THESIS PRESENTED FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

SCALAR PERTURBATIONS OF SCHWARZSCHILD BLACK HOLES IN MODIFIED GRAVITY

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Scalar Perturbations of Schwarzschild Black Holes in Modified Gravity

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As the candidate's supervisors, we have approved this dissertation for submission.

Dr Rituparno Goswami Prof. Sunil Maharaj

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Declaration I

I, Dan B. Sibandze, declared that this thesis titled, "Scalar Perturbations of Schwarzschild Black Holes in Modified Gravity" and the work presented in it are my own. I affirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

D B Sibandze

Declaration II - Publications

The work presented in this thesis is partly based on collaboration with my supervisors Dr. Rituparno Goswami and Prof. Sunil Maharaj in collaboration with Prof. Peter Dunsby and Dr. Anne-Marie Nzioki.

Part of the work has been accepted for publication and part of it has been published in the papers

- Publication 1

  Sibandze, Dan B., *et al.*, "Scattering of Ricci scalar perturbations from Schwarzschild black holes in modified gravity." *The European Physical Journal C* **77**, 364 (2017).

- Publication 2

  Sibandze, Dan B., *et al.*, "Echoes from the black holes: Evidence of higher order corrections to General Relativity in strong gravity regime." *The European Physical Journal C*, submitted (2017).

D B Sibandze

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To Sellinah S'phiwe Sibandze

Abstract

This thesis is concerned with the physics related to scalar perturbations in the Schwarzschild geometry that arise in modified gravity theories. It has already been shown that the gravitational waves emitted from a Schwarzschild black hole in $f(R)$ gravity have no signatures on the modification of gravity from General Relativity, as the Regge-Wheeler equation remains invariant. In this thesis we consider the perturbations of the Ricci scalar in a vacuum Schwarzschild spacetime, which is unique to higher order theories of gravity and is absent in General Relativity. We show that the equations that govern these perturbations can be reduced to a Volterra integral equation. We explicitly calculate the reflection coefficients for the Ricci scalar perturbations, when they are scattered by the black hole potential barrier. Our analysis shows that a larger fraction of these Ricci scalar waves are reflected compared to the gravitational waves. This may provide a novel observational signature for fourth order gravity. We also show that higher order curvature corrections to General Relativity, in the strong gravity regime on scales of the order of the near horizon, produce a rapidly oscillating and infalling Ricci scalar fireball just outside the horizon. These fluctuations behave like an infalling extra massive scalar field that can generate the ringdown modes of gravitational waves having the same natural frequency as those that are generated by black hole mergers. Our analysis provides a viable classical or semi-classical explanation for the echoes in the ringdown modes without invoking the existence of any exotic structures at the horizon.

"The illiterate of the 21st century will not be those who cannot read and write, but those who cannot learn, unlearn, and relearn"

– Alvin Toffler

Preface

The inspiration for the work carried out in this thesis came from a PhD thesis of one of Peter Dunsby's students, Anne-Marie Nzioki. Having looked at solutions and perturbations of spherically symmetric spacetimes in fourth order gravity, she concluded a section of her thesis by raising an interesting but intrinsic question, "*At observational level, what are the properties of the extra degree of freedom that manifests itself in the Ricci scalar of the spacetime, due to the higher order modifications in the theory of gravity?*"

Having taken up the challenge partially to elucidate the answer to this question during the study of a PhD degree, the audacity of my thinking has resulted in the work presented in this thesis. The enthusiasm to carry out the work has been largely influenced by the recent detection of gravitational waves from binary black hole merger by LIGO and an imagination of remarkable vivacity. A Norwegian mathematician, Marius Sophus Lie, in the last third of the nineteenth century is quoted as having written *"It was the audacity of my thinking".* Indeed it was his ability to think outside the conventional mode that has uncovered the so-called sophisticated techniques that were used to solve differential equations.

I have always maintained the belief that there was something special but yet uncovered about scattering of Ricci scalar waves. The series of these surprising facts are contained in chapters 4 and 5 of this thesis. Major parts of this thesis have been accepted for publication in the cited journals and some are in preprint form. I still maintain that we have yet to experience the far-reaching beauty and ability of modified theories of gravity.

Conventions and Abbreviations

A problem that often arises when one does research in astrophysics is the many conventions that are in use at the same time. Different authors, even in the same topic, use different symbols and units to describe and talk about the same concept. One of the first difficulties that we encountered in preparing this thesis was that such conventions were not always explicit and in some cases completely unclear. In order to not put the reader through the same ordeal, the conventions used in this work will be presented before anything else. Sign conventions follow Ellis (1971) and Ellis *et al.* (1999).

Sign conventions

Signature:	$[-,+,+,+]$.
Geometrised units:	$8\pi G = c = 1$.
Latin indices:	$0, 1, 2, 3$.

The Riemann tensor is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de},$$

and $\Gamma^a{}_{bd}$ are the Christoffel symbols (i.e. symmetric in the lower indices), defined by

$$\Gamma^a{}_{bd} = \frac{1}{2} g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}).$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd} R_{acbd}.$$

Abbreviations

BH	Black hole
GR	General relativity
QNM	Quasi-normal mode(s)
ECO	Exotic compact object(s)
GW	Gravitational wave(s)
SNR	Signal to noise ratio
QFT	Quantum field theory
PPN	Parametrized post Newtonian
NP	Newman-Penrose
CMB	Cosmic microwave background
AdS	Anti-de Sitter
CFT	Conformal field theory
BAO	Baryon acoustic oscillations
LLR	Lunar laser ranging
VIE	Volterra integral equation
LIGO	Laser Interferometer Gravitational-Wave Observatory
ODE	Ordinary differential equation
LRS	Local rotational symmetry

List of Symbols

g^{ab}	Lorentzian metric
g	Determinant of g^{ab}
$\Gamma^a{}_{bc}$	General affine connection
R_{ab}	Ricci tensor of g_{ab}
R	Ricci scalar of g_{ab}
$\mathcal{R}$	$\equiv g^{ab}\mathcal{R}_{ab}$
$\mathcal{S}$	Matter action
T_{ab}	Stress energy tensor
ψ	Matter fields (collectively)
Δ^a_{bc}	Hypermomentum
$\{^{\,a}_{bc}\}$	Levi-Civita connection
$\bar{\nabla}_a$	Covariant derivative with respect to $\Gamma^a{}_{bc}$
∇_a	Covariant derivative with respect to $\{^{\,a}_{bc}\}$
$\mathcal{L}_g$	Lagrangian density
$\Box$	$\equiv \nabla_c\nabla^c$
T^M_{ab}	Energy momentum tensor

$'$	Differentiation with respect to R
(ab)	Symmetrisation over the indices a and b
$[ab]$	Anti-symmetrisation over the indices a and b
$M_\odot$	Solar mass

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Chapter 1

Introduction

1.1 History of General Relativity

Galileo Galilei was the first to introduce pendulums and inclined planes to the study of terrestial gravity at the end of the 16th century. However, it was not until 1665, when Sir Isaac Newton introduced the now renowned "inverse-square gravitational force law", that terrestrial gravity was actually united with celestial gravity in a single theory. Newton's theory made correct predictions for a variety of phenomena at different scales, including both terrestrial experiments and planetary motion.

Newton's contribution to gravity is not restricted to the expression of the inverse square law. Much attention should be paid to the conceptual basis of his gravitational theory, which incorporates two ideas:

1. The idea of absolute space, *i.e.* the view of space as fixed, unaffected structure; a rigid arena in which physical phenomena take place.

2. The idea of what was later called the equivalence principle which, expressed in the language of Newtonian theory, states that the inertial and the gravitational masses coincide.

Asking whether Newton's theory, or any other physical theory for that matter, is right or wrong, would be ill-posed to begin with, since any consistent theory is apparently "right". A more appropriate way to pose the question would be to ask how suitable is this theory for describing the physical world or, even better, how large a portion of the physical world is sufficiently described by this theory. It was obvious in the first 20 years after the introduction of Newtonian gravity that it did manage to explain all of the aspects of gravity known at that time.

In 1893 Ernst Mach stated what was later called by Albert Einstein "*Mach's principle*". This is the first constructive attack on Newton's idea of absolute space after the 17th century debate between Gottfried Wilhelm Leibniz and Samuel Clarke[1] known as the Leibniz-Clarke Correspondence. Mach's idea can be considered as rather vague in its initial formulation, and it was essentially brought into mainstream physics later on by Einstein along the following lines: "*...inertia originates in a kind of interaction between bodies...*". This is obviously in contradiction with Newton's ideas, according to which inertia was always relative to the absolute frame of space. There exists also a later, probably clearer interpretation of Mach's Principle, which, however, also differs in substance. This was given by Dicke: "*The gravitational constant should be a function of the mass distribution in the universe*". This is different from Newton's idea of the gravitational constant as being universal and unchanging.

But it was not until 1905, when Albert Einstein completed Special Relativity, that Newtonian gravity would face a serious challenge. Einstein's new theory, which managed to explain a series of phenomena related to non-gravitational physics, appeared to be incompatible with Newtonian gravity. Relative motion and all the linked concepts had gone well beyond the ideas of Galileo and Newton, and it seemed that Special Relativity should somehow be generalised to include non-inertial frames. In 1907, Einstein introduced the equivalence between gravitation and inertia and successfully used it to predict the gravitational redshift. Finally, in 1915, he completed the theory of

[1]Clarke was acting as Newton's spokeman.

General Relativity, a generalisation of the Special Relativity which included gravity. Remarkably, the theory matched perfectly experimental findings.

General Relativity replaced Newtonian gravity and continues to be, up to now, an extremely successful and well accepted theory for gravitational phenomena. It was realised that Newtonian gravity is of limited validity compared to General Relativity but it is still sufficient for most applications related to gravity. General Relativity is bound to face similar questions as were faced by Newtonian gravity and many would agree that it is facing them now. In the forthcoming chapters, experimental facts and theoretical problems will be presented which justify that this indeed is the case. Remarkably, there exists a striking similarity to the problems which Newtonian gravity faced, *i.e.* difficulty in explaining particular observations, incompatibility with other well established theories and lack of uniqueness.

1.2 Gravitational Waves

Gravitational waves are introduced later as solutions of the linearised Einstein equations around flat spacetime. These waves are shown to propagate at the speed of light and to possess two polarization states. Gravitational waves can interact with matter, allowing for their direct detection by means of laser interferometers. Einstein's quadrupole formulae are derived and used to show that non-spherical compact objects moving at relativistic speeds are powerful gravitational wave sources.

The existence of gravitational radiation is first shown to be a natural consequence of any relativistic description of the gravitational interaction. Together with black holes and the expansion of the Universe, the existence of gravitational radiation is one of the key predictions of Einstein's general theory of relativity (Einstein (1918a) and Einstein (1916a)). The discovery of the binary pulsar PSR B1913+16 by Hulse and Taylor (1975), and the subsequent observation of its orbital decay, as well as that of other binary pulsars, have provided strong evidence for the existence of gravitational waves (Weisberg and Huang (2016), Lorimer (2008)). These observations have triggered an ongoing international effort to detect gravitational waves directly, mainly by using kilometer-scale laser interferometric antennas such as the LIGO and Virgo detectors (Aasi (2015), Acernese *et al.* (2014)). During the months of September and October 2015, the Advanced LIGO antennas have detected, for the first time, gravitational waves generated by two distinct cosmic sources. These waves were emitted, more than a billion years ago, during the coalescence of two binary black hole systems of $65M_{\odot}$ and $22M_{\odot}$, respectively (Abbott *et al.* (2016a,b)). The gravitational wave radiation from a perturbed black hole can, in general, be divided into three components:

1. An initial pulse emitted directly by the perturbation source depending on the initial conditions,

2. An exponentially damped oscillation (ringing) at intermediate times characterised by a single complex frequency, which doesn't depend on the initial conditions,

3. A power-law tail that develops after the ringing at very late times.

The ringing phase is due to a superposition of quasinormal modes of the black hole. The sources of GWs could be classified into two categories roughly. One is called cosmological origin, the other is of relativistic astrophysical origin. In the cosmological case, GWs can be produced in the early stages of the Universe, for example, during the inflation and reheating epochs. Such GWs are called primordial GWs, and they will leave a unique imprint on the cosmic microwave background (CMB), the so-called B-mode. The detection of GWs in Abbott *et al.* (2016a,b) and Abbott *et al.* (2016c) opens a new window to explore the Universe.

1.3 Quasinormal Modes of Black Holes

It is known that most objects around us, like a bell or a drum, produce very specific sounds when excited appropriately. These sounds are characteristic to each object which respond to such excitations with a superposition of different oscillatory modes. Take, for example, a guitar string. No matter how you pull it, the sound it produces will always be recognizable as a specific note.

Black holes are not much different in that respect. They also have a set of natural frequencies. However, what is now oscillating are fields in the BH vicinity or even spacetime itself rather than pressure in a gas producing sound waves. As such, these frequencies characterize the behaviour of fields in the region immediate to a BH. This has been verified both by numerical simulations of BH systems as well as theoretically at a perturbative linearized level. As a result, they are a unique characteristic of BH.

These oscillations are called "*quasi-normal modes*" and the frequencies associated to them "*quasi-normal frequencies*". Their name is inspired from ordinary normal modes and frequencies from which, however, they do differ substantially. A system that oscillates in a purely normal mode, is never going to stop, *i.e.* normal modes are stationary states. Many basic systems can be adequately modelled by such a scheme. The modelling of a pendulum, for example, can usually be quite accurate without taking into account that friction will eventually stop it. However, this is not always the case. Quasinormal modes are exponentially damped due to the system's energy loss. No matter the mechanism, which in the case of BH is emission of radiation (gravitational or other), they offer a much more realistic and precise picture of reality.

Quasinormal modes model the late time behaviour of perturbed compact objects. In our case, it is the BH spacetime as well as fields in its vicinity that are excited, and which we study at a linearised level. This thesis concentrates on scalar field perturbations. What is said in this section, though, is generally applicable to other fields as well. In most, if not all cases, the study of the field in question can be reduced to a second order differential equation of the form

$$\frac{d^2\phi_s}{dx^2} + Q_s^2(\omega, x)\phi_s = 0, \tag{1.1}$$

where x is related to a spatial variable (usually radial distance from the centre of the BH), ω is the (quasinormal) frequency and s is the spin of the field under study[2]. Time dependence is assumed to be of the form $\exp(-i\omega t)$ which, though seemingly restrictive, it is not due to time translational invariance.

Q_s^2 will be referred to as the generalized potential, due to its relation to the Schrödinger equation (where, usually, $Q^2 = E - V$). Its form, as well as the relation between ϕ_s and the actual field density are dependent on the specifics of the BH spacetime as well as the type of the field itself (scalar, spinor, vector, etc). In special cases, such as the Schwarzschild BH, it takes the simpler form $Q_s^2 = \omega^2 - V_s(x)$, see section 3.1. The variable x has similar dependencies and usually ranges in all $\mathcal{R}$ with $-\infty$ being the BH's event horizon and $+\infty$ the actual spatial infinity.

The physical problem studied requires that there are no other sources of waves. The settling down of the excitation we are studying is the only source. Mathematically, this means that at spatial infinity, only outgoing wave solutions should be allowed, *i.e.*

$$\phi_s \sim e^{(+i\Omega_+ x)}, \quad \text{for} \quad x \to +\infty \quad \text{and} \quad Q_s(+\infty) = \Omega_+. \tag{1.2}$$

A similar argument applies on the other boundary of our problem, at $x \to -\infty$. The very nature of a BH's event horizon along with preservation of causality disallows any outgoing solutions. By definition, matter and energy (which includes any field, either massive or massless) can only go further into the BH once they cross it. Similar to before, mathematically, this means that at the horizon only ingoing wave solutions should be allowed, *i.e.*

$$\phi_s \sim e^{(-i\Omega_- x)}, \quad \text{for} \quad x \to -\infty \quad \text{and} \quad Q_s(-\infty) = \Omega_-. \tag{1.3}$$

The frequencies $\Omega_\pm$ depend on the frequencies ω and potentially on the rest of the model parameters. It is only a discrete set of complex frequencies ω that give solutions satisfying the aforementioned boundary conditions. These are the ones called

[2] $s = 0$ for a scalar, $s = \pm 1$ vector, $s = \pm 2$ for a tensor and $s = \pm 1/2$ for a spinor

quasinormal frequencies. From the field's time dependence $\exp(-i\omega t)$, it is easy to see that it is the imaginary part of ω that models the damping (exponential decay) and thus the dissipative effect. Furthermore, it is evident that $\Im\omega$ must be strictly not positive since in the opposite case, the field will diverge for large times (which is of course unphysical).

1.3.1 Physical significance

The study of quasi-normal modes of black holes is very important in physics. They are considered to be the very basic objects of GR, much like the hydrogen atom is in quantum mechanics.

BH parameter estimation

Historically, the first BH related QNM studied where those of gravitational waves. The study of the binary pulsar PSR B1913+16 by Weisberg and Taylor (2005) was the first experimental indication to their existence. The observed increase in the pulsar's frequency could very well be explained by the spiralling in of the binary due to energy loss from radiating gravitational waves. Gravity being so weak, compared to the rest of the fundamental forces, makes the detection of gravitational waves an extremely delicate process. With current detector technology, it is only but the most violent gravitational phenomena that we expect to see, such as black hole collisions, stellar collapses, etc. Theoretical considerations along with numerical simulations indicate that the late time behaviour of such processes (even though they are not stationary) can very well be approximated by a superposition of QNM. As a result, since QNM depend on BH parameters only, detecting gravitational waves and fitting them to QNM models will, in principle, allow us to measure these parameters. More on the specifics of such computations can be found in Pitkin *et al.* (2011) and references therein.

However, in this thesis, we only study scalar perturbations, *i.e.* $s = 0$, and the resulting QNM are not directly applicable to gravitational waves ($s = 2$). Nevertheless, the form of Q_s^2 in both cases is not much different (see 3.1), and the techniques presented in this thesis are still in principle applicable.

Gauge-gravity duality

Another important field of interest for the application of QNM is string theory and the AdS/CFT correspondence, also known as the Maldacena duality. The correspondence is the conjectured equivalence between string theory and gravity on a spacetime of N dimensions with negative cosmological constant and a conformal quantum field theory defined on an $(N-1)$ dimensional space without gravity. It has been useful for the calculation of many quantities of strongly coupled systems which would have otherwise been next to impossible to study.

According to the duality, a black hole in AdS spacetime corresponds approximately to a thermal state of a strongly coupled system in the CFT. As a result, knowledge of the BH's QNM allows us to model the behaviour of the thermal state, something that would otherwise be much more difficult owing to its strongly coupled nature. More specifically, QNM coincide with the poles of correlation functions. Effectively, they correspond to quasi-particles in the CFT side.

BH area quantization

String theory is not the only candidate for a theory of quantum gravity. Attempts are still made for the study of BH in the context of QFT. Bekenstein conjectured (Bekenstein (1998)) that the area of a BH's event horizon takes on a discreet value spectrum resulting in the quantization of the BH's mass as well.

Semi-classical arguments suggest that $\Delta M = \Delta\omega$ in the highly damped limit. What is more, within loop quantum gravity, an alternative approach to a theory of quantum gravity, knowledge of the QNM spectrum may allow one to fix an otherwise unknown parameter (known as Barbero-Immirzi parameter) which shows up in the formula for the area of a BH. All these point to a potentially fundamental relation between QNM and a theory of quantum gravity. However, such suggestions are still highly theoretical and more research is required.

1.4 Recent Historical Developments

The detection of gravitational waves from binary black hole mergers (Abbott *et al.* (2016a)) was a historical event that established general relativity (GR) on a stronger footing as the classical theory of gravitational interactions. On 11 February 2016, the LIGO Scientific Collaboration and the Virgo Collaboration (Abbott *et al.* (2016c)) announced that on 14 September 2015 at 09 : 50 : 45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory (LIGO) simultaneously observed a transient gravitational wave (GW) signal. The GW event was named GW150914. The GW signal was consistent with the one predicted by general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole. This is the first direct detection of GW and the first observation of a binary black hole merger. On 15 June 2016, the second GW event, GW151226, was announced by the same team Abbott *et al.* (2016b). This time, the observed signal lasts approximate 1 second, the frequency increases from 35 to 450. The source is also the merger of two black holes.

The GW was predicted by Albert Einstein in 1916 (Einstein (1916b, 1918b)), 1 year later after he finally formulated his theory on gravitation, general relativity. But the physical reality of the GW solution of the Einstein field equations was not showed until the Chapel Hill conference in 1957 (Saulson (2011)). In Bondi (1957) and Bondi *et al.* (1959), it has been shown that GW carries energy and when passing through the spacetime in a form of a sandwich, it affects test particles. More than one century has passed since Einstein's proposal of GR, although it passed various precise tests, some alternatives still survive, for example, scalar-tensor gravity theory, $f(R)$ gravity, modified gravity with higher curvature terms, etc. Now we understand well that GW exists not only in general relativity, but also in other relativistic covariant gravity theories.

1.5 Modification of Gravity

Over the past hundred years General Relativity (GR) has matured into what is now arguably one of the most successful theories of modern physics. It has allowed us to explain gravitational phenomena from solar system scales (Capozziello and Tsujikawa (2008), Clifton (2008), Guo (2014) and Hu and Sawicki (2007),Berry and Gair (2011)) all the way to some of the largest scales in the observable universe. It provides us with full control of gravitational phenomena at terrestrial, solar and galactic scale, in a range between 10^{-5}m and 108parsec (Capozziello *et al.* (2006)). Some of GR predictions have been confirmed with an astonishing precision, which is comparable or better than the celebrated precision in perturbative quantum electrodynamics. Commenting on this fact, Roger Penrose provokingly stated that, since GR is such a precise theory, we should extend our knowledge of quantum field theories in order to accommodate them within GR and not viceversa (Mendoza and Rosas-Guevara (2007)).

Therefore the investigation of alternative theories of gravity seems at least a peripheral problem, due to the enormous success that GR has reached. However, modifications to GR are pursued vigorously for two main reasons. First, from a theoretical standpoint, an ultraviolet completion of GR is highly desirable. Such a completion, arising from quantum gravity theories such as String Theory or Loop Quantum Gravity, would lead to higher curvature corrections in the action, *i.e.* higher powers of scalar invariants constructed from the Riemann tensor. Although quantum gravity effects could be negligible for practical purposes, nevertheless it is quite disappointing that we know a *priori* the existence of an energy scale - presumably the Planck scale - at which our understanding of the Laws of Nature fails.

Secondly, from an experimental standpoint there are strong evidences that the deep infrared gravity regime is dominated by some form of dark energy (Weinberg, 1972,Ghosh and Narasimha (2009)). With the first two direct detections of gravitational waves from coalescing black holes by LIGO (Abbott *et al.* (2016a,b)), the past year has been a particularly triumphant period for GR. Despite these successes, most well established tests of GR still only involve weak gravitational fields and motions with speeds much less that the speed of light. While the recent LIGO events represented the first real strong field tests of the theory and were consistent with GR, many more such observations will be needed to probe the dynamical features of the strong field regime, before we can be certain that all extensions of Einstein gravity can be ruled out. Some of the most natural and promising extensions to GR are those which appear as the low energy limit of fundamental theories such as String or M-theory (e.g., Damour and Esposito-Farese (1992)). Examples of such modifications of GR can be found in a particularly popular and now very extensively studied class of fourth order theories of gravity, the socalled $f(R)$ theories of gravity. In these theories, the modification to the gravitational action is described by the addition of a general function of the Ricci scalar R which leads to field equations which are fourth order in the metric tensor g_{ab} (in GR the field equations are second order in g_{ab}). This implies that the gravitational interaction is generated by the usual spin-2 graviton degrees of freedom together with a scalar degree of freedom. These deviations from GR derive from the work on scalar-tensor theory by Brans and Dicke (1961a), Fierz (1956) and Jordan (1959).

On cosmological scales, we require that $f(R)$ theories reproduce cosmological dynamics consistent with type Ia supernovae, BAO, Large Scale Structure and CMB measurements. They should be free from tachyonic instabilities, sudden singularities and ghosts, and they should have valid Newtonian and post-Newtonian limits (de la Cruz-Dombriz *et al.* (2016)). We should also expect that well defined solutions found in GR, such as the Schwarzschild solution, are stable against generic perturbations in this more general context. Failure to satisfy the aforementioned criteria disfavours the theory as a viable alternative to GR.

Alternative theories of gravity are developed with the aim to extend the region of validity for GR, eventually resolving its infrared and ultraviolet regimes, but without giving any observable modification in the range where GR has been tested with excellent precision. Hence, in these theories the weak gravity regime is the same as in GR, and it is difficult to tell an alternative theory from Einstein's gravity by means of, for example, Solar System experiments. More precisely, in the weak gravity regime the Newtonian gravitational potential, velocities and related variables are much smaller than unity. In this regime a parametrized post-Newtonian (PPN) expansion (Hořava (2009),Carmichael and AS (1925)) is usually appropriate. Therefore alternative theories of gravity usually have the same PPN expansion as in GR, at first order. However, observable differences may presumably arise when strong curvature effects are taken into account (Carloni *et al.* (2005)). This is the case for cosmology or for strongly relativistic objects, such as black holes, whose astrophysical imprints in the framework of gravity theories beyond GR are the main topic of the present discussion.

Black holes (BHs), probing the strong curvature regime of any gravity theory, provide a means of possible high energy corrections to GR. Unfortunately, the majority of quantum gravity theories are vastly more complex than GR in their full-fledged form. It is thus not surprising that progress in understanding the exact differences between one and the other (and specially differences one can measure experimentally) has been slow and mostly focusing on the weak, far-field behaviour. Therefore our approach will be different. We shall focus on selected and well established modifications of GR, and we investigate effective actions arising as low energy approximations of more fundamental quantum gravity theories. These effective theories are much more tractable than their exact versions, and the imprint of their modifications to GR can already leave some signature in astrophysical phenomena, such as strong gravity effects taking place around astrophysical BHs. One of the most important of such effects is the emission of gravitational waves, whose detection is one of the main scientific achievements of current experimental physics. During the 20th century spectroscopy has opened a new era in quantum physics, via the precise detection of electromagnetic radiation from atoms, molecules and quantum systems. In the same way the detection of gravitational waves from BHs, neutron stars and other astrophysical objects has opened a new era in gravitational physics and will enhance our knowledge of gravity to unprecedented levels.

At present there are several gravitational wave observatories worldwide: LIGO in the U.S. (Capozziello (2002)), VIRGO (Maartens and Bassett (1998)) and GEO600 in Europe, TAMA300 in Japan. They have reached (or are approaching to reach) the design sensitivity and, recently, LIGO opened a new opportunity to probe the strong curvature regime of gravity via gravitational wave detection. Gravitational wave detection will provide us with high precision tests of GR and hopefully with evidence of physics beyond it. Thus it is of fundamental importance to investigate astrophysical properties of BHs in alternative theories of gravity and, in particular, to infer corrections to GR from the gravitational wave imprint of BHs (Levi-Civita (1927), Szekeres (1966)).

Generally speaking, this thesis gives further confirmation of the prominent role played by black holes in modern physics. In particular, the investigation of black hole perturbations provides us with fundamental insights both in theoretical physics and in astrophysics. These modern applications were perhaps anticipated in John Archibald Wheeler's autobiography in 1998:

Black holes teach us that space can be crumpled like a piece of paper into an infinitesimal dot, that time can be extinguished like a blown-out flame, and that the laws of *physics that we regard as 'sacred', as immutable, are anything but.*

Sitting on the shoulders of giants, we still have much to learn from this lesson.

1.6 Thesis outline

This Thesis is organized as follows: Chapter 2 is aimed to be self contained. We introduce $f(R)$ theories of gravity and present the general equations for these theories.

In Chapter 3, we investigate in detail how the Ricci scalar waves from infinity get scattered by the black holes in $f(R)$ gravity. To study the problem of reflection and transmission of the perturbations of Ricci scalar, we use the method of Jost functions. This is a powerful mathematical tool that enables us to model the problem in terms of a Volterra integral equation of the second kind. We explicitly calculate the reflection coefficient for the Ricci scalar perturbations for wavelengths much smaller than the ratio of the second order coefficient to the first order coefficient of the Taylor expansion of the function f around $R = 0$, and compare them to that of the gravity waves.

In Chapter 4, we show that the higher order curvature corrections to general relativity in the strong gravity regimes of near horizon scales produce a rapidly oscillating and infalling Ricci scalar fireball just outside the horizon, that can generate the ringdown modes of the gravitational waves having the same natural frequency as those which are generated by the black hole mergers.

Chapter 5 contains our conclusion.

Chapter 2

$f(R)$ Gravity

There are numerous ways to deviate from GR. Setting aside the early attempts to generalize Einstein's theory, most of which have been shown to be non-viable (Will (1981)), and the most well known alternatives to GR, the scalar-tensor theories of (Brans and Dicke (1961a,b) and Faraoni (2004)), there are still numerous proposals for modified gravity in the contemporary literature. In this chapter we introduce $f(R)$ theories of gravity and present the general equations for these theories (see Clifton (2008) and Sotiriou and Faraoni (2012) for detailed reviews).

2.1 Action and field equations

In general relativity (GR) the Einstein-Hilbert action is given as

$$\mathcal{S} = \frac{1}{2}\int dV\,\left[\sqrt{-g}\,(R-2\Lambda) + 2\,\mathcal{L}_M(g_{ab},\psi)\right]\,, \tag{2.1}$$

where $\mathcal{L}_M$ is the Lagrangian density of the matter fields ψ, R is the Ricci scalar and Λ is the cosmological constant. The invariant 4-volume element is given by the expression $\sqrt{-g}\,dV$ and the gravitational Lagrangian density as $\mathcal{L}_g = \sqrt{-g}\,(R-2\Lambda)$, where g is the determinant of the metric tensor g_{ab}. A generalisation of this action is done by replacing R in (2.1) with a C^2 function of the quadratic contractions of the Riemann curvature tensor R^2, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$ and $\varepsilon^{klmn}R_{klst}R^{st}{}_{mn}$ where ε^{klmn}

is the antisymmetric 4-volume element. In fact, in the quantum field picture, the effects of renormalisation are expected to add such terms to the Lagrangian in order to give a first approximation to some quantised theory of gravity (DeWitt (1967), Birrell and Davies (1982)). The Lagrangian density that can be constructed from the generalisation of the form

$$\mathcal{L}_g = \sqrt{-g}\; f(R, R_{ab}\,R^{ab}, R_{abcd}\,R^{abcd}) \ . \tag{2.2}$$

It is a well known result that (DeWitt and Mullin (1966), Buchdahl (1970), Barth and Christensen (1983)),

$$(\delta/\delta g_{ab}) \int dV \left(R_{abcd}\,R^{abcd} - 4R_{ab}\,R^{ab} + R^2\right) = 0 \ , \tag{2.3}$$

$$(\delta/\delta g_{ab}) \int dV\, \varepsilon^{klmn} R_{klst}\, R^{st}{}_{mn} = 0 \ , \tag{2.4}$$

that is, the functional derivative of the Gauss-Bonnet invariant $R_{abcd}\,R^{abcd} - 4R_{ab}\,R^{ab} + R^2$ and $\epsilon^{iklm} R_{ikst}\, R^{st}{}_{lm}$ vanish with respect to g_{ab}. If we consider the function f to be linear in $R_{abcd}R^{abcd}$, we can use this symmetry to rewrite $R_{abcd}R^{abcd}$ in terms of the other two invariants and as a result the action for FOG can be written as

$$\mathcal{S} \;=\; \frac{1}{2}\int dV \left\{\sqrt{-g}\left(c_0\, R + c_1\, R^2 + c_2\, R_{ab}\, R^{ab}\right) + 2\,\mathcal{L}_M(g_{ab}, \psi)\right\} \ , \tag{2.5}$$

where the coefficients c_0, c_1 and c_2 have the appropriate dimensions. Similarly, if the spacetime is homogeneous and isotropic, then because of the following identity,

$$(\delta/\delta g_{ab}) \int dV\; \left(3R_{ab}\,R^{ab} - R^2\right) = 0 \ , \tag{2.6}$$

the term $R_{ab}\,R^{ab}$ can always be rewritten in terms of the variation of R^2. Though in the present chapter we are not discussing isotropic spacetimes, nevertheless even for spherically symmetric case we can safely assert that a sufficiently general and "effective" fourth order Lagrangian for highly symmetric spacetimes contain only powers of R. Also this makes the problems more physically realistic as it has been shown that the theories that contain the square of Ricci tensor in the action, suffer from several instabilities.

Therefore we can write, without loss of generality, the action as

$$\mathcal{S} = \frac{1}{2} \int dV \left[\sqrt{-g}\, f(R) + 2\, \mathcal{L}_M(g_{ab}, \psi) \right] \ . \tag{2.7}$$

This action represents the simplest generalisation of the Einstein-Hilbert density. Demanding that the action be invariant under some symmetry ensures that the resulting field equations also respect that symmetry. That being the case, since the Lagrangian is a function R only, and R is a generally covariant and locally Lorentz invariant scalar quantity, then the field equations derived from the action (2.7) are generally covariant and Lorentz invariant.

There are different variational principles that can be applied to the action $\mathcal{S}$ in order to obtain the field equations. One approach is the *standard metric formalism* where variation of the action is with respect to the metric g_{ab} and the connection $\Gamma^a{}_{bc}$ in this case is the Levi-Civita one, that is, the metric connection

$$\Gamma^a{}_{bc} = \frac{1}{2}\, g^{ad} \left(g_{bd,c} + g_{dc,b} - g_{bc,d} \right) \ . \tag{2.8}$$

In the Palatini[1] formalism, the metric and the connection are assumed to be independent fields and one varies the action with respect to each of them (we will see how this variation leads to Einstein's equations shortly), under the important assumption that the matter action does not depend on the connection. The choice of the variational principle is usually referred to as a formalism, so one can use the terms metric (or second order) formalism and Palatini (or first order) formalism. Finally, there is actually even a third version: the metric-affine approach (Sotiriou and Liberati (2007)). This comes about if one uses the Palatini variation but abandons the assumption that the matter action is independent of the connection as well as the metric. Clearly, the metric-affine approach is the most general of these theories and reduces to the metric or Palatini formalism if further assumptions are made.

[1] Even though it was Einstein and not Palatini who introduced it (Ferraris *et al.* (1982))

In this section we will present the actions and field equations of all three versions of gravity and point out their differences.

2.1.1 Metric formalism

Varying the action (2.7) with respect to the metric g_{ab} over the 4-volume yields

$$\delta\mathcal{S} \;=\; -\frac{1}{2}\int dV\,\sqrt{-g}\left\{\frac{1}{2}f\,g_{ab}\,\delta g^{ab} - f'\,\delta R + T^M_{ab}\,\delta g^{ab}\right\}\;, \qquad (2.9)$$

where $'$ denotes differentiation with respect to R, and T^M_{ab} is the matter *energy momentum tensor* (EMT) defined as

$$T^M_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta g^{ab}}\;. \qquad (2.10)$$

Writing the Ricci scalar as $R = g^{ab}\,R_{ab}$ and assuming the connection is the Levi-Civita one, we can write

$$f'\,\delta R \simeq \delta g^{ab}\,(f'\,R_{ab} + g_{ab}\,\Box f' - \nabla_a\nabla_b f')\;, \qquad (2.11)$$

where the $\simeq$ sign denotes equality up to surface terms and $\Box \equiv \nabla_c\nabla^c$. By requiring that $\delta\mathcal{S} = 0$ with respect to variations in the metric, ergo a stationary action, one has finally

$$\begin{aligned} f'\left(R_{ab} - \frac{1}{2}g_{ab}\,R\right) \;&=\; \frac{1}{2}g_{ab}\,(f - R\,f') + \nabla_a\nabla_b f' \\ &\quad -g_{ab}\,\Box f' + T^M_{ab}\;. \end{aligned} \qquad (2.12)$$

The special case $f = R$ gives us the standard Einstein field equations.

It is convenient to write (2.12) in the form of the effective Einstein equations as

$$G_{ab} = \left(R_{ab} - \frac{1}{2}g_{ab}\,R\right) = \tilde{T}^M_{ab} + T^R_{ab} = T_{ab}\;, \qquad (2.13)$$

where we define T_{ab} as the total EMT with

$$\tilde{T}^M_{ab} = \frac{T^M_{ab}}{f'}, \qquad (2.14)$$

and

$$T^R_{ab} = \frac{1}{f'}\left[\frac{1}{2}g_{ab}\left(f - R\, f'\right) + \nabla_a\nabla_b f' - g_{ab}\,\Box f'\right] \ . \tag{2.15}$$

The field equations (2.13) contain fourth order derivatives of the metric functions, which can be seen from the existence of the $\nabla_a\nabla_b f'$ term in (2.15). This result also follows from a corollary of Lovelock's theorem (Lovelock (1971, 1972)) stated in Theorem 2.1.1:

Theorem 2.1.1. *In a four-dimensional Riemannian manifold, the construction of a metric theory of modified gravity must admit higher than second order derivatives in the field equations.*

This is an undesirable feature in a Lagrangian based theory as it can lead to *Ostrogradsk instabilities*[2] (Ostrogradsky (1850)) in the solutions of the field equations. The $f(R)$ theories are special as there instabilities can be avoided (Woodard (2007)), due to the existence of an equivalence with scalar-tensor theories.

2.1.2 Palatini formalism

We have already mentioned that the Einstein equations can be derived using, instead of the standard metric variation of the Einstein-Hilbert action, the Palatini formalism. In the Palatini formalism, the metric g_{ab} and connection $\Gamma^a{}_{bc}$ are treated as independent fields and the variation of the action is performed with respect to each of them separately. An independent variation with respect to the metric and the connection is called Palatini variation. Note that this should not be confused with the term Palatini formalism, which refers not only to the Palatini variation, but also to having the matter action being independent of the connection. Varying the action (2.7) independently with respect to the metric and the connection, respectively, over a 4-volume and using

[2]This is a consequence of a theorem of Mikhail Ostrogradsky in classical mechanics according to which a non-degenerate Lagrangian dependent on time derivatives of higher than the first corresponds to a linearly unstable Hamiltonian associated with the Lagrangian via a Legendre transform (Motohashi and Suyama (2014)).

the formula

$$\delta\mathcal{R}_{ab} \;\; = \;\; \nabla_c\delta\Gamma^c{}_{ab} - \nabla_d\delta\Gamma^d{}_{bc}. \tag{2.16}$$

yields

$$f'\mathcal{R}_{(ab)} - \frac{1}{2}g_{ab}f = T^M_{ab}, \tag{2.17}$$

$$-\nabla_c\left(\sqrt{-g}g^{ab}f'\right) + \nabla_\delta\left(\sqrt{-g}f'(\mathcal{R})g^{\delta(a}\right)\delta^{b)}_c = 0, \tag{2.18}$$

where T^M_{ab} is defined in the usual way as in equation (2.10), and the covariant derivative is taken with the independent connection $\Gamma^a{}_{bc}$, and (ab), and $[ab]$ show symmetrization or anti-symmetrization over the indices a and b, respectively. Taking the trace of equation (2.21), it can be easily shown that

$$\nabla_\delta\left(\sqrt{-g}f'(\mathcal{R})g^{\delta(a}\right)\delta^{b)}_c = 0, \tag{2.19}$$

which implies that we can bring the field equations into the more economical form

$$f'\mathcal{R}_{ab} - \frac{1}{2}g_{ab}f = T^M_{ab}, \tag{2.20}$$

$$\nabla_c\left(\sqrt{-g}g^{ab}f'\right) = 0, \tag{2.21}$$

We see here how the Palatini formalism leads to GR[3] when $f(\mathcal{R}) = \mathcal{R}$ which implies $f'(\mathcal{R}) = 1$ and equation (2.21) becomes the definition of the Levi-Civita connection for the initially independent connection $\Gamma^a{}_{bc}$. It follows that, $\mathcal{R}_{ab} = R_{ab}$, $\mathcal{R} = R$ and from equation (2.20) we recover Einstein's field equations. These reproduced results can be found in the textbooks by (Misner *et al.* (1973) and Wald (1984)).

Serious short comings of the Palatini formalism include the introduction of non-perturbative corrections to the matter fields and strong couplings between gravity and matter at low energies (Flanagan (2004), Iglesias *et al.* (2007)). Moreover, the nature of the Cauchy problem for $f(R)$ gravity in the Palatini formalism is not well

[3]In the Palatini formalism for GR, the fact that the connection turns out to be the Levi-Civita one is a dynamical feature instead of an *a priori* assumption

formulated in the presence of matter. Without a well-posed initial value problem, the Palatini $f(R)$ gravity lacks the predictive power that is required of any physical theory (Lanahan-Tremblay and Faraoni (2007)).

2.1.3 Metric-affine formalism

As we already pointed out that in the Palatini formalism of $f(R)$ gravity, the matter action $\mathcal{S}_{\mathcal{M}} = \int \mathcal{L}_M(g_{ab}, \psi)$ is assumed to be dependent only on the metric and matter fields and not on the independent connection. This assumption relegates this connection to the role of some sort of auxiliary field and the connection carrying the usual geometrical meaning - parallel transport and definition of the covariant derivative - remains the Levi-Civita connection of the metric. We would define the covariant derivatives of the matter fields with this connection and, therefore, we would have $\mathcal{S}_{\mathcal{M}} = \mathcal{S}_{\mathcal{M}}(g_{ab}, \Gamma^a{}_{bc}, \psi)$. The action of this theory, dubbed metric-affine $f(R)$ gravity (Sotiriou and Liberati (2007)), then becomes[4]

$$\mathcal{S} = \frac{1}{2}\int dV\sqrt{-g}f(\mathcal{R}) + 2\mathcal{L}_M(g_{ab}, \Gamma^a_{bc}, \psi). \tag{2.22}$$

where $\mathcal{R} = g^{ab}\mathcal{R}_{ab}$ and the Ricci tensor $\mathcal{R}_{ab}$ is constructed with an independent connecton as in the Palatini approach. Since now the matter action depends on the connection, we should define a quantity representing the variation of $\mathcal{S}_M$ with respect to the connection, which Hehl and Kerlick (1978) mimics the definition of the stress-energy tensor, as

$$\Delta_a^{bc} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta \mathcal{S}_M}{\delta \Gamma^a_{bc}}. \tag{2.23}$$

Since the connection is a completely independent field, it is interesting to consider not placing any restrictions on it. Therefore, besides dropping the assumption that the connection is related to the metric, we will also drop the assumption that the connection is symmetric. In this theory, as well as in other theories with an independent connection, some part of the connection is still related to the metric. In our case, the

[4]Note the difference with respect to the action (2.7).

connection is left completely unconstrained and is to be determined by the field equations. Metric-affine gravity with the linear version of the action (2.22) was initially proposed in Hehl and Kerlick (1978).

But leaving the connection completely unconstrained comes with complications. If we consider that the Ricci scalar is invariant under the projective transformation

$$\Gamma^c{}_{de} \to \Gamma^c{}_{de} + \lambda_d \delta^c{}_e, \tag{2.24}$$

where λ_d is an arbitrary covariant vector field, then any action built from a function of $\mathcal{R}$, and this includes the Einstein-Hilbert action, is projective invariant in metric-affine gravity. However, since the matter fields do not exhibit this type of invariance, this can lead to inconsistency in the field equations. One way to get around this problem is by generalizing the gravitational action in order to break projective invariance. This can be done in several ways, such as allowing for the metric to be non-symmetric as well, adding higher order curvature invariants or terms including the Cartan torsion tensor [see Sotiriou (2006) and Sotiriou and Liberati (2007) for a more detailed discussion].

However, if one wants to stay within the framework of $f(R)$ gravity, which is our subject here, then there is only one way to cure this problem: to somehow constrain the connection. In fact, it is evident from equation (2.24) that, if the connection were symmetric, projective invariance would be broken. Breaking this invariance can therefore come by fixing some degrees of freedom of the field, similarly to gauge fixing (Sandberg (1975)). The number of degrees of freedom which we need to fix is obviously the number of the components of the four-vector used for the transformation, *i.e.*, simply four. However, one does not have to take such a drastic measure. Adding an action term containing a Lagrange multiplier term B^a which has the form

$$\mathcal{S}_L = \int dV \sqrt{-g}\, B^a \Gamma^b{}_{ba}, \tag{2.25}$$

is the most general metric-affine $f(R)$ theory of gravity. Varying the action with respect to the metric, the connection and the Lagrange multiplier, respectively, results in

$$f'\mathcal{R}_{ab} - \frac{1}{2}g_{ab}f = T^M_{ab}, \tag{2.26}$$

$$\Gamma^a{}_{ab} = 0, \tag{2.27}$$

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\left[\nabla_c\left(\sqrt{-g}f'g^{ac}\right)\delta^b{}_d - \nabla_d\left(\sqrt{-g}f'g^{ab}\right)\right] \\ &\quad +2f'\left(g^{ab}\Gamma^c{}_{dc} - g^{a\rho}\Gamma^c{}_{\rho c}\delta^b{}_d + g^{ac}\Gamma^b{}_{cd}\right) \\ &\qquad\qquad = \frac{\chi}{2}\left(\Delta^{ab}_d - B^{[a}\delta^{b]}{}_d\right). \end{aligned} \tag{2.28}$$

Taking the trace of equation (2.28) over the indices a and d and using equation (2.27) yields

$$B^a = \frac{2}{3}\Delta^{ca}_c. \tag{2.29}$$

Therefore the final form of the field equations is

$$f'\mathcal{R}_{ab} - \frac{1}{2}g_{ab}f = T^M_{ab}, \tag{2.30}$$

$$\Gamma^a{}_{ab} = 0, \tag{2.31}$$

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\left[\nabla_c\left(\sqrt{-g}f'g^{ac}\right)\delta^b{}_d - \nabla_d\left(\sqrt{-g}f'g^{ab}\right)\right] \\ &+2f'g^{ac}\Gamma^b{}_{[cd]} = \frac{\chi}{2}\left(\Delta^{ab}_d - \frac{2}{3}\Delta^{c[a}_c\delta^{b]}{}_d\right) \end{aligned} \tag{2.32}$$

By splitting equation (2.32) into a symmetric and an antisymmetric part and performing contractions and manipulations, it can be shown that (Sotiriou and Liberati (2007))

$$\Delta^{bc}_a = 0 \implies \Gamma^a{}_{bc} = 0. \tag{2.33}$$

This means torsion is introduced by matter fields for which $\Delta^{[bc]}_a \neq 0$ and $\Delta^{[bc]}_a = 0$ corresponds to the vanishing of torsion. It is not propagating since it is given algebraically in terms of the matter fields through $\Delta^{[bc]}_a$. In the absence of the latter, spacetime will have no torsion. Metric-affine $f(R)$ gravity appears to be the most general case of $f(R)$ gravity. It is not a metric theory, hence the name.

The metric approach to the $f(R)$ theories will be the focus of the thesis.

Chapter 3

Potential Scattering using Jost functions

In the Schwarzschild spacetime the wave equation for the scalar field reduces to the following Schrodinger-type equation. We initiate our discussion of one dimensional potential scattering by considering the solution of this time independent equation for $\mathcal{R}$:

$$\frac{d^2\mathcal{R}}{dr_*^2} + \left[\kappa^2 - V(r)\right]\mathcal{R} = 0, \tag{3.1}$$

where we have the so-called tortoise coordinate that was first introduced by Wheeler *et al* (1955) and is related to the standard Schwarzschild radial coordinate r by

$$\frac{d}{dr_*} = \left(1 - \frac{2M}{r}\right)\frac{d}{dr_*}. \tag{3.2}$$

This integrates to

$$r_* = r + 2M\log\left(\frac{r}{2M} - 1\right), \tag{3.3}$$

and we can see that introducing the tortoise coordinate corresponds to pushing the event horizon of the black hole away to $-\infty$. The effective potential $V(r)$ is positive definite everywhere and has a single peak in the range $r_* \in [-\infty, \infty]$ and is also of 'short range' effect in the sense that

$$\int_{-\infty}^{\infty} V(r)dr \quad \text{is finite.} \tag{3.4}$$

A BH is distinguished by the fact that no information can escape from within the event horizon. Hence, any physical solution to (3.1) must be purely ingoing at the event horizon, that is, at $r = 2M(r_* \to -\infty)$. Problems involving waves scattered from a

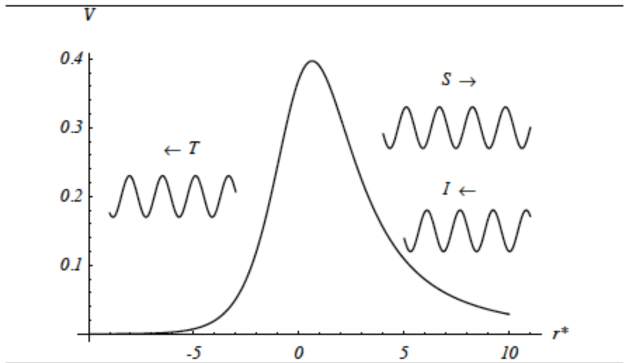

Figure 3.1: A schematic description of the scattering of waves in the Schwarzschild background. The effective potential of equation (3.1) is shown as a function of r. The event horizon of the black hole is located at $r = -\infty$. An incident wave I is decomposed into a transmitted component T and a scattered component S.

Schwarzschild black hole share many features with scattering problems in quantum theory. Hence we adopt standard techniques to study the resolvent. The nature of the scattered waves can be understood from the following observations: For $\kappa << 2M$ the wavelength of the infalling wave is so large that the wave is essentially unaffected by the presence of the black hole. It is only if we “aim” the wave straight at the black hole that we can get an appreciable effect. So one would expect waves of short wave length to be easily transmitted through the barrier. Hence, we expect to have scattered waves approach unity as $\kappa \to 0$.

For large frequencies $\kappa >> 2M$, the situation is the opposite and we expect to find that the scattered wave approaching 0 as $\kappa \to \infty$. Thus, high frequency waves will be absorbed unless they are aimed away from the black hole. Finally, waves with $\kappa \sim 2M$ will be partly transmitted and partly reflected.

3.1 Jost solutions

Equation (3.1) is an ODE integrable over the entire range $(-\infty, \infty)$ of r_*. The potential is a smooth function of r_*. Moreover, all polynomials constructed out of V_S and of its derivatives of all orders, are integrable over the entire range, $(-\infty, +\infty)$ of r_*. If we let $r_* \to \pm\infty$ in equation (3.1), we obtain two particular solutions with the asymptotic behaviours

$$\mathcal{R}_1(r_*, \kappa) \sim e^{-i\kappa' r_*} \sim e^{-i\kappa r_*}, \qquad (r_* \to +\infty),$$

and

$$\mathcal{R}_2(r_*, \kappa) \sim e^{i\kappa r_*}, \qquad (r_* \to -\infty),$$

which are linearly independent because their Wronskian

$$\begin{aligned} [\mathcal{R}_1(r_*, \kappa), \mathcal{R}_2(r_*, \kappa)] &= (i\kappa) e^{i\kappa r_*} e^{-i\kappa r_*} - (-i\kappa) e^{-i\kappa r_*} e^{i\kappa r_*} \\ &= +2i\kappa \neq 0. \end{aligned} \tag{3.5}$$

For real κ, the solution represents ingoing and outgoing waves at $\pm\infty$. This problem becomes one of reflection and transmission of incident waves by the one dimensional potential barrier V_S. We seek solutions satisfying of the wave equation (3.1) and the boundary conditions[1],

$$\mathcal{R}_2(r_*, \kappa) = \frac{R_1(\kappa)}{T_1(\kappa)} \mathcal{R}_1(r_*, \kappa) + \frac{1}{T_1(\kappa)} \mathcal{R}_1(r_*, -\kappa), \tag{3.6}$$

and

$$\mathcal{R}_1(r_*, \kappa) = \frac{R_2(\kappa)}{T_2(\kappa)} \mathcal{R}_2(r_*, \kappa)) + \frac{1}{T_2(\kappa)} \mathcal{R}_2(r_*, -\kappa), \tag{3.7}$$

where $R_1(\kappa)$, $R_2(\kappa)$, $T_1(\kappa)$, $T_2(\kappa)$ are distinct functions that exist if $\kappa \neq 0$. Here we can easily see that $T_1(\kappa)\mathcal{R}_2(r_*, \kappa)$ corresponds to an *incident wave* of unit amplitude from $+\infty$ giving rise to a *reflected wave* of amplitude $R_1(\kappa)$ and a transmitted wave of amplitude $T_1(\kappa)$.

[1] In conformity with physical requirements, the boundary conditions we have imposed do not allow for waves emerging from the event horizon.

Similarly, $T_2(\kappa)\mathcal{R}_1(r_*, \kappa)$ corresponds to an *incident wave* of unit amplitude from $-\infty$ giving rise to *reflected* and a transmitted waves of amplitude $R_1(\kappa)$ and $T_2(\kappa)$, respectively. For $\kappa \neq 0$, we define the *scattering* or $\mathbf{S} - \mathbf{matrix}$ as

$$\mathbf{S}(\kappa) = \begin{vmatrix} T_1(\kappa) & R_2(\kappa) \\ R_1(\kappa) & T_2(\kappa) \end{vmatrix}.$$

In the theory of potential scattering, the Jost functions are defined by

$$m_1(r_*, \kappa) = e^{+i\kappa r_*}\mathcal{R}_1(r_*, \kappa), \tag{3.8}$$

and

$$m_2(r_*, \kappa) = e^{-i\kappa r_*}\mathcal{R}_2(r_*, \kappa), \tag{3.9}$$

which satisfy the boundary conditions

$$m_1(r_*, \kappa) \to 1 \quad \text{as} \quad r_* \to +\infty, \tag{3.10}$$

and

$$m_2(r_*, \kappa) \to 1 \quad \text{as} \quad r_* \to -\infty. \tag{3.11}$$

The Jost solution is *holomorphic* in the upper complex κ-plane, where $\Im(\kappa) > 0$. In our approach, the most important quantity is the Jost function which has the following properties:

$$T(\kappa)m_2(r_*, \kappa) = R_1(\kappa)e^{-2i\kappa r_*}m_1(r_*, \kappa) + m_1(r_*, -\kappa), \tag{3.12}$$

and

$$T(\kappa)m_1(r_*, \kappa) = R_2(\kappa)e^{+2i\kappa r_*}m_2(r_*, \kappa) + m_2(r_*, -\kappa), \tag{3.13}$$

where $T_1(\kappa) = T_2(\kappa) = T(\kappa)$. From the conditions imposed in (3.10) and (3.11), it follows that

$$m_1(r_*, \kappa) = \frac{R_2(\kappa)}{T(\kappa)}e^{+2i\kappa r_*} + \frac{1}{T(\kappa)} + o(1) \quad (r_* \to -\infty), \tag{3.14}$$

and

$$m_2(r_*, \kappa) = \frac{R_1(\kappa)}{T(\kappa)}e^{-2i\kappa r_*} + \frac{1}{T(\kappa)} + o(1) \quad (r_* \to +\infty). \tag{3.15}$$

We also note that the Jost functions satisfy the differential equations

$$\frac{d^2 m_1}{dx^2} - 2i\kappa \frac{dm_1}{dx} = V m_1, \tag{3.16}$$

and

$$\frac{d^2 m_2}{dx^2} + 2i\kappa \frac{dm_2}{dx} = V m_2. \tag{3.17}$$

3.2 Volterra Integral equation

We now obtain an integral equation for $m(r_*, \kappa)$. We let

$$\mathcal{R}_2(r_*, \kappa) = e^{i\kappa r_*} + \psi(r_*, \kappa). \tag{3.18}$$

We note that $\psi \to 0$ as $r_* \to -\infty$, and ψ satisfies the differential equation

$$\left(\frac{d^2}{dr_*^2} + \kappa^2\right)\psi = \left(e^{i\kappa r_*} + \psi\right) V_S. \tag{3.19}$$

Now we know that, given any linear ODE of the form $L\psi(x) = -f(x)$, where L is the linear harmonic differential operator, the solution is given by Green's function

$$\psi(x) = \int G(x, x') f(x') dx', \tag{3.20}$$

where

$$G(x, x') = \frac{1}{\kappa}\left[\frac{1}{2i}\left(e^{i\kappa(x-x')} - e^{-i\kappa(x-x')}\right)\right]. \tag{3.21}$$

Therefore we can write the solution $\psi(x)$ in the form

$$\begin{aligned}\psi(r_*, \kappa) =& \frac{1}{2i\kappa}\int_{-\infty}^{r_*}\left[e^{i\kappa(r'_* - r_*)} - e^{-i\kappa(r'_* - r_*)}\right] V_S(r'_*) \\ & \times \left[e^{i\kappa r_*} + \psi(r_*, \kappa)\right] dr'_*. \end{aligned} \tag{3.22}$$

Using the above equations we now get an integral equation for the Jost function

$$\begin{aligned} m_2(r_*, \kappa) &= e^{-i\kappa r_*}\mathcal{R}_2(r_*, \kappa) \\ &= 1 + e^{-i\kappa r_*}\psi(r_*, \kappa), \end{aligned} \tag{3.23}$$

as

$$m_2(r_*, \kappa) = 1 - \frac{1}{2i\kappa} \int_{-\infty}^{r_*} \left(e^{2i\kappa(r'_* - r_*)} - 1 \right) \times V_s(r'_*) m_2(r'_*, \kappa) dr'_*, \tag{3.24}$$

which is a Volterra integral equation of the second kind, for all $\kappa \neq 0$. It can be shown that $m_2(r_*, \kappa)$ is an analytic function in κ in the lower half-plane $\Im(\kappa) < 0$ and is continuous in κ up to the real axis with a possible exception of the point $\kappa = 0$. It's solution can be obtained by the method of successive approximations. In the next chapter we give a numerical scheme to solve this equation, which will then provide us the required expressions for reflected and transmitted waves.

Chapter 4

Scattering of Ricci Scalar perturbations from Schwarzschild black holes in modified gravity

4.1 Introduction

In GR, linear perturbations of Schwarzschild black holes were first studied in detail by Chandrasekhar using the metric approach together with the Newman-Penrose formalism (Chandrasekhar (1983)). More recently, the standard results of Black Hole perturbation theory were reproduced using the 1+1+2 covariant approach (Clarkson and Barrett (2003)). In the metric approach, perturbations are described by two wave equations, *i.e.*, the Regge-Wheeler equation for odd parity modes and the Zerilli equation in the even parity case. These wave equations are described by functions (and their derivatives) in the perturbed metric which are not gauge-invariant, as general coordinate transformations do not preserve the form of the wave equation. However, using the 1+1+2 covariant approach, Clarkson and Barrett (2003) demonstrated that both the odd and even parity perturbations may be unified in a single covariant wave equation, which is equivalent to the Regge-Wheeler equation. This wave equation is governed by a single covariant, gauge and frame-independent, transverse-traceless tensor. These results were extended to include couplings (at second order) to a homogeneous magnetic field leading to an accompanying electromagnetic signal alongside the standard tensor (gravitational wave modes) by Clarkson *et al.* (2004a), and to electromagnetic perturbations on general locally rotationally symmetric spacetimes by Burston and Lun (2008).

The 1+1+2 covariant approach was later applied to $f(R)$ gravity in Nzioki *et al.* (2017a) and Pratten (2015) where all calculations were performed in the Jordan frame. The dynamics of the extra gravitational degree of freedom inherent in these fourth order theories were determined by the trace of the effective Einstein equations, leading to a linearised scalar wave equation for the Ricci scalar. One of the key results that came out of this analysis was: at the linearised level, the Regge-Wheeler equation in general $f(R)$ gravity (that admits the Schwarzschild solution), for gravitational perturbations around a black hole is exactly same the as in GR. Therefore, any measurement of gravitational waves emitted from a black hole will not have any signatures of the modification of gravity. This brings us to the following important question:

At the observational level, what are the properties of the extra degree of freedom that manifests itself in the Ricci Scalar of the spacetime, due to the higher order modifications in the theory of gravity? The answer to this question may then provide us with observational templates that can be used to verify GR at strong gravity regimes near the black hole horizon.

In this Chapter we address the above question in the following way:

1. We consider a small perturbation in the Ricci scalar from it's zero value for a Schwarzschild spacetime in $f(R)$-gravity. We note that this is unique to higher order gravity and is not possible in GR, where the Ricci scalar must be zero in vacuum. We then study the scattering of this disturbance of the Ricci scalar by the black hole. Since all the calculations are done in the Jordan frame, the results can be directly linked to observables.

2. We would like to emphasize the following important point here: We know that at the action level and in the Einstein frame, $f(R)$ gravity is equivalent to a scalar tensor theory (GR with a massive scalar field) (De Felice and Tsujikawa (2010)). Hence studying the propagation of the scalar perturbations on a Schwarzschild background should be equivalent to studying the Klein-Gordon equation for a massive scalar field on that background (see for example Décanini *et al.* (2011) and the references therein). However this equivalence may miss certain important features in the observational level, as in this case there is no *real* scalar field, but the geometry of spacetime behaving like a scalar field. Therefore, it will be unwise to assume aforehand, that this geometrical effect will obey all physically realistic conditions (e.g. energy conditions) like a real massive scalar field. Hence in this chapter we perform all our calculations in the Jordan frame (the physical frame), to find out what fraction of the in-falling Ricci scalar perturbation would be reflected by the black hole potential barrier.

3. To study the problem of reflection and transmission of the perturbations of Ricci scalar, we use the method of Jost functions. This is a powerful mathematical tool that enables us to model the problem in terms of a Volterra integral equation of the second kind. It is interesting to note that in the context of the Ricci scalar perturbations, the convergence of the numerical solution to this equation is guaranteed.

4. We explicitly calculate the reflection coefficient for the Ricci scalar perturbations for wavelengths much smaller than the ratio of the second order coefficient to the first order coefficient of the Taylor expansion of the function f around $R = 0$, and compare them to that of the gravity waves. Our analysis brings out certain interesting features which may provide a novel observational signature for modified gravity.

Furthermore, we also explicitly calculate the reflection coefficients for these Ricci scalar perturbations in a vacuum Schwarzschild spacetime, when they are scattered by the BH potential barrier.

4.2 Schwarzschild solution and it's stability

We know that in general relativity, the rigidity of spherically symmetric vacuum solutions of Einstein's field equations continues even in the perturbed case. Particularly, almost spherical symmetry and or almost vacuum implies almost static or almost spatially homogeneous (Goswami and Ellis (2011, 2012), Ellis and Goswami (2013)). This result emphasises the stability of Schwarzschild solution in general relativity. In $f(R)$-gravity, the extension of this result is not so obvious due to the presence of an extra scalar degree of freedom.

4.2.1 Jebsen Birkhoff's Theorem

Birkhoff's theorem[1] is of great significance for the weak field limit of General Relativity (Jebsen (1921)). The Theorem states:

All spherically symmetric solutions of Einstein's equations in vacuum must be static and asymptotically flat (in the absence of Λ).

Strictly speaking, there are very few situations in the real Universe in which Birkhoff's theorem is of direct applicability: Exact spherical symmetry and true vacuums are rarely, if ever, observed. Nevertheless, Birkhoff's theorem is very influential in how we understand the gravitational field around (approximately) isolated masses. It provides strong support for the relativistic extension of our Newtonian intuition that far from such objects their gravitational influence should become negligible, or, equivalently, spacetime should be asymptotically flat. Birkhoff's theorem also tells us that certain types of gravitational radiation (from a star that pulsates in a spherically symmetric fashion, for example) are not possible.

[1] This theorem is commonly attributed to Birkhoff, although it was already published two years earlier by Birkhoff and Langer (1923). It is not to be confused with Birkhoff's pointwise ergodic theorem.

Birkhoff's theorem does not hold in many alternative theories of gravity. We therefore have less justification, aside from our own intuition, in treating the weak field limit of these theories as perturbations about Minkowski space. We must instead be more careful, as the spacetime we perform our expansion around can have asymptotic curvature, leading to either time or space-dependence of the background (or some combination of the two). What is more, the perturbations themselves may be time-dependent, and their form can be sensitive to the type of asymptotic curvature that the background exhibits. Behaviours such as these are not expected in General Relativity (Lue and Starkman (2004)).

However, it has been shown recently that a *Birkhoff-like theorem* does exist in these theories (Nzioki *et al.* (2014)), that states the following:

Theorem 4.2.1. *(Birkhoff-like theorem)*
For $f(R)$ gravity, where the function f is of class C^3 at $R = 0$, with $f(0) = 0$ and $f'_0 \neq 0$, the only spherically symmetric solution with vanishing Ricci scalar in empty space in an open set $\mathcal{S}$, is one that is locally equivalent to part of maximally extended Schwarzschild solution in $\mathcal{S}$.

The stability of this local theorem in the perturbed case has been formulated as:

Theorem 4.2.2. *For $f(R)$ gravity, where the function f is of class C^3 at $R = 0$, with $f(0) = 0$ and $f'_0 \neq 0$, any almost spherically symmetric solution with almost vanishing Ricci scalar in empty space in an open set $\mathcal{S}$, is locally almost equivalent to part of maximally extended Schwarzschild solution in $\mathcal{S}$.*

The important point to note here is that the size of the open set $\mathcal{S}$ depends on the parameters of the theory (namely the quantity $f''(0)$) and the Schwarzschild mass) and they can be always tuned such that the perturbations continue to remain small for a time period which is greater than the age of the universe. This clearly indicates that the local spacetime around almost spherical stars will be stable in the regime of linear perturbations in these modified gravity theories. A more direct perturbative analysis of Schwarzschild black holes in $f(R)$ gravity (Myung *et al.* (2011)), does establish the stability in a more rigorous way.

4.3 Linear perturbation of Schwarzschild black hole in $f(R)$ gravity

In general relativity, the two fundamental second order wave equations that govern the gravitational perturbations of the Schwarzschild black holes are the Regge Wheeler equation Regge and Wheeler (1957) and the Zerilli equation Zerilli (1970). The former equation describes the odd perturbations and the latter the even perturbations. Both equations satisfy a Schrödinger-like equation and the effective potential of these equations is shown to have the same spectra (Chandrasekhar and Detweiler (1975)). These waves are tensorial, and are sourced by small deviation from the spherical symmetry of the Schwarzschild black hole in vacuum.

For $f(R)$ gravity, we can easily see from the almost *Birkhoff-like theorem* stated in the previous section that there can be two types of perturbations. The first is the tensor perturbation driven by small departure from the spherical symmetry (like GR), whereas the second one is the scalar perturbation that is sourced by perturbations in the Ricci scalar, which vanishes in the unperturbed background. This is an extra mode, that is generated by the extra scalar degree of freedom in these theories and is absent in GR. The detection of these modes are of a crucial importance in asserting the validity or otherwise of GR as the theory of gravity. We will now briefly discuss the wave equations that govern these two different kind of perturbations in $f(R)$ gravity.

4.3.1 Tensor perturbations

It has been shown (Nzioki *et al.* (2017a)) that in $f(R)$ gravity, one can construct a transverse, traceless gauge independent 2 dimensional tensor M_{ab} which can be harmonically decomposed and obeys equation (4.1):

$$\kappa^2 M - \frac{2m}{r^2}\left[\frac{2m-r}{r}\right]\frac{dM}{dr} + \left(\frac{2m-r}{r}\right)^2 \frac{d^2M}{dr^2} + \left(\frac{2m-r}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6m}{r^3}\right] M = 0. \tag{4.1}$$

We then make a change to the tortoise coordinate r_* which is related to the usual radial coordinate r as shown in (3.3) and so we can write (4.1) in the form

$$\left(\frac{d^2}{dr_*^2} + \kappa^2 - V_T\right) M_T = 0 \ , \tag{4.2}$$

with the effective potential V_T given by

$$V_T = \left(1 - \frac{2m}{r}\right)\left[\frac{\ell\,(\ell+1)}{r^2} - \frac{6m}{r^3}\right] \ , \tag{4.3}$$

and we have factored out the harmonic time dependence part of M_T, which is $\exp(i\kappa t)$. As V_T is the *Regge - Wheeler potential* for *gravitational perturbations* this clearly indicates that the tensorial modes of the gravitational perturbations in $f(R)$ gravity have the same spectrum as in GR, and hence observationally it is impossible to differentiate between the two through these modes.

4.3.2 Perturbations of Ricci scalar

Taking the trace of the equation (2.13) in vacuum we get

$$3\Box f' + Rf' - 2f = 0 \ , \tag{4.4}$$

which is a wave equation in terms of the Ricci scalar R associated with scalar modes. These modes are not present in GR as can be seen by substituting $f(R) = R$ in the above equation, which gives $R = 0$. Hence in vacuum spacetimes in GR there cannot be any perturbations in the Ricci scalar. However, this is possible in $f(R)$ gravity.

Necessary condition for existence of solutions with vanishing Ricci scalar

The function f must be of class C^3 at $R = 0$, which implies,

$$|f'(0)| < +1, \quad |f''(0)| < +1, \quad |f'''(0)| < +1. \tag{4.5}$$

Also, we impose the conditions

$$f(0) = 0, \quad R = 0. \tag{4.6}$$

Now there are two possibilities:

1. $f'(0) \neq 0$: Solving for the metric using the definition of the geometrical quantities we get (Clarkson *et al.* (2004b)) the metric of a Schwarzschild exterior

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2. \tag{4.7}$$

   In this case it is also interesting to note that the above result is consistent with the conditions $f' > 0$ and $f'' > 0$, which guarantee the attractive nature of the gravitational interaction and the absence of tachyons (Starobinsky (2007a)). This shows that there may be a connection between this solution and the very nature of the gravitational interaction.

   The presence of this solution can have interesting consequences on the validity of these models on the Solar System level. In particular if one concludes that the Sun behaves very close to a Schwarzschild solution, the experimental data of the solar system would help to constrain these models.

2. $f'(0) = 0$, $f(0) = 0$: In this case, for all models with $f'(0) = 0$, any solution with vanishing Ricci Scalar in GR would be a solution to the above system. This is interesting as it shows that fourth order gravity in this context can present the same solutions of GR plus additional solutions.

The presence of solutions of the type in paragraph (2) shows that when the conditions given in paragraph (1) are not satisfied the Schwarzschild solution is not a unique static spherically symmetric solution. Such results hint towards a violation of the general Jebsen-Birkhoff's Theorem in its classical form for fourth order gravity.

Necessary condition for existence of solutions with constant scalar curvature

Solutions with constant Ricci scalar are characterised by the fact that $R = R_0 = const$. A first solution exists if

$$f_0' \neq 0, \quad f_0 \neq 0, \quad 2f_0 - R_0 f_0' = 0. \tag{4.8}$$

If we take instead $f_0' \neq 0$, $f_0 = 0$ one obtains again the Schwarzschild solution ($R_0 = 0$). Finally another solution can be achieved if

$$f_0' = 0, \quad f_0 = 0, \quad R = R_0, \tag{4.9}$$

is satisfied. In this case also, any constant Ricci scalar solution in GR would identically be a solution of the system. The relation (4.9) was already found by Barrow and Ottewill (1983) in the cosmological context and later rediscovered in Clifton and Barrow (2006). It relates the value of the constant Ricci scalar with the universal constants in the action. For example if we have the Lagrangian as $R - 2\Lambda$, which is the Lagrangian for GR with the cosmological constant, we must have, the relation $R_0 = 4\Lambda$.

Now we can Taylor expand the function f around $R = 0$ using $f(0) = 0$ to get

$$f(R) = f_0' R + \frac{f_0''}{2} R^2 + \dots \,. \tag{4.10}$$

Using the tortoise coordinates, rescaling $R = r^{-1}\mathcal{R}$, and factoring out the time dependence part $\exp(i\kappa t)$ from $\mathcal{R}$ we get,

$$\left(\frac{d^2}{dr_*^2} + \kappa^2 - V_S\right)\mathcal{R} = 0, \tag{4.11}$$

where

$$V_S = \left(1 - \frac{2m}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2m}{r^3} + U^2\right] \,, \tag{4.12}$$

is the Regge Wheeler potential for massive scalar perturbations on LRS background spacetimes in GR with

$$U^2 = \frac{f_0'}{3f_0''}. \tag{4.13}$$

as the effective mass of the scalar.

Potential profile

The form of the wave equations (4.11) describing BH perturbation is similar to a one dimensional Schrödinger equation and hence the potential corresponds to a single potential barrier. This equation can be made dimensionless by multiplying through with the square of the black hole mass m. In this way the potential (4.12) becomes

$$V_S = \left(1 - \frac{2}{r}\right)\left[\frac{\ell\,(\ell+1)}{r^2} + \frac{2}{r^3} + u^2\right] , \tag{4.14}$$

where we have defined (and dropped the tildes),

$$\tilde{r} = \frac{r}{m} , \qquad \tilde{u} = m\,U , \qquad \tilde{\kappa} = m\kappa . \tag{4.15}$$

For scalar perturbations with $u = 0$, the derivative of the potential has two extrema, one in the unphysical region $r < 0$ and the other in $r > 0$ corresponding to the maximum of the potential. In the case of the scalar perturbations with $u \neq 0$, for a certain range of u, the potential has three extrema: one in the unphysical region $r < 0$, a local maximum at r_{max} and local minimum at r_{min} in the region $r > 0$ such that $2 < r_{max} < r_{min}$.

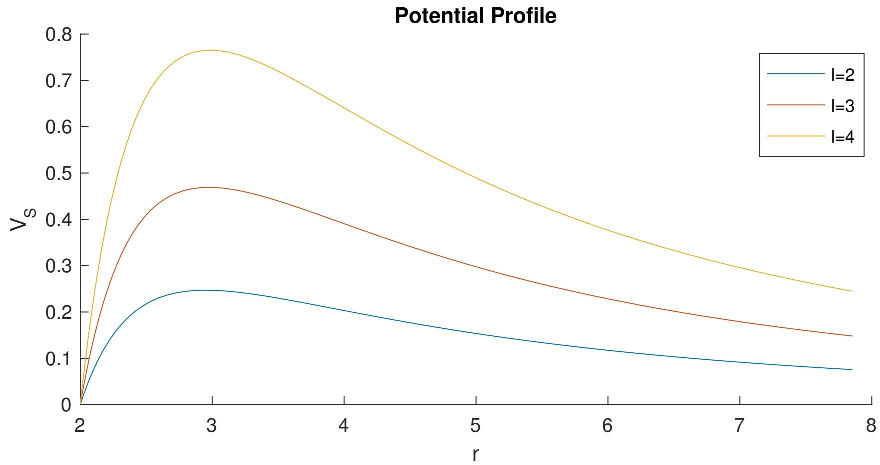

Figure 4.1: The potential profile for the scalar field for $l = 2, 3, 4$ as a function of r.

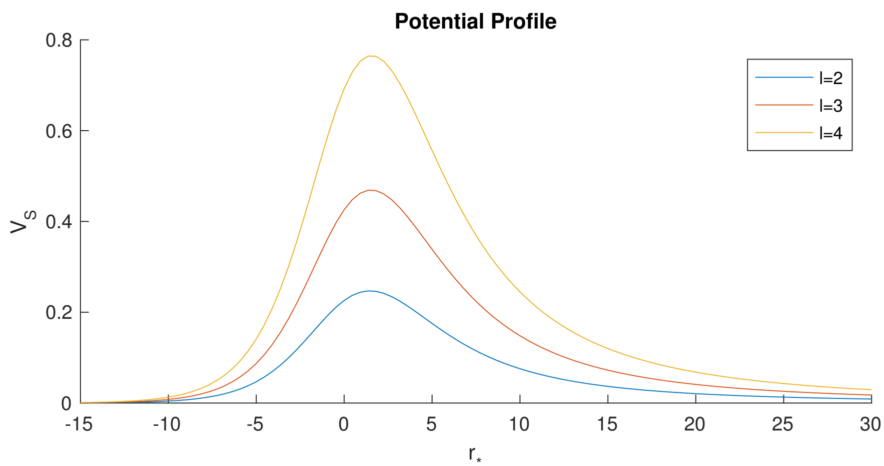

Figure 4.2: The potential profile for the scalar field for $l = 2, 3, 4$ as a function of r_*.

The potential decays exponentially near the horizon and is $\frac{1}{r^2}$ at spatial infinity. Figure 4.1 and Figure 4.2, respectively, show a plot of the potential for the scalar field for different l as a function of the Schwarzschild radial coordinate r and the tortoise coordinates r_*.

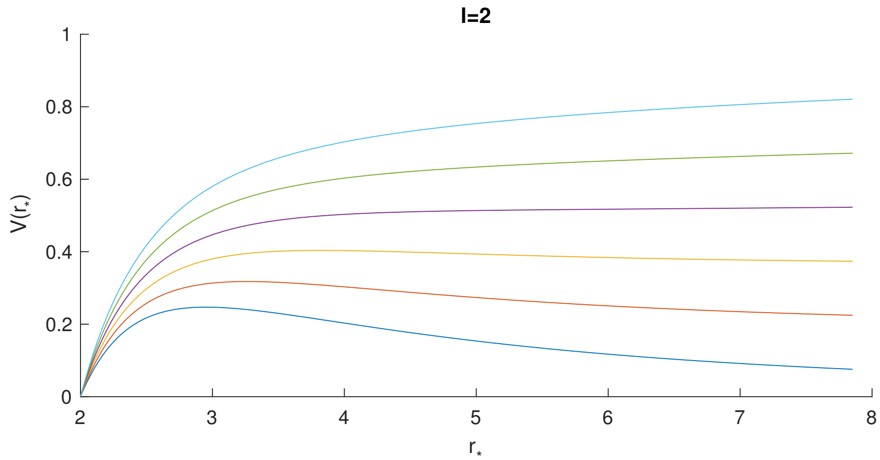

Figure 4.3: The potential profile for the scalar field for different u as a function of r_* for $l = 2$ and $l = 3$, respectively.

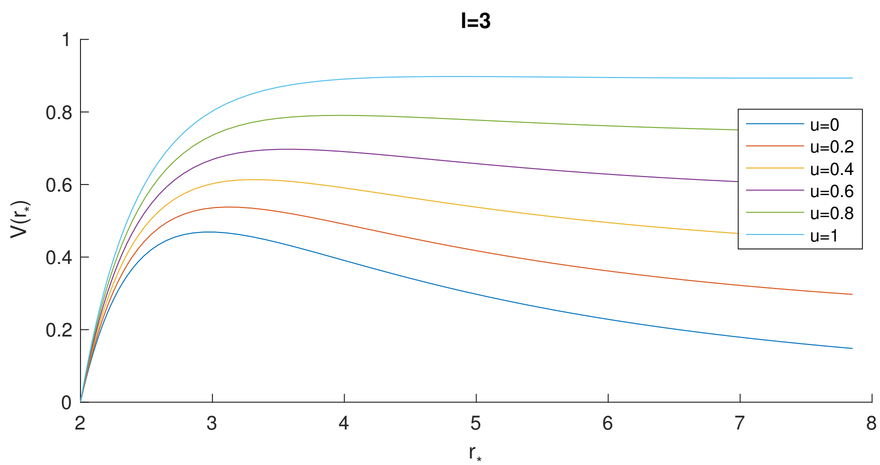

Figure 4.4: The potential profile for the scalar field for different u as a function of r_* for $l = 2$ and $l = 3$, respectively.

Figures 4.3 and 4.4 show the potential profile for the scalar field for several values of u at $l = 2$ and $l = 3$, respectively. We observe that the effect of the massive term $\tilde{U}$ is to shift the asymptotic value of the potential of scalar perturbations up by u^2 and to cause the potential to approach the asymptotic value slowly. Over and above that, increasing the value of u causes the peak of the potential to broaden as the peak value decreases relative to the asymptotic value.

4.4 Infra-red cutoff for incoming waves of disturbance of Ricci scalar

Let us now look at the equation governing the Ricci scalar perturbations (4.11) and the form of the potential (4.12), to study the limiting behaviour of the waves generated by these perturbations. This will help us specify the physically realistic boundary conditions. At $r_* \to -\infty$, (which implies the horizon at $r = 2$), we have $V_S = 0$, and equation (4.11) becomes

$$\left(\frac{d^2}{dr_*^2} + \kappa^2 \right) \mathcal{R} = 0 \; , \tag{4.16}$$

which is a usual harmonic equation with two linearly independent solutions

$$\mathcal{R} \sim C_1 \exp\left(i\kappa r_*\right) + C_2 \exp\left(-i\kappa r_*\right) \, . \tag{4.17}$$

Since we do not have any *outgoing* mode at the horizon, this implies $C_2 = 0$. On the other hand, at $r_* = +\infty$, equation (4.11) becomes

$$\left(\frac{d^2}{dr_*^2} + \kappa^2 - u^2 \right) \mathcal{R} = 0, \tag{4.18}$$

with

$$\mathcal{R} \sim C_3 \exp\left(i\sqrt{\kappa^2 - u^2} r_*\right) + C_4 \exp\left(-i\sqrt{\kappa^2 - u^2} r_*\right) \, . \tag{4.19}$$

At this point, we come to a very important proposition which we state as follows:

Proposition 4.4.1. *The parameters of the theory in $f(R)$ gravity provide a cut-off for long wavelength spherical incoming Ricci scalar waves from infinity.*

Proof. When $u^2 > \kappa^2$, we can immediately see for the incoming modes,

$$\lim_{r_* \to \infty} \mathcal{R}_{in} = C_3 \exp\left(-\sqrt{-\kappa^2 + u^2} r_*\right) \to 0. \tag{4.20}$$

Hence, there are no incoming scalar waves at $r_* \to \infty$ for $\kappa < u$. □

As we are interested in the scattering of incoming Ricci scalar waves from infinity by the black hole potential barrier, in the following sections we choose the parameters of the theory, such that $u^2 << \kappa^2$. Hence for all practical purposes we have $\kappa' \equiv \sqrt{\kappa^2 - u^2} = \kappa$.

4.5 Numerical solution

Given a Volterra Integral Equation of the second kind (3.24), which is of the form

$$u(x) = f(x) + \lambda \int_a^x K(x,y)u(t)dt, \tag{4.21}$$

we divide the interval of integration (a, x) into n equal subintervals, $\Delta t = \frac{x_n - a}{n}$, where $n \geq 1$ and $x_n = n$. Also let $y_0 = a$, $x_0 = t_0$, $x_n = t_n = x$, $t_j = a + j\Delta t = t_0 + j\Delta t$, $x_0 + i\Delta t = a + i\Delta t = t_i$. Using the trapezoid rule[2], for simplicity, the integral can now be written as

$$\begin{aligned}
&\int_a^x K(x,t)u(t)dt \\
&\approx \Delta t \left[\frac{1}{2}K(x,t_0)u(t_0) + K(x,t_1)u(t_1) + \ldots\right. \\
&\left. + K(x,t_{n-1})u(t_{(n-1)}) + \frac{1}{2}K(x,t_n)u(t_n)\right],
\end{aligned} \tag{4.22}$$

where $\Delta t = \frac{t_j - a}{j} = \frac{x-a}{n}$, $t_j \leq x$, $j \geq 1$, $x = x_n = t_n$.

Using the above, equation (4.21) can be discretised as

$$\begin{aligned}
u(x) = f(x) + \lambda\Delta t &\left[\frac{1}{2}K(x,t_0)u(t_0) + K(x,t_1)u(t_1) + \ldots\right. \\
&\left. + K(x,t_{n-1})u(t_{(n-1)}) + \frac{1}{2}K(x,t_n)u(t_n)\right] .
\end{aligned} \tag{4.23}$$

Since $K(x,t) \equiv 0$ when $t > x$ (the upper limit of the integration ends at $t = x$), then $K(x_i, t_j) = 0$ for $t_j > x_i$. Numerically, equation (4.23) becomes

$$\begin{aligned}
u(x_i) = f(x_i) + \lambda\Delta t &\left[\frac{1}{2}K(x_i,t_0)u(t_0) + K(x_i,t_1)u(t_1)\right. \\
&\left. + \ldots + K(x_i,t_{j-1})u(t_{(j-1)}) + \frac{1}{2}K(x_i,t_j)u(t_j)\right],
\end{aligned} \tag{4.24}$$

[2]Linz (1971) has shown that should we use better methods for numerical integration, we will get more accurate results.

where $i = 1, 2, \ldots, n \quad t_j \leq x_i$ and $u(x_0) = f(x_0)$. Denoting $u_i = u(x_i)$, $f_i = f(x_i)$ and $K_{ij} = K(x_i, t_j)$, we can write the numeric equation in a simpler form as

$$\begin{aligned} u_0 &= f_0 \\ u_i &= f_i + \lambda \Delta t \left[\frac{1}{2} K_{i0} u_0 + K_{i1} u_1 + \ldots + K_{i(j-1)} u_{j-1} + \frac{1}{2} K_{ij} u_j \right], \end{aligned} \tag{4.25}$$

with $i = 1, 2, \ldots, n$ and $j \leq i$. Therefore there are $n+1$ linear equations

$$\begin{aligned} u_0 &= f_0 \\ u_1 &= f_1 + \lambda \Delta t \left[\frac{1}{2} K_{10} u_0 + K_{11} u_1 \right] \\ u_2 &= f_2 + \lambda \Delta t \left[\frac{1}{2} K_{20} u_0 + K_{21} u_1 + \frac{1}{2} K_{22} u_2 \right] \\ &\vdots \qquad \vdots \\ u_n &= f_n + \lambda \Delta t \left[\frac{1}{2} K_{n0} u_0 + K_{n1} u_1 + \ldots \right. \\ &\qquad \left. + K_{n(n-1)} u_{n-1} + \frac{1}{2} K_{nn} u_n \right] . \end{aligned} \tag{4.26}$$

Hence a general equation can be written in compact form as

$$u_i = \frac{f_i + \lambda \Delta t \left[\frac{1}{2} K_{i0} u_0 + K_{i1} u_1 + \ldots + K_{i(i-1)} u_{i-1} \right]}{1 - \frac{\lambda \Delta t}{2} K_{ii}} \tag{4.27}$$

and can be evaluated by substituting $u_0, u_1, \ldots, u_{i-1}$ recursively from previous calculations. A MATLAB code was written to evaluate this system of linear equations for (3.24) and the results were used to evaluate the reflexion and transmission coefficients by coding the numerical solution for $m_2(x, \kappa)$ with the potential (4.12) for different values of u.

κ	$\boldsymbol{R}$
0.10	1.0000
0.20	0.9991
0.30	0.9945
0.32	0.8895
0.34	0.7929
0.36	0.6491
0.40	0.3102
0.50	0.0154

Table 4.1: The *reflection amplitude* ($\boldsymbol{R}$) of gravitational waves for $l = 2$, for various frequencies (κ) as calculated in Chandrasekhar (1983)

	$\boldsymbol{R}$		
κ	$u = 0$	$u = 0.001$	$u = 0.01$
0.10	0.67452	0.67452	0.67495
0.20	0.090579	0.090584	0.91000
0.30	0.0063914	0.0063916	0.0064152
0.32	0.0037518	0.0037520	0.0037651
0.34	0.0022053	0.0022051	0.0021880
0.36	0.0012993	0.0012994	0.0013105
0.40	4.5259e-04	4.5254e-04	4.483e-04
0.50	3.3182e-05	3.3173e-05	3.2280e-05

Table 4.2: The *reflection amplitude* ($\boldsymbol{R}$), where $l = 0$, for various frequencies (κ) and for different values of u.

	$\boldsymbol{R}$		
κ	$u = 0$	$u = 0.001$	$u = 0.01$
0.10	0.99951	0.99951	0.99952
0.20	0.96450	0.96450	0.96462
0.30	0.46717	0.46718	0.46746
0.32	0.31583	0.31583	0.31611
0.34	0.19749	0.19750	0.19785
0.36	0.11622	0.11622	0.11627
0.40	0.037427	0.037428	0.037515
0.50	0.0020066	0.0020066	0.0020092

Table 4.3: The *reflection amplitude* ($\boldsymbol{R}$), where $l = 1$, for various frequencies (κ) and for different values of u.

	$\boldsymbol{R}$		
κ	$u = 0$	$u = 0.001$	$u = 0.01$
0.10	1.0000	1.0000	1.0000
0.20	0.9995	1.0000	1.000
0.30	0.9690	0.9989	0.9991
0.32	0.9382	0.9974	0.9980
0.34	0.8837	0.9946	0.9955
0.36	0.7920	0.9886	0.9903
0.40	0.5441	0.9698	0.9589
0.50	-	0.5028	0.5028

Table 4.4: The *reflection amplitude* ($\boldsymbol{R}$), where $l = 2$, for various frequencies (κ) and for different values of u.

4.6 Results of the Reflection of Ricci scalar perturbations

It is well known (Chandrasekhar (1983)) that the solution to the Volterra integral equation (3.24) is analytic in the lower half of the complex κ-plane and is continuous for $\Im(\kappa) \leq 0$. In this case, the solution obtained by repeated iterations always converges and $m_2(r_*, \kappa)$ can be expanded as a power series in $1/\kappa$. Following from (3.24), these facts indicate the following:

$$\begin{aligned} m_2(r_*, \kappa) = &- e^{2i\kappa r_*} \frac{1}{2i\kappa} \int_{-\infty}^{r_*} e^{2i\kappa r'_*} V_S(r'_*) m_2(r'_*, \kappa) dr' \\ &+ \frac{1}{2i\kappa} \int_{-\infty}^{r_*} V_S(r'_*) m_2(r'_*, \kappa) dr'_* + 1. \end{aligned} \tag{4.28}$$

For $r_* \to \infty$,

$$\begin{aligned} m_2(r_*, \kappa) =& 1 - e^{-2i\kappa r_*} \frac{1}{2i\kappa} \int_{-\infty}^{+\infty} e^{2i\kappa r'_*} V_S m_2 dr'_* \\ &+ \frac{1}{2i\kappa} \int_{-\infty}^{+\infty} V_S m_2 dr'_* + o(1). \end{aligned} \tag{4.29}$$

Comparing the above result with equation (3.15) immediately gives the relation between reflexion and transmission coefficients and the Jost function as

$$\frac{R_1(\kappa)}{T(\kappa)} = -\frac{1}{2i\kappa} \int_{-\infty}^{+\infty} e^{2i\kappa r'_*} V_S m_2 dr'_*, \tag{4.30}$$

$$\frac{1}{T(\kappa)} = 1 + \frac{1}{2i\kappa} \int_{-\infty}^{+\infty} V_s m_2 dr'_*. \tag{4.31}$$

From the above expression, the following conservation of flux condition can be verified easily:

$$\boldsymbol{R} + \boldsymbol{T} \equiv |R_1|^2 + |T|^2 = 1. \tag{4.32}$$

Thus the part of the incident wave that is not absorbed by the BH is reflected back to infinity. From equation (4.32) it follows that

$$\boldsymbol{R} \leq 1 \tag{4.33}$$

and

$$\boldsymbol{T} \leq 1. \tag{4.34}$$

The reflection wave amplitude $\boldsymbol{R}$ for various frequencies and for different values of l and u are summarised in Table 4.2 - 4.4.

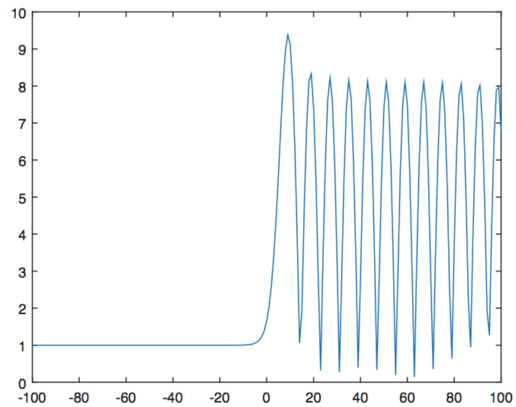

Figure 4.5: Jost function for $l = 2; u = 0.001$

In Figure 4.5, we have plotted the nature of the Jost function $m_2(r'_*, \kappa)$.

4.7 Discussion

From this analysis we find a few interesting insights, which are as follows:

1. First of all, the Ricci waves has $l = 0, 1$ modes, which are absent for the gravitational waves. It is interesting to note that for the monopole term ($l = 0$ mode) the reflection coefficients are quite less than those with higher values of l for all wavelengths and for all values of the parameter u. This shows that a large fraction of monopole modes gets transmitted through the black hole potential barrier.

2. This analysis also provides a nice observational template to constraint the parameters of the higher order gravity theory. Assuming in the near future we will have an interferometer to detect scalar waves that are backscattered from an

astrophysical black hole, we can in principle constrain the parameter u through the observation of the amplitude of these waves. We recall that the parameter u is linked to the parameters of the theory as $u^2 = m\sqrt{\frac{f_0'}{3f_0''}}$, where m is the black hole mass.

3. If we compare the reflection coefficients of the tensor waves for $l = 2$ in GR from Chandrasekhar (1983) (tabulated in Table 4.1), which will be the same in $f(R)$ gravity, we see that for all wavelengths, larger fraction of the scalar waves get reflected (in comparison to tensor waves) from the black hole potential barrier. This may provide a novel observational signature for modified gravity or otherwise.

4. Furthermore from the Table 4.2-4.4 we can immediately see that for all values of l, as u increases, the tendency of reflection increases for long wavelength scalar waves. This trait continues until the infra-red cut off happens for a given frequency.

5. Also these calculations indicate that for $l = 2$, as u increases, the reflection wave amplitude attains a plateau near $\boldsymbol{R} = 1$ for long wavelengths that suddenly drop off for higher frequencies, which is not the case for tensor waves tabulated in Table 4.1.

We would like to emphasise here that these results are only applicable in the scenario where the frequency of the scalar waves are much larger than u (which is given by the parameters of the theory of gravity considered). An interesting limiting case happens when $\kappa \to u$. We can immediately see from the Ricci wave equation that the inner boundary condition at the black hole horizon remains unchanged, whereas for the outer boundary condition both ingoing and outgoing modes reach a non-oscillating constant value at spatial infinity, that can be rescaled to zero without any loss of generality. A detailed analysis of this limiting case was performed in Starobinskii (1973) for rotating Kerr black holes. A similar Jost function analysis as presented in this chapter with the modified outer boundary condition would replicate the results in this chapter for the special case of vanishing rotation. For $\kappa >> u$ there will be a completely different scenario in terms of localisation of the scalar waves, which will be reported in the next chapter.

Chapter 5

Evidence of higher order corrections to GR in strong gravity regime

In this chapter we show that the higher order corrections to GR in the strong gravity regime of near horizon scales produce a rapidly oscillating and infalling Ricci scalar fireball just outside the horizon. The advantage of this approach is two fold: firstly this result indicates the existence of a more general theory of gravity in the strong gravity regime, of which GR is a weak field approximation. Secondly, we do not need to invoke any exotic objects to explain the echoing effects during a black hole merger. In Abedi *et al.* (2016) the authors demonstrated, by building a phenomenological template for successive echoes from exotic quantum structures expected in firewall or fuzzball models or exotic compact objects (ECO's) (Cardoso *et al.* (2016d)), and after marginalizing over it's parameters, tentative evidence for these echoes were reported at 2.9σ.

5.1 Gravitational wave template for successive echoes

At the linear order, perturbed black holes are described by QNM's which satisfy the boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves at the horizon. The transition (from ingoing to outgoing) takes place continuously at the peak of the angular momentum potential barrier of the black hole. In this case, the ingoing modes of the ringdown reflect back from the membrane (e.g., fuzzball or firewall) near horizon and passes back through the potential barrier (Abedi *et al.* (2016)). Part of the wave goes to infinity with a time delay. We call this the 1st echo (see Figure 5.1). This time delay corresponds to twice the tortoise coordinate distance between the peak of the angular momentum barrier (r_{max}) and the membrane (which diverges logarithmically if the membrane approaches the horizon) (Cardoso *et al.* (2016b)). The remaining part of the 1st echo returns back towards the membrane and the process repeats itself. In spite of its simplicity, this picture is

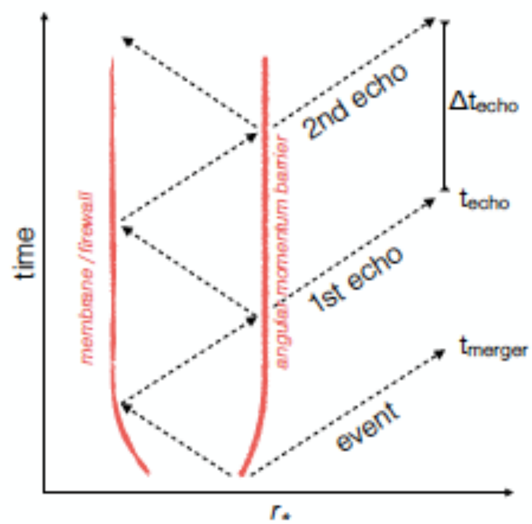

Figure 5.1: Spacetime representation of gravitational wave echoes from a firewall on the stretched horizon, following a black hole merger event.

quite robust. As first noticed in Cardoso *et al.* (2016b,c), introduction of structure near the event horizon leads to late, repeating, echoes of the ringdown phase of the black hole merger, due to waves trapped between the near-horizon structure and the angular momentum barrier (Figure 5.1). This is relatively insensitive to the nature of the structure, or how one defines the Planck length, l_p, as the time for reflection from the stretched horizon is only logarithmically dependent on its distance from the event horizon, i.e. $\Delta t_{echo} = 8M\log(M/l_p)$, where M is the black hole mass in Planck units. While t_{echo} is determined by linear physics, the time between the main merger event and the first echo could be further affected by non-linear physics during merger, i.e. $t_{echo} - t_{merger} = t_{echo} + O(M)$ (see Fig. 5.1), or equivalently:

$$\frac{t_{echo} - t_{merger}}{\Delta t_{echo}} = 1 + O(1\%) \tag{5.1}$$

where t_{echo} is predicted from the final (red shifted) mass and spin measurements for each event. Statistical evidence for these delayed echoes in LIGO events: GW150914, GW151226, and LVT151012 were reported at a combined significance of $2,9\sigma$ by Abbott *et al.* (2016a,b) and Abbott *et al.* (2016d).

The *ad hoc* nature of the echoe template construction is not entirely satisfactory and could lead to some ambiguity in interpreting the statistical significance of the finding. In particular, the fact that the combined signal to noise ratio (SNR) is maximized on the edge of the parameter range points to a need for a better physical prior on parameters, or simply a more physical echo template. This does not change the statistical significance of the SNR peaks, but suggests higher peaks may lie beyond this range. However, extending the analysis beyond this range requires analysing a much larger portion of LIGO data. These higher order corrections to GR are not *ad hoc* in nature, but would be expected from any attempt to create a re-normalisable theory of gravity like string theory.

5.2 Sources of echoes

At the outset it is important to note that there can be at least two distinct sources of echoes. One source is the spacetime itself, and more specifically the curvature potential through which waves propagate (Zerilli (1970), Cardoso *et al.* (2016a)). A second source of echoes is some sort of a "wall" that forms an inner boundary of the wave propagation problem, and that replaces the horizon as the boundary (Prescod-Weinstein *et al.* (2009)). These walls are typically associated with speculations, or specific models, of quantum effects.

For the remainder of the chapter, we transparently show that if we consider higher order curvature corrections to the general relativistic Lagrangian in the near horizon scales, this will produce a fireball of very high frequency fluctuations of the Ricci scalar near the horizon.

5.3 Higher order curvature corrections to General Relativity

In GR, the Einstein-Hilbert Lagrangian density of gravitational interaction is given as

$$\mathcal{L}_{EH} = \sqrt{-g}\,(R - 2\Lambda) \tag{5.2}$$

We can generalise the above Lagrangian density by adding the higher order curvature correction terms generated by the Riemann curvature tensor:

$$\begin{aligned} \mathcal{L}_g &= \sqrt{-g}\left[R - 2\Lambda + \alpha R^2 + \beta R_{ab}R^{ab} + \gamma R_{abcd}R^{abcd} \right. \\ &\quad \left. + \nu \varepsilon^{klmn} R_{klst} R^{st}{}_{mn} + \cdots \mathcal{O}(N)\right], \end{aligned} \tag{5.3}$$

where α, β, γ and ν are the coupling constants. In fact, as discussed earlier, in the quantum field picture the effects of re-normalisation are expected to add such terms to the Lagrangian in order to give a first approximation to some quantised theory of gravity (DeWitt (1967), Birrell and Davies (1982)). Keeping up to the quadratic terms and using the very well known results (DeWitt and Mullin (1966), Buchdahl (1970), Barth and Christensen (1983)),

$$(\delta/\delta g_{ab}) \int dV \left(R_{abcd}\, R^{abcd} - 4R_{ab}\, R^{ab} + R^2\right) = 0 \;, \tag{5.4}$$

$$(\delta/\delta g_{ab}) \int dV \,\varepsilon^{klmn} R_{klst}\, R^{st}{}_{mn} = 0 \;, \tag{5.5}$$

we can eliminate the Kretchman scalar term and $\epsilon^{iklm} R_{ikst}\, R^{st}{}_{lm}$ from the Lagrangian density. Also it has been shown that the theories that contain the square of Ricci tensor in the action, suffer from several instabilities like the Ostrogradsky instability (Ostrogradsky (1850)). Therefore the Lagrangian density, up to the quadratic order, of a stable gravitational theory will only contain the square of the Ricci scalar and the corresponding gravitational action can be written as

$$\mathcal{S} = \frac{1}{2} \int dV \sqrt{-g} \left(R + \alpha R^2\right) \tag{5.6}$$

A few exact static vacuum solutions are known for the *Starobinsky model* of $R + \alpha R^2$ gravity, for which there exists the following theorem (Whitt (1984), Mignemi and Wiltshire (1992)):

Theorem 5.3.1. *(Uniqueness theorem)*
For all functions f(R) which are of class C^3 at $R = 0$ and $f(0) = 0$ while $f'(0) \neq 0$, the only static spherically symmetric asymptotically flat solution with a regular horizon in these models is the Schwarzschild solution, provided that the coefficient of the R^2 term in the Lagrangian polynomial is positive.

Since we require $\alpha > 0$ to avoid ghosts in the theory and also require the solution to describe a well defined Black Hole with a regular horizon, the Schwarzschild solution is the only possible exact asymptotically flat exterior. This is a very well known result which follows the famous BH *no-scalar-hair theorems*. It states that the stationary BH solution is the same as those in general relativity, namely Schwarzschild for the non-rotating case. It was proved by Barrow and Ottewill (1983) and Sudarsky (1995) for a quintessence field with convex potential, which corresponds to the Starobinsky model in the Einstein frame.

5.3.1 Constraints on the coupling constant

As discussed in detail by Ganguly *et al.* (2014), solar system experiments as well as cosmological observations give a strong bound on the coupling constant α. Perhaps the strongest constraint in given by the latest dataset from Planck, which do not rule out the above curvature corrected gravitational theory (5.6) as a viable candidate for the early acceleration phase of the universe.

Let us first consider the experimental bound that comes from the solar system tests of the equivalence principle (LLR). For any chameleon theory with a scalar field (ϕ) we can define a thin-shell parameter ϵ (Khoury and Weltman (2004)), which for the earth gives

$$\epsilon \equiv \sqrt{6}\frac{\phi_\infty - \phi_\oplus}{M_{pl}\Phi_\oplus} < 2.2 \times 10^{-6}, \tag{5.7}$$

where (ϕ_∞, $\phi_\oplus$) are respectively the minimum of the effective potential at infinity and inside the planet and $\Phi_\oplus$ the Newtonian potential for the earth. Notice that the constraint on the post-Newtonian parameter γ gives $\epsilon < 2.3 \times 10^{-5}$. Using the value $\Phi_\oplus \simeq 7 \times 10^{-10}$, the previous bound translates into $\phi_\infty/M_{pl} < 10^{-15}$. The LLR bound leads to the result of Gannouji *et al.* (2012),

$$\left| f'\left(\frac{\rho_\infty}{M_{pl}^2}\right) - 1\right| < 10^{-15}. \tag{5.8}$$

For the Starobinsky model and with the density $\rho_\infty \simeq 10^{-24}\text{g cm}^{-3}$, equation (5.8) tells us that $\alpha < 10^{-15} M_{pl}^2/\rho_m$ which gives $\alpha < 10^{45}\text{eV}^{-2}$. But the tightest local constraint comes from the Eöt-Wash experiments, which use torsion balances. We know that a point mass has a Yukawa gravitational potential (see e.g. Stelle (1978)),

$$V(r) = \frac{GM}{r}\left(1 + \frac{1}{3}e^{-r/\sqrt{6\alpha}}\right), \tag{5.9}$$

which gives (Kapner *et al.* (2007)) $\alpha < 4 \times 10^4\text{eV}^{-2}$. Notice that according to the bound from Big Bang nucleosynthesis and CMB physics, we have $\alpha << 10^{35}\text{eV}^{-2}$ (Zhang (2007)). We turn now to the inflation, and according to the latest dataset from Planck, the Starobinsky model is a viable candidate for the early acceleration phase of the Universe. We have from Starobinsky (1981) and Starobinsky (2007b) that $\alpha \simeq 10^{-45}(N/50)^2\text{eV}^{-2}$ where N is the number of e-folds. Notice that it may not be compatible with the classicality condition of the field (Gannouji *et al.* (2012), Upadhye *et al.* (2012)). Hence for all practical purposes we will consider $\alpha \simeq 10^{-45}\text{eV}^{-2}$ from the cosmological constraints or $\alpha < 4 \times 10^4\text{eV}^2$ from the laboratory tests. It is evident that for such a small value of coupling constant GR should be the best fitted theory in the weak gravity regime.

5.3.2 Curvature corrected field equations in vacuum

Varying the action (5.6) with respect to the metric g_{ab} over a 4-volume in vacuum yields

$$\begin{aligned} \delta\mathcal{S} &= -(1/2)\int dV\,\sqrt{-g}\,\big\{(1/2)\left(R+\alpha R^2\right)(\,g_{ab}\,\delta g^{ab} \\ &\quad -(1+2\alpha R)\;\delta R\,\delta g^{ab}\big\}\;, \end{aligned} \tag{5.10}$$

Since $R = g^{ab}\,R_{ab}$ and the connection is the Levi-Civita one, we can write

$$\begin{aligned} (1+2\alpha R)\;\delta R &\simeq \delta g^{ab}\,[(1+2\alpha R)\;R_{ab} + 2\alpha g_{ab}\,\Box R \\ &\quad -2\alpha\nabla_a\nabla_b R]\,, \end{aligned} \tag{5.11}$$

where the $\simeq$ sign denotes equality up to surface terms and $\Box \equiv \nabla_c\nabla^c$. By requiring that $\delta\mathcal{S} = 0$ with respect to variations in the metric, we finally get the required field equations:

$$\begin{aligned} (1+2\alpha R)\,G_{ab} &= -\frac{1}{2}g_{ab}\,\alpha R^2 + 2\alpha\nabla_a\nabla_b R \\ &\quad -2\alpha g_{ab}\,\Box R\,. \end{aligned} \tag{5.12}$$

Here G_{ab} is the Einstein tensor, and we can easily see that when $\alpha = 0$, we regain the Einstein field equations in vacuum. Taking the trace of the field equations above, we get

$$6\alpha\Box R - R = 0\;, \tag{5.13}$$

This is a non-trivial equation that determines the evolution of Ricci scalar in vacuum.

5.3.3 Comparison with GR

We would now like to highlight the key similarities and differences from GR, when we consider the curvature corrected field equations in vacuum.

- **Similarities**: From the field equations (5.12) it is evident that all the Ricci flat ($R = 0$) vacuum solutions of GR are solutions of the curvature corrected theory. This implies that in the background level the Schwarzschild or Kerr geometries remain a solution to this theory. Since these geometries encompass all the possible astrophysical black hole spacetimes, therefore there will be absolutely no difference in the properties of the black holes at the background level.

- **Differences**: The key difference arises when we consider small perturbations around these background geometries. In GR we know that the Ricci scalar has to vanish in vacuum. Hence any small geometrical perturbations of background geometry will not affect the Ricci scalar. However in the curvature corrected theory, because of the non-trivial trace equation (5.13), we see that there can be small perturbation of Ricci scalar around it's zero value in the background. This will then generate a Ricci scalar wave along with the usual tensor gravitational waves.

5.4 Ricci Wave fireball around perturbed black holes

For a more detail analysis of the Ricci wave phenomenon, let us consider a Schwarzschild black hole perturbed from it's usual background geometry (as one would expect just after the black hole merger). This will then perturb the Ricci scalar from it's zero background value and it's evolution will be governed by the trace equation (5.13). Seeking the solution of this equation of the form $R(r,t) \equiv e^{i\kappa t}R(r)$, and performing the usual harmonic decomposition for the d'Alembert operator in Schwarzschild geometry and using the tortoise coordinates $r*$, the trace equation takes the form (Nzioki *et al.*

(2017b))

$$\left(\frac{d^2}{dr_*^2} + \kappa^2 - V_S\right)\mathcal{R} = 0, \tag{5.14}$$

where we have rescaled $R = r^{-1}\mathcal{R}$, and

$$V_S = \left(1 - \frac{2m}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2m}{r^3} + \frac{1}{6\alpha}\right], \tag{5.15}$$

is the Regge-Wheeler potential for the Ricci scalar perturbations, with m being the black hole mass. The form of the wave equation (5.14) is similar to a one dimensional Schrödinger equation and hence the potential correspond to a single potential barrier. This equation can be made dimensionless by multiplying through with the square of the black hole mass m. In this way the potential (5.15) becomes

$$V_S = \left(1 - \frac{2}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2}{r^3} + \frac{1}{6\alpha}\right], \tag{5.16}$$

where we have defined (and dropped the tildes),

$$\tilde{r} = \frac{r}{m}, \qquad \tilde{\alpha} = \frac{\alpha}{m}, \qquad \tilde{\kappa} = m\kappa . \tag{5.17}$$

It is interesting to note that the equation (5.14) is exactly the same as a massive scalar wave equation in Schwarzschild background with the scalar field mass $\mathcal{M} = \sqrt{\frac{1}{6\alpha}}$. Now let us look the property of this equation carefully. At the horizon ($r_* \to -\infty$), we have $V_S = 0$, and equation (5.14) becomes

$$\left(\frac{d^2}{dr_*^2} + \kappa^2\right)\mathcal{R} = 0 , \tag{5.18}$$

with two linearly independent solutions

$$\mathcal{R} \sim C_1 \exp\left(i\kappa r_*\right) + C_2 \exp\left(-i\kappa r_*\right) . \tag{5.19}$$

Since there cannot be any outgoing modes at the horizon, this implies $C_2 = 0$. At spatial infinity ($r_* = +\infty$), equation (5.14) becomes

$$\left(\frac{d^2}{dr_*^2} + \kappa^2 - \mathcal{M}^2\right)\mathcal{R} = 0. \tag{5.20}$$

The solution for ingoing modes is given as

$$\mathcal{R} \sim C_3 \exp\left(i\sqrt{\kappa^2 - \mathcal{M}^2}\, r_*\right). \tag{5.21}$$

Now since from the previous section we know that $\alpha << 1$, which means $\mathcal{M} >> 1$. Hence this problem reduces to the problem of in-falling massive scalar field into the black hole. Now for all frequencies $\kappa < \mathcal{M}$, from equation (5.21) we can immediately see that the solution goes to zero exponentially at the spatial infinity. We can now solve

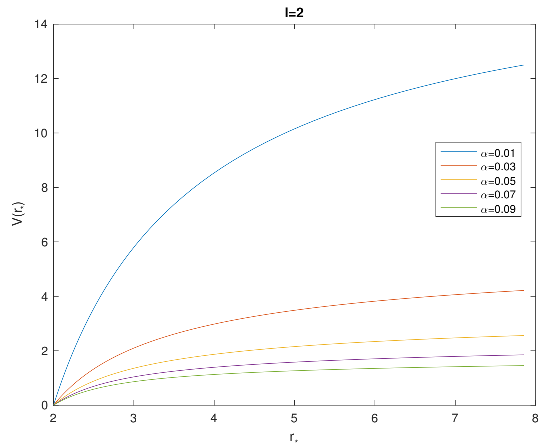

Figure 5.2: The effective scattering potential given by Equation (5.16) for $\alpha = 0.01, 0.03, 0.05, 0.07$ and 0.09

the Ricci wave equation (5.14) numerically, for a realistic black hole with $\mathcal{M} >> k$, using the following boundary conditions:

$$\mathcal{R} \sim 0 \quad at \quad (r_* = +\infty), \tag{5.22}$$

and

$$\mathcal{R} \sim e^{i\kappa r_*} \quad at \quad (r_* = -\infty). \tag{5.23}$$

In Figure **??** we have plotted the nature of the Ricci scalar perturbations around the black hole. It has an interesting behaviour: as $\mathcal{M}$ increases the Ricci scalar fluctuates with extremely high frequency near the horizon and rapidly dies down to zero value within $r_* = 0 \Rightarrow r \sim 2.5$. Thus we can conclude that a perturbed black hole in a curvature corrected theory is surrounded by rapidly oscillating and in falling Ricci scalar field just outside the horizon. Thus without invoking any quantum phenomenon we can get a massive scalar *fireball* surrounding the black hole horizon.

5.5 Quasinormal modes due to massive scalar accretion

In this section we established that a higher order correction to general relativity at the near horizon scales, gives rise to a rapidly oscillating Ricci scalar just outside the horizon, and it exactly behaves like an in falling massive scalar field.

5.5.1 Methods for computing quasinormal frequencies

There have been numerous attempts to calculate QNMs to high accuracy using numerical and semi-analytical methods. Difficulties arise from, for example, the admixture of the solutions such that the exponentially growing required solution gets contaminated by traces of the unwanted solution which decreases exponentially as we approach the boundaries. In 1975, Chandrasekhar and Detweiler (1975) computed numerically the first few modes and in 1985, Leaver (1985) proposed the most accurate method to date. We present here some of the methods that have been employed:

- Continued fraction method by Leaver (1985), which was later improved by Nollert (1999), to cater for quasinormal frequencies with very large imaginary parts. This is based on the observation that the Teukolsky equation is a special case of a class of spheroidal wave equations that appear in the determination of the eigenvalues of the H_2^+ ion. The quasinormal frequencies are calculated from the recurrence relations constructed for the coefficients of the series representation of the solutions of the equations governing the perturbations.

- Laplace transforms approach by Nollert (1999) where the QNMs are regarded as the poles of the Green's function for the Laplace transformed solution of the time-dependent equations governing the perturbations.

- The inverted BH effective potentials approach by Mashhoon (1983), Ferrari and Mashhoon (1984a,b). They provided an analytical approach to the problem by approximating the Regge-Wheeler potential in the wave equation governing the perturbations with other potentials. The parameters of these potentials are adjusted to obtain a good fit to the Regge -Wheeler potential near its maximum. This method doesn't allow for the determination of frequencies with large imaginary parts as these highly damped modes are more sensitive to changes in the potential far away from its maximum.

- WKB approach by Schutz, Will and Iyer (1987), Iyer and Will (1987) and Schutz and Will (1985). This semi-analytical procedure is based on reducing QNM problem into the standard JWKB treatment of scattering of waves on the peak of the potential barrier in quantum mechanics. It involves relating matching of the asymptotic WKB solutions at spatial infinity and the event horizon with the Taylor expansion near the top of the potential barrier across the turning points. A QNM is expected to have a frequency such that the square of the frequency is approximately equal to the peak of the potential. The method works best for modes with relatively small imaginary parts.

Other methods include the phase integral approach Fröman *et al.* (1992) and the monodromy technique Motl and Neitzke (2003).

5.5.2 Results on Scalar field quasinormal modes

For the scalar field perturbations, studies have shown that the mass of the field has crucial influence on the damping rate of the QNMs. Using the WKB approximation (Iyer *et al.* (1989), Simone and Will (1992), Konoplya (2002)), it was found that when the massive term u of the scalar field increases, the damping rate decreases. The WKB method that was used in this analysis is valid for $n < l$ and within this restriction, the approximation breaks down for large u. This is due to the potential losing its maximum as it drops relative to the asymptotic value (see Figure 5.3). The procedure requires modification (Gal'tsov and Matiukhin (1992)) to avoid this problem. Later calculations using Leaver's method showed that as a result of the decreasing damping rates,

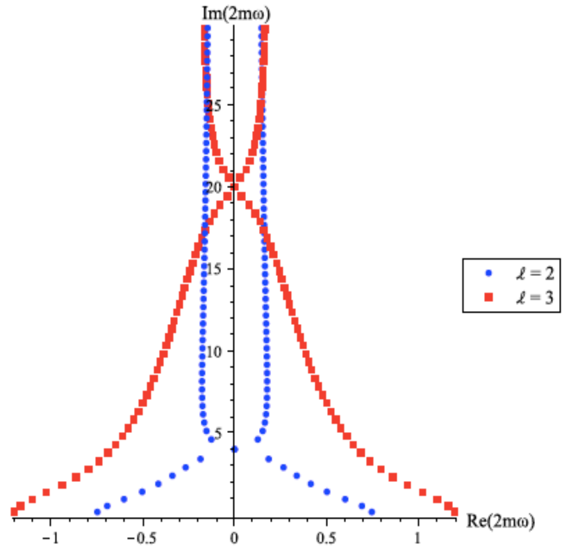

Figure 5.3: Plot of the spectrum of quasinormal modes for $l = 2$ and $l = 3$ for a Schwarzschild BH. [The modes were calculated using the method of Andersson and Linnæus (1992).]

for certain values of u, there are QNM oscillations with arbitrary long life (Ohashi and Sakagami (2004), Konoplya and Zhidenko (2005)). These "almost" purely real modes are called *quasiresonant modes*, a term originally coined by Ohashi and Sakagami (2004). It has also been found that there is a threshold value of u above which the QNMs may disappear, at least for the lower overtones only. The higher overtones will continue to decay with time (Konoplya and Zhidenko (2005)).

It is important to note that the massive term u affects the lower QNMs only as observed in Konoplya and Zhidenko (2005). They showed that for asymptotically high overtones ($n \to \infty$), the real part of the frequencies approaches the same asymptotical value $\ln 3(8\pi m)^{-1}$ as in the gravitational field case.

In GR the possible sources of massive scalar QNMs are from the collapse of objects made up of self-gravitating scalar fields ("boson" stars) (Colpi *et al.* (1986), Friedberg *et al.* (1987), Seidel and Suen (1991)), in situations where the massless field gains an effective mass (Konoplya and Fontana (2008)) or as scalar field dark matter (Cruz-Osorio *et al.* (2011)). In order to illustrate what these results mean for $f(R)$ theories of gravity we restrict our attention to the $l = 0$ multipole of the field. From Ohashi and Sakagami (2004), the cut-off mass at which the QNMs disappear for these modes is approximately at $m\tilde{U} = 0.4 - 0.5$ and from PPN constraints (Clifton (2008)) for these theories we obtain the bound for $\tilde{U}$ as

$$\tilde{U} = \frac{C_1}{3C_2} \gg \frac{2}{L^2}, \tag{5.24}$$

where L is the smallest length scale on which Newtonian gravity has been observed. Recent results (Geraci *et al.* (2008)) place at $L \approx 10\mu m$ and using this we can set (5.24) as

$$\tilde{U} \gg 1.4 \times 10^5 m^{-1}. \tag{5.25}$$

Given these details, we can estimate that the mass of the BH associated with the disappearance of the QNMs,

$$\text{BH} \quad \text{mass} \ll 4\mu m. \tag{5.26}$$

Such a BH could only have been formed from density fluctuations in the early universe (Hawking (1975), MacGibbon and Webber (1990)). If more of these primordial BH are to be detected now, they would have to have an initial mass of subatomic scales ($\sim 10^{-16} m$) (Hawking (1971)). These results apply to QNMs at lower overtones and even then, QNMs are short-ranged, making their detection currently unfeasible (Konoplya and Zhidenko (2011)).

Now our problem reduces to the following: *We have a Schwarzschild black hole with an infalling massive scalar test field in the exterior Schwarzschild geometry. We would like to know the nature of gravitational waves produced by the black hole which is perturbed due the the presence of this accreting massive test field.* Fortunately a very detailed analysis of the above problem has already been performed in Núñez *et al.*

(2011), and it generalized all the important earlier works (Burko and Khanna (2004), Koyama and Tomimatsu (2001)). The key findings of these analysis are as follows:

1. The gravitational wave generated by the infall of the massive scalar field has some unique features that differentiates it from those generated black hole mergers or by the infall of dust. The most interesting feature is that the ring-down part of the gravitational wave in case of massive scalar accretion has the same values of the quasinormal frequencies as those obtained in the case of a binary black hole collision. Hence just from the ring down part it is very hard to differentiate between these processes.

2. The above point is really interesting as it shows that although the frequency of the scalar field propagating on a Schwarzschild background is different from the one associated with the gravitational perturbation, the gravitational signal preserves its characteristic ring-down frequency. This is despite the fact that the scalar wave travels together with the gravitational one.

3. However the late time tails of the gravitational waves generated due to the infalling massive scalar field do differ from that of a binary black hole merger and this gives a nice observational test for differentiating these processes.

4. The amplitude of the emitted gravitational waves due to the massive scalar accretion increases as the mass increases.

Therefore we can safely claim that the infalling Ricci waves due to the curvature corrected theory will generate gravitational waves with the same natural frequency as the binary black hole merger and that is exactly what causes the echoes in the ringdown modes.

5.6 Discussion

In this chapter we provided a viable explanation to the *echoes* of the ringdown modes from the binary black hole mergers as detected by LIGO, without invoking any exotic structures near the black hole horizon. Inspired by the available renormalisable quantum gravity theories, we conjectured that there should be higher order curvature corrections to GR at the near horizon scales. We rigorously showed that these corrections produce rapidly oscillating and infalling Ricci scalar waves near the horizon that behaves exactly like accreting massive scalar field. As already known, the perturbed black holes due to this massive scalar accretion produces gravitational waves that has exactly the same natural characteristic ringdown frequency as those that are created by the binary black hole merger. It is exactly these waves that are detected as the *echoes from the abyss.*

Chapter 6

Conclusion

In this thesis we have explored several aspects of black hole physics, both from a theoretical and from an astrophysical point of view. Although black holes have many faces, some approaches of investigation turn out to be useful in fairly different areas of black hole physics. In particular, the study of black hole perturbations can provide insights on several topics including gauge/gravity duality, astrophysical imprints of strong curvature corrections to GR and possible methods to discriminate between astrophysical black holes and other ultra-compact objects. We discussed some theoretical and astrophysical aspects of this connection.

In Chapter 2 we introduced $f(R)$ theories of gravity and presented the general equations for these theories. We have explored all three versions of $f(R)$ gravity: metric, Palatini and metric-affine. In the Palatini formalism, there is an independent variation with respect to the metric and an independent connection. The action is formally the same but now the Riemann tensor and the Ricci tensor are constructed with the independent connection. Note that the matter action is assumed to depend only on the metric and the matter fields, and not on the independent connection. This assumption is crucial for the derivation of Einstein's equations from the linear version of the action and is the main feature of the Palatini formalism.

In Chapter 3 we studied potential scattering in one dimension and showed that the Schrödinger-like equation reduces to a VIE of the second kind which is analytic in the lower half plane.

In Chapter 4 a Jost function analysis of Ricci scalar perturbations was presented. This showed that for scenarios where the frequency of the scalar waves are much higher than u, for the monopole term, the reflection coefficients are quite less than those with higher values of l for all wavelengths and for all values of the parameter u. This shows that a large fraction of monopole modes gets transmitted through the black hole potential barrier. We observed that a larger fraction of the scalar waves get reflected (in comparison to tensor waves) from the black hole potential barrier.

In Chapter 5 we provided a viable explanation to the echoes of the ringdown modes from the binary black hole mergers as detected by LIGO, without invoking any exotic structures near the black hole horizon. Inspired by the available re-normalisable quantum gravity theories, we conjectured that there should be higher order curvature corrections to GR at the near horizon scales. We rigorously showed that these corrections produce rapidly oscillating and infalling Ricci scalar waves near the horizon that behaves exactly like an accreting massive scalar field. As already known, the perturbed black holes due to this massive scalar accretion produces gravitational waves that has exactly the same natural characteristic ringdown frequency as those that is created by the binary black hole merger. We concluded it is exactly these waves that are detected as the echoes from the abyss.

Final Experiential Comment

It is often said that mathematicians make horrible chefs. Of course I make an exception to the person reading this Thesis. However, writing this Thesis has been a rather peculiar experience. As the Mask of Zeus Dobie thinks to himself in page **1089** volume **30** of the Notices of the American Mathematical Society:

"It's funny... (but also sad), how many people imagine that Mathematics consists of interminably applying fixed formulae to clearly define problems and so 'working them out'. Because it's not like that at all. Half the time you don't even know what you are looking for until you've found it. A great deal more than half the time you spend looking at a blank sheet of paper and chewing the end of a pencil - the blunt end, hopefully - while you are trying to see what the bloody problem is. You know it is just there all right, but no, you can't grasp it, you can't quite perceive how to formulate it....mathematicians block..."

I am a true bystander to this rather unpleasant fact. It happened to me countless times throughout this journey. Many times I never knew what I was looking for until...

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