Students' understanding of elementary differential calculus concepts in a computer laboratory learning environment at a University of Technology.

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Declaration

I, the undersigned, Kristie Naidoo, declare that "Students' understanding of differential calculus concepts in a in a computer laboratory teaching environment at a University of Technology." is my own work and that all sources I have used or quoted have been indicated and acknowledged by means of complete references.

Signed: ______________________ **25 March 2007**

Kristie Naidoo

Statement by supervisor:

This thesis is submitted with/without my approval.

Signed: ______________________ **25 March 2007**

Dr R. Naidoo

ACKNOWLEDGEMENTS

I wish to dedicate my work to my sons Sarushen Naidoo and Eushen Naidoo and my dear wife Thilakavathi Naidoo who have supported me throughout my preparation.

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ABSTRACT

This thesis investigates the mathematical cognitive errors made in elementary calculus concepts by first-year University of Technology students. A sample of 34 first year students, the experimental group, from the Durban University of Technology Faculty of Engineering were invited to participate in project in elementary calculus using computer technology (CT).

A second group, the control group, also consisted of 34 first year engineering students from the same University were given a conventional test in elementary calculus concepts. The experimental group was then given the same conventional test as the control group on completion of the project in elementary calculus using computer technology (CT).

The purpose of the analysis was to study the effect of technology on the understanding of key concepts in elementary calculus. The major finding was that technology helps students to make connections, analyse ideas and develop conceptual frameworks for thinking and problem solving.

The implications include:

- Improvement of curriculum in mathematics at tertiary level;
- New strategies for lecturers of elementary calculus;
- An improved understanding by students taking the course in elementary calculus.
- Redesign of software to improve understanding in elementary calculus

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CHAPTER 1: INTRODUCTION AND STATEMENT OF PROBLEM

This chapter gives an overview of aspects found by researchers that have a bearing on the understanding of basic concepts in differential calculus in a traditional learning environment and how computer technology can affect the cognitive process.

1.1 THE PROBLEM

The primary problem is that there is a high failure rate in mathematics at first year level at the Durban University of Technology, where this study is located. This difficulty can be attributed to a lack of understanding of differential calculus concepts. Having taught mathematics, in particular calculus, for over 15 years at secondary school level gave rise to probing questions about the state of mathematics learning in the country. In my experience most secondary school learners have difficulty in working differential calculus. In South Africa Calculus forms 40 % of the Algebra component of the grade twelve national examinations. This contributes to at least 20 % of the overall assessment in the grade 12 mathematics examinations.

Many researchers have also been concerned with the failure rate in other countries (Burton, 1989; Fullilove & Treisman, 1990; Tall, 1997; Acherman-Chor, Aladro & Gupta, 2003) and students' conceptual understanding in elementary calculus (Heid, 1988; Tall, 1992; White & Mitchelmore, 1996) at university level. The study narrowed the problem to differential calculus as it is the most important concept for first year engineering students.

Naidoo (1998), in his study found that at tertiary level the majority of the students study by rules. They do not enjoy mathematics and are de-motivated. Lecturers tend to teach mechanistically and do standard type solutions to standard type problems. Students' found rate of change, differentiation as limit and the use of symbolism difficult. He concluded that rate of change needed to be studied intensely. He draws attention to the fact that mathematics at the Technikon (now University of Technology) level is not a specialist subject. This contributes to the "poor" understanding of critical concepts that are essential for extended learning – a type of understanding that is needed to support an increasingly technological world. Consequently the time and attention given to study mathematics is limited. This contributes to failure in making a distinction between process and the concepts integral to the process.

Bezuidenhout (2003) suggests that students' ability to interpret a mathematical symbol as representing both a process and an object is more likely to develop if it is the direct focus of teaching rather than if the development is left to chance. If mathematics educators comprehend student's understanding, they can develop specific mathematical tasks and teaching strategies to assist students in dealing with limitations in their understanding of mathematical symbols.

A mathematics research group had been established at the Durban University of Technology for over a decade. The aim of the local calculus reform research group was to research alternate ways of teaching elementary calculus. Students had access to mathematics laboratory sessions where project work in a computer-learning environment was encouraged. The learning environment was used by the students to investigate and explore concepts in calculus under the guidance of lecturers. These include function, limit and rate of change. In my opinion, this environment would help students to build their mental models to connect with aspects they meet during traditional lessons. These attempts hoped to develop interest in mathematics study and improve throughput rate within the University.

Another calculus reform group was reported by Silverberg (2004) at a University where the traditional approach was supported by weekly computer laboratory sessions. The goals were to improve fundamental concepts in calculus for application in the natural sciences and engineering. These attempts hoped to reduce failure rate, withdrawals from the course and narrow the gap in performance between the better and weaker students. Here the computer was not used for practice or drill but for creating mathematical objects and processes. Much thought had been put into the material used in the project activity students do during the laboratory sessions. They used a collaborative environment where students worked in groups of 3 or 4. The groups worked together both in and out of the class. Positive results were achieved with students from the reform section. After some time their overall assessment scores in examinations were improved. There was also an increase in student confidence levels as well. Furthermore there were zero withdrawals from the course.

Zandieh (2000) studied the understanding of the derivative by students in a typical USA university. She viewed the concept of the derivative as ratio, function and limit as process-object pairs. These layers can be viewed as dynamic processes and as static objects. When a student lacks a structural conception of one of the layers the pseudo-structural term is used to describe an object with no internal structure. The graphic interpretation of the derivative is seen in three layers namely, the slope of the secant line, limit of the sequence of slope values of secant lines and the instantaneous slope (limit).

Due to the high frequency of errors made by first year students at the University of Technology and the failure of the traditional lecturing methods this study sought to investigate whether students will fare better in a computer teaching and learning environment. This leads to the secondary problem. This research intends finding out what impedes students understanding of calculus, what errors students' make and why they are making these errors. The results will feature as an important aspect for curriculum planning purposes.

Some factors that must be considered about the learners at school level: All grade twelve learners doing mathematics must study calculus irrespective of their background knowledge, ability and motivation. This complicates the design of the curriculum and research evaluating its effectiveness. Some learners appear to make connections while others do not. Given the wide spectrum of approaches by such a diverse range of learners, the method appropriate to teach some learners may be inappropriate for others. Consequently a course designed found to be of positive help to some may be a failure for others.

To gain a better perspective on the teaching and learning of calculus concepts a review of pertinent literature was performed. It was hoped that the literature survey would give us a handle on the difficulty of the teaching and learning of elementary concepts in calculus.

Davis (1984: 3) raises an important concern: "If a person wants to learn certain mathematics, we are less inclined to accept the verdict that he or she cannot do so. We want more specific information; we want to know exactly what obstacles impede this person's progress, exactly what they cannot seem to do, exactly what errors they are making and why they make them."

The need for alternate methods of instruction to enhance teaching and understanding of calculus is essential. Hughes-Hallet (1989) found that students can differentiate complicated functions analytically but could not interpret differentiation graphically. In order to achieve this, she suggested that students' need to learn through discovery, visualization and experimentation.

Cipra (1988) and White & Michael (1996) in their studies show that students enrolled in the traditional university calculus class have a very superficial and incomplete understanding of many of the basic concepts in calculus. This was attributed to the rote and manipulative learning that takes place in an introductory course.
Smith & Moore (1991: 85) explains:
"*Much of what our students have actually learned ………more precisely, what they have invented for themselves is a set of 'coping skills' for getting past the next assignment,*

the next quiz, the next exam. When their coping skills fail them, they invent new ones. The new ones don't have to be consistent with the old ones; the challenge is to guess right among the available options and not get faked out by the teacher's tricky questions……. We see some of the 'best' students in the country; what makes them 'best' is that their coping skills have worked better than most for getting them past the various testing barriers by which we sort students. We can assure you that does not necessarily mean our students have any real advantage in terms of understanding mathematics."

Tall (1992) identifies some difficulties that students encounter with calculus. These include:

- Algebraic manipulation or lack of it;
- Preference for procedural methods rather than conceptual understanding;
- Difficulty in translating real-world problems into calculus formulation;
- Restricted mental images of functions;
- Difficulty in absorbing complex new ideas in limited time;
- Difficulty with notation.

These difficulties need to be addressed to improve understanding of elementary concepts in calculus. He advocated the use of the "zoom" function to teach the derivative. The zoom graph method is mainly a computer laboratory experience where the curve is approximated to a straight line. It is effective because the student is dealing only with gradients of a straight line. When the domain intervals are made very small the curve can be approximated as a straight line. Instead of using secants we zoom to get a straight line. Here we establish the idea of the gradient of a curved graph. Using suitable software the graph can be drawn and a part of it can be selected and magnified. The magnified part looks "straight". This method frees the student from cognitive overload. The student does not have to deal with tangents, secants and complex geometry.

1.2 THE TRADITIONAL MATHEMATICS CLASSROOM

Currently much of the focus in mathematics at secondary school is based on wanting to make mathematics as simpler as possible. Boaler (1995: 280) in her research mentions that mathematics lessons at Amber Hill were algorithmic, focusing on standard methods, rules and procedures. A similar situation exists in mathematics learning environments in a majority of situations. Little or no attention is given to understanding.

Understanding is twofold. It is based on: how we do? And why we do? Learners need to know not only why they do something but also how to do something. Skemp (1976) gives an account of relational understanding and instrumental understanding. Students enter the University of Technology with a goal to understand instrumentally while the lecturer wants them to understand relationally or vice versa. Attempts by the lecturer to explain in detail will have no bearing on a group that is only interested in learning by rules. It is essential that students understand why a method works. This is crucial since the understanding of a particular concept turns out to be fundamental to the study of a new concept. Thus relational understanding makes it possible to connect concepts in areas of mathematics that are interrelated. In a sense attempts can be made to extend students existing knowledge schemas in a particular area of knowledge. For example to form differential equations, schemas such as functions, average rate of change and limits must be viewed relationally in order for the concept of the derivative to be understood.

In coming to understand the derivative it is imperative that the student poses two questions to himself namely, how? And why? The how question alone cannot reconstruct the why question. This is supported by a response from a learner in Boaler's research (1995:281): "It's like, you have to work it out and you get the right answers but you don't know what you did, you don't know how you got them, you know?"

In coming to understand the derivative it is necessary to review research performed on students' whilst they were at school. Focus (1990) has found that calculus courses at Grade 12 level are pretty much freewheeling – they emphasize the mechanical techniques to the extent that drill is necessary and they contain certain illustrations and applications that the educator is competent to explain and the learners ready to receive. Student preparation is a key factor to how understanding unfolds itself. This trend, from my experience at secondary school, is brought to tertiary level from students' previous learning experience. We also learnt from Naidoo's (1998) study at the same institution that lecturers too tend to perpetuate teaching that promotes rote learning.

Orton (1983) in his investigation on the understanding of differentiation by students at high school and training college students concluded that "both groups found the same items difficult and the same items easy". He concluded that students had little intuitive understanding as well as fundamental misconceptions about the derivative.

Clearly traditional methods do not focus on real world problems. This accounts for the shortcoming in students' ability to think in realistic situations and inhibits their ability to design their own solutions. The discovery approach starting with real world problems that can be modelled mathematically, and students learning at their own pace becomes necessary. The environment in which this can be performed is a computer laboratory.

1.3 TECHNOLOGY IN THE CLASSROOM

The requirement for students taking mathematics at the Durban University of Technology is a pass in mathematics at least on the standard grade at grade 12 level. The course requirement enables students with average mathematics ability to take mathematics as a subject in their engineering studies. The majority of the student population is second language and most students come from primarily traditional mathematics settings; chalk and talk with no technology.

Having worked with tertiary students in their first course in Mathematics, both in a traditional setting and a computer learning environment to learn calculus, I deduced that there was a need for further research. Many researchers have indicated that changes are necessary in the way in which mathematics is taught. Traditional methods do not fully relate to the real world situation and engineering subject concepts. Students need to be helped think and solve problems – even those related to the real world. The computer can offer support in carrying out tedious calculations quickly. Heid (1988) also found that using computer technology in the calculus class encourages students to reason deeply from and about the graphs. Traditional teachings methods do not cater adequately for this type of interaction since students spend a lot of time doing calculations. Students are unable to make the connection between algebraic and graphical representations.

Traditional learning and teaching methods are preferred by educators (lecturers) that have little or no interest in using technology to enhance learning. Educators need to find out what is different about the new technology and what effect these would have on cognition, teaching and learning (Kaput, 1992).

Kober (1992) found that computers are used more often in mathematics than any other subject, and the use of computing technology has fundamentally changed how mathematical research is conducted. Cohen (1995: 63) quotes Henry Pollak from Bell Laboratories as saying "With technology – some mathematics become more important, some mathematics become less important, and some mathematics possible".

De Villiers (1993) shows how computing technology can be used in mathematical modelling to solve practical problems with great success. He challenges the emphasis of technical and manipulative skills in traditional teaching at the cost of model construction and interpretation. Here the computer is seen as an essential tool in modern applied mathematics.

Tall (1991) in his research on visualization in calculus found that in traditional lectures formal definitions (even if remembered) are long and complex and usually need to be written down to be able to grasp them as a whole. Visual ideas prove to be easier to discuss in everyday language. Visual ideas can be demonstrated and discovered using the computer laboratory. Heid (1988) found that traditional calculus courses offer little opportunity for students to develop deep conceptual understanding of the graph. This in turn hinders the connection between algebraic and graphical representation.

De Ting Wu (2004) found that the limit concept was confusing to students. Task group meetings at "The 9th International Congress on Mathematical Education (ICME) in Japan in 2000" and "The 10th ICME in Denmark in 2004," revealed that while the teaching and learning of the limit concept continues to be a much-discussed topic, it is also a topic that is both important and difficult.

Different thinking and study methods are needed for the advancement of elementary mathematics. Secondary school learners lack background to advanced mathematics and hence the understanding the limit concept. Students need new approaches and powerful tools to help them to overcome the difficulty in studying limit concepts and to realize a smoother transition from their secondary mathematics education to learning a more advanced level of post-secondary mathematics. De Ting Wu (2004) advocates that computer technology is a powerful tool and a helpful aid in teaching and learning mathematics. The components: computation, visualization, and animation could be helpful in developing new approaches to the teaching of the limit concept and ultimately to help students to overcome their difficulty in understanding this important concept.

Sierpinska (1992) describes how subtle changes to meaning resulted in conceptual obstacles that needed to be overcome. An example of this is given: A learner whose experience of function in terms of formulae and computation will find it difficult to accept a definition which does not have these attributes. Sfard (1992) showed how the operational view of mathematics in terms of processes to be carried out preceded the structural view using objects and formal definitions, both in history and cognitive development. The set theoretic definition was less successful in practice and in courses where the formal definition had less emphasis it lost its application. The computer provides a new environment to explore the function concept. Cuoco (1994) found that an approach to functions through programming in Logo gave different insights from a traditional approach. They were able to think of a function as an object in its own right as well as seeing the relationship in terms of input and output.

Similar observations were made in structured BASIC which includes procedural functions (Li & Tall, 1993) and in ISETL (Breidenbach et al, 1992; Cuoco, 1994).
Wilson (1995) indicates that there is a lack of consensus on why and how technology should be integrated into the educational environment, what students should be taught and how to train educators to use technology. Kaput (1992) indicates that educators need

to discern what is different about the new technology, and what those differences mean in terms of cognition, learning, teaching and education in general.

1.4 PEDAGOGICAL SHIFT

Krantz (1991) refers to the pedagogical shift on the part of mathematics educators from a point where "only the best students make it through a course" to a new attitude that mathematical knowledge should be available to all students.

The characteristics of this pedagogy include:

- Cooperative work; the emphasis on "getting help" rather than individualised student effort;
- Exploratory study; the emphasis on exploration by the individual rather than chalk-board exploration;
- Multiple representations of the subject; the graphical, algebraic and numerical representations are emphasized;
- Alternate assessments of students progress; includes review of portfolios of student effort and project work.

In a sense there is a shift from an "instructional" paradigm to a "learning" paradigm. In this "learning" paradigm, opportunity must be created for students to interact with concepts in a meaningful way. This to a certain extent would mean that this opportunity will give these students chance to learn with more insight than students from a traditional class. The computer laboratory environment is an interactive one. It creates opportunity for cooperative work, exploration, multiple representations of the subject and alternate forms of assessments.

Comparing how learning took place previously will give an indication of the potential to understand in this "learning" paradigm. Learning took place by reading, writing, listening and discussing. In this technological era, learning can be supported by technological development.

It is quite evident that each educator prefers his or her own technique or style. This is similar when it comes to the use of technology. Cartwright (1993) hints that ".... in the hands of many educators it can be very useful." Hence technology is a flexible tool in the hands of the lecturer.

The use of the computer as a tool in the learning environment can enhance learner participation since students naturally tend to become automatically engrossed in a learning situation as compared to a chalk and talk event. In a computer learning environment, students are required to make inputs on an ongoing basis. Learning can only take place if the learner becomes involved in the learning. This involvement would result in a search for some solution whether right or wrong. The students in the study done by Heid (1988) enjoyed the computer work since it freed them from doing the tedious manipulations. It can minimise the problems experienced in algebra which is found to be a stumbling block in differentiation.

Dreyfus and Halevi (1990/91) mention topics which lend themselves to computer implementation having visual aspects which can be well represented on a computer screen. They have transformational aspects which necessitate a dynamic implementation and technical aspects that are taken care of by the computer and connect two different representations of the same concept. These two representations can be dealt with by the computer program.

Students' mathematics abilities fall below the needs of the technological advancement of society. This would suggest that something must be done in the teaching and learning of mathematics so as to "catch up" with technological advancement.

1.5 MATHEMATICAL MICROWORLDS WITH *MATHEMATICA*

At the Durban University of Technology *mathematica* is used by a few lecturers for research in mathematics and for teaching and learning of functions. The majority lecturers prefer the traditional method of teaching. One classroom has been converted to a computer laboratory using *Mathematica* software. *Mathematica* has both graphics and symbol manipulation capabilities. *Mathematica* is usually used for project work in calculus. In *Mathematica*, notebooks can be created in which selected mathematical concepts are grouped together and then closed with only a heading visible. It gives opportunity to introduce a topic, develop it more and then ask the students to do an example or think about some particular aspect or example on their own before opening the next part of the lesson to see details. Students get a feeling for geometric properties of functions and plotting of graphs and many of the calculus concepts as well.

Papert (1980) advocates the use computer-based tools to encourage students to make conjectures and explore them. Concerns were raised on how computers affect the way people think and learn. Using computational technology, like *Mathematica* and computational ideas can provide new possibilities for learning, thinking and growing emotionally as well as cognitively. The tools available enhance thinking and change the patterns of access to knowledge resulting in different experience from that experienced in a traditional learning environment. Papert (1980: 120) describes microworlds as "incubators for knowledge……First, relate what is new and to be learned to something you already know. Second, take what is new and make it your own: Make something new with it, play with it, build with it."

Kent et al (1996) have had success in creating mathematical microworlds using *Mathematica* in a chemistry undergraduate class at the Imperial College, University of London. The program aimed at encouraging students to explore mathematical relationships by experimentation in a mathematics laboratory setting.

Mokros and Tinker (1987), in their study, on how middle school students learn graphing skills using micro-computer laboratories with *Mathematica*, found a significant improvement in students' ability to interpret and use graphs.

Mathematicians found software like *Mathematica* offered a wide range of possibilities in teaching and learning calculus. Some of the motivating factors included selfless desires to make calculus more understandable for a wider range of learners and a growing aspiration to research the learning process and to understand how individuals are able to conceptualize concepts in calculus.

Engineers are increasingly using computers with *Mathematica* to solve mathematics problems. Teaching mathematics using computers with *Mathematica* therefore also trains the student engineer to use a tool to solve appropriate mathematics problems in industry.

Modern calculus reform seeks to use these computer representations such as *Mathematica* to make calculus more practical and meaningful. The computer laboratory with *Mathematica* offers the student the opportunity to perform these procedures quickly.

There is a spectrum of possible approaches to teaching and learning calculus in a computer laboratory with *Mathematica.* These include intuitions from real-world calculus, using numeric, symbolic and graphical representations and formal definition-theorem-proof-illustration of analysis. The computer with *Mathematica* allows both a numeric quantitative approach to do calculations as well as graphical representations offering a possible conceptual approach based on visualization so that the student can be motivated to do more mathematics.

Research shows that the limit concept to have embedded cognitive difficulties. *Mathematica* overcomes the difficulty by appealing to Cornu's summary (1991: 154): "The enactive real-world approach deals with this at a practical approximation level. The graphical approach allows the limit the notion to be handled implicitly, like, by magnifying the graph using computer technology, to see it looking "locally straight" so that the required gradient is that of a straight line graph. This enables moving through elementary calculus with ease but requires further reconstruction to cope with formal concepts."

Krutetskii (1976: 178) found that learners exhibit relative preferences for verbal-logical and visual thinking that he classified as "analytic", "geometric" and "harmonic". Such factors imply that research into calculus must take into account what is applicable to one group of learners in one context may turn out different for another group. This leads to the question of theoretical and philosophical issues which may lack in the development of learners long before they began their study in calculus. The discovery approach using the computer laboratory with *Mathematica* offers an additional learning strategy.

Cotton, J (1995) describes active learning as learning by doing, student-centred, experiential learning. Students want to engage in their learning when learning is interesting, motivating and rewarding. Active participation builds confidence to attempt more difficult problems and applications. Using computers to teach and learn assist in taking the learner through his or her thought process. It stimulates learning in that certain prompts from the computer is a means of communication with the learner. In this environment, the learner gets an immediate response to his /her actions whether right or wrong. This allows the learner to proceed to the next step of the problem immediately (provided that the computer jargon is understood). Students that are engaged have more opportunity to talk about their experiences even if they are not successful. It is through this interaction that the learner is able to practice techniques through "hands on" practical experience. Technology, like *Mathematica*, can be used to foster understanding which may be more difficult to realize when using the traditional approach to solve problems.

Dubinski (1991) has developed a framework for research and curriculum development in mathematics that he calls the Action-Process-Object-Schema (APOS) theoretical perspective. Students seeing a concept for the first time are limited to an action conception of that concept. For example, beginning calculus students may understand differentiation as an action on polynomials, in which rules are applied in sequence. As the student reflects upon a particular action, he / she begins to view the concept as a process. In the case of differentiation, the student would understand that it is a more general process, not limited to a set of rules applied to individual functions. In the computer laboratory environment with *Mathematica* the student is offered the graphical approach to understanding the derivative.

According to Naidoo (1998), in a computer learning environment with *Mathematica* a student begins to grasp a process as a cognitive object through reflection. The student builds a schema that links actions, processes, objects, and other schema into a coherent framework. For a complex subject such as calculus, this is not easily described, and no two schema would be alike. Furthermore, the connections within any student's mind include both conscious and unconscious links. What we should expect is that the student would understand that an important class of functions, have associated with them derived functions and derivatives and integrals have an "inverse" relationship. Furthermore students must realize that calculus is a study of the properties and the behaviour of functions.

1.6 RESEARCH OUTPUT

Research based on aspects of this study, have been presented in 3 conferences; two local conferences and one international conference. The details are as follows:

- Naidoo, K. and Naidoo, R. (2007). First year students understanding of elementary concepts in differential calculus in a computer laboratory teaching environment. Double peer reviewed. In *Proceedings of the College Teaching and Learning Conference*, Oahu, Hawaii USA, 2-5 January. ISSN: 1539-8757.
- Naidoo, K. and Naidoo, R. (2007). First year students understanding of elementary concepts in differential calculus in a computer laboratory teaching environment. To be published in the *Journal of College Teaching and Learning* (TLC).

- Naidoo, K. and Naidoo, R. (2005). On Errors in Differential Calculus. In *Proceedings of the 48th Annual Congress of the South African Mathematical Society*, Rhodes University, 2- 4 October.
- Naidoo, K. and Naidoo, R. (2004). Teaching Elementary Calculus using computer Technology: A Case Study at a Technikon. *Proceedings of the 14th Annual KwaZulu-Natal Mathematics Conference,* Durban Institute of Technology, 8 May.

CHAPTER 2: THEORY OF TEACHING AND LEARNING AND ELEMENTARY DIFFERENTIAL CALCULUS

CHAPTER 2: THEORY OF TEACHING AND LEARNING AND ELEMENTARY DIFFERENTIAL CALCULUS

2.1 LANGUAGE AND LEARNING

Language is fundamental to learning. Aiken (1972) mentioned that linguistic factors affect performance in mathematics and that mathematics is a specialized language with its own vocabulary and syntax. Wittengenstein (1976) showed that mathematics is a language and it follows language rules. Vygotsky (1962) argued that language is learnt in a social context. From the viewpoint of mathematics as a language, it follows that project work and group discussions should be given prominence in the learning of mathematics.

In order to understand students' mathematical language concepts deep and surface structures are elicited. Chomsky (1957) described syntax in terms of its *surface* and *deep structure*. Here the surface structure in print refers to the visual information on the page or the actual words and word order represented in graphic symbols. The deep structure referred to the underlying structure of the language, where the component phrases of a complex sentence are identified and their relationships specified to result in meaning.

2.2 DEEP AND SURFACE LEARNING

Many studies have identified deep and surface approaches to learning in a wide range of contexts, (Biggs, 1979; Entwistle & Ramsden, 1983). They identified two qualitatively different “levels of processing”. Surface-level processing focused on the text itself and memorizing. Deep-level processing focused on the underlying meaning of the text. (Marton & Saljo: 1984) used the term “approach to learning” to describe strategy (what students do) and intention (why they do it).

Deep and surface approaches are related to motivation. The deep approach to study is derived from intrinsic motivation and the surface approach from extrinsic motivation. The deep or surface approach can be adopted by an individual with either motivation. These approaches are not attributes of individuals. An individual may use both approaches at different times although they may have a preference for one or the other.

In terms of Bloom’s Taxonomy (1956), the “deep” approach requires higher order thinking skills that includes analysis and synthesis. “Deep” learners incorporate new ideas that they learn with existing knowledge and personal experience. In my opinion, deep learning is encouraged by extending individual study time and time given for projects since it gives learners more opportunity to practice.

Entwistle & Ramsden (1983) and Marton & Saljo (1984), note the importance of group work and problem-solving as a means of fostering the deep approach to learning. These are similar to the “active learning”, “cooperative learning” and “problem-based instruction”.

Atherton (2003) makes the following remarks on deep and surface approaches:

In the surface approach learning is viewed as:

- a quantitative increase in knowledge (acquiring information or "knowing a lot");
- memorizing (storing information that can be reproduced);
- acquiring facts, skills and methods that can be retained and used as necessary.

In the deep approach learning is viewed as:

- making sense or abstract meaning (learning involves relating parts of the subject matter to each other and to the real world);
- interpreting and understanding reality in a different way (learning involves comprehending the world by re-interpreting knowledge).

Depending on how the computer is used, it is a potential means of getting students to use deep approaches in their search for solutions. In the case of calculus solutions it also provides a visual aid to enhance comprehension.

Case & Marshall (2004) refer to two intermediate approaches to learning, the procedural surface approach and the procedural deep approach. These approaches lie between the deep and surface approach. The table below shows how these approaches are applied:

STRATEGY	INTENTION	
	Passing the test	Understanding
Memorization	Surface	
Problem Solving	Procedural surface	Procedural deep
Concepts	Conceptual surface	Conceptual deep

Table: 1 Intermediate Approaches to Learning

Both these approaches focus on problem solving. The deep approach involves the intention to understand and the surface approach not.

2.3 COGNITIVIST THEORY

Davis (1984) proposed a cognitive theory as a language to describe mathematical behaviours. Here thought processes are regarded as fundamental. The theory relates observations to a postulated theory of 'metaphoric' processes with information of how the individual thinks about some mathematical problem. The theory borrows its basic concepts from the field of artificial intelligence.

Cognitivists also focus on what the student is thinking. Focus is on the learning process that takes place in the students' mind. The cognitivist tries to identify ideal learning strategies for students whereby the student is active in the learning process. In this theory, errors are viewed as an unsuccessful attempt to understand, order and act upon their environment in ways that make sense to them. Such an error analysis is necessary for learning when using computers. The curriculum can be adjusted to accommodate students' development stages.

2.4 CONSTRUCTIVIST THEORY

Constructivism is a theory of learning based on constructing knowledge, not receiving it (Marlowe & Page, 1998: 2). It is about constructing knowledge to get more knowledge. Constructivism is concerned with the thinking and the thinking process. This would mean uncovering and discovering knowledge for one's self rather than receiving the knowledge from an instructor. Children learn best when they find out for themselves the specific knowledge they need (Papert: 1993). In his work with logo, Papert believed that programming was fundamental to problem solving in mathematics calling for both "convergent" and "divergent" thinking which he refers to as "logic" and "intuition".

In terms of the constructivist learning is:

- both the process and the result of questioning, interpreting and analyzing information;
- using this information and thinking process to develop, build and alter our meaning and understanding of concepts and ideas; and
- integrating current experiences with our past experiences and what we already know about a given subject. (Marlowe & Page: 1998)

They concluded that:

- Students learn more when they are actively engaged in their own learning;
- By investigating and discovering for themselves, by creating and re-creating, and by interacting with the environment, students build their own knowledge structure;
- Learning actively leads to an ability to think critically and to solve problems;
- Through an active learning approach, students learn content and process at the same time (Marlowe & Page: 1998).

2.5 FRAMES

A frame is an abstract formal structure that is stored in memory and somehow encodes and represents a sizeable amount of knowledge. This collection of knowledge representation structures or "frames" grows as more complex frames are built on the existing ones.

We focus on the sequential processes which guide mathematical problem solving activity, the critique which is an information processing operator that is capable of detecting certain of frames, information in one's mind must be typically organized into quite large chunks (Davis & Mc Knight, 1979, Minsky, 1975). Minsky (1975) states "when one encounters a new situation ..... one selects from memory a substantial structure called a frame. This is a remembered framework to be adapted to fit reality by changing details as necessary".

The four basic concepts of Davis's theory include: sequential process, critique, frames and deeper-level procedures. An expert possesses an abundance of critiques and this attribute distinguishes an expert from a novice.

Davis (1984: 276/7) lists six possible frame selection procedures:

- Bootstrapping - deals with what one sees in the given. It leads to certain associations, frames that involve such things;
- Not knowing too much - deals with the limited knowledge on a topic or concept;

- Focus on some key cue - deals with the presence of a small number of cues that lead to the retrieval of some specific frame;
- Using context - deals with how the context influences student's choice;
- Using systematic search - deals with the student learning things in a systematic way and develops systematic procedures for searching his/her memory;
- Parameter-adjusting or spreading activation - deals with how certain frames or assimilation patterns acquire high expectation values.

The types of frames necessary for the concept derivative include pre-differentiation frames, e.g. to understand composition functions ($f \circ g$), we need the pre-frames of functions f and g. If these pre-differentiation frames are brought to bear on a differentiation problem then solution to the problem can be possibly sought. The problem with most students is that these pre-differentiation frames are incomplete or inadequate.

In order to be successful one needs to build on pre-differentiation frames and synthesize an adequate knowledge representation to what we recognize as mathematical thought.

2.6 CATEGORISATION OF ERRORS

Errors in calculus can be categorized as structural errors, executive errors and arbitrary errors as described in Donaldson (1963). Structural errors are those which arise from some failure to appreciate the relationships involved in a problem or group of principles essential to the solution of the problem. Failure to tackle relationships in a problem arises from a false expectation of the problem. Structural errors may arise in connection with variable interaction. These errors occur in the deductive mode when the subject reasons deductively but fallaciously. One may expect that failure to perceive inconsistency or consistency would be a common source of structural error (Donaldson, 1963). An incorrect frame may be retrieved or the frame maybe not developed adequately. Structural errors are caused by incorrect frame retrieval, sketchy or incomplete frames, deep-level procedures and sub-procedures.

The second type of error is the executive error. Executive errors occur when there is a failure to carry out manipulations, although the principles may have been understood. Some defect of concentration, attention or immediate memory lie at their origin. The most prevalent of this class of errors is loss of hold on reasoning (Donaldson: 1963). A correct frame maybe retrieved but a sub-frame responsible for calculations maybe underdeveloped.

The third type of error is the arbitrary error. Arbitrary errors are those in which the subject behaves arbitrarily and fails to take account of the constraints laid down in what was given. These are errors which have as their outstanding common feature a lack of loyalty to the given. Sometimes the subject appears to be constrained by knowledge of what is 'true' by some considerations drawn from 'real- life' experience. Sometimes there is no constraint of any kind. The subject simply decided 'it is so' (Donaldson: 1963). Incorrect inputs maybe assigned to the retrieved frame. "Arbitrary" errors are caused by mapping incorrect inputs to the retrieved frame (surface structures).

2.7 PROBLEM SOLVING STRATEGIES

To gain a better handle on how some students produce correct or incorrect answers we appeal to Larkin (1980). She refers to the difference between the expert students and novice students. Experts know a great many things and a novice does not. Experts tend to use a knowledge-development (forward-working) approach. In this approach the student begins with the "givens" of the problem applying successive equations that could be solved with the givens (Larkin et al., 1980). A computer learning environment aims to do precisely this, i.e. increase the knowledge base of the student. The expert's knowledge of the field is more hierarchically arranged, and this is stored in larger functional units, or chunks, for more coordinated access. (Larkin, Heller, & Greeno, 1980).

In contrast novices tend to sequentially access principles in a more "piece-meal" approach. They use a means-end analysis or "working backward." In this approach, the "givens" of the problem are compared with the desired result. Differences between the two are recognized and the solver attempts to "transform one or the other to reduce, and finally eliminate, these differences" (Simon & Paige, 1979).

An intermediate response is used to describe a response that is neither forward nor backward. It is believed that the mental frame is in the process of being developed.

2.8 CALCULUS AND COMPUTERS

This brings us to the fact that there are many tools available to teach mathematics. To mention a few, the chalkboard, overhead projector, displays, videos and audio cassettes. The computer was chosen in this project because it is an interactive tool and encompasses all the above functions. It is an environment that allows the student to explore examples of mathematical processes and concepts. The characteristics of the examples can be abstract. Students are at liberty to give input without having to be scared of what the outcome would be. Student responses may be viewed privately and this would cause little or no embarrassment in the case of one having to respond to a question in a large lecture theatre or classroom.

A typical computer learning environment is shown below:

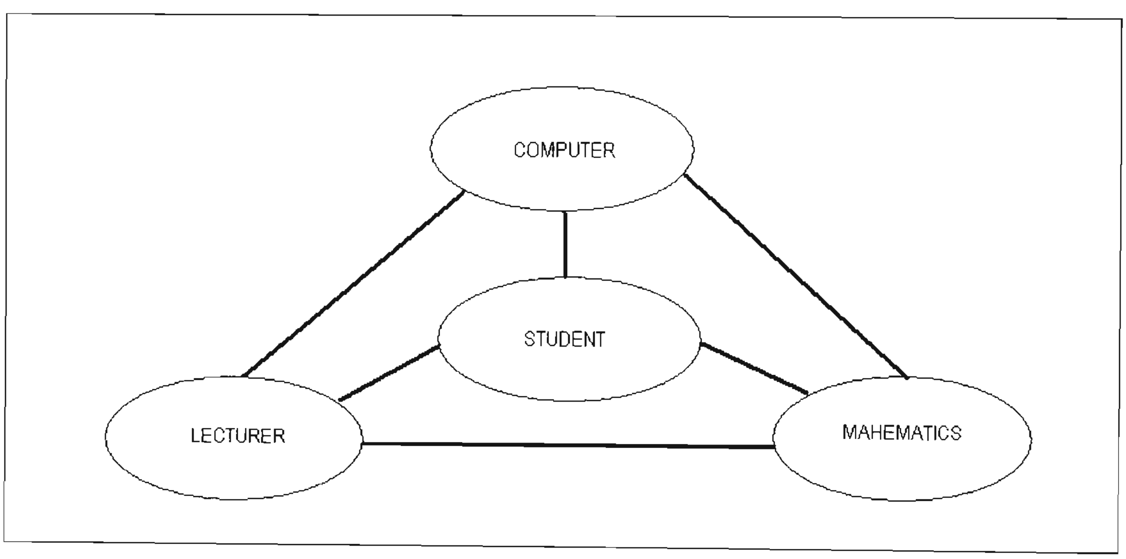


Fig. 1 Computer Learning Environment

The element of teaching and discussion is to demonstrate examples and slow down the action of lecturing, to explain what is happening and pausing on occasions when an important point is reached that is worthy of discussion.

The lecturer and student need to negotiate the meaning of a concept. Furthermore the lecturer must help students form their own concept images in a way that is in agreement with mathematicians. A dialogue must ensue between lecturer and student. The mathematics must be an external representation on the computer as a dynamic process under the control of the users. Tall (1986) states that concepts may be built by seeing examples in action and tested by predicting what will happen on artfully chosen examples before letting the computer carry them out.

Naidoo (1998) also mentions that pre-knowledge frames for which concepts such as functions and algebra can be enhanced and corrected in the computer laboratory environment. Many types of graphs can be quickly drawn. Analyzing a graph is like analyzing a painting. Everything is there but the student must know what to look for. Students need to understand the mathematics of the graphs such as slope, concavity, asymptotes, zoom, scaling etc.

Ramsden (1992) found that *Mathematica* gave opportunity to set up sophisticated models in a way that students understood them and they were able to set up their own projects.

Some advantages of lecturing using computers as cited by Cotton (1995: 112) are:

- it encourages shy learners to build confidence;
- each learner would be allowed to work independently at their own pace; and
- arouses learners through active participation.

Further advantages of lecturing using computers include:

- as the learner is in involved in the activity at hand, attention span is improved;
- long-term memory is enhanced by the fact that the learner takes a personal stake in getting through the steps that are required to succeed;
- students that are engaged in activity are motivated;
- focus is on individual attention: the student is able to ask tutor for assistance with any computer jargon (like syntax errors);
- stimulation of cognitive drive; and
- improves self enhancement and affiliation.

Disadvantages of lecturing using computers include:

- some knowledge of working in a computer environment is necessary;
- knowledge of syntax in *Mathematica* (or other software) is an essential tool; and
- resources that are needed have financial implications.

Colgan (2000) refers to the use of computers in the curriculum design and delivery in undergraduate courses in engineering. These studies are published by The International Commission on Mathematical Instruction (ICMI). The use of the computer has been an area of concern regarding not only the teaching and learning but the enhancement of student learning.

There are many sections of the elementary calculus that automatically lend themselves to computer demonstration. Colgan (2000) in his study using MATHLAB in first year engineering mathematics suggested that these should be illustrated in lectures as well in the form of a software guide that students could use.

Coetzee & du Bruyn (2003) in their study of the students' perspective on the benefits of incorporating practical computer training in auditing software package found that students are willing to spend more time to include practical training classes because they are aware of the benefit on their understanding of the subject.

2.9 THEORY OF THE ELEMENTARY DERIVATIVE

We review theoretical issues in the literature which explore some of the concepts and processes associated with differentiation. The derivative can be seen as a concept which is built from other concepts. Particularly the derivative can be seen as a function, a number if evaluated at a point, limit of the sequence of secant slopes or rate of change. Differentiation assumes the understanding of function or more generally a curve (not all curves can be formulated by a function). There do not seem to be clear-cut characteristics that set advanced mathematical concepts from those in elementary mathematics. Each advanced concept is based on elementary concepts and cannot be grasped without a solid and sometimes very specific understanding of these elementary concepts. Thus the concepts of advanced mathematics carry an intrinsic complexity. For example students cannot grasp what is meant by a differential equation or interpret its solution unless they have understood the derivative concept and not just the techniques of differentiation. Mathematicians explain the derivative using pre-concepts such as elementary algebra, rates of change, limits and infinity and tangents. The network or sequence leads to interrelated ideas, each idea integrating some of the more elementary ones into an added structure. It is precisely the complexity of concepts that make differentiation difficult for students to grasp (Naidoo: 1998).

There is a distinction between the mathematical concepts as formally defined and the cognitive processes by which they are conceived. The term concept image describes the total cognitive structure that is associated with the concept. Tall (1981) indicates that the concept image includes all mental pictures and associated properties and processes. In coming to understand mathematical concepts at school students evolve mental pictures at a concrete level. For example, to understand rate of change students evoke pictures of a moving car. The mental pictures which served the students well at school level may now become an impediment. Bruner (1986) suggested that iconic processing limited ideas and urged a movement onto the symbolic level. The student with an inadequate concept image may find such a development difficult to achieve.

To build an adequate concept image of the derivatives lecturers write the derivative as 'the gradient of the graph of a function or curve'. This interpretation, basic to the understanding of calculus, deals with the slope of the line tangent at a point on a curve.

Conventionally we consider two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the figure 2 below. The slope of the line through these points is given by $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ which according to Skemp (1970) is the ratio of a pair of corresponding changes. This, however, represents the slope of the line through P and Q and no other line. If we now allow Q_1 to be a point closer to P, the slope of PQ_2 will more closely approximate the slope of a line drawn tangent to the curve at P in figure 2 below. In fact, the closer Q is to P, the better this approximation becomes. It is not possible to allow Q to coincide with P, for then it would not be possible to define the slope of PQ in terms of two points. The slope of the tangent line, often referred to as the slope of the curve, is the limiting value of the slope of PQ as Q approaches P.

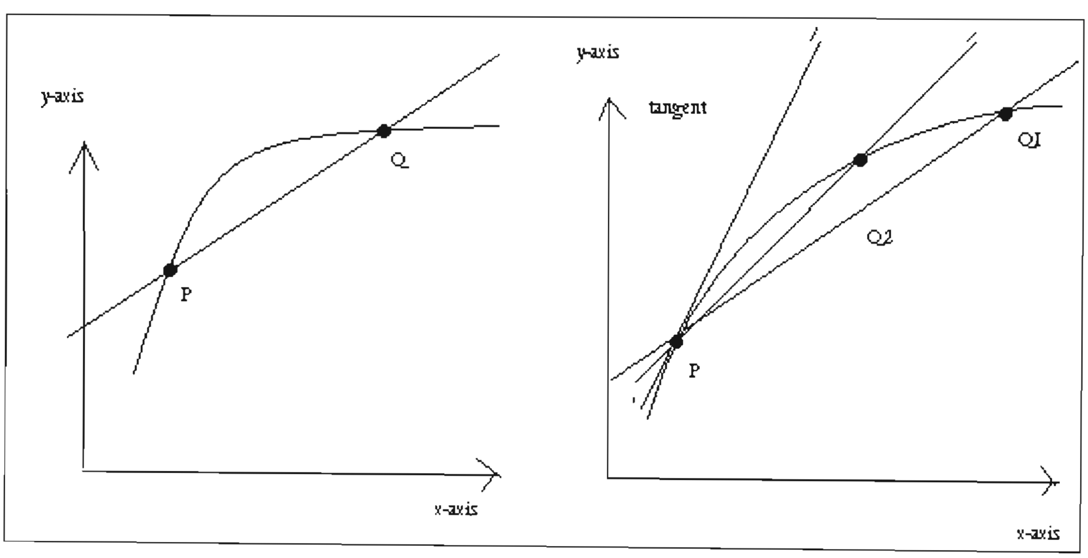


Fig. 2 Gradient of the graph of a function or curve

The derivative is defined as the limit of the ratios $\frac{\Delta y}{\Delta x}$ as $\Delta x \xrightarrow{approaches} 0$. Therefore the derivative is the gradient of the line tangent to the curve. Since the slopes of the secants form a Cauchy sequence the derivative exists and it is unique. The average rate of change is given by $\frac{\Delta y}{\Delta x}$ which is important in engineering applications such as material testing in a laboratory. The derivative is then a measure of the rate of change of y with respect to x at a point P which is the measure of the instantaneous rate of change, which is applied in engineering concept formulations. The notation $\frac{dy}{dx}$ is used for a derivative. This is given as:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\Delta y}{\Delta x}\end{aligned}$$

The concept image of the limit may evoke a mental frame of a chord (secant) tending to a tangent which is a form of a metaphor as described by Oerthman (2003). The metaphor is an integral part of the qualitative theoretical frame work used in this study. The Cauchy concept of the limit is employed where the limit is interpreted as a sequence of elements which is a well known theorem in Real Analysis. Hence the limit $\varepsilon-\delta$ formalism was not required. A well known theorem by Cauchy states that convergence implies uniqueness of the existence of the limit. The non-convergence of the sequence suggests the non-existence of the limit which would imply that the derivative does not exist. Tall & Vinner (1981) conjectured that if a student displays a concept image that does not allow $s_n = s$ *in* $s_n \to s$ then a student may not absorb a concocted example when it is presented to him.

A qualitative theoretical framework was constructed for the analysis of errors in differential calculus (Naidoo: 1998) who uses mainly the cognitive theory that regards mathematical thought processes as fundamental. The theory relates observations to a postulated theory of 'metaphoric' process with information of how the individual thinks about some mathematical problem.

The three types of errors were linked to the sequential processes, critique, frames and the deeper level procedures of Davis (1984) by Naidoo (1998). The learning of differentiation does not require verbatim repeating of verbal statements but the appropriate mental frames to represent the concepts and procedures of differentiation. The qualitative theoretical framework refers to the ways students are thinking with respect to the mathematical tasks. This necessitates that one has to get information from students whilst they are engaged in specific mathematical tasks. The frame theory includes metaphors, collages or chunks embedded in the frames. Engineers typically use metaphors, collages or chunks of cognition to explain design or mechanisms. Oertmans's (2003) research on metaphors used in the understanding of the derivative exhibit a particular aspect of the theoretical frame structure designed by Donaldson 1963.

Tall (1996) used the "local straightness" of the graph as his "good" cognitive root to build calculus. His Graphic Calculus software enabled the student to magnify a portion of the graph to observe the straightness by tracing the gradient numerically along the graph. Additional software allowed the student to point the mouse at a given place in the plane and draw a line segment of the given gradient. An approximate solution could be constructed physically and visually by sticking segments from end to end. This is a means of encouraging deep approaches to learning. The student is motivated further by adding reality to his/her solution. In this way the student's meaning can be extended to real-world problems that society needs solutions to.

The zoom graph approximates the curve to a straight line. When the domain intervals are made very small the curve can be approximated as a straight line. Instead of using secants we zoom to get a straight line. Here we establish the idea of the gradient of a curved graph. Using *Mathematica* a graph can be drawn and a part of the graph can be selected and magnified. The magnified part looks "straight". This method frees the student from cognitive overload. The student does not have to deal with tangents, secants and complex geometry. Tall (2002) agrees that calculus software should be programmed to assist the user to explore graphs with corners and wrinkled graphs. Fig. 3 (adapted from Visual Calculus software programmed by Teresinha Kawasaki) shows how computer software can be used to zoom over a small interval on a curve. The rate of change can be found from both directions.

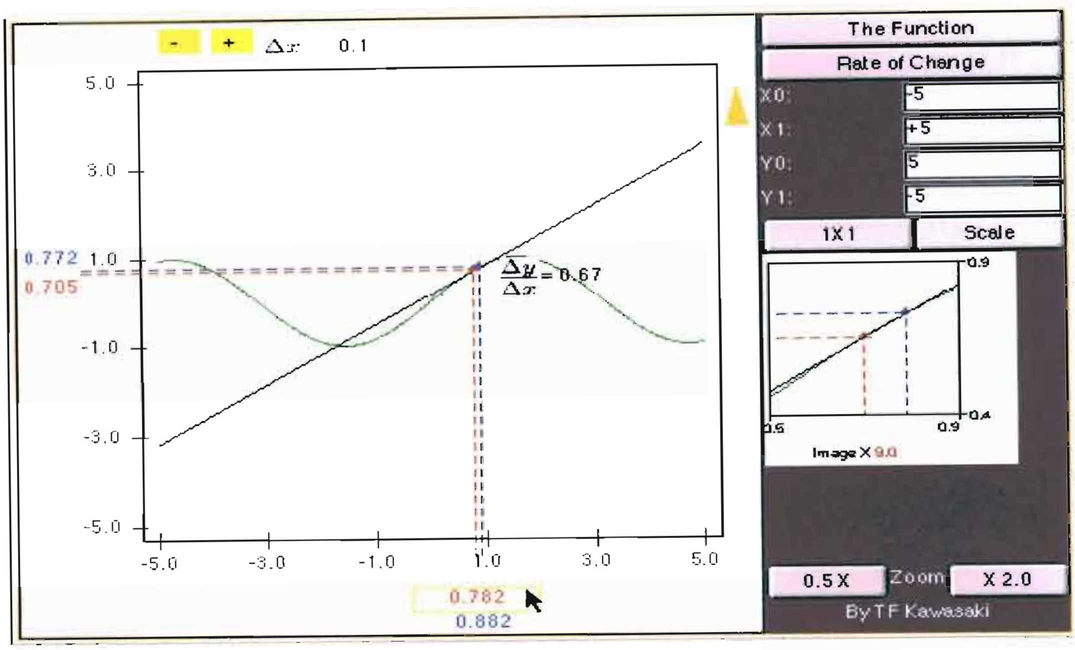


Fig. 3 Zoom Graph

It maybe easier for students to explore the derivative using Scaling or Zooming: Below is an exhibition of the concept.

Consider a function $f(x) = y$. Let $y = x^2$. We may want more detail to see how the curve touches its tangent line or we may want a big picture to check on asymptotes. The axes can be scaled as follows:

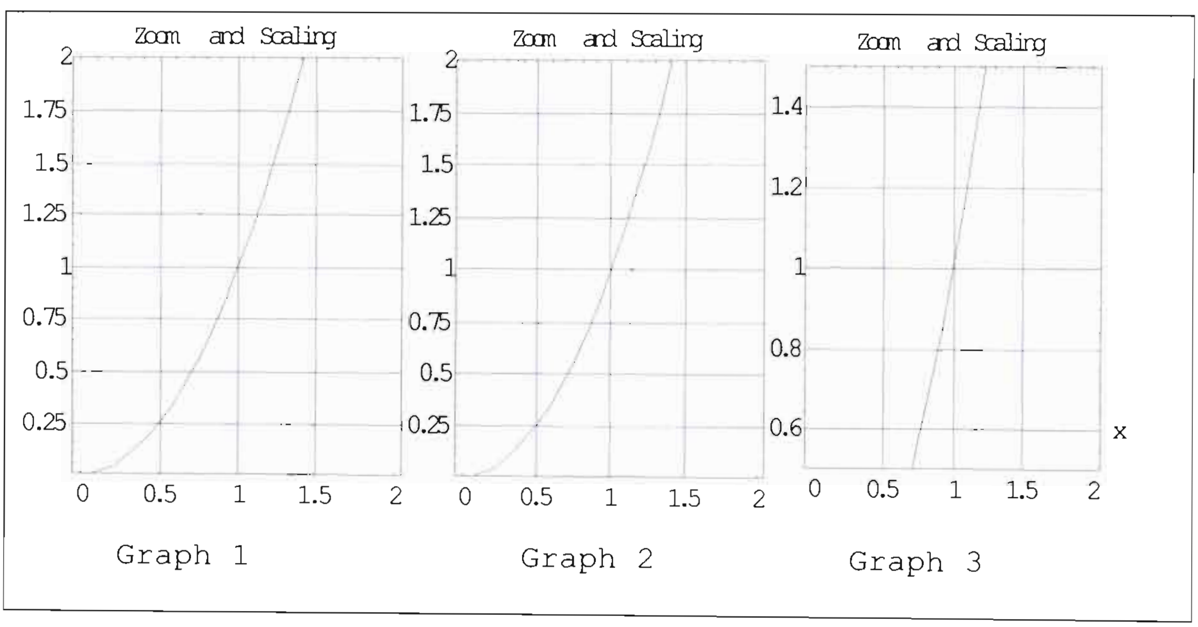
Stretch the horizontal axes by C: The new x is CX =x

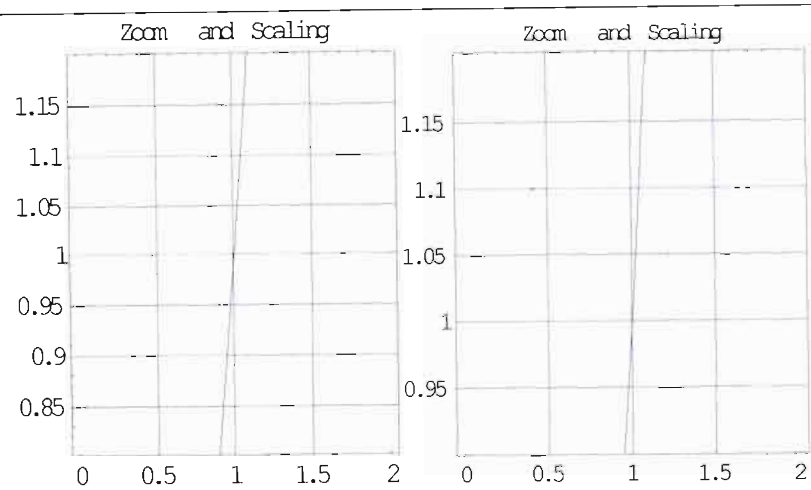
Stretch the vertical axes by D: The new y is DY=y

Therefore $f(X) = Y$ becomes $y = D\left[f\left(\frac{x}{C}\right)\right]$. If we want to magnify the graph at a point ten times we take C=D=10 which gives a Zoom Transform. To determine the slope of the

Zoom Transform we may use the chain rule to get $y' = \frac{D}{C}\left[f'\left(\frac{x}{C}\right)\right]$. This means the derivative or slope is multiplied by $\frac{D}{C}$. If we let D=C the slope remains the same. The following graphs exhibit the calculation of derivatives using Zoom or rescaling: The following plot commands can be used in *Mathematica* to generate different scaled/zoomed graphs. As an exemplar we chose a simple quadratic function $y = x^2$ to demonstrate the Zoom function. The slope was approximated at $x = 1$. The following is the *mathematica* command:

The aspect ratio represents the Zoom transform and the PlotRange was used to approach the point $(1,1)$ as close as possible.





Graph 4 Graph 5

Fig. 4 Demonstration of Zoom Function

Graph	1	2	3	4	5
Ratio= $\frac{\Delta y}{\Delta x}$	$\frac{1}{0.4}=2.5$	$\frac{1}{0.4}=2.5$	$\frac{0.5}{0.22}=2.27$	$\frac{0.2}{0.1}=2$	$\frac{0.2}{0.1}=2$

Table 2 Zoom Graphs with corresponding gradients

Graphs 4 and 5 visually seem to be a straight line and therefore represent the tangent at the point. Table 1 indicates that as we zoom closer to the point where the derivative may be determined if the Cauchy sequence exists. It can be easily verified by taking the derivative $\frac{dy}{dx} = 2x\big|_{x=1} = 2$. Hence the Zoom function method in determining the derivative is simpler and faster to calculate the derivative than the "secant becoming the tangent" method.

CHAPTER: 3 METHODOLOGY

3.1 DATA COLLECTION PLAN

Data collection was performed in the Engineering Science and Built Environment Faculty at the Durban University of Technology.

Four staff members from the institution assisted in the orientation of the experimental group to the *Mathematica* software. This was done over three weeks in two hourly sessions weekly. A two-tier design, combining qualitative methods (control group) in an exploratory phase and quantitative methods (task on computer) in a more focused learning environment, was used. We made complementary use of the qualitative and quantitative methods (Punch: 1998). The quantitative data would give an indication of student errors and the qualitative data would give more meaning to how students' think during their interaction. Qualitative methods are used in the study of human behaviour and behaviour changes (Stevens: 2003). This study wanted to find out the errors students were making and why they were making these errors.

After determining that there was reasonable competency with syntax and other computer related aspects, the students were asked to complete the compulsory mathematics project from the Department of Mathematics at the University of Technology. This task was also part of their course fulfillment requirements. This also ensured that students would find the experience beneficial in that it also contributed to their course mark.

The mathematical laboratory type of intervention in which students could experiment and test mathematical knowledge was particularly suitable to link numeric, symbolic and graphical computations (Wolfram: 1991). The use of computers would ensure that the student would be able to work at his/her own pace (Cotton: 1995).

3.2 PROJECT WORK

According to Vithal (2004) projects or project work form a "progressive" approach to mathematics education and advocates more "open-ended", "problem-centered" activities in which learners are given greater independence in their learning, in contexts relevant to them. In terms of the outcomes based approach to learning, project work is extensively used as an assessment strategy in modern South African Schools. Not much research exists to test the effect and use of project work. In countries like Scandinavia and Denmark project work had been introduced for decades.

In this study only the experimental group was given a project to do. Students had to perform the task in groups at the mathematics laboratory using *Mathematica*. The aim of the tasks was to assist students to understand the elementary concepts in calculus. The tasks also contributed to their assessment for the mathematics module MATH101. This would also ensure that the participants in the experimental group would carry out the project tasks meaningfully.

The following are the project tasks and discussion. These tasks were used to determine errors, deep, intermediate and surface thinking.

3.3 PROJECT TASKS

Use *Mathematica* to solve the following problems. Please show clear programming techniques and explain your answers fully. Show commands, numerical tables and graphs.

TASK A

Find the limit of the following numerically and graphically. Discuss your results. For the numeric values show explicitly whether the sequence is converging or diverging.

$$\lim_{x \to 4} \frac{2-\sqrt{x}}{4-x}$$

DISCUSSION:

This task tested the understanding of the limit as a converging sequence. Students were exposed to calculating the limit from the left and the limit from the right. The graphical solution using *Mathematica* would give them an indication that the sequence of values of x as it approaches 4 converged to a particular value. The numerical solution can be used to identify the converging value.

TASK B

Let $f(x) = 3x - 2x^2$

2.1 Find the average rate of change of $f(x)$ from $x = 0.5$ to $x = 0.9$

2.2 Find the equation of the corresponding secant line.

2.3 Plot the graphs of *f(x)* and the secant line on the same axes.

2.4 Repeat parts 2.1 to 2.3 for $x = 0.5$ and $x = 0.51$. Explain what you observe.

2.5 Zoom in on the graph around the point $(0.5; f(0.5))$. Show your plot and explain what you observe about the two graphs in 2.4.

2.6 Re-plot the graph *f(x)* over the interval [0; 1]. Now zoom in on the graph around the point $(0.5; f(0.5))$ until the graph looks like a straight line. Show your plot and explain how you can use this graph to estimate the slope of this line. (Hint: Move the mouse pointer to the line and click at two different points on it; then observe the first and second coordinates of the points you clicked on.)

DISCUSSION:

Students' had to use the frame 'average rate of change' and 'slopes' to solve this problem. The frame 'straight line' had to be used to obtain the equation of the secant line. A visual representation of the graphs would enable students to see, that as the interval between the corresponding x- values were made smaller, then the secant became a tangent.

TASK C

Let $f(x) = 3x - 2x^2$

3.1 Find the instantaneous rate of change of *f(x)* at x =0.5 using the definition of the derivative.

3.2 Find the equation of the corresponding tangent line.

3.3 Plot the graphs of *f(x)* and the tangent line on the same system of axes. Zoom in on the graph around the point (0.5; f (0.5)) until the two graphs are indistinguishable. How close did you have to get?

3.4 Evaluate $\dfrac{f(0.5+h)-f(0.5)}{h}$

Explain how you can use this to estimate the derivative of *f(x)* at 0.5 from the graph.

DISCUSSION:

This task needed students to retrieve the frames 'tangent lines' and 'rates of change'. Students would have to apply the fact that the instantaneous rate of change is the limit of the average rate of change of f as the width of the interval x tends to zero. The frame '$\lim_{h\to 0}\dfrac{f(0.5+h)-f(0.5)}{h}$' had to be retrieved.

3.4 THE QUALITATIVE THEORETICAL FRAMEWORK

These modified Orton Tasks are used extensively in the mathematics syllabus. The object of this study was to determine the errors engineering students make in coming to understand the derivative. Furthermore the derivative is highly contextualized within the engineering disciplines and therefore requires tasks such as elementary algebra, limits and infinity, average rates of change, rates of change at a point and a high emphasis on graphics or curves. We distinguish between average rates of change and rate of change at a point as they represent two separate concepts in engineering. Average rate of change is applied extensively in the engineering laboratories. Zandieh (2000) constructed an object-process qualitative framework, where the derivative can be taken as velocity or acceleration. There are literally hundreds of engineering -derivative -derived concepts such as velocity, acceleration, stress ($\frac{df}{dA}$), fluid flow ($\frac{dV}{dt}$), current flow ($\frac{dq}{dt}$) etc which are derived from first principles. Using all these derived concepts in the qualitative frame is impracticable. Expert engineers derive these derivatives using algebra, limits and infinity, average rate of change and rates of change at a point. Furthermore expert engineers apply the abstract (definition of the derivative) first before concretization (velocity, acceleration, current, fluid flow etc).

Using the clinical method (using verbal and written responses), responses to the tasks was elicited. The focus on this study was on identifying errors and the types of errors engineering students were making. If there were more than one type of error in a task both errors were reported. If the error could not be easily categorized responses from two experts were sought to give their opinion. The tasks on differentiation were listed and discussed as to relevance and type of the frame retrieved. The tasks were then itemized according to required skills and concepts. There were 8 items, listed in Table: 3.

The testing instrument used consisted of a battery of tests (Orton, 1983). Naidoo (1998) used the same modified tasks for his data collection. The experimental group and control group were asked to do the battery of tests, based on basic concepts in differential calculus. The students' scores were then graded.

These tasks were grouped into eight items each accounting for a score according to a marking scheme designed by Orton (1983).

Item No	Description
1	Infinite geometric sequence
2	Limit of geometric sequence
3	Rate of change from straight line graph
4	Rate, average rate and instantaneous rate
5	Average rate of change from curve
6	Carrying out differentiation
7	Differentiation as a limit
8	Use of delta symbolism

Table: 3 Item No and Description

3.5 THE SUBJECTS

The experimental group consisted of 34 students from the Faculty of Engineering and Built Environment. The control group also consisted of 34 students also from the Faculty of Engineering and Built Environment. Both groups were randomly chosen and represented students with mixed abilities. The first group was the control group and the second group the experimental group. The sample size was determined by the number of students in that particular class group. It represented a convenience sample (Rose: 1991). In each group the students needed to achieve an E symbol on higher grade or D symbol on standard grade to gain access into the Engineering programme. The average symbol for both groups was a D on the standard grade. Both groups were made up of heterogeneous students who accounted for learners from all race groups. Selection of the number of female and male students was purely determined by the class groupings as determined by the University structures. All students have studied calculus at secondary school as part of their Mathematics algebra component. Data collection was done during the end of the first semester at the University of Technology.

The instruments used for the data collection included: the compulsory mathematics project and Orton's battery of tests. It was felt that multiple ways should be used to collect the data so that during data analysis the researcher would have adequate material to refer to when drawing inferences.

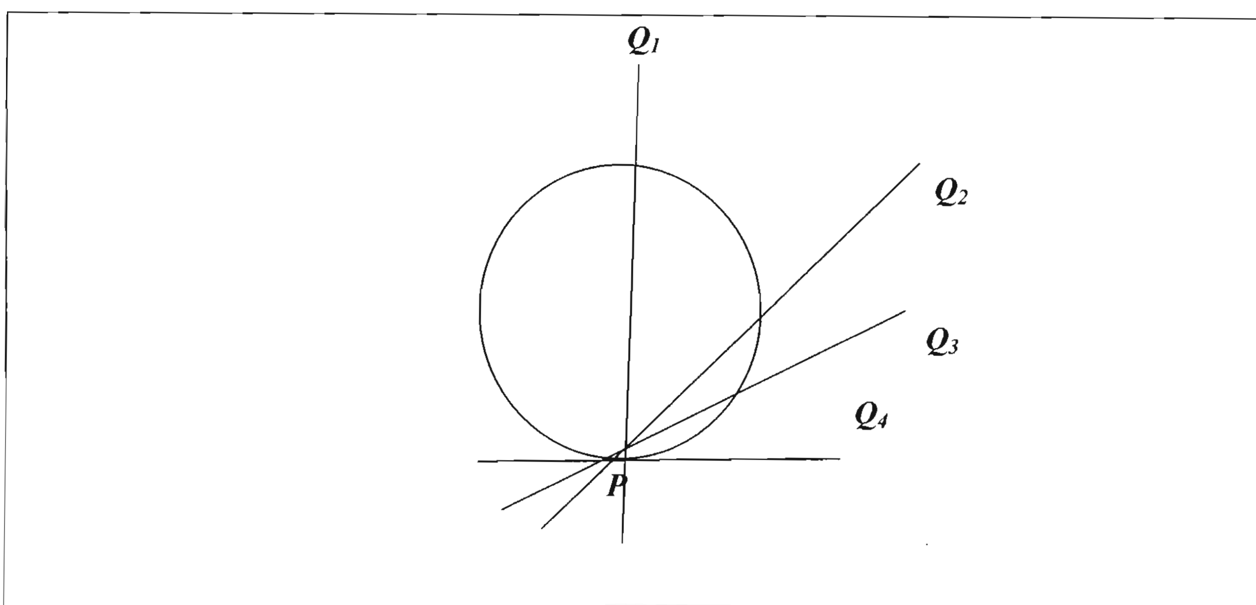
The pilot study consisted of the questionnaire designed by the researcher. Responses were audio recorded so that the researcher could see if students really understood the compulsory project that they completed for their course requirement. The questions were modified and applied to the experimental group.

3.6 THE TASKS

The tasks were selected and modified from Orton's instruments for the understanding of differentiation (Orton: 1983). Below are the actual tasks that were used for the study to determine the differences between the experimental and control group in terms of errors, deep structures, intermediate structures and surface structures and forward, intermediate and backward inferences.

TASK 1

The diagram shows a circle and a fixed point P on the circle. Secant lines PQ are drawn from P to points Q on the circle and are extended in both directions.



1.1 How many different secants could be drawn in addition to the ones already in the diagram?

1.2 As Q gets closer and closer to P, what happens to the secant?

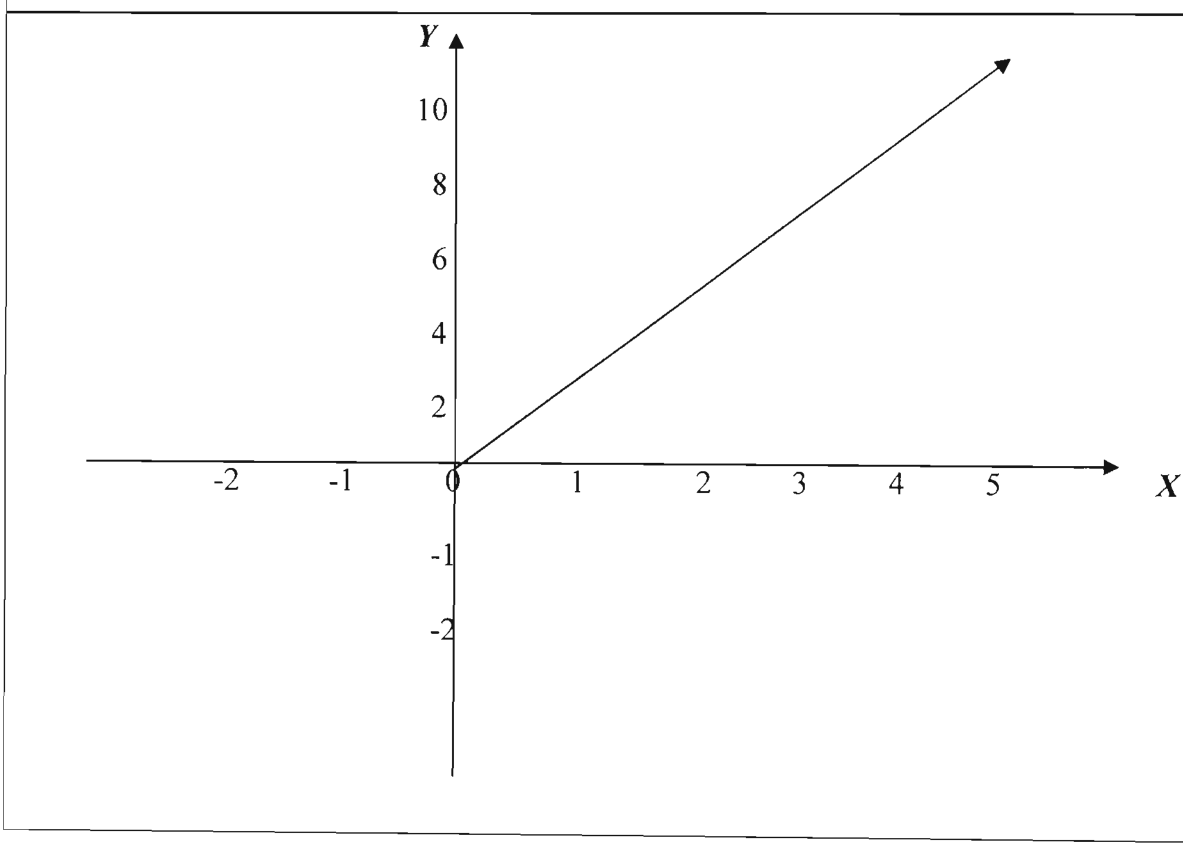
DISCUSSION

The purpose of the task was to determine if students could perceive that as the moving point approached the fixed point, the secant approaches the tangent at the fixed point. Consequently the slope of the tangent at a fixed point can be considered as the limit of the sequence of slopes through the same fixed point. The frame to be retrieved involved 'a secant cutting two points on a curve' and 'a sequence of secants through ' and P with Q approaching P'. The student could synthesize the above frames welded into a single frame or construct each frame from assemblies.

TASK 2

Water is flowing into a tank at a constant rate, such that for each unit increase in time the depth of the water increases by two units. The graph illustrates the situation.

Time *(x)*	0	1	2	3	4	5
Depth *(y)*	0	2	4	6	8	10
1st difference (depth)		2	2	2	2	2



2.1 What is the rate of increase in the depth when $x = 2\frac{1}{2}$?

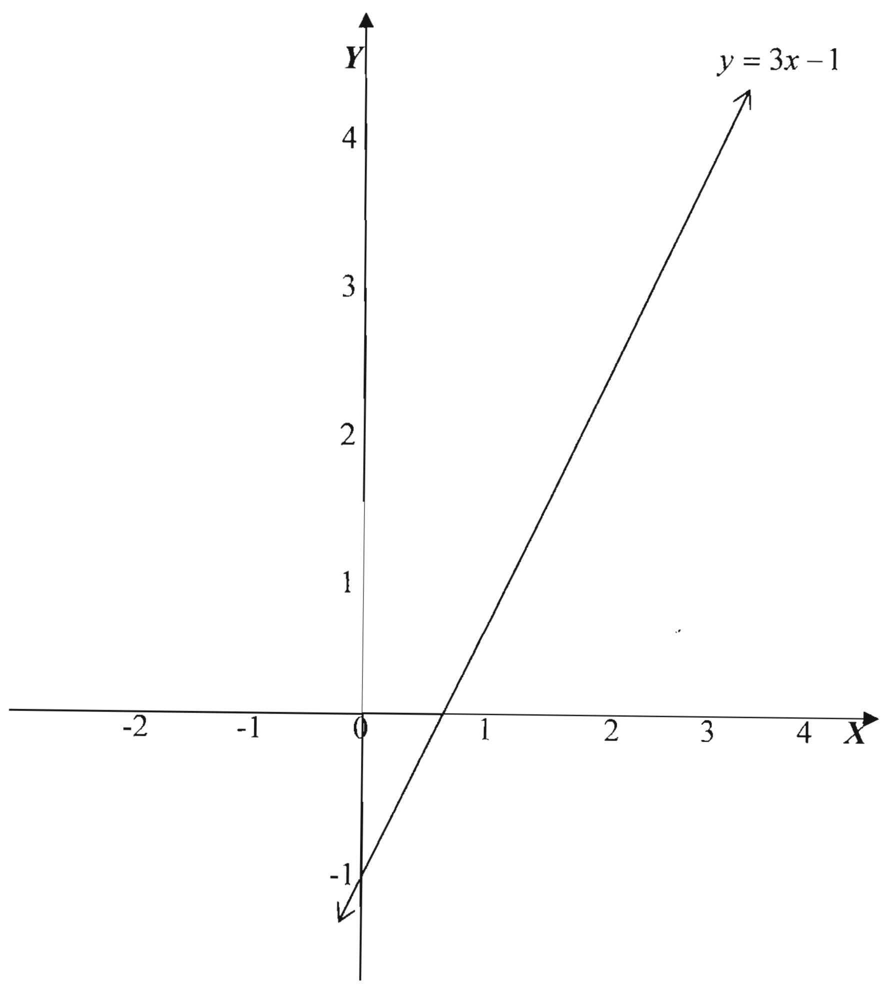
2.2 What is the rate of increase in the depth when $x = T$?

DISCUSSION

Questions on rate of change were based on the same graphical situation. The student must retrieve the frame 'a tank being filled with water', 'a straight line graph with gradient 2', and 'rate of change equal to gradient'. This task was based on a real world problem. The 'tank being filled' can be taken as a pre-mathematical frame or collages for the synthesized frames. The procedure of the frame is to see that the constant rate relates to a straight line graph and every point on the X-axis gives the same rate of change.

TASK 3

The graph below represents $y = 3x - 1$



3.1 What is the value of y when $x = a$? [a is a real number]

3.2 What is the value of y when $x = a + h$? [h is any increment]

3.3 What is the increase in y as x increases from a to $a = h$?

3.4 What is the rate of increase of y as x increases from a to $a + h$?

3.5 What is the rate of increase of y at $x = 2\frac{1}{2}$?

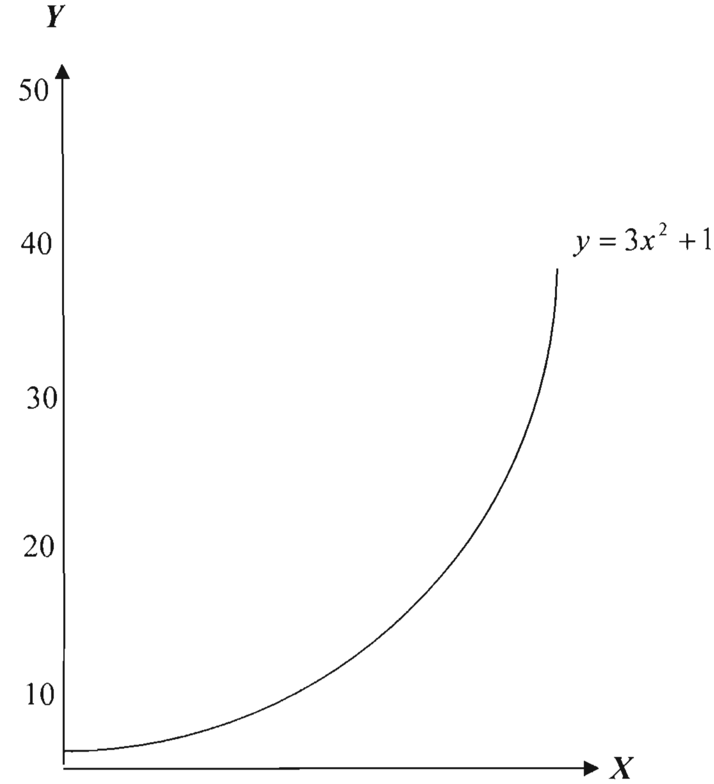
and at $x = X$?

DISCUSSION

Both Task 3 and Task 2 included questions on the theme of rate. This task also required the student to retrieve similar frames as the previous task. The previous task is usually found in engineering courses. The frame required inputs for a function $y = f(x)$, change of y, change of x, rate of change = $\frac{\Delta y}{\Delta x}$ and this is constant throughout the X-axis.

TASK 4

The graph below represents $y = 3x^2 + 1$, from $x = 0$ to $x = 4$



4.1 What is the value of y when $x = a$? [a is any real number]

4.2 What is the value of y when $x = a + h$? [h is any increment on the x-axis]

4.3 What is the change in y as x increases from a to $a + h$?

4.4 What is the average rate of change in the x-interval a to $a + h$?

4.5 Can you use the result in (4.4) to obtain the rate of change of y at $x = 2\frac{1}{2}$?

At $x = T$? If so, how?

DISCUSSION

This task complemented the previous one. A similar graph was presented but with a different function. A frame of a curved graph was sought indicating different tangent points. This task aimed at extracting information concerning students' capabilities and understanding relating to rate of change based on graphs. The required frames are similar to the previous task except that the input function is a quadratic and the average rate of change is now $\frac{\Delta y}{\Delta x}$. In the linear graph the rate of change is the same as the average rate of change. Using a super-procedure within the frame, $\lim_{h \to 0} \frac{\Delta y}{\Delta x}$ = the rate of change, the student will be able to determine the rate of change at $x = 2\frac{1}{2}$ and at $x = T$. The sub-procedures involve determining Δy, Δx and the limit. These sub-procedures can also be taken as assemblies.

TASK 5

5.1 What is the formula for the rate of change for the equation $y = x^n$?

[n is an element of the natural numbers]

5.2 What is the rate of change formula for each of the following equations?

a) $y = 3x^3$?

b) $y = 4$?

c) $y = \frac{2}{x^2}$?

DISCUSSION

These tasks required students to retrieve the rules for differentiation frame. Both tasks were typical problems found in mathematics at first year level at the Durban University of Technology.

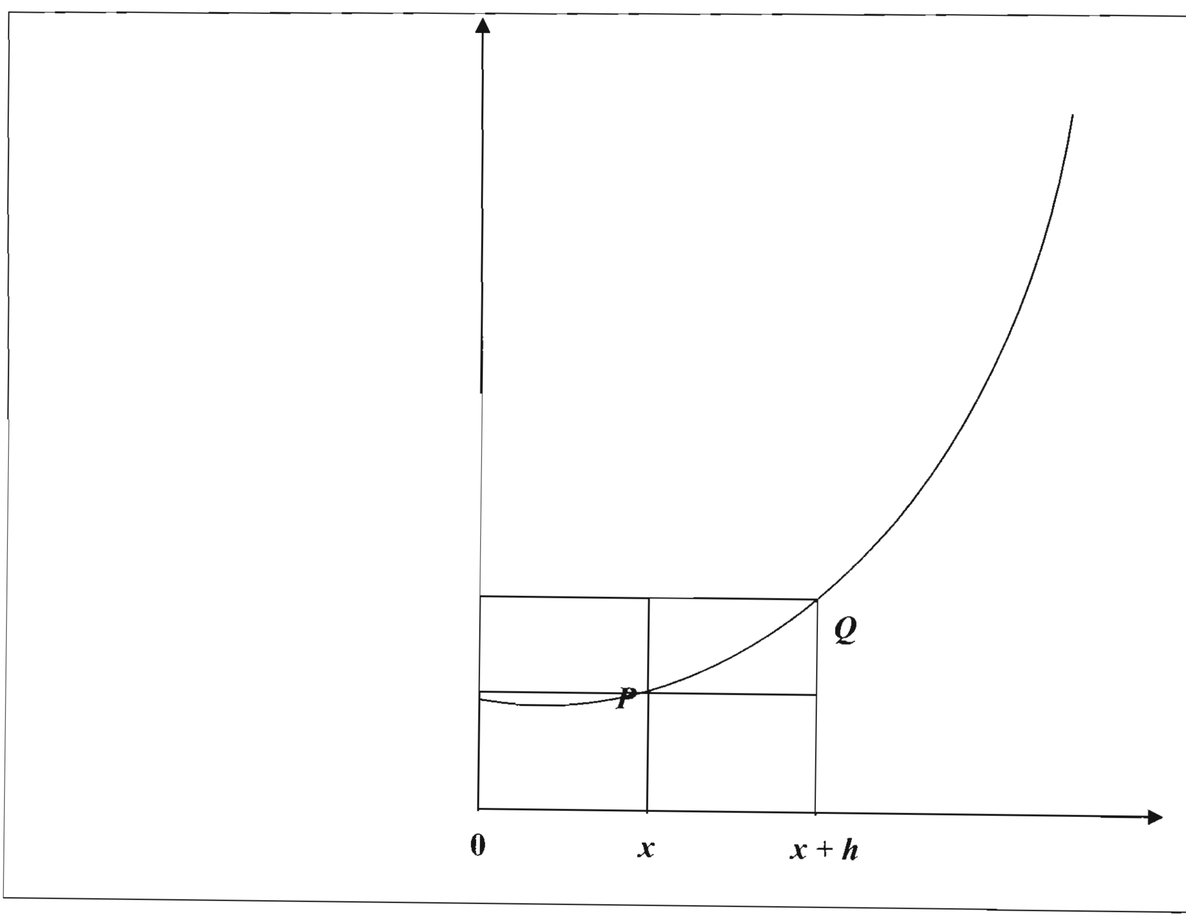
TASK 6

The diagram below is used to introduce the definition of the derivative, viz.

$\frac{dy}{dx} = = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h}$ in engineering mathematics, where y is any function and h is an increment in x.

6.1 At which point or points of the graph does the formula measure the rate of change?

6.2 Explain why the formula defines this rate of change?

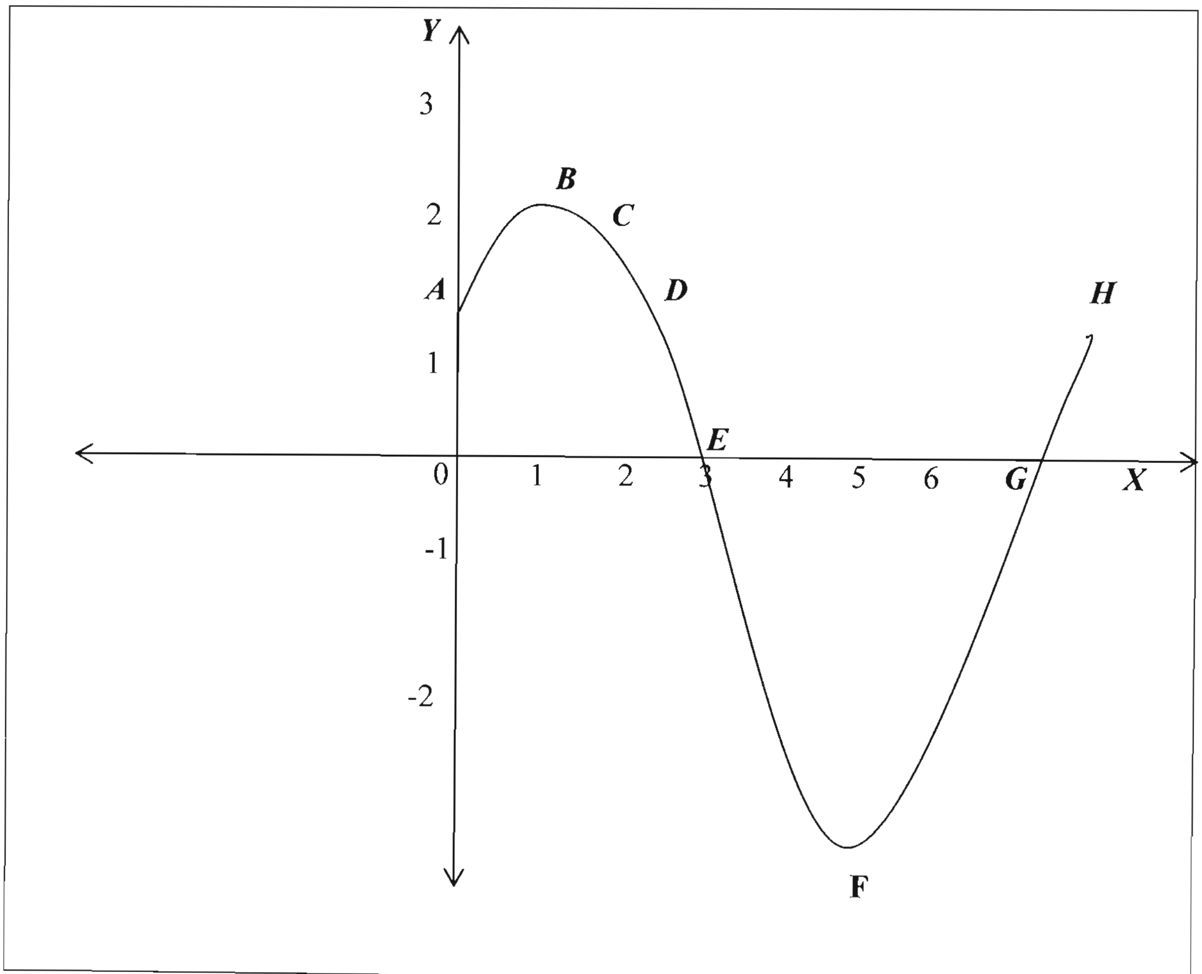


DISCUSSION

The frame to be retrieved could be the sequential secant tending towards a tangent to the curve at a point and the slope of the tangent is a representation of a rate of change at that point. Further the instantaneous rate of change represents the derivative at that instant.

TASK 7

The graph of y for a certain equation, for $x = 0$ to $x = 6$ is shown.



What is the average rate of change of y with respect to x?

7.1 From A to B?

7.2 From B to E?

7.3 From F to H?

DISCUSSION

The purpose of this task was to introduce the idea of rate of change in the sinusoidal curve, which is often encountered by students in engineering. The frame to be retrieved is that the average rate of change can be calculated from any two points irrespective of the curve.

TASK 8

Explain the meaning of the following symbols:-

8.1 δx,

8.2 δy,

8.3 $\dfrac{\delta y}{\delta x}$

8.4 dx

8.5 dy

8.6 $\dfrac{dy}{dx}$

8.7 What is the relationship between $\delta y/\delta x$ and dy/dx?

DISCUSSION

This task probed the understanding of the various symbols used in connection with differentiation. The frame retrieved gives meaning to each symbol and the relationship between the symbols and related concepts such as differentiation or limits or average rate of change.

3.7 THE ITEMS AND THE TASKS

Tasks were regrouped to form items (Orton, 1983). Each item represented one aspect of elementary differential calculus. The item number, item description and related tasks are given in Table: 4

Item No	Item Description	Related Tasks
1	Infinite geometric sequences	1.1
2	Limits of geometric sequence	1.2
3	Rate of change from straight line graph	2; 3.5
4	Rate, average rate and instantaneous rate	3.4; 4.4 4.5
5	Average rate of change from curve	7.1; 7.2; 7.3
6	Carrying out differentiation	5.1; 5.2
7	Differentiation as a limit	6.1; 6.2
8	Use of δ - symbolism	8.1; 8.2; 8.3; 8.4; 8.5; 8.6; 8.7

Table: 4 Item Description and Related Tasks

3.8 THE ITEMS AND THE SCORING PROCEDURE

Scrutiny of the students' protocols suggested a five-point scale be used to assess responses to the items given. A score of 4 was given for a response that was judged nearly correct as one would expect to achieve after a study of elementary calculus.

A score of 0 was given for no response or for an incorrect attempt. Criteria were defined for the scores for each item by noting common levels of the responses of the 66 subjects. A provisional rating scale was prepared and scores were tabulated.

The criteria for the scores were amended where deficiencies had been observed and the revised scales were used to obtain the table of scores.

The grading procedure for the items also took into consideration the following:

- equivalent answers or methods were accepted
- correct answers were given full credit
- understanding of a method was the main criterion used rather than penalizing for carelessness

ITEM 1: INFINITE GEOMETRIC SEQUENCES

Only one question constituted this item, based on the idea of "How many?"

Task 1 (1.1), "How many different secants could be drawn, in addition to the ones already in the diagram?

Answers: "An infinite number", or "No limit to the number", or "You could go on for ever", or equivalent were accepted without explanation. Vague answers like, "innumerable", "Any number", "As many as you like", "Almost infinite", "You can't say because there are too many" were not accepted.

Criteria for levels of response:

4: Answer correct

0: Answer incorrect

ITEM 2: LIMIT OF A GEOMETRIC SEQUENCE

Only one question, Task 1 (1.2), "As Q gets closer and closer to P, what happens to the secant?

Answers: Only "The secant becomes a tangent", or "a tangent is formed" were accepted.

Criteria for levels of response:

4: Answer correct

0: Answer incorrect

ITEM 3: RATE OF CHANGE FROM STRAIGHT LINE GRAPH

This item was based on Task 2 and Task 3 (3.5). There were four numerical answers and explanations were not required.

Answers:	Task 2:	2 and 2
	Task 3 (3.5)	3 and 3

Criteria for levels of response:

4: All four answers correct

3: Three answers correct

2: Two answers correct

1: One answer correct

0: No answer correct

ITEM 4: RATE, AVERAGE RATE AND INSTANTANEOUS RATE

The items for this task were Task 3 (3.4) and Task 4 (4.4) and (4.5). Item 3 was concerned only with straight lines; item 4 involved similar questions but led to rate of change at an instant and introduced the complication of a curve rather than a straight line.

Answers:

3.4 3

4.4 $6a + h$

4.5 "yes, put $a = 2\ ½$ and $h = 0$ in the answer to 4.4; no further explanation needed.

Criteria for levels of response:

4: All three parts fully correct and

3: All three parts fully correct but without $a = 2\ ½$ and $h = 0$

2: Two parts ultimately correct

1: One part ultimately correct

0: No parts correct.

ITEM 5: AVERAGE RATE OF CHANGE FROM CURVE

Task 7 was used for this item. Explanations were not required. Responses were assessed according to a simple marking scheme.

Answers		Coding Scheme
7.1	1	1 point
7.2	- 3	1 point for -, 1 point for 3
7.3	0	1 point

Criteria for levels of response:

4: All four points obtained

3: Three points obtained

2: Two points obtained

1: One points obtained

0: No points obtained

ITEM 6: CARRYING OUT DIFFERENTIATION

Task 5 constituted this item. Only answers were required.

Answers:

5.1 nx^{n-1}

5.2 $9x^2$, 0 , $-4x^{-3}$

Criteria for levels of response:

4: All four answers correct

3: Three answers correct

2: Two answers correct

1: One answer correct

0: No answer correct

ITEM 7: DIFFERENTIATION AS A LIMIT

This item was based on task 6. Two responses were required, the second one being an explanation. The criteria for levels of response had to take into account the fact that some students could only answer part 6.1 correctly as a result of thinking about part 6.2.

Answers:

6.1 At P.

6.2 In essence, "k/h measures the gradient of the line PQ, the limit as h -> 0 implies Q -> P and in the limit Q coincides with P and the line has become a tangent at P, the formula therefore measures the gradient of the tangent at P."

Criteria for levels of response:

4: Answers correct with acceptable explanation including both k/h and h -> 0. Also acceptable was "Anywhere on the curve" or similar in 6.1 only if it was clear from 6.2 that the earlier response simply took account of the fact that P could have been chosen anywhere on the curve, though 6.2 must have bee correct before the complete response to task 6 could be accepted. Note that "All points P to Q", or similar was not acceptable.

3: Able to explain 6.2 correctly but 6.1 only corrected in the course of explaining 6.2. Also acceptable was "Anywhere on the curve" if 6.2 was correct but it was apparent that there was some confusion over 6.1. Also acceptable in this category was "very near P" in 6.1 if 6.2 was correctly explained.

2: Part 6.1 correct, though perhaps not immediately; some progress in 6.2 but only partial explanation achieved.

1: Either part 6.1 correct but no acceptable progress in 6.2,
or part 6.1 incorrect and part 6.2 partially answered as for level 2.

0: Neither part answered correctly.

ITEM 8: USE OF δ- SYMBOLISM

Task 8 was used for this item. Responses were assessed by using a marking scheme which put particular emphasis on the ability to explain $\delta y/\delta x$ and dy/dx.

Answers		Coding Scheme
8.1	A small x-increment	1 point if (i) or both (i) and (ii) correct
8.2	A small y-increment	
8.3	The y-increment/x-increment, or answer given in terms of rate of change.	1 point

8.4 Not usually meaningful but may be thought of as "with respect to x".

8.5 Not usually meaningful but may be thought of as "with respect to y". 1 point if either or both correct

8.6 Derivative or rate of change of a function y with respect to x, or gradient at a point, (answer may have been given as $\lim_{\delta x \to 0} \delta y / \delta x$) 1 point

8.7 $dy / dx = \lim_{\delta x \to 0} \delta y / \delta x$ 1 point for lim $\delta y/\delta x$,

1 point for $\delta x \rightarrow 0$.

(or more informal statements of the same e.g. dy/dx is $\delta y/\delta x$ as $\delta x \rightarrow 0$

Criteria for levels of response:

4: All six points obtained.

3: Five points obtained.

2: Three or four points obtained.

1: One or two points obtained.

0: No points obtained.

The above criteria acknowledge that from inspection of the protocols few students had one or exactly three points. Again, few students scored one point or three points, so it seemed appropriate to arrange criteria based on six, five, four, two and zero points.

Scoring was done for the modified Orton's task only. A graphical representation of scores for both groups is shown in Figure 6 and Figure 7 (Chapter 4).

CHAPTER 4

4.1 ANALYSIS OF DATA

Six sections of elementary calculus were considered: sequences, limits and infinity, rate of change, average rate of change, differentiation and δ - symbolism to classify the errors made by students.

Further analysis was done to find out the strategy used by the students with respect to the use of deep and surface structures to relate to the tasks presented to them. One item was also analyzed to find out the type of problem-solving strategy used by the students.
Table 4 represents the classification for the errors in the various items used in the instrument.

The first is the structural error which arises from some failure to appreciate the relationships involved in a problem or group of principles essential to the solution of the problem. These errors occur with the deductive mode when the subject reasons deductively but fallaciously. It is caused by incorrect frame retrieval, sketchy or incomplete frames, deep-level procedures and sub-procedures.

The second type of error is the executive error. These errors occur when there is a failure to carry out manipulations, although the principles may have been understood. Some defect of concentration, attention or immediate memory lie at their origin. A correct frame may be retrieved but a sub-frame responsible for calculations may be underdeveloped.

The third type of error is the arbitrary error. Arbitrary errors are those in which the subject behaves arbitrarily and fails to take account of the constraints laid down in what was given. These are errors which have as their outstanding common feature a lack of loyalty to the given. Sometimes the subject appears to be constrained by knowledge of what is 'true' by some considerations drawn from 'real- life' experience. Sometimes there is no constraint of any kind.

Classification of items	Structural errors		Executive Errors		Arbitrary Errors	
	Experimental	Control	Experimental	Control	Experimental	Control
Sequence	26 (79 %)	32 (96 %)	2 (6 %)	1 (3 %)	0	0
Limit	19 (56 %)	25 (74%)	2 (6 %)	0	13 (38 %)	7 (21 %)
Average rate of change	8 (24 %)	24 (71 %)	3 (9 %)	3 (9 %)	2 (6 %)	2 (6 %)
Rate of change straight line	20 (60 %)	27 (81 %)	0	1 (3 %)	0	0
Rate of change straight curve	22 (67 %)	28 (84 %)	2 (6 %)	3 (9 %)	0	0
Derivative	4 (12 %)	11 (32 %)	9 (27 %)	11 (32 %)	0	2 (6%)
Symbolism	14 (42 %)	16 (48 %)	0	0	0	0

Table: 5 Classification of errors

4.2 ANALYSIS OF ERRORS

Item: 1 and Item: 2 were based on the limit of an infinite geometric sequence. The idea of the rotating secant was intended to relate to the approach to differentiation. This item would give evidence concerning the level of understanding of the tangent as a limit. 79 % of the experimental group failed to make the relationship. Table 4 shows that these errors were primarily structural errors. 6 % of the students from the experimental group made executive errors; they displayed a loss of hold of reasoning. According to Donaldson (1963), this results from a defect in concentration or attention.

A larger percentage of the control group, 96 %, displayed structural errors in this item. Table 5 shows the classification for this group. The frame ‘sequences’, ‘tangent line’ and ‘limit’ could not be retrieved. Vague answers like “as many as you want”, “as many as possible” and “many of them” were characteristic of the responses that were to vague to classify. There was no opportunity to gauge their understanding further. The required frames were sketchy and incomplete. Clearly students needed help in understanding the tangent as the limit of the set of secants.

This task was a sub-problem of Task 6. Comparing the responses from both the groups in each of these tasks revealed that there was a correlation between the poor performance in both the experimental group and the control group. This confirmed that an incomplete frame in one sub-frame would reflect incomplete in another related frame.

EXEMPLARS FROM EXPERIMENTAL GROUP

Item: 1 Task: 1 (1.1)

Structural error sequences:

"one." [Cannot relate secant to circle - structural error]

Executive error sequences:

"Becomes less because its angle decreases" [takes secant as angle in Semi-circle -executive error]

EXEMPLARS FROM CONTROL GROUP

Item: 2 Task: 1 (1.2)

Structural error sequences:

"I say it is converging because it is getting smaller and smaller." [cannot relate secant to circle, considers length of secant - structural error]

Executive error sequences:

"No secant, only a tangent can be drawn." [Loss of hold of reasoning – executive error]

Item: 3, was based on the rate of change from the straight line graph. Task 2 and Task 3.5 were grouped for this item. Students were informed that water was flowing into a tank at constant rate; the rate was given as 2 units of depth per unit of time. It was apparent that both groups did not grasp this meaning.

For Question 2.1, at $x = 2\ ½$, a large number of the subjects gave a response of $y = 5$, and not with the rate. At the general point $x = T$ in task 3.5, the responses were worse. Below are some exemplars. There was a significant misunderstanding between the rate of change and the y-value at that point. It is also possible that the students had no conception of rate of change at all. This is why they worked out the y-value, given the x-value. It could also be that they didn't read the question properly and just thought that was what was being asked – this often happens. Part of the problem may also be that students are procedural, and want to work with formulas, plugging values in, etc., and the only available formula for them is the "formula" suggested by the graph. How often are they expected to interpret & give meaning to what they are doing? It is also indictment, I think, of how we teach & often evaluate only rote procedures.

A fairly large amount of structural errors were recorded. Below are exemplars of such errors. This represented 60 % from the experimental group and 81 % from the control group. Clearly many students were unable to retrieve the frame 'a tank being filled with water', ' a straight line graph with gradient 2' and 'rate of change equal to gradient'. In particular the frame 'straight line graph' was incomplete. Within this frame the algebraic sub-frame was also not developed. This task represented a real world problem. Another explanation that could be afforded is that the students were not subject to real world problems during their lecture and tutorial sessions. These responses represent the experience of the students, a type of experience that is characterized by doing problems by "drill" or using the mechanistic approach. Tall (1992) mentioned that students have a preference for procedural methods. Students' relational and instrumental understanding, Skemp (1976) was tested here. A further consideration by De Villiers (1993) was that the traditional "theory first – applications late" approach had certainly not been successful. He agrees that the modelling approach is not easy and like anything in education provides no guarantee, but is certainly more educationally sound.

EXEMPLARS FROM EXPERIMENTAL GROUP

Item: 3 Task: 2 (2.1)

Structural error rate of change:

$y = 5$ [Considers range when domain is $2\frac{1}{2}$ - structural error]

EXEMPLARS FROM CONTROL GROUP

Item: 3 **Task: 3 (3.5)**

Executive error rate of change:

$$\frac{y\left(2\frac{1}{2}\right)-y(1)}{2\frac{1}{2}-1}=\frac{3\left(2\frac{1}{2}\right)-1-3(1)-1}{1\frac{1}{2}}=2$$

[Defect in concentration resulted in incorrect computation – executive error]

Item: 4 was based on the rate of change from a curve. Task 3 (3.4), Task 4(4.4) and Task 4(4.5) were grouped for this item. Item 3 was concerned only with straight lines; item 4 involved similar questions but led to rate of change at an instant and introduced the complication of a curve rather than a straight line. As compare to Item: 3, the students made more errors in this item. The experimental group made 67 % structural errors and the control group made 84 % structural errors. It would appear that the sub-procedures involve in determining Δy, Δx and the limit were lacking.

An interesting observation is that students made similar errors in both item 3 and item 4.

EXEMPLARS FROM CONTROL GROUP

Item: 4 **Task: 4 (4.4)**

Structural error and executive error rate of change from curve:

$$y=3(a+h)^2+1=\frac{3a\ \ +6ah+3h^2+1-3a^2-1}{h}=\frac{6ah+3h}{h}=\frac{h(6a+3)}{h}=6a+3$$

[Equates function to gradient, omits the limit - structural error]

[Writes *3h* instead of $3h^2$- executive error]

Item: 5 Task 7.1, 7.2 and 7.3 was used for this item. This item was based on the average rate of change from a curve. It demanded calculation of *y*-difference/*x*-difference to obtain average rates of change for a curve.

24 % of the experimental group and 71 % of the control group made structural errors. A greater percentage of students from the experimental group were able to retrieve the frame required for the solution of this task 'the average rate of change can be calculated from any two points irrespective of the curve'. This seems to indicate that their interaction with the computer may have reinforced this frame. The students from the control group were baffled. An interesting observation was that this is a typical real world problem encountered in engineering.

Figure 5 below shows the scores for both groups for this item.

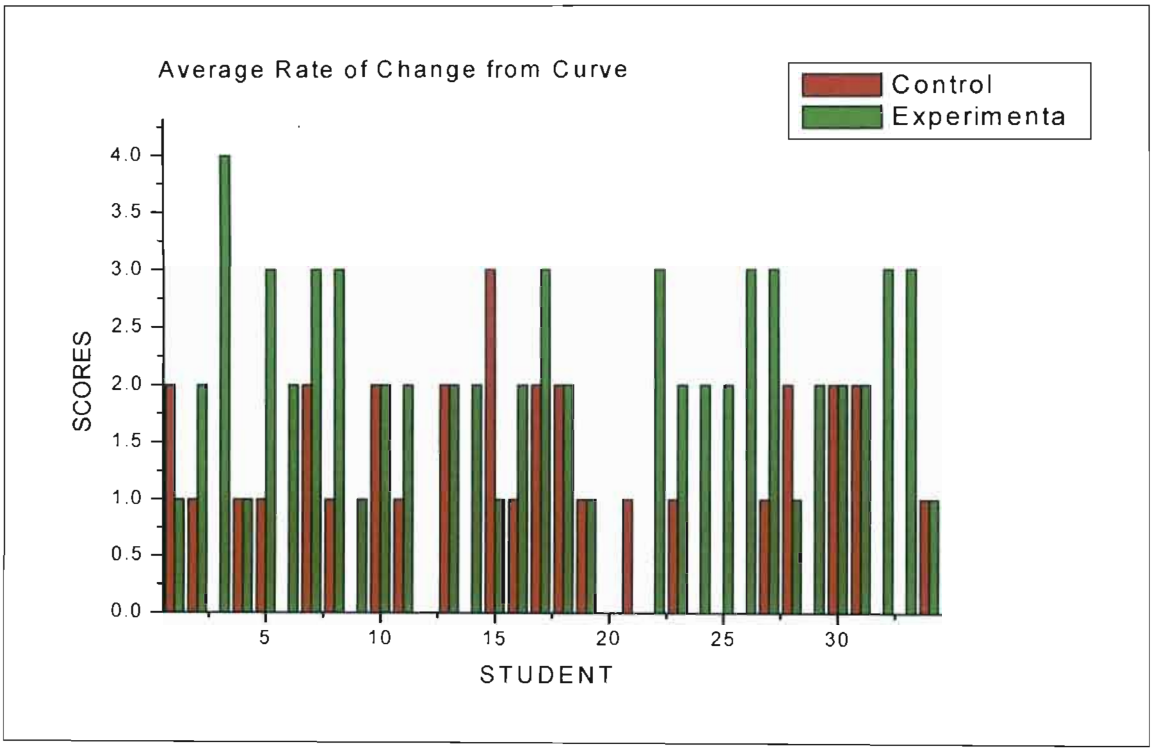


Fig. 5 **Average Rate of Change from Curve**

EXEMPLARS FROM EXPERIMENTAL GROUP

Task: 7 (7.1)

Executive error average rate of change:

Average rate = $\frac{\Delta y}{\Delta x} = \frac{5-0}{6-1} = 1$ [takes length OA and divides by B and point 1 on x –axis - executive error]

EXEMPLARS FROM CONTROL GROUP

Task: 7 (7.1)

Structural error and executive average rate of change:

$\Delta y = H - A$

[Takes rate of change to be Δy

– structural error]

[and substitutes H and A instead of the

y – values – executive error]

Task: 7 (7.2)

Arbitrary error average rate of change:

$$\frac{dy}{dx} = \frac{6-5}{1-0} = 1$$

[The Δy and Δx values have no relevance to

the co-ordinates of B and E

- arbitrary error]

Item: 6 dealt with differentiation. 12 % of the experimental group recorded structural errors and 32 % of the control group recorded structural errors. 26 % of the experimental group made executive errors and 32 % of the control group made executive errors. Students have lost track of the algorithm that they were trying to use. Davis (1984) refers to this as a control error. The student has memorized a rule he/she has been following or they behave in a certain way because they know from experience that this is an effective or appropriate way to tackle the problem.

The majority of the students were able to employ the mechanistic methods that were needed to solve the task. It is clear from the data that students have mastered the "rules" required to undertake this task. This confirms that frame 'rules for differentiation' were easily accessible to these students.

EXEMPLARS FROM EXPERIMENTAL GROUP

Task: 5 (5.1)

Structural error differentiation:

$y = x^n$ [log frame is surface and differentiation frame is surface

$y = n\log x$ evidence of rote learning – structural error]

Task: 5(5.2)

Structural error differentiation

$y = 3x^2$

$y' = 3\log x$ [log frame is surface and differentiation frame is surface

evidence of rote learning – structural error]

Task: 5(5.2)

Structural error and executive error differentiation

$y = \frac{2}{x^2} = 2x^{-2} = -4x^{-1}$ [writes function equal to derivative

- structural error]

[computes (-2-1) incorrectly

- executive error]

Task: 5.2 a

Executive error differentiation:

$y' = 3x^2$ [Failure to apply nx^{n-1} for differentiation

- executive error]

EXEMPLARS FROM CONTROL GROUP

Task: 5.1

Structural error differentiation:

$y = \frac{x^n}{x^{-n}}$ [Fallacious reasoning

- structural error]

Task: 5.2 a

Structural error differentiation:

$y = \frac{3x^3}{x^{-3}}$ [Fallacious reasoning

- structural error]

Task: 5.2 b

Structural error differentiation:

$y = \frac{4}{4^{-1}}$ [Fallacious reasoning

- structural error]

Task: 5.2 a

Structural and arbitrary error differentiation:

$y = 2x^2$ [Writes function as the derivative

- structural error]

[Writes *2* instead of *3* for *n* in nx^{n-1}

- arbitrary error]

Task: 5.2 a

Executive error differentiation:

$y' = 3x^2$ [Failure to carry out manipulation

- structural error]

Item: 7 was based on differentiation defined as a limit. 56 % of the experimental group made structural errors. A high percentage made arbitrary errors (38%). It is evident that these students did not understand the definition for the derivative. 74 % of the control group made structural errors and 21% made arbitrary errors. The percentage of arbitrary errors is less than that of the experimental group. This can be attributed to the fact that a single answer response was needed for this task and it became a problem to classify a wrong answer, like *Q*, for instance. The majority of the students were unable to retrieve the frame 'instantaneous rate of change'. The 'congruent motive-strategy package' described by Biggs (1986) is prevalent here. A larger percentage of the experimental group gave a correct response. They were able to show sound reasoning based on understanding.

EXEMPLARS FROM EXPERIMENTAL GROUP

Task: 6 (6.1)

Structural error differentiation as a limit:

"$x = h$

It includes both height and the time taken to reach that height"

[Fails to link to basic principle of problem - structural error]

Arbitrary error differentiation as a limit:

" P and Q " [student interpreted as P fixed or Q fixed - arbitrary error]

EXEMPLARS FROM CONTROL GROUP

Task: 6 (6.1)

Structural error differentiation as a limit:

" Point of inflection."

"Because it is when the graph is stationery." [Fails to grasp the basic principle of the problem - structural error]

Task: 6 (6.1)

Arbitrary error differentiation as a limit:

"At (x;y) and [(x+y);(y+k)]" [responds arbitrarily - arbitrary error]

"It is a curve graph."

Item: 7 was based on the use of symbolism. Task 8 was used for this item. The symbols that were given represented standard notation used in elementary calculus and those that must be understood by students. 42 % of the experimental group made structural errors and 48 % of the control group exhibited structural errors. It showed that a large percentage of the students were unable to connect the various symbols meaningfully. Clearly these symbols were confusing to both groups. These may not have been explained adequately in the lectures or the frame 'symbolic images' is lacking in both groups. A number of students were able to say that δx and δy represented small increments in the x –direction and y-direction respectively. It would appear that students have met these symbols before. However students were not able to explain the quotient $\delta y/\delta x$ correctly. The symbols dx and dy caused many problems. It seemed that students could not make sense of these symbols if they were not written as a quotient dy/dx.

EXEMPLARS FROM EXPERIMENTAL GROUP

Task: 8 (8.1)

Structural error use of symbolism:

"Specific change in x"

Task: 8 (8.2)

Structural error use of symbolism

"Specific change in y"

Task: 8 (8.3)

Structural error use of symbolism

"Specific derivative of y with respect to x"

EXEMPLARS FROM CONTROL GROUP

Task: 8 (8.1)

Structural error use of symbolism:

"Function of x"

Task: 8 (8.2)

Structural error use of symbolism:

"Function of y"

Task: 8 (8.3)

Structural error use of symbolism:

"Change in y"

Task: 8 (8.4)

Structural error use of symbolism:

"Change in x"

4.3 GRAPHICAL REPRESENTATION OF CUMULATIVE SCORES FOR EACH ITEM FOR CONTROL GROUP AND EXPERIMENTAL GROUP

Figure 6 and Figure 7 shows the overall scores in the conventional test for both groups.

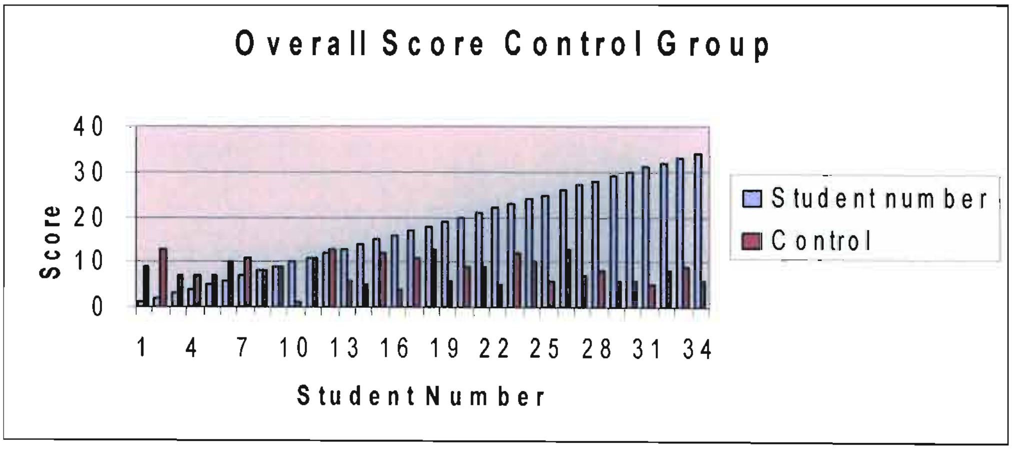


Fig. 6 Overall Scores for Control Group

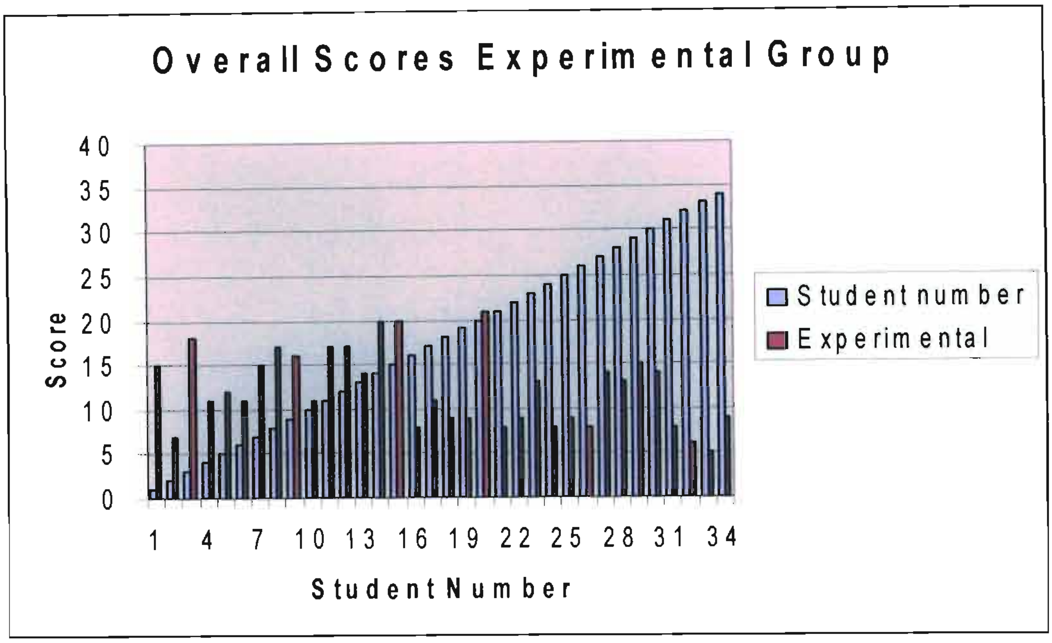


Fig. 7 Overall Scores Experimental Group

The graph shows a clearly distinction in the improved learning of the experimental group as compared to the control group. It seems that the overall performance of the experimental group was enhanced by their use of the computer in their teaching and learning. It is apparent that students who used the computer to perform tasks for their compulsory project had an advantage of using constructive interactive methods and co-operative learning strategies to aid their understanding of concepts. The null hypothesis was used to determine if there was a difference in understanding at a 95 % confidence level. Results indicate that there was a difference. The graph shows the scores of all the experimental group and control group. Particular students were not compared.

4.4 GRAPHICAL REPRESENTATION OF THE MANN-WHITNEY TEST

Software packages used were Mathematica Statistics and SigmaStatistic

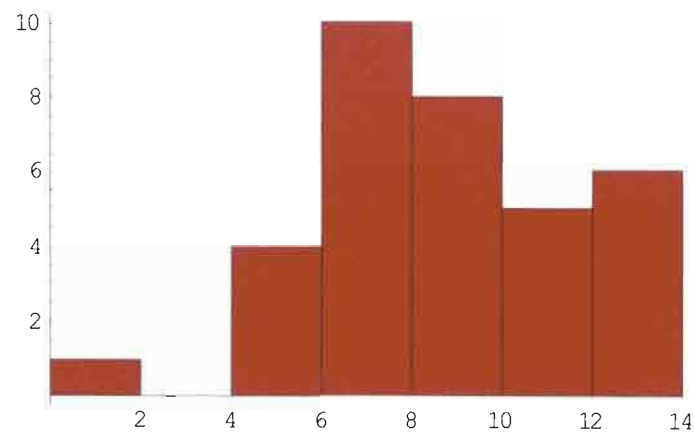


Fig. 8 Mann-Whitney Test for Control Group

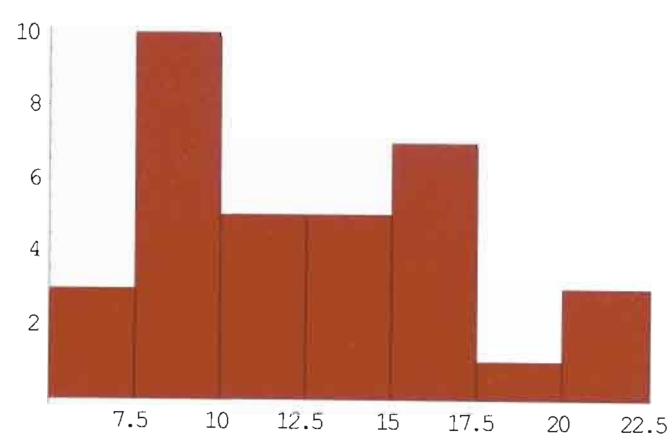


Fig. 9 Mann-Whitney Test for Experimental Group

Normality test fails at (p<0.050). Data do not follow a normal distribution. We then use the Mann Whitney test which can be performed on none normal data. The formula for the Mann Whitney test is $z = \dfrac{U - \dfrac{n_1 n_2}{2}}{\sqrt{\left(\dfrac{n_1 n_2}{N(N-1)}\right)\left(\dfrac{N^3 - N}{12} - \Sigma T\right)}}$

Group	N	Missing	Median	25%	75%
Control	34	0	8.0	6.0	11.0
Experimental	34	0	11.5	9.0	15.0

Table 6 Analysis of Mann-Whitney Test

We use $\alpha = 0.01$, UStatistic=884.5 and T=866.5 .

Decision: The difference in the median values between the two groups is greater than would be expected by chance. There is a statistically significant difference at p<0.001.

However both groups struggled with the items presented to them. Five notable peaks in the graphical representation show that the experimental group had an advantage over the control group. Their experience in their project work using the computer assisted them in their responses to the paper and pencil task presented to all the students. It must be noted that the experimental group in some cases had their very first experience in working on a computer. Responses from the experimental group showed that this was a positive experience for their learning.

4.5 ANALYSIS OF DEEP, SURFACE AND INTERMEDIATE LEARNING

The following principle was used in the classification of deep, surface and intermediate learning:

Deep approach: correct principle used in solution to problem.

Intermediate approach: partially correct principle used in solution to problem.

Surface approach: no principle or incorrect principle used in solution to problem.

CLASSIFICATION OF ITEMS	DEEP APPROACH		SURFACE APPROACH		INTERMEDIATE APPROACH	
	Experimental	Control	Experimental	Control	Experimental	Control
Sequence	82 %	6 %	4 %	88 %	6 %	6 %
Limit	12 %	6 %	76 %	74 %	12 %	20 %
Average rate of change	47 %	26 %	41 %	71 %	12 %	3 %
Rate of change	38 %	6 %	44 %	88 %	6 %	6 %
Derivative	79 %	50 %	12 %	41 %	9 %	9 %
Symbolism	2 %	1 %	86 %	90 %	12 %	9 %

Table: 7 Analysis of Deep, Surface and Intermediate Learning

* where there are less than 100 % in the total for the experimental group indicates no response from certain students.

In Item: 1, 82 % of the experimental group used the deep approach in finding solution to the problem This as compared to only 6 % of the control indicates that the experimental group had an advantage of the computer to aid their visual appreciation of the problem. It also shows that the concept of the secant converging to a tangent was reasonably well developed in the experimental group and only partially developed in the control group. The experimental group was better able to relate theoretical ideas to everyday experience (Ramsden, 1988).

Of the control group, 88 % did the task using a surface approach. This according to Ramsden (1988) corresponds to facts and concepts are being associated unreflectively. It also shows that the frame "limit" was poorly developed in the control group.

There was no significant difference in both groups with respect to the intermediate approach.

In Item 4, 47 % of the experimental group used the deep approach and 41 % used the surface approach as compared to 26 % of the control group using the deep approach and 71 % using the surface approach. A larger percentage of the experimental group used the deep approach. Again their experience with the tasks done on the computer gave them an improved understanding to solve the task at hand.

Item: 5 was concerned with the differentiation. It is apparent that 79 % of the experimental group used the deep approach and 50 % of the control group did the same. 12 % of the experimental group used surface structures as compared to 41 % of the control group who did the task using surface structures only. Neither group showed preference for the intermediate approach.

In item: 6, 12 % of the experimental group used the deep approach and 6 % of the control group used the deep approach. A larger percentage approached the task using the surface approach: 74 % in the experimental group and 76 % in the control group respectively. This did not show any significant difference in how both groups approached differentiation as a limit. The frame "instantaneous rate of change" in both the group was sketchy and incomplete.

Item: 7 dealt with the use of symbolism. Here task 8 was used. Task 8.7 was analyzed for deep and surface structures. Both the experimental group and control group resorted to the surface approach in analyzing the task. An interesting note was that none of the students in the entire group used the deep approach. They were not able to see the relationship $dy / dy = \lim_{\delta x \to 0} \delta y / \delta x$.

An analysis of task 2 (the tank problem) where students had to use problem-solving strategies is shown below:

Experimental Group			**Control Group**		
Forward	**Backward**	**Intermediate**	**Forward**	**Backward**	**Intermediate**
74 %	3 %	23 %	68 %	3 %	29 %

Table: 8 **Comparison of Forward, Backward and Intermediate problem solving strategies**

The experimental group showed slightly more occasions of the forward-working approach to seek solution to the task presented to them. This can be attributed to the fact that they had opportunity to interact in a computer learning environment and enjoyed benefit of a more enriching experience. According to Larkin (1980) forward-working is related to the student having expert knowledge. It was also found that experts tend to use the forward working strategy. They worked from the givens to the unknowns. The task presented was similar to a problem a physics student would encounter. Larkin did her experiment with physics students. Here we wanted to see how students would approach a physics problem in mathematics.

A large percentage of students wrote down an incorrect response. It would have been ideal to interview a sample of the students to determine what they were thinking when writing such responses.

EXEMPLAR FROM EXPERIMENTAL GROUP

$(0;0)$ and $(T:2T)$	[Working from the givens
$\dfrac{2T-0}{T-0}=2$	- forward working strategy]

EXEMPLAR FROM EXPERIMENTAL GROUP

The gradient is constant for any value of x.	[Working backwards
m=2	- back working strategy]

Project Task	Deep	Intermediate	Surface
A	8	14	6
B	9	13	5
C	6	11	5

Table: 9 ANALYSIS OF DEEP, INTERMEDIATE AND SURFACE LEARNING IN PROJECT WORK

- 28 students responded to task A
- 27 responded to task B
- 22 responded to task C.

Task A was based on the convergence of a sequence and the limit concept

Deep: Responses must include "sequences", "converges to a point" and "limit".

Intermediate: Responses included "sequences", and "converges to a point" but neglects the "limit"

Surface: Responses have "sequences" but does not mention convergence.

(See appendix 2 Exemplar for student S1 Task A)

Task B was based on the average rate of change

Deep: Responses includes "function", "change in function$(f(x+\Delta x)-f(x)$", "points (x, y) and $(x+\Delta x, y+\Delta y)$ on the graph and represents a secant line"

Intermediate: Responses includes "function", "change in function$\left(f(x+\Delta x)-f(x)\right.$" but does not mention a secant line.

Surface: Does mention change in function, not able to indicate points (x, y) and $(x+\Delta x, y+\Delta y)$ on the graph and show that it represents a secant line"

(See appendix 2 Exemplar for student S5 Task B)

Task C was based on instantaneous rate of change

Deep: Able to show "sequence of secants converge to a point to become a tangent", and "slopes of secants converging to a slope of the tangent", and "$\frac{dy}{dx}=\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$", $\frac{dy}{dx}$ is the slope of the tangent and instantaneous rate of change, and $\frac{\Delta y}{\Delta x}$ is the average rate of change which is the slope of the secants.

Intermediate: Able to show "sequence of secants converge to a point to a tangent", and "slopes of secants converging to a slope of the tangent" but not responses to and "$\frac{dy}{dx}=\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$", $\frac{dy}{dx}$ is the slope of the tangent and instantaneous rate of change, and $\frac{\Delta y}{\Delta x}$ is the average rate of change which is the slope of the secants.

Surface: No distinction made between slopes of secants and tangents.

(See appendix 2 Exemplar for student S8 Task C)

4.6 FINDINGS PR0JECT WORK

Even though students were making structural errors less frequently than observed by Naidoo (1998) in a traditional lecturing environment, the majority of the students still made serious structural errors. This suggests that the interaction with the software did not reinforce certain frames adequately. The assumptions made when using *Mathematica* were:

- to promote versatile thinking
- to give students opportunity to work at their own pace
- to allow students to diagnose their own errors

The *Mathematica* project work gave students numerical, symbolic and visual representation of the tasks. Hughes Hallet (1991) supports the 'Rule of Three' in which topics must be taught graphically and numerically as well as analytically with the aim of allowing students to be able to see a major idea from several angles. Many students' accept the numerical data without connecting these to the graphical representation and vice versa. *Mathematica* also assumes that students can proceed from algebraic to numerical and graphical with ease. Only students with established pre-knowledge frames such as rate of change, graphs and algebra were able to switch from one frame to another without difficulty. Students also accepted the computer generated graph without analysis and interpretation. Students believe that the computer is right. Similar findings were recorded by Giraldo, Carvalho &, Tall (1987), in research done in Brazil. I agree with Tall and Sheath (1983) that see the gradient of the graph as an intermediate stage in calculus. Visualization gives a metaphoric image of the derivative. It does not account for complete understanding of the processes. Analytical rigour is lost in the process.

Students exhibiting surface structures experienced difficulty in using *Mathematica* commands. For instance, a command like, *Plot [{F[x], G[x]}, {x, 0.49, 0.51},* required understanding of function, variable and domain. The plot command assumes that the student possess deep understanding of the concepts of function, variable, domain and ranges i.e. it assumes that students' pre-knowledge frames are already in place to do programming of this type. The logic in the language of the commands in the programming is quite different from paper-pencil type applications. Due to the weak pre-knowledge frames some rely on an algorithmic approach to solve problems. The computer software favours objects rather than processes. The computer does everything for the student. If the programming language is correct, all the student needs to do is "press one button" and the output is generated. For example when finding the derivative the concept function, rate of change and instantaneous rate of change (limit) are needed. Students' sub-frames must be well developed to link concepts needed to find the derivative. Hence the mathematical meaning is lost at each stage of the task. *Mathematica* makes many assumptions about student frames establishment. Tall (1986) agrees that the computer has built in functions to represent the mathematics explicitly but must also show the processes of the mathematics with results. This is necessary to connect to students' pre-knowledge frames.

Hence there is little mathematical thinking for the student to do. It loses the students mathematical cognitive processes in the output stage of its program. Although the software provides an environment where students can discover for themselves certain mathematical phenomena such as maximum and minimum of functions, students cannot analytically prove why at these points the derivative is zero. As claimed by Wolfram (1999) the mathematical processes are done internally in the CPU which acts as a black box. This suggests students can carry out the procedure mechanistically and generate the required graphs without assigning meaning to the result. For example a study of a student interview protocol suggests (task C) surface thinking due to the software influence. Software commands generated results and some students could not draw conclusions from their output. However, students who exhibited deep structures tended to flourish in the microworld's environment.

Other problems such as students had to multitask in that they had to concentrate on getting the programming right and simultaneously pay attention to conceptual aspects. In some instances students gave a correct output but were unable to make cognitive connections due to cognitive overload. Socratic activities could possibly be used to engage students in discussion at various stages of the computer interaction.

Environmental considerations referred to by Piaget (1972), Dubinsky (1991) and Sfard (1991) was not catered for adequately in the *Mathematica* environment. A proportion of students had difficulties transforming from a traditional to a computer laboratory environment and many of these students exhibited surface structures.

Although students enter the mathematics class with a specified grade – at least 50 % at the grade 12 examinations in mathematics (higher grade), they have serious problems with manipulation in algebra, calculating the gradient of a straight line passing through two points, and sketching quadratic graphs, let alone cubic curves. The computer simplifies tedious calculations and the process by which the results are obtained is not clear. The student is not able to cope with the speed at which these are generated. It can also be described as an overload of information and students are not able to contextualize the results as a gestalt.

When comparing the protocols from the project work to the conventional test it was apparent that students made similar errors in related tasks. The observation showed that students possessed weak sub-frames.

For instance in Task 3, students found it extremely difficult to conclude that $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. The observation, according to table 9, students exhibited under developed frames in limits, rates of change and derivative, indicating that students possessed a series of connecting underdeveloped frames. This suggests that the software does not help understanding this concept and direct teaching would be more valuable. Bezuidenhout (1998) suggests that students' ability to interpret a mathematical symbol as representing both a process and an object is more likely to develop if it is the direct focus of teaching rather than if the development is left to chance. At this moment in time no mathematical software is designed to improve on the teaching and learning of underdeveloped frames.

In table 9, the analyses for deep, intermediate and surface structures reveal that the majority of the students used surface and intermediate structures in the construction of their answers in both written (conventional test) and verbal responses (interview questions). A small percentage of students drew on deep structures in reasoning and analyzing results. The software design must cater for deep, intermediate and surface structures to allow access to a group of mixed abilities. Lecturers ought to modify or refine their courses periodically. Certain key concept processes may be included in the software. Much can be achieved if the software allows students to find the limit of functions at different values and plot graphs to verify whether the limit exists or not. Descriptions and definitions of elementary concepts must be included as a process so that concept images of the students are meaningfully established during the interaction with the software.

Tall (1985) advocated the zoom function of the software for students understanding of the derivative at a point. Although the computer depicted the graph as a series of straight lines the student could not see the meaning behind the magnification. It seem abstract for students although it was concretely visualized using the software. It mathematically brought to the teaching and learning, a microworld, a new concept scaling which requires transformation mathematics, thus exacerbating the cognitive processes involved in the derivative. This dynamic interpretation of the graph creates new patterns of thought that students find difficult to assimilate. It creates a web of confusion in the students mind. The software instead should be designed to include calculation of ratios, gradients and tangents of graph using explicit coordinate geometry.

CHAPTER 5: CONCLUSION AND RECOMMENDATIONS

5.1 CONCLUSION

The frequency of errors made by the students indicates that their pre-knowledge frames were not well developed. With regards to elementary differential calculus, the poor understanding of pre-calculus concepts, contribute to a host of difficulties in the mind of the learner. Some of these difficulties were observed during the application of the modified battery of tests of Orton (1983). Many factors need to be considered when referring to students understanding of differentiation.

The first factor relates to weak pre-knowledge frames. Students' had a poor mental image of rate of change, average rate of change and the limit concept. They were unable to find the rate of change from a straight line graph. Their problems were compounded when dealing with rate, average rate and instantaneous rate and average rate of change from a curve. The analyses for deep, intermediate and surface structures show a clear distinction between the learning strategies employed by each group. It clearly showed that a sub-frame that was poorly developed in one task, reflected poorly again in a related task. This gives an indication that concepts in elementary calculus are difficult to grasp.

Despite generally performing better than the control group, the experimental group still made a significant amount of errors. This shows that the software by itself is not sufficient to address the pre-knowledge deficiencies that were prevalent and also certain concepts.

The second factor deals with reliance on algorithmic means to solve problems. This was evident in the derivative questions. Students develop coping strategies as described by Smith and Moore (1991) to overcome their difficulty. In this way meaning is “lost”. It is important that concepts be seen from several points of view. They must relate to the student’s ‘own environment’ and ‘world view’. The student in turn must build a web of connections to tackle real world problems.

The third factor deals with errors made by students. A classification of the errors revealed that there were more structural and executive errors as compared to arbitrary errors. The experimental group made fewer errors in both categories as compared with the findings of Naidoo (1998). It is suggested that appropriately designed academic systems software be used to assist learning in aspects of calculus, in particular, elementary calculus. Such software must allow for flexibility and cater for students’ pre-knowledge frame deficiencies.

The fourth factor deals with symbolism in elementary calculus. Students lacked ability to interpret symbols. Bezuidenhout (2003) study also found students focusing on superficial aspects of symbols and ignoring the meanings behind the symbols. The analysis for deep, intermediate and surface structures using the Orton instrument showed that both groups struggled to connect meaningfully with symbols. They had a very superficial understanding of the symbols in elementary differential calculus. This was consistent in the project task done by the experimental group as well. Only 18% of the experimental group were able to identify with symbols using both process and object conceptions.

It is clear from the graphical representation of the overall scores, for the experimental group and the control group, that the experimental group had a slight advantage of more developed frames in each of the tasks presented to them. However the Mann Whitely test suggests that there is a significant difference between the experimental and control groups. We believe that by modifying the *Mathematica* course greater improvements in learning of calculus concepts may be achieved.

In the learning of elementary calculus it is essential that a mechanistic application of a set of rules is not sufficient, rather the synthesis of the appropriate mental frames is needed to represent concepts and the procedures necessary to seek solutions.

The shortcomings of the study include:

- Failure to tackle the larger issue of curriculum reform itself, where the content, the ordering of topics, the emphasis on certain aspects that have to be revised in the light of the availability of computing software - a costly issue and will take long to be introduced successfully;
- Consideration on how lecturers themselves needed to integrate computing technology into their normal classes tests and examinations – changing the mind set of those that prefer traditional teaching approaches would be a daunting task too;

- Technology was presented as a "remedial" learning tool or "supplementary" learning tool not much thought had been given to fundamentally changing the focus of a technology rich curriculum, which ought to be on the things which a computer can't do well.
- Lack of consideration for students that do not have access to graphic calculators and computer technology outside the ambit of the University of Technology and also the lack of use of such technology prior to study at the University of Technology.
- There was a lack of matching of groups for ability, sex and prior knowledge.

5.2 RECOMMENDATIONS

It is recommended that:

- the mathematics instructional programme ought to be redesigned to allow for the inclusion of academic systems software in all elementary calculus courses at a University of Technology;
- attempts ought to be made to allow for further development in understanding of concepts in elementary calculus by using software that addresses deficiencies in students pre-knowledge frames;
- analysis be done in students examination scripts to determine retention of reform efforts in subsequent study of calculus;
- ongoing revision and evaluation be done to improve teaching and learning strategies and the monitoring of success;

- ongoing research is done to reconstruct learning material to address pre-knowledge deficiencies.

Colgan (2000) refers to changes in The University of South Australia first year engineering mathematics course that had to demonstrate outcomes in terms of that specified by The Institute of Engineers, Australia. The syllabus and teaching methodology of the first year mathematics subjects had to include:

- innovations based on information technology;
- opportunities for problem-based group work;
- opportunities for students to undertake self-learning of material deliberately not covered in lectures;
- a mixture of supervised and unsupervised learning activities;
- alternative pathways for students with less than a predetermined minimum prior knowledge.

This would suggest that the University of Technology should use their Advisory Board forums to reorganize mathematics curriculum to suit the needs of industry as well. In doing so some of the countries needs in terms of training in mathematics and technology will be addressed.

Finally an important consideration is that of the software design. Human-computer interaction must be supported by an improvement in the quality of software products. Digital information and communication technologies have become an important group of artifacts in today's information and knowledge societies (Kassgard, 2000). As suggested by this study, software can and should be used to address cognitive shortcomings in the learning experience. It can effectively be used to create an environment for average and below average students to engage with the mathematics. In doing so students might become more interested in making their attempts meaningful.

Instructional designers need to recognize their personal philosophies of learning and instruction, because these philosophies ultimately account for the instructional products they produce (Rieber: 1994). In this way the deficiencies in pre-knowledge frames can be considered to minimize errors in student learning.

This in turn would suggest that:

- design must be aimed at that which will work for individuals in a specific context making it possible for the production of quality results and a satisfying experience;
- there is a great cognitive distance between the mental model of the designers and that of the users;
- social relationships evolve over time while computer infrastructure stays static until there is a big investment to make changes (Kaasgard : 2000).

It is recommended that further research be done to investigate how the software can be modified to improve learning. Students were given an investigative task in the project work. In a sense the questions were designed to assist them in solving problems. In the tasks, student's problem solving ability was questionable. Since students had varying abilities the software must be designed accordingly. Software design must cater adequately for student pre-knowledge frames. The novice student would be provided with an inbuilt facility proving the validity of certain key concepts. The software must also prompt the student when an incorrect programming statement is made so that students are able to revise statements with ease and concentrate on the mathematics. Another factor that comes to mind is the students' level of ICT (information and communication technology) skills. These were not taken into consideration in the project work.

We propose the following algorithm for the design of software to support computer interaction in mathematics:

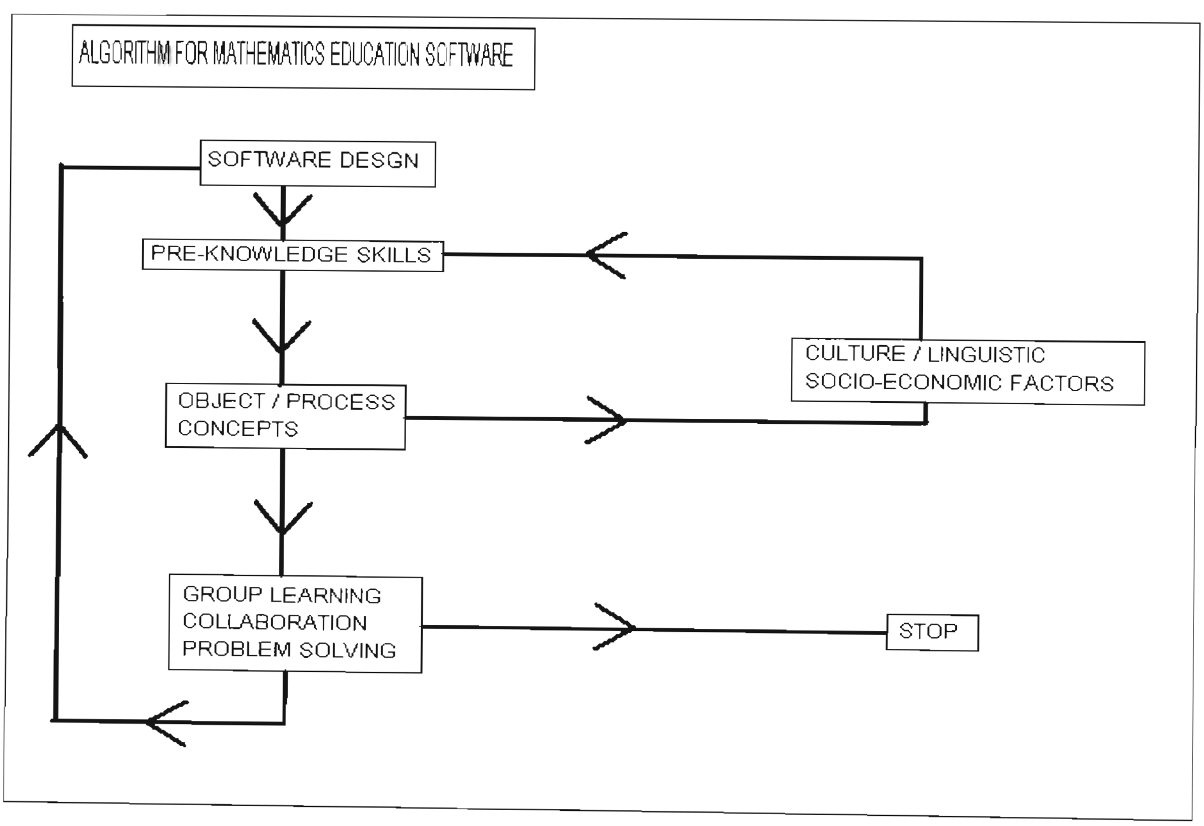


Fig. 10 Algorithm for Mathematics Education Software

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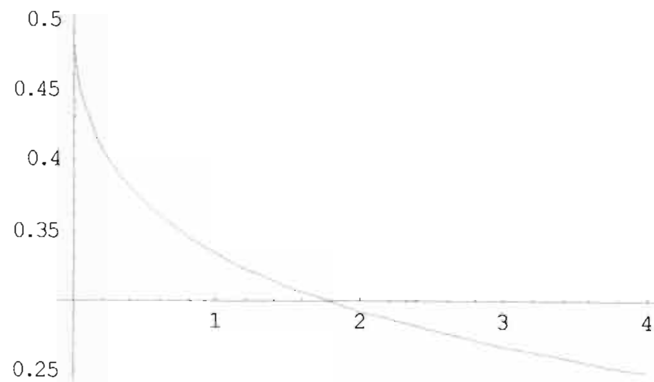
APPENDIX: 1 EXPERIMENTAL GROUP PROJECT

Here we exhibit suitable exemplars from a student response in the project.

Task A: Finding the limit graphically and verifying numerically:

Mathematica command: **Mathematica graphical solution:**

Plot [*F*[*x*], {*x*, 0, 3.99}]



{The student determines left hand limit graphically by deducing as $x \to 4^{-}$ *(from the left)*

$f(x) \to f(4^{-}) \to 0.25\}$

Mathematica command:

Table[*N*[{*x*,*f*[*x*]}],{*x*,1,3.99,0.1}]//**Table Form**

Mathematica numerical solution:

1.	0.333333	2.6	0.27682
1.1	0.327997	2.7	0.274486
1.2	0.323055	2.8	0.272233
1.3	0.318454	2.9	0.270056
1.4	0.314148	3.	0.267949
1.5	0.310102	3.1	0.265909
1.6	0.306287	3.2	0.263932
1.7	0.302678	3.3	0.262014
1.8	0.299254	3.4	0.260152
1.9	0.295998	3.5	0.258343
2.	0.292893	3.6	0.256584
2.1	0.289928	3.7	0.254872
2.2	0.287089	3.8	0.253206
2.3	0.284368	3.9	0.251582
2.4	0.281754		
2.5	0.279241		

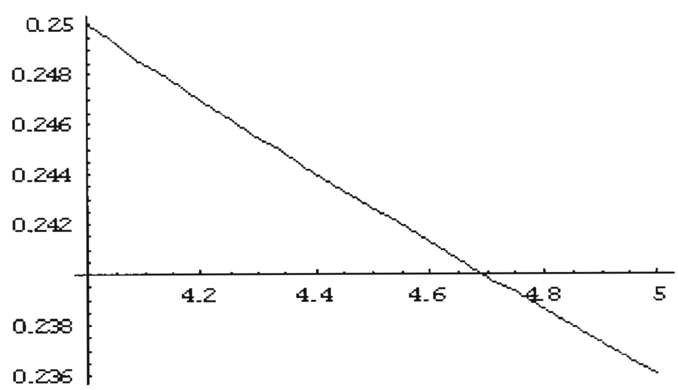
{Student finds the left hand limit numerically by viewing the converging sequence and concludes that

As $x \to 4^-$ *(from the left)* $f(x) \to f(4^-) \to 0.25$*}*

Mathematica command:

Plot [*F*[*x*], {*x*, 4.0001, 5}]

Mathematica graphical solution:



{Student plots graph and finds limit from the right and deduces that. as $x \to 4^+$ *(from the right)* $f(x) \to f(4^+) \to 0.25$*}*

Table*[*N*[{*x*,*F*[*x*]}],{*x*,4.001,5,0.1}]//*Table Form

Mathematica numerical solution:

4.001	0.249984
4.101	0.248441
4.201	0.246936
4.301	0.245466
4.401	0.24403
4.501	0.242627
4.601	0.241255
4.701	0.239913
4.801	0.2386
4.901	0.237314

{Student finds the right t hand limit numerically by viewing the converging sequence and concludes that

As $x \to 4^+$ *(from the right)* $f(x) \to f(4^+) \to 0.25\}$

Task B: Finding average rate of change:

F [x_]:3x-2x^2

f[0.5])/0.4 {student finds gradient of the secant line from f (0.9) to f (0.5) with increment h=0.4 with slope =0.2}

=0.2

To view the function and the secant line on the same system of axes the following commands were performed:

Equation of a secant line

y=mx+c [straight line]

{Since the gradient is calculated above the intercept is calculated using the solve command}

Solve [*f* [0.9] = =0.2*0.9+*c*, *c*]

{{*c*=0.9}} *{Student calculates the value of c}*

Graph of secant line $y = 0.2x+0.9$ *{equation of secant line obtained}*

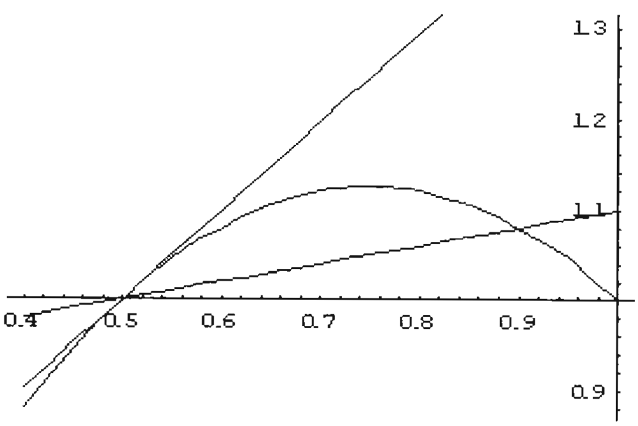
To plot secant and initial function on same axes

L [*x*_]:=0.2*x* + 0.9 [secant line equation]

f [x_]:=3x-2x^2 [original function]

8x+0.51 {*student finds of the secant line from f (0.5) to f (0.51) with increment h=0.01 with slope =0.98}*

Plot [{*f*[x], *L*[*x*], *S*[x]}, {*x*, 0.4,1.0}]



Task C: Find instantaneous rate of change:

F [x_]:=3x-2x^2

Limit [(*F*[0.5+*h*]-*F*[0.5])/*h*, *h* = 0] {the definition of the derivative is used, $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$}

Solve [*F* [0.5] =0.5+*c*, *c*] {Calculates the *y* intercept for the equation of the secant at $x = 0.5$}

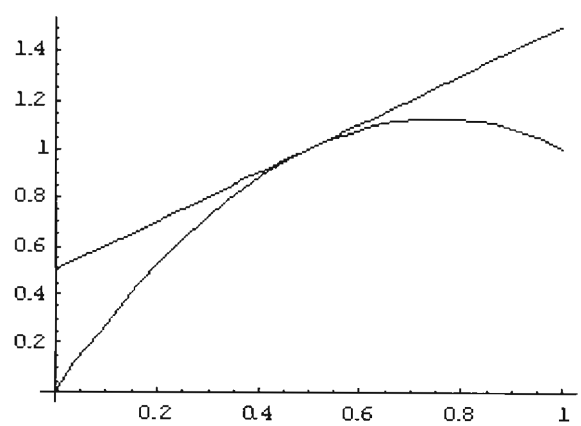
Mathematica solution:

{{c=0.5}}

G [*x*_]:=*x*+0.5

Plot [{*F*[*x*], *G*[*x*]}, {*x*, 0,1}] {Plots the graphs of the function and the secant on the same system of axes}

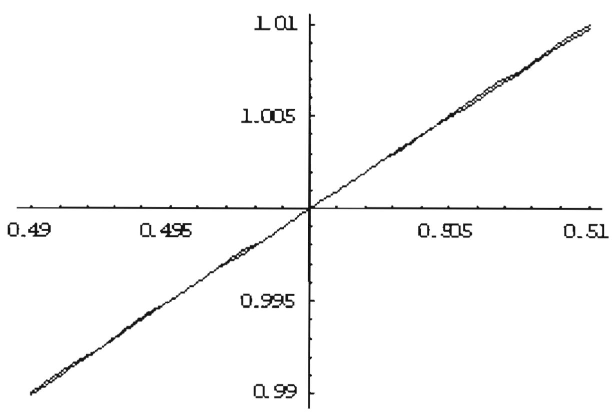
Mathematica graphical solution:



Mathematica command:

Plot [{F[x], G[x]}, {x, 0.49,0.51}] {Zooming into a small section of the graph indicating that tangent and curve coincides}

Mathematica graphical solution:



{Over a smaller domain, the secant within the region of the domain becomes a tangent}

APPENDIX: 2 EXEMPLARS FROM STUDENT INTERVIEWS

Exemplar for student S1 Task 1.

I: Explain your results (output) obtained for question 1.

S1 "I plot the graph and found the limit from the left and verified this by showing that the sequence converges at 0.25 by getting the numerical solution as well."

I: What happens at 0.25?

S1: The sequence converges.

{Student response does not connect the limit concept to convergence}

I: Explain in your own words what this means?

S1: It means that the limit exists and the point to which the sequence converges is the limit of the sequence.

{Deep structure response}

Exemplar for student S5 Task 2

I: Explain your output for Task 2.

S5: We were asked to find the average rate of change.

I: Tell me what you did to get the average rate of change?

S5: I found the difference in the function over a small interval and plotted the graph.

I: Why did you plot the graph?

S5: I wanted to get a visual representation.

{Intermediate structure response – no mention of secant line}

Exemplar for student S8 Task 3

I: Explain your output for Task 3.

S8: I wanted to calculate the instantaneous rate of change at $x = 0.5$

I: Explain what you did?

S8: I plotted the graph of the tangent and curve at observed what happened at 0.5

{Surface response – repeated features of the question}

APPENDIX 3 CUMULATIVE SCORES CONTROL GROUP AND EXPERIMENTAL GROUP

Student number	Control	Experimental
1	9	15
2	13	7
3	7	18
4	7	11
5	7	12
6	10	11
7	11	15
8	8	17
9	9	16
10	1	11
11	11	17
12	13	17
13	6	14
14	5	20
15	12	20
16	4	8
17	11	11
18	13	9
19	6	9
20	9	21
21	9	8
22	5	9
23	12	13
24	10	8
25	6	9
26	13	8
27	7	14
28	8	13
29	6	15
30	6	14
31	5	8
32	8	6
33	9	5
34	6	9