UNIVERSITY OF KWAZULU-NATAL

EXACT SOLUTIONS FOR SPHERICAL RELATIVISTIC MODELS

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This dissertation is submitted in fulfilment of the academic requirements for the degree of Master in Science to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and Science, University of KwaZulu-Natal, Durban.

December 2011

As the candidate's supervisors, we have approved this dissertation for submission.

Signed:	Professor K S Govinder	December 2011
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Abstract

In this thesis we study relativistic models of gravitating fluids with heat flow and electric charge. Firstly, we derive the model of a charged shear-free spherically symmetric cosmological model with heat flow. The solution of the Einstein-Maxwell equations of the system is governed by the pressure isotropy condition. This condition is a highly nonlinear partial differential equation. We analyse this master equation using Lie's group theoretic approach. The Lie symmetry generators that leave the equation invariant are found. We provide exact solutions to the gravitational potentials using the first symmetry admitted by the equation. Our new exact solutions contain the earlier results of Msomi *et al* (2011) without charge. Using the second symmetry we are able to reduce the order of the master equation to a first order highly nonlinear differential equation.

Secondly, we study a shear-free spherically symmetric cosmological model with heat flow in higher dimensions. We establish the Einstein field equations and find the governing pressure isotropy condition. We use an algorithm due to Deng (1989) to provide several new classes of solutions to the model. The four-dimensional case is contained in our general result. Solutions due to Bergmann (1981), Maiti (1982), Modak (1984) and Sanyal and Ray (1984) for the four-dimensional case are regained. We also establish a new class of solutions

that contains the results of Deng (1989) from four dimensions.

ALLAH

You are worthy of all praise

The Most Gracious, Most Merciful.

To

Muhammad Mutebi and Amina Kabongoya: my late parents,

Hajjat Minsa Nalweyiso: my late grand mother.

May the Almighty have mercy on your souls.

Declaration

I declare that the contents of this dissertation are my original work except where due reference has been made. It has not been submitted before for any degree to any other institution.

Yusuf Nyonyi

December 2011

Declaration - Plagiarism

I, Yusuf Nyonyi, student number: 210556520, declare that

1. The research done in this thesis, except where otherwise indicated, is my original research.
2. This thesis has not been submitted for any degree or examination at any other university.
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Acknowledgments

Firstly, I would like to thank my supervisors, Professor K S Govinder and Professor S D Maharaj for giving me this opportunity to do a masters degree with them. Their constructive criticism, mentorship, friendship and encouragement has certainly brought out the best in me. They have also enabled me to rejuvenate my love for Physics as a window for applying Mathematics.

For funding my research, I am duly indebted to the African Institute for Mathematical Sciences, University of KwaZulu-Natal and the National Research Fund; not to mention the additional assistance from my supervisors. I am also grateful to the former head of school Professor D Baboolal for funding my trip from AIMS in July 2010, the school administrator Miss N C Mkhwanazi and the entire staff of the School of Mathematics, Statistics and Computer Science for providing me with assistance whenever I needed it. The school has also allowed me to use their equipment freely. Thank you all for making my stay a success.

My best friends Miss E W Nayiga, Mr J Jumba, Mr A S Bamunoba and Miss W M M Altuieb have been a rock in my life. I thank them for lending me their ear whenever I needed someone to speak to. Finally, I thank my family for being there for me.

Contents

Chapter 1

Introduction

At the beginning of the 20^{th} century, general relativity came to life to explain the astronomical observations that Newtonian gravity had failed to account for. This theory, developed by Einstein in 1916, also provided a new avenue to our understanding of gravity and the important role it plays in shaping the universe. This model of gravity not only described the interaction between objects but also defined the interaction that arises as gravitational fields of the various massive bodies come into play. It is this model of gravity for the curvature of spacetime on a four–dimensional manifold that enables us to study and understand the gravitational nature and behaviour of various cosmological bodies. Understanding the gravitational field behaviour of these bodies gives us an insight into their evolution and the impact they have on the universe. In this respect, several particular models have been suggested to interpret observational data obtained from galactic bodies and even predict their evolution in time. There is vast literature available to understand basic general relativity and its importance as far as studying astrophysical and cosmological structures like stars, galaxies and black holes. For more detailed information, the reader is referred to Narlikar (1993), Stephani (1990), Wald (1984) and Wolfgang (2006).

A variety of models on relativistic bodies have been suggested with the aim of getting a deeper insight into the behaviour of these stellar and galactic systems. The assumption of spherical symmetry aids in developing such models. Most of the exact solutions to the Einstein field equations obtained are mathematically feasible but only a few of them are physically acceptable, making it difficult to fully describe the dynamics of the problem. However, any exact solution found provides a clue to the behaviour of the gravitational field thus enabling us to suggest physically viable parameters for more complex models. Therefore exact solutions are essential in studying the physical features of the model and help to make original predictions on the evolution of these bodies.

We intend to find several exact classes of solutions of the Einstein field equations and Einstein-Maxwell equations for the models stipulated in our study. Several techniques of obtaining exact solutions are available including the *ad hoc* Deng (1989) approach. Some methods involve making viable assumptions to the matter distribution, gravitational potentials and imposing a particular equation of state. We can also take advantage of the symmetries of the manifold and apply the Lie analysis of differential equations, Noether symmetries, Lie-Bäcklund transformations, among others. Several texts describe many of these techniques. The reader is referred to Bluman and Anco (2002), Bluman and Kumei (1989), Cantwell (2002), Olver (1986, 1995) and Stephani (1989) for more details. In this thesis, we use Lie's group theoretic approach to provide solutions to spherically symmetric gravitating fluids with heat flux in the presence of an electromagnetic field. We also use Deng's algorithm (Deng 1989) to provide solutions to uncharged spherically symmetric cosmological models with heat flow in higher dimensions.

Schwarzschild (1916a) pioneered the quest to provide exact solutions to the Einstein field equations for a gravitating body in the exterior. He also provided a model that describes the gravitational field in the interior spacetime with constant density (Schwarzschild 1916b). Investigations were also carried out by Nordström (1918) and Reissner (1916) to give the Reissner-Nordström solution for a charged body. Since then many authors have provided several physically viable solutions to describe the interior stellar matter distribution. Some of the recent treatments include studies on compact stars by Thirukkanesh and Maharaj (2006, 2009), the charged Tikekar superdense star solutions by Komathiraj and Maharaj (2007b) and the analytical models for quark stars by Komathiraj and Maharaj (2007a). Shear-free models have also been extensively studied and early solutions were provided by Kustaanheimo and Qvist (1948) to the Einstein field equations. Shear-free models in which heat flux is incorporated across the boundary of a radiating star have also been proposed. Recently, Msomi *et al* (2011) provided solutions to such relativistic models. Conformally flat radiating solutions provided by Banerjee *et al* (1989) have been applied to radiating stars by Herrera *et al* (2004, 2006), Maharaj and Govender (2005) and Misthry *et al* (2008) among others. More general models involving shearing, accelerating and expanding spacetimes, though not extensively studied, have been proposed to describe cosmological processes in the absence of heat flux. Here we only highlight the known solutions by Bradley and Marklund (1999) and Maharaj *et al* (1993). A shearing model in the relativistic astrophysical context was found by Naidu *et al* (2006) for an anisotropic star.

The desire to gain further insight in the behaviour of radiating and charged gravitating bodies motivates our quest to provide more new classes of exact solutions. The rest of the thesis is arranged as follows:

Chapter 2: We review the basic features in differential geometry and their use in formulating and studying problems arising in general relativity. We highlight the main definitions and formulæ used in our quest. We then derive the Einstein field equations and Einstein-Maxwell field equations. The example of spherically symmetric spacetimes is used to illustrate the key ideas.

Chapter 3: In this chapter, we review the ideas behind Lie's group theoretic approach. We briefly describe the ideas of infinitesimal transformations, symmetry generators and reduction of order and indicate how they are used to study differential equations. We give a general outline of how the infinitesimal transformations and extended symmetry generators appear in studying a differential equation involving one independent variable and m dependent variables.

Chapter 4: We study the model of a charged spherically symmetric relativistic fluid with heat flow. The pressure isotropy condition is studied using Lie's group theoretic approach. Two symmetries of the problem are obtained that we subsequently use to provide solutions. We believe that our treatment is the first systematic investigation of a charged radiating fluid using the geometric Lie method. The first symmetry gives a new class of solutions. The second symmetry is used to reduce the order of the initial equation to a first order highly nonlinear differential equation.

Chapter 5: We construct the model of a shear-free spherically symmetric relativistic model defined on a higher dimensional manifold. Using Deng's algorithm (Deng 1989), we first provide the solutions obtained for a four–dimensional manifold. We then derive the Einstein field equations for the higher dimensional problem. The pressure isotropy condition obtained from the system is transformed to another form that becomes our master equation. We then apply Deng's approach to our master equation from which we obtain three classes of solutions. We finally show how our third class of solutions in higher dimensions contains the one obtained by Deng (1989) in four dimensions as a special case.

Chapter 6: We give a summary of our results and discuss them in more depth. Possible avenues of future work are also discussed.

Chapter 2

General relativity

2.1 Introduction

Differential geometry is a branch of mathematics that utilises the ideas of integral and differential calculus coupled with multilinear algebra to study geometric problems. In general relativity we apply the results of Riemannian geometry to study astrophysical, cosmological and relativistic problems. In this chapter, we give a brief description of the main ideas from differential geometry and the field equations relevant to this thesis. In §2.2, we introduce the metric connection and quantities associated with the curvature, in particular the Einstein tensor. The energy momentum tensor and the Einstein field equations are defined in §2.3 for a neutral fluid. This is then extended to a charged gravitating fluid with the corresponding Einstein-Maxwell equations. The connection coefficients, Ricci tensor components, Ricci scalar and Einstein tensor components are explicitly determined for a shear-free spherically symmetric metric in §2.4. A particular form of the four-potential is chosen on physical grounds which enables us to specify the electromagnetic field. Then the Einstein and Einstein-Maxwell field equations are explicitly derived. The conditions of pressure isotropy, for both neutral and charged matter, are transformed to simpler forms. We indicate some of the areas of applications for neutral and charged gravitating relativistic models.

2.2 Differential geometry

In general relativity, we assume that spacetime **M** is a four–dimensional differentiable manifold endowed with an invertible, symmetric metric tensor field **g**. The local neighbourhood of the manifold possesses the same structure as that of an open neighbourhood of a point in $\mathbb{R}^n$. Even though the local structure of $\mathbb{R}^n$ may be similar to **M**, it should be noted that the global structure of the manifold may be very different. Local coordinates represent points on the manifold and are given by $(x^a) = (x^0, x^1, x^2, x^3)$ where x^0 is timelike and x^1, x^2, x^3 are spacelike. The metric tensor field **g** represents the gravitational field since its components contain the gravitational potential. We take the signature of **M** to be $(-+++)$.

The invariant distance between neighbouring points on the manifold **M** is given by the line element

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b \tag{2.1}$$

where g_{ab} are covariant components of **g**. The connection coefficients, defined in terms of the components of the metric tensor and its derivatives, are given by

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(g_{cd,b} + g_{db,c} - g_{bc,d}\right) \tag{2.2}$$

where commas indicate partial differentiation. By the fundamental theorem of Riemannian geometry there exists a unique connection Γ that preserves inner products under parallel transport (do Carmo 1976).

The Riemann tensor $\mathbf{R}$ is generated from the noncommutativity of the covariant derivative. The Riemann tensor is also called the Riemann-Christoffel or curvature tensor. In terms of the metric connection and its derivatives we obtain

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec} \tag{2.3}$$

The Ricci tensor is obtained by contraction of the Riemann tensor. We obtain

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^e{}_{ab}\Gamma^c{}_{ec} - \Gamma^e{}_{ac}\Gamma^c{}_{eb} \end{aligned} \tag{2.4}$$

which is symmetric. On contracting the Ricci tensor we obtain the Ricci scalar R. This quantity is given by

$$\begin{aligned} R &= R^a{}_a \\ &= g^{ab}R_{ab} \end{aligned} \tag{2.5}$$

We are now in a position to construct the Einstein tensor $\mathbf{G}$ in terms of the Ricci tensor, Ricci scalar and metric tensor. This tensor is defined by

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} \tag{2.6}$$

By definition, the Einstein tensor is symmetric. It can be shown that the divergence of the Einstein tensor vanishes so that

$$G^{ab}{}_{;b} = 0 \tag{2.7}$$

(Semicolons denote covariant derivatives.) This property is sometimes called the Bianchi identity. It is a necessary condition for generating the conservation laws via the Einstein field equations.

2.3 Matter and electromagnetic fields

In cosmological and astrophysical applications, the matter distribution is described as a relativistic fluid. The energy momentum tensor of uncharged matter is represented by the tensor **T**. This symmetric tensor is defined by

$$T_{ab} = (\rho + p)U_a U_b + p g_{ab} + q_a U_b + q_b U_a + \pi_{ab} \tag{2.8}$$

where ρ is the energy density, p is the kinetic (or isotropic) pressure, $\mathbf{q}$ is the heat flux vector ($q_a U^a = 0$) and π_{ab} is the stress (or anisotropic pressure) tensor (with $\pi^{ab}U_a = 0 = \pi^a{}_a$). The quantities given above are measured relative to a comoving four-velocity vector $\mathbf{U}$ that is taken to be unit and timelike ($U^a U_a = -1$). For perfect fluids, there is no heat conduction and the anisotropic pressure is absent. Thus the energy momentum tensor becomes

$$T_{ab} = (\rho + p)U_a U_b + p g_{ab} \tag{2.9}$$

which is widely applied.

For many physical applications, we require that the matter distribution satisfies the barotropic equation of state

$$p = p(\rho) \tag{2.10}$$

In cosmology, we often assume the particular equation of state

$$p = (\gamma - 1)p \tag{2.11}$$

This is a linear equation of state with $1 \leq \gamma \leq 2$. The parameter γ takes on different values that describe different matter distributions: we regain dust ($\gamma = 1$), radiation ($\gamma = \frac{4}{3}$) and stiff matter ($\gamma = 2$). In relativistic astrophysics, we apply the polytropic equation of state

$$p = k\rho^{1+\frac{1}{n}} \tag{2.12}$$

where k and n are constants. This is a highly nonlinear equation of state.

The Einstein field equations

$$G^{ab} = T^{ab} \tag{2.13}$$

govern the interaction between the curvature of spacetime and matter distribution in the absence of charge. We have set the coupling constant to be unity. Taking into account the divergence of the Einstein tensor in (2.7) we obtain the result

$$T^{ab}{}_{;b} = 0 \tag{2.14}$$

Equation (2.14) defines the conservation of matter.

We define the electromagnetic field tensor or the Faraday tensor $\mathbf{F}$ as a function of a four-potential $\mathbf{A}$ by

$$F_{ab} = A_{b;a} - A_{a;b} \tag{2.15}$$

The Faraday tensor is anti-symmetric and can be written as a matrix in terms of the electric field $\mathbf{E} = (E^1, E^2, E^3)$ and the magnetic field $\mathbf{B} = (B^1, B^2, B^3)$ as shown below

$$F_{ab} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{2.16}$$

The contribution of the electromagnetic field to the total energy momentum of matter is given by

$$E_{ab} = F_{ac}F^{c}{}_{b} - \frac{1}{4}g_{ab}F_{cd}F^{cd} \tag{2.17}$$

The need to study the effect of **E** on the gravitational field necessitates defining Maxwell's equations of electromagnetism in covariant form. The Maxwell equations in tensorial form are

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{2.18a}$$

$$F^{ab}{}_{;b} = J^{a} \tag{2.18b}$$

The quantity **J** is the four-current vector usually expressed in terms of the four-velocity vector **U** as

$$J^{a} = \sigma U^{a} \tag{2.19}$$

with σ being the proper charge density.

For a charged relativistic fluid in a gravitational field, the total momentum tensor is the sum of **E** and **T**. Therefore, the Einstein-Maxwell field equations for such a charged gravitating system are of the form

$$G^{ab} = T^{ab} + E^{ab} \tag{2.20a}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{2.20b}$$

$$F^{ab}{}_{;b} = J^{a} \tag{2.20c}$$

The field equations (2.20) describe the interactions between **g**, **E** and **T**. In the absence of charge equations (2.20) reduce to (2.13). For a detailed review of general relativity and all the ideas involved in formulating both the Einstein field equations and the Einstein-Maxwell field equations, the reader is referred to Narlikar (1993), Stephani (1990), Wald (1984) and Wolfgang (2006).

2.4 Spherically symmetric spacetimes

In this section, we consider the spacetime geometry corresponding to a spherically symmetric metric. We take $(x^a) = (t, r, \theta, \phi)$ because of spherical symmetry. These coordinates simplify the equations considerably. Then the line element can be written in the form

$$\mathrm{d}s^2 = -D^2\mathrm{d}t^2 + \frac{1}{V^2}\left[\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right] \tag{2.21}$$

The shear vanishes for the spacetime (2.21). However, the fluid is expanding and accelerating in general. The functions $D = D(t, r)$ and $V = V(t, r)$ represent gravitational potentials.

The nonvanishing connection coefficients (2.2) for the line element (2.21) become

$$\Gamma^0{}_{00} = \frac{D_t}{D} \qquad \Gamma^0{}_{10} = \frac{D_r}{D}$$

$$\Gamma^0{}_{11} = -\frac{V_t}{D^2V^3} \qquad \Gamma^0{}_{22} = -\frac{r^2V_t}{D^2V^3}$$

$$\Gamma^0{}_{33} = -r^2\sin^2\theta\frac{V_t}{D^2V^3} \qquad \Gamma^1{}_{00} = V^2DD_r$$

$$\Gamma^1{}_{01} = -\frac{V_t}{V} \qquad \Gamma^1{}_{11} = -\frac{V_r}{V}$$

$$\Gamma^1{}_{22} = r\left(r\frac{V_r}{V} - 1\right) \qquad \Gamma^1{}_{33} = r\sin^2\theta\left(r\frac{V_r}{V} - 1\right)$$

$$\Gamma^2{}_{02} = -\frac{V_t}{V} \qquad \Gamma^2{}_{12} = \frac{1}{r} - \frac{V_r}{V}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \qquad\qquad \Gamma^3{}_{03} = -\frac{V_t}{V}$$

$$\Gamma^3{}_{13} = \frac{1}{r} - \frac{V_r}{V} \qquad\qquad \Gamma^3{}_{23} = \cot\theta$$

By substituting the connection coefficients given above into (2.4), we obtain the nonvanishing Ricci tensor components given below

$$R_{00} = 3\left(\frac{V_{tt}}{V} - \frac{D_t V_t}{DV} - 2\frac{V_t^2}{V^2}\right) + V^2 D^2 \left(\frac{D_{rr}}{D} + 2\frac{D_r}{rD} - \frac{D_r V_r}{DV}\right) \tag{2.22a}$$

$$R_{01} = 2\left(\frac{V_{tr}}{V} - \frac{V_t}{V}\left(\frac{V_r}{V} + \frac{D_r}{D}\right)\right) \tag{2.22b}$$

$$\begin{aligned} R_{11} = &\frac{1}{D^2V^2}\left(\frac{-V_{tt}}{V} + \frac{D_t V_t}{DV} + 4\frac{V_t^2}{V^2}\right) + 2\left(\frac{V_{rr}}{V} - \frac{V_r^2}{V^2} + \frac{V_r}{rV}\right) \\ &- \left(\frac{D_{rr}}{D} + \frac{D_r V_r}{DV}\right) \end{aligned} \tag{2.22c}$$

$$\begin{aligned} R_{22} = &\frac{r^2}{D^2V^2}\left(\frac{-V_{tt}}{V} + \frac{D_t V_t}{DV} + 4\frac{V_t^2}{V^2}\right) + r^2\left(\frac{V_{rr}}{V} - 2\frac{V_r^2}{V^2} + \frac{D_r V_r}{DV}\right) \\ &+ r\left(3\frac{V_r}{V} - \frac{D_r}{D}\right) \end{aligned} \tag{2.22d}$$

$$R_{33} = \sin^2\theta R_{22} \tag{2.22e}$$

From (2.22) and the definition of the Ricci scalar, given by (2.5), we obtain

$$\begin{aligned} R = &\frac{6}{D^2}\left(\frac{-V_{tt}}{V} + \frac{D_t V_t}{DV} + 3\frac{V_t^2}{V^2}\right) - 2V^2\left(\frac{D_{rr}}{D} + 2\frac{D_r}{rD} - \frac{D_r V_r}{DV}\right) \\ &+ 2V^2\left(2\frac{V_{rr}}{V} - 3\frac{V_r^2}{V^2} + 4\frac{V_r}{rV}\right) \end{aligned} \tag{2.23}$$

The Ricci tensor components (2.22) and the Ricci scalar (2.23) enable us to generate the nonvanishing Einstein tensor components from (2.6). These components are given by

$$G_{00} = 3\frac{V_t^2}{V^2} + V^2D^2\left(2\frac{V_{rr}}{V} - 3\frac{V_r^2}{V^2} + 4\frac{V_r}{rV}\right) \tag{2.24a}$$

$$G_{01} = 2\left(\frac{V_{tr}}{V} - \frac{V_t}{V}\left(\frac{V_r}{V} + \frac{D_r}{D}\right)\right) \tag{2.24b}$$

$$G_{11} = \frac{1}{D^2V^2}\left(2\frac{V_{tt}}{V} - 2\frac{D_tV_t}{DV} - 5\frac{V_t^2}{V^2}\right) + \frac{V_r^2}{V^2} - 2\frac{V_r}{rV} - 2\frac{D_rV_r}{DV} + 2\frac{D_r}{rD} \tag{2.24c}$$

$$G_{22} = \frac{r^2}{D^2V^2}\left(\frac{2V_{tt}}{V} - 2\frac{D_tV_t}{DV} - 5\frac{V_t^2}{V^2}\right) - r^2\left(\frac{V_{rr}}{V} - \frac{V_r^2}{V^2} + \frac{V_r}{rV} - \frac{D_r}{rD} - \frac{D_{rr}}{D}\right) \tag{2.24d}$$

$$G_{33} = \sin^2\theta G_{22} \tag{2.24e}$$

for a shear-free spacetime.

2.4.1 Uncharged matter distribution

We are now in a position to determine the energy momentum tensor. For the four-velocity vector

$$u^a = \left(\frac{1}{D}, 0, 0, 0\right)$$

and the heat flux vector

$$q^a = (0, q, 0, 0)$$

the components of the perfect fluid energy momentum tensor (2.9) are

$$T_{00} = D^2\rho \tag{2.25a}$$

$$T_{01} = -\frac{D}{V^2}q \tag{2.25b}$$

$$T_{11} = \frac{1}{V^2}p \tag{2.25c}$$

$$T_{22} = \frac{r^2}{V^2}p \tag{2.25d}$$

$$T_{33} = \sin^2\theta T_{22} \tag{2.25e}$$

By equating the Einstein tensor components (2.24) and the energy momentum tensor components (2.25), we obtain the Einstein field equations (2.13) in the form

$$\rho = 3\frac{V_t}{D^2V^2} + V^2\left(2\frac{V_{rr}}{V} - 3\frac{V_r^2}{V^2} + 4\frac{V_r}{rV}\right) \tag{2.26a}$$

$$\begin{aligned} p = {} & \frac{1}{D^2}\left(2\frac{V_{tt}}{V} - 2\frac{D_tV_t}{DV} - 5\frac{V_t^2}{V^2}\right) + V_r^2 \\ & - 2\frac{VV_r}{r} - 2\frac{VD_rV_r}{D} + 2\frac{V^2D_r}{rD} \end{aligned} \tag{2.26b}$$

$$\begin{aligned} p = {} & \frac{1}{D^2}\left(2\frac{V_{tt}}{V} - 2\frac{D_tV_t}{DV} - 5\frac{V_t^2}{V^2}\right) - VV_{rr} + V_r^2 \\ & - \frac{VV_r}{r} + \frac{V^2D_r}{rD} + \frac{V^2D_{rr}}{D} \end{aligned} \tag{2.26c}$$

$$q = -2\frac{V^2}{D}\left(\frac{V_{tr}}{V} - \frac{V_t}{V}\left(\frac{V_r}{V} + \frac{D_r}{D}\right)\right) \tag{2.26d}$$

for a neutral spherically symmetric shear-free fluid.

On equating (2.26b) and (2.26c), we obtain

$$-\frac{VV_r}{r} - 2\frac{VD_rV_r}{D} + \frac{V^2D_r}{rD} + VV_{rr} - \frac{V^2D_{rr}}{D} = 0 \tag{2.27}$$

By making the transformation

$$u = r^2$$

equation (2.27) becomes

$$VD_{uu} + 2V_uD_u - DV_{uu} = 0 \tag{2.28}$$

This gives the pressure isotropy condition for a relativistic model without charge. The form (2.28) was first suggested by Bergmann (1981) and Glass (1981) independently.

2.4.2 Charged matter distribution

For a charged gravitating matter distribution, we need to incorporate the electromagnetic field. Taking

$$A_a = (\phi(t,r), 0, 0, 0)$$

because of spherical symmetry (where ϕ is the nonzero potential component of A_a) and using (2.15), we obtain the nonvanishing Faraday tensor components

$$F_{10}\,(= -F_{01}) = \phi_r \tag{2.29}$$

Also, knowledge of **F** enables us to write Maxwell's equations of electromagnetism explicitly. The nonvanishing equations for a shear-free charged fluid are

$$\frac{V^2}{D^2}\left(\phi_{rr}+\left(\frac{2}{r}-\frac{V_r}{V}-\frac{D_r}{D}\right)\phi_r\right)=\frac{1}{D}\sigma \tag{2.30a}$$

$$-\frac{V^2}{D^2}\left(\phi_{rt}-\left(\frac{V_t}{V}+\frac{D_t}{D}\right)\phi_r\right)=0 \tag{2.30b}$$

from equations (2.18).

We obtain the nonvanishing electromagnetic field components

$$E_{00}=\frac{1}{2}V^2\phi_r^2 \tag{2.31a}$$

$$E_{11}=-\frac{1}{2D^2}\phi_r^2 \tag{2.31b}$$

$$E_{22}=\frac{1}{2}\frac{r^2}{D^2}\phi_r^2 \tag{2.31c}$$

$$E_{33}=\sin^2\theta E_{22} \tag{2.31d}$$

from (2.17) and (2.29). Then the total contribution to the energy momentum tensor is represented by

$$T_{00}+E_{00}=D^2\rho+\frac{1}{2}V^2\phi_r^2 \tag{2.32a}$$

$$T_{01}+E_{01}=-\frac{D}{V^2}q \tag{2.32b}$$

$$T_{11} + E_{11} = \frac{1}{V^2}p - \frac{1}{2D^2}\phi_r^2 \tag{2.32c}$$

$$T_{22} + E_{22} = \frac{r^2}{V^2}p + \frac{1}{2}\frac{r^2}{D^2}\phi_r^2 \tag{2.32d}$$

$$T_{33} + E_{33} = \sin^2\theta\,(T_{22} + E_{22}) \tag{2.32e}$$

from (2.25) and (2.31). The Einstein-Maxwell equations for a charged gravitating relativistic fluid are obtained as the system

$$\rho = 3\frac{V_t}{D^2V^2} + V^2\left(2\frac{V_{rr}}{V} - 3\frac{V_r^2}{V^2} + 4\frac{V_r}{rV}\right) - \frac{V^2}{2D^2}\phi_r^2 \tag{2.33a}$$

$$\begin{aligned} p = {} & \frac{1}{D^2}\left(2\frac{V_{tt}}{V} - 2\frac{D_tV_t}{DV} - 5\frac{V_t^2}{V^2}\right) + V_r^2 - 2\frac{VV_r}{r} \\ & - 2\frac{VD_rV_r}{D} + 2\frac{D_rV^2}{rD} + \frac{1}{2}\frac{V^2}{D^2}\phi_r^2 \end{aligned} \tag{2.33b}$$

$$\begin{aligned} p = {} & \frac{1}{D^2}\left(2\frac{V_{tt}}{V} - 2\frac{D_tV_t}{DV} - 5\frac{V_t^2}{V^2}\right) - VV_{rr} + V_r^2 \\ & - \frac{VV_r}{r} + \frac{V^2D_r}{rD} + \frac{V^2D_{rr}}{D} - \frac{1}{2}\frac{V^2}{D^2}\phi_r^2 \end{aligned} \tag{2.33c}$$

$$q = -2\frac{V^2}{D}\left(\frac{V_{tr}}{V} - \frac{V_tV_r}{V^2} - \frac{D_rV_t}{DV}\right) \tag{2.33d}$$

$$\sigma = \frac{V^2}{D}\left(\phi_{rr} + \left(\frac{2}{r} - \frac{V_r}{V} - \frac{D_r}{D}\right)\phi_r\right) \tag{2.33e}$$

$$0 = -\frac{V^2}{D^2}\left(\phi_{rt} - \left(\frac{V_t}{V} + \frac{D_t}{D}\right)\phi_r\right) \tag{2.33f}$$

from (2.24) and (2.32).

We can integrate (2.33f) to obtain

$$\phi_r = VDf(r) \tag{2.34}$$

where $f(r)$ is arbitrary. Looking back at the system (2.33), only one other condition has to be satisfied to complete the model.

Equating equations (2.33b) and (2.33c), we obtain

$$-\frac{VV_r}{r} - 2\frac{VD_rV_r}{D} + \frac{D_rV^2}{rD} + VV_{rr} - \frac{V^2D_{rr}}{D} + \frac{V^2}{D^2}\phi_r^2 = 0 \tag{2.35}$$

By letting

$$u = r^2$$

and taking (2.34) into consideration, we can show that (2.35) simplifies to

$$VD_{uu} + 2V_uD_u - DV_{uu} - \frac{V^2}{4u}f(u) = 0 \tag{2.36}$$

This is the pressure isotropy condition for a charged spherically symmetric model with heat flux. When the charge vanishes (2.36) reduces to (2.28) for neutral fluids.

Several models have been put forward that are special cases of the equations derived in this section describing several astrophysical and cosmological systems. The solutions of (2.26) have been used in the study of relativistic stars that emit null radiation in the form of radial heat flow. This was made possible by Santos *et al* (1985) who showed that the interior spacetime must contain a nonzero heat flux to match at the boundary with the exterior

Vaidya spacetime. Some of the early studies were carried out by Bergmann (1981), Maiti (1982), Modak (1984) and Sanyal and Ray (1984) in their quest to provide exact solutions to shear-free spherically symmetric radiating bodies with heat flow. Deng (1989), using his special algorithm, regained earlier results and provided a new class of solutions. Recently, Msomi *et al* (2011) studied the radiating model and used Lie's group theoretic approach to provide a five-parameter family of transformations that mapped known solutions into new ones. Msomi *et al* (2011) also obtained new classes of solutions using Lie infinitesimal generators. Such models also enable us to understand the evolution of radiating stars undergoing gravitational collapse, the formation of singularities, the cosmic censorship hypothesis, and the formation of superdense matter. Herrera *et al* (2004), when studying radiative collapse, proposed a model where they highlighted the utility of the Weyl tensor in simplifying the form of the Einstein field equations. By introducing a transformation that linearizes the boundary condition, Maharaj and Govender (2005) solved the field equations and the junction conditions exactly, expressing these solutions in terms of elementary functions. These solutions contain the Friedmann dust solution as a special case. Later Herrera *et al* (2006) provided other classes of solutions. Recently, studies made by Misthry *et al* (2008) involved transforming the boundary condition to an Abel equation thence providing a variety of explicit nonlinear solutions.

Models in which charge is incorporated, so that (2.33) is valid, have also been extensively studied. Komathiraj and Maharaj (2007b) showed that by considering a linear equation of state, exact analytical solutions to the Einstein-Maxwell equations can be obtained that contain the Mak and Harko (2004) model. They obtained solutions that relate to quark matter in the presence of an electromagnetic field. They also obtained a second class of solutions that can be used to describe charged quark stars with physically acceptable interiors, a feature absent in previous models. Some other recent charged stellar models include the results of Komathiraj and Maharaj (2007a), Lobo (2006), Maharaj and Thirukkanesh (2009), Sharma and Maharaj (2007) and Thirukkanesh and Maharaj (2009).

These examples further cement the importance of spherically symmetric models, both charged and neutral, in describing and widening our understanding of gravitating stellar structures.

Chapter 3

Review of Lie analysis methods

3.1 Introduction

The union between the algebraic concept of a group and the differential geometric notion of a manifold may seem rather surreal but it is this marriage that Sophus Lie sought to create in his theory of continuous symmetry transformations. By symmetry we mean transformations that keep the function invariant (Bluman and Kumei 1989). These transformations are sometimes called invariants. Even though Lie's initial drive was studying the integration of partial differential equations, a mission he viewed as the extension of Galois theory to differential equations, it is these ground breaking ideas that led to the birth of this new area of mathematics that was consequently named after him (Olver 1986, 1995). The versatility of this theory has made it possible to be applied to numerous applications in engineering, quantum mechanics, theoretical mechanics, among others. In this chapter, we seek to explore some of the ideas of Lie analysis used in solving differential equations arising in astrophysical and cosmological models. In §3.2.1, we describe point transformations and symmetry generators on the x–y plane and define what is meant for a function to be invariant under a point transformation. The way derivatives transform is reviewed and a general expression derived by Mahomed *et al* (1990) is stated. The extended generator for a function involving derivatives of the dependent variable is also described. Reduction of order as a technique used in Lie analysis is explained in §3.2.2 and we highlight its importance as far as solving higher order differential equations is concerned. In §3.2.3, we give both the Lie point transformations and the extended symmetry generator for a system involving one independent variable and m dependent variables.

3.2 Lie symmetries

3.2.1 Point transformations and generators

In order to understand what a symmetry of a function means, there is need for us to have a good understanding of transformations and their generators.

Consider a pair of transformations in the x–y plane

$$\tilde{x} = \tilde{x}(x, y; \varepsilon) \qquad \tilde{y} = \tilde{y}(x, y; \varepsilon) \tag{3.1}$$

We call (3.1) a one parameter Lie group of transformations if the transformations are invertible, the identity transformation holds and a successive application of the transformation yields a transformation within the same class (Stephani 1989). The pair (3.1) represents a point transformation. Examples of point transformations include the rotation group in the plane, the translational group and the stretching group.

A Taylor series expansion of (3.1) about $\varepsilon = 0$ gives the infinitesimal transformations as

$$\tilde{x} = x + \varepsilon\xi(x, y) + O(\varepsilon^2) \tag{3.2a}$$

$$\tilde{y} = y + \varepsilon\eta(x, y) + O(\varepsilon^2) \tag{3.2b}$$

where

$$\xi(x, y) = \frac{\partial \tilde{x}}{\partial \varepsilon}\Big|_{\varepsilon=0} \qquad \eta(x, y) = \frac{\partial \tilde{y}}{\partial \varepsilon}\Big|_{\varepsilon=0} \tag{3.3}$$

and the operator X is given by

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{3.4}$$

Operator (3.4) defines what is called the symmetry generator of the infinitesimal transformation (3.2) with (3.3) as its tangent vectors. A simple example to illustrate this is that of a rotation group with infinitesimal form given by

$$\tilde{x} = x\cos\varepsilon - y\sin\varepsilon \qquad \tilde{y} = x\sin\varepsilon + y\cos\varepsilon \tag{3.5}$$

Taking (3.3) into account, it easily follows that the symmetry generator is given by

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{3.6}$$

Suppose that a function $f(x, y)$ is transformed via (3.1). By applying the Taylor series expansion about $\varepsilon = 0$, it can easily be shown that

$$f(\tilde{x}, \tilde{y}) = (1 + \varepsilon X)\, f(x, y) \tag{3.7}$$

The requirement

$$Xf(x,y) = 0 \tag{3.8}$$

means that the function $f(x,y)$ is invariant under the transformation (3.1). Therefore X is a symmetry of f.

To envision how differential equations transform under point transformations, we need to first know how their derivatives are affected by the same transformations. Suppose that the infinitesimal transformation (3.2) with symmetry generator (3.4) is applied to the differential equation

$$E(x,y,y',y'',\ldots,y^{(n)}) = 0 \tag{3.9}$$

where

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x},\ y'' = \frac{\mathrm{d}^2y}{\mathrm{d}x^2},\ \ \ldots,\ y^{(n)} = \frac{\mathrm{d}^ny}{\mathrm{d}x^n} \tag{3.10}$$

Now, in terms of $\tilde{x}$ and $\tilde{y}$, the first derivative becomes

$$\begin{aligned}\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{x}} &= \frac{\mathrm{d}(y+\varepsilon\eta)}{\mathrm{d}(x+\varepsilon\xi)} \\ &= \frac{\dfrac{\mathrm{d}y}{\mathrm{d}x} + \varepsilon\dfrac{\mathrm{d}\eta}{\mathrm{d}x}}{1+\varepsilon\dfrac{\mathrm{d}\xi}{\mathrm{d}x}} \\ &= (y' + \varepsilon\eta')\left(1 - \varepsilon\xi' + \varepsilon^2\xi'' - \cdots\right) \\ &\approx y' + \varepsilon(\eta' - y'\xi')\end{aligned}$$

where we have ignored terms of the order $O(\varepsilon^2)$. Note that primes stand for total derivatives with respect to x. The second derivative becomes

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}\tilde{x}^2} = \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\left(\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{x}}\right)$$

$$= \frac{\mathrm{d}\left[y' + \varepsilon(\eta' - y'\xi')\right]}{\mathrm{d}\left[x + \varepsilon\xi\right]}$$

$$= \frac{\dfrac{\mathrm{d}y'}{\mathrm{d}x} + \varepsilon\dfrac{\mathrm{d}}{\mathrm{d}x}(\eta' - y'\xi')}{1 + \varepsilon\xi'}$$

$$= y'' + \varepsilon(\eta'' - 2y''\xi' - y'\xi'')$$

In the same way, we obtain the third and fourth derivatives as

$$\frac{\mathrm{d}^3\tilde{y}}{\mathrm{d}\tilde{x}^3} = y''' + \varepsilon(\eta''' - 3y'''\xi' - 3y''\xi'' - y'\xi''')$$

$$\frac{\mathrm{d}^4\tilde{y}}{\mathrm{d}\tilde{x}^4} = y^{iv} + \varepsilon(\eta^{iv} - 4y^{iv}\xi' - 6y'''\xi'' - 4y''\xi''' - y'\xi^{iv})$$

Inductively, we generalise the derivatives to transform as (Mahomed *et al* 1990)

$$\frac{\mathrm{d}^n\tilde{y}}{\mathrm{d}\tilde{x}^n} = y^{(n)} + \varepsilon\left[\eta^{(n)} - \sum_{i=0}^{n}\binom{n}{i}y^{(i+1)}\xi^{(n-i)}\right]$$

$$= y^{(n)} + \varepsilon\eta_n \tag{3.11}$$

The symmetry generator applied to a function involving derivatives therefore extends as

$$X^{[1]} = X + (\eta' - y'\xi')\frac{\partial}{\partial y'} \tag{3.12}$$

$$X^{[2]} = X^{[1]} + (\eta'' - 2y''\xi' - y'\xi'')\frac{\partial}{\partial y''} \tag{3.13}$$

where (3.12) and (3.13) indicate the first and second extensions of X. The generator is extended until all the derivatives appearing in the differential equation are contained in X. Hence for an n^{th} order differential equation, Mahomed and Leach (1990) showed that the prolongation of X becomes

$$X^{[n]} = X^{[n-1]} + \eta_n \frac{\partial}{\partial y^{(n)}}$$

$$= X + \sum_{i=1}^{n}\left(\eta^i - \sum_{j=1}^{i-1}\binom{i}{j} y^{(i-j+1)}\xi^{(j)}\right)\frac{\partial}{\partial y^{(i)}} \tag{3.14}$$

Since η and ξ are functions of x and y, and primes indicate total derivatives, we have

$$\xi' = \frac{\partial \xi}{\partial x} + y'\frac{\partial \xi}{\partial y} \tag{3.15}$$

as the first derivative and

$$\xi'' = \frac{\partial^2 \xi}{\partial x^2} + 2y'\frac{\partial^2 \xi}{\partial x \partial y} + y'^2\frac{\partial^2 \xi}{\partial y^2} y''\frac{\partial \xi}{\partial y} \tag{3.16}$$

is the expansion of the second derivative of ξ. The derivatives of η are obtained in the same way.

Definition 3.2.1. A differential equation

$$E(x, y, y', \ldots, y^{(n)}) = 0 \tag{3.17}$$

possesses a symmetry

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \tag{3.18}$$

if and only if

$$X^{[n]} E|_{E=0} = 0 \tag{3.19}$$

■

That is to say, the n^{th} extension of X on E vanishes when the original equation is taken into consideration.

Equation (3.19) yields a partial differential equation for ξ and η. We observe that this equation contains derivatives of y, even though these derivatives do not appear in the arguments of ξ and η. This enables us to obtain a system of linear partial differential equations in ξ and η by equating the coefficients of the different functions of the corresponding derivatives of y to zero. This system can then be solved to obtain ξ and η explicitly thence providing the symmetries (3.18) of (3.17).

3.2.2 Reduction of order

It is well documented that one of the great uses of symmetries as a method of solving differential equations lies in the reduction of the order of the differential equation (Olver 1986). Therefore, once the symmetries have been obtained, they can be used in this respect.

If an n^{th} order differential equation

$$E(x, y, y', \ldots, y^{(n)}) = y^{(n)} - F(x, y, y', \ldots, y^{(n-1)}) = 0 \tag{3.20}$$

possesses a symmetry

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \tag{3.21}$$

its order can be reduced to a differential equation of order $(n-1)$ by obtaining the group invariant and first differential invariant associated with the first extension of X. This involves solving the following Lagrange's system

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}y}{\eta} = \frac{\mathrm{d}y'}{\eta' - y'\xi'} \tag{3.22}$$

When we consider the first and second term, the solution to this gives the group invariant or zeroth invariant $u = g(x, y)$ while the solution to the remaining pair gives the first differential invariant $v = h(x, y, y')$. The names zeroth invariant arise from $Xu = 0$ and first differential invariant from $X^{[1]}v = 0$. Now, using u, v and the derivatives of v with respect to u, the original equation (3.20) can be written as

$$G(u, v, v', \ldots, v^{(n-1)}) = v^{(n)} - H(u, v, v', \ldots, v^{(n-2)}) = 0 \tag{3.23}$$

where primes indicate total differentiation with respect to u. If the resulting equation (3.23) possesses a symmetry, then its order can be reduced further and so on.

3.2.3 Extended transformations and their infinitesimal generator

The situation of one independent and m dependent variables arises in the study of our system and as such there is need to stress the generalisation of both the Lie point transformation and the extended symmetry generator.

Consider a one parameter Lie group of transformations

$$\tilde{x} = \tilde{x}\,(x, y_i; \varepsilon)$$

$$\tilde{y}_i = \tilde{y}_i\,(x, y_i; \varepsilon)$$

for a system of one independent variable and m dependent variables such that each $y_i = y_i(x)$. Its infinitesimal transformation about $\varepsilon = 0$ is defined by

$$\tilde{x} = x + \varepsilon\xi(x, y_i) + O(\varepsilon^2) \tag{3.24a}$$

$$\tilde{y}_i = y_i + \varepsilon\eta_i(x, y_i) + O(\varepsilon^2) \tag{3.24b}$$

with symmetry generator

$$X = \xi(x, y_i)\frac{\partial}{\partial x} + \eta_i(x, y_i)\frac{\partial}{\partial y_i} \tag{3.25}$$

where $i = 1, 2, \ldots, m$.

The k^{th} extension of (3.24), given by

$$\tilde{x} = x + \varepsilon\xi(x, y_i) + O(\varepsilon^2)$$

$$\tilde{y}_i = y_i + \varepsilon\eta_i(x, y_i) + O(\varepsilon^2)$$

$$\begin{aligned} \tilde{y}_i' &= {y_i}' + \varepsilon{\eta_i}^1(x, y_i, {y_i}') + O(\varepsilon^2) \\ &\vdots \\ \tilde{y}_i^{(k)} &= {y_i}^{(k)} + \varepsilon{\eta_i}^k(x, y_i, {y_i}', \ldots, {y_i}^{(k)}) + O(\varepsilon^2) \end{aligned}$$

has its symmetry generator in the form

$$X = \xi(x, y_i)\frac{\partial}{\partial x} + \eta_i(x, y_i)\frac{\partial}{\partial y_i} + {\eta_i}^1(x, y_i, {y_i}')\frac{\partial}{\partial {y_i}'} + \cdots$$

$$+ {\eta_i}^k(x, y_i, \ldots, {y_i}^{(k)})\frac{\partial}{\partial {y_i}^{(k)}} \tag{3.27}$$

with $k = 1, 2, \ldots$ and

$${\eta_i}^k(x, y_i, \ldots, {y_i}^{(k)}) = \frac{\mathrm{D}{\eta_i}^{k-1}}{\mathrm{D}x} - {y_i}^{(k)}\frac{\mathrm{D}\xi(x, y_i)}{\mathrm{D}x} \tag{3.28}$$

It is important to note that in (3.28) D indicates total differentiation with respect to the independent variable x.

Readers interested in expanding their knowledge on the ideas discussed are referred to Bluman and Anco (2002), Bluman and Kumei (1989), Cantwell (2002), Olver (1986, 1995) and Stephani (1989).

Chapter 4

Charged spherically symmetric fluids with heat flow

4.1 Introduction

Heat flux is of great importance in astrophysical problems involving singularity formation, gravitational collapse and black hole physics, among others. This is manifested in many models including the treatment of Wagh *et al* (2001) who chose a barotropic equation of state and gave solutions to the Einstein field equations for a shear-free spherically symmetric spacetime. Maharaj and Govender (2005), when studying radiating collapse with vanishing Weyl stresses, provided exact solutions to both the Einstein field equations and the junction conditions. Herrera *et al* (2006) showed that analytic solutions can be obtained from the study of the field equations arising from radiating and collapsing spheres in the diffusion approximation. They showed that heat flow is a requirement in thermal evolution of the collapsing sphere modelled in causal thermodynamics.

Govinder *et al* (1995), Kweyama *et al* (2011), Leach and Maharaj (1992) and Msomi *et al* (2010) used Lie point symmetries to study the underlying nonlinear partial differential equations that arise in the study of gravitating fluids. In so doing, they provided several families of solutions while generalising already known solutions. Msomi *et al* (2011) studied shear-free spherically symmetric models with heat flow and used Lie analysis to obtain a five parameter family of transformations that can be used to map existing solutions to new solutions. They also generated several classes of solutions using symmetry generators. In our treatment, we incorporate charge into the model and obtain the underlying symmetries in §4.2. This is a complex calculation and we provide all the details thereof. We then use one of the symmetries obtained to provide new solutions for an arbitrary form of charge in §4.3.1. The gravitational potentials can be found explicitly. In §4.3.2, we show how the second symmetry is used to reduce the order of the governing equation. The nonlinearity of the resultant equation makes it difficult to perform the integration.

4.2 Lie analysis of the problem

We rewrite the master equation (2.36) as

$$K \equiv 4uVD_{uu} + 8uV_uD_u - 4uDV_{uu} - V^2 f(u) = 0 \tag{4.1}$$

We seek Lie point transformations of the form

$$\tilde{u} = u + \varepsilon\xi(u, V, D) + O(\varepsilon^2)$$

$$\tilde{D} = D + \varepsilon\eta^1(u, V, D) + O(\varepsilon^2)$$

$$\tilde{V} = V + \varepsilon\eta^2(u, V, D) + O(\varepsilon^2)$$

with symmetry generator given by

$$X = \xi\frac{\partial}{\partial u} + \eta^1\frac{\partial}{\partial D} + \eta^2\frac{\partial}{\partial V} \tag{4.2}$$

It is important to note that both D and V are functions of u and t but t does not appear explicitly in equation (4.1). As a result we treat (4.1) as a second order nonlinear ordinary differential equation only in u. However, we will ultimately let the constants of integration be functions of t.

Using definition 3.2.1, we now require

$$X^{[2]}K|_{K=0} = 0 \tag{4.3}$$

This expands to

$$\begin{aligned} X^{[2]}K = \xi\frac{\partial}{\partial u} + \eta^1\frac{\partial}{\partial D} + \eta^2\frac{\partial}{\partial V} + (\eta^{1'} - D'\xi')\frac{\partial}{\partial D'} + (\eta^{2'} - V'\xi')\frac{\partial}{\partial V'} \\ + (\eta^{1''} - 2D''\xi' - D'\xi'')\frac{\partial}{\partial D''} + (\eta^{2''} - 2V''\xi' - V'\xi'')\frac{\partial}{\partial V''} \end{aligned} \tag{4.4}$$

and we need to take (4.1) into account. We then combine the coefficients of the corresponding derivatives of V and D, after substituting for (4.1), to obtain the system of thirteen differential equations

$$-\xi f'(u)V^2 + \xi\frac{f(u)}{u}V^2 - 2Vf(u)\eta^2 + \frac{V^2}{D}f(u)\eta^1 + V^2 f(u)\eta^2_V - 2\xi_u V^2 f(u) - \frac{V^3}{D}f(u)\eta^1_V + 4uV\eta^1_{uu} - 4uD\eta^2_{uu} = 0 \tag{4.5a}$$

$$8u\eta^1_u - 8uD\eta^2_{uV} - 3V^2 f(u)\xi_V + 4uD\xi_{uu} + 8uV\eta^1_{uV} = 0 \tag{4.5b}$$

$$8u\eta^1_V - 4uD\eta^2_{VV} + 8uD\xi_{uV} + 4uV\eta^1_{VV} = 0 \tag{4.5c}$$

$$4uD\xi_{VV} = 0 \tag{4.5d}$$

$$-4uV\xi_{DD} = 0 \tag{4.5e}$$

$$8u\eta^2_D - 4uD\eta^2_{DD} - 8uD\xi_{uD} + 4uV\eta^1_{DD} = 0 \tag{4.5f}$$

$$8u\eta^2_u + 8uD\eta^2_{uD} + 8uV\eta^1_{uD} - 4uV\xi^1_{uu} - 2V^2 f(u)\xi_D + \frac{V^3}{D}f(u)\xi_V = 0 \tag{4.5g}$$

$$4u\eta^2 - 4u\frac{V}{D}\eta^1 - 4uV\eta^2_V - 4uD\eta^2_D + 8uV\xi_u + 4u\frac{V^2}{D}\eta^1_V + 4uV\eta^1_D - 8uV\xi_u = 0 \tag{4.5h}$$

$$-8\frac{u}{D}\eta^1 + 8u\eta^2_V - 8u\xi_u + 8u\eta^1_D - 8u\xi_u - 8uD\eta^2_{VD} - 8u\eta^2_V + 16u\xi_u + 8uD\xi_{uD} + 8uV\eta^1_{VD} + 8u\frac{V}{D}\eta^1_V - 8uV\xi_{uV} = 0 \tag{4.5i}$$

$$-8u\xi_V - 8u\xi_V + 16u\xi_V + 8uD\xi_{VD} + 8u\xi_V - 4uD\xi_{VV} = 0 \tag{4.5j}$$

$$-8u\xi_D - 8u\xi_D + 16u\xi_D + 4uD\xi_{DD} - 8uV\xi_{VD} - 8u\frac{V}{D}\xi_V = 0 \tag{4.5k}$$

$$8uV\xi_V + 4uV\xi_V + 4uD\xi_D - 8uV\xi_V = 0 \tag{4.5l}$$

$$8uV\xi_D - 8uV\xi_D + 4u\frac{V^2}{D}\xi_V - 4uV\xi_D = 0 \tag{4.5m}$$

By using equations (4.5d)–(4.5e) and (4.5j)–(4.5m), we obtain the expression for ξ to be

$$\xi(u, V, D) = C^0(u) \tag{4.6}$$

Setting

$$G(u, V, D) = \eta^2 - \frac{V}{D}\eta^1 \tag{4.7}$$

for convenience, we find, via (4.5h) that

$$G(u, V, D) = Vg\left(u, \frac{V}{D}\right) \tag{4.8}$$

where g is arbitrary.

By using (4.6)–(4.8) the system (4.5) reduces to

$$rD^2\left(r\left[\xi f' + \left(-\frac{1}{u}\xi + g - rg_r + 2\xi_u\right)f\right] + 4ug_{uu}\right) = 0 \tag{4.9a}$$

$$\xi_{uu} - 2g_u - 2rg_{ur} = 0 \tag{4.9b}$$

$$rg_{rr} + 2g_r = 0 \tag{4.9c}$$

$$r^2\left(rg_{rr} + 4g_r\right) + 4r\left(\frac{1}{D}\eta^1 - \eta^1_D\right) = 0 \tag{4.9d}$$

$$2rg_{ur} + 2g_u + \frac{4}{D}\eta^1_u - \xi_{uu} = 0 \tag{4.9e}$$

$$\frac{2}{D}\eta^1_r + 2g_r + rg_{rr} = 0 \tag{4.9f}$$

where

$$\xi = C^0(u) \tag{4.10}$$

$$\eta^2 = r\eta^1 + Vg\left(u, r\right) \tag{4.11}$$

$$r = \frac{V}{D} \tag{4.12}$$

We solve equation (4.9c) to obtain $g(u, r)$ as

$$g(u, r) = -\frac{C^2(u)}{r} + C^3(u) \tag{4.13}$$

Using equations (4.9b), (4.9e)–(4.9f) and (4.13), it can easily be shown that

$$\eta^1 = h(D) \tag{4.14}$$

By substituting (4.13) and (4.14) into (4.9d) we obtain

$$2C^2(u)\frac{D}{V} + 4\left(\frac{h}{D} - h_D\right) = 0 \tag{4.15}$$

Since h is independent of V, we compare corresponding coefficients of V in (4.15) from which two equations are obtained. Solving these equations gives

$$C^2(u) = 0 \tag{4.16}$$

$$h(D) = c_1 D \tag{4.17}$$

We then solve equation (4.9b) to obtain

$$C^3 = \frac{1}{2} C_u^0 + c_2 \tag{4.18}$$

Equation (4.9a) therefore becomes

$$2VDC_{uuu}^0 + V^2 \left[f \left(-\frac{C^0}{u} + \frac{5}{2} C_u^0 + c_2 \right) + C^0 f' \right] = 0 \tag{4.19}$$

For arbitrary f, (4.19) can only be satisfied if

$$C_{uuu}^0 = 0 \tag{4.20a}$$

$$-\frac{C^0}{u} + \frac{5}{2} C_u^0 + c_2 = 0 \tag{4.20b}$$

$$C^0 = 0 \tag{4.20c}$$

By inspection, we can deduce from equations (4.20) that

$$C^0(u) = 0 \tag{4.21}$$

and

$$c_2 = 0 \tag{4.22}$$

Using (4.6)–(4.7), (4.14) and (4.21)–(4.22) while making the necessary substitutions, we obtain the coefficient functions for the symmetry generator, when $f(u)$ is arbitrary, as

$$\xi(u) = 0 \tag{4.23}$$

$$\eta^1(D) = c_1 D \tag{4.24}$$

$$\eta^2(V) = c_1 V \tag{4.25}$$

From the above coefficient functions, we obtain the symmetry

$$X_1 = D\frac{\partial}{\partial D} + V\frac{\partial}{\partial V} \tag{4.26}$$

We now take (4.19) to be a restriction on f. As C^0 and f are functions of u, this implies that both

$$C^0_{uuu} = 0 \tag{4.27}$$

and

$$f\left(-\frac{C^0}{u} + \frac{5}{2}C^0_u + c_2\right) + C^0 f' = 0 \tag{4.28}$$

must hold. Solving equation (4.27) gives

$$C^0(u) = c_3u^2 + c_4u + c_5 \tag{4.29}$$

Using equations (4.28) and (4.29) we solve for $f(u)$ to obtain

$$f(u) = \frac{c_6 u}{(c_3u^2 + c_4u + c_5)^{5/2}} \exp\left[\frac{2c_2}{\sqrt{-c_4 + 4c_3c_5}} \arctan\left(\frac{c_4 + 2c_3u}{\sqrt{-c_4 + 4c_3c_5}}\right)\right] \tag{4.30}$$

where c_6 is a constant of integration.

In summary, (4.1) admits a Lie point symmetry with coefficient functions given by

$$\xi(u) = c_3u^2 + c_4u + c_5 \tag{4.31}$$

$$\eta^1(D) = c_1D \tag{4.32}$$

$$\eta^2(u, V) = V\left(c_1 + c_2 + c_3u + \frac{c_4}{2}\right) \tag{4.33}$$

provided

$$f(u) = \frac{c_6u}{(c_3u^2 + c_4u + c_5)^{5/2}} \exp\left[\frac{2c_2}{\sqrt{-c_4^2 + 4c_3c_5}} \arctan\left(\frac{c_4 + 2c_3u}{\sqrt{-c_4^2 + 4c_3c_5}}\right)\right] \tag{4.34}$$

For convenience, we rewrite the arbitrary constants as

$$\begin{aligned}
c_1 &= \bar{c}_1 - \frac{\bar{c}_3}{2} + \bar{c}_5 \\
c_2 &= -\bar{c}_1 \\
c_3 &= \bar{c}_4 \\
c_4 &= \bar{c}_3 \\
c_5 &= \bar{c}_2
\end{aligned}$$

and obtain

$$\xi(u) = \bar{c}_2 + \bar{c}_3u + \bar{c}_4u^2 \tag{4.35}$$

$$\eta^1(D) = \left(\bar{c}_1 - \frac{\bar{c}_3}{2} + \bar{c}_5\right) D \tag{4.36}$$

$$\eta^2(u, V) = V\left(\bar{c}_4u + \bar{c}_5\right) \tag{4.37}$$

with

$$f(u) = \frac{c_6 u}{(\bar{c}_2 u^2 + \bar{c}_3 u + \bar{c}_4)^{5/2}} \exp\left[\frac{2\bar{c}_1}{\sqrt{-\bar{c}_3^2 + 4\bar{c}_2\bar{c}_4}} \arctan\left(\frac{\bar{c}_3 + 2\bar{c}_4 u}{\sqrt{-\bar{c}_3^2 + 4\bar{c}_2\bar{c}_4}}\right)\right] \tag{4.38}$$

Henceforth, the bars on the constants are dropped for convenience.

As a final remark, we observe that, when $f = 0$, (4.19) reduces to

$$C^0_{uuu} = 0 \tag{4.39}$$

which yields

$$C^0(u) = c_3 u^2 + c_4 u + c_5 \tag{4.40}$$

As a result, we obtain the five symmetries

$$X_1 = \frac{\partial}{\partial u} \tag{4.41a}$$

$$X_2 = u\frac{\partial}{\partial u} \tag{4.41b}$$

$$X_3 = D\frac{\partial}{\partial D} \tag{4.41c}$$

$$X_4 = V\frac{\partial}{\partial V} \tag{4.41d}$$

$$X_5 = u^2\frac{\partial}{\partial u} + uV\frac{\partial}{\partial V} \tag{4.41e}$$

as obtained by Msomi *et al* (2011). They performed a full analysis of this case and we do not repeat their results here.

We are now in a position to list the symmetries of equation (4.1). In general, (4.1) admits

$$X_1 = D\frac{\partial}{\partial D} + V\frac{\partial}{\partial V} \tag{4.42}$$

When f takes on the form (4.38), (4.1) also admits

$$X_2 = \left(c_2u^2 + c_3u + c_4\right)\frac{\partial}{\partial u} + \left(c_1 - \frac{c_3}{2}\right)D\frac{\partial}{\partial D} + c_4uV\frac{\partial}{\partial V} \tag{4.43}$$

Therefore, in addition to X_1, we obtain another symmetry X_2 whose nature is dictated by the form of $f(u)$. Equations (4.35)–(4.38) match the ones obtained using SYM (Dimas *et al* 2005) which serves as a usual check of our results. As far as we are aware the two symmetries (4.42)–(4.43) are new and have not been found before for a charged gravitating fluid.

4.3 New solutions using symmetries

Usually, after obtaining the symmetries of a differential equation, we use the associated differential invariants to determine the solution(s) of the equation. In our case we will use a partial set of invariants in our quest to provide solutions.

4.3.1 Generator X_1

We obtain the invariants of X_1 by taking its first extension. The associated Lagrange's system becomes

$$\frac{\mathrm{d}u}{0} = \frac{\mathrm{d}D}{D} = \frac{\mathrm{d}V}{V} = \frac{\mathrm{d}D'}{D'} = \frac{\mathrm{d}V'}{V'} \tag{4.44}$$

We obtain the invariants of the system as

$$p = u$$

$$q(p) = \frac{V}{D}$$

$$r(p) = \frac{D'}{D}$$

$$s(p) = \frac{V'}{D}$$

However, for our purposes, we only use p, q and r. Invoking these differential invariants, (4.1) reduces to

$$q'' = -\frac{1}{4}q^2\frac{f(p)}{p} + 2qr^2 \tag{4.46}$$

which can be written as

$$r = \pm\sqrt{\frac{q''}{2q} + \frac{1}{8}q\frac{f(p)}{p}} \tag{4.47}$$

or

$$\frac{D'}{D} = \pm\sqrt{\frac{q''}{2q} + \frac{1}{8}q\frac{f(p)}{p}} \tag{4.48}$$

On integrating both sides we have

$$D = \exp\left[\pm k\int\sqrt{\frac{q''}{2q} + \frac{1}{8}q\frac{f(u)}{u}}\quad \mathrm{d}u\right] \tag{4.49}$$

where k is a constant of integration.

From solution (4.49), we can see that whenever we are given any ratio of the gravitational potentials $\frac{V}{D}$ and an arbitrary function $f(u)$ representing charge, we can explicitly obtain the exact expression of the potentials. This is a new result that to the best of our knowledge has not been obtained before. We observe that when we set $f(u) = 0$ in (4.49), we obtain the uncharged solution of Msomi *et al* (2011).

4.3.2 Generator X_2

We hope that by using the symmetry X_2, we can obtain a solution to (4.1) for the specified function (4.38) without relying on a ratio of potentials as a requirement for obtaining solutions. We now seek the invariants of X_2 by taking its second extension. The resulting Lagrange's system then becomes

$$\frac{\mathrm{d}u}{c_2 + c_3 u + c_4 u^2} = \frac{\mathrm{d}D}{D(c_1 - \frac{1}{2}c_3)} = \frac{\mathrm{d}V}{c_4 V u}$$

$$= \frac{\mathrm{d}D'}{D'\left(c_1 - \frac{3}{2}c_3 - 2c_4 u\right)} = \frac{\mathrm{d}V'}{(V c_4 - V'(c_3 + c_4 u))}$$

$$= \frac{\mathrm{d}D''}{D''\left(c_1 - \frac{5}{2}c_3 - 4c_4 u\right) - 2c_4 D'} = \frac{\mathrm{d}V''}{-V''\left(2c_3 + 3c_4 u\right)} \tag{4.50}$$

By coupling the first term of the Lagrange's system with the successive terms, we evaluate the resulting equations to obtain the invariants of the system as

$$p = (2c_1 - c_3)T(u) - \ln D \tag{4.51a}$$

$$q(p) = \ln(c_2 + c_3 u + c_4 u^2)^{\frac{1}{2}} - c_3 T(u) - \ln V \tag{4.51b}$$

$$r(p) = (2c_1 - c_3)T(u) - \ln(c_2 + c_3u + c_4u^2) - \ln D' \tag{4.51c}$$

$$s(p) = V'(c_2 + c_3u + c_4u^2)^{\frac{1}{2}} \exp\left[c_3T(u)\right] - c_4u \exp\left[-q\right] \tag{4.51d}$$

$$t(p) = V''(c_2 + c_3u + c_4u^2)^{\frac{3}{2}} \exp\left[c_3T(u)\right] \tag{4.51e}$$

with

$$T(u) = \frac{1}{\sqrt{-c_3 + 4c_2c_4}} \arctan\left[\frac{c_3 + 2c_4u}{\sqrt{-c_3 + 4c_2c_4}}\right]$$

Using Mathematica (Wolfram 2008), we reduce (4.1) to

$$r_p(p) = \frac{\exp(r)\left[4c_1 - 6c_3 - c_6 \exp(r-q) + 8s\exp(q) - 4t\exp(-p+q+r)\right]}{\exp(r)\left(4c_1 - 2c_3\right) - 4\exp(p)} \tag{4.52}$$

Equation (4.52) is a highly nonlinear first order differential equation which cannot be reduced any further. It is also difficult to integrate (4.52) and demonstrate an explicit solution.

Chapter 5

Higher dimensional shear-free relativistic models

5.1 Introduction

In this chapter, we consider shear-free spherically symmetric relativistic models in higher dimensional manifolds. Shear-free spacetimes provide avenues for modelling of relativistic stars that emit null radiation in the form of radial heat flow. Heat flow is a necessary ingredient in the study of radiating bodies because it provides avenues for the complete and proper description of these bodies. The analysis of Santos *et al* (1985) shows that the interior spacetime should contain a nonzero heat flux to match at the boundary with the exterior Vaidya spacetime. This junction condition is also applicable to relativistic models in higher dimensions. Bhui *et al* (1995) derived the Einstein field equations in higher dimensions and used them to study the non-adiabatic gravitational collapse with junction conditions. The existence of heat flux is a vital element in this study. Banerjee *et al* (2005) provided conditions under which a spherical heat conducting fluid in higher dimensions collapses without the appearance of the horizon. In our study, we aim to provide new classes of solutions in higher dimensional radiating models with heat flow without making assumptions on the pressure isotropy condition. In doing so, we can then investigate how the dimensionality affects the physical features of the system. In §5.2, we outline the algorithm first suggested by Deng (1989). We then illustrate this technique by generating the classes of solutions obtained by Bergmann (1981), Deng (1989), Maiti (1982), Modak (1984) and Sanyal and Ray (1984) on a four dimensional manifold in §5.3. In §5.4, the Einstein field equations of the higher dimensional problem are derived, and the pressure isotropy condition is transformed to a simpler form. We then use the algorithm due to Deng to provide new solutions. We show that the Deng (1989) class of solutions to the model in four dimensions is contained in our new higher dimensional solutions.

5.2 The Deng method

In this section, we seek heat conducting solutions to Einstein's equations for the line element

$$\mathrm{d}s^2 = -D^2\mathrm{d}t^2 + \frac{1}{V^2}\left[\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right] \tag{5.1}$$

and with the resulting pressure isotropy condition

$$VD_{uu} + 2V_uD_u - DV_{uu} = 0 \tag{5.2}$$

already introduced in §2.4. Deng (1989) provided a general recipe for generating a series of solutions of the isotropy condition (5.2). In solving this equation, Deng recognised that the isotropy condition is an ordinary differential equation in u (since no time derivatives appear) that can be reduced to a simple differential equation in D if V is known and vice versa. In this technique, simple forms of either V or D are chosen that are in turn substituted into the equation to be solved so as to obtain the form of the remaining term. Below we give a brief outline of the method.

- Take a simple form of V, say $V = V_1$, and substitute it into the master equation to find the most general solution of D, say $D = D_1$. The pair $V = V_1$ and $D = D_1$ provides the first class of solutions to the master equation.
- Take $D = D_1$ and substitute it into the master equation. This gives an equation in V with $V = V_1$ already a solution. We are now in a position to obtain a second solution $V = V_2$ linearly independent of V_1. The linear combination $V_3 = aV_2 + bV_1$ gives the general solution that satisfies the master equation. The pair $V = V_3$ and $D = D_1$ is the second class of solutions to the master equation.
- Take $V = V_3$ and substitute it into the master equation. We obtain an equation in D with $D = D_1$ already a solution. We are then in a position to obtain $D = D_2$ in the same way we obtained V_2. The pair $V = V_3$ and $D = cD_1 + dD_2$ is the third class of solutions to the master equation.
- Repeat this process above to obtain an infinite sequence of solutions.

It is important to note that, in principle, this is a nonterminating process of obtaining solutions, and as such an infinite number of solutions can be listed. The difficulty arises in trying to obtain the subsequent solutions because the equations become more complicated.

In other words, the problem now becomes that of integration using the available techniques. However, this procedure proves to be a powerful mechanism for generating these solutions.

We illustrate this technique by considering the four–dimensional problem before using it for the higher dimensional case.

5.3 The four–dimensional problem

In four dimensions the equation to be solved is

$$VD_{uu} + 2V_uD_u - DV_{uu} = 0 \tag{5.3}$$

Following Deng's algorithm, we set

$$\begin{aligned} D(t,u) &= D_1 \\ &= 1 \end{aligned} \tag{5.4}$$

Then (5.3) simplifies to

$$V_{uu} = 0 \tag{5.5}$$

This equation can be directly solved to give

$$\begin{aligned} V(t,u) &= V_1 \\ &= a(t)u + b(t) \end{aligned} \tag{5.6}$$

The pair of solutions D_1 and V_1 provides the first class of solutions. This model was first obtained by Bergmann (1981) and Maiti (1982) independently.

We then substitute V_1 and its corresponding derivatives back into (5.3) to obtain

$$D_{uu} + \frac{2a}{au+b} D_u = 0 \tag{5.7}$$

Equation (5.7) can be easily solved to obtain

$$\begin{aligned} D &= D_2 \\ &= \frac{cu+d}{au+b} \end{aligned} \tag{5.8}$$

with a, b, c, d being functions of t. The pair of solutions V_1 and D_2 provides the second class of solutions. This model is credited to Modak (1984) and Sanyal and Ray (1984).

We then go further to substitute D_2 with its corresponding derivatives back into (5.3). We obtain the second order equation in V

$$V_{uu} - 2\left(\frac{(bc-ad)/(au+b)^2}{(cu+d)/(au+b)}\right) V_u + 2\left(\frac{a(bc-ad)/(au+b)^3}{(cu+d)/(au+b)}\right) V = 0 \tag{5.9}$$

Since V_1 is already a solution to equation (5.9), we utilise the method of reduction of order to obtain its second solution. Suppose

$$V_2 = V_1 \alpha \tag{5.10}$$

where V_2 is the second solution and α is an arbitrary function of u and t. Then α has to satisfy the second order differential equation

$$\alpha_{uu} + 2\left(\frac{a}{au+b} - \frac{(bc-ad)}{(cu+d)(au+b)}\right)\alpha_u = 0 \tag{5.11}$$

This can be integrated once to give

$$\alpha_u = e\left(\frac{cu+d}{(au+b)^2}\right)^2 \tag{5.12}$$

where e is a constant of integration (taken to be an arbitrary function of t). We therefore obtain α in the form

$$\alpha = \int^u e\frac{(cs+d)^2}{(as+b)^4}\mathrm{d}s \tag{5.13}$$

The integral in (5.13) can be evaluated and we obtain

$$\alpha = -\frac{e}{3a(au+b)}\left(\frac{c^2}{a^2} + \frac{c}{a}\left(\frac{au+b}{cu+d}\right) + \left(\frac{cu+d}{au+b}\right)^2\right) + f \tag{5.14}$$

where f is a constant of integration (taken to be an arbitrary function of t). Therefore the second solution to (5.9) is

$$V_2 = -\frac{e}{3a}\left[\frac{c^2}{a^2} + \frac{c}{a}\left(\frac{au+b}{cu+d}\right) + \left(\frac{cu+d}{au+b}\right)^2\right] + f\,[au+b] \tag{5.15}$$

The general solution to (5.9), say V_3, is a linear combination of V_1 and V_2.

The third pair of solutions

$$D_2 = \frac{cu+d}{au+b} \tag{5.16a}$$

$$V_3 = g(t)\,(au+b) - h(t)\frac{1}{3a}\left(\frac{c^2}{a^2} + \frac{c}{a}\frac{au+b}{cu+d} + \left(\frac{cu+d}{au+b}\right)^2\right) \tag{5.16b}$$

is attributed to Deng (1989).

As already stated when outlining Deng's algorithm, this method can be extended indefinitely but difficulty arises in practice when evaluating the evolved integrals. After seeing how Deng's method can be used to provide several classes of solutions, we are now in a position to utilise this approach in solving the higher dimensional problem.

5.4 The higher dimensional problem

We consider the line element of a shear-free, spherically symmetric $(n+2)$–dimensional manifold in the form

$$\mathrm{d}s^2 = -D^2\mathrm{d}t^2 + \frac{1}{V^2}\left(\mathrm{d}r^2 + r^2\mathrm{d}X_n^2\right) \tag{5.17}$$

where $n \geq 2$. The gravitational potential components D and V are functions of r and t with

$$X_n^2 = \mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\theta_2^2 + \cdots + \sin^2\theta_1\sin^2\theta_2\ldots\sin^2\theta_{n-1}\mathrm{d}\theta_n^2 \tag{5.18}$$

For a shear-free heat conducting fluid, the energy momentum tensor is given by

$$T_{ab} = (\rho + p)U_aU_b + pg_{ab} + q_aU_b + q_bU_a \tag{5.19}$$

where ρ is the energy density, p is the kinetic pressure and q^a is the heat flux tensor. The Einstein field equations are

$$G_{ab} = T_{ab} \tag{5.20}$$

For our model we take

$$q^a = (0, q, 0, \ldots, 0) \tag{5.21}$$

and $\mathbf{U}$ is the $(n+2)$–velocity vector such that

$$U^a = \left(\frac{1}{D}, 0, 0, \ldots, 0\right) \tag{5.22}$$

which is timelike.

With reference to (5.19), we obtain the nonvanishing energy momentum tensor components as

$$T_{00} = D^2\rho \tag{5.23a}$$

$$T_{11} = \frac{1}{V^2} p \tag{5.23b}$$

$$T_{22} = \frac{r^2}{V^2} p \tag{5.23c}$$

$$T_{33} = \sin^2 \theta_1 T_{22} \tag{5.23d}$$

$$T_{44} = \sin^2 \theta_2 T_{33} \tag{5.23e}$$

$$\vdots \qquad \vdots$$

$$T_{nn} = \sin^2 \theta_{(n-2)} T_{(n-1)(n-1)} \tag{5.23f}$$

$$T_{01} = -\frac{D}{V^2} q \tag{5.23g}$$

Using (2.6), we find that the nontrivial Einstein tensor components with line element (5.17) are

$$G_{00} = \frac{n(n+1)V_t^2}{2V^2} + nD^2V^2 \left(\frac{V_{rr}}{V} - \frac{(n+1)V_r^2}{2V} + n\frac{V_r}{rV} \right) \tag{5.24a}$$

$$\begin{aligned} G_{11} = &-\frac{nD_rV_r}{VD} + \frac{nD_r}{rD} + \frac{n(n-1)V_r^2}{2V^2} - \frac{n(n-1)V_r}{rV} \\ &+ \frac{nV_{tt}}{D^2V^3} - \frac{n(n+3)V_t^2}{2D^2V^4} - \frac{nD_tV_t}{D^3V^3} \end{aligned} \tag{5.24b}$$

$$
\begin{aligned}
G_{22} = &\frac{r^2 D_{rr}}{D} - \frac{(n-1)r^2 V_{rr}}{V} + \frac{n(n-1)r^2 V_r^2}{2V^2} + \frac{(n-1)rD_r}{D} - \frac{(n-1)^2 rV_r}{V} \\
&- \frac{(n-2)r^2 D_r V_r}{DV} + \frac{nr^2 V_{tt}}{D^2 V^3} - \frac{n(n+3)r^2 V_t^2}{2D^2 V^4} - \frac{nr^2 D_t V_t}{D^3 V^3}
\end{aligned} \tag{5.24c}
$$

$$
G_{33} = \sin^2\theta_1 G_{22} \tag{5.24d}
$$

$$
G_{44} = \sin^2\theta_2 G_{33} \tag{5.24e}
$$

$$
\vdots \qquad\qquad \vdots
$$

$$
G_{nn} = \sin^2\theta_{(n-2)} G_{(n-1)(n-1)} \tag{5.24f}
$$

$$
G_{01} = n\left(\frac{V_{tr}}{V} - \frac{V_t}{V}\left(\frac{V_r}{V} + \frac{D_r}{D}\right)\right) \tag{5.24g}
$$

With the help of (5.20) we obtain the Einstein field equations. Equating the energy momentum tensor components (5.23) to the Einstein tensor components (5.24) we find that

$$
\rho = \frac{n(n+1)V_t^2}{2D^2V^2} - \frac{n(n+1)VV_r^2}{2} + nVV_{rr} + \frac{n^2 VV_r}{r} \tag{5.25a}
$$

$$
\begin{aligned}
p = &-\frac{nD_r VV_r}{D} + \frac{nD_r V^2}{rD} + \frac{n(n-1)V_r^2}{2} - \frac{n(n-1)VV_r}{r} \\
&+ \frac{nV_{tt}}{D^2 V} - \frac{n(n+3)V_t^2}{2D^2V^2} - \frac{nD_t V_t}{D^3 V}
\end{aligned} \tag{5.25b}
$$

$$
\begin{aligned}
p = &\frac{D_{rr}V^2}{D} - (n-1)VV_{rr} + \frac{n(n-1)V_r^2}{2} + \frac{(n-1)D_r V^2}{rD} - \frac{(n-1)^2 VV_r}{r} \\
&- \frac{(n-2)D_r VV_r}{D} + \frac{nV_{tt}}{D^2 V} - \frac{n(n+3)V_t^2}{2D^2V^2} - \frac{nD_t V_t}{D^3 V}
\end{aligned} \tag{5.25c}
$$

$$q = -\frac{nVV_{tr}}{D} + \frac{nV_r V_t}{D} + \frac{nD_r VV_t}{D^2} \tag{5.25d}$$

which is consistent with the derivation of Bhui *et al* (1995).

By equating equations (5.25b) and (5.25c), we obtain the pressure isotropy condition

$$-\frac{D_{rr}V^2}{D} + (n-1)VV_{rr} + \frac{D_r V^2}{rD} - 2\frac{D_r VV_r}{D} - \frac{(n-1)VV_r}{r} = 0 \tag{5.26}$$

By using the transformation

$$u = r^2$$

equation (5.26) simplifies to

$$VD_{uu} + 2D_u V_u - (n-1)DV_{uu} = 0 \tag{5.27}$$

Solving equation (5.27) is enough to determine the parameters stipulated in the system (5.25). Thus it becomes our master equation for the gravitation fluid in $(n+2)$–dimensions. It is worth noting that (5.27) reduces to the four–dimensional equation (5.3) with heat flow when $n = 2$.

Following Deng's approach, we begin with the simple case

$$\begin{aligned} D &= D_1 \\ &= 1 \end{aligned} \tag{5.28}$$

Substituting (5.28) into (5.27) gives

$$V_{uu} = 0 \tag{5.29}$$

Equation (5.29) is solved to give

$$\begin{aligned} V(u,t) &= V_1 \\ &= au + b \end{aligned} \tag{5.30}$$

with a and b being arbitrary functions of t. Therefore the pair of equations (5.28) and (5.30) solves (5.27). Thus D_1 and V_1 give the first class of solutions to our master equation (5.27). It is important to highlight that our class of solutions are independent of the dimension n. They take the same form as the solutions first obtained by Bergmann (1981) and Maiti (1982) for $n = 2$.

By substituting (5.30) into (5.27), we obtain

$$(au + b)D_{uu} + 2aD_u = 0 \tag{5.31}$$

which is solved to give

$$\begin{aligned} D(u,t) &= D_2 \\ &= \frac{cu + d}{au + b} \end{aligned} \tag{5.32}$$

with c and d arbitrary functions of t. The second class of solutions is therefore given by

$$V_1 = au + b \tag{5.33a}$$

$$D_2 = \frac{cu + d}{au + b} \tag{5.33b}$$

As for the first class, the second class of solutions is independent of the dimension n. This class is a generalisation of the solutions obtained by Modak (1984) and Sanyal and Ray (1984) when $n = 2$.

The next class of solutions is obtained by first substituting (5.32) with its corresponding derivatives into (5.27). We obtain

$$V_{uu} - \frac{2}{n-1}\left(\frac{(bc - ad)/(au + b)^2}{(cu + d)/(au + b)}\right)V_u + \frac{2}{n-1}\left(\frac{a(bc - ad)/(au + b)^3}{(cu + d)/(au + b)}\right)V = 0 \tag{5.34}$$

Equation (5.34) is a second order differential equation in V that can be solved by the method of reduction of order. We know that V_1 is a solution of (5.27). Therefore we suppose that

$$V_2 = \alpha V_1 \tag{5.35}$$

is a second solution where α is an arbitrary function of t and u. Then α has to satisfy the second order equation

$$\alpha_{uu} + \left[2\frac{V_1'}{V_1} - \frac{2}{n-1}\left(\frac{(bc-ad)/(au+b)^2}{(cu+d)/(au+b)}\right)\right]\alpha_u = 0 \tag{5.36}$$

By substituting for V_1, (5.36) becomes

$$\alpha_{uu} = \frac{2}{n-1}\left(\frac{c}{cu+d} - n\frac{a}{au+b}\right)\alpha_u \tag{5.37}$$

Integrating (5.37) gives

$$\alpha_u = e\left(\frac{cu+d}{(au+b)^n}\right)^{2/(n-1)} \tag{5.38}$$

with e being an arbitrary function of t. We can express α in the form

$$\alpha = \int^u e\left(\frac{cs+d}{(as+b)^n}\right)^{2/(n-1)} \mathrm{d}s \tag{5.39}$$

To evaluate this integral we need to consider two cases: $ad = bc$ and $ad \neq bc$.

5.4.1 Case I

Considering the special case when

$$\frac{c}{d} = \frac{a}{b} \tag{5.40}$$

(5.39) reduces to

$$\alpha = k\int \left(1 + \frac{a}{b}u\right)^{-2} du \tag{5.41}$$

where we have set

$$k = e\left(\frac{d}{b^n}\right)^{\frac{2}{n-1}} \tag{5.42}$$

Evaluating the integral in (5.41) gives

$$\alpha = \frac{a^2gu + (abg - kb^2)}{a(au+b)} \tag{5.43}$$

where g is an arbitrary function of t. Therefore V_2 becomes

$$\begin{aligned} V_2 &= \alpha V_1 \\ &= agu + \left(bg - lb^2\right) \end{aligned} \tag{5.44}$$

with $l = \frac{k}{a}$. A closer look at solution (5.44) shows that V_2 takes the form of V_1 and therefore we do not have a second linearly independent solution. This is expected as condition (5.40) is degenerate with respect to (5.39).

5.4.2 Case II

For this case $ad \neq bc$. In (5.39) we let

$$au + b = s \tag{5.45}$$

This simplifies the integral to

$$\alpha = \frac{e}{a}\int s^{-2}\left(p + ms^{-1}\right)^{\frac{2}{n-1}} \mathrm{d}s \tag{5.46}$$

where we let

$$p = \frac{c}{a}\,, \quad m = \frac{ad - bc}{a}$$

By defining

$$p + ms^{-1} = k \tag{5.47}$$

we further simplify the integral (5.46) to

$$\alpha = -\frac{e}{am} \int k^{2/(n-1)} \mathrm{d}k \tag{5.48}$$

On evaluating (5.48), we obtain the expression for α as

$$\alpha = -\frac{e}{am} \left(\frac{n-1}{n+1}\right) \left(p + ms^{-1}\right)^{\frac{n+1}{n-1}} + g \tag{5.49}$$

where g is an arbitrary function of t. Substituting for m, p and s in (5.49) gives

$$\alpha = \frac{e}{ad-bc} \left(\frac{1-n}{n+1}\right) \left(\frac{cu+d}{au+b}\right)^{\frac{n+1}{n-1}} + g \tag{5.50}$$

We therefore obtain the second solution V_2 using (5.35) as

$$V_2 = \left(\frac{e}{ad-bc} \left(\frac{1-n}{n+1}\right) \left(\frac{cu+d}{au+b}\right)^{\frac{n+1}{n-1}} + g\right)(au+b) \tag{5.51}$$

The general solution to equation (5.34) is given by the linear combination of V_1 and V_2

$$V_3 = h(t)V_1 + j(t)V_2$$

$$= \left(h(t) + j(t)\left(\frac{e}{ad-bc} \left(\frac{1-n}{n+1}\right) \left(\frac{cu+d}{au+b}\right)^{\frac{n+1}{n-1}} + g\right)\right)(au+b) \tag{5.52}$$

The third class of solutions to equation (5.27) is therefore given by

$$D_2(u,t) = \frac{cu+d}{au+b} \tag{5.53a}$$

$$V_3(u,t) = \left(h(t) + j(t)\left(\frac{e}{ad-bc} \left(\frac{1-n}{n+1}\right) \left(\frac{cu+d}{au+b}\right)^{\frac{n+1}{n-1}} + g\right)\right)(au+b) \tag{5.53b}$$

The class of solutions (5.53) is a new pair of exact solutions to the Einstein field equations in higher dimensions. Unlike the two earlier classes, this third class of solutions depends on the parameter n. Hence the dimension of the spacetime does directly affect the dynamics of the gravitational field. The next class of solutions can be obtained by substituting V_3 into equation (5.27) and then solve the resulting equation for D_3. The integration is not performed in this thesis because of the complexity of the process.

It appears that with $n = 2$ in (5.53b), we obtain a result different from the known result of Deng (1989). This is not the case and we can show the equivalence between the two solutions. However, this is not a straightforward exercise. It is important to note that when we substitute $n = 2$ in (5.53), we obtain a solution that satisfies (5.27) for $n = 2$. This solution however possesses a subtle difference to that provided by Deng (1989). This observation is illustrated by obtaining the difference between F_2 (in Deng's notation) and V_2 for the case when $n = 2$. We know that

$$F_2 = -\frac{1}{3a}\left[\frac{c^2}{a^2} + \frac{c}{a}\left(\frac{au+b}{cu+d}\right) + \left(\frac{cu+d}{au+b}\right)^2\right] \tag{5.54}$$

from Deng (1989). For $n = 2$, we obtain V_2 in the form

$$V_2 = \left(-\frac{e}{3(ad-bc)}\left(\frac{cu+d}{au+b}\right)^3 + g\right)(au+b) \tag{5.55}$$

from (5.53b). Now the difference between V_2 and F_2 gives

$$\begin{aligned} V_2 - F_2 = &-\left[\frac{c^3(ea^3u^3 + 3a^2bu^2 + 3ab^2u + b^3) + a^3d(e-1)\left(3cu(cu+d) + d^2\right)}{3a^2(ad-bc)(au+b)^2}\right] \\ &+ g\left[au+b\right] \end{aligned} \tag{5.56}$$

When $e = 1$, equation (5.56) simplifies to

$$V_2 - F_2 = \left(g - \frac{c^3}{3a^3(ad-bc)}\right)(au+b) \tag{5.57}$$

This therefore means that

$$V_2 = F_2 + \kappa\,(au+b) \tag{5.58}$$

where we have set

$$\kappa = \left(g - \frac{c^3}{3a^3(ad - bc)} \right) \tag{5.59}$$

It is actually amazing how a cubic expression (5.55) with the term $\left[(cu+d)/(au+b)\right]^3$ differs from a quadratic expression (5.54) with the term $\left[(cu+d)/(au+b)\right]^2$ by the linear factor $au+b$. The quantity (5.59) essentially plays no part in the general solutions to the master equation (5.27) when $n = 2$. We show this by first obtaining the general solution to (5.34). In this case V_3 becomes

$$\begin{aligned} V_3 &= hV_1 + jV_2 \\ &= h(au+b) + j\left(F_2 + \kappa\,(au+b)\right) \\ &= M(au+b) - \frac{j}{3a}\left(\frac{c^2}{a^2} + \frac{c}{a}\left(\frac{au+b}{cu+d}\right) + \left(\frac{cu+d}{au+b}\right)^2\right) \end{aligned} \tag{5.60}$$

where $M = h + \kappa$ and h, j, κ and M are functions of t.

Then the pair of solutions to (5.27) is

$$D_2 = \frac{cu+d}{au+b} \tag{5.61a}$$

$$V_3 = M\,(au+b) - \frac{j}{3a}\left(\frac{c^2}{a^2} + \frac{c}{a}\frac{au+b}{cu+d} + \left(\frac{cu+d}{au+b}\right)^2\right) \tag{5.61b}$$

for $n = 2$. We can therefore conclude that the final solution due to Deng shown by equations (5.16) has the same form as our solution (5.61).

Finally, we note that when V is taken to be linear in u, n does not appear in equation

(5.27). Thus all solutions with linear V do not contain the dimension n. It is interesting that such a result is possible in higher dimensions.

Chapter 6

Conclusion

In this dissertation we aimed at providing a comprehensive study of shear-free spherically symmetric spacetimes with charge and heat flux, applicable in various relativistic and cosmological studies of gravitating bodies.

In our first study we obtained new exact solutions to the Einstein-Maxwell system of charged relativistic fluids in the presence of heat flux. Solutions to this highly nonlinear system were obtained by essentially solving the pressure isotropy condition. A suitable transformation reduced the master equation to a second order nonlinear differential equation. We solved the resulting equation by using Lie's group theoretic approach. The new class of solutions contains the result obtained by Msomi *et al* (2011) as a special case.

In our second study, we obtained new generalised classes of exact solutions to the Einstein field equations for a neutral relativistic fluid in the presence of heat flow in a higher dimensional manifold. We found solutions to the coupled Einstein system by solving the higher dimensional pressure isotropy condition which is a second order nonlinear differential equation. We solved the master equation by making use of Deng's algorithm (Deng 1989) and obtained three classes of solutions. These classes of solutions in higher dimensions generalise the results obtained previously by Bergmann (1981), Maiti (1982), Modak (1984) and Sanyal and Ray (1984) in four dimensions. The solutions of Deng (1989) are contained in this new class of solutions.

In chapter 2, we introduced the essential definitions and expressions underlying differential geometry that are relevant to our study. We generated the Einstein field equations and the Einstein-Maxwell equations for both a neutral matter distribution and a charged matter distribution with heat flow in spherically symmetric spacetimes. We provided reasons for pursuing this project by listing relevant examples from previous investigations.

In chapter 3, we reviewed the relevant ideas underlying Lie's group theoretic approach as a technique used for solving differential equations. The aspect of order of reduction was emphasized as a necessary method that we use in our research. We also highlighted how this method can be used to solve a differential equation involving one independent variable and m dependent variables by generalising this approach to suit our problem.

In chapter 4, we aimed at providing exact solutions to a charged shear-free spherically symmetric model with heat flow. The pressure isotropy condition

$$VD_{uu} + 2V_uD_u - DV_{uu} - \frac{V^2}{4u}f(u) = 0 \tag{6.1}$$

was derived. This equation was analysed using Lie's group theoretic approach. For an arbitrary form of charge, we obtained

$$X_1 = D\frac{\partial}{\partial D} + V\frac{\partial}{\partial V}$$

When f is of the form

$$f(u) = \frac{c_6 u}{(c_2 u^2 + c_3 u + c_4)^{\frac{5}{2}}} \exp\left[\frac{2c_1}{\sqrt{-c_3^2 + 4c_2 c_4}} \arctan\left(\frac{c_3 + 2c_4 u}{\sqrt{-c_3^2 + 4c_2 c_4}}\right)\right]$$

we obtained the additional symmetry

$$X_2 = \left(c_2 u^2 + c_3 u + c_4\right)\frac{\partial}{\partial u} + \left(c_1 - \frac{c_3}{2}\right) D\frac{\partial}{\partial D} + c_4 V u\frac{\partial}{\partial V}$$

The function $f(u)$ dictated the form of the second symmetry X_2. We then used the first symmetry X_1 to obtain new classes of solutions represented by

$$D = \exp\left[\pm k \int \left(\frac{q''}{2q} + \frac{1}{8} q \frac{f(u)}{u}\right)^{1/2} \mathrm{d}u\right]$$

where k is a constant of integration and q is the ratio of the gravitational potentials $\frac{V}{D}$. This result means that we can explicitly obtain the exact expressions of the potentials whenever we are given any ratio of the gravitational potentials for any arbitrary form of charge $f(u)$. We emphasize that this is a new result that to the best of our knowledge has not been obtained before. We regain the results of Msomi *et al* (2011) when $f(u) = 0$. Using the second symmetry X_2, we reduced the pressure isotropy condition to

$$r_p(p) = \frac{\exp(r)\left[4c_1 - 6c_3 - c_6 \exp(r - q) + 8s\exp(q) - 4t\exp(-p + q + r)\right]}{\exp(r)\left(4c_1 - 2c_3\right) - 4\exp(p)}$$

where, q, r, s and t are functions of p. The integration of this equation is difficult and we have not established an explicit solution.

In chapter 5, we constructed the model of a shear-free spherically symmetric spacetime with heat flow defined in a higher dimensional manifold. We derived the Einstein field equations from which we obtained the pressure isotropy condition

$$VD_{uu} + 2D_uV_u - (n-1)DV_{uu} = 0 \tag{6.2}$$

which is a highly nonlinear second order differential equation. We have solved this higher dimensional equation to provide three classes of solutions. The first class of solution we obtained was

$$D_1 = 1 \qquad\qquad V_1 = au + b$$

This class of solution generalises the one obtained by Bergmann (1981) and Maiti (1982). The second class of solution we obtained was

$$V_1 = au + b \qquad\qquad D_2 = \frac{cu+d}{au+b}$$

This class of solution generalises the results obtained by Modak (1984) and Sanyal and Ray (1984). It is important to note that both classes of solutions are independent of the dimension of the manifold. We obtained the third class of solutions as

$$D_2 = \frac{cu+d}{au+b}$$

$$V_3 = \left(h(t) + j(t)\left(\frac{e}{ad-bc}\left(\frac{1-n}{n+1}\right)\left(\frac{cu+d}{au+b}\right)^{\frac{n+1}{n-1}} + g\right)\right)(au+b)$$

We believe that this result has not been found previously and is not contained in models that appear in the literature. This new class of solution depends on the dimension n of the spacetime. This class of solution also contains the one obtained by Deng (1989).

Given the success of Lie symmetry analysis in handling (6.1), we intend to apply the method to the charged analogue of (6.2) in future work.

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