EXTENSIONS AND GENERALISATIONS OF LIE ANALYSIS

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Abstract

The Lie theory of extended groups applied to differential equations is arguably one of the most successful methods in the solution of differential equations. In fact, the theory unifies a number of previously unrelated methods into a single algorithm. However, as with all theories, there are instances in which it provides no useful information. Thus extensions and generalisations of the method (which classically employs only point and contact transformations) are necessary to broaden the class of equations solvable by this method.

The most obvious extension is to generalised (or Lie–Bäcklund) symmetries. While a subset of these, called contact symmetries, were considered by Lie and Bäcklund they have been thought to be curiosities. We show that contact transformations have an important role to play in the solution of differential equations. In particular we linearise the Kummer–Schwarz equation (which is not linearisable via a point transformation) via a contact transformation. We also determine the full contact symmetry Lie algebra of the third order equation with maximal symmetry ($y''' = 0$), *viz* $sp(4)$.

We also undertake an investigation of nonlocal symmetries which have been shown to be the origin of so–called hidden symmetries. A new procedure for the determination of these symmetries is presented and applied to some examples. The impact of nonlocal symmetries is further demonstrated in the solution of equations devoid of point symmetries. As a result we present new classes of second order equations solvable by group theoretic means.

A brief foray into Painlevé analysis is undertaken and then applied to some physical examples (together with a Lie analysis thereof). The close relationship between these two areas of analysis is investigated.

We conclude by noting that our view of the world of symmetry has been clouded. A more broad–minded approach to the concept of symmetry is imperative to successfully realise Sophus Lie's dream of a single unified theory to solve differential equations.

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

K S Govinder

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Dedication

To the Diceros bicornis

May they once again roam our land in numbers aplenty

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Preface

I am often asked the question: 'What possesses anyone to study Mathematics?'. One could easily wax lyrical about the great applications of Mathematics to problems in the real world. This would immediately suggest that one studies Mathematics for some philanthropic reason – a need to be able to contribute to the improvement of life on earth. As attractive a reason as this is, I cannot, in all honesty, use it as my excuse. Indeed, I do what I do, for purely selfish reasons – I enjoy it! As hedonistic as this sounds, I take comfort in the knowledge that I am not alone in this regard. For example, Poincaré observes: *The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful.* [176, p v]

That is precisely the reason I am enamoured with the study of differential equations in general and Lie's method for their analysis in particular. There can be no greater harmonious order than Lie's method which encompasses many divers (often *ad hoc*) methods of solution of differential equations. To Lie I am imminently grateful for drawing me into the world of invariance and differential equations. He provided the impetus for, what I hope to be, a long and lasting romance with differential equations.

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Contents

Chapter 1

Historical Overview

We present a brief overview of the historical events leading to and including Lie's development of the theory of continuous groups.

1.1 Introduction

It is an oft-related tale [173] that Sophus Marius Lie (1842–1899) and Felix Christian Klein (1849–1925) met in Paris in 1870, and learned group theory from Camille Jordan (1838–1922) and his *Traité des substitutions et des équations algèbriques*. Thereafter, the story goes, they divided the subject between them; Lie took continuous and Klein discrete groups [179, p 271]. Rowe [173], however, points out that at best this can be regarded as 'a first-order approximation of the truth'. Indeed, Lie had studied a semester of group theory under Ludwig Sylow (1832–1918) in 1862/1863 [173]. There is also ample evidence to support the view that Klein was well aware of group theory before his arrival in Paris [95]. Further, both Lie and Klein had difficulty digesting Jordan's book. Klein referred to it as a 'book sealed with seven seals' [30, p 411]. Nonetheless, Klein, unlike Lie, was greatly impressed with the book [95].

Whatever the origin of their introduction to group theory, today it is accepted that the work of Lie and Klein formed one of the three founding pillars of group theory [173]. (The other two were Gauss' theory of binary quadratic forms and the work of Lagrange, Cauchy, Abel and Galois on permutations.) As Galois' work had the greatest influence on Lie, we begin our exposition with him.

1.2 A tragedy of errors

The life of Evariste Galois (1811–1832) was plagued with misfortune [189, p 1]. As a young boy he was easily bored with exercises in the Classics and subsequently did not pass his examinations. This setback was the first experienced in his short life. His early lessons in mathematics sparked his interest despite the fact that it was taught as an 'aside' to Latin and Greek. Luckily, during his first year of proper mathematical studies at school he came into possession of Legendre's geometrical work. He was mastered it 'as easily as other boys read a pirate yarn' [25, p 408]. Galois had found his niche.

At the age of sixteen Galois repeated Abel's mistake of obtaining the general solution for the quintic. (This was just one of the many parallels in their lives.) Hotheaded and stubborn, Galois' conviction of his own worth led him to ignore kindly advice from friends and teachers. He took the entrance examination to the École Polytechnique in Paris without preparation and was promptly failed. (This downfall has often been ascribed to the ineptitude of his examiners [189, p 1].)

The following year Galois came into contact with Louis–Paul–Émile Richard (1795–1849), a teacher of advanced mathematics at Louis-le-Grand. Under Richard's tutelage, Galois flourished; leading Richard to hail him as 'the Abel of France' [25, p 412]. This was his greatest year, with the publication of his first paper on continued fractions; his fundamental discoveries, however, were saved for a memoir he wished the Paris Academy to publish. This manuscript was sent to Cauchy to present, but Galois never heard from him thereafter.

Augustin Louis Cauchy (1789–1857), for all his contributions to the advancement of mathematics, was most irresponsible in neglecting Galois' manuscript.

This carelessness was a repeat of his abysmal treatment of Abel's memoir on transcendental functions. (In that respect he shares the blame with Adrien–Marie Legendre (1752–1833). However, it is acknowledged that the latter attempted to make amends [157].) Such treatment caused both the young mathematicians considerable distress. In Abel's case it exacerbated his depression. For Galois, it eventually led to his abandonment of mathematics.

Galois' eighteenth year brought more illfortune. It was the year of his second failure at entering the École Polytechnique. During this entrance examination he demonstrated his only use for a blackboard eraser – he threw it at (and hit) an examiner who irritated him [25, p 414]. Sadly, his father's death and subsequent slandering soon followed. These incidents, together with the loss of his entry for the Grand Prix in mathematics of the Academy of Sciences proved too much for the young Galois who renounced mathematics for politics (The secretary who took his entry home died that evening. No trace was ever found of the manuscript.) [25, p 416].

His path through politics (on the side of republicanism) was no less tumultuous. Arrested repeatedly, he was eventually given a six month prison sentence for illegally wearing a uniform. This period in gaol was one of reflection. He resolved, upon his release, to retire to the country to meditate. Unfortunately this was not to be.

On the day of his release, he agreed to a duel with some apparent loyalists. Perhaps anticipating his end, he spent the night feverishly noting his discoveries in the theory of algebraic equations. He was fatally wounded in the duel and died in his brother's company on 31 May 1832 [189, p 1].

Galois' furious scribblings that fateful evening were destined to play a fundamental role in nineteenth and twentieth century mathematics. Niels Hendrik Abel (1802–1829) had already provided the first (correct) proof that the general quintic equation could not be solved in radicals (a paper Carl Friedrich Gauss (1777–1855) dismissed without reading [25, p 353]). This was a special case of Lagrange's conjecture that no general method of solution existed for algebraic equations of degree greater than four. Joseph Louis Lagrange (1736–1814) came to this conclusion in his long memoir *Réflexions sur la résolution algébrique des équations* wherein he reviewed all previous attempts to solve the quintic [189, pp 6–7]. Noting that the solutions to the quadratic, cubic and quartic were obtained via resolvents, he attempted the same for the quintic. He soon realised that this required the solution of a sixth degree equation and conjectured that, generally, the solution of an nth degree equation required the solution of an $(n+1)$th degree equation. Lagrange's work relied heavily on permutations of the equation's roots and led to his prophetic statement that the theory of permutations was the key to the proof of his conjecture [189, p 7]. Although he was unaware of the term 'group', Lagrange was certainly led in that direction by his study of permutations of roots. As a result of this work one of the first theorems in group theory was named after him [189, p 7].

The proof of Lagrange's conjecture (using the theory of permutation of roots) was contained in Galois' writings. Galois, already aware of (and impressed by) Abel's work, clearly saw the route to the solution of the problem. He expressed the fundamental properties of the transformation group belonging to the roots of an algebraic equations and showed that the field of rationality of these roots was determined by the group. Galois' work was based on the idea of the 'degree of symmetry' of an algebraic equation [189, p 10]. According to him, the degree of symmetry of an nth degree equation $f(x) = 0$ (with rational coefficients) is described by the set of permutations of its roots $x_1, x_2, \ldots, x_n$ that preserve all the algebraic relationships between them. This set, G say, contained

(i) the unique identity permutation,

(ii) the product of every two permutations already contained in G and

(iii) the inverse of every permutation already in G.

Such a set Galois called a *group* of permutations. (While this *notion* is evident in the work of Lagrange and Abel, Galois was the first to clearly delineate the idea. The more abstract approach to the notion of a group (without specifying the nature of the elements or the meaning of the group operation) is due to Cauchy.) He showed this transformation group, leaving the algebraic relationships between the roots of the equation invariant, determined the field of rationality of those roots. Galois is also credited with introducing the concept of a field (again already known to others in some vague form), subgroup, order of a group, normal subgroup, simple groups, solvable groups and many other group theoretic terms in use today [189, p 14].

Galois' final writings did not lead directly to the popularising of his results. In an ironic twist, it was Cauchy's work that spurred development in Galois' area. In his 'theory of substitutions', Cauchy grasped at the *notion* of a group (though he did not use the word) [189, p 139]. He flooded mathematical literature during 1844–1846 with notes and memoirs on his theory [189, p 139]. (Indeed his legendary productivity was responsible for the restriction in *Comptes Rendus hebdomadaires des Séances de l'Académie des Sciences* of the length of articles to a maximum of four pages [25, p 324].) This paved the way for Joseph Louiville (1809–1882) to publish most of Galois' papers in his *Journal de mathématiques pures et appliqués* in 1846. The pivotal role played by Cauchy in popularising Galois' work was yet to come.

After Cauchy's death, Jordan was given the task of overseeing publication of his collected works. Searching for unpublished papers among Cauchy's writings, Jordan came across Galois' neglected letter to Cauchy [189, p 2]. Given his own interest in group theoretic–type work [95] Jordan immediately recognised the importance of Galois' letter. After collecting and assimilating all Galois' work he clarified and elaborated on the ideas in his *Traité* of 1870. The importance of this work lies not only in popularising Galois' main result of the solvability of algebraic equations in radicals, but in making Galois' methods accessible to a wide audience.

Building on Jordan's dissemination of Galois' work, Lie and Klein were responsible for introducing group theoretic concepts into various branches of mathematics. In the sequel we provide a brief biography of Lie's life and describe some of his results.

1.3 A lonely wanderer

Sophus Lie was born in 1842 in a seaside town near Bergen, Norway. Even as a young boy he enjoyed travelling the length and breadth of his beloved country. This passion for his homeland fostered many returns through his life. Unlike Galois [25, p 408], Lie flourished at school and soon had his pick of professions. Like Abel [157] he came from a theological background and, after initial hesitation [38], seized upon mathematics as his vocation.

His first forays into mathematics were not very encouraging. However, he eventually came across the geometers Victor Poncelet (1789–1867) and Julius Plücker (1801–1868). Their work greatly influenced the young Lie and led to his first publications. To continue his education he travelled to Berlin in 1870 where he met Klein (a former student of Plücker). Both were immediately drawn to each other (somewhat encouraged by the cold reception they received by the other mathematicians). Here they cemented what was to become a long and lasting friendship.

Desiring a meeting with Jordan and Gaston Darboux (1824–1917) Lie and Klein set off for Paris in 1871. They planned to acquaint themselves with the French school before moving on to London. However, the Franco-Prussian war soon put an end to these ambitions – Klein had to leave for Germany in some haste. Alone again, Lie reverted to his favourite pastime, beginning a hike through the French countryside on his way to the Alps and Italy. His abstract wanderings in France, coupled with feverish mathematical scribblings in his notebook (in Norwegian of course), were sufficient to rouse the suspicions of the locals. There is little doubt that his poor French did not help. He was arrested as a German spy and spent a month in the prison of Fontainebleau [189, p 24].

Despite this setback, the time in prison was not wasted. Lie spent the month pondering Plücker's line geometry and his discussions with Klein in Paris. (Indeed these musings eventually brought him his greatest fame [189, p 125].) Upon gaining his freedom (through Darboux's intervention) he continued his hike through France and Italy.

Lie's first task on returning to Norway was the completion of his doctoral degree (1871). Subsequently, Klein used his extensive scientific connections to secure a professorship for Lie in Oslo (then Christiana). With the exception of a year teaching at Lund (Sweden) Lie spent the next decade in his beloved Norway. During this period he met and married Anna Sophie Birch; a marriage which was by all accounts a happy one.

Much as Norway appealed to Lie's nature, the setting posed considerable limitations. These included a lack of vibrant scientific community and competent students. Once again, Klein came to his rescue, suggesting that Lie take his (Klein's) chair of geometry at Leipzig University - Klein having been called to Göttingen. The years in Germany were Lie's most productive and saw fruitful collaboration with his students Freidrich Engel (1861–1941), Georg Scheffers (1866–1945) and Felix Hausdorff (1868–1949). However, sensitive and highly strung, Lie was unhappy in Leipzig's austere environment. He suffered bouts of depression towards the end of his stay, and ended up attending a psychiatric clinic in Hannover.

The only blemish on Lie's conduct occurred during this period. He rather bluntly pointed out in [135] that *he* was not Klein's pupil, but that the opposite was true. This tactless remark in a scientific work hurt Klein deeply. It is to his great tribute that he did not respond and the friendship survived this mishap intact.

In 1898, Lie received the first International Lobachevsky prize [189, p 127] to add to his many titles and awards [38]. Later that year he returned to Norway, no doubt hoping to spend his days in quiet retirement. This was terminated prematurely by his death the following year (18 February 1899) of cerebral anæmia.

1.4 The Group theoretic work of Lie

In this section, we are primarily concerned with Lie's (and to a lesser extent Klein's) contributions to group theory. Nonetheless, it would be inaccurate to suggest that this was the only subject consuming his attention. His background was in geometry and he considered himself a representative of that field [173].

The work of Lie and Klein grew from their first collaborative effort on W–curves. This arose from Lie's considerations on tetrahedral complexes before arriving in Berlin. His interest lay in invariance (or symmetry) and led to the introduction of the totality of all projective transformations of space (denoted by G) which leave the vertices of a fixed tetrahedron fixed. In his study of the geometry of tetrahedral complexes Lie made use of the properties of G. These properties can be seen to be related to the structure of G as a continuous group [95]. Together with Klein, Lie continued these deliberations to consider the determination of each curve of the plane that has a 'complex group of projective motions along itself' [189, p 110]. These curves were called W–curves. This eventually led to his discovery of the correspondence between systems of straight lines and spheres [95].

In fact, in the work on W–curves, Lie had been studying one–parameter subgroups of a given continuous group. This played a great role in the construction of Lie algebras and was pivotal in Klein's *Erlangen Program* [189, p 110]. For Lie it marked the beginning of his remarkable work on continuous groups.

Lie's main result is the proof that it is always possible to assign to a continuous (Lie) group a corresponding (Lie) algebra. He also worked out the construction to obtain a (Lie) group from each (Lie) algebra [189, p 103]. Thus he was able to carry into the theory of (Lie) algebra all previously group theoretic notions. Today the distinction is often blurred.

In the early to middle 1870s, Lie observed that special types of differential equations with known methods of solution admitted known infinitesimal transformations. He believed it was the existence of these transformations that was the 'real' reason behind the integration of these equations [95]. Greatly inspired by Galois, he wrote in 1874:

> In the theory of algebraic equations before Galois only these questions were posed: Is an equation solvable by radicals, and how is it to be solved? Since Galois, among other questions proposed is this: How is an equation to be solved by radicals in the *simplest* way possible? I believe the time is come to make a similar progress in differential equations. [25, p 436]

Lie conceived of a variant of the role of the Galois (finite) group method for algebraic equations to that of a continuous group of infinitesimal transformations of a differential equation whose properties enabled one to solve the equation.

Lie's theory of continuous groups applied to differential equations soon became the standard in many texts [108, 65]. However, it was eventually lost in the mathematical advances of the early twentieth century. Interest in the theory today is largely due to the work of Lev Vasil'evich Ovsyannikov (1919–) and his group in Novosibirsk in the mid twentieth century. The term 'modern group analysis' used widely today is due to one of his students, Nail Hairullovich Ibragimov (1939–) [104]. Interest in Lie groups did not undergo a similar wane, primarily due to applications in particle physics.

Another topic in continuous groups that attracted Lie's attention was that of contact transformations. (Contact transformations are those that transform curves into curves leaving their degree of contact invariant.) He believed that they had an important role to play in the solution of partial differential equations. This work flowed from his previous studies of W-curves and appeared as early as 1873 [38]. Lie was also grasping towards a group theoretic approach to Huygen's principle [136, p 49][102]. In addition to these applications contact transformations provided the key to understanding Hamiltonian dynamics as a part of group theory [179, p 274]. (See also [31].)

A testament to Lie's research is his collected works – spanning *ten* volumes. The compilation of his work was started in 1900. With the help of the Leipzig Scientific Society and the Teubner publishing house, this effort endured World War I to be produced in 1920–1934 [189, p 128]. These 15 books did not include his three volume work on transformation groups with Engel [133, 134, 135] or his three books on differential equations [132], continuous groups [136] and contact transformations [137] with Scheffers. It is said that Lie was one of the last great mathematicians of the nineteenth century. He had a touch of Gauss and Riemann in his scientific profile [189, p 12].

For those interested in biographies of mathematicians we recommend [105, 25, 26, 179, 64, 157, 92, 114, 176, 189, 30, 95, 173, 113, 158]. Those interested in Lie and/or Klein are particularly encouraged to read [189].

Chapter 2

Lie Theory of Differential Equations

We present a brief introduction to the classical Lie theory of extended groups applied to differential equations. Simple examples are used to illustrate different aspects of the theory.

2.1 Definitions

We present some definitions that will aid in the clarification of the subsequent analysis.

Group: A group G is a set of elements with a law of composition ϕ between elements satisfying the following axioms [29]:

(i) Closure property: For any element a and b of G, $\phi(a,b)$ is an element of G.

(ii) Associative property: For any elements a, b and c of G,

$$\phi(a,\phi(b,c)) = \phi(\phi(a,b),c).$$

(iii) IDENTITY ELEMENT: There exists a unique identity element I of G such that, for any element a of G,

$$\phi(a, I) = \phi(I, a) = a.$$

(iv) INVERSE ELEMENT: For any element a of G there exists a unique inverse element a^{-1} in G such that

$$\phi(a, a^{-1}) = \phi(a^{-1}, a) = I.$$

Abelian group: A group G is Abelian if $\phi(a, b) = \phi(b, a)$ holds for all elements a and b in G.

Group of transformations: The set of transformations

$$\bar{\mathrm{x}} = \mathbf{X}(\mathrm{x}; \varepsilon)$$

defined for each x in the space $D \subset R$, depending on the parameter ε lying in the set $S \subset R$, with $\phi(\varepsilon, \delta)$ defining a composition of parameters ε and δ in S, forms a group of transformations on D if:

(i) For each parameter ε in S the transformations are one–to–one onto D.

(ii) S with the law of composition ϕ forms a group.

(iii) $\bar{\mathrm{x}} = \mathrm{x}$ when $\varepsilon = I$, *ie*

$$\mathbf{X}(\mathrm{x}; I) = \mathrm{x}.$$

(iv) If $\bar{\mathrm{x}} = \mathbf{X}(\mathrm{x}; \varepsilon)$, $\bar{\bar{\mathrm{x}}} = \mathbf{X}(\bar{\mathrm{x}}; \delta)$, then

$$\bar{\bar{\mathrm{x}}} = \mathbf{X}(\mathrm{x}; \phi(\varepsilon, \delta)).$$

Lie group of transformations: A one–parameter Lie group of transformations is a group of transformations which, in addition to the above, satisfies the following:

(i) ε is a continuous parameter, *ie* S is an interval in R. (Without loss of generality $\varepsilon = 0$ corresponds to the identity element I.)

(ii) $\mathbf{X}$ is infinitely differentiable with respect to $\mathbf{x}$ in D and an analytic function of ε in S.

(iii) $\phi(\varepsilon, \delta)$ is an analytic function of ε and δ, $\varepsilon \in S$, $\delta \in S$.

Subgroup: A subgroup of G is a group formed by a subset of elements of G with the same law of composition.

Special linear group: The complex general linear group $GL(n, C)$ and the real general linear group $GL(n, R)$ consist of all nonsingular complex and real $n \times n$ matrices respectively [24]. (The latter may be considered as a subgroup of the former.) The complex special linear group $SL(n, C)$ is the subgroup of $GL(n, C)$ consisting of matrices with determinant one. The real special linear group $SL(n, R)$ is the intersection of these two subgroups

$$SL(n, R) = SL(n, C) \cap GL(n, R).$$

Rotation group: The rotation group $SO(n, R)$ is the special or proper real orthogonal group given by the intersection of the group of orthogonal matrices $O(n, R)$ and the complex special linear group, *ie*

$$SO(n, R) = O(n, R) \cap SL(n, C).$$

Lie algebra: A Lie algebra $\mathcal{L}$ is a vector space together with a product $[x, y]$ that:

(i) is BILINEAR (*ie* linear in x and y separately),

(ii) is ANTICOMMUTATIVE (antisymmetric):

$$[x, y] = -[y, x],$$

(iii) satisfies the JACOBI IDENTITY

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all vectors x, y, z in the Lie algebra.

Abelian algebra: A Lie algebra $\mathcal{L}$ is called Abelian (equivalently commutative) if $[x, y] = 0 \,\forall\, x, y \in \mathcal{L}$ [159].

Solvable algebra: A Lie algebra $\mathcal{L}$ is called solvable if the derived series

$$\begin{aligned} \mathcal{L} \supseteq \mathcal{L}' &= [\mathcal{L}, \mathcal{L}] \\ \supseteq \mathcal{L}'' &= [\mathcal{L}', \mathcal{L}'] \\ \supseteq \cdots & \\ \supseteq \mathcal{L}^{(k)} &= [\mathcal{L}^{(k-1)}, \mathcal{L}^{(k-1)}] \end{aligned}$$

terminates with a null ideal, *ie* $\mathcal{L}^{(k)} = 0, k > 0$ [106]. Note: Any Abelian algebra is solvable and indeed any Lie algebra of dimension ≤ 3 is solvable except when $\dim \mathcal{L} = 3 = \dim \mathcal{L}'$.

A few comments about Lie algebras are now in order. The Jacobi identity plays the same role for Lie algebras that the associative law plays for associative algebras. While we can define a Lie algebra over any field, in practice it is usually considered over real and complex fields. We define the product associated with the Lie algebra as that of commutation, *ie*

$$[X, Y] = XY - YX.$$

If a differential equation admits the operators X and Y, it also admits their commutator $[X, Y]$. Lie's main result [189] is the proof that it is always possible to assign to a continuous group (Lie group) a corresponding Lie algebra and vice versa. Thus for the real special linear group $SL(n, R)$ the corresponding Lie algebra is $s\ell(n, R)$ and for $SO(n, R)$, $so(n, R)$. Here we will be primarily concerned with Lie algebras and accept that the transformations we employ may not always be globally valid.

2.2 The Algorithm

We present an introduction to the Lie analysis of differential equations. More comprehensive details can be found in [55, 18, 19, 28, 21, 159, 172, 101, 29, 171, 178, 20, 102, 156, 103].

An nth order ordinary differential equation

$$N(x, y, y', \ldots, y^{(n)}) = 0 \tag{2.2.1}$$

admits the one–parameter Lie group of transformations

$$\bar{x} = x + \varepsilon\xi \tag{2.2.2}$$

$$\bar{y} = y + \varepsilon\eta \tag{2.2.3}$$

with infinitesimal generator

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \tag{2.2.4}$$

if

$$G^{[n]}N\Big|_{N=0} = 0, \tag{2.2.5}$$

where $G^{[n]}$ is the nth extension of G needed to transform the derivatives in (2.2.1) and is given by [146]

$$G^{[n]} = G + \sum_{i=1}^{n}\left\{\eta^{(i)} - \sum_{j=0}^{i-1}\begin{pmatrix} i \\ j \end{pmatrix} y^{(j+1)}\xi^{(i-j)}\right\}\frac{\partial}{\partial y^{(i)}}. \tag{2.2.6}$$

Note that the superscripts in (2.2.6) refer to total differentiation with respect to the independent variable. (In the case of partial differential equations or systems, ξ, η, x and y acquire suitable subscripts.) We say that (2.2.1) possesses the symmetry (group generator) (2.2.4) iff (2.2.5) holds.

The classical approach requires that the coefficient functions ξ and η depend on the independent and dependent variables only (in this case x and y). The operation of (2.2.6) on (2.2.1) produces an overdetermined system of linear partial differential equations, the solution of which gives ξ and η. We illustrate the procedure with a simplified form of the Ermakov–Pinney equation [58, 166]

$$y'' = \frac{1}{y^3}. \tag{2.2.7}$$

(See [79] for a generalisation of (2.2.7) and the use of Lie analysis in its solution.) The operation of $G^{[2]}$ on (2.2.7) produces the equation

$$\begin{aligned}\frac{\partial^2\eta}{\partial x^2} + 2y'\frac{\partial^2\eta}{\partial x\partial y} + y'^2\frac{\partial^2\eta}{\partial y^2} + \frac{1}{y^3}\frac{\partial\eta}{\partial y} - \frac{1}{2y^3}\left(\frac{\partial\xi}{\partial x} + y'\frac{\partial\xi}{\partial y}\right)& \\ - y'\left(\frac{\partial^2\xi}{\partial x^2} + 2y'\frac{\partial^2\xi}{\partial x\partial y} + y'^2\frac{\partial^2\xi}{\partial y^2}\right) &= -\frac{3\eta}{y^4}.\end{aligned} \tag{2.2.8}$$

We observe that, while ξ and η do not depend on derivatives of y, these derivatives do appear in (2.2.8). This allows us to equate different powers of y' to zero to obtain a system of linear partial differential equations. In the case of (2.2.8) the system is

$$\begin{aligned}\frac{\partial^2\xi}{\partial y^2} &= 0\\ \frac{\partial^2\eta}{\partial y^2} - 2\frac{\partial^2\xi}{\partial x\partial y} &= 0\\ 2\frac{\partial^2\eta}{\partial x\partial y} - \frac{3}{y^3}\frac{\partial\xi}{\partial y} - \frac{\partial^2\xi}{\partial x^2} &= 0\\ \frac{\partial^2\eta}{\partial x^2} + \frac{1}{y^3}\frac{\partial\eta}{\partial y} - \frac{2}{y^2}\frac{\partial\xi}{\partial x} &= -\frac{3\eta}{y^4}.\end{aligned} \tag{2.2.9}$$

We solve (2.2.9) to obtain

$$\begin{aligned}\xi &= A_0 + A_1x + A_2x^2\\ \eta &= \left(\tfrac{1}{2}A_1 + A_2x\right)y\end{aligned} \tag{2.2.10}$$

which gives the three symmetries of (2.2.7) as

$$\begin{aligned}G_1 &= \frac{\partial}{\partial x}\\ G_2 &= x\frac{\partial}{\partial x} + \tfrac{1}{2}y\frac{\partial}{\partial y}\\ G_3 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.\end{aligned} \tag{2.2.11}$$

(Note that (2.2.10) gives a three parameter symmetry. The usual practice is to set each of the constants in an n–parameter symmetry to one in turn and the

We have verified using *Mathematica* [185] that (2.2.7) is invariant under the finite transformation (2.2.18).

Having obtained the symmetries we are now in a position to utilise them in the solution of differential equations.

2.3 Reduction of Order

The most important use of symmetries is in the reduction of order of an equation. If an equation is invariant under the symmetry

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{2.3.1}$$

the variables for the reduction of order are obtained by requiring

$$G^{[1]}z = 0, \tag{2.3.2}$$

where $z = z(x,y,y')$ is an arbitrary function of its arguments. The operation (2.3.2) results in the equation

$$\xi\frac{\partial z}{\partial x} + \eta\frac{\partial z}{\partial y} + (\eta' - y'\xi')\frac{\partial z}{\partial y'} = 0 \tag{2.3.3}$$

which has the associated Lagrange's system

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}y}{\eta} = \frac{\mathrm{d}y'}{\eta' - y'\xi'}. \tag{2.3.4}$$

The integration of the first two terms gives the zeroth order differential invariant (which we denote u) and that of the second two terms the first order differential invariant (which we denote v). The terminology follows from

$$Gu(x,y) = 0 \tag{2.3.5}$$

and

$$G^{[1]}v(x,y,y') = 0. \tag{2.3.6}$$

In the case of G_1, (2.3.4) is

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}y'}{0} \tag{2.3.7}$$

We have verified using *Mathematica* [185] that (2.2.7) is invariant under the finite transformation (2.2.18).

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from which

$$u = y \qquad v = y'. \tag{2.3.8}$$

To find the reduced equation we determine v' and substitute from the original equation as follows:

$$\begin{aligned}
v' &= \frac{\mathrm{d}v}{\mathrm{d}u} \\
&= \frac{\mathrm{d}v}{\mathrm{d}x} \Big/ \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= \frac{y''}{y'} \\
&= \frac{1}{y'}\frac{1}{y^3} \\
&= \frac{1}{vu^3}.
\end{aligned} \tag{2.3.9}$$

Thus the transformation (2.3.8) reduces (2.2.7) to

$$vv' = \frac{1}{u^3}. \tag{2.3.10}$$

The symmetries G_2 and G_3 transform as follows:

$$\begin{aligned}
G_2^{[1]} &= x\frac{\partial}{\partial x} + \tfrac{1}{2}y\frac{\partial}{\partial y} - \tfrac{1}{2}y'\frac{\partial}{\partial y'} \\
&= x\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \tfrac{1}{2}y\frac{\partial u}{\partial y}\frac{\partial}{\partial u} - \tfrac{1}{2}y'\frac{\partial u}{\partial y'}\frac{\partial}{\partial u} + x\frac{\partial v}{\partial x}\frac{\partial}{\partial v} + \tfrac{1}{2}y\frac{\partial v}{\partial y}\frac{\partial}{\partial v} - \tfrac{1}{2}y'\frac{\partial v}{\partial y}\frac{\partial}{\partial v} \\
&= \tfrac{1}{2}u\frac{\partial}{\partial u} - \tfrac{1}{2}v\frac{\partial}{\partial v}
\end{aligned} \tag{2.3.11}$$

$$\begin{aligned}
G_3^{[2]} &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (y - xy')\frac{\partial}{\partial y'} \\
&= xu\frac{\partial}{\partial u} + (u - xv)\frac{\partial}{\partial v}.
\end{aligned} \tag{2.3.12}$$

Now, from (2.3.8) we can write

$$\begin{aligned}
v &= \frac{\mathrm{d}y}{\mathrm{d}x} \\
&= \frac{\mathrm{d}u}{\mathrm{d}x} \\
\mathrm{d}x &= \frac{\mathrm{d}u}{v} \\
x &= \int \frac{\mathrm{d}u}{v}.
\end{aligned} \tag{2.3.13}$$

The symmetry G_3 then becomes

$$G_3^{[1]} = u\int \frac{\mathrm{d}u}{v}\frac{\partial}{\partial u} + \left(u - v\int \frac{\mathrm{d}u}{v}\right)\frac{\partial}{\partial v}. \tag{2.3.14}$$

As the symmetry now contains an integral of the dependent variable, it is no longer a point symmetry, but falls under the class of nonlocal symmetries which we discuss in the next chapter. This result can be inferred from the Lie Bracket relations of (2.2.11), *viz*

$$[G_1, G_2] = G_1 \qquad [G_1, G_3] = 2G_2 \qquad [G_2, G_3] = G_3. \tag{2.3.15}$$

Is can be easily proved [156, p 148] that, if the Lie Bracket relation between two symmetries is

$$[X_1, X_2] = \lambda X_1 \tag{2.3.16}$$

($\lambda = 0$ or a constant usually scaled to 1), then reduction via X_1 will result in $X_2^{[1]}$ being a point symmetry of the reduced equation. Reduction via X_2 will result in $X_1^{[1]}$ being a nonlocal symmetry of the reduced equation.

Eq (2.3.10) contains (at least) one point symmetry of the form

$$X_1 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}. \tag{2.3.17}$$

When a first order equation possesses a symmetry, it is usual to convert that symmetry into one of translation in the independent variable thereby transforming the equation into autonomous form [5]. It is thereafter a trivial matter to write down the quadrature. In this case we wish to transform (2.3.17) to

$$Z_1 = \frac{\partial}{\partial Q} \tag{2.3.18}$$

under the point transformation

$$Q = F(u, v) \qquad P = G(u, v). \tag{2.3.19}$$

We require the operation of (2.3.17) on (2.3.19) to produce (2.3.18). This results in the system

$$\begin{aligned} u\frac{\partial F}{\partial u} - v\frac{\partial F}{\partial v} &= 1 \\ u\frac{\partial G}{\partial u} - v\frac{\partial G}{\partial v} &= 0 \end{aligned} \tag{2.3.20}$$

which is easily solved to give

$$F = \ln u + H(uv) \qquad G = G(uv), \tag{2.3.21}$$

where G and H are arbitrary functions of their arguments. If we take G to be the identity and H as zero, (2.3.19) becomes

$$Q = \ln u \qquad P = uv. \tag{2.3.22}$$

Then we have

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}Q} &= \frac{v + uv'}{1/u} \\ &= uv + u^2(1/(vu^3)) \\ &= P + P^{-1}\end{aligned} \tag{2.3.23}$$

from which the quadrature

$$Q - Q_0 = \int \frac{P\mathrm{d}P}{P^2 + 1} \tag{2.3.24}$$

follows naturally. The solution to (2.2.7) is obtained from (2.3.24) by inverting the transformations (2.3.22) and (2.3.8) respectively.

Remarks: i) Eq (2.2.7) is trivially integrable. The above procedure is merely illustrative. ii) In the case of (2.3.10) it is also easily integrable as it is in variables separable form. However, in practice this is rarely the case and the subsequent procedure need be followed to make any progress. iii) We mentioned that (2.3.10) has at least one point symmetry, *viz* (2.3.17). In fact, it is well-known [132, p 114][29, p 103] that all first order equations possess an infinite number of point symmetries, the determination of which requires the solution of the original equation! The standard method to determine symmetries of first order equations is to examine the fate of symmetries of higher order equations under successive reductions of order. Convenient tables of symmetries for specific first order equations exist in [102]. iv) The transformation of (2.3.10) to autonomous form illustrates another use of symmetries – that of transforming equations to recognisable and/or solvable equations by a change of basis of the algebra of the symmetries.

2.4 First Integrals of Equations

In addition to solving equations, symmetries can be used to determine first integrals of systems of equations. The physical importance of first integrals is that, in some instances, they represent conserved quantities.

The differential equation (2.2.1) possesses a first integral

$$I = f(x, y, y', \ldots, y^{(n-1)}) \tag{2.4.1}$$

in which the dependence on $y^{(n-1)}$ is nontrivial if

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{N=0} = 0. \tag{2.4.2}$$

(The subscript $N = 0$ is used to emphasise that the differential equation must be invoked. This underscores the fact that first integrals cannot exist in isolation of their equations of motion.) To calculate a first integral, I, associated with a symmetry, G, two linear partial differential equations need be solved. The first is obtained from the restriction that the integral satisfies the requirement of annihilation under the action of the $(n-1)$th extension of the symmetry and the second from its vanishing total derivative with respect to the independent variable, x (taking the equation into account). If the first integral is (2.4.1), we require

$$G^{[n-1]}f = 0. \tag{2.4.3}$$

There are n characteristics of (2.4.3) (which we denote u_i, $i = 1 \ldots n$) and we have

$$f = g(u_i). \tag{2.4.4}$$

The second partial differential equation results from the further requirement

$$\left.\frac{\mathrm{d}g}{\mathrm{d}x}\right|_{N=0} = 0 \tag{2.4.5}$$

for which there are $n-1$ characteristics (we which denote v_i, $i = 1 \ldots (n-1)$). (Obviously the method does not work for first order equations.) The first integral is now

$$I = h(v_i), \tag{2.4.6}$$

where h is an arbitrary function of its arguments. In particular, each of the v_i (which (must) include $y^{(n-1)}$ nontrivially) are first integrals. It may be advantageous to take combinations. However, it must be noted that only $n-1$ independent first integrals can be obtained from any one symmetry.

Let us consider the equation [89]

$$y''' + y'' + yy' = 0 \tag{2.4.7}$$

which arises in the study of shear–free, spherically symmetric perfect fluid spacetimes which admit a conformal symmetry. Eq (2.4.7) possesses the sole Lie point symmetry [96]

$$G_1 = \frac{\partial}{\partial x}. \tag{2.4.8}$$

Reduction of order via

$$u = y \qquad\qquad v = y' \tag{2.4.9}$$

results in

$$vv'' + v' + v'^2 + u = 0 \tag{2.4.10}$$

which integrates to

$$vv' + v + \tfrac{1}{2}u^2 = K. \tag{2.4.11}$$

We now investigate the occurrence of first integrals for (2.4.7). A first integral of (2.4.7) will have the form

$$I = f(x, y, y', y''). \tag{2.4.12}$$

To find f we first require

$$G_1^{[2]} I = 0. \tag{2.4.13}$$

The associated Lagrange's system is

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}y'}{0} = \frac{\mathrm{d}y''}{0} \tag{2.4.14}$$

from which

$$I = f(u, v, w) = f(y, y', y''). \tag{2.4.15}$$

The further requirement that

$$\left.\frac{\mathrm{d}I}{\mathrm{d}x}\right|_{(2.4.7)} = 0 \tag{2.4.16}$$

results in

$$\frac{\mathrm{d}u}{v} = \frac{\mathrm{d}v}{w} = \frac{\mathrm{d}w}{-w-uv}. \tag{2.4.17}$$

The normal way to proceed is to take suitable combinations of the terms in (2.4.17) and integrate them to obtain the characteristics [105, p 49]. Here we take $u \times (2.4.17)_1 + (2.4.17)_2 + (2.4.17)_3$ to obtain

$$\frac{u\mathrm{d}u + \mathrm{d}v + \mathrm{d}w}{0} \tag{2.4.18}$$

from which we obtain

$$\begin{aligned} p &= \tfrac{1}{2}u^2 + v + w \\ &= \tfrac{1}{2}y^2 + y' + y''. \end{aligned} \tag{2.4.19}$$

It is not obvious that any other characteristic can be obtained from (2.4.17). This example depicts clearly that while $n-1$ first integrals are expected for a given symmetry, this is merely an 'in principle' existence. We have

$$I = f(p) \tag{2.4.20}$$

which is a first integral of (2.4.7). For convenience f is taken to be the identity.

The combinations that are required to integrate the associated Lagrange's system are not always obvious. However, we can make some progress by following a purely formal route without knowledge of the appropriate combination. Consider $(2.4.17)_1$ and $(2.4.17)_2$, *viz*

$$\frac{\mathrm{d}u}{v} = \frac{\mathrm{d}v}{w} \tag{2.4.21}$$

which we can formally integrate to obtain

$$q_1 = \int w\mathrm{d}u - \tfrac{1}{2}v^2. \tag{2.4.22}$$

Substitution of (2.4.22) into $(2.4.17)_1$ and $(2.4.17)_3$ produces

$$\frac{\mathrm{d}u}{\sqrt{2(\int w\mathrm{d}u - q_1)}} = \frac{\mathrm{d}w}{-w - u\sqrt{2(\int w\mathrm{d}u - q_1)}}. \tag{2.4.23}$$

This is easily integrated to obtain (after substitution from (2.4.22) again)

$$q_2 = w + v + \tfrac{1}{2}u^2 \tag{2.4.24}$$

which is (2.4.19) again. Further progress could be made by substituting (2.4.24) back into (2.4.17). However, this results in the equivalent of (2.4.11) and does not facilitate further analysis.

2.5 Generalised and Contact Symmetries

We have thus far been primarily concerned with Lie point symmetries of the form

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}, \tag{2.5.1}$$

in which the coefficient functions ξ and η depend on (x, y) only. It is natural to consider the extension of the dependence of the coefficient functions to derivatives of the dependent variable. These symmetries are called generalised symmetries and have the form [156, p 289]

$$G = \xi(x, y, y', \ldots)\frac{\partial}{\partial x} + \sum_{i=0}^{n} \eta_i(x, y, y', \ldots)\frac{\partial}{\partial y^{(i)}}. \tag{2.5.2}$$

Symmetries of the form (2.5.2) are also referred to as Lie–Bäcklund symmetries [21, 29]. Olver [156, p 375] quite emphatically points out that both Lie and Bäcklund did not make the transition from contact symmetries to the consideration of truly generalised symmetries and discourages the use of that terminology. We use both terms interchangeably as there is reason [144] to dispute Olver's choice of terminology as well.

The determination of generalised symmetries, while algorithmically the same as that of point symmetries, does contain some intricacies not present in the latter. In general, the coefficient functions depend on (up to) the $(n-1)$th derivative of y. For the practical determination of these symmetries one has to place a restriction on the order of the derivatives. Thus we can consider, for example, the third order generalised symmetries of Burgers' equation (in potential form) [156, p 295]

$$u_t - u_x^2 - u_{xx} = 0 \tag{2.5.3}$$

or the fifth order Lie–Bäcklund symmetries of the Korteweg–de Vries equation [29, p 279]

$$u_t + uu_x + u_{xxx} = 0. \tag{2.5.4}$$

In the determination of generalised symmetries it must be borne in mind that differential consequences need to be taken into account. In the case of (2.5.3) we replace the time derivatives that arise in the extension of (2.5.2) via the equation, *ie*

$$u_t \longrightarrow u_{xx} + u_x^2 \tag{2.5.5}$$

$$u_{xt} \longrightarrow u_{xxx} + 2u_x u_{xx} \tag{2.5.6}$$

$$\vdots$$

This leads to a great simplification as the coefficient functions are then free of time derivatives. The technique now proceeds as in the case of point symmetries. While their greatest application is to partial differential equations Adam *et al* [17] show that it is possible to successfully apply the concept to ordinary differential equations. (See [21] for a detailed exposition of Lie–Bäcklund symmetries and [180] for a recent contribution.)

A subset of generalised symmetries, called contact symmetries, has proven to have important application in the study of higher order ordinary differential equations [22, 137]. That Lie believed in their importance is evidenced by the fact that two [134, 137] of his six books are devoted to them.

A scalar nth order ordinary differential equation

$$E(x, y, y', \ldots, y^{(n)}) = 0 \tag{2.5.7}$$

possesses the contact symmetry

$$G = \xi(x, y, y')\frac{\partial}{\partial x} + \eta(x, y, y')\frac{\partial}{\partial y} + \zeta(x, y, y')\frac{\partial}{\partial y'} \tag{2.5.8}$$

provided there exists a characteristic function [137, p 95] W such that

$$\xi = \frac{\partial W}{\partial y'} \tag{2.5.9}$$

$$\eta = y'\frac{\partial W}{\partial y'} - W \tag{2.5.10}$$

$$\zeta = -\frac{\partial W}{\partial x} - y'\frac{\partial W}{\partial y}. \tag{2.5.11}$$

Note that W is defined by [29]

$$W = \xi y' - \eta. \tag{2.5.12}$$

The implication of (2.5.8) is that

$$\frac{\partial \eta}{\partial y'} - y'\frac{\partial \xi}{\partial y'} = 0. \tag{2.5.13}$$

This requirement forces the space of transformations to close and obviates assumptions about the y' dependence of the coefficient functions. Again the determination of these symmetries follows the same algorithm of that of point symmetries.

In the case of a second order ordinary differential equation an infinite number of contact symmetries exists [137, p 84]. While their form and relationship to first integrals are of some interest [147], their existence for second order equations is analogous to that of point symmetries for first order equations. The importance of contact symmetries is evidenced by their presence and applicability in third order equations [15, 59, 9, 60]. We provide an instance of third order equations equivalent under a contact transformation but not under a point transformation.

It is easy to verify that

$$y''' = 0 \tag{2.5.14}$$

has the ten contact symmetries [137, p 242]

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial y} \\
G_2 &= x\frac{\partial}{\partial y} + \frac{\partial}{\partial y'} \\
G_3 &= x^2\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial y'} \\
G_4 &= \frac{\partial}{\partial x} \\
G_5 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\
G_6 &= x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial y'} \\
G_7 &= y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'} \\
G_8 &= y'\frac{\partial}{\partial x} + \tfrac{1}{2}y'^2\frac{\partial}{\partial y} \\
G_9 &= 2(xy' - y)\frac{\partial}{\partial x} + xy'^2\frac{\partial}{\partial y} + y'^2\frac{\partial}{\partial y'} \\
G_{10} &= (x^2y' - 2xy)\frac{\partial}{\partial x} + (\tfrac{1}{2}x^2y'^2 - 2y^2)\frac{\partial}{\partial y} + (xy'^2 - 2yy')\frac{\partial}{\partial y'},
\end{aligned}
\tag{2.5.15}
$$

the commutation relations of which are given in Table 2.1.

The Lie algebra generated by these symmetries is a 10–dimensional simple Lie algebra, isomorphic to the Lie algebra of the symplectic group $Sp(4)$. This isomorphism can be realised as follows. Write a general element of the Lie algebra in (2.5.15) as $\sum_{j=1}^{10} g_j G_j$. Then this element corresponds to the matrix below in the Lie algebra of $sp(4)$:

$$
\left[\begin{array}{cc|cc}
\frac{1}{2}(g_5 - g_7) & g_9 & g_{10} & g_6 \\
-g_2 & \frac{1}{2}(g_5 + g_7) & g_6 & -2g_3 \\
\hline
2g_2 & -g_4 & -\frac{1}{2}(g_5 - g_7) & g_2 \\
-g_4 & -g_2 & -g_9 & -\frac{1}{2}(g_5 + g_7)
\end{array}\right].
\tag{2.5.16}
$$

Note that the first seven symmetries in (2.5.15) are really first extensions of point symmetries. (Obviously all point symmetries are contact symmetries.) It is conventional to call the remaining three symmetries in (2.5.15) intrinsic or 'purely' contact symmetries. The ten contact symmetries of (2.5.14) are

Table 2.1: Commutation table for the ten contact symmetries of $y''' = 0$. The table has been subdivided to highlight the $3A_1$, $s\ell(2,R) \oplus A_1(\sim g\ell(2,R))$ and $3A_1$ subalgebras.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}
G_1	0	0	0	0	G_1	$2G_2$	G_1	0	$-2G_4$	$-2(G_5+G_7)$
G_2	0	0	0	$-G_1$	0	G_3	G_2	G_4	$2G_7$	$-G_6$
G_3	0	0	0	$-2G_2$	$-G_3$	0	G_3	$2(G_5-G_7)$	$2G_6$	0
G_4	0	G_1	$2G_2$	0	G_4	$2G_5$	0	0	$2G_8$	G_9
G_5	$-G_1$	0	G_3	$-G_4$	0	G_6	0	$-G_8$	0	G_{10}
G_6	$-2G_2$	$-G_3$	0	$-2G_5$	$-G_6$	0	0	$-G_9$	$-2G_{10}$	0
G_7	$-G_1$	$-G_2$	$-G_3$	0	0	0	0	G_8	G_9	G_{10}
G_8	0	$-G_4$	$-2(G_5-G_7)$	0	G_8	G_9	$-G_8$	0	0	0
G_9	$2G_4$	$-2G_7$	$-2G_6$	$-2G_8$	0	$2G_{10}$	$-G_9$	0	0	0
G_{10}	$2(G_5+G_7)$	G_6	0	$-G_9$	$-G_{10}$	0	$-G_{10}$	0	0	0

the maximum number for third order ordinary differential equations [137, p 241]. In addition to the equivalence of all third order equations with seven point symmetries to (2.5.14) under a point transformation we now have the equivalence of all third order equations with ten contact symmetries to (2.5.14) under a contact transformation. Consider the Kummer–Schwarz equation [27]

$$2Y'Y''' - 3Y''^2 = 0 \tag{2.5.17}$$

which has the ten contact symmetries [96]

$$\begin{aligned}
Z_1 &= \frac{\partial}{\partial X} \\
Z_2 &= X\frac{\partial}{\partial X} - Y'\frac{\partial}{\partial Y'} \\
Z_3 &= X^2\frac{\partial}{\partial X} - 2XY'\frac{\partial}{\partial Y'} \\
Z_4 &= \frac{\partial}{\partial Y} \\
Z_5 &= Y\frac{\partial}{\partial Y} + Y'\frac{\partial}{\partial Y'} \\
Z_6 &= Y^2\frac{\partial}{\partial Y} + 2YY'\frac{\partial}{\partial Y'} \\
Z_7 &= 2Y'^{-1/2}\frac{\partial}{\partial X} - 2Y'^{1/2}\frac{\partial}{\partial Y} \\
Z_8 &= 2XY'^{-1/2}\frac{\partial}{\partial X} - 2XY'^{1/2}\frac{\partial}{\partial Y} - 4Y'^{1/2}\frac{\partial}{\partial Y'} \\
Z_9 &= 2YY'^{-1/2}\frac{\partial}{\partial X} - 2YY'^{1/2}\frac{\partial}{\partial Y} - 4Y'^{3/2}\frac{\partial}{\partial Y'} \\
Z_{10} &= 2XYY'^{-1/2}\frac{\partial}{\partial X} - 2XYY'^{1/2}\frac{\partial}{\partial Y} - 4\left(XY'^{3/2} + YY'^{1/2}\right)\frac{\partial}{\partial Y'}.
\end{aligned} \tag{2.5.18}$$

The last four symmetries in (2.5.18) are intrinsic contact symmetries. The first six are point with the Lie algebra that decomposes into $s\ell(2,R) \oplus s\ell(2,R)$. (Although $s\ell(2,C) \oplus s\ell(2,C) \sim so(4,C)$ there does not exist a similar characteristic of the real form [15].) As the Lie algebras of the point symmetries of (2.5.14) and (2.5.17) differ these equations are not equivalent under a point transformation. That the dimension of the Lie algebras of the contact symmetries of (2.5.14) and (2.5.17) are equal is suggestive. We find that the Lie Bracket relations of (2.5.18) are the same as those of (2.5.15) with Zs replaced by Xs if we make the identifications

$$
\begin{aligned}
X_1 &= G_3 \\
X_2 &= -\tfrac{1}{4}G_{10} \\
X_3 &= -G_6 \\
X_4 &= \tfrac{1}{2}G_8 \\
X_5 &= G_5 - G_2 \\
X_6 &= \tfrac{1}{2}G_9 \\
X_7 &= -\left(G_2 + G_5\right) \\
X_8 &= 2G_4 \\
X_9 &= G_7 \\
X_{10} &= -2G_1.
\end{aligned} \tag{2.5.19}
$$

Noting the equivalence of the contact Lie algebras we look for a contact transformation between (2.5.14) and (2.5.17). This must have the form

$$X = F(x,y,y') \qquad Y = G(x,y,y') \qquad Y' = H(x,y,y'). \tag{2.5.20}$$

We pick on three symmetries from (2.5.14) to transform to three symmetries of (2.5.17). The choice is aided by the knowledge that $sp(4)$ decomposes into the $3A_1$, $s\ell(2,R)$, A_1 and $3A_1$ subalgebras. We choose those symmetries which form the $s\ell(2,R)$ subalgebras, *viz*

$$G_4 = \frac{\partial}{\partial x} \tag{2.5.21}$$

$$G_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{2.5.22}$$

$$G_6 = x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial y'} \tag{2.5.23}$$

and

$$X_4 = XY'^{-1/2}\frac{\partial}{\partial X} - XY'^{1/2}\frac{\partial}{\partial Y} - 2Y'^{1/2}\frac{\partial}{\partial Y'} \tag{2.5.24}$$

$$X_5 = -X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + 2Y'\frac{\partial}{\partial Y'} \tag{2.5.25}$$

$$X_6 = YY'^{-1/2}\frac{\partial}{\partial X} - YY'^{1/2}\frac{\partial}{\partial Y} - 2Y'^{3/2}\frac{\partial}{\partial Y'}. \tag{2.5.26}$$

We obtain the three sets of quasi-linear partial differential equations

$$\frac{\partial F}{\partial x} = \frac{F}{H^{1/2}} \tag{2.5.27}$$
$$\frac{\partial G}{\partial x} = -FH^{1/2} \tag{2.5.28}$$
$$\frac{\partial H}{\partial x} = -2H^{1/2} \tag{2.5.29}$$

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = -F \tag{2.5.30}$$
$$x\frac{\partial G}{\partial x} + y\frac{\partial G}{\partial y} = G \tag{2.5.31}$$
$$x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} = 2H \tag{2.5.32}$$

and

$$x^2\frac{\partial F}{\partial x} + 2xy\frac{\partial F}{\partial y} + 2y\frac{\partial F}{\partial y'} = \frac{G}{H^{1/2}} \tag{2.5.33}$$
$$x^2\frac{\partial G}{\partial x} + 2xy\frac{\partial G}{\partial y} + 2y\frac{\partial G}{\partial y'} = -GH^{1/2} \tag{2.5.34}$$
$$x^2\frac{\partial H}{\partial x} + 2xy\frac{\partial H}{\partial y} + 2y\frac{\partial H}{\partial y'} = -2H^{3/2}, \tag{2.5.35}$$

which we solve to obtain

$$X = \frac{2}{xy' - 2y}, \qquad Y = \frac{2x}{y'}, \qquad Y' = \left(\frac{2y - xy'}{y'}\right)^2. \tag{2.5.36}$$

(It is easily verified that $Y' = \mathrm{d}Y/\mathrm{d}X$.) Thus (2.5.14) is transformed to (2.5.17) via the contact transformation (2.5.36). This result is consistent with Lie's [137, p 86] that it is necessary for third order equations to be of the form

$$y''' = Ay''^3 + By''^2 + Cy'' + D, \tag{2.5.37}$$

where A, B, C and D are arbitrary functions of x, y and y', to be equivalent to (2.5.14) under a contact transformation. This further underscores the importance of contact symmetries for third order equations. The solution of (2.5.17) follows from that of (2.5.14), *viz*

$$y = A + Bx + Cx^2 \tag{2.5.38}$$

which, via (2.5.36), is

$$Y = \frac{4(1 + AX)}{4C + (4AC - B^2)X}. \tag{2.5.39}$$

We note that both contact and generalised symmetries can be calculated using `PROGRAM LIE` [96]. The calculation of the former is straightforward and a standard input file can easily be constructed. However, in the case of the latter the input file depends on the equation under consideration. See Appendix A for further details.

Chapter 3

Nonlocal symmetries

We explore the implications of extending the concept of Lie symmetries to include nonlocal symmetries. A method is presented for the calculation of nonlocal symmetries for equations possessing a single point symmetry [83]. Classes of second order equations not possessing Lie point symmetries but possessing a rich structure of nonlocal symmetries (and thereby enabling their solution) are calculated [8, 87]. We also demonstrate the equivalence of linear third order ordinary differential equations under nonlocal transformations [88].

3.1 Introduction

In a number of papers Abraham–Shrauner and co-workers [10, 4, 11, 13, 91, 5, 12] have discussed what they term hidden symmetries. Hidden symmetries are those that arise unexpectedly in the change of order of an equation. Consider the three-parameter group

$$\begin{aligned} G_1 &= \frac{\partial}{\partial x} \\ G_2 &= x\frac{\partial}{\partial x} \\ G_3 &= x^2\frac{\partial}{\partial x} \end{aligned} \tag{3.1.1}$$

with the Lie Bracket relations

$$\begin{aligned}
[G_1, G_2] &= G_1 \\
[G_1, G_3] &= 2G_2 \\
[G_2, G_3] &= G_3.
\end{aligned} \tag{3.1.2}$$

If an equation invariant under only this group is reduced in order via G_1, then [156, p 148] only the reduced form of G_2 is expected as a point symmetry of the reduced equation. All other point symmetries of the reduced equation that are found by direct calculation are called hidden symmetries of the reduced equation. (While the initial occurrences of hidden symmetries were as point symmetries, there is no reason to exclude contact symmetries [14].)

There are two varieties. Hidden symmetries of Type I arise when the order of an equation is increased and Type II when the order of an equation is decreased. Equivalently we could say that Type I (II) hidden symmetries are lost (gained) when the order of an equation is reduced. Consider the second order ordinary differential equation [9]

$$y^2y'' + yy'^2 - x = 0 \tag{3.1.3}$$

with the single Lie point symmetry [96]

$$G_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{3.1.4}$$

If we increase the order of (3.1.3) via

$$v = x \qquad v' = y, \tag{3.1.5}$$

we obtain the linear third order equation

$$v''' - v = 0 \tag{3.1.6}$$

which has, in addition to the ascendant of (3.1.4), *viz*

$$X_1 = v\frac{\partial}{\partial v} \tag{3.1.7}$$

and the symmetry created by (3.1.5), *viz*

$$X_2 = \frac{\partial}{\partial u}, \tag{3.1.8}$$

the three 'solution' symmetries of (3.1.6) [82], *viz*

$$\begin{aligned} X_3 &= \mathrm{e}^{u}\frac{\partial}{\partial v} \\ X_4 &= \mathrm{e}^{-\omega u}\frac{\partial}{\partial v} \\ X_5 &= \mathrm{e}^{-\bar{\omega} u}\frac{\partial}{\partial v}, \end{aligned} \tag{3.1.9}$$

where $\omega = (1+i\sqrt{3})/2$. The three symmetries (3.1.9) are called Type I hidden symmetries of (3.1.6) as they were gained in the increase of order of (3.1.3) to (3.1.6).

If we now reduced the order of (3.1.6) using X_3, *ie* via

$$X = u, \qquad Y = v' - v, \tag{3.1.10}$$

we obtain the linear second order equation

$$Y'' + Y' + Y = 0. \tag{3.1.11}$$

As (3.1.11) is linear, it possesses, in addition to the descendants of X_1, X_2, X_4 and X_5, the symmetries

$$\begin{aligned} U_5 &= \sin\left(\sqrt{3}X\right)\frac{\partial}{\partial X} - Y\sin\left(\sqrt{3}X - \tfrac{\pi}{3}\right)\frac{\partial}{\partial Y} \\ U_6 &= \cos\left(\sqrt{3}X\right)\frac{\partial}{\partial X} - Y\cos\left(\sqrt{3}X - \tfrac{\pi}{3}\right)\frac{\partial}{\partial Y} \\ U_7 &= Y\mathrm{e}^{\omega X}\frac{\partial}{\partial X} + Y^2\mathrm{e}^{\omega X}(\omega - 1)\frac{\partial}{\partial Y} \\ U_8 &= Y\mathrm{e}^{\bar{\omega} X}\frac{\partial}{\partial X} + Y^2\mathrm{e}^{\bar{\omega} X}(\bar{\omega} - 1)\frac{\partial}{\partial Y} \end{aligned} \tag{3.1.12}$$

for a total of eight. The four symmetries (3.1.12) are called Type II hidden symmetries of (3.1.11) as they were gained in the reduction of order of (3.1.6) to (3.1.11).

As our primary concern is the reduction of order of equations, hidden symmetries of Type II are the ones of obvious interest to us. (We show later the importance of Type I hidden symmetries.)

The obvious question to consider is that of the origin of hidden symmetries. The route to the answer is simple. If we consider (3.1.11), the origin of its hidden symmetries (3.1.12) is obtained via the transformation (3.1.10). A symmetry of (3.1.6) must have the form

$$X_6 = \xi\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial v}, \tag{3.1.13}$$

where the arguments of ξ and η are as yet unspecified. Under (3.1.10) $X_6^{[1]}$ transforms as

$$\begin{aligned} X_6^{[1]} &= \left(\xi\frac{\partial X}{\partial u} + \eta\frac{\partial X}{\partial v} + (\eta' - v'\xi')\frac{\partial X}{\partial v'}\right)\frac{\partial}{\partial X} \\ &\quad + \left(\xi\frac{\partial Y}{\partial u} + \eta\frac{\partial Y}{\partial v} + (\eta' - v'\xi')\frac{\partial Y}{\partial v'}\right)\frac{\partial}{\partial Y} \\ &= \xi\frac{\partial}{\partial X} + (\eta' - \eta - v'\xi')\frac{\partial}{\partial Y}. \end{aligned} \tag{3.1.14}$$

In the case of U_5 in (3.1.12),

$$\begin{aligned} \xi &= \sin\left(\sqrt{3}X\right) \\ \eta' - \eta - v'\xi' &= -Y\sin\left(\sqrt{3}X - \tfrac{\pi}{3}\right). \end{aligned} \tag{3.1.15}$$

In terms of the variables of (3.1.6) ξ and η are (via (3.1.10))

$$\xi = \sin(\sqrt{3}u) \tag{3.1.16}$$

$$\eta = \mathrm{e}^{u}\int \mathrm{e}^{-u}\left\{v\sin\left(\sqrt{3}u - \tfrac{\pi}{3}\right) - v'\left[\sin\left(\sqrt{3}u - \tfrac{\pi}{3}\right) + \sqrt{3}\cos(\sqrt{3}u)\right]\right\}du. \tag{3.1.17}$$

We observe that X_6 has integrals of the dependent variable in its coefficient functions. (Note that the v' in (3.1.17) can be removed by integration by parts.) As integrals cannot be defined at a point (*ie* locally), we call symmetries involving integrals of the dependent variable with respect to the independent variable nonlocal symmetries. (See also [103, p 68].)

The origin of Type II hidden symmetries then is found in nonlocal symmetries of the higher order equation. In a similar manner we find that Type I hidden symmetries originate from nonlocal symmetries of the lower order equation. It is therefore natural to investigate nonlocal symmetries and attempt to find a means of calculating them for a given differential equation. (We note that some hidden symmetries originate in contact symmetries [9]. However, this occurrence is rare and we can confine our attention to those hidden symmetries that originate in nonlocal symmetries.)

For the purposes of reduction of order we are interested in nonlocal symmetries which become point for the equation of reduced order. This type of nonlocal symmetry we call first order and we have

$$G = \xi(x, y, I)\frac{\partial}{\partial x} + \eta(x, y, I)\frac{\partial}{\partial y}, \tag{3.1.18}$$

where

$$I = \int f(x, y)\mathrm{d}x. \tag{3.1.19}$$

We shall see in §3.3 that even (3.1.19) is too general, but it shall suffice for the present.

In the following sections we investigate the implications of allowing the generalisation to nonlocal symmetries on the reduction of differential equations to quadratures.

3.2 Determination of nonlocal symmetries

We recall, if

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{3.2.1}$$

is a symmetry of some differential equation, then X is the generator of the group of infinitesimal point transformations

$$\begin{aligned}\bar{x} &= x + \varepsilon\xi \\ \bar{y} &= y + \varepsilon\eta\end{aligned} \tag{3.2.2}$$

which leave the equation invariant. We wish to broaden this class of transformations to nonlocal transformations.

For our purposes we call the set of infinitesimal transformations

$$\begin{aligned}\bar{x} &= x + \varepsilon\xi \\ \bar{y} &= y + \varepsilon\eta \\ \bar{I} &= I + \varepsilon\gamma,\end{aligned} \tag{3.2.3}$$

where

$$I = \int f(x, y)\mathrm{d}x, \tag{3.2.4}$$

a first order one–parameter Lie group of nonlocal infinitesimal transformations. (Here order refers to the number of integrals in I. If I involved double integrals, we would call (3.2.3) a second order one–parameter Lie group of nonlocal infinitesimal transformations.) The generator of these nonlocal transformations is

$$G = \xi(x, y, I)\frac{\partial}{\partial x} + \gamma(x, y, I)\frac{\partial}{\partial I} + \eta(x, y, I)\frac{\partial}{\partial y}, \tag{3.2.5}$$

where

$$\frac{\partial\gamma}{\partial y} - y\frac{\partial\xi}{\partial y} = 0. \tag{3.2.6}$$

We require (3.2.6) to remove the possibility of derivatives in η and thereby that the space of transformations closes as in the case of contact symmetries. The above notation is given to show the link between our concept of nonlocal symmetries and that of the classic Lie point symmetries.

Remark: In general we could require ξ, γ and η in (3.2.5) to depend on x, y, y' and $I = \int f(x, y, y')\mathrm{d}x$. We address this possibility in §3.4.

Note that we need not include η in (3.2.3) and (3.2.5) as the first extension of

$$\tilde{G} = \xi(x, y, I)\frac{\partial}{\partial x} + \gamma(x, y, I)\frac{\partial}{\partial I} \tag{3.2.7}$$

defines η as

$$\eta = \frac{\mathrm{d}\gamma}{\mathrm{d}x} - y\frac{\mathrm{d}\xi}{\mathrm{d}x}. \tag{3.2.8}$$

However, in practice one knows ξ and η and works backwards to determine γ. In the actual calculation of nonlocal symmetries we ignore $\gamma\partial/\partial I$ in (3.2.5) as we are only concerned with differential equations. Abraham–Shrauner derives the expressions for γ in [7].

We are now in a position to calculate the nonlocal symmetries of a given equation. The procedure is similar to that of determining its Lie point symmetries. We require the equation to be invariant under the nth extension of (3.2.5). The main difference between the resulting calculation and that for point symmetries is the introduction of $\partial/\partial I$ terms. The determining equations form a system of linear *ordinary* differential equations as opposed to the *partial* differential equations of the classic Lie theory.

3.3 Nonlocal symmetries of second order ordinary differential equations

While it is of mathematical interest to determine nonlocal symmetries of differential equations in general, the important occurrence of these symmetries is in second order ordinary differential equations. Recall that nonlocal symmetries of a differential equation manifest themselves as Type II hidden symmetries of the reduced equation. Thus a simple reduction of order and subsequent calculation of the Lie point symmetries (*eg* using `PROGRAM LIE` [96]) will determine the 'useful' nonlocal symmetries of any equation. (We define 'useful' nonlocal symmetries as those that reduce to point symmetries under a single reduction of order of the equations.) While this is true in general, it does not apply for second order equations as there is no direct method to determine the infinite number of point symmetries that arise in the reduced first order equation [29]. In the instance that the second order equation possesses more than one point symmetry, reduction of order via the appropriate point symmetry (*ie* one which does not annihilate the others as point symmetries [156, p 149]) will result in a first order equation with at least one known point symmetry. Thus the case of second order equations possessing just one point symmetry is the one of paramount importance. The determination of at least one 'useful' nonlocal symmetry for such equations provides a systematic route to the solution of these equations via the classical Lie theory of extended groups [29, 156].

We analyse the equation

$$E(y, y', y'') = y'' - g(y, y') = 0 \tag{3.3.1}$$

with the sole Lie point symmetry

$$G_1 = \frac{\partial}{\partial x} \tag{3.3.2}$$

for the existence of nonlocal symmetries. The restriction to (3.3.1) causes no loss of generality as all ordinary differential equations (not just those of second order) with at least one symmetry can always be transformed to autonomous form.

In the case of (3.3.1) possessing two point symmetries the Lie Bracket relationship

$$[G_1, G_2]_{LB} = \lambda G_1, \tag{3.3.3}$$

where λ is a constant (either 0 or scaled to 1), guarantees G_2 as a point symmetry of the reduced equation. If G_1 is defined as in (3.3.2) and

$$G_2 = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{3.3.4}$$

(3.3.3) implies that G_2 must have the form

$$G_2 = (\lambda x + k(y))\frac{\partial}{\partial x} + a(y)\frac{\partial}{\partial y}. \tag{3.3.5}$$

The reduction of (3.3.1) by the transformation generated by (3.3.2), *viz*

$$u = y \qquad v = y' \tag{3.3.6}$$

results in the first order equation

$$vv' = g(u, v). \tag{3.3.7}$$

Thus G_2 will reduce (using (3.3.6) and the first extension of (3.3.5)) to

$$Y = a(u)\frac{\partial}{\partial u} + v(a'(u) - \lambda - k'(u)v)\frac{\partial}{\partial v}. \tag{3.3.8}$$

(See also [29, p 129].)

The structure of a nonlocal symmetry will be, in general, (disregarding the $\partial/\partial I$ term)

$$G_{nl} = \xi(x, y, I)\frac{\partial}{\partial x} + \eta(x, y, I)\frac{\partial}{\partial y}, \tag{3.3.9}$$

where

$$I = \int f(x, y)\mathrm{d}x. \tag{3.3.10}$$

However, by noting the form of (3.3.5) we deduce that

$$\eta = a(y) \qquad \text{and} \qquad \xi = \lambda x + k(y, I). \tag{3.3.11}$$

This follows from the requirement

$$[G_1, G_{nl}] = \lambda G_1. \tag{3.3.12}$$

The nonlocal symmetry of (3.3.1) is then

$$G_{nl} = (\lambda x + k(y, I))\frac{\partial}{\partial x} + a(y)\frac{\partial}{\partial y}. \tag{3.3.13}$$

The requirement that ξ be free of y' and ξ' be free of x and I gives

$$G_{nl} = (\lambda x + I)\frac{\partial}{\partial x} + a(y)\frac{\partial}{\partial y}, \tag{3.3.14}$$

where

$$I = \int c(y)\mathrm{d}x. \tag{3.3.15}$$

This form is more restrictive than just being of first order (*ie* more restrictive than (3.1.18)). (We are fortunate that I has this specific form. If not, our prolongation formulæ would not have the simple form we have assumed [127, p 163].) Higher order nonlocal symmetries (with multiple integrals) cannot be useful in the sense defined above as they will not reduce to point symmetries under a single reduction of order.

To determine the coefficient functions in (3.3.14) we require

$$G_{nl}^{[2]} E\Big|_{E=0} = 0, \tag{3.3.16}$$

ie

$$-a\frac{\partial g}{\partial y} + ((\lambda + c)y' - y'a')\frac{\partial g}{\partial y'} + ((a' - 2(\lambda + c))g + y'^2(a'' - c')) = 0. \tag{3.3.17}$$

The solution of the associated Lagrange's system reduces to that of two linear first order ordinary differential equations which we solve [105] for g to obtain

$$g = \mathrm{e}^{-\int \phi \mathrm{d}y}\left[\int \psi \mathrm{e}^{\int \phi \mathrm{d}y}\mathrm{d}y + L(u)\right], \tag{3.3.18}$$

where

$$\begin{aligned}
\phi &= \frac{a' - 2(\lambda + c)}{a}, \\
\psi &= \frac{a(c' - a'')}{u^2}\exp\left(-2\int\frac{\lambda + c}{a}\mathrm{d}y\right), \\
u &= \frac{a}{y'}\exp\left(-\int\frac{\lambda + c}{a}\mathrm{d}y\right).
\end{aligned} \tag{3.3.19}$$

To make this implicit result clearer it is useful to look at a practical example.

Consider the equation [61]

$$R'' + \frac{\varepsilon R^3}{(\rho^2 + 1)^3} - \frac{K^2}{R^3} = 0, \tag{3.3.20}$$

where K is a constant and ε a parameter, which is a reduced form of the complex Lorenz system under certain assumptions about its parameters. Eq (3.3.20) has the single Lie point symmetry

$$G = (\rho^2 + 1)\frac{\partial}{\partial \rho} + \rho R\frac{\partial}{\partial R}. \tag{3.3.21}$$

We use (3.3.21) to rewrite (3.3.20) in autonomous form, *viz*

$$y'' + \varepsilon y^3 + y - \frac{K^2}{y^3} = 0 \tag{3.3.22}$$

via the transformation

$$x = \tan^{-1}\rho \qquad\qquad y = R(\rho^2 + 1)^{-1/2}. \tag{3.3.23}$$

We can now analyse (3.3.22) for nonlocal symmetries of the form (3.3.14). We find that

$$\begin{aligned} a &= \frac{1}{p}(A_1 + A_0 \int p\mathrm{d}y) \\ c &= -\frac{p'}{p^2}(A_1 + A_0 \int p\mathrm{d}y) + \bar{A}_0 \\ p &= \varepsilon y^3 + y - \frac{K^2}{y^3} \qquad \bar{A}_0 = \tfrac{1}{2}A_0 - \lambda, \end{aligned} \tag{3.3.24}$$

where A_0 and A_1 are constants and the $'$ on $p(y)$ means differentiation with respect to its argument, y.

It is significant that (3.3.22) has at least two Lie symmetries of the form (3.3.14) that commute. This guarantees the reduction of (3.3.20) to quadratures. It has already been shown that (3.3.20) possesses the Painlevé property [170] and is hence conjectured to be integrable [168]. (See Chapter 4.) The reduction to quadrature was previously performed [61] without knowledge of the nonlocal symmetry.

3.4 Generalised nonlocal symmetries

We have thus far only considered 'point-like' nonlocal symmetries, *ie* the dependence of the coefficient functions was free from derivatives. This was due to the fact that we were working from a knowledge of point symmetries. However, there is no reason to exclude more general nonlocal symmetries as they can reduce to point symmetries under a single reduction of order and are also 'useful'. We call these symmetries generalised nonlocal symmetries in keeping with Olver's notation [156] for the local variant. However, they could equally be called Lie–Bäcklund nonlocal symmetries.

Consider the generator of nonlocal infinitesimal transformations

$$Z_{nl} = \xi(x, y, y', \textstyle\int f(x, y, y')\mathrm{d}x)\frac{\partial}{\partial x} + \eta(x, y, y', \textstyle\int f(x, y, y')\mathrm{d}x)\frac{\partial}{\partial y}. \tag{3.4.1}$$

For (3.4.1) to be a 'useful' nonlocal symmetry it has to reduce to

$$G_{red} = a(u, v)\frac{\partial}{\partial u} + b(u, v)\frac{\partial}{\partial v} \tag{3.4.2}$$

under

$$u = y, \qquad v = y'. \tag{3.4.3}$$

This restriction, together with the analogue of (3.3.12), confines the analysis to generalised nonlocal symmetries of the form

$$Z_{nl} = \int c(y, y')\mathrm{d}x \frac{\partial}{\partial x} + a(y, y')\frac{\partial}{\partial y}. \tag{3.4.4}$$

(Here we have taken $\lambda = 0$ for simplicity.) With c and a being arbitrary functions of y and y' the analysis can only proceed to writing down the equation to be solved. (The problem is similar to calculating contact symmetries of second order ordinary differential equations.) Further progress can only be made by assuming *a priori* a form for the nonlocal symmetry (3.4.4). We look at some special cases.

3.4.1 $a(y, y') = 0$

We require

$$E(y, y', y'') = y'' - g(y, y') = 0 \tag{3.4.5}$$

to be invariant under (3.4.4) with $a(y, y') = 0$. This results in g having the following form

$$g = \frac{1}{cy'^2}\left[F(y) - y^2\frac{\partial}{\partial y}\left(\int cy'\mathrm{d}y'\right)\right] \tag{3.4.6}$$

for (3.4.5) to have a generalised nonlocal symmetry. Given g we can determine c via (3.4.6) to obtain (3.4.4) with $a(y, y') = 0$. ($F(y)$ is an arbitrary function of y.)

3.4.2 $a(y, y') = 0, g = g(y)$

In this example we require

$$E(y, y'') = y'' - g(y) = 0 \tag{3.4.7}$$

to be invariant under

$$Z_{nl} = \int c(y, y')\mathrm{d}x \frac{\partial}{\partial x}. \tag{3.4.8}$$

For a given g, c has the form

$$c = \frac{F\left(\frac{1}{2}y'^2 - \int g(y)\mathrm{d}y\right)}{y'^2}, \tag{3.4.9}$$

where F is an arbitrary function of its argument. We note that the Lorenz system (3.3.22) falls into the class (3.4.7). This implies that (3.3.22) has 'an' additional nonlocal symmetry of the form (3.4.8). (In fact (3.4.8), together with (3.4.9), constitutes an infinite class of symmetries.) The occurrence of the additional symmetry is unsurprising as the first order equation obtained from (3.3.22) under the reduction (3.4.3) will have an infinite number of point symmetries. We expect to find, in principle, an infinite number of 'useful' nonlocal symmetries of (3.3.22). The form of these symmetries will depend on our *ansätze* for $a(y, y')$ and $c(y, y')$.

The class of equations considered by Guo and Abraham–Shrauner [91] is also contained in (3.4.7). They found that (3.4.7) was invariant under

$$Z_{nl} = \left(\int \frac{1}{y'^2}\mathrm{d}x\right)\frac{\partial}{\partial x} \tag{3.4.10}$$

by first considering the first order equation that results from the reduction of (3.4.7) via (3.4.3). Our method has resulted in a generalisation of their results.

3.4.3 $a = a(y)$, $c = c(y')$, $g = g(y)$

We look at

$$y'' = g(y) \tag{3.4.11}$$

again. This time we require it to be invariant under

$$Z_{nl} = \int c(y')\mathrm{d}x\frac{\partial}{\partial x} + a(y)\frac{\partial}{\partial y}. \tag{3.4.12}$$

For a given g, a and c are related via

$$g = A_0 a \exp\left[-\left(2c + y'\frac{\mathrm{d}c}{\mathrm{d}y'}\right)\int \frac{1}{a}\mathrm{d}y\right] + y'^2 \int\left[\frac{1}{a^2}\frac{\mathrm{d}^2 a}{\mathrm{d}y^2}\exp\left(-\int \frac{1}{a}\mathrm{d}y\right)\mathrm{d}y\right], \tag{3.4.13}$$

where A_0 is an arbitrary constant of integration.

3.4.4 Equations linear in y'

Equations that are at most linear in y', *eg*

$$y'' = \alpha y' + g(y), \tag{3.4.14}$$

where α is a constant, are of some interest as they reduce to an Abel's equation [109, p 24] under (3.4.3). Unfortunately there is no simple way to compute its nonlocal symmetry, *eg* nonlocal symmetries of the forms

$$Z_{nl} = \int c(y')\mathrm{d}x \frac{\partial}{\partial x} \tag{3.4.15}$$

and

$$\tilde{Z}_{nl} = \int c(y)\mathrm{d}x \frac{\partial}{\partial x} \tag{3.4.16}$$

do not leave (3.4.14) invariant. We obviously need to make a more complicated *ansatz*. It is whimsical to note that the choice $c = c(y, y')$ and $a = 0$ reduces the problem to that of solving an Abel's equation to determine the nonlocal symmetry! The interested reader is referred to [17] for a discussion of methods to obtain solutions for Abel's equation and the impact of hidden symmetries.

Similar observations should aid in the choice of *ansätze* for a and c in (3.4.4). It is apparent that these symmetries are most appropriate to third and higher order equations in the same manner in which contact symmetries have application to those equations.

3.5 Equations not possessing point symmetries

The motivation for what we report here lies in an example of a class of equations given by González–Gascón and González–López [76] which was easily integrable and yet did not possess any Lie point symmetries. They wished to illustrate that it was not true that every system of ordinary differential equations which can be reduced to quadrature possesses at least one Lie point symmetry.

The system González–Gascón and González–López used to demonstrate their point was

$$u'' = c(u) \tag{3.5.1}$$

$$v'' = f(x, u, u')v' + f'v + g(x, u, u') \tag{3.5.2}$$

in which

$$f' := \frac{\partial f}{\partial x} + u'\frac{\partial f}{\partial u} + c(u)\frac{\partial f}{\partial u'}. \tag{3.5.3}$$

(This system could be thought of as describing some Newtonian system.) González–Gascón and González–López showed that there existed at least one instance of choices of f and g for which the system (3.5.1) and (3.5.2) did not possess a Lie point symmetry of the form

$$G = \xi(x, u, v)\frac{\partial}{\partial x} + \eta(x, u, v)\frac{\partial}{\partial u} + \zeta(x, u, v)\frac{\partial}{\partial v} \tag{3.5.4}$$

(thereby in effect a doubly denumerable infinity) and yet (3.5.1) and (3.5.2) are always integrable. The argument for integration is that (3.5.1) is always integrable to give

$$u = u(x, k_1, k_2). \tag{3.5.5}$$

With (3.5.5) (3.5.2) can be integrated since it now becomes a linear second order equation for v. González–Gascón and González–López do show how (3.5.2) can be integrated once to a first order linear equation and the final quadrature performed. However, we are content to agree with Painlevé [160] that reduction to a linear equation is a sufficient criterion for integrability or, as Conte [47] quotes him, the integration is 'parfaite'.

In their concluding remarks González–Gascón and González–López point out that the system (3.5.1) and (3.5.2), as far as their knowledge extends, is the first one to be presented which is reducible to quadratures and yet devoid of point symmetries. They also express the hope that a simpler example, say of a single nonlinear second order equation, could be found. We explore this avenue of investigation and demonstrate the role of nonlocal symmetries in the solution of these equations.

We note that the system of González–Gascón and González–López does possess the symmetry

$$G = (v - j)\frac{\partial}{\partial v}, \tag{3.5.6}$$

where j is a solution of

$$j'' = j'f + jf' + g. \tag{3.5.7}$$

Given a solution of (3.5.7), which can be found once (3.5.1) is solved, (3.5.6) is a point symmetry. That we have to be able to solve the system to determine the symmetry is nothing new. Indeed, this is the case for all linear equations [146].

Recently [181, p 105] a more convincing example has been provided which also fulfills the hope of González–Gascón and González–López that an example comprising a single equation would be found. The equation

$$y'' = y^{-1}y'^2 + pg(x)y^py' + g'(x)y^{p+1}, \tag{3.5.8}$$

where p is a nonzero constant and $g(x)$ a nonzero arbitrary function, does not possess a Lie point symmetry unless [181, p 105]

$$g(x) = \begin{cases} c_1\exp(c_2x^2) \\ c_3(c_5 + c_6x)^{c_7}\exp(c_4x). \end{cases} \tag{3.5.9}$$

However, trivially a first integral is found to be

$$I = \frac{y'}{y} - g(x)y^p. \tag{3.5.10}$$

Eq (3.5.10) is a Bernoulli equation and is easily integrated [109, p 19] to give

$$y = \frac{e^{Ix}}{\left(J - p\int g(x)e^{pIx}\mathrm{d}x\right)^{1/p}}, \tag{3.5.11}$$

where J is the second constant of integration.

The ease of integration of (3.5.8) is sufficient to suggest that an inherent property of the equation has been exploited (without knowledge thereof). The equation may not have any point symmetries, but that is not necessarily an obstacle to solution by group theoretic techniques. Normally one uses the possession of a point symmetry to reduce the order of a differential equation. However, there is no reason why one cannot reverse the procedure and increase the order of a differential equation to create a point symmetry in the higher order equation. This technique has been used to good effect in the case of Abel's equation [17]. There is always the hope that by increasing the order in this manner more than one point symmetry can be gained, the so-called Type I hidden symmetries introduced earlier. Our choice of transformation is suggested by the success of the generalised Riccati transformation in transforming the first order nonlinear generalised Riccati equation [105, p 23]

$$y' + a(x) + b(x)y + c(x)y^2 = 0 \tag{3.5.12}$$

to the second order linear equation

$$z'' + \left(b - \frac{c'}{c}\right) z' + acz = 0 \tag{3.5.13}$$

via

$$y(x) = \frac{z'(x)}{c(x)z(x)} \tag{3.5.14}$$

without a knowledge of the solution of (3.5.13). The transformation (3.5.14) is associated with the homogeneity symmetry, G_2, for which appropriate variables are found from the solution of the associated Lagrange's system

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}w}{w} = \frac{\mathrm{d}w'}{w'}. \tag{3.5.15}$$

The invariants of (3.5.15) are

$$u = x \qquad v = \frac{w'}{w} \tag{3.5.16}$$

and so suitable new variables are

$$X = F\left(x, \frac{w'}{w}\right) \qquad W = G\left(x, \frac{w'}{w}\right), \tag{3.5.17}$$

where F and G are independent functions. The well-known transformation (3.5.14) falls obviously into the class given by (3.5.17).

In the case of (3.5.8) we write the transformation as

$$y^p = \frac{-w'}{pgw} \tag{3.5.18}$$

to obtain the third order nonlinear equation

$$w'w''' - w''^2 - \left(\frac{g'}{g}\right)' w'^2 = 0 \tag{3.5.19}$$

which does have a second symmetry in addition to the one produced by the transformation (3.5.18). The symmetries are

$$G_1 = \frac{\partial}{\partial w} \qquad\qquad G_2 = w\frac{\partial}{\partial w} \tag{3.5.20}$$

and, since $[G_1, G_2] = G_1$, it is evident that the reduction of (3.5.19) to (3.5.8) was via the non-normal subgroup [156, p 148], *ie* using (3.5.18), and so the first symmetry was lost as a point symmetry of the reduced equation. We use G_1 to reduce the order of (3.5.19) by means of the transformation

$$W = \log\left(\frac{w'}{g}\right) \qquad\qquad X = x \tag{3.5.21}$$

and obtain

$$\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} = 0 \tag{3.5.22}$$

which is easy enough to solve and possesses the eight element algebra of point symmetries $s\ell(3, R)$.

It is possible to obtain the solution of (3.5.8) from that of (3.5.22) by reversing the transformations. Thus from the solution of (3.5.22) we have in turn

$$\begin{aligned}
W &= A + BX \\
w' &= g(x)\exp(A + Bx) \\
w &= C + K\int g(x)\exp(Bx)\mathrm{d}x \\
y &= \left(\frac{-K\exp(Bx)}{p(C + K\int g(x)\exp(Bx)\mathrm{d}x)}\right)^{\frac{1}{p}}
\end{aligned} \tag{3.5.23}$$

which is in agreement with (3.5.11) when constants are suitably relabelled. When the eight point symmetries of (3.5.22) are transported to (3.5.8) via

(3.5.19), they become, as one would expect, quite nonlocal. As González–Gascón and González–López suspected, there are nonlocal symmetries behind easy integrability. The main problem is to find a co–ordinate system in which they can take point (or contact if at the third order level) form so that they can be readily determined. Unfortunately the point symmetries of (3.5.22) become horribly nonlocal by the time (3.5.8) is reached. To give some idea of their appearance we list

$$
\begin{aligned}
X_2 &= \left[\frac{y}{p}\exp\left(p\int gy^p\mathrm{d}x\right)\int\exp\left(-p\int gy^p\mathrm{d}x\right)\mathrm{d}x - xy\frac{p+1}{p}\right]\frac{\partial}{\partial y}\\
X_6 &= y\left[\log\left(-py^p\right) - p\int gy^p\mathrm{d}x + \exp\left(p\int gy^p\mathrm{d}x\right)\times\right.\\
&\quad\left.\int gy^p\left(\log\left(-py^p\right) - p\int gy^p\mathrm{d}x\right)\exp\left(-p\int gy^p\mathrm{d}x\right)\mathrm{d}x\right]\frac{\partial}{\partial y}
\end{aligned}
\tag{3.5.24}
$$

in the standard numbering of the $2A_1 \oplus_s (s\ell(2,R) \oplus_s A_1) \oplus_s 2A_1$ decomposition of $s\ell(3,R)$ [142]. Thus X_2 is the second element of the first two–dimensional Abelian subalgebra and X_6 is the sole element of the one–dimensional Abelian subalgebra. However, G_1 in (3.5.20) becomes the exponential nonlocal symmetry [5]

$$U_1 = y\exp\left(\int g(x)y^p\mathrm{d}x\right)\frac{\partial}{\partial y} \tag{3.5.25}$$

of (3.5.8) and the first integral associated with U_1 is just (3.5.10). Thus, if the integration of (3.5.8) had not been obvious, it could easily have been performed by group theoretical methods. It is true that G_1 is not a point symmetry, but exponential nonlocal symmetries are just as useful in the determination of first integrals.

3.6 A Group Theoretic classification of equations not possessing Lie point symmetries

The example above contains the essence of what we wish to develop in generality. The central principle is to increase the order of a differential equation by a transformation which produces a point symmetry in the higher order equation. To recast (3.5.8) into more accessible form we set

$$z = y^p \tag{3.6.1}$$

to obtain

$$z'' = \frac{z'^2}{z} + azz' + a'z^2. \tag{3.6.2}$$

Using the transformation

$$z = -\frac{w'}{aw} \tag{3.6.3}$$

we obtain the third order equation

$$w'w''' - w''^2 - \left(\frac{a'}{a}\right)' w'^2 = 0 \tag{3.6.4}$$

which has the two point symmetries

$$G_1 = \frac{\partial}{\partial w} \qquad G_2 = w\frac{\partial}{\partial w}. \tag{3.6.5}$$

We wish to generate classes of second order ordinary differential equations not possessing point symmetries but solvable by group theoretic means using the above procedure.

The constraint placed on the original second order equation is that the third order equation have two symmetries so that there is an alternate route to reduce the third order equation to second order. The best approach is to start with a third order equation which is required to have two Lie point symmetries and that one of them persists in the second order equation obtained by reduction using the normal subgroup. (We envisage that the second order equation we are interested in was obtained by the reduction of this third order equation via the nonnormal subgroup and so will not have any Lie point symmetries.) As the existence of one point symmetry is not sufficient for integrability of the new second order equation we determine those which have two point symmetries according to the classification of two-dimensional algebras by Lie [132, p 412]. Note that the existence of two point symmetries at the second order level does not imply the existence of three point symmetries at the third order level. The second symmetry *could* be a Type II hidden symmetry. Although the two–dimensional algebra is sufficient for integrability we provide a listing of the second order equations with three and eight point symmetries which fit into our scheme. For all of these we give the representative second order equation with no point symmetries from which the process commences. These will comprise the new classes of second order equations solvable by the Lie method even though they do not possess Lie point symmetries.

3.7 General Form of the Third Order Equation

We require that the third order equation be invariant under the action of a two–dimensional algebra and that, due to the origin of the equation, one of the symmetries be the homogeneity symmetry

$$G_2 = y\frac{\partial}{\partial y}. \tag{3.7.1}$$

(The labelling is with hindsight.) There are four two–dimensional Lie algebras [132, p 412]. We are not interested in the two Abelian algebras

$$\begin{array}{llll} \text{Type I} & G_1 = \frac{\partial}{\partial X} & G_2 = \frac{\partial}{\partial Y} & [G_1, G_2] = 0 \\ \text{Type II} & G_1 = \frac{\partial}{\partial Y} & G_2 = X\frac{\partial}{\partial Y} & [G_1, G_2] = 0 \end{array} \tag{3.7.2}$$

since both elements are normal subgroups and reduction by one does not mean the loss of the other as a point symmetry of the reduced equation [156, p 148]. In our approach the original second order equation can be imagined as coming from the third order equation as a result of reduction using G_2 (in (3.7.1)). The other symmetry is lost because G_2 is not the normal subgroup. Our method then utilises the normal subgroup to reduce the third order equation to another second order equation. This equation will inherit G_2. To solve the new second order equation we seek those equations which admit a second Lie point symmetry. This procedure imposes the minimum constraint on the third order equation. We do not require that it have a third point symmetry, only that it have a nonlocal symmetry which becomes point in the variables of the reduced equation.

The two solvable algebras (Types III and IV of Lie's classification [132, p 425]) have the canonical forms

$$\begin{array}{llll} \text{Type III} & G_1 = \dfrac{\partial}{\partial Y} & G_2 = X\dfrac{\partial}{\partial X} + Y\dfrac{\partial}{\partial Y} & [G_1, G_2] = G_1 \\ \text{Type IV} & G_1 = \dfrac{\partial}{\partial Y} & G_2 = Y\dfrac{\partial}{\partial Y} & [G_1, G_2] = G_1. \end{array} \tag{3.7.3}$$

We shall determine the class of third order equations invariant under each canonical realisation and then transform the nonnormal subgroup to the desired form of G_2, *viz* (3.7.1). In an attempt to avoid confusion we use the subscripts '2' and '3' to denote the two–dimensional Lie algebras associated with the second and third order equations respectively. We consider each canonical realisation in turn.

3.7.1 Type III$_3$

We begin with the Type III$_3$ two–dimensional algebra in (3.7.3). It is evident that the general form of third order equation invariant under G_1 is

$$Y''' = f(X, Y', Y''). \tag{3.7.4}$$

Under G_2 the function, f, satisfies the first order linear partial differential equation

$$X\frac{\partial f}{\partial X} - Y''\frac{\partial f}{\partial Y''} = -2f \tag{3.7.5}$$

with associated Lagrange's system

$$\frac{\mathrm{d}X}{X} = \frac{\mathrm{d}Y'}{0} = \frac{\mathrm{d}Y''}{-Y''} = \frac{\mathrm{d}f}{-2f}. \tag{3.7.6}$$

The characteristics are

$$\begin{aligned} u &= XY' \\ v &= Y' \\ w &= \frac{f}{Y''^2} \end{aligned} \tag{3.7.7}$$

and (3.7.4) becomes

$$Y''' = Y''^2 f(Y', XY''). \tag{3.7.8}$$

Under the transformation

$$x = F(X,Y) \qquad y = G(X,Y) \tag{3.7.9}$$

G_2 takes the required form (3.7.1) if

$$F = A\left(\frac{Y}{X}\right) \qquad G = YB\left(\frac{Y}{X}\right), \tag{3.7.10}$$

where A and B are arbitrary functions of their arguments. If we take A to be the identity and B to be one over the identity, the transformation (3.7.10) is

$$x = \frac{Y}{X} \qquad y = X \tag{3.7.11}$$

which represents all classes of equations up to the transformation

$$x \longrightarrow a(x) \qquad y \longrightarrow yb(x). \tag{3.7.12}$$

Under the transformation (3.7.11) (3.7.8) becomes

$$y''' = 3y''\left(\frac{yy''-y'^2}{yy'}\right) - y\left(\frac{yy''-2y'^2}{yy'}\right)^2 f\left[\frac{xy'+y}{y'}, -\frac{y^2}{y'^2}\left(\frac{yy''-2y'^2}{yy'}\right)\right] \tag{3.7.13}$$

or, equivalently,

$$y''' = 3y''\left(\frac{yy''-y'^2}{yy'}\right) - \frac{y'^4}{y^3} f\left[\frac{xy'+y}{y'}, -\frac{y^2}{y'^2}\left(\frac{yy''-2y'^2}{yy'}\right)\right]. \tag{3.7.14}$$

The symmetry G_1 is now

$$G_1 = \frac{1}{y}\frac{\partial}{\partial x}. \tag{3.7.15}$$

3.7.2 Type IV_3

In a similar manner we find the third order equation invariant under the canonical representation of the Type IV_3 algebra in (3.7.3), *viz*

$$Y''' = Y'' f\left(X, \frac{Y''}{Y'}\right). \tag{3.7.16}$$

As G_2 is in the desired form, the only admissible transformations are those which transform it to itself. These are of the class (3.7.12). Hence up to this equivalence class the normal form of the equation is

$$y''' = y''f\left(x, \frac{y''}{y'}\right) \tag{3.7.17}$$

and

$$G_1 = \frac{\partial}{\partial y}. \tag{3.7.18}$$

While both (3.7.8) and (3.7.17) are given in [75] we present the above detail to aid the reader in applying the method to equations other than second order. (There are also errors and omissions in [75] which have only recently been rectified [144].)

3.8 The Reduced Second Order Equation: Type III_3 Lie Algebra

We first consider the reduction of the third order equation invariant under the Type III_3 two–dimensional Lie algebra, *viz* (3.7.14). The transformation which the symmetry (3.7.15) naturally suggests is

$$u = y \qquad v = x + \frac{y}{y'} \tag{3.8.1}$$

which reduces (3.7.14) to the second order equation

$$u^2v'' = f(v, uv'). \tag{3.8.2}$$

The symmetry is

$$G = u\frac{\partial}{\partial u}. \tag{3.8.3}$$

Neither equation nor symmetry is in the most suitable form. We make the further transformation

$$U = v \quad V = \log u \quad g(U, V') = -V'^3 f\left(U, \frac{1}{V'}\right) - V'^2 \tag{3.8.4}$$

Table 3.1: Canonical forms of $v'' = g(u, v')$ admitting two Lie point symmetries.

Type	$[G_1, G_2]$	Canonical forms of G_1 and G_2	Form of equation
I_2	0	$G_1 = \frac{\partial}{\partial u}$ $G_2 = \frac{\partial}{\partial v}$	$v'' = F(v')$
II_2	0	$G_1 = \frac{\partial}{\partial v}$ $G_2 = u\frac{\partial}{\partial v}$	$v'' = F(u)$
III_2	G_1	$G_1 = \frac{\partial}{\partial v}$ $G_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$	$uv'' = F(v')$
IV_2	G_1	$G_1 = \frac{\partial}{\partial v}$ $G_2 = v\frac{\partial}{\partial v}$	$v'' = v'F(u)$

and obtain

$$\frac{\mathrm{d}^2 V}{\mathrm{d}U^2} = g(U, V') \tag{3.8.5}$$

which has the symmetry

$$G = \frac{\partial}{\partial V} \tag{3.8.6}$$

so that we can make direct comparison with the four types of equations with two symmetries (cf Table 3.1 with $U \leftrightarrow u$ and $V \leftrightarrow v$.). Equations invariant under the Type I_2 algebra must have g free of U which means that (3.7.14) is autonomous and so has the extra symmetry $\partial/\partial x$ which is not lost under the reduction of order using $y\partial/\partial y$. Hence the original second order equation has one symmetry and is not within the class considered here.

For (3.8.5) to be invariant under Type II_2 it must be free of V' which means that

$$f\left(U, \frac{1}{V'}\right) = -\frac{F(U) + V'^2}{V'^3}. \tag{3.8.7}$$

The third order equation takes the form

$$y''' = 3y''\left(\frac{y''}{y'} - \frac{y'}{y}\right) + \frac{y'^3}{y^2}\left(2 - \frac{yy''}{y'^2}\right) + y'\left(2 - \frac{yy''}{y'^2}\right)^3 F\left(x + \frac{y}{y'}\right). \tag{3.8.8}$$

Eq (3.8.8) has just the two point symmetries and so the original second order equation obtained by means of the Riccati transformation

$$z = x \qquad w = \frac{y'}{y}, \tag{3.8.9}$$

viz

$$w'' = \frac{3w'^2}{w} - ww' + w\left(1 - \frac{w'}{w^2}\right)^3 F\left(z + \frac{1}{w}\right) \tag{3.8.10}$$

has no symmetry for nontrivial F.

In the case of the Type III_2 algebra the function in (3.8.5) has a very specific U dependence [142], *viz*

$$g(U, V') = \frac{1}{U} F(V') \tag{3.8.11}$$

and so

$$f\left(U, \frac{1}{V'}\right) = -\frac{1}{V'^3}\left(\frac{1}{U} F(V') + V'^2\right) \tag{3.8.12}$$

$$f(v, uv') = -(uv')^3\left(\frac{1}{v} F\left(\frac{1}{uv'}\right) + \frac{1}{(uv')^2}\right). \tag{3.8.13}$$

The third order equation is

$$y''' = 3y''\left(\frac{y''}{y'} - \frac{y'}{y}\right) + \frac{y'}{y^2}(2y'^2 - yy'') + \frac{(2y'^2 - yy'')^3}{y'^4(xy' + y)} F\left(\frac{y'^3}{y(2y'^2 - yy'')}\right) \tag{3.8.14}$$

and the original second order equation (via (3.8.9)) is

$$w'' = \frac{3w'^2}{w} - ww' + \frac{w^2 - w'}{w^4(1 + zw)} F\left(\frac{w^3}{w^2 - w'}\right). \tag{3.8.15}$$

The third order equation has just the two symmetries for general $F(V')$. There is a technical difficulty with this case in that the integration of

$$UV'' = F(V') \tag{3.8.16}$$

will give V' as an implicit function of U. In principle this is not a problem, but there are certain to be practical difficulties for an unspecified $F(V')$. The solution to this problem is addressed in part in §3.10.1 where we consider the three–dimensional and eight–dimensional symmetry cases for Type III_2 in both this and the Type IV_3 realisation.

Type IV_2 is linear and the solution readily expressed as a quadrature. From (3.8.4)

$$f\left(U, \frac{1}{V'}\right) = -\frac{F(U)}{V'^2} - \frac{1}{V'} \tag{3.8.17}$$

and the third order equation is

$$y''' = 3y''\left(\frac{y''}{y'} - \frac{y'}{y}\right) + \frac{y'^3}{y^2}\left(2 - \frac{yy''}{y'^2}\right) + \frac{y'^2}{y}\left(2 - \frac{yy''}{y'^2}\right)^2 F\left(x + \frac{y}{y'}\right). \tag{3.8.18}$$

The general dependence of F on U which contains x means that for arbitrary F there is not an additional symmetry and the original second order equation

$$w'' = \frac{3w'^2}{w} - ww' + \left(\frac{w'}{w} - w\right)^2 F\left(z + \frac{1}{w}\right) \tag{3.8.19}$$

obtained by the Riccati reduction (3.8.9) will not have a point symmetry.

3.9 The Reduced Second Order Equation: Type IV_3 Lie Algebra

We now turn our attention to the third order equation invariant under the Type IV_3 two–dimensional Lie algebra, *viz* (3.7.17). The standard transformation for the reduction of order of (3.7.17) with the symmetry (3.7.18) is to set

$$u = x \qquad v = y'. \tag{3.9.1}$$

However, the reduced equation is made a little simpler in appearance if the transformation

$$u = x \quad v = \log y' \quad g(u, v') = v'f(u, v') - v'^2 \tag{3.9.2}$$

is used for then (3.7.17) becomes

$$v'' = g(u, v'). \tag{3.9.3}$$

In the new co–ordinates G_2 becomes the symmetry

$$X_1 = \frac{\partial}{\partial v}. \tag{3.9.4}$$

It then becomes a very simple matter to read off the canonical forms of (3.9.3) under the four algebras of dimension two since the symmetry (3.9.4) appears in each one of them (cf Table 3.1).

For Type I_2 the canonical form has g independent of u and (3.7.17) has the additional symmetry

$$G_3 = \frac{\partial}{\partial x} \tag{3.9.5}$$

which has zero Lie Bracket with G_2. Hence the original second order equation has at least one symmetry and is not of the class sought. For Type II_2

$$f(u, v') = \frac{F(u)}{v'} + v' \tag{3.9.6}$$

and (3.7.17) has the form

$$y''' = y' \left\{ \left(\frac{y''}{y'}\right)^2 + F(x) \right\}. \tag{3.9.7}$$

For general $F(x)$ (3.9.7) has no additional symmetry. The Type II_2 canonical form has eight symmetries and is integrable by quadrature. Hence the solution of the original second order equation derived from (3.9.7) by reduction using (3.8.9), *viz*

$$w'' = \frac{w'^2}{w} - ww' + wF(z) \tag{3.9.8}$$

is always integrable. Any equation related to (3.9.8) by an invertible point transformation is also integrable.

For Type III_2 the actual solution of the equation requires an inversion after an integration. It is best to look at the particular instances in which this is possible. (See §3.10.2.) For now we give the third order equation, *viz*

$$y''' = y' \left\{ \frac{1}{x} F\left(\frac{y''}{y'}\right) + \left(\frac{y''}{y'}\right)^2 \right\} \tag{3.9.9}$$

and the original second order equation via (3.8.9), *viz*

$$w'' = \frac{w'^2}{w} - ww' + \frac{w}{z} F\left(\frac{w'}{w} + w\right). \tag{3.9.10}$$

In the case of a Type IV_2 equation the solution follows from a straight forward quadrature. The third order equation becomes

$$y''' = y'\left[\left(\frac{y''}{y'}\right)^2 + \frac{y''}{y'}F(x)\right] \tag{3.9.11}$$

and for general $F(x)$ this has no other point symmetries than G_1 and G_2. Reduction by the Riccati transformation gives an integrable second order equation, *viz*

$$w'' = \frac{w'^2}{w} - ww' + (w' + w^2)F(z) \tag{3.9.12}$$

which has no point symmetries and is representative of a whole class of equations equivalent to it under a point transformation.

3.10 The Reduced Second Order Equation: Type III_2 Lie Algebra

For both Type III_3 and Type IV_3 symmetries of the third order equation we have seen that suitable equations, *ie* ones without point symmetry, can lead via the increase in order with the use of the Riccati transformation and the reduction via the normal subgroup to equations which have either eight symmetries (Types II_2 and IV_2) and are readily reduced to quadratures or equations which have at least two point symmetries (Type III_2). As we expect some difficulties in the integration of the latter, it is well to look at the special cases of Type III_2 equations which have either three or eight symmetries.

The equations which have three symmetries are going to be either of Type I_2 or Type III_2 with some constraint due to the extra symmetry. As the Type I_2 equations do not lead to the case of the second order equation not possessing symmetry, we need consider only those equations which belong to Type III_2. There are only four equivalence classes of equations of Type III_2 [143] with three symmetries and we list them together with their symmetries in Table 3.2.

Table 3.2: Equivalence classes of equations of Type III_2 with an additional symmetry. (In each case $K \neq 0$.)

Equation	Symmetries	Algebra
$uv'' = v'^3 + v' + K(1+v'^2)^{3/2}$	$G_1 = \frac{\partial}{\partial v}$ $G_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$ $G_3 = 2uv\frac{\partial}{\partial u} + (v^2 - u^2)\frac{\partial}{\partial v}$	$A_{3,8}$ $(s\ell(2,R))$ $[G_1, G_2] = G_1$ $[G_1, G_3] = 2G_2$ $[G_2, G_3] = G_3$
$uv'' = Kv'^3 - \frac{1}{2}v'$	$G_1 = \frac{\partial}{\partial v}$ $G_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$ $G_3 = 2uv\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v}$	$A_{3,8}$ $(s\ell(2,R))$ $[G_1, G_2] = G_1$ $[G_1, G_3] = G_2$ $[G_2, G_3] = G_3$
$uv'' = (a-1)v' + Kv'^{\frac{2a-1}{a-1}}$ $a \neq 0, \frac{1}{2}, 1, 2$	$G_1 = \frac{\partial}{\partial v}$ $G_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$ $G_3 = u^{1-a}\frac{\partial}{\partial u}$	$A^a_{3,5}$ $[G_1, G_2] = G_1$ $[G_1, G_3] = 0$ $[G_2, G_3] = (1-a)G_3$
$uv'' = 1 + K\exp(v')$	$G_1 = \frac{\partial}{\partial v}$ $G_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$ $G_3 = \frac{\partial}{\partial u} + \log u\frac{\partial}{\partial v}$	$A_{3,2}$ $[G_1, G_2] = G_1$ $[G_1, G_3] = 0$ $[G_2, G_3] = G_1 - G_3$

3.10.1 Type III$_3$

We consider the Type III$_3$ reduction first. (Note that the variables in Table 3.2 are read the same way as those for Table 3.1 in §3.8.) We recall that the third order equation is (3.8.14), *viz*

$$y''' = 3y''\left(\frac{y''}{y'} - \frac{y'}{y}\right) + \frac{y'}{y^2}(2y'^2 - yy'') + \frac{(2y'^2 - yy'')^3}{y'^4(xy' + y)}F\left(\frac{y'^3}{y(2y'^2 - yy'')}\right), \tag{3.10.1}$$

where $F(\cdot)$ is one of the functions listed in Table 3.2. The symmetry G_1 in Table 3.2 corresponds to the symmetry G_2 (in (3.7.1)) which is to be used to reduce the third order equation to the second order equation with no point symmetries. The Lie Brackets show that G_2 in Table 3.2 is a possible symmetry of the original equation in all cases since G_1 is the normal subgroup for the pair. However, unravelling the various transformations at the third order level we find that

$$G_2 = \left(x + \frac{1}{y}\int y\mathrm{d}x\right)\frac{\partial}{\partial x} + y\log y\frac{\partial}{\partial y} \tag{3.10.2}$$

which is nonlocal and reduction via $y\partial/\partial y$ leaves it that way. In the case of the two representations of $s\ell(2,R)$ in Table 3.2 G_3 is not possible as a point symmetry of the original equation since $[G_1, G_3] = G_2$. However, in the cases of $A_{3,2}$ and $A_{3,5}^a$ it must be considered. In both cases the symmetry is nonlocal at the third order since for $A_{3,5}^a$

$$\xi = \frac{1}{y}\int y'\left(x + \frac{y}{y'}\right)^{1-a}\mathrm{d}x \qquad \eta = 0 \tag{3.10.3}$$

and for $A_{3,2}$

$$\xi = 1 - \frac{1}{y}\int\left(\eta - \frac{y}{y'}\eta'\right)\mathrm{d}x \quad \eta = y\log\left(x + \frac{y}{y'}\right) \tag{3.10.4}$$

and the nonlocal property persists with the $y\partial/\partial y$ reduction. Thus all four of the equivalence classes of Type III$_2$ with the additional symmetry do not produce a point symmetry in the original second order equation. It is a simple matter to determine the original second order equation in all four cases. We use (3.8.15) with F replaced by the expressions on the right hand side of the equations in Table 3.2 and v' replaced by the argument of F. The third order equation is obtained in a similar manner using (3.10.1).

3.10.2 Type IV_3

The symmetry G_2 of Table 3.2 becomes nonlocal at the third order level and is

$$G_2 = x\frac{\partial}{\partial x} + \left(y + \int y' \log y' \mathrm{d}x\right)\frac{\partial}{\partial y}. \tag{3.10.5}$$

The nonlocal nature persists after reduction using $y\partial/\partial y$. For $A^a_{3,5}$

$$G_3 = x^{1-a}\frac{\partial}{\partial x} + \left\{(1-a)yx^{-a} + a(1-a)\int yx^{-a-1}\mathrm{d}x\right\}\frac{\partial}{\partial y} \tag{3.10.6}$$

at the third order level and this remains nonlocal under reduction by $y\partial/\partial y$. For $A_{3,2}$

$$G_3 = \frac{\partial}{\partial x} + \left(y\log x - \int \frac{y}{x}\mathrm{d}x\right)\frac{\partial}{\partial y} \tag{3.10.7}$$

which also remains nonlocal when reduction via $y\partial/\partial y$ is performed. (Recall that $a \neq 0, 1$ and so the integral in (3.10.6) is always present.)

Concern was expressed in §3.8 as to the feasibility of Type III_2 equations because of problems of inversion. All integrals for these four classes can be inverted. The original second order equation is (3.9.10) where we replace F with the expressions on the right hand side of the equations in Table 3.2. We also replace v' in Table 3.2 with the argument of F. The third order equation is obtained in a similar manner via (3.9.9).

3.10.3 Linear equations

The final case to consider is when Type III_2 admits eight symmetries. The representative equation is [143]

$$uv'' = v'^3 + v' \tag{3.10.8}$$

with the symmetries

$$G_1 = \frac{\partial}{\partial v}$$

$$
\begin{aligned}
G_2 &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \\
G_3 &= \left(u + \frac{v^2}{u}\right)\frac{\partial}{\partial u} \\
G_4 &= \frac{1}{u}\frac{\partial}{\partial u} \\
G_5 &= \frac{v^3}{u}\frac{\partial}{\partial u} - \tfrac{1}{2}(u^2 + 3v^2)\frac{\partial}{\partial v} \\
G_6 &= \left(\frac{v^4}{4u} - \frac{u^3}{4}\right)\frac{\partial}{\partial u} - \tfrac{1}{2}\left(vu^2 + v^3\right)\frac{\partial}{\partial v} \\
G_7 &= \frac{v}{u}\frac{\partial}{\partial u} \\
G_8 &= \left(uv - \frac{v^3}{u}\right)\frac{\partial}{\partial u} + 2v^2\frac{\partial}{\partial v}.
\end{aligned} \tag{3.10.9}
$$

The Lie Brackets with G_1 are

$$
\begin{aligned}
&[G_1, G_2] = G_1, && [G_1, G_3] = 2G_7, && [G_1, G_4] = 0 \\
&[G_1, G_5] = -3(G_2 - G_3), && [G_1, G_6] = G_5, && [G_1, G_7] = G_4 \\
&[G_1, G_8] = 4G_2 - 3G_3
\end{aligned} \tag{3.10.10}
$$

and so possible candidates are just G_2 and G_4. However, we have already seen that G_2 does not lead to a point symmetry and so we need only consider G_4. For the Type III$_3$ equations we find that at the third order level

$$
G_4 = \frac{1}{y}\int\frac{y'^2\mathrm{d}x}{xy' + y}\frac{\partial}{\partial x} \tag{3.10.11}
$$

and consequently is nonlocal under the reduction via $y\dfrac{\partial}{\partial y}$. The third order equation is

$$
\begin{aligned}
y''' = {} & 3y''\left(\frac{y''}{y'} - \frac{y'}{y}\right) + \frac{y'}{y^2}(2y'^2 - yy'') \\
& + \frac{(2y'^2 - yy'')^3}{y'^4(xy' + y)}\left[\left(\frac{y'^3}{y(2y'^2 - yy'')}\right)^3 + \frac{y'^3}{y(2y'^2 - yy'')}\right]
\end{aligned} \tag{3.10.12}
$$

and the original second order equation is

$$
w'' = \frac{3w'^2}{w} - ww' + \frac{w^2 - w'}{w^4(1 + zw)}\left[\left(\frac{w^3}{w^2 - w'}\right)^3 + \frac{w^3}{w^2 - w'}\right]. \tag{3.10.13}
$$

For Type IV$_3$ equations we find that G_4 becomes

$$
G_4 = \frac{1}{x}\frac{\partial}{\partial x} + \left\{-\frac{y}{x^2} - 2\int\frac{y}{x^3}\mathrm{d}x\right\}\frac{\partial}{\partial y} \tag{3.10.14}
$$

which remains nonlocal when the reduction via $y\frac{\partial}{\partial y}$ is performed. The third order equation is

$$y''' = y'\left\{\frac{1}{x}\left[\left(\frac{y''}{y'}\right)^3 + \frac{y''}{y'}\right] + \left(\frac{y''}{y'}\right)^2\right\} \tag{3.10.15}$$

and the original second order equation is

$$w'' = \frac{w'^2}{w} - ww' + \frac{w}{z}\left\{\left(\frac{w'+w^2}{w}\right)^3 + \frac{w'+w^2}{w}\right\}. \tag{3.10.16}$$

Hence in all cases the Type III_2 equations lead to no point symmetry in the original second order equation.

3.11 Equivalence of linear third order equations

As a final endorsement of the importance of nonlocal symmetries we investigate the equivalence of linear third order ordinary differential equations. Lie [137, p 298] proved that the maximum number of point symmetries for an nth order ($n \geq 3$) ordinary differential equation is $n+4$. (In the case of $n=2$ there are eight.) He further showed that if an nth order equation was equivalent to

$$y^{(n)} = 0 \tag{3.11.1}$$

under a point transformation, *viz*

$$X = F(x,y) \qquad Y = G(x,y), \tag{3.11.2}$$

it possessed $n+4$ point symmetries. (In the case of linear equations, (3.11.2) reduces to the Kummer–Liouville [120, 138] transformation where $F = F(x)$.) Recently [118, 119, 146] there has been some interest in the algebraic properties of linear nth order equations. Krause and Michel [118, 119] proved that the maximum number of point symmetries for nth order ($n \geq 3$) equations is $n+4$ iff the equation is iterative, *ie* it can be written in the form

$$L[y] \equiv r(x)y' + q(x)y = 0, \qquad L^n[y] = L^{n-1}[L[y]]. \tag{3.11.3}$$

They also strengthened Lie's result by proving that equations possessing $n+4$ point symmetries were equivalent to (3.11.1) under (3.11.2). Mahomed and Leach [146] showed that all nth order linear equations were not equivalent to (3.11.1) under a point transformation. There exist linear equations with $n+1$ and $n+2$ point symmetries. The corresponding Lie algebras are $nA_1 \oplus_s A_1$ and $nA_1 \oplus_s A_1 \oplus A_1$. For equations of maximal symmetry the Lie algebra is $(A_1 \oplus s\ell(2,R)) \oplus_s nA_1$.

In the case of second order equations all linear or linearisable (under a point transformation) equations are equivalent to [132, p 405]

$$y'' = 0. \tag{3.11.4}$$

If the class of transformation is broadened to include contact, *viz*

$$X = F(x,y,y') \qquad Y = G(x,y,y') \qquad Y' = H(x,y,y'), \tag{3.11.5}$$

all second order ordinary differential equations are equivalent to (3.11.4) [137, p 84].

For third order equations there exist linear equations with four, five and seven (the maximum) point symmetries. We take

$$y''' + f(x)y'' + y' + f(x)y = 0, \tag{3.11.6}$$

$$y''' - y = 0 \tag{3.11.7}$$

and

$$y''' = 0 \tag{3.11.8}$$

with the symmetries

$$\begin{aligned} G_1 &= \sin x \frac{\partial}{\partial y} \\ G_2 &= \cos x \frac{\partial}{\partial y} \\ G_3 &= z(x) \frac{\partial}{\partial y} \\ G_4 &= y \frac{\partial}{\partial y}, \end{aligned} \tag{3.11.9}$$

where $z(x) = \int_0^x \exp[-\int f(u)\mathrm{d}u]\sin(x-u)\mathrm{d}u$,

$$\begin{aligned}
X_1 &= \mathrm{e}^{x}\frac{\partial}{\partial y} \\
X_2 &= \mathrm{e}^{-\omega x}\frac{\partial}{\partial y} \\
X_3 &= \mathrm{e}^{-\bar{\omega} x}\frac{\partial}{\partial y} \\
X_4 &= \frac{\partial}{\partial x} \\
X_5 &= y\frac{\partial}{\partial y},
\end{aligned} \tag{3.11.10}$$

where $\omega = (1+i\sqrt{3})/2$, and

$$\begin{aligned}
U_1 &= \frac{\partial}{\partial y} \\
U_2 &= x\frac{\partial}{\partial y} \\
U_3 &= x^2\frac{\partial}{\partial y} \\
U_4 &= \frac{\partial}{\partial x} \\
U_5 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\
U_6 &= x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} \\
U_7 &= y\frac{\partial}{\partial y},
\end{aligned} \tag{3.11.11}$$

respectively, to be the representative equations of each class. As equations of four, five and seven point symmetries (with the Lie algebras $3A_1 \oplus_s A_1$, $3A_1 \oplus_s A_1 \oplus_s A_1$ and $(A_1 \oplus s\ell(2,R)) \oplus_s 3A_1$ respectively) are equivalent to (3.11.6), (3.11.7) and (3.11.8) respectively under a point transformation, there is no loss in generality in considering only these equations individually. All results obtained will be up to a point transformation.

The natural question to contemplate is the equivalence of (3.11.6), (3.11.7) and (3.11.8) under contact transformation. It is simple (using `PROGRAM LIE` [96]) to confirm that the only contact symmetries of (3.11.6) and (3.11.7) are first extensions of the point symmetries in (3.11.9) and (3.11.10) respectively.

In the case of (3.11.8) there are three irreducible contact symmetries (in addition to the first extensions of those in (3.11.11)), *viz*

$$
\begin{aligned}
U_8 &= y'\frac{\partial}{\partial x} + \tfrac{1}{2}y'^2\frac{\partial}{\partial y} \\
U_9 &= 2(xy' - y)\frac{\partial}{\partial x} + xy'^2\frac{\partial}{\partial y} + y'^2\frac{\partial}{\partial y'} \\
U_{10} &= (x^2y' - 2xy)\frac{\partial}{\partial x} + (\tfrac{1}{2}x^2y'^2 - 2y^2)\frac{\partial}{\partial y} + (xy'^2 - 2yy')\frac{\partial}{\partial y'}.
\end{aligned}
\tag{3.11.12}
$$

In spite of the extension to contact transformation, (3.11.6), (3.11.7) and (3.11.8) are not equivalent. It remains to consider nonlocal transformations. While some method to calculate nonlocal symmetries (and thereby find nonlocal transformations) has been suggested [5, 83] the complexities of finding the full Lie algebra of nonlocal symmetries suggest that an alternate route be sought. Our approach involves the reduction of (3.11.6) and (3.11.7) to second order equations, the linearisation of these equations to (3.11.4) and finally an increase of order to (3.11.8).

In the case of (3.11.6) we reduce the order using G_1. This implies that, under the nonlocal transformation

$$u = x \qquad v = y'\sin x - y\cos x, \tag{3.11.13}$$

(3.11.6) reduces to

$$v'' + (f(u) - \cot u)v' = 0. \tag{3.11.14}$$

Eq (3.11.14) is linear and, under the point transformation

$$
\begin{aligned}
t &= -\left\{\int \sin u \exp[-\textstyle\int f(u)\mathrm{d}u]\mathrm{d}u\right\}^{-1} \\
q &= v\left\{\int \sin u \exp[-\textstyle\int f(u)\mathrm{d}u]\mathrm{d}u\right\}^{-1},
\end{aligned}
\tag{3.11.15}
$$

becomes

$$\ddot{q} = 0. \tag{3.11.16}$$

The nonlocal transformation

$$X = t \qquad Y' = q \tag{3.11.17}$$

increases the order of (3.11.16) to

$$Y''' = 0. \tag{3.11.18}$$

Eq (3.11.6) is equivalent to (3.11.8) under the nonlocal transformation

$$\begin{aligned} Y &= \int \frac{(y'\sin x - y\cos x)\sin x \exp[-f(x)\mathrm{d}x]}{\{\int\{\sin x \exp[-f(x)\mathrm{d}x]\mathrm{d}x\}\}^3}\,\mathrm{d}x \\ X &= -\left\{\int \sin x \exp[-\textstyle\int f(x)\mathrm{d}x]\mathrm{d}x\right\}^{-1}. \end{aligned} \tag{3.11.19}$$

We remark that (3.11.19) is not a nonlocal contact transformation [84] as the derivative in the integrand for Y can be removed by integration by parts. We maintain the compact structure of (3.11.19) for purely æsthetic reasons.

In the case of (3.11.7) we reduce the order using X_1, *ie* via the nonlocal transformation

$$u = x \qquad v = y' - y, \tag{3.11.20}$$

to obtain the linear second order equation

$$v'' + v' + v = 0. \tag{3.11.21}$$

Under the point transformation

$$t = \tfrac{2}{\sqrt{3}}\tan\tfrac{\sqrt{3}u}{2} \qquad q = v\exp(\tfrac{1}{2}u)\sec\tfrac{\sqrt{3}u}{2} \tag{3.11.22}$$

(3.11.21) is transformed to (3.11.16). Invoking (3.11.17) we obtain (3.11.18). Eq (3.11.7) is equivalent to (3.11.8) under the nonlocal transformation

$$X = \tfrac{2}{\sqrt{3}}\tan\tfrac{\sqrt{3}x}{2} \qquad Y = \int (y' - y)\exp\left(\tfrac{1}{2}x\right)\sec^3\left(\tfrac{\sqrt{3}x}{2}\right)\mathrm{d}x. \tag{3.11.23}$$

From the above it can be seen that all linear third order equations are equivalent to the simplest third order equation (3.11.8) under nonlocal transformations.

Chapter 4

Painlevé Analysis

We depart, for the moment, from the Lie approach to consider the Painlevé analysis. As the Lie analysis is an algebraic process and the approach of Painlevé an analytic one, the results obtained from the former should encompass those obtained from the latter. It is this interplay that we wish to exploit and use as a further testament to the importance of considering a more general class of symmetries than solely point.

4.1 Introduction

After the pioneering work of Cauchy on complex variable theory, it was natural for the analysis of differential equations to be extended into the complex domain. Once local solutions (in the complex plane of the independent variable) were established more global results could be obtained by analytic continuation [105]. It is important to note that, even if a differential equation has real coefficients and is defined on R, it is necessary to extend the solution to the complex plane to obtain a global picture of its behaviour. Consider, for example, the equations

$$w' - w = 0 \tag{4.1.1}$$

and

$$w'^2 - 4w^3 + c_1 w + c_2 = 0, \tag{4.1.2}$$

where c_1 and c_2 are complex constants. The full periodic behaviour of the solutions of (4.1.1) and (4.1.2) is only observable in the complex plane [47].

It was Paul Painlevé [160, 161] who sought to classify all ordinary differential equations the solutions of which were free of movable critical points in an attempt to determine those equations which could be solved analytically. Note that by critical points we mean branch points and essential singularities. Critical points which depend upon parameters that appear in the equations are termed fixed, *eg* in the solution of

$$(z-c)w' = \lambda w, \tag{4.1.3}$$

viz

$$w = K(z-c)^{\lambda}, \tag{4.1.4}$$

for $\lambda \in \mathbb{Z} < 0$, $z = c$ is a fixed pole and for λ rational it is an algebraic branch point. When the critical points depend on the initial conditions, they are termed movable, *eg*

$$w' - \lambda w^{1-1/\lambda} = 0 \tag{4.1.5}$$

has the solution

$$w = (z-z_0)^{\lambda}, \tag{4.1.6}$$

where z_0 is an arbitrary constant obviously determined from the initial conditions. Depending on the value of λ, $z = z_0$ is an analytic point, movable pole or movable branch point.

As a result of Painlevé's initial work, equations whose solutions are free of movable critical points are said to possess the *Painlevé property.* It was found that, of all first order equations, the only one possessing this property is the Riccati equation [168, 47]. Painlevé started the classification of second order equations of the form

$$w'' = f(w', w, z), \tag{4.1.7}$$

where f is rational in w', algebraic in w and analytic in z. For the case of f being rational in w, the problem was completed by Gambier [71] with refinements by Ince [105] and Bureau [34]. Painlevé also considered the case of

f being rational in w to some depth [161]. Note that the class of equations (4.1.7) are of the first degree in the highest derivative. The classification problem has recently been extended to second order equations of arbitrary degree [35, 37, 52, 51], but the simplifications imposed in these works have resulted in the full classification problem remaining incomplete. For third order equations some progress has been made [39, 74, 40, 33, 36], but the work is ongoing.

For the class of equations (4.1.7) it was found that only fifty (Bureau [34] and Cosgrove [51] argue that more exist.) possessed the Painlevé property, most of which could be solved in terms of the (then known) elementary and semi–transcendental functions [105]. However, for the six equations

$$w'' = 6w^2 + z \tag{4.1.8}$$

$$w'' = 2w^3 + zw + a \tag{4.1.9}$$

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{aw^2 + b}{z} + cw^3 + \frac{d}{w} \tag{4.1.10}$$

$$w'' = \frac{1}{2w}w'^2 + \tfrac{3}{2}w^3 + 4zw^2 + 2\left(z^2 - a\right)w + \frac{b}{w} \tag{4.1.11}$$

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)w'^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(aw + \frac{b}{w}\right) + \frac{cw}{z} + \frac{dw(w+1)}{w-1} \tag{4.1.12}$$

$$\begin{aligned} w'' = {} & \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)w'^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w' \\ & + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(a + \frac{bz}{w^2} + \frac{c(z-1)}{(w-1)^2} + \frac{dz(z-1)}{(w-z)^2}\right) \end{aligned} \tag{4.1.13}$$

new functions, called the Painlevé transcendants, had to be introduced. The first three equations were discovered by Painlevé [160], the next two by Gambier [71] and the final one by Fuchs [70] [105, pp 344–345]. The literature on these and related equations is vast. Fortunately a convenient bibliography has been compiled by Peter Clarkson [42].

4.2 The ARS Algorithm

Post 1980 interest in the Painlevé analysis was primarily due to the relationship observed by Ablowitz *et al* [1, 2, 3] between partial differential equations and their reduced ordinary differential equations. They conjectured that all ordinary differential equations obtained by reductions of partial differential equations which were solvable by inverse scattering possessed the Painlevé property. Considering the wealth of results available, the conjecture has been extended to stating that equations possessing the Painlevé property are integrable [49]. Note that the conjecture is assumed to be sufficient, but is, by no means, necessary. (See [168] for a review of results using the ARS algorithm.)

The attraction of the work of Ablowitz *et al* [1, 2, 3] is that they provided a convenient algorithm (called the ARS algorithm) to test whether an equation possessed the Painlevé property. The method, that of 'pole–like' expansions, is in fact due to Kowalevski [115, 116]. She first applied the method to the equations of motion of the spinning top and thereby found another integrable case (in addition to those of Euler and Lagrange). We carry out our analysis using this method with the caution that the price for its algorithmic simplicity is that it contains some pitfalls. (We mention these as they are encountered.)

The method essentially involves assuming a Laurent series expansion for the dependent variable about some point $z - z_0$. The procedure has three accepted parts. The first is the determination of the leading order behaviour by the substitution of

$$w = \alpha\chi^p, \tag{4.2.1}$$

where α and p are constants to be determined and

$$\chi = z - z_0, \tag{4.2.2}$$

with z_0 being the location of the movable pole, into the equation of interest. Note that, for the purposes of the algorithm, we insist on $p \in \mathbb{Z} < 0$. If not, we invoke the homographic transformation

$$(u,x) \longrightarrow (U,X) : u = \frac{\alpha(x)U+\beta(x)}{\gamma(x)U+\delta(x)}, \quad X=\xi(x) \qquad \alpha\delta-\beta\gamma \neq 0 \quad (4.2.3)$$

to force $p \in \mathbb{Z} < 0$. (We use (4.2.3) as it is the only transformation that preserves the Painlevé property[71, 47]. As result the first step in analysing an equation is to attempt to simplify it using a transformation of the form (4.2.3).) In the determination of leading order behaviour the term involving the highest derivative *must* be taken to be one of the dominant terms. This term is then balanced with other terms in the equation to obtain p and α. More than one family of solutions for p and α is possible. All have to be considered.

Thereafter the expression

$$w = \alpha\chi^p + \beta\chi^{i+p} \qquad (4.2.4)$$

is substituted into the dominant terms of the equation to determine the indices i (at which the remaining constants of integration arise) by requiring that the coefficient of terms linear in β is zero. (The terms resonances [168] and Kowalevski exponents [190, 191], which are equivalent to indices, can also be found in the literature.) In addition to -1, which must occur, it is usual for the remaining indices to be positive integers. However, it has recently been shown that some information can be obtained from fractional [168] and negative indices [48].

Finally, the truncated Laurent expansion

$$w = \alpha\chi^p + \delta_1\chi^{p+1} + \cdots + \delta_{p+i}\chi^{p+i} \qquad (4.2.5)$$

is substituted into the original equation. This serves to verify that arbitrary constants arise at the indices. A summation of the Laurent expansion (which includes all arbitrary constants), when possible, produces the general solution of the equation. In the case that less than the required number of constants arise, logarithmic terms need be inserted at those indices where the constants did not arise. The introduction of these logarithmic terms is normally an indication of non–integrability [168]. (In the case of families of solutions for p and α every expansion of the form (4.2.5) must be considered. If there are any inconsistencies in any one of the expansions the equation does not possess the Painlevé property [48].)

We illustrate the procedure with the well–known Emden–Fowler equation of index two

$$w'' = w^2. \tag{4.2.6}$$

A substitution of (4.2.1) into (4.2.6) gives

$$\alpha p(p-1)\chi^{p-2} = \alpha^2\chi^{2p} \tag{4.2.7}$$

from which it is evident that

$$p = -2 \qquad\qquad \alpha = 6 \tag{4.2.8}$$

since both terms have to be dominant. We determine the indices by substituting (4.2.4), using (4.2.8), into (4.2.6) and equating the coefficients of terms linear in β (These terms will involve equal powers of χ.), *viz*

$$(i-2)(i-3)\chi^{i-4} = 12\chi^{i-4}. \tag{4.2.9}$$

From (4.2.9) the indices are determined as

$$i = -1, 6. \tag{4.2.10}$$

It now remains to substitute the truncated Laurent expansion

$$w = 6\chi^{-2} + a_{-1}\chi^{-1} + a_0 + a_1\chi + a_2\chi^2 + a_3\chi^3 + a_4\chi^4 \tag{4.2.11}$$

into (4.2.6) to check whether any incompatibilities arise at the indices. We accomplish this by equating the coefficients of like powers of χ and solving for the a_j ($j = -2, \ldots, i$). The results are summarised in Table 4.1. It is unsurprising that the terms between the pole and index are zero as both terms in the equation are dominant. An extension of the Laurent series reveals that

Table 4.1: Calculation of terms in the Laurent expansion for the solution of $w'' = w^2$

χ^{-4}	χ^{-3}	χ^{-2}	χ^{-1}	χ^{0}	χ^{1}	χ^{2}
$0 \equiv 0$	$a_{-1} = 0$	$a_0 = 0$	$a_1 = 0$	$a_2 = 0$	$a_3 = 0$	a_4–arbitrary

the solution to (4.2.6) is, in fact, the Weierstrass $\mathcal{P}$ function. This, of course, could have been obtained by integrating (4.2.6).

Remarks: i) The first step in the procedure is the requirement of negative p for the order of the pole and suggests an immediate transformation of the equation under consideration when p is positive. While the resulting equation may possess the Painlevé property, it must be noted that it could well fall outside the class of equations listed by Gambier [71] and Ince [105]. The reason is simple: these lists of equations do *not* insist on negative p! An example is the Ermakov–Pinney [58, 166] equation

$$y'' = y^{-3}. \tag{4.2.12}$$

Eq (4.2.12) possesses the full Painlevé property, but does not occur in [71, 105] in its present form. A simple transformation of the form (4.2.3) results in an equation that *does* arise in [71, 105] with p now being positive.

ii) In the case of systems with arbitrary coefficients (cf the Lotka–Volterra and Quadratic Systems [100]) the determination of the indices can lead to an obscuring of the results. We note that the determination of these indices is merely a convenient mechanism to aid the computations. In principle one could substitute a Laurent expansion from $-\infty$ to ∞ for the dependent variable(s) into the equation being studied and determine poles, indices and series expansions for the solution. However, the algorithm provides us with a feel for what 'infinity' can be taken to be.

iii) The ARS algorithm cannot contend with negative (apart from -1) and fractional indices. It was only recently that negative indices have been further analysed and their presence used to advantage in the so–called perturbative Painlevé approach [48]. The presence of fractional indices can also be linked to integrability in what has been termed the 'weak' Painlevé property [168]. (This excludes the case in which the fractional indices can be transformed away by means of a homographic transformation (4.2.3) [168].) However, there is only partial evidence to support this [168]. It suffices to mention that the occurrence of fractional and negative indices should not be a cause for dismay as some information can still be obtained from them.

iv) In the case that irrational or complex indices are present, this can point to non–integrability. In the case of homogeneous systems Yoshida [190, 191] has shown that, for the system to be integrable, all the Kowalevski exponents (KE) must be rational numbers. If at least one KE is irrational or imaginary, then the system does not have algebraic integrals. [168].

v) It is inevitable, when dealing with systems, that extra care need be taken in any analysis. This is also true of the Painlevé analysis. (See [121] for an in-depth analysis of coupled nonlinear oscillator systems.) A particularly delicate aspect of the ARS algorithm is the determination of leading order behaviour which necessitates considerable attention.

Thus due caution must be observed in the implementation of the algorithm and the analysis of the results. It is little wonder that Painlevé did not see the need for 'le procédé connu de Madame Kowaleski' [161].

4.3 The Partial Painlevé Property

In their analysis of the gravitational field of the Mixmaster Universe [149] Cotsakis and Leach [53] observed that a Laurent expansion for the system of equations describing the model possessed fewer constants that arose at the indices than was expected. They conjectured that such systems can be interpreted as being integrable on an p–dimensional submanifold in the q–dimensional phase space, where p is the number of constants found and q the degrees of freedom of the system ($p < q$). This property was called the partial Painlevé property. (We emphasise that this does not relate to the nature of the critical points in the system as the Painlevé property does. It is a (conjectured) measure of integrability). While it is normal to introduce logarithmic terms into the Laurent expansion at this stage, the attraction of the partial Painlevé property is that it suggests that partial solutions can still be found.

We have observed [89] a similar property in another physical system. As the nature of the occurrence is not quite that of [53] we dub this the pseudo partial Painlevé property. The difference is that an arbitrary constant that arises at one index becomes fixed when the next arbitrary constant appears in the Laurent expansion. (See §5.3 for more details.)

4.4 Partial Differential Equations

Considering the success of the Painlevé analysis applied to ordinary differential equations via the ARS approach, it was natural to speculate on its applicability to partial differential equations. After all the ARS algorithm was originally applied to ordinary differential equations which were obtained by the reduction of partial differential equations. In cases where this reduction was 'too trivial' to obtain any information the ARS approach failed [168]. This encouraged Weiss *et al* [183] to extend the algorithm to partial differential equations using the concept of a singular manifold. This concept has also been used to develop an invariant version of the analysis applicable to both ordinary and partial differential equations [44, 45]. In fact the singular manifold concept has been used to find Lax pairs and (auto–)Bäcklund transformations of differential equations [182]. (See [54] for an unusually clear account of the application of the singular manifold method to the analysis of the inhomogeneous spherically symmetric Heisenberg ferromagnet.)

Considering the usefulness of the classification performed by Painlevé and co–workers for ordinary differential equations one would expect some benefit from a classification of partial differential equations. Work on this problem has been started. In particular Hlavaty [98, 99] and Cosgrove [49, 50] have completed preliminary classifications. The length of these contributions and the partial results obtained suggests that the complete classification is far from close to being realised.

4.5 The WTC Algorithm

Weiss *et al* [183] assert that, for a partial differential equation to possess the Painlevé property, its solutions must be single–valued about a movable singularity manifold [168]. If we consider a $(1+1)$–dimensional partial differential equation, the singularity manifold is given by

$$\phi(x,t) = 0 \tag{4.5.1}$$

and the solution of the equation

$$X(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt},\ldots) = 0 \tag{4.5.2}$$

is assumed to be an expansion of the form

$$u(x,t) = \phi(x,t)^p \sum_{j=0}^{\infty} u_j(x,t)\phi^j(x,t). \tag{4.5.3}$$

The determination of the poles is along the same manner as the ARS method for ordinary differential equations. However, the indices are calculated by determining a recursion relationship for u_n in (4.5.3). The verification that no incompatibilities arise at the indices is of much greater complexity than for ordinary differential equations. We illustrate the procedure with Burger's equation

$$u_t + uu_x + u_{xx} = 0. \tag{4.5.4}$$

The leading singularity behaviour is determined by substituting

$$u = u_0\phi^p \tag{4.5.5}$$

into (4.5.4) to obtain

$$u_{0t}\phi^p + u_0 p\phi^{p-1}\phi_t + u_0\phi^p(u_{0x}\phi^p + u_0 p\phi^{p-1}\phi_x) + u_{0xx}\phi^p + 2u_{0x}p\phi^{p-1}\phi_x$$

$$+ u_0 p(p-1)\phi^{p-2}\phi_x^2 + u_0 p\phi^{p-1}\phi_{xx} = 0, \tag{4.5.6}$$

ie

$$u_{0t}\phi^p + pu_0\phi^{p-1}\phi_t + u_0 u_{0x}\phi^{2p} + pu_0^2\phi^{2p-1}\phi_x + u_{0xx}\phi^p + 2pu_{0x}\phi^{p-1}\phi_x$$

$$+ p(p-1)u_0\phi^{p-2}\phi_x^2 + pu_0\phi^{p-1}\phi_{xx} = 0 \tag{4.5.7}$$

from which it is evident that the last two terms in (4.5.4) are dominant and

$$p = -1 \qquad u_0 = 2\phi_x. \tag{4.5.8}$$

The indices are obtained by substituting

$$u = \phi^{-1}\sum_{j=0}^{n} u_j\phi^j \tag{4.5.9}$$

into (4.5.4), *viz*

$$-\phi^{-2}\phi_t\sum_{j=0}^{n} u_j\phi^j + \phi^{-1}\sum_{j=0}^{n}\left(u_{jt}\phi^j + ju_j\phi^{j-1}\phi_t\right)$$

$$+ \phi^{-1}\sum_{j=0}^{n} u_j\phi^j\left[-\phi^{-2}\phi_x\sum_{j=0}^{n} u_j\phi^j + \phi^{-1}\sum_{j=0}^{n}\left(u_{jx}\phi^j + ju_j\phi^{j-1}\phi_x\right)\right]$$

$$+ 2\phi^{-3}\phi_x^2\sum_{j=0}^{n} u_j\phi^j - \phi^{-2}\phi_{xx}\sum_{j=0}^{n} u_j\phi^j - 2\phi^{-2}\phi_x\sum_{j=0}^{n}\left(u_{jx}\phi^j + ju_j\phi^{j-1}\phi_x\right)$$

$$+ \phi^{-1}\sum_{j=0}^{n}\left(u_{jxx}\phi^j + 2ju_{jx}\phi^{j-1}\phi_x + j(j-1)u_j\phi^{j-2}\phi_x^2 + ju_j\phi^{j-1}\phi_{xx}\right) = 0 \tag{4.5.10}$$

and establishing the recurrence relationship

$$\phi_x^2(2-j)(1+j) = F(u_0, \ldots, \phi_t, \phi_x, \ldots) \tag{4.5.11}$$

from which the indices are -1 and 2. To verify the compatibility condition at index 2 we substitute

$$u = \phi^{-1}\sum_{j=0}^{2} u_j\phi^j \tag{4.5.12}$$

into (4.5.4). For $j = 1$ we have

$$\phi_t + u_1\phi_x^2 + \phi_{xx} = 0 \tag{4.5.13}$$

which defines u_1. For $j = 2$ we have

$$0 = \frac{\partial}{\partial x}\left(\phi_t + u_1\phi_x^2 + \phi_{xx}\right) \tag{4.5.14}$$

which, given (4.5.13), is identically satisfied. This implies u_2 is arbitrary and (4.5.4) possesses the Painlevé property.

The simple example of Burger's equation is sufficient to discourage most practitioners from applying the Painlevé analysis to partial differential equations. Fortunately Kruskal has proposed an improvement [107]. Noting that (4.5.1) can be solved for one of the variables, x say, he suggested replacing $\phi(x,t)$ by $x + \psi(t)$. The analysis is greatly simplified and proceeds along the lines of the ARS algorithm. To take the example of Burger's equation (4.5.4) again we substitute

$$u = u_0(x + \psi(t))^p, \tag{4.5.15}$$

where $u_0 = u_0(t)$, into (4.5.4) and obtain

$$u_0p(x+\psi)^{p-1}\psi_t + u_{0t}(x+\psi)^p + u_0^2p(x+\psi)^{2p-1} + u_0p(p-1)(x+\psi)^{p-2}. \tag{4.5.16}$$

The last two terms are dominant and give

$$p = -1 \qquad u_0 = 2. \tag{4.5.17}$$

The indices are calculated by the substitution of

$$u = 2(x+\psi)^{-1} + \beta(x+\psi)^{i-1} \tag{4.5.18}$$

into the last two terms of (4.5.4). (We could substitute an expression of the form (4.5.9) and establish another recurrence relationship, but that is not necessary is this case.) Collection of terms linear in β and solution for i gives $i = -1, 2$. To verify that no incompatibilities arise we substitute

$$u = (x+\psi)^{-1}\sum_{j=0}^{2} u_j(x+\psi)^j \tag{4.5.19}$$

into (4.5.4) and obtain, for $j = 1$,

$$u_1 = -\psi_t \tag{4.5.20}$$

and, for $j = 2$,

$$0 = \frac{\partial}{\partial x}(\psi_t + u_1) \tag{4.5.21}$$

which, given (4.5.20), is identically satisfied. The usefulness of the reduced *ansatz* is emphasised in the example of the Boussinesq equation [183].

Thus Kruskal's refinement greatly reduces the complexity while preserving the salient features of the original WTC algorithm. It must be noted, however, that this requirement is only adequate for the purposes of the Painlevé test. We cannot use this form of the singular manifold to search for solutions of the equation as the simplifying assumption is too restrictive [44, 47]. It is best to resort to the invariant Painlevé analysis of Conte [44, 45] to obtain the most information about a particular system. The interested reader is referred to [47] for a clear introduction to the method.

Chapter 5

Applications

Having presented the necessary tools for the analysis of differential equations we apply these techniques to equations that arise in the modelling of physical phenomena [81, 89]. We restrict our analysis to two (related) equations. Space prevents us from reporting our results on the pivotal equation in shear–free motion of spherically symmetric charged perfect fluid distributions in general relativity and an equation arising in population dynamics.

5.1 Integrability analysis of a conformal equation arising in relativity

Here we analyse a nonlinear third order differential equation and a nonlinear second order differential equation that arise in general relativity. These equations arise in the study of spherically symmetric gravitational fields that possess a conformal symmetry in the t–r plane [56]. The third order equation possesses the pseudo partial Painlevé property. We can integrate it once to obtain a second order first integral. For a particular value of the first integral this equation has the Painlevé property, has two symmetries and can be reduced to quadratures. It is remarkable that an equation that possesses the pseudo partial Painlevé property can be integrated to one that possesses the

Painlevé property. This demonstrates that the pseudo partial Painlevé property is significant in the solution of differential equations that arise in practical applications. The second order equation is analysed using both the Lie and Painlevé methods. We show how this gives further evidence of the close relationship between these two methods of solving differential equations. We finally note that the problem of solving the two field equations of Dyer *et al* [56] is essentially that of solving a single Emden–Fowler equation of index two.

5.2 The equations of Dyer, McVittie and Oattes

Spherically symmetric gravitational fields are important in relativistic astrophysics and cosmology [117, Ch 14]. Such gravitational fields with vanishing shear and admitting a conformal Killing vector in the t–r plane have been investigated by Dyer *et al* [56]. They generated the third order field equation

$$\mu^2 T_{\mu\mu\mu} + \mu(2m-1)T_{\mu\mu} + (m^2 - 2m + 2T)T_\mu = 0, \tag{5.2.1}$$

where T is related to the gravitational potential, $\mu = r/t$ is a self-similarity variable and m is a constant. (For more information on conformal symmetries and their relationship to mathematical physics the reader is referred to [41].) Solutions to (5.2.1) are important because they help to generate solutions to the Einstein field equations. Dyer *et al* [56] did not present any solutions to (5.2.1). It was only recently that Maharaj *et al* [140] found solutions to (5.2.1) in the form of Weierstrass elliptic functions. Their solution has the form

$$y = a^2 C_1^2 \mathrm{e}^{-2ax} \mathcal{P}(C_1 \mathrm{e}^{-ax} + C_2, 0, -1), \tag{5.2.2}$$

where

$$\begin{aligned} T(\mu) &= \gamma y(x) + T_0, & x &= \ln(\mu)/\beta, \\ 2(m-2)\beta &= 5a, & \bar{k}\beta^2 &= 6a^2, \\ \beta^2\gamma &= -6, & \bar{k} &= \pm[(m-1)^2(m-3)^2 + 4k]^{1/2}, \end{aligned} \tag{5.2.3}$$

$\mathcal{P}$ is the Weierstrass elliptic function, k the value of a first integral of (5.2.1) and T_0, C_1, C_2 are arbitrary constants. Note that the assumption implicit in (5.2.2) is that $m \neq 2$. For the case $m = 2$ a solution can be found by simple integration. This is given by Havas [94]

$$y = \frac{6\lambda^2}{-\beta^2\gamma}\mathcal{P}\left(\lambda x + \lambda\left(-\frac{6}{\beta^2\gamma}\right)^{1/2}, \lambda^{-4}\left(\frac{\beta^2\gamma}{6}\right)^2\frac{3\bar{k}^2}{\gamma^2}, \lambda^{-6}\left(-\frac{\beta^2\gamma}{6}\right)^3 b\right) - \frac{\bar{k}}{2\gamma}, \tag{5.2.4}$$

where he used the homogeneity property of the Weierstrass elliptic function and its evenness in x, (5.2.3), $b = (\bar{k}/\gamma)^3 + 12s/(\beta^2\gamma)$ and s a constant.

It is interesting to note that other well-known solutions of the Einstein field equations may be related to (5.2.1) as was observed by Havas. For example, we regain some of the Wyman solutions [187, 188] for particular values of m and the first integral of (5.2.1). These are given by

$$m = -3 \qquad T = -24\mu^4\mathcal{P}(\mu^2 + a, 0, b) \tag{5.2.5a}$$

$$m = 7 \qquad T = -24\mu^{-4}\mathcal{P}(\mu^{-2} + a, 0, b) \tag{5.2.5b}$$

$$m = \frac{9}{7} \qquad T = -\frac{24}{49}\left[\mu^{4/7}\mathcal{P}(\mu^{2/7} + a, 0, b) + 1\right] \tag{5.2.5c}$$

$$m = \frac{19}{7} \qquad T = -\frac{24}{49}\left[\mu^{-4/7}\mathcal{P}(\mu^{-2/7} + a, 0, b) + 1\right]. \tag{5.2.5d}$$

We have independently verified that the functions (5.2.5) are solutions of the field equation (5.2.1).

5.3 Painlevé Analysis

Using the transformation

$$\mu = \mathrm{e}^{x/(2m-4)} \qquad T(\mu) = 2(m-2)^2 y(x) - \tfrac{1}{2}(m^2 - 4m + 3) \tag{5.3.1}$$

we rewrite (5.2.1) in the autonomous form

$$y''' + y'' + yy' = 0. \tag{5.3.2}$$

We proceed with the Painlevé analysis as explained in Chapter 4. Firstly, setting

$$y = \alpha\chi^p, \tag{5.3.3}$$

where

$$\chi = x - x_0, \tag{5.3.4}$$

we find that the singularity is a pole of second order and $\alpha = -12$. Now setting

$$y = -12\chi^{-2} + \beta\chi^{i-2} \tag{5.3.5}$$

we obtain the indices $i = -1$, 4 and 6. To verify that (5.3.2) passes the Painlevé test we substitute the truncated Laurent expansion

$$y = -12\chi^{-2} + a_{-1}\chi^{-1} + a_0 + a_1\chi + a_2\chi^2 + a_3\chi^3 + a_4\chi^4 \tag{5.3.6}$$

into (5.3.2) and solve for the $a_j\,(j = -1, \ldots, 4)$. This procedure results in

$$\begin{aligned}
a_{-1} &= \frac{1}{125}\\
a_0 &= \frac{1}{25}\\
a_1 &= \frac{12}{5}\\
a_2 &= -\frac{1}{12500}\\
a_3 &= -\frac{1}{187500}\\
a_4 &\text{ is arbitrary.}
\end{aligned} \tag{5.3.7}$$

As there are only two arbitrary constants as opposed to the required three corresponding to the three degrees of freedom for (5.3.2), this suggests that (5.3.2) possesses the partial Painlevé property of Cotsakis and Leach [53]. However, if we truncate the expansion at the first index, *viz* make the substitution

$$y = -12\chi^{-2} + \cdots + a_2\chi^2 \tag{5.3.8}$$

into (5.3.2), we obtain

$$a_{-1} = \frac{1}{125}$$

$$\begin{aligned} a_0 &= \frac{1}{25} \\ a_1 &= \frac{12}{5} \\ a_2 &\text{ is arbitrary.} \end{aligned} \tag{5.3.9}$$

This indicates that the constant at the first index is initially arbitrary, but is restricted to a particular value when the arbitrary constant at the second index is introduced. This system is said to possess the pseudo partial Painlevé property. The partial solution to (5.3.2) can be written as

$$\begin{aligned} y = \frac{-12}{(x-x_0)^2} + \frac{1}{125}\frac{1}{(x-x_0)} + \frac{1}{25} + \frac{12}{5}(x-x_0) - \frac{1}{12500}(x-x_0)^2 \\ - \frac{1}{187500}(x-x_0)^3 + a_4(x-x_0)^4 + \cdots \end{aligned} \tag{5.3.10}$$

and we say that (5.3.2) is integrable on a surface in three–dimensional parameter space.

Our analysis would normally continue by introducing logarithmic terms into the Laurent expansion (5.3.10). However, we note that (5.3.2) can be easily integrated to obtain the first integral

$$y'' + y' + \tfrac{1}{2}y^2 = K, \tag{5.3.11}$$

where K is a constant of integration and thus a parameter. In general, (5.3.11) does not possess the Painlevé property (or any degree thereof). Reduction via the only symmetry [96], $\partial/\partial x$, results in an Abel's equation of the second kind the solution of which, unsurprisingly, is not obvious. However, for $K = 18/625$ (5.3.11) *does* possess the Painlevé property and has the solution

$$\begin{aligned} y = \frac{-12}{(x-x_0)^2} + \frac{12}{5}\frac{1}{(x-x_0)} + \frac{1}{25} + \frac{1}{125}(x-x_0) - \frac{1}{12500}(x-x_0)^2 \\ - \frac{1}{187500}(x-x_0)^3 + a_4(x-x_0)^4 + \frac{1-7500000}{9375000}a_4(x-x_0)^5 + \cdots. \end{aligned} \tag{5.3.12}$$

To make the solution of (5.3.11) more transparent we use the transformation

$$Y = \mathrm{e}^{x/5}\left(\frac{y}{5} - \frac{2}{25}\right) \qquad X = \mathrm{e}^{-x/5}. \tag{5.3.13}$$

This transformation is suggested by the fact that, when $K = 18/625$, (5.3.11) has the two symmetries

$$G_1 = \frac{\partial}{\partial x} \tag{5.3.14a}$$

$$G_2 = e^{x/5}\frac{\partial}{\partial x} + e^{x/5}\left(-\frac{2}{5}y + \frac{12}{125}\right)\frac{\partial}{\partial y}, \tag{5.3.14b}$$

where we call (5.3.14b) a 'conditional' symmetry in the spirit of configurational invariants [175]. We can now rewrite (5.3.11) as

$$Y'' + 25Y^2 = 0 \tag{5.3.15}$$

which has the symmetries

$$G_1 = \frac{\partial}{\partial X} \tag{5.3.16a}$$

$$G_2 = X\frac{\partial}{\partial X} - 2Y\frac{\partial}{\partial Y}. \tag{5.3.16b}$$

Since the transformation (5.3.13) is homographic, (5.3.15) still has the Painlevé property. We write its solution as

$$\begin{aligned} Y(\chi) &= -\frac{6}{25}\chi^{-2} + a_4\chi^4 - \frac{25}{78}a_4^2\chi^{10} + \frac{625}{8892}a_4^3\chi^{16} - \frac{3125}{231192}a_4^4\chi^{22} + \cdots \\ &= -\frac{6}{25}\left(\chi^{-2} + b_4\chi^4 - \frac{b_4^2}{13}\chi^{10} + \frac{b_4^3}{247}\chi^{16} - \frac{3b_4^4}{16055}\chi^{22} + \cdots\right) \\ &= -\frac{6}{25}\mathcal{P}(\chi), \end{aligned} \tag{5.3.17}$$

where $\chi = X - X_0$ and $\mathcal{P}(\chi)$ is the Weierstrass $\mathcal{P}$ function with $c_2 = 0$ and $c_3 = b_4 = (25/6)a_4$ [16, Ch 18]. The form of $Y(\chi)$ is not surprising as (5.3.15) is essentially the defining differential equation for the Weierstrass $\mathcal{P}$ function. Note that (5.3.17) is the solution to (5.3.11) (and hence (5.3.2)) for a particular value of the first integral K, but for all values of m, in contrast to Wyman's solutions (5.2.5) which hold only for particular values of m.

5.4 The solution via point transformations

We can transform (5.3.11) into the standard form of the Emden–Fowler equation [122, 57, 66, 67, 68, 69]. Setting

$$y = 2z(x) + b, \tag{5.4.1}$$

where $K = \frac{1}{2}b^2$, we write (5.3.11) as

$$z'' + z' + bz + z^2 = 0. \tag{5.4.2}$$

We remove the z' and z terms in (5.3.11) using the well-known Kummer–Liouville transformation [120, 138]

$$z(x) = u(x)v(t) \qquad t = t(x) \tag{5.4.3}$$

where, in our case,

$$\begin{aligned} u(x) &= \mathrm{e}^{-(1+\alpha)x/2} \\ t(x) &= \frac{1}{|\alpha|}\mathrm{e}^{\alpha x} \\ \alpha &= \sqrt{1-4b}. \end{aligned} \tag{5.4.4}$$

Eq (5.4.2) then becomes

$$\ddot{v} + (|\alpha|t)^{-(1+5\alpha)/(2\alpha)}v^2 = 0. \tag{5.4.5}$$

For our particular value of K, 18/625, α has the four values $\pm 1/5, \pm 7/5$. This gives the set of equations

$$\ddot{v} + v^2 = 0 \qquad \alpha = -\frac{1}{5} \tag{5.4.6a}$$

$$\ddot{v} + \left(\frac{t}{5}\right)^{-5} v^2 = 0 \qquad \alpha = \frac{1}{5} \tag{5.4.6b}$$

$$\ddot{v} + \left(\frac{t}{5}\right)^{-15/7} v^2 = 0 \qquad \alpha = -\frac{7}{5} \tag{5.4.6c}$$

$$\ddot{v} + \left(\frac{t}{5}\right)^{-20/7} v^2 = 0 \qquad \alpha = \frac{7}{5}. \tag{5.4.6d}$$

It is easy to eliminate the constant coefficients of v^2 and consider the transformed set of equations

$$\ddot{v} + v^2 = 0 \tag{5.4.7a}$$

$$\ddot{v} + t^{-5}v^2 = 0 \tag{5.4.7b}$$

$$\ddot{v} + t^{-15/7}v^2 = 0 \tag{5.4.7c}$$

$$\ddot{v} + t^{-20/7}v^2 = 0 \tag{5.4.7d}$$

with the corresponding symmetries

$$G_1 = \frac{\partial}{\partial t} \qquad G_2 = t\frac{\partial}{\partial t} - 2v\frac{\partial}{\partial v} \tag{5.4.8a}$$

$$X_1 = t^2\frac{\partial}{\partial t} + vt\frac{\partial}{\partial v} \qquad X_2 = -t\frac{\partial}{\partial t} - 3v\frac{\partial}{\partial v} \tag{5.4.8b}$$

$$Y_1 = 343t^{6/7}\frac{\partial}{\partial t} + (147t^{-1/7}v - 12)\frac{\partial}{\partial v} \qquad Y_2 = 7t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v} \tag{5.4.8c}$$

$$Z_1 = 343t^{8/7}\frac{\partial}{\partial t} + (196t^{1/7}v + 125)\frac{\partial}{\partial v} \qquad Z_2 = -7t\frac{\partial}{\partial t} - 6v\frac{\partial}{\partial v}. \tag{5.4.8d}$$

Since the Lie Bracket of each pair is

$$[G_1, G_2] = G_1, \tag{5.4.9}$$

we can reduce each equation in (5.4.7) to quadratures [156, p 148] using first G_1 then G_2. Eq (5.4.7a) can be reduced to the elliptic integral

$$t - t_0 = \int \frac{\mathrm{d}v}{(C_0 - (2v^3)/3)^{1/2}} \tag{5.4.10}$$

or regarded as

$$v = \mathcal{P}(\sqrt{-1/6}t + a, 0, b), \tag{5.4.11}$$

where $\mathcal{P}$ is again the Weierstrass $\mathcal{P}$ function [90, p 631] and a and b are arbitrary constants.

We can relate the solutions of (5.4.7b) to (5.4.11) by the mapping

$$t \longrightarrow \frac{25}{t} \tag{5.4.12a}$$

$$v \longrightarrow \frac{5v}{t}, \tag{5.4.12b}$$

of (5.4.7c) by

$$t \longrightarrow (49t)^7 \tag{5.4.13a}$$

$$v \longrightarrow vt^3 + 6t \tag{5.4.13b}$$

and of (5.4.7d) by

$$t \longrightarrow -\frac{1}{(49t)^7} \tag{5.4.14a}$$

$$v \longrightarrow \frac{1}{49^7 t^4}\left(v + \frac{6}{t^2}\right). \tag{5.4.14b}$$

Note that Leach *et al* [128] showed that (5.4.7d) can be reduced to

$$7(t_0^{-1/7} - t^{-1/7}) = \int_{p_0}^{p} \frac{d\eta}{(2I - (2\eta^3)/3)^{1/2}}, \tag{5.4.15}$$

where p was one of the two integral invariants of Z_2, *ie* one of

$$q_1 = t^{-4/7}v - \frac{6}{49}t^{2/7} \tag{5.4.16a}$$

$$q_2 = t^{4/7}v - \frac{4}{7}t^{-3/7} - \frac{12}{343}t^{3/7}, \tag{5.4.16b}$$

and also related (5.4.7c) to (5.4.7d) using the mapping

$$t \longrightarrow t^{-1} \tag{5.4.17a}$$

$$v \longrightarrow \frac{v}{t}. \tag{5.4.17b}$$

We note that the Painlevé analysis provided the value $\frac{18}{625}$ for K. This translated into α (in (5.4.5)) having the values $\pm\frac{1}{5}, \pm\frac{7}{5}$ which are the *only* values for which (5.4.5) possesses two Lie point symmetries.

5.5 Analysis of the second field equation

Havas [94] showed that the second field equation of Dyer *et al* [56] could be written in the form

$$w'' - 6x^{-5n}w^2 = 0. \tag{5.5.1}$$

Following the idea of Mellin *et al* [148] we analyse (5.5.1) by requiring that it possess two point symmetries and hence be integrable.

In accordance with Mellin *et al* [148] we require that (5.5.1) be invariant under a point symmetry of the form

$$G = a(x)\frac{\partial}{\partial x} + (b(x)w + c(x))\frac{\partial}{\partial w}. \tag{5.5.2}$$

(While the equation considered by Mellin *et al* [148] is more general than (5.5.1) it can be easily shown that a point symmetry of (5.5.1) must have the form (5.5.2).) Separation by powers of w' and w results in the following system of ordinary differential equations:

$$2b' = a'', \tag{5.5.3a}$$

$$(b + 2a')x^{-5n} = 5anx^{-5n-1}, \tag{5.5.3b}$$

$$b'' = 12cx^{-5n}, \tag{5.5.3c}$$

$$c'' = 0. \tag{5.5.3d}$$

Eqq (5.5.3d) and (5.5.3c) give c and hence b as

$$c = C_0 + C_1 x, \tag{5.5.4}$$

$$b = B_0 + B_1 x + \frac{12C_0 x^{-5n+2}}{(-5n+1)(-5n+2)} + \frac{12C_1 x^{-5n+3}}{(-5n+2)(-5n+3)}. \tag{5.5.5}$$

In general, we can write a, via (5.5.3b), as

$$a = A_1 x^{5n/2} - \frac{B_0 x}{-5n+2} - \frac{B_1 x^2}{-5n+4} - \frac{6C_0 x^{-5n+3}}{(-5n+1)(-5n+2)(-15n/2+3)}$$
$$- \frac{6C_1 x^{-5n+4}}{(-5+2)(-5n+3)(-15n/2+4)}. \tag{5.5.6}$$

The final step in determining the form of (5.5.2) is to satisfy the consistency condition given by (5.5.3a) which we can now write as

$$2B_1 + \frac{24C_0 x^{-5n+1}}{(-5n+1)} + \frac{24C_1 x^{-5n+2}}{(-5n+2)} = \frac{5n}{2}\left(\frac{5n}{2} - 1\right) A_1 x^{5n-2} - \frac{B_1}{-5n/2+2}$$
$$- \frac{6(-5n+3)C_0 x^{-5n+1}}{(-5n+1)(-15n/2+3)} - \frac{6(-5n+4)C_1 x^{-5n+2}}{(-5n+2)(-15n/2+4)}. \tag{5.5.7}$$

Note that B_0 does not appear in (5.5.7) implying that we always have at least one symmetry which has the form

$$G_1 = x\frac{\partial}{\partial x} + (5n-2)w\frac{\partial}{\partial w}. \qquad (5.5.8)$$

This implies that (5.5.1) can always be reduced to a first order equation. In addition, we have a possible second symmetry by equating coefficients of the powers of x in (5.5.7) to zero. Thereafter, setting each of the constants B_1, C_0, C_1 except one (which is set equal to one), in turn, to zero we obtain appropriate values for n, *ie* for B_1, $n = 1$; C_0, $n = 3/7$ and for C_1, $n = 4/7$. Further for A_1 we have $n = 0, 2/5$. However, $n = 2/5$ makes the coefficient of C_1 infinite and is therefore invalid. These values of n imply that the equations

$$w'' - 6w^2 = 0 \qquad n = 0, \qquad (5.5.9a)$$

$$w'' - 6x^{-15/7}w^2 = 0 \qquad n = \frac{3}{7}, \qquad (5.5.9b)$$

$$w'' - 6x^{-20/7}w^2 = 0 \qquad n = \frac{4}{7}, \qquad (5.5.9c)$$

$$w'' - 6x^{-5}w^2 = 0 \qquad n = 1 \qquad (5.5.9d)$$

have the corresponding pairs of symmetries

$$G_1 = -x\frac{\partial}{\partial x} + 2w\frac{\partial}{\partial w} \qquad G_2 = \frac{\partial}{\partial x}, \qquad (5.5.10)$$

$$X_1 = 7x\frac{\partial}{\partial x} + w\frac{\partial}{\partial w} \qquad X_2 = 343x^{6/7}\frac{\partial}{\partial x} + (147x^{-1/7}w + 2)\frac{\partial}{\partial w}, \qquad (5.5.11)$$

$$Y_1 = 7x\frac{\partial}{\partial x} + 6w\frac{\partial}{\partial w} \qquad Y_2 = 343x^{8/7}\frac{\partial}{\partial x} + (196x^{1/7}w - 2)\frac{\partial}{\partial w}, \qquad (5.5.12)$$

$$Z_1 = x\frac{\partial}{\partial x} + 3w\frac{\partial}{\partial w} \qquad Z_2 = x^2\frac{\partial}{\partial x} + xw\frac{\partial}{\partial w}. \qquad (5.5.13)$$

The constant coefficients of the nonlinear terms in (5.5.9) can be transformed away. This reduces (5.5.9) to the system (5.4.7) and the discussion following (5.4.7) applies equally in this case. Thus we have reduced the problem of solving the two field equations of Dyer *et al* [56] to that of solving the single equation

$$y'' + x^n y^2 = 0. \qquad (5.5.14)$$

Mellin *et al* [148] showed that the equation

$$y'' + p(x)y' + r(x)y = f(x)y^2 \tag{5.5.15}$$

can be transformed to the autonomous form

$$Y'' + 2C_0Y' + (M + C_0^2)Y + N = KY^2, \tag{5.5.16}$$

where C_0 is given by our B_0 in (5.5.5),

$$M = \tfrac{1}{2}aa'' - \tfrac{1}{4}a'^2 - \tfrac{1}{2}(p' + \tfrac{1}{2}p^2 - 2r)a^2 - 2K\int \frac{d}{a^{3/2}} \exp\left[\tfrac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right] \tag{5.5.17}$$

and

$$\begin{aligned} N = K&\left\{\int \frac{d}{a^{3/2}} \exp\left[\tfrac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right]\right\}^2 - \int\left\{\left[\tfrac{1}{2}aa''' - \left(p' + \tfrac{1}{2}p^2 - 2r\right)aa'\right.\right. \\ &\left.\left. - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)a^2\right]\left[\int \frac{d}{a^{3/2}} \exp\left[\tfrac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right]\right]\right\}, \end{aligned} \tag{5.5.18}$$

where a is given in (5.5.2), K is a constant of integration and d is our c in (5.5.2), provided $f(x)$ is given by

$$f(x) = Ka^{-5/2} \exp\left[\tfrac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right]. \tag{5.5.19}$$

The transformation that takes (5.5.15) to (5.5.16) is

$$X = \int \frac{\mathrm{d}x}{a} \tag{5.5.20a}$$

$$Y = y\exp\left(-\int\frac{c}{a}\right) - \int\left[\frac{d}{a}\exp\left(-\int\frac{c}{a}\right)\right], \tag{5.5.20b}$$

where c is our b in (5.5.2) and is given by

$$c = C_0 + \tfrac{1}{2}(a' - ap). \tag{5.5.21}$$

Mellin *et al* [148] further showed that (5.5.16) had two symmetries iff

$$\left(M + \frac{C_0^2}{25}\right)\left(M + \frac{49C_0^2}{25}\right) + 4KN = 0. \tag{5.5.22}$$

Our analysis is much simpler as we do not have the functions $p(x)$ and $r(x)$ of [148] and $f(x)$ is explicitly given by

$$f(x) = x^{-5n}. \tag{5.5.23}$$

The condition (5.5.22) restricts n in (5.5.23), via (5.5.17), to $0, 3/7, 4/7$ and 1. We note that (5.5.16) is also integrable when $C_0 = 0$. (While the equation only possesses a single Lie point symmetry it also possesses a 'useful' nonlocal symmetry of the form (3.4.8) and so can be reduced to quadratures in a group theoretic manner.) This gives $n = 1/2$ and (5.5.1) becomes

$$w'' - 6x^{-5/2}w^2 = 0. \tag{5.5.24}$$

Eqq (5.5.9) and (5.5.24) are exactly those for which Wyman [187] found solutions.

Noting the parallels between the two field equations one is tempted to find an analogy for $C_0 = 0$ in the case of the first equation. This would require setting the coefficient of z' in (5.4.2) to zero. However, we do not have that latitude as the coefficient of z' is fixed.

It is interesting to analyse our version of (5.5.16) using the Painlevé method to determine if further solutions can be found. We find that our version of (5.5.16) has the Painlevé property only if $n = 0, 3/7, 1/2, 4/7$ or 1. This is equivalent to (5.5.22) or $C_0 = 0$ for (5.5.16). While not providing new solutions this is further evidence of the close relationship between the Lie and Painlevé analyses of differential equations.

5.6 Discussion

In the above analysis we introduced the pseudo partial Painlevé property. We emphasise that the possession of this property merely suggests that the equation is integrable on a subspace of the space of initial conditions [53]. We cannot predict the behaviour of the solution off the subspace. In fact, it has recently been shown that the third order system that possesses the pseudo partial Painlevé property on a known subspace is chaotic in the sense of Lyapunov for sets of initial conditions off that surface [169]. This is supported by the fact that we are only able to find solutions to the first field equation for a particular value of its first integral, *ie* this value determines the subspace on which the equation is integrable. This is in agreement with the ARS conjecture which holds only for this value of the first integral. Thus we expect solutions of the first field equation to be found, at best, for particular values of the first integral and all m or vice versa. This explains Havas' results [94] which are only valid for particular values of the first integral and Wyman's results [187, 188] which only hold for particular values of m.

The forms of (5.4.5) and (5.5.1) suggest that the search for further solutions of the Dyer–McVittie–Oattes field equations [56] should be confined to finding further values of n for which

$$y'' + x^n y^2 = 0 \tag{5.6.1}$$

is integrable.

5.7 The generalised Emden–Fowler equation

The generalised Emden–Fowler equation [122, 57, 66, 67, 68, 69] (of which (5.6.1) is a special case) has attracted much attention over the years. Wong, in his review of 1976 [186], contains over 100 references, but even these were selective. Subsequent to Wong's paper a further plethora of papers has appeared devoted to a study of this ubiquitous equation. (See [85] for a recent review.) The most general form studied today is

$$Y'' + p(X)Y' + q(X)Y = r(X)Y^n. \tag{5.7.1}$$

However, a Kummer–Liouville transformation [120, 138] converts (5.7.1) into standard form, *viz*

$$y'' = f(x)y^n. \tag{5.7.2}$$

It is this form of the equation to which we confine our analysis. Eq (5.7.2) has become increasingly important as it arises in the modelling of many physical systems. It is perhaps best known for its occurrence as the pivotal equation in the study of the shear-free spherically symmetric perfect fluid motion in Cosmology when $n = 2$ [177, 140, 94, 89, 141]. More recently [139] it has been shown to have application in the study of cosmic strings.

We study (5.7.2) from the viewpoints of Lie symmetries and the Painlevé analysis. In general, (5.7.2) does not possess any Lie point symmetries nor can one easily say anything about its possession of the Painlevé property. However, for an appropriate $f(x)$, (5.7.2) does possess at least one Lie point symmetry. We analyse (5.7.2) for these instances and also consider the conditions for it to possess more than one Lie point symmetry, thereby enabling the reduction to quadratures. (See also in this respect [27].) In addition we show under what further conditions (5.7.2) (with only one Lie point symmetry) can be reduced to quadratures.

We also undertake a Painlevé analysis of (5.7.2) (suitably transformed) and discuss its possession of the full Painlevé property. We comment on a possible link between possession of the Painlevé property and explicit integration of the equation. Our interest is in the relationship between the equation possessing the Painlevé property and the evaluation of the quadrature it is reduced to via the Lie analysis.

5.8 Lie Analysis

In the case of (5.7.2) it is easily verified that a point symmetry must have the form

$$G = a(x)\frac{\partial}{\partial x} + (c(x)y + d(x))\frac{\partial}{\partial y}. \tag{5.8.1}$$

The action of $G^{[2]}$ on (5.7.2) results in a nonlinear partial differential equation. Equating different powers of y' and y in this equation to zero results in the system

$$-2fa' + cf = af' + ncf \tag{5.8.2}$$

$$nfd = 0 \tag{5.8.3}$$

$$c'' = 0 \tag{5.8.4}$$

$$d'' = 0 \tag{5.8.5}$$

$$2c' - a'' = 0. \tag{5.8.6}$$

We immediately integrate (5.8.6) to obtain

$$c = \tfrac{1}{2}(a' + \alpha) \tag{5.8.7}$$

and observe that d in (5.8.3) is zero. Thus (5.8.5) is identically satisfied. Note that (5.8.3) and (5.8.4) coalesce in the case $n = 2$ and c and f are related via d (See (5.10.8).). Note also that (5.8.2) can be rewritten as

$$af' + \left(\frac{n-1}{2}(a' + \alpha) + 2a'\right) f = 0 \tag{5.8.8}$$

which is special in the case $n = -3$ and $\alpha = 0$. We return to these cases later. The cases $n = 0, 1$ are equivalent as the equation is then linear. Linear second order differential equations have eight Lie point symmetries which form the Lie algebra $s\ell(3, R)$. (See [80] for a recent proof and references therein.)

For general n we write (5.8.8) as

$$\frac{f'}{f} = -\left(\frac{n+3}{2}\right)\frac{a'}{a} - \left(\frac{n-1}{2}\right)\frac{\alpha}{a} \tag{5.8.9}$$

from which

$$f = Ka^{-(n+3)/2} \exp\left[-\frac{(n-1)\alpha}{2}\int \frac{\mathrm{d}x}{a}\right]. \tag{5.8.10}$$

The differential equation for a is (from (5.8.4) and (5.8.7))

$$a''' = 0, \tag{5.8.11}$$

whence

$$a = A_0 + A_1 x + A_2 x^2. \tag{5.8.12}$$

Eq (5.7.2) has the symmetry

$$G_1 = a\frac{\partial}{\partial x} + \tfrac{1}{2}(a' + \alpha)y\frac{\partial}{\partial y} \tag{5.8.13}$$

if $f(x)$ is given by (5.8.10).

Using the transformation

$$X = \int \frac{\mathrm{d}x}{a} \tag{5.8.14}$$

$$Y = \frac{y}{a^{1/2}} \exp\left(-\alpha \int \frac{\mathrm{d}x}{2a}\right) \tag{5.8.15}$$

we rewrite (5.7.2) in autonomous form, *viz*

$$Y'' + \alpha Y' + \left(\Delta + \frac{\alpha^2}{4}\right) Y = KY^n, \tag{5.8.16}$$

where

$$\Delta = A_0 A_2 - \tfrac{1}{4} A_1^2. \tag{5.8.17}$$

Reduction via

$$u = Y \qquad\qquad v = Y' \tag{5.8.18}$$

results in an Abel's equation of the second kind, *viz*

$$vv' = Ku^n - \alpha v - \left(\Delta + \frac{\alpha^2}{4}\right) u, \tag{5.8.19}$$

the solution of which (though it exists in principle) is unobvious.

Eq (5.7.2) can be reduced to a first order equation provided f is given by (5.8.10) and a by (5.8.12).

To reduce (5.8.16) to quadratures we require that the equation which arises after the first reduction of order possesses at least one Lie point symmetry. If (5.8.16) possesses two Lie point symmetries, G_1 and G_2 say, and $[G_1, G_2] = \lambda G_1$ (λ an arbitrary constant usually 1 or 0), the reduction via G_1 will result in a first order equation with G_2 (suitably transformed) as a point symmetry [156, p 148]. We therefore further examine (5.8.16) to determine under what circumstances it possesses two point symmetries.

Setting

$$G = \tilde{a}(X)\frac{\partial}{\partial X} + \tilde{c}(X) Y \frac{\partial}{\partial Y}, \tag{5.8.20}$$

where we have removed $\tilde{d}(X)$ in the coefficient of $\partial/\partial Y$ since the form of (5.8.16) implies $\tilde{d}(X) = 0$, we require

$$G^{[2]} N_{\big|_{N=0}} = 0, \tag{5.8.21}$$

where we have rewritten (5.8.16) as $N(Y, Y', Y'') = 0$. The operation (5.8.21) results in the system

$$\tilde{c} - 2\tilde{a}' = n\tilde{c} \tag{5.8.22}$$

$$2\tilde{c}' + \alpha\tilde{a}' - \tilde{a}'' = 0 \tag{5.8.23}$$

$$\tilde{c}'' + 2M\tilde{a}' + \alpha\tilde{c}' = \tilde{c}, \tag{5.8.24}$$

where

$$M = \Delta + \frac{\alpha^2}{4}. \tag{5.8.25}$$

The function $\tilde{c}$ is determined from (5.8.22), *viz*

$$\tilde{c} = -\frac{2\tilde{a}'}{n-1}. \tag{5.8.26}$$

The differential equation for $\tilde{a}$ now becomes (via (5.8.23) and (5.8.26))

$$\frac{n+3}{n-1}\tilde{a}'' - \alpha\tilde{a}' = 0 \tag{5.8.27}$$

and so $\tilde{a}$ is given by

$$\tilde{a} = \tilde{A}_0 + \tilde{A}_1 \exp\left(\left(\frac{n-1}{n+3}\right)\alpha X\right). \tag{5.8.28}$$

Eq (5.8.24) becomes the consistency condition

$$\frac{\tilde{a}'''}{n-1} - M\tilde{a}' + \frac{\alpha\tilde{a}''}{n-1} = 0. \tag{5.8.29}$$

When (5.8.27) is invoked, (5.8.29) is satisfied only if

$$M = \frac{2\alpha^2(n+1)}{(n+3)^2} \tag{5.8.30}$$

from which (via (5.8.25))

$$\Delta = -\left[\frac{\alpha(n-1)}{2(n+3)}\right]^2. \tag{5.8.31}$$

The substitution of (5.8.31) into (5.8.17) results in

$$\tfrac{1}{4}A_1^2 - A_0A_2 = \left[\frac{\alpha(n-1)}{2(n+3)}\right]^2 \tag{5.8.32}$$

which requires the equation for a, *viz* (5.8.12), to have real roots!

From (5.8.20) and (5.8.28), (5.8.16) has the two Lie point symmetries

$$G_1 = \frac{\partial}{\partial X} \tag{5.8.33}$$

$$G_2 = \exp\left(\left(\frac{n-1}{n+3}\right)\alpha X\right)\left(\frac{\partial}{\partial X} - \frac{2\alpha Y}{n+3}\frac{\partial}{\partial Y}\right) \tag{5.8.34}$$

provided (5.8.32) holds.

Under the transformation

$$\mathcal{X} = \alpha\left(\frac{n+3}{n-1}\right)\exp\left(\left(\frac{n-1}{n+3}\right)\alpha X\right) \qquad \mathcal{Y} = Y\exp\left(\left(\frac{2\alpha}{n+3}\right)X\right) \tag{5.8.35}$$

(5.8.16) becomes

$$\mathcal{Y}'' = K\mathcal{Y}^n, \tag{5.8.36}$$

(5.8.33)-(5.8.34) transform to

$$X_1 = (1-n)\mathcal{X}\frac{\partial}{\partial \mathcal{X}} + 2\mathcal{Y}\frac{\partial}{\partial \mathcal{Y}} \tag{5.8.37}$$

$$X_2 = \frac{\partial}{\partial \mathcal{X}} \tag{5.8.38}$$

and

$$[X_1, X_2] = (1-n)X_2. \tag{5.8.39}$$

We now evaluate f as

$$f = KA_2^{-\frac{n+3}{2}}\left(x + \frac{A_1}{2A_2} - \frac{\alpha}{2A_2}\frac{n-1}{n+3}\right)^{-(n+3)}. \tag{5.8.40}$$

Eq (5.8.36) can be reduced using X_2 and then X_1 to the quadrature

$$\mathcal{X} - \mathcal{X}_0 = \pm\int \frac{\mathrm{d}\mathcal{Y}}{\left(\frac{K}{n+1}\mathcal{Y}^{n+1} + K_1\right)^{\frac{1}{2}}}, \tag{5.8.41}$$

where $\mathcal{X}_0$ and K_1 are arbitrary constants of integration.

Remark: We observe that (5.8.16) can also be reduced to quadratures in the case $\alpha = 0$. An extension of the Lie theory to nonlocal symmetries [83] reveals that

$$y'' + My = Ky^n \tag{5.8.42}$$

possesses an additional (nonlocal) symmetry of the form (3.4.8). The function f now becomes

$$f = K(A_0 + A_1 x + A_2 x^2)^{-\frac{n+3}{2}}. \tag{5.8.43}$$

(See also [111] for a treatment of the two symmetry case for $f = x^m$.)

Thus for (5.7.2) to be reduced to quadratures f must be given by (5.8.10), a by (5.8.12) and (5.8.32) must hold.

5.9 Painlevé Analysis

From §5.8 it would seem that the Lie theory of differential equations (and its extensions) is rather exhaustive in its treatment of the Emden–Fowler equation. We now investigate its possession of the Painlevé property to determine whether any further interesting information can be obtained. We note that (5.8.36) does not naturally fall into the classes of equations listed in [71, 105] as i) n can be rational and ii) these lists are complete up to a homographic transformation. Thus a Painlevé analysis of (5.8.36) should highlight interesting properties of this equation.

We study the equation in the form

$$y'' = y^n, \tag{5.9.1}$$

where the K in (5.8.36) has been removed through the rescaling of y. It is easily verified that the pole arises at

$$p = -\frac{2}{n-1}, \tag{5.9.2}$$

and the coefficient of the pole is

$$\alpha^{n-1} = \frac{2(n+1)}{(n-1)^2}, \tag{5.9.3}$$

obviously with both terms in (5.9.1) being dominant. The indices arise at

$$i = -1, \frac{2(n+1)}{n-1}. \tag{5.9.4}$$

For the implementation of the ARS algorithm p in (5.9.3) must be a negative integer. This arises for the values $n = 2, 3$. The corresponding indices are $i = 6, 4$ respectively. It is not necessary to substitute the truncated Laurent expansion into (5.9.1) to check for incompatibilities at the index. This follows from both the terms in (5.9.1) being dominant. We note that (5.8.41) can be easily evaluated for these values of n.

For $n \neq 2, 3$ p and i are rational and the possibility that the solution of (5.9.1) possesses algebraic branch points exists. However, we can transform the denominator of p away by setting either

$$Y = y^{n-1} \qquad\qquad X = x \tag{5.9.5}$$

or

$$Y = y \qquad\qquad X = x^{1/(n-1)}. \tag{5.9.6}$$

The transformation (5.9.6) is homographic (and thereby preserves the Painlevé property [71, 47]) and results in

$$Y'' = (n-2)\frac{Y'}{X} + (n-1)^2 X^{2(n-2)} Y^n. \tag{5.9.7}$$

Our analysis of (5.9.7) yields the two cases

i) $p = -2$, $i = 1-n, 2(1+n)$, $\alpha^{n-1} = \frac{2(n+1)}{(n-1)^2}$

ii) $p = n-1$, $i = -1, 0$, α - arbitrary.

The first case does not have $i = -1$ and so the standard analysis halts (See [43] for a detailed discussion of the absence of $i = -1$ in the Painlevé analysis.). An alternative route need be sought.

While (5.9.5) is not homographic, it does preserve the polynomial form of (5.9.1) (for integer n) and so is an acceptable transformation [47]. The equation becomes

$$Y'' = \left(\frac{n-2}{n-1}\right)\frac{Y'^2}{Y} + (n-1)Y^2. \tag{5.9.8}$$

Again two cases arise:

i) $p = -2$, $i = -1, \frac{2(n+1)}{(n-1)}$, $\alpha = \frac{2(n+1)}{(n-1)^2}$

ii) $p = n-1$, $i = -1, 0$, α - arbitrary.

Case ii) arises in the instance that the two derivative terms in (5.9.8) are dominant only. However, this is equivalent to only the first term in (5.9.1) being dominant. In case i) i is a positive integer only when $n = -3, -1, 2, 3, 5$ (with the corresponding i values $1, 0, 6, 4, 3$ respectively). The case $n = -1$ can be immediately discounted as $i = 0$ implies α is arbitrary. However, we note that α, in case i), is fixed (and is in fact zero!). This implies that the expected arbitrary constant corresponding to $i = 0$ is fixed. This can only be resolved by introducing logarithmic terms in the Laurent expansion for Y [168].

We need to examine the family $p = n-1$ for the remaining values of n. For $n = 2, 3, 5$, $p > 0$. Ordinarily this would suggest the transformation

$$\mathcal{Y} = \frac{1}{Y} \tag{5.9.9}$$

to make p negative. However, as we have specific values for n we can resort to looking up the appropriate equations in [71, 105]. We find that the equations corresponding to $n = -3, 2, 3, 5$ are eqq (22), (2), (18) and (21) of [71] respectively. Thus (5.9.1) has the Painlevé property for $n = -3, 2, 3, 5$.

Note that, as n is a physical constant related to the ratio of specific heats in the astrophysical context [124], it can be rational. In the subsequent analysis we expressly ignore integer n. By inspection we see that (5.9.3) implies p a negative integer for $1 < n < 5/3$. For $p = -2/(n-1) \in \mathbb{Z} < 0$, $i = 2(n+1)/(n-1) = 2-2p \in \mathbb{Z} > 0$ (in (5.9.4)). This points to (5.9.1) possessing the full Painlevé property. In this instance we do not have recourse to the lists in [71, 105] as these are concerned with rational functions of the dependent variable. It should be noted that no such restriction was originally intended by Painlevé [160, 161]. Again, we do not have to substitute the truncated Laurent expansion into (5.9.1) to verify that no incompatibilities arise at the index as both terms are dominant. Thus we introduce equations of the form

$$y'' = y^{(p+2)/p}, \qquad p \in \mathbb{N} > 2 \tag{5.9.10}$$

into the literature as part of the class of second order ordinary differential equations possessing the Painlevé property.

5.10 The special cases $n = -3, 2$

We have seen above that in the case $n = -3, 2$, (5.7.2) can be reduced to a quadrature that can be evaluated. However, these cases have a deeper significance that the Lie analysis in §2 did not reveal.

For $n = -3$ and $\alpha = 0$ we solve (5.8.8) to obtain

$$f = \tilde{K}, \tag{5.10.1}$$

where $\tilde{K}$ is an arbitrary constant. The solution for a (obtained from (5.8.4) and (5.8.7)) is

$$a = A_0 + A_1 x + A_2 x^2 \tag{5.10.2}$$

with c given by

$$c = \frac{A_1}{2} + A_2 x \tag{5.10.3}$$

and $d = 0$ as before. Eq (5.7.2) now has the form

$$y'' = \tilde{K} y^{-3} \tag{5.10.4}$$

(which is the well-known Ermakov–Pinney equation [58, 166]) and has the three Lie point symmetries

$$G_1 = \frac{\partial}{\partial x} \tag{5.10.5}$$

$$G_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{5.10.6}$$

$$G_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \tag{5.10.7}$$

which form the Lie algebra $s\ell(2, R)$. (See also [110, 128, 111, 131].) This Lie algebra is not solvable, but as we are only concerned with a second order equation, it is sufficient to reduce (5.10.4) to quadratures.

The 'new' point symmetry G_3' is not a descendant of any of the point symmetries in (5.10.17), but comes from the nonlocal symmetry [6]

$$G_4 = 3\left(\int a\mathrm{d}x\right)\frac{\partial}{\partial x} + 2a^2\frac{\partial}{\partial a} \tag{5.10.21}$$

and is hence a Type II hidden symmetry. This hidden symmetry is the appropriate one for further reduction of (5.10.18). Using

$$t = vu^{-1/2} \qquad w = \tfrac{1}{2}(v'u^{3/2} - \tfrac{1}{2}vu^{-1/2})^2 \tag{5.10.22}$$

we obtain

$$w'' + 3w' + 2w = 0 \tag{5.10.23}$$

which is trivially solved. Reversing the transformations we obtain

$$x - x_0 = \int \frac{\mathrm{d}u}{(-Ku^3/6 - Lu^2/2 - 2Pu - 2Q)^{3/2}}, \tag{5.10.24}$$

where K, L, P and Q are constants of integration and

$$u = \int \frac{1}{a^{3/2}}. \tag{5.10.25}$$

It is interesting to note that the trivial cases of setting all except one (in turn) of K, L, P and Q to zero produce special cases of the function f given by Srivastava [177] for the reduction of the Emden–Fowler equation of index two to quadrature.

When $\alpha \neq 0$, eq (5.10.11) has only the two point symmetries

$$\begin{aligned} G_1 &= \frac{\partial}{\partial x} \\ G_2 &= x\frac{\partial}{\partial x} + a\frac{\partial}{\partial a}. \end{aligned} \tag{5.10.26}$$

Unfortunately reduction using G_1 does not produce any hidden symmetries. However, it is of interest to test (5.10.11) for integrability using the Painlevé analysis. After transforming (5.10.11) to a suitable form for the analysis, we find that one of the *indices* which occurs is arbitrary. In fixing this index it turns out that the expected arbitrary constant at the other index becomes fixed. The analysis can only be continued by introducing logarithmic terms into the expansion. Thus (5.10.11) does not possess the full Painlevé property. It has been observed [78] that some information about the partial solution of (5.10.11) can be obtained from considering the different families of expansions for y. However, this information is naturally contained in (5.10.24).

5.11 Discussion

The Emden–Fowler equation

$$y'' = f(x)y^n \tag{5.11.1}$$

has been shown to be integrable (for certain functions $f(x)$) for all n (including rational values) by considering a Lie analysis. It was further shown that, if (5.11.1) possessed two Lie symmetries, it could always be transformed to

$$Y'' = Y^n. \tag{5.11.2}$$

(Of course (5.11.1) can be reduced to quadratures if it has two point symmetries. We mention the transformation to (5.11.2) so that comparison can be made with the Painlevé property.) This point must be emphasised. All Emden–Fowler equations with two symmetries G_1 and G_2 such that $G_1 \neq \rho(x,y)G_2$ (which form the Lie algebra A_2) can be transformed to (5.11.2) and the solution to the original equation is obtained from the solution of (5.11.2) via the same transformation.

In this work, the Painlevé analysis was restricted to equations of the form (5.11.2) by requiring that (5.11.1) possess two Lie point symmetries. However, noting that some of the equations in [71, 105] do not possess at least two Lie point symmetries, the Painlevé analysis of the equation in the form (5.11.1) would be of some interest (See [46].). This could point the way to nonlocal symmetries of (5.11.1).

Chapter 6

Conclusion

The case is made for a less constrained approach to the definition of symmetries of functions and differential equations.

6.1 The determination of nonlocal symmetries

In §§3.2–3.4 we presented a systematic approach to finding first order and generalised nonlocal symmetries of second order ordinary differential equations. While the solutions of the resulting equations may look complicated their determination is surprisingly straightforward (We need to solve linear first order ordinary differential equations as opposed to the linear partial differential equations of the classical method.). The analysis was confined to the determination of those nonlocal symmetries that reduce to point symmetries under reduction of order via

$$G = \frac{\partial}{\partial x}, \tag{6.1.1}$$

where x represents the independent variable. This restriction is valid as we are only interested in those nonlocal symmetries of the second order equation that allow us to reduce the resulting first order equation to quadratures.

To properly utilise the nonlocal symmetry some restrictions had to be placed on the coefficient functions. The restrictions imposed are justified by the requirement that the nonlocal symmetry commutes with (6.1.1) and that it becomes a point symmetry under the reduction of order. In spite of these restrictions we have been able to make some progress. In particular we have been able to classify all second order equations possessing a first order nonlocal symmetry in addition to the point symmetry (6.1.1). The further classification of second order equations using nonlocal symmetries lies in making an appropriate *ansatz* for the integrand in the coefficient functions. A few examples (not meant to be exhaustive) were given to illustrate the principle.

We remarked in Chapter 3 that the search for nonlocal symmetries should be confined to second order equations. In the case that one is dealing with higher order systems where the reductions are nontrivial it may be of some benefit to analyse those systems for first and higher order nonlocal symmetries. We leave it to the practitioner to decided which of the two approaches is optimal.

We note that differential equations have a rich structure of nonlocal symmetries. This was amply illustrated by the reduced form of the complex Lorenz system. A simple example is the analysis of

$$y'' = 0 \tag{6.1.2}$$

for nonlocal symmetries of the form

$$G_{nl} = \xi(x, y, \int y \mathrm{d}x)\frac{\partial}{\partial x} + \eta(x, y, \int y \mathrm{d}x)\frac{\partial}{\partial y}. \tag{6.1.3}$$

Even with this severe restriction on the integrand we find a large number of nonlocal symmetries.

We have only concentrated on linear nonlocal symmetries. A method for the systematic search for other possible nonlocal symmetries (*eg* involving exponential [5] or other functions) would be of interest. The determination of exponential nonlocal symmetries is of particular interest as they allow us to reduce the order of equations that do not necessarily possess any point symmetries [5]. They can also be used to determine first integrals. However, in the calculation of these symmetries we need to solve a system of nonlinear partial differential equations [5].

A final remark, in order to place this work in its proper perspective, is that hidden symmetries have recently been explained from a geometric viewpoint [93] using the concept of solvable structures [23]. It is hoped that our treatment complements that approach.

6.2 Equations not possessing Lie point symmetries

The Lie method of extended groups is attractive in that it provides an algorithmic method to solve differential equations. However, in those instances that the equations being studied do not possess Lie point (contact) symmetries the method is inapplicable. In §§3.5–3.10 we showed that, by reversing the standard procedure, some progress can still be made. Due to their proliferation in applications we have concentrated on second order equations and presented those which do not have Lie point symmetries and yet are solvable using a variant of the classic Lie analysis. The method applies *mutatis mutandis* to higher order equations.

In this work we have considered the two–dimensional algebras of symmetries. We have thereby ignored the fact that the second order equations invariant under Types II_2 and IV_2 are linear and hence have eight Lie point symmetries. Thus there exists a point transformation to take the second order equation invariant under Type II_2 to that invariant under Type IV_2. This suggests a relationship between the original second order equations not possessing Lie point symmetries. In the case of (3.9.8) the transformation

$$w = W\int F(z)\mathrm{d}z, \qquad Z = \int \frac{1}{\int F(z)}\mathrm{d}z \tag{6.2.1}$$

yields (3.9.12) (with w, z in (3.9.12) replaced by W, Z respectively). A similar result will hold for (3.8.10) and (3.8.19).

We note that the solutions of eqq (3.9.8) and (3.9.12) are easily obtained. Both integrate trivially to a first integral that can be rewritten as a Riccati equation. This integration is related to the presence of exponential nonlocal symmetries of the form [9]

$$G = \exp\left[\int P(x,y)\mathrm{d}x\right]\left(\xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}\right). \tag{6.2.2}$$

(Thus the route to the linear second order equation is obtainable by this elementary method and that detailed earlier.) The reason we still consider the solution of these equations via our method is to show that those equations which are solvable by elementary methods are so because of the group theoretic basis. The practical usefulness of our approach is evidenced by the solutions of (3.8.10), (3.8.15), (3.8.19) and (3.9.10) which are not obviously integrable.

This approach opens up a number of new avenues. The first is the extension to second order equations possessing one Lie point symmetry. Here, one would require that the third order equation possess three Lie point symmetries and that reduction using the third symmetry result in a new second order equation with more than one symmetry. A further aspect that would need investigation in this case is that of the original second order equation possessing, in addition to the one Lie point symmetry, a 'useful' nonlocal symmetry, *ie* a nonlocal symmetry that reduces to a new point symmetry under the reduction of the second order equation via the point symmetry. In fact that option can also be applied to the new second order equations obtained earlier – it is not necessary for these equations to possess two Lie point symmetries to be reducible to quadratures.

Another possible line of research is the possession of contact symmetries by the third order equation. As it has been shown [14] that contact symmetries can reduce to point symmetries, this could result in new classes of third order equations that reduce to second order equations which are solvable without the imposition of further restrictions, *ie* they may naturally possess more than one Lie point symmetry.

Contact symmetries can also be considered for the original second order equations. While there may be technical difficulties associated with finding these contact symmetries, it is possible to assume that the second order equation has no Lie point symmetries but one contact symmetry of some specific form (These difficulties have been largely overcome by the work of Mahomed and Leach [147].). Then an increase of order could lead to a third order equation with more than one Lie contact symmetry *etc.*

From the above it can be seen that the ideas presented here can be used successfully to extend the number of solvable equations. They can also be used to explain the integration of equations not possessing Lie point symmetries via group theory. It is worthwhile to note that we do not expect 'fundamental equations' like the six Painlevé equations [160, 161, 71] to be solvable using our method. These equations are 'irreducible' [105, p 345].

6.3 The Lie–Painlevé link

In Chapter 5 we presented two applications of Lie and Painlevé analysis. Both demonstrated clearly a possible link between these two types of analyses (Another example can be found in [130].). Take, for example, the equation

$$Y'' = Y^n. \tag{6.3.1}$$

The Painlevé analysis of (6.3.1) reveals integrability only for restricted values of n. It is not surprising that, only for these restricted integer values, the quadrature (6.3.1) reduces to, *viz* (5.8.41), can be evaluated. In the case of rational n, it was shown that, for specific values of n in the range $(1, \frac{5}{3}]$, (6.3.1) possessed the Painlevé property. While the quadrature (5.8.41) cannot, as yet, be evaluated for these values, noting the results in [184, p 82] and [32] we believe that the evaluation thereof is only a matter of time and effort.

As expected, the results of the Painlevé analysis are a subset of those obtained from the Lie analysis (extended in some cases to nonlocal symmetries). The Painlevé analysis isolates those cases in which the quadrature that the Lie analysis provides can be easily evaluated, thereby identifying those cases with analytic solutions (as opposed to the algebraic results of the Lie analysis).

These examples are a clear indication that both Lie and Painlevé analyses provide important information about equations and should both be attempted for a given problem.

6.4 A proliferation of nonlocal symmetries

Consider the third order equation

$$y''' = 0 \tag{6.4.1}$$

and the second order equation

$$Y'' = 0 \tag{6.4.2}$$

which is obtained from (6.4.1) by means of the reduction of order transformation

$$X = x \qquad Y = y' \tag{6.4.3}$$

which is a consequence of the symmetry $\partial/\partial y$ of (6.4.1). Consider the fates and sources of the symmetries of the two equations which are summarised in Table 6.1. There is nothing esoteric about this example. It demonstrates clearly and precisely just how closely local and nonlocal symmetries are related.

Why then should one restrict the type of functional dependence of the coefficient functions of a symmetry? If we think of the symmetry as an operator

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}, \tag{6.4.4}$$

we do not have to constrain the nature of the dependence of the coefficient functions on the variables. In an application we may be forced to impose restrictions in order to make any progress. However, we must always be sensitive to the fact that we are imposing a restriction. The restriction we impose should be the minimum possible so that we obtain the widest possible information about the symmetry of the equation or function. Our approach marks a departure from the classical approach which started from point transformations and sought generalisations. (See [86] for a comprehensive account of this

Table 6.1: Fates of the symmetries of $y''' = 0$ and sources of the symmetries of $Y'' = 0$. In the reduction of $G_1 - G_{10}$ any terms in $\partial/\partial y$ are omitted as $Y'' = 0$ is independent of $y = \int Y \mathrm{d}X$.

$y''' = 0$	Fate	$Y'' = 0$	Source
G_1	annihilated	U_1	G_2
G_2	U_1	U_2	G_3
G_3	U_2	U_3	G_4
G_4	U_3	U_4	$x\frac{\partial}{\partial x} + \frac{3}{2}y\frac{\partial}{\partial y} = G_5 + \frac{1}{2}G_7$
G_5	$X\frac{\partial}{\partial X} = U_4 - \frac{1}{2}U_6$	U_5	$x^2\frac{\partial}{\partial x} + \frac{3}{2}(xy - \int y\mathrm{d}x)\frac{\partial}{\partial y}$
G_6	$X^2\frac{\partial}{\partial X} + 2(\int Y\mathrm{d}X)\frac{\partial}{\partial Y}$	U_6	G_7
G_7	U_6	U_7	G_8
G_8	U_7	U_8	$xy'\frac{\partial}{\partial x} + \frac{1}{2}(xy'^2 + 3\int y'^2\mathrm{d}x)\frac{\partial}{\partial y}$
G_9	$2(\int Y\mathrm{d}X - XY)\frac{\partial}{\partial X} - Y^2\frac{\partial}{\partial Y}$		
G_{10}	$(2X\int Y\mathrm{d}X - X^2Y)\frac{\partial}{\partial X}$ $+(2Y\int Y\mathrm{d}X - XY^2)\frac{\partial}{\partial Y}$		

concept.) Consequently the restriction imposed was usually greater than one needs for practical purposes. In fact it often reduced the applicability of the theory. We envision the existence of a sublevel of usable symmetries (effectively point, at which most work has been done in the past). In this context usable means being able to construct a transformation from the symmetry. In our approach it is not a matter of generalising the concept of symmetry, but of realising that in the past our thinking has been too confined.

It is hoped that our endeavours further enlarge the classes of equations solvable via the Lie method and bring us closer to realising Lie's dream of the solution of all differential equations in a unified manner.

Appendix A

Noether's Theorem

A.1 Introduction

Symmetries of differential equations are not the only symmetries which are used to provide integrals and solutions. The invariance of the Action Integral under infinitesimal transformation leads to the celebrated Noether's theorem [154, 155] which relates a first integral to each symmetry obtained for the Action Integral. The classical Noether's theorem provides for the infinitesimal transformation to be a generalised transformation in that it can depend upon derivatives as well as the dependent and independent variables [62, 63] despite more recent opinions to the contrary [72, 73]. (See [174] for an excellent review of generalisations of Noether's Theorem.)

Under an infinitesimal transformation

$$\bar{x} = x + \varepsilon\xi \qquad \bar{y} = y + \varepsilon\eta, \tag{A.1.1}$$

where ε is the parameter of smallness and ξ and η are differentiable, the Action Integral

$$A = \int_{x_0}^{x_1} L(x, y, y')\mathrm{d}x \tag{A.1.2}$$

is invariant if

$$\begin{aligned} \bar{A} &= \int_{\bar{x}_0}^{\bar{x}_1} L(\bar{x}, \bar{y}, \bar{y}')\mathrm{d}\bar{x} \\ &= A, \end{aligned} \tag{A.1.3}$$

where now $'$ denotes $\mathrm{d}/\mathrm{d}\bar{x}$. However, it is not necessary for $\bar{A}$ to be equal to A since Hamilton's Principle requires that the variation in the functional be zero under a zero endpoint variation. We impose the restriction on ξ and η that they conform to the requirements for Hamilton's Principle, but allow for the freedom of a gauge function which does not contribute to the variation. Thus we may write

$$\begin{aligned} \bar{A} &= \int_{\bar{x}_0=x_0}^{\bar{x}_1=x_1} L(\bar{x},\bar{y},\bar{y}')\mathrm{d}\bar{x} \\ &= A + \int_{x_0}^{x_1} \frac{\mathrm{d}F}{\mathrm{d}x}\mathrm{d}x. \end{aligned} \tag{A.1.4}$$

Invoking the transformation (A.1.1) and requiring that (A.1.4) be the identity for $\varepsilon = 0$ we have

$$\int_{x_0}^{x_1} \left(L + \varepsilon\left(\xi\frac{\partial L}{\partial x} + \eta\frac{\partial L}{\partial y} + \zeta\frac{\partial L}{\partial y'} + \xi' L\right)\right)\mathrm{d}x = \int_{x_0}^{x_1} (L + \varepsilon f')\,\mathrm{d}x \tag{A.1.5}$$

to the first order in ε, where $F = \varepsilon f$ and $\zeta = \eta' - y'\xi'$. Consequently

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \tag{A.1.6}$$

is a Noether symmetry of (A.1.2) if

$$\xi\frac{\partial L}{\partial x} + \eta\frac{\partial L}{\partial y} + \zeta\frac{\partial L}{\partial y'} + \xi' L = f'. \tag{A.1.7}$$

We note that nothing has been said about the functional dependence of ξ, η and f. We also note that there is no mention of a first integral. The solution of (A.1.7) will provide the infinitesimal transformation under which the variation of the Action Integral will be invariant.

Eq (A.1.7) can be rewritten as

$$0 = \left\{f - \left[\xi L + (\eta - y'\xi)\frac{\partial L}{\partial y'}\right]\right\}' - (\eta - y'\xi)\left[\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right] \tag{A.1.8}$$

so that when the variational principle is imposed on (A.1.2) for it to take a stationary value and consequently the Euler–Lagrange equation applies, (A.1.8) leads to the Noetherian integral

$$I = f - \left[\xi L + (\eta - y'\xi)\frac{\partial L}{\partial y'}\right]. \tag{A.1.9}$$

(We refer the interested reader to [145, 112] for interesting results on Noether symmetries.)

In the event that a Hamiltonian is known, it is useful to be able to use Noether's Theorem directly. (This is of particular benefit to physicists who wish to avoid the mathematical intricacies involved in transforming the Hamiltonian to a Lagrangian.) In the sequel we cast Noether's Theorem in Hamiltonian form [77]. We also show that nonlocal transformations have a role to play in Noether's Theorem [84].

A.2 Hamiltonian Formulation

If we define a Hamiltonian, $H(q,\dot{q},t)$, via the Legendre transformation

$$H(q,p,t) = p\dot{q} - L(q,\dot{q},t), \tag{A.2.1}$$

where the momentum is

$$p = \frac{\partial L}{\partial \dot{q}}, \tag{A.2.2}$$

the Action Integral can be written as

$$A\int_{t_0}^{t_1}(p\dot{q} - H(q,p,t))\mathrm{d}t. \tag{A.2.3}$$

In transformed coordinates, $\bar{q},\bar{p}$ and $\bar{t}$,

$$\bar{A} = \int_{\bar{t}_0}^{\bar{t}_1}\left(\bar{p}\dot{\bar{q}} - H\left(\bar{q},\bar{p},\bar{t}\right)\right)\mathrm{d}\bar{t}, \tag{A.2.4}$$

where $\dot{\bar{q}} = \mathrm{d}\bar{q}/\mathrm{d}\bar{t}$. Suppose that q,p and t and $\bar{q},\bar{p}$ and $\bar{t}$ are related by the infinitesimal transformation

$$\begin{aligned}\bar{t} &= t + \varepsilon\tau \\ \bar{q} &= q + \varepsilon\eta \\ \bar{p} &= p + \varepsilon\xi,\end{aligned} \tag{A.2.5}$$

where ε is the parameter of smallness and τ,η and ξ are differentiable functions of their as yet unspecified arguments, generated by the differential operator

$$G = \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial q} + \xi\frac{\partial}{\partial p}. \tag{A.2.6}$$

As we eventually intend to apply the variational principle in its zero end–point variation form, we equate $\bar{t}_0$ and $\bar{t}_1$ to t_0 and t_1 respectively.

We have

$$\begin{aligned}\bar{A} - A &= \int_{t_0}^{t_1} \left\{ [\bar{p}\dot{\bar{q}} - H(\bar{q},\bar{p},\bar{t})] \frac{\mathrm{d}\bar{t}}{\mathrm{d}t} - [p\dot{q} - H(q,p,t)] \right\} \mathrm{d}t \\ &= \varepsilon \int_{t_0}^{t_1} \left\{ p\dot{\eta} + \xi\dot{q} - H\dot{\tau} - \eta\frac{\partial H}{\partial q} - \xi\frac{\partial H}{\partial p} - \tau\frac{\partial H}{\partial t} \right\} \mathrm{d}t, \qquad \text{(A.2.7)}\end{aligned}$$

where we have used (A.2.5),

$$\dot{\bar{q}} = \dot{q} + \varepsilon(\dot{\eta} - \dot{q}\dot{\tau}) \qquad \text{(A.2.8)}$$

and

$$H(\bar{q},\bar{p},\bar{t}) = H(q,p,t) + \varepsilon\left[\eta\frac{\partial H}{\partial q} + \xi\frac{\partial H}{\partial p} + \tau\frac{\partial H}{\partial t}\right], \qquad \text{(A.2.9)}$$

to the first order in ε. The change in A induced by the infinitesimal transformation (A.2.5) will be independent of the path in configuration space between t_0 and t_1 provided the integrand in (A.2.7) is expressed as the total time derivative of some function, *ie* we require

$$p\dot{\eta} + \xi\dot{q} - H\dot{\tau} - \eta\frac{\partial H}{\partial q} - \xi\frac{\partial H}{\partial p} - \tau\frac{\partial H}{\partial t} = \dot{f}. \qquad \text{(A.2.10)}$$

Note that the effect of the presence of the gauge function, f, is to change the value of the Action Integral from A to $\bar{A}$ when the transformation (A.2.5) is applied and reduces to zero when $\varepsilon = 0$, *ie* the transformation is the identity.

If we take τ, η, ξ and f to be functions of q, p and t, (A.2.10) becomes

$$p\left(\frac{\partial\eta}{\partial t} + \dot{q}\frac{\partial\eta}{\partial q} + \dot{p}\frac{\partial\eta}{\partial p}\right) + \xi\dot{q} - H\left(\frac{\partial\tau}{\partial t} + \dot{q}\frac{\partial\tau}{\partial q} + \dot{p}\frac{\partial\tau}{\partial p}\right) - \eta\frac{\partial H}{\partial q} - \xi\frac{\partial H}{\partial p} - \tau\frac{\partial H}{\partial t} = \frac{\partial f}{\partial t} + \dot{q}\frac{\partial f}{\partial q} + \dot{p}\frac{\partial f}{\partial p}. \qquad \text{(A.2.11)}$$

Since the variational principle has not been invoked, Hamilton's equations of motion do not apply and $\dot{q}$ and $\dot{p}$ are independent. Thus (A.2.11) may be separated to give

$$\frac{\partial f}{\partial p} = p\frac{\partial\eta}{\partial t} - H\frac{\partial\tau}{\partial p} \qquad \text{(A.2.12)}$$

$$\frac{\partial f}{\partial q} = p\frac{\partial \eta}{\partial q} + \xi - H\frac{\partial \tau}{\partial q} \tag{A.2.13}$$

$$\frac{\partial f}{\partial t} = p\frac{\partial \eta}{\partial t} - H\frac{\partial \tau}{\partial t} - \eta\frac{\partial H}{\partial q} - \xi\frac{\partial H}{\partial p} - \tau\frac{\partial H}{\partial t} \tag{A.2.14}$$

and these are the equations to be solved to determine the admissible symmetries.

If we regroup some terms in (A.2.10), we have

$$(f + H\tau - p\eta)^{\dot{}} = 0 \tag{A.2.15}$$

when Hamilton's equations of motion are invoked. Hence it follows that the symmetry G as determined by the solution of eqq (A.2.12–A.2.14) has associated with it a first integral

$$I = f + H\tau - p\eta \tag{A.2.16}$$

for the system described by the Hamiltonian, $H(q,p,t)$. We emphasise that the symmetry exists independently of the integral which follows from the additional requirement that the variational principle be applied.

For problems in higher dimensions the relevant expressions are

$$G = \tau\frac{\partial}{\partial t} + \eta_i\frac{\partial}{\partial q_i} + \xi_i\frac{\partial}{\partial p_i} \tag{A.2.17}$$

$$I = f + H\tau - p_i\eta_i, \tag{A.2.18}$$

where τ, η_i, ξ_i and f satisfy

$$\frac{\partial f}{\partial p_i} = p_j\frac{\partial \eta_j}{\partial p_i} - H\frac{\partial \tau}{\partial p_i} \tag{A.2.19}$$

$$\frac{\partial f}{\partial q_i} = p_j\frac{\partial \eta_j}{\partial q_j} - H\frac{\partial \tau}{\partial q_i} + \xi_i \tag{A.2.20}$$

$$\frac{\partial f}{\partial t} = p_j\frac{\partial \eta_j}{\partial t} - H\frac{\partial \tau}{\partial t} - \eta_j\frac{\partial H}{\partial q_j} - \xi_j\frac{\partial H}{\partial p_j} - \tau\frac{\partial H}{\partial t} \tag{A.2.21}$$

and summation from 1 to n is implied by repeated indices and a free index indicates one of n equations.

In general eqq (A.2.19–A.2.21) are to be solved for τ, η_i, ξ_i and f and this cannot be done without some further information or *ansatz.* Although in Hamiltonian mechanics $\mathbf{q}$ and $\mathbf{p}$ are equivalent, in practice $\mathbf{p}$ occurs in a preferred mode. Hence one would normally specify the $\mathbf{p}$ dependence in τ, η_i and ξ_i. That of f follows from (A.2.19).

A simplification does occur if the transformation in space and time is not restricted to a point transformation. Differentiating (A.2.18) with respect to p_j one finds that, given (A.2.19),

$$\frac{\partial I}{\partial p_j} = \frac{\partial H}{\partial p_j}\tau - \eta_j. \tag{A.2.22}$$

If the η_j are permitted to be momentum dependent, there is no loss of generality in setting

$$\tau = 0 \tag{A.2.23}$$

which does simplify the appearance of (A.2.19–A.2.21). The relationship

$$\eta_j = -\frac{\partial I}{\partial p_j}, \tag{A.2.24}$$

or equivalently

$$I = -\int \eta_i \mathrm{d}p_i + g(\mathbf{q}, t), \tag{A.2.25}$$

does emphasise the close relationship between the momentum dependence of the first integral and the symmetry since the $\mathbf{p}$-dependence of f and the ξ_i follows from (A.2.19) and (A.2.20).

A.3 Noetherian integrals via nonlocal transformation

The derivation in §A.1 was completely independent of the functional dependence of f, ξ and η. Consequently Noether's Theorem applies equally well to nonlocal symmetries as it does to the generalised symmetries used by Noether [154, 155].

To take an illustrative, and so trivial, example we consider the free particle with Lagrangian

$$L = \tfrac{1}{2}y'^2. \tag{A.3.1}$$

The corresponding equation of motion, *viz*

$$y'' = 0, \tag{A.3.2}$$

has many nonlocal symmetries one of which is

$$G = \left(\int y\mathrm{d}x\right)\frac{\partial}{\partial x} + \tfrac{1}{2}y^2\frac{\partial}{\partial y}. \tag{A.3.3}$$

The substitution of (A.3.1) and (A.3.3) into (A.1.7) leads to

$$\begin{aligned} f' &= \tfrac{1}{2}yy'^2 \\ f &= \tfrac{1}{2}\int yy'^2\mathrm{d}x \end{aligned} \tag{A.3.4}$$

so that (A.1.9) yields the first integral

$$I = \tfrac{1}{2}y'^2\int y\mathrm{d}x - \tfrac{1}{2}y^2y' + \tfrac{1}{2}\int yy'^2\mathrm{d}x \tag{A.3.5}$$

$$= \tfrac{1}{2}y'^2\int y\mathrm{d}x - \tfrac{1}{4}y^2y' \tag{A.3.6}$$

in which the equation of motion (A.3.2) has been taken into account in the integration of the second quadrature of (A.3.5). Although (A.3.6) satisfies the formal requirement for a first integral in that

$$\left.\frac{\mathrm{d}I}{\mathrm{d}x}\right|_{y''=0} = 0, \tag{A.3.7}$$

it could be regarded as being somewhat unuseful since its evaluation requires a knowledge of $y(x)$. This is so even though (A.3.3) is a useful nonlocal symmetry since under reduction of order of (A.3.2) by the obvious symmetry $\partial/\partial x$ it becomes

$$G_r = \tfrac{1}{2}u^2\frac{\partial}{\partial u}, \tag{A.3.8}$$

where $u = y$ and $v = y'$ are the zeroth order and first order differential invariants [55]. However, this example was more for the illustration of the idea than an attempt at a serious treatment of the solution of (A.3.2).

For a more compelling example consider the nonlinear differential equation

$$y'' = \frac{y'^2}{y} + a(x)yy' + a'(x)y^2 \tag{A.3.9}$$

which we encountered earlier. A nonlocal Lagrangian for (A.3.9) is readily found and is

$$L = \tfrac{1}{2}\left(\frac{y'}{y} - ay\right)^2 \exp\left[-\int ay\mathrm{d}x\right]. \tag{A.3.10}$$

A nonlocal symmetry of (A.3.9) is [8]

$$G_1 = y\exp\left[\int ay\mathrm{d}x\right]\frac{\partial}{\partial y}. \tag{A.3.11}$$

Substitution of (A.3.11) into (A.1.7) gives

$$f' = 0 \tag{A.3.12}$$

and so f may be taken as zero in (A.1.9). The first integral associated with the nonlocal symmetry (A.3.11) is calculated from (A.1.9) to be

$$I = \frac{y'}{y} - ay. \tag{A.3.13}$$

Eq (A.3.9) does have another exponential nonlocal symmetry [8], *viz*

$$G_2 = y\exp\left[-\int ay\mathrm{d}x\right]\frac{\partial}{\partial y} \tag{A.3.14}$$

which is a Noether symmetry for the Lagrangian when there is the nonlocal gauge function

$$f = \left(\frac{y'}{y} - ay\right)\left(\exp\left[-2\int ay\mathrm{d}x\right] + C\right). \tag{A.3.15}$$

However, the first integral obtained is the same as that with G_1, *ie* (A.3.13).

There are several points to be considered in the use of nonlocal symmetries in Noether's theorem. The example of equation (A.3.9) shows that a nonlocal Noether symmetry can be used to obtain a regular first integral from Noether's Theorem. This must be regarded as useful. That this did not occur in the illustrative example of the free particle indicates that one cannot always expect to be successful in obtaining useful first integrals. That the nonlocal symmetries which produced the first integral (A.3.13) are exponential nonlocal symmetries [11] is suggestive. We are quite prepared to accept that the standard concept of a first integral does not encompass the function in (A.3.6) even though it does satisfy the normal requirement that its first derivative be zero when the differential equation is taken into account.

In the context of differential equations it does not seem to be of material import whether a Lagrangian be of standard form or contain nonlocal aspects. However, it becomes another matter if one thinks of the Lagrangian as the pathway to Hamiltonian Mechanics and the implications in various branches of Physics. There is no doubt that a Hamiltonian formulation is possible. Proceeding formally we find that the momentum for (A.3.10) is

$$\begin{aligned} p &= \frac{\partial L}{\partial y'} \\ &= \frac{1}{y}\left(\frac{y'}{y} - ay\right)\exp\left[-\int ay\mathrm{d}x\right] \end{aligned} \tag{A.3.16}$$

and consequently the Hamiltonian is

$$\begin{aligned} H &= py' - L \\ &= apy^2 + \tfrac{1}{2}p^2y^2\exp\left[\int ay\mathrm{d}x\right] \end{aligned} \tag{A.3.17}$$

which could be interpreted as the Hamiltonian of a particle in one degree of freedom with a mass which depends upon both position and time. (The expression $\int ay\mathrm{d}x$ must be regarded as a function of the independent variable, x, whenever total differentiation with respect to it is being performed.) Hamilton's equations of motion lead to (A.3.9) in the usual way.

What we have done is to demonstrate that nonlocal symmetries can be used in the context of Noether's theorem to produce realistic first integrals. What is needed is a systematic procedure for the determination of Noetherian nonlocal symmetries as has been proposed for second order ordinary differential equations [83]. It would be interesting to consider the development of a complete Hamiltonian formalism for nonlocal Hamiltonians. Whilst at first sight it may seem to be impractical to require a knowledge of the solution (to evaluate the integral in the nonlocal symmetry) to obtain the solution, this may not be as strange as it seems. We recall, for example, that for some time–dependent Hamiltonian systems it is possible to obtain formal solutions which require just the solution of a single ordinary differential equation to obtain the numerical results. A typical example is in the calculation of expectation values [123] or Berry's phase [125] for nonautonomous Hamiltonian systems.

Appendix B

Computer Methods

B.1 Introduction

A few manual calculations of Lie point symmetries will easily convince one of the need to implement the procedure via computer algebra packages. Due to its algorithmic nature, the calculation of Lie point symmetries can be programmed. To this end a number of packages do exist. The recent trend has been to move away from the stand alone packages of the past to routines running under the more general, broadly applicable computer algebra systems like *Mathematica*, *Maple*, *Macsyma*, *Reduce etc.* Hereman [97] has composed an excellent survey of currently available packages, discussing their strengths, weaknesses and ways to obtain them. It is reassuring to note that almost all these packages are free.

We are concerned with one package in particular, that of Alan Head [96]. The package, `PROGRAM LIE` (hereinafter referred to as `LIE`), was written utilising the (now defunct) package *Mumath* that ran under DOS. However, it is distributed as a stand alone package without the need for *Mumath* or its manuals. The main criticism of `LIE` is that it only accesses 256Kb RAM – a deficiency of *Mumath* itself – and so cannot cope with 'large' problems. However, 'large' here is relative. It is remarkable what can be achieved in only 256Kb.

LIE is, to some extent, an interactive package. The user is encouraged to 'help' the program along. However, the degree of automation is superior to most available packages. Essentially the user creates a file containing the equation(s) and reads it into LIE. Thereafter a series of commands is invoked which i) calculate the determining equations, ii) solve these equations, iii) check the solution, iv) calculate the symmetries and v) determine the non-zero Lie Bracket relationships. While the files `lie.doc`, `morelie.doc` and `readme.1st` contain some documentation, we will embark on a detailed exposition of how to use LIE based on our experiences.

B.2 The Input File

Let us first consider a standard input file:

```
ECHO:TRUE $
NIND#:1   $
NDEP#:1   $
DE#:DV#:{} $
DE#[1]:  $
DV#[1]:  $
ECHO:FALSE $
RDS() $
```

The first and penultimate commands turn the output of LIE to the standard console on and off, respectively. The first 'proper' command is `NIND#` which is the variable that stores the number of independent variables. Similarly `NDEP#` stores the number of dependent variables. The fourth line is an internal *Mumath* command that sets up the variables `DE#` and `DV#` as arrays. The differential equation is entered after the command `DE#[1]` and the variable to be eliminated (usually the highest derivative) is inserted after `DV#[1]`. The equation *must* be linear in this term. Obviously one would have n pairs of `DE#` and `DV#` for a system of n equations. The final command simply returns `LIE` to accepting input from the keyboard.

Remarks: Mumath is an uppercase system. All expressions *must* be entered in uppercase. The colon, ':', is the *Mumath* assignment sign. Thus `NDEP#: 2` is read as 'the number of dependent variables equals two'. The percent sign, '%', brackets comments. Each line must be terminated with a '$'. (This applies to files in general. It is usual to use ';' for the interactive procedure.)

Equations must be written in the form

$$y^{(n)} - f\left(x, y, y', \ldots, y^{(n-1)}\right) = 0 \tag{B.2.1}$$

(with obvious extensions to systems and partial differential equations) and entered without the '= 0' part. `LIE` interprets `U(i)` as a dependent variable and `Xi` as an independent variable. Thus the dependent variables u and w will be entered as `U(1)` and `U(2)` while the independent variables x and t will be entered as `X1` and `X2`. There are two equivalent techniques to enter derivatives. The first is to use the `DIF` command in the following way: $\partial^3 u/\partial x \partial y^2$ is entered as `DIF(U(1),X1,X2,2)`. The second, in our opinion the more convenient, way is to write $\partial^3 u/\partial x \partial y^2$ as `U(1,1,2,2)`. The first number in the argument of `U` is used to determine the dependent variable. The remaining numbers determine the order of differentiation (by the number of integers excluding the first) and the number of occurrences of a particular number determines the number of times the dependent variable is differentiated with respect to that independent variable.

Other functions and constants that can be used are `#PI`, `#E`, `#I`, `SIN`, `COS`, `LOG` (or `LN`), `ERF` and `EI`. However, `TAN`, `SEC` or inverse trigonometric functions cannot be used. While it is strongly suggested that square roots, fractional powers of derivatives and fractions not be used (and it *is* preferable to avoid them), this is not, in our opinion, a rigid rule.

The commands '+', '−', '/' and '∗' are self-explanatory. We do stress that '∗' should be used as a rule to indicate multiplication even though it is not always necessary. This serves to prevent a misinterpretation of input by `LIE`. Consider the expression $A \times (x+y)$, where A is an arbitrary constant. Entering the expression as `A (X1 + X2)` causes the program to interpret this as A being a function of $x+y$. The correct way to enter the expression is `A * (X1 + X2)`.

We illustrate the above with some examples.

1. Equation:

$$y'' + y' + y = 0$$

LIE input file:

```
ECHO:TRUE $
NIND#:1   $
NDEP#:1   $
DE#:DV#:{} $
DE#[1]: U(1,1,1) + U(1,1) + U(1) $
DV#[1]: U(1,1,1) $
ECHO:FALSE $
RDS() $
```

2. Equations:

$$u_{xyz} + v_{xx} = 0$$

$$w_{xz} + Kz = v_x + u_y$$

$$v_x + w_y = 0$$

LIE input file:

```
ECHO:TRUE $
NIND#:3   $
NDEP#:3   $
DE#:DV#:{} $
DE#[1]: U(1,1,2,3) + U(2,1,1) $
DV#[1]: U(1,1,2,3) $
DE#[2]: U(3,1,3) + K * X3 - U(2,1) - U(1,2) $
DV#[2]: U(3,1,3) $
DE#[3]: U(2,1) + U(3,2) $
DV#[3]: U(2,1) $
ECHO:FALSE $
RDS() $
```

3. Equation:

$$y''' + f(x)y^2 = 0$$

LIE input file:

```
ECHO:TRUE $
NIND#:1   $
NDEP#:1   $
DE#:DV#:{} $
DE#[1]: U(1,1,1,1) + F(X1) * U(1)^2 $
DV#[1]: U(1,1,1,1) $
ECHO:FALSE $
RDS() $
```

Note that some complications may arise in the initial stages of compiling the input file when systems of equations are being studied. The main cause of difficulty is due to cyclic substitution via DV#. A simple rearrangement of the order of the equations normally solves the problem. If not, different terms should be substituted in DV#.

B.3 Processing the Input

Now that we have created the input file we need to compile it using LIE. To help illustrate the procedure we use the equation

$$y''' + y' - x = 0. \tag{B.3.1}$$

We execute LIE by typing MULIE at the DOS prompt. The user is then presented with the Mumath prompt '?'. The normal order of commands is as follows:

(i) RDS(FILENAME,DAT);

(ii) DOLIE();

(iii) A#; (optional)

(iv) `DOSOLV();`

(v) `DOCHECK();`

(vi) `DOVEC();`

(vii) `DONZC();`

The first command reads in the input file and checks whether the *syntax* is correct. Obviously `LIE` cannot determine if the differential equation it has read in conforms to the one the user wishes to analyse. If there is a problem with the input, the user can exit the program using either the command `SYSTEM();` at the prompt or `Ctrl-C` and then `Y`. The input file can then be re-edited.

We read in our example file using `RDS(EG1,DAT);`

```
? RDS(EG1,DAT);
: EG1
?
?
NIND#:1   $
?
NDEP#:1   $
?
DE#:DV#:{} $
?
DE#[1]: U(1,1,1,1) + U(1,1) - X1 $
?
DV#[1]: U(1,1,1,1) $
?
ECHO:FALSE $
?
```

As there are no problems we can immediately invoke `DOLIE();`

```
? DOLIE();
Program LIE  v. 4.3 (c) 1994  A K Head

(22)
(21)
(6 13)
```

```
: Def Eqns =  (6, 13)
```

This command calculates the determining (or defining) equations and gives the user an update on its progress. The first number, 22, indicates the number of terms resulting from the operation of the third extension of the generator on `DE#[1]`. The second, 21, is the number of terms in the expression after substituting for `DV#[1]`. The expression is then split into 6 equations in 13 terms. LIE attempts to simplify them further (but fails in this example) and saves the result (still 6 equations in 13 terms) as `A#`. (This simplification process can be turned off with `SCUT#:  FALSE;`)

We inspect the equations (not necessary but useful) using `A#;`

```
? A#;
: {DIF (F# (1, U1, X1), U1, 2),
    DIF (F# (1, U1, X1), U1, 3),
    DIF (F# (1, U1, X1), U1, 2, X1),
    DIF (F# (1, U1, X1), U1, X1) - DIF (F# (2, X1), X1, 2),
    3*DIF (F# (1, U1, X1), U1, X1, 2) + 2*DIF (F# (2, X1), X1)
- DIF (F# (2, X1), X1, 3),
    X1*DIF (F# (1, U1, X1), U1) - 3*X1*DIF (F# (2, X1), X1) +
DIF (F# (1, U1, X1), X1) + DIF (F# (1, U1, X1), X1, 3) - F# (2, X1),
    UUU#1 == F# (1, U1, X1),
    XXX#1 == F# (2, X1)}
?
```

The first six expressions are the equations to be solved. The last two are the forms of η (`UUU#1`) and ξ (`XXX#1`) respectively. Note that some simplification has already taken place: the function representing ξ, `F# (2, X1)`, only depends on x. In instances where `DOSOLV();` cannot solve the equations, one is at least presented with the determining equations. This is, in fact, the most that some packages do.

We attempt to solve these equations using `DOSOLV();`

```
? DOSOLV();
```

```
(3 12 3 1)
(4 12 1 3)
(4 13 4 1)
(4 14 4 1)
(4 15 4 3)
(4 16 5 2)
(3 17 2 2)
(3 18 4 1)
(2 13 2 2)
(3 13 1 1)
(2 10 2 1)
(1 8 2 1)
(1 9 4 1)
(1 13 5 1)
(0 0 2 1)
```

```
: {UUU#1 == -U1*COS (X1)*F# (9) - U1*SIN (X1)*F# (8) - U1*F# (6)
+ X1*COS (X1) *F# (8) - X1*SIN (X1)*F# (9) - X1*F# (10)
+ X1^2*COS (X1)*F# (9)/2 + X1^2*SIN (X1)*F# (8)/2 + X1^2*F# (6)/2
- 3/4*COS (X1)*F# (9) - COS (X1)*F# (13) - SIN (X1)*F# (8)/4
- SIN (X1)*F# (12) - F# (6) - F# (11),
    XXX#1 == COS (X1)*F# (8) - SIN (X1)*F# (9) - F# (10)}
```

The four numbers $(a\ b\ c\ d)$ are interpreted as follows: there are a equations containing b terms after applying operation c to what was the dth equation. (There are eight operations that `LIE` attempts on any system of differential equations as explained in the file `morelie.doc`) The final lines contain the explicit expressions for η and ξ. Note that the terms `F# (9)` *etc* are arbitrary constants. Sometimes unsolved equations precede these expressions.

It must be noted that the operations in `DOSOLV();` are heuristic and so it is advisable to check the calculations. This is accomplished by `DOCHECK();`

```
? DOCHECK();
```

```
: Check OK, proceed to DOVEC
```

In the event that the checking is not `OK` the user is prompted to `Do DOCHECK again`. This message may be repeated. If there is no end to this recursion in a finite time, one cannot accept the results of the calculation. A manual verification has to be undertaken.

The explicit form of the symmetries (or vectors as Head calls them in the documentation) is given by DOVEC();

```
? DOVEC();

Vectors
VEC# (1) == -D# (U1)
VEC# (2) == -SIN (X1)*D# (U1)
VEC# (3) == -COS (X1)*D# (U1)
VEC# (4) == -X1*D# (U1) - D# (X1)
VEC# (5) == -U1*D# (U1) + X1^2*D# (U1)/2 - D# (U1)
VEC# (6) == -U1*SIN (X1)*D# (U1) + X1*COS (X1)*D# (U1)
+ X1^2*SIN (X1)*D# (U1)/2 + COS (X1)*D# (X1) - SIN (X1)*D# (U1)/4
VEC# (7) == -U1*COS (X1)*D# (U1) - X1*SIN (X1)*D# (U1)
+ X1^2*COS (X1)*D# (U1)/2 - 3/4*COS (X1)*D# (U1) - SIN (X1)*D# (X1)
:
```

We observe that there are seven symmetries. Noting that D# (U1) and D# (X1) represent $\partial/\partial y$ and $\partial/\partial x$ respectively makes reading off these symmetries trivial, *eg*

$$G_5 = \left(\tfrac{1}{2}x^2 - y - 1\right)\frac{\partial}{\partial y}. \tag{B.3.2}$$

It is advisable to check that no symmetries are repeated.

The Lie algebra is determined from the Lie Bracket relationships. The nonzero relationships are found using DONZC();

```
? DONZC();

Non Zero Commutators
NZCOM (1, 5) == -VEC# (1)
NZCOM (1, 6) == -VEC# (2)
NZCOM (1, 7) == -VEC# (3)
NZCOM (2, 4) == VEC# (3)
NZCOM (2, 5) == -VEC# (2)
NZCOM (2, 6) == -VEC# (1)
NZCOM (3, 4) == -VEC# (2)
NZCOM (3, 5) == -VEC# (3)
```

```
NZCOM (3, 7) == -VEC# (1)
NZCOM (4, 6) == U1*COS (X1)*D# (U1) + X1*SIN (X1)*D# (U1)
- X1^2*COS (X1)*D# (U1)/2 + COS (X1)*D# (U1)/4 + SIN (X1)*D# (X1)
NZCOM (4, 7) == -U1*SIN (X1)*D# (U1) + X1*COS (X1)*D# (U1)
+ X1^2*SIN (X1)*D# (U1)/2 + COS (X1)*D# (X1) - 3/4*SIN (X1)*D# (U1)
NZCOM (5, 6) == -3/4*VEC# (2)
NZCOM (5, 7) == -VEC# (3)/4
NZCOM (6, 7) == VEC# (4)
:
```

Note that, while LIE attempts to express the result in terms of the original symmetries, it cannot do so when the result is a combination of symmetries. It is easy to work out those that LIE cannot handle, *eg*

$$[G_4, G_7] = \tfrac{1}{2}G_3 - G_7. \tag{B.3.3}$$

B.4 Lie–Bäcklund and contact symmetries

A recent addition to LIE is the ability to calculate Lie–Bäcklund and contact symmetries. The former depends on the order of the Lie–Bäcklund symmetries required. The input file changes with each equation considered. We refer the interested reader to the README.1ST file of version 4.3 for further details. In the case of contact symmetries we enter the equation as a system. Consider the equation

$$y''' + y = 0. \tag{B.4.1}$$

If we set

$$v = y', \tag{B.4.2}$$

we can write (B.4.1) as the system

$$v = y' \tag{B.4.3}$$

$$v'' + y = 0. \tag{B.4.4}$$

If we enter the above system into LIE verbatim, the results are incorrect as LIE treats v' and y'' as independent quantities. The correct method is to add the equation

$$v' = y'' \tag{B.4.5}$$

into the input file as well. The procedure is as before. The only difference will be reading one of the dependent variables as y'.

B.5 Troubleshooting and Helpful Hints

The most frequently encountered problem in using LIE is that of the exhaustion of memory. (At the `Abort, DOS?` prompt it is recommended that `D` be entered and LIE started afresh.)

If the problem arises in the process of `DOLIE();`, two options are available. The first is to rearrange the equations so that the simpler ones are analysed first. However, what may seem simpler to the user may not seem so to LIE and so, if this does not work, different combinations should be taken. The other option is to introduce new dependent variables to lower the order of the system. Caution must be exercised in extrapolating the results to the original equation.

If the memory problem occurs in `DOSOLV();`, there are further options available than those mentioned above. If the equations contain more `COS` functions than `SIN` functions, setting `TRGSQ:-1;` at the start of a LIE session (before reading in the input file) will replace $\sin^2$ by $1 - \cos^2$ instead of vice versa which is the default setting.

The lowest priority operation is number 8 which adds the integrability conditions to the determining equations. This is often long and tedious, but it sometimes helps to add these conditions before attempting the other operations in `DOSOLV();`. This is accomplished by entering `DOINTCON();` after `DOLIE();` has run its course. Thereafter `DOSOLV();` *should* have an easier time.

The remaining occurrence of memory problems is during `DOCHECK();`. Fortunately the latest version of LIE has a new command `TESTVEC();`. After obtaining the symmetries using `DOVEC();` we can test if one, say G_2, is indeed a symmetry of the equation using `TESTVEC( V#[2] );`.

Sometimes, during the solution of the determining equations, LIE has trouble carrying out some of the integrations. This is especially true when the differential equation contains arbitrary functions. In some instances a new route to the solution is found, but in others not. Consider the Emden–Fowler equation of index two [122, 57, 66, 67, 68, 69]

$$y'' + g(x)y^2 = 0. \tag{B.5.1}$$

It is not advisable to enter the equation into LIE in the form `U(1,1,1) + G(X1) * U(1)^2` as LIE cannot integrate `G(X1)^(2/5)`. The problem is circumvented by replacing the arbitrary function `G(X1)` with another, equally arbitrary, function that is easily integrated, *viz* `DIF( H(X1), X1)^(5/2)`. Now LIE reports that the equation has no symmetries, as expected.

In spite of one's (and LIE's) best efforts there are often equations that are returned as unsolved in the route to determining the symmetries. If one can solve these manually (as in the case of Euler equations), the solution can be inserted into LIE using `EVSA#();`. Consider the output

```
: {6*X1*DIF (F# (4, X1), X1, 3) - 3*X1^2*DIF (F# (4, X1), X1, 4)
+ X1^3*DIF (F# (4, X1), X1, 5) - 6*DIF (F# (4, X1), X1, 2),
   UUU#1 == -2*U1*F# (5) + F# (4, X1),
   XXX#1 == -X1*F# (5)}
```

The equation that LIE cannot solve is the Euler equation

$$6x\frac{\mathrm{d}^3 f_4(x)}{\mathrm{d}x^3} - 3x^2\frac{\mathrm{d}^4 f_4(x)}{\mathrm{d}x^4} + x^3\frac{\mathrm{d}^5 f_4(x)}{\mathrm{d}x^5} - 6\frac{\mathrm{d}^2 f_4(x)}{\mathrm{d}x^2} = 0 \tag{B.5.2}$$

which is easily solved to give

$$f_4(x) = a + bx + cx^3 + dx^4 + ex^5. \tag{B.5.3}$$

We substitute this solution into LIE using

```
EVSA#( F#(4,X1), F#(10) + F#(11) X1 + F#(12) X1^3 + F#(13) X1^4
+ F#(14) X1^5);
```

We then proceed with DOCHECK(); *etc.*

The command EVSA#(); could also be used to verify whether a given symmetry is a symmetry of a particular equation. However, a more convenient method is to use the command TESTVEC();. If we want to verify whether

$$\frac{\partial}{\partial x} \tag{B.5.4}$$

is a symmetry of

$$y'' + x^2 y = 0 \tag{B.5.5}$$

we simply enter (after DOLIE();) TESTVEC(D# (X1)); which returns FALSE. This technique is useful for single symmetries.

Experience dictates that the expressions for ξ and η are often polynomials. LIE has the commands DOPOLYALL(); and DOPOLY(); which assumes polynomial expansions for all the unknowns and specified unknowns respectively. They can only be invoked once the determining equations have been calculated, *ie* after DOLIE();. We can assume that all the unknown functions are polynomials of degree four by entering DOPOLYALL(4);. Running through DOSOLV(); discards the invalid terms in the expressions. For specific functions, F#(2,U1,X1), say, we use DOPOLY(F#(2,U1,X1),4);. Obviously any symmetries found will belong to the full Lie algebra of symmetries. However, there is no way to verify that these are all the symmetries that exist for the equation.

B.6 Miscellaneous

Recent versions of LIE (since 4.1) have been bundled with the DOS program PRN2FILE. This public domain package is useful to save the output of LIE to a file. PRN2FILE is run with an option indicating the name of the file to which output should be directed, *viz* PRN2FILE LIE.OUT. This will redirect printer directed output into the file LIE.OUT. In LIE, Ctrl-P toggles printing. Thus one can ensure that only the relevant parts of the calculation are saved, *eg* the symmetries and nonzero Lie Brackets. Once the printing is complete, running PRN2FILE (with no options) rechannels printer directed output to the printer.

LIE is obtainable from all SIMTEL repositories by anonymous ftp. It is normally in a `math` or `education` subdirectory on most of the popular anonymous ftp servers. The package is free and the author requires no registration.

Given the success of Head's routines for solving linear partial differential equations and the problem of memory constraints for LIE it is desirable to implement the algorithm on an unrestricted system. Discussions with this goal in mind are currently underway [153]. The idea is to implement LIE on *Mathematica* [185].

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