AN INVESTIGATION OF GRADE 11 LEARNERS' UNDERSTANDING OF THE *COSINE* FUNCTION WITH *SKETCHPAD*

BY

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ABSTRACT

AN INVESTIGATION OF GRADE 11 LEARNERS' UNDERSTANDING OF THE *COSINE* FUNCTION WITH *SKETCHPAD*

This study investigated how Grade 11 learners from a school in KwaNdengezi, near Pinetown, in Durban, understood the *cosine* function with software known as *The Geometer's Sketchpad*. This was done on the basis of what they had learnt in Grade 10. The timing was just before they had covered the topic again in their current grade.

The researcher hoped, by using *The Geometer's Sketchpad,* to contribute in some small way to teaching and learning methods that are applicable to the subject. This may also, hopefully, assist and motivate both teachers and learners to attempt to recreate similar learning experiences in their schools with the same or similar content and concepts appropriate to them.

In this research project, data came from learners through task-based interviews and questionnaires. The school was chosen because of the uniqueness of activities in most African schools and because it was easily accessible. Most learners do not have access to computers both in school and at home. This somehow alienates them from modern learning trends. They also, in many occasions, find it difficult to grasp the knowledge they receive in class since the medium of instruction is English, a second language to them.

Another reason is the nature of the teaching and learning process that prevails in such schools. The Primary Education Upgrading Programme, according to Taylor and Vinjevold (1999), found out that African learners would mostly listen to their teacher through-out the lesson. Predominantly, the classroom interaction pattern consists of oral input by teachers where learners occasionally chant in response. This shows that questions are asked to check on their attentiveness and that tasks are oriented towards information acquisition rather than higher cognitive skills. They tend to resort to

memorisation.

Despite the fact that trigonometry is one of the topics learners find most challenging, it is nonetheless very important as it has a lot of applications. The technique of triangulation, which is used in astronomy to measure the distance to nearby stars, is one of the most important ones. In geography, distances between landmarks are measured using trigonometry. It is also used in satellite navigation systems. Trigonometry has proved to be valuable to global positioning systems. Besides astronomy, financial markets analysis, electronics, probability theory, and medical imaging (CAT scans and ultrasound), are other fields which make use of trigonometry.

A study by Blackett and Tall (1991), states that when trigonometry is introduced, most learners find it difficult to make head or tail out of it. Typically, in trigonometry, pictures of triangles are aligned to numerical relationships. Learners are expected to understand ratios such as *Cos A= adjacent/hypotenuse*. A dynamic approach might have the potential to change this as it allows the learner to manipulate the diagram and see how its changing state is related to the corresponding numerical concepts. The learner is thus free to focus on relationships that are of prime importance, called the principle of selective construction (Blackett & Tall, 1991). It was along this thought pattern that the study was carried-out.

Given a self-exploration opportunity within *The Geometers' Sketchpad*, the study investigated learners' understanding of the *cosine* function from their Grade 10 work in all four quadrants to check on:

- ❖ What understanding did learners develop of the *Cosine* function as a function of an angle in Grade 10?
- ❖ What intuitions and misconceptions did learners acquire in Grade 10?
- ❖ Do learners display a greater understanding of the *Cosine* function when using *Sketchpad*?

In particular,

- ❖ As a ratio of sides of a right-angled triangle?

- As a functional relationship between input and output values and as depicted in graphs?

The use of *Sketchpad* was not only a successful and useful activity for learners but also proved to be an appropriate tool for answering the above questions. It also served as a learning tool besides being time-saving in time-consuming activities like sketching graphs. At the end, there was great improvement in terms of marks in the final test as compared to the initial one which was the control yard stick.

However, most importantly, the use of a computer in this research revealed some errors and misconceptions in learners' mathematics. The learners had anticipated the ratios of sides to change when the radius of the unit circle did but they discovered otherwise. In any case, errors and misconceptions are can be understood as a spontaneous result of learner's efforts to come up with their own knowledge. According to Olivier (1989), these misconceptions are intelligent constructions based on correct or incomplete (but not wrong) previous knowledge. Olivier (1989) also argues that teachers should be able to predict the errors learners would typically make. They should explain how and why learners make these errors and help learners to correct such misconceptions. In the analysis of the learners' understanding, correct understandings, as well as misconceptions in their mathematics were exposed. There also arose some cognitive conflicts that helped learners to reconstruct their conceptions.

DECLARATION

I, Calisto Majengwa (209529036), declare that the research involved in my dissertation submitted for the Masters of Education degree in Mathematics, entitled "An investigation of Grade 11 learners' understanding of the *cosine* function with *Sketchpad* " represents my own and original work.

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Date:

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CONTENTS

CHAPTER ONE

Introduction

1.1 Background to the study

Curriculum reform appears to be pushing teachers to a paradigm shift in their practices towards more participatory and inquiry-based methods (Pournara, 2001) where learners' meanings are given greater credence. This places demand on teachers since they are expected to understand learners' meanings and mediate between learners' personal meanings and public mathematical meanings. It is therefore important, according to Pournara (2001, p.2) *"that we understand how learners make sense of mathematical concepts in order to support teachers in making the transition to new pedagogical approaches in the teaching of mathematics."*

My broad personal experience of teaching trigonometry at Grade 10 to 12 level, observation and discussion with other teachers support the findings that the mathematical knowledge of secondary schools learners is dominated by content and teacher- centred pedagogies. Learners have constantly indicated difficulty in learning trigonometric functions whenever the topic is being done. Many a mathematics teacher also see it an uphill task to aid learners to make sense out this topic. It is hoped that giving learners some sort of visual intuition about circles, angles, and graphs, might help create more meaningful relational understanding and, possibly, eliminate some unnecessary misconceptions.

As a teacher of mathematics in high school, I am aware of the struggles learners face when trying to understand this subject. I was looking for a way which would help learners create concrete knowledge on their own. The idea was to expose the learners to a picture of the unit circle with a right angle in it when they thought about the *cosine* function. It was hoped that such a dynamic sketch would help learners gain first-hand experience and conviction of relationships in trigonometry. A concrete example would be why the *cosine* of a given angle changes depending on the given quadrant. This research focused on learners' understanding of the *cosine* function, after some initial introduction, hence the choice of Grade 11 level.

According to the constructivist perspective, the teacher is not able to pass on knowledge as something complete without any flaws to learners. They ought to construct or reconstruct concepts for themselves, as they do not easily accommodate or assimilate knew ideas (Olivier 1989). Learning involves the interaction between a learner's previously acquired knowledge and new knowledge or concepts. This internal activity involves two interrelated processes according to Olivier (1989) called assimilation and accommodation. Accordingly, the use of dynamic software accords the learners an opportunity to experience varying conditions of aspects as they construct concepts themselves.

Many learners appear to have little understanding (Pournara, 2001) of underlying trigonometric principles. They resort to memorizing and applying procedures and rules even though many are able to do this successfully. They tend to ignore conceptual aspects of its objects. Skemp (1976), states that this has led many novice trigonometry learners to develop an instrumental rather than a relational understanding. They concentrate on trigonometry algorithms and learning 'how to' rather than 'why'. This study (2010) was mainly based on trying to find a way of changing the above scenario by using *Sketchpad.*

In most cases, to grasp an idea is basically to have it fit into an appropriate existing group of ideas, a schema. Consequently, if the new idea is very different from any of the available schemas, it would be impossible to stick to any of them. In such cases assimilation and accommodation is impossible as the learner creates a new "box" and tries to memorize the idea. This, according to Olivier (1989), is *rote-learning*. It is not related to any previously acquired knowledge. It will be difficult for the learner to understand this kind of knowledge. Such knowledge is isolated and cannot be recalled whenever necessary. It can be argued then, that the cause of many mistakes in mathematics is rote-learning. Learners try to recall things that they cannot fully remember. In most cases they seem to fail to link the concept of trigonometry to any of the previously learnt ones. It was then hoped that the use of dynamic computer software could assist them in linking it with other functions.

The second International Mathematics Study suggests that secondary school learners have not mastered the elementary pre-calculus topics of function, graphing and "teacher centred'' problem solving (Waits & Demana, 1998). Looking through recent research, it is possible to note the increase of studies based on the constructivist point of view, using the computer (Wenzelburger, 1992, Matthews, 2002, Powell & Kalina 2009). Also documented are increased studies in mathematics in everyday life (Johnston-Wilder & Pimm, 2005, Taylor, 2000) as well as studies encompassing both contexts (Magina, 1994). Graphing using *Geometer's Sketchpad* is a lot easier and faster and gives a learner the opportunity to concentrate on other aspects of graphs rather than the sketching itself using free hand.

In trigonometry teaching, according to Hart (1981, p.22), *"there have been attempts to move away from a process-oriented style of teaching and learning which may have prevented leaner understanding of important concepts".* Recent research has sought to use computer software to improve understanding and simplification of concepts. This research attempts to cement this and further spread it to other mathematical topics. In the present curriculum, the topic of trigonometry has many aspects and takes a long time to complete. Some of the time-consuming aspects are the static sketches and graphs made by hand on chalk boards. This could easily be alleviated by the use of the computer.

Teacher education around the new curriculum has emphasized learner activity, participation and group work as central aspects of classrooms (Brodie, 1998). Teachers are encouraged to facilitate learning rather than provide instruction. A paradigm shift from practices is urged. The past practice is characterized by being teacher-centred and encouraging passive learners. The learners engage in individualized rote-learning rather creative and flexible thinking (National Department of Education, 2002). Although e-learning and e-classrooms are now a common sight in most private schools, they are still rare species in most government schools.

In recent years, according to the NCTM (1989), mathematics educators have focused attention on rethinking the process of mathematics education. Teachers and faculty are urged to improve not only the cognitive side of instruction, but also the emphasis on non-cognitive issues. These include learners' feelings, attitudes, beliefs, interests, expectations and motivations. Learners are most likely to change their attitudes towards mathematics and could be motivated when given a chance to use the computer in class as they are familiar with and enjoy playing games on computer, mobile phones and other gadgets.

Machado (1996, p.34) highlights that the teaching process might be contributing to errors and failure in mathematics. The emphasis is on *''formal procedures (algorithms and rules)",* unrelated to the concept that supports them. This prevents the flexibility of thought that is necessary for success in mathematics. The computer intervention comes in handy as an alternative teaching process that might avert and alleviate the errors and failure in this subject.

The importance of the use computers in mathematics is well researched (Tall, 1989; Leinhardt, Zaslavsky et, al 1990; Duren 1991). In particular, the benefit of the use of the computer software on learners' understanding of the concept (Breindenbach, Dubinsky et al 1992) and in developing a visual approach to transformation and graphs (Bloom, Comber et, al 1986; 1992), have been demonstrated. Mudaly (2004) further supports the use of computer software. He outlines that *Sketchpad* could be used effectively to answer mathematical questions. Trigonometry is one of the areas of mathematics most convenient for the use of a computer.

In most cases, in the day to day classroom teaching, when learners make errors, corrections are handed down by the teacher as an external authority. Usually learners do not use their own abilities to evaluate and correct their own work nor are they encouraged to do so. Von Glasersfeld (1987, p.14) has noted that this kind of correction is *"not completely satisfactory"* because it denies learners the opportunity to restructure their own conceptual schemas. This does give them the opportunity to have meaningful learning take place in their minds. They tend to give more importance to the answer than the working procedure.

From a constructivist point of view, according to Von Glasersfeld (1987), it cannot be assumed that simply telling someone that he/she has done something right results in powerful cognitive satisfaction, as long as rightness is assessed by someone else. 'Rightness' should be viewed as something that comes from self-introspection if it is to become a source of real satisfaction. It is argued that such cognitive satisfaction could be gained through investigative work in learner-centred teaching, which is most effective when mediated by a computer.

This study also sought to address the gap in the research literature on learners' understanding on trigonometric concepts. Not much research has been done on various content areas of mathematics internationally and locally. According to Pournara (2001), a survey of Dissertation Abstracts internationally identified only two master's dissertations/doctorates in the area of trigonometry in the period 1995-1999. There are some articles in mathematics teaching journals on methods of teaching trigonometry (e.g. Dooley, 1968; Satty, 1976) which are generally based on personal opinions and experiences rather than on empirical research, he adds. These have given little or no attention given to learners' thinking about trigonometry

1.2 Research questions

The purpose of this study was to find out whether or not *Sketchpad* could be of some importance as a mathematical tool for learners to better understand trigonometry. It also sought to find out if learners are not laboring under a misapprehension of the concept. This was done such that the researcher would not necessarily adopt an adversarial position as the researcher was not teaching these learners in that grade. Since the topic is introduced in Grade 10, the study does not completely throw cold water on efforts previously undertaken, but as a matter of necessity, tries to demystify a topic that deserves more than just a thoughtful consideration. *Sketchpad* was used to see if it could make a significant difference, to provide an important contribution to mathematics education.

The theoretical and empirical part of this research is focused on the following major research questions:

1. What understanding did learners develop of the *Cosine* function as a function of an angle in Grade 10?
2. What intuitions and misconceptions did learners acquire in Grade 10?
3. Do learners display a greater understanding of the *Cosine* function when using *Sketchpad*?

More specifically, given the self-exploration opportunity within *The Geometers' Sketchpad*, the study investigates the development of learners' understanding and misconceptions of the *cosine* function regards the following:

4. As a ratio of sides of a right angled triangles?
5. As a relation between the angle as input and a function value as output in the specific context of graphs?

The study tried to answer the research questions 1 and 2 using a preliminary test which was used as a control level. As the learners worked with *Sketchpad* to go through their tasks, more answers to research questions 1 and 2 also emerged. *Geometer's Sketchpad* was used to answer research question 3 which provided data on the quality of responses as the learners went through their research tasks, during probing and when they wrote the final test. The study tools, the tests and *Sketchpad* were used in relation to 4 and 5.

1.3 Outline of the report

Chapter two briefly discusses the importance of the history of mathematics for understanding how human beings or mathematicians learnt mathematics in general. More particularly, this chapter looks at the history of trigonometry and how it possibly provides some guidelines to designing a trigonometry curriculum. It serves to highlight the potential socio-cultural role a historical perspective of mathematics can have. It also provides information on broad educational and social policy, as a guideline to the South African curriculum, its guiding principles and frames of reference. The Revised National Curriculum Statement (RNCS) (DoE, 2002, p.13) also acknowledges the importance of the history of mathematics as it indicates that a Mathematics Learning Area should develop *"an appreciation for the diverse historical, cultural and social practices of Mathematics"*.

In chapter three some of the different approaches to trigonometry in the curriculum, are discussed and analysed in order to provide a background to the research. The difficulties of learning trigonometry are also discussed. Chapter four develops the theories of learning and the theoretical frame work for this study which are constructivism and the Van Hiele theory.

Chapter five deals with the review of the literature related to this study. Chapter six addresses the research design and methodology while Chapter seven provides an analysis and the results of the research. Chapter eight deals with a summary of the main points of the analyses and looks at the implications of the findings for the teaching and learning of trigonometry in some South African schools.

CHAPTER TWO

The history of trigonometry

2.1 Overview

This chapter mainly focuses on how trigonometry and the concept of a function evolved to be what they are today. The importance of knowing the history of these two and how they emerged is also discussed. This is relevant to the teaching and learning process of this topic and its aspects as the computer intervention in isolation would not make much sense.

The definition of trigonometry is basically from the Greek words *"trigono"* which is triangle, while *"metria"* is measure (Bressoud, 2010). The term trigonometry was probably invented by the German mathematician Bartholomaeus Pitiscus whose work was first published in 1595. According to De Villiers (2010, unpublished lecture notes), the use of trigonometric functions arose from the use of chords of a circle in mathematics and astronomy. The term trigonometry means *"the study of triangles"*. It was first used by the ancient Greeks to aid in the study of astronomy. De Villiers (2010) also states that evidence has been found in works from many other countries, including China and India. Trigonometry was used as long ago as over 2000 years to calculate the height of mountains, to navigate across seas, to survey large areas for farming and to determine the distance between the earth and the moon.

The history of mathematics, however, cannot answer directly routine questions in the teaching of mathematics (Fauvel, 1991). However, it normally serves to shed light the relationship between mathematics and social policy in general. This approach serves to uncover the relevant frames of reference. The use of the history of mathematics can also illuminate guiding principles as well as other theoretical aspects which routine questions might raise. In addition, the history of education provides some necessary knowledge for the background principles, basic understanding and routine action in education.

Planning the curriculum involves more than choosing the facts and theories to be taught. We must also foresee in what sequence and by what methods those facts and theories should or could be taught (Polya, 1981). This shows that it is sometimes important for learners to know some facts and theories of some mathematics topics for them to better understand these topics.

Polya (1981) states that the learner should retrace the paths followed by the original discoverers and rediscover what he/she has to learn. He further states that teaching can be stated in various ways. In teaching a branch of science (a theory or a concept) we should let the learner retrace the great steps of the mental evolution of human race. This helps the educator to anticipate how the learner might assimilate the same knowledge.

There are thus two issues regarding the history of mathematics and trigonometry. Hull (1969) states that the direct use of historical material can give learners a better cultural, socio historical perspective on why and how trigonometry was developed. He also supports the view that it may give an idea or good guideline on how the curriculum might be structured. This suggests that the teaching of trigonometry has to be linked to its historical background when imparted to learners in class.

According to De Villiers (2008), the history of mathematics can also lead to the identification of some general patterns and trends by which mathematical content evolved and was invented. These patterns and processes could then be utilized as possible teaching approaches without any direct reference to the history of the particular content being taught. He asserts that there are at least four ways in which a teacher can use the history of mathematics:

1) As a concept, algorithm or theorem looking at its historical development.
2) As a historical development of the most significant moments in chronological order.
3) As an analysis of the historical development, with no historical material, of the particular concepts, algorithms and theorems (the indirect genetic method)

4) As an attempt to simulate with the advantage of hindsight, how particular concepts, algorithms and theorems might have been discovered and/or invented through typical mathematical processes or ways of thinking.

The history of mathematics is not clearly presented in the current curricula. However, there are some mathematics textbooks that do include information on important historical figures and outstanding events. This information is not compulsory and is mostly considered not that important by teachers when imparting mathematical knowledge. Thus, this chapter highlights the importance of using the history of mathematics in the classroom and explores the historical path of the emergence of the *cosine* function.

The National Council of Teachers of Mathematics (NCTM, 1989) views the history of mathematics as important in the classroom. To that effect, material on the history of mathematics has been produced in accordance to classroom needs. This is also supported by John Fauvel (1991). He came up with a list of reasons to support the use of history of mathematics in classrooms. The list had guide-lines on how a teacher could effectively use history in mathematics.

Normally, a learner might not be in a position to understand some phrases and thought patterns that uphold mathematical knowledge. In such cases the historical background becomes a necessity. The learner might get it from a learning process that is directly linked to the ancient way by which humankind worked its way up to mathematical knowledge. Mathematics history can also motivate some learners through its beauty and logical structure.

History also shows us how some of the definitions used today were developed. For instance, the definition of a function developed as follows (De Villiers, 1984);

- The first definition only appeared after the Renaissance, when Jean Bernoulli in 1718 stated it as a unit comprised of a variable and constants.
- Then Euler in 1748 stated it as any analytic expression whatsoever made up of a variable quantity including numbers or constants

- Euler in 1750 stated that quantities that are dependent on others, such that as the second changes, so does the first, are said to be functions.
- Then Dirichlet in 1837, talked of a relationship between two variables, where numerical values assigned to one will affect the other one. These definitions include the idea of functional dependence; however the following does not.
- The formal set-theoretic definition of around 1880 used by John Venn, George Boole, Auguste' de Morgan and others that we use today (De Villiers 1984), where a function is a seen as an ordered pair (*a; b*). The domain is represented by *a,* and the range by *b*. Each element of *a,* belongs precisely to one ordered pair of the function and is thus uniquely related to a single element of *b.* There are numerous other definitions that may be traced backwards. This might be helpful to both educators and learners to take longer periods when dealing with some topics and aspects of mathematics as they see the time it took them to be where they are today.

A function can also be represented by tables and graphs. De Villiers (2010, unpublished lecture notes) asserts that tables of values and the Cartesian graph did not exist at the time of the Greeks because they did not have the co-ordinate system. This does not mean that the concept of a function did not exist. Maybe it was not explicated and they did not formulate it, but they certainly had an intuitive understanding. This is the whole distinction by Tall (1989) between concept image and concept definition. Newton and Leibniz did not have a concept definition for limits and functions, but they had a good understanding of what a limit of a function is, even though they did not have a formal definition.

Euclid, 300 BC, saw trigonometry as part of geometry. From the 1600s onwards, people battled with the trigonometric/ algebraic function. The late historical development of the co-ordinate system suggests that it may not be such an easy idea (De Villiers, 2010, unpublished lecture notes). The co-ordinate system developed from physics, mechanics, and astronomy. The more problems that involved periodic motion required the use of functions, the more they needed to further concept of trigonometry.

The idea for the definition and concept of a function also developed later (De Villiers, 2010, unpublished lecture notes). There was a need for the concept of a function as people were beginning more and more to apply mathematics and science to phenomena that involved periodic function. This shows that the motivation for the development of the function definition was from different kinds of practical consideration.

The quest to formally clarify what a function is arose from the dramatically increasing application from the 1600s and onwards, of mathematical functions and calculus to scientific problems of motion and forces (De Villiers, 2010, unpublished). In turn, this had been made possible by the development in the 1600s of the algebraic symbolism and nation. The Cartesian co-ordinate system, which simplified the antiquated methods of the ancient Hindus, Greeks, and Arabs, also emerged. On the other hand, this late development of the formalisation of the concept, also suggests it may be conceptually a subtle and deep idea.

At times, we might have all the characteristics of a particular concept listed, but that would not be an economical definition. A definition only selects a small subset of that, as necessary and sufficient conditions, which become a concept definition (De Villiers, 1984). A definition does not include all the properties. One of the dangers and problems of teaching is that people think that if they use the formal definition then they would have covered all aspects. The circle definition, which was formalized over a number of stages, includes a small portion of the kind of concept image that learners should have.

When learning trigonometry, learners should certainly have the concept image of a ratio and of a right triangle. This would make it easier for them to solve application questions, since that is the most useful concept for applications. A good example of the use of this concept would be the need to model periodic functions in physics which we do not even deal with at school. We do not deal with the practical aspects of periodic functions either. This can only be possible maybe if we deal with pendulums that are regular or rotating wheels, tides or the cycles of the moon and so forth. Evidently then, one must question the idea of starting with the circle definition as it is a limited from a practical perspective.

It only abstracts and selects certain aspects that are useful for a certain perspective.

Drawing on the history of mathematics, it becomes clear that trigonometry was initially used for practical applications. In terms of the teaching and learning process we come up with the problem-centred approach which states that one should start with a practical problem that motivates the development of new content. Historically, this is how trigonometry developed. There was a practical need to build buildings, to find out the time of the seasons and for astronomy and for that they needed some apparatus, the tool they developed was trigonometry. This tells us that mathematics does not develop on its own; it develops to solve practical and theoretical problems.

Much of the work of abstract algebra for example, field theory and ring theory as we know them today, developed to solve some problems in ordinary algebra of the real number system (Bressoud, 2010). For the solving of polynomials of higher order to understand why they could not find the general form, it was necessary to develop abstract algebra. There were theoretical reasons which tell us that if we want to follow the problem-centred approach, we need to choose and select good starting problems that can similarly motivate a learner to see the need for trigonometry and beyond.

According to Kennedy (1991) these developments originated in the general region of the eastern Mediterranean, were recorded by people writing in Greek, and were well established by the second century of the era. The centre of activity then shifted to India (where the chord function was transformed into varieties of the *sine*), and thence it moved part of the way back. In the region stretching from Syria to central Asia, and from the ninth century up to the fifteenth, trigonometric functions were elaborately tabulated in the form of sexadecimals. This development helped the emergence of the first real trigonometry in the sense that only then did the object of study become the sides and angles of spherical or plane triangles.

Kennedy (1991, p.359) also states that the *Almagest* is of interest to the mathematician because of the trigonometric identities Ptolemy devised to help him in compiling his table of chords (which is roughly equivalent to the *sine* table). Subsequently, as the locus of activity in astronomy moved to Europe, so also did the new trigonometry. According to Kennedy (1991), the same type of work occupied Oriental scientists whereas development of tables and functions from the triangle continued in the West.

By the end of the eighteenth century, according to Fuhrer (1987), Leonard Euler and the others had exhibited all the theorems of trigonometry as corollaries of complex function theory. As a school subject, however, especially useful for surveyors and navigators, trigonometry still keeps its separate identity.

Here the account is confined to the leaders in the field of working with triangles; their predecessors and rank-and-file contemporaries operated on a more primitive level, but they created the background without which these leaders could not have existed. According to Kennedy (1991), knowledge of the subject was not smooth in terms of growth. There was a lot of discontinuity though in series. Important advances made at one time and place sometimes only spread slowly, sometimes not at all, sometimes disappearing only to be rediscovered later.

2.2 Birth of the *Cosine* function

According to De Villiers (2010, unpublished lecture notes), the *cosine* function is a co-function of the *sine* function. The *sine* function itself emanated from the applications of a chord (plane as well as spherical). Eventually they thought of calculating and using half the chord of double an arc. Once this was done, the *sine* function had been born. He further states that in the earliest days a scale diagram was used. This is the kind of *Sketchpad* approach used today. Then in time of Euclid, they used the chord method. The use of Ptolemy's theorem later allowed them to calculate the *sine* ratios far more efficiently and quickly, and to more decimal places.

The earliest sine tables turned up in India, where they originated (De Villiers (2010, unpublished lecture notes). The *Surya Siddhanta* is a set of rules in Sanskrit verse. It was composed around the fifth century A.D., but has been revised many times to the extent that it is no longer easy to say which sections have withstood change and are still as they were originally.

2.3 A function orientation

A function orientation is based on the processing-output notion, similar to algebraic functions (Pournara, 2001). A strong function orientation, he states, makes explicit that the process links the output, and vice versa, whereas a weak function orientation does not make the connection explicit.

A function orientation focuses on three aspects: the angle, the trigonometric operator (e.g. *sine, cosine, & tangent*) and the function value. This orientation is dependent on an understanding that the trigonometric operator maps an angle to a real number in a many-to-one relationship. The trigonometric operator, according to Pournara (2001), is seen as exactly that, an operator. In the function definition, function values are not related to the sides of a triangle. He further goes on to say that a function orientation is more likely to promote a dynamic view of trigonometry than would a ratio orientation. A function orientation assumes that the independent variable, the angle in this case, can take on many values. The resulting function value reflects clearly the effect of changing the angle.

As stated by Pournara (2001), the mathematical elements of a function orientation include the notions of periodicity, amplitude, asymptotes and discontinuity. Its other elements are the representation of trigonometric functions by means of tables, equations, or graph. It is possible that South African learners may develop a distorted view of trigonometric functions because the trigonometric curriculum places a great deal of emphasis on algebraic solutions of trigonometric equations and only studies the graph of *sine, cosine* and *tangent*. As a result, learners may develop a function orientation that is limited to the graphical representation of these functions. Pournara (2001) also argues that this is too limited if learners need to develop a broader understanding of functions. More so, they should be able to draw links between trigonometric functions and linear, quadratic, cubic and exponential functions

According to Bressoud (2010), beginning with the ninth century, the number of people working in trigonometry increased markedly. Astronomers lived and travelled widely over a region reaching from India to Spain: the Iranian plateau, Iraq, Syria, Egypt, North Africa and Spain. Indian scientific books were the first to receive the attention of Moslem scholarship. Some were translated into doggerel Arabic verses in imitation of the Sanskrit slokas. Later the available Greek works were translated. The *sine* function was quickly adopted in preference to the chord. In fact, the etymology of the word *"sine"* indicates the wide variation in background of those who dealt with the function it designates. The Indians called the function *ardhajya,* Sanskrit for "half chord". This was shortened to *jya* and translated into three Arabic characters, *jhb.* This can be read as *jayb,* Arabic for "pocket" or "gulf". It was so read by Europeans, who translated it into Latin *sinus,* whence English *"sine"* and its co-function, *cosine.*

2.4 More functions and tables

The subject matter of the previous section is primarily geometrical. Its development, according to Kennedy (1991), was accompanied by an accumulation of numeral and computational materials and techniques. In the ninth century, tables of the (horizontally) extended shadow were common. Al-Biriuni, a great scientist who lived in central Asia in the eleventh century, wrote an exhaustive treatise on shadow lore. Among Orientals, he asserts, it was customary to use a gnomon of a hand span of length. Rarely tabulated, but explicitly defined and applied in Sanskrit as well as Arabic works, were relations called the "hypotenuse of the shadow".

Jugmohan (2004) points out that the motivation for the development of the function definition of trigonometry was different from the original one. She further elucidates that originally it was used within land surveying where simple triangulation sufficed, but by the time it came to the Renaissance it was a different scenario. Then the practical considerations being addressed with investigations were of the pendulum, and of the movement of the planets around the sun. All these were periodic, and for that they needed to come up with a more abstract definition for the trigonometric functions to model periodicity. The concept of a function therefore became more developed and further removed from the Greek view, which did not have any sort of formal or written definition.

If we were to follow a historical approach, then maybe we should not start with the definition of a function. The fact that it developed later suggests that it may be sophisticated and might be more subtle for learners to understand. Furthermore,

- it would be against the historical order
- its usefulness is on modeling periodic functions which are not in the curriculum
- most applications require the right triangle definition

The practical problem is one aspect. The other aspect is the idea that the definition of a function and the concept of a function were needed as people were beginning more and more to apply mathematics within sciences. It was also necessary to have trigonometry applied in cases that involved periodic functions.

CHAPTER THREE

Teaching trigonometry

3.1 Introduction

This chapter looks at the teaching of trigonometry and the different methods that are used. The basic idea on which the whole of trigonometry is based on is that of similarity. Triangles can have the same shape but different sizes. Two triangles can be equal if their angles are, and consequently their corresponding sides would be in proportion. Such triangles are then said to be *similar triangles.* Trigonometry starts with a right-angled triangle for which the side lengths are related by *Pythagoras' theorem.*

In a right-angled triangle, trigonometric functions relate the size of any angle to the ratio of any two sides. *Sine*, *cosine*, and *tangent* are the basic functions of trigonometry. They are based on right triangles with one common angle and are hence similar.

3.2 Two methods of introducing trigonometry

Different approaches to trigonometry in the curriculum are discussed to provide a background to the study.

In most countries like Canada (De Kee et al, 1996), the United States (Satty, 1976), Australia (Willis, 1966), the United Kingdom (Collins, 1973) and in South Africa, school trigonometry has traditionally been introduced by means of ratios and right-angled triangles. According to Jugmohan (2004), the introduction of the "new Mathematics" in the 1960's called for a shift in school trigonometry, from a ratio to a function approach with particular emphasis on the unit circle. The unit circle swallows up the function approach (Pournara, 2001) to the extent that one frequently reads about "*unit circle approaches"* rather than the function approach to trigonometry.

When it all started, the ratio method was used to introduce trigonometry. In this case trigonometric functions were demonstrated as the ratio of sides in a right triangle.

According to Trende (1962) around the early 1960s another "modern" way was introduced and approved by some educationalists. This was viewed as more user friendly for the understanding of learners (Willis, 1966). This method advocated having trigonometric functions defined in terms of x and y. This was called the unit circle approach, where a point with coordinates x and y is used. Most textbooks stick to solely one method although there are some that try to blend both methods.

3.2.1 The ratio method

<u>The sides and angles of a right triangle:</u>

The sides can be named in 3 ways:

1) Using two capital letters
2) Using the small letters corresponding to the angle opposite the side
3) Using the terms opposite, adjacent or hypotenuse.

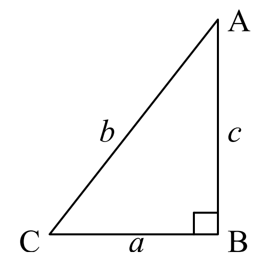


Figure 3.1

In the triangle above (figure 3.1), AC = b = hypotenuse. The other two sides are named opposite or adjacent, depending on the angle to which we are referring.

i.e. CB is <u>opposite</u> to $\hat{A}$, but <u>adjacent</u> to $\hat{C}$.

<u>Summary</u>:

$$\text{Sine } \theta = \frac{\text{opp}}{\text{hyp}}$$

$$\text{Cosine } \theta = \frac{\text{adj}}{\text{hyp}}$$

$$\text{Tangent } \theta = \frac{\text{opp}}{\text{adj}}$$

These are abbreviated to *sin* θ, *cos* θ and *tan* θ.

<u>Calculator usage:</u>

a) <u>To find the values of ratios</u>

Scientific calculators have been programmed with the trigonometric ratios of all angles.

E.g. *Cos* 10, 5° = 0.9832

b) <u>To calculate an angle</u>

When we need to find an angle, we use the inverse functions represented by the symbols sin^{-1}, cos^{-1} or tan^{-1}, i.e. the **second function** of the *sin*, *cos* and *tan* buttons.

If we are given the value of the ratio, we simply enter the appropriate 2^{nd} function, the given value and then =.

$Cos\ \theta = 0.612,\ \therefore \theta = \underline{52, 27°}$

Sine θ = opposite/hypotenuse,

Cosine θ = adjacent/hypotenuse,

Tangent θ = opposite/adjacent.

All this is summarized by *SOHCAHTOA*

3.2.2 The unit circle method

The unit circle method, initially, emphasizes the nature of the trigonometry functions "*as function taking real numbers to real numbers*" (Kendal, 1992, p.77).

If we draw an angle in one quadrant, with a radius *r*, the triangle formed will have *x* and *y* as sides (figure 3.2).

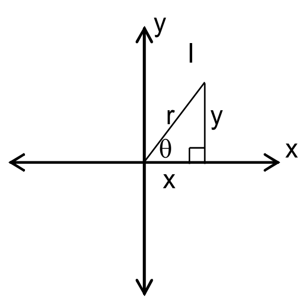


Figure 3.2

The unit circle method made solving triangle problems easier, "*an interesting and useful outcome*" (Dooley, 1968, p.30). Kendal (1992) describes how unit circle approaches have evolved since they were first introduced. Three different unit circle methods are described. Kendal (1992, p.87) refers to these as *functions of real variable, angle based definition, and scale factor technique.*

Most importantly, this method is not mainly centred on angles and triangles when working with it. This fact is also supported by Kendal (1992, p.89) when he says that *"One of the aims of new Mathematics was to use mathematical language more precisely, so this was thought to be a desirable feature. Cosine and tangent are similarly defined as lengths"*

Practical Applications

Trigonometry enables us to calculate heights and angles that we would not be able to reach. The angle of **elevation** starts from the horizontal **upwards**, and the angle of **depression** is the angle measured from the horizontal **downwards** (figure 3.3).

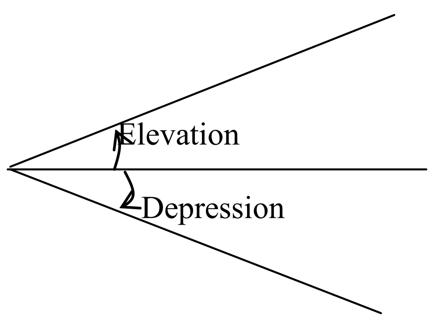


Figure 3.3

When standing 5m away from the base of the Nelson Mandela Statue (picture 3.1) in Johannesburg, the angle of elevation to the top of its head is 31°. Calculate the height of the statue, to the nearest metre.

Picture 3.1

$\frac{TB}{PB} = \tan \hat{P}$

TB = 5 x *tan* 31°
= 3m

Using figure 3.4

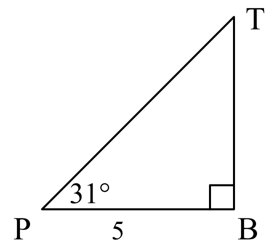


Figure 3.4

TRIGONOMETRIC GRAPHS

Terminology:

Domain: This is the set of values of *x* shown in the graph. The maximum domain required for these graphs is $x \in [-360°; 360°]$.

Range: This is the set of *y*-values used, and usually extends from the minimum value to the maximum value, where they exist. E.g. for y=*sin x* the range is [-1; 1]

Amplitude: This is half the distance between the maximum and minimum values. In the case of the basic graphs, it is the distance from the *x*-axis to the highest or lowest point. **NB:** Since it is a distance, it cannot be negative.

Period: The interval over which the graph completes one cycle of its basic shape. In other words, how often the graph repeats itself. It is expressed in degrees, and does not have a starting point and endpoint, as it can be measured anywhere along the graph. It is important to distinguish between domain and period, and between range and amplitude.

Asymptote: A line which a graph approaches but never intersects.

Function of a real variable

Initially the unit circle method referred the trigonometric functions to functions of real variable. According to Kendal (1992), learners had difficulty understanding these definitions. This, combined with the need for angle based definitions to solve triangles, led to the second unit circle method.

Scale factor method and angle-based method in trigonometry

There are two important differences (Pournara, 2001), between the scale factor method and the angle-based method. Firstly, the scale factor method does not require learners to transpose equations; hence the algebraic demands are reduced. The second difference lies in the way the learner works with the two triangles. In the angle-based method, the learner looks for corresponding sides and sets up equivalent ratios. In the scale factor method, (Pournara, 2001), the learner views each triangle as a whole and treats the one triangle as an enlargement of the other, hence the term *scale factor*. The other significant mathematical difference between the two definitions is that the angle is measured in radians in the first method and in degrees in the second method. There are vast conceptual differences because learners work with reference triangles, derived from the unit circles in the angle-based method. In the other method they use reference triangles. Learners must focus on the lengths of the sides of triangles rather than on arc lengths as in the previous method. This shows that it was necessary to come up with the second unit method

3.3 The international debate

According to (Pournara, 2001), there exist only two research studies documented in the literature that compares the ratio and function approaches, one conducted in Australia (Kendal, 1992) and the other in Canada (De Kee et al, 1996).

De Kee et al (1996), Kendal and Stacey (1996) and Markel (1982), maintain that a ratio approach is best. Others prefer a function approach based on the unit circle. Dooley (1968) argues for the function-of-a-real-variable method because it does not depend on angles or triangles. On the other hand Willis (1966) proposes the angle-based method because of learners' difficulties in working with the function of a real variable in the context of circular functions. Others propose an approach that combines both methods (Satty, 1976). However, according to Pournara (2001), most of the debate seems to have been based on personal preference and the individual experiences of participants in the debate, with little reference to empirical research on teaching and learning trigonometry.

3.4 The function vs. unit circle approach

Quite often the term *cosine* function is used synonymously with the circle definition or its graph. However, one has to acknowledge the fact that the *cosine* function can also develop within the right-triangle orientation. According to De Villiers (2010, unpublished lecture notes), although the Greeks did not formalize the concept of a function or did not use *y over r* and *x over r,* this did not mean that they did not intuitively understand the *cosine* function.

We understand a function as something which relates input to output values, domain and range. Functions can usually be represented by some kind of formula for example, $y=\cos x$ or by $\cos \theta$ = adjacent/hypotenuse. Therefore, it would be limiting to restrict the *cosine* function term to only the circle definition as we talk about the *cosine* function within the right triangle context as well.

When the function approach is used in this research, it refers to the unit circle approach, and when the terminology is used it simply refers to the *cosine* function as a whole. The function value as a ratio, also changes as θ changes, so it has the idea of variability; functional dependence. The graphs that the learners drew and the tables they completed are just some of the methods of representing a function. Generally, the three methods of representing a function are: graphically, tables and formula. This view of function developed late. From mathematics history we note that it took 2000 years for the function approach to trigonometry to develop, suggesting, to conclude, that maybe it is a lot more complicated.

3.5 Difficulties in the learning of trigonometry

Difficulties in learning trigonometry are closely linked to learners' inability to understand algebraic manipulations (Dwyer, 2010). As a mathematics educator, my experience of teaching trigonometry has shown that the sources of learners' difficulties in trigonometry are more than just meet the eye. They range from the curriculum to the teaching and learning, assessment, and from the teacher to the learners.

The present curriculum documents (Pournara, 2001) do not reflect a properly conceived trigonometry that does pay sufficient attention to a notion of trigonometric functions. It does not develop appropriate links between trigonometric ratio and trigonometric function. Current assessment practices, particularly at grade 12 level reward procedural and rule based thinking (Pournara, 2001). The teaching in trigonometry may not review learners' poor conceptual understanding in this area. These factors, combined with learners' inability to perform to perform algebraic manipulation such as factorizing and solving equations, and their under-developed spatial skills (Pournara, 2001), lead to generally poor performance and difficulty in trigonometry. It would be rather unjust to lay blame squarely on learners as at times the teachers themselves do not quite understand the concept and tend to read it off textbooks, some of which also contain errors!

According to Hart (1981), ratios in general prove to be very hard for learners to understand. Changes have been made in some textbooks to try and lessen the burden of learners by writing the *sine* of an angle only. In this case the radius is used as the hypotenuse where learners are expected to be able to identify the triangle even if it is rotated.

According to Blackett and Tall (1991, p. 13) "*As an acute angle in the triangle is increased and the hypotenuse remains fixed, so the opposite side increases while adjacent side decreases*" and "*As the angles remains constant, the enlargement of the hypotenuse by a given factor changes the other two sides by the same factor*". These are some of the concepts learners are faced with when going through the topic of trigonometry. They also state (1991, p. 15) "*The traditional approach uses pictures in two different ways, each of which had its drawbacks*" This downgrades the role of pictures in the minds of the learners. They will tend to think that rough sketches lead to wrong answers and dedicate all their energy to accurate diagrams and not to changing relationships of the triangle.

The computer approach (Blackett and Tall, 1991) has the capability of changing this kind of thinking as it gives the learner a chance to move the diagram anyhow. It allows the learner to relate the shape to its randomly changing form and to the related numeral concepts. This way the learner might understand better. The learner can focus on important things since the computer would not take time to draw the diagram in any state.

Bruna (1996) says that the strengths of the learners can be played around with as another way of facilitating learning. Nowadays most learners play around with cell phones and even computers, this means that the use of the computer in learning mathematics could alleviate problems learners have in some mathematical concepts. Bruna further goes on to denote that learning is mostly through participation not being a spectator. It clearly shows that learners need to participate as much as they can in order to understand most mathematical concepts. This can be by working-out questions on their own which might lead to discovering best methods to solve problems.

CHAPTER FOUR

Theoretical framework: Theories of learning related to mathematics

This chapter seeks: a) to examine and outline two opposing learning theories, which will illustrate different approaches to handling learners' understanding as well as their misconceptions in mathematics and b) to discuss the theoretical framework for this study.

4.1 Learning theories

There have been different perspectives that have been put in place and adopted as regards teaching and learning of mathematics for quite some time. Some learning theories that have influenced mathematics teaching and learning in South African classrooms are Behaviourism and Constructivism. More emphasis, however, is on constructivism which is part of the theoretical framework of this research.

4.1.1 The Behaviourist theory

The behaviourist theory of learning is based on the empiricist philosophy of science. It claims that knowledge entirely comes from experience. De Villiers (2010, unpublished lecture notes) argues that even though experience plays a role, it is affected by what is in the mind. The traditional empiricist motto was "*there is nothing in the mind that was not first in the senses",* according to Olivier (1989, p.37). The empiricists believe that it is possible for a learner to acquire direct and complete knowledge of anything that is real. They say that through the senses, the image of that reality corresponds exactly with reality.

Behaviourism presumes that learners learn what is delivered to them by teachers, or part of it. According to Olivier (1989, p.38), they claim that "*knowledge can be transferred intact from one person to another"*. Jugmohan (2004) also writes that the behaviourists see learning as the forming of habit, based on reinforcement. Something has to be repeated over and over again in order for the learners remember what they would have learnt for a long time. This suggests that rote-learning, drill and practice are important factors in the learning mathematical knowledge according to them.

Behaviourists see the minds of learners as empty, waiting to be filled by knowledge, transmitted by their teachers (De Villiers 2010, unpublished lecture notes). The learners are seen as *"a sponge absorbing the mathematical structures invented by others"* (Clements & Battista, 1990, p.33). Behaviourists, therefore, see knowledge as something that learners are able to acquire from experience. They take it that what learners have already acquired is unimportant to learning.

This type of acquiring knowledge does not allow for application of knowledge according to Penchalia (1997). Skills acquired in this manner are not transferable and learners become mathematically illiterate. Furthermore, Alder (1992, p.264) argues that school mathematics is an activity having its own goals and means and cannot be *"simply transplanted into another activity"*. The organisation of learning, according to behaviourists' principles must proceed from the simple to the complex, and exercise through drill and practice (De Villiers 2010, unpublished lecture notes). He goes on to say that from a behaviourist point of view, errors and misconceptions are not an issue since previously acquired knowledge does not come into play when it comes to learning new concepts.

4.1.2 Constructivism

According to De Villiers (2010, unpublished lecture notes), constructivism is a type of learning theory which assumes that learners construct meaning and that their understanding is dependent on their pre-knowledge. Concepts are actively constructed by learners and the teacher acts as a facilitator. However, this does not eradicate the mushrooming of misconceptions. De Villiers (2010, unpublished lecture notes) asserts that it is important that learners must be given experiences which conflict with their learning as it is far more important than the rules. This shows that cognitive conflict is an important aspect of assimilation in the education process.

From a constructivist's view point, errors and misunderstandings by learners are of great value to education, because they address a section of a learner's conceptual structure.

Olivier (1989, p.18) points out that, *"errors and misconceptions are considered an integral part of the learning process"*. Misconceptions combine with new knowledge, and play an important role in new learning mostly, according to him, in a negative way, because they are the root cause of mistakes. The theory has its roots in a view that "*knowledge is made and not given; it is constructed by an active cognizing subject rather than transmitted by a teacher or a text*" (Adler 1992, p.29). Nickson and Noddings (1997), state that since learners are internally motivated, they interpret and adjust information to their personal mathematical schemas thereby constructing their own mental representations of situations and concepts. The learner's ability to learn depends on the ideas the learner brings to the experience. According to Muthukrishna and Rocher (1999), the learners' pre-existing knowledge will influence the type of knowledge gained.

Socio-constructivists believe that learning is something that is based on communal and personal activity. Olivier (1989) points out that there is an awareness of interaction between a learner's current schema and learning experience. The learner's point of view is taken into consideration. This also implies that mathematics teaching consists primarily of mathematical interaction between the teacher and the learners. Learners at times also communicate their ideas and interpretations with each other. An active self-reliant attitude to learning is inculcated within the learner through discovery, negotiation and reflection. Most learners develop their own methods rather than rely on methods taught by the teacher.

According to De Villiers (2010, unpublished lecture notes), the character of a learner's existing schemas determines what the learner gets from previous activities or acquired knowledge and how it is grasped. Constructivists uphold the interaction between a learner's current schema and past experiences with high esteem. Discussion, communication, reflection, and negotiation are components of a constructivist approach to teaching (De Villiers 2010, unpublished lecture notes). The constructivist also looks at knowledge acquisition from the learner's point of view in order for the teacher to come up with suitable methods (Olivier 1989). The teacher has to consider the mental process by which new knowledge is acquired.

This then shows why it was necessary to use *Sketchpad* in this study on what learners had done in Grade 10. In a way it served to authenticate what they had previously learnt. Even if they had any form of misconceptions, the intervention by dynamic software would help with some form of cognitive conflict. Some of the things they thought they had understood previously will be in contradiction with what they will discover from the computer. This forms an interaction between the learner's schema and current learning experience with the computer which would be an important part in the learner's assimilation of the concept.

The researcher considered using constructivism as the theoretical framework for this study because the study was based on the way in which individual learners constructed knowledge. For learning to take place, the learners should reconstruct and transform external, social activity into internal individual activity through a process of internalisation (Wersch & Stone, 1986, p. 169). The formation of such consciousness, according to Wersch and Stone, depends on social interaction and on *"mastering semiotically mediated processes and categories"*.

4.1.2.1 Misconceptions in mathematics

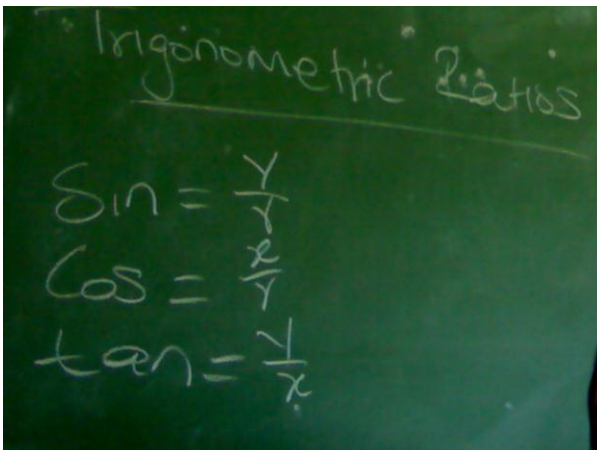
When learning takes place, the new ideas that the learners' are exposed to, need to be linked to the learners' previous conceptions. This is what determines what the learner understands. In this research, the analysis of the learners' understanding, correct intuitions as well as misconceptions in their mathematics were exposed.

The following points need to be further considered (Olivier, 1989, p. 18):

- correct new learning depends on previous correct learning
- incorrect new learning is often the result of previous incorrect learning
- incorrect learning is mostly the result of previous correct learning

Every misconception has its origin in some form of correct learning. Every misconception is correct learning at least for some earlier activity previously done or some previously worked on domain of the curriculum. Some misconceptions emanate from what teachers say and some from "sloppy notation" during the teaching and learning process. In some cases, teachers tend to use words loosely without considering the negative impact it will have on the learners.

In one trigonometry introductory lessons observed in the Pinetown district, just outside of Durban in May 2010, a qualified mathematics teacher wrote the following on the chalkboard as he presented a Grade 12 revision lesson (picture 4.1):



Picture 4.1

As a result learners will frequently write things like $cos = \frac{1}{2}$. They then find it hard to relate their findings to an angle as it cannot be located anywhere. Sloppy notation like this obscures the functional combination of the independent variable $sin\ \theta$ with the dependent variable θ. At times instead of writing $cos^2\theta$, learners can write $cos\ \theta^2$. Another frequent error is where learners confuse *cos* 60° for *cos* multiplied by 60°.

Misconceptions mostly arise from an *over-generalization* of previously learnt information (correct in that set of values), to an extended new set of values, where the former is not valid. Information, states Olivier (1989), assimilated earlier and well cemented is not easy to change. A learner will not easily accept and assimilate new knowledge, nor is it easy to add new things to existing knowledge. Learners find it hard to alter their already acquired knowledge. This is how errors are normally created. Learners would try to fit new knowledge to what they already know and it becomes distorted (Olivier, 1989). There is a blame-shifting for poor teaching methods right from the university down to high school, to high school, to junior primary, down to the family. It is not clear exactly where the problem really lies. Either learning basics must be changed, states Olivier (1989), so as not to alter ideas later, or special effort must be made later, to prevent or remediate learners' misconceptions. Neither, according to Olivier (1989), is easy.

Learners are not able to unpack knowledge on their own; they just categorize it into related big units all with similar concepts. Olivier (1989) defined such a unit as a *schema.* These, he states, are of vital importance intellectually as they can be accessed and applied whenever necessary. At the end of the day the combination of the learner's schemas and new knowledge is important as it allows the learner to assimilate and accommodate knowledge.

Constructivists have a very different way of looking at learning as compared to that of behaviourists. They do not see it a matter of piling up new information on previously acquired one. According to Olivier (1989), learning leads to changes in our schema. During early learning, e.g. "multiplication makes bigger" is a result of expressing it as another form of addition for easier understanding. Mostly the teacher is aware that the learners are familiar with addition and in order to clearly explain this new concept of multiplication begins with addition as it is the only easy way of introducing it (Olivier, 1989). Unfortunately this is not universal in all number dominions like fractions and decimals and it could be the root of numerous other errors. Learners then begin to try and relate all new concepts to one previously done.

4.1.2.2 The learning of the *cosine* function

The conceptualization of the *cosine* function using *Sketchpad* provided a unique way of dealing with this section and allowed for experimentation, questioning, reflecting, discovering, inventing and discussing. According to De Villiers (2010, unpublished lecture notes), any use of a system which denies the opportunity for reflection, discussion and posing own questions must be seriously questioned. He further indicates that from a practical point of view, teachers should be on guard against designing lessons that aim to develop skills strictly through repetitive practice. This practice, according to Artigue (1991), may have no meaning if learners are encouraged to use a computer, organized in pairs, groups, or whole classes. They should be designed with both the mathematics in mind and the learners' developing conceptions of mathematics.

4.1.3 Van Hiele theory

4.1.3.1 Introduction

This theory was developed in the doctoral dissertations of two students, Dina and Pierre Van Hiele from Netherlands in 1957 (De Villiers, 1996). Pierre was mainly concerned about why learners found it difficult to explain and describe shapes whilst Dina was mainly worried about arranging geometric knowledge. The most outstanding characteristics of this theory are the different categories they came up with. Four of them are summarized as follows by (De Villiers 1996):

- **Fixed order** - The way in which learners move from one level to another. A learner can only move to the next level after having completed the previous one.
- **Adjacency** – Ideas are inter-related according to levels. One concept that was very important in some level becomes less important in the next one.
- **Distinction** – Every level is different from another in terms of terminology, relationships and symbols.
- **Separation** – The reasoning is quite distinct at different levels.

In an outstanding way, the Van Hieles showed that the curriculum was operating at a far higher level than that of learners (De Villiers, 1996). This led to the learners' failure to understand the geometric concepts, and leaving the teachers wondering why! The general characteristics of each level, according to De Villiers (1996) are elaborated below:

Level 1: Recognition

Learners are able to see shapes and give the correct name but might not be in a position to indicate the characteristics correctly.

Level 2: Analysis

Learners can be in a position to name and describe the properties of the shape but might fail to establish the relationship between these and the shape in general.

Level 3: Ordering

Learners can arrange characteristics of figures in an orderly manner and generalize them (e.g. class inclusions).

Level 4: Deduction

Learners can now understand proofs about shapes, basic theorems, and axioms about shapes.

Level 5: Rigour

The learner is comfortable with an axiomatic system such as those for the non-Euclidean geometries and different systems can be compared. They can analyse the consequences of and manipulate different axioms and definitions. The learner understands the formal aspects of deductions.

4.1.4 This study

In this research two assumptions come into play. Firstly, learners actively construct meaning and can change any form of misconceptions on their own as they engage with mathematics. Secondly, knowledge construction must occur individually first and then socially.

The Van Hiele and constructivist theories provide an appropriate tool for the investigation of learners' thinking in this study. The strategy used was based on a constructivist point of view which describes human beings as builders of theory and structures (Balacheff, 1996; Schoenfeld, 1987). The Van Hiele theory served as a yard stick to see the extent to which the learners could visualize, interpret and draw out meaning from trigonometric shapes and graphs. An attempt is made below to conjecture what levels 1, 2, 3, and 4 for trigonometry would be.

4.1.4.1 Van Hiele in relation to this study

Level 1: Visualisation

The learner can identify a right-angled triangle in whatever form or stance and is able to distinguish the difference between different forms of right-angled triangles in and out of the unit circle. The ability to identify the opposite side, adjacent sides and the hypotenuse of a triangle also involves visualisation. This includes being able to identify the *cosine* graph.

Level 2: Analysis

In different right-angled triangles, if angles are the same, then the ratios between any two sides would remain the same, no matter how big or small the triangle might be (which is the concept of similarity). Learners are able to solve practical and theoretical problems related to right angles. They should also be able to identify shifts of graphs.

Level 3: Definition

The discoveries given above are now formalized definitions in terms of the sides of right triangles as ratios. Understanding develops of the changing nature of the trigonometric functions in all four quadrants, as well as of their non-linear nature. The understanding of the inverse also develops. In terms of graphs learners should be able to know the effects of a constant in a given function.

Level 4: Circle definition

Conceptualising the definition of trigonometry in its abstract form develops in terms of the unit circle and in terms of the trigonometric function. The unit circle to be defined as function, which is independent of the right angled triangle, and its trigonometric functions, are extended into the other three quadrants. In graphical concepts the learner begins to understand the period and the shifts without plotting the graph.

The Van Hiele theory will therefore clearly show which level the learners are at with the use of *Sketchpad.*

CHAPTER FIVE

Review of studies on trigonometry and teaching

5.1 Research studies

De Kee et al (1996) used in-depth qualitative interviews with five Canadian learners who were at the equivalent of the South African Grade 11 level. De Kee explored the learners' understandings of *sine* and *cosine* as they relate to both trigonometric ratios and functions. Overall, her findings showed that the learners had difficulties with both approaches but were more comfortable with the ratio approach. The learners found the work on the functions of real variables confusing. De Kee et al, as quoted in Pournara (2001) identified four concept images of *sine* and *cosine* revealed by learners:

- A procedure whereby the length of two sides of a right angled triangle is divided by each other, thus producing the *sine* or *cosine* of the triangle.
- The *sine or cosine* functions of a calculator.
- The typical undulating curves of the *sine and cosine* functions.
- The Cartesian coordinates of a point. Learners referred to these as the *sine* or *cosine* of the point.

In a study by Kendal (1992), where the scale factor method and ratio method of introducing trigonometry was compared, it was found that learners who were taught by the ratio method were more successful in solving problems involving the solution of a triangle. However, he argues that the focus of trigonometry in Australia is the solution of right triangles and therefore the method employed to introduce trigonometry should support this goal. He acknowledges that the study did not investigate conceptual development in learners nor the extent to which either method laid foundations for future work in trigonometry

Pournara (2001) observes that in recent years the focus in school mathematics has shifted from formalist approaches with their emphasis on mathematical rigour, to approaches that prioritize mathematical meaning. He argues that the general curriculum changes demand a shift in focus in school trigonometry, from an emphasis on ratio and triangles to a focus on trigonometric functions and modeling. However their call for a function is not related to the unit circle (Pournara, 2001). Their focus is on the periodic *sin* and *cos* curves that provide tools for analysing periodic phenomena, and hence applications in modeling. They argue that this type of the function approach will broaden and deepen learners' understanding of the concept of a function in general and hence strengthen connections with algebraic functions.

5.2 Symbols as process and objects

Pournara (2001) focuses on ways of working with trigonometric ratio and trigonometric function, as well as the ways in which learners see these as processes and objects. The notion of "procept" (Gray & Tall, 1994, p.53) provides a starting point for seeing symbols in different ways. A procept is a "*cognitive construct, in which the symbol can act as a pivot, switching from a focus on process to compute and manipulate"*. There are many examples of procepts in mathematics; for example, Pournara (2001), mentions that *3\4* represents division of numbers and the notion of fractions; *3x+2* represents an expression as the object and the process of multiplying *3* by *x* and then adding 2.

Gray and Tall (1994) consider all the trigonometric ratios to be "procepts". The symbol *sin A= opposite/hypotenuse* involves both the process of dividing the length of two sides, and the product, which is the ratio of the two lengths. The symbol *opposite/hypotenuse* (without *sin A*) is a process or an object (Pournara, 2001). As a process, it indicates a method for calculating the ratio and as an object; it represents a ratio that can be used in other calculations. The symbol of *cos A* can be taken to be either a ratio or a function. It can be seen as a ratio because it is equivalent to *adjacent/hypotenuse,* but it can also be seen as a function, it bears no relation to the fraction *adjacent/hypotenuse*. Within each of these possibilities – ratio and function – the symbol can be seen as a process or an object. It then follows, according to Pournara (2001), that *cos A* can be seen in four different ways; as in Table 5.1.

	PROCESS	OBJECT
***Cos A* seen as a ratio**	A process for calculating a ratio	A ratio describing the relationship between the hypotenuse and the side adjacent to A.
***Cos A* seen as a function**	A process whereby *cos* operates on A to produce an answer.	The result of *cos* operating on A. e.g. the coordinate of a point: (*rcosA; rsinA*)

Table 5.1

As quoted by Jugmohan (2004), Sfard (2000) argues that the introduction of a symbol constitutes the "conception" of a mathematical object and not its birth. The symbol of *cos A* can be viewed in multiple ways; firstly as a ratio or as function then as process or as object. These views influence and are influenced by the operations that learners perform with and on the symbol *cos A*.

5.3 The impact of methods and procedures on learners' conceptions of ratio and function

Trigonometry is a sub-domain of school mathematics that also relies on procedures and methods (Pournara 2001). In most cases learners score high marks in the trigonometry section of the Grade 12 examination if they apply correctly the procedures they have been taught (De Villiers 2010, unpublished lecture notes). Thus the use of procedures when teaching trigonometry should not be downplayed. However, Pournara (2001) argues that some procedures are better than others in supporting a conceptual understanding of trigonometric principles.

Pournara (2001) found that the conceptions of trigonometry ratio are closely tied to the methods they use, particularly their methods for solving triangles. He stated that in some cases, learners appeared to treat the ratio simply like part of working in the procedure for solving triangles. The first step of the procedure is to set up a ratio of two sides: "what I want over what I know"- and this he stated reflected the way in which they worked with ratio.

Procedures and methods for solving trigonometric tasks provide an efficient means of solving problems (Pournara, 2001), but learners do not necessarily understand the meaning behind the procedures. However, at times, they are able to execute the procedures successfully. Methods and procedures, he states are therefore both necessary and problematic. He further states that learners need to "*re-appropriate these and on a personal level and they do so through participation in the mathematical culture of the classroom"*. Without the appropriate participation, the procedures will have no meaning to the learners.

5.4 The metaphor of a converter

According to Pournara (2001), the metaphor of the trigonometric operator as a converter is one possible means for helping learners to shift orientations. Learners need to see *cos* operating on an angle and converting it to a ratio. The idea of a converter, according to Pournara (2001), may also help to deal with the cognitive discontinuity where the learners expect the input and output numbers to be the same type of number. The notion of a converter, according to Pournara (2001), has many physical applications and is embodied in the slider-crank mechanism which converts between linear and rotary motion.

Jugmohan (2004) suggests that some toys for learners provide an excellent illustration of how circular movement is translated into vertical and horizontal movement. The "popper" which consists of a dome-shaped chamber on wheels, is an illustrative example. She goes on to say that the rotation of this popper can be related to the trigonometric circle. When the wheel axle hits the spring-loaded mechanism, it has rotated through 90º and is at its maximum displacement. This illustrates the conversion of circular movement to linear motion – a change in angle (of rotation of the axe) produces a change in vertical distance (of spring loaded mechanism and balls). In a similar way, Pournara (2001) states, *tan* converts an angle of 41º to a ratio of 0.87. The *tan* button (or more correctly, its second function) can also be used to convert from ratio of 0.87 to angle of 41º. If the triangle contains an angle of 41º, then the ratio of the vertical to the horizontal side is 0.87. This notion of a converter (Pournara, 2001) may help learners when solving triangles to see the how the angle and the ratio of sides are related.

5.5 The scale factor method for solving triangles

Pournara (2001) suggests the scale factor method for solving triangles. He states that although the scale method is very efficient, the role of the ratio in the algebraic manipulation requires a deeper understanding of the fundamental principles of trigonometry, the link between angle and ratio of sides. He also suggests that there are two advantages of using the ratio as a scale factor. It promotes a structural conception of ratio, and it requires that the learners shift between functions and ratio orientations.

Pournara (2001) states that this approach makes explicit the equivalence of *cos* 38º and *AB/AC*. In doing so, it helps learners to see the ratio as an object; a scale factor that gives the proportion of the sides. The method still requires learners to work with an operational notion of ratio in doing the multiplication. The only algebraic manipulation required in this method, he states, is to isolate the unknown in the ratio. This manipulation required may not be essential because learners can reason "what over 12 gives me 0.788" and then carry out the manipulation without actually doing the algebraic manipulation. Another advantage of this approach, he states, is that it avoids the need for the reciprocal ratios in the introductory stages of trigonometry.

5.6 The teaching of mathematics using a computer

5.6.1 Micro worlds

In this study the micro world was provided by *Sketchpad,* which encompassed the necessary data to provide a way and means not easily accessible to learners. A micro world represents mathematical concepts in a peculiar way that can be close or far away from the school mathematics. Hoyles and Noss (1993, p.84) had observed that *"learners frequently construct and articulate mathematical relationships which are general within* *the micro world yet are interpretable and meaningful only by reference to the specific (computational) setting".* One might conclude that simulations, micro worlds and modeling are powerful implementations. They have enormous potential for the enrichment of learning processes. Each one in its own way is capable of offering a computer environment which supports exploration of the user's ideas. Exploration may happen at different levels. The nature of the software and the knowledge domain of the user are likely to determine the kinds of exploration that can take place.

The notion of a computer-based micro-world for exploring mathematics in the classroom situation appears to be the most attractive. The attraction lies in the ability to focus upon a limited number of related concepts. Exploration of these concepts can take place without the user having to waste time and effort in overcoming difficulties presented by the computer language used. At the same time some access to the computer language is allowed in order to change relationships or rules. It would be argued that a small programme on a calculator, which generates a sequence from a given rule, is a micro world in its simplest form.

Micro worlds are basically computational environments which embody mathematical concepts and ideas. Yerushalmy et al (1990 suggest that learning mathematics should be mainly centred on maintaining a climate of learner decision-making and exploration. A micro world consists of software together with careful sequenced sets of activities on and off the computer (Yerushalmy et al, 1990). It is organized in pairs and whole classes designed with both the mathematics and learners' developing concepts in mind. This came into play in this study as obviously the group, even though familiar with the computer, tends to live in a world almost completely divorced from its use.

However, learning processes can be enriched enormously through micro worlds, simulations and modeling (Mudaly, 2004). Each one can offer a computer environment which supports exploration of new ideas at different levels although mainly determined by the nature of software and the user's knowledge domain. Basically, these explorations of concepts take place without the user having to waste time and effort in overcoming difficulties presented by the computer language used, and alternatively small programmes like those on calculators also come in handy.

5.6.2 Visual reasoning

"The aim of the mathematics department is to provide interesting lessons for all learners, in order to develop their mathematical skills and knowledge. A central way of achieving understanding of mathematics is by talking, reading about it. In order to do this we must provide learners with appropriate mathematics vocabulary and appropriate stimulus for the use of language to take place" (Cox, Gammon et al, 1993, p.9). They believe that the ease, with which the computer produces a visual image of function, and the need to retain a picture of this image, pushes the learners into talking and describing, and hence using "*appropriate mathematical language".* Recent research in mathematics and especially in trigonometry has shown that the concept of a function is most difficult to understand (Pournara, 2001).

The use of *Sketchpad* or computers makes it possible to represent visual trigonometry or mathematics more than any other visual display. Graphs are simple to plot and all their attributes are easy to see. Vertical and horizontal shifts are easy to determine. Moreover, the situation can be inverted. It is possible to also investigate the question as to which actions will lead to a given change in the relationships. The result of such action often can be dynamically implemented. Actions can be repeated at liberty, with or without changing parameters of the action. Conclusions can be drawn on the bases of the feedback given by the computer programme. The power of the computer for learning visual reasoning in mathematics derives from these possibilities.

Like most, Cox, Gammon et al (1993, p.11) were impressed by the potential of technology to make visual representations of mathematics widely available. At the same time they were aware of learners difficulties with graphs described in the mathematical education literature. Rather than approach learner difficulties as "*misconceptions to be uprooted",* they approached them as ideas they could change in the normal course of learning and instruction, and as indications of "*conventions in which their training blinds them.*"

The graph and the data represented on the computer seemed to enable some learners to develop a better understanding of their graphs and so eventually to be able to give a fuller interpretation of their meaning. Recent research on visualisation is concerned with the effects of a visual versus a symbolic approach and how learners relate both (Dreyfus & Eisenberg, 1991). There are studies that show the positive effects of visualising in mathematical concept formation (Bishop, 1989) and give convincing arguments for emphasizing visual components in the introduction of concepts in school. *"There are dangers in doing this carelessly because visual presentations have their own ambiguities"* (Goldenberg, 1988, p.122).

Tall (1989) reports on using the computer to encourage visually based concept formation on calculus. He stresses that the goal is not only to provide solid visual intuitive support, but to sow the seeds of understanding of the formal subtleties that later occur. This implies that learners learn to reason visually with the details of screen representations of concepts such as function, secant, tangent, gradient etc. Kaput and Thompson (1994) have used concrete visual computer representations to build on natural actions in the learners' world with the aim of supporting the learning and application of multiplicative reasoning, ratio and proportion. In particular, they aim to tie the visually concrete and enactive operations on objects on the screen with more formal and abstract representations of these operations. Thus learners' visual operations are directly used in the learning process.

Yerushalmy and Chazzan (1990) see it fit that learners should empirically generate the geometric information and visually infer conjectures. Shama and Dreyfus (1991) have used computer screen presentations of linear programming to allow learners to develop their own solution strategies. Learners need to analyse the problems in terms of the visually presented information. They should also aim for detailed analysis of the relationship contained in the visual screen presentations and form reasoning based on such analysis.

In computerized learning environments it is possible to directly address and overcome some of the problems associated with visualisation (Tall, 1989). Some could be related to lack of flexibility in the learners thinking. It is also possible to transfer a large measure of control over the mathematical actions to the learner. The potential of computers for visual mathematics does not by itself solve the more important problems which were mentioned in the introduction. In every case, visual representations need to be carefully constructed and their cognitive properties for learners need to be investigated in detail (Tall, 1989). The adaptation and correction of features of these visual representations on the basis of learner reaction to them is an integral part of the development. Tall's choice of local straightness rather than a limiting process for the derivative is a case in point.

Similarly, Kaput and Thompson (1994) describe how they have found dissonances between learners' visual experience and the semantic structure of the situation being modeled and have consequently designed a way to avoid such difficulties. These difficulties associated with visual representations can be overcome, but only if they are systematically searched for, analysed and dealt with. In this endeavour, the design of learner activities within the learning environment plays at least as important a part as the design of the computerized environment itself (Dreyfus 1990).

5.6.3 Computer-aided instruction

According to Papert (1980) as quoted by Ainley (1994) instruction and reference to programming are somewhat out of fashion in educational discussion. The tension expressed between computers being seen primarily as rigid and mechanistic tools for teaching and as tools for learning is the current norm. Although the developments in the technology have been enormous, the same ambiguity still causes anxiety for many practitioners.

According to Ainley (1994), the above scenario is very complex in at least two-ways:

1. A lack of clarity about relative roles of teacher and computer (and, of course, learner) is only one of a long list of factors which affect the extent and quality of the use of computers in mathematics classrooms.
2. Issues to do with access to appropriate hardware and software, curriculum constraints and assessment requirements, attitudes to technology and management issues at both classroom and school level are all extremely significant.

Even when high levels of access are available, and curriculum pressure relaxed, teachers' confidence in integrating technology within their existing classroom practice remains a key issue.

A common teaching strategy in mathematics, according to Dugdale (1992), is the use of graphical representations, mostly on the blackboard, but also on worksheets, textbooks homework assignments or written examinations. Since microcomputers are more and more accessible, there exists a new powerful tool to represent graphs and functions and thus to study mathematics. The study by Dugdale (1992) is based on the "*development of graphical environments with computers",* which enable learners to discover and acquire the concept of functions. The approach and rationale behind Dugdale's study was attempted in this research.

This study involved the *cosine* function. According to Dugdale (1992, p.28), "*the function concept is a central one in mathematics because of its potential to tie together seemingly unrelated subjects like geometry, algebra and trigonometry".* It is also a very complex concept which has various sub-concepts associated (Dreyfus, 1990, p.33). In spite of efforts to teach functions by means of multiple representations, high school learners show limited concept images of functions (Vinner & Dreyfus, 1989).

Wenzelburger, (1990, p.118) states that, "*the computer plays an important role in mathematics education, since it is considered a valuable tool to aid in the teaching learning process in mathematics."* Tedious and complex computations can be done on the computer. The learners remain free to concentrate on essential aspects of concepts. Carefully designed graphing software, used thoughtfully, presents opportunities to teach functions successfully. Such software, according to Goldenberg, (1988, p.17), makes use of this possibility: "C*omputer environments seem to be an ideal to build a curriculum from a constructivist point of view, which help learners with transitions between algebraic and geometry representations."* Pea (1994, p.22) puts computers in the context of "*interactive cognitive technologies".* Computers can provide functions that promote mathematical thinking. They fulfill the process functions of being a tool to integrate different mathematical representations.

Garancon et al (1983) undertook a study to find out the use of the computer in a specific activity. Their aim was to introduce the idea of line graphs in two ways, one making use of the computer, and one relying on more traditional resources. The conjecture was that learners who had used the computer will be better able to produce their own graphs by hand, and to interpolate for them. Garancon et al (1983) gave the whole class a pre-test in which data was presented in a tabular form and a graph of a learner's growth drawn by free hand; learners were asked to recognise specific points, and to interpolate. The results from the pre-test were used to establish a base line of skills, and to divide the class into two groups of matched pairs.

On reflection, Garancon et al (1983, p.385) conjectured that the learners were "*able to interpolate, handle scale, plot points and construct sensibly scaled axes because they did not attempt to teach them these skills".* They see these as the process by which skills reach a level at which we are able to function with them automatically, when they are encountered in contexts and at levels subordinated to other tasks.

More traditional approaches to teaching line graphs would necessarily begin by teaching construction skills; constructing suitably scaled axes and plotting points. If attention is focused on these, it could be difficult for learners to keep in mind why the graph is being drawn in the first place. Indeed the skills of constructing graphs are often taught in isolation from meaningful context, and so appear to learners to be an end in them.

Using computers allows learners to have control: to select the data, which is appropriate for their work, and to produce graphical images of that data quickly and easily. Garancon et al (1983, p.387), state that "*their experience suggests that, given that opportunity young learners' ability to work with line graphs is far greater than is generally understood".* Another related study by Garancon et al. (1983, p.54), focused on *"a functional approach to the teaching of early algebra"*. It made extensive use of computer-assisted graphical representations as tools for solving a variety of problems. The aim was to uncover areas of ease/difficulty experienced by seventh graders in learning how to produce, interpret and modify graphs. They worked in pairs at a computer during approximately 25 problem-solving sessions. Garancon et al. (1983, p.387), describes the ways in which learners coped with the two types of infinity they encountered in a dynamic graphing environment that plotted intervals of discrete points rather than continuous curves. In addition to helping learners to become aware of the use of graphical representations as problem solving tools, "*the environment provided a rich context for learning about density of points, infinity, continuity and other issues that tend to be ignored until calculus".*

McDermott et al. (1987) conjectured that the computer plays a significant role in enabling learners to gain access to work with line graphs. It allows learners to build on their intuitive understanding to come up with the skills required to draw such graphs by hand. In this study, learners were able to produce graphs without worrying much about the problems of scaling axes and plotting points. This allowed learners to focus their attention on using the graph in a meaningful way. One feature of the software seemed to be potentially important: if the size of the frame within which the graph is drawn is changed, the scale is altered to fit the new frame. McDermott et al. (1987) had a sense that this might be powerful in implicitly drawing learners' attention to significant features of the graph, which did not change under these conditions.

Although there appears to be considerable difference in the results McDermott et al. (1987) had obtained and those reported by these two papers, it is worth pointing out two factors which they recognize as having considerable significance. The learners and their project class were caring out within the context of a project they had been closely involved with for some weeks. According to McDermott et al. (1987), the data they were working with was, although artificial in the sense it referred to imaginary learners, real and meaningful to them. This would not be the case in either of the studies referred to above. Secondly, the line graphs the learners produced were ones in which the appearance of the graph matched the phenomenon which was being graphed; the graph goes up as the learner grows up. Kerslake (1981, p.132) suggests that "*graphs of this type are the easiest for learners to interpret, and it is not clear whether Padilla's or Swanton's test items contained graphs of this kind".*

In traditional classroom teaching, corrections are handed down by external authority. There is no way that learners can use their own abilities to correct their own work. Dugdale (1992) has pointed out the principles that should be followed in designing learning environments for mathematics, which were used in this study:

- ❖ The environment should consist of a "working model" of the concepts to be learned, in which the mathematics is intrinsic. Learners should be able to explore and manipulate this model.
- ❖ This environment should include a set of inherently-interesting problems which can be explored by learners of varying abilities and inclinations.

This type of learning, according to Edwards, (1991), is constructivist, in that the learner must build upon his or her existing knowledge, and the micro-world provides the tools needed to correct and refine this knowledge. These environments also have the potential to allow learners more independent and self-directed exploration of mathematical patterns, in which learners can go beyond the goals of the game and continue to satisfy their own desire to find meaning and order in their educational experiences.

What is significant about much of the learners' activities in a computer environment is the very much reduced traditional role of the teacher. It is not by design or a conscious act on the part of the teacher to stay more in the background, it appears a thing to do under the circumstances, which comes to prevail. Linked to this role change of the teacher is an equal and opposite role change of the learner. "*Comparisons of computer use and conventional instruction reveal a 39% to 88% reduction in time taken to complete a task"* (Kulik, et al 1983, p.24). This may be due to the software itself, how content is presented and solutions pursued, or it may simply be due to increased work by the learners. A novelty effect may also contribute to an increased working rate. On the face of it there seems to be sufficient evidence to support the use of computers as instructional aides. At the same time, we should not over-estimate their effectiveness for learning; neither should we equate reduced time on task with an increase in conceptual knowledge.

According to Yerushalmy (1998, p.167), *"the use of computers where there is some control over graphic output is an area where it is difficult to argue that there are any better ways of learning. Functions and their graphs, raw numerical data and bar or pie charts, scatter diagrams or just manipulating shapes, all fall into this category".* The essence of this work is in the control which the user has over the computer environment, and the control being exercised by the teacher in demonstration-mode or by learners in a workshop-mode. Learners can now draw graphs accurately, super-impose one on another, change parameters to see the effect zoom in, zoom out, 'see' a limiting value, understand what it is to talk about a point of inflexion. All manner of things can be presented in an interesting way so that learners feel that they need to know about what is going on. A balance needs to be maintained between what is explored, appreciated and expressed using computers and how mathematics is encouraged, expressed and refined.

Proponents of computer-based group-work suggest that potential benefits include the externalisation of ideas through interaction. The other benefits are the consideration of alternative perspectives, a greater diversity of skills and knowledge enabling exchange of information and ideas, and increased attentiveness and on-task behaviour. According to Healy et al (1990), research has indicated higher levels of discussion in computer–based mathematical environments as compared to paper and pencil environments. Research studies into learning resulting from computer-based group-work have however produced conflicting evidence.

5.6.4 Negative factors in computer implementation

Some of the factors at present militating against computers realising their full potential are (Yerushalmy, 1998, p.170):

- ❖ Lack of potential in managing the resource
- ❖ Identification of areas of the curriculum which can be enhanced by the use of computers.
- ❖ Integration into non-computer mathematics work
- ❖ Status of mathematical programming and choice of languages

Evidently, there is also a possibility that over use of computer algorithms for solving problems will retard or even eliminate some of the possible mathematical and critical thinking essential in the process of learning. There is a possibility that learners might get the notion that "only a computer can do it".

CHAPTER SIX

Research design and methodology

6.1 Methodological framework

Research in mathematics education, that focuses on trigonometry, in particular learners' thinking about trigonometry, is limited. This study provides clear details rather than generalities because it explores issues that have not yet been widely researched (Erickson, 1986). The methods of data collection were determined by the research questions and thus it was decided to use the method of qualitative analysis by means of one-to-one- task- based interviews (Goldin, 2000) and interview schedules.

Research question 1: What understanding did learners develop of the *Cosine* function as a function of an angle in Grade 10?

A test was given at the beginning of the research to find out what understanding learners had developed of the *Cosine* function in Grade 10. The use of *Sketchpad*, task sheets and probing also gave more information on where the learners stood in terms the aspect.

Research question 2: What intuitions and misconceptions did learners acquire in Grade 10?

The initial test showed some of the intuitions and some misconceptions the learners acquired in Grade 10. More of them also surfaced when *Sketchpad* was used, task sheet had been completed, and some probing had been done.

Research question 3: Do learners display a greater understanding of the *Cosine* function when using *Sketchpad*?

When the learners used *Sketchpad,* answered the interview questions, and wrote the final test, the results showed some improvement from the initial test in terms of marks.

Although this does not in the least sense demonstrate understanding it reveals some degree of learners' clarity in terms of the aspect.

This method made it possible to document most of the necessary information that individual learners reveal about their sense making of situations and contexts. It was also beneficial to the researcher as it allowed greater control to observe and take note of, how each learner went through the task sheet.

6.2 The sample

This research is based on a case study of a class of Grade 11 learners from a school situated in KwaNdengezi, an African high density township west of Durban, and west of Pinetown. The aim was to obtain insight into how learners at Grade 11 understood different aspects of the cosine function from Grade 10 and what their misconceptions were, if any. It was expected to see similarities and differences between the learners and it was hoped that these would illuminate different aspects of learning and provide a deeper understanding of issues surrounding the important issues of the cosine function. *Sketchpad* was used as a tool and a context to probe their understanding.

The school was chosen due to the convenience of having easy access to the computer laboratory. Arrangements could easily be made to interview the learners since the researcher works there as a Grade 12 mathematics teacher. These learners were selected by their mathematics teacher who chose those who are doing Computer Application Technology (C.A.T.). They were randomly chosen from a group of 123 learners in May 2010. These learners were of different ability levels and no screening was done in this respect although it was taken into consideration that they were not repeating Grade 11. Six learners, all girls, were chosen. All the learners doing Mathematics and C.A.T. in this grade were girls. The purpose of the research was explained to the learners before the research was carried out.

The learners were mainly worried if any marks would be recorded and form part of the end of term report. After a while we reached a mutual understanding that even no marks were recorded and used in their school assessment, they were going to learn mathematics in a more exciting way than the is the norm. Even though this forms part of their learning activities in school, letters to inform their parent/guardians of the research and to obtain permission to participate were given. In turn, permission was also sought from the Principal who further instructed the Mathematics Head of Department, the Grade 11 Mathematics teacher and the C.A.T. teacher to assist me in every possible way. Besides, the participants were given a choice of withdrawing at any stage of the investigation.

All of them were very enthusiastic and willing to be part of the research although as the day neared one of them was developing cold feet. I then realized that I had to spend more time with them for them to feel sufficiently relaxed. Eventually I had to draft in a seventh participant as reserve, in case something happened.

A pilot test for the instruments was carried-out and several adaptations made thereafter. This was done in the second term of the academic year; the learners had not yet done trigonometry again in Grade 11. These learners were ideal for this study as the questions were suited to their level of understanding, taking into account that the topic was dealt with in their previous year. Learners had done trigonometry before and this was their first experience with a computer in trigonometric concepts. Everything was well within the capabilities of the Grade 11 learners.

Learners were not previously exposed to using computers when learning mathematics and therefore also not to the use of *Sketchpad.* Thus the learners involved were brought together for a period of 60 minutes in order to familiarize them with the general use and application of this software before the resumption of the actual investigation. The fact that the learners were not exposed to *Sketchpad* did not affect the experiment because minimal knowledge was expected from the learners about the software. Each learner was made to feel at ease before the interview commenced, in order to ensure that they would respond in a way that would reflect their understanding of the task provided. They were given some time to ponder and write down their answers first before being interviewed and probed but were then not allowed to change or alter their answers later.

6.3 The interview and microteaching experiment

This study relates to action research, as it was more a teaching experiment done in an interview setting. This study was also different, as it was not done in a full classroom setting; it also used an interview format. A microteaching experiment was designed, using psychological interview techniques to study how each learner experiences and conceptualizes each activity. They were given some time to ponder and write down their answers first before being interviewed and probed but were then not allowed to change or alter their answers later. The objective was to see if some learning took place, and to analyse the nature and quality of that learning. These questions were relevant: Have the learners managed to form some concepts of the *cosine* ratio and function? What is the nature and quality of their understanding? What intuitions and misconceptions do learners bring to the learning situation and what is their role in their learning? To what extent did *Sketchpad* assist in their conceptualization?

The learner interviews involved mathematical problems and task-oriented interviews (Goldin, 2000). One of the salient features of such interviews is that the interviewer and interviewee(s) interact in relation to a task(s) that is/are presented by the interviewer in a pre-planned way. This method of interview is, according to Goldin (2000), particularly well-suited for exploring conceptual understanding, complex problem-solving and the construction of meaning in mathematics. The "*structured mathematical environment"*, according to Goldin (2000), can be controlled to some extent but also adapted where necessary. Two important advantages of structured, task-based interviews are that they provide access to the learners' *processes* of thinking about a predetermined task, and consequently provide opportunities to investigate some complex mathematical topics in greater depth.

An important point to consider is that, although the interview setting provides a means of exploring learner thinking in a controlled and systematic way that is not possible in the classroom, the interview setting is not the classroom setting. A significant difference is in the power relation between interviewer and interviewees that is considerably different to the power relations between the teachers and learners. The interactions between the interview and the interviewees are also different. In a task-based interview setting, the focus is generally on learners' thinking and so the correctness or not of the answer may be of little consequence to the interviewer. In classroom interaction, however, the teacher's focus is to usually obtain the correct answer and so the teacher provides the learner with appropriate response to indicate whether their answers are correct or not. The researcher, playing the role of the interviewer, continuously probed the learners' understanding but did not necessarily reveal whether their responses were correct or not and avoided at all costs pushing the participants to an answer.

The purpose in selecting interviews as the means of collecting data was to gain deeper insight into the learners' initial conceptualization of the *cosine* function, in a bid to make inferences about their thinking at a particular point in time.

It must also be pointed out that learning took place in the interview, as can be evidenced in the analysis in chapter seven. It is also possible that the probing by the researcher influenced the learners' thinking and that this may have led to learning. When and how learning took place and how it impacted on learner thinking in the interview is a very important aspect of the analysis undertaken in chapter seven.

6.4 The interview task

The interview focused on three types of task: procedural, conceptual and applicable.

Procedural: tasks that are generally solved by applying a particular method, which is usually, taught by the teacher, for example, the majority of tasks in the existing text books.

Conceptual: tasks that probe learners' understanding of the fundamental principles of mathematics.

Applicable: tasks that require learners to make use of their knowledge of mathematical principles to solve them successfully.

A pilot interview was carried out. Several adaptations were made thereafter. The reason for the choice by the researcher of different types of tasks was that it enabled him to explore the extent to which learners could work correctly with trigonometric concepts.

The study also checked if learners understood the algorithms they learnt and practised, and if they could apply their existing knowledge to a unique situation.

The introductory task to the *cosine* function that the learners had to work through was based on a circle within a Cartesian-coordinate system. The learners were also given ready-made sketch (figure 6.1), which they manipulated as required in terms of the first four tasks, but they also had to make their own constructions in the other remaining three tasks (graphs) using *Sketchpad*. This sketch was of a unit circle where the radius could be changed in length and could be moved right round the circle using a mouse to change the size of the angle. The sketch was drawn using *Sketchpad*. It would simultaneously draw the graph of *cosine,* as participants changed the angles to fill in the tables. There was a small table in the sketch that showed the value of the angle selected, its *cosine* value, the radius, the side *x*, and the ratio of sides. The sketch also gave the learners a chance to realize that as they would be changing the input to get an output it would automatically translate that into a graph. This would give them a visual definition of a function which most are familiar with and can easily identify. At the same time, this will somehow make them more familiar with the *cosine* graph.

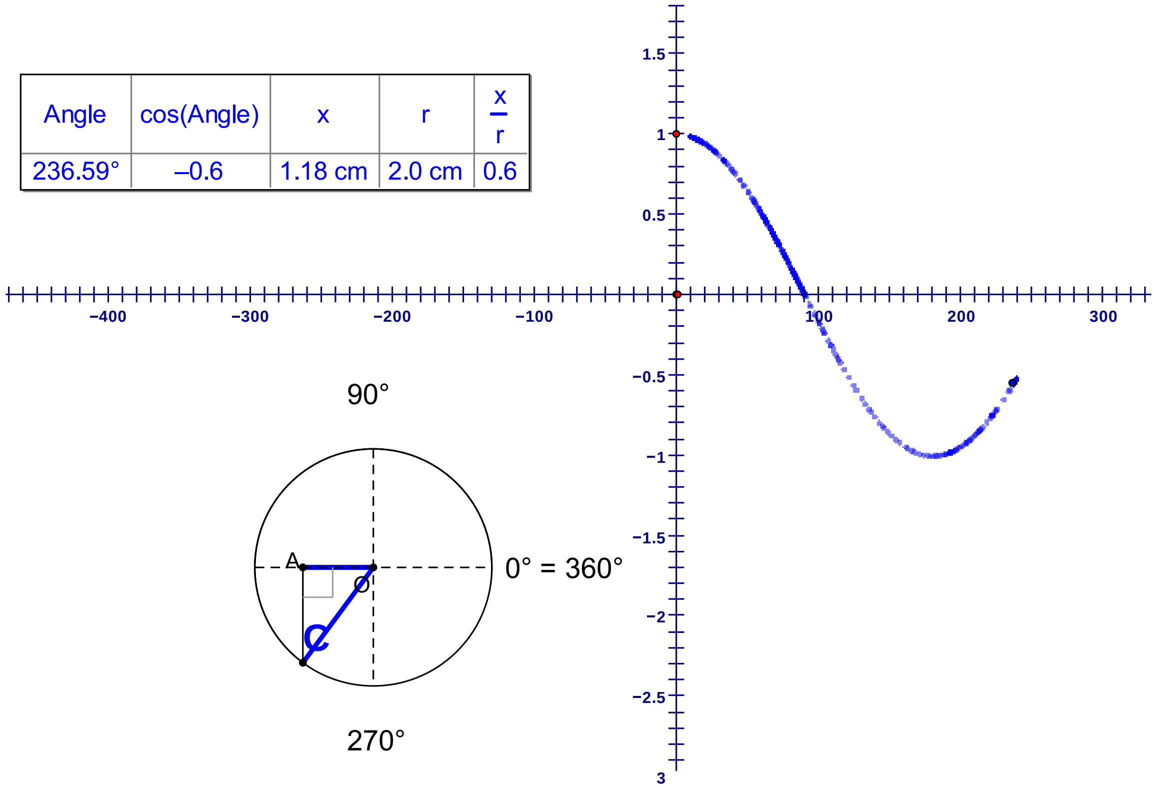


Figure 6.1

The learners needed some guidance in getting to know *Sketchpad*. To build their intuition, they needed to observe, reflect on and conjecture about their experiments.

The decision to present the diagram to them was based on the following reasons:

- ❖ It would take each learner a long time to figure out how to construct a right triangle dynamic in a circle within a Cartesian coordinate system because they were not familiar with *Sketchpad*.
- ❖ The construction of the sketch was not one of the objectives of this experiment. So presenting the construction to them did not affect the essence of the experiment.

At the commencement of the interview, learners were put at ease by the researcher. They were asked whether they understood the task and if they had any question at that stage.

The empirical part of this research focused on the understanding, intuitions, and misconceptions the learners had in Grade 10. Given a self-exploration opportunity within the *Geometer's Sketchpad,* it also sought to see if the learners gained some understanding of the *cosine* function in **all the quadrants,** during a first introductory activity. The study was done in relation to the *cosine* function;

- ❖ As a ratio of sides of a right-angled triangle
- ❖ As a functional relationship between input and output values and as depicted in graphs

In order to evaluate their understanding of the last category above (the functional relationship), this study checked to see if learners could estimate the value of the *cosine* function for an angle and if they could draw a rough sketch of some *cosine* function. It also checked to see if learners had somehow improved their conceptualization of the *cosine* function.

Learners in this investigation were introduced to the *cosine* function in a purely mathematical way, without a real context. This situation may go against outcomes-based education, which proposes that learning should start with a problem in the real–world and then move on to the more theoretical, abstract aspects. In a modeling approach, scale drawing could first be used to solve the problem and to introduce similarity (the constant ratio) of corresponding sides as the basis of trigonometry. Thereafter, a formal definition of the trigonometric functions in terms of a circle may be introduced. In this study, however, my purpose was not to investigate modeling, but to concentrate on the learning of the *cosine* function during a more formal stage, using *Sketchpad.*

The interview protocol that follows was redesigned after a trial run. This is what it finally looked like:

Interview Question 1.1: Do learners understand the *cosine* as a relation between input and output values by filling in tables of values and comparing these values?

No

Yes

Probe and redirect thinking

Interview Question 1.2: Do learners see *cos θ* as a ratio of two sides i.e. *x/r*?

No

Yes

Probe and redirect thinking

Interview Question 2 and 3: Do learners see that *cos θ* is independent of *r* and that it is a function of *θ*?

No

Yes

Probe and redirect thinking

Interview question 4: Are learners able to approximate or estimate the value of a *cosine* function for an angle not included in the data or vice versa? (Using the table of values from *Sketchpad*)

No

Yes

Probe and redirect thinking

Interview question 5: Are learners able to determine the range, domain, period and amplitude of graphs of the *cosine* functions drawn using *Sketchpad*?

No

Yes

Probe and redirect thinking

Interview question 6: Are learners able to determine the effects of a coefficient on *x*-intercepts and range of graphs of the *cosine* functions drawn using *Sketchpad*?

No

Yes

Probe and redirect thinking

Interview question 7: Are learners able to determine the effects of a constant on range and amplitude of graphs of the *cosine* functions drawn using *Sketchpad*?

No

Yes

Probe and redirect thinking

Interview question 8: Are learners able to draw graphs of the *cosine* function without using *Sketchpad*?

No

Yes

Probe and redirect thinking

Each interview was approximately 60 to 90 minutes long and each was audio-taped. Although these questions were structured around the critical questions, it also allowed for variation in expected responses from the learners. Further probing was done in particular cases where learners wrote out the answers to questions at each step of the experiment.

In the final stage, the data was analysed and tabulated. This required the systematic grouping and summarizing of the responses. It also provided a coherent organising framework that explained the way each learner produced meaning whilst working through the tasks provided.

6.5 The study

The learners were given a one-hour test, at the beginning of this study, on Grade 10 trigonometric concepts, to identify some of the misconceptions and gaps they have. The test was not sufficient enough to establish all of these as that would be beyond the scope of this study. At the end they were given again another similar test to the first one to see if the activity had had a positive impact on their conceptualization of the *cosine* function. The two tests together with the instrument were solely based on the main aspects covered in Grade 10, which are basically the trigonometric ratios and the graphs only. The results were tabulated.

Geometer's Sketchpad was used for the task of creating visual intuition. Initially the researcher showed the learners some of the basic tools of the programme such as how to drag the mouse to change the radius, to move the radius to find different angles and ratios. A demonstration on how to draw graph had to be done for them to understand and be able to do it on their own. The exercise is quite simple for the computer does almost everything as long as one follows the correct commands to draw graphs.

Participants were provided with a series of question for them to explore. They were then asked to generalize from their findings. They were "rediscovering" the law of *cosine* for themselves when they manipulated the sketch which automatically measured sides and computed ratios. As they moved the radius to get to the desired angle, a graph of the *cosine* function would be simultaneously drawn.

It was not an original discovery in the strictest sense of ownership. It felt like discovery to them when they realized that from their own calculations it came out a constant ratio for a given angle. This came out as they filled in the tables with different radii; for the same angle the ratio was the same respective of the radius. In this study I also partially took on the role of the teacher, in some instances, in order for me to guide learners through the task. On the other hand, I also assumed the role of the researcher during and after the problem-solving session and analysing the results. My interest was in what the learners did and their conceptual understanding, not analysing the learning objectives.

The six learners' first task was to complete a set of tables for r=1, up to r=4, see table 6.1. All the tasks were completed individually; they would write them down first and then were each interviewed and probed to get their thought patterns and assist them to conceptualize wherever possible. The interview protocol was based on the interview schedule (appendix B) and was merely a guideline to important questions, as some of the questions were not written down.

Relationship between *cos θ* and *x/r*

R=1

Θ	*x/r*	*cos θ*
10º		
20º		
30º		
100º		
150º		
200º		
250º		
300º		
350º		

Table 6.1

Completing the table

The researcher had to explain the following to each learner initially in a brief session, which described the clicking and dragging to use *Sketchpad*:

- ANGLE: Move the mouse until the tip of the cursor is over the end of the radius and drag it to the desired size and take a reading.
- RADIUS: Press the mouse right button on the centre and drag it then release quickly if you reach the desired length.
- GRAPH: Go to "File" and select "New sketch" and from "Graph" choose "Grid form" drag the *x–axis* to 100 and the *y-axis* to 1, 2 and 3 and then from "Graph" select "Plot new function" and select *cosine* function.

The learners were asked questions to ensure that they understood exactly what was expected of them. The learners seemed to quickly grasp the clicking and dragging operations of *Sketchpad* since they all do C.A.T. as a subject.

After the introduction, they were asked to complete a set of tables for $r=1$, $r=2$, $r=3$, and $r=4$.

The *Geometer' Sketchpad* Screen

The learners first completed table 6.1. Based on the information in this table and the exposure to the software, they were then further interviewed (see question 1 in the interview schedule, appendix B).

Before completing the table for $r=2$, they were also interviewed and required to complete question two in the interview schedule (appendix B). Thereafter, they then completed the rest of the tables, were interviewed and probed for their understanding and answered questions in the interview schedule (appendix B).

6.6 Transcripts of interview

The transcripts of the interview in this research form the primary source of the data for the analysis of the learners' understanding. The probing part was mostly in isiZulu and

English to unpack the questions and get what the participants were actually pondering on. The interviews were recorded and the transcriptions of the interviews were done completely by the researcher. According to Jugmohan (2004), it is important that one does not assume that a transcript is an accurate reflection of the interview as there is a great deal of information in the interview situation that an audio-recording cannot capture, for example learners' emotions, the power relations between the interviewer and the interviewee, physical movement and facial expression. An important consideration in transcripts of interviews is that it not a written down version of an audio-recording, it is an interpretation of the audio-recording. There is great information, such as intonation, length of pause, and verbal expressions that cannot be captured easily in a transcript. The transcriber makes a decision about the manner of information that is transcribed by giving meaning from tone of the speaker on the recording.

CHAPTER SEVEN

Analysis and results

7.1 Introduction

In this chapter, the focus is on the level of understanding of the *cosine* function by learners and how they engage conceptualization and visualize the *cosine* function while working with the *Geometer's Sketchpad* during an activity involving a formal circle definition. Also discussed will be how learners' procedures impacted on their thinking.

7.2 Theoretical framework

The Van Hiele theory of geometric thought and aspects of the existence of levels, properties of levels and moment from one level to the next as well as constructivism are used as a framework. Tall and Vinner (1992) and their notion of concept definition are also used in the analysis.

7.3 A ratio orientation

According to Pournara (2001), the mathematical symbol most central to ratio orientation is the right-angled triangle. Other mathematical elements, he states, include: definitions of trigonometric ratios as the lengths of sides of a right-angled triangles; the relationship between the ratios particularly the quotient ratios such as $\tan\theta = \sin\theta/\cos\theta$ and the inverse ratios such as $\sec\theta = 1/\cos\theta$; and typical Grade 10 tasks where learners are given a point in the Cartesian plane and are asked to determine values of particular trigonometric ratios and expressions involving these ratios.

Such problems, he states, usually require learners to set up a right-angled triangle and make use of the theorem of Pythagoras. The angle is grounded in a ratio orientation and it merely serves as a reference point to locate the opposite and adjacent sides of the triangle. It must be positioned in the triangle before the opposite and adjacent sides are assigned. Thereafter the angle plays no further part in the problem.

An important point made by Pournara is that the use of the phrase "*the cosine ratio*" may cause misunderstandings on the part of learners. This statement is often used for simplicity. In referring to *"the cosine ratio"* we hide the role of the angle and so when learners work with notation such as *cos 30º*, they may not use the *cos 30º* as a single object, a ratio. They tend to treat *cos 30º* separately (Pournara, 2001). They know that 30º is an angle, so they treat *cos* as the ratio, hence the *cos* ratio. He further goes on to say that such misconceptions are reinforced when we speak of "*the cosine of an angle"* if *cos* is a ratio and 30º is an angle, then learners see no problem in the "*cos of 30º"* as "*the ratio of an angle"*.

7.4 Methods and procedures

In this section the methods learners employed to solve the interview tasks and how their answers relate to their understanding of trigonometric ratio and function (see appendices A, B, and C) are discussed. The instruments used were an initial test, *Sketchpad*, and a final test, in that order. Using two different tests to test progress on learning related to concept is the main instrument used in the education system of education in South Africa and the world over. Similar instruments on *Sketchpad* have been used in mathematics (Mudaly, 1999, 2004 & 2007, De Villiers & Govender, 2006, De Villiers, 2008) and in trigonometry (Jugmohan, 2004). This shows that the instruments used in this study were valid and reliable for they have been extensively tried and tested.

7.4.1 Analysis of the initial trigonometry test results

The initial trigonometry test was meant to assess the participants' level of understanding before going into the entire research process. It comprised of five questions. Question one was meant to test Van Hiele Theory's level one, whilst two and three were for level two and four and five were for level three. The analysis is divided into 3 categories, correct, wrong and partially correct (where a learner shows that she knows what the question is all about and comes up with correct working but fails to get the correct answer because of an error of some sort)

Question 1: Which side is the hypotenuse, which one is the adjacent and which one is the opposite in relation to the given angle in table 7.1, with reference to the triangles in figures 7.1 & 7.2?

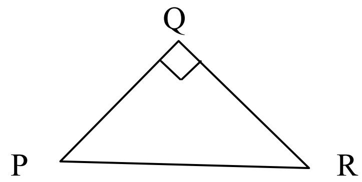


Figure 7.1: Non-standard right triangle

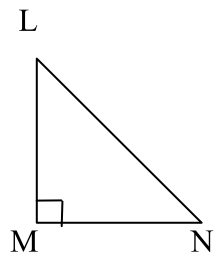


Figure 7.2: Standard right triangle

Angle	Hypotenuse	Adjacent side	Opposite side
$\hat{P}$	$PR = q$	$PQ = r$	$QR = p$
$\hat{R}$			
$\hat{L}$			
$\hat{N}$			

Table 7.1

Results analysis: **Question 1**

Standard right triangle		
Hypotenuse	**Adjacent**	**Opposite**
All 6 learners managed to identify the correct hypotenuse.	All 6 learners identified the correct adjacent side.	All 6 learners identified the correct opposite side.

Table 7.2

Non-standard right triangle		
Hypotenuse	**Adjacent**	**Opposite**
- Only 1 learner identified correctly the hypotenuse side - 5 failed to identify the hypotenuse	- Only 1 learner was able to identify the correct adjacent side - 5 failed to identify the correct adjacent side	- 2 learners identified the correct opposite side - 4 failed to identify the correct opposite side

Table 7.3

This question was not well done, and indicates that these learners have problems with visualization, as shown in the tables of results above (tables 7.2 & 7.3). It was easier for the learners to answer questions from the standard right triangle (table 7.2). Out of the three questions on hypotenuse, only seven responses out of 18 were correct. The same applied to the question on the adjacent side, and all were from the standard right triangle. The opposite side had better results scoring 14 correct responses out of 18 though with a mere two from the non-standard right triangle (table 7.3). No answer was partially correct in the entire questions. Those who did not answer correctly just indicated that they could not remember a thing from the topic, as shown by their verbal responses. If the learners can identify the attributes of a right triangle only if it is in standard form, then it means that the concept was not mastered to a great extent. The possibility of not having understood the question does not come into play at this juncture because all the learners were able to fill in correctly values for the standard triangle. This means that the learners are struggling to visualize at Van Hiele theory's level one.

Busi' response (table 7.4)

Researcher	I see you wrote that the hypotenuse is RQ=p when dealing with angle R in triangle PQR. How did you arrive at the answer?
Busi	I changed the vertices and moved them around…. (*Giggles*) θ was missing...(pause)……it makes it clear

Table 7.4

A possible explanation for their poor visualisation could be the standard, prototypical way in which right triangles are mostly presented in textbooks. It also proves beyond reasonable doubt that the interaction between the learner and the object of study, in this case the right triangle, was not sufficient to allow learners to construct meaning out of it. The use of dynamic geometry software would give the learner varied positions of the shape in a short space of time. The learner would also have the opportunity to move it to desired positions and stance.

Question 2: Given 15 $\sin\theta = 12$ and $\tan\theta < 0$. Calculate the value of $1 - 15\cos\theta$ with the aid of a diagram

Results analysis: Question 2

Sketching a right triangle with hypotenuse 15 and side 12	**Missing Side**	**Ratios**	**Substitution**
3 learners managed to sketch and label correctly and 1 learner had a correct sketch but did not label the sides. 2 learners did not respond	All the learners failed to calculate the missing side using Pythagoras theorem	No learner managed to employ the trigonometric ratios	All the learners seemed confused about the context

Table 7.5

In order to answer correctly question two it was necessary for the learners to sketch a diagram and be able to come up with the answers (table 7.5). Three diagrams were properly drawn, one was partially correct and the other two did not attempt to do anything at all. In response they indicated that they were not familiar with the use of a diagram when responding to such questions. However, they may have simply forgotten the work they had done in the previous year.

Researcher	You didn't answer question 2.
Busi	I'm slow. Mathematics is difficult. I tell myself it is and so does everybody.

Table 7.6

This negative view (table 7.6) shows that learners tend to believe that they do not have to do anything other than listen to the teacher as he/she speaks in class. This coincides with Brousseau (1997)'s "didactical contract" where learners take the process of teaching and learning to be tantamount to the tea-pot tea-cup relationship. They see themselves as empty vessels that have to be filled up by the teacher with them watching passively.

Another participant, Thabisile had the following to say (table 7.7)

Researcher	You got 19.2 as the length of the missing side in question 2. How did you arrive at that answer?
Thabisile	I said Adjacent=15^2+12^2
Researcher	Which method did you employ?
Thabisile	Theorem of …eh…can't remember….ah (*sigh*)…Pythagoras.
Researcher	Do you remember exactly when to use it?
Thabisile	Yes. When you want an answer using square roots….If you add 2 sides you get the other side…..when you want the hypotenuse…I'm really not sure how to use it.

Table 7.7

Question 3: Given $5 \cos A + 3 = 0$ and $A \in [180°; 360°]$. Calculate the values of the following with an aid of a diagram;

i) $Tan\ A$

ii) $3 \tan A + 25 \sin^2 A$

Results analysis: Question 3

Sketching a right triangle with hypotenuse 5 and side 3	**Missing side**	**Ratios**	**Substitution**
1 learner drew the correct diagram and managed to label it accurately whilst another had a sketch with no labeled sides. The other 4 learners did not attempt to answer the questions.	No learner managed to use the Pythagoras theorem to find the remaining side.	No learner managed to employ the trigonometric ratios.	All the learners seemed confused about the context.

Table 7.8

Even though question two and three are similar, the learners failed to connect them (table 7.8). They were actually surprised to discover that they are similar and had no reasons as to why they found it difficult to draw the sketch in question three when they had managed it in question two. It does appear that less attention might have been given to problems in quadrants other than the first in Grade 10.

However, questions two and three indicate that though some of the participants were able to identify right triangles, several still fall short when it came to Van Hiele Level 2 thinking; not seeing relationships between properties.

Question 4:

4) P is the point (-5; $\sqrt{11}$). Determine (figure 7.3):

i) OP
ii) $1 - cos^2\ \alpha$
iii) $Sin^2\ \alpha$

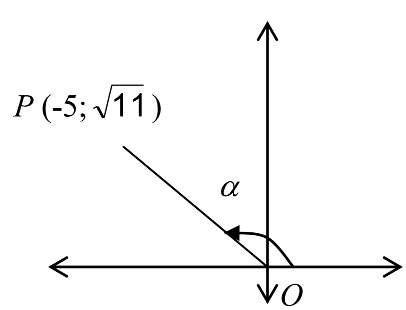


Figure 7.3

Results analysis: Question 4

Sketching a right triangle by drawing a line perpendicular to the x-axis from point P	**Label sides**	**Ratios**	**Substitution**
None of the learners indicted that a line could be dropped from the point to form a right triangle with the axes.	All the learners did not indicate any dimensions of sides x and y.	No learner managed to employ the trigonometric ratios.	All the learners seemed confused about the context.

Table 7.9

Performance on this question was poor (table 7.9). None of the learners were able to solve the items correctly. Evidently, the learners could not come up with any method to tackle it. All the learners just skipped the question with no attempt at all. After the test the learners were even convinced that they had never come across this type of question at all. Perhaps such questions were never done with them even though they are in the curriculum. This could be so, because, at times when teachers see that their learners are failing to understand some basic concepts they see no reason to take them to the next level and postpone them to some later date.

Question 5: If $\alpha = 63.7°$ and $\beta = 28.2°$, use a calculator and give your answers correct to 2 decimal places to evaluate:

Cos $(\alpha - \beta)$

Results analysis: Question 5

Substitution	Simplification	Use of Calculator
Only 1 learner managed to substitute the values into the formula	The same learner failed to simplify the expression	The same learner was the only one who realized that a calculator had to be used in order to answer the question. The rest of the learners were confused by the context.

Table 7.10

Only one learner was able to substitute correctly (table 7.10). This showed that she had understood the question even though she did not actually get the correct answer through a computational error. She tried to subtract the angles on her own first and got a wrong answer. She then referred to the calculator to find the *cosine* of the angle. This is quite common as learners even with a calculator in hand, tend to prefer simplifying numbers on their own and only use the calculator where they think the numbers are complicated.

The rest of the learners did not attempt to answer the question as they found it complicated. This serves to show that the theory is correct in terms of fixed order as defined above.

7.5.2 Analysis of *Sketchpad* activity

7.5.2.1 Interview Question 1.1: Do learners understand the *cosine* function as a relationship between input and output values and as a ratio of sides of a right-angled triangle in different quadrants?

After the tables for r=1 was completed, the learners were asked interview question 1.1 (appendix B): *What do you notice about the x and r value respectively as the angle θ changes in size?"* The reader is reminded that the table referred to all four quadrants.

This question tested if learners, for a given *r*, could observe and understand that *x* changes as the angle changes also.

Four of the learners made a correct observation in this regard, while two learners could not. Their responses are now discussed in more detail in two categories below:

7.5.2.1.1 Category 1: *x* changes and *r* stays the same

After probing, four learners felt that *x* changes and *r* remains the same. When they had previously written down their answers they had thought otherwise. They said they had not understood the question initially. In most cases learners have a tendency of rushing through their work without reading questions carefully.

Only Samkelisiwe was accurate in her answer when she wrote and then explained: *"As it increases, x gets smaller, bigger, smaller, and bigger again in each quadrant but r stays the same".* Busi, Thabisile and Bongekile had to refer to their tables when answering this question. Noxolo mistook the *x/r* in her table with the *x* and *r* value respectively and got more confused the more questions were asked. When questioned about *x* and *r*, she realized that they are not angles, but still failed to make the correct conjecture. Referring to her completed table she replied: "*x and r decrease as the angle increases"*. When asked what she meant, she replied *"I don't know; Maths is difficult"*.

Thandeka correctly dragged the angle and answered: "*r remains the same, but the x value decreases".* She first described, *for the r value: there are changes happening…. The ratio is going further up from the x…. And the degrees and the ratio change.* When asked directly about the *r* and *x* value as the angle changes she responded: (smiling) "….*x is decreasing yes and r remaining the same"*, but failed to realize the difference in the other quadrants.

What do you notice about the values of x and r respectively as the angle θ increases?

Busi's answer to interview question 1.1 (table 7.11): when first asked this question it was clear she had not understood the question as evidenced by what she had written above.

Researcher	Ok, Busi, what do you notice about the x value and the value of r as the angle θ increases
Busi	I don't understand. They are decimal fractions.

Table 7.11

The researcher further probed the learner by focusing attention on only one variable first (table 7.12).

Researcher	Ok, what can you say about the x value as the angle increases?
Busi	x is that (pointing to the column x in the table)
Researcher	Yes.
Busi	The angle is changing. Yeah. Ah... x changes…I see, oh that is the question! Ok.

Table 7.12

The researcher further questioned Busi to answer the original question (table 7.13):

Researcher	Ok, what do you notice about x value as the angle is increasing?
Busi	The x value decreases, increases, decreases and then increases again. (dragging the radius in different quadrants, somewhat surprised that she had failed to notice it)
Researcher	What do you notice about the r value as the angle is increasing?

Busi	Same.
Researcher	As the angle is going up?
Busi	Same.
Researcher	Are you convinced?
Busi	Yes, because this diagram is the one for r=1.

Table 7.13

Interestingly all the learners, except Samkelisiwe, referred back to *Sketchpad* and to the table when answering this question. Conclusively, four out of the six learners correctly observed and understood that, for a given r value, x decreases in the first quadrant, increases in the second, decreases in the third and then increase again the fourth quadrant as the angle increases. Thus they were successful in the conjectured Level 2 of the Van Hiele Theory. The four learners, who answered correctly, referred to *Sketchpad* (Busi) and to the tables (Thabisile and Bongekile) whilst Samkelisiwe relied on her memory and what she had done on her own.

Noxolo replied that both decrease, not realizing that r did not. Thandeka replied that x decreases and r remains the same only considering the first quadrant. Time permitting more probing could have been done with these two learners. They also noted is that they did not take time to check their responses using either *Sketchpad* or the table, they just rushed through everything.

7.5.2.1.2 Category 2- Both *x* and *r* decrease, or *x* decreases and *r* remains the same

Noxolo felt that both x and r decrease.

Example: She was quite convinced that both x and r decrease, for example, she responded by saying that: "…..*as the degrees get bigger, they both decrease………x and r decrease. Haw, Maths is difficult, sir*". This is again the didactical contract discussed earlier on. Learners expect teachers to give answers as they sit and watch. They do not have to figure out anything.

Thandeka felt that x only decreases and r remains the same.

Example: Though she correctly dragged the angle, she failed to realize the difference in the other quadrants. She answered: "*r remains the same, but the x value decreases"*. She first described, for the *r* value*: "There are changes happening…. The ratio is going further up from the x…The degrees and the ratio change…x is decreasing yes and r remaining the same"*.

These two learners do not seem to have mastered Level 1 of the Van Hiele Theory yet, namely correct visual observation of the displayed lengths. It is difficult to discover the generalization from the constant ratio to the functional relationship, which is characteristic of Van Hiele Level 2 without correct observation. However, at times the learners can observe correctly but can have difficulty in expressing their observations in words. Accurate description of one's observations is a skill on its own and more of a challenge for second language learners than for first language ones.

7.5.2.2 Interview Question 1.2: Do learners see *cos θ* as a ratio of two sides *x* and *r*?

The learners were more confident and seemed clear about what was asked. Their responses showed that they did not have any problems with answering this question.

7.5.2.2.1 *x/r* and *cos θ* are the same

All the learners correctly observed that *x/r* and *cos θ* are the same or almost the same and that in two quadrants *cos θ* would be negative. Most learners had gained confidence and did not need to go to the table. Interestingly, the learners did not pick up on or chose to ignore the small differences in the decimal displays. Even though the decimal differed in some cases in the table because of the improper placement of the diagram when changing the radius, learners observed the "sameness".

The immediate reply was *"They are the same"* without looking at the table or computer screen. When questioned why they did not refer back to the sketch on the computer screen or the table, they said that they knew that from when they completed the table and did not need to look. Busi also immediately said that: "*answer for x/r and cos θ are* *almost the same, only that in some quadrants cos θ is negative".* Thandeka also replied: "*…it's like the same…. It's not exactly the same, some are exactly the same…..but some are like below or above the value."*

Noxolo, Thabisile and Busi answered that the values were almost the same as their values differed in one decimal in some cases. Below are Busi's tables (tables 7.14 & 7.15):

r=1

Θ	*x/r*	*Cos θ*
10º	0.95	1.0
20º	0.92	0.9
30º	0.86	0.9
100º	0.10	-0.2
150º	0.79	-0.9
200º	0.92	-0.9
250º	0.39	-0.3
300º	0.42	0.5
350º	0.92	1.0

Table 7.14

Obviously in this case she had resorted to using her calculator in dividing the actual value by *r* hence the 2 decimals after the comma. This must have emanated from the fact that they were not that aware of that the calculation could be done by *Sketchpad*. Thandeka actually had 0.93cm/1.0cm in her first table. It came out different in table 2 for all except Samkelisiwe who only changed in the last one. The researcher felt that personal discovery at this juncture was more appropriate.

r=2

Θ	*x/r*	*Cos θ*
10º	1.0	1.0
20º	1.0	0.9
30º	0.9	0.9
100º	0.1	-0.2
150º	0.8	-0.9
200º	1.0	-0.9
250º	0.4	-0.3
300º	0.5	0.5
350º	1.0	1.0

Table 7.15:

This question demonstrated that all six learners, with the aid of a visual representation (*Sketchpad),* were able to correctly observe that *x/r* and *cos θ* are the same where *cos θ* is positive. Where it is negative, in the other 2 quadrants, the values would be distinctly different because of the negative sign only. This is not to say that they could not have discovered it in the same way using paper and pencil, but they might not have done it in Grade 10. Through observation and experimentation the learners observed the relationship between the "*cosine of an angle*" and the displayed values. All the learners who were interviewed therefore achieved Van Hiele Level 2 (Analysis), with respect to this task.

7.5.2.2 Interview Question 2: Do learners see that *cos θ* is independent of *r*?

The main purpose of the following question was to establish if the learners were able to make a conjecture regarding their observations, and generalize that the ratio would remain unchanged for $r = 2$. The learners were asked in question 2 in the interview schedule (Appendix B): *"What do you think will happen to the above ratios if we increase r to 2? Why?"*

The reader must note that this question was asked after the first table for $r = 1$ and the interview Question 2 were completed, and before r was dragged to 2 with *Sketchpad*.

7.5.2.2.1 The ratio *x/r* will increase

Initially all learners replied that the ratio *x/r* will increase. This shows that despite having been introduced to the cosine and other trigonometric functions in Grade 10, they firstly did not know that the ratios remained constant. Secondly, this means that they probably did not understand the underlying similarity of right triangles with the same reference angle, which forms the basis of trigonometry.

Bongekile wrote: *The x/r for example will be, if $\theta = 10^o$ - x/r will be 2.95cm because we added 2, before it a 0.95.*

In justification of her answer she said: *"…the circle will increase so the answers for the ratios will increase the total degrees will get bigger by 2; x/r will get bigger".*

Similar responses were given by Noxolo, Thabisile, Busi, Thandeka, and Samkelisiwe: "*If we change r to 2, the cos value and the x/r value will increase to that because the circle will be bigger"*

None of the six learners knew that the *cosine* of a given angle will remain constant, irrespective of *r, and* all of them expected it to increase as *r* is increased. The researcher had to allow and guide the learners discover for themselves the conjecture. The learners did not see the relationship initially, but after completing the second table the room was filled with giggles and whispers that the tables are the same. It seemed that they were surprised by their finding which contradicted their expectations. This is similar to the method of "cognitive conflict' where meaningful learning requires learners' false conceptions to be contradicted by observed experience.

7.5.2.3 Interview Question 3: Are learners are able to generalize that *cosine θ* is independent of *r*?

Interview question 3 (Appendix B) was given to them after completion of tables for $r = 2$, $r = 3$ and $r = 4$ using *Sketchpad*: "*for any given angle, what do you notice about the corresponding values of x/r in each table for r=2, r=3 and r= 4***?"** The learners answered that the values remained constant without any form of hesitation. This can be evidenced by their responses as shown below:

Bongekile replied: *"The values are similar; they all begin and end with same number".* So she correctly refers to the degree of accuracy in the decimals. Noxolo concurred. Thabisile answered: *"They are almost the same."* So did Thandeka and Samkelisiwe. This again takes into account the correct values of decimals. Asked if they had observed anything about the graph the answer was the similar: *"It was the same through-out, it never changed".* Asked if they thought *cosine θ* was a function: Noxolo answered: *"It drew a graph, it can be"* and Bongekile *"It has a graph".* Thandeka said: *"The graph shows it".* Samkelisiwe said she knew it from class.

Notably, these learners seem to have the conception that a graph is necessarily a function. One wonders if they would take a bar graph of, say, income distribution grouped in geographical areas as a function.

Busi had a slightly different answer: "*They are the same. The values of x/r in each table are the same".* She did not even look at her table to answer this question but rather referred to the graph which was simultaneously drawn. When asked to explain this, she replied: *"Even the graph traced remains the same, it does not change".* When she was asked if r=100, and for any *r* she replied: *"yes".* Asked if she thought *cosine θ* was a function she replied: *"yes".* When she was asked to elaborate: *"You can draw a table of values of x and y and draw a graph"* Are all graphs functions? *"Most, but this one I know it, it's a function".*

All six learners answered correctly that for any given angle, the corresponding values of *x/r* were the same in the table for *r*=2, *r*=3 and *r*=4. According to the levels of geometric thought in Van Hiele Theory, they had achieved level 3, called informal deduction (Ordering), where learners can come up with meaningful definitions.

7.5.2.4 Interview Question 4: Are learners able to estimate the size of an angle given a ratio only?

7.5.2.4.1 If *cos* (angles) = ½ then the angles are _____and______?

7.5.2.4.1.1 The use of *Sketchpad*

This question gave the learners a lot of problems. They did not know what to find and how to go about it. Only two learners, Thandeka and Samkelisiwe, asked if they could use the computer for they felt they would do it better the second time around but still failed to interpret the question to a level they understand. The truth of the matter was that they were not sure of what exactly they were supposed to look for.

Samkelisiwe: *Can I use the computer? I want to check my answer first?*
Interviewer*: How?*
Samkelisiwe: *I'll drag the radius till I get 0.5 and then check the corresponding angle.*
Interviewer*: Show me then*.
Samkelisiwe: *There it i*s. (Pointing at 60º)
Interviewer*: Is it the only one?*
Samkelisiwe: (she then dragged the radius round all the way, with a lot of scrutiny, to the fourth quadrant) *there is the other one …300º* (smiling triumphantly)

It was almost the same with Thandeka.
Thandeka: *I prefer to use the computer. I have to check my answer again. I am not sure*
Interviewer*: How?*
Thandeka: *I'll use diagram and see where there is 0.5 and then check the angle of it.*
Interviewer*: Show me then.*

Thandeka: *There it i*s. (Pointing at the diagram indicating 60º)

Interviewer*: Is it the only one?*

Thandeka: *There could be 2 or more. Let me just find out…. There is the other one …so they are 2.*

7.5.2.4.1.2 The use of a table

Two learners needed some assistance before going on to use their tables. The other two got so confused by the question that the assistance by the researcher made no difference and they gave up answering it without any further attempt.

Thabisile realized her mistake early but could not find a remedy at first.

Thabisile: *... it's difficult sir*

Interviewer*: Can you try it using the previous exercise?*

Thabisile: (checked her table first and located 0.5 in line with 300º)*...ah... It's easy*

Interviewer*: Is it the only angle?*

Thabisile: *Let me check…..it's not there but I think since these values are decreasing* (pointing at 20º and 30º) *it will be there.*

Busi*: I think it is 90º.*

Interviewer*: How did you get the angle?*

Busi*: I am thinking 90º, because that is the angle I know in a right triangle".*

She was not aware that she needed to refer to the previous exercise to come up with the correct answer. The interviewer had to ask her if she thought the question was related to the previous exercise. After that she went back to the exercise and then referred to her table and eventually came up with 300º even though she failed to find the other value.

Noxolo and Bongekile found the question demanding and gave up without any attempt regardless of the convincing verbal persuasion from the researcher. Once again we see the didactical contract creeping in from the side of the learners, as explained earlier on. Most learners are comfortable with the teacher telling them only the answer and not just giving them clues.

Noxolo: *Must I change it to decimal?*

Interviewer*: Are we referring to an angle here?*

Noxolo: *No we are talking about a fraction ……. So ……you can say ½ =0.5 so will give you cos 0.5".*

Interviewer*: Read the question again, this time slowly.*

Noxolo: *I have to find an angle! Oh so no … (silence)….. (Thinking)….. There could be a way of doing it, but I don't know.*

Samkelisiwe and Thandeka realized that they needed to use the computer to find the angles. This demonstrates that the learners had acknowledged its usefulness of a computer in solving mathematical problems. Even though Thabisile and Busi used their tables, it was still an indirect way of accepting the computer as a useful tool in the subject for they could have tried to use calculators.

7.5.2.4.2 Estimating the value of the angles if *x/r* = 0.55

The answers the learners gave emanated from their answers to the previous question. Those who had a correct answer found it simple to proceed to another correct one for it was a simple continuation of whatever they had as the answer to the question. All the four learners who had managed to make the correct conclusion made a conclusion that ½ was similar to 0.55 and that the angles would be different by a few degrees (57º and 303º).

One of the misconceptions that the learners had is that they had the belief, particularly at Grade 10 level, that ALL functions are linear. This came out very nicely where they had to estimate an angle of 0.55. Most of them used the assumption that the *cosine* function is linear. They got an answer close to the correct one because within the small interval, it is approximately linear. As a teacher, it is very important that one is aware that the learners could be using incorrect reasoning. Teachers should be aware of this and should develop strategies to alert them that the *cosine* function is not linear. Over a small interval, yes, but over a larger interval, learners are bound to make mistakes. It is not only applicable to the *cosine* function, but to the quadratic function as well.

In this question Noxolo and Thabisile were the only learners who were initially confused by angles and ratios. The other learners at this point did not confuse the angles and ratios. Thandeka and Samkelisiwe found it fit to use the *Sketchpad* even though they were only geared to find only one angle initially. However when told that the ratios corresponded to the *cos* (angle) they had in the tables, they quickly found the two values and correctly answered the questions. Four out of six learners, that is Busi, Thandeka, Thabisile and Samkelisiwe correctly noticed this and used the computer to find the two angles that corresponded to the two given ratios each. According to the conjectured levels of geometric thought, they have achieved Van Hiele Level 2. Here the emphasis is on ratio.

7.5.2.5 Interview Question 5: Are learners able to determine range, domain, period and amplitude of a graph of *cosine*?

All six learners had the correct *cosine* graph on the computer. The graph had been drawn using *Sketchpad* to add some flavour to the study even though it was not the main task. The researcher had to assist the learners through the steps in most cases as they had not drawn any graphs before using this software.

The researcher had to guide the learners by reminding them of the definition of domain, range, amplitude, and period. The learners had to identify these on their own from the graphs without the use of *Sketchpad.* Some learners had written down wrong answers, however, they were able to identify correct ones after the explanation.

7.5.2.6 Interview Question 6: Are learners able to determine the effect of the coefficient of *cosine* ($y = a \cos x$) on x- intercepts and range if it is increased, decreased, less than 0?

The dynamic software, *Sketchpad*, was used to put together a group of different graphs for the learners to clearly observe differences between them. The learners had to observe the diagram and come up with the corresponding effects on their own without the use of *Sketchpad.* The learners had varied ways of expressing themselves. Some preferred to describe the effects in terms of amplitude and not range, others even used *y*. The researcher had to guide them through the steps at times as they could not remember the next step or so.

Bongekile had the following:
As the co-efficient of *cos x* increases: *"Range increases"*
As the co-efficient of *cos x* decreases: *"Decreases"*

Asked if she could show it from her graphs she replied: *"There they are".* She referred to the correctly drawn diagrams. Thandeka used *y* and so did Noxolo and Busi. Samkelisiwe and Thabisile talked of amplitude instead.

It can be concluded that it seems as if all the learners succeeded in identify all the attributes of the *cosine* graph but not the correct terminology. Some confused range with amplitude. Perhaps not much had been done in Grade 10 along those lines. However, according to levels of geometric thought in Van Hiele Theory, they have achieved Van Hiele Level 2

7.5.2.7 Interview Question 7: Are learners able to determine the effect of a constant (*y = cos x + q)* on amplitude and range of a *cosine* graph if it is greater or less than 0?

Experience from the previous exercise helped as the problems in this one were very minimal and they all managed to correctly observe the *y* or vertical shift of the graph. This shows that the learners were able to benefit from the use of *Sketchpad* as a learning tool. The fact that a number of similar exercises can be done in a short space of time is yet another advantage of using dynamic software as shown by these two similar activities. To summarize: it can be concluded that it seems as if all the learners succeeded in identifying all the attributes of the *cosine* graph. According to levels of geometric thought in Van Hiele Theory, they have achieved Van Hiele Level 2.

7.5.2.8 Interview Question 8: Are learners able to draw the graphs of *y=cos x* -2 and *y*=-2*cos x* without the aid of the computer?

This question was intended to check whether the learners could generalize from the above graphs and draw the graphs on a piece of paper without using the computer. No probing was done. The researcher just looked at the diagrams and asked a few questions where necessary, to draw some conclusions.

Thandeka failed to draw the sketches as she claimed that she could only do it using the computer. Busi and Thabisile had only one wrong one. Noxolo had one wrong and another correct. Bongekile and Samkelisiwe had their diagrams correct.

Of the six learners, it can be concluded that three of them succeeded without using *Sketchpad* in drawing the *cosine* graph and identifying all the attributes.

7.5.3 Analysis of the final trigonometry test results

The main objective of this test was to see if the intervention by *Geometer's Sketchpad* had in any way filled in some of the missing gaps in the learner's understanding of the *cosine* function and had given them a better conceptualization. The analysis is divided into three categories, correct, wrong and partially correct (where a learner shows that she knows what the question is all about and comes up with correct working but fails to get the correct answer because of an error of some sort).

Question 1: Use the diagram (figure 7.4) (no calculator) to determine the value of:

Cos A

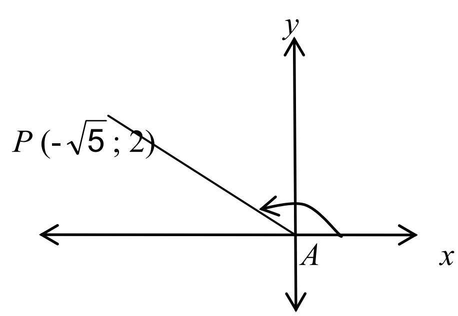


Figure 7.4

Results analysis: Question 1

Sketching a right triangle by drawing a line perpendicular to the x-axis from point P	***Cos A***
All the learners indicted that a line could be dropped from the point to form a right triangle with the axes	2 learners managed to get the correct value whilst 1 had the hypotenuse/adjacent instead of vice-versa. 3 other learners did not indicate any dimensions of sides *x* and *y*

Table 7.16

This question regarding the use of a diagram was correctly done by all (7.16). The ratio part had half getting it right and the other incorrect. Somehow the use of *Geometer's Sketchpad* had helped the learners observe that as the radius moved, from each point a right triangle could be drawn. Clearly the learners showed that they were able to assimilate the information given to them in the problem taking into account the relevant data.

Question 2: $17 \sin \theta = -15$, $\theta \in (90°; 270°)$. Use a diagram to evaluate the value of $\cos^2 \theta$. Do not use a calculator.

Results analysis: Question 2

Sketching a right triangle with hypotenuse 17 and side 15	***Cos θ***	***Cos²θ***
5 learners drew the correct diagram and managed to label it accurately whilst only 1 failed to so.	2 learners managed to get the correct value. 2 other learners had hypotenuse/adjacent instead whist the other 2 seemed confused by the question.	2 learners went on to get the correct value. 1 learner squared the wrong hypotenuse/adjacent whilst the other 3 did not attempt the question at all.

Table 7.17

Only one learner failed to come up with a diagram which indicated that they somehow tried to relate to the exercise with *Sketchpad* where they had to use the ratio of sides of a right triangle (table 7.17). Four learners showed that they could find the values of $cos\ \theta$ and three of $cos^2\theta$. Two learners had no idea on how to calculate $cos\ \theta$ and 3 failed to find $cos^2\theta$. It appears that these 3 learners are still struggling at Van Hiele Level 2 and still require more time and more practice in order to grasp the concept a little more. However, there is some upward movement in terms of the attempt as compared to the similar question in the first test.

Question 3: Calculate the values of:

$Cos\,(123.4° - 86.1°)$

Results analysis: Question 3

Simplification to *Cos* 37. 3º	**Use of Calculator to get correct fraction**
5 learners managed to simplify what was inside the brackets to 37. 3º whereas only 1 learner failed to do so.	2 learners got the correct value whilst 2 others showed that they could not correctly use their calculators. The other 2 learners thought it was enough to get ***Cos* 37. 3º** and saw no reason to proceed to finding the fraction.

Table 7.18

In this question the learners showed that they were able to assimilate the information given to them in the problem and were able to use the calculator effectively (table 7.18). Even though some committed some errors which impeded them from getting correct answers, their working showed that they knew what they were doing and only 2 did not do it the correctly.

Question 4: Solve for x: $x \in [0°; 90°]$

a) $2 \cos x = 0.766$

Results analysis: Question 4 (a)

Simplification to *Cos* x = 0.383	**Use of Calculator to get correct angle**
5 learners managed to simplify the equation to ***Cos* x = 0.383** whereas only 1 learner failed to do so.	3 learners got the correct value and the other 2 used *cos* 0.383 instead of the inverse. The other learner did not attempt the question.

Table 7.19

Only one learner failed to simplify the equation to get $\cos x = 0.383$ (table 719). Their algebraic manipulation showed that they were able to work correctly with the information given to them in the problem. Some learners used *cos* 0.383 to find the value of x instead of the inverse. This shows that the learners have problems in using the calculator effectively. However half of the learners correctly found the value of x.

(b) $Cos\ 2x = 0.766$

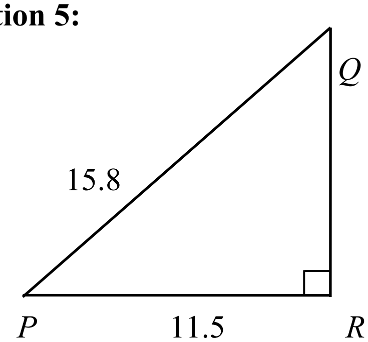
Results analysis: Question 4 (b)

Simplifying to $\mathbf{2\,x = cos^{-1}\ 0.766}$	**Simplifying to** $\mathbf{x = (cos^{-1}\ 0.766) \div 2}$	**Finding the value of** $\mathbf{x}$
4 learners managed to simplify the equation to $\mathbf{2\ x = cos^{-1}\ 0.766}$ whereas 2 learners failed to do so.	4 learners managed to get the correct value. The other 2 seemed confused by the question.	Only 1 learner went on to get the correct value of x. The other 2 did not simplify their answers to get x as they did not divide by the angle by 2. The other 1 went on to multiply the answer by 2 whilst 2 learners did not attempt the question at all.

Table 7.20

Only two learners struggled with the algebraic manipulations (7.20). Only one dropped off as she failed to obtain the value of x in the end. The other learner seemed to have failed to notice the difference between this question and the previous one. However, some minor calculations were made which cost the other two the correct answers. This problem could be emanating from the fact that most learners in African schools do not calculators and even if they do they only start using them in Grade 10. At times they borrow them to use in mathematics tests and examinations and fail to operate them properly.

Question 5:



Calculate $\hat{Q}$ (figure 7.5).

Figure 7.5

Results analysis: Question 5

Correctly identifying side PR as the opposite side (11.5-opp)	**Coming up with the correct ratio** ***sin θ*=(11.5/15.8)**	**Finding the value of *θ*= *sin*$^{-1}$ (11.5/15.8)**
5 learners managed to identify the opposite side with only 1 taking it for the adjacent one.	5 learners managed to get the correct ratio whilst the other used the wrong opposite side.	Only 2 learners went on to get the correct value of ***θ*= *sin*$^{-1}$ (11.5/15.8)**. The other 2 did not simplify their answers. Whilst 2 learners did not attempt this part of the question at all.

Table 7.21

Five learners were able to identify the sides with respect to the given angle but failed to get to the bottom line of the question (table 7.21). One learner though could not figure out that the side *PR* was opposite to the angle *Q*. Only two of them managed to make the angle the subject.

Question 6: A boy stands at *A* on top of a building *AE*, looking up at an airplane at *B*, through an angle of elevation of 22.3°.

He then looks down at a car at *D* through an angle of depression of 37.8° at precisely the moment that the airplane is directly above the car. Calculate the height of the airplane above the car, if the car is 200m from the foot of the building (figure 7.6).

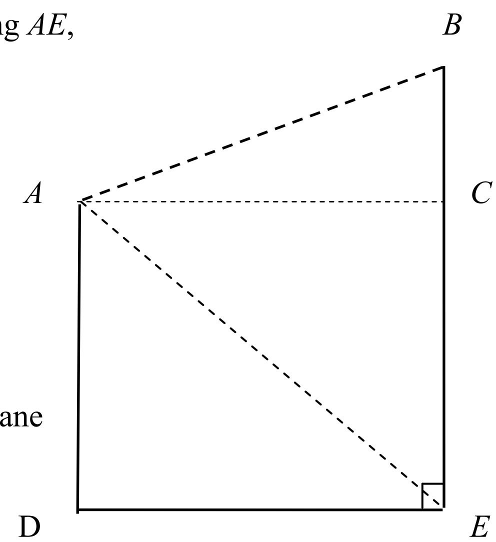


Figure 7.6

Results analysis: Question 6

Calculating BC=200*tan*22. 3º	**Calculating DC=200*tan* 37. 8º**
3 learners managed to get BC whereas the other 3 found the question confusing and did not attempt it.	3 learners got the correct value of DC and even went on to it to that of BC. The other 3 learners did not attempt the question.

Table 7.22

Question 6 was basically on the application of the concept in a real life situation (table 7.22). Only three learners did it correctly and the other three did not even attempt it. Maybe this was because of the lexical density or readability or plain language problems as they seemed to be comfortable with those questions with fewer words.

The learners had improved marks in the final test to show some improvement from where the learners had started from. The fact that the learners knew that the activity was not examinable could have had a negative impact on their performance and attitude. However, half of the learners clearly had achieved Van Hiele Level 3; they were able to deduce ratios when angles were given, angles when ratios were given, and diagrams from given statements which shows that the use of *Sketchpad* helped in learning and in reminding them of some concepts of the *cosine* function.

7.5.4 Analysis of the research questions

<u>7.5.4.1 Research Question 1:</u>

What understanding did learners develop of the *Cosine* function as a function of an angle in Grade 10?

From the test that was given at the beginning of the research learners showed that they had understood very little of the *Cosine* function as a function of an angle in Grade 10. However, the researcher could not fully establish if all the new understanding displayed in the final test had emanated from the use of *Sketchpad*, task sheets and probing. It could have been that they were now recalling some of the things they had done previously.

The first test showed that the learners could not identify satisfactorily the sides of a right triangle given an angle. This test demonstrated that all six learners were struggling at Van Hiele Level 1 and clearly needed some assistance to move to level 2 and 3. The learners demonstrated that, from their Grade 10 trigonometry, they had mastered working with a right triangle, only when it is standard.

Three of the learners could sketch a right triangle to answer trigonometric questions given a point in a Cartesian plane, but they could not proceed to find the ratios necessary to answer the questions. This showed that the learners had problems in finding a side or an angle when given a point in a Cartesian plane or a trigonometric equation. Only one learner was able to substitute values of angles into a given statement correctly, even though she failed to carry-out the correct calculations to get to the required answer.

During the *Sketchpad* activity the learners showed that they were familiar with the ratio of the cosine of an angle given as a fraction and not as a decimal. This was displayed when they struggled to relate the given decimal to a ratio. The learners also knew that a graph could be plotted using the *cosine* function, although not at a click of a button, as they found out.

The use of visual dynamic software was expected to fill in these gaps as it did with the different forms of the right triangle. It was then evident that the use of *Sketchpad* had accorded the learners an opportunity to construct their own meaning of the *cosine* function and improve their visualisation by working with non-rigid diagrams. The study also gave the learners an opportunity of organizing and structuring what they had learnt in Grade 10.

<u>7.5.4.2 Research question 2:</u>

What intuitions and misconceptions did learners acquire in Grade 10?

The initial test showed some of the intuitions and some misconceptions the learners had acquired in Grade 10. More of them also surfaced when *Sketchpad* was used, task sheets had been completed, and some probing had been done.

Most learners showed that they thought that trigonometry was solely confined to a standard triangle as they found it an uphill task to identify the sides of a non-standard one. If they did, maybe, they just thought that it was something difficult to identify. This also comes up as one of the misconceptions exposed by the test that the learners seemed to assume that all right triangles should be standard only. The other possibility could be that they have the notion of thinking that trigonometry deals with the standard right triangle all the time and that it could be impossible to apply it to other forms of right triangles. This is where the use of *Sketchpad* came in handy as working with the unit circle gave them exposure to different forms of the right triangle as they dragged the radius around in different quadrants.

The learners had also assumed that if the radius of a unit circle was changed, then so should the ratios of sides. This clearly emanates from the fact that when the topic is done in class the rigid shapes used to do not give room for change of radius. The relationship between function of an angle and the ratio of sides was something not very clear to them. Some thought that the function of an angle was one thing and the ratio of sides another with no relationship whatsoever.

The learners had their own intuitions and misconceptions from the previous grade. This made the entire study a meaningful learning process for them. Olivier (1989, p.18) points out that, *"errors and misconceptions are considered an integral part of the learning process".* It is the starting point of knowledge acquisition by learners. A conflict is created from within and they capitalize on that.

7.5.4.3 Research question 3:

Did learners display a greater understanding of the *Cosine* function when using *Sketchpad*?

7.5.4.3.1 Were learners able to use the provided Sketchpad sketch effectively to arrive at reasonable solutions?

The learners were quite comfortable with the sketch provided. It never came out at any stage, during the interview, that there were signs of not being able to use or understand the *Sketchpad* sketch provided. The fact that they were able to fill in the tables showed that they were able to work with the sketch to a reasonable extent. At times, of course, they could not hold the cursor steadily to get the actual value of the angle but the error was minimal.

After filling in the first table for r=1, when the learners were asked what they anticipated would happen to the ratios when then radius was increased, they were quick to realize that what they thought was not correct. The fact that they could recognize that the ratios did not change when the radius did gives credence to the effectiveness of the *Sketchpad* sketch.

7.5.4.3.2 Did learners display greater understanding of the Cosine function when using *Sketchpad*?

The researcher was able to draw some substantial general conclusions from the interviews conducted because the preliminary test was used as the control level and yard stick. Their performance in the final test was improved as compared to the first one. It can be said that in the second test the learners were able to sketch the appropriate triangle for the *Cosine* function. They did not just see a point in a Cartesian plane but were able to relate it to trigonometric functions.

There are a few ways in which *Sketchpad* could have assisted in increasing their understanding of the *Cosine* function:

- *Sketchpad*, without any doubt, helped in the visualisation of the unit circle and how it is related to the *Cosine* function as the graph was simultaneously drawn.
- The fact that learners could move around the radius of the unit circle and see the values of the angle and of the ratio change gave them a sufficiently good idea on the relation between angle and sides of a right-angled triangle.

7.5.4.3.3 Did learners acquire knowledge about trigonometric concepts and graphs from *Sketchpad* without being told?

When the learners went to the task sheet, using *Sketchpad*, the learners informally acquired some information relating to the radius, the angle, ratios, graphs, and so on. The learners became convinced that the ratios did not change when the radius did. They also discovered that the sign of the *cosine* of an angle changed as the quadrant did. Learners were also exposed to working with ratios in decimal form as opposed to the fraction regularly used. They also learnt the graph of the *Cosine* function as it was drawn simultaneously when they moved the radius of the circle.

The general usefulness of *Sketchpad* can be summarized as follows:

- The ease with which the diagram was manipulated and graphs constructed. This allowed the researcher and the learners the freedom to drag, change the radius and manipulate the figures as and when required. This may have been impossible to achieve if pencil and paper were used.
- The use of buttons saved a lot of time and allowed learners to see changes at the simple click of a button. Tedious and cumbersome constructions were avoided by the use of the mouse.
- The graphs constructed using *Sketchpad* were clear and made misinterpretation less likely. The use of pencil and paper might have resulted in many errors besides the fact that they most likely could have failed to plot them all.
- The shifts of graphs and the movement of the radius were clearly visible. In many instances this was essential to their understanding. It could have been time consuming if the learners had to show all the shifts on paper using a pencil. Besides, the possibility of coming up with incorrect ratios was eliminated.
- The manipulation and drawing of diagrams on the screen allowed the learners to grasp properties and understand relationships easily.

It can be concluded that the study managed to answer all the research questions successfully and that the instruments used were suitable and appropriate. The timing of the research was also appropriate. The level of questions was up to standard as they are similar to those found in mathematics textbooks used in schools in Grade 10. The situation and site of the study also gave the learners the liberty to participate at ease as they were in the familiar territory of their own school.

CHAPTER EIGHT

Conclusions and recommendations

8.1 Introduction

The focus in this study was on Grade 11 learners' understanding of the *cosine* function this was probed with some Sketchpad activities. In this chapter, the findings from the interview schedules are summarized. Further, some issues and difficulties in trigonometry in general are discussed and recommendations are made

8.2 Summary of findings

8.3 Overall findings

This research came up with some valuable results that could be used in the process of teaching and learning of trigonometry, functions and mathematics in general. The mode of instruction employed gave learners a greater and better understanding of the *cosine* function. This research concentrated on the *cosine* function as a ratio and its graphical representation. It also managed to expose some of the deeper misconceptions and intuitions learners have on the *cosine* function after their first encounter with the topic at Grade 10. The use of *Geometer's Sketchpad* helped in exposing more of these whilst at the same time working as a remedial and valuable tool for the learners to better grasp the concept of trigonometry.

The learners continually asked for questions to be elaborated or to have the question read for them. They seemed to be very dependent on the researcher for the direction of their cognitive processes. I found that, if the learners were given time and probed further about their thinking, it gave them an opportunity to think, even to correct themselves and come up with their own answers. Most of the time the learners did not have the patience and perseverance that are conducive and necessary to problem-solving. This also came through when they read a question. They did not read it carefully enough and rather read what they expected the question to ask. Another thing that was absent was the zeal to get correct answers since they knew that the exercise did not, at the end of the day, have any effect on their end of term mark.

The research also showed that some learners can not use a calculator effectively and efficiently. This is evidenced by the fact that they could still get wrong answers in their addition and subtraction even though they all had calculators. They would try to add or subtract numbers on their own and only use calculators to find the *cosine* of angles (table 6.1). This had a negative effect on the performance of the learners. The main reason is that most of them cannot afford to buy calculators and when they do, they opt for cheap ones. They depend on borrowing a calculator when they find they cannot do an exercise without one. In most cases their calculators do not last long and do not work properly as they go through many hands. In this study it is possible that the learners could have borrowed calculators from other learners and were therefore not familiar with how to operate them properly.

However, past experiences with computers and previous knowledge about computers helped the learners to feel comfortable with the use of GSP (global positioning systems) in the study. The visual function of GSP helped to bring about a better understanding of the abstract that they were asked in the interview and the questionnaire. In the beginning of course, the use of GSP and the learners understanding seemed to be separate as shown by the values they used in the first table. They were able to solve most of the problems as they became more familiar with GSP. They could even change the value of *r* on their own.

The use of the computer can change a learner's understanding as it allows the learner to move the picture and relate its changing state to the relevant numerical concepts (Blackett and Tall, 1991, p.146). It is, by no doubt capable of improving understanding. This is referred to by Blackett and Tall (1991) as the "*principle of selective constructions",* employing the computer to perform tedious and cumbersome constructions whilst the learner concentrates on more important aspects.

The following statistics reveal the significant level of success that the learners obtained in each test and the interview questions:

8.3.1 Initial Test Questions

Test Questions 1: Out of the 3 questions on hypotenuse, 7 (38%) answers from 18 were correct. Out of the 3 questions on the adjacent side, 7 (38%) answers from 18 were correct. Answers on the problem related to the opposite side were better with 11 (62%) out of 18 correct.

Test Question 2: Only 3 (50%) learners out of 6 managed to draw the correct diagram and 1 (16,7%) partially correct and 2 (33,3%) incorrect. No one managed to come up with the remaining side. No ratio was correct neither was the substitutions.

Test Question 3: Only 1 (16,7%) learners out of 6 managed to draw the correct diagram and 1 (16,7%) partially correct and 4 (66,7%) incorrect. No one managed to come up with the remaining side. No ratio was correct and only 1 (16,7%) partially correct in the substitutions.

Test Question 4: None of the learners could do this question correctly. Two did not attempt the question as they found it very difficult. There was no correct diagram. No one managed to come up with the remaining side. No ratio was correct and neither was there any correct in the substitutions.

Test Question 5: All 6 (100%) learners did not attempt the question as they found it very difficult. There was no correct side. No one managed to come up with the other remaining sides. No ratio was correct and neither was there any correct in the substitutions.

Test Question 6: Only 1 (16,7%) learners out of 6 managed to do it correctly and 1 (16, 7%) partially correct and 4 (66,7%) did not attempt to do it.

8.3.2 Interview Questions during *Sketchpad*

Interview Questions 1.1: Four (66,7%) of the learners interviewed correctly observed and understood that as the angle increases the ratio changes and r stays the same.

Interview Question 1.2: All the learners (100%) correctly observed and understood that x/r and $\cos\theta$ are the same or almost the same and that in two quadrants $\cos\theta$ would be negative.

Interview Question 2: None of the learners (0%) interviewed at this stage of the interview, could conjecture, without the use of *Sketchpad* or tables that the *cosine* of a given angle would be independent of the radius.

Interview Question 3: The activity seemed to have addressed the noted misconceptions in question 2, all six learners (100%) answered correctly that the value for any given angle, the corresponding values of *x/r* were the same in the table for *r*=2, *r*=3 and *r*=4 after using *Sketchpad*.

Interview Question 4:

a) Four out of six learners (66, 7%) correctly observed and used the computer to find the two angles that corresponded to the given ratio. However learners struggled with ½ since all the other values they had used were in decimal form. Those who obtained some other answer misinterpreted the ½ in *cos* (angle) =½, as an angle or a fraction that needed to be changed to a decimal.

b) After some probing of the previous question, four out of six learners (66, 7%) correctly observed that the answer to this question will be a few degrees different from the previous one.

Interview Question 5: All six learners (100%) correctly drew the graph of *cosine* as it is an easy exercise when using *Sketchpad*. The researcher had to guide them through the steps in most cases at times as they could not remember the next step or so.

Interview Question 6: All the learners (100%) succeeded in using *Sketchpad* to draw the graph of $y=acos\ x$ and identify all the attributes.

Interview Question 7: All the learners (100%) succeeded in using *Sketchpad* to draw the graph of $y=cos\ x+q$ and identify all the attributes.

Interview Question 8: It can be concluded that 3 (50%) of the 6 learners succeeded without using *Sketchpad* in drawing the *cosine* graph and identify all the attributes.

It seemed that away from the computer the learners seemed at a loss of what to do, and completely blank; it took a lot of probing for them to seem to understand the question.

8.3.3 Final test questions

Test Question 1: All 6 (100%) of the learners managed to correctly draw or use the diagram. Two learners (33%) came up with the correct ratio, 1 (16,7%) was partially correct and 3 (66,7%) were incorrect.

Test Question 2: Five of the 6 learners (82,3%), managed to correctly draw or use the diagram with only one failing to do so. Two learners (33%) came up with the correct ratio, 1 (16, 7%) was partially correct and 3 (66,7%) were incorrect.

Test Question 3: Five of the 6 learners (82,3%), managed to correctly simplify the values inside brackets with only one failing to do so. Two learners (33,3%) came up with the correct fraction, 2 (33,3%) were partially correct and 2 (33,3%) were incorrect.

Test Question 4 a): Five (82,3%) of the learners managed to correctly simplify the expression with only one failing to do so. Three learners (50%) came up with the correct solution, 1 (16,7%) partially correct and 2 (33,3%) were incorrect.

Test Question 4 b): Four (66,7%) of the learners managed to correctly simplify the expression with only 2 (33,3%) failing to do so. Four learners (66,7%) came up with the correct angle, and 2 the correct solution, 1 (16,7%) partially correct and 3 (50%) were incorrect.

However learners struggled to transpose terms correctly. This question was quite simple as it only tested their understanding and recognition of the algebraic expressions and not the deeper trigonometric equations which are required in Grade 11 and 12.

Test Question 5: Five of the 6 learners (82,3%), managed to correctly come up with the opposite side with only one failing to do so. Five of the 6 learners (82,3%), managed to come up with the correct ratio, where only one failing to do so. Two learners (33, 3%) came up with the correct angle, 2 (33,3%) partially correct and 2 (33,3%) were incorrect.

Test Question 6: Three (50%) of the learners got the correct length for BC and 3 (50%) again obtained the correct length of DC. The other half failed to do so.

If we look at the three sets of results we see a gradual upward movement from the preliminary test results to the final one, even though the development was not that remarkable. At its conclusion, the study managed to answer successfully all the research questions. The methodology and the study instruments used proved to be appropriate and suitable for the investigation. However, one of the factors which could have had negative effect on the study is that the participants were aware that the exercise would not carry marks for the final term mark or any meaningful evaluation. In any case it is still evident that the use of GSP and understanding seem to go hand in hand for better understanding of mathematics. Most of the *Sketchpad* activity was designed towards relational understanding as opposed to the instrumental understanding from the chalk and talk method they were exposed to in Grade 10. As the study came to an end, the learners' attitude changed dramatically in favour of this type of exercise in their daily classrooms (even mine as well, I now use *Sketchpad* in my lessons on trigonometry and graphs). The learners were convinced that it is more convenient and easy to explore the trigonometric questions with the aid of a dynamic sketch.

When learners get right answers in a test it could be because of understanding, but unfortunately learners also got right answers with incorrect reasoning. One thing this research showed is the pervasiveness of the idea that all functions are linear, for example, when learners subtracted 3º from 60º to get the corresponding angle for *cosine* of 0.55 since that of 0.5 was 60º. Telling learners the correct answer will not help; activities such as those used in the interview should be designed to place learners in cognitive conflict. This has important implications for teaching. We need to do follow up for a deeper understanding.

Since we are in a computer age, the computer environment is significant in changing the traditional mathematical environment. Freed from routine performing mathematical techniques, the problem solver can now focus on mathematical meaning, methods and explanations (Pournara, 1991). By combining various representations of mathematical problems, teachers can invent new ones.

Even though using the computer is proving to be very useful in mathematics instruction, language remains a very serious problem in most black schools. In this study it was evident that the learners were not fully engaged in the process as they looked somehow distanced from the activity. It is different from their behaviour when you see them playing outside, even in the presence of an educator. There are far too many things they struggle to grasp, thereby hindering their full involvement and ownership, not only of this one single activity; one has the impression that the entire system which is somewhat divorced from their everyday way of life. Although this goes beyond the scope of this study, there is a need for research which links this study with the broader problems of the South African schooling system.

In general, computers give learners room for generalization. They are powerful problem-solving tools in the hands of a proficient user, and learners need to acquire new skills in order to work proficiently with them.

8.4 Misconceptions and constructivism

Even though at the beginning of the study it had been proved beyond any reasonable doubt that the learners had errors of misconceptions the study did not focus on uprooting them (Olivier, 1989). The *Geometer's Sketchpad* helped the learners' change and correct their misconception that "$cos\ \theta$ was dependent on r". By changing or increasing r and seeing that x also changed or increased with it, and that the ratio x/r remained constant, participants made a realisation that was a surprise to them, and that resulted in an important conceptual change. For example, in this study, when learners were asked to calculate a ratio for an angle that was not in their completed tables (see analysis) this produced some form of conflict. They were used to fractions. Again, given their mathematics experience at Grade 10, some obtained incorrect answers by assuming that the *cosine* function was linear, as shown, after a short interval, the *cosine* function is approximately linear.

Learners also revealed gaps in their knowledge; especially their ratio and function orientation needs to be improved. There is need for teachers to shift from implicit notions of ratios and functions by assisting learners to develop a strong function orientation which is explicit. The teacher needs to make the input-process-output mechanism explicit so that learners can use it (Pournara, 2001). This may be done with the aid of a calculator, showing how it takes an input and operates on it to produce the output.

Learners should be able to shift between ratio and function in order to solve trigonometric tasks. Pournara (2001), states that teachers need to understand that sin 35º can be seen as number (ratio) and that y=sin 35º may be considered a function, and that the orientation which they adopt, will depend on the task, or sub-task at hand. It is therefore the duty of teachers to make learners know that there are two orientations, both equally valid, and then make use of this resource in their thinking. The orientations become explicit tools that learners can draw on consciously. Pournara (2001) states that in making the orientations explicit, they become objects of attention and therefore may become too visible (Lave & Wenger, 1991) which leads to the dilemma of transparency (Alder, 2001). In making the orientations visible, learners may focus on the orientations as ends, not means. Thus they may see the orientations, but not see them to be trigonometry (Pournara, 2001). Through continuous use of the orientations in a variety of different tasks, learners become familiar with them (Pournara, 2001), and ultimately the orientations become implicit again. I believe this kind of state creates a conflict which in turn makes them inquisitive and active participants with the desire of quenching their curiosity.

8.5 Van Hiele Theory

Since my participants were at Van Hiele Level 1 at the start, they needed a ready-made sketch to work with. In this study, the learners relied a lot on visualisation as they progressed from the first difficulty, which were, for example, recognizing angles and ratios, (Van Hiele Level 1, visualisation), to looking at embedded properties (Van Hiele Level 2) and then eventually to the generalizations that occurred in Question 3 and 8, when the learners were asked about the *cosine* of an angle when r=2 to r=4. All the learners seemed to understand that for a fixed angle, it does not matter what value r assumed. That seems to indicate that the learners have progressed to Van Hiele Level 3 where they made a generalization from particular cases that were documented in the tables that for $r = 1$ and so on, to the general case of any given value of r, the *cosine* of a fixed angle will always be the same ratio.

8.6 Learners' understanding of ratio

Learners worked with the ratio in the initial test, the study, and the final test thus was in different contexts. It seems as if the learners struggled each time the ratio changed from decimal to fraction form. Noxolo and Thabisile had 0.5 for the angle from *cos* (angle) =½ and had most likely wrongly divided 0.55 by 2 to get the 20.5. Noxolo asked: "*Must I change it to decimal?*" When asked if we were talking about an angle she replied: "*No we are talking about a fraction ……. So ……you can say ½ =0.5 so will give you cos 0.5*". If the question was in decimal fraction form, like the value in their table I believe they could have come up with correct answers.

It is clear that some learners also did not fully understand ratio and proportion at the beginning. Learners could not link the word to the relationship between two sides of a triangle and thus could not explain why its value increased or decreased as the angle changed in question 3.

The learners in the study, initially were taking the opposite side and the adjacent side to be fixed, and did not move them even when the angle changes. Even though all the learners made use of the theorem of Pythagoras, still this problem reared its head. These tasks reinforce an operational view of ratio since learners focused on individual sides of a triangle. Thus learners did not view their answers as ratios of an angle, but as common fractions and got mesmerized when these were changed to decimal. Understood in this manner, the numerator and the denominator are treated as separate entities that have meanings independent of each other. This weakens the development of a concept of ratio.

The first activity illustrates "*pseudo-structural conception*" because, according to Sfard and Linchevski (1994), learners were able to calculate the ratios for angle measurement. They gave ratios in decimal form in the study; here the ratio was seen in the lengths of sides of the right triangle. This suggests that the learners did not fully understand the meaning of the ratio and its relationship to the angle or to the sides of the triangle. It seems many learners did not understand that the process of dividing the length of the adjacent side by the hypotenuse is equivalent to keying in an angle and pressing the *cos*-button. Thus they were unable to take the output from the calculator and relate it to the appropriate sides of the triangle.

On the *Sketchpad* screen display, the values for *x/r* and *cos θ* were in decimal form. When learners worked with the ratio in decimal form, they may have got confused because they had only one side in their view. However, *the Geometer's Sketchpad* helped the learners develop some better understanding of ratio and proportion. Finding an angle whose *cosine* is equal to ½ which they had to change to 0.5, resulted in an important conceptual understanding.

8.7 Difficulties with learning trigonometry

Some of the factors that make trigonometry difficult to learn are: poor understanding of trigonometric notation due to some sloppy notation form teachers themselves, difficulties in the use of the calculator which the teacher should strive to explain to the learners, a poor concept of ratio, and difficulties with algebraic manipulation, inadequate pre-knowledge, confusing the ratio of sides with the actual length of sides, and the need to understand the conversion between angle and ratio.

8.7.1 Converting between angle and ratio

Most learners experience problems if different types of numbers are used (question 4 a) in the interview schedule). Consider cos (angle) =½; the output can be seen either as a ratio or simply a fraction that has to be changed to a decimal number, depending on the orientation that is adopted (Pournara, 2001). This emanates from the fact that they have been using decimal fractions in all the other exercises. This can cause *cognitive discontinuity* (Tall et al, in press) if the learner is not firm in the concept knowledge.

Some recognized this as the question clearly asks for the angle. Thandeka subsequently answers this question correctly: "*I prefer to use the computer. I have to check my answer again. I am not sure"* When asked what how, she replied: "*I'll use diagram and see where there is 0.5 and then check the angle of it". "Show me then". "There it i*s.

(Pointing)" Asked if it was the only one she replied: *"There could be 2 or more. Let me just find out. There is the other one …so they are 2".*

None of the learners clearly distinguished between input and output values to use the inverse function in this question, but eventually the point got home since this method was employed in the final test.

8.8 Recommendations

8.8.1 Classroom practice

8.8.1.1 Computer software

The use of dynamic geometry software, such as *Sketchpad,* in the research paid some valuable dividends in conceptualizing the *cosine* function. The instruction method used in the research provided learners a greater, and more meaningful, understanding of the *cosine* function and other functions.

The use of computers in mathematics instruction has several important benefits. Teaching will take far less time than usual. Imagine if a teacher has to illustrate to learners how to draw a *cosine* graph, how long it will take? Using *Sketchpad* is faster. The rest of the time will be left to explanation and questions from the learners. More graphs can also be plotted on the same axes for comparison's sake and different colours used.

8.8.1.2 Classroom strategies

The findings of this research have positive implications for the use of textbooks in the classroom as well. Some textbooks are written at Van Hiele levels different from that of learners and teachers unknowingly and trustingly use them without considering their learners. Therefore teachers should become aware of these potential gaps in some textbooks and exercises carried-out in lessons. Teachers should help develop strategies to get as much as possible from the available textbooks.

Some suggested strategies are given below:

a) The teacher should be alert to possible misconceptions formed as a result of limited visual examples.

b) The teacher should help learners understand trigonometric concepts where text book presentations can be done dynamically by the computer or by manipulative models.

c) Teachers can use the textbook to reconcile more exploratory activities in trigonometry

d) To help learners progress to Level 1 thought, the teacher can use dynamic geometry software and some exercises from the textbook to encourage learners to test many examples to determine if properties are true or false. Teachers should ensure that learners have exposure to a wide range of right triangles.

e) To help learners progress to Level 2 or 3 thought, the teacher can raise the level required in many routine exercises by asking “why?”, and “explain your answer.”

8.8.3 Changes to the curriculum

Hirsch et al (1991) proposed a trigonometry curriculum that is built around the graphing calculator. Now that there is computer software like *Geometers' Sketchpad, Geogebra* and others, this would be the most appropriate tool to assist in the learning of this subject. Thanks to various sponsors, most schools in South Africa now have computers; so this intervention would be very possible in most schools.

The chalk and talk technique does not seem to be yielding any positive results in terms of improving the pass-rate, so, we need to work harder to try other methods to assist in the understanding of mathematics. When a learner discovers something on his/her own, it is easier to recall and apply the concept as compared to just taking results for granted. Computer added software like *Sketchpad*, provides visuals and easy to use techniques to enhance discovery learning.

Of late, there have been many calls to abandon the OBE curriculum but none for change in mathematical textbooks to involve more Level 1 and Level 2 thinking, and which are more consistent with the Van Hiele model. The teachers' guide might be more explicit in identifying Van Hiele levels in some parts of the text, and in helping teacher's plan instruction to fill in levels and lead to higher levels of thinking. Still more attention should be given to the selection of visual examples in lessons involving Level 1 thought. There is a need to be more innovative in trying to use methods that will aid the understanding of mathematics, making it easier to visualize that which is abstract.

8.9 Short-comings of my research

- The study was done in one school, which provided a reasonably homogenous group, with only six learners who happened to be all girls.
- The study did not focus on modeling trigonometry functions and how this might motivate learners to learn trigonometry and aid conceptualization.
- It is difficult to test understanding since getting the correct answer does not necessarily mean understanding.
- The study used task-based interviews with individual learners, which is very different from a classroom context. This was just an introductory activity and what is needed a longitudinal study.
- The learners had some familiarity with computers, so the findings may not necessarily generalize to learners who were not familiar with computers let alone those who do not have them.
- IsiZulu and English had to be used in the interviews which are not what might happen in a normal classroom situation. This might be a hindrance when it comes to exam questions.
- From facial expressions and general body language one could sense a reasonable degree of reluctance of some sort which is not common in a normal classroom situation or test. Some gave up far too soon. Most of them would actually take action only after a lot of probing which, which indicates a lack of seriousness of some sort in some cases. This could have emanated from the fact that they knew that the exercise did not carry any marks for their term-end evaluation and was not examinable.
- Culturally most Blacks take *"not being open"* as synonymous with "*respect"*, which is not useful at this juncture. In most cases answers one can get are just "*I don't know or I don't know Mathematics*". The learners' mindset is that they view

themselves as "passive recipients" according to G. Brousseau's (1997) didactical contract. This also has a negative impact even in their learning activities in mathematics as the element of critical thinking is very remote, or missing. They mainly rely on being spoon-fed and copying from the nearest learner and are basically concerned about the "*answer*". In most cases one has to do a lot of examples in order to "*kick start them*" and still will get the usual request "*Can you please do number so and so for us, we don't understand?*" Some have even asked to have questions explained to them during examinations!

The researcher had to spend more time talking to the learners to establish a mutual understanding of each other at a social level to begin with. The researcher also took the entire group to a Ministry of Transport schools competition on "Road Safety", where public presentation skills were tested in order for them to participate more freely.

8.10 Further research

- Further research would indicate whether similar results could be obtained with a classroom of learners plus non- homogeneous groups, instead of one-to-one interviews.
- An investigation to ascertain whether these results are also true for the graphing calculator environment would be helpful. This would perhaps be more relevant to the present classroom situation in many South African schools.
- In African schools it would be appropriate to conduct further research to indicate whether similar or better results could be obtained when using the mother tongue in trigonometry.

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Appendix A:

GRADE 11 LEARNERS' UNDERSTANDING OF THE COSINE FUNCTION WITH SKETCHPAD

INTERVIEW SCHEDULE

1. (a) What do you notice about the values of x and r respectively as the angle θ increases?

1. (b) What do you notice about the values of x/r and cosine in table 1?

2. What do you think will happen to the above ratios if we increase r to 2? Why?

3. For any given angle, what do you notice about the corresponding values of x/r in each table for r=2, r=3, and r=4?

____

____

____

____

4. Answer the following questions:

a) If *cos* (angles) = ½ then the angles are _____and______?

b) Estimate the value of the angle if $x/r = 0.55$

____

5. <u>***y = cos x***</u>

a. Draw the graph of $y = \cos x$ ($-180^\circ \leq x \leq 180^\circ$)

The characteristics of $y = \cos x$ are:

<u>Domain</u>: $x \in$ [______°; ______°]

<u>Range</u>: $y \in$ [_____; _____]

<u>Amplitude</u>: ________

<u>Period</u>: ________

6. <u>***y = a cos x***</u>

a. Draw the graphs of $y = 2 \cos x$, $y = \frac{1}{2} \cos x$ and $y = -\cos x$ on the same set of axes, labeling each graph. ($-180^\circ \leq x \leq 180^\circ$)

<u>Conclusion</u>: As the co-efficient of *cos x* increases,

____

As the co-efficient of *cos x* decreases,

____

When $a < 0$,

____

The effect of a i) on the x-intercepts:

ii) on the range:

7. ***$y = cos\ x + q$***

a. Draw the graphs of $y = cos\ x$, $y = cos\ x + ½$ and $y = cos\ x – 1$ on the same set of axes, labeling each graph.

<u>Conclusion</u>: When $q > 0$,

When $q < 0$,

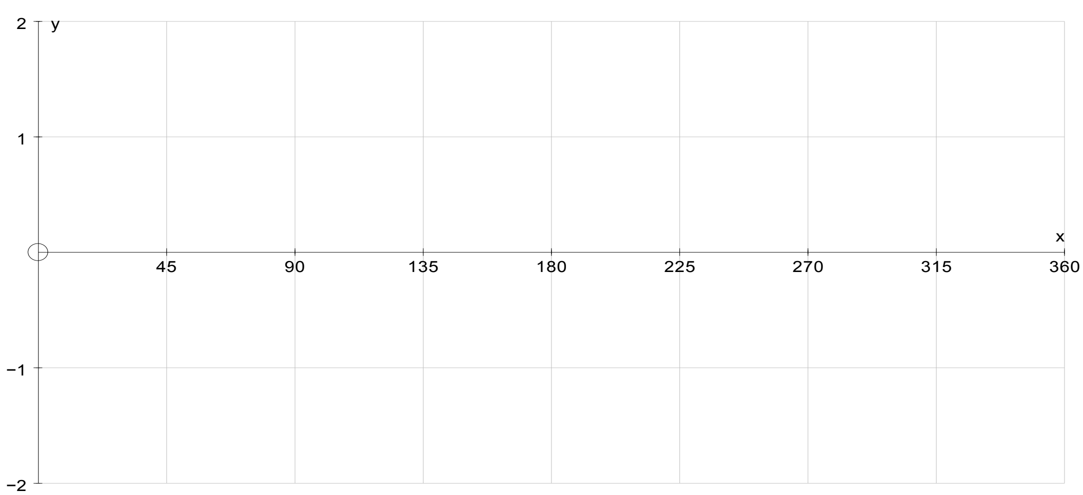
The effect of q i) on the amplitude:

ii) On the range:

8) Use what you have learned to draw graphs of the following for $x \in [0°; 180°]$ below

a) $y = cos\ x – 2$

b) $y = -2\ cos\ x$



Appendix B:

Relationship between *cos* θ **and** *x/r*

r=1

θ	*x/r*	*Cos* θ
10º		
20º		
30º		
100º		
150º		
200º		
250º		
300º		
350º		

r=2

θ	*x/r*	*Cos* θ
10º		
20º		
30º		
100º		
150º		
200º		
250º		
300º		
350º		

r=3

θ	x/r	$Cos\ \theta$
10º		
20º		
30º		
100º		
150º		
200º		
250º		
300º		
350º		

r=4

θ	x/r	$Cos\ \theta$
10º		
20º		
30º		
100º		
150º		
200º		
250º		
300º		
350º		

Appendix C:

Participants' thought patterns

Busi:

Researcher	Ok, Busi, what do you notice about the *x* value and the value of *r* as the angle θ increases?
Busi	… (Silence)…. I don't understand. They are decimals fractions.
Researcher	Ok, what do you notice about the *x* value as the angle increases?
Busi	*x* is that (pointing to the column *x* in the table)
Researcher	Yes.
Busi	The angle is changing. Yeah. Ah... *x* changes…I see, or that is the question. Ok.
Researcher	Ok, what do you notice about *x* value as the angle is increasing?
Busi	The *x* value decreases, increases, decreases and then increases again. (Dragging the radius in different quadrants, somewhat surprised that she had failed to notice it)
Researcher	What do you notice about the *r* value as the angle is increasing?
Busi	Same
Researcher	As the angle is going up?
Busi	Same
Researcher	Are you convinced?
Busi	Yes, because the this diagram is the one for *r*=1
Researcher	Thank you. Now what do you notice about the values of *x/r* and *cos* θ in your table?
Busi	Answer for *x/r* and *cos* θ are almost the same, only that *cos* θ at times is negative
Researcher	Where exactly?
Busi	Let me check… (Referring to the table)…Here and here.
Researcher	Can you be more specific?
Busi	I don't understand.
Researcher	Ok, how about in term of quadrants?
Busi	I think it is negative…2 and 3. I am not very sure…Let me check again. Ok. It's correct.
Researcher	What do you think will happen to the above ratios if we increase *r* to 2?
Busi	They will increase by 2 because *r* value is increasing (before tables 2, 3 and 4)
Researcher	(After completing tables 2, 3 and 4) For any given angle, what do you notice the corresponding values of *x/r* in each table for r=2, r=3 and r=4?

Busi	They are the same. The values of x/r in each table are the same.
Researcher	Why?
Busi	Even the graph traced remains the same, it does not change. The radius does not affect the angle
Researcher	Even if r=100?
Busi	Yes
Researcher	For any r?
Busi	Yes (confidently)
Researcher	Can you say *Cosine* is a function?
Busi	Yes
Researcher	Why?
Busi	You can draw a table of values of x and y and draw a graph
Researcher	Are all graphs functions then?
Busi	Most, but this one I know it, it's a function
Researcher	If *cos* (angle) =½ then the angle is?
Busi	I think is 90º
Researcher	Why do you say 90º?
Busi	I am thinking 90º, because that is the angle I know in a right triangle
Researcher	Do you think the question is related to the previous exercise in any way?
Busi	……(silence)…Let me read again
Researcher	What are we talking about here?
Busi	So I made a mistake. Let me use the table to find the angle with ½. …There... (pointing at 300º)
Researcher	Is it the only one?
Busi	Let me continue…. (Searching)…would be here or there…ah…no…I don't know.
Researcher	Ok, let us draw the *cosine* graph from -180º to 180º
Busi	(Enthusiastically)…. …. (Working on the computer)…There.
Researcher	Look carefully at the diagram. What is the domain?
Busi	… Domain? ….the values of *x?* (rhetoric)…-180º to 180º
Researcher	Range?
Busi	Range?...-1 to 1
Researcher	And amplitude?
Busi	..½ this...1.
Researcher	Period?
Busi	360º. It's easy with computer.
Researcher	Let us draw the graph of $y=2\cos x$, $y=\frac{1}{2}\cos x$ and $y=-\cos x$ on the same axes?
Busi	Yes. ….(Working on the computer)
Researcher	I would like you to check on the coefficients of *cos x.* 2, ½ and -1 and then check on their effect on the original *cos x* you drew earlier on… (Pause)... What changes occur as the coefficient increase?
Busi	It gets taller…Like the y value increases up and down.
Researcher	What if it decreases?
Busi	Yes… it decreases
Researcher	Now, what if the coefficient is negative, doe it have an effect on the x-intercepts?

Busi	No
Researcher	And on the range?
Busi	No... (Pause)If we are talking about this one, but it was only one.
Researcher	Can you draw the graph of y=*cos x*, y= *cos x* +½ and y=*cos x*-1 on the same axes?
Busi	Yes. ….(Working on the computer)
Researcher	I would like you to check on the numbers being added to *cos x,* ½ and -1 and then check on their effect on the graph of *cos x*… (Pause)... What changes occur if the number is positive, for instance ½?
Busi	……(silence)…….mmmmm…the graph goes up by the same
Researcher	And if negative?
Busi	Goes down by same…Oh yeh.
Researcher	What do you think about the amplitude?
Busi	Does not change
Researcher	And range?
Busi	No change.
Researcher	Use what you have learnt to do question 8
Researcher	Ok (she only managed to draw one diagram which was not correct)
Researcher	Thank you Busi
Busi	Ok, it's my pleasure!

Thabisile:

Researcher	Ok, Thabisile, what do you notice about the x value and the value of r as the angle θ increases?
Thabisile	…………. (Silence) …. They are all less than 1
Researcher	Ok, what do you notice about the x value as the angle increases?
Thabisile	(Referring to the table)…. Like if this is 1.0….. 0.34… (Pointing at the ratios)….then this increases …. Here 0.2…. 0.4 er …it's decreasing and here increase again. So it change here and here, and here …it change 4 times….. Ya!
Researcher	….oh ….ok what do you notice about the r value? If you increase the angle
Thabisile	I did not change r ….it's the same
Researcher	Now what do you notice about the values of x/r and $cos\ \theta$ in your table?
Thabisile	(Referring to the table)…x/r and $cos\ \theta$ are almost the same, only that $cos\ \theta$ at times is negative
Researcher	Where exactly?
Thabisile	(Looking at the table)…I don't understand
Researcher	Where are the values of $cos\ \theta$ at times is negative and positive, say, in terms of quadrants?
Thabisile	(Looking at the table)….. I cannot say…Let me try the diagram
Researcher	Ok
Thabisile	(Dragging the radius in different quadrants) …Here…positive 1…negative 2…negative 3 and positive 4.
Researcher	Ok, what do you think will happen to the above ratios if we increase r to 2? Why?

Thabisile	(Looking at the table)…. Each figure will increase about 2….yeh because *r* increased.

Thabisile completed the table by increasing *r* to 2, 3 and 4 respectively.

Researcher	For any given angle, what do you notice the corresponding values of *x /r* in each table for *r*=2, *r*=3, and *r*=4.
Thabisile	(Looking at tables) …. They not the same …no let me round off… it's the same… certain points are the same …. Three points….. It's different because of the decimal …. Can I check this one?....(rounds off the decimal fractions for table 2)
Researcher	What made you do that?
Thabisile	Because all these are the same … the decimal… *x /r* and the *cosine* … I wanted to come up with the same.
Researcher	Ok, coming back to our question, what do you notice about the corresponding values of y/r in each table for r=2, r=3 and r=4.
Thabisile	They are the same figures.
Researcher	Now Thabisile, if *r*=100?
Thabisile	They will be same.
Researcher	For any *r*?
Thabisile	Yes (smiling)
Researcher	Can you say *Cosine* is a function?
Thabisile	Yes
Researcher	Ok, good. Any reason?
Thabisile	But I don't know it, can't remember it….
Researcher	…. (Long pause)…. How do you define a function?
Thabisile	….No, not easy…. I don't know. Maybe if you put one value in an equation and then you get a value or you draw a graph.
Researcher	Are all graphs functions then?
Thabisile	Can't say.
Researcher	What do you think?
Thabisile	(silence)……(smiling)…This one I know it
Researcher	Let us look at question 4. If *cos* (angle) =½ then the angle is?
Thabisile	That is 0.5
Researcher	Right, Why do you say 0.5?
Thabisile	Because it is ½ and the decimal is 0.5. It's difficult sir.
Researcher	Can your try it using the previous exercise?
Thabisile	(Checked her table first and located 0.5 in line with 300º)...ah... It's easy
Researcher	Is it the only angle?
Thabisile	… Let me check…..it's not there but I think since these values are decreasing (pointing at 20º and 30º) it will be there.
Researcher	Ok, let us draw the *cosine* graph from -180º to 180º
Thabisile	…. …. (Working on the computer)…There.
Researcher	Look carefully at the diagram. What is the domain?

Thabisile	Here to here (Pointing -180º and 180º)
Researcher	Very good. Now what is the range?
Thabisile	……. (Silence)…….mmmmm… there (Pointing -1 and 1) Is it right?
Researcher	Is it right (smiling)….. What do you think?
Thabisile	It is... Now?
Researcher	Now? Amplitude.
Thabisile	Now …… there...1
Researcher	Nice, ok, Period?
Thabisile	…….. (Pointing it with cursor)
Researcher	Yes, try y=2*cos x*, y=½ *cos x* and y=-*cos x* on the same axes. File-New sketch.
Thabisile	……. (Working on the computer)…There
Researcher	I would like you to check on the coefficients of *cos x.* 2, ½ and -1 and then check on their effect on the original *cos x* you drew earlier on… (Pause)... What changes occur as the coefficient increase?
Thabisile	It gets bigger here (Pointing at the range)
Researcher	What if it decreases?
Thabisile	….er … it decreases.
Researcher	Now, what if the coefficient is negative, doe it have an effect on the x-intercepts?
Thabisile	No.
Researcher	And on the range?
Thabisile	No.
Researcher	Let us draw the graph of y=*cos x*, y= *cos x* +½ and y=*cos x*-1 on the same axes?
Thabisile	(Working on the computer)…There
Researcher	Oh….OK. I would like you to check on the numbers being added to *cos x,* ½ and -1 and then check on their effect on the graph of *cos x*… (Pause)... What changes occur if the number is positive, for instance ½?
Thabisile	It goes up ……..
Researcher	Negative?
Thabisile	Down
Researcher	What do you think about the amplitude?
Thabisile	Same.
Researcher	And range?
Thabisile	Same
Researcher	Use what you have learnt to do question 8
Thabisile	Ok (she drew one diagram which was not correct)
Researcher	….. Ok, thank you Thabisile.

Samkelisiwe:

Researcher	Ok, Samkelisiwe, what do you notice about the *x* value and the value of *r* as the angle θ increases?
Samkelisiwe	(Silence)…. Can I use the table?
Researcher	You want to use the table… ok
Samkelisiwe	They decrease as well (writes angles *x* and *r* increases)
Researcher	Samkelisiwe, do you think *x* and *r* are angles
Samkelisiwe	… (Silence)….. Er they are lines……
Researcher	So what do you notice about *x* and *r* as the angle increases?
Samkelisiwe	(Turns to the computer) can I try?
Researcher	Yes you should.
Samkelisiwe	(Drags to make the angle larger) *r* value remains the same, and *x* value decreases, increases, decreases and increases.
Researcher	Ok, Good, what do you notice about the values of *x /r* and *cos* θ in your table?
Samkelisiwe	(Immediately without looking at the table). They change……. As the angle increases *x /r* and *cos* θ decrease as well…here
Researcher	Ok, Samkelisiwe, if you look at the question it says: what do you notice about the values of *x /r* and *cos* θ in table 1? Which means all of them
Samkelisiwe	(Looking at the table, then smiling) they are the same, only that some *cos* θ are negative but equal.
Researcher	You are smiling … why?
Samkelisiwe	No …... because, when you read the question over and over, then you realize what they are really asking.
Researcher	What do you think will happen to the above ratios if we increase *r* to 2? Why?
Samkelisiwe	Must I try it out or just give an answer?
Researcher	Ok, but first, what do you think will happen?
Samkelisiwe	Er ….. Increase by 2. The circle is getting bigger.

Samkelisiwe then continued to complete the table by increasing *r* to 2, halfway through the second table, she said (surprised):

Samkelisiwe	Now I realize that the *x /r* will still have the same ratio, because when you increase the *r* to 2, for example, *x* will increase as well.
Researcher	That's a very good observation (Samkelisiwe now completed the table by increasing *r* to 2, 3 and 4 respectively)
Researcher	For any given angle, what do you notice about the corresponding values of *x /r* in each table for *r* =2, *r* =3 and 4?
Samkelisiwe	… (Silence)….. They are the same. The values of *x /r*, in each table is the same
Researcher	If I find it interesting, you know, when I ask you a question on the table, you don't look at it to answer. Why is that?
Samkelisiwe	I don't know ……. I am not sure. I assume I take it for granted. The *x /r* in each table, I know, is equal from the completion of the table, I remember. They are not talking about *x* alone and *r* alone? They are not talking about the ratio. I

	know that *r* is increasing, and *x* will increase too.
Researcher	So ….
Samkelisiwe	It will increase …. It will remain the same. The value of *x* /*r* in each table is the same
Researcher	Can you say *Cosine* is a function?
Samkelisiwe	Yeah… I know that from class.
Researcher	Ok. Let us look at question 4. If *cos* (angle) =½ then the angle is?
Samkelisiwe	Can I use the computer? I want to check my answer first?
Researcher	Hmmm …
Samkelisiwe	….. (Silence) …… I'll drag the radius till I get 0.5 and then check the corresponding angle.
Researcher	Ok, tell me what you notice?
Samkelisiwe	There it is. (Pointing at 60º)
Researcher	What else do you have?
Samkelisiwe	(Dragging the radius all the way, with a lot of scrutiny, to the fourth quadrant)…there is the other one …300º (smiling triumphantly)
Researcher	Ok, let us draw the *cosine* graph from -180º to 180º
Samkelisiwe	…. …. (Working on the computer)…Ok.
Researcher	Look carefully at the diagram. What is the domain?
Samkelisiwe	-180º; 180º
Researcher	Range?
Samkelisiwe	Ya,…..-1; 1
Researcher	Amplitude?
Samkelisiwe	Mmmmm, this (smiling)…1
Researcher	Period?
Samkelisiwe	360º
Researcher	Yes, try *y*=2*cos x*, *y*=½ *cos x* and *y*=-*cos x* on the same axes. File-New sketch.
Samkelisiwe	(Working on the computer)…Ok.
Researcher	I would like you to check on the coefficients of *cos x.* 2, ½ and -1 and then check on their effect on the original *cos x* you drew earlier on… (Pause)... What changes occur as the coefficient increase?
Samkelisiwe	Amplitude increases
Researcher	Yes, so and if it decreases?
Samkelisiwe	So will it
Researcher	…… (Long pause) …. Ok fine, if it is negative, any effect on the *x*-intercepts?
Samkelisiwe	The graph is upside down but no effect
Researcher	On range?
Samkelisiwe	Ya ….. Ya.
Researcher	Let us draw the graph of *y*=*cos x*, *y*= *cos x* +½ and *y*=*cos x*-1 on the same axes?
Samkelisiwe	Ok…(Working on the computer)
Researcher	I would like you to check on the numbers being added to *cos x,* ½ and -1 and then check on their effect on the graph of *cos x*… (Pause)... What changes occur if the number is positive, for instance ½?
Samkelisiwe	It goes up.
Researcher	Ok …. If negative?

Samkelisiwe	Down
Researcher	What do you think about the amplitude?
Samkelisiwe	……. (Silence)…… it will not stay the same. (Measuring from x-axis)
Researcher	Do you take it from x-axis all the time or what is it?
Samkelisiwe	No…half this (pointing at the range)…so there it is no effect.
Researcher	On range?
Samkelisiwe	No
Researcher	Yes. Use what you have learnt to do question 8
Samkelisiwe	Ok. (she drew 2 diagrams, 1 was correct and the other partially correct)
Researcher	Ok Thank you.
Samkelisiwe	Ok, thanks.

Noxolo:

Researcher	Ok, Noxolo, what do you notice about the x value and the value of r as the angle θ increases?
Noxolo	I don't understand
Researcher	Ok, what do you notice about the x value as the angle increases?
Noxolo	x is that (pointing to x in the table)
Researcher	Yes
Noxolo	They are all less than 1 and when the angle increases value of x and r decreases.
Researcher	Ok, now, what do you notice about the x value as the angle increases?
Noxolo	x value? Increasing or decreasing?
Researcher	You must tell me, what do you notice the x value as the angle is increasing?
Noxolo	As the angle is going up or down?
Researcher	As the angle going up.(There was a total communication breakdown)
Noxolo	Can I check it from diagram?
Researcher	Yes
Noxolo	So this is x
Researcher	Yes
Noxolo	So when it goes up (dragging point up), the x value is decreasing.
Researcher	What do you notice about the r value?
Noxolo	The r value as the angle increases?
Researcher	As the angle increases
Noxolo	(Working on the computer) stays the same. When the angle increases r stays the same
Researcher	What do you notice about the values of x/r and $\cos\theta$ in your table?
Noxolo	They are equal. Only that $\cos\theta$ is negative from here to there. (pointing at values in the table)
Researcher	What do you think will happen to the above ratios if we increase r to 2? Why?
Noxolo	The x/r will increase to 2. If $\theta = 10^{\circ}$ then to x/r will be 2.95 because we added 2 before it was 0.95
Researcher	Ok, complete the tables for r=2, 3, and 4

Noxolo	(Works on the computer)
Researcher	What do you think will happen to the above ratios if we increase the value of *r*? Why?
Noxolo	What I notice is that it starts with a value 0, 1 and end with 0, 1 that I notice about r = 2, r=3, and r=4.
Researcher	I see 1.0 in your table
Noxolo	Sorry, 1.0
Researcher	So for any given angle, what do you notice about the corresponding values of *x/r* in each table for r=2, r=3 and r=4?
Noxolo	As *x/r* increases the value of *cosine* θ increase with it.
Researcher	So the *x/r* for each angle?
Noxolo	They don't increase by much.
Researcher	Do you think *cosine* θ is a function?
Noxolo	Ya. It could be. It is... I know it
Researcher	In no.4 it says answer the following question if cos (angle) =½, then the Angle =____?
Noxolo	Must I change it to decimal? 0.5
Researcher	Then the angle = 0.5?
Noxolo	Er ….. (Silence)….. If *cos* is half ….. Then the angle …… you are asking how many degrees the angle will be?
Researcher	Are we talking about length or angle?
Noxolo	No we are talking about a fraction ……. So ……you can say ½ =0.5 so will give you cos 0.5
Researcher	Can θ be in 0.5?
Noxolo	Ya.
Researcher	*x* and *r* represent length or angles?
Noxolo	Length
Researcher	So?
Noxolo	*r* is the hypotenuse? Can I use the Pythagoras Theorem?
Researcher	What do you think?
Noxolo	Yes, you use Pythagoras to find the hypotenuse, but I don't remember….. (silence)
Researcher	Can you use Pythagoras to find an angle of a right angled triangle, given the hypotenuse and another side?
Noxolo	Our teacher showed us how to find the hypotenuse; he did not show us how to find an angle.
Researcher	Ok. Can you please read the question again, slowly?
Noxolo	I have to find an angle! Oh so no … (silence)….. (Thinking)….. There could be a way of doing it, but I don't know
Researcher	What do you think?
Noxolo	I can't do it.
Researcher	Ok, all right Noxolo…. That was interesting
Noxolo	It was bad
Researcher	It was bad? It happens
Noxolo	Because I didn't get the answer

Researcher	Ok, let us draw the *cosine* graph from -180º to 180º
Noxolo	Fine. (Works on the computer)
Researcher	Look carefully at the diagram. What is the domain?
Noxolo	….. (Long silence) ……….(thinking)……can't ….I don't know
Researcher	……. (Long pause)……Ok… What if I tell you it is from where your graph begins to where it ends along the *x*-axis
Noxolo	Then this will be it (showing -180º to 180º)
Researcher	Right, thank you. Now what do you think will be the range?
Noxolo	This? (point at it) Is that what you are asking?
Researcher	Yes
Noxolo	We are talking about Y. It is here and here (pointing at -1 to 1)
Researcher	And amplitude?
Noxolo	½?...1
Researcher	Period?
Noxolo	Ok, it is this (pointing at the period with the cursor)
Researcher	Yes, try $y=2\cos x$, $y=\frac{1}{2}\cos x$ and $y=-\cos x$ on the same axes. File-New sketch.
Noxolo	Ok. (Works on the computer)
Researcher	I would like you to check on the coefficients of *cos x.* 2, ½ and -1 and then check on their effect on the original *cos x* you drew earlier on… (Pause)... What changes occur as the coefficient increase?
Noxolo	Coefficient?
Researcher	Yeah
Noxolo	Er ….. (Silence)….. What is coefficient of *cos x*?
Researcher	The number multiplying it
Noxolo	(Checking on the computer)…..mmmmm, it goes up. It gets bigger, taller I mean
Researcher	If it decreases?
Noxolo	Becomes smaller
Researcher	If negative?
Noxolo	It was like this (showing using a hand) and now this
Researcher	Yes. Any effect on the *x*-intercepts? Where it cuts the *x*-axis?
Noxolo	Ok. No
Researcher	On range?
Noxolo	(looking at the graph) No.
Researcher	Let us draw the graph of $y=\cos x$, $y= \cos x +\frac{1}{2}$ and $y=\cos x-1$ on the same axes? Remember to go to File, and then New sketch
Noxolo	Ok. (Works on the computer)
Researcher	I would like you to check on the numbers being added to *cos x,* ½ and -1 and then check on their effect on the graph of *cos x*… (Pause)... What changes occur if the number is positive, for instance ½?
Noxolo	It goes up
Researcher	If negative?
Noxolo	It goes down
Researcher	What do you think would be the effect on the amplitude?
Noxolo	No effect
Researcher	On range?

Noxolo	Nothing. It is the same
Researcher	Yes. Use what you have learnt to do question 8
Noxolo	Ok. (she drew 2 diagrams, 1 was not correct and the other partially correct)
Researcher	Yes ….. thank you Noxolo
Noxolo	Thank you.

Thandeka:

Researcher	Ok, Thandeka, what do you notice about the *x* value and the value of *r* as the angle θ increases?
Thandeka	It is decreasing
Researcher	You mean the *x* value and the value of *r*?
Thandeka	No, because I won't be able to move the radius up and down
Researcher	Ok. What do you notice about the values of *x/r* and *cos* θ in your table?
Thandeka	(Immediately without looking at the table). They change……. As the angle increases *x /r* and *cos* θ decrease as well…here
Researcher	Is it like that through-out the table?
Thandeka	At times they increase and like here *cos* θ value is negative but are the same.
Researcher	Ok, you are happy with your answer?
Thandeka	Ok
Researcher	What do you think will happen to the above ratios if we increase *r* to 2? Why?
Thandeka	They increase because *r* will increase
Researcher	Go to No. 3…. What do you think will happen to *x /r* and *cos* θ if we increase the value of *r*? Why?
Thandeka	Same. Because you only change the circle.
Researcher	If $r=\pi$? Will it be still the same?
Thandeka	….. (silence) Ya it will.
Researcher	Do you think *cos* θ is a function?
Thandeka	Ya it is
Researcher	Ok, Thandeka, now for number 4.answer the following questions: if *cos* (angle) = ½, then angle =
Thandeka	60º
Researcher	How did you get it?
Thandeka	I used my calculator. I prefer to use the computer. I have to check my answer again. I am not sure. There it is. (Pointing at the diagram indicating 60º)
Researcher	Is it the only one?
Thandeka	There could be 2 or more. Let me just find out. There is the other one …so they are 2
Researcher	Ok, let us draw the *cosine* graph from -180º to 180º
Thandeka	Ok. (Works on the computer)
Researcher	Look carefully at the diagram. What is the domain?
Thandeka	This?
Researcher	Yes

Thandeka	-180º to 180º
Researcher	Range?
Thandeka	-1 to 1
Researcher	Amplitude?
Thandeka	….. (silence) Ya it will be 1.
Researcher	Why (Explain or justify your reasoning.)
Thandeka	Because it is half this (indicating with the cursor).
Researcher	And the period?
Thandeka	Here to there.
Researcher	Write it in your answer space
Thandeka	….. (silence) ….(she wrote 360)
Researcher	Yes, try $y=2\cos x$, $y=\frac{1}{2}\cos x$ and $y=-\cos x$ on the same axes. File-New sketch.
Thandeka	Alright. (Works on the computer)
Researcher	I would like you to check on the coefficients of *cos x.* 2, ½ and -1 and then check on their effect on the original *cos x* you drew earlier on… (Pause)... What changes occur as the coefficient increase?
Thandeka	Y increases
Researcher	Right. If it decreases?
Thandeka	Y decreases
Researcher	Hmmmm….Any effect on the *x*-intercepts if it is negative?
Thandeka	No
Researcher	On range?
Thandeka	No
Researcher	Let us draw the graph of $y=\cos x$, $y=\cos x+\frac{1}{2}$ and $y=\cos x-1$ on the same axes? Remember to go to File, and then New sketch
Thandeka	Ok. (Works on the computer)
Researcher	I would like you to check on the numbers being added to *cos x,* ½ and -1 and then check on their effect on the graph of *cos x*… (Pause)... What changes occur if the number is positive, for instance ½?
Thandeka	It moves up positive for y
Researcher	If negative?
Thandeka	It moves down negative for y
Researcher	Any effect on amplitude?
Thandeka	No
Researcher	On range?
Thandeka	No
Researcher	Yes. Use what you have learnt to do question 8
Thandeka	Ok. (she starred on the blank paper for quite some and then shook her head) I can't
Researcher	Ok, good, thank you very much
Thandeka	Ok. Thank you.

Bongekile:

Researcher	Ok, Bongekile, what do you notice about the *x* value and the value of *r* as the angle θ increases?
Bongekile	When the angle θ increases, values of *x* and r decrease.
Researcher	Look at one value at a time
Bongekile	(Checking the table)… stays the same. When the angle increases *r* stays the same. Only *x* decreases, increases, decreases, and increases again
Researcher	Ok. What do you notice about the values of *x /r* and *cos* θ in each table?
Bongekile	Mmmmm, it's like they are equal….it is not exactly the same; some are exactly the same …. But some like below or above the value and some *cos* θ are negative but it's like the same
Researcher	What do you think will happen to the above ratios if we increase *r* to 2? Why?
Bongekile	Mmmmm, the ratios will be more than what they are. The *x /r* for example will be, if θ =10º - *x /r* will be 2.95 cm because we added 2, before it was 0.95cm. The circle will increase so the answers for the ratios will increase the total degrees will get bigger by 2; *x/r* will get bigger
Researcher	For any given angle, what do you notice about the corresponding values of *x /r* in each in each table for *r*= 2, *r*=3 and *r*=4?
Bongekile	(After completing the table 2)… won't it be the same? …because every time I hold it to 10º for example, I notice both values are the same. (Bongekile now completed the table by increasing r to 2, 3 and 4 respectively) The values are similar; it's either one below one above.
Researcher	Do you think *cos* θ is a function?
Bongekile	(Silence)….. Yeah it is.
Researcher	Ok, Bongekile, now for number 4.answer the following questions: if *cos* (angle) = ½, then angle =
Bongekile	½? ½.
Researcher	Why?
Bongekile	These are the same (pointing to *x /r* and *cos* θ in the table), so this will be the same too.
Researcher	You happy with the answer?
Bongekile	You want angles, degrees? …. 50º, coz it's half. (She must have been thinking of %)
Researcher	What about 0.55?
Bongekile	It is 55.
Researcher	Ok …..
Bongekile	(smiling) I am not too good at Maths.
Researcher	That's Ok, just tell me what you are thinking, it's not a test.
Bongekile	I am not sure, maybe, er….. θ is not 50
Researcher	So you know θ is not 50?
Bongekile	(smiling), but I don't know what it should be………..I have no idea what to do
Researcher	You have no idea what to do? Ok…..

Bongekile	I know it is not right. .. I'll just guess anything ….. this is ½ and this is 0.55 (pointing at figures in the question), then, this is ½ or it will be 0.5
Researcher	So what you think it will be?
Bongekile	It looks small……. I mean it looks too small for an angle, so I will go with 50 and this one will be 55.
Researcher	Ok, thank you. Let us draw the *cosine* graph from -180º to 180º
Bongekile	Ok. (Works on the computer)
Researcher	Look carefully at the diagram. What is the domain?
Bongekile	This?
Researcher	Yes
Bongekile	-180º to 180º
Researcher	Range?
Bongekile	-1 to 1
Researcher	Amplitude?
Bongekile	1.
Researcher	And the period?
Bongekile	This. (Indicating with the cursor, correct one, even though she had 180º written).
Researcher	I would like you to check on the coefficients of *cos x.* 2, ½ and -1 and then check on their effect on the original *cos x* you drew earlier on… (Pause)... What changes occur as the coefficient increase?
Bongekile	The range increases
Researcher	Right. If it decreases?
Bongekile	Decreases
Researcher	Any effect on the *x*-intercepts if it is negative?
Bongekile	No
Researcher	On range?
Bongekile	Yeah
Researcher	Let us draw the graph of *y=cos x*, *y= cos x* +½ and *y=cos x*-1 on the same axes? Remember to go to File, and then New sketch
Bongekile	Ok. (Works on the computer)
Researcher	I would like you to check on the numbers being added to *cos x,* ½ and -1 and then check on their effect on the graph of *cos x*… (Pause)... What changes occur if the number is positive, for instance ½?
Bongekile	It moves up
Researcher	If negative?
Bongekile	It moves down
Researcher	Any effect on amplitude?
Bongekile	No

Researcher	On range?
Bongekile	No
Researcher	Yes. Use what you have learnt to do question 8
Bongekile	Ok. (she came up with 2 correct diagrams with minimal errors)
Researcher	Ok, good, thank you very much Bongekile
Bongekile	Ok. Thank you, sir.