

ERMAKOV SYSTEMS
A Group Theoretic Approach

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Abstract

The physical world is, for the most part, modelled using second order ordinary differential equations. The time-dependent simple harmonic oscillator and the Ermakov–Pinney equation (which together form an Ermakov system) are two examples that jointly and separately describe many physical situations.

We study Ermakov systems from the point of view of the algebraic properties of differential equations. The idea of generalised Ermakov systems is introduced and their relationship to the Lie algebra $sl(2, R)$ is explained. We show that the ‘compact’ form of generalized Ermakov systems has an infinite-dimensional Lie algebra. Such algebras are usually associated only with first order equations in the context of ordinary differential equations. Apart from the Ermakov invariant which shares the infinite-dimensional algebra of the ‘compact’ equation, the other three integrals force the dimension of the algebra to be reduced to the three of $sl(2, R)$.

Subsequently we establish a new class of Ermakov systems by considering equations invariant under $sl(2, R)$ (in two dimensions) and $sl(2, R) \oplus so(3)$ (in three dimensions). The former class contains the ‘generalized Ermakov system’ as a special case in which the force is velocity-independent. The latter case is a generalization of the classical equation of motion of the magnetic monopole which is well known to possess the conserved Poincaré vector. We demonstrate that in fact there are three such vectors for all equations of this type.

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

A handwritten signature in dark ink, consisting of several loops and a long horizontal stroke at the end.

K S Govinder

November 1993

Dedication

To the Cetaceans

Acknowledgments

I thank my supervisor, Professor P G L Leach, for his academic, financial and personal help during the course of this work. He has done more than is expected of a supervisor, colleague or close friend in shaping this dissertation into a polished academic document, improving and refining my techniques of approaching and solving problems and developing my personal self into a mature and worldly person. His infinite patience and sincere dedication to his research and students have played an influential role in moulding my ideals.

I also thank him for introducing me to the captivating world of differential equations and the superb Lie method for the solutions thereof. It is apparent that I have now found my niche. On the technical side all the computational facilities and many of the technical and general references used in this work have been generously provided by him. He is also thanked for introducing me to the wordprocessing package \LaTeX and symbolic manipulation packages `Program LIE` and `Mathematica` which have been used extensively in this work. Finally I thank him for his careful and painstaking proof reading of the manuscript, a task that has introduced us both to a slightly different side of the other.

I thank Drs S D Maharaj and F M Mahomed for explaining and clarifying many details that were intrinsic to this work. A special word of thanks to Dr Maharaj for setting me on the path to differential equations by introducing me to Professor Leach in early 1991 and providing both financial help and personal advice as and when the need arose.

Finally I thank Ryan Lemmer whose unique skills at making coffee have ensured that sanity prevailed during the final days of this work and David White whose innumerable emails (many of which are non-publishable) have never failed to place an expression of bemusement, joy or ecstasy on an oft tired countenance.

Preface

Since the mathematical models of many phenomena of the real world are formulated in the form of differential equations, it becomes clear that its use in the general theory of differential equations is one of the most essential applications of Lie group theory.

Applying his theory to the differential equations of mechanics, Lie stated: "The principles of [Newtonian] mechanics originate in the theory of [continuous] groups. ... Kinematics and its theorems are partly subsumed as very special cases in my general theorems. My researches on geodesics and the general problem of equivalence in differential equations, show by what principles mechanics can be successfully handled." [10].

One could wax lyrical about the many and varied uses and applications of differential equations. However, it would be dishonest to imply that they gave impetus to this study. Differential equations possess an innate beauty that, once coaxed out, has an æsthetic appeal second to none. This, coupled with the desire to seek solutions to them, provided the main motivation behind this work. It is hoped that many more will fall, as I did, under the charm of differential equations and, in feeding their personal desires for knowledge, will help us all reach a better understanding of the world around us.

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Chapter 1

Introduction

We look at the origin of Ermakov systems and provide a survey of work done in the field.

1.1 Differential equations

Solutions (both numeric and analytic) to differential equations have been sought since they were introduced in the late seventeenth century. Naturally it is infinitely more desirable to obtain analytic solutions as opposed to numeric. While this is not always possible many techniques have been developed to help in the former process.

Inspired by Galois' theory of algebraic equations, M S Lie sought an analogue for the theory of differential equations. While he was not succesful in this endeavour, his theory of the invariance of differential equations under infinitesimal transformations has collected the various techniques used previously into a concise method for the construction of solutions of differential equations. Indeed his theory eventually led to Vessiot's structural theory of linear differential equations.

Besides this excellent theory of differential equations, two other techniques stand out. The first, closely associated with Lie's work, is that of Noether. Amalie Emmy Noether, who is currently better known for her work in ab-

stract algebra, made an important contribution to theoretical physics. While the work was mathematical, the applications in physics led to her being discussed under ‘Intellectual contributions of Women to Physics’ in *Women of Science*[47]. Noether worked on the theory of invariants all her life, but, under the influence of Ernst Fischer, she focussed on the field of algebraic invariant theory[16] and neglected the differential invariant theory which was the basis of her dissertation. She went to the extent of dismissing her earlier work on differential invariants as “crap” and was harsh in admonishing others who worked in the same field[22]. It is ironic that this work has given rise to the famous Noether’s Theorem of variational calculus that theoretical physicists use extensively, especially in modern elementary particle physics. Simply stated it says that, if the action integral is invariant under a group of mappings of functions into functions, the stationary functions satisfy corresponding restrictive conditions, which in the case of simple-integral problems take the form of first integrals of the Euler–Lagrange equations¹. These first integrals include the conservation laws of physics. As knowledge of conserved quantities enables one to better understand the evolution of the state of a system, this theorem has obvious applications to physical phenomena.

In the study of nonlinear equations it is found that solutions may have moveable singularities, ie., the location is based on initial conditions. Painlevé² and his co-workers performed an exhaustive study of all second order ordinary differential equations of the form

$$\frac{d^2w}{dz^2} = F(z, w, w') \quad (1.1.1)$$

with F rational in w' , algebraic in w and analytic in z , the critical points of which were fixed, ie., they were able to identify all equations which have their moveable singularities poles. In essence, a system of ordinary differential equations (in the complex plane) is said to have the *Painlevé property* when every

¹The technical details are left to Appendix A.

²Whose life is remarkably undocumented.

solution is single-valued, except at the fixed singularities of the coefficients[27]. Thus the Painlevé property requires that all of its solutions be free of moveable critical points³.

These methods have proven to be successful in either providing solutions to or determining properties of differential equations. We utilise them extensively (especially the Lie method) in our analysis of Ermakov systems.

1.2 Invariance

“Invariance is changelessness in the midst of change, permanence in a world of flux, the persistence of configurations that remain the same despite the swirl and stress of countless hosts of curious transformations.”[10]

The concept of invariance, first seriously considered in 1841, became one of the major influences of mathematical thought only thirty years later. The main ideas were considered earlier by Lagrange in the eighteenth century in the course of his arithmetical investigation of binary quadratic forms. Thereafter Gauss, Boole and Cayley (amongst others) made many and various contributions to this trend, but it was Sophus Lie who used this concept to make a significant impact to the world of differential equations in his theory of transformation groups. There has been much written on the Lie theory[4, 12, 86] and on Lie himself, eg [77, 118], and we refer the interested reader to these works for further information.

The idea of symmetry, integrally linked to that of invariance, only evolved into a fundamental idea of mathematics and science in the nineteenth century (It had long been the sole concern of artists.). The very origins of mathematical thought dating back to the Paleolithic period contain evidence indicating the allure of symmetry even at that early stage. More recently the theorems

³See Appendix B for a more detailed treatment of the Painlevé analysis of ordinary differential equations.

of Thales involving congruences of triangles and circles have been based on symmetry[42]. One even finds evidence of symmetry in the last books of Euclid's *Elements*[26] which are devoted to polyhedra⁴. However, it was not until the nineteenth century that Sophus Lie and Felix Klein used the notion of symmetry to lay the foundation for ideas which have had a profound impact on twentieth century mathematics and physics. The ideas behind Lie's theory of differential equations are discussed in §2.2.

1.3 The Harmonic Oscillator

Our work originates with the study of the well-known harmonic oscillator. The equation of motion for a harmonic oscillator with variable frequency is represented by

$$\ddot{q} + \omega^2(t)q = 0, \tag{1.3.1}$$

where the overdot represents differentiation with respect to time. In 1979 Korsch[49] stated: "Over the past decade a considerable amount of work has been devoted to the study of the time-dependent classical oscillator ...". Fourteen years on the interest has not diminished.

The most notable occurrence was at the first Solvay Conference[71] where Lorentz conjectured that

$$I := \frac{\dot{q}^2}{\omega} + \omega q^2 \tag{1.3.2}$$

was approximately constant provided $\omega(t)$ was slowly varying, ie., I is an adiabatic invariant[114]. Over fifty years later there was resurgence of interest[71, 72] specifically in applications to controlled thermonuclear fusion[66, 67]. However, it is only now becoming known that the problem of the time-dependent oscillator was first solved some thirty years before Lorentz's conjecture by V Ermakov[28] in 1880. (Ermakov's interest was in finding conditions of complete integrability for second order ordinary differential equations. He first showed

⁴Although subsequent Greek scholars scoffed at his techniques.

that any linear second order differential equation of the form

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + By = 0 \quad (1.3.3)$$

could be transformed to

$$\frac{d^2z}{dx^2} = \left(\frac{1}{4} + \frac{1}{2} \frac{da}{dx} - B \right) z \quad (1.3.4)$$

by setting

$$y = ze^{-\frac{1}{2} \int a dx}.$$

Eq (1.3.4) is now of a form that makes it easy to discover the integrability conditions and is essentially just (1.3.1).) Ermakov obtained a first integral for (1.3.1) by introducing the auxiliary equation

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3}, \quad (1.3.5)$$

eliminating the ω^2 terms, multiplying by the integrating factor

$$\rho \dot{q} - \dot{\rho} q$$

and integrating the resulting differential equation. (Eq (1.3.5) is often called the Pinney equation after Edmund Pinney who, in a delightfully concise paper, first provided solutions to it in 1950[88]. However, since it is now known that the first documented occurrence of the equation is due to Ermakov, we shall refer to it as the Ermakov–Pinney equation⁵ in future. This equation is endemic to time-dependent systems[61, 64, 78] and will be considered in some detail later. Its generalisation has more recently attracted attention[39, 93]⁶.) We observe that emphasis has shifted from just the one equation (1.3.1) to the combination of (1.3.1) and (1.3.5) which we call the Ermakov system⁷.

⁵It has also been called the Lewis–Pinney equation[79]. Most Russians, of course, omit any reference to anyone bar Ermakov. The question of appropriate names of equations, invariants, transformations and the like crops up with some regularity in this work. However, to paraphrase, “What’s in a name? That which we call an equation by any other word would remain as difficult to solve.” (with apologies to William Shakespeare[109, Act II Sc II]).

⁶Does this perhaps suggest a suitable renaming?

⁷Sometimes called Ray–Reid systems[104].

The first integral obtained by Ermakov is

$$I = \frac{1}{2} [(\rho\dot{q} - \dot{\rho}q)^2 + (q/\rho)^2], \quad (1.3.6)$$

which is usually called the Lewis invariant⁸. Lewis rediscovered this invariant in 1966[66, 67]⁹ in his study of the motion of charged particles moving in magnetic fields. In plasma physics one proves the adiabatic invariance of the orbital magnetic moment of a charged particle in a magnetic field[108]. Lewis proved, using the asymptotic method of Kruskal[52], that (1.3.6) was an exact invariant, a special case of which is the adiabatic invariance of the orbital magnetic moment. The existence of this invariant (or family of invariants) enables progress with some time-dependent quantum mechanical problems. Lewis and Reisenfeld[68] have used this feature in the development of a method whereby the eigenstates of the invariant are used to construct the solution of the Schrödinger equation for the system. It has also been shown that the Feynman propagator can be written in terms of the eigenfunctions of (1.3.6)[48].

Several other derivations of (1.3.6) have been proposed in the literature: Leach[53, 54, 55, 56, 57, 58] used time-dependent canonical transformations, Lutzky[73] derived it from Noether's theorem and Ray and Reid[92] presented a direct proof of its dynamical invariance. However, the last approach assumes that the auxilliary equation (1.3.5) is already known. This may be an obstacle in the more general case. Eliezer and Gray[24] provided a more physical interpretation in terms of an auxilliary rotational motion. Finally, Korsch[49] developed a simplified algebraic derivation based on the concept of the dynamical algebra.

Since the work of Ermakov was published in Russian in the Reports of the University of Kiev (in the nineteenth century) it was largely lost to the aca-

⁸Or more correctly the Ermakov-Lewis invariant. There are also instances in the literature[6, 7], where (1.3.6) has been called the Lewis-Ray-Reid invariant.

⁹The papers date to '67 and '68. However the work was done during 1966 whilst Lewis was on study leave in Heidelberg.

demic world¹⁰ except, perhaps, for a few select academics in the University of Kiev[11]. The significance of his work was only realised when the possible physical applications of eqq (1.3.1) and (1.3.5) became apparent. As pointed out above the equations of motion of a charged particle in certain types of magnetic fields may be reduced to the form (1.3.1). In a semiclassical theory of radiation one considers the notion of an oscillator with variable frequency. There are also mechanical problems of oscillating systems where the equations of motion lead to (1.3.1), eg, the lengthening pendulum[103]. Lewis and Riesenfeld[68] presented an exact quantum theory for the time-dependent Shrödinger equation corresponding to (1.3.1). Leach *et al* [64], in a most appropriately titled paper¹¹, pointed out further applications including the Schrödinger equation for the signal passing along an optical fibre and the Schrödinger equation for the particle distribution function in a Fabry–Pérot cavity.

1.4 The Ermakov–Pinney equation and time-dependent systems

Ermakov’s auxilliary equation

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3} \quad (1.4.1)$$

deserves more attention as it arises in all sorts of physical and mathematical situations. Besides being intrinsic to time-dependent systems, it is of great importance in reduction of order of differential equations. More attention will be paid to this property in the next chapter.

The time-dependent oscillator

$$\ddot{q} + \omega^2(t)q = 0 \quad (1.4.2)$$

¹⁰Some may be tempted to argue that it was only lost to the West. However, the well-known Russian academic N H Ibragimov only learned of it in his visit to Sicily in 1992.

¹¹The ubiquitous time-dependent simple harmonic oscillator.

is transformed to the time-independent oscillator

$$Q'' + \Omega^2 Q = 0 \quad (1.4.3)$$

by the transformation

$$Q = \frac{q}{\rho} \quad T = \int \rho^{-2}(t) dt, \quad (1.4.4)$$

where $\rho(t)$ satisfies the equation

$$\ddot{\rho} + \omega^2(t)\rho = \frac{\Omega^2}{\rho^3}. \quad (1.4.5)$$

Taking Ω^2 in eq (1.4.5) as unity we can write the solution to (1.4.3) as

$$Q(T) = A \sin T + B \cos T, \quad (1.4.6)$$

where A and B are constants of integration. The solution of (1.4.2) is then

$$q(t) = A\rho(t) \sin(T(t)) + B\rho(t) \cos(T(t)). \quad (1.4.7)$$

The solution to (1.4.5), given by Pinney[88] (and derived by Athorne [7] using projective geometry), is

$$\rho^2(t) = au^2 + bv^2 + 2cuv, \quad ab - c^2 = 1, \quad (1.4.8)$$

where a, b and c are constants and u and v are two linearly independent solutions of

$$\ddot{u} + \omega^2(t)u = 0, \quad (1.4.9)$$

the original time-dependent oscillator! To quote Leach *et al* [64] "... we have a situation in which the problem posed can be solved provided we can solve the posed problem". There are comparatively few instances in the literature where solutions to (1.4.5) are provided[24, 64, 66] and we must resort to numerical techniques in general.

Thus far the development has been less than satisfactory. However, we have only considered the simplest scenario to which the transformation (1.4.4)

applies and more often than not simplicity implies duplicity. The same transformation and its Hamiltonian version (also known as the Lewis–Leach variables[69, 70]),

$$Q = \frac{q - \alpha}{\rho}, \quad P = \rho(q - \dot{\alpha}) - \dot{\rho}(q - \alpha), \quad T = \int \rho^{-2}(t)dt, \quad (1.4.10)$$

where α is an arbitrary function of time, can be and is used in both classical and quantum mechanical problems¹². The Hamiltonian[69]

$$H = \frac{1}{2}p^2 - F(t) + \frac{1}{2}\omega^2(t)q^2 + \frac{1}{\rho^2}U\left(\frac{q - \alpha}{\rho}\right) \quad (1.4.11)$$

can be transformed to autonomous form, viz.

$$\bar{H} = \frac{1}{2}P^2 + U(Q) \quad (1.4.12)$$

provided ρ satisfies (1.4.5) and α is a solution of

$$\ddot{\alpha} + \omega^2(t)\alpha = F(t). \quad (1.4.13)$$

Finally we note that (1.4.5) reduces to the time-dependent simple harmonic oscillator when $\Omega = 0$.

The relationship between the equation of motion for the time-dependent harmonic oscillator and the Ermakov–Pinney equation has been expounded upon in[64] in some detail and we just mention that Eliezer and Günther[25] focus on the solution of

$$\ddot{\rho}(t) + \omega^2(t)\rho(t) + \lambda/\rho^3(t) = 0$$

which is the radial equation of motion for a harmonic oscillator in a uniform magnetic field.

1.5 Generalised Ermakov systems

Ray and Reid, who were primarily interested in the relationship between solutions of eq (1.3.1) and certain non-linear equations, (jointly and separately)

¹²For a detailed treatment of the Schrödinger equation for the signal passing along an optical fibre see[64].

popularised Ermakov systems in the western literature[92, 93, 94, 95, 96, 97, 100] and introduced generalised Ermakov systems of the form

$$\ddot{x} + \omega^2(t)x = \frac{1}{x^3}f(y/x) \quad (1.5.1a)$$

$$\ddot{y} + \omega^2(t)y = \frac{1}{y^3}g(y/x), \quad (1.5.1b)$$

where f and g are arbitrary functions of their arguments. (Athorne[8] introduced an interesting variation of (1.5.1) with his Kepler–Ermakov system

$$\ddot{x} = -\frac{x}{r^3}H\left(\frac{y}{x}\right) + \frac{1}{x^3}X\left(\frac{y}{x}\right)$$

$$\ddot{y} = -\frac{y}{r^3}H\left(\frac{y}{x}\right) + \frac{1}{y^3}Y\left(\frac{y}{x}\right)$$

which reduces to a class of (autonomous) Ermakov systems when H is taken to be identically zero, and a Kepler problem with an angle-dependent central force when X and Y are identically zero.) One can obtain a first integral for the system (1.5.1) in a manner similar to the one used in §1.3[92], ie eliminating the ω^2 term and multiplying by an integrating factor, this time

$$x\dot{y} - \dot{x}y.$$

The first integral thus obtained is

$$I = \frac{1}{2}(x\dot{y} - \dot{x}y)^2 + \int^{y/x} [uf(u) - u^{-3}g(u)] du. \quad (1.5.2)$$

Leach[61] showed that the presence of the ω^2 terms in (1.5.1) is misleading. Firstly, we use the transformation

$$T = \cot\left(\int \rho^{-2}dt\right)$$

$$X = \rho^{-1}x \csc T, \quad Y = \rho^{-1}y \csc T,$$

together with (1.3.5) to transform (1.5.1) to

$$\ddot{X} = \frac{1}{X^3}f\left(\frac{Y}{X}\right) \quad (1.5.3a)$$

$$\ddot{Y} = \frac{1}{Y^3}g\left(\frac{Y}{X}\right). \quad (1.5.3b)$$

Secondly, the ω^2 can be replaced by *anything* [34, 61, 93, 96] and the terms can still be eliminated giving

$$x\ddot{y} - \ddot{x}y = \frac{x}{y^3}g\left(\frac{y}{x}\right) - \frac{y}{x}f\left(\frac{y}{x}\right). \quad (1.5.4)$$

It is this compact form of the Ermakov system that we use in our Lie analysis.

Leach[61] further showed that the generalised Ermakov systems of Ray and Reid (1.5.1) were really just Cartesian forms of a system of equations that have the polar form

$$\ddot{r} - r\dot{\theta}^2 = \frac{F(\theta)}{r^3} \quad (1.5.5a)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{G(\theta)}{r^3}. \quad (1.5.5b)$$

We change to plane polars to understand better the nature of generalised Ermakov systems. This change is suggested by the fact that the Ermakov system has an integral which is of angular momentum type[61].

The Ermakov invariant is obtained from the angular component of (1.5.5) and is

$$\begin{aligned} I &= \frac{1}{2} \left(r^2 \dot{\theta} \right)^2 - \int G(\theta) d\theta \\ &= \frac{1}{2} [p_\theta^2 + F(\theta)] \end{aligned} \quad (1.5.6)$$

if the system is Hamiltonian.

This suggests an obvious generalisation to higher dimensions which, using polar coordinates, is straightforward. We express our system in three dimensions as

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = \frac{F(\theta, \phi)}{r^3} \quad (1.5.7a)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta = \frac{G(\theta, \phi)}{r^3} \quad (1.5.7b)$$

$$r\sin\theta\ddot{\phi} + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta = \frac{H(\theta, \phi)}{r^3\sin\theta}. \quad (1.5.7c)$$

$$(1.5.7d)$$

We obtain a first integral for (1.5.7c), viz.

$$J = \frac{1}{2} \left(r^2 \sin^2 \theta \dot{\phi} \right)^2 + \frac{1}{2} U(\phi). \quad (1.5.8)$$

provided

$$H(\theta, \phi) = -\frac{\frac{1}{2} U'(\phi)}{\sin^2 \theta}. \quad (1.5.9)$$

Eq (1.5.8) is just the type of Ermakov invariant we met earlier (1.5.6). We treat eqq (1.5.7b) and (1.5.7c) in an Ermakovian manner¹³ to obtain the first integral

$$J' = \frac{1}{2} \left[\left(r^2 \dot{\theta} \right)^2 + \frac{\left(r^2 \sin^2 \theta \dot{\phi} \right)^2}{\sin^2 \theta} + W(\theta, \phi) \right] \quad (1.5.10)$$

provided that

$$G = -\frac{1}{2} \frac{\partial W}{\partial \theta}, \quad H = -\frac{1}{2} \frac{\partial W}{\partial \phi}. \quad (1.5.11)$$

Eq (1.5.10) is the Ermakov invariant appropriate to three dimensions in that J' is the sum of the square of the angular momentum plus an angle dependent term just as it was in (1.5.6).

We have so far indicated the usefulness of solving *any* equation of Ermakov–Pinney type in terms of an equation of time-independent harmonic oscillator type. It is this feature of Ermakov systems that have contributed to the search for similar systems in classical and quantum systems.

Lee[65] studied a multi-dimensional Ermakov theory applicable to separable and non-separable time-independent quantum mechanical systems. He obtained an exact invariant for multi-dimensional quantum systems and showed how this reduced to the Ermakov–Lewis invariant in one-dimension by considering the harmonic oscillator as a one-dimensional Schrödinger equation. In addition he analysed the hydrogen atom as a quantum Ermakov system. Thereafter Nassar[84] presented a unified protocol to treat the quantum time-dependent harmonic oscillator with friction. He discussed *inter alia* a time-dependent linear Schrödinger equation for which he also derived an Ermakov

¹³See §1.3

system with an Ermakov–Lewis type invariant. At about the same time Pedrosa also considered Ermakov systems with friction and the corresponding Ermakov–Lewis invariant[87]. These works were preceded by Ray and Hartley who, in a series of papers[40, 98, 99], applied the Ermakov technique to more general time-dependent systems, including the n -dimensional Schrödinger equation.

The above applications of Ermakov systems imply that they are an important part of the study of differential equations and that further investigation thereof is of relevance to dynamical systems in particular and the theory of differential equations in general.

1.6 Outline

We devote the next chapter to the Lie analysis of differential equations. A short introduction to this essential theory is provided and the generalised Ermakov system (1.5.4) is used to illustrate the technique. We also point out the usefulness of this technique in providing various routes to solving differential equations.

An alternative use of the Lie method is introduced in Chapter 3. This technique which, in essence, is the exact opposite implementation of the standard Lie analysis, has provided interesting insights into the generalised equations obtained.

We extend this alternative approach to three dimensions in Chapter 4 and explain the occurrence of Poincaré vectors appropriate to three-dimensional Ermakov systems.

Chapter 2

Lie Theory of Differential Equations

We introduce Lie's theory for the solution of differential equations and explain the concept of Lie algebras. The theory is then utilised to investigate the behaviour of Ermakov systems.

2.1 Definitions

Many of the concepts in the ensuing discussion are not familiar to those who are normally interested in the solutions of differential equations. We pause to explain some of the terminology that will arise later. However, we do not elaborate on these concepts in any great detail as it would detract from the tenor of this work.

Group: A group G is a set of elements with a law of composition ϕ between elements satisfying the following axioms[12]:

- (i) CLOSURE PROPERTY: For any element a and b of G , $\phi(a, b)$ is an element of G .
- (ii) ASSOCIATIVE PROPERTY: For any elements a, b and c of G ,

$$\phi(a, \phi(b, c)) = \phi(\phi(a, b), c).$$

(iii) IDENTITY ELEMENT: There exists a unique identity element I of G such that, for any element a of G ,

$$\phi(a, I) = \phi(I, a) = a.$$

(iv) INVERSE ELEMENT: For any element a of G there exists a unique inverse element a^{-1} in G such that

$$\phi(a, a^{-1}) = \phi(a^{-1}, a) = I.$$

Abelian group: A group G is Abelian if $\phi(a, b) = \phi(b, a)$ holds for all elements a and b in G .

Group of transformations: The set of transformations

$$\bar{x} = \mathbf{X}(\mathbf{x}; \varepsilon)$$

defined for each \mathbf{x} in $D \subset R$, depending on the parameter ε lying in the set $S \subset R$, with $\phi(\varepsilon, \delta)$ defining a composition of parameters ε and δ in S , forms a group of transformations on D if:

(i) For each parameter ε in S the transformations are one-to-one onto D .

(ii) S with the law of composition ϕ forms a group.

(iii) $\bar{x} = x$ when $\varepsilon = I$, ie.

$$\mathbf{X}(\mathbf{x}; I) = \mathbf{x}.$$

(iv) If $\bar{x} = \mathbf{X}(\mathbf{x}; \varepsilon)$, $\bar{\bar{x}} = \mathbf{X}(\bar{x}; \delta)$, then

$$\bar{\bar{x}} = \mathbf{X}(\mathbf{x}; \phi(\varepsilon, \delta)).$$

Lie group of transformations: A one-parameter Lie group of transformations is a group of transformations which, in addition to the above, satisfies the following:

- (i) ε is a continuous parameter, ie. S is an interval in R . (Without loss of generality $\varepsilon = 0$ corresponds to the identity element I .)
- (ii) \mathbf{X} is infinitely differentiable with respect to \mathbf{x} in D and an analytic function of ε in S .
- (iii) $\phi(\varepsilon, \delta)$ is an analytic function of ε and δ , $\varepsilon \in S$, $\delta \in S$.

Subgroup: A subgroup of G is a group formed by a subset of elements of G with the same law of composition.

Special linear group: The complex general linear group $GL(n, C)$ and the real general linear group $GL(n, R)$ consist of all nonsingular complex and real $n \times n$ matrices respectively[9]. (The latter may be considered as a subgroup of the former.) The complex special linear group $SL(n, C)$ is the subgroup of $GL(n, C)$ consisting of matrices with determinant one. The real special linear group $SL(n, R)$ is the intersection of these two subgroups

$$SL(n, R) = SL(n, C) \cap GL(n, R).$$

Rotation group: The rotation group $SO(n, R)$ is the special or proper real orthogonal group given by the intersection of the group of orthogonal matrices¹ $O(n, R)$ and the complex special linear group, ie.

$$SO(n, R) = O(n, R) \cap SL(n, C).$$

Lie algebra:² A Lie algebra \mathcal{L} is a vector space together with a product $[x, y]$ that:

- (i) is BILINEAR (ie, linear in x and y separately),

¹Recall that a matrix is orthogonal if its transpose is its inverse.

²The term 'Lie algebra' was coined by H Weyl in 1934. Previously mathematicians simply spoke of "infinitesimal transformations $X_1 f, \dots, X_r f$ " of the group which Lie and Engel often abbreviated to "the group $X_1 f \dots, X_r f$ " [13].

(ii) is ANTICOMMUTATIVE (antisymmetric):

$$[x, y] = -[y, x],$$

(iii) satisfies the JACOBI IDENTITY

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all vectors x, y, z in the Lie algebra.

A familiar Lie algebra is the real three-dimensional vector space with the vector cross product as multiplication.

Abelian algebra: A Lie algebra \mathcal{L} is called Abelian (equivalently commutative) if $[x, y] = 0 \forall x, y \in \mathcal{L}$ [86].

Solvable algebra: A Lie algebra \mathcal{L} is called solvable if the derived series

$$\begin{aligned}\mathcal{L} &\supseteq \mathcal{L}' = [\mathcal{L}, \mathcal{L}] \\ &\supseteq \mathcal{L}'' = [\mathcal{L}', \mathcal{L}'] \\ &\supseteq \dots \\ &\supseteq \mathcal{L}^{(k)} = [\mathcal{L}^{(k-1)}, \mathcal{L}^{(k-1)}]\end{aligned}$$

terminates with a null ideal, ie. $\mathcal{L}^{(k)} = 0, k > 0$ [45]. Note: Any Abelian algebra is solvable and indeed any Lie algebra of dimension ≤ 3 is solvable except when $\dim \mathcal{L} = 3 = \dim \mathcal{L}'$.

A few comments about Lie algebras are now in order. The Jacobi identity plays the same role for Lie algebras that the associative law plays for associative algebras. While we can define a Lie algebra over any field, in practice it is usually considered only over real and complex fields. We define the product associated with the Lie algebra as that of commutation, ie.

$$[X, Y] = XY - YX.$$

If a differential equation admits the operators X and Y , it also admits their commutator $[X, Y]^3$. Lie's main result[118] is the proof that it is always possible to assign to a continuous group (Lie group) a corresponding Lie algebra and vice versa. Thus for the real special linear group $SL(n, R)$ the corresponding Lie algebra is $sl(n, R)$ and for $SO(n, R)$, $so(n, R)$.

2.2 The Lie analysis

We now present a brief introduction to this method of analysing differential equations. For a more detailed approach we refer the reader to [12, 77, 78, 85]. While the subsequent development will encompass only ordinary differential equations, the results apply *mutatis mutandis* to partial differential equations.

An ordinary differential equation

$$F(x, y, y', \dots, y^{(n)}) = 0 \quad (2.2.1)$$

admits the one-parameter Lie group of (point) transformations⁴

$$\bar{x} = x + \varepsilon \xi(x, y) \quad (2.2.2a)$$

$$\bar{y} = y + \varepsilon \eta(x, y) \quad (2.2.2b)$$

with infinitesimal generator

$$G = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \quad (2.2.3)$$

if and only if

$$G^{[n]} F(x, y, y', \dots, y^{(n)}) \Big|_{F=0} = 0, \quad (2.2.4)$$

where

$$G^{[n]} = G^{[n-1]} + \eta^{[n]} \frac{\partial}{\partial y^{(n)}}, \quad G^{[0]} = G, \quad (2.2.5)$$

³See[86] for a detailed discussion of the properties of commutators.

⁴We are only concerned here with the study of classical point symmetries. More recently different types of symmetries, eg. hidden[1, 2, 3], potential[12], contact[80], internal, external and generalised[4] symmetries have come under focus. Many, such as hidden and potential, are just variations of point symmetries that appear under different circumstances.

$$\eta^{[n]} = \frac{d\eta^{[n-1]}}{dx} - y^{(n)} \frac{d\xi}{dx}, \quad (2.2.6)$$

or equivalently,

$$\eta^{[n]} = \eta^{(n)} - \sum_{j=0}^{n-1} \binom{n}{j} y^{(j+1)} \xi^{(n-j)} \quad (2.2.7)$$

and

$$\frac{d}{dx} = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + \cdots + y^{(n)} \frac{\partial}{\partial y^{(n-1)}}. \quad (2.2.8)$$

G , defined by (2.2.3), is called a symmetry of (2.2.1) and (2.2.5) the n th extension or prolongation of G ⁵. We say that F is invariant under the n th extension of G if (2.2.4) holds. The symmetry is usually written in its unextended form for compactness.

For a second order ordinary differential equation

$$f(x, y, y', y'') = 0 \quad (2.2.9)$$

we use the second extension (or prolongation) of G , viz.

$$G^{[2]} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + (\eta' - y' \xi') \frac{\partial}{\partial y'} + (\eta'' - 2y'' \xi' - y' \xi'') \frac{\partial}{\partial y''} \quad (2.2.10)$$

and require that

$$\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + (\eta' - y' \xi') \frac{\partial f}{\partial y'} + (\eta'' - 2y'' \xi' - y' \xi'') \frac{\partial f}{\partial y''} = 0. \quad (2.2.11)$$

Given ξ and η we can find $f = F(u, v, w)$ where u, v and w are the three characteristics determined from the associated Lagrange's system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\eta' - y' \xi'} = \frac{dy''}{\eta'' - 2y'' \xi' - y' \xi''}.$$

Using the first two expressions we obtain $u(x, y)$, the zeroth order invariant, the second two $v(x, y, y')$, the first order invariant and the final two $w(x, y, y', y'')$, the second order invariant. However, if f is given, (2.2.11) is an equation for ξ and η and we rewrite it as

$$\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \left(\frac{\partial \eta}{\partial x} + y' \frac{\partial \eta}{\partial y} - y' \frac{\partial \xi}{\partial x} - y'^2 \frac{\partial \xi}{\partial y} \right) \frac{\partial f}{\partial y'}$$

⁵We need to extend the symmetry to take care of the higher derivatives in the equation.

$$\begin{aligned}
& + \left(\frac{\partial^2 \eta}{\partial x^2} + 2y' \frac{\partial^2 \eta}{\partial x \partial y} + y'^2 \frac{\partial^2 \eta}{\partial y^2} + y'' \frac{\partial \eta}{\partial y} - 2y'' \frac{\partial \xi}{\partial x} - 2y' y'' \frac{\partial \xi}{\partial y} \right. \\
& \quad \left. - y' \frac{\partial^2 \xi}{\partial x^2} - 2y'^2 \frac{\partial^2 \xi}{\partial x \partial y} - y'^3 \frac{\partial^2 \xi}{\partial y^2} - y' y'' \frac{\partial \xi}{\partial y} \right) \frac{\partial f}{\partial y'} = 0. \quad (2.2.12)
\end{aligned}$$

Since η and ξ are functions of x and y only, we use (2.2.9) to eliminate y'' , collect terms by powers of y' and set each coefficient separately equal to zero. The resulting equations form an overdetermined system of partial differential equations which are solved to obtain the explicit forms of ξ and η , and thereby G . The commutation relationships of the G_i 's form a Lie algebra.

It is evident that the calculation of ξ and η involves tedious manipulation which is best left to a computer. To this end many packages have been developed to determine the symmetries of differential equations⁶. We have made use of the packages `Program LIE`[41] and `Mathematica`[116] to reduce the effort and increase the accuracy in calculations.

2.3 Using symmetries to solve differential equations

Symmetry algebras calculated using Lie's infinitesimal method have numerous applications. These include integration of ordinary differential equations, group invariant solutions of partial differential equations, conservation laws, bifurcation theory etc. We illustrate the usefulness of the knowledge of symmetries in analysing differential equations.

2.3.1 Point transformations

We can relate differential equations of similar order with the same symmetry algebras by a point transformation. The knowledge of the solution of just one equation of a particular class is required to determine the solutions of the

⁶See Hereman's comprehensive review of existing software[43].

remaining equations as they are also related via point transformations. This is illustrated most aptly by second order ordinary differential equations which have the maximal symmetry algebra $sl(3, R)$ (with eight symmetries).

Consider the nonlinear second order ordinary differential equation

$$y'' + 3yy' + y^3 = 0, \quad (2.3.1)$$

which arises in the study of the modified Emden equation, and the simple equation

$$Y'' = 0, \quad (2.3.2)$$

both of which have the eight element Lie algebra $sl(3, R)$ [76]. The solution of (2.3.2) is

$$Y = A + BX,$$

while that of (2.3.1) not exactly obvious. However, one can transform eq (2.3.1) to eq (2.3.2) using

$$Y = x - \frac{1}{y} \quad X = -\frac{1}{2}x^2 + \frac{x}{y}.$$

Thus the solution to (2.3.1) is

$$x - \frac{1}{y} = A + B \left(-\frac{1}{2}x^2 + \frac{x}{y} \right),$$

ie.

$$y = \frac{Bx + 1}{\frac{1}{2}Bx^2 + x - A}.$$

2.3.2 Reduction of order

One of the preferred routes to the solution of differential equations is via reduction of order. One can always reduce the order of an ordinary differential equation if it admits a one-parameter Lie group of transformations. In the instance of first order ordinary differential equations this results in a reduction to quadrature. This reduction can always be accomplished by using canonical coordinates associated with the group.

For higher order ordinary differential equations ($n \geq 2$) the reduction can also be accomplished by using differential invariants (invariants of the n th extended group). The attraction of this method is that the order of an ordinary differential equation can be reduced by m if it is invariant under an m -parameter Lie group of transformations whose Lie algebra is solvable.

The general second order linear homogeneous ordinary differential equation

$$y'' + p(x)y' + q(x)y = 0 \quad (2.3.3)$$

admits the one-parameter (α) Lie group of transformations

$$\bar{x} = x \quad (2.3.4a)$$

$$\bar{y} = \alpha y. \quad (2.3.4b)$$

We first reduce (2.3.3) by canonical coordinates: The canonical coordinates corresponding to (2.3.4) are

$$u(x, y) = x$$

$$v(x, y) = \log y.$$

This implies

$$\frac{dv}{du} = \frac{y'}{y},$$

ie.

$$y' = y \frac{dv}{du}.$$

Thus

$$\begin{aligned} y'' &= y' \frac{dv}{du} + y \frac{d^2v}{du^2} \\ &= y \left(\frac{dv}{du} \right)^2 + y \frac{d^2v}{du^2}. \end{aligned} \quad (2.3.5)$$

We can now make the form of (2.3.3) more transparent by using (2.3.5) and setting

$$z = \frac{dv}{du}.$$

Thus (2.3.3) becomes

$$\frac{dz}{du} + z^2 + p(u)z + q(u) = 0,$$

the first order ordinary differential equation of Riccati type.

We now consider the reduction using differential invariants. The invariants of the first extension of (2.3.4) are

$$\begin{aligned} u(x, y) &= x \\ v(x, y, y') &= \frac{y'}{y}. \end{aligned}$$

The corresponding differential invariant⁷ is

$$\frac{dv}{du} = \frac{y''}{y} - \left(\frac{y'}{y}\right)^2 = \frac{y''}{y} - v^2$$

which reduces to

$$y'' = y \frac{dv}{du} + v^2 y.$$

Hence (2.3.3) once again reduces to the Riccati equation

$$\frac{dv}{du} + v^2 + p(u)v + q(u) = 0.$$

The existence of a sufficient number of symmetries can help in determining an appropriate path for reduction. If a second order equation has two symmetries G_1 and G_2 with the Lie bracket

$$[G_1, G_2] = kG_2,$$

reduction via G_2 will lead to a first order equation which will have G_1 as a symmetry. Reduction via G_1 causes the reduced equation to lose G_2 , so care must be taken in the selection. Note that, while reduction is desirable, one can not always solve the reduced equation.

Alternatively, increasing the order of an equation sometimes does offer advantages. It is often lamented that an increase in order reduces symmetry. However, one does obtain Type I Hidden symmetries[2] which provide alternative routes for reduction via these ‘new’ symmetries.

⁷In the notation of Bluman and Kumei[12].

2.4 Lie algebra of Ermakov systems

Knowledge of Lie symmetries has enabled previous researchers[100, 102] to generate new Ermakov systems. However, it was only recently that the Lie algebra of the Ermakov system (1.5.3):

$$\ddot{X} = \frac{1}{X^3} f\left(\frac{X}{Y}\right) \quad (2.4.1a)$$

$$\ddot{Y} = \frac{1}{Y^3} g\left(\frac{X}{Y}\right) \quad (2.4.1b)$$

was computed[61]. We illustrate the procedure by considering the compact form (1.5.4) of our system (2.4.1), viz.

$$N = x\ddot{y} - \ddot{x}y - \frac{x}{y^3}g(y/x) + \frac{y}{x^3}f(y/x) = 0. \quad (2.4.2)$$

We require invariance of (2.4.2) under the second extension of the group generator G , viz.

$$\begin{aligned} G^{[2]} = & \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y} + \left(\dot{\eta} - \dot{x}\dot{\xi}\right) \frac{\partial}{\partial \dot{x}} + \left(\dot{\gamma} - \dot{y}\dot{\xi}\right) \frac{\partial}{\partial \dot{y}} \\ & + \left(\ddot{\eta} - 2\dot{x}\dot{\xi} - \dot{x}\ddot{\xi}\right) \frac{\partial}{\partial \ddot{x}} + \left(\ddot{\gamma} - 2\dot{y}\dot{\xi} - \dot{y}\ddot{\xi}\right) \frac{\partial}{\partial \ddot{y}}, \end{aligned}$$

ie.

$$\begin{aligned} G^{[2]}|_{N=0} N = & \eta \left(\ddot{y} - \frac{g}{y^3} - \frac{x}{y^3} \frac{\partial g}{\partial x} - \frac{3yf}{x^4} + \frac{y}{x^3} \frac{\partial f}{\partial x} \right) \\ & + \gamma \left(-\ddot{x} + \frac{3xg}{y^4} - \frac{x}{y^3} \frac{\partial g}{\partial y} + \frac{f}{x^3} + \frac{y}{x^3} \frac{\partial f}{\partial y} \right) \\ & + \left(\frac{\partial^2 \eta}{\partial t^2} + 2\dot{x} \frac{\partial^2 \eta}{\partial t \partial x} + \dot{x}^2 \frac{\partial^2 \eta}{\partial x^2} + \ddot{x} \frac{\partial \eta}{\partial x} + 2\dot{y} \frac{\partial^2 \eta}{\partial t \partial y} + 2\dot{x}\dot{y} \frac{\partial^2 \eta}{\partial x \partial y} \right. \\ & \left. + \dot{y}^2 \frac{\partial^2 \eta}{\partial y^2} + \ddot{y} \frac{\partial \eta}{\partial y} - 2\dot{x} \frac{\partial \xi}{\partial t} - 2\dot{x}\ddot{x} \frac{\partial \xi}{\partial x} - 2\dot{x}\dot{y} \frac{\partial \xi}{\partial y} - \dot{x} \frac{\partial^2 \xi}{\partial t^2} - 2\dot{x}^2 \frac{\partial^2 \xi}{\partial t \partial x} \right) \end{aligned}$$

$$\begin{aligned}
& -\dot{x}^3 \frac{\partial^2 \xi}{\partial x^2} - \dot{x} \ddot{x} \frac{\partial \xi}{\partial x} - 2\dot{x} \dot{y} \frac{\partial^2 \xi}{\partial x \partial y} - 2\dot{x}^2 \dot{y} \frac{\partial^2 \xi}{\partial x \partial y} - \dot{x} \dot{y}^2 \frac{\partial^2 \xi}{\partial y^2} - \dot{x} \ddot{y} \frac{\partial \xi}{\partial y} \Big) (-y) \\
& + \left(\frac{\partial^2 \gamma}{\partial t^2} + 2\dot{x} \frac{\partial^2 \gamma}{\partial t \partial x} + \dot{x}^2 \frac{\partial^2 \gamma}{\partial x^2} + \ddot{x} \frac{\partial \gamma}{\partial x} + 2\dot{y} \frac{\partial^2 \gamma}{\partial t \partial y} + 2\dot{x} \dot{y} \frac{\partial^2 \gamma}{\partial x \partial y} \right. \\
& + \dot{y}^2 \frac{\partial^2 \gamma}{\partial y^2} + \ddot{y} \frac{\partial \gamma}{\partial y} - 2\ddot{y} \frac{\partial \xi}{\partial t} - \dot{x} \ddot{y} \frac{\partial \xi}{\partial x} - \dot{y} \ddot{x} \frac{\partial \xi}{\partial y} - \dot{y} \frac{\partial^2 \xi}{\partial t^2} - 2\dot{x} \dot{y} \frac{\partial^2 \xi}{\partial t \partial x} \\
& \left. - \dot{x}^2 \dot{y} \frac{\partial^2 \xi}{\partial x^2} - \ddot{x} \dot{y} \frac{\partial \xi}{\partial x} - 2\dot{y}^2 \frac{\partial^2 \xi}{\partial x \partial y} - 2\dot{x} \dot{y}^2 \frac{\partial^2 \xi}{\partial x \partial y} - \dot{y}^3 \frac{\partial^2 \xi}{\partial y^2} - \dot{y} \ddot{y} \frac{\partial \xi}{\partial y} \right) x = 0,
\end{aligned}$$

which, after solving for \ddot{x} (or equivalently for \ddot{y}) in (2.4.2) and substituting into (2.4.3), becomes

$$\begin{aligned}
& -y \ddot{y} \frac{\partial \eta}{\partial y} - \ddot{x} y \frac{\partial \eta}{\partial x} + 2\ddot{x} y \frac{\partial \xi}{\partial t} + 2\dot{x} \ddot{x} y \frac{\partial \xi}{\partial x} + 2\ddot{x} y \dot{y} \frac{\partial \xi}{\partial y} + \dot{x} \ddot{x} y \frac{\partial \xi}{\partial x} + \dot{x} y \ddot{y} \frac{\partial \xi}{\partial y} + x \ddot{x} \frac{\partial \gamma}{\partial x} \\
& + x \ddot{y} \frac{\partial \gamma}{\partial y} - 2x \ddot{y} \frac{\partial \xi}{\partial t} - 2x \dot{x} \ddot{y} \frac{\partial \xi}{\partial x} - 2x \dot{y} \ddot{y} \frac{\partial \xi}{\partial y} - x \ddot{x} \dot{y} \frac{\partial \xi}{\partial x} - x \dot{y} \ddot{y} \frac{\partial \xi}{\partial y} = \\
& -y \ddot{y} \frac{\partial \eta}{\partial y} + 2 \left(x \ddot{y} - \frac{xg}{y^3} + \frac{yf}{x^3} \right) \frac{\partial \xi}{\partial t} + 2\dot{x} \left(x \ddot{y} - \frac{xg}{y^3} + \frac{yf}{x^3} \right) \frac{\partial \xi}{\partial x} \\
& + 2\dot{y} \left(x \ddot{y} - \frac{xg}{y^3} + \frac{yf}{x^3} \right) \frac{\partial \xi}{\partial y} + \dot{x} \left(x \ddot{y} - \frac{xg}{y^3} + \frac{yf}{x^3} \right) \frac{\partial \xi}{\partial x} + \dot{x} y \ddot{y} \frac{\partial \xi}{\partial y} \\
& + x \left(\frac{x \ddot{y}}{y} - \frac{xg}{y^4} + \frac{f}{x^3} \right) \frac{\partial \gamma}{\partial x} + x \ddot{y} \frac{\partial \gamma}{\partial y} - 2x \ddot{y} \frac{\partial \xi}{\partial t} - 2x \dot{x} \ddot{y} \frac{\partial \xi}{\partial x} - 2x \dot{y} \ddot{y} \frac{\partial \xi}{\partial y} \\
& - x \dot{y} \left(\frac{x \ddot{y}}{y} - \frac{xg}{y^4} + \frac{f}{x^3} \right) \frac{\partial \xi}{\partial x} - x \dot{y} \ddot{y} \frac{\partial \xi}{\partial y} - y \left(x \ddot{y} - \frac{xg}{y^3} + \frac{yf}{x^3} \right) \frac{\partial \eta}{\partial x} - y \frac{\partial^2 \eta}{\partial t^2} \\
& + x \frac{\partial^2 \gamma}{\partial t^2} + \eta \left(\ddot{y} - \frac{g}{y^3} - \frac{x}{y^3} \frac{\partial g}{\partial x} - \frac{3yf}{x^4} + \frac{y}{x^3} \frac{\partial f}{\partial x} \right) \\
& + \gamma \left(\frac{3xg}{y^4} - \frac{x}{y^3} \frac{\partial g}{\partial y} + \frac{f}{x^3} + \frac{y}{x^3} \frac{\partial f}{\partial y} \right) + \gamma \left(\frac{xg}{y^4} - \frac{f}{x^3} - \frac{x \ddot{y}}{y} \right).
\end{aligned}$$

Separating by powers of \dot{x} , \dot{y} and \ddot{y} we obtain the following thirteen determining equations:

$$\dot{x}^3 : y \frac{\partial^2 \xi}{\partial x^2} = 0,$$

$$\dot{x}^2 \dot{y} : 2y \frac{\partial^2 \xi}{\partial x \partial y} - x \frac{\partial^2 \xi}{\partial x^2} = 0,$$

$$\dot{x} \dot{y}^2 : y \frac{\partial^2 \xi}{\partial y^2} - 2x \frac{\partial^2 \xi}{\partial x \partial y} = 0,$$

$$\dot{y}^3 : -x \frac{\partial^2 \xi}{\partial y^2} = 0,$$

$$\dot{x}^2 : -y \frac{\partial^2 \eta}{\partial x^2} + 2y \frac{\partial^2 \xi}{\partial t \partial x} + x \frac{\partial^2 \gamma}{\partial x^2} = 0,$$

$$\dot{y}^2 : -y \frac{\partial^2 \eta}{\partial y^2} + x \frac{\partial^2 \gamma}{\partial y^2} - 2x \frac{\partial^2 \xi}{\partial t \partial y} = 0,$$

$$\dot{x} \dot{y} : -y \frac{\partial^2 \eta}{\partial x \partial y} + y \frac{\partial^2 \xi}{\partial t \partial y} + x \frac{\partial^2 \gamma}{\partial x \partial y} - x \frac{\partial^2 \xi}{\partial t \partial x} = 0,$$

$$\dot{x} : -2y \frac{\partial^2 \eta}{\partial t \partial x} + y \frac{\partial^2 \xi}{\partial t^2} + 2x \frac{\partial^2 \gamma}{\partial t \partial x} + \left(-\frac{3xg}{y^3} + \frac{3yf}{x^3} \right) \frac{\partial \xi}{\partial x} = 0,$$

$$\dot{y} : -2y \frac{\partial^2 \eta}{\partial t \partial y} + 2x \frac{\partial^2 \gamma}{\partial t \partial y} - x \frac{\partial^2 \xi}{\partial t^2} + \left(-\frac{2xg}{y^3} + \frac{2yf}{x^3} \right) \frac{\partial \xi}{\partial y}$$

$$+ \left(-\frac{x^2 g}{y^4} - \frac{f}{x^2} \right) \frac{\partial \xi}{\partial x} = 0,$$

$$\dot{x} \ddot{y} : x \frac{\partial \xi}{\partial x} + y \frac{\partial \xi}{\partial y} = 0,$$

$$\dot{y} \ddot{y} : x \frac{\partial \xi}{\partial x} + y \frac{\partial \xi}{\partial y} = 0,$$

$$\ddot{y} : -y \frac{\partial \eta}{\partial y} - x \frac{\partial \eta}{\partial x} + \frac{x^2}{y} \frac{\partial \gamma}{\partial x} + x \frac{\partial \gamma}{\partial y} + \eta - \frac{x}{y} \gamma = 0,$$

$$\begin{aligned} \dot{x}^0 \dot{y}^0 \ddot{y}^0 : & 2 \left(-\frac{xg}{y^3} + \frac{yf}{x^3} \right) \frac{\partial \xi}{\partial t} + \left(-\frac{x^2 g}{y^4} + \frac{f}{x^2} \right) \frac{\partial \gamma}{\partial x} - y \frac{\partial^2 \eta}{\partial t^2} + x \frac{\partial^2 \gamma}{\partial t^2} \\ & + \eta \left(-\frac{g}{y^3} - \frac{x}{y^3} \frac{\partial g}{\partial x} - \frac{3yf}{x^4} + \frac{y}{x^3} \frac{\partial f}{\partial x} \right) + \gamma \left(\frac{4xg}{y^4} - \frac{x}{y^3} \frac{\partial g}{\partial y} + \frac{y}{x^3} \frac{\partial f}{\partial y} \right) \\ & + \left(\frac{xg}{y^3} - \frac{yf}{x^3} \right) \frac{\partial \eta}{\partial x} = 0. \end{aligned}$$

After a lengthy calculation⁸ we obtain the explicit forms of ξ, η and γ as

$$\xi = a(t)$$

$$\eta = \frac{1}{2} \dot{a}(t) x$$

$$\gamma = \frac{1}{2} \dot{a}(t) y.$$

Thus the equation (2.4.2) has the symmetry

$$G = a \frac{\partial}{\partial t} + \frac{1}{2} \dot{a} x \frac{\partial}{\partial x} + \frac{1}{2} \dot{a} y \frac{\partial}{\partial y}. \quad (2.4.3)$$

As a is an arbitrary function of time, (2.4.2) admits an infinite dimensional Lie algebra (and not merely $sl(2, R)$ as stated by Leach). Recalling that (2.4.2) was obtained by eliminating the $\omega^2(t)$ terms from the equations

$$\ddot{x} + \omega^2(t)x = \frac{1}{x^3} f\left(\frac{y}{x}\right) \quad (2.4.4a)$$

$$\ddot{y} + \omega^2(t)y = \frac{1}{y^3} g\left(\frac{y}{x}\right), \quad (2.4.4b)$$

we examine these equations for invariance under the symmetry (2.4.3). We do this by applying the second extension of G , viz.

$$\begin{aligned} G^{[2]} = & a \frac{\partial}{\partial t} + \frac{1}{2} \dot{a} x \frac{\partial}{\partial x} + \frac{1}{2} \dot{a} y \frac{\partial}{\partial y} + \left(\frac{1}{2} \ddot{a} x - \frac{1}{2} \dot{a} \dot{x} \right) \frac{\partial}{\partial \dot{x}} + \left(\frac{1}{2} \ddot{a} y - \frac{1}{2} \dot{a} \dot{y} \right) \frac{\partial}{\partial \dot{y}} \\ & + \left(\frac{1}{2} \ddot{a} x - \frac{3}{2} \dot{a} \ddot{x} \right) \frac{\partial}{\partial \ddot{x}} + \left(\frac{1}{2} \ddot{a} y - \frac{3}{2} \dot{a} \ddot{y} \right) \frac{\partial}{\partial \ddot{y}} \end{aligned}$$

⁸We disagree with Leach who dismissed it as merely ‘routine’ in his paper in the manner to which he is accustomed[29].

to (2.4.4) and obtain that $a(t)$ must be a solution of the equation

$$\ddot{a} + 4\omega^2(t)\dot{a} + 4\omega\dot{\omega}a = 0 \quad (2.4.5)$$

(This is really the third order form of the Ermakov–Pinney equation, but more on this later.). We note that Leach pointed out that, in effect, one could consider two types of Ermakov systems: the oscillator system and a free particle type system, the latter corresponding to ω^2 in (2.4.4) being zero. It was also pointed out in §1.5 that the ω terms in (2.4.4) could be easily transformed away. Thus we can take ω in (2.4.5) to be zero without any loss of generality to obtain

$$\ddot{a} = 0.$$

Hence our symmetry (2.4.3) now becomes

$$G_1 = \frac{\partial}{\partial t} \quad (2.4.6a)$$

$$G_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad (2.4.6b)$$

$$G_3 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} \quad (2.4.6c)$$

as reported by Leach[61].

G_1 represents invariance under time translation which implies that the equations are explicitly independent of time t . Invariance under $2t\partial/\partial t + \alpha(x\partial/\partial x + y\partial/\partial y)$ means that the equations are invariant under scale transformations or similarity transformations[81]. When $\alpha = 2$, we talk of uniform or homogeneous dilations. Note that in these two instances the one parameter group of transformations is globally defined. Invariance under G_3 on the other hand means that the equations are invariant under a true *local* one parameter group of point transformations. While there exist instances in the literature[61] where the term *conformal* is used we note that this has a more general meaning[75] and can refer to any one of 10 symmetries. A proper classification thereof requires further thought.

The Lie brackets of the symmetries in (2.4.6) are

$$[G_1, G_2] = 2G_1, \quad [G_1, G_3] = G_2, \quad [G_2, G_3] = 2G_3 \quad (2.4.7)$$

which is the non-Abelian Lie algebra $A_{3,8}$ ⁹, better known as $sl(2, R)$. Equally we could use the solutions of (2.4.5) which also give the algebra $sl(2, R)$ [78].

In the previous chapter we saw that the Ermakov invariant (in two dimensions) was obtained from the angular component (2.4.1b). This suggested a natural extension to higher dimensions if polar coordinates were used. However, this is not the only pathway for generalisation to higher dimensions. Lutzky[74] (whose results were extended by Sarlet[104]) kept the auxilliary equation (1.3.5) while increasing the number of Cartesian equations. The main criterion is invariance under the second extension of the symmetries in (2.4.6).

The above development would have been greatly simplified if we had used the plane polar forms of (2.4.4), viz.

$$\ddot{r} - r\dot{\theta}^2 = \frac{F(\theta)}{r^3} \quad (2.4.8a)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{G(\theta)}{r^3}. \quad (2.4.8b)$$

We proceed with our analysis in this coordinate system and rewrite our symmetries (2.4.6) as

$$G_1 = \frac{\partial}{\partial t} \quad (2.4.9a)$$

$$G_2 = 2t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \quad (2.4.9b)$$

$$G_3 = t^2\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r}. \quad (2.4.9c)$$

2.5 First integrals for the Ermakov system

Recall that the restriction on $a(t)$ in (2.4.3) was obtained by the necessity of consistency of the invariance of (2.4.2) and (2.4.4) under the same symmetries. We will now investigate the possibilities of restrictions imposed on $a(t)$ by the the alternative method of finding the first integrals associated with the point

⁹ $A_{i,j}$ means the j th algebra in i dimensions. We adopt the Lie algebra classification of Mubarakzyanov as explained in [77].

symmetry (2.4.3) (and in so doing attempt to find all (four) first integrals for (2.4.8)).

We calculate the first integrals for an n th order system having the symmetry G by taking the $(n - 1)$ th extension of G and applying it to some arbitrary function. In this way we obtain the functional form of the first integrals and by requiring its total derivative to be zero we obtain the first integrals. For our problem we require the first extension of our symmetry (2.4.3), viz.

$$G^{[1]} = a \frac{\partial}{\partial t} + \frac{1}{2} \dot{a} r \frac{\partial}{\partial r} + 0 \frac{\partial}{\partial \theta} + \left(\frac{1}{2} \ddot{a} r - \frac{1}{2} \dot{a} \dot{r} \right) \frac{\partial}{\partial \dot{r}} - \dot{a} \dot{\theta} \frac{\partial}{\partial \dot{\theta}}$$

to operate on the function

$$f(t, r, \theta, \dot{r}, \dot{\theta}).$$

We find that the first integrals have the functional form

$$I = f(u, v, w, x)$$

where u, v, w and x are the characteristics

$$u = r a^{-\frac{1}{2}}, \quad v = \theta, \quad w = \dot{r} a^{\frac{1}{2}} - \frac{1}{2} \dot{a} a^{-\frac{1}{2}} r, \quad x = r^2 \dot{\theta}.$$

The requirement

$$\dot{I} = 0$$

results in a partial differential equation with associated Lagrange's system

$$\frac{du}{w/a} = \frac{dv}{x/r^2} = \frac{dw}{a^{1/2} \ddot{r} - \frac{1}{2} a^{-1/2} \ddot{a} r + \frac{1}{4} a^{-3/2} \dot{a}^2 r} = \frac{dx}{G(\theta)/r^2}. \quad (2.5.1)$$

Using the second and fourth terms we obtain one first integral as

$$I_1 = \frac{1}{2} \left(r^2 \dot{\theta} \right)^2 - \int G(\theta) d\theta \quad (2.5.2)$$

which is just our Ermakov–Lewis invariant (1.5.6). This implies that (2.5.2) has an infinite-dimensional Lie algebra as no restriction on a is imposed. It is the calculation of another first integral (using the first and third terms of (2.5.1))

$$J = a \left(\frac{1}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - \frac{\int G(\theta)}{r^2} \right) - \frac{1}{2} \dot{a} r \dot{r} + \frac{1}{4} r^2 \ddot{a} \quad (2.5.3)$$

that imposes a restriction on a , viz.

$$\begin{aligned} (a^{1/2})^{\cdot\cdot} &= \frac{\alpha}{a^{\frac{3}{2}}} \\ \Rightarrow \quad \ddot{a} &= 0. \end{aligned}$$

Hence

$$a = \{1, t, t^2\}.$$

(Note that this (and indeed any further) development is only possible if the system is Hamiltonian (ie. $F = -2 \int G$) as was observed in the search for first integrals for the three-dimensional Ermakov system in §1.5.) If we set

$$\ddot{r} - r\dot{\theta}^2 = \frac{F(\theta)}{r^3} = \frac{1}{r^3} \left(-2 \int G(\theta) \right) = -\frac{\partial V}{\partial r} \quad (2.5.4a)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r^3} G(\theta) = -\frac{1}{r} \frac{\partial V}{\partial \theta}, \quad (2.5.4b)$$

we find the Hamiltonian as

$$\begin{aligned} H &= \frac{1}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - \frac{\int G}{r^2} \\ &= \frac{1}{2} \dot{r}^2 + \frac{I}{r^2}. \end{aligned} \quad (2.5.5)$$

We pause for a moment to consider the physical interpretations of both the Ermakov–Lewis invariant (2.5.2) and the Hamiltonian (2.5.5). The former has always been considered as an expression for the energy of a particle (moving in a plasma). However, if we consider it to be (more correctly) the equation for the angular momentum of the particle we realise that we need to rewrite it as

$$I' = 2I = \left(r^2 \dot{\theta} \right)^2 - 2 \int G(\theta) d\theta. \quad (2.5.6)$$

(Later calculations also bear this out.) Our Hamiltonian can be rewritten as

$$\begin{aligned} H' &= 2H = \dot{r}^2 + \frac{I'}{r^2} \\ &= \left(\dot{r}^2 - \frac{2 \int G}{r^2} \right) + \frac{L^2}{r^2} \end{aligned} \quad (2.5.7)$$

(However, we do not call H' the ‘new’ Hamiltonian.) which is just the equation for the Newton–Cotes spiral[115, p 83]. Replacing J by $\frac{1}{2}J'$ gives

$$\begin{aligned} J' &= a \left[\left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - \frac{2 \int G}{r^2} \right] - \dot{a} r \dot{r} + \frac{1}{2} \ddot{a} r^2 \\ &= 2aH - \dot{a} r \dot{r} + \frac{1}{2} \ddot{a} r^2. \end{aligned} \quad (2.5.8)$$

The expression for our third first integral (obtained from the first and second terms of (2.5.1)) is

$$K = \frac{1}{2\sqrt{2I}} \arcsin \frac{Jr^2/a - 2I}{(r^2/a)\sqrt{J^2 - 2I\alpha}} - \int \frac{1}{r^2} dt, \quad (2.5.9)$$

which we can rewrite as

$$K' = 2K = \frac{1}{\sqrt{I'}} \arcsin \frac{J'r^2 - 2aI'}{r^2 \sqrt{J'^2 - 4\alpha I'}} - 2 \int \frac{1}{r^2} dt. \quad (2.5.10)$$

Note that both J' and K' depend on $a(t)$. We therefore have seven first integrals, viz.

$$I' = \left(r^2 \dot{\theta} \right)^2 - 2 \int G(\theta) d\theta \quad (2.5.11)$$

$$J'_1 = \dot{r}^2 + \frac{I'}{r^2} = H' \quad (2.5.12)$$

$$J'_2 = tH' - r\dot{r} \quad (2.5.13)$$

$$J'_3 = t^2 H' - 2tr\dot{r} + r^2 \quad (2.5.14)$$

$$K'_1 = \frac{1}{\sqrt{I'}} \arcsin \left(1 - \frac{2I'}{r^2 J'_1} \right) - 2 \int \frac{1}{r^2} dt \quad (2.5.15)$$

$$K'_2 = \frac{1}{\sqrt{I'}} \arcsin \left[\frac{1}{\sqrt{J'_1 J'_3}} \left(J'_2 - \frac{2I't}{r^2} \right) \right] - 2 \int \frac{1}{r^2} dt \quad (2.5.16)$$

$$K'_3 = \frac{1}{\sqrt{I'}} \arcsin \left(1 - \frac{2I't^2}{r^2 J'_3} \right) - 2 \int \frac{1}{r^2} dt. \quad (2.5.17)$$

(Note that J'_1 is twice the Hamiltonian.) However, we can relate three of these first integrals to the others in the following manner:

$$J'_3 = \frac{J'^2_2 + I'}{J'_1}, \quad (2.5.18)$$

$$K'_2 = K'_1 - \sqrt{I'} \arcsin \sqrt{\frac{I'}{J'_1 J'_3}} \quad (2.5.19)$$

$$K'_3 = K'_1 - \sqrt{I'} \arcsin \frac{2\sqrt{I'} J'_2}{J'_1 J'_3}. \quad (2.5.20)$$

Thus the system (2.4.8), with $F(\theta) = -2 \int G(\theta) d\theta$, possesses the four independent first integrals I', J'_1, J'_2 and K'_1 where I' has an infinite-dimensional Lie algebra given by (2.4.3) and J'_1, J'_2 and K'_1 possess the Lie algebra $sl(2, R)$ given by the generators (2.4.9). We remark that while there have been other attempts to find first integrals for (2.4.8) (or its cartesian equivalent) [33, 92, 94], these efforts impose artificial constraints on the system which the above approach does not.

2.6 The Lie Algebra $sl(2, R)$

The Lie algebra $A_{3,8}$ has two familiar forms – $sl(2, R)$ with Lie brackets

$$[G_1, G_2] = G_1, \quad [G_1, G_3] = 2G_2, \quad [G_2, G_3] = G_3$$

and $so(2, 1)$ with Lie brackets

$$[G_1, G_2] = G_3, \quad [G_1, G_3] = -G_2, \quad [G_2, G_3] = -G_1$$

which are isomorphic. The $so(2, 1)$ form implies non-compact rotations (It is a non-compact form of $so(3)$.) and is more appropriate to three dimensions. For our purposes it is desirable that we use $sl(2, R)$ as the problems of interest to us are based in $1 + 1$ dimensions.

Firstly, $sl(2, R)$ is found in higher order equations of maximal symmetry . For an n th ($n > 2$) order equation the algebra is [78, 79]

$$nA_1 \oplus_s (A_1 \oplus sl(2, R)),$$

where nA_1 is the n -dimensional decomposable Abelian algebra. For $n = 2$ the maximal algebra is $sl(3, R)$. Two of the realizations of $sl(2, R)$ in terms of operators in the plane are subalgebras of $sl(3, R)$ [35, 79].

We also observe the occurrence of $sl(2, R)$ in the study of quasi exact solutions to the Schrödinger equation. Turbiner[111, 112] treated the problem of the sextic potential via the representation

$$J^+ = z^2 \frac{d}{dz} - 2jz, \quad J^0 = z \frac{d}{dz} - j, \quad J^- = \frac{d}{dz} \quad (2.6.1)$$

of $sl(2, R)$, where $z = x^2$ and $j(j+1)$ is the eigenvalue of the Casimir operator. Subsequently, de Souza Dutra and Boschi Filho[20] used the algebra $so(2, 1)$ to obtain polynomial potentials of higher degree. The link between $sl(2, R)$ and quasi exact problems is such that Chhajlany[19] defined quasi exact problems as those wherein a partial set of exact solutions with a dynamical symmetry (viz. $sl(2, R)$) could be found. A more detailed and clearer discussion is to be found in[121].

As a final remark we point out that Boyer and Wolf[14] provide a detailed analysis of the canonical transformation between coordinate-momentum and number-phase descriptions for systems possessing the $sl(2, R)$ dynamical algebra on the quantum mechanical level.

Thus it is useful to consider the general form of the equation invariant under $sl(2, R)$. This will be determined in §3.

2.7 Transformation of symmetries

It sometimes occurs that the Lie algebra obtained from the symmetries of a particular differential equation cannot be readily identified as belonging to a particular Lie group. To better identify the Lie algebra one needs to often make a change of basis. We illustrate the procedure with a simple example.

In §2.6 it was noted that the Lie algebra $sl(2, R)$ has the Lie brackets

$$[G_1, G_2] = G_1, \quad [G_1, G_3] = 2G_2, \quad [G_2, G_3] = G_3. \quad (2.7.1)$$

However, in our analysis of Ermakov systems (see also [6, 61]) we found that the Lie brackets of the generators were

$$[X_1, X_2] = 2X_1, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = 2X_3. \quad (2.7.2)$$

and remarked that this was also $s\ell(2, R)$. We use the procedure of change of basis to show that the two forms are equivalent and do not represent different Lie algebras.

We can proceed in the following way. We easily show that the two forms (2.7.1) and (2.7.2) are equivalent by setting

$$Y_1 = X_1$$

$$Y_2 = \frac{1}{2}X_2$$

$$Y_3 = X_3$$

(by inspection) which yields

$$[Y_1, Y_2] = Y_1, \quad [Y_1, Y_3] = 2Y_2, \quad [Y_2, Y_3] = Y_3. \quad (2.7.3)$$

which is isomorphic to (2.7.1).

The second (more rigorous) method is to proceed as follows: We want the generators

$$G_1 = \frac{\partial}{\partial t} \quad (2.7.4a)$$

$$G_2 = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} \quad (2.7.4b)$$

$$G_3 = t^2 \frac{\partial}{\partial t} + 2tr \frac{\partial}{\partial r} \quad (2.7.4c)$$

to be transformed to

$$X_1 = \frac{\partial}{\partial T} \quad (2.7.5a)$$

$$X_2 = 2T \frac{\partial}{\partial T} + R \frac{\partial}{\partial R} \quad (2.7.5b)$$

$$X_3 = T^2 \frac{\partial}{\partial T} + TR \frac{\partial}{\partial R}. \quad (2.7.5c)$$

We set

$$T = F(t, r) \quad R = G(t, r).$$

Invoking eq (2.7.4a) we have

$$\frac{\partial F}{\partial t} = 1 \quad \frac{\partial G}{\partial t} = 0, \quad (2.7.6)$$

eq (2.7.4b)

$$t \frac{\partial F}{\partial t} + r \frac{\partial F}{\partial r} = F \quad (2.7.7)$$

$$t \frac{\partial G}{\partial t} + r \frac{\partial G}{\partial r} = \frac{1}{2} G \quad (2.7.8)$$

and eq (2.7.4c)

$$t^2 \frac{\partial F}{\partial t} + 2tr \frac{\partial F}{\partial r} = F^2 \quad (2.7.9)$$

$$t^2 \frac{\partial G}{\partial t} + 2tr \frac{\partial G}{\partial r} = FG \quad (2.7.10)$$

(If one transforms two symmetries to two new symmetries, then the procedure is straightforward and one would obtain $2F$ on the RHS of (2.7.7) and G on the RHS of (2.7.8). However, as we have three symmetries, eqq (2.7.9–2.7.10) impose a restriction on the transformation and the RHS of eqq (2.7.7) and (2.7.8) are as above.) whence

$$T (= F) = t \quad R (= G) = r^{1/2}. \quad (2.7.11)$$

Thus we can transform (2.7.4) to (2.7.5) using (2.7.11).

2.8 The Ermakov–Pinney equation revisited

Now that we have developed the concept of symmetries we return to the Ermakov–Pinney equation.

2.8.1 The $sl(2, R)$ connection

The symmetries of the Ermakov–Pinney equation (1.4.5), viz.

$$\ddot{\rho} + \omega^2 \rho = \frac{\Omega^2}{\rho^3} \quad (2.8.1)$$

are[63]

$$G_1 = u_1(t) \frac{\partial}{\partial t} + \frac{1}{2} \dot{u}_1(t) \rho \frac{\partial}{\partial \rho} \quad (2.8.2a)$$

$$G_2 = u_2(t) \frac{\partial}{\partial t} + \frac{1}{2} \dot{u}_2(t) \rho \frac{\partial}{\partial \rho} \quad (2.8.2b)$$

$$G_3 = u_3(t) \frac{\partial}{\partial t} + \frac{1}{2} \dot{u}_3(t) \rho \frac{\partial}{\partial \rho}, \quad (2.8.2c)$$

where the u_i 's are the linearly independent solutions of

$$\ddot{u} + 4\omega^2 \dot{u} + 4\omega \dot{\omega} u = 0. \quad (2.8.3)$$

(This is the same equation obtained in §2.4.) Using the method explained in [79] we find that the Lie algebra of (2.8.2) is $sl(2, R)$.

We can integrate (2.8.3), after multiplying by u , to obtain

$$u\ddot{u} - \frac{1}{2}\dot{u}^2 + 2\omega^2 u^2 = \text{cst} \quad (2.8.4)$$

which can be rewritten as

$$\ddot{\rho} + \omega^2 \rho = \frac{\Omega^2}{\rho^3} \quad (2.8.5)$$

if we let

$$u = \rho^2, \quad \text{cst} = 2\Omega^2.$$

Thus we can say that the Ermakov–Pinney equation is a first integral of (2.8.3). Note that (2.8.3) is a third order ordinary differential equation with the maximal number (seven) of symmetries which have the algebra $A_1 \oplus (3A_1 \oplus sl(2, R))$. The Ermakov–Pinney equation is the integral associated with the $sl(2, R)$ part.

To solve the Ermakov–Pinney equation (2.8.1) we proceed as follows: Introduce two new variables

$$V = \rho u^{-1/2} \quad T = \int u^{-1} dt$$

so that (2.8.1) becomes

$$V'' + \Omega^2 V = \frac{\Omega^2}{V^3} \quad (2.8.6)$$

which can be solved in terms of the solutions of [88]

$$Y'' + \Omega^2 Y = 0 \quad (2.8.7)$$

as

$$V^2 = \alpha \sin^2 \Omega T + 2\beta \sin \Omega T \cos \Omega T + \gamma \cos^2 \Omega T \quad (2.8.8)$$

where

$$\alpha\gamma - \beta^2 = \frac{1}{\Omega^2}.$$

2.8.2 Generalization of the Ermakov–Pinney equation

Reid[101] has shown that the more general form of (2.8.1), viz.

$$\ddot{\rho} + p(t)\rho = q_m(t)/\rho^{2m-1} \quad (2.8.9)$$

has the exact solution

$$\rho = [u^m + c(m-1)^{-1}W^{-2}v^m]^{1/m} \quad (2.8.10)$$

for $m \neq 0, 1$, where u and v are the linearly independent solutions of

$$\ddot{y} + p(t)y = 0 \quad (2.8.11)$$

provided that

$$q_m(t) = c(uv)^{m-2}, \quad (2.8.12)$$

W is the Wronskian of u and v and c is a constant.

Recently we obtained a wider class of solutions to (2.8.9)[39] from a group theoretic approach. Eq (2.8.9) has the symmetry

$$G = uv \frac{\partial}{\partial t} + \frac{1}{2}(uv) \frac{\partial}{\partial \rho}$$

when $q_m(t)$ has the form (2.8.12). However, if we first require (2.8.9) to have a symmetry, it is of the form

$$a(t) \frac{\partial}{\partial t} + \frac{1}{2}(\dot{a}(t) + \alpha) \rho \frac{\partial}{\partial \rho}, \quad (2.8.13)$$

where α is a constant and $a(t)$ a solution of

$$\ddot{a} + 4p\dot{a} + 2\dot{p}a = 0, \quad (2.8.14)$$

which is again the third order form of the Ermakov–Pinney equation. It is required that $q_m(t)$ have the form

$$q_m(t) = K [a(t)]^{m-2} \exp \left[\int \frac{m\alpha}{a(t)} dt \right], \quad (2.8.15)$$

where K is a constant.

Using the transformation

$$X = \int \frac{1}{a(t)} dt, \quad Y = \rho a^{-1/2} \exp \left[\frac{\alpha}{2} \int \frac{1}{a(t)} dt \right]$$

we can rewrite (2.8.9) in autonomous form, viz.

$$Y'' + \alpha Y' + \left(M + \frac{1}{4} \alpha^2 \right) Y = \frac{K}{Y^{2m-1}}, \quad (2.8.16)$$

where M is obtained in the same manner as the constant in (2.8.5). If $\alpha = 0$, the solution of (2.8.16) is always reducible to a quadrature. When, in addition, $M < 0$, (2.8.16) reduces to the form of one of Reid's solutions, viz.

$$\ddot{y} - \omega^2 y = c_1 / y^{2m-1}. \quad (2.8.17)$$

However, using (2.8.15) and the above transformation we obtain a wider class of equations that have the solution

$$\rho = \left[\delta^m e^{m\omega t} + c'_m \sigma^m e^{-m\omega t} \right]^{1/m} \quad (2.8.18)$$

given by Reid, where

$$c'_m = \frac{c_1}{4\omega^2 (\delta\sigma)^m (m-1)},$$

c_1 an arbitrary constant, δ and σ are constants determined by the initial conditions and $\omega^2 = p(t)$.

When $\alpha \neq 0$, we can reduce the order of the equation (by one) using the symmetry

$$G = \frac{\partial}{\partial X}.$$

This results in an Abel's equation of the second kind, which, unsurprisingly, cannot be solved for general α . We can find a second symmetry if

$$M = \frac{-\alpha^2 m^2}{[2(m-2)]^2}.$$

We can now reduce the solution to a quadrature which invalidates the solution given by Reid.

2.9 Classification of Ermakov systems

Sarlet and Ray[106] proposed an extensive classification scheme for Ermakov-type systems involving Pfaffian and standard forms. The central feature of their scheme is the existence of a so-called nonlinear superposition of the general solution of the Ermakov–Pinney equation and its extensions. This superposition is in terms of the linearly independent solutions of the associated linear equation. This property of Ermakov systems has attracted considerable interest[34, 93, 97, 104], but is beyond the scope of this work.

Leach[61] based the concept of a Ermakov–Lewis invariant on the invariance of (1.5.4) under the representation (2.4.6) of the algebra $sl(2, R)$. This led him to define generalised Ermakov systems as systems wherein the original equations of motion, eq (1.5.3), possessed $sl(2, R)$ symmetry and weak generalised Ermakov systems as systems where only (1.5.4) held. We can easily extend this notion to higher dimensions. Consider the three-dimensional system

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = S(t, r, \theta, \phi) \quad (2.9.1a)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta = -\frac{1}{2r^3}\frac{\partial W(\theta, \phi)}{\partial\theta} \quad (2.9.1b)$$

$$r\sin\theta\ddot{\phi} + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta = -\frac{1}{2r^3\sin\theta}\frac{\partial W(\theta, \phi)}{\partial\phi}, \quad (2.9.1c)$$

where S can be any function of r, θ, ϕ and t . This system need have no symmetries at all, but does possess an Ermakov–Lewis invariant, and so falls into the class of weak generalised Ermakov systems. The reason for this is that eqq (2.9.1b) and (2.9.1c) do have $sl(2, R)$ symmetry and it is from these equations that we obtain the invariant(s).

Athorne[6], although not disagreeing with the distinction, noted that other classifications – such as Hamiltonian and non-Hamiltonian – were also important. Indeed, the point of that letter was that those (non-Hamiltonian) systems described, and which had only *one* global invariant, could be understood as ‘linear extensions’ of an underlying Hamiltonian system with appropriate choice of time-variable.

We follow Leach's classification scheme, but take note of Athorne's comments in later calculations.

Chapter 3

The Inverse problem

We find a wider class of Ermakov systems by considering the general form of the equation invariant under $sl(2, R)$.

3.1 Canonical forms of $sl(2, R)$

We have shown that the Lie algebra of Ermakov systems admits $sl(2, R)$. The natural question to contemplate is “What is the class of second order differential equations invariant under $sl(2, R)$?”[38]. This gives rise to the so-called ‘inverse problem’ – the finding of the equation given the algebra. Mahomed[77] showed that the Lie algebra $sl(2, R)$ has three canonical forms of operators in two variables, viz:

$$G_1 = \frac{\partial}{\partial t} \tag{3.1.1a}$$

$$G_2 = t \frac{\partial}{\partial t} \tag{3.1.1b}$$

$$G_3 = t^2 \frac{\partial}{\partial t}, \tag{3.1.1c}$$

$$G_1 = \frac{\partial}{\partial t} \tag{3.1.2a}$$

$$G_2 = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} \tag{3.1.2b}$$

$$G_3 = (t^2 - r^2) \frac{\partial}{\partial t} + 2tr \frac{\partial}{\partial r} \quad (3.1.2c)$$

and

$$G_1 = \frac{\partial}{\partial t} \quad (3.1.3a)$$

$$G_2 = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} \quad (3.1.3b)$$

$$G_3 = t^2 \frac{\partial}{\partial t} + 2tr \frac{\partial}{\partial r}. \quad (3.1.3c)$$

Taking the Lie brackets of each set of these generators we easily verify that the non-zero commutators are

$$[G_1, G_2] = G_1, \quad [G_2, G_3] = G_3 \quad \text{and} \quad [G_3, G_1] = -2G_2. \quad (3.1.4)$$

We investigate the behaviour of the general second order ordinary differential equation

$$f(t, r, \theta, \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta}) = 0 \quad (3.1.5)$$

under the restriction of invariance under (3.1.1–3.1.3) in turn. It will then be determined whether the resulting equations can be interpreted physically and/or reduced to the form of Ermakov systems.

3.2 Equations invariant under $sl(2, R)$

We need the second extensions of the generators of the first realization of $sl(2, R)$ (3.1.1), viz.

$$G_1^{[2]} = \frac{\partial}{\partial t} \quad (3.2.1a)$$

$$G_2^{[2]} = t \frac{\partial}{\partial t} + 0 \frac{\partial}{\partial r} + 0 \frac{\partial}{\partial \theta} - \dot{r} \frac{\partial}{\partial \dot{r}} - \dot{\theta} \frac{\partial}{\partial \dot{\theta}} - 2\ddot{r} \frac{\partial}{\partial \ddot{r}} - 2\ddot{\theta} \frac{\partial}{\partial \ddot{\theta}} \quad (3.2.1b)$$

$$G_3^{[2]} = t^2 \frac{\partial}{\partial t} + 0 \frac{\partial}{\partial r} + 0 \frac{\partial}{\partial \theta} - 2t\dot{r} \frac{\partial}{\partial \dot{r}} - 2t\dot{\theta} \frac{\partial}{\partial \dot{\theta}} - (4t\ddot{r} + 2\dot{r}) \frac{\partial}{\partial \ddot{r}} - (4t\ddot{\theta} + 2\dot{\theta}) \frac{\partial}{\partial \ddot{\theta}}. \quad (3.2.1c)$$

G_1 is its own second extension (3.2.1a) and requires (3.1.5) to be independent of t , viz.

$$f = g(r, \theta, \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta}) = 0. \quad (3.2.2)$$

The invariance of (3.2.2) under (3.2.1b) results in a partial differential equation with associated Lagrange's system

$$\frac{dr}{0} = \frac{d\theta}{0} = \frac{d\dot{r}}{-\dot{r}} = \frac{d\dot{\theta}}{-\dot{\theta}} = \frac{d\ddot{r}}{-2\ddot{r}} = \frac{d\ddot{\theta}}{-2\ddot{\theta}}, \quad (3.2.3)$$

where we keep the first two terms to remind us that r and θ are characteristics. (Note: When we use the notation (3.2.3_{*i*}) we refer to the i -th term of equation (3.2.3). This allows us to easily indicate which terms of (3.2.3) we are dealing with to find the associated characteristics.) (3.2.3₃), with (3.2.3₄) requires that $u = \dot{\theta}/\dot{r}$ be a characteristic, with (3.2.3₅) $v = \ddot{r}/\dot{r}^2$ and with (3.2.3₆) $w = \ddot{\theta}/\dot{r}^2$. Eq (3.2.2) now reduces to

$$g = h(r, \theta, u, v, w). \quad (3.2.4)$$

Using (3.2.1c) in a similar manner to (3.2.1b) further reduces the number of characteristics and the general form of the second order differential equation invariant under (3.1.1) is

$$k \left(r, \theta, \frac{\dot{\theta}}{\dot{r}}, \frac{\ddot{r}\dot{\theta} - \dot{r}\ddot{\theta}}{\dot{r}^3} \right) = 0. \quad (3.2.5)$$

We observe that (3.2.5) has coupled acceleration terms, and thus has no further interest for us (in two dimensions at least). This should not really be surprising as, if we consider the scalar form of (3.1.5), viz.

$$F(t, x, \dot{x}, \ddot{x}) = 0, \quad (3.2.6)$$

invariance under (3.1.1a) removes the t dependence, (3.1.1b) gives the form of the characteristics as

$$u = x, \quad \text{and} \quad v = \frac{\ddot{x}}{\dot{x}^2}$$

while (3.1.1c) removes the dependence of f on \ddot{x} !

We now consider the invariance of (3.1.5) under the second extension of the generators (3.1.2). Once again G_1 is its own second extension and requires that (3.1.5) be free of t , viz.

$$f = g(r, \theta, \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta}) = 0. \quad (3.2.7)$$

The second extension of (3.1.2b) requires that (3.2.7) have the form

$$h(\theta, u, v, w, x) = 0 \quad (3.2.8)$$

where u, v, w and x are given by

$$\begin{aligned} u &= \dot{r}, & v &= r\dot{\theta} \\ w &= r\ddot{r} & \text{and} & \quad x = r^2\ddot{\theta}. \end{aligned}$$

Imposing the invariance of (3.2.8) under the second extension of (3.1.2c) gives the final form of (3.1.5) as

$$k(\theta, \alpha, \beta, \gamma) = 0 \quad (3.2.9)$$

with

$$\alpha = \frac{r^2\dot{\theta}^2}{1 + \dot{r}^2}, \quad \beta = \frac{r\ddot{r}}{(1 + \dot{r}^2)^{3/2}} + \frac{1}{(1 + \dot{r}^2)^{1/2}}$$

and

$$\gamma = \frac{r^2\ddot{\theta} - r\dot{r}\dot{\theta}}{1 + \dot{r}^2} - \frac{r^2\dot{r}\ddot{\theta}}{(1 + \dot{r}^2)^2}.$$

While this case is slightly better than the previous one in that we can obtain two equations, the physical applications thereof are not immediately obvious.

After our previous successes we approach the third and final realization of $s\ell(2, R)$ (3.1.3) with more than a little trepidation. In this case, however, we need not to have worried excessively. To prevent the onset of boredom¹ we omit the invariance of (3.1.5) under the second extension of G_1 and move on to the far more interesting $G_2^{[2]}$ which requires that (3.1.5) have the form

$$g(\theta, \dot{r}, r\dot{\theta}, r\ddot{r}, r^2\ddot{\theta}) = 0. \quad (3.2.10)$$

The second extension of (3.1.3c) reduces the number of characteristics to four, and (3.2.10) becomes

$$h\left(\theta, r\dot{\theta}, r\ddot{r} - \frac{1}{2}\dot{r}^2, r^2\ddot{\theta} + r\dot{r}\dot{\theta}\right) = 0. \quad (3.2.11)$$

¹Equally *déjà vu*.

The acceleration terms are nicely separated, but we do not observe an immediate resolution of (3.2.11) to Ermakov systems. However, we have taken (3.1.4) to be $sl(2, R)$ while in §2.4 we used the form (2.4.7). We noted in §2.7 that (2.7.4) can be transformed to (2.7.5) which satisfy the commutation relations (2.4.7). Hence we can transform (3.1.3) to (2.7.5) by

$$T = t \quad R = r^{1/2}.$$

Thus we can rewrite (3.2.11) as

$$H(\theta, R^2\dot{\theta}, R^3\ddot{R}, R^4\ddot{\theta} + 2R^3\dot{R}\dot{\theta}) = 0. \quad (3.2.12)$$

(In the ensuing discussion we use (3.2.12) with R replaced by r for convenience.) We note that in spherical polar co-ordinates the characteristics $\phi, r^2\dot{\phi}$ and $r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi}$ would also be arguments of H and the general form of the equation is

$$f(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}, r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi}) = 0 \quad (3.2.13)$$

The generalisation to hyperspherical co-ordinates is obvious.

In two dimensions we require two equations which, in general, would be

$$F(\theta, r^2\dot{\theta}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}) = 0 \quad (3.2.14a)$$

$$G(\theta, r^2\dot{\theta}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}) = 0 \quad (3.2.14b)$$

We note that a particular case of the general result represented by eqq (3.2.14) was reported by Ray and Reid[97] who used a quaint combination of Noether's theorem and an *ad hoc* method.

We confine our attention to the subset of (3.2.14) which can be written in the normal forms

$$r^3\ddot{r} = f_1(\theta, r^2\dot{\theta}) \quad (3.2.15a)$$

$$r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta} = g(\theta, r^2\dot{\theta}). \quad (3.2.15b)$$

The implicit function theorem[15, p 171] does impose some restrictions on F and G for this inversion to be possible. In particular (3.2.15) may only have

local validity. In such a case the development below would be confined to a collection of neighbourhoods. However, in view of possible physical applications based on the ideas of Newtonian mechanics, we assume that eqq (3.2.15) have more than local validity. Eqq (3.2.15) can be recast in the form of the equation of motion of a classical particle if we take $f_1 = f + (r^2\dot{\theta})^2$ so that eqq (3.2.15) become

$$\ddot{r} - r\dot{\theta}^2 = \frac{1}{r^3}f(\theta, r^2\dot{\theta}) \quad (3.2.16a)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r^3}g(\theta, r^2\dot{\theta}). \quad (3.2.16b)$$

The usual generalised Ermakov systems (2.4.8) have f and g free of $r^2\dot{\theta}$.

In order to make the structure of these equations more transparent we introduce ‘new time’² τ defined by

$$\tau = \int r^{-2} dt \quad (3.2.17)$$

and the inverse radial distance $\chi = 1/r$. If derivatives with respect to τ are denoted by $'$, $''$ etc, eqq (3.2.16) become

$$\chi'' + [\theta'^2 + f(\theta, \theta')]\chi = 0 \quad (3.2.18a)$$

$$\theta'' = g(\theta, \theta'). \quad (3.2.18b)$$

(We have previously found[36] a system resulting from a more generalised form of (3.1.3). That system reduces to (3.2.18) naturally.) Eq (3.2.18b) is effectively a first order equation for θ' with θ as dependent variable. In terms of the Lie theory for the integration of a first order ode an integrating factor can be found such that

$$g(\theta, \theta') = -\theta' \frac{\partial M(\theta, \theta')}{\partial \theta} \bigg/ \frac{\partial M(\theta, \theta')}{\partial \theta'} \quad (3.2.19)$$

so that (3.2.18b) integrates to

$$M(\theta, \theta') = h, \quad (3.2.20)$$

²The specific form of the concept was first introduced by Jean-Robert Burgan in 1978[17, 18].

where h is a constant. (Note: We are concerned here with principle. In practice there may be technical difficulties!) Given the structure assumed for g the implicit function theorem[15, p 165] guarantees inversion of (3.2.20) to

$$\theta' = N(\theta, h)$$

(at least locally) so that

$$\tau - \tau_0 = \int \frac{d\theta}{N(\theta, h)}. \quad (3.2.21)$$

This can also be inverted (locally) to give

$$\theta = J(\tau, h, \tau_0). \quad (3.2.22)$$

Now that θ is known, (3.2.18a) becomes the differential equation in (χ, τ) space of the classical time-dependent linear oscillator if the coefficient of χ is positive, the free particle if zero and the linear repulsor if negative. We remark that the new time defined in (3.2.17) is almost familiar except that $r(t)$ is used instead of $\rho(t)$ [60, 69]. Another way to look at the definition of new time is as

$$\tau = \int r^{-2} dt = \int r^{-2} \frac{dt}{d\theta} d\theta = \int (r^2 \dot{\theta})^{-1} d\theta \quad (3.2.23)$$

so that τ is the measure of time in which the time rate of change of angle is the angular momentum. This angular momentum interpretation and resulting oscillator equation (3.2.18a) remind one of the interpretation of the Ermakov–Pinney equation by Eliezer and Gray[24].

Eqq (3.2.16) reduce to the equations for a Newton–Cotes spiral in the case $f = \text{constant}$ and $g = 0$ [115, p 83]. The qualitative features of a spiral are maintained for the generalised Ermakov system in the cases that (i) $\theta'^2 + f(\theta, \theta') < 0$ since $\chi(\tau)$ is unbounded and so $r \rightarrow 0$ and (ii) $\theta'^2 + f(\theta, \theta') > 0$ since $\chi(\tau)$ passes through zero and so $r \rightarrow \infty$. However, it is possible to obtain closed orbits[38].

The reduction of the nonlinear equation (3.2.16a) to that of the linear time-dependent oscillator combines the method of Whittaker[115, p 78] and the introduction of the ‘new time’ τ . In the case that the angular momentum

$L(:= r^2\dot{\theta})$ is conserved, the new time is just the $L\theta$ which Whittaker uses. In the general case the procedure adopted here is very similar to that found in Athorne *et al* [5]. In the two-dimensional case the connection between the angular variable and new time is obvious. We consider the set of equations in spherical polars corresponding to eqq (3.2.15) in the next chapter.

We have concentrated on generalised Ermakov systems (as defined in[5]) in the preceding analysis. It is apparent that weak systems can be included if in (3.2.16a) we allow f to include, say, additional radial forces. The integration of (3.2.16b) to obtain an Ermakov invariant is unaffected. However, in general (3.2.18a) would become nonlinear in χ and the possibility of closed form integration much reduced, except in such special cases as the Kepler inverse-square force. Further attention will be given to this in chapter 4.

For completeness we add that the general equation invariant under the generators of $so(2,1)$, viz.

$$G_1 = \frac{\partial}{\partial \phi} \quad (3.2.24a)$$

$$G_2 = \sin \phi \frac{\partial}{\partial \theta} + \coth \theta \cos \phi \frac{\partial}{\partial \phi} \quad (3.2.24b)$$

$$G_3 = \cos \phi \frac{\partial}{\partial \theta} - \coth \theta \sin \phi \frac{\partial}{\partial \phi} \quad (3.2.24c)$$

is

$$\begin{aligned} f \left(r, \dot{\phi}^2 \sinh^2 \theta + \dot{\theta}^2, \dot{\theta} \ddot{\phi} \sinh \theta + 2\dot{\theta}^2 \dot{\phi} \cosh \theta - \ddot{\theta} \dot{\phi} \sinh \theta + \dot{\phi}^3 \sinh^2 \theta \cosh \theta, \right. \\ \left. \ddot{\phi}^2 \sinh^2 \theta + 4\dot{\theta} \dot{\phi} \ddot{\phi} \sinh \theta \cosh \theta + 4\dot{\theta}^2 \dot{\phi}^2 \cosh^2 \theta + \ddot{\theta}^2 - 2\ddot{\theta} \dot{\phi}^2 \sinh \theta \cosh \theta \right. \\ \left. + \dot{\phi}^4 \sinh^2 \theta \cosh^2 \theta \right) = 0. \quad (3.2.25) \end{aligned}$$

3.3 Equations invariant under a generalised similarity symmetry

We finally look at the class of general second order ordinary differential equations invariant under the symmetry

$$G = a \frac{\partial}{\partial t} + \frac{1}{2} \dot{a} r \frac{\partial}{\partial r} \quad (3.3.1)$$

of the Ermakov–Lewis invariant. The by-now-familiar routine results in

$$f \left(r a^{-1/2}, \theta, \dot{r} a^{1/2} - \frac{1}{2} r \dot{a} / a^{1/2}, \dot{\theta} a, \ddot{r} a^{3/2} - \frac{1}{2} (a \ddot{a} - \frac{1}{2} \dot{a}^2) r \dot{a}^{1/2}, a^2 \ddot{\theta} + a \dot{a} \dot{\theta} \right) = 0. \quad (3.3.2)$$

If we set

$$a = \rho^2,$$

we have

$$f(u, v, w, x, y, z) = 0, \quad (3.3.3)$$

where

$$\begin{aligned} u &= r/\rho, & v &= \theta \\ w &= \rho \dot{r} - \dot{\rho} r, & x &= \rho^2 \dot{\theta} \\ y &= \rho^3 \ddot{r} - \rho^2 \ddot{\rho} r & \text{and} & \quad z = \rho^4 \ddot{\theta} + 2\rho^3 \dot{\rho} \dot{\theta}. \end{aligned}$$

It is apparent³ that, if we insist on an autonomous form for (3.3.3) to obtain equations of the type (1.5.4), then we just have (3.2.12), the equation invariant under the ‘natural’ representation of $sl(2, R)$.

³Well, perhaps not immediately so.

Chapter 4

Three-dimensional Ermakov systems

Properties of Ermakov systems (invariant under $sl(2, R)$) have so far been studied primarily in two dimensions. We now look at the structure of systems invariant under $sl(2, R)$ in three dimensions[37] and attempt to obtain Ermakov-type systems. We also establish the existence of three vectors of Poincaré type. We complete the analysis by considering some weak systems.

4.1 Equations invariant under $sl(2, R) \oplus so(3)$

In spherical polar co-ordinates, the class of equations corresponding to (3.2.12) is

$$F(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}, r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi}) = 0. \quad (4.1.1)$$

We consider (4.1.1) as a system of second differential equations invariant under $sl(2, R)$ (3.1.3) by considering three equations of the form (4.1.1), viz.

$$r^3\ddot{r} = f_1(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}) \quad (4.1.2a)$$

$$r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta} = g_1(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}) \quad (4.1.2b)$$

$$r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi} = h_1(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}) \quad (4.1.2c)$$

which is just (3.2.15) in spherical polars. (One could conceive of variations on this. By way of example (4.1.2c) could be replaced by

$$H(\theta, \phi, r^2 \dot{\theta}, r^2 \dot{\phi}, I) = 0, \quad (4.1.3)$$

where I is a parameter which may be taken to be the value of a first integral. If I has a particular value, I_0 , in which case it could just as well be omitted from (4.1.3), we are in the realm of configuration invariants. To keep the discussion concise we do not digress into this specialized area. The reader is referred to Sarlet *et al* [107] for a discussion of the relationship between systems of second order equations, first integrals and configurational invariants.)

We recall that in two dimensions (3.2.15) the connection between the angular variable and new time was obvious. We attempt to proceed in a similar manner by again using the transformations

$$T = \int r^{-2} dt \quad \chi = r^{-1}. \quad (4.1.4)$$

However, the new system

$$\chi'' = -f(\theta, \phi, \theta', \phi')\chi \quad (4.1.5a)$$

$$\theta'' = g(\theta, \phi, \theta', \phi') \quad (4.1.5b)$$

$$\psi'' = h(\theta, \phi, \theta', \phi'). \quad (4.1.5c)$$

is much more complex than in the two-dimensional case. An immediate problem is to determine which of the two angles should be chosen.

An obvious approach is to impose a further constraint on the system (4.1.2). Considering the possible physical applications, we require that eqq (4.1.2) also have rotational invariance, ie, the system of equations also be invariant under the action of the generators of the Lie algebra $so(3)$, viz.

$$G_4 = \frac{\partial}{\partial \phi} \quad (4.1.6a)$$

$$G_5 = \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \quad (4.1.6b)$$

$$G_6 = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \quad (4.1.6c)$$

with Lie brackets

$$[G_4, G_5] = G_6 \quad [G_5, G_6] = G_4 \quad [G_6, G_4] = G_5.$$

The application of $(\partial/\partial\phi)$ to eqq (4.1.2)¹ is simple enough: f_1 , g_1 and h_1 must be ϕ free. The second extension of G_5 mixes $\ddot{\theta}$ and $\ddot{\phi}$ terms requiring the treatment of (4.1.2b) and (4.1.2c) as a coupled system whereas (4.1.2a) can be treated by itself. The algebraic manipulation involved we have seen in the previous chapter and we omit the details except to point out that, once we have used $G_5^{[2]}$, requiring invariance under $G_6^{[2]}$ makes no difference to the result. This is to be expected since $G_6 = [G_4, G_5]$.

We find that the most general system of the form (4.1.2) invariant under $\mathfrak{sl}(2, R) \oplus \mathfrak{so}(3)$ is

$$r^3 \ddot{r} = A_1(L) \quad (4.1.7a)$$

$$r^4 \ddot{\theta} + 2r^3 \dot{r} \dot{\theta} = r^4 \dot{\phi}^2 \sin \theta \cos \theta + B(L) r^2 \dot{\theta} - C(L) r^2 \dot{\phi} \sin \theta \quad (4.1.7b)$$

$$r^4 \ddot{\phi} + 2r^3 \dot{r} \dot{\phi} = -2r^4 \dot{\theta} \dot{\phi} \cot \theta + \frac{1}{\sin \theta} [B(L) r^2 \dot{\phi} \sin \theta + C(L) r^2 \dot{\theta}], \quad (4.1.7c)$$

where A_1 , B and C are arbitrary functions of their argument L , the magnitude of the angular momentum given by

$$L^2 := r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta). \quad (4.1.8)$$

Equations (4.1.7) may be written in the compact vectorial form

$$\ddot{\mathbf{r}} = \frac{1}{r^3} \left\{ A(L) \hat{\mathbf{r}} + B(L) \hat{\boldsymbol{\omega}} + C(L) \hat{\mathbf{L}} \right\}, \quad (4.1.9)$$

where we have replaced $A_1(L)$ by $A(L) + L^2$. In an obvious notation $\hat{\mathbf{r}}$ and $\hat{\mathbf{L}}$ are the unit vectors in the direction of the radius vector and the angular momentum vector $\mathbf{L} (:= \mathbf{r} \times \dot{\mathbf{r}})$. The unit vector $\hat{\boldsymbol{\omega}} := \hat{\mathbf{L}} \times \hat{\mathbf{r}}$ is in the direction of the rate of change of $\hat{\mathbf{r}}$ and is the natural generalisation of $\dot{\theta}$ in plane polar co-ordinates.

¹We could also use eqq (4.1.5), but subsequent calculations have proved the first route is the better.

In terms of the definition of generalised and weak generalised Ermakov systems (4.1.2) represents the three-dimensional form of the generalised Ermakov system. The addition of some extra term to (4.1.2a) would be in the spirit of the meaning of weak generalised Ermakov system as given by [61]. However, two points should be made. The first is that under suitable (for example analyticity) conditions (4.1.2) have integrals, ie, constants of integration, defined over some local neighbourhood. The existence of one or more global first integrals for (4.1.2b), (4.1.2c) or a combination of (4.1.2b) and (4.1.2c) would require some constraints on the functions g_1 and h_1 . The second is that we have chosen the radial equation to be the one which leads to the symmetry breaking. It made sense in two dimensions as we were guaranteed the ‘in principle’ existence of an Ermakov–Lewis invariant provided that the system maintained $sl(2, R)$ symmetry in the angular equation. This of course is lost in the general three-dimensional case and further thought needs to be given to a correct terminology.

To conclude this section we make some observations about (4.1.9). For B and C zero and $A(L)$ a constant (L is conserved) we have the equation for a Newton–Cotes spiral [115] which, in essence, is the free particle in the plane with an excess or deficit of angular momentum. For A and B zero and $C(L)$ proportional to L ($= \lambda L$) a constant (L is again conserved) we have the classic equation of a particle moving in the field of a magnetic monopole. In this case it is well-known that there exists the first integral

$$\mathbf{P} = \mathbf{L} + \lambda \hat{\mathbf{r}} \quad (4.1.10)$$

and the motion is on the surface of a cone of semi-vertex angle given by $\arccos(C/PL)$ [90]. It is only more recently that Moreira *et al* [83] demonstrated that the algebra of the equation of motion was $so(2, 1) \oplus so(3)$ (isomorphic to $sl(2, R) \oplus so(3)$). We note that the classical monopole is a Hamiltonian system and the components of the Poincaré vector possess the algebra $so(3)$ under the operation of taking the Poisson Bracket [82].

4.2 Poincaré vector for equations invariant under $sl(2, R) \oplus so(3)$

The combination of the existence of the Poincaré vector (4.1.10) and the symmetry algebra $sl(2, R) \oplus so(3)$ for the classical monopole equation

$$\ddot{\mathbf{r}} = \frac{C(L)\hat{\mathbf{L}}}{r^3} \quad (4.2.1)$$

suggests that it may be fruitful to look for a similar vector for the general equation (4.1.9). Elementary manipulation of (4.2.1) produces (4.1.10) more or less without trying. This is not the case with (4.1.9). However, an equally simple-minded approach does yield interesting results. We assume the existence of a vector of Poincaré type given by

$$\mathbf{P} := I\hat{\mathbf{r}} + J\hat{\boldsymbol{\omega}} + K\hat{\mathbf{L}}, \quad (4.2.2)$$

where I , J and K are functions to be determined. Requiring that $\dot{\mathbf{P}}$ be zero when (4.1.9) is satisfied leads to the system of equations

$$\frac{d}{dt} \begin{pmatrix} I \\ J \\ K \end{pmatrix} = r^{-2} \begin{pmatrix} 0 & L & 0 \\ -L & 0 & C/L \\ 0 & -C/L & 0 \end{pmatrix} \begin{pmatrix} I \\ J \\ K \end{pmatrix} \quad (4.2.3)$$

which in terms of new time

$$T = \int r^{-2} dt$$

is

$$\begin{pmatrix} I \\ J \\ K \end{pmatrix}' = \begin{pmatrix} 0 & L & 0 \\ -L & 0 & C/L \\ 0 & -C/L & 0 \end{pmatrix} \begin{pmatrix} I \\ J \\ K \end{pmatrix}. \quad (4.2.4)$$

Equations (4.2.4) have a geometrical interpretation. They are the formulæ of Frenet or the Serret-Frenet formulæ² [110, p 70] associated with a curve of

²They were first published in the Toulouse dissertation of F Frenet in 1847 (An abstract later appeared in *Journal de Mathématique* in 1852.). J A Serret also published his paper in the same journal but in 1851. It therefore appeared after Frenet's thesis, but before he made his results more widely known.

curvature L and torsion $C(L)/L$, parametrized by T . An orthonormal triad of solution vectors represents the principle triad of the curve, consisting of tangent, normal and binormal vectors.

As an aside we note that this approach is not feasible for the two dimensional system of equations since then $\dot{\mathbf{r}}$ and $\dot{\boldsymbol{\theta}}$ are multiples of $\boldsymbol{\theta}$ and each multiple is a property of the geometry of the plane and is independent of the mechanics. The only way to make progress would be to specify the \dot{r} and $\dot{\theta}$ dependence in \mathbf{P} . This has not been necessary in the present case because the dynamics is introduced via $\hat{\omega}$.

The scalar product of (4.1.9) with $\hat{\mathbf{r}}$ is

$$\ddot{r} - r(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) = r^{-3} A(L) \quad (4.2.5)$$

which in terms of χ and T is

$$\chi'' + \{L^2 + A(L)\} \chi = 0. \quad (4.2.6)$$

The vector product of \mathbf{r} with (4.1.9) gives

$$\dot{\mathbf{L}} = r^{-2} \{B\hat{\mathbf{L}} - C\boldsymbol{\omega}\} \quad (4.2.7)$$

so that

$$L\dot{L} = r^{-2}BL \quad (4.2.8)$$

or

$$L' = B(L) \quad (4.2.9)$$

which gives the first integral

$$M = T - \int \frac{dL}{B(L)}. \quad (4.2.10)$$

This can be interpreted as an equation defining T in terms of L or L in terms of T . Naturally, if B is zero, the magnitude of the angular momentum is constant.

By virtue of (4.2.10), (4.2.6) becomes the by now familiar time-dependent oscillator which characterizes the radial equation (3.2.18a) for generalised Ermakov systems expressed in the appropriate co-ordinates.

In like fashion (4.2.4) is now a three-dimensional nonautonomous first order system of differential equations. Its structure is suggestive of a time-dependent oscillator written as a system of first order equations. However, the analogy only helps for constant L . Before going into the details of the method of solution of (4.2.4) a short remark is in order. As a three-dimensional first order linear system it has three linearly independent solutions. This means that there are in fact three ‘Poincaré’ vectors.

In view of the geometric interpretation of equations (4.2.4) some natural examples to consider would be ones which have standard properties of curves. The simplest one is a curve of constant curvature which means that B is zero. The solution of (4.2.4) is

$$\begin{pmatrix} I \\ J \\ K \end{pmatrix} = \exp \left\{ \begin{pmatrix} 0 & L & 0 \\ -L & 0 & C/L \\ 0 & -C/L & 0 \end{pmatrix} T \right\} \begin{pmatrix} I_0 \\ J_0 \\ K_0 \end{pmatrix}$$

$$= \begin{bmatrix} \frac{C}{L\Omega} \\ 0 \\ \frac{L}{\Omega} \end{bmatrix} I_0 + \begin{bmatrix} \frac{L}{\Omega} \sin \Omega T \\ \cos \Omega T \\ -\frac{C}{L\Omega} \sin \Omega T \end{bmatrix} J_0 + \begin{bmatrix} \frac{L}{\Omega} \cos \Omega T \\ -\sin \Omega T \\ -\frac{C}{L\Omega} \cos \Omega T \end{bmatrix} K_0, \quad (4.2.11)$$

where $\Omega^2 := L^2 + C^2/L^2$ and the scaling has been chosen so that the norm of each vector is one. The standard magnetic monopole has $C = \lambda L$ and is associated with a curve of constant torsion. The usual Poincaré vector of the literature has $I_0 = \Omega$, $J_0 = 0$ and $K_0 = 0$ and is

$$\mathbf{P} = \mathbf{L} + \lambda \hat{\mathbf{r}}, \quad (4.2.12)$$

but we emphasize that there are in fact three vectors.

The solution (4.2.11) also applies in the case that there is a constant ratio of torsion to curvature, ie. $C = \lambda L^2$. There is a Poincaré vector $\hat{\mathbf{L}} + \lambda \hat{\mathbf{r}}$ (*regardless*

of A and B) which is time-independent. Then the general solution of (4.2.4) is written using $\int^T L(\tau) d\tau$.

Since L is constant, the solution of the radial equation (4.2.6) is simply

$$\chi(T) = E \sin \omega T + F \cos \omega T, \quad (4.2.13)$$

where $\omega^2 := L^2 + A(L)$ and E and F are constants of integration. (We consider only ω real and nonzero. The other two possibilities can be treated in a similar fashion.) From (4.2.13) and the definition of T we have

$$t = \int \frac{dT}{(E \sin \omega T + F \cos \omega T)^2} \quad (4.2.14)$$

which is easily evaluated and inverted to give T in terms of t and, through (4.2.13), we have

$$r(t) = \left[\frac{1}{E^2 + F^2} + \frac{2\omega^2}{F^2} (t - t_0)^2 \right]^{\frac{1}{2}}. \quad (4.2.15)$$

With the general solution of (4.2.1) inverted to give $\hat{\mathbf{r}}$, multiplication by r gives $\mathbf{r}(t)$ with six constants of integration and hence the general solution. This solution applies to all problems associated with constant curvature.

We, of course, would expect to find three conserved vectors as the form posited for \mathbf{P} spans the space (except for exceptional points where degeneracy occurs). One is reminded of the work of Fradkin[31, 32] and Yoshida[119, 120] on the existence of Laplace-Runge-Lenz vectors for central force and other three-dimensional problems.

The procedure described in detail above for the constant curvature case applies *mutatis mutandis* for the general equation (4.1.9). Usually the equations become nonautonomous with a consequent increase in the degree of difficulty of solution. This is particularly the case with (4.2.4) which in the autonomous case is solved by a straightforward exponentiation of the coefficient matrix by time. Nevertheless the general equation can be treated.

By construction \mathbf{P} is a constant vector and I , J and K are not independent when the magnitude of \mathbf{P} is specified. (In the case of the single vector

there is not much point to it, but, when there are three vectors spanning the space, there is no small appeal in specifying unit vectors.) Only two dependent variables are needed and we introduce the transformation (the so-called Weierstrass transformation³ of Forsyth[30, p19]; see also[46, eq 8.50]),

$$\xi = \frac{I + iJ}{1 - K} \quad \eta = -\frac{I + iJ}{1 + K}. \quad (4.2.16)$$

Together with the normalization of \mathbf{P} , (4.2.16) leads to a common differential equation for ξ and η which is of Riccati form, viz.

$$w' + iLw + \frac{iL}{2C}(1 - w^2) = 0, \quad (4.2.17)$$

where w stands for ξ and η in turn. The transformation

$$w = \frac{2iCy'}{Ly} \quad (4.2.18)$$

yields the linear second order equation

$$y'' + \left(\frac{C'}{C} - \frac{L'}{L} + iL \right) y' + \frac{L^2}{4C^2} y = 0 \quad (4.2.19)$$

which is trivially related via

$$y = \left(\frac{L}{C} \right)^{1/2} u e^{-i/2 \int L} \quad (4.2.20)$$

to the standard time-dependent harmonic oscillator

$$u'' + \left\{ \frac{1}{4} \left(\frac{C'}{C} - \frac{L'}{L} + iL \right)^2 - \frac{1}{2} \left(\frac{C'}{C} - \frac{L'}{L} + iL \right)' + \frac{L^2}{4C^2} \right\} u = 0. \quad (4.2.21)$$

Given the solution for u , ξ and η follow through (4.2.18) and (4.2.20). The components of \mathbf{P} are given by

$$I = \frac{1 - \xi\eta}{\xi - \eta}, \quad J = \frac{i(1 + \xi\eta)}{\xi - \eta}, \quad K = \frac{\xi + \eta}{\xi - \eta}. \quad (4.2.22)$$

Needless to remark the tricky business is always the solution of the time-dependent harmonic oscillator equation (4.2.21). We illustrate this with what

³The transformation was apparently first effectively introduced by Weierstrass, though it has occurred earlier in the work of Lagrange and of Gauss on the conformal representation of a spherical surface upon a plane.

appears to be a fairly innocuous set of functions B and C being proportional to L , ie.

$$B = \alpha L \quad C = \beta L. \quad (4.2.23)$$

Then

$$L = L_0 e^{\alpha T} \quad (4.2.24)$$

and (4.2.21) becomes

$$u'' - \frac{1}{4} [L_0^2 e^{2\alpha T} + 2i\alpha L_0 e^{\alpha T} - \beta^{-2}] u = 0 \quad (4.2.25)$$

which is Whittaker's differential equation in slightly disguised form. With the solution to (4.2.25) the route back to ξ and η via (4.2.18) and (4.2.20) is straightforward. To keep things simple we take $A(L)$ to be zero. The solution of the radial equation (4.2.6) is

$$\chi = EJ_0(L_0 e^{\alpha T}) + FY_0(L_0 e^{\alpha T}), \quad (4.2.26)$$

where J_0 and Y_0 are zeroth order Bessel's functions of the first and second kinds and, as before, E and F are constants. However, the determination of t is via

$$t - t_0 = \int \frac{dT}{\{EJ_0(L_0 e^{\alpha T}) + FY_0(L_0 e^{\alpha T})\}^2} \quad (4.2.27)$$

for which a closed form expression is not known.

4.3 Some 'weak' considerations

As mentioned in §2.9 Leach[61] proposed that systems with Ermakov invariants which did not possess $s\ell(2, R)$ symmetry should be termed 'weak'. Here we wish to consider a few examples of systems for which only the angular equations possess $s\ell(2, R)$ symmetry. We maintain $so(3)$ symmetry overall so that the radial equation has the form

$$\ddot{r} - \frac{L^2}{r^3} = \frac{1}{r^3} A(L) + f(r, L), \quad (4.3.1)$$

where $f(r, L)$ is the symmetry-breaking term. The analysis of the angular equations is the same which means that 'in principle' we have $L = L(T)$ and the three Poincaré vectors. In terms of the inverse radial variable χ and new time (4.3.1) is

$$\chi'' + [A(L) + L^2] \chi + \frac{1}{\chi^2} f\left(\frac{1}{\chi}, L\right) = 0. \quad (4.3.2)$$

When f is zero, (4.3.2), as the equation for the time-dependent harmonic oscillator, is transformed to autonomous form by the transformation

$$J = \frac{\chi}{\rho} \quad \tau = \int \rho(T)^{-2} dT, \quad (4.3.3)$$

where ρ is a solution of the Ermakov–Pinney equation

$$\frac{d^2 \rho}{dT^2} + [A(L) + L^2] \rho = \rho^{-3} \quad (4.3.4)$$

and $L = L(T)$ through (4.2.10). One could hope that for some functions f the transformation (4.3.3) would render it autonomous. For this to happen it is necessary for $\rho = g(L)$ and the argument of f to be $\chi^{-1}g(L)$, where g is a solution to an Ermakov–Pinney-type equation with L as independent variable containing $A(L)$ and $B(L)$.

Such constraints are not required in a few cases. If the additional force is due to a Newton–Cotes potential, (4.3.2) is as if f were zero and $A(L)$ changed. For a Kepler type potential

$$f\left(\frac{1}{\chi}, L\right) = \mu(L)\chi^2 \quad (4.3.5)$$

(4.3.2) is just the inhomogeneous time-dependent oscillator and is solved in standard fashion. For an oscillator type potential

$$f\left(\frac{1}{\chi}, L\right) = \mu(L)\chi^{-1} \quad (4.3.6)$$

which makes (4.3.2) an Ermakov–Pinney equation with time-dependent coefficients. If μ is independent of L or $B = 0$, this can be treated as if it were the standard time-dependent harmonic oscillator problem. If such is not the case, the best that one can do is to introduce a time-dependent transformation which converts (4.3.2) to a generalised Emden–Fowler equation of order three.

Chapter 5

Conclusion

Be either consequent or inconsequent, never both together.

(An unknown moralist)[21]

We have demonstrated that the Lie algebra of generalised Ermakov systems is $sl(2, R)$. However, the Ermakov–Lewis invariant obtained from Ermakov systems has an infinite-dimensional algebra. The implications thereof are not yet apparent, though it is possible that the presence of the Wronskian causes this effect. This is just conjecture at this stage and needs to be further investigated. It would also be of interest to perform this calculation in conjunction with determining the Lie algebra of other angular momentum type first integrals.

The Ermakov–Pinney equation was shown to be a first integral for, what was called, its third order form

$$\ddot{a} + 4\omega^2 \dot{a} + 4\omega \dot{\omega} a = 0. \quad (5.1)$$

Now (5.1) has maximal symmetry and is also self-adjoint¹ since

$$\left(-\frac{d}{dt}\right)^3 u + \left(-\frac{d}{dt}\right)(4\omega^2 u) + \left(-\frac{d}{dt}\right)^0(4\omega \dot{\omega} u) = 0 \quad (5.2)$$

gives

$$-\left[\ddot{u} + 4\omega^2 \dot{u} + 4\omega \dot{\omega} u\right] = 0. \quad (5.3)$$

¹See[44] for a clear definition of self-adjoint equations and the usefulness of this property in solving differential equations.

Eq (5.1) is the first of the hierarchy of higher order equations which possess the maximal symmetry $nA_1 \oplus_s (sl(2, R) \ominus A_1)$. They are all self-adjoint and can be transformed to autonomous form using

$$T = \int u^{-1} dt \quad V = \frac{v}{u^{(n-1)/2}}.$$

There should be an equation of order $(n - 1)$ which has the essential feature of the Ermakov–Pinney equation, viz. the Lie algebra $sl(2, R)$. This proposition needs further investigation as does the question of the imposition of other conditions.

Ermakov systems were shown to be a subclass of equations invariant under $sl(2, R)$. In extending this to three dimensions we insisted that the equations be invariant under $sl(2, R) \oplus so(3)$ and demonstrated the existence of three Poincaré type vectors. The general form of generalised Ermakov systems in three dimensions can be written compactly in the vector form

$$\ddot{\mathbf{r}} = \frac{1}{r^3} \left\{ A(L)\hat{\mathbf{r}} + B(L)\hat{\boldsymbol{\omega}} + C(L)\hat{\mathbf{L}} \right\}. \quad (5.4)$$

In the case that (5.4) has a Hamiltonian representation the Poincaré vectors will have the Lie algebra $so(4)$ under the operation of taking the Poisson Bracket. The question is under what circumstances does it have a Hamiltonian? (One would not expect the usual Poisson Bracket relation $[z_\mu, z_\nu]_{PB} = J_{\mu\nu}$, ($z_i = q_i, z_{n+1} = p_i$ and J is the $2n \times 2n$ symplectic matrix) but more the monopole type of relation, ie, seek $H : \dot{q} = [q, H]_{PB}$ and $\dot{p} = [p, H]_{PB}$ lead to the equation of motion). There are two cases of (5.4) to consider: (i) when (5.4) is itself Hamiltonian, and (ii) when (5.4) possesses a global invariant which is not, however, a Hamiltonian function for the system. In the latter case the possibility arises that this invariant is a Hamiltonian function for a subsystem on an appropriate phase space, as in[6].

We have attempted to appease the moralist quoted at the beginning of this section. We hope that we have been the former.

Appendix A

Noether's Theorem

The version of Noether's Theorem normally encountered is based on the invariance of the action functional under infinitesimal transformation. Sarlet and Cantrijn[105] have compiled an excellent survey of the different versions including comparisons and their work is today regarded as *the* definitive work on generalisations of Noether's Theorem in classical mechanics.

The usefulness of this theorem has been widely criticised (See eg.[59, 89].) primarily for two reasons: Firstly, it only applies to problems where a Lagrangian exists or can be constructed¹ and, secondly, it yields an incomplete algebra of symmetries, eg. in the case of point symmetries² for one-dimensional second order linear systems Noether's Theorem provides five generators while the Lie method eight. However, Mahomed[77] argues strongly for the continued use of this theorem pointing out that it has the advantage of possessing an explicit formula for the first integrals once the symmetries have been calculated. Knowledge of the symmetries in the Lie method is just one step in the (often tortuous) route to finding first integrals. Mahomed also showed how the complete algebra for a problem can be constructed by investigating the Lie

¹This criticism has lost some of the force of its validity as Djukić and Vujanović[23, 113] have extended the application of Noether's Theorem to equations of motion for dissipative systems.

²One should note that Noether's Theorem applies equally well to velocity dependent symmetries.

algebra of the Noether symmetries. Thus Noether's Theorem is the obvious path to finding first integrals of dynamical systems.

We illustrate the procedure by considering the time-dependent simple harmonic oscillator[62]

$$\ddot{q} + \omega^2(t)q = 0 \quad (\text{A.1})$$

with Lagrangian

$$L(q, \dot{q}, t) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2(t)q^2. \quad (\text{A.2})$$

We can find a Noether point symmetry

$$G = \tau(q, t)\frac{\partial}{\partial t} + \eta(q, t)\frac{\partial}{\partial q} \quad (\text{A.3})$$

if the action integral

$$A = \int_{t_0}^{t_1} L(q, \dot{q}, t)dt \quad (\text{A.4})$$

is invariant under the infinitesimal transformation generated by (A.3). The functions $\tau(q, t)$ and $\eta(q, t)$ satisfy[105]

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q}\eta + \frac{\partial L}{\partial \dot{q}}(\dot{\eta} - \dot{q}\dot{\tau}) + L\dot{\tau} = \frac{df}{dt},$$

where the overdot represents the total derivative with respect to time and $f(q, t)$ is a gauge term. We proceed with the calculation in the same manner as in the Lie method to obtain

$$G_1 = b_1 \frac{\partial}{\partial q} \quad (\text{A.5a})$$

$$G_2 = b_2 \frac{\partial}{\partial q} \quad (\text{A.5b})$$

$$G_3 = \rho_1^2 \frac{\partial}{\partial t} + \rho_1 \dot{\rho}_1 q \frac{\partial}{\partial q} \quad (\text{A.5c})$$

$$G_4 = \rho_2^2 \frac{\partial}{\partial t} + \rho_2 \dot{\rho}_2 q \frac{\partial}{\partial q} \quad (\text{A.5d})$$

$$G_5 = \rho_3^2 \frac{\partial}{\partial t} + \rho_3 \dot{\rho}_3 q \frac{\partial}{\partial q}, \quad (\text{A.5e})$$

where the b_i 's and ρ_i 's are the linearly independent solutions of

$$\ddot{b} + \omega^2(t)b = 0 \quad (\text{A.6})$$

and

$$\ddot{a} + 4\omega^2 \dot{a} + 4\omega \dot{\omega} a = 0 \quad (\text{A.7})$$

$[a = \rho^2]$ respectively.

Now we can associate a first integral with G if the first variation of the action integral (A.4) is also invariant under G . The formula for this first integral is

$$I = f - \left(L\tau - \frac{\partial L}{\partial \dot{q}} (\eta - \dot{q}\tau) \right).$$

The first integrals associated with the symmetries (A.5) are

$$I_1 = \dot{b}_1 q - b_1 \dot{q} \quad (\text{A.8a})$$

$$I_2 = \dot{b}_2 q - b_2 \dot{q} \quad (\text{A.8b})$$

$$I_3 = \frac{1}{2} (\rho_1 \dot{q} - \dot{\rho}_1 q)^2 + \frac{1}{2} (\rho_1 \ddot{\rho}_1 + \rho_1^2 \omega^2) q^2 \quad (\text{A.8c})$$

$$I_4 = \frac{1}{2} (\rho_2 \dot{q} - \dot{\rho}_2 q)^2 + \frac{1}{2} (\rho_2 \ddot{\rho}_2 + \rho_2^2 \omega^2) q^2 \quad (\text{A.8d})$$

$$I_5 = \frac{1}{2} (\rho_3 \dot{q} - \dot{\rho}_3 q)^2 + \frac{1}{2} (\rho_3 \ddot{\rho}_3 + \rho_3^2 \omega^2) q^2, \quad (\text{A.8e})$$

where the b_i 's and ρ_i 's³ are as in (A.5).

Using the Lie method we find three further symmetries (as expected[117])

$$G_6 = q \frac{\partial}{\partial q} \quad (\text{A.9a})$$

$$G_7 = b_1 q \frac{\partial}{\partial t} + \dot{b}_1 q^2 \frac{\partial}{\partial q} \quad (\text{A.9b})$$

$$G_8 = b_2 q \frac{\partial}{\partial t} + \dot{b}_2 q^2 \frac{\partial}{\partial q}, \quad (\text{A.9c})$$

where the b_i 's again satisfy (A.6).

³We could integrate (A.7) to obtain

$$\ddot{\rho} + \omega^2 \rho = \text{cst}/2\rho^3$$

and substitute accordingly for the coefficients of the q^2 in (A.8). However, this introduces a new constant (that of integration) which is integrally linked to the (nonlinear) Ermakov-Pinney equation. As this, or (A.7), generally cannot be solved in closed form, we must resort to numerical integration. The advantage of (A.7) is that its general solution is a linear combination of three linearly independent solutions and this is straight forward numerically.

Appendix B

Painlevé Analysis

The origin of the Painlevé analysis can be traced back to Sonia Kowalevskaya[50, 51] who, amongst other things, investigated the connection between integrability and the presence of only poles in the solutions of the equations of the spinning top. Painlevé and his group, in their singularity analysis of second order differential equations, found fifty different types. Forty four of them were shown to be integrable in terms of elementary functions, by quadratures or by linearization. The remaining six equations

$$w'' = 6w^2 + z \quad (\text{B.1a})$$

$$w'' = 2w^3 + zw + a \quad (\text{B.1b})$$

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{aw^2 + b}{z} + cw^3 + \frac{d}{w} \quad (\text{B.1c})$$

$$w'' = \frac{1}{2w}w'^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - a)w + \frac{b}{w} \quad (\text{B.1d})$$

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1} \right) w'^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2} \left(aw + \frac{b}{w} \right) + \frac{cw}{z} + \frac{dw(w+1)}{w-1} \quad (\text{B.1e})$$

$$w'' = \frac{1}{2} \left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z} \right) w'^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z} \right) w' + \frac{w(w-1)(w-z)}{z^2(z-1)^2} \left(a + \frac{bz}{w^2} + \frac{c(z-1)}{(w-1)^2} + \frac{dz(z-1)}{(w-z)^2} \right) \quad (\text{B.1f})$$

defined new transcendental functions known as the Painlevé Transcendents¹. We say that a system of n th order ordinary differential equations of the form

$$\frac{dw}{dz} = g(w) \quad (\text{B.2})$$

where g is rational in w , passes the Painlevé Test only if there is a Laurent expansion

$$w_k = (z - z_1)^m \sum_{j=0}^{\infty} a_{kj} (z - z_1)^j$$

with $(n - 1)$ arbitrary expansion coefficients (in addition to the pole position z_1).

We verify that the Ermakov–Pinney equation

$$y^3 \ddot{y} + \omega^2 y^4 = 1 \quad (\text{B.3})$$

passes the Painlevé Test by employing the ARS algorithm explained in [91]². Thereafter we attempt to construct its solution using the Painlevé analysis. (It was noted earlier (§2.4) that the ω^2 term in (B.3) can be transformed away and so without any loss of generality we can take it to be a constant.) Setting

$$y = \alpha \chi^p, \quad (\text{B.4})$$

where

$$\chi = t - t_0,$$

in (B.5) obtains

$$p = \frac{1}{2}$$

and

$$\alpha^4 = -4.$$

Now the dominant terms in (B.3) are $y^3 \ddot{y}$ and 1. Hence we ignore the $\omega^2 y^4$ term and only consider

$$y^3 \ddot{y} = 1 \quad (\text{B.5})$$

¹In actual fact, only the first three were discovered by Painlevé. The last three were added later by Gambier [44].

²We refer the reader to this excellent review for a more detailed and comprehensive treatment of the Painlevé analysis of ordinary and partial differential equations.

in the next stage of the analysis. The fractional value of p is suggestive of the ‘weak’ Painlevé property[91]. However, we do not consider this to be a new type of Painlevé property as fractional powers can, in general, be transformed away. We proceed by setting

$$y^2 = u$$

so that (B.3) becomes

$$u\ddot{u} - \frac{1}{2}\dot{u}^2 + 2\omega^2 u^2 = 2 \quad (\text{B.6})$$

and (B.4)

$$\begin{aligned} u &= \alpha^2 \chi^{2p} \\ &= \gamma \chi^q, \end{aligned}$$

where $\gamma^2 = -4$ and $q = 1$. The first two terms in (B.6) are due to the first term in (B.3) and are dominant with 2 (We can also confirm this by analysing (B.6) in isolation.). The $2\omega^2 u^2$ term in (B.6) can be ignored in the initial stages of the analysis. To find the resonances we assume the form

$$\begin{aligned} u &= \gamma \chi^q + \beta \chi^{q+r} \\ &= \gamma \chi + \beta \chi^{1+r}. \end{aligned} \quad (\text{B.7})$$

Substitution into (B.6) (less $2\omega^2 u^2$) and taking terms linear in β gives r as

$$r = \pm 1.$$

The form of (B.7) is

$$u = \gamma_1 \chi + \gamma_2 \chi^2, \quad (\text{B.8})$$

where $\gamma_1 \equiv \gamma$. In our analysis we expect an arbitrary constant to occur at χ^2 , the resonance³. Substituting (B.8) into (B.6) and solving for γ_1 and γ_2 gives

$$\gamma_1^2 = -4$$

$$\gamma_2 = \text{arbitrary constant}$$

³We have taken r to be 1 as $r = -1$ is expected. The occurrence of the latter can easily be explained using perturbation theory.

as expected. Hence (B.6) (and therefore (B.3)) possesses the Painlevé property.

To construct the full solution for (B.6) we need a few more terms and thus set

$$u = \gamma_1 \chi + \gamma_2 \chi^2 + \cdots + \gamma_6 \chi^6.$$

Substituting into (B.6), collecting by powers of χ and setting the coefficients thereof to zero we can solve for the γ_i 's to obtain

$$\begin{aligned}\gamma_1^2 &= -4 \\ \gamma_2 &= \text{arbitrary constant} \\ \gamma_3 &= \frac{8\omega^2}{3\gamma_1} \\ \gamma_4 &= -\frac{\omega^2\gamma_2}{3} \\ \gamma_5 &= -\frac{8\omega^4}{15\gamma_1} \\ \gamma_6 &= \frac{2\omega^4\gamma_2}{45}.\end{aligned}$$

Thus the solution to (B.6) is

$$\begin{aligned}u &= \gamma_1 \chi + \gamma_2 \chi^2 + \frac{8\omega^2}{3\gamma_1} \chi^3 - \frac{\omega^2\gamma_2}{3} \chi^4 - \frac{8\omega^4}{15\gamma_1} \chi^5 + \frac{2\omega^4\gamma_2}{45} \chi^6 + \cdots \\ &= \frac{\gamma_1}{2\omega} \left[(2\omega\chi) - \frac{1}{3!} (2\omega\chi)^3 + \frac{1}{5!} (2\omega\chi)^5 + \cdots \right] \\ &\quad - \frac{\gamma_2}{2\omega^2} \left[1 - \frac{1}{2!} (2\omega\chi)^2 + \frac{1}{4!} (2\omega\chi)^4 - \frac{1}{6!} (2\omega\chi)^6 + \cdots \right] + \frac{\gamma_2}{2\omega^2} \\ &= \frac{\gamma_1}{2\omega} \sin(2\omega\chi) - \frac{\gamma_2}{2\omega^2} \cos(2\omega\chi) + \frac{\gamma_2}{2\omega^2}.\end{aligned}\tag{B.9}$$

Recall that

$$\chi = t - t_0$$

whence

$$y^2 = u = \left[\left(\frac{\gamma_1}{\omega} \sin \omega t_0 + \frac{\gamma_2}{\omega^2} \cos \omega t_0 \right) \cos \omega t_0 \right] \sin^2 \omega t$$

$$\begin{aligned}
& + \left[\left(\frac{\gamma_2}{\omega^2} \sin \omega t_0 - \frac{\gamma_1}{\omega} \cos \omega t_0 \right) \cos \omega t_0 \right. \\
& - \left. \left(\frac{g_1}{\omega} \sin \omega t_0 + \frac{\gamma_2}{\omega^2} \cos \omega t_0 \right) \sin \omega t_0 \right] \sin \omega t \cos \omega t \\
& + \left[\left(\frac{\gamma_2}{\omega^2} \sin \omega t_0 - \frac{\gamma_1}{\omega} \cos \omega t_0 \right) \sin \omega t_0 \right] \cos^2 \omega t \quad (B.10)
\end{aligned}$$

with

$$AC - \frac{1}{4}B^2 = \frac{1}{\omega^2} \equiv \frac{1}{W^2}$$

as given by Eliezer and Gray[24], where A, B and C are the coefficients of $\sin^2 \omega t$, $\sin \omega t \cos \omega t$ and $\cos^2 \omega t$ respectively and W is the Wronksian. Eq (B.10) is just Pinney's solution to (B.3).

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