\sigma}}\left(2y_\eta + y\right) - \frac{{\phi_\eta}^2}{2e^{2\nu+2\sigma}} \tag{4.15c}$$

$$y_{\eta\eta} = \left(y + 2y_\eta\right)\sigma_\eta \tag{4.15d}$$

$$g_\eta - 2g = -(\sigma_\eta + 1)(f + g)$$

$$- \frac{{\phi_\eta}^2}{e^{2\nu+2\sigma}}\left(\frac{\phi_{\eta\eta}}{\phi_\eta} + \sigma_\eta + (\sigma_\eta + 1) - \frac{2(y + y_\eta}{y}\right) \tag{4.15e}$$

$$f_\eta = -(f + g)\left(\sigma_\eta + \frac{2y_\eta}{y}\right)$$

$$+ \frac{{\phi_\eta}^2}{e^{2\nu+2\sigma}}\left(\frac{2y_\eta}{y} - 2 + \frac{\phi_{\eta\eta}}{\phi_\eta} - (\sigma_\eta + 1) - \sigma_\eta\right). \tag{4.15f}$$

We now investigate the integration of (4.15) in general. The objective is to obtain specific forms for the variables ν, σ, y, ϕ (the potential functions) and f, g (the matter functions), a total of six variables. This is not possible in general as (4.15) is a system of only four independent equations as we have included the conservation equations (4.15e) and (4.15f). However it is possible to obtain a differential equation containing only the gravitational potential y, which then enables us to obtain a particular explicit solution to the system (4.15). We eliminate $y_{\eta\eta}$ from (4.15b), with the help of (4.15d), to obtain

$$g = -\frac{1}{y^2} + \frac{2}{e^{2\sigma}y}\left(2y_\eta + \frac{3y}{2}\right) + \frac{{\phi_\eta}^2}{2e^{2\nu+2\sigma}}. \tag{4.16}$$

Then it is possible to eliminate the variable g by taking a combination of (4.15c) and (4.16) to give

$$e^{2\sigma} = 2y_\eta y + 2y^2 + \frac{{\phi_\eta}^2(y^2+y)}{2e^{2\nu}}. \tag{4.17}$$

To proceed we need to make a choice for the electromagnetic gauge potential. An appropriate choice is

$${\phi_\eta}^2 = \frac{2Ce^{2\nu}}{(y^2+y)}, \tag{4.18}$$

where C is a positive arbitrary constant. Then equation (4.17) becomes

$$e^{2\sigma} = 2y_\eta y + 2y^2 + C, \tag{4.19}$$

with the help of (4.18). Differentiating (4.19) with respect to η leads to

$$2e^{2\sigma}\sigma_\eta = 2{y_\eta}^2 + 2yy_{\eta\eta} + 4yy_\eta. \tag{4.20}$$

We substitute for σ_η from (4.15d), and $e^{2\sigma}$ from (4.19), in equation (4.20) to obtain a second order differential equation

$$y_{\eta\eta}(y^2+C) - 5{y_\eta}^2 y - 2{y_\eta}^3 - 2y_\eta y^2 = 0. \tag{4.21}$$

Hence we have succeeded in isolating the gravitational potential y. Equation (4.21) is a nonlinear, second order differential equation; it is homogeneous and accordingly integrable. Two cases arise in the integration of the master equation (4.21) which we consider in turn:

CASE I: $C = 0$

Note that *CASE I* corresponds to uncharged matter and is accordingly related to

the Einstein equations (2.24). With $C = 0$, (4.21) becomes

$$y_{\eta\eta}y^2 - 5{y_\eta}^2 y - 2{y_\eta}^3 - 2y_\eta y^2 = 0. \tag{4.22}$$

The substitution

$$q = y$$

$$Q = y_\eta$$

converts (4.22) to the Riccati equation

$$Q_q = \left(\frac{2}{q^2}\right) Q^2 + \left(\frac{5}{q}\right) Q + 2.$$

This first order equation is integrable, and consequently the solution to (4.22) is given by

$$1 - c_1 + \ln y^2 = c_2 e^{2\eta} y^2, \tag{4.23}$$

where c_1 and c_2 are constants. The solution (4.23) for uncharged matter was also obtained by Govender (1997).

CASE II: $C \neq 0$

Note that *CASE II* corresponds to charged matter and is accordingly related to the Einstein–Maxwell equations (2.21)-(2.22). Again we make the substitution

$$q = y \tag{4.24a}$$

$$Q = y_\eta. \tag{4.24b}$$

Then (4.21) is converted to the Riccati equation

$$Q_q \left(1 + \frac{C}{q^2}\right) = \left(\frac{2}{q^2}\right) Q^2 + \left(\frac{5}{q}\right) Q + 2. \tag{4.25}$$

We have failed to integrate (4.25), in the given form, with $C \neq 0$; however progress can be made if we convert this equation to second order. Let

$$Q = \alpha \left(\frac{w_q}{w}\right),$$

where $\alpha = \alpha(q)$. Then (4.25) becomes

$$\alpha \left(q^2 + C\right) \frac{w_{qq}}{w} - \alpha \left(q^2 + C + 2\alpha\right) \frac{{w_q}^2}{w^2} + \alpha_q \left(q^2 + C\right) \frac{w_q}{w} - 2q^2 = 0.$$

The choice

$$\alpha = -\frac{1}{2}\left(q^2 + C\right)$$

eliminates the term containing the nonlinearity ${w_q}^2$. Hence the transformation

$$Q = -\frac{1}{2}\left(q^2 + C\right) \frac{w_q}{w} \tag{4.26}$$

converts the first order equation (4.25) to the second order equation

$$\left(q^2 + C\right)^2 w_{qq} - 3q\left(q^2 + C\right) w_q + 4q^2 w = 0, \tag{4.27}$$

which is linear in w. Equation (4.27) is simplified if we let

$$C = \alpha^2 \tag{4.28a}$$

$$q = \alpha x \tag{4.28b}$$

$$w = (x^2 + 1)^{\frac{1}{2}} W. \tag{4.28c}$$

Then (4.27) becomes

$$(x^2 + 1)W_{xx} - xW_x + W = 0, \tag{4.29}$$

which is a linear second order differential equation. The general solution of (4.29) is given by Kamke (1983) in the form

$$W(x) = d_1 x + d_2 x \left[-\frac{\sqrt{x^2+1}}{x} + \ln\left(x + \sqrt{x^2+1}\right) \right],$$

where d_1, d_2 are constants. Then in terms of the intermediate variables w and q in (4.28) we can write

$$w = q\sqrt{q^2 + C} \left[D_1 + D_2[\ln(q + \sqrt{q^2 + C})] - \frac{\sqrt{q^2+C}}{q} \right],$$

where

$$D_1 = \frac{d_1}{\alpha}$$

$$D_2 = \frac{d_2}{\alpha}$$

are constants.

Then utilising the original transformation (4.24), and (4.26), gives

$$\frac{dy}{d\eta}\sqrt{y^2+C}\left\{D_1 y + D_2 y\left[\ln\left(y+\sqrt{y^2+C}\right) - \frac{\sqrt{y^2+C}}{y}\right]\right\}$$

$$= -\frac{1}{2}(y^2+C)\left\{\frac{y^2}{\sqrt{y^2+C}}\left(D_1 + D_2\left[\ln\left(y+\sqrt{y^2+C}\right) - \frac{\sqrt{y^2+C}}{y}\right]\right)\right.$$

$$\left.+\sqrt{y^2+C}\left(D_1 + D_2\left[\ln\left(y+\sqrt{y^2+C}\right) - 2\frac{\sqrt{y^2+C}}{y}\right]\right)\right\}. \tag{4.30}$$

Thus we have reduced the problem to generating a solution of (4.30). Note that equation (4.30) is a separable first order equation which is in principle integrable. However this is a nontrivial task in practice. The solution to (4.30) can be expressed as

$$\ln\frac{r}{t} - \eta_0 = \eta - \eta_0$$

$$= \int^{y=Y/r} I(y)dy \tag{4.31}$$

where η_0 is a constant and the integrand is given by

$$I(y) = \sqrt{y^2+C}\left\{D_1 y + D_2 y\left[\ln\left(y+\sqrt{y^2+C}\right) - \frac{\sqrt{y^2+C}}{y}\right]\right\} \div$$

$$-\frac{1}{2}(y^2+C)\left\{\frac{y^2}{\sqrt{y^2+C}}\left(D_1 + D_2\left[\ln\left(y+\sqrt{y^2+C}\right) - \frac{\sqrt{y^2+C}}{y}\right]\right)\right.$$

$$\left.+\sqrt{y^2+C}\left(D_1 + D_2\left[\ln\left(y+\sqrt{y^2+C}\right) - 2\frac{\sqrt{y^2+C}}{y}\right]\right)\right\}.$$

Hence we have a solution to the system (4.15) (and consequently the original system of partial differential equations (4.11)-(4.12)) in principle. The behaviour of the

gravitational potential Y is specified by (4.31). The remaining gravitational potential (ν, λ), the electromagnetic gauge potential (ϕ), and the matter variables (μ, p) are then easy to obtain, as these quantities are defined in terms of y (and therefore Y).

Thus we have demonstrated a solution to the Einstein–Maxwell system where we have utilised the Lie analysis of differential equations. This cosmological model is expanding, accelerating and shearing. We believe that our solution corresponding to (4.31) is new and has not been published before. Our solution extends the Govender (1997) solution to include charge (that is $C \neq 0$ in (4.21)). Note that it would appear that by setting $D_2 = 0$ in (4.30), a simple analytic expression for the potential Y is obtainable. However this is not the case because then (4.30) simplifies to

$$y\frac{dy}{d\eta} = -y^2 - \frac{1}{2}C.$$

This then gives $e^{2\sigma} = 0$ from (4.19) which is an inconsistency. Thus we require $D_1 \neq 0$ and $D_2 \neq 0$ in (4.30) which implies that there will be no simple analytic expression arising from (4.31). This is not surprising because the presence of the electromagnetic field increases the nonlinearity of the field equations and allows for a wider range of behaviour of the gravitational field.

5 Solutions with Constant Potential

5.1 Introduction

In chapter 4 we utilised the Lie analysis of differential equations on the Einstein–Maxwell system *ab initio* to generate a solution. It is important to note that an *ad hoc* choice of one of the potential functions sometimes greatly simplifies the integration procedure. In this chapter we investigate a new class of solutions to the Einstein–Maxwell field equations. By assuming that the metric potential λ is constant in §5.2 we are able to simplify the field equations. In this case the line element is expressed in terms of the gravitational potential Y only. A suitable choice for Y yields a charged spherically symmetric nonvacuum model in §5.3. This particular charged solution reduces to a vacuum model found previously. In order to find more physically relevant solutions we utilise the Lie analysis in §5.4. A particular Lie point symmetry reduces the relevant partial differential equation to a third order ordinary differential equation. This ordinary differential equation is reduced to quadratures for inhomogeneous cosmological models with shear and charge. We regain the solution of Govender (1997) in the uncharged limit. In §5.5 we consider the case with $\lambda =$ constant and $p = \mu$. The field equations are converted into a third order ordinary differential equation with the help of the Lie analysis. This is solved in general in terms of elementary functions.

5.2 Field Equations for Constant λ

In this section our intention is to find a solution of the system (2.21) that gives a new class of shearing solutions that are expanding and accelerating. We make the assumption

$$\lambda = \text{constant},$$

and then the factor $e^{2\lambda}$ in the line element (2.8) can be absorbed by redefinition of the radial coordinate r. With this value for λ the field equation (2.21d) reduces to

$$\dot{Y}' - \dot{Y}\nu' = 0.$$

On integration the above equation gives

$$e^{2\nu} = \dot{Y}^2,$$

where we have set the function of integration to unity. Thus the line element, for this class of solution, is

$$ds^2 = -\dot{Y}^2 dt^2 + dr^2 + Y^2(d\theta^2 + \sin^2\theta d\Phi^2). \tag{5.1}$$

Here we need $\dot{Y} \neq 0$. From (2.10) the acceleration vector, the expansion scalar and the magnitude of the shear are given, respectively, by

$$\dot{u}^a = \left(0, \frac{\dot{Y}'}{\dot{Y}}, 0, 0\right)$$

$$\Theta = \frac{2}{Y}$$

$$\sigma_1^1 = \sigma_2^2 = -\frac{1}{2}\sigma_3^3 = \frac{1}{3Y}.$$

Therefore the assumption λ = constant is restrictive but allows the kinematical quantities (shear, expansion, acceleration) to remain nonvanishing. Note that the line element (5.1) is the simplest form possible with the assumption λ = constant.

With the simplified form of the line element (5.1), the field equations (2.21) reduce to the system of partial differential equations

$$\mu = \frac{2}{Y^2} - \frac{2}{Y}\left(Y'' + \frac{Y'^2}{2Y}\right) - \frac{\phi'^2}{2e^{2\nu}} \tag{5.2a}$$

$$p = -\frac{2}{Y^2} + \frac{2}{Y}\left(\frac{\dot{Y}'}{\dot{Y}}Y' + \frac{Y'^2}{2Y}\right) + \frac{\phi'^2}{2e^{2\nu}} \tag{5.2b}$$

$$p = \frac{\dot{Y}''}{\dot{Y}} + \frac{\dot{Y}'}{\dot{Y}}\frac{Y'}{Y} + \frac{Y''}{Y} - \frac{\phi'^2}{2e^{2\nu}}. \tag{5.2c}$$

On equating equations (5.2b) and (5.2c) we generate the partial differential equation

$$\frac{Y''}{Y} + \frac{\dot{Y}''}{\dot{Y}} - \frac{\dot{Y}'}{\dot{Y}}\frac{Y'}{Y} - \frac{Y'^2}{Y^2} + \frac{2}{Y^2} - \frac{\phi'^2}{e^{2\nu}} = 0. \tag{5.3}$$

If $\phi = \phi(t)$, then (5.3) becomes

$$\frac{Y''}{Y} + \frac{\dot{Y}''}{\dot{Y}} - \frac{\dot{Y}'}{\dot{Y}}\frac{Y'}{Y} - \frac{Y'^2}{Y^2} + \frac{2}{Y^2} = 0. \tag{5.4}$$

Equation (5.4) was generated by Govender (1997) for uncharged matter. This is sometimes called the condition of pressure isotropy.

5.3 A Simple Solution

Equation (5.3) is difficult to integrate because of the nonlinearity and the appearance of two functions, namely Y and ϕ, in one equation. However, it is possible to generate a simple solution by assuming an *ad hoc* form for Y. We make the assumption

$$Y = g(t) + h(r),$$

where g and h are functions of t and r respectively. Then we observe that (5.3) is simplied if h is linear:

$$h = ar + b,$$

where a and b are constants. We therefore have

$$Y = g(t) + ar + b \tag{5.5}$$

as our choice for the gravitational potential.

With the help of (5.5), equation (5.3) generates the expression

$$\phi' = \frac{\dot{g}\sqrt{2-a^2}}{g+ar+b},$$

which may be immediately integrated to give

$$\phi = \frac{\dot{g}\sqrt{2-a^2}\ln(g+ar+b)}{a}$$

for the electromagnetic gauge potential. The function of integration has been set equal to zero as it does not contribute to the dynamics. The energy density and pressure are given respectively by

$$\mu = \frac{(2-a^2)}{2(g+ar+b)^2}$$

$$p = -\frac{(2-a^2)}{2(g+ar+b)^2}.$$

We observe that this solution has the equation of state

$$p = -\mu,$$

and our model corresponds to the line element

$$ds^2 = -\left[\dot{g}(t)\right]^2 dt^2 + dr^2 + \left[g(t) + ar + b\right]^2 (d\theta^2 + \sin^2\theta d\Phi^2). \tag{5.6}$$

As the pressure may be negative (μ and p are of opposite sign) we do not claim that the choice (5.5) for Y, corresponding to the metric (5.6), is physically significant; however the model found does indicate the existence of nonvacuum solutions with nonvanishing electromagnetic field. This is in contrast to the case for neutral matter. When $a = \sqrt{2}$, $b = 0$, (5.6) becomes

$$ds^2 = -[\dot{g}(t)]^2 dt^2 + dr^2 + [g(t) + \sqrt{2}r]^2 \left(d\theta^2 + \sin^2\theta d\Phi^2\right) \tag{5.7}$$

which was found by Govender (1997). With this form of the line element we have

$$\phi = \mu = p = 0.$$

Thus the line element (5.7) corresponds to vacuum with no electromagnetic field. We have demonstrated the existence of a nonvacuum model (5.6) (with $\mathbf{F} \neq 0$), which reduces to the model of Govender (1997) in the relevant limit.

5.4 The Lie Analysis

In order to find a general solution to (5.3) we utilise the now familiar method of Lie symmetries of differential equations. We recall that (5.3) is said to possess the Lie point symmetry

$$G = \xi(r, Y)\frac{\partial}{\partial r} + \eta(r, Y)\frac{\partial}{\partial Y}$$

if the following equation

$$G^{[2]}\left(Y''Y + \frac{\dot{Y}''}{\dot{Y}}Y^2 - \frac{\dot{Y}'}{\dot{Y}}YY' - Y'^2 + 2 - \frac{\phi'^2 Y^2}{e^{2\nu}}\right) = 0,$$

taking (5.3) into account, is satisfied, where $G^{[2]}$ is the $2nd$ extension of G. It can be checked that (5.3) admits the three Lie point symmetries

$$G_1 = \frac{\partial}{\partial r}$$

$$G_2 = \tilde{g}(t)\frac{\partial}{\partial t}$$

$$G_3 = r\frac{\partial}{\partial r} + Y\frac{\partial}{\partial Y},$$

where $\tilde{g}$ is an arbitrary function of t. It is unlikely that another similarity reduction (to reduce (5.3) to an ordinary differential equation) will reduce the resulting equations to quadratures given the number and forms of the symmetries.

The nature of the similarity generators G_1 and G_2 suggests that we introduce a new variable u such that

$$u = g(t) + r,$$

(where $g(t) = -\int \left(\frac{1}{\tilde{g}}\right) dt$) with the gravitational potential of the form

$$Y = y(u).$$

Then (5.3) can be reduced to

$$y^2 y_{uuu} - y_u^3 + 2y_u - \frac{y^2 y_u \phi'^2}{e^{2\nu}} = 0, \tag{5.8}$$

where the subscript denotes differentiation with respect to u. Thus the partial differential equation (5.3) has been reduced to the simpler differential equation (5.8). This is a third order highly nonlinear differential equation. To integrate (5.8) we need to choose a form for ϕ:

$$\frac{\phi'^2}{e^{2\nu}} = \frac{C_1}{y^2},$$

where C_1 is a constant. Equation (5.8) can now be written as

$$(y^2 y_{uu})_u - (y y_u^2)_u + (2 - C_1) y_u = 0.$$

This is easily integrated to give

$$y^2 y_{uu} - y {y_u}^2 + (2 - C_1) y = A, \tag{5.9}$$

where A is an integration constant. We can rewrite (5.9) as

$$\frac{y^2}{2}\frac{dy_u^2}{dy} - y y_u^2 + (2 - C_1) y = A,$$

which is of first order. Upon integration we obtain

$$y_u^2 = -\frac{2A}{3y} + By^2 + (2 - C_1),$$

where B is an integration constant. The last equation is of first order and is separable. On integration the above equation is reduced to the quadrature

$$u - u_0 = \int \frac{\sqrt{3y}dy}{\sqrt{3By^3 + 3y(2 - C_1) - 2A}},$$

where u_0 is an integration constant. In terms of the radial coordinate r we have that

$$r - r_0 = \int^Y \frac{\sqrt{y}dy}{\sqrt{By^3 + (2 - C_1)y - (2/3)A}} - g(t). \tag{5.10}$$

We can simplify (5.10) if we make the substitution

$$v = \frac{1}{y}.$$

Then equation (5.10) can then be written as

$$r - r_0 = -\int \frac{dv}{v\left[B + (2 - C_1)v^2 - (2/3)Av^3\right]^{1/2}} - g(t). \tag{5.11}$$

Note that when $C_1 = 0$, (5.11) reduces to

$$r - r_0 = -\int \frac{dv}{v\left[B + 2v^2 - (2/3)Av^3\right]^{1/2}} - g(t),$$

which is the quadrature of Govender (1997) in his analysis of inhomogeneous cosmological models with shear for neutral matter. A general solution of the Einstein field equations (5.2), with λ = constant, is given by the line element (5.1). The potential Y is related to the spacetime coordinates t and r by (5.10). Observe that for the special case

$$A = B = 0,$$

we regain the metric potential Y as considered in §5.3.

We have solved the partial differential equation (5.3) by the Lie analysis and introduction of a new variable. The solution has been reduced to the quadrature (5.11). The integral in (5.11) may be expressed in terms of elliptic functions for particular values of the constants A, B, C_1. We do not pursue this here as Govender

(1997) has demonstrated the existence of real solutions for the special case $C_1 = 0$. Clearly other solutions are generated for different values of C_1.

5.5 Stiff Fluid Solutions

In this section we investigate the existence of solutions to the field equations for constant λ with

$$p = \mu,$$

which corresponds to a stiff equation of state. Here a second differential equation arises in addition to (5.3). If we let $\mu = p$, then (5.2a) and (5.2b) can be equated to give

$$\frac{Y''}{Y} + \frac{\dot{Y}'Y'}{\dot{Y}Y} + \frac{Y'^2}{Y^2} - \frac{2}{Y^2} + \frac{\phi'^2}{2e^{2\nu}} = 0. \tag{5.12}$$

The electromagnetic gauge potential ϕ may be eliminated from (5.3) and (5.12) to yield

$$3\frac{Y''}{Y} + \frac{\dot{Y}''}{\dot{Y}} + \frac{\dot{Y}'Y'}{\dot{Y}Y} + \frac{Y'^2}{Y^2} - \frac{2}{Y^2} = 0. \tag{5.13}$$

We seek solutions to the partial differential equation (5.13). Once a form for Y is obtained from (5.13) then expressions for ϕ, μ and p follow immediately. Equation (5.13) is the governing equation in the case for stiff matter with λ = constant.

The Lie analysis of differential equations is utilised to generate closed form

solutions. It is easy to see that (5.13) admits the three point symmetries:

$$G_1 = \tilde{a}(t)\frac{\partial}{\partial t}$$

$$G_2 = \frac{\partial}{\partial r}$$

$$G_3 = r\frac{\partial}{\partial r} + Y\frac{\partial}{\partial Y},$$

where $\tilde{a}(t)$ is an arbitrary function of t. The associated characteristic system for the Lie symmetry G_3 is

$$\frac{dt}{0} = \frac{dr}{r} = \frac{dY}{Y}$$

from which we obtain the characteristics

$$u = t$$

$$v = \frac{Y}{r}.$$

Hence we can take

$$Y = ry(t), \tag{5.14}$$

where y is an arbitrary function of t. However (5.14) is not useful because (5.13) implies that y is constant. This situation arises because of the use of the Lie symmetry G_3. Another choice for the Lie symmetry may lead to a more useful solution as we now demonstrate.

A linear combination of G_1 and G_2 creates the generator

$$G = \tilde{a}(t)\frac{\partial}{\partial t} + \frac{\partial}{\partial r}.$$

The associated characteristic system for the Lie symmetry G is

$$\frac{dt}{\tilde{a}(t)} = \frac{dr}{1} = \frac{dY}{0},$$

from which we obtain the characteristics

$$u = r - \int \frac{dt}{\tilde{a}(t)} = r - a(t),$$

$$v = Y.$$

Hence we can take

$$Y = y(u), \tag{5.15}$$

where y is an arbitrary function of u. Substituting (5.15) in (5.13) gives

$$y^2 y_{uuu} + 4yy_u y_{uu} + y_u^3 - 2y_u = 0, \tag{5.16}$$

which is a third order equation in y. Note that (5.16) admits the Lie point symmetry $\frac{\partial}{\partial u}$. Hence we take

$$r = z,$$

$$s = y_u$$

Then (5.16) becomes

$$z^2 \left(ss'' + s'^2\right) + 4zss' + s^2 - 2 = 0, \tag{5.17}$$

which is second order differential equation. Primes denotes differentiation with respect to z.

It is possible to further simplify the form of the differential equation (5.17) by introducing a new variable q given by

$$q = \frac{1}{2}s^2.$$

Then equation (5.17) becomes

$$z^2q'' + 4zq' + 2q - 2 = 0,$$

which is a linear, second order Euler equation. It has solution

$$q = \frac{2\tilde{A}}{z} + \frac{\tilde{B}}{z^2} + 1,$$

where $\tilde{A}$ and $\tilde{B}$ are constants. In terms of the variables y and u we have

$$y_u = \left[2\left(\frac{2\tilde{A}}{y} + \frac{\tilde{B}}{y^2} + 1\right)\right]^{\frac{1}{2}}. \tag{5.18}$$

We can rewrite (5.18) in the form

$$u - u_0 = r - a(t)$$

$$= \int^Y \frac{ydy}{(2y^2 + Ay + B)^{\frac{1}{2}}} \tag{5.19}$$

where we have set

$$A = 4\tilde{A}$$

$$B = 2\tilde{B},$$

for convenience and u_0 is a constant. The integral in (5.19) is of standard form and we find the general solution

$$r - a(t) = \frac{1}{2}\ln\left[2Y^2 + AY + B\right]. \tag{5.20}$$

Hence we have generated an expression, namely (5.20), governing the behaviour of the gravitational potential Y. The remaining quantities (ϕ, μ, p) now easily follow as the solution of the master equation (5.13) has been specified. Thus we have exhibited an exact solution for the Einstein–Maxwell system of equations for an inhomogeneous shearing and charged relativistic fluid corresponding to a stiff equation of state. We believe that this is a new charged model and has not been published before.

6 Conclusion

Our objective in this thesis was to find new solutions to the Einstein–Maxwell field equations for spherically symmetric gravitational fields. We were primarily concerned with the case of nonvanishing shear. A number of new classes of shearing solutions extended those of Govender (1997) who performed a comprehensive analysis in the uncharged case. A number of simple solutions to the Einstein–Maxwell field equations, which we believe to be physically reasonable, were obtained explicitly. We employed the following techniques in generating new solutions: the Lie analysis of differential equations, imposing an equation of state, *ad hoc* choices for the gravitational potentials, converting Riccati equations to second order linear equations, and converting nonlinear equations to simpler equations using the appropriate Lie symmetry.

We now highlight the main points and conclusions obtained in this thesis. We give only those items of principal interest:

- After reviewing the differential geometry applicable to general relativity we obtained the Einstein–Maxwell field equations for a perfect fluid source in the presence of charge in spherically symmetric spacetimes.

- The class of nonaccelerating spacetimes was comprehensively analysed. Two cases arose: $Y' = 0$, and $Y' \neq 0$. A particular charged, dust solution was presented when $Y' = 0$. When $Y' \neq 0$ we assumed that $p =$ constant. The evolution of the model was reduced to a second order nonlinear equation which has solution in terms of quadratures. The solutions of Govender (1997) are

regained when the charge vanishes. For nonexpanding solutions we reduced the solution to quadratures. We also imposed an equation of state $p = (\gamma - 1)\mu$. With λ = constant we find that the solution was valid for all values of the constant γ. When $\lambda \neq$ constant then we set $\gamma = 2$ and the line element then depended on the gravitational potential λ only.

- The Lie symmetries of differential equations was used to reduce the Einstein-Maxwell field equations to a system of ordinary differential equations. This reduction depended on the choice of a suitable infinitesimal generator, which generates a self-similar variable. After a number of simplifications the solution was reduced to a Riccati equation depending on the constant C. When $C = 0$ we regained the uncharged solution of Govender (1997). When $C \neq 0$ we reduced the behaviour of the model to a separable first order equation.

- We also found new solutions to field equations by assuming that the metric potential λ is constant. We generated a nonvacuum charged solution which has a simple form. This solution reduced to the vacuum solution of Govender (1997) in the uncharged limit. We choose a form for the electromagnetic gauge potential and performed the Lie analysis. The evolution of model was reduced to quadratures. For the stiff equation of state $p = \mu$, the solution was expressed in terms of elementary functions.

We have considered spherically symmetric spacetimes in the presence of charge. The governing equations are the Einstein-Maxwell partial differential equations which we showed admit a variety of solutions. Therefore we have demonstrated that it is possible to handle the complex system including charge and nonvanishing shear. Researchers in the past avoided this case because of the perceived difficulty in

making any progress. We have provided a strong case for the comprehensive analysis of the Einstein–Maxwell system in general by demonstrating the existence of explicit solutions.

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