Mean-square fractional calculus and some applications

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by

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Abstract

The fractional calculus of deterministic functions is well known and widely used. Mean-square calculus is a calculus that is suitable for use when dealing with second-order stochastic processes. In this dissertation we explore the idea of extending the fractional calculus of deterministic functions to a mean-square setting. This exploration includes the development of some of the theoretical aspects of mean-square fractional calculus – such as definitions and properties – and the consideration of the application of mean square fractional calculus to fractional random differential and integral equations. The development of mean-square calculus follows closely that of the calculus of deterministic functions making mean square calculus more accessible to a large audience. Wherever possible, our development of mean-square fractional calculus is done in a similar manner to that of ordinary fractional calculus so as to make mean-square fractional calculus more accessible to people with some exposure to ordinary fractional calculus.

Keywords

Fractional calculus, mean-square calculus

Contents

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Chapter 1

Introduction

Fractional calculus is a branch of mathematics in which integrals and derivatives can be of non-integer order. Since its first appearance in the late 17^{th} century it has become popular (especially amongst mathematicians and engineers) because many problems are described by, and can be solved using, fractional calculus. It has been used in areas such as finance (Gorenflo et al. (2001)), viscoelasticity (Glöckle & Nonnenmacher (1991)), electromagnetism (Engheia (1997)), signal processing (Tseng (2001)), control theory (Podlubny (1999)) and in the biomedical field (Magin (2006)). Modelling real-life phenomena with mathematical equations involving differential and integral equations (of both integer and fractional order) may be limited by the fact that the uncertainties inherent in real-life may not be taken into account. As a result of this, random differential and integral equations have been used more and more over the past few decades. It therefore makes sense to develop a fractional calculus that takes into account the "randomness" of real situations. In this dissertation we aim to do this by translating the deterministic fractional calculus to a mean-square (m.s.) setting. The reason we choose to use a m.s. setting is that important information about a stochastic process (s.p.) can be found from its first and second moments. That m.s. calculus is a well-developed subject with methods that follow, in a general way, those of ordinary calculus only makes it more attractive.

There is currently a small body of work dedicated to m.s. fractional calculus (see Hafiz et al. (2001), Hafiz (2004) and El-Sayed et al. (2005)). We will use this body of work as a base from which to launch an exploration of the m.s. fractional calculus. In Chapter 2 we will give some important definitions and results from m.s. calculus. In Chapter 3 we will introduce several definitions for the m.s. fractional integral and derivative based on some of the common definitions from the deterministic fractional calculus. In Chapter 4 we will consider various properties of m.s. fractional integrals and derivatives – properties such as the m.s. continuity of the integrals and derivatives. A large portion of Chapter 4 is dedicated to finding expressions for the fractional integral (derivative) of the fractional derivative (integral) and similar expressions. In Chapter 5 we will find expressions for the fractional integral and derivative of the product of a deterministic function and a (second-order) stochastic process. Chapters 6 and 7 are dedicated to solving m.s. fractional integral and differential equations. In Chapter 6 we do this using the properties developed in Chapter 4 and in Chapter 7 we do this by introducing a transform of m.s. fractional integrals and derivatives.

Chapter 2

Background material

Throughout this thesis we will use the following terminology and notation:

$\mathbb{N}$	=	$\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	=	$\{0, 1, 2, \ldots\}$
$\mathbb{R}$	=	the set of all real numbers
$[a, b]$	=	the set of all real numbers between a and b

We will use "$\beta > 0$" to mean "all positive real values of β" and similarly "$\beta \in [a, b]$" to mean "all real values of β between a and b". We will also use "$\triangleq$" to mean "by definition, equals to" and "iff" to mean "if, and only if". Other notation and terminology in this dissertation is commonly used.

Mean-square calculus

A random variable, X, is called a second-order random variable if its second moment, $E[X^2]$, is finite. The norm of the second order random variable X is defined as follows:

$$\|X\| \triangleq \sqrt{E[X^2]} \, .$$

Consider a stochastic process (s.p.) with index set $T \subset \mathbb{R}$ for which $X(t_1), X(t_2), \ldots, X(t_m)$ are elements of L_2-space for every set $t_1, t_2, \ldots, t_m$. Such an s.p. is called a second-order s.p. and is characterized by

$$\|X(t)\|^2 = E[X(t)\ X(t)] = \Gamma_{XX}(t, t) < \infty \ , \quad t \in T \ .$$

Before introducing m.s. integrals and derivatives we must consider the concept of m.s. convergence. A sequence of random variables $\{X_m(t)\}$, $t \in T$, is said to converge in mean-square to a second-order s.p. $X(t)$ as $m \to \infty$, written $\underset{m\to\infty}{\text{l.i.m.}}\, X_m(t) = X(t)$, if

$$\lim_{m\to\infty} \|X_m(t) - X(t)\| = 0 \ .$$

Let m vary over some index set M and let n be a limit point of M. The sequence $\{X_m(t)\}$, $t \in T$, converges to a second order s.p. $X(t)$, $t \in T$, as $m \to n$

iff the functions $E[X_m(t)X_{\acute{m}}(t)]$ converge to a finite function on T as $m, \acute{m} \to n$ in any manner whatever. Then

$$\Gamma_{X_m X_m}(s,\theta) \ \rightarrow \ \Gamma_{XX}(s,\theta)$$

on $T \times T$. This is called the convergence in mean-square criterion. We note that the operators "E" and "l.i.m." commute (Loève (1955), sec. 34).

Mean-square integrals

Let p_n be a finite partition of the interval $[a,b]$ defined by the partition points $s_0, s_1, s_2, \ldots, s_n$ such that $a = s_0 < s_1 < s_2 < \ldots < s_n = b$. Letting s_k^* be an arbitrary point within the interval $[s_{k-1}, s_k]$ and letting

$$\Delta_n = \max_k (s_k - s_{k-1}) \ , \tag{2.1}$$

we have the following definition:

Definition 2.1. Let $X(t)$, $t \in T$, be a second-order stochastic process and let $f(t,s)$ be a deterministic function defined on $T \times T$. The mean-square Riemann integral of $f(t,s)X(s)$ over the interval $[a,b] \subset T$ is defined by

$$\int_a^b f(t,s)X(s)\,ds \ \triangleq \ \underset{\substack{n\to\infty \\ \Delta_n \to 0}}{\text{l.i.m.}} \sum_{k=1}^{n} f(t,s_k^*)X(s_k^*)(s_k - s_{k-1}) \tag{2.2}$$

if the limit in mean exists for all partitions p_n.

Using the convergence in m.s. criterion we see that the m.s. Riemann integral in (2.2) exists iff the following double Riemann integral exists and is finite:

$$\int_a^b \int_a^b f(t,s)f(t,u)\Gamma_{XX}(u,s)\,du\,ds \ . \tag{2.3}$$

(We have not shown the detail of how the convergence in m.s. criterion is used to get the above condition because next, and in Chapter 4, we will give these details for other m.s. integrals.) A stochastic process $X(t)$, $t \in T$, is said to be m.s. integrable on $[a,b] \subset T$ if $\int_a^b X(s)\,ds$ exists.

As with deterministic Riemann integrals, there are improper m.s. Riemann integrals. Suppose that $f(t,s)$ has a vertical asymptote at $s = b$. Then the m.s. Cauchy-Riemann integral of $f(t,s)X(s)$ over the interval $[a,b]$ is defined as

$$\text{CR -}\int_a^b f(t,u)\,X(u)\,du \ \triangleq \ \underset{h\to 0}{\text{l.i.m.}} \int_a^{b-h} f(t,s)\,X(s)\,ds \ , \quad h > 0 \tag{2.4}$$

provided the limit in mean exists. (For a discussion of the deterministic Cauchy-Riemann integral see Kestelman (1960).) Due to the convergence in m.s. criterion, in order to show that the right hand side does converge in mean-square we need only show that, for $h_1 > 0$ and $h_2 > 0$,

$$\lim_{h_1,h_2\to 0} E\left[\int_a^{b-h_1} f(t,u)\,X(u)\,du \cdot \int_a^{b-h_2} f(t,s)\,X(s)\,ds\right]$$

converges to a finite limit as h_1 and h_2 converge to zero in any manner whatever. If we let

- $a = u_0 < u_1 < u_2 < \ldots < u_n = b - h_1$,
- $u_k^* \in [u_{k-1}, u_k]$ for $k = \{1, 2, \ldots, n\}$
- $\Delta_n = \max_k(u_k - u_{k-1})$

and

- $a = s_0 < s_1 < s_2 < \ldots < s_m = b - h_2$
- $s_j^* \in [s_{j-1}, s_j]$ for $j = \{1, 2, \ldots, m\}$
- $\Delta_m = \max_j(s_j - s_{j-1})$,

then

$$\begin{aligned}
& \lim_{h_1,h_2\to 0} E\left[\int_a^{b-h_1} f(t,u)X(u)du \cdot \int_a^{b-h_2} f(t,s)X(s)ds\right] \\
= \;& \lim_{h_1,h_2\to 0} E\left[\underset{\substack{n,m\to\infty \\ \Delta_n,\Delta_m\to 0}}{\textbf{l.i.m.}} \sum_{k=1}^{n} f(t,u_k^*)X(u_k^*)\,(u_k - u_{k-1}) \sum_{j=1}^{m} f(t,s_j^*)X(s_j^*)\,(s_j - s_{j-1})\right] \\
= \;& \lim_{h_1,h_2\to 0} \lim_{\substack{n,m\to\infty \\ \Delta_n,\Delta_m\to 0}} \left[\sum_{k=1}^{n}\sum_{j=1}^{m} f(t,u_k^*)f(t,s_j^*)\,\Gamma_{XX}(u_k^*,s_j^*)\,(u_k - u_{k-1})(s_j - s_{j-1})\right] \\
= \;& \lim_{h_1,h_2\to 0} \int_a^{b-h_1}\int_a^{b-h_2} f(t,u)f(t,s)\,\Gamma_{XX}(u,s)\,du\,ds \\
= \;& \text{CR -} \int_a^b\int_a^b f(t,u)f(t,s)\,\Gamma_{XX}(u,s)\,du\,ds \;. \qquad (2.5)
\end{aligned}$$

So we see that the m.s. Cauchy-Riemann integral given in (2.4) will exist iff the deterministic double Cauchy-Riemann integral in (2.5) exists and is finite. Note that as with the deterministic case where the Cauchy-Riemann integral will equal the Riemann integral when the Riemann integral exists, the m.s. Cauchy-Riemann integral will equal the m.s. Riemann integral when the m.s. Riemann integral exists. *For convenience, in the remainder of this dissertation we will drop the "CR" when writing Cauchy-Riemann integrals.*

Soong (1973) shows that

$$\left|\left| \int_a^b X(t)\, dt \right|\right| \leq \int_a^b ||X(t)||\, dt$$

where $[a,b] \subset T$ and the s.p. $X(t)$ is m.s. continuous. Here we will try to show under what conditions – if any – the following holds:

$$\left|\left| \int_a^t f(t,s)\, X(s)\, ds \right|\right| \leq \int_a^t f(t,s)\, ||X(s)||\, ds \qquad (2.6)$$

where $X(t)$ is a second-order s.p. defined on T (not necessarily m.s. continuous), $t \in [a,b] \subset T$ and $f(t,s) = \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}$ for $\beta > 0$. The reason for interest in $f(t,s)$ of this form will become apparent in the following chapters. We note that $f(t,s)$ is not defined for $s = t$ when $\beta \in (0,1)$ so we will consider the $\beta \geq 1$ and $\beta \in (0,1)$ cases separately.

Let $\beta \geq 1$ and let

$$Y_n = \sum_{k=1}^{n} \frac{(t - s_k^*)^{\beta-1}}{\Gamma(\beta)}\, X(s_k^*)\, (s_k - s_{k-1})$$

where

$$\begin{aligned}
&\Delta_n = \max_k (s_k - s_{k-1})\ , \\
&a = s_0 < s_1 < \ldots < s_n = t \quad \text{and} \\
&s_k^* \in [s_{k-1}, s_k]\ .
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{\substack{n\to\infty \\ \Delta_n \to 0}} ||Y_n|| &= \lim_{\substack{n\to\infty \\ \Delta_n \to 0}} \left|\left| \sum_{k=1}^{n} \frac{(t - s_k^*)^{\beta-1}}{\Gamma(\beta)}\, X(s_k^*)\, (s_k - s_{k-1}) \right|\right| \\
&= \left|\left| \underset{\substack{n\to\infty \\ \Delta_n \to 0}}{\text{l.i.m.}} \sum_{k=1}^{n} \frac{(t - s_k^*)^{\beta-1}}{\Gamma(\beta)}\, X(s_k^*)\, (s_k - s_{k-1}) \right|\right| \\
&= \left|\left| \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\, X(s)\, ds \right|\right| \qquad (2.7)
\end{aligned}$$

provided that the integral inside the norm exists as a m.s. Riemann integral. Also,

$$\begin{aligned}
||Y_n|| &\leq \sum_{k=1}^{n} \left|\left| \frac{(t - s_k^*)^{\beta-1}}{\Gamma(\beta)}\, X(s_k^*)\, (s_k - s_{k-1}) \right|\right| \\
&= \sum_{k=1}^{n} \frac{(t - s_k^*)^{\beta-1}}{\Gamma(\beta)}\, ||X(s_k^*)||\, (s_k - s_{k-1})
\end{aligned}$$

so that

$$
\begin{aligned}
\lim_{\substack{n\to\infty\\ \Delta_n\to 0}} ||Y_n|| &\leq \lim_{\substack{n\to\infty\\ \Delta_n\to 0}} \sum_{k=1}^{n} \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)} \,||X(s_k^*)||\,(s_k - s_{k-1}) \\
&= \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \,||X(s)||\, ds \qquad (2.8)
\end{aligned}
$$

provided that the integral above exists as an ordinary Riemann integral.
Using (2.7) and (2.8) we thus have for $\beta \geq 1$

$$
\left|\left| \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \right|\right| \leq \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \,||X(s)||\, ds
$$

provided that the integrals involved exist.
If $X(t)$ is m.s. continuous on $[a, b]$ then $\Gamma_{XX}(t, s)$ is continuous on $[a, b] \times [a, b]$. Also, $f(t, s)$ is continuous so that $f(t, s)f(t, u)\Gamma_{XX}(s, u)$ is continuous. Thus

$$
\int_a^t \int_a^t f(t, s) f(t, u) \Gamma_{XX}(s, u)\, ds\, du
$$

exists as a double Riemann integral and so the integral in the last line of (2.7) will exist in a m.s. sense. Similarly, if $X(t)$ is m.s. continuous then $\|X(s)\|$ is continuous so that $f(t, s)\|X(s)\|$ is also continuous and the integral in (2.8) will exist as an ordinary Riemann integral. Thus, if $X(t)$ is m.s. continuous on $[a, b]$ then equation (2.6) holds for $\beta \geq 1$. The result that is given in Soong (1973) is the case when $\beta = 1$ and $X(t)$ is m.s. continuous.

The derivation for the $\beta \in (0, 1)$ case is only slightly different to that of the $\beta \geq 1$ case. Following the method given for the $\beta \geq 1$ case but with

$$
a = s_0 < s_1 < \ldots < s_n = t - h\,, \qquad h > 0
$$

we see that

$$
\left|\left| \int_a^{t-h} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \right|\right| \leq \int_a^{t-h} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \,||X(s)||\, ds \qquad (2.9)
$$

provided that the integrals involved exist.
Taking the limit as h tends to zero of the left hand side of (2.9) we have

$$
\begin{aligned}
\lim_{h\to 0} \left|\left| \int_a^{t-h} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \right|\right| &= \left|\left| \underset{h\to 0}{\text{l.i.m.}} \int_a^{t-h} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \right|\right| \\
&= \left|\left| \int_a^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \right|\right|
\end{aligned}
$$

provided that the m.s. Cauchy-Riemann integral exists.
Taking the limit as h tends to zero of the right hand side of (2.9) we have

$$
\lim_{h\to 0} \int_a^{t-h} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \,||X(s)||\, ds = \int_a^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \,||X(s)||\, ds
$$

provided that this Cauchy-Riemann integral exists.
Thus, (2.6) will hold for $\beta \in (0,1)$ if the deterministic Cauchy-Riemann integral and the m.s. Cauchy-Riemann integral involved exist. Note that unlike the $\beta \geq 1$ case, for the $\beta \in (0,1)$ case equation (2.6) does not necessarily hold when $X(t)$ is m.s. continuous – when $X(t)$ is m.s. continuous we still need to check that the limits as $h \to 0$ of both sides of (2.9) do indeed exist.

Combining all the preceding pieces we have the following result:

Theorem 2.1. *Let $X(t)$ be a second-order stochastic process. Then, for $\beta > 0$,*

$$\left|\left| \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \right|\right| \leq \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \, ||X(s)||\, ds \tag{2.10}$$

provided that the integrals involved exist as deterministic/m.s. Riemann integrals when $\beta \geq 1$ and as deterministic/m.s. Cauchy-Riemann integrals when $\beta \in (0,1)$.

Corollary 2.1. *If (2.10) holds then*

$$\left. \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \right|_{t=a} = 0 \; .$$

Proof:
For $\beta > 0$

$$\left|\left| \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \right|\right| \leq \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \, ||X(s)||\, ds \leq M \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \, ds$$

where $M = \max_{s \in [a,t]} \|X(s)\|$.
Thus

$$\left. \left|\left| \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \right|\right| \right|_{t=a} \;\; \leq \;\; (M)(0) \;\; = \;\; 0 \; .$$

Since $\|X(t)\| = 0$ iff $X(t) = 0$,

$$\left. \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(t)\, dt \right|_{t=a} = 0 \; , \quad \beta > 0 \; .$$

■

Mean-square derivatives

Definition 2.2. Let $X(t)$, $t \in T$, be a second-order stochastic process. The mean square derivative of $X(t)$ at t, $\dot{X}(t)$, is defined by

$$X^{(1)}(t) \equiv \dot{X}(t) \ \triangleq \ \underset{\tau \to 0}{\text{l.i.m.}} \frac{1}{\tau}[X(t) - X(t-\tau)] \tag{2.11}$$

provided this limit exists.

Using the convergence in m.s. criterion i.e. letting $X_\tau(t) = \frac{1}{\tau}[X(t) - X(t-\tau)]$ and considering $E[X_\tau(t)X_\epsilon(s)]$, we see that the m.s. derivative of $X(t)$ exists at t iff the second generalized derivative – given by equation (2.12) below – exists at (t,t) and is finite:

$$\begin{aligned}\lim_{\tau,\,\acute{\tau} \to 0} \Delta_\tau \Delta_{\acute{\tau}} \Gamma_{XX}(t,s) \ &= \ \lim_{\acute{\tau},\,\acute{\tau} \to 0} \frac{1}{\tau\acute{\tau}} \big[\Gamma_{XX}(t,s) - \Gamma_{XX}(t-\tau,s) \\ &\quad -\Gamma_{XX}(t,s-\acute{\tau}) + \Gamma_{XX}(t-\tau,s-\acute{\tau})\big] \,. \end{aligned} \tag{2.12}$$

We can similarly define the second mean-square derivative of $X(t)$ at t by

$$X^{(2)}(t) \ \triangleq \ \underset{\tau \to \infty}{\text{l.i.m.}} \frac{1}{\tau^2}[X(t) - 2X(t-\tau) + X(t-2\tau)] \tag{2.13}$$

provided this limit exists. Using the convergence in m.s. criterion we see this limit will exist if the following second generalized derivative exists:

$$\begin{aligned}\lim_{\tau,\,\acute{\tau} \to 0} \Delta_\tau \Delta_{\acute{\tau}} \Gamma_{X^{(1)}X^{(1)}}(t,s) \ &= \ \lim_{\acute{\tau},\,\acute{\tau} \to 0} \frac{1}{\tau\acute{\tau}} \big[\Gamma_{X^{(1)}X^{(1)}}(t,s) - \Gamma_{X^{(1)}X^{(1)}}(t-\tau,s) \\ &\quad -\Gamma_{X^{(1)}X^{(1)}}(t,s-\acute{\tau}) + \Gamma_{X^{(1)}X^{(1)}}(t-\tau,s-\acute{\tau})\big] \,. \end{aligned}$$

In general we can define the n^{th} m.s. derivative at t of a second-order s.p. $X(t)$ in the following way:

$$X^{(n)}(t) = \underset{\tau \to 0}{\text{l.i.m.}} \left[\frac{1}{\tau^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} X(t - j\tau)\right] \tag{2.14}$$

provided this limit exists.

In deterministic fractional calculus relationships between integrals and derivatives are given by integration by parts (IBP) and Leibniz's Rule. Next we give the m.s. versions of these two important results.

Integration by parts (IBP)
Suppose that $f(t,s)$ is a deterministic function defined on $T \times T$ whose partial

derivative $\frac{\partial f(t,s)}{\partial s}$ exists and suppose that the second-order stochastic process $X(t)$ is mean-square differentiable on T. Then

$$\int_a^t f(t,s)\,\dot{X}(s)\,ds = f(t,s)\,X(s)\,\big|_a^t - \int_a^t \frac{\partial f(t,s)}{\partial s}\,X(s)\,ds\ .$$

Leibniz's Rule

Suppose that $f(t,s)$ is a continuous deterministic function defined on $T \times T$ whose partial derivative $\frac{\partial f(t,s)}{\partial t}$ exists and suppose that the second-order stochastic process $X(t)$ is mean-square integrable on T. Then

$$\frac{d}{dt}\int_a^t f(t,s)\,X(s)\,ds = \int_a^t \frac{\partial f(t,s)}{\partial t}\,X(s)\,ds + f(t,t)X(t)\ .$$

One last concept that we will need in this dissertation is that of m.s. continuity. The s.p. $X(t)$, $t \in T$, is m.s. continuous at t (a fixed point in T) if, for $t, t+h \in T$,

$$\lim_{h\to 0} \|X(t+h) - X(t)\| = 0\ .$$

Using the convergence in mean-square criterion we see that if $\Gamma_{XX}(s,\theta)$ is continuous at (t,t) then $X(t)$ is m.s. continuous at t. It can be shown that m.s. differentiability implies m.s. continuity which in turn implies m.s. integrability.

For the interested reader, many of the results covered in this chapter are developed for fuzzy stochastic processes in Feng et al. (2001).

Chapter 3

The mean-square fractional integral and derivative

In the deterministic theory of fractional calculus there are many different definitions, and almost as many different notations and names, for both the fractional integral and the fractional derivative. That there are many different definitions should not be surprising as there are many different kinds of functions to which you may want to apply a fractional integral or derivative. The choice of definition in any particular situation depends not only on the function involved but also on the properties associated with the definition. For example, what Loverro (2004) refers to as the Left-Hand definition (of a fractional derivative) is a common choice for the fractional derivative because when solving a fractional differential equation with this definition, the initial conditions take the form of integer order derivatives – initial conditions of this kind being relatively easy to find and interpret. In some situations other definitions for the fractional derivative are used because physical meaning can be given to initial conditions that are non-integer order derivatives (Heymans & Podlubny (2006) demonstrate this using examples from the field of viscoelasticity). What these definitions have in common is that they reduce to ordinary repeated integrals and derivatives when the fractions are in fact integers. As this property – reducing to ordinary repeated integrals or derivatives when the parameter is of integer order – is so desirable, we will start our search for definitions for the m.s. fractional integral and the m.s. fractional derivative by building up expressions for the n^{th} integral and n^{th} derivative of a second-order s.p. $X(t)$, $t \in T$, where $n \in \mathbb{N}$. Using these expressions we will then explore various definitions based on some common definitions from the deterministic fractional calculus.

In deterministic calculus Cauchy's formula for repeated integrals gives an expression for the n^{th} repeated integral of a (suitable) deterministic function $f(t)$. Below we find an expression for the n^{th} integral of $X(t)$, $t \in T$, over the interval $[a,t]$ where $t \in [a,b]$ by briefly showing that Cauchy's formula for repeated integrals holds for mean-square integrals.

Suppose $\int_a^t \int_a^s X(u)\,du\,ds$ exists in a m.s. sense. Using mean-square IBP we have

$$\begin{aligned}\int_a^t \left[\int_a^s X(u)\,du\right] ds &= s\int_a^s X(u)\,du\Big|_a^t - \int_a^t sX(s)\,ds \\ &= t\int_a^t X(u)\,du - \left[s\int_a^s X(u)\,du\right]_{s=a} - \int_a^t sX(s)\,ds\ .\end{aligned}$$

Using Corollary 2.1

$$\left[t\int_a^t X(u)\,du\right]_{t=a} = \left[t\int_a^t \frac{(t-a)^{1-1}}{\Gamma(1)}X(u)\,du\right]_{t=a} = 0\ .$$

Thus

$$\begin{aligned}\int_a^t \left[\int_a^s X(u)\,du\right] ds &= \int_a^t tX(u)\,du - \int_a^t uX(u)\,du \\ &= \int_a^t (t-u)X(u)\,du \\ &= \int_a^t \frac{(t-u)^{2-1}}{\Gamma(2)}X(u)\,du\ .\end{aligned}$$

Continuing in this manner we are able to show that

$$\begin{aligned}&\int_a^t \int_a^{s_{n-1}} \cdots \int_a^{s_3}\int_a^{s_2} X(s_1)\,ds_1\,ds_2\ldots ds_{n-2}\,ds_{n-1} \\ =\ &\int_a^t \frac{(t-s)^{n-1}}{\Gamma(n)}X(s)\,ds \qquad (3.1)\\ \triangleq\ & I_a^n X(t)\end{aligned}$$

Equation (3.1) leads us to our first definition for the m.s. fractional integral. The m.s. Riemann-Liouville fractional integral is found by replacing $n \in \mathbb{N}$ by $\beta > 0$ in equation (3.1). Doing so we have the following definition:

Definition 3.1. Let $X(t)$, $t \in T$, be a second-order stochastic process and let $\beta > 0$. The mean-square Riemann-Liouville (R-L) fractional integral to order β of $X(t)$ is given by

$$I_a^\beta X(t) \triangleq \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}X(s)\,ds\ , \qquad t \in [a,b] \subset T\ . \qquad (3.2)$$

According to m.s. theory the integral $\int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}X(s)\,ds$ will exist in a m.s. sense iff the following ordinary double Riemann integral exists and is finite:

$$\int_a^t \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\frac{(t-\theta)^{\beta-1}}{\Gamma(\beta)}\Gamma_{XX}(s,\theta)\,ds\,d\theta\ . \qquad (3.3)$$

Remembering that if $E[X(t)X(t)] < \infty$ then $X(t)$ is a second order s.p. and noting that

$$\int_a^t \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(t-\theta)^{\beta-1}}{\Gamma(\beta)} \Gamma_{XX}(s,\theta)\, ds\, d\theta = E[I_a^\beta X(t) I_a^\beta X(t)] \ ,$$

we see that if $I_a^\beta X(t)$ exists in a m.s. sense then it is a second-order stochastic process. It is often easy to check if the double Riemann integral in (3.3) exists and is finite as the following example demonstrates.

Let Φ be a second-order random variable with $E(\Phi) = 0$ and finite variance $Var(\Phi) = \sigma^2$. Then the s.p. $X(t)$ defined by $X(t) = \Phi\, t$ will be a second-order s.p. for finite t and will have

$$\Gamma_{XX}(s,\theta) = \sigma^2 s\theta \ .$$

Then

$$\begin{aligned}
& \int_a^t \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(t-\theta)^{\beta-1}}{\Gamma(\beta)} \Gamma_{XX}(s,\theta)\, ds\, d\theta \\
= \ & \sigma^2 \left[\int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} s\, ds \right] \left[\int_a^t \frac{(t-\theta)^{\beta-1}}{\Gamma(\beta)} \theta\, d\theta \right] \\
= \ & \sigma^2 \left[\frac{a(t-a)^\beta}{\Gamma(\beta+1)} + \frac{(t-a)^{\beta+1}}{\Gamma(\beta+2)} \right]^2
\end{aligned}$$

where ordinary integration by parts was used to get the last line. Clearly $I_a^\beta X(t)$ will exist and be finite for $0 < \beta < \infty$.

Under some circumstances it is not necessary to evaluate (3.3) in order to check the existence of the m.s. integral. For example, when $X(t)$ is m.s. continuous for $t \in [a,b]$ and $1 \le \beta < \infty$, the integral in (3.2) will exist in a m.s. sense (for a more detailed explanation see the discussion preceding Theorem 2.1 in the previous chapter).

In the previous chapter we found that the n^{th}, $n \in \mathbb{N}$, m.s. derivative of the second-order s.p. $X(t)$ at t is given by

$$X^{(n)}(t) = \underset{\tau \to 0}{\text{l.i.m.}} \left[\frac{1}{\tau^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} X(t - j\tau) \right] \tag{3.4}$$

provided this limit exists. Like the R-L fractional integral, if $X^{(n)}(t)$ exists it will be a second-order stochastic process. To see this we note that $\dot{X}(t)$ will exist iff

$$\lim_{\tau,\acute{\tau} \to 0} E[X_\tau(t) X_{\acute{\tau}}(t)]$$

exists and is finite for all (t, t) in $T \times T$ where $X_\tau(t) = \frac{1}{\tau}[X(t) - X(t-\tau)]$.
But

$$\begin{aligned}
\lim_{\tau,\acute{\tau}\to 0} E[\,X_\tau(t)X_{\acute{\tau}}(t)\,] &= E\left[\underset{\tau,\acute{\tau}\to 0}{\text{l.i.m.}}\ \frac{1}{\tau}[\,X(t) - X(t-\tau)\,]\,\frac{1}{\acute{\tau}}[\,X(t) - X(t-\acute{\tau})\,]\right] \\
&= E[\,\dot{X}(t)\dot{X}(t)\,]\ .
\end{aligned}$$

Thus if $\dot{X}(t)$ exists then $E[\,\dot{X}(t)\dot{X}(t)\,] < \infty$ and so $\dot{X}(t)$ is a second order stochastic process. Since $X^{(n)}(t) = \frac{d}{dt}X^{(n-1)}(t)$, by considering

$$\lim_{\tau,\acute{\tau}\to 0}\ E[X_\tau^{(n-1)}(t)X_{\acute{\tau}}^{(n-1)}(t)]$$

we can similarly show that if $X^{(n)}(t)$ exists then it is a second-order stochastic process.

To find a definition for the m.s. fractional derivative we can try the same approach that we used for the fractional integral – replacing $n \in \mathbb{N}$ by $\beta > 0$ in (3.4). However, before we try this we are going to use $X^{(n)}(t)$, in combination with the m.s. R-L fractional integral to find two definitions for the m.s. fractional derivative.

When deterministic calculus is taught, the Riemnann integral is often presented in terms of the anti-derivative giving the impression that differentiation and integration are inverse operations. This is not entirely true in deterministic calculus, nor is it the case in m.s. calculus. For example, although

$$\frac{d}{dt}\int_a^t X(s)\ ds = X(t)\ ,$$

we have

$$\int_a^t \dot{X}(s)\ ds = X(t) - X(a)\ .$$

Also,

$$\frac{d^2}{dt^2}\int_a^t X(s)\ ds = \frac{d}{dt}X(t) = \dot{X}(t)$$

but

$$\int_a^t \ddot{X}(s)\ ds = \dot{X}(t) - \dot{X}(a)$$

and in general, integration and differentiation will only be inverse operations if $X^{(n)}(a) = 0$ for $n \in \mathbb{N}_0$. Despite this fact, the Right-Hand and Left-Hand definitions of the m.s. fractional derivative are based on the idea that integration and differentiation are (roughly) inverse operations.

Before giving the definitions let us consider an example. Suppose we want to find the 3.5^{th} derivative of $X(t)$. We could take the 4^{th} m.s. derivative of $X(t)$

and "undo" some of the differentiation by taking the 0.5^{th} R-L fractional integral to arrive at $X^{(3.5)}(t)$. Or we could reverse the order of the operations – first taking the 0.5^{th} R-L fractional integral of $X(t)$ then taking the 4^{th} m.s. derivative of the resulting integral to arrive at $X^{(3.5)}(t)$.

Below we give the definitions for the Right-Hand and Left-Hand definitions of the m.s. fractional derivative.

Definition 3.2. Let $X(t)$, $t \in T$, be a second-order stochastic process and let $\beta > 0$ be such that $\beta \in (m-1, m]$, $m \in \mathbb{N}$. The mean square Left-Hand (LH) fractional derivative of $X(t)$ at t, $t \in [a,b] \subset T$, is given by

$$ {}_*D_a^{\beta} X(t) = \begin{cases} \dfrac{d^m}{dt^m} I_a^{m-\beta} X(t) \ , & \beta \in (m-1, m) \\ \\ \dfrac{d^m}{dt^m} X(t) \ , & \beta = m. \end{cases} \tag{3.5} $$

(3.6)

Definition 3.3. Let $X(t)$, $t \in T$, be a second-order stochastic process and let $\beta > 0$ be such that $\beta \in (m-1, m]$, $m \in \mathbb{N}$. The mean square Right-Hand (RH) fractional derivative of $X(t)$ at t, $t \in [a,b] \subset T$, is given by

$$ D_a^{\beta} X(t) = \begin{cases} I_a^{m-\beta} X^{(m)}(t) \ , & \beta \in (m-1, m) \\ \\ \dfrac{d^m}{dt^m} X(t) \ , & \beta = m \end{cases} \tag{3.7} $$

(3.8)

When $\beta = m \in \mathbb{N}$ both ${}_*D_a^{\beta} X(t)$ and $D_a^{\beta} X(t)$ give us ordinary repeated m.s. derivatives and so will exist if a suitable generalized derivative holds.

Clearly, for (3.5) to exist, $I_a^{m-\beta} X(t)$ must exist as a m.s. Cauchy-Riemann integral i.e. the (deterministic) repeated Cauchy-Riemann integral

$$ \int_a^t \int_a^t \frac{(t-s)^{m-\beta-1}}{\Gamma(m-\beta)} \frac{(t-\theta)^{m-\beta-1}}{\Gamma(m-\beta)} \Gamma_{XX}(s,\theta) \, ds \, d\theta \ , $$

must exist. Assuming this is the case, we then require the m^{th} m.s. derivative of $I_a^{m-\beta} X(t)$ to exist – something that can be checked using a suitable second generalized derivative.

Now consider (3.7). For this fractional derivative to exist we need $X^{(m)}(t)$ to exist – the existence can be checked by the use of a suitable second generalized derivative. If it does exist, since (3.7) is a m.s. Cauchy-Riemann integral, we then require the following (deterministic) repeated Cauchy-Riemann integral to exist:

$$ \int_a^t \int_a^t \frac{(t-s)^{m-\beta-1}}{\Gamma(m-\beta)} \frac{(t-\theta)^{m-\beta-1}}{\Gamma(m-\beta)} \Gamma_{X^{(m)}X^{(m)}}(s,\theta) \, ds \, d\theta \ . $$

The two definitions that we have just given for the m.s. fractional derivative are based on the R-L fractional integral. We will now consider a third definition – this one based on the form given in given in (3.4) for the n^{th} m.s. derivative of $X(t)$. Equation (3.4) is repeated below for convenience:

$$X^{(n)}(t) = \underset{\tau\to 0}{\text{l.i.m.}} \left[\frac{1}{\tau^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} X(t - j\tau) \right] .$$

Let us replace $n \in \mathbb{N}$ by $\beta > 0$. If we do this we encounter two problems. The first problem is only a minor problem – the expression $(-1)^j\binom{n}{j}$ must be written in terms of the gamma function in order to make sense when we replace n by β. The second problem involves the upper limit of the sum – the upper limit must be a non-negative integer. Sorting out this second problem requires a bit of creativity and unfortunately requires some unwanted changes to equation (3.4). In (3.4) τ tends to zero through values that are unrestricted. If this limit exists then so too does the limit when τ tends to zero through particular values. Further, this restricted limit will then be equal to the unrestricted limit. So letting τ tend to zero through the values given by $\delta t_N = \frac{(t-a)}{N}$ for $N = 1, 2, \ldots$ where $a < t$ and using the notation

$$(-1)^j \binom{n}{j} = \frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)} = \frac{\psi(j,n)}{\Gamma(-n)} ,$$

we can re-write equation (3.4) as follows:

$$\begin{aligned} X^{(n)}(t) &= \underset{\delta t_N\to 0}{\text{l.i.m.}} \left[\frac{1}{(\delta t_N)^n} \sum_{j=0}^{N-1} (-1)^j \binom{n}{j} X(t - j\,\delta t_N) \right] \\ &= \underset{N\to\infty}{\text{l.i.m.}} \left[\frac{(\delta t_N)^{-n}}{\Gamma(-n)} \sum_{j=0}^{N-1} \psi(j,n)\, X(t - j\,\delta t_N) \right] . \end{aligned}$$

Replacing $n \in \mathbb{N}$ by $\beta > 0$ we thus have the following definition for the fractional derivative:

Definition 3.4. Let $X(t)$, $t \in T$, be a second-order stochastic process and let $a \in T$. The mean-square Grünwald fractional derivative of order $\beta > 0$ of a second-order stochastic process $X(t)$, is given by

$$X_a^{(\beta)}(t) = \underset{N\to\infty}{\text{l.i.m.}} \left[\frac{(\delta t_N)^{-\beta}}{\Gamma(-\beta)} \sum_{j=0}^{N-1} \psi(j,\beta)\, X(t - j\,\delta t_N) \right] \tag{3.9}$$

where, for $N = \{1, 2, 3, \ldots\}$,

$$\delta t_N = \frac{(t-a)}{N} \qquad \text{and} \qquad \psi(j,\beta) = \frac{\Gamma(j-\beta)}{\Gamma(j+1)} .$$

Using the convergence in mean-square criterion we see that the above limit will exist iff

$$\frac{(\delta t_N\,\delta s_{N'})^{-\beta}}{\Gamma^2(-\beta)}\sum_{j=0}^{N-1}\sum_{k=0}^{N'-1}\psi(j,\beta)\psi(k,\beta)\;\Gamma_{XX}(t-j\delta t_N\,,\,s-k\,\delta s_{N'}) \tag{3.10}$$

tends to a finite limit as N and N' tend to infinity in any manner whatever.
It is important to note that the limit in (3.9) is a restricted limit so that for $\beta = m \in \mathbb{N}$, (3.9) will give us integer order derivatives only when those derivatives exist i.e. when the unrestricted limit in (3.4) exists.

A second definition for the m.s. fractional integral can be found by allowing negative values of β in the formula for the Grünwald m.s. fractional derivative. Doing so we have the following definition:

Definition 3.5. Let $X(t)$, $t \in T$, be a second-order stochastic process and let $a \in T$. The mean-square Grünwald fractional integral to order $\beta > 0$ of a second-order stochastic process $X(t)$, is given by

$$[I_a^\beta X(t)]_G = \underset{N\to\infty}{\text{l.i.m.}}\left[\frac{\left(\frac{t-a}{N}\right)^\beta}{\Gamma(\beta)}\sum_{j=0}^{N-1}\frac{\Gamma(j+\beta)}{\Gamma(j+1)}X\Big(t-j\Big[\frac{t-a}{N}\Big]\Big)\right]. \tag{3.11}$$

By replacing β by $-\beta$ in the existence condition for the Grünwald m.s. fractional derivative, we have the existence condition for the Grünwald m.s. fractional integral.

Now, it is not clear from looking at equation (3.11) that the Grünwald m.s. fractional integral reduces to ordinary repeated m.s. integrals for integer β. To see that it does, we will show that $[I_a^\beta X(t)]_G$ equals the m.s. R-L fractional integral not only for integer values of β but also for non-integer values of β that are greater than 2. The method that we will use to show the equality of the two definitions is the one Oldham & Spanier (1974) use to determine the equality in the deterministic case. We will start by finding an expression for the R-L fractional integral that allows us to compare the two definitions.

Letting $[I_a^\beta X(t)]_{R-L}$ represent the m.s. R-L fractional integral of order $\beta > 0$, we have

$$\begin{aligned}
[I_a^\beta X(t)]_{R-L} &= \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}X(s)\,ds \\
&= \int_0^{t-a}\frac{s^{\beta-1}}{\Gamma(\beta)}X(t-s)\,ds \\
&= \underset{\substack{n\to\infty\\ \Delta_n\to 0}}{\text{l.i.m.}}\sum_{k=1}^{n}\frac{(s_k^*)^{\beta-1}}{\Gamma(\beta)}X(t-s_k^*)(s_k-s_{k-1})
\end{aligned} \tag{3.12}$$

where

- $0 = s_0 < s_1 < s_2 < \ldots < s_n = t - a \le b$,
- $s_k^* \in [s_{k-1}, s_k]$,
- $\Delta_n = \max_k (s_k - s_{k-1})$.

The limit in mean in (3.12) is independent of both the sequence of subdivisions and the position of the s_k^*'s within the intervals $[s_{k-1}, s_k]$. The existence of this limit implies the existence of the limit when a particular sequence of subdivisions is chosen and the s_k^*'s are chosen to be particular points in the intervals $[s_{k-1}, s_k]$. Choosing a particular sequence of subdivisions – one in which the intervals are of equal length – and letting the s_k^*'s be the upper points of each interval, we can write

$$\begin{aligned}
[I_a^\beta X(t)]_{R-L} &= \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{\text{l.i.m.}} \sum_{k=1}^{n} \frac{(s_k^*)^{\beta-1}}{\Gamma(\beta)} X(t - s_k^*)(s_k - s_{k-1}) \\
&= \underset{N\to\infty}{\text{l.i.m.}} \sum_{k=0}^{N-1} \frac{(k\,\delta t_N)^{\beta-1}}{\Gamma(\beta)} X(t - k\delta t_N)\delta t_N \\
&= \underset{N\to\infty}{\text{l.i.m.}} \sum_{k=0}^{N-1} \frac{k^{\beta-1}}{\Gamma(\beta)} (\delta t_N)^\beta X(t - k\,\delta t_N)\ . \qquad (3.13)
\end{aligned}$$

Note that due to the way we have defined our subdivisions, letting $N \to \infty$ implies that $\Delta_n \to 0$.

Now, again letting

$$\psi(j,\beta) = \frac{\Gamma(j+\beta)}{\Gamma(j+1)}$$

we have

$$\begin{aligned}
\Delta &\triangleq \big[I_a^\beta X(t)\big]_G - \big[I_a^\beta X(t)\big]_{R-L} \\
&= \underset{N\to\infty}{\text{l.i.m.}} \left[\frac{(\delta t_N)^\beta}{\Gamma(\beta)} \sum_{j=0}^{N-1} \psi(j,\beta)\, X(t - j\,\delta t_N)\right] \\
&\quad - \underset{N\to\infty}{\text{l.i.m.}} \left[\frac{(\delta t_N)^\beta}{\Gamma(\beta)} \sum_{k=0}^{N-1} k^{\beta-1} X(t - k\,\delta t_N)\right] \\
&= \underset{N\to\infty}{\text{l.i.m.}} \left(\frac{(\delta t_N)^\beta}{\Gamma(\beta)} \sum_{j=0}^{N-1} X(t - j\,\delta t_N)\big[\psi(j,\beta) - j^{\beta-1}\big]\right) \\
&= \frac{(t-a)^\beta}{\Gamma(\beta)} \underset{N\to\infty}{\text{l.i.m.}} \left(\sum_{j=0}^{N-1} \frac{1}{N^\beta} X(t - j\,\delta t_N)\big[\psi(j,\beta) - j^{\beta-1}\big]\right)\ .
\end{aligned}$$

(3.14)

In (3.14) we will group the first J terms together and then group the remaining $(N-J)$ terms. We require J to be independent of N and large enough to allow us to use the asymptotic expansion

$$\psi(j,\beta) = \frac{\Gamma(j+\beta)}{\Gamma(j+1)} \quad \sim \quad j^{\beta-1}\left[1+\frac{\beta(\beta-1)}{2j}+\mathrm{O}(j^{-2})\right] \qquad j\to\infty\ .$$

We thus have

$$\begin{aligned}\Delta &= \frac{(t-a)^\beta}{\Gamma(\beta)} \operatorname*{l.i.m.}_{N\to\infty} \sum_{j=0}^{J-1} X(t-j\,\delta t_N)\frac{1}{N^\beta}\left[\psi(j,\beta)-j^{\beta-1}\right] \\ &+ \frac{(t-a)^\beta}{\Gamma(\beta)} \operatorname*{l.i.m.}_{N\to\infty} \frac{1}{N}\sum_{j=J}^{N-1} X(t-j\,\delta t_N)\left[\frac{j}{N}\right]^{\beta-2}\left[\frac{\beta(\beta-1)}{2N}+\frac{\mathrm{O}(j^{-1})}{N}\right] .\end{aligned}$$

We note that in the first summation, for $\beta > 1$

$$\frac{1}{N^\beta}\left[\psi(j,\beta)-j^{\beta-1}\right] \to 0 \quad \text{as } N\to\infty$$

so that since $X(t)$ is a second-order s.p. the first summation will tend to zero as N tends to infinity. Also, in the second summation

$$\left[\frac{\beta(\beta-1)}{2N}+\frac{\mathrm{O}(j^{-1})}{N}\right] \to 0 \quad \text{as } N\to\infty,$$

$$\left[\frac{j}{N}\right]^{\beta-2} < 1 \quad \text{provided } \beta \geq 2 \text{ and}$$

$$\frac{1}{N} \to 0 \quad \text{as } N\to\infty,$$

so that since $X(t)$ is a second-order s.p. the second summation will tend to zero as N tends to infinity. Thus, we have shown the equality of the two definitions for the $\beta \geq 2$ case.

Looking at the specific case when $\beta = 1$ we see that

$$\begin{aligned}\Delta &\triangleq \left[I_a^1 X(t)\right]_G - \left[I_a^1 X(t)\right]_{R-L} \\ &= \operatorname*{l.i.m.}_{N\to\infty}\left[\delta t_N \sum_{j=0}^{N-1} X(t-j\,\delta t_N)\right] - \operatorname*{l.i.m.}_{N\to\infty}\left[\delta t_N \sum_{k=0}^{N-1} X(t-k\,\delta t_N)\right] \\ &= \operatorname*{l.i.m.}_{N\to\infty} \delta t_N \sum_{j=0}^{N-1} X(t-j\,\delta t_N)\left[1-1\right] \\ &= 0\end{aligned}$$

since $X(t)$ is a second-order stochastic process. Thus the two definitions are also equal for $\beta = 1$.

The first step we did when showing the equality of the two definitions highlights an important point – the limit used in the Grünwald m.s. fractional integral is a restricted limit and so will only actually reduce to integer order repeated m.s. integrals when those repeated m.s. integrals exist i.e. when $[I_a^n X(t)]_{R-L}$, $n \in \mathbb{N}$, exists.

We have considered the m.s equivalents for some of the more common definitions of the deterministic fractional integral and fractional derivative. We have seen that they can be derived in the same way, and have the same form as, the deterministic fractional integrals and derivatives. This should not be surprising – the development of m.s. calculus is similar to that of deterministic calculus and so it seems reasonable that the development of m.s. fractional calculus is similar to that of deterministic fractional calculus. There are many more definitions that we could consider here but since this dissertation is not aimed at giving a complete working of the subject we will not do so. For this same reason, for the remainder of this dissertation we will restrict ourselves to the use of only one m.s. fractional integral – the m.s. R-L fractional integral – and one m.s. fractional derivative – the m.s. RH fractional derivative. The reason we will not be using the Grünwald m.s. fractional integral and derivative is that the current body of work (given in Hafiz et al. (2001), Hafiz (2004) and El-Sayed et al. (2005)) only introduces the R-L fractional derivative and the RH and LH fractional derivatives and a fair exploration of the m.s. fractional calculus can be done simply by extending and adding to the current body of work. The reason that the m.s. RH fractional derivative is being used instead of the m.s. LH fractional derivative is that for any particular value of $t \in T$, say t_1,

$$D_a^\beta X(t_1) \quad = \quad \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \underbrace{\frac{d^m}{dt^m} X(t_1)}_{0} \, ds \quad = \quad 0 \, .$$

Intuitively this seems correct. By contrast

$$\begin{aligned}
{}_*D_a^\beta X(t_1) &= \frac{d^m}{dt^m} \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(t_1)\, ds \\
&= X(t_1) \frac{d^m}{dt^m} \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \, ds \\
&= X(t_1) \frac{d^m}{dt^m} \frac{(t-a)^\beta}{\Gamma(\beta+1)} \\
&= X(t_1) \frac{(t-a)^{\beta-m}}{\Gamma(\beta-m+1)} \, .
\end{aligned}$$

In general $X(t_1) \neq 0$ so in general ${}_*D_a^t X(t_1) \neq 0$.

One drawback of our choice for the definition of the m.s. fractional derivative is that it requires the stochastic process to be m.s. differentiable to order $m \in \mathbb{N}$. This means that we will not be able to consider the m.s. fractional derivatives of some commonly used stochastic processes. The Poisson process, the Wiener process and binary noise are three such processes.

For the remainder of this dissertation, unless the situation calls for the use of the full names of the fractional integrals or derivatives to avoid confusion, we will refer to the m.s. Riemann-Liouville fractional integral simply as the m.s. fractional integral and the m.s. RH fractional derivative simply as the m.s. fractional derivative. We will also define $I_a^0 X(t)$ and $D_a^0 X(t)$ as follows:

$$\begin{aligned} I_a^0 X(t) &\triangleq X(t) \\ D_a^0 X(t) &\triangleq X(t) \, . \end{aligned}$$

Chapter 4

Properties of mean-square fractional integrals and derivatives

Having defined the m.s. fractional integral and derivative, we are now in a position to consider some of their properties. Ideally we would like to show that many of the properties of the deterministic fractional calculus have m.s. equivalents. There are some properties in the deterministic fractional calculus that are used to solve deterministic fractional integral and differential equations. As such, these are properties that once transferred to a m.s. setting may possibly be used to solve m.s. fractional integral and differential equations. Presumably for this reason Hafiz et al. (2001), Hafiz (2004) and El-Sayed et al. (2005) consider these properties as well as some basic properties of m.s. fractional derivatives and integrals. Solving m.s. fractional integral and differential equations is the topic of Chapter 6 and so in this chapter we too will consider these properties using more general conditions in several places and extending the properties to include the $\beta > 1$ cases which are not considered in Hafiz et al. (2001), Hafiz (2004) and El-Sayed et al. (2005) when fractional derivatives are involved.

There is a large overlap of the content in Hafiz et al. (2001), Hafiz (2004) and El-Sayed et al. (2005). The proofs of the properties that are common are remarkably alike despite Hafiz (2004) using different conditions to Hafiz et al. (2001) and El-Sayed et al. (2005). Due to this overlap we will, for convenience, often only make reference to Hafiz et al. (2001).

4.1 Basic properties

Theorem 4.1. *Let $X(t)$ and $Y(t)$ be second-order stochastic processes for which $I_a^\beta X(t)$ and $I_a^\beta Y(t)$, $\beta > 0$, exist for $t \in [a, b] \subset T$. Then* *(a) (Linearity)*

$$I_a^\beta[X(t) + Y(t)] = I_a^\beta X(t) + I_a^\beta Y(t).$$

(b) (Homogeneity)

$$I_a^\beta[cX(t)] = c \cdot I_a^\beta X(t)$$

where c is a constant.

(c) $I_a^\beta X(t)|_{t=a} = 0$

Proof: Letting

- $a = s_0 < s_1 < s_2 < \ldots < s_n = t \leq b$,
- $s_k^* \in [s_{k-1}, s_k]$ for $k = 1, 2, \ldots, n$ and
- $\Delta_n = \max\limits_k (s_k - s_{k-1})$

we have

(a)

$$\begin{aligned}
I_a^\beta[X(t) + Y(t)] &= \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}[X(s) + Y(s)]\, ds \\
&= \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{\text{l.i.m.}} \sum_{k=1}^n \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)}[X(s_k^*) + Y(s_k^*)](s_k - s_{k-1}) \\
&= \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{\text{l.i.m.}} \sum_{k=1}^n \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)} X(s_k^*)\,(s_k - s_{k-1}) \\
&\quad + \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{\text{l.i.m.}} \sum_{k=1}^n \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)} Y(s_k^*)\,(s_k - s_{k-1}) \\
&= \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds + \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Y(s)\, ds \\
&= I_a^\beta X(t) + I_a^\beta Y(t)\ .
\end{aligned}$$

(b) Similarly,

$$\begin{aligned}
I_a^\beta[cX(t)] &= \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{\text{l.i.m.}} \sum_{k=1}^n \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)}\ c\, X(s_k^*)\,(s_k - s_{k-1}) \\
&= c\ \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{\text{l.i.m.}} \sum_{k=1}^n \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)} X(s_k^*)\,(s_k - s_{k-1}) \\
&= c \cdot I_a^\beta X(t)\ .
\end{aligned}$$

(c) Since $I_a^\beta X(t)$ is defined as follows

$$I_a^\beta X(t) \triangleq \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \ ,$$

we see that (c) is proved in Corollary 2.1 in Chapter 2. ■

Corollary 4.1. *Let $\beta \in (m-1, m)$ where $m \in \mathbb{N}$ and let $X(t)$ be a second-order stochastic process such that $D_a^\beta X(t)$ exists for $t \in [a,b] \subset T$. Then*

$$D_a^\beta X(t)|_{t=a} = 0 \ .$$

Proof:
For $\beta \in (m-1, m)$ $D_a^\beta X(t)$ is defined as follows

$$D_a^\beta X(t) \triangleq I_a^{m-\beta} X^{(m)}(t) \ .$$

This is of the form $I_a^\alpha Y(t)$ where $\alpha = m - \beta$ and $Y(t)$ is a second-order stochastic process. Thus, using Part (c) of Theorem 4.1 we have

$$D_a^\beta X(t)|_{t=a} = I_a^{m-\beta} X^{(m)}(t)|_{t=a} = 0 \ .$$

■

Note: Later in this chapter we will come across terms like $I_a^\beta X(a)$. By this we mean $I_a^\beta[X(a)]$ and not $I_a^\beta X(t)|_{t=a}$. If there is cause for confusion we will specify what is meant.

In the following theorem we consider the continuity of $I_a^\beta X(t)$.

Theorem 4.2. *Let $\beta > 0$ and let $X(t)$ be a second-order stochastic process such that $I_a^\beta X(t)$ exists for $t \in [a,b] \subset T$. Then $I_a^\beta X(t)$ is mean-square continuous provided that, for $h > 0$,*

$$\int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} \|X(s)\|\, ds$$

and

$$\int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} \|X(s)\|\, ds$$

exist as Riemann integrals when $\beta \geq 1$ and as CR integrals when $\beta \in (0,1)$.

Proof:
To show that $I_a^\beta X(t)$ is m.s. continuous we must show that, for $t, t+h \in T$ where $h > 0$,

$$\|I_a^\beta X(t+h) - I_a^\beta X(t)\| \to 0 \quad \text{as } h \to 0. \tag{4.1}$$

Now,

$$\begin{aligned}
\|I_a^\beta X(t+h) - I_a^\beta X(t)\| &= \Bigg\| \int_a^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \\
&\quad - \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \Bigg\| \\
&= \Bigg\| \int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \\
&\quad + \int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \Bigg\| \\
&\leq \Bigg\| \int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \Bigg\| \\
&\quad + \Bigg\| \int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \Bigg\| \\
&= \| \mathfrak{I}_1\| + \| \mathfrak{I}_2\|
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{I}_1 &= \int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \quad \text{and} \\
\mathfrak{I}_2 &= \int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \,.
\end{aligned}$$

Consider $\mathfrak{I}_1$.

$$\|\mathfrak{I}_1\| \leq \int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} \; \|X(s)\|\, ds$$

provided the integral on the right exists as a Riemann integral when $\beta \geq 1$ and as a CR integral when $\beta \in (0,1)$.
Then, letting $M = \max_{s\in[t,t+h]} \|X(s)\|$ we have

$$\|\mathfrak{I}_1\| \leq M \int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)}\, ds \qquad \to 0 \text{ as } h \to 0.$$

Similarly,

$$\|\mathfrak{I}_2\| \leq \int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} \|X(s)\|\, ds$$

provided the integral on the right exists as a Riemann integral when $\beta \geq 1$ and as a CR integral when $\beta \in (0,1)$.
Then, letting $N = \max_{s\in[a,t]} \|X(s)\|$ we have

$$\|\mathfrak{I}_2\| \leq N \int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)}\, ds \qquad \to 0 \text{ as } h \to 0.$$

Thus

$$\|I_a^\beta X(t+h) - I_a^\beta X(t)\| \quad \leq \quad \|\mathfrak{I}_1\| + \|\mathfrak{I}_2\| \quad \to 0 \quad \text{as } h \to 0.$$

So we see that $I_a^\beta X(t)$ will be m.s. continuous. ■

In Theorem 4.2 we do not restrict ourselves to m.s. continuous second-order stochastic processes. If $X(t)$ is m.s. continuous and $\beta \geq 1$ then $I_a^\beta X(t)$ and the integrals

$$\int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} \|X(s)\| \, ds$$

and

$$\int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} \|X(s)\| \, ds$$

will exist and so the statement of the above theorem becomes:

> If $X(t)$ is mean-square continuous for $t \in [a,b] \subset T$, then $I_a^\beta X(t)$ is mean-square continuous for $\beta \geq 1$.

Using a similar method Hafiz et al. (2001) arrive at the same result – if $X(t)$ is mean-square continuous then $I_a^\beta X(t)$ is mean-square continuous – but include the region $\beta \in (0,1)$. The reasoning for this being that, according to Hafiz et al. (2001), the continuity of $X(t)$ insures the existence of $I_a^\beta X(t)$ for $\beta > 0$. However, as we have discussed previously, this is not the case when $\beta \in (0,1)$.

Up till now we have emphasized the importance of checking the existence of CR integrals when they are used. This is quite tedious so in the following work – *not only in this chapter but also in following chapters – it will go without saying that the existence of all CR integrals should be checked.*

In the next few theorems we take the focus off of $t \in T$ and instead focus on $\beta > 0$.

Theorem 4.3. *Let $X(t)$ be a second-order stochastic process such that $I_a^\gamma X(t)$, $t \in [a,b] \subset T$, exists for all $\gamma > 0$. Then, for $\alpha > 0$ where $\alpha \neq \beta$,*

$$\underset{\beta\to\alpha}{\textit{l.i.m.}}\, I_a^\beta X(t) = I_a^\alpha X(t)$$

Proof:
Using the convergence in mean-square criterion we see that to prove

$$\underset{\beta\to\alpha}{\text{l.i.m.}}\, I_a^\beta X(t) = I_a^\alpha X(t)$$

we need only show that $E[I_a^\beta X(t) I_a^{\beta'} X(t)]$ tends to a finite limit as β and β' tend to α (in any manner whatever) and that the limit is $E[I_a^\alpha X(t) I_a^\alpha X(t)]$.
Now,

$$\begin{aligned}
&\lim_{\beta,\beta'\to\alpha} E[I_a^\beta X(t) I_a^{\beta'} X(t)] \\
= &\lim_{\beta,\beta'\to\alpha} \int_a^t \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(t-\theta)^{\beta'-1}}{\Gamma(\beta')} \Gamma_{XX}(s,\theta)\, ds\, d\theta \\
= &\int_a^t \int_a^t \left[\lim_{\beta,\beta'\to\alpha} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(t-\theta)^{\beta'-1}}{\Gamma(\beta')} \right] \Gamma_{XX}(s,\theta)\, ds\, d\theta \\
= &\int_a^t \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \Gamma_{XX}(s,\theta)\, ds\, d\theta < \infty \\
= &E[I_a^\alpha X(t) I_a^\alpha X(t)] \; .
\end{aligned}$$

■

By looking at the proof of Theorem 4.3 it is clear that we could not consider the case $\beta \to \alpha$ where $\alpha = 0$. In the following theorem we consider this case.

Theorem 4.4. *Let $X(t)$ be a second-order stochastic process such that $\dot{X}(t)$ exists and is mean-square continuous on $[a,b] \subset T$. Then*

$$\underset{\beta\to 0}{\textit{l.i.m.}}\; I_a^\beta X(t) = X(t).$$

Proof:
Using integration by parts we have for $\beta > 0$

$$\begin{aligned}
I_a^{\beta+1} \dot{X}(t) &= \left[\frac{(t-s)^\beta}{\Gamma(\beta+1)} X(s) \right]_a^t + \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \\
&= -\frac{(t-a)^\beta}{\Gamma(\beta+1)} X(a) + I_a^\beta X(t) \; .
\end{aligned}$$

So

$$\begin{aligned}
\underset{\beta\to 0}{\text{l.i.m.}}\; I_a^\beta X(t) &= \underset{\beta\to 0}{\text{l.i.m.}} \left[I_a^{\beta+1} \dot{X}(t) + \frac{(t-a)^\beta}{\Gamma(\beta+1)} X(a) \right] \\
&= \underset{\beta\to 0}{\text{l.i.m.}}\; I_a^{\beta+1} \dot{X}(t) + \underset{\beta\to 0}{\text{l.i.m.}} \left[\frac{(t-a)^\beta}{\Gamma(\beta+1)} X(a) \right] \\
&= \underset{\gamma\to 1}{\text{l.i.m.}}\; I_a^\gamma \dot{X}(t) + X(a) \\
&= I_a^1 \dot{X}(t) + X(a) \\
&= X(t) - X(a) + X(a) \\
&= X(t) \; .
\end{aligned}$$

■

We have followed the example of deterministic fractional calculus defining $I_a^0 X(t)$ to be $X(t)$. This definition makes sense intuitively but Theorem 4.4 further motivates it (despite Theorem 4.4 requiring the existence of $\dot{X}(t)$).

Theorem 4.5. *Let $\beta \in (m-1, m)$ where $m \in \mathbb{N}$. Let $X(t)$ be a second order stochastic process such that $X^{(m)}(t)$ exists and is mean square continuous on $[a,b] \subset T$. Then*

(a)

$$\underset{\beta\to m}{\textit{l.i.m.}}\ D_a^\beta X(t) = X^{(m)}(t)\ .$$

(b)

$$\underset{\beta\to(m-1)}{\textit{l.i.m.}}\ D_a^\beta X(t) = X^{(m-1)}(t) - X^{(m-1)}(a)\ .$$

Proof:

(a)

$$\begin{aligned}\underset{\beta\to m}{\text{l.i.m.}}\ D_a^\beta X(t) &= \underset{\beta\to m}{\text{l.i.m.}}\ I_a^{m-\beta} X^{(m)}(t) \\ &= \underset{\gamma\to 0}{\text{l.i.m.}}\ I_a^\gamma X^{(m)}(t) \\ &= X^{(m)}(t)\end{aligned}$$

where Theorem 4.4 has been used in the last step.

(b)

$$\begin{aligned}\underset{\beta\to(m-1)}{\text{l.i.m.}}\ D_a^\beta X(t) &= \underset{\beta\to(m-1)}{\text{l.i.m.}}\ I_a^{m-\beta} X^{(m)}(t) \\ &= \underset{\gamma\to 1}{\text{l.i.m.}}\ I_a^\gamma X^{(m)}(t) \\ &= I_a^1 X^{(m)}(t) \\ &= X^{(m-1)}(t) - X^{(m-1)}(a)\end{aligned}$$

where IBP has been used in the last step. ∎

From Part (a) of Theorem 4.5 we know that

$$\underset{\beta\to m}{\text{l.i.m.}}\ D_a^\beta X(t) = X^{(m)}(t)$$

and from Part (b) we see that we will have

$$\underset{(\beta+1)\to m}{\text{l.i.m.}}\ D_a^{\beta+1} X(t) = X^{(m)}(t) - X^{(m)}(a)\ .$$

In general $X^{(m)}(a)$ does not equal 0. Thus Theorem 4.5 shows the necessity of explicitly defining $D_a^\beta X(t)$ to be $X^{(m)}(t)$ when $\beta = m \in \mathbb{N}$ in the definition of the mean-square fractional derivative. $D_a^\beta X(t)$ could as easily have been defined as $X^{(m)}(t) - X^{(m)}(a)$ when $\beta = m \in \mathbb{N}$, but this seems counter-intuitive.

4.2 The composition rule

Since $I_a^\beta X(t)$ and $D_a^\beta X(t)$, when they exist, are second-order stochastic processes, we can consider expressions such as $I_a^\alpha I_a^\beta X(t)$ and $D_a^\alpha I_a^\beta X(t)$. These expressions, and those like it, make up what we will call the composition rule. They will play an important role in later chapters. This section deals with parts of the composition rule.

The following theorem is used to derive many of the remaining properties in this chapter. In deterministic fractional calculus the proof of this theorem is often left out. We include the proof for this m.s. version – as do Hafiz et al. (2001) and Hafiz (2004), albeit in a less detailed manner – for the sake of completeness.

Theorem 4.6. *Let $\beta > 0$ and $\alpha > 0$ and let $X(t)$ be a second-order stochastic process such that $I_a^\alpha I_a^\beta X(t)$ exists for $t \in [a, b] \subset T$. Then*

$$I_a^\alpha I_a^\beta X(t) = I_a^{\alpha+\beta} X(t) \ .$$

Proof:

$$\begin{aligned} I_a^\alpha I_a^\beta X(t) &= \int_a^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \int_a^\theta \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds\, d\theta \\ &= \int_a^t X(s) \left[\int_s^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)}\, d\theta \right] ds \ . \end{aligned}$$

Consider

$$\mathfrak{I} = \int_s^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)}\, d\theta \ .$$

Letting $v = \theta - s$ we have

$$\mathfrak{I} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{t-s} (t-v-s)^{\alpha-1} v^{\beta-1}\, dv \ .$$

Now letting $v = (t-s)u$ we have

$$\begin{aligned} \mathfrak{I} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (t-s)^{\alpha-1}(1-u)^{\alpha-1}(t-s)^{\beta-1}u^{\beta-1}(t-s)\, du \\ &= \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-u)^{\alpha-1}u^{\beta-1}\, du \\ &= \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ &= \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \ . \end{aligned}$$

So

$$\begin{aligned} I_a^\alpha I_a^\beta X(t) &= \int_a^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} X(s)\, ds \\ &= I_a^{\alpha+\beta} X(t)\ . \end{aligned}$$ ■

In Theorem 4.6 we considered the m.s. fractional integral of a m.s. fractional integral. In the next few theorems we find expressions for $I_a^\alpha D_a^\beta X(t)$ and $D_a^\alpha I_a^\beta X(t)$. The proofs used when finding these expressions will rely heavily on Leibniz's rule and IBP. This is not surprising as Leibniz's rule deals with the derivative of an integral and IBP deals with the integral of a derivative (derivatives and integrals being defined in the m.s. sense for second-order stochastic processes). We will see that the proofs are very similar and will often rely on parts of previously proved theorems. It would be cumbersome to constantly refer to previous theorems when they are used so, in most situations, we will not do so.

Theorem 4.7. *Let $\alpha > 0$ and $\beta \in (m-1, m]$ where $m \in \mathbb{N}$. Let $X(t)$ be a second-order stochastic process such that $D_a^\beta X(t)$ exists for $t \in [a,b] \subset T$. Then*

$$I_a^\alpha D_a^\beta X(t) = I_a^{\alpha-\beta} X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^{\alpha-\beta+j}}{\Gamma(\alpha-\beta+j+1)} X^{(j)}(a)\ , \qquad \alpha \geq \beta.$$

Proof:
Using integration by parts we have

$$\begin{aligned} I_a^1\, X^{(1)}(t) &= \int_a^t X^{(1)}(s)\ ds \\ &= [\,X(s)\,]_a^t + 0 \\ &= X(t) - X(a)\ , \end{aligned}$$

$$\begin{aligned} I_a^2\, X^{(2)}(t) &= \int_a^t \frac{(t-s)}{\Gamma(2)} X^{(2)}(s)\ ds \\ &= \left[\frac{(t-s)}{\Gamma(2)} X^{(1)}(s) \right]_a^t + \underbrace{\int_a^t X^{(1)}(s)\ ds}_{I_a^1\, X^{(1)}(t)} \\ &= -\frac{(t-a)^1}{\Gamma(2)} X^{(1)}(a) - X(a) + X(t)\ , \end{aligned}$$

and

$$\begin{aligned}
I_a^3\, X^{(3)}(t) &= \int_a^t \frac{(t-s)^2}{\Gamma(3)} X^{(3)}(s)\, ds \\
&= \left[\frac{(t-s)^2}{\Gamma(3)} X^{(2)}(s) \right]_a^t + \underbrace{\int_a^t \frac{(t-s)}{\Gamma(2)} X^{(2)}(s)\, ds}_{I_a^2\, X^{(2)}(t)} \\
&= -\frac{(t-a)^2}{\Gamma(3)} X^{(2)}(a) - \frac{(t-a)^1}{\Gamma(2)} X^{(1)}(a) - X(a) + X(t)\ .
\end{aligned}$$

Continuing in the same manner we have, for $\alpha = \beta = m$,

$$I_a^m\, X^{(m)}(t) = X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a)\ . \tag{4.2}$$

For $\alpha = \beta \in (m-1, m)$

$$\begin{aligned}
I_a^\beta D_a^\beta X(t) &= I_a^\beta\, I_a^{m-\beta} X^{(m)}(t) = I_a^m X^{(m)}(t) \\
&= X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a)
\end{aligned}$$

where we have used equation (4.2) to find the last line.

Let $n \in \mathbb{N}$. When $\alpha = n$ and $\beta = m$

$$\begin{aligned}
I_a^n D_a^m X(t) &= I_a^{n-m} I_a^m X^{(m)}(t) \\
&= I_a^{n-m} \left[X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a) \right] \\
&= I_a^{n-m} X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^{n-m+j}}{\Gamma(n-m+j+1)} X^{(j)}(a)\ .
\end{aligned}$$

When $\alpha \in (n-1, n)$ and $\beta = m$

$$\beta = m \ \le\ n-1\ < \alpha\ < n$$

so that $\alpha - m > 0$. Under these conditions we have

$$\begin{aligned}
I_a^\alpha X^{(m)}(t) &= I_a^{m-m} I_a^\alpha X^{(m)}(t) \\
&= I_a^{\alpha-m} I_a^m X^{(m)}(t) \\
&= I_a^{\alpha-m} \left[X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a) \right] \\
&= I_a^{\alpha-m} X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^{\alpha-m+j}}{\Gamma(\alpha-m+j+1)} X^{(j)}(a)\ .
\end{aligned}$$

For $\alpha \in (n-1, n]$ and $\beta \in (m, m-1)$

$$\begin{aligned}
I_a^\alpha D_a^\beta X(t) &= I_a^\alpha I_a^{m-\beta} X^{(m)}(t) \\
&= I_a^{\alpha-\beta} I_a^m X^{(m)}(t) \\
&= I_a^{\alpha-\beta}\Big[X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a)\Big] \\
&= I_a^{\alpha-\beta} X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^{\alpha-\beta+j}}{\Gamma(\alpha-\beta+j+1)} X^{(j)}(a)\ .
\end{aligned}$$

■

In Theorem 4.7 we have considered the fractional integral of a fractional derivative when the integral is of an order greater than, or equal to, that of the fractional derivative. In the following theorem, Theorem 4.8, we will look at the case when the order of the integral is smaller than that of the derivative.

Theorem 4.8. *Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ and let $0 < \alpha < \beta$ where $\alpha \in (n-1, n]$ and $\beta \in (m-1, m]$. Let $X(t)$ be a second-order stochastic process such that $D_a^\beta X(t)$ exists for $t \in [a, b] \subset T$. Then*

(a)

$$I_a^\alpha D_a^\beta X(t) = D_a^{\beta-\alpha} X(t)$$

when $\beta - \alpha \in (m-1, m)$ and $\beta \neq m$.

(b)

$$I_a^\alpha D_a^\beta X(t) = D_a^{\beta-\alpha} X(t) - \frac{(t-a)^{m-\beta-1+\alpha}}{\Gamma(m-\beta+\alpha)} X^{(m-1)}(a)$$

when $\alpha \in (0, 1)$, $\beta - \alpha \in (m-2, m-1]$ and $\beta \neq m$.

(c)

$$I_a^n D_a^\beta X(t) = D_a^{\beta-n} X(t) - \sum_{j=0}^{n-1} \frac{(t-a)^{m-\beta+j}}{\Gamma(m-\beta+j+1)} X^{(m-n+j)}(a)$$

for $\beta \in (m-1, m]$

(d)

$$I_a^\alpha D_a^\beta X(t) = D_a^{\beta-\alpha} X(t) - \sum_{j=1}^{n-1} \frac{(t-a)^{m-\beta-n+\alpha+j}}{\Gamma(m-\beta-n+\alpha+j+1)} X^{(m-n+j)}(a)$$

when $\alpha \in (n-1, n)$, $\alpha > 1$.

Proof

(a)

$$\begin{aligned} I_a^\alpha D_a^\beta X(t) &= I_a^\alpha I_a^{m-\beta} X^{(m)}(t) \\ &= I_a^{m-(\beta-\alpha)} X^{(m)}(t) \\ &= D_a^{\beta-\alpha} X(t) \; . \end{aligned}$$

(b) Let $\beta - \alpha = m - 1$.

$$\begin{aligned} I_a^\alpha D_a^\beta X(t) &= I_a^\alpha I_a^{m-\beta} X^{(m)}(t) \\ &= I_a^{m-(\beta-\alpha)} X^{(m)}(t) \\ &= I_a^1 X^{(m)}(t) \\ &= X^{(m-1)}(t) - X^{(m-1)}(a) \\ &= D_a^{\beta-\alpha} X(t) - \frac{(t-a)^{m-1-\beta+\alpha}}{\Gamma(m-\beta+\alpha)} X^{(m-1)}(a) \; . \end{aligned}$$

Now let $\beta - \alpha \in (m-2, m-1)$.

$$\begin{aligned} I_a^\alpha D_a^\beta X(t) &= I_a^{m-(\beta-\alpha)} X^{(m)}(t) \\ &= I_a^{(m-1)-(\beta-\alpha)} I^1 X^{(m)}(t) \\ &= I_a^{(m-1)-(\beta-\alpha)} X^{(m)}(t) - I_a^{(m-1)-(\beta-\alpha)} X^{(m-1)}(a) \\ &= I_a^{(m-1)-(\beta-\alpha)} X^{(m-1)}(t) - \frac{(t-a)^{(m-1)-(\beta-\alpha)}}{\Gamma(m-\beta+\alpha)} X^{(m-1)}(a) \\ &= D_a^{\beta-\alpha} X(t) - \frac{(t-a)^{m-\beta-1+\alpha}}{\Gamma(m-\beta+\alpha)} X^{(m-1)}(a) \; . \end{aligned}$$

(c) Using equation (4.2) and recalling that we are working under the assumption that $\beta > \alpha$, we have, for $\beta = m$ and $\alpha = n$,

$$\begin{aligned} I_a^n D_a^m X(t) &= I_a^n \frac{d^n}{dt^n} \left[X^{(m-n)}(t) \right] \\ &= X^{(m-n)}(t) - \sum_{j=0}^{n-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(m-n+j)}(a) \; . \end{aligned} \tag{4.3}$$

Using equation (4.3) we have, for $\beta \in (m-1, m)$ and $\alpha = n$,

$$\begin{aligned}
I_a^n D_a^\beta X(t) &= I_a^n I_a^{m-\beta} X^{(m)}(t) \\
&= I_a^{m-\beta} I_a^n X^{(m)}(t) \\
&= I_a^{m-\beta}\left[X^{(m-n)}(t) - \sum_{j=0}^{n-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(m-n+j)}(a)\right] \\
&= I_a^{m-\beta} X^{(m-n)}(t) - \sum_{j=0}^{n-1} \frac{(t-a)^{m-\beta+j}}{\Gamma(m-\beta+j+1)} X^{(m-n+j)}(a) \\
&= D_a^{\beta-n} X(t) - \sum_{j=0}^{n-1} \frac{(t-a)^{m-\beta+j}}{\Gamma(m-\beta+j+1)} X^{(m-n+j)}(a) \; .
\end{aligned}$$

(d) Using equation (4.3) we have for $\beta = m$ and $\alpha \in (n-1, n)$ where $\alpha > 1$,

$$\begin{aligned}
I_a^\alpha X^{(m)}(t) &= I_a^{\alpha-n+1} I_a^{n-1} X^{(m)}(t) \\
&= I_a^{\alpha-n+1}\left[X^{(m-n+1)} - \sum_{j=0}^{n-2} \frac{(t-a)^j}{\Gamma(j+1)} X^{(m-n+1+j)}(a)\right] \\
&= I_a^{\alpha-n+1} X^{(m-n+1)} - \sum_{j=0}^{n-2} \frac{(t-a)^{\alpha-n+1+j}}{\Gamma(\alpha-n+j+2)} X^{(m-n+1+j)}(a) \\
&= D_a^{m-\alpha} X(t) - \sum_{j=1}^{n-1} \frac{(t-a)^{\alpha-n+j}}{\Gamma(\alpha-n+j+1)} X^{(m-n+j)}(a) \; .
\end{aligned}$$

Let $\beta \in (m-1, m)$ and $\alpha \in (n-1, n)$ where $\alpha > 1$. If $n = m$, then $\beta - \alpha \in (0, 1)$. Under these conditions we have

$$\begin{aligned}
I_a^\alpha D_a^\beta X(t) &= I_a^{\alpha+m-\beta} X^{(m)}(t) \\
&= I_a^{1-(\beta-\alpha)} I_a^{m-1} X^{(m)}(t) \\
&= I_a^{1-(\beta-\alpha)}\left[X^{m-m+1}(t) - \sum_{j=0}^{m-2} \frac{(t-a)^j}{\Gamma(j+1)} X^{(m-m+1+j)}(a)\right] \\
&= I_a^{1-(\beta-\alpha)} X^1(t) - \sum_{j=0}^{m-2} \frac{(t-a)^{1-\beta+\alpha+j}}{\Gamma(j+2-\beta+\alpha)} X^{(j+1)}(a) \\
&= D_a^{\beta-\alpha} X(t) - \sum_{j=1}^{m-1} \frac{(t-a)^{m-\beta-n+\alpha+j}}{\Gamma(m-\beta-n+\alpha+j+1)} X^{(m-n+j)}(a) \; .
\end{aligned}$$

Let $\beta \in (m-1, m)$ and $\alpha \in (n-1, n)$ where $\alpha > 1$ and $n \neq m$. Using equation (4.3) we have

$$\begin{aligned}
I_a^{\alpha} D_a^{\beta} X(t) &= I_a^{\alpha}\, I_a^{m-\beta} X^{(m)}(t) \\
&= I_a^{m-\beta+\alpha-n+1} I_a^{n-1} X^{(m)}(t) \\
&= I_a^{m-\beta+\alpha-n+1} \left[X^{(m-n+1)}(t) - \sum_{j=0}^{n-2} \frac{(t-a)^j}{\Gamma(j+1)} X^{(m-n+j+1)}(a) \right] \\
&= I_a^{m-\beta+\alpha-n+1} X^{(m-n+1)}(t) \\
&\quad - \sum_{j=0}^{n-2} \frac{(t-a)^{m-\beta-n+\alpha+1+j}}{\Gamma(m-\beta+\alpha-n+j+2)} X^{(m-n+j+1)}(a) \\
&= D_a^{\beta-\alpha} X(t) - \sum_{j=1}^{n-1} \frac{(t-a)^{m-\beta-n+\alpha+j}}{\Gamma(m-\beta-n+\alpha+j+1)} X^{(m-n+j)}(a)\ .
\end{aligned}$$

■

Hafiz et al. (2001) and Hafiz (2004) are only able to consider a situation like that in Part (a) of the theorem that we have just completed because they only define the fractional derivative for orders $\beta \in (0,1]$. Parts (b), (c) and (d) are extensions that result from our mean-square fractional derivative being defined for $\beta > 0$. In Part (a) we have used the exact method that is used to prove Theorem 4.5(i) of Hafiz et al. (2001) and Theorem 3.3(1) of Hafiz (2004) where it states – incorrectly – that $I_a^{\alpha} D_a^{\beta} X(t) = D_a^{\beta-\alpha} X(t)$ for $\alpha \in (0,1]$ and $\beta \in (0,1]$ when $\alpha \leq \beta$. The reason this is incorrect is that it includes the case when $\alpha = \beta = 1$. This leads back to the discussion at the end of Section 4.1 where we looked at the necessity of defining $D_a^{\beta} X(t)$ to be $X^{(m)}(t)$ when $\beta = m \in \mathbb{N}$. In Hafiz et al. (2001), Hafiz (2004) and El-Sayed et al. (2005) the RH m.s. fractional derivative (referred to as the Caputo fractional derivative in said papers) is defined as

$$D_a^{\beta} X(t) = I_a^{1-\beta} \dot{X}(t)\ , \qquad \beta \in (0,1]. \tag{4.4}$$

Note the inclusion of $\beta = 1$. If we simply allow β to be 1 in equation (4.4) we get

$$D_a^{\beta} X(t) = I_a^{1-\beta} \dot{X}(t) = I_a^{0} \dot{X}(t) = \dot{X}(t).$$

So the definition given in equation (4.4) *seems* reasonable. Using this definition we find that, for $\alpha \in (0,1]$ and $\beta \in (0,1]$ where $\alpha \leq \beta$,

$$I_a^{\alpha} D_a^{\beta} X(t) = I_a^{\alpha} I_a^{1-\beta} \dot{X}(t) = I_a^{1-(\beta-\alpha)} \dot{X}(t) = D_a^{\beta-\alpha} X(t)\ . \tag{4.5}$$

If we let $\alpha = \beta = 1$ in (4.5) we get

$$I_a^{\alpha} D_a^{\beta} X(t) = D_a^{\beta-\alpha} X(t) = D_a^{1-1} X(t) = D_a^{0} X(t) = X(t)\ .$$

However, using IBP we know that

$$I_a^\alpha D_a^\beta X(t) = I_a^1 D_a^1 X(t) = X(t) - X(a)\ .$$

This does not, in general, equal $X(t)$. The mistake that Hafiz et al. (2001) and Hafiz (2004) have made has been to treat the cases when α and/or β are positive integers the same as the cases when they are not integers – a mistake that is probably, in part, caused by not using the definition $D_a^\beta X(t) = X^{(m)}(t)$ when $\beta = m \in \mathbb{N}$. Our definition of $D_a^\beta X(t)$ gives separate expressions for $\beta = m \in \mathbb{N}$ and $\beta \notin \mathbb{N}$ and so for the remainder of the chapter we will do as we have done for Theorem 4.7 and Theorem 4.8 – we will, when necessary, treat the cases when α and/or β are positive integers separately from the cases when they are not.

Theorem 4.9. *Let $\beta > 0$, $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $X(t)$ be a stochastic process such that $I_a^\beta X(t)$ exists for $t \in [a,b] \subset T$. Then*

$$D_a^\alpha I_a^\beta X(t) = I_a^{\beta-\alpha} X(t)$$

when $\alpha \in (n-1, n]$ and

(a) $\beta = m$ and $n \leq \beta$, or

(b) $\beta \in (m-1, m)$ and $n < \beta$.

Proof

(a) Let $\alpha = n$.
Using Leibniz's rule for $\beta = 1$ we have

$$\frac{d}{dt} I_a^1 X(t) = X(t)\ .$$

For $\beta = 2$ we have

$$\begin{aligned}
\frac{d}{dt} I_a^2 X(t) &= \frac{d}{dt} \int_a^t \frac{(t-s)}{\Gamma(2)} X(s)\,ds = \int_a^t X(s)\,ds + 0 = I_a^1 X(t)\ . \\
\frac{d^2}{dt^2} I_a^2 X(t) &= \frac{d}{dt}\frac{d}{dt} I_a^2 X(t) = \frac{d}{dt} I_a^1 X(t) = X(t)\ .
\end{aligned}$$

For $\beta = 3$ we have

$$\begin{aligned}
\frac{d}{dt} I_a^3 X(t) &= \frac{d}{dt} \int_a^t \frac{(t-s)^2}{\Gamma(3)} X(s)\,ds \\
&= \int_a^t \frac{(t-s)}{\Gamma(2)} X(s)\,ds + 0 \\
&= I_a^2 X(t)\ . \\
\frac{d^2}{dt^2} I_a^3 X(t) &= \frac{d}{dt}\frac{d}{dt} I_a^3 X(t) = \frac{d}{dt} I_a^2 X(t) = I_a^1 X(t) \\
\frac{d^3}{dt^3} I_a^3 X(t) &= \frac{d}{dt}\frac{d^2}{dt^2} I_a^3 X(t) = \frac{d}{dt} I_a^1 X(t) = X(t)\ .
\end{aligned}$$

Continuing in this manner we have, for $\beta = m$ and $\alpha = n$

$$D_a^n I_a^m X(t) \quad = \quad I_a^{m-n} X(t) \qquad \text{for } n \leq m. \tag{4.6}$$

Now let $\alpha \in (n-1, n)$ and $\beta = m$. Using equation (4.6) we have for $n = m$

$$\begin{aligned} D_a^\alpha I_a^\beta X(t) &= I_a^{n-\alpha} \frac{d^n}{dt^n} I_a^m X(t) \\ &= I_a^{m-\alpha} \frac{d^m}{dt^m} I_a^m X(t) \\ &= I_a^{m-\alpha} X(t) \end{aligned}$$

and for $n < m$

$$\begin{aligned} D_a^\alpha I_a^\beta X(t) &= I_a^{n-\alpha} \frac{d^n}{dt^n} I_a^m X(t) \\ &= I_a^{n-\alpha} I_a^{m-n} X(t) \\ &= I_a^{m-\alpha} X(t) \; . \end{aligned}$$

(b) Here we have the conditions

$$n - 1 \; < \; \alpha \; \leq \; n \; \leq \; m - 1 \; < \; \beta \; < \; m \; .$$

Since $n \geq 1$ we have $\beta > 1$.

Let $\alpha = n$. Using Leibniz's rule we have

$$\begin{aligned} \frac{d}{dt} I_a^\beta X(t) &= \frac{d}{dt} \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \\ &= \int_a^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)} X(s)\, ds + \left[\frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s) \right]_{s=t} \\ &= I_a^{\beta-1} X(t) \; , \qquad \beta > 1. \end{aligned} \tag{4.7}$$

Taking the derivative of (4.7) we have

$$\begin{aligned} \frac{d^2}{dt^2} I_a^\beta X(t) &= \frac{d}{dt} \frac{d}{dt} I_a^\beta X(t) \\ &= \frac{d}{dt} I_a^{\beta-1} X(t) \\ &= I_a^{\beta-2} X(t) \; , \qquad \beta > 2 \end{aligned} \tag{4.8}$$

where (4.7) has been used – with β replaced by $(\beta - 1)$ – in the last step.

Taking the derivative of (4.8) we have

$$\begin{aligned}\frac{d^3}{dt^3} I_a^{\beta} X(t) &= \frac{d}{dt}\,\frac{d^2}{dt^2} I_a^{\beta} X(t)\\ &= \frac{d}{dt} I_a^{\beta-2} X(t)\\ &= I_a^{\beta-3} X(t)\ , \qquad\qquad \beta > 3\end{aligned}$$

where (4.7) has been used – with β replaced by $(\beta - 2)$ – in the last step.

Continuing in this manner we have

$$\frac{d^n}{dt^n} I_a^{\beta} X(t) = I_a^{\beta-n} X(t)\ , \qquad\qquad \beta > n. \tag{4.9}$$

Now let $\alpha \in (n-1, n)$ and $\beta \in (m-1, m)$ where $\beta > n$. Using equation (4.9) we have

$$\begin{aligned}D_a^{\alpha} I_a^{\beta} X(t) &= I_a^{n-\alpha} \frac{d^n}{dt^n} I_a^{\beta} X(t)\\ &= I_a^{n-\alpha} I_a^{\beta-n} X(t)\\ &= I_a^{\beta-\alpha} X(t)\ .\end{aligned}$$

■

Corollary 4.2. *Let $\beta \in (m-1, m]$, $m \in \mathbb{N}$, and let $X(t)$ be a second order stochastic process such that $I_a^{\beta} X(t)$ exists for $t \in [a, b] \subset T$. Then for $j \in \{0, 1, 2, ..., m-1\}$*

$$\frac{d^j}{dt^j} I_a^{\beta} X(t)|_{t=a} = 0\ .$$

Proof:
Using the definition of $D_a^0 X(t)$ we have $D_a^0 I_a^{\beta} X(t)|_{t=0} = I_a^{\beta} X(t)|_{t=0} = 0$.
Using Parts (a) and (b) of Theorem 4.9 we have

$$\frac{d^j}{dt^j} I_a^{\beta} X(t) = I_a^{\beta-j} X(t) \qquad\qquad j \in \{1, 2, ..., m-1\}\ .$$

Using Part (c) of Theorem 4.1 we have

$$\frac{d^j}{dt^j} I_a^{\beta} X(t)|_{t=a} = I_a^{\beta-j} X(t)|_{t=a} = 0 \qquad\qquad j \in \{1, 2, ..., m-1\}\ .$$

■

Using Corollary 4.2 we see that the result in Theorem 4.7 becomes $I_a^{\alpha} D_a^{\beta} X(t) = I_a^{\alpha-\beta} X(t)$ if $X(t)$ is itself a mean-square fractional integral or derivative.

Theorem 4.10. *Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ where $m < n$. Let $X(t)$ be a second-order stochastic process such that $X^{(n-m)}(t)$ exists and is mean-square continuous on $[a,b] \subset T$. Then, for $\alpha \in (n-1, n]$,*

$$D_a^{\alpha} I_a^m X(t) = D_a^{\alpha-m} X(t) \; .$$

Proof

Let $\alpha = n$. Using Leibniz's rule we have for $m = 1$

$$\begin{aligned}
\frac{d^2}{dt^2} I_a^1 X(t) &= \frac{d}{dt}\frac{d}{dt} I_a^1 X(t) = \dot{X}(t) \\
\frac{d^3}{dt^3} I_a^1 X(t) &= \frac{d}{dt}\frac{d^2}{dt^2} I_a^1 X(t) = \frac{d}{dt}\dot{X}(t) = \ddot{X}(t) \\
\frac{d^4}{dt^4} I_a^1 X(t) &= \frac{d}{dt}\frac{d^3}{dt^3} I_a^1 X(t) = \frac{d}{dt}\ddot{X}(t) = X^{(3)}(t) \\
\vdots \quad & \quad \vdots \\
\frac{d^n}{dt^n} I_a^1 X(t) &= X^{(n-1)}(t) \; , \qquad\qquad n \geq 1.
\end{aligned}$$

Using equation (4.6) we have for $m = 2$

$$\begin{aligned}
\frac{d^3}{dt^3} I_a^2 X(t) &= \frac{d}{dt}\frac{d^2}{dt^2} I_a^2 X(t) = \frac{d}{dt} X(t) = \dot{X}(t) \\
\frac{d^4}{dt^4} I_a^2 X(t) &= \frac{d}{dt}\frac{d^3}{dt^3} I_a^2 X(t) = \frac{d}{dt}\dot{X}(t) = \ddot{X}(t) \\
\frac{d^5}{dt^5} I_a^2 X(t) &= \frac{d}{dt}\frac{d^4}{dt^4} I_a^2 X(t) = \frac{d}{dt}\ddot{X}(t) = X^{(3)}(t) \\
\vdots \quad & \quad \vdots \\
\frac{d^n}{dt^n} I_a^2 X(t) &= X^{(n-2)}(t) \; , \qquad\qquad n \geq 2.
\end{aligned}$$

Continuing in this manner we have for $n \geq m$

$$\frac{d^n}{dt^n} I_a^m X(t) = X^{(n-m)}(t) \; . \tag{4.10}$$

When $\alpha \in (n-1, n)$ and $n > m$ we have

$$\begin{aligned}
D_a^{\alpha} I_a^m X(t) &= I_a^{n-\alpha} \frac{d^n}{dt^n} I_a^m X(t) \\
&= I_a^{n-\alpha} X^{(n-m)}(t)
\end{aligned}$$

where we have used equation (4.10) to find the last line.

Since $\alpha - m \in (n-m-1, n-m)$ we have

$$I_a^{n-\alpha} X^{(n-m)}(t) = I_a^{(n-m)-(\alpha-m)} X^{(n-m)}(t) = D_a^{\alpha-m} X(t)$$

and so $D_a^{\alpha} I_a^m X(t) = D_a^{\alpha-m} X(t)$ when $\alpha \in (n-1, n)$. ∎

The result we derive next is an extension of Theorem 3.4 of Hafiz et al. (2001). It, like our Theorems 4.10 and 4.9, deals with m.s. derivatives of m.s. fractional integrals.

Let $X(t)$ be a stochastic process such that $X^{(n)}(t)$, $n \in \mathbb{N}$, exists and is mean-square continuous. Since $X^{(n)}(t)$ exists, all $X^{(j)}(t)$, $j \in \{0, 1, 2, ..., n-1\}$, exist and are mean-square continuous. Thus $I_a^\gamma X(t)$ and $I_a^\gamma X^{(j)}(t)$ exist and are mean square continuous for all $\gamma \geq 1$ and $j \in \{0, 1, 2, ..., n-1\}$.

Let $\beta > 1$. Using IBP we have

$$\begin{aligned} I_a^\beta \dot{X}(t) &= \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\Big|_a^t + \int_a^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)} X(s)\, ds \\ &= I_a^{\beta-1} X(t) - \frac{(t-a)^{\beta-1}}{\Gamma(\beta)} X(a) . \end{aligned} \tag{4.11}$$

Using Leibniz's rule we have

$$\begin{aligned} \frac{d}{dt} I_a^\beta X(t) &= \frac{d}{dt} \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\ ds \\ &= \int_a^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)} X(s)\ ds + \left[\frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\right]_{s=t} \\ &= I_a^{\beta-1} X(t) . \end{aligned} \tag{4.12}$$

Substituting (4.11) into (4.12) we have

$$\frac{d}{dt} I_a^\beta X(t) = I_a^\beta \dot{X}(t) + \frac{(t-a)^{\beta-1}}{\Gamma(\beta)} X(a) . \tag{4.13}$$

Taking the derivative of (4.13) we have

$$\begin{aligned} \frac{d^2}{dt^2} I_a^\beta X(t) &= \frac{d}{dt} I_a^\beta \dot{X}(t) + \frac{d}{dt}\left[\frac{(t-a)^{\beta-1}}{\Gamma(\beta)} X(a)\right] \\ &= I_a^\beta \ddot{X}(t) + \frac{(t-a)^{\beta-1}}{\Gamma(\beta)} \dot{X}(a) + \frac{(t-a)^{\beta-2}}{\Gamma(\beta-1)} X(a) \end{aligned} \tag{4.14}$$

where we have used (4.13) – with $X(t)$ replaced by $\dot{X}(t)$ – to find an expression for $\frac{d}{dt} I_a^\beta \dot{X}(t)$.
Taking the derivative of (4.14) we have

$$\begin{aligned} \frac{d^3}{dt^3} I_a^\beta X(t) &= \frac{d}{dt} I_a^\beta \ddot{X}(t) + \frac{d}{dt}\left[\frac{(t-a)^{\beta-1}}{\Gamma(\beta)} \dot{X}(a) + \frac{(t-a)^{\beta-2}}{\Gamma(\beta-1)} X(a)\right] \\ &= I_a^\beta X^{(3)}(t) + \frac{(t-a)^{\beta-1}}{\Gamma(\beta)} \ddot{X}(a) + \frac{(t-a)^{\beta-2}}{\Gamma(\beta-1)} \dot{X}(a) \\ &\quad + \frac{(t-a)^{\beta-3}}{\Gamma(\beta-2)} X(a) \end{aligned}$$

where we have used (4.13) – with $X(t)$ replaced by $\ddot{X}(t)$ – to find an expression for $\frac{d}{dt} I_a^\beta \ddot{X}(t)$.

Continuing in this manner we have the following result.

Theorem 4.11. *Let $n \in \mathbb{N}$ and let $X(t)$ be a second-order stochastic process such that $X^{(n)}(t)$ exists and is mean-square continuous on $t \in [a,b] \subset T$. Then, for $\beta > 1$,*

$$I_a^\beta X^{(n)}(t) = \frac{d^n}{dt^n} I_a^\beta X(t) - \sum_{j=0}^{n-1} \frac{(t-a)^{\beta-n+j}}{\Gamma(\beta-n+j+1)} X^{(j)}(a) \ .$$

■

Note that if $\frac{d^j}{dt^j} X(t)|_{t=a} = 0$ for $j \in \{0,1,2,\ldots,n-1\}$, as is the case when $X(t)$ is itself a m.s. fractional integral or derivative, the result will be $I_a^\beta X^{(n)}(t) = \frac{d^n}{dt^n} I_a^\beta X(t)$.

Theorem 4.11 gives an expression for $\frac{d^n}{dt^n} I_a^\beta X(t)$ when $\beta \in \{1,2,3,\ldots,n\}$ – something for which an expression had not previously been found. It also gives alternative expressions for $\frac{d^n}{dt^n} I_a^\beta X(t)$ when $\beta \geq n$ and for $I_a^\beta X^{(n)}(t)$ when $\beta > 1$.

In Theorem 4.11 $\beta = 1$ has not been included. This is because the result does not, in general, hold for $\beta = 1$ as we show next. The method we will use is the same as that used to prove Theorem 4.11.

Let $\beta = 1$. Using IBP we have

$$I_a^1 \dot{X}(t) = X(t) - X(a) \tag{4.15}$$

and using Leibniz's rule we have

$$\frac{d}{dt} I_a^1 X(t) = \frac{d}{dt} \int_a^t X(s)\, ds = X(t) \ . \tag{4.16}$$

Substituting equation (4.15) into equation (4.16) we have

$$\frac{d}{dt} I_a^1 X(t) = I_a^1 \dot{X}(t) + X(a) \ . \tag{4.17}$$

Now, taking the derivative of equation (4.17) we get

$$\begin{aligned} \frac{d^2}{dt^2} I_a^1 X(t) &= \frac{d}{dt} I_a^1 \dot{X}(t) + \frac{d}{dt} X(a) \\ &= I_a^1 \ddot{X}(t) + \dot{X}(a) + 0 \end{aligned} \tag{4.18}$$

where we have used equation (4.17) to find an expression for $\frac{d}{dt} I_a^1 \dot{X}(t)$.
Taking the derivative of equation (4.18) we have

$$\begin{aligned} \frac{d^3}{dt^3} I_a^1 X(t) &= \frac{d}{dt} I_a^1 \ddot{X}(t) + \frac{d}{dt} \dot{X}(a) \\ &= I_a^1 X^{(3)}(t) + X^{(2)}(a) + 0 \end{aligned}$$

where we have used equation (4.17) to find an expression for $\frac{d}{dt} I_a^1 \ddot{X}(t)$. Following this pattern we see that

$$\frac{d^n}{dt^n} I_a^1 X(t) = I_a^1 X^{(n)}(t) + X^{(n-1)}(a) \; .$$

Clearly, only when $n = 1$ will the expression in Theorem 4.11 hold for $\beta = 1$.

IBP is used in the proof of Theorem 4.11. Due to this Theorem 4.11 will not hold for $\beta \in (0, 1)$. Hafiz et al. (2001) include the $\beta \in (0, 1)$ range despite stating that IBP is used to prove the result.
Since Theorem 4.11 is consistent with previous work for non-integer values of β that are greater than 1, we will formally extend it to include the range $\beta \in (0, 1)$.

Theorem 4.11 and its formal extension, can be used to derive many other properties – one of which we show next.

Theorem 4.12. *Let $n \in \mathbb{N}$ and let $X(t)$ be a second-order stochastic process such that $X^{(n)}(t)$ exists and is mean-square continuous on $[a, b] \subset T$. Then for $\alpha \in (n - 1, n)$*

$$D_a^\alpha I_a^\alpha X(t) = X(t) \; .$$

Proof

$$\begin{aligned} D_a^\alpha I_a^\alpha X(t) &= I_a^{n-\alpha} \frac{d^n}{dt^n} \, I_a^\alpha X(t) \\ &= I_a^{n-\alpha} \left[I_a^\alpha X^{(n)}(t) + \sum_{j=0}^{n-1} \frac{(t-a)^{\alpha-n+j}}{\Gamma(\alpha - n + j + 1)} X^{(j)}(a) \right] \\ &= I_a^n X^{(n)}(t) + \sum_{j=0}^{n-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a) \\ &= \left[\, X(t) - \sum_{j=0}^{n-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a) \, \right] + \sum_{j=0}^{n-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a) \\ &= X(t) \; . \end{aligned}$$

■

In Theorem 4.9 we proved that $D_a^n I_a^n X(t) = X(t)$. Therefore, by combining Theorem 4.9 with Theorem 4.12 we see that $D_a^\alpha I_a^\alpha X(t) = X(t)$ for $\alpha > 0$.

In the following two theorems we find expressions for $D_a^\alpha D_a^\beta X(t)$. In Theorem 4.13 we will consider the case when $\beta = m$, $m \in \mathbb{N}$, and in Theorem 4.14 we will consider the case when $\beta \in (m - 1, m)$.

Theorem 4.13. *Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ and let $X(t)$ be a second order stochastic process such that $X^{(m+n)}(t)$ exists for $t \in [a, b] \subset T$. Then for $\alpha > 0$*

$$D_a^\alpha X^{(m)}(t) = D_a^{\alpha+m} X(t) \; .$$

Proof:
When $\alpha = n$ the result clearly holds.

$$\text{If} \quad n-1 < \alpha < n \qquad \text{then} \qquad n-1+m \, < \, \alpha+m \, < \, n+m$$

and so

$$D_a^{\alpha+m} X(t) = I_a^{(n+m)-(\alpha+m)} X^{(n+m)}(t) = I_a^{n-\alpha} X^{(n+m)}(t) \; .$$

Using this we have for $n-1 < \alpha < n$

$$D_a^{\alpha} X^{(m)}(t) = I_a^{n-\alpha} \frac{d^n}{dt^n} X^{(m)}(t) = I_a^{n-\alpha} X^{(n+m)}(t) = D_a^{\alpha+m} X(t) \; .$$

■

Theorem 4.14. *Let $\alpha \in (n-1, n]$ and $\beta \in (m-1, m)$ where $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $X(t)$ be a second-order stochastic process such that $X^{(m+n)}(t)$ exists and is mean-square continuous on $[a, b] \subset T$. Then*

$$(a) \quad D_a^{\alpha} D_a^{\beta} X(t) = D_a^{\alpha+\beta} X(t)$$

for $\alpha \in (0,1)$ and $\alpha + \beta \in (m-1, m]$.

$$(b) \quad D_a^{\alpha} D_a^{\beta} X(t) = D_a^{\alpha+\beta} X(t) + \frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)} X^{(m)}(a)$$

for $\alpha \in (0,1)$ and $\alpha + \beta \in (m, m+1)$.

$$(c) \quad D_a^{\alpha} D_a^{\beta} X(t) = D_a^{\alpha+\beta} X(t) + \sum_{j=0}^{n-1} \frac{(t-a)^{m-\beta-n+j}}{\Gamma(m-\beta-n+1+j)} X^{(m+j)}(a)$$

for $\alpha = n$.

Proof:

(a)
For $\alpha \in (0,1)$

$$\begin{aligned} D_a^{\alpha} D_a^{\beta} X(t) &= I_a^{1-\alpha} \left[\frac{d}{dt} \, I_a^{m-\beta} X^{(m)}(t) \right] \\ &= I_a^{1-\alpha} \left[I_a^{m-\beta} X^{(m+1)}(t) + \frac{(t-a)^{m-\beta-1}}{\Gamma(m-\beta)} X^{(m)}(a) \right] \\ &= I_a^{m-(\alpha+\beta)} I_a^{1} X^{(m+1)}(t) + \frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)} X^{(m)}(a) \; . \end{aligned} \tag{4.19}$$

Thus, when $\alpha+\beta=m$ we have

$$\begin{aligned}
D_a^{\alpha} D_a^{\beta} X(t) &= I_a^1 X^{(m+1)}(t) + X^{(m)}(a) \\
&= X^{(m)}(t) - X^{(m)}(a) + X^{(m)}(a) \\
&= X^{(m)}(t) \\
&= D_a^{\alpha+\beta} X(t) \ .
\end{aligned}$$

When $\alpha+\beta \in (m-1,m)$ the first term on the right hand side of (4.19) becomes

$$\begin{aligned}
I_a^{1-\alpha+m-\beta} X^{(m+1)}(t) &= I_a^{m-(\alpha+\beta)} \left[X^{(m)}(t) - X^{(m)}(a)\right] \\
&= I_a^{m-(\alpha+\beta)} X^{(m)}(t) - \frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)} X^{(m)}(a) \\
&= D_a^{\alpha+\beta} X(t) - \frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)} X^{(m)}(a) \ .
\end{aligned}$$

Substituting this into equation (4.19) we have, for $\alpha+\beta \in (m-1,m)$,

$$\begin{aligned}
D_a^{\alpha} D_a^{\beta} X(t) &= D_a^{\alpha+\beta} X(t) - \frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)} X^{(m)}(a) \\
&\quad + \frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)} X^{(m)}(a) \\
&= D_a^{\alpha+\beta} X(t) \ .
\end{aligned}$$

(b)
If $\alpha \in (0,1)$ and $\alpha+\beta \in (m,m+1)$ then

$$\begin{aligned}
D_a^{\alpha} D_a^{\beta} X(t) &= I_a^{1-\alpha} \left[\frac{d}{dt} I_a^{m-\beta} X^{(m)}(t)\right] \\
&= I_a^{1-\alpha} \left[I_a^{m-\beta} X^{(m+1)}(t) + \frac{(t-a)^{m-\beta-1}}{\Gamma(m-\beta)} X^{(m)}(a)\right] \\
&= I_a^{(m+1)-(\alpha+\beta)} X^{(m+1)}(t) + \frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)} X^{(m)}(a) \\
&= D_a^{\alpha+\beta} X(t) + \frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)} X^{(m)}(a) \ .
\end{aligned}$$

(c)
When $\alpha = n$ we have

$$\begin{aligned}
D_a^\alpha D_a^\beta X(t) &= \frac{d^n}{dt^n} I_a^{m-\beta} X^{(m)}(t) \\
&= I_a^{m-\beta} \frac{d^n}{dt^n} X^{(m)}(t) + \sum_{j=0}^{n-1} \frac{(t-a)^{m-\beta-n+j}}{\Gamma(m-\beta-n+j+1)} X^{(m+j)}(a) \\
&= I_a^{(m+n)-(\beta+n)} X^{(m+n)}(t) \\
&\quad + \sum_{j=0}^{n-1} \frac{(t-a)^{m-\beta-n+j}}{\Gamma(m-\beta-n+j+1)} X^{(m+j)}(a) \\
&= D_a^{n+\beta} X(t) + \sum_{j=0}^{n-1} \frac{(t-a)^{m-\beta-n+j}}{\Gamma(m-\beta-n+j+1)} X^{(m+j)}(a) \ .
\end{aligned}$$

■

Theorem 4.14 completes our exploration of the composition rule. The properties in Sections 4.1 and 4.2 are only a selection of the properties of m.s. fractional integrals and derivatives. In the following chapter we will consider one last property.

Chapter 5

Integrals and derivatives of products

In Section 5.5 of Oldham & Spanier (1974) expressions are found for the fractional integral and the fractional derivative of the product of two functions (where the two functions possess certain qualities). The aim of this chapter is to find expressions for $I_a^\beta[f(t)X(t)]$ and $D_a^\beta[f(t)X(t)]$ where $f(t)$ is a deterministic fuction and $X(t)$ is a second-order stochastic process. In Section 5.1 we will use $\beta \in \mathbb{N}$ and $f(t)$ a continuous function of t. We will see that these expressions have the same form as those of the fractional integral and derivative of the product of two deterministic functions. In Sections 5.2 and 5.3 we will build on the results of Section 5.1 extending $\beta \in \mathbb{N}$ to $\beta > 0$. In Sections 5.2 and 5.3 we will, however, only work with $f(t)$ a polynomial in t.

5.1 The case in which $\beta \in \mathbb{N}$

From integration by parts we know that for $X_1(t)$ a m.s. differentiable s.p. and $f(t)$ a deterministic function for which $\frac{d^j}{dt^j}f(t)$ exists and is m.s. continuous for $j \in \mathbb{N}$, we have

$$\int_a^t f(s)\dot{X}_1(s)\,ds = f(s)X_1(s)\Big|_a^t - \int_a^t \frac{df(s)}{ds}X_1(s)\,ds. \tag{5.1}$$

Let

$$X_1(t) = \int_a^t X(s)\,ds = I_a^1 X(t) \tag{5.2}$$

where $X(t)$ is m.s. continuous. Clearly $X_1(t)$ thus defined is indeed a second-order m.s. continuous stochastic process. Futher, using Lebniz's rule we find that $\dot{X}_1(t) = X(t)$. Substituting (5.2) into (5.1) we thus have

$$\begin{aligned}\int_a^t f(s)X(s)\,ds &= \left[f(s)\int_a^s X(y)\,dy\right]_{s=a}^{s=t} - \int_a^t \frac{df(s)}{ds}\left[\int_a^s X(y)\,dy\right]ds \\ &= f(t)\int_a^t X(y)\,dy - 0 - \int_a^t \frac{df(s)}{ds}\left[\int_a^s X(y)\,dy\right]ds\,.\end{aligned}$$

Now, we notice that in the above expression

$$\text{LHS} = I_a^1[f(t)X(t)]$$

and

$$\text{RHS} = f(t)I_a^1X(t) - I_a^1\left[\frac{df(t)}{dt}I_a^1X(t)\right]$$

so that we have

$$I_a^1[f(t)X(t)] = f(t)I_a^1X(t) - I_a^1\left[\frac{df(t)}{dt}I_a^1X(t)\right]. \tag{5.3}$$

If we replace $X(t)$ by $I_a^1X(t)$ and $f(t)$ with $\frac{df(t)}{dt}$ in (5.3) we have

$$\begin{aligned}I_a^1\left[\frac{df(t)}{dt}I_a^1X(t)\right] &= \frac{df(t)}{dt}I_a^1[I_a^1X(t)] - I_a^1\left[\frac{d^2f(t)}{dt^2}I_a^1[I_a^1X(t)]\right] \\ &= \frac{df(t)}{dt}I_a^2X(t) - I_a^1\left[\frac{d^2f(t)}{dt^2}I_a^2X(t)\right].\end{aligned} \tag{5.4}$$

Substituting (5.4) back into (5.3) we have

$$I_a^1[f(t)X(t)] = f(t)I_a^1X(t) - \frac{df(t)}{dt}I_a^2X(t) + I_a^1\left[\frac{d^2f(t)}{dt^2}I_a^2X(t)\right]. \tag{5.5}$$

If we replace $X(t)$ with $I_a^2X(t)$ and $f(t)$ with $\frac{d^2f(t)}{dt^2}$ in (5.3) we have

$$I_a^1\left[\frac{d^2f(t)}{dt^2}I_a^2X(t)\right] = \frac{d^2f(t)}{dt^2}I_a^3X(t) - I_a^1\left[\frac{d^3f(t)}{dt^3}I_a^3X(t)\right].$$

Using this, (5.5) becomes

$$I_a^1[f(t)X(t)] = f(t)I_a^1X(t) - \frac{df(t)}{dt}I_a^2X(t) + \frac{d^2f(t)}{dt^2}I_a^3X(t) - I_a^1\left[\frac{d^3f(t)}{dt^3}I_a^3X(t)\right].$$

Continuing in this manner we arrive at the following expression for the m.s. integral of the product $f(t)X(t)$ where $X(t)$ is m.s. continuous and $f(t)$ is such that $\frac{d^j}{dt^j}f(t)$ exists and is continuous for $j \in \mathbb{N}$:

$$I_a^1[f(t)X(t)] = \sum_{j=0}^{\infty}(-1)^j\,\frac{d^jf(t)}{dt^j}I_a^{j+1}X(t)\,. \tag{5.6}$$

Using the fact that (Feller, 1957)

$$\binom{j-\beta-1}{j} = (-1)^j \binom{\beta}{j}$$

we can express (5.6) as

$$I_a^1[f(t)X(t)] = \sum_{j=0}^{\infty} \binom{-1}{j} \frac{d^j f(t)}{dt^j}\, I_a^{j+1} X(t)\ . \tag{5.7}$$

In order to find an expression for $I_a^2[f(t)X(t)]$ we take the integral of both sides of (5.7). Doing this we have

$$\begin{aligned} \text{LHS} &= I_a^1\Big[I_a^1[f(t)X(t)]\Big] = I_a^2[f(t)X(t)] \\ \text{RHS} &= \sum_{j=0}^{\infty} \binom{-1}{j} I_a^1\left[\frac{d^j f(t)}{dt^j}\, I_a^{j+1} X(t)\right] . \end{aligned}$$

Using (5.7) – with $f(t)$ replaced by $\frac{d^j}{dt^j} f(t)$ and $X(t)$ replaced by $I_a^{j+1} X(t)$ – we have

$$\begin{aligned} \text{RHS} &= \sum_{j=0}^{\infty} \binom{-1}{j} I_a^1\left[\frac{d^j f(t)}{dt^j}\, I_a^{j+1} X(t)\right] \\ &= \sum_{j=0}^{\infty} \binom{-1}{j} \sum_{k=0}^{\infty} \binom{-1}{k} \left[\frac{d^k}{dt^k} \frac{d^j f(t)}{dt^j}\right] I_a^{k+1}\, I_a^{j+1} X(t) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{-1}{k} \binom{-1}{j} \frac{d^{k+j} f(t)}{dt^{k+j}}\, I_a^{k+j+2} X(t)\ . \end{aligned}$$

Letting $k + j = m$ we then have

$$\text{RHS} = \sum_{j=0}^{\infty} \sum_{m=j}^{\infty} \binom{-1}{j} \binom{-1}{m-j} \frac{d^m f(t)}{dt^m}\, I_a^{m+2} X(t)\ .$$

Noting that

$$\sum_{j=0}^{\infty} \sum_{m=j}^{\infty} = \sum_{m=0}^{\infty} \sum_{j=0}^{m}$$

and

$$\sum_{j=0}^{m} \binom{\beta}{j} \binom{\alpha}{m-j} = \binom{\beta+\alpha}{m}$$

we have

$$\begin{aligned} \text{RHS} &= \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{-1}{j} \binom{-1}{m-j} \frac{d^m f(t)}{dt^m}\, I_a^{m+2} X(t) \\ &= \sum_{m=0}^{\infty} \binom{-2}{m} \frac{d^m f(t)}{dt^m}\, I_a^{m+2} X(t)\ . \end{aligned}$$

Letting LHS=RHS we then have

$$I_a^2[f(t)X(t)] \quad = \quad \sum_{m=0}^{\infty} \binom{-2}{m} \frac{d^m f(t)}{dt^m} \, I_a^{m+2} X(t) \ .$$

Continuing in this fashion we have the following expression for the repeated integral of the product $f(t)X(t)$:

$$I_a^n[f(t)X(t)] = \sum_{j=0}^{\infty} \binom{-n}{j} \frac{d^j f(t)}{dt^j} \, I_a^{n+j} X(t) \ , \qquad n \in \mathbb{N}. \tag{5.8}$$

Now, for $X(t)$ a m.s. differentiable stochastic process, the derivative of $f(t)X(t)$ is given by the expression

$$\frac{d}{dt}[f(t)X(t)] = f(t)\dot{X}(t) + \frac{df(t)}{dt} X(t) \ . \tag{5.9}$$

A derivation of this result is given in Soong (1973). If $\ddot{X}(t)$ exists we can take the derivative of both sides of this expression to get

$$\begin{aligned} \frac{d^2}{dt^2}[f(t)X(t)] \quad &= \quad \frac{d}{dt} f(t)\dot{X}(t) + \frac{d}{dt}\left[\frac{df(t)}{dt} X(t)\right] \\ &= \quad \frac{d^2 f(t)}{dt^2} X(t) + 2\frac{df(t)}{dt} \ \dot{X}(t) + f(t)\ddot{X}(t) \end{aligned} \tag{5.10}$$

where we have used (5.9) to find expressions for both of the terms on the RHS of (5.10). Continuing in this manner we find that, for $X(t)$ a s.p. for which $X^{(n)}(t)$ exists,

$$D_a^n[f(t)X(t)] = \sum_{j=0}^{n} \binom{n}{j} \frac{d^{n-j} f(t)}{dt^{n-j}} X^{(j)}(t) \ .$$

Since $\binom{n}{j} = 0$ when $j > n$ we have

$$D_a^n[f(t)X(t)] = \sum_{j=0}^{\infty} \binom{n}{j} \frac{d^{n-j} f(t)}{dt^{n-j}} X^{(j)}(t) \ , \qquad n \in \mathbb{N},$$

or equivalently

$$D_a^n[f(t)X(t)] = \sum_{j=0}^{\infty} \binom{n}{j} \frac{d^{j} f(t)}{dt^{j}} X^{(n-j)}(t) \ , \qquad n \in \mathbb{N} \ . \tag{5.11}$$

Had we chosen to denote integrals as negative derivatives i.e. instead of using $I_a^n[\cdot]$ to denote our n^{th} repeated integral we used $D_a^{-n}[\cdot]$, then (5.8) could be written as

$$D_a^{-n}[f(t)X(t)] = \sum_{n=0}^{\infty} \binom{-n}{j} \frac{d^j f(t)}{dt^j} X^{(-n-j)}(t) \ , \qquad n \in \mathbb{N}$$

so that we could combine the integral and derivative cases together to give the expression

$$D_a^n[f(t)X(t)] = \sum_{n=0}^{\infty} \binom{n}{j} \frac{d^j f(t)}{dt^j} X^{(n-j)}(t) , \qquad n \in \mathbb{Z}$$

where $D_a^0[f(t)X(t)] = f(t)X(t)$. This is consistent with the form given by Oldham & Spanier (1974) for the n^{th} integral and derivative of the product of two deterministic functions.

5.2 The case in which $\beta > 0$

We can now attempt to generalize (5.8) and (5.11) to arbitrary order $\beta > 0$ when $f(t)$ is a polynomial.

Let $f(t) = t$ and $\beta > 0$.

$$\begin{aligned}
I_a^\beta[tX(t)] &= \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\, sX(s)\,ds \\
&= \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\, sX(s)\,ds + t\,I_a^\beta X(t) - t\,I_a^\beta X(t) \\
&= t\,I_a^\beta X(t) - \left[t\int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\,ds - \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\, sX(s)\,ds\right] \\
&= t\,I_a^\beta X(t) - \frac{1}{\Gamma(\beta)}\left[\int_a^t (t-s)^{\beta-1}\, tX(s)ds - \int_a^t (t-s)^{\beta-1}\, sX(s)ds\right] \\
&= t\,I_a^\beta X(t) - \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^\beta X(s)\,ds \\
&= t\,I_a^\beta X(t) - \frac{\beta}{\Gamma(\beta+1)}\int_a^t (t-s)^\beta X(s)\,ds \\
&= t\,I_a^\beta X(t) - \beta\, I_a^{\beta+1} X(t) .
\end{aligned}$$

Above we have shown that for $f(t) = t$

$$\begin{aligned}
I_a^\beta[f(t)X(t)] &= f(t)\,I_a^\beta X(t) - \beta\,\frac{df(t)}{dt}\,I_a^{\beta+1}X(t) \qquad (5.12)\\
&= \sum_{j=0}^{\infty} \binom{-\beta}{j} \frac{d^j f(t)}{dt^j}\, I_a^{\beta+j} X(t) .
\end{aligned}$$

Now let us consider $f(t) = t^2$.

$$I_a^\beta[t^2X(t)] = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\, s^2X(s)\,ds .$$

Adding and subtracting $t^2 I_a^\beta X(t)$ to the RHS above we then get

$$\begin{aligned}
I_a^\beta[t^2 X(t)] &= t^2 I_a^\beta X(t) + \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} s^2 X(s) ds - t^2 \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s) ds \\
&= t^2\, I_a^\beta X(t) - \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, [t^2 - s^2]\, ds \\
&= t^2\, I_a^\beta X(t) - \mathfrak{I}_1
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{I}_1 &= \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, [t^2 - s^2]\, ds \\
&= \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)(t-s)(t+s)\, ds \\
&= \int_a^t \frac{(t-s)^{\beta}}{\Gamma(\beta)} X(s)(t+s)\, ds \\
&= \beta \int_a^t \frac{(t-s)^{\beta}}{\Gamma(\beta+1)} X(s)(t+s)\, ds \\
&= \beta \Big[\, t\, I_a^{\beta+1} X(t) + I_a^{\beta+1}[tX(t)] \,\Big]\, .
\end{aligned}$$

Thus we have

$$I_a^\beta[t^2 X(t)] = t^2\, I_a^\beta X(t) - \beta t\, I_a^{\beta+1} X(t) - \beta\, I_a^{\beta+1}[tX(t)]\, .$$

Using (5.12) on the third term on the RHS of the above we have

$$I_a^\beta[t^2 X(t)] = t^2\, I_a^\beta X(t) - 2\beta t\, I_a^{\beta+1} X(t) + \beta(\beta+1)\, I_a^{\beta+2} X(t)\, .$$

So for $f(t) = t^2$ this is simply

$$\begin{aligned}
I_a^\beta[f(t)X(t)] &= f(t)\, I_a^\beta X(t) - \beta \frac{df(t)}{dt}\, I_a^{\beta+1} X(t) + \frac{\beta(\beta+1)}{2}\, \frac{d^2 f(t)}{dt^2} I_a^{\beta+2}[X(t)] \\
&= \sum_{j=0}^{\infty} \binom{-\beta}{j} \frac{d^j f(t)}{dt^j}\, I_a^{\beta+j} X(t)\, .
\end{aligned}$$

Continuing in this manner we can show that for $f(t) = t^p$, $p \in \mathbb{N}$, and $\beta > 0$ we have

$$I_a^\beta[f(t)X(t)] = \sum_{j=0}^{\infty} \binom{-\beta}{j} \frac{d^j f(t)}{dt^j}\, I_a^{\beta+j} X(t)\, . \tag{5.13}$$

Due to the linearity property of $I_a^\beta[\cdot]$, we can use (5.13) to find $I_a^\beta[f(t)X(t)]$ where $f(t)$ is a polynomial in t. Clearly the stochastic process, $X(t)$, in (5.13) can be replaced by a fractional integral or derivative if those integrals or derivatives are m.s. continuous.

We will now find an expression for $D_a^\beta[f(t)X(t)]$ when $\beta \in (m-1, m)$, $m \in \mathbb{N}$, $f(t)$ is a polynomial of order p and $X(t)$ is a second-order s.p. for which $X^{(m)}(t)$ exists. An expression (or, expressions) for the case(s) when $p > m$ can be found but we will restrict ourselves to the case $p \leq m$. Let us start by considering $f(t) = t$. Using (5.11) and (5.13) we have

$$
\begin{aligned}
D_a^\beta[\,tX(t)\,] &= I_a^{m-\beta}\,\frac{d^m}{dt^m}\,[\,tX(t)\,] \\
&= I_a^{m-\beta}\,[\,tX^{(m)}(t) + mX^{(m-1)}(t)\,] \\
&= I_a^{m-\beta}\,[\,tX^{(m)}(t)\,] + m\,I_a^{m-\beta}X^{(m-1)}(t) \\
&= t\,I_a^{m-\beta}X^{(m)}(t) - (m-\beta)\,I_a^{m-\beta+1}X^{(m)}(t) \\
&\quad +m\,I_a^{m-\beta}X^{(m-1)}(t)\;.
\end{aligned}
$$

Applying integration by parts to $I_a^{m-\beta+1}X^{(m)}(t)$ we have

$$
\begin{aligned}
D_a^\beta[\,tX(t)\,] &= t\,I_a^{m-\beta}X^{(m)}(t) - (m-\beta)\,I_a^{m-\beta}X^{(m-1)}(t) \\
&\quad +(m-\beta)\,X^{(m-1)}(a)\,\frac{(t-a)^{m-\beta}}{\Gamma(m-\beta+1)} + m\,I_a^{m-\beta}X^{(m-1)}(t) \\
&= t\,I_a^{m-\beta}X^{(m)}(t) + \beta\,I_a^{m-\beta}X^{(m-1)}(t) \\
&\quad +(m-\beta)\,X^{(m-1)}(a)\,\frac{(t-a)^{m-\beta}}{\Gamma(m-\beta+1)}\;. \qquad (5.14)
\end{aligned}
$$

Equation (5.14) can be written in the following form:

$$
D_a^\beta[\,f(t)X(t)\,] = \sum_{j=0}^{p} \binom{\beta}{j} \frac{d^j f(t)}{dt^j}\, I_a^{m-\beta}X^{(m-j)}(t) + c_1 \qquad (5.15)
$$

where $f(t) = t$ and

$$
c_1 = (m-\beta)\,X^{(m-1)}(a)\,\frac{(t-a)^{m-\beta}}{\Gamma(m-\beta+1)}\;. \qquad (5.16)
$$

Now let us consider $D_a^\beta[f(t)X(t)]$ when $f(t) = t^2$. Using (5.11) we have

$$
\begin{aligned}
D_a^\beta\big[\,t^2X(t)\,\big] &= I_a^{m-\beta}\frac{d^m}{dt^m}\,\big[\,t^2X(t)\,\big] \\
&= I_a^{m-\beta}\big[\,t^2X^{(m)}(t)\,\big] + 2m\,I_a^{m-\beta}[\,tX^{(m-1)}(t)\,] \\
&\quad +m(m-1)\,I_a^{m-\beta}X^{(m-2)}(t)\;. \qquad (5.17)
\end{aligned}
$$

Using (5.13) and IBP, we have

$$
\begin{aligned}
I_a^{m-\beta}[\,tX^{(m-1)}(t)\,] &= t\,I_a^{m-\beta}X^{(m-1)}(t) - (m-\beta)\,I_a^{m-\beta+1}X^{(m-1)}(t) \\
&= t\,I_a^{m-\beta}X^{(m-1)}(t) - (m-\beta)\,I_a^{m-\beta}X^{(m-2)}(t) \\
&\quad +(m-\beta)\,X^{(m-2)}(a)\,\frac{(t-a)^{m-\beta}}{\Gamma(m-\beta+1)}
\end{aligned}
$$

and

$$\begin{aligned}
I_a^{m-\beta}\left[\, t^2 X^{(m)}(t) \,\right] &= t^2\, I_a^{m-\beta} X^{(m)}(t) - 2t(m-\beta)\, I_a^{m-\beta+1} X^{(m)}(t) \\
&\quad +(m-\beta)(m-\beta+1)\, I_a^{m-\beta+2} X^{(m)}(t) \\
&= t^2\, I_a^{m-\beta} X^{(m)}(t) - 2t(m-\beta)\, I_a^{m-\beta} X^{(m-1)}(t) \\
&\quad +(m-\beta)(m-\beta+1)\, I_a^{m-\beta} X^{(m-2)}(t) \\
&\quad +2t(m-\beta) X^{(m-1)}(a)\, \frac{(t-a)^{m-\beta}}{\Gamma(m-\beta+1)} \\
&\quad -(m-\beta)(m-\beta+1)\, X^{(m-1)}(a)\, \frac{(t-a)^{m-\beta+1}}{\Gamma(m-\beta+2)} \\
&\quad -(m-\beta)(m-\beta+1)\, X^{(m-2)}(a)\, \frac{(t-a)^{m-\beta}}{\Gamma(m-\beta+1)}\ .
\end{aligned}$$

Substituting these into (5.17) and collecting like terms we have

$$\begin{aligned}
D_a^{\beta}\left[t^2 X(t)\right] &= t^2\, I_a^{m-\beta} X^{(m)}(t) + \left[\, 2mt - 2t(m-\beta) \,\right] I_a^{m-\beta} X^{(m-1)}(t) \\
&\quad +[m(m-1) - 2m(m-\beta) \\
&\quad +(m-\beta)(m-\beta+1)]\ I_a^{m-\beta} X^{(m-2)}(t) + 2\, m\, c_2 + c_3 \\
&= t^2\, I_a^{m-\beta} X^{(m)}(t) + 2t\beta\, I_a^{m-\beta} X^{(m-1)}(t) \\
&\quad +\beta(\beta-1)\, I_a^{m-\beta} X^{(m-2)}(t) + 2\, m\, c_2 + c_3
\end{aligned} \tag{5.18}$$

where

$$c_2 = (m-\beta)\, X^{(m-2)}(a)\, \frac{(t-a)^{m-\beta}}{\Gamma(m-\beta+1)} \tag{5.19}$$

and

$$\begin{aligned}
c_3 &= 2t(m-\beta) X^{(m-1)}(a)\, \frac{(t-a)^{m-\beta}}{\Gamma(m-\beta+1)} \\
&\quad -(m-\beta)(m-\beta+1)\, X^{(m-1)}(a)\, \frac{(t-a)^{m-\beta+1}}{\Gamma(m-\beta+2)} \\
&\quad -(m-\beta)(m-\beta+1)\, X^{(m-2)}(a)\, \frac{(t-a)^{m-\beta}}{\Gamma(m-\beta+1)}\ .
\end{aligned} \tag{5.20}$$

We may rewrite (5.18) in the form:

$$D_a^{\beta}[\, f(t) X(t)\,] = \sum_{j=0}^{p} \binom{\beta}{j} \frac{d^j f(t)}{dt^j}\, I_a^{m-\beta} X^{(m-j)}(t) + 2\, m\, c_2 + c_3$$

where c_2 and c_3 are given by (5.19) and (5.20) respectively and $f(t) = t^2$.

For both the cases $f(t) = t$ and $f(t) = t^2$ we thus have

$$D_a^{\beta}[\, f(t) X(t)\,] = \sum_{j=0}^{p} \binom{\beta}{j} \frac{d^j f(t)}{dt^j}\, I_a^{m-\beta} X^{(m-j)}(t) \quad + \quad \text{extra terms.} \tag{5.21}$$

Clearly $D_a^\beta[\,f(t)X(t)\,]$ will have this general form for any $f(t)=t^p$, $p\in\mathbb{N}$. In order to find an explicit form for the "extra terms" we can consider $D_a^\beta[\,f(t)X(t)\,]$ with $f(t)=t^3$ as we will then have more terms from which we can infer a pattern. This is done in Section 5.3 and the results are as follows:

For $f(t)=t^p$, $p\le m$, $\beta\in(m-1,m]$ and $X^{(m)}(t)$ a m.s. continuous second-order stochastic process,

$$D_a^\beta[f(t)X(t)] \;=\; \sum_{j=0}^{p}\binom{\beta}{j}\frac{d^jf(t)}{dt^j}\,I_a^{m-\beta}X^{(m-j)}(t)+C_D \tag{5.22}$$

where C_D is given by

$$C_D=\sum_{n=0}^{p-1}\binom{m}{n}\sum_{k=1}^{p}\sum_{j=0}^{k-1}\binom{-(m-\beta)}{k}\frac{d^{k+n}f(t)}{dt^{k+n}}\,I_a^{m-\beta+j}X^{(m-k-n+j)}(a)\;. \tag{5.23}$$

Note that we include the case $\beta=m$ above. This is because if we set $\beta=m$ in (5.23) then

$$\binom{-(m-\beta)}{k}=\binom{0}{k}=0\qquad\forall\,k\neq 0\;,\qquad\binom{0}{0}=1$$

so that $C_D=0$. Also, if we set $\beta=m$ in (5.22) then

$$I_a^{m-\beta}X^{(m-j)}(t)=I_a^0X^{(m-j)}(t)=X^{(m-j)}(t)$$

and so (5.22) reduces to the form previously found for the integer order case.

Due to the linearity property of $D_a^\beta[\cdot]$, we can use (5.22) to find $D_a^\beta[f(t)X(t)]$ where $f(t)$ is a polynomial in t. Clearly the stochastic process, $X(t)$, can be replaced by a mean square fractional integral or derivative if those integrals or derivatives are m.s. continuous. By considering Corollary 4.2 we see that $C_D=0$ when this is done.

5.3 Appendix: An expression for $D_a^\beta[t^3X(t)]$

In the previous section we derived formulas for $D_a^\beta[f(t)X(t)]$ where $f(t)=t^p$, $p\in\{1,2\}$. Here we will consider the case $p=3$ in order to find the form of the terms we have called "extra terms" in Section 5.2. Before starting we recall that we have made the assumption $p\le m$ where $\beta\in(m-1,m)$, $m\in\mathbb{N}$.

$$\begin{aligned}
D_a^\beta[\,t^3X(t)\,] \;&=\; I_a^{m-\beta}\frac{d^m}{dt^m}[t^3X(t)]\\
&=\; I_a^{m-\beta}\big[\,t^3X^{(m)}(t)\,\big]+3m\,I_a^{m-\beta}\big[\,t^2X^{(m-1)}(t)\,\big]\\
&\quad +3m(m-1)\,I_a^{m-\beta}\big[\,tX^{(m-2)}(t)\,\big]\\
&\quad +m(m-1)(m-2)\,I_a^{m-\beta}X^{(m-3)}(t)\;.
\end{aligned} \tag{5.24}$$

Using (5.13) and IBP we have

$$\begin{aligned} I_a^{m-\beta}[\,tX^{(m-2)}(t)\,] &= t\,I_a^{m-\beta}X^{(m-2)}(t) - (m-\beta)\,I_a^{m-\beta+1}X^{(m-2)}(t) \\ &= t\,I_a^{m-\beta}X^{(m-2)}(t) - (m-\beta)\,I_a^{m-\beta}X^{(m-3)}(t) + c_4 \end{aligned} \tag{5.25}$$

where

$$c_4 = (m-\beta)I_a^{m-\beta}X^{(m-3)}(a)\ .$$

Similarly,

$$\begin{aligned} I_a^{m-\beta}\left[\,t^2X^{(m-1)}(t)\,\right] &= t^2\,I_a^{m-\beta}X^{(m-1)}(t) - 2t(m-\beta)\,I_a^{m-\beta+1}X^{(m-1)}(t) \\ &\quad +(m-\beta)(m-\beta+1)\,I_a^{m-\beta+2}X^{(m-1)}(t) \\ &= t^2\,I_a^{m-\beta}X^{(m-1)}(t) - 2t(m-\beta)\,I_a^{m-\beta}X^{(m-2)}(t) \\ &\quad +(m-\beta)(m-\beta+1)\,I_a^{m-\beta}X^{(m-3)}(t) + c_5 \end{aligned} \tag{5.26}$$

where

$$\begin{aligned} c_5 &= 2t(m-\beta)I_a^{m-\beta}X^{(m-2)}(a) - (m-\beta)(m-\beta+1)I_a^{m-\beta+1}X^{(m-2)}(a) \\ &\quad -(m-\beta)(m-\beta+1)I_a^{m-\beta}X^{(m-3)}(a)\ . \end{aligned}$$

Using (5.13) we have

$$\begin{aligned} I_a^{m-\beta}\left[\,t^3X^{(m)}(t)\,\right] &= t^3\,I_a^{m-\beta}X^{(m)}(t) - 3t^2(m-\beta)\,I_a^{m-\beta+1}X^{(m)}(t) \\ &\quad +3t(m-\beta)(m-\beta+1)\,I_a^{m-\beta+2}X^{(m)}(t) \\ &\quad -(m-\beta)(m-\beta+1)(m-\beta+2)\,I_a^{m-\beta+3}X^{(m)}(t)\ . \end{aligned}$$

Applying IBP to this we have

$$\begin{aligned} I_a^{m-\beta}\left[\,t^3X^{(m)}(t)\,\right] &= t^3\,I_a^{m-\beta}X^{(m)}(t) \\ &\quad -3t^2(m-\beta)\left[I_a^{m-\beta}X^{(m-1)}(t) - I_a^{m-\beta}X^{(m-1)}(a)\right] \\ &\quad +3t(m-\beta)(m-\beta+1)\left[I_a^{m-\beta}X^{(m-2)}(t)\right. \\ &\qquad \left.-I_a^{m-\beta+1}X^{(m-1)}(a) - I_a^{m-\beta}X^{(m-2)}(a)\right] \\ &\quad -(m-\beta)(m-\beta+1)(m-\beta+2)\left[I_a^{m-\beta}X^{(m-3)}(t)\right. \\ &\qquad -I_a^{m-\beta+2}X^{(m-1)}(a) - I_a^{m-\beta+1}X^{(m-2)}(a) \\ &\qquad \left.-I_a^{m-\beta}X^{(m-3)}(a)\right] \\ &= t^3\,I_a^{m-\beta}X^{(m)}(t) - 3t^2(m-\beta)\,I_a^{m-\beta}X^{(m-1)}(t) \\ &\quad +3t(m-\beta)(m-\beta+1)\,I_a^{m-\beta}X^{(m-2)}(t) \\ &\quad -(m-\beta)(m-\beta+1)(m-\beta+2)\,I_a^{m-\beta}X^{(m-3)}(t) + c_6 \end{aligned} \tag{5.27}$$

where

$$\begin{aligned}
c_6 &= 3t^2(m-\beta)I_a^{m-\beta}X^{(m-1)}(a) \\
&\quad -3t(m-\beta)(m-\beta+1)\left[I_a^{m-\beta+1}X^{(m-1)}(a)+I_a^{m-\beta}X^{(m-2)}(a)\right] \\
&\quad +(m-\beta)(m-\beta+1)(m-\beta+2)\Big[I_a^{m-\beta+2}X^{(m-1)}(a) \\
&\quad +I_a^{m-\beta+1}X^{(m-2)}(a)+I_a^{m-\beta}X^{(m-3)}(a)\Big] .
\end{aligned}$$

Thus, substituting (5.25), (5.26) and (5.27) into (5.24) we have

$$\begin{aligned}
D_a^\beta\big[t^3X(t)\big] &= t^3\, I_a^{m-\beta}X^{(m)}(t) \\
&\quad +\big[3mt^2-3t^2(m-\beta)\big]\, I_a^{m-\beta}X^{(m-1)}(t) \\
&\quad +\big[3t(m-\beta)(m-\beta+1)-6tm(m-\beta)+3tm(m-1)\big]\, I_a^{m-\beta}X^{(m-2)}(t) \\
&\quad +\big[3m(m-\beta)(m-\beta+1)-(m-\beta)(m-\beta+1)(m-\beta+2)+ \\
&\qquad -3m(m-1)(m-\beta)+m(m-1)(m-2)\big]\, I_a^{m-\beta}X^{(m-3)}(t) \\
&\quad +3m(m-1)\,c_4+3m\,c_5+c_6 \,. \\
&= \sum_{j=0}^{3}\binom{\beta}{j}\frac{d^j\,t^3}{dt^j}\, I_a^{m-\beta}X^{(m-j)}(t)+3m(m-1)\,c_4+3m\,c_5+c_6 \,.
\end{aligned}$$

Now, c_6 can be written as follows:

$$\begin{aligned}
c_6 &= 3t^2(m-\beta)\sum_{j=0}^{0} I_a^{m-\beta+j}X^{(m-1+j)}(a) \\
&\quad -3t(m-\beta)(m-\beta+1)\sum_{j=0}^{1} I_a^{m-\beta+j}X^{(m-2+j)}(a) \\
&\quad +(m-\beta)(m-\beta+1)(m-\beta+2)\sum_{j=0}^{2} I_a^{m-\beta+j}X^{(m-3+j)}(a) \\
&= \binom{m}{0}\sum_{k=1}^{3}\sum_{j=0}^{k-1}\binom{-(m-\beta)}{k}\frac{d^k\,t^3}{dt^k}\; I_a^{m-\beta+j}X^{(m-k+j)}(a) \,.
\end{aligned}$$

Similarly

$$\begin{aligned}
3mc_5 &= 6mt(m-\beta)\sum_{j=0}^{0} I_a^{m-\beta+j}X^{(m-2+j)}(a) \\
&\quad -3m(m-\beta)(m-\beta+1)\sum_{j=0}^{1} I_a^{m-\beta+j}X^{(m-3+j)}(a) \\
&= \binom{m}{1}\sum_{k=1}^{2}\sum_{j=0}^{k-1}\binom{-(m-\beta)}{k}\frac{d^{k+1}\,t^3}{dt^{k+1}}\; I_a^{m-\beta+j}X^{(m-k-1+j)}(a)
\end{aligned}$$

and

$$\begin{aligned} 3m(m-1)c_4 &= 3m(m-1)(m-\beta)\sum_{j=0}^{0} I_a^{m-\beta+j} X^{(m-3+j)}(a) \\ &= \binom{m}{2} \sum_{k=1}^{1}\sum_{j=0}^{k-1} \binom{-(m-\beta)}{k} \frac{d^{k+2}\, t^3}{dt^{k+2}}\; I_a^{m-\beta+j} X^{(m-k-2+j)}(a)\ . \end{aligned}$$

Thus

$$\begin{aligned} &c_6 + 3mc_5 + 3m(m-1)c_4 \\ &\qquad = \sum_{n=0}^{2}\binom{m}{n} \sum_{k=1}^{3}\sum_{j=0}^{k-1} \binom{-(m-\beta)}{k} \frac{d^{k+n}\, t^3}{dt^{k+n}}\; I_a^{m-\beta+j} X^{(m-k-n+j)}(a)\ . \end{aligned}$$

Clearly, for $f(t) = t^p$ where $p \leq m$ and $m, p \in \mathbb{N}$

$$\begin{aligned} D_a^\beta[t^p X(t)] &= \sum_{j=0}^{p} \binom{\beta}{j} \frac{d^j\, t^p}{dt^j}\; I_a^{m-\beta} X^{(m-j)}(t) \\ &\quad + \sum_{n=0}^{p-1}\binom{m}{n} \sum_{k=1}^{p}\sum_{j=0}^{k-1} \binom{-(m-\beta)}{k} \frac{d^{k+n}\, t^p}{dt^{k+n}}\; I_a^{m-\beta+j} X^{(m-k-n+j)}(a)\ . \end{aligned}$$

Chapter 6

Application of the composition rules

In deterministic fractional calculus several methods are used to solve fractional integral and differential equations. One of the common methods involves the use of the composition rules. In this chapter we will demonstrate this method using several examples. Throughout the chapter we will assume that the stochastic processes involved are such that the steps used are allowed.

We will start by considering the equation

$$I_a^\beta X(t) = Y(t) \tag{6.1}$$

where $\beta > 0$, $X(t)$ is an unknown second-order stochastic process and $Y(t)$ is a known second-order stochastic process. When $\beta \in (0,1]$ equation (6.1) is the mean-square Abel integral of the first kind. We note that this equation could instead be considered in the form $c_1 I_a^\beta X(t) = c_2 Y(t)$ where c_1 and c_2 are arbitrary constants, but since the presence of the constants will have no effect on the method we will let $c_1 = 1$ and $c_2 = 1$.

Applying D_a^β to equation (6.1) we have

$$D_a^\beta I_a^\beta X(t) = D_a^\beta Y(t) \ .$$

Using Theorem 4.9 when $\beta \in \mathbb{N}$, or Theorem 4.12 when $\beta \notin \mathbb{N}$, we see that the LHS will equal $X(t)$. Thus, provided $D_a^\beta Y(t)$ exists,

$$X(t) = D_a^\beta Y(t)$$

will be a potential solution to equation (6.1).

Using only $\beta \in (0,1]$, Hafiz et al. (2001) and Hafiz (2004) solve equation (6.1) in a slightly different manner – first by applying $I_a^{1-\beta}$ and then by applying D_a^1.

Below we show the case where $\beta \in (m-1, m)$, $m \in \mathbb{N}$.

$$\begin{aligned} I_a^\beta X(t) = Y(t) \quad &\Rightarrow \quad \underbrace{I_a^{m-\beta} I_a^\beta X(t)}_{I_a^m X(t)} = I_a^{m-\beta} Y(t) \\ &\Rightarrow \quad D_a^m I_a^m X(t) = D_a^m I_a^{m-\beta} Y(t) \; . \end{aligned}$$

Using Theorem 4.9 the LHS will be $X(t)$. Using the formal extension of Theorem 4.11 along with the fact that $Y(t)|_{t=a} = I_a^\beta X(t)|_{t=a} = 0$, the RHS will be $D_a^\beta X(t)$. Thus we get

$$X(t) = D_a^\beta Y(t) \; .$$

This is the same potential solution as that found with the previous method.

We can now check if this potential solution is a valid solution by substituting it back into equation (6.1).

$$\begin{aligned} LHS &= I_a^\beta X(t) \\ &= I_a^\beta D_a^\beta Y(t) \\ &= Y(t) \\ &= RHS \end{aligned}$$

where Theorem 4.7 was used in conjunction with Corollary 4.2 in the last step. So $X(t) = D_a^\beta Y(t)$ is a valid solution to equation (6.1).

If we replace the mean-square fractional integral to order β in equation (6.1) by the mean-square fractional derivative to order β, we have the following mean square fractional differential equation:

$$D_a^\beta X(t) = Y(t) \; . \tag{6.2}$$

To solve this equation we can apply I_a^β to both sides and use Theorem 4.7 to get the following potential solution:

$$X(t) = I_a^\beta Y(t) + \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a) \; . \tag{6.3}$$

To check if this is a valid solution we substitute it back into (6.2).

$$\begin{aligned} LHS &= D_a^\beta X(t) \\ &= D_a^\beta \left[I_a^\beta Y(t) + \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a) \right] \\ &= D_a^\beta I_a^\beta Y(t) + \sum_{j=0}^{m-1} X^{(j)}(a) \left[D_a^\beta \frac{(t-a)^j}{\Gamma(j+1)} \right] \; . \end{aligned} \tag{6.4}$$

Using Theorem 4.9 and Theorem 4.12 we see that the first term on the RHS of equation (6.4) will equal $Y(t)$. Now

$$\frac{d^m}{dt^m}\left[\frac{(t-a)^j}{\Gamma(j+1)}\right] = 0 \qquad \text{for } j \in \{0,1,2,...,m-1\}$$

so when $\beta = m \in \mathbb{N}$ the sum on the RHS of equation (6.4) equals zero. Since

$$D_a^\beta\left[\frac{(t-a)^j}{\Gamma(j+1)}\right] = I_a^{m-\beta}\left[\frac{d^m}{dt^m}\frac{(t-a)^j}{\Gamma(j+1)}\right]$$

the sum will also equal zero when $\beta \in (m-1,m)$. Thus equation (6.4) will be

$$LHS = Y(t) = RHS\ .$$

So the solution given in equation (6.3) is valid.

If $\beta \in (m-1,m)$ an alternative way to solve equation (6.2) is to first apply $D_a^{m-\beta}$ to both sides and use Part (a) of Theorem 4.14 to get

$$\underbrace{D_a^{m-\beta}D_a^\beta X(t)}_{D_a^m X(t)} = D_a^{m-\beta}Y(t)\ .$$

Applying I_a^m to this we have

$$I_a^m D_a^m X(t) = I_a^m D_a^{m-\beta}Y(t)\ .$$

Using Theorem 4.7 on the LHS and the formal extension of Theorem 4.11 in conjunction with Corollary 4.2 on the RHS, we arrive at the same solution as that found previously.

The mean-square fractional equations $I_a^\beta X(t) = Y(t)$ and $D_a^\beta X(t) = Y(t)$ are easy to solve because the unknown stochastic process, $X(t)$, appears in only one term. It is more difficult to solve equations in which $X(t)$ appears in more than one term. What can be done in these situations is to apply the composition rules to the equation in order to manipulate it into the form of an ordinary m.s. random differential equation. The well known methods for solving ordinary mean-square random differential equations (see, for example, Soong (1973)) can then be used. We will demonstrate using two simple examples.

Consider

$$I_a^{\alpha+Q}X(t) + I_a^\alpha X(t) = Y(t) \tag{6.5}$$

where $n, Q \in \mathbb{N}$, $\alpha \in (n-1,n)$, $X(t)$ is an unknown second-order stochastic process and $Y(t)$ is a known second-order stochastic process. As with the simple fractional integral and differential equations that we solved previously, we will use two different approaches.

For the first approach we start by applying $I_a^{n-\alpha}$.

$$\underbrace{I_a^{n-\alpha} I_a^{\alpha+Q} X(t)}_{I_a^{n+Q} X(t)} + \underbrace{I_a^{n-\alpha} I_a^{\alpha} X(t)}_{I_a^{n} X(t)} = I_a^{n-\alpha} Y(t) \ .$$

Applying D_a^{n+Q} to this we have

$$D_a^{n+Q} I_a^{n+Q} X(t) + D_a^{n+Q} I_a^{n} X(t) = D_a^{n+Q} I_a^{n-\alpha} Y(t) \ .$$

Using Theorem 4.9 the first term on the LHS is $X(t)$. Using Theorem 4.10 the second term on the LHS is $X^{(Q)}(t)$. Using Theorem 4.11 in conjunction with Corollary 4.2 the term on the RHS is

$$D_a^{n+Q} I_a^{n-\alpha} Y(t) = I_a^{n-\alpha} D_a^{n+Q} Y(t) = D_a^{\alpha+Q} Y(t) \ .$$

Thus we have

$$X^{(Q)}(t) + X(t) = D_a^{\alpha+Q} Y(t) \ .$$

This is clearly a Q^{th} order mean-square random differential equation.

A second approach that can be used to solve equation (6.5) starts with the application of D_a^{α} to both sides of the equation. This gives

$$D_a^{\alpha} I_a^{\alpha+Q} X(t) + D_a^{\alpha} I_a^{\alpha} X(t) = D_a^{\alpha} Y(t) \ .$$

Using Part (a) of Theorem 4.9 the first term on the LHS is $I_a^Q X(t)$ and using Theorem 4.12 the second term on the LHS is $X(t)$. Thus

$$I_a^Q X(t) + X(t) = D_a^{\alpha} Y(t) \ .$$

Applying D_a^Q to this we get

$$\underbrace{D_a^Q I_a^Q X(t)}_{X(t)} + X^{(Q)}(t) = \underbrace{D_a^Q D_a^{\alpha} Y(t)}_{D_a^{\alpha+Q} Y(t)}$$

where Part (b) of Theorem 4.9 was used on the first term on the LHS and Theorem 4.14 was used in conjunction with Corollary 4.2 on the RHS. This is the same Q^{th} order mean-square random differential equation as that found using the previous approach.

If we replace the fractional integrals by fractional derivatives in equation (6.5) we have the following mean square fractional differential equation:

$$D_a^{\alpha+Q} X(t) + D_a^{\alpha} X(t) = Y(t) \ . \tag{6.6}$$

Let us work with $\alpha \in (0, 1)$. Applying $I_a^{\alpha+Q}$ gives

$$I_a^{\alpha+Q} D_a^{\alpha+Q} X(t) + I_a^{\alpha+Q} D_a^{\alpha} X(t) = I_a^{\alpha+Q} Y(t) \ .$$

Using Theorem 4.7 and rearranging we have

$$X(t) + I_a^Q X(t) = I_a^{\alpha+Q} Y(t) + \sum_{j=0}^{Q} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a) + \frac{(t-a)^Q}{\Gamma(Q+1)} X(a) \ . \tag{6.7}$$

To this we can apply D_a^Q. Doing so, the first term on the LHS will be $X^{(Q)}(t)$ and the second term will be $X(t)$. For the first term on the RHS we will have

$$D_a^Q I_a^{\alpha+Q} Y(t) = D_a^Q I_a^Q \, I_a^\alpha Y(t) = I_a^\alpha Y(t) \ .$$

Since

$$\frac{d^n}{dt^n}\left[\frac{(t-a)^j}{\Gamma(j+1)}\right] \quad = \quad \begin{cases} 0 \ , & \text{for } j \in \{0,1,2,\ldots,n-1\} \\ 1 \ , & \text{for } j = n \end{cases}$$

when D_a^Q is applied to equation (6.7), the second and third terms on the RHS will be, respectively, $X^{(Q)}(a)$ and $X(a)$. Thus we have the ordinary m.s. random differential equation

$$X^{(Q)}(t) + X(t) = I_a^\alpha Y(t) + X^{(Q)}(a) + X(a) \ .$$

An alternative approach to reducing equation (6.6) to an ordinary m.s. random differential equation is to start by applying $D_a^{1-\alpha}$ to both sides of the equation. Doing this we have

$$D_a^{1-\alpha} D_a^{\alpha+Q} X(t) + D_a^{1-\alpha} D_a^\alpha X(t) = D_a^{1-\alpha} Y(t) \ .$$

Now, $\alpha \in (0,1)$ implies that $\alpha + Q \in (Q, Q+1)$. Also, $(1-\alpha) + (\alpha + Q) = (1+Q)$ and $(1-\alpha) + \alpha = 1$ so that using Part (a) of Theorem 4.14 the first term on the LHS is $X^{(Q+1)}(t)$ and the second term is $X^{(1)}(t)$. Thus we have the following ordinary m.s. random differential equation

$$X^{(Q+1)}(t) + X^{(1)}(t) = D_a^{1-\alpha} Y(t) \ .$$

This does not look like the same equation that we found using the previous approach. However, by manipulating it we can show it is the same. Applying I_a^1 we have

$$I_a^1 X^{(Q+1)}(t) + I_a^1 X^{(1)}(t) = I_a^1 D_a^{1-\alpha} Y(t) \ .$$

Using IBP the LHS is

$$X^{(Q)}(t) - X^{(Q)}(a) + X(t) - X(a) \ .$$

Using Theorem 4.9 and noting that $Y(t)|_{t=a} = 0$ (see equation (6.6)), the RHS is $I_a^\alpha Y(t)$. So again we have the ordinary m.s. random differential equation

$$X^{(Q)}(t) + X(t) = I_a^\alpha Y(t) + X^{(Q)}(a) + X(a) \ .$$

In Section 4.2 we did not give an exhaustive account of the composition rules but instead considered only certain rules. Due to this, using different approaches when solving mean square fractional equations in which the unknown stochastic process occurs more than once, different ODE's may be found. In the example we have just seen, we could manipulate the ODE's using the composition rules so that they were the same. Using our limited set of composition rules we may not always be able to do this.

In El-Sayed et al. (2005) conditions for the existence of a solution to the following random differential equation are considered:

$$\frac{d}{dt} I_a^{1-\beta} X(t) = f(t, X(t)) \; . \tag{6.8}$$

In equation (6.8) $\beta \in (0,1)$, $f(t, X(t))$ is a "sufficiently nice" function and $X(t)$ is an unknown second-order stochastic process. Since ${}_*D_a^{\beta} X(t) = \frac{d}{dt} I_a^{1-\beta} X(t)$, equation (6.8) is clearly a mean-square fractional differential equation where the derivative is the Left-Hand fractional derivative. Since we are focussing on the RH definition the problem presented in equation (6.8) may, at first glance, seem off topic. However, n^{th}order random differential equations – like those found when applying the composition rules to equations (6.5) and (6.6) – can be converted into a set of first-order random differential equations. So instead of working with a single stochastic process, as is done in El-Sayed et al. (2005), we can work with a vector stochastic process.

The fractional equations we have considered here are very basic. We have not considered equations in which fractional integrals and derivatives both occur, nor have we considered equations in which $X(t)$ occurs more than once but the orders of the integrals/derivatives differ by a non-integer. We have also not considered the case when the equation involves terms of the form $I_a^{\beta}[f(t)X(t)]$ or $D_a^{\beta}[f(t)X(t)]$ where $f(t)$ is a deterministic function. We will not consider these types of equations as the examples we have considered demonstrate both the simplicity and the restrictions of this method that is common in deterministic fractional calculus.

Chapter 7

A transform of mean-square fractional integrals and derivatives

In deterministic fractional calculus the Laplace transform can be used to solve some fractional integral and differential equations (see, for example, Podlubny (1999) and Glöckle & Nonnenmacher (1991)). It therefore seems reasonable to try use a transformation to solve m.s. fractional integral and differential equations.

Let us define the following:

$$\begin{aligned} \mathcal{L}_*[X(t)] &\triangleq \int_0^\infty e^{-st} X(t)\, dt \qquad (7.1) \\ \mathcal{L}_*[f(t)] &\triangleq \int_0^\infty e^{-st} f(t)\, dt \triangleq \mathcal{L}[f(t)] \end{aligned}$$

where $f(t)$ is a deterministic function for which the Laplace transorm, $\mathcal{L}[f(t)]$, exists and $X(t)$ is a second-order s.p. that is defined for $t \in [0, \infty)$. Using the integration in m.s. criterion we know that the integral in (7.1) will exist iff the improper Riemann integral below exists and is finite:

$$\int_0^\infty \int_0^\infty e^{-st} e^{-s\tau}\, \Gamma_{XX}(t, \tau)\, dt\, d\tau$$

For the rest of the chapter we will assume that $\beta \in (m-1, m]$, $m \in \mathbb{N}$, unless otherwise stated.

7.1 Expressions for $\mathcal{L}_*[I_a^\beta X(t)]$ and $\mathcal{L}_*[D_a^\beta X(t)]$

If $I_0^\beta X(t)$ and $D_0^\beta X(t)$ exist then they are second-order stochastic processes and so we can consider $\mathcal{L}_*[I_0^\beta X(t)]$ and $\mathcal{L}_*[D_0^\beta X(t)]$.

In order to find an expression for $\mathcal{L}_*[I_0^\beta X(t)]$ we note that

$$I_0^\beta X(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds$$

is the convolution of the stochastic process $X(t)$ and the deterministic function $f(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$. It can easily be shown that the $\mathcal{L}_*$ transform of the convolution of a stochastic process and a deterministic function is the product of their individual $\mathcal{L}_*$ transforms. Thus we have

$$\mathcal{L}_*[I_0^\beta X(t)] = \mathcal{L}_*[f(t) * X(t)] = \mathcal{L}[f(t)] \cdot \mathcal{L}_*[X(t)] \; . \tag{7.2}$$

Since $\beta > 0$ we have

$$\mathcal{L}_*[f(t)] \triangleq \mathcal{L}\left[\frac{t^{\beta-1}}{\Gamma(\beta)}\right] = \frac{1}{s^\beta} \; , \qquad s \neq 0$$

so that for $\beta > 0$ and $s \neq 0$

$$\mathcal{L}_*[I_0^\beta X(t)] = \frac{\mathcal{L}_*[X(t)]}{s^\beta} \; .$$

Now consider $\mathcal{L}_*[D_0^\beta X(t)]$.

$$\begin{aligned} \mathcal{L}_*[D_0^\beta X(t)] &= \mathcal{L}_*\left[\int_0^t \frac{(t-u)^{m-\beta-1}}{\Gamma(m-\beta)} X^{(m)}(u)\, du\right] \\ &= \mathcal{L}\left[\frac{t^{m-\beta-1}}{\Gamma(m-\beta)}\right] \mathcal{L}_*[X^{(m)}(t)] \\ &= \frac{1}{s^{m-\beta}} \cdot \mathcal{L}_*[X^{(m)}(t)] \; , \qquad s \neq 0. \end{aligned} \tag{7.3}$$

Below we derive an expression for $\mathcal{L}_*[X^{(m)}(t)]$.

Using IBP we have

$$\begin{aligned} \mathcal{L}_*[\dot{X}(t)] &= \int_0^\infty e^{-st} \dot{X}(t)\, dt \\ &= \underset{T\to\infty}{\text{l.i.m.}} \int_0^T e^{-st} \dot{X}(t)\, dt \\ &= \underset{T\to\infty}{\text{l.i.m.}} \left[e^{-st} X(t) \Big|_0^T + s \int_0^T e^{-st} X(t)\, dt \right] \\ &= \underset{T\to\infty}{\text{l.i.m.}} \left[e^{-sT} X(T) \right] - \underset{T\to\infty}{\text{l.i.m.}} \left[X(0) \right] + \underset{T\to\infty}{\text{l.i.m.}} \left[s \int_0^T e^{-st} X(t)\, dt \right] \\ &= \underset{T\to\infty}{\text{l.i.m.}} \left[e^{-sT} X(T) \right] - X(0) + s\, \mathcal{L}_*[X(t)] \; . \end{aligned}$$

Thus if $\underset{T\to\infty}{\text{l.i.m.}} \left[e^{-sT} X(T)\right] = 0$

$$\mathcal{L}_*[\dot{X}(t)] = s\, \mathcal{L}_*[X(t)] - X(0)\ . \tag{7.4}$$

Now consider $\mathcal{L}_*[\ddot{X}(t)]$. Using IBP we have

$$\begin{aligned}
\mathcal{L}_*[\ddot{X}(t)] &= \underset{T\to\infty}{\text{l.i.m.}} \int_0^T e^{-st} \ddot{X}(t)\, dt \\
&= \underset{T\to\infty}{\text{l.i.m.}} \left[e^{-st}\dot{X}(t)\Big|_0^T + s\int_0^T e^{-st}\dot{X}(t)\, dt \right] \\
&= \underset{T\to\infty}{\text{l.i.m.}} \left[e^{-sT}\dot{X}(T)\right] - \underset{T\to\infty}{\text{l.i.m.}} \left[\dot{X}(0)\right] + s\ \underset{T\to\infty}{\text{l.i.m.}} \left[\int_0^T e^{-st}\dot{X}(t)\, dt\right] .
\end{aligned}$$

Substituting (7.4) into the last term of the above and assuming that $\underset{T\to\infty}{\text{l.i.m.}} \left[e^{-sT}\dot{X}(T)\right] = 0$, we have

$$\begin{aligned}
\mathcal{L}_*[\ddot{X}(t)] &= s\,[s\,\mathcal{L}_*[X(t)] - X(0)] - \dot{X}(0) \\
&= s^2\,\mathcal{L}_*[X(t)] - sX(0) - \dot{X}(0)\ .
\end{aligned}$$

In general, assuming that $\underset{T\to\infty}{\text{l.i.m.}}[e^{-sT}X^{(j)}(T)] = 0$ for $j = \{0, 1, 2, \ldots, m-1\}$, we have

$$\mathcal{L}_*[X^{(m)}(t)] = s^m\ \mathcal{L}_*[X(t)] - \sum_{j=0}^{m-1} s^{m-1-j} X^{(j)}(0)\ . \tag{7.5}$$

We now need to show that our assumption that $\underset{T\to\infty}{\text{l.i.m.}}[e^{-sT}X^{(j)}(T)] = 0$ for $j \in \{0, 1, 2, \ldots, m-1\}$ is valid. To show the assumption holds for $j = 0$ we must show that

$$\lim_{T\to\infty} ||e^{-sT}X(T) - 0|| = 0\ .$$

Now,

$$\begin{aligned}
\lim_{T\to\infty} ||e^{-sT}X(T) - 0|| &= \lim_{T\to\infty} \sqrt{E[e^{-sT}e^{-sT}X(T)X(T)]} \\
&= \sqrt{\lim_{T\to\infty} e^{-2sT}\Gamma_{XX}(T,T)}\ .
\end{aligned}$$

Since $X(t)$ is a second-order stochastic process $\Gamma_{XX}(t,\tau)$ is finite for all (t,τ). Thus $\underset{T\to\infty}{\text{l.i.m.}}[e^{-sT}X(T)] = 0$ for $s > 0$. We can similarly show that the assumption holds for $j \in \{1, 2, 3, \ldots, m-1\}$ when $s > 0$.

Returning to (7.3) we have

$$\begin{aligned}
\mathcal{L}_*[D_0^\beta X(t)] &= \frac{1}{s^{m-\beta}}\left[s^m\mathcal{L}_*[X(t)] - \sum_{j=0}^{m-1} s^{m-j-1}X^{(j)}(0)\right] \\
&= s^\beta\mathcal{L}_*[X(t)] - \sum_{j=0}^{m-1} s^{\beta-j-1}X^{(j)}(0)\ , \qquad s > 0.
\end{aligned}$$

We have considered the case where the lower limit of integration for the fractional integral and derivative is zero. Let us now consider the case where the lower limit is some constant $a > 0$.

Using the Heaviside function

$$H(t-u) = \begin{cases} 0 & \text{for } t < u \\ 1 & \text{for } t \geq u \end{cases}$$

we have the following expression:

$$\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s) H(s-a)\, ds = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds\ .$$

So

$$\begin{aligned} \mathcal{L}_*[I_a^\beta X(t)] &= \mathcal{L}_*\left[\left(\frac{t^{\beta-1}}{\Gamma(\beta)}\right) * X(t)H(t-a)\right] \\ &= \mathcal{L}\left[\frac{t^{\beta-1}}{\Gamma(\beta)}\right] \cdot \mathcal{L}_*[X(t)H(t-a)] \\ &= \left[\frac{1}{s^\beta}\right] \cdot \mathcal{L}_*[X(t)H(t-a)] \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}_*[X(t)H(t-a)] &= \int_0^\infty e^{-st} X(t) H(t-a)\, dt \\ &= \int_a^\infty e^{-st} X(t)\, dt \\ &= \int_0^\infty e^{-s(t+a)} X(t+a)\, dt \\ &= e^{-as} \int_0^\infty e^{-st} X(t+a)\, dt \\ &= e^{-as} \mathcal{L}_*[X(t+a)] \end{aligned}$$

Note that we have used a change of variable to get from the second to the third line.

Therefore,

$$\mathcal{L}_*[I_a^\beta X(t)] = \frac{e^{-as}\, \mathcal{L}_*[X(t+a)]}{s^\beta}\ , \qquad s \neq 0$$

and similarly

$$\begin{aligned} \mathcal{L}_*[D_a^\beta X(t)] &= \mathcal{L}\left[\int_0^\infty \frac{(t-u)^{m-\beta-1}}{\Gamma(m-\beta)} X^{(m)}(u) H(u-a)\, du\right] \\ &= \frac{e^{-as}\, \mathcal{L}_*[X^{(m)}(t+a)]}{s^\beta}\ , \qquad s > 0. \end{aligned}$$

7.2 Solving fractional integral and differential equations using the $\mathcal{L}_*$ transform

It is clear that the $\mathcal{L}_*$ transform works in a m.s. setting like the Laplace transform works in a deterministic setting so it seems reasonable to try use the $\mathcal{L}_*$ transform to solve m.s. fractional integral and differential equations. In order to do this we will need to be able to recover the solution process from the transformed equation. For this reason we will need to define an inverse $\mathcal{L}_*$ transform.

Definition 7.1. Suppose $X(t)$ is a second order stochastic process for which $\mathcal{L}_*[X(t)] \triangleq X^*(s)$ exists. Then $X(t)$ is called an inverse $\mathcal{L}_*$ transform of $X^*(s)$. Symbolicaly we will write $X(t) = \mathcal{L}_*^{-1}[X^*(s)] \triangleq \mathcal{L}_*^{-1}\mathcal{L}_*[X(t)]$.

Provided the validity of the solution process is checked, this simple definition will suffice. The m.s. fractional integral and differential equations that we will solve in this section using the $\mathcal{L}_*$ transform are ones to which we know the solutions. By comparing the solutions we find here to the known solutions, we will be able to decide if using $\mathcal{L}_*$ transforms is a viable method for solving some m.s. fractional integral and differential equations.

Consider the following m.s. fractional integral equation:

$$I_a^\beta X(t) = Y(t) \ , \qquad \beta \in (m-1, m] \tag{7.6}$$

where $X(t)$ is an unknown second-order s.p. and $Y(t)$ is a known second-order stochastic process. In Chapter 6 we solved this equation using the composition rule. Here we will solve it using $\mathcal{L}_*$ transforms and so add the conditions that both $X(t)$ and $Y(t)$ must be defined on $[0, \infty)$ and that their $\mathcal{L}_*$ transforms must exist.

Setting $a = 0$ in (7.6) and applying $\mathcal{L}_*$ transforms we have for $\beta \in (m-1, m)$

$$\begin{aligned}
\mathcal{L}_*[I_0^\beta X(t)] &= \mathcal{L}_*[Y(t)] \\
\Rightarrow \frac{\mathcal{L}_*[X(t)]}{s^\beta} &= \mathcal{L}_*[Y(t)] \\
\Rightarrow \mathcal{L}_*[X(t)] &= s^\beta\, \mathcal{L}_*[Y(t)] \\
&= s^m\, \frac{\mathcal{L}_*[Y(t)]}{s^{m-\beta}} \\
&= s^m\, \mathcal{L}_*[I_0^{m-\beta} Y(t)] \ .
\end{aligned}$$

Now, using (7.5) and Corollary 4.2

$$\begin{aligned}
\mathcal{L}_*\left[\frac{d^m}{dt^m}\, I_0^{m-\beta} Y(t)\right] &= s^m\, \mathcal{L}_*[I_0^{m-\beta} Y(t)] - \sum_{j=0}^{m-1} s^{m-1-j}\, \frac{d^j}{dt^j} I_0^{m-\beta} Y(0) \\
&= s^m\, \mathcal{L}_*[I_0^{m-\beta} Y(t)] \ .
\end{aligned}$$

So

$$\begin{aligned} I_0^\beta X(t) &= Y(t) \\ \Rightarrow \mathcal{L}_*[X(t)] &= \mathcal{L}_*\left[\frac{d^m}{dt^m} I_0^{m-\beta} Y(t)\right] . \end{aligned}$$

Applying $\mathcal{L}_*^{-1}$ i.e. inverting this equation, we have

$$\begin{aligned} X(t) &= \frac{d^m}{dt^m} I_0^{m-\beta} Y(t) \\ &= D_0^\beta Y(t) \end{aligned}$$

where Theorem 4.11 and Corollary 4.2 were used in the last step.

For the $\beta = m \in \mathbb{N}$ case we have

$$\begin{aligned} I_0^m X(t) = Y(t) \quad &\Rightarrow \quad \mathcal{L}_*[X(t)] = s^m \mathcal{L}_*[Y(t)] = \mathcal{L}_*[Y^{(m)}(t)] \\ &\Rightarrow \quad X(t) = D_0^m Y(t) \ . \end{aligned}$$

Thus

$$X(t) = D_0^\beta Y(t) \ , \qquad \beta \in (m-1, m] \tag{7.7}$$

is the solution to (7.6). This is the same solution as that found in the previous chapter.

In Chapter 6 we also considered the m.s. fractional differential equation

$$D_a^\beta X(t) = Y(t) \ , \qquad \beta \in (m-1, m].$$

We can solve this using $\mathcal{L}_*$ transforms by simply noting that for $\beta \in (m-1, m)$ this equation can be written as

$$I_0^{m-\beta} X^{(m)}(t) = Y(t) \ .$$

Clearly this is of the form of (7.6) so using (7.7) we have

$$X^{(m)}(t) = D_0^{m-\beta} Y(t) \ , \qquad \beta \in (m-1, m).$$

Applying I_0^m to both sides and using Theorem 4.7 and Corollary 4.2 we have

$$X(t) - \sum_{j=0}^{m-1} \frac{t^j}{\Gamma(j+1)} X^{(j)}(0) = I_0^\beta Y(t) - \frac{t^\beta}{\Gamma(\beta+1)} Y(0) \ .$$

Thus the solution when $\beta \in (m-1, m)$ will be

$$X(t) = I_0^\beta Y(t) + \sum_{j=0}^{m-1} \frac{t^j}{\Gamma(j+1)} X^{(j)}(0) \ .$$

Using the same method it can be shown that the solution is also valid for $\beta = m$. Again we see that this is the same solution as that found in the previous chapter.

Now consider the following m.s. fractional integral equation:

$$X(t) + \lambda I_a^\beta X(t) = Y(t)\ , \qquad \beta \in (m-1, m] \tag{7.8}$$

where $X(t)$ is an unknown second-order s.p. and $Y(t)$ is a known stochastic process. When $\beta \in (0,1]$ equation (7.8) is the stochastic Abel integral of the second kind. If we assume that $\mathcal{L}_*[X(t)]$ and $\mathcal{L}_*[Y(t)]$ exist and let $a = 0$, then (7.8) can be solved using $\mathcal{L}_*$ transforms.

Applying $\mathcal{L}_*$ transforms to (7.8) we have

$$\begin{aligned}
\mathcal{L}_*[Y(t)] &= \mathcal{L}_*[X(t)] + \frac{\lambda}{s^\beta}\mathcal{L}_*[X(t)] \\
&= \left[1 + \frac{\lambda}{s^\beta}\right]\mathcal{L}_*[X(t)] \\
&= \left[\frac{s^\beta}{s^\beta + \lambda}\right]^{-1}\mathcal{L}_*[X(t)]\ .
\end{aligned}$$

So

$$\begin{aligned}
\mathcal{L}_*[X(t)] &= \left[s\cdot\frac{s^{\beta-1}}{s^\beta + \lambda}\right]\mathcal{L}_*[Y(t)] \\
&= \left[s\cdot\frac{s^{\beta-1}}{s^\beta + \lambda} - 1\right]\mathcal{L}_*[Y(t)] + \mathcal{L}_*[Y(t)]\ .
\end{aligned} \tag{7.9}$$

In order to find a solution to the above problem we will need to introduce the Mittag-Leffler function which we will denote by $M_\alpha(z)$:

$$M_\alpha(z) \triangleq \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n + 1)}\ , \qquad \alpha > 0\ , z \in \mathbb{C}.$$

Taking the $\mathcal{L}_*$ transform of $M_\alpha(-\lambda t^\alpha)$ we have (Gorenflo & Mainardi, 2000)

$$\mathcal{L}_*[M_\alpha(-\lambda t^\alpha)] = \mathcal{L}[M_\alpha(-\lambda t^\alpha)] = \frac{s^{\alpha-1}}{s^\alpha + \lambda}\ , \qquad \Re(s) > |\lambda|^{\frac{1}{\alpha}}.$$

Now,

$$\begin{aligned}
\mathcal{L}_*\left[\frac{d}{dt}M_\alpha(-\lambda t^\alpha)\right] &= \mathcal{L}\left[\frac{d}{dt}M_\alpha(-\lambda t^\alpha)\right] \\
&= s\,\mathcal{L}[M_\alpha(-\lambda t^\alpha)] - [M_\alpha(-\lambda t^\alpha)]_{t=0} \\
&= s\cdot\frac{s^{\alpha-1}}{s^\alpha + \lambda} - \left[1 + \sum_{j=1}^{\infty}\frac{(-\lambda)^j t^{\alpha j}}{\Gamma(\alpha j + 1)}\right]_{t=0} \\
&= s\cdot\frac{s^{\alpha-1}}{s^\alpha + \lambda} - 1\ , \qquad \Re(s) > |\lambda|^{\frac{1}{\alpha}}
\end{aligned}$$

so that (7.9) becomes

$$\mathcal{L}_*[X(t)] \quad = \quad \mathcal{L}\left[\frac{d}{dt}M_\beta(-\lambda t^\beta)\right] \cdot \mathcal{L}_*[Y(t)] + \mathcal{L}_*[Y(t)] \; .$$

Taking the inverse $\mathcal{L}_*$ transform of this we have the following solution to (7.8):

$$X(t) \quad = \quad \left[\frac{d}{dt}\, M_\beta(-\lambda t^\beta)\right] * Y(t) + Y(t) \quad , \qquad \Re(s) > |\lambda|^{\frac{1}{\beta}}. \tag{7.10}$$

This solution is of the same form as that found by Loverro (2004) for the deterministic version of (7.8). Loverro (2004) uses two methods. One method involves the use of Laplace transforms – it is on this method that we have based our $\mathcal{L}_*$ transform method. The other method is that adopted in Hafiz et al. (2001). For $\beta \in (0, 1]$ and $|\lambda| < \frac{\Gamma(1+\beta)}{(b-a)^\beta}$, Hafiz et al. (2001) solve (7.8) as follows:

$$\begin{aligned} Y(t) &= X(t) + \lambda I_a^\beta X(t) \\ &= (1 + \lambda I_a^\beta)X(t) \\ \Rightarrow X(t) &= (1 + \lambda I_a^\beta)^{-1} Y(t) \\ &= \sum_{j=0}^{\infty} (-\lambda)^j I_a^{j\beta} Y(t) \; . \end{aligned} \tag{7.11}$$

Equations (7.10) and (7.11) are equivalent. To see this we note that

$$\begin{aligned} \frac{d}{dt}M_\beta(-\lambda t^\beta) &= \frac{d}{dt}\left[\; 1 + \sum_{j=1}^{\infty} \frac{(-\lambda t^\beta)^j}{\Gamma(\beta j + 1)} \;\right] \\ &= \sum_{j=1}^{\infty} \frac{(-\lambda)^j \;\; t^{\beta j - 1}}{\Gamma(\beta j)} \end{aligned}$$

so that

$$\begin{aligned} \left[\frac{d}{dt}M_\beta(-\lambda t^\beta)\right] * Y(t) &= \int_0^t \sum_{j=1}^{\infty} \frac{(-\lambda)^j \;\; (t-u)^{\beta j - 1}}{\Gamma(\beta j)}\, Y(u)\, du \\ &= \sum_{j=1}^{\infty} (-\lambda)^j \int_0^t \frac{(t-u)^{\beta j - 1}}{\Gamma(\beta j)}\, Y(u)\, du \\ &= \sum_{j=1}^{\infty} (-\lambda)^j \; I_0^{\beta j} Y(t) \; . \end{aligned}$$

Since $Y(t) = (-\lambda)^0 \, I_0^0 Y(t)$, our solution given by (7.10) becomes

$$
\begin{aligned}
X(t) &= \left[\frac{d}{dt} M_\beta(-\lambda t^\beta)\right] * Y(t) + Y(t) \\
&= \sum_{j=1}^{\infty} (-\lambda)^j \, I_0^{\beta j} Y(t) \; + \; (-\lambda)^0 I_0^0 Y(t) \\
&= \sum_{j=0}^{\infty} (-\lambda)^j \, I_0^{\beta j} Y(t) \; .
\end{aligned}
$$

The solutions to the three equations we have considered in this section are the same as the solutions found using other methods. This indicates that using $\mathcal{L}_*$ transforms may be useful in finding potential solutions to some m.s. fractional integral and differential equations.

Chapter 8

In Closing

The aim of this dissertation was to transfer deterministic fractional calculus – a powerful tool in deterministic calculus – to a m.s. setting. Our contribution to the current body of work comes predominantly from Chapters 4, 5 and 7. In Chapter 4 we considered some properties of m.s. fractional integrals and derivatives. We saw that the inclusion of the $\beta > 1$ range often leads to results that are expected even when different methods are required to prove them. When we say the results are expected we mean that they are the same as those for the $\beta \in (0, 1]$ range that is used most often in the current body of work, or that they follow a similar pattern. Including the $\beta > 1$ range also highlighted the need to define the m.s. (Right-Hand) fractional derivative carefully so that the results are consistent with one another. In Chapter 5 we found expressions for $I_a^\beta[f(t)X(t)]$ and $D_a^\beta[f(t)X(t)]$ which, although derived simply by following the basic steps used when working with deterministic fractional calculus, had not yet been given for m.s. fractional integrals and derivatives. In Chapter 7 the simple transform that we introduced allowed us to find potential solutions to simple m.s. fractional integral and differential equations in the same way that the Laplace transform allows us to find solutions to deterministic fractional integral and differential equations. We have also contributed by considering definitions other than the m.s. RH and LH fractional derivatives and the m.s. R-L fractional integral, and by demonstrating that the composition rule method for solving m.s. fractional integral and differential equations is only useful for simple equations.

The work in this dissertation extends the existing body of work but is still only a start to a topic that may become as useful as deterministic fractional calculus. Work can be put into translating the more complex properties of deterministic fractional calculus to a m.s. setting. Mean fourth order calculus is similar to m.s. calculus. Instead of using second-order random variables fourth order random variables – those for which the fourth moment is finite – are used and instead of using the norm $\|X\| = \sqrt[2]{E(X^2)}$, the norm $\|X\|_4 = \sqrt[4]{E(X^4)}$ is used. Using mean fourth-order calculus Villafuerte et al. (2010) found a m.s.

product rule for two stochastic processes that are not necessarily independent. They also found a m.s. chain rule. These rules involve integer order derivatives so it may be possible to generalize them to non-integer orders. In deterministic fractional calculus Grünwald integrals and derivatives are often used for numerical computations of fractional integrals and derivatives. Here we have restricted ourselves to the use of only the m.s. R-L fractional integral and the m.s. RH fractional derivative but it might be interesting to look into the use of m.s. Grünwald integrals and derivatives. These and other ideas can be extensions for further study.

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