Combined Impulse Control and Optimal Stopping in Insurance and Interest Rate Theory

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by

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As the candidate's supervisor, I have approved this thesis for submission

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Disclaimer

The study described in this thesis was carried out in the School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, during the period February 2013 to July 2015. This thesis was completed under the supervision of Dr. E. Chikodza and Professor S. Mukwembi.

This study represents original work by the author and has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

Abstract

In this thesis, we consider the problem of portfolio optimization for an insurance company with transactional costs. Our aim is to examine the interplay between insurance and interest rate. We consider a corporation, such as an insurance firm, which pays dividends to shareholders.

We assume that at any time t the financial reserves of the insurance company evolve according to a generalized stochastic differential equation. We also consider that these liquid assets of the firm earn interest at a constant rate. We consider that when dividends are paid out, transaction costs are incurred. Due to the presence of transactions costs in the proposed model, the mathematical problem becomes a combined impulse and stochastic control problem.

This thesis is an extension of the work by Zhang and Song [69]. Their paper considered dividend control for a financial corporation that also takes reinsurance to reduce risk with surplus earning interest at the constant force $\rho > 0$.

We will extend their model by incorporating jump diffusions into the market with dividend payout and reinsurance policies. Jump-diffusion models, as compared to their diffusion counterpart, are a more realistic mathematical representation of real-life processes in finance.

The extension of Zhang and Song [69] model to the jump case will require us to reduce the analytical part of the problem to Hamilton-Jacobi-Bellman Qausi-Variation Inequalities for combined impulse control in the presence of jump diffusion. This will assist us to find the optimal strategy for the proposed jump diffusion model while keeping the financial corporation in the solvency region. We will then compare our results in the jump-diffusion case to those obtained by Zhang and Song [69] in the no jump case.

We will then consider models with stochastic volatility and uncertainty as a means of extending the current theory of modeling insurance reserves.

Keywords

Quasi-variational inequality, Impulse Control, Uncertainty Theory, Uncertain Stochastic Processes

Contents

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0.1 Glossary of Notation

$\mathbb{R}$	the set of all real numbers.
$\mathbb{R}^n$	n-dimensional real Euclidean space.
$\mathcal{C}(U, V)$	continuous functions from U into V.
$\mathcal{C}(U)$	the same as $\mathcal{C}(U, \mathbb{R})$.
$\mathcal{C}^k(U)$	functions in $\mathcal{C}(U, \mathbb{R})$ with continuous derivatives up to order k.
I_f	the indicator function of the set f.
$(\Omega, \mathcal{F}, P)$	probability space.
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$	filtered probability space.
$(\Gamma, \mathcal{L}, \mathcal{M})$	uncertainty space.
$(\Gamma, \mathcal{L}, \{\mathcal{L}_t\}_{t\in[0,T]}, \mathcal{M})$	filtered uncertainty space.
$(\Gamma \times \Omega, \mathcal{L} \otimes \mathcal{F}, \mathcal{M} \times P)$	uncertain probability space.
$(\Gamma \times \Omega, \mathcal{L} \otimes \mathcal{F}, \{\mathcal{L}_t \otimes \mathcal{F}_t\}_{t\in[0,T]}, \mathcal{M} \times P)$	filtered uncertain probability space.
E_p	expectation under probability space.
$E_{\mathcal{M}}$	expectation under uncertainty space.
$E[\Lambda] = E_p\big[E_{\mathcal{M}}[\Lambda]\big]$	expectation of uncertain random variable.
$X(t) = X_t$	real valued uncertain stochastic process.
$X(t_-) = X_{t-}$	left limit of Lévy process X at time t.
Δ, ΔX	jump process of X.
$\mathbb{L}$	infinitesimal operator.
$\mathbb{M}$	maximum utility operator.
$a \wedge b$	minimum of two real numbers a and b.
càdlàg	left continuous with right limits.
$P\ a.s$	almost surely for the probability measure P.
$\mathcal{M}\ a.s$	almost surely for the uncertainty measure $\mathcal{M}$.
$P \times \mathcal{M}\ a.s$	almost surely for the uncertain random measure $P \times \mathcal{M}$.
$a.a.$, $a.e.$, $a.s.$	almost all, almost everywhere, almost surely for P in chapter 2-4, and $P \times \mathcal{M}$ for chapter 5-8.

Chapter 1

Introduction

The protection of insurance companies against the impact of claims, reinsurance is a practice that has been adopted for centuries with the oldest known reinsurance contract being the 12th of July 1370, Goods Shipment Genoa, contract. The goal of reinsurance is to reduce and eliminate the risk of an insurance company. This thesis aims at examining the problem of reducing the risk of an insurance company while keeping shareholders dividend optimal. We present and investigate the optimal control problem for an insurance firm under different forms of indeterminacy. We consider an insurance firm whose reserves are driven by jump-diffusions, stochastic volatility, uncertain stochastic processes and uncertain stochastic processes with uncertain jumps.

1.1 Literature Review

In recent years the problem of determining optimal reinsurance and dividend policies has attracted the attention of many mathematicians, see for example Asmussen et al. [3], Assmussen and Taksar [4], Cadenillas et al. [10], Chikodza [16]. The work by Lundberg [43] constitutes one of the classical masterpieces on risk control in insurance. The classical collective Lundberg risk model describes the free surplus process of an insurance portfolio. It can be shown (Asmussen [2], Liang and Huang [37], Liang and Sun [38], Schmidli [54]) that the limiting Lundberg model for large portfolios can be approximated by the diffusion process.

Motivated by the need to improve the Lundeberg model, several extensions have been proposed and investigated under different forms of indeterminacy, see for example Asmussen et al. [3], Choulli et al. [17], Højgaard and Taksar [28], Taksar et al. [61], Zhang [68]. An excellent survey of recent works on optimal dividend control policies can be found in Asmussen and Taksar [4], Choulli et al. [17], Hojgaard and Taksar [27] and references therein.

A large number of researchers, over the past five years, have applied the optimal stochastic control theory in different fields of engineering, economics, operations research, production planning, investment and medicine, see for example Gueriero and Olivito [25], Liang and Huang [37], Liang and Sun [38], Øksendal and Sulem [51], Soni and Patel [58] and Tsoularis [60]. The most recent and interesting application of optimal control theory include the work by Guerriero and Olivito [25], where an optimal control problem for a car rental agency is studied in order to optimize the agency's revenue through acceptance and rejection of booking request. Trabelsi [62] considers a solution to a nonlinear optimal multiple stopping problem for the valuation of perpetual American style fixed strike discretely random monitoring Asian put options. Soni and Patel [58] investigate a single-vendor single buyer production inventory model involving defective items. They develop an effective iterative procedure to identify an optimal solution for the vendor-buyer problem.

The paper by Cadenillas et al. [10], in an attempt to improve the Lundeberg model, assumes that the reserve process follows a diffusion process with proportional reinsurance. The model also considers that there is fixed and proportional cost each time a dividend is paid out. The presence of transaction costs makes the problem an impulse control problem and its solution relies on impulse control theory (Øksendal and Sulem [49]). As pointed out by Øksendal [47], fixed costs, however small they are, can have a big effect on the value function. Zhang and Song [69] extend the results of Cadenillas et al. [10] by involving the interest rate into their diffusion model.

Traditionally, indeterminacy has been measured by randomness and fuzziness.

It has, however, emerged that randomness and fuzziness are not the only forms of indeterminacy. By applying probability theory and fuzzy set theory, the stochastic optimal control problem, the fuzzy optimal control problem, and their combination have been developed by many researchers, such as [12], [22], [33], [35], [46], [57] and references therein. The major difference between randomness, fuzziness and uncertainty are in the additivity axiom applied in classical measure theory. A probability measure satisfies countable additivity axiom, a credibility measure satisfies maximality axiom while an uncertain measure satisfies the countable subadditivity axiom. Human language like "about $100km$", "approximately $60kg$", "fast", and "heavy" behave neither as randomness nor as fuzziness [42]. In order to model these imprecise quantities, uncertainty theory was founded by Liu [39] in 2007 and refined by Liu [41] in 2010. The uncertain optimal control problem was presented and investigated by Zhu [70]. Many researchers nowadays are interested in further developing uncertain optimal control theory and its applications.

The paper by Yao and Qin [67] extends uncertain optimal control theory by proposing an uncertain linear quadratic control model. Optimal control of uncertain stochastic systems with Markovian switching and its application is presented by Fei [20], where indeterminacy is measured by a combination of randomness and uncertainty. Deng and Zhu [19] propose an extension of the uncertain optimal control problem by considering a model driven by both uncertain V jump process and uncertain canonical process. As its application, they consider an optimal control problem of pension funds.

1.2 Thesis Contribution

In this thesis, we extend the results of Zhang and Song [69] whose work was a simple extension of that of Cadenillas et al. [10]. Due to the complexity of the world, the thesis extends Zhang and Song [69] work by considering different forms of indeterminacy an insurance firm may face when controlling its reserves. The thesis first extends the problem examined, by Zhang and Song [69] to the jump-diffusion case. The major difference between the work by Cadenillas et al. [10] and the paper by Zhang and Song [69] is that the former uses convexity of $V'(x)$ whereas the latter resorts to the convexity of $V'(x)$ to solve their respective control problems, where $V(x)$ is the value of the optimal control.

The first part of the thesis considers that the reserves of the insurance corporation evolve according to an Itô-Lévy process. It is also assumed that the reserves earn interest at a constant rate. The major contribution of the jump-diffusion case of indeterminacy is the integrodifferential quasi-variational inequalities for the impulse-classical control problem. It is also important to note that the jump component in the model gives rise to quasi-variational inequalities that involve an integrodifferential equation. The integrodifferential equation is more difficult to solve as compared to the quasi-variational inequalities in the Zhang and Song [69] paper. An additional aspect of the novelty of the thesis is the construction of an explicit impulse control.

Generally speaking, jump diffusion models do not yield explicit solutions for the control parameter or the value function. The presence of the jump component and the interest rate term makes the problem unique and difficult to solve. However, we manage to prove the existence of the reinsurance policy.

The disadvantages of the first extension are that the model presented fails to incorporate stochastic volatility. It is well known that the exclusion of stochastic volatility into the model of the stock price has it own biases. To address the issue of stochastic volatility the second part of the paper considers an extension of the model presented by Zhang and Song [69] to a stochastic volatility model. We assume that the stochastic volatility coefficient follows a mean-reverting volatility process, where volatility strives to reach a certain level in the long run. For this reason we assume, as in Stein and Stein [59], that the volatility coefficient follows an arithmetic Ornstein-Uhlenbeck process. Under the risk neutral assumptions for the proposed model, we explicitly solve the problem and construct its value function with the optimal policy. We also present and prove the verification theorem for the stochastic volatility optimal classical and impulse control problem.

A lot of surveys showed that in many cases, randomness is not the only form of indeterminacy as assumed by the jump diffusion model extension and the stochastic volatility extension. Based on uncertainty theory, Itô-Liu calculus and the need to evaluate the belief degree on the occurrence of an event, we extend the model by Zhang and Song [69] to an uncertain stochastic model. We consider a model driven by both randomness and uncertainty, where randomness is measured by a one-dimensional Brownian motion and uncertainty is measured by a one-dimensional canonical process. The canonical process is an uncertainty process representing incurred but not reported reserves and the belief degree at which uncertain events will occur. We present and prove the equation of optimality for the classical and impulse control problem. Due to the present of uncertain indeterminacy, the principle of optimality and the equation of optimality for uncertain stochastic processes are essential in solving the control problem studied in the thesis. It is also important to note that the uncertainty component in our model gives rise to the equation of optimality that involves a partial differential equation (PDE). A partial differential equation is more difficult to solve compared to the ordinary differential equation in the Zhang and Song [69] paper. An additional aspect of novelty in this section of the thesis is in the method employed in solving the derived (PDE) for the value function. We also manage to construct an explicit classical and impulse control when indeterminacy is measured by the combination of uncertainty and randomness.

The uncertain stochastic model extension is a reasonable model for continuous uncertain random systems without jumps. Nevertheless, in a real world, uncertain systems do exhibit jumps. In many cases, for example, stock prices may jump at scheduled or unscheduled times because of economic crises, war, announcements of economic statistics and monetary policies, and so on. This factors should be incorporated into the reserves process model. We, therefore, consider an extension of the optimal control problem by proposing a model driven by Brownian motion, canonical process and V jump process. The Brownian motion term measures random indeterminacy while the canonical process and the V jump process measure uncertainty and uncertain jumps, respectively. The major contribution of the thesis is the equation of optimality for an uncertain stochastic process with uncertain jumps. An additional aspect of novelty in the thesis is the presentation of the linear combination of the first and second moment for uncertain stochastic processes with uncertain jumps. This linear combination of moments plays an important role when proving the equation of optimality. As an application, we make use of the equation of optimality theorem for uncertain stochastic processes with uncertain jump to investigate the optimal control problem for insurance reserves. A closed-form solution for the optimal control and consumption is presented when reserves are assumed to be modeled by uncertain stochastic processes with a V jump.

Recently Bahlali et al. [5] provided a proof of the existence of optimal controls of nonlinear forwards-backwards differential equations. Motivated by their paper, we present for the first time the existence and uniqueness theorem of forward-backward uncertain stochastic differential equations. We also present for the first time a maximum principal for combined impulse and classical control problem with partial information. An application to the dividend and reinsurance control problem for insurance firms is presented. The optimal reinsurance policy and the optimal dividend distribution policy for an insurance firm is obtained.

1.3 Thesis Outline

The thesis is organized as follows. In Chapter 2 we present some key results in probability theory and uncertainty theory. The chapter and the thesis as a whole assume that the basic probability theory concept is known and focuses on the Lévy theory part of probability theory.

The objective of Chapter 3 is to find an optimal stopping time for an insurance company before any intervention to the reserves is made. We consider a jump-diffusion model whose surplus earns interest at a constant force. We give a mathematical foundation of the optimal stopping problem and the formulation of the main result of the chapter. The verification theorem for the optimal stopping problem under jump diffusions is presented and applied to the optimal stopping problem for insurance reserves.

The next chapter, Chapter 4, considers the extension of the problem presented by Zhang and Song [69] to the jump-diffusion case. The verification theorem for classical and impulse optimal control of insurance reserves is presented and proved. We make use of the verification theorem and the quasi-variational inequalities to find the optimal value function for insurance reserves under the smooth pasting condition. The optimal strategy for an insurance firm, in order to optimize its dividend distribution and reinsurance policy, is presented.

In Chapter 5 we extend the classical and impulse control theory by considering models with stochastic volatility. Motivated by Heston [26], Hull and White [31] and other state of the art research papers, notably Barndorff-Nielsen and Shephard [7], Barndorff-Nielsen et. al. [8] and Ball and Roma [6], we extend the results of Zhang and Song [69] by including stochastic volatility into their reserve model. We assume that volatility follows an Ornstein-Uhlenbeck process. A verification theorem for the problem is constructed and proved. The optimal control structure is conjectured and an explicit expression of the value function is given.

In Chapter 6 we consider the optimal control problem under uncertain stochastic processes. We present and prove for the first time the Principle of Optimality Theorem and the Equation of Optimality Theorem for the proposed problem. A closed form solution of the value function for an insurance control problem is given and the optimal reinsurance policy is derived.

In Chapter 7 we present an optimal control model driven by uncertain stochastic processes with uncertain jumps. The Principle of Optimality Theorem and the Equation of Optimality Theorem for uncertain stochastic processes with V jump process is derived and proved. We also present for the first time the linear combination of first and second moment for uncertain stochastic processes with jump. We then use the derived Equation of Optimality to solve the optimal control problem for an insurance firm that reinsures a proportion of its reserves to reduce risk.

In Chapter 8 we present for the very first time forward-backward uncertain stochastic differential equation (FBUSDE) and the existence and uniqueness theorem for FBUSDE. We then formulate a maximum principal for the optimal control problem of forward-backwards uncertain stochastic systems. The necessary and sufficient conditions for the local critical points is given for optimal control of uncertain stochastic processes with partial information. An application to dividend and reinsurance problem with partial information is presented. The optimal reinsurance policy and the optimal dividend distribution policy for an insurance firm is derived.

1.4 Publication and Conference Proceeding

The material presented in this thesis has resulted in the following research papers.

(i) S.W. Mgobhozi and E. Chikodza. Optimal Combined Dividend and Reinsurance Policies Under Interest Rate in Lèvy Markets. Accepted for publication by International Journal of Mathematics in Operational Research (IJMOR). The paper was also presented at the 2014 SAMS conference in Johannesburg, South Africa.

(ii) S.W. Mgobhozi and E. Chikodza. Optimal Proportional Reinsurance Policies Under Interest Rates in Ito-Liu Markets with Jump. Submitted for publication in the Journal of Uncertainty Analysis and Application. The paper has been accepted for presentation at the Quantitative Methods in Finance (QMF 2015) conference to be held in Sydney, Australia.

(iii) S.W. Mgobhozi and E. Chikodza. Impulse Control and Optimal Stopping Under Stochastic Volatility Model. Submitted for publication in the Journal of Uncertain Systems (JUS).

(iv) S.W. Mgobhozi and E. Chikodza. Optimal Dividend and Reinsurance Policies for Uncertain Stochastic Processes. Submitted for publication to Afrika Matematika.

(v) S.W. Mgobhozi and E. Chikodza. A Maximum Principle for Partial Information Forward-Backward Uncertain Stochastic Control with Application to Insurance and Finance. Working paper.

Chapter 2

Review of Lévy Processes and Uncertainty Theory

In this chapter, we present a brief introduction to Lévy processes and uncertain processes as building blocks for the problems examined in the thesis.

2.1 Lévy Processes

The term "Lévy process" was named in honour of the French mathematician Paul Lévy, who played an instrumental role in bringing an understanding and characteristic of processes with stationary and independent increments. There are numerous books giving a detailed theory on Lévy processes such as Applebaum [1], Bertoin [9], Kyprianou [36], Protter [52], and Sato [53] and references there in.

2.1.1 Basic Definitions and Results

We begin this section by defining a probability space $(\Omega, \mathcal{F}, P)$, where Ω is a set containing scenarios, equipped with a σ-algebra $\mathcal{F}$. In finance Ω represents different elementary outcomes that can be observed in the market, with each outcome $\omega \in \Omega$ describing a possible scenario in the economy. The σ-algebra $\mathcal{F}$ is the collection of subsets of Ω and P is the probability measure on $(\Omega, \mathcal{F})$.

Definition 2.1. A σ-algebra $\mathcal{F}$ is a collection of subsets of Ω such that

(i) $\emptyset, \Omega \in \mathcal{F}$,

(ii) $A \in \mathcal{F} \Rightarrow \ A^c \in \mathcal{F}$, where A^c is the complement of A,

(iii) If $A_1, A_2, \ldots, A_n, \cdots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Any subset B of Ω that belongs to $\mathcal{F}$ is called a measurable set.

Definition 2.2. The set function P is called a probability measure if it satisfies the following axioms:

Axiom 1. (Normality) $P\{\Omega\} = 1$ for the universal set Ω.

Axiom 2. (Nonnegativity) $P\{A\} \geq 0$ for any event A.

Axiom 3. (Additivity) For every countable sequence of mutually disjoint events $\{A_i\}$, $i = 1, 2, \ldots$, we have

$$P\Big\{\bigcup_{i=1}^{\infty} A_i\Big\} = \sum_{i=1}^{\infty} P\{A_i\}.$$

Definition 2.3. A random variable is a function from a probability space $(\Omega, \mathcal{F}, P)$ to the set of real numbers such that $\{\eta \in \mathbf{A}\}$ is an event for any Borel set $\mathbf{A}$.

A stochastic process is essentially a sequence of random variables indexed by time.

Definition 2.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let T be a totally ordered set (e.g. time). A stochastic process is a function $X(t)$ from $T \times (\Omega, \mathcal{F}, P)$ to the set of real numbers such that $\{X(t) \in \mathbf{A}\}$ is an event for any Borel set $\mathbf{A}$ at each time t.

Remark 2.1. A stochastic process can be written in the form X_t or $X(t)$. These notations will be used interchangeably, depending on the situation. The notation X_t is convenient when multiple variables are present, while the notation $X(t)$ is helpful when an emphasis is desired on the indexing variable.

Definition 2.5. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space. An $\mathcal{F}_t$-adapted process $X = \{X(t), t \geq 0\}$ is said to be a Lévy process if

(i) Each $X(0) = 0 \quad (a.s)$,

(ii) X has independent and stationary increments,

(iii) X is stochastically continuous, $i.e.$, for all $a \neq 0$ and for all $s \geq 0$

$$\lim_{t \to s} P\Big[|X(t) - X(s)| > a\Big] = 0.$$

Thus, a stochastic process $X(t)$ satisfying the above definition is called a Lévy process in law. The following simple result was proved in [52, 53].

Theorem 2.1. *Let $\{X(t)\}$ be a Lévy process. Then $X(t)$ has a càdlàg version (right continuous with left limits) which is also a Lévy process.*

The jump of $X(t)$ at time $t \geq 0$ is defined by

$$\Delta X(t) = X(t) - X(t_-). \tag{2.1}$$

Moreover, the jump process of $X(t)$, namely $\Delta X(t) = (\Delta X_t, t \geq 0)$, is a Poisson process. In this thesis we are going to use two types of Lévy processes to model the risk of an insurance company's reserves, which are the Brownian motion and the Poisson process.

The Brownian Motion

The Brownian motion was first introduced by Robert Brown to describe the random movement of pollen grains immersed in a container filtered with liquid such as water. It was first used in the modelling of the dynamics of stock prices by Louis Bachelier in the 1900s. Since then it has been used to model the evolution of many financial instruments, including the reserves of an insurance company. The Brownian motion is the dynamic counterpart of a Normal distribution. In this thesis we will use the Brownian motion to measure the continuous risk of the reserve process for an insurance company.

Definition 2.6. Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ be a filtered probability space. A stochastic process $B = \{B(t), t \geq 0\}$ defined on the probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ is a Brownian motion if it satisfies the following conditions

(i) $B(0) = 0 \quad (a.s.)$,

(ii) it has independent increments,

(iii) it has stationary increments,

(iv) an increment of the process over a period of $[s, s+t]$, $s, t \geq 0$ is Normally distributed with mean zero and variance t: $B(s+t) - B(s) \sim N(0, t)$.

It is thus very easy to see that a Brownian motion is a Lévy process. Another well known example of a Lévy process is the Poisson process. Besides the Brownian motion with drift, all other Lévy processes, except the deterministic case, have discontinuous paths.

Poisson Process

Many processes in everyday life that count events up to a particular point in time can be accurately described by the so called Poisson process, which was named after the French scientist Siméon Poisson. We will use the Poisson process to measure the risk associated with the jumps of the reserve process for an insurance company.

Definition 2.7. A counting process $\{C(t), t \geq 0\}$ is a stochastic process that keeps count of the number of events that have occurred up to time t. $C(t)$ is a non-negative and integer-valued for all $t \geq 0$. Furthermore, $C(t)$ is non-decreasing in t. $C(t) - C(s)$ equals the number of events in the time interval $(s, t]$, $s < t$.

Definition 2.8. A Poisson process $\{N(t), t \geq 0\}$ is a counting process with the following properties

(i) $N(0) = 0 \quad (a.s.)$,

(ii) The process has independent and stationary increments,

(iii) $P[N(t) = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \qquad n = 0, 1, 2, \ldots.$

We can therefore conclude that Poisson processes are Lévy process with intensity $\lambda > 0$. The Poisson process has mean λt and variance λt.

The Compound Poisson Process

Let $X(n), n \in \mathbb{N}$ be a sequence of identically independently distributed (i.i.d) random variables taking values in $\mathbb{R}$ with common distribution $\mu_{X(1)} = \mu_X$ and let $N(t)$ be a Poisson process of intensity λ, independent of all the $X(n)$'s.
The Compound Poisson process $Y(t)$ is defined by

$$Y(t) = X(1) + X(2) + \cdots + X(N(t)), \qquad t \geq 0. \tag{2.2}$$

An increment of this process is given by

$$Y(s) - Y(t) = \sum_{k=N(t)+1}^{N(s)} X(k), \qquad s > t.$$

This is independent of $X(1), X(2), \ldots, X(N(t))$ and its distribution depends only on the difference $(s - t)$ and on the distribution of $X(1)$. Thus $Y(t)$ is a Lévy process.

2.2 Uncertainty Theory

Probability theory is applied when indeterminacy is only measured by randomness. However, sufficient data may not be available to estimate a probability distribution, thus a domain expert needs to be invited to evaluate the belief degree that each event will happen. To deal with this belief degree, uncertainty theory was founded in 2007 by Liu [39] and subsequently studied by many researchers, see [14], [20], [41], [65], [70] and references therein. Uncertainty theory has become a new branch of axiomatic mathematics.

There exist two mathematical systems for modeling indeterminacy, one is probability theory (Kolmogorov [34]) and the other is uncertainty theory (Liu [39]). In general probability is interpreted as frequency, while uncertainty is interpreted as personal belief degree. The fundamental basis of applying probability theory is that the estimated probability distribution is close enough to the long-run cumulative frequency. The law of large numbers is no longer valid when there is not enough sample size and probability theory is no longer applicable. In many cases, for example, modeling insurance claims, sample are not available to estimate a probability distribution and domain experts needs to be invited to evaluate the belief degree that each event will happen.

A belief degree represents the strength which we believe the event will happen. For insurance firms, some claims may have been observed at the end of the accounting period but the insurance company has no information about these claims' presence and cost. Actuaries for these insurance firms need to have some belief degree on the existence of uncaptured claims when modeling the evolution of insurance reserves to avoid solvency problems. The belief degree depends heavily on the personal knowledge concerning the event. When the personal knowledge changes, the belief degree changes as well. For more information on belief degree and belief degree functions, the reader is referred to Liu [39].

2.2.1 Basic Definitions

We begin this section by defining an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, where Γ is a set containing scenarios, equipped with a σ-algebra $\mathcal{L}$. In finance Γ represents uncertain events that can occur in the market, with each event $\gamma \in \Gamma$ describing the evolution of prices of different instruments. The σ-algebra $\mathcal{L}$ contains all events we are concerned about and $\mathcal{M}$ is the uncertain measure on σ-algebra $\mathcal{L}$.

Definition 2.9. Let Γ be a nonempty set (sometimes called universal set). A collection $\mathcal{L}$ consisting of subsets of Γ is called an algebra over Γ if the following three conditions hold:

(i) $\emptyset, \Gamma \in \mathcal{L}$;

(ii) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$;

(iii) if $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \in \mathcal{L}$, then $\bigcup_{i=1}^{n} \Lambda_i \in \mathcal{L}$.

The collection $\mathcal{L}$ is called a σ-algebra over Γ. If the third condition is replaced with closure under countable union, i.e., when $\Lambda_1, \Lambda_2, \Lambda_3, \dots \in \mathcal{L}$, we have $\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}$.

Definition 2.10. The set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality) $\mathcal{M}\{\Gamma\} = 1$,

Axiom 2. (Self-Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ,

Axiom 3. (Countable Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\Big\{\bigcup_{i=1}^{\infty} \Lambda_i\Big\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

It is clear to see that the difference between randomness and uncertainty is in the additivity axiom applied in classical measure theory. The probability measure, Definition 2.2, satisfies maximality axiom while an uncertain measure, Definition 2.10, satisfies the countable subadditivity axiom.

Definition 2.11. An uncertain variable is a function Λ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\Lambda \in \mathbf{A}\}$ is an event for any Borel set $\mathbf{A}$.

Definition 2.12. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and let T be a totally ordered set (e.g., time). An uncertain process is a function X_t from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in \mathbf{A}\}$ is an event for any Borel set $\mathbf{A}$ at each time t.

The following section which gives the main difference between probability theory and uncertainty theory is from Liu [39].

The Difference Between Probability Theory and Uncertainty Theory

The main difference between probability theory (Kolomogorov [34]) and uncertainty theory (Liu [39]) is that the probability measure of a product of event is the product of the probability measures of the individual events, *i.e.*,

$$P\{A \times B\} = P\{A\} \times P\{B\},$$

and the uncertain measure of a product of events is the minimum of the uncertain measures of the individual events, *i.e.*,

$$\mathcal{M}\{A \times B\} = \mathcal{M}\{A\} \wedge \mathcal{M}\{B\}.$$

This difference implies that random variables and uncertain variables obey different operational laws.
Probability theory and uncertainty theory are complementary mathematical models to deal with the indeterminate world. Probability is interpreted as frequency, while uncertainty is interpreted as personal belief degree.

Liu Process

The term "Liu process" was named by the academic community in honour of Baoding Liu due to its importance and usefulness. A detailed theory on Liu processes can be found in [39], [41], [42], and references therein.

Definition 2.13. An uncertain process C_t is said to be a canonical Liu process if
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

It is clear that a canonical Liu process C_t is a stationary independent process and has a normal uncertain distribution with expected value 0 and variance t^2.

The uncertainty distribution of C_t is

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R}.$$

We state the following theorem without proof; its proof can be found in Liu [41].

Theorem 2.2. *(Linearity of Expected Value Operator)*
Let Λ and η be independent uncertain variables with finite expected values. Then for any real number a and b, we have

$$E\big[a\Lambda + b\eta\big] = aE\big[\Lambda\big] + bE\big[\eta\big]. \tag{2.3}$$

2.3 Chance Theory

Chance theory is the hybrid between probability theory and uncertainty theory. Indeterminacy is thus measured by the combination of randomness and uncertainty. Chance theory was pioneered by Yuhan Liu in 2013 for modeling complex systems which simultaneously exhibit randomness and uncertainty.
For the optimal control problem of uncertain stochastic systems, a filtered uncertain probability space $(\Gamma \times \Omega, \mathcal{L} \otimes \mathcal{F}, (\mathcal{L}_t \otimes \mathcal{F}_t)_{t\in[0,T]}, \mathcal{M} \times P)$ is constructed, on which the following related concepts are defined Fei [21].

Definition 2.14. (Fei [21]) (i) An uncertain random variable is a measurable function $\Lambda \in \mathbb{R}^p$ (resp. $\mathbb{R}^{p\times m}$) from uncertainty probability space $(\Gamma \times \Omega, \mathcal{L} \otimes \mathcal{F}, \mathcal{M} \times P)$ to the set in $\mathbb{R}^p$ (resp. $\mathbb{R}^{p\times m}$), $i.e.$, for any Borel set $\mathbf{A} \in \mathbb{R}^p$ (rep. $\mathbb{R}^{p\times m}$), the set
$\{\Lambda \in \mathbf{A}\} = \{(\gamma, \omega) \in \Gamma \times \Omega : \Lambda(\gamma, \omega) \in \mathbf{A}\} \in \mathcal{L} \otimes \mathcal{F}.$
(ii) The expected value of an uncertain random variable is defined by

$$\begin{aligned} E[\Lambda] &= E_p\big[E_{\mathcal{M}}[\Lambda]\big] \\ &\triangleq \int_{\Omega}\Big[\int_0^{+\infty} \mathcal{M}\{\Lambda > \gamma\}d\gamma\Big]P(d\omega) - \int_{\Omega}\Big[\int_{-\infty}^{0} \mathcal{M}\{\Lambda < \gamma\}d\gamma\Big]P(d\omega), \end{aligned}$$

where E_p and $E_{\mathcal{M}}$ denote expected values under the probability space and uncertainty space, respectively.

It is clear that if a and b are constant, then $E[aC_t + bB_t] = 0$, where C_t is a canonical process and B_t is a Brownian motion.

Definition 2.15. (Fei [21]) (i) A hybrid process $X(t)$ is called an uncertain stochastic process if for each $t \in [0,T]$, $X(t)$ is an uncertain random variable. An uncertain stochastic process $X(t)$ is called continuous if the sample paths of $X(t)$ are all continuous functions of t for almost all $(\gamma,\omega) \in \Gamma \times \Omega$.
(ii) An uncertain stochastic process $X(t)$ is called $\mathcal{F}_t$-adapted if $X(t,\gamma)$ is $\mathcal{F}_t$-measurable for all $t \in [0,T]$, $\gamma \in \Gamma$. Moreover, a hybrid process $X(t)$ is called $\mathcal{L}_t \otimes \mathcal{F}_t$-adapted (or adapted) if $X(t)$ is $\mathcal{L}_t \otimes \mathcal{F}_t$-measurable for all $t \in [0,T]$.
(iii) An uncertain stochastic process is called progressively measurable if it is measurable with respect to the σ-algebra

$$\mathfrak{F}(\mathcal{L}_t \otimes \mathcal{F}_t) = \{\mathbf{A} \in \mathcal{B}([0,T]) \otimes \mathcal{L} \otimes \mathcal{F} : \mathbf{A} \cap ([0,t] \times \Gamma \times \Omega) \in \mathcal{B}([0,T]) \otimes \mathcal{L}_t \otimes \mathcal{F}_t\}.$$

Moreover, an uncertain stochastic process $X(t) : \Gamma \times \Omega \to \mathbb{R}^p$ $\big($rep. $X(t) : \Gamma \times \Omega \to \mathbb{R}^{p\times m}\big)$ is called L^2-progressively measurable if it is progressively measurable and satisfies $E\big[\int_0^T |X(t)|^2 dt\big] < \infty$. The set, $\mathbf{M}^2(0,T,\mathbb{R}^p)$ $\big($rep. $\mathbf{M}^2(0,T,\mathbb{R}^{p\times m})\big)$ denote the set of L^2-progressively measurable uncertain random processes.

Definition 2.16. (Itô-Liu integral) Let $X(t) = (Y(t), Z(t))^T$ be an uncertain stochastic process, where $Y(t) \in \mathbb{R}^{p\times m}$ and $Z(t) \in \mathbb{R}^{p\times n}$. For any partition of closed interval $[a,b]$ with $a = t_1 < t_2 < \ldots < t_{N+1} = b$, the mesh is written as $\delta = \max_{1\le i\le N} |t_{i+1} - t_i|$. Then the Itô-Liu integral of $X(t)$ with respect to (B_t, C_t) denoted by $\int_a^b X^T(s)d(B_s, C_s)$ is defined as follows,

$$\int_a^b X(s)d(B_s, C_s) = \lim_{\delta\to 0} \sum_{i=1}^{N} \Big[Y(t_i)(B_{t_{i+1}} - B_{t_i}) + Z(t_i)(C_{t_{i+1}} - C_{t_i})\Big], \qquad (2.4)$$

provided that the limit on the right hand side of (2.4) exists in mean square and is an uncertain random variable, where C_t and B_t are n-dimensional canonical process and m-dimensional Brownian motion, respectively. In this case, $X(t)$ is called Itô-Liu integrable. Specially, when $Y(t) \equiv 0$, $X(t)$ is called Liu integrable.

Remark 2.2. The Itô-Liu integral for a one-dimensional uncertain stochastic process can be written as follows

$$\int_a^b X(s)d(B_s, C_s) = \int_a^b \big(Y_t dB_t + Z_t dC_t\big) \tag{2.5}$$

Example 2.1. Let B_t be a one-dimensional Brownian motion, and C_t a one-dimensional canonical process. Then

$$\int_0^t \big(\sigma_1 dB_s + \sigma_2 dC_s\big) = \sigma_1 B_t + \sigma_2 C_t$$

where σ_1 and σ_2 are constants, random variables, uncertain variables, or uncertain random variables.

Example 2.2. Let B_t be a one-dimensional Brownian motion, and C_t a one-dimensional canonical process. Then

$$\int_0^t \big(B_s dB_s + C_s dC_s\big) = \frac{1}{2}\big(B_t^2 + C_t^2 - t\big),$$

and

$$\int_0^t \big(C_s dB_s + B_s dC_s\big) = B_t C_t.$$

The following Itô-Liu formula for the case of multi-dimensional uncertain stochastic processes is given [20].

Theorem 2.3. *(Itô-Liu Formula) Let $B = \{B_t, t \in [0,T]\}$ be an m-dimensional standard Brownian motion and let $C = \{C_t, t \in [0,T]\}$ be an n-dimensional canonical process. Assume that the uncertain processes $X_1(t), X_2(t), \ldots, X_p(t)$ are given by*

$$dX_k(t) = u_k(t)dt + \sum_{l=1}^{m} v_{kl}(t)dB_t^l + \sum_{l=1}^{n} w_{kl}(t)dC_t^l \qquad k = 1, 2, \ldots, p;$$

where $u_k(t)$ are all absolute integrable uncertain stochastic processes, $v_{kl}(t)$ are all square integrable uncertain stochastic processes and $w_{kl}(t)$ are all Liu integrable uncertain stochastic processes. For $k, l = 1, \ldots, p$; let $\frac{\partial F}{\partial t}(t, x_1, \ldots, x_p)$ and

$\frac{\partial^2 F}{\partial x_k \partial x_l}(t, x_1, \ldots, x_p)$ *be continuous functions. Then we have*

$$dF(t, X_1(t), \ldots, X_p(t))$$

$$= \frac{\partial F}{\partial t}(t, X_1(t), \ldots, X_p(t))dt + \sum_{k=1}^{p} \frac{\partial F}{\partial x_k}(t, X_1(t), \ldots, X_p(t))dX_k(t)$$

$$+\frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{p} \frac{\partial^2 F}{\partial x_k \partial x_l}(t, X_1(t), \ldots, X_p(t))dX_k(t)dX_l(t),$$

where $dB_t^k dB_t^l = \delta_{kl} dt$ *and*

$dB_t^k dt = dC_t^i dC_t^j = dC_t^i dt = dB_t^k dC_t^i = 0$

for $k, l = 1, \ldots, m$ *and* $i, j = 1, \ldots, n$

$$\delta_{kl} = \begin{cases} 0, & \text{if} \quad k \neq l, \\ 1 & \text{otherwise} . \end{cases}$$

Proof. Since $F(t, X_1(t), \ldots, X_p(t))$ is a continuously differentiable function, we have

$$\Delta F(t, X_1(t), \ldots, X_p(t))$$

$$= \frac{\partial F}{\partial t}(t, X_1(t), \ldots, X_p(t))\Delta t + \sum_{k=1}^{p} \frac{\partial F}{\partial x_k}(t, X_1(t), \ldots, X_p(t))\Delta X_k(t)$$

$$+\frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{p} \frac{\partial^2 F}{\partial x_k \partial x_l}(t, X_1(t), \ldots, X_p(t))\Delta X_k(t)\Delta X_l(t)$$

$$+\frac{1}{2}\frac{\partial^2 F}{\partial t^2}(t, X_1(t), \ldots, X_p(t))(\Delta t)^2 + \sum_{k=1}^{p} \frac{\partial^2 F}{\partial x_k \partial t}(t, X_1(t), \ldots, X_p(t))\Delta t \Delta X_k(t)$$

$$+\varepsilon_1(\Delta t)^2 + \sum_{k=1}^{p}\sum_{l=1}^{p} \varepsilon_{kl}\Delta X_k(t)\Delta X_l(t) + \sum_{k=1}^{p} \varepsilon_k \Delta t \Delta X_k(t)$$

where $\varepsilon_1 \to 0$, $\varepsilon_{kl} \to 0$, $\varepsilon_k \to 0$ for $k, l = 1, \ldots, p$ as $\Delta t \to 0$.

Since $\Delta B_k(t) \to 0$, $\Delta C_l(t) \to 0$

$$\Delta X_k(t) = u_k \Delta t + \sum_{l=0}^{m} v_{kl} B + l(t) + \sum_{l=1}^{n} w_{kl} \Delta C_l(t) \to 0 \quad \text{as } \Delta t \to 0$$

On the other hand, since $\Delta B_l(t) \to 0$, $\Delta C_k(t) \to 0$, $(\Delta B_l(t))^2 \to \Delta t$, $(\Delta C_k(t))^2 \to (\Delta t)^2$,
$(\Delta X_k(t))^2 \to (\Delta t)$, we obtain the chain rule. □

The following theorem can be found in Liu [40].

Theorem 2.4. *Let B_t be a one-dimensional standard Brownian motion, C_t a one-dimensional standard canonical process, and $f(t,b,c)$ a twice continuously differentiable function. Define $X_t = f(t, B_t, C_t)$. Then we have the following chain rule,*

$$dX_t = \frac{\partial f}{\partial t}(t,B_t,C_t)dt + \frac{\partial f}{\partial b}(t,B_t,C_t)dB_t + \frac{\partial f}{\partial c}(t,B_t,C_t)dC_t + \frac{1}{2}\frac{\partial^2 f}{\partial b^2}(t,B_t,C_t)dt \tag{2.6}$$

Proof. Since the function f is twice continuously differentiable, by using Taylor series expansion, the infinitesimal increment of X_t has a second-order approximation,

$$\begin{aligned}\Delta X_t \quad = \quad & \frac{\partial f}{\partial t}(t,B_t,C_t)\Delta t + \frac{\partial f}{\partial b}(t,B_t,C_t)\Delta B_t + \frac{\partial f}{\partial c}(t,B_t,C_t)\Delta C_t \\ & + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(t,B_t,C_t)(\Delta t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial b^2}(t,B_t,C_t)(\Delta B_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial c^2}(t,B_t,C_t)(\Delta C_t)^2 \\ & + \frac{\partial^2 f}{\partial t \partial b}(t,B_t,C_t)\Delta t\Delta B_t + \frac{\partial^2 f}{\partial t \partial c}(t,B_t,C_t)\Delta t\Delta C_t + \frac{\partial^2 f}{\partial b \partial c}(t,B_t,C_t)\Delta B_t\Delta C_t.\end{aligned}$$

Since we can ignore the term $(\Delta t)^2$, $(\Delta C_t)^2$, $\Delta t\Delta B_t$, $\Delta t\Delta C_t$, $\Delta B_t\Delta C_t$ and replace $(\Delta B_t)^2$ with Δt, the chain rule is obtained because it makes

$$X_s = X_0 + \int_0^s \frac{\partial f}{\partial t}dt + \int_0^s \frac{\partial f}{\partial b}dB_t + \int_0^s \frac{\partial f}{\partial c}dC_t + \frac{1}{2}\int_0^s \frac{\partial^2 f}{\partial b^2}dt$$

for any $s \geq 0$. □

Definition 2.17. (Liu [40]) Suppose B_t is a standard Brownian motion, C_t is a standard canonical process, and μ, σ, γ are some functions. Then

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dB_t + \gamma(t,X_t)dC_t \tag{2.7}$$

is called an uncertain stochastic differential equation. A solution is a hybrid process X_t that satisfies (2.7) identically in t.

Remark 2.3. A theorem on the existence and uniqueness of solution of (2.7) are proved in Fei [21] for an m-dimensional Brownian motion and a d-dimensional canonical process.

Example 2.3. Let B_t be a one-dimensional Brownian motion, and C_t a one-dimensional canonical process. Then the uncertain stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dB_t + \lambda X_t dC_t$$

has a solution

$$X_t = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t + \lambda C_t\right) \tag{2.8}$$

which is just a geometric uncertain stochastic process.

Definition 2.18. [41] Let ξ be an uncertain random variable. Then the chance distribution of ξ is defined by

$$\Phi(x) = Ch(\xi \leq x)$$

for any $x \in \mathbb{R}$.

As a special uncertain random variable, the chance distribution of a random variable η is just its probability distribution, that is,

$$\Phi(x) = Ch(\xi \leq x) = P\{\eta \leq x\},$$

while the chance distribution of an uncertain variable τ is just its uncertainty distribution, that is,

$$\Phi(x) = Ch(\xi \leq x) = \mathcal{M}\{\tau \leq x\}.$$

2.4 Conclusion

The first three models considered in this thesis are Lévy type models and a basic knowledge of Lévy processes will be required for reading this thesis. The remainder of the models presented in this thesis are uncertain stochastic models, and the knowledge and understanding of Theorem 2.3 and Theorem 2.4 is critical for solving problems under uncertain stochastic theory.

Chapter 3

Optimal Stopping Rules Under Interest Rate in Lévy Markets.

3.1 Introduction

In this chapter we examine the problem of determining the optimal time to stop before bankruptcy for an insurance company when its surplus earns interest at a constant rate $\rho > 0$. The problem is solved by making use of the intergrovariational inequalities theorem for optimal stopping under jump diffusions. The value function and the optimal stopping policy are constructed for the insurance company.

This chapter is the building block of the thesis and its objective is to find an optimal stopping time for an insurance company before any intervention to the reserves is made. An impulse control problem, which is the main study of the thesis, can be thought of as a sequence of stopping time problems. There is therefore need to study the stopping problem before the impulse control problem. We therefore consider the model presented by Zhang and Song [69] in the Lévy diffusion setting excluding reinsurance control and dividend policy and find the optimal time to stop before bankruptcy or any harvesting is made. The optimal value function for the reserves is constructed at the stopping time using quasi-variational inequalities (QVI).

3.2 Preliminaries and Problem Formulation

For the mathematical foundation of the optimal stopping problem, fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, a standard Brownian motion $\{B(t)\}_{t\geq 0}$ and a compensated Poisson random measure $\{\widetilde{N}(t,\cdot)\}_{t\geq 0}$ defined on the real probability space $\{P_x, x \in \mathbb{R}\}$. We denote by $\{P_x, x \in \mathbb{R}\}$ the family of probability measures corresponding to the real-valued Lévy process $X = \{X(t)\}_{t\geq 0}$ which is a stochastic process with stationary and independent increment with $P_0 = P$. We further define E_x to be the expectation with respect to P_x.
We assume that the Lévy process $X = \{X(t)\}_{t\geq 0}$ is $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted, increasing and right-continuous with left limits, where $\mathcal{F}_t$ represents the information available at time t and any decision is made based on this information.
By the Lévy-Khintchine Theorem the laws of the Lévy process are characterized by the characteristic exponent Ψ defined through $E[e^{izX_t}] = e^{-t\Psi(z)}$ for all $t \geq 0$ and $z \in \mathbb{R}$. The characteristic exponent $\Psi(z)$ is given by

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + iaz + \int_{\mathbb{R}} (1 + e^{izx} + \mathcal{X}_{\{|x|<1\}} izx\nu(dx),$$

where $a \in \mathbb{R}$, $z^2 \geq 0$ and ν is a σ-finite measure on $\mathbb{R} - \{0\}$ satisfying

$$\nu(0) = 0 \quad \text{and} \quad \int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty.$$

The triple (σ, z, ν) is usually referred to as the Lévy triplet. For a further discussion on the filtration and Lévy processes see, for example, [1], [9], [36], [53] and references given therein. The Lévy process $X = \{X(t)\}_{t\geq 0}$ which we assume to be the reserve process represents the liquid assets of the insurance company and evolves according to

$$dX(t) = \mu dt + \sigma dB(t) + \int_{\mathbb{R}} \gamma z \widetilde{N}(dt, dz), \tag{3.1}$$

where $\mu > 0$, $\gamma > 0$ and $\sigma > 0$ are constant, while $\{B(t)\}_{t\geq 0}$ is a standard one-dimensional Brownian motion and $\{\widetilde{N}(t,\cdot)\}_{t\geq 0}$ is the compensated Poisson random measure with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ given by

$$\widetilde{N}(dt,dz) = N(dt,dz) - dt\nu(dz),$$

where $N(.,.)$ is a Poisson random measure. With the assumption that reserves of the insurance company also earn interest at the constant force $\rho > 0$, the reserve process $X(t)$ evolves according to the following jump diffusion process

$$dX(t) = [\mu + \rho X(t)]dt + \sigma dB(t) + \int_{\mathbb{R}} \gamma z \widetilde{N}(dt,dz), \tag{3.2}$$

where $X(0) = x > 0$ is the initial reserves of an insurance company. We assume that

$$-1 < \gamma z \leq 0 \; a.s. \; \nu. \tag{3.3}$$

The time to bankruptcy, which is formally defined in the following section, is the stopping time defined by

$$\tau := \inf\{t \geq 0 : X(t) = 0\}.$$

We assume that the process $X(t)$ vanishes for $t \geq \tau$ as we are only dealing with optimal problem during the time interval $[0, \tau)$. Define the performance function $J^\tau(s,x)$ by

$$J^\tau(s,x) = E^x\left[e^{-q(s+t)}(X(\tau) - b)\mathcal{X}_{\{\tau<\infty\}}\right], \tag{3.4}$$

where the constant $b > 0$ represent fixed transaction cost paid at intervention. We define an infinitesimal operator $\mathbb{L}$ acting on a sufficiently smooth function ϕ to be given by

$$\mathbb{L}\phi(x) = \frac{\sigma^2}{2}\phi''(x) + [\mu + \rho x]\phi'(x) + \int_{\mathbb{R}} \{\phi(x+\gamma z) - \phi(x) - \phi'(x)\gamma z\}\nu(dz).$$

Problem 3.2.1. *The insurance company wants to find an optimal stopping time τ^* for the reserve process in order to maximize its expected discounted net payoff*

J. We thus need to find a value function $\Phi(s,x)$ and an optimal stopping time $\tau^* \in \mathcal{T}$ such that

$$\Phi(s,x) = \sup_{\tau \in \mathcal{T}} J^{\tau}(s,x) = J^{\tau^*}(s,x). \tag{3.5}$$

We consider the following general formulation from Øksendal and Sulem [49] to solve our problem.

3.3 Formulation and Verification Theorem

The formulation in this section and the theorems presented are taken from Øksendal and Sulem [49]. This section will assist us in solving the proposed problem, Problem 3.2.1, for an insurance company. The idea behind this formulation is to take advantage of the Markovian property of the underlying process. If the underlying process is Markov we should at any point in time be able to decide whether to stop or continue without considering the history of the process. The Markovian approach of our optimal stopping problem will translate the problem into a free-boundary problem.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered complete probability space satisfying the usual conditions. If we fix an open set $\mathbf{S} \subset \mathbb{R}^k$ (which we will call the solvency region) and let $Y(t)$ be a jump diffusion in $\mathbb{R}^k$ given by

$$\begin{aligned} dY(t) &= b(Y(t))dt + \sigma(Y(t))dB(t) + \int_{\mathbb{R}^l} \gamma(Y(t^-), z)\tilde{N}(dt, dz) \\ &Y(0) = y \in \mathbb{R}^k, \end{aligned} \tag{3.6}$$

where $b : \mathbb{R}^k \to \mathbb{R}^k$, $\sigma : \mathbb{R}^k \to \mathbb{R}^{k\times l}$, and $\gamma : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^{k\times l}$ are given functions satisfying the conditions for the existence and uniqueness of a solution $Y(t)$. Define the bankruptcy (stopping time) by

$$\tau_{\mathbf{S}} = \tau_{\mathbf{S}}(y, \omega) = \inf\{t > 0; Y(t) \notin \mathbf{S}\} \tag{3.7}$$

and let $\mathcal{T}$ denote the set of all stopping time $\tau \leq \tau_{\mathbf{S}}$.

As in Øksendal and Sulem [49], we allow $\mathbf{S}$ to be any Borel set such that $\mathbf{S} \subset \bar{\mathbf{S}}^0$ where $\mathbf{S}^0$ denote the interior of S and $\bar{\mathbf{S}}^0$ denote its closure.

Let $f : \mathbb{R}^k \to \mathbb{R}$ and $g : \mathbb{R}^k \to \mathbb{R}$ be continuous functions satisfying the conditions

$$E^y\Big[\int_0^{\tau_{\mathbf{S}}} f^-(Y(t))dt\Big] < \infty \qquad \text{for all } y \in \mathbb{R}^k. \tag{3.8}$$

The family $\{g^-(Y(\tau)) \cdot \mathcal{X}_{\{\tau<\infty\}}, \tau \in \mathcal{T}\}$ is uniformly integrable, for all $y \in \mathbb{R}^k$.

The optimal stopping problem consist of finding $\Phi(y)$ and $\tau^* \in \mathcal{T}$ such that

$$\Phi(y) = \sup_{\tau\in\mathcal{T}} J^\tau(y) = J^{\tau^*}(y), \qquad y \in \mathbb{R}^k$$

where the performance function J is given by

$$J^\tau(y) = E^y\Big[\int_0^{\tau} f(Y(t))dt + g(Y(\tau)) \cdot \mathcal{X}_{\{\tau<\infty\}}\Big], \quad \tau \in \mathcal{T}.$$

We refer to Φ as the value function. We define the infinitesimal operator $\mathbb{L}$ for some function $\phi(y) \in \mathcal{C}^2$ as

$$\begin{aligned}\mathbb{L}\phi(y) \quad &= \quad \sum_{i=1}^k b_i(y)\frac{\partial\phi}{\partial y_i} + \frac{1}{2}\sum_{i,j=1}^k (\sigma\sigma^T)_{ij}(y)\frac{\partial^2\phi}{\partial y_i\partial y_j} \\ &\quad + \sum_{j=1}^l \int_{\mathbb{R}} \Big\{\phi(y+\gamma^{(j)}(y,z_j)) - \phi(y) - \nabla\phi(y)\cdot\gamma^{(j)}(y,z_j)\Big\}\nu_j(dz_j).\end{aligned}$$

for all $\phi : \mathbb{R}^k \to \mathbb{R}$ and $y \in \mathbb{R}^k$. The following results play a critical role in the optimal stopping problem. We state the following theorem without proof, its proof can be found in Øksendal [48].

Theorem 3.1. *(Approximation Theorem) Let D be an open set such that $D \subset$ $\mathbf{S}$. Assume that $Y(\tau_{\mathbf{S}}) \in \partial\mathbf{S}$ a.s. on $\{\tau_{\mathbf{S}} < \infty\}$ and*

$$\partial D \quad \textit{is a Lipschitz surface} \tag{3.9}$$

($i.e.$, ∂D is locally the graph of a Lipschitz continuous function) and let $\varphi : \bar{\mathbf{S}} \to \mathbb{R}$ be a function with the following properties:

$$\varphi \in \mathbf{C}^1(\mathbf{S}) \cap \mathbf{C}(\bar{\mathbf{S}}) \tag{3.10}$$

and

$$\varphi \in \mathbf{C}^2(\mathbf{S} \setminus \partial D) \tag{3.11}$$

and the second-order derivative of φ are locally bounded near ∂D.

Then there exists a sequence $\{\varphi_m\}_{m=1}^{\infty} \subset \varphi \in \mathbf{C}^1(\mathbf{S}) \cap \mathbf{C}(\bar{\mathbf{S}})$ such that, with the infinitesimal operator $\mathbb{L}$ of $Y(t)$,

$$\varphi_m \to \varphi \quad \textit{pointwise dominatingly in } \bar{\mathbf{S}} \textit{ as } m \to \infty \tag{3.12}$$

$$\frac{\partial \varphi_m}{\partial x_i} \to \frac{\partial \varphi}{\partial x} \quad \textit{pointwise dominatingly in } \mathbf{S} \textit{ as } m \to \infty \tag{3.13}$$

$$\frac{\partial^2 \varphi_m}{\partial x_i \partial x_j} \to \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \textit{ and } \mathbb{L}\varphi_m \to \mathbb{L}\varphi$$

$$\textit{pointwise dominatingly in } \mathbf{S} \setminus \partial D \textit{ as } m \to \infty. \tag{3.14}$$

Theorem 3.2. *(Integrovariational Inequalities for Optimal Stopping)* *(a) Suppose we can find a function* $\phi : \bar{\mathbf{S}} \to \mathbb{R}$ *such that*

(i) $\phi \in \mathbf{C}^1(\mathbf{S}) \cap \mathbf{C}(\bar{\mathbf{S}})$.

(ii) $\phi \geq g \quad on \quad \mathbf{S}$.

$Define$

$$D = \{y \in \mathbf{S}; \phi(y) > g(y)\} \qquad \textit{(the continuation region)}.$$

$Suppose$

(iii) $E^y\left[\int_0^{\tau_{\mathbf{S}}} \mathcal{X}_{\partial D}(Y(t))dt\right] = 0.$

(iv) ∂D *is a Lipschitz surface*.

(v) $\phi \in \mathbf{C}^2(\mathbf{S} \setminus \partial D)$ *with locally bounded derivatives near* ∂D.

(vi) $\mathbb{L}\phi + f \leq 0$ *on* $\mathbf{S} \setminus \partial D$.

(vii) $Y(\tau_{\mathbf{S}}) \in \partial \mathbf{S}$ $a.s.\ on$ $\{\tau_{\mathbf{S}} < \infty\}$ and $\lim_{t \to \tau_{\mathbf{S}}^-} \phi(Y(t)) = g(Y(\tau_{\mathbf{S}})).\mathcal{X}_{\{\tau_{\mathbf{S}} < \infty\}}$

$(viii)$ $E^y\Big[\mid \phi(Y(\tau)) \mid + \int_0^{\tau_{\mathbf{S}}} \mid \mathbb{L}\phi(Y(t)) \mid dt\Big] < \infty$ *for all* $\tau \in \mathcal{T}$.

Then $\phi(y) \geq \Phi(y)$ *for all* $y \in \bar{\mathbf{S}}$.

(b) Moreover, assume

(ix) $\mathbb{L}\phi + f = 0$ *on* D.

(x) $\tau_D := \inf t > 0; Y(t) \notin D < \infty$ $a.s.$ *for all* y.

(xi) $\{\phi(Y(\tau)); \tau \in \mathcal{T}, \tau \leq \tau_D\}$ *is uniformly integrable, for all* y.

Then

$$\phi(y) = \Phi(y)$$

and

$\tau^* = \tau_D$ *is an optimal stopping time*.

Proof. (a) Let $\tau \leq \tau_{\mathbf{S}}$ be a stopping time. By the Approximation theorem we can assume that $\phi \in \mathbf{C}^2(\mathbf{S})$. Then by (vii) and $(viii)$ and the Dynkin formula applied to $\tau_n := \min(\tau, n), n = 1, 2, \ldots$ we have by (vi),

$$\begin{aligned} E^y[\phi(Y(\tau_n))] &= \phi(y) + E^y\Big[\int_0^{\tau_n} \mathbb{L}\phi(Y(t))dt\Big] \\ &\leq \phi(y) - E^y\Big[\int_0^{\tau_n} f(Y(t))dt\Big]. \end{aligned}$$

Hence by (ii) and the Fatou lemma,

$$\begin{aligned} \phi(y) &\geq \liminf_{n\to\infty} E^y\Big[\int_0^{\tau_n} f(Y(t)) + \phi(Y(\tau_n))dt\Big] \\ &\geq E^y\Big[\int_0^{\tau} f(Y(t)) + g(Y(\tau))\mathcal{X}_{\tau<\infty}dt\Big] = J^{\tau}(y). \end{aligned}$$

Hence

$$\phi(y) \geq \Phi(y). \tag{3.15}$$

(b) Moreover, if we apply the above argument to $\tau = \tau_D$ then by $(ix) - (xi)$ and the definition of D we get equality, so that

$$\phi(y) = J^{\tau_D}(y) = \Phi(y) \tag{3.16}$$

and τ_D is the optimal stopping time. □

3.4 Optimal Stopping Rules Under Interest Rate in Lévy Markets.

Suppose that we are given a reserve process $\{X(t); t \geq 0\}$, representing liquid assets of the insurance company evolving according to (3.2). With $Y(t) = (s + t, X(t))$, $b(Y(t)) = [\mu + \rho X(t)]$, $\sigma(Y(t)) = \sigma$ and $\gamma(Y(t^-), z) = \gamma z$ from (3.6), we thus make use of Theorem 3.2 in order to solve our problem.

3.4.1 Solution of the Reserves Problem

Consider the stochastic process with jump diffusion given by (3.2) with expected net payoff given by (3.4).

Proposition 3.1.

The optimal stopping time for the reserve process (3.2) is given by

$$\tau^* = \inf\Big\{t > 0, X(t) \geq \frac{2(\ln|\alpha_2| - \ln\alpha_1)}{\alpha_1 - \alpha_2}\Big\}$$

where α_1 and α_2 solve

$$h(\alpha) = -q + \frac{1}{2}\sigma^2\alpha^2 + (\mu + \rho x)\alpha + \int_{\mathbb{R}} \{e^{\gamma z\alpha} - 1 - \gamma z\alpha\}\nu(dz),$$

and the function

$$\phi(s,x) = \begin{cases} A_1 e^{-qs}(e^{\alpha_1 x} - e^{\alpha_2 x}) & \text{if} \quad 0 < x < x^*, \\ e^{-qs}(x-b) & \text{if} \quad x \geq x^*, \end{cases} \tag{3.17}$$

where $x^* = \frac{2(\ln|\alpha_2| - \ln\alpha_1)}{\alpha_1 - \alpha_2}$ and $\phi(s,x)$ satisfies all the requirements of Theorem 3.2.

Proof. If we let $\phi(s,x) = e^{-qs}\psi(x)$, then

$$\mathbb{L}\phi(s,x) = e^{-qs}\mathbb{L}_0\psi(x),$$

where

$$\begin{aligned} \mathbb{L}_0\psi(x) &= -q\psi(x) + \frac{1}{2}\sigma^2\psi''(x) + [\mu + \rho x]\psi'(x) \\ &\quad + \int_{\mathbb{R}} \{\psi(x+\gamma z) - \psi(x) - \psi'(x)\gamma z\}\nu(dz). \end{aligned}$$

If we try $\psi(x) = e^{\alpha x}$ for some constant $\alpha > 0$, then

$$\begin{aligned}\mathbb{L}_0\psi(x) &= -qe^{\alpha x} + \frac{1}{2}\sigma^2\alpha^2 e^{\alpha x} + (\mu + \rho x)\alpha e^{\alpha x} \\ &\quad + \int_{\mathbb{R}} \{e^{\alpha(x+\gamma z)} - e^{\alpha x} - \alpha e^{\alpha x}\gamma z\}\nu(dz) \\ &= e^{\alpha x}h(\alpha),\end{aligned}$$

where

$$h(\alpha) = -q + \frac{1}{2}\sigma^2\alpha^2 + (\mu + \rho x)\alpha + \int_{\mathbb{R}} \{e^{\gamma z\alpha} - 1 - \gamma z\alpha\}\nu(dz).$$

It is very easy to see that $h(0) = -q < 0$. On the other hand, since we have

$$e^{\alpha(x+\gamma z)} - e^{\alpha x} - \alpha e^{\alpha x}\gamma z \geq 0 \qquad \forall x \in \mathbb{R},$$

provided that $\alpha\gamma z \leq -1$, it follows that $\lim_{\alpha\to\pm\infty} h(\alpha) = \infty$, which implies that there exist at least two solutions α_1 and α_2 such that $\alpha_2 < 0 < \alpha_1$ and $h(\alpha_1) = h(\alpha_2) = 0$. With this value of α_1 and α_2, we try

$$\psi(x) = A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x} \tag{3.18}$$

for some constant A_1 and A_2 to be determined. Since $\psi(0) = 0$, we have $A_1 + A_2 = 0$. We can thus rewrite (3.18) as

$$\psi(x) = A_1(e^{\alpha_1 x} - e^{\alpha_2 x}), \quad 0 < x < x^*, \tag{3.19}$$

where we have guessed the continuation region D to have the form

$$D = \{x; 0 < x < x^*\},$$

so that

$$\psi(x) = \begin{cases} A_1(e^{\alpha_1 x} - e^{\alpha_2 x}) & \text{if} \quad 0 < x < x^*, \\ x - b & \text{if} \quad x \geq x^*, \end{cases} \tag{3.20}$$

where $A_1 > 0$ and $x^* > 0$ are constant. Since ψ is continuous at $x = x^*$, $\psi \in \mathcal{C}^1$ and $\psi \in \mathcal{C}^2$ we have that

$$A_1(e^{\alpha_1 x^*} - e^{\alpha_2 x^*}) = x^* - b, \tag{3.21}$$

$$A_1(\alpha_1 e^{\alpha_1 x^*} - \alpha_2 e^{\alpha_2 x^*}) = 1 \tag{3.22}$$

and

$$A_1(\alpha_1^2 e^{\alpha_1 x^*} - \alpha_2^2 e^{\alpha_2 x^*}) = 0. \tag{3.23}$$

From (3.23) it is easy to see that

$$x^* = \frac{2(\ln|\alpha_2| - \ln \alpha_1)}{\alpha_1 - \alpha_2}, \tag{3.24}$$

and from (3.22) we can see that

$$A_1 = \frac{1}{\alpha_1 e^{\alpha_1 x^*} - \alpha_2 e^{\alpha_2 x^*}}. \tag{3.25}$$

We therefore have that the function

$$\phi(s,x) = \begin{cases} A_1 e^{-qs}(e^{\alpha_1 x} - e^{\alpha_2 x}) & \text{if} \quad 0 < x < x^*, \\ e^{-qs}(x-b) & \text{if} \quad x \geq x^*, \end{cases} \tag{3.26}$$

where the constant x^* and A_1 are given by (3.24) and (3.25) and satisfies all the requirements of Theorem 3.2. Hence

$$\phi(x) = \Phi(x)$$

and

$$\tau^* = \inf\left\{t > 0, X(t) \geq \frac{2(\ln|\alpha_2| - \ln \alpha_1)}{\alpha_1 - \alpha_2}\right\}$$

is an optimal stopping time.

(ii) By construction, we have that $\phi = g$ whenever $x > x^*$. If on the other hand $x < x^*$ we need to check if

$$A_1(e^{\alpha_1 x} - e^{\alpha_2 x}) \geq x - b.$$

If we put the function

$$G(x) = A_1(e^{\alpha_1 x} - e^{\alpha_2 x}) - x + b,$$

then $G(x^*) = G'(x^*) = 0$ and

$$G''(x) = A_1(\alpha_1^2 e^{\alpha_1 x} - \alpha_2^2 e^{\alpha_2 x}) > 0 \quad \text{for } x \leq x^*.$$

This implies that $G'(x) < 0$ for $x < x^*$ and we therefore have that $G(x) > 0$ for $x < x^*$. Hence condition (ii) holds.

(vi) From the construction of $\phi(x)$

$$\mathbb{L}\phi(x) + f = \mathbb{L}\phi(x) = 0 \quad \text{for } x < x^*.$$

For $x > x^*$ we have that

$$\begin{aligned}
& \mathbb{L}\phi(x) \\
= \; & e^{-qs}\Big[-q(x-b) + (\mu + \rho x) + \\
& \int_{x+\gamma z < x^*} \Big\{A_1(e^{\alpha_1(x+\gamma z)} - e^{\alpha_2(x+\gamma z)}) - (x-b) - \gamma z\Big\}\nu(dz)\Big] \\
= \; & e^{-qs}\Big[x(\rho - q) + (\mu + qb) + \\
& \int_{x+\gamma z < x^*} \Big\{A_1(e^{\alpha_1(x+\gamma z)} - e^{\alpha_2(x+\gamma z)}) - (x + \gamma z - b)\Big\}\nu(dz)\Big] \\
\leq \; & e^{-qs}\Big[x^*(\rho - q) + (\mu + qb) + \\
& \int_{\mathbb{R}} \Big\{A_1(e^{\alpha_1(x^*+\gamma z)} - e^{\alpha_2(x^*+\gamma z)}) - (x^* + \gamma z - b)\Big\}\nu(dz)\Big] \\
\leq \; & e^{-qs}\Big[x^*(\rho - q) + (\mu + qb) + \qquad (3.27) \\
& \int_{\mathbb{R}} \Big\{A_1(e^{\alpha_1(x^*)} - e^{\alpha_2 x^*}) - (x^* + \gamma z - b)\Big\}\nu(dz)\Big] \\
\leq \; & e^{-qs}\Big[x^*(\rho - q) + (\mu + qb) - \int_{\mathbb{R}} \gamma z \nu(dz)\Big], \qquad (3.28)
\end{aligned}$$

where we have used (3.3), $\alpha_2 < 0$ and that $|\alpha_2| > \alpha_1$ to get (3.28). The last line (3.28) is obtained by using (3.21). We therefore see that condition (vi) holds if

$$\int_{\mathbb{R}} z\nu(dz) \geq A_4$$

for some constant $\quad A_4 = x^*(\rho - q) + (\mu + qb)$

$(viii)$ For this condition to hold we consider

$$\begin{aligned}
de^{-\rho t}X(t) &= -\rho e^{-\rho t}X(t)dt + e^{-\rho t}dX(t) \\
&= \mu e^{-\rho t}dt + \sigma e^{-\rho t}dB(t) + \int_{\mathbb{R}} e^{-\rho t}\gamma z\widetilde{N}(dt, dz).
\end{aligned}$$

Integrating the above equation will give us

$$\begin{aligned} X(t) &= xe^{\rho t} + \mu \int_0^t e^{\rho(t-s)} ds + \sigma \int_0^t e^{\rho(t-s)} dB(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} e^{\rho(t-s)} \gamma z \widetilde{N}(dt, dz). \end{aligned}$$

Which after discounting gives us

$$\begin{aligned} e^{-qt}X(t) &= xe^{t(\rho-q)} + \mu \int_0^t e^{t(\rho-q)-\rho s} ds + \sigma \int_0^t e^{t(\rho-q)-\rho s} dB(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} e^{t(\rho-q)-\rho s} \gamma z \widetilde{N}(dt, dz). \end{aligned}$$

Condition $(viii)$ will hold if and only if

$$E\Big[e^{-2\rho t} X^2(t)\Big] < \infty,$$

which will hold if

$$E\left[\int_0^t e^{2t(\rho-q)} \left\{\sigma^2 e^{-2\rho s} + \int_{\mathbb{R}} e^{-2\rho s} \gamma^2 z^2 \nu(dz)\right\} ds\right] < \infty.$$

(x) For this condition to hold, we need to check if $\tau_D < \infty$ $a.s$, which for the proposed solution $\phi(x)$ is given by

$$\tau_D = \inf\{t > 0; X(t) > x^*\} < \infty \quad a.s.,$$

where

$$D = \Big\{(s, x) \in \mathbb{R}^2; x < x^*\Big\}.$$

We know that

$$\begin{aligned} X(t) &= xe^{\rho t} + \mu \int_0^t e^{\rho(t-s)} ds + \sigma \int_0^t e^{\rho(t-s)} dB(s) + \int_0^t \int_{\mathbb{R}} e^{\rho(t-s)} \gamma z \widetilde{N}(dt, dz). \\ &= e^{\rho t} \left\{ x + \mu \int_0^t e^{-\rho s} ds + \sigma \int_0^t e^{-\rho s} dB(s) + \int_0^t \int_{\mathbb{R}} e^{-\rho s} \gamma z \widetilde{N}(dt, dz) \right\}. \\ &= e^{\rho t} F(t). \end{aligned}$$

It thus suffices to check if

$$\lim_{t \to \infty} X(t) = \infty.$$

We can see that if

$$\lim_{t \to \infty} \int_0^t \int_{\mathbb{R}} e^{-\rho s} \gamma z \widetilde{N}(dt, dz) \geq 0 \; a.s.,$$

then

$$\lim_{t \to \infty} X(t) = \lim_{t \to \infty} e^{\rho t} F(t) = \infty \quad a.s.$$

and in particular $\tau_D < \infty \quad a.s.$

(xi) For this condition to hold it suffices to check that

$$\sup_{\tau \in \mathcal{T}} E^x \left[e^{-2q\tau} X^2(\tau) \right] < \infty.$$

This will hold if and only if condition (viii) holds.

The other conditions hold trivially and by our chosen function $\phi(x)$, so Theorem 3.2 holds for the proposed model. We therefore have that

$$\Phi(s, x) = \sup_{\tau \in \mathcal{T}} J^{\tau}(s, x) = J^{\tau^*}(s, x). \tag{3.29}$$

as required. □

3.5 Conclusion

A closed form solution of the value function for the optimal stopping problem was constructed and its optimal stopping time determined. We proved that the constructed value function and the optimal stopping policy satisfies the condition of the Integro-variational Inequalities for Optimal Stopping. We can, therefore, conclude that the optimal strategy for an insurance firm is to stop the process at the optimal stopping time τ^*. In the following chapters, we consider the type of controls that this insurance firm could consider at the stopping time τ^*. We also consider a different form of indeterminacy that can be faced by such a firm.

Chapter 4

Optimal Combined Dividend and Reinsurance Policies Under Interest Rate in Lévy Markets.

4.1 Introduction

A combined dividend and risk control problem is presented and investigated in this chapter. The risk of the insurance firm is controlled by using a proportional reinsurance policy. It is considered that the evolution of the cash reserves of the firm is driven by a generalized Itô-Lévy process. The surplus cash reserves earns interest at a constant rate. The objective of the firm is to maximize the total expected discounted dividends paid out to shareholders. The situation is modeled as an impulse-classical control problem. We manage to construct the value function and the optimal impulse control. The existence and uniqueness of an optimal classical control is proved.

Motivated by the need to extend the optimal control theory and its application in insurance, we present an extension of the model proposed by Zhang and Song [69]. We propose a model driven by jump-diffusion process. It is well known that claims process is an important component of the reserve process for an insurance company. The reserve process for an insurance company experiences a drop in value whenever a large claim occurs. It is, therefore, appropriate to add the claims process into the diffusion reserve process proposed by Zhang and Song [69].

Itô-Levý processes provide a more realistic paradigm for describing the evolution of cash reserves of insurance companies. For more details and examples on the application of jumps diffusion processes in finance and insurance, the reader is referred to Applebaum [1], Framstad et al. [23], Øksendal and Sulem [51] and Zou et al. [71].

An Itô-Lévy model consists of a drift term, a Brownian motion component, and a jump term. The drift term represents the average growth rate of a financial asset whereas the Brownian motion part models the riskiness of the assets. The jump component is a mathematical representation of sudden changes in the value of the liquid assets of the firm. Such sudden changes in the value of assets are due to, for example, a shift in policy by central monetary authority, arrival of unexpected news on financial market, breakout of war or discovery of new natural resource. In most cases, it is not easy to determine an explicit solution of stochastic control problem where the underlying process is an Itô-Lévy process.

4.2 The Model Formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered complete probability space. Consider the extension of the model by Zhang and Song [69], by including the jump into their proposed reserve process $R = \{R(t)\}_{t\geq 0}$. The jump term represents the impact of claims and important news for an insurance company. An example of such important news is the market crash of 1987 and the latest market crash of 2008. The amounts of funds claimed from an insurance company are a significant component of the reserve process as they occur discretely and unpredictably over time. For our considered model, the amount of dividends received by shareholders over the time interval $[0, t)$ is given by $L(t) = \sum_{k=1}^{\infty} I_{\{\tau_k < t\}} \xi_k$, where ξ_k represents the amount of dividends paid at some stopping time $\tau_k \in \mathcal{T}$

and $I_{\{.\}}$ is the indicator function. Dividend distribution for an insurance company are assumed to be controlled by a sequence of increasing stopping times $\{\tau_i; i = 1, 2, \ldots\}$ and a sequence of nonnegative random variables $\{\xi_i; i = 1, 2, \ldots\}$ which are associated with times and the amount of dividends paid out to shareholders. We assume in the absence of intervention that the reserve process $R(t)$ evolves, according to

$$dR(t) = \mu dt + \sigma dB(t) + \int_{\mathbb{R}} \gamma z \tilde{N}(dt, dz),$$

where $\mu > 0$, $\sigma > 0$, $\gamma > 0$ are constants we assume that $z \leq 0$ $a.s$ ν. Note that $\{B(t)\}_{t\geq 0}$ is a standard Brownian motion with respect to $\{\mathcal{F}(t)\}_{t\geq 0}$ and $\tilde{N}(dt, dz)$ is a compensated Poisson random measure. The compensated Poisson random measure is given by

$$\tilde{N}(dt, dz) = N(dt, dz) - dt\nu(dz),$$

where $\nu(.)$ is a Lévy measure associated with the Poisson random measure $N(.,.)$. An application of dynamic proportional reinsurance to the reserve process which is governed by the parameter $a(t) \in [0, t]$ gives

$$dR(t) = a(t)\Big(\mu dt + \sigma dB(t) + \int_{\mathbb{R}} \gamma z \tilde{N}(dt, dz)\Big).$$

After applying proportional reinsurance, dividend control and considering constant interest rate $\rho > 0$ for reserves, the dynamics of the controlled surplus process, $R(t)$ at time t is described by the equation

$$R_t = x + \int_0^t (\mu a + \rho R_s) ds + \int_0^t \sigma a dB(s) + \int_0^t \int_{\mathbb{R}} \gamma a z \tilde{N}(ds, dz) - \sum_{k=1}^{\infty} I_{\{\tau_k < t\}} \xi_k, \quad (4.1)$$

where $R(0) = x > 0$ is the initial reserve.

Definition 4.1. A classical control $a = \{a(t)\}_{t\geq 0}$ is an $\mathcal{F}_t$-adapted process such that $a(t) \in [0, 1]$.

Definition 4.2. An impulse control for a stochastic process is a double sequence

$$\vartheta = (\mathcal{T}, \xi) = (\tau_1, \tau_2, \dots, \tau_n, \dots; \xi_1, \xi_2, \dots, \xi_n, \dots)$$

where $0 \leq \tau_1 < \tau_2 < \tau_3 < \dots$ is an $\mathcal{F}_t$-adapted sequence of increasing stopping times and $\xi_1, \xi_2, \dots$ are $\mathcal{F}_t$ measurable random variables with $\xi_i \in [0, R_{\tau_i^-}]$; for $i = 1, 2, 3, \dots$.

Judgments on how much needs to be paid to shareholders as dividends are mathematically described by impulse control. The insurance company pays ξ_i as the i-th dividend at some stopping time τ_i, which implies $R(\tau_i) = R(\tau_i^-) - \xi_i$. On the other hand decisions on how much needs to be reinsured are made by classical control $a(t)$.

Definition 4.3. The combined classical control a and impulse control ϑ given by the triple

$$\pi := (a, \vartheta) = (a, \mathcal{T}, \xi)$$

is called an admissible control. The class of all admissible controls is denoted by $\mathcal{A}(x)$.

We define the stopping (bankruptcy) time by

$$\tau \equiv \tau^\pi := \inf\{t \geq 0 : R(t) = 0\}.$$

We assume that $R(t)$ vanishes for $t \geq \tau$ as we are only dealing with the optimization problem during the time interval $[0, \tau)$. At time $t \in [0, \infty)$, the controlled reserve process R_t^π is given by

$$R_t^\pi = \begin{cases} x + \int_0^t (\mu a + \rho R_s^\pi) ds + \int_0^t \sigma a dB_s + \int_0^t \int_{\mathbb{R}} \gamma a z \tilde{N}(ds, dz) - \sum_{k=1}^\infty I_{\{\tau_k < t\}} \xi_k, & \text{if } t < \tau \\ 0, & \text{if } t \geq \tau. \end{cases}$$

Define the performance functional $J(x, \pi)$ by

$$J(x, \pi) := E\left[\sum_{k=1}^\infty e^{-c\tau_k} g(\xi_k) I_{\{\tau_k < \tau\}} \Big/ R_0^\pi = x \right], \tag{4.2}$$

where $c > \rho$ and the function $g : [0,\infty) \mapsto (-\infty,\infty)$ is given by

$$g(\eta) = k^*\eta - K^*, \tag{4.3}$$

where $k^* \in (0,1)$ and $K^* \in (0,\infty)$ are constants, with $1-k^*$ being interpreted as tax rate and K^* as a fixed cost when dividends are paid.

Problem 4.2.1. *The problem is to determine the value function* $\Phi(x)$ *and the optimal control* $\pi^* = (a^*, \mathcal{T}^*, \xi^*) \in \mathcal{A}$ *such that*

$$\Phi(x) = \sup_{\pi\in\mathcal{A}} J(x,\pi) = J(x,\pi^*). \tag{4.4}$$

4.3 The Value function

For every $x \geq 0$ denote the value function by $V(x)$ where

$$V(x) = \sup\{J(x,\pi); \pi \in \mathcal{A}(x)\} = \sup_{\pi\in\mathcal{A}(x)} E\left[\sum_{k=1}^{\infty} e^{-c\tau_k} g(\xi_k) I_{\{\tau_k<\tau\}}\right] \tag{4.5}$$

and define the maximum utility operator $\mathbb{M}$ by

$$\mathbb{M}\phi(x) = \sup\{\phi(x-\eta) + g(\eta) : \eta > 0, x > \eta\}, \tag{4.6}$$

where ϕ is a twice continuously differential function from $[0,\infty)$ to $(-\infty,\infty)$ and g is given by (4.3). Define the differential operator $\mathbb{L}^a$ by

$$\begin{aligned}\mathbb{L}^a\psi(x) \quad &= \quad \frac{a^2\sigma^2}{2}\psi''(x) + [\mu a + \rho x]\psi'(x) - c\psi(x) \\ &\quad + \int_{\mathbb{R}} \{\psi(x+a\gamma z) - \psi(x) - \psi'(x)a\gamma z\}\nu(dz).\end{aligned}$$

We consider the method used in Cadenillas et al. [10] and in Zhang and Song [69] in order to solve the problem in association with the quasi-variational inequalities. An explicit value function and a corresponding optimal control for Problem 4.2.1 is obtained when jumps are considered.

Definition 4.4. (QVI). A function $W : [0, \infty) \mapsto [0, \infty)$ is said to satisfy the quasi-variational inequalities (QVI) of Problem 4.2.1 if for every $x \in [0, \infty)$ and $a \in [0, 1]$

$$\mathbb{L}^a W(x) \leq 0, \tag{4.7}$$

$$\mathbb{M}W(x) \leq W(x), \tag{4.8}$$

$$(W(x) - \mathbb{M}W(x))\Big(\max_{a\in[0,1]} \mathbb{L}^a W(x)\Big) = 0, \tag{4.9}$$

$$W(0) = 0. \tag{4.10}$$

A solution W of the QVI splits the interval $[0, \infty)$ into two regions: a continuation region

$$\mathcal{C} := \left\{ x \in (0, \infty) : \mathbb{M}W(x) < W(x) \text{ and } \max_{a\in[0,1]} \mathbb{L}^a W(x) = 0 \right\}$$

and an intervention region

$$\Sigma := \left\{ x \in (0, \infty) : \mathbb{M}W(x) = W(x) \text{ and } \max_{a\in[0,1]} \mathbb{L}^a W(x) < 0 \right\}.$$

Given a solution W to the QVI, we define the following policy associated with this solution.

Definition 4.5. The control $\pi^W = (a^W, \mathcal{T}^W, \xi^W) = (a^W; \tau_1^W, \tau_2^W, \dots, \tau_n^W, \dots; \xi_1^W \dots)$ is called the QVI control associated with W if the associated state process R^W given by (4.1) satisfies

$$P\Big\{a^W(t) \neq \arg\max_{a\in[0,1]} \mathbb{L}^a W(R_t^W), R_t^W \in \mathcal{C}\Big\} = 0, \tag{4.11}$$

$$\tau_1^W := \inf\{t \geq 0 : W(R^W(t)) = \mathbb{M}W(R^W(t))\}, \tag{4.12}$$

$$\xi_1^W := \arg \sup_{\eta>0, \eta\leq R^W(\tau_1^W)} \Big\{ W(R^W(\tau_1^W) - \eta) + g(\eta) \Big\}, \tag{4.13}$$

and for $n \geq 2$

$$\tau_n^W := \inf\{t \geq \tau_{n-1} : W(R^W(t)) = \mathbb{M}W(R^W(t))\}, \tag{4.14}$$

$$\xi_n^W := \arg \sup_{\eta > 0, \eta \leq R^W(\tau_n^W)} \Big\{ W(R^W(\tau_n^W) - \eta) + g(\eta) \Big\}, \tag{4.15}$$

with $\tau_0^W := 0$ and $\xi_0^W = 0$.

Under this control the intervention takes place whenever W and $\mathbb{M}W$ coincide, and the amount of the liquid assets withdrawn at these times is determined from the solution to the one-dimensional optimization problem associated with the operator $\mathbb{M}W$.

Theorem 4.1. *Let $W \in \mathcal{C}^1((0, \infty))$ be a solution of the QVI (4.7)-(4.10). Suppose there exists $U > 0$ such that W is twice continuously differentiable on $(0, U)$ and W is linear on $[U, \infty)$. Then for every $x \in (0, \infty)$*

$$V(x) \leq W(x). \tag{4.16}$$

Further, if the QVI control $\pi^W = (a^W, \mathcal{T}^W, \xi^W)$ associated with W is admissible, then W coincides with the value function and the QVI control associated with W is the optimal policy, hence

$$V(x) = W(x) = J(x; a^W, \mathcal{T}^W, \xi^W). \tag{4.17}$$

Proof. The proof of the above theorem is similar to the proof of Theorem 3.4 of Cadenillas et al. [10]. Consider an arbitrarily chosen impulse control $\vartheta = (\tau_1, \tau_2, \ldots, \tau_j, \ldots; \xi_1, \xi_2, \ldots, \xi_j, \ldots)$ and let $\tau_0 = 0$. Choose $a \in [0, 1]$ and put $R(t) = R^{(\pi)}(t)$. Noting that the function W is bounded on $[0, U]$ due to its continuity while W' is bounded on $(0, \infty)$ due to its differentiability and continuity. The linearity of W and the Lebesgue dominated convergence theorem implies that

$$\lim_{t \to \infty} E\big[e^{-c(t)} W(R(t))\big] = 0. \tag{4.18}$$

The boundedness of W' implies that

$$E\Big[\int_0^\infty e^{-2ct}\big(\sigma^2 W'(R_t)^2+\int_{\mathbb{R}}\big(a\gamma z W'(R_t)\big)^2\nu(dz)\big)dt\Big]<\infty. \tag{4.19}$$

For $n=1,2,3,\dots$, we can write

$$\begin{aligned}
& e^{-c(\tau_j\wedge n)}W(R_{\tau_j\wedge n})-W(x)\\
= & \sum_{j=1}^m I_{\{\tau_j\le t\}}\Big[e^{-c(\tau_j\wedge n)}W(R_{\tau_j\wedge n})-e^{-c(\tau_{j-1}\wedge n)}W(R_{\tau_{j-1}\wedge n})\Big]\\
= & \sum_{j=1}^m I_{\{\tau_j\le t\}}\Big[e^{-c(\tau_j\wedge n)}W(R_{\tau_j^-\wedge n})-e^{-c(\tau_{j-1}\wedge n)}W(R_{\tau_{j-1}\wedge n})\Big]\\
& +\sum_{j=1}^m I_{\{\tau_j\le t\wedge n\}}e^{-c\tau_j}\big[W(R_{\tau_j})-W(R_{\tau_j^-})\big].
\end{aligned}$$

An application of Itô's formula for jump processes (see, $IV.45$ in Rogers and Williams 1987) yields

$$\begin{aligned}
& e^{-c(\tau_j\wedge n)}W(R_{\tau_j^-\wedge n})-e^{-c(\tau_{j-1}\wedge n)}W(R_{\tau_{j-1}\wedge n})\\
= & \int_{\tau_{j-1}\wedge n}^{\tau_j\wedge n}e^{-cs}\mathbb{L}W(R_s)ds+\int_{\tau_{j-1}\wedge n}^{\tau_j\wedge n}e^{-cs}\sigma a W'(R_s)dB_S\\
& +\int_{\tau_{j-1}\wedge n}^{\tau_j\wedge n}\int_{\mathbb{R}}e^{-cs}a\gamma z W'(R_s)\tilde{N}(dt,dz).
\end{aligned}$$

In view of inequality (4.7), we have that

$$\begin{aligned}
& e^{-c(\tau_j\wedge n)}W(R_{\tau_j^-\wedge n})-e^{-c(\tau_{j-1}\wedge n)}W(R_{\tau_{j-1}\wedge n})\\
\le & \int_{\tau_{j-1}\wedge n}^{\tau_j\wedge n}e^{-cs}\sigma a W'(R_s)dB_S+\int_{\tau_{j-1}\wedge n}^{\tau_j\wedge n}\int_{\mathbb{R}}e^{-cs}a\gamma z W'(R_s)\tilde{N}(dt,dz).
\end{aligned}$$

Note that if τ_{j-1} and τ_j are intervention times as defined in (4.12) and (4.13), then

$W(R_s)>\mathbb{M}W(R_s)$ for $s\in[\tau_{j-1},\tau_j)$, so $\mathbb{L}W(R_s)=0$ by Definition 4.4. The inequality above thus becomes an equality for the QVI-control associated with W. Also noting that $R(\tau_j) = R(\tau_j^-) + \xi_j$, according to (4.8), we have that

$$e^{-c\tau_j}\Big[W(R(\tau_j)) - W(R(\tau_j^-))\Big] \leq -e^{-c\tau_j}g(\xi_j).$$

We should also note that this inequality becomes an equality for the QVI-control associated with W, since $g(\xi_j) + W(R(\tau_j^-)) = \mathbb{M}(R(\tau_j^-)) = W(R(\tau_j))$ if (τ_j, ξ_j) are the impulse control defined by Definition 4.5. Combining the above two inequalities gives us,

$$\begin{aligned} & W(x) - e^{-c(\tau_j \wedge n)}W(R_{\tau_j \wedge n}) \\ \geq \ & \sum_{j=1}^{m} I_{\{\tau_j \leq t \wedge n\}} e^{-c\tau_j \wedge n} g(\xi_j) - \int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs}\sigma a W'(R_s)dB_S \\ & - \int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs}\int_{\mathbb{R}} a\gamma z W'(R_s)\tilde{N}(dt, dz). \end{aligned}$$

Taking expectation of both sides we get

$$\begin{aligned} & W(x) - E\Big[e^{-c(\tau_j \wedge n)}W(R_{\tau_j \wedge n})\Big] \\ \geq \ & E\Big[\sum_{j=1}^{m} I_{\{\tau_j \leq t \wedge n\}} e^{-c\tau_j \wedge n} g(\xi_j)\Big] - E\Big[\int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs}\sigma a W'(R_s)dB_S\Big] \\ & - E\Big[\int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs}\int_{\mathbb{R}} a\gamma z W'(R_s)\tilde{N}(dt, dz)\Big]. \end{aligned}$$

The above inequality becomes an equality for the QVI-control associated with W. Now

$$\begin{aligned} & \lim_{n\to\infty}\Big\{W(x) - E\Big[e^{-c(\tau_j \wedge n)}W(R_{\tau_j \wedge n})\Big]\Big\} \\ = \ & W(x) - E\Big[e^{-c(\tau_j)}W(R_{\tau_j})\Big]. \end{aligned}$$

From growth condition 4.19 we obtain

$$\lim_{n\to\infty} E\left[\int_{\tau_{j-1}\wedge n}^{\tau_j\wedge n} e^{-cs}\sigma a W'(R_s)dB_S + \int_{\tau_{j-1}\wedge n}^{\tau_j\wedge n} e^{-cs}\int_{\mathbb{R}} a\gamma z W'(R_s)\tilde{N}(dt,dz)\right] = 0.$$

Thus,

$$W(x) - E\Big[e^{-c(\tau_j)}W(R_{\tau_j})\Big] \geq E\Big[\sum_{j=1}^{m} I_{\{\tau_j\leq t\}}e^{-c\tau_j}g(\xi_j)\Big].$$

According to the growth condition (4.18) and letting $m \to \infty$ we get

$$W(x) \geq E\Big[\sum_{j=1}^{\infty} I_{\{\tau_j\leq \tau\}}e^{-c\tau_j}g(\xi_j)\Big] = J(x,\pi).$$

Since this is true for any control π, we have that

$$W(x) \geq V(x).$$

Noting that the above inequality becomes an equality for the QVI-control associated with W.

□

4.4 Smooth Solution to The QVI and The Optimal Policy

In the continuation region $\mathcal{C}$, we have for any $x \in \mathcal{C}$,

$$\max_{a\in[0,1]}\Big[\quad \frac{a^2\sigma^2}{2}W''(x) + [\mu a + \rho x]W'(x) - cW(x) \\ + \int_{\mathbb{R}}\{W(x+a\gamma z) - W(x) - W'(x)a\gamma z\}\nu(dz)\Big] = 0,$$

where

$$\begin{aligned}
\mathbb{L}_0^a W(x) &= \frac{1}{2}a^2\sigma^2 W''(x) + [\mu a + \rho x]W'(x) - cW(x) \\
&\quad + \int_{\mathbb{R}} \{W(x + a\gamma z) - W(x) - W'(x)a\gamma z\}\nu(dz).
\end{aligned}$$

We will follow the optimization technique applied by Schmidli [54] and Framstad et al [23] of initially guessing the solution and then establishing if the proposed solution is in fact correct. Inspired by Jeanblanc-Picque and Shiryaev [32], we try a function of the form, $W(x) = A_1 e^{rx}$ for $x \in D$ (the continuation region) and some constant A_1. This yields

$$\begin{aligned}
&\mathbb{L}_0^a W(x) \\
= \;& -cA_1 e^{rx} + \frac{1}{2}a^2\sigma^2 r^2 A_1 e^{rx} + [\mu a + \rho x]A_1 r e^{rx} + \\
& \int_{\mathbb{R}} \{A_1 e^{r(x+a\gamma z)} - A_1 e^{rx} - A_1 r e^{rx} a\gamma z\}\nu(dz) \\
= \;& A_1 e^{rx}\left[-c + \frac{1}{2}a^2\sigma^2 r^2 + [\mu a + \rho x]r + \int_{\mathbb{R}} \{e^{ra\gamma z} - 1 - ra\gamma z\}\nu(dz)\right].
\end{aligned}$$

If we let

$$h(a, r) = -c + \frac{1}{2}a^2\sigma^2 r^2 + [\mu a + \rho x]r + \int_{\mathbb{R}} \{e^{ra\gamma z} - 1 - ra\gamma z\}\nu(dz),$$

we get that

$$\frac{\partial h(a, r)}{\partial a} = a\sigma^2 r^2 + \mu r + \int_{\mathbb{R}} \{r\gamma z e^{a\gamma z r} - r\gamma z\}\nu(dz) = 0,$$

which gives us

$$\Lambda(a) = a\sigma^2 r + \mu + \int_{\mathbb{R}} \{e^{a\gamma z r} - 1\}\gamma z \nu(dz) = 0.$$

The above equation can be written as

$$\Lambda(a) = \mu + a\sigma^2 r - \int_{\mathbb{R}} \{1 - e^{a\gamma z r}\}\gamma z \nu(dz) = 0. \tag{4.20}$$

For $a = 0$, we have

$$\Lambda(0) = \mu > 0$$

and for $a = 1$

$$\Lambda(1) = \mu + \sigma^2 r - \int_{\mathbb{R}} \{1 - e^{\gamma z r}\}\gamma z \nu(dz).$$

We require $\Lambda(1) \leq 0$ so that

$$\mu \leq \int_{\mathbb{R}} \{1 - e^{\gamma z r}\}\gamma z \nu(dz) - \sigma^2 r.$$

With this choice of μ there exists an optimal control $a = \hat{a} \in (0, 1]$. With this $a = \hat{a}$ (constant) we require that

$$\mathbb{L}_0^{\hat{a}} W(x) = 0,$$

that is

$$\begin{aligned} & \mathbb{L}_0^{\hat{a}} W(x) \\ = \;& -cA_1 e^{rx} + \frac{1}{2}\hat{a}^2\sigma^2 r^2 A_1 e^{rx} + [\mu\hat{a} + \rho x]A_1 r e^{rx} + \\ & \int_{\mathbb{R}} \{A_1 e^{r(x+\hat{a}\gamma z)} - A_1 e^{rx} - A_1 r e^{rx}\hat{a}\gamma z\}\nu(dz) = 0 \end{aligned}$$

or

$$-c + \frac{1}{2}\hat{a}^2\sigma^2 r^2 + [\mu\hat{a} + \rho x]r + \int_{\mathbb{R}} \{e^{\hat{a}\gamma z} - 1 - r\hat{a}\gamma z\}\nu(dz) = 0.$$

So condition (4.7) of Definition 4.4 is satisfied for this choice of $W(x)$. We also note that for

$$h(a, r) = -c + \frac{1}{2}a^2\sigma^2 r^2 + [\mu a + \rho x]r + \int_{\mathbb{R}} \{e^{ra\gamma z} - 1 - ra\gamma z\}\nu(dz), \tag{4.21}$$

$h(a,0) = -c < 0$ and $\lim_{r\to\infty} h(a,r) = \infty$ since $\{e^{ra\gamma z} - 1 - ra\gamma z\} \geq 0$. This implies that there exist two solutions r_1 and r_2 of $h(a,r) = 0$ such that $r_2 < 0 < r_1$. With this value of r_1 and r_2 it is easy to see that condition (4.7) of Definition 4.4 is again satisfied and

$$W(x) = A_2 e^{r_1 x} + A_3 e^{r_2 x} \qquad \text{for constant } A_2 \text{ and } A_3.$$

But $W(x)$ is a value function and as such $W(0) = 0$. We therefore have that

$$A_2 = A = -A_3 > 0.$$

Thus

$$W(x) = A(e^{r_1 x} - e^{r_2 x}) \qquad \text{for } x \in D.$$

Outside D we consider

$$\mathbb{M}W_0(x) = \sup\{W_0(x(1+\rho) - \xi) + \xi; \quad 0 < \xi < x(1+\rho)\}.$$

If we let

$$h_1(\xi) = W_0(x(1+\rho) - \xi) + \xi,$$

then

$$-W_0'(x(1+\rho) - \xi) + 1 = 0.$$

Now suppose that there exists a unique point $\bar{x} \in (0, x^*)$ subject to

$$W_0'(x(1+\rho) - \xi) = 1.$$

Then

$$\bar{x} = x(1+\rho) - \xi^*.$$

Thus

$$W(x) = W_0(\bar{x}) + x(1+\rho) - \bar{x}.$$

In particular

$$W'(x^*) = 1,$$

and $$W(x^*) = W_0(\bar{x}) + x^*(1+\rho) - \bar{x}.$$

We therefore have

$$W(x) = \begin{cases} A(e^{r_1 x} - e^{r_2 x}) & \text{if} \quad 0 < x < x^*, \\ W_0(\bar{x}) + x(1+\rho) - \bar{x} & \text{if} \quad x \geq x^*. \end{cases} \tag{4.22}$$

Theorem 4.2. *Define the function $\phi(s,x) : [0,\infty) \times (0,\infty) \to [0,\infty)$ by*

$$\phi(s,x) = \begin{cases} Ae^{-cs}(e^{r_1 x} - e^{r_2 x}) & \textit{if} \quad 0 < x < x^*, \\ e^{-cs}\Big(W_0(\bar{x}) + x(1+\rho) - \bar{x}\Big) & \textit{if} \quad x \geq x^*, \end{cases} \tag{4.23}$$

where r_1 and r_2 solves (4.21) and

$$A = \frac{W_0(\bar{x}) + x^*(1+\rho) - \bar{x}}{e^{r_1 x^*} - e^{r_2 x^*}}.$$

If

$$\mu \leq \int_{\mathbb{R}} \{1 - e^{\gamma z r}\} \gamma z \nu(dz) - \sigma^2 r,$$

then $\phi(s,x)$ is the value function of Problem 4.2.1, that is

$$\phi(s,x) = V(x) = \sup\big\{J(x,\pi)\big\}. \tag{4.24}$$

Proof. We observe that $W(x)$ is a solution of the QVI of Definition 4.4 if and only if $\phi(s,x)$ is a solution and according to Theorem 4.1 is a value function with the optimal strategy given by (4.12)-(4.15). With this value function $\phi(s,x)$ we thus only need to check if $W(x)$ satisfies all the conditions of Definition 4.4 and Theorem 4.1. We note that condition (4.8) of Definition 4.4 is satisfied by the construction of $W(x)$ when $x \geq x^*$. For $0 < x < x^*$ we have

$$\begin{aligned}
\mathbb{M}W(x) &= \sup_{\xi}\Big\{W(x(1+\rho)-\xi)+\xi; \quad 0 \leq \xi \leq x(1+\rho)\Big\} \\
&= \sup_{\xi}\Big\{A\Big(e^{r_1(x(1+\rho)-\xi)} - e^{r_2(x(1+\rho)-\xi)}\Big)+\xi; 0 \leq \xi \leq x(1+\rho)\Big\} \\
&\leq A\Big(e^{r_1 x}e^{x r_1 \rho} - e^{r_2 x}\Big) \\
&\leq A\Big(e^{r_1 x}e^{-r_2 x \rho} - e^{r_2 x}\Big) \qquad \text{since } |r_2| > r_1 \\
&\leq A\Big(e^{r_1 x} - e^{r_2 x}\Big) \\
&= W(x).
\end{aligned}$$

Therefore condition (4.8) is again satisfied. Continuity at x^* gives

$$\begin{aligned}
A(e^{r_1 x^*} - e^{r_2 x^*}) &= W_0(\bar{x}) + x^*(1+\rho) - \bar{x}, \\
A &= \frac{W_0(\bar{x}) + x^*(1+\rho) - \bar{x}}{e^{r_1 x^*} - e^{r_2 x^*}}.
\end{aligned}$$

Condition (4.19) is satisfied if and only if

$$E\Big[e^{-2\rho t}X^2(t)\Big] < \infty,$$

which will hold if

$$E\bigg[\int_0^t e^{2t(\rho-c)}\bigg\{\sigma^2 a^2(s)e^{-2\rho s} + \int_{\mathbb{R}} e^{-2\rho s}a^2(s)z^2\nu(dz)\bigg\}ds\bigg] < \infty.$$

If condition (4.19) holds then all the condition of Theorem 4.1 will also hold as this condition will require

$$\sup_{\tau\in\mathbb{T}} E^x\Big[e^{-2c\tau}X^2(\tau)\Big] < \infty.$$

$\square$

We note that with these values of x^*, $\bar{x}$ and A, our value function $\phi(s,x)$ satisfies all the requirement of the QVI and the verification theorem. We can thus describe the solution to the optimal control Problem 4.2.1 as follows.
As long as the reserve process $R(t) < x^*$ we do nothing. If the reserve process $R(t)$ reaches the value x^*, we immediately make an intervention to bring the reserves $R(t)$ down to the level $\bar{x}$ by distributing dividends to shareholders.

4.5 Conclusion

We managed to analytically derive the optimal policy for an insurance firm when claim processes, which are modeled by jump diffusion, are considered in the model proposed by Zhang and Song [69]. The existence and uniqueness of an optimal classical impulse control was proved. In the paper by Cadenillas et al. [10], convexity is required for the existence of result, while in the paper by Zhang and Song [69] monotonicity is essential. As seen in our solutions and analysis of the value function, the existence of the optimal policy of Problem 4.2.1 is independent of the monotonicity or concavity of the value function. Even though our optimal control policy differs from the ones obtained by Cadenillas et al. [10], Zhang and Song [69], they are quiet similar, in the continuation region, to the result obtained by Chikodza [16], Framstad et al. [23], Øksendal and Sulem [49] and Zou et al. [71]. The major difference being in the intervention region.

Chapter 5

Impulse Control and Optimal Stopping Under Stochastic Volatility Model

5.1 Introduction

In this chapter, we present and investigate the dividend optimization problem when the volatility coefficient is allowed to be stochastic. Due to the presence of transactions costs in the proposed model, the mathematical problem becomes a combined impulse and stochastic control problem. Under the risk neutral assumptions for the proposed model, we explicitly solve the problem and construct its value function with the optimal policy.

Motivated by Heston [26], where the option price of a European call on an asset that has a stochastic volatility is examined, Hull and White [31], and other state of the art research papers on stochastic volatility models, we extend the results of Zhang and Song [69] by including stochastic volatility into their reserve model. It is common practice to assume that the stochastic volatility coefficient follows a mean-reverting volatility process, where volatility strive to reach a certain level in the long run. For this reason, we assume as in the Stein and Stein [59] that the volatility coefficient follows an arithmetic Ornstein-Uhlenbeck process.

5.2 Preliminaries and The Mathematical Model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered complete probability space satisfying the usual conditions endowed with a standard Brownian motion $B = \{B(t); t \geq 0\}$ adapted to the filtration. The sub-σ-algebra $\mathcal{F}_t$ represents the information available at time t and any control that is made is based on this information. Our state variable is the reserve process $R = \{R(t); t \geq 0\}$, representing the liquid assets of an insurance company. Insurance companies often adopt proportional reinsurance to reduce risk. Let $a(t) \in [0, 1]$ be a classical control which represents the reinsurance rate at time t. We describe the distribution of dividends as a sequence of increasing stopping times $\{\tau_i; i = 1, 2, \ldots\}$ and a sequence of random variables $\{\xi_i; i = 1, 2, \ldots\}$, which are associated with the times and amount of dividends paid out to shareholders. We also assume that the reserves earn interest at a constant force $\rho > 0$ and that the diffusion coefficient $\sigma(t)$ is stochastically distributed. The dynamics of our controlled process is thus given by

$$dR(t) = [a(t)\mu + \rho R(t)]dt + \sigma(t)a(t)dB_1(t) - dL(t), \tag{5.1}$$

with

$$\frac{d\sigma(t)}{\sigma(t)} = \delta(\sigma(t) - \theta)dt + \kappa dB_2(t), \tag{5.2}$$

where $R(t)$ describes the reserves of an insurance company,
$L(t) = \sum_{k=1}^{\infty} I_{\{\tau_k < t\}}\xi_k$ is the impulse control and $a(t)$ is the stochastic control for the reserve process. We have assumed that the volatility $\sigma(t)$ is governed by an arithmetic Ornstein-Uhlenbeck process, with fixed constants, δ being the speed of σ's reversion to the long-run mean θ and κ being the volatility of σ.

5.3 Change of Measure for the Stochastic Volatility Model

Consider a combined stochastic control and impulse control process for insurance reserves given by (5.1) and (5.2). We are going to apply Girsanov's transformation to change the probability measure of our model from the real world measure P to an equivalent martingale measure Q in order to simplify our calculation. In a stochastic volatility market, it is well known that the market is incomplete. The Girsanov's theorem gives an explicit representation of market price of risk which induces the equivalent martingale measure used for pricing. In incomplete markets there are infinitely many such equivalent martingale measures, leaving researchers looking for what could be a good candidate measure for pricing [45]. An application of Girsanov's theorem to (5.1) and (5.2) states that the processes

$$d\tilde{B}_1(t) = \frac{\mu a(t) + \rho R(t)}{\sigma a(t)} dt + dB_1(t) \tag{5.3}$$

and

$$d\tilde{B}_2(t) = \frac{\delta}{\kappa}(\sigma - \theta)dt + dB_2(t) \tag{5.4}$$

are one-dimensional Brownian motions with respect to the equivalent martingale measure Q, with the Radon-Nikodym derivative given by

$$\begin{aligned} \frac{dQ(\omega)}{dP(\omega)} &= \exp\Bigg\{ - \int_0^t \frac{\mu a(t) + \rho R(t)}{\sigma a(t)} dB_1(t) - \int_0^t \frac{\delta}{\kappa}(\sigma - \theta) dB_2(t) \\ &\quad - \frac{1}{2}\int_0^t \Big(\frac{\mu a(t) + \rho R(t)}{\sigma a(t)}\Big)^2 + \Big(\frac{\delta}{\kappa}(\sigma - \theta)\Big)^2 dt \Bigg\}. \end{aligned} \tag{5.5}$$

Now, it is possible to represent equations (5.1) and (5.2) under an equivalent martingale measure Q as follows

$$dR(t) = \sigma(t)a(t)d\tilde{B}_1(t) - dL(t) \tag{5.6}$$

and

$$\frac{d\sigma(t)}{\sigma(t)} = \kappa d\tilde{B}_2(t). \tag{5.7}$$

If we let $X(t) = \frac{R(t)}{\sigma(t)}$ we get that

$$\begin{aligned} \frac{dX(t)}{X(t)} &= \frac{dR(t)}{R(t)} - \frac{d\sigma(t)}{\sigma(t)} - \frac{dR(t)}{R(t)}\frac{d\sigma(t)}{\sigma(t)} + \left(\frac{d\sigma(t)}{\sigma(t)}\right)^2 \\ &= \left(\kappa^2 - \alpha\sigma a(t)\kappa\right)dt + \frac{\sigma(t)a(t)}{R(t)}d\tilde{B}_1(t) - \kappa d\tilde{B}_2(t) - \frac{1}{R(t)}dL(t). \end{aligned}$$

We have assumed that $\tilde{B}_1(t)$ and $\tilde{B}_2(t)$ are positively correlated with,

$$d\tilde{B}_1(t)d\tilde{B}_2(t) = \alpha dt \tag{5.8}$$

for some constant $\alpha > 0$. The following equation is obtained

$$dX(t) = X(t)(\kappa^2 - \alpha\sigma a(t)\kappa)dt + a(t)d\tilde{B}_1(t) - X(t)\kappa d\tilde{B}_2(t) - \frac{1}{\sigma(t)}dL(t). \tag{5.9}$$

5.4 The Value Function for the Stochastic Volatility Model

Without loss of generality we consider the model

$$dX(t) = X(t)(\kappa^2 - \alpha\sigma a(t)\kappa)dt + a(t)dB_1(t) - X(t)\kappa dB_2(t) - \frac{1}{\sigma(t)}dL(t). \tag{5.10}$$

with $X(0) = x$ and $L(0) = 0$. As in previous chapter, chapter 4, and in Cadenillas et al. [10] we define the function $g : [0,\infty) \to (-\infty,\infty)$, which is the net amount of money that shareholders receive, by

$$g(\eta) := -K + k\eta \tag{5.11}$$

where the constant $K > 0$ is a fixed setup cost incurred every time a dividend is paid, while $(1-k) \in (0,1)$ is the tax rate at which dividends are taxed, and η is the amount of liquid assets withdrawn.
The performance functional J, with each admissible control $\pi = (a, \mathcal{T}, \xi) \in \mathcal{A}(x)$, is defined by

$$J(x,\pi) = E\left[\sum_{k=1}^{\infty} e^{-c\tau_k} g(\xi_k) I_{\{\tau_k<\tau\}}\right] \tag{5.12}$$

where $c > \rho$ and $\mathcal{A}(x)$ represents the class of all admissible controls. The performance functional represents the expected present value of the dividends received by shareholders until the time of bankruptcy.
We define the value function $V(x)$ by

$$V(x) = \sup\{J(x,\pi); \pi \in \mathcal{A}(x)\} = \sup_{\pi \in \mathcal{A}(x)} E\left[\sum_{k=1}^{\infty} e^{-c\tau_k} g(\xi_k) I_{\{\tau_k < \tau\}}\right]. \tag{5.13}$$

The problem for an insurance company is to select the triple control $\pi = (a, \mathcal{T}, \xi)$ that maximizes the performance functional J. The optimal control policy $\pi^* = (a^*, \mathcal{T}^*, \xi^*)$ is a policy for which the following equality is satisfied,

$$V(x) = J(x, \pi^*). \tag{5.14}$$

Define the maximum utility operator $\mathbb{M}$ by

$$\mathbb{M}\phi(x) := \sup\{\phi(x-\eta) + g(\eta) : \eta > 0, x \geq \eta\}, \tag{5.15}$$

where g is given by (5.11) and ϕ is a function defined on $[0, \infty)$ into $\mathbb{R}$.

Cadenillas et al. [10] heuristically argue that if the payment of dividends occurs at time 0 and the amount of it equals η, then the reserve decreases from initial position x to $x - \eta$. After that, if the optimal policy is followed then the total expected utility is $k\eta - K + V(x-\eta)$. Consequently, under such a policy, the total maximal expected utility would be equal to $\mathcal{M}V(x)$. On the other hand, for each initial position x, if there exists an optimal policy, which is optimal for the whole domain, then the expected utility associated with this optimal policy is $V(x)$. This value function $V(x)$ is greater or equal to any expected utility associated with another different policy. It follows that

$$V(x) \geq \mathcal{M}V(x), \tag{5.16}$$

where equality holds if x is the position process where it is optimal to intervene.
If we define the infinitesimal operator $\mathcal{L}^a$ of $X(t)$ by

$$\mathcal{L}^a\psi(x) = \frac{1}{2}a^2\psi''(x) - \frac{1}{2}x^2\kappa^2\psi''(x) + x(\kappa^2 - \alpha\sigma\kappa a)\psi'(x) - c\psi(x), \tag{5.17}$$

then by the dynamic programming principle applied on $V(x)$, yields

$$\max_{a\in[0,1]} \mathcal{L}^a V(x) = 0. \tag{5.18}$$

This leads us to Definition 5.1 and Definition 5.2 given below.

Definition 5.1. A function $W : [0,\infty) \to [0,\infty)$ satisfies the quasi-variational inequalities (QVI) of the control problem if for $x \in [0,\infty)$ and $a \in [0,1]$

$$\mathbb{L}^a W(x) \leq 0, \tag{5.19}$$

$$\mathbb{M}W(x) \leq W(x), \tag{5.20}$$

$$(W(x) - \mathbb{M}W(x))\Big(\max_{a\in[0,1]} \mathbb{L}^a W(x)\Big) = 0 \tag{5.21}$$

and

$$W(0) = 0. \tag{5.22}$$

We observe that a solution W of the QVI separates the interval $[0,\infty)$ into two regions: a continuation region

$$\mathcal{C} := \left\{ x \in (0,\infty) : \mathbb{M}W(x) < W(x) \text{ and } \max_{a\in[0,1]} \mathbb{L}^a W(x) = 0 \right\}$$

and an intervention region

$$\Sigma := \left\{ x \in (0,\infty) : \mathbb{M}W(x) = W(x) \text{ and } \max_{a\in[0,1]} \mathbb{L}^a W(x) < 0 \right\}.$$

Given a solution W to the QVI, we define the following policy associated with this solution.

Definition 5.2. The control $\pi^W = (a^W, \mathcal{T}^W, \xi^W) = (a^W; \tau_1^W, \tau_2^W, \ldots, \tau_n^W, \ldots; \xi_1^W \ldots)$ is called the QVI control, associated with the state process X^W given by (5.10), if it satisfies the following conditions

$$P\Big\{a^W(t) \neq \arg\max_{a\in[0,1]} \mathbb{L}^a W(X_t^W), \quad X_t^W \in \mathcal{C}\Big\} = 0, \tag{5.23}$$

$$\tau_1^W := \inf\{t \geq 0 : W(X^W(t)) = \mathbb{M}W(X^W(t))\}, \tag{5.24}$$

$$\xi_1^W := \arg \sup_{\eta>0, \eta\leq X^W(\tau_1^W)} \Big\{W(X^W(\tau_1^W) - \eta) + g(\eta)\Big\}, \tag{5.25}$$

and for $n \geq 2$

$$\tau_n^W := \inf\{t \geq \tau_{n-1} : W(X^W(t)) = \mathbb{M}W(X^W(t))\}, \tag{5.26}$$

$$\xi_n^W := \arg \sup_{\eta>0, \eta\leq X^W(\tau_n^W)} \Big\{W(X^W(\tau_n^W) - \eta) + g(\eta)\Big\}, \tag{5.27}$$

with $\tau_0^W := 0$ and $\xi_0^W = 0$.

Under this control the intervention takes place whenever W and $\mathbb{M}W$ coincide, and the amount of the liquid assets withdrawn at these times is determined from the solution to the one-dimensional optimization problem associated with the operator $\mathbb{M}W$.

Theorem 5.1. *Let $W \in \mathcal{C}^1((0,\infty))$ be a solution of the QVI (5.19)-(5.22). Suppose there exists $U > 0$ such that W is twice continuously differentiable on $(0,U)$ and W is linear on $[U,\infty)$. Then for every $x \in (0,\infty)$*

$$V(x) \leq W(x). \tag{5.28}$$

Further, if the QVI control $\pi^W = (a^W, \mathcal{T}^W, \xi^W)$ associated with W is admissible, then W coincides with the value function and the QVI control associated with W is the optimal policy, hence

$$V(x) = W(x) = J(x; a^W, \mathcal{T}^W, \xi^W). \tag{5.29}$$

Proof. The proof of the above theorem is similar to the proof of Theorem 3.4 of Cadenillas et al. [10]. Define $\tau^*(t) = \max\{\tau_j : \tau_j < t\}$ for $j = 0,1,2,\ldots$, where

$\tau^*(t) \to \infty$ as $t \to \infty$ $a.s.$. Consider an arbitrarily chosen impulse control $\vartheta = (\tau_1, \tau_2, \dots, \tau_j, \dots; \xi_1, \xi_2, \dots, \xi_j, \dots)$ and let $\tau_0 = 0$. Choose $a \in [0,1]$ and put $X(t) = X^{(\pi)}(t)$. The function W is bounded on $[0, U]$ due to its continuity while W' is bounded on $(0, \infty)$ due to its differentiability and continuity. The linearity of W and the Lebesgue dominated convergence theorem implies that

$$\lim_{t\to\infty} E\big[e^{-ct} W(X(t))\big] = 0. \tag{5.30}$$

The boundedness of W' implies that

$$E\left[\int_0^\infty \Big(e^{-ct} W'(X(t))\Big)^2 \Big(a^2(t) - \kappa^2 X^2(t)\Big) dt\right] < \infty. \tag{5.31}$$

For $n = 1, 2, 3, \dots$, we can write

$$\begin{aligned}
& e^{-c(\tau^*(t)\wedge n)} W(X_{\tau^*(t)\wedge n}) - W(x) \\
= \; & \sum_{j=1}^{m} I_{\{\tau_j \le t\}} \Big[e^{-c(\tau_j \wedge n)} W(X_{\tau_j \wedge n}) - e^{-c(\tau_{j-1}\wedge n)} W(X_{\tau_{j-1}\wedge n})\Big] \\
= \; & \sum_{j=1}^{m} I_{\{\tau_j \le t\}} \Big[e^{-c(\tau_j \wedge n)} W(X_{\tau_j^- \wedge n}) - e^{-c(\tau_{j-1}\wedge n)} W(X_{\tau_{j-1}\wedge n})\Big] \\
& + \sum_{j=1}^{m} I_{\{\tau_j \le t \wedge n\}} e^{-c\tau_j} \big[W(X_{\tau_j}) - W(X_{\tau_j^-})\big].
\end{aligned}$$

An application of Itô's formula for jump processes (see, $IV.45$ in Rogers and Williams 1987) yields

$$\begin{aligned}
& e^{-c(\tau_j \wedge n)} W(X_{\tau_j^- \wedge n}) - e^{-c(\tau_{j-1}\wedge n)} W(X_{\tau_{j-1}\wedge n}) \\
= \; & \int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs} \mathbb{L} W(X_s) ds + \int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs} a(t) W'(X_s) dB_s^{(1)} + \\
& \int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs} \kappa X_s W'(X_s) dB_s^{(2)}.
\end{aligned}$$

In view of inequality (5.19), we have

$$\begin{aligned} & e^{-c(\tau_j \wedge n)} W(X_{\tau_j^- \wedge n}) - e^{-c(\tau_{j-1} \wedge n)} W(X_{\tau_{j-1} \wedge n}) \\ \leq \; & \int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs} a(t) W'(X_s) dB_s^{(1)} + \int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs} \kappa X_s W'(X_s) dB_s^{(2)}. \end{aligned}$$

Note that if τ_{j-1} and τ_j are intervention times as defined in (5.24) and (5.25), then

$W(X_s) > \mathbb{M}W(X_s)$ for $s \in [\tau_{j-1}, \tau_j)$. It follows that $\mathbb{L}W(X_s) = 0$ by definition 5.1. The inequality above becomes an equality for the QVI-control associated with W. Also noting that

$X(\tau_j) = X(\tau_j^-) + \xi_j$, according to (5.20), we have

$$e^{-c\tau_j}\Big[W(X(\tau_j)) - W(X(\tau_j^-))\Big] \leq -e^{-c\tau_j} g(\xi_j).$$

We should also note that this inequality becomes an equality for the QVI-control associated with W, since $g(\xi_j) + W(X(\tau_j^-)) = \mathbb{M}W(X(\tau_j^-)) = W(X(\tau_j))$ if (τ_j, ξ_j) is the impulse control defined by definition 5.2. Combining the above two inequalities gives,

$$\begin{aligned} & W(x) - e^{-c(\tau_j \wedge n)} W(X_{\tau_j \wedge n}) \\ \geq \; & \sum_{j=1}^{m} I_{\{\tau_j \leq t\wedge n\}} e^{-c\tau_j \wedge n} g(\xi_j) - \int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs} a(t) W'(X_s) dB_s^{(1)} \\ & - \int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs} \kappa X_s W'(X_s) dB_s^{(2)}. \end{aligned}$$

Taking expectation of both sides we get

$$
\begin{aligned}
& W(x) - E\Big[e^{-c(\tau_j \wedge n)} W(X_{\tau_j \wedge n})\Big] \\
\geq \;& E\Big[\sum_{j=1}^{m} I_{\{\tau_j \leq t \wedge n\}} e^{-c\tau_j \wedge n} g(\xi_j)\Big] - \\
& E\Big[\int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs} a(t) W'(X_s) dB_s^{(1)} + \int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs} \kappa X_s W'(X_s) dB_s^{(2)}\Big].
\end{aligned}
$$

The above inequality becomes an equality for the QVI-control associated with W. Now

$$
\begin{aligned}
& \lim_{n\to\infty} \left\{ W(x) - E\Big[e^{-c(\tau_j \wedge n)} W(X_{\tau_j \wedge n})\Big] \right\} \\
= \;& W(x) - E\Big[e^{-c(\tau_j)} W(X_{\tau_j})\Big].
\end{aligned}
$$

From the growth condition 5.61 we have

$$
\lim_{n\to\infty} E\left[\int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs} a(t) W'(X_s) dB_s^{(1)} + \int_{\tau_{j-1}\wedge n}^{\tau_j \wedge n} e^{-cs} \kappa X_s W'(X_s) dB_s^{(2)}\right] = 0.
$$

Thus

$$
W(x) - E\Big[e^{-c\tau_j} W(X_{\tau_j})\Big] \geq E\Big[\sum_{j=1}^{m} I_{\{\tau_j \leq t\}} e^{-c\tau_j} g(\xi_j)\Big].
$$

According to the growth condition (5.30) letting $m \to \infty$ we get

$$
W(x) \geq E\Big[\sum_{j=1}^{\infty} I_{\{\tau_j \leq \tau\}} e^{-c\tau_j} g(\xi_j)\Big] = J(x, \pi).
$$

Since this is true for any control π, we have

$$W(x) \geq V(x).$$

The above inequality becomes an equality for the QVI-control associated with W. This completes the proof of the theorem. $\square$

5.5 Solution to the QVI for Stochastic Volatility Model

In the continuation region we have for any $x \in \mathcal{C}$,

$$\max_{a\in[0,1]} \left\{ \frac{1}{2}(a^2 - x^2\kappa^2)W''(x) + x\kappa(\kappa - \alpha\sigma a)W'(x) - cW(x) \right\} = 0. \tag{5.32}$$

If we let $a(x)$ be the maximizer of the expression inside the set braces on the left-hand side of (5.32) for all$a(x) \in (-\infty, \infty)$. In such a case we have,

$$a(x) = \frac{x\kappa\alpha\sigma W'(x)}{W''(x)}. \tag{5.33}$$

For those values of x for which the argmaximum of the left-hand side of (5.32) does not coincide with the endpoints of the interval $[0, 1]$, we can substitute the expression (5.33) into (5.32) to get

$$\frac{1}{2}\left(\frac{(x\kappa\alpha\sigma)^2 W'(x)^2}{W''(x)^2} - x^2\kappa^2\right)W'(x) +$$

$$\left(x\kappa^2 - \frac{(x\kappa\alpha\sigma)^2 W'(x)}{W''(x)}\right)W'(x) - cW(x) = 0. \tag{5.34}$$

A general solution subject to the boundary condition (5.61) is given by

$$W_1(x) = A_2 e^{rx}, \tag{5.35}$$

where A_2 is a free constant and $r \in (0, 1)$ is given by

$$r = \frac{x\kappa^2 - c}{x^2\kappa^2} + \sqrt{\left(\frac{x\kappa^2 - c}{x^2\kappa^2}\right)^2 - (\alpha\sigma)^2}, \tag{5.36}$$

which is a positive solution of $f(r) = 0$, where

$$f(r) = r^2 - 2\Big(\frac{x\kappa^2 - c}{x^2\kappa^2}\Big)r + (\alpha\sigma)^2.$$

This solution is only valid when $a(x)$ given by (5.33) belongs to the interval $(0, 1)$. From (5.33) and (5.35) it follows that,

$$a(x) = \frac{x\kappa\alpha\sigma}{r}.$$

This is an increasing function and $a(x) \leq 1$ if and only if $x \leq x^*$, where

$$x^* = \frac{r}{\kappa\alpha\sigma}. \tag{5.37}$$

Thus if $x^* \leq x < x_1$ then $a(x) \geq 1$. However, the maximization in (5.32) must be taken over $a(x) \in [0, 1]$. Hence, for $x^* \leq x < x_1$ we must have $a(x) = 1$, and (5.32) becomes

$$\frac{1}{2}(1 - x^2\kappa^2)W''(x) + x\kappa(\kappa - \alpha\sigma)W'(x) - cW(x) = 0. \tag{5.38}$$

Inspired by Høgaard and Taksar [29], we need to find a solution to (5.38) subject to $W'(u_1) = -1$ and $W''(u_1) = 0$ such that W is twice continuously differentiable on the whole real line, where $u_1 > x^*$.

Lemma 5.1. *Let $u > 0$ and $W(x)$ be a solution of (5.38) with $W'(u) = -1$ and $W''(u_1) = 0$. Then $W(x)$ is strictly concave for $x < u$.*

Proof. Clearly $W(x)$ is an analytic function and differentiating (5.38) gives us

$$(1 - x^2\kappa^2)W'''(x) - 2x\kappa\alpha\sigma W''(x) + 2(\kappa^2 - \alpha\sigma\kappa - c)W'(x) = 0. \tag{5.39}$$

For $x = u$ we get that

$$W'''(u) = \frac{2(\kappa^2 - \alpha\sigma\kappa - c)}{1 - u^2\kappa^2}. \tag{5.40}$$

From (5.38) we can see that $W(u) = \frac{u\kappa(\alpha\sigma - \kappa)}{c}$, implying that $\alpha\sigma > \kappa$. We therefore have $W'''(u) > 0$ for every $u > \frac{1}{\kappa}$ if $\kappa \in \Big(0, \frac{\sigma\alpha + \sqrt{(\alpha\sigma)^2 + 4c}}{2}\Big)$ or $W'''(u) > 0$

for every $u \in (0, \frac{1}{\kappa})$ if $\kappa > \frac{\sigma\alpha+\sqrt{(\alpha\sigma)^2+4c}}{2}$. If $W'''(u) > 0$ then $W''(x) < 0$ in the left neighborhood of u. Assuming that $\tilde{x} = \sup\{x < u; W''(x) = 0\} > -\infty$, then $W'(x) > -1$ for $x > \tilde{x}$. Since (5.38) yields $W(u) = \frac{u\kappa(\alpha\sigma-\kappa)}{c}$, we must have $W(x) \leq \frac{x\kappa(\alpha\sigma-\kappa)}{c}$ for $x \in [\tilde{x}, u]$. On the other hand, inserting $x = \tilde{x}$ in (5.38), we get

$$0 = \tilde{x}\kappa(\kappa - \alpha\sigma)W'(\tilde{x}) - cW(\tilde{x}) > \tilde{x}\kappa(\alpha\sigma - \kappa) - cW(\tilde{x})$$

which implies that

$$W(\tilde{x}) > \frac{\tilde{x}\kappa(\alpha\sigma - \kappa)}{c}.$$

This contradict the existence of $\tilde{x}$ and the concavity of $W(x)$ follows. □

Lemma 5.2. *There exist constants $A_2 > 0$, $u_1 > x^*$ and a twice continuously differentiable function W which is given by (5.35) for $x < x^*$ and satisfies (5.38) for $x^* < x < u_1$ with $W'(u_1) = -1$ and $W''(u_1) = 0$.*

Proof. Let $u > x^*$ be a candidate for u_1 and let $W_u(x)$ be the solution to (5.38) with $W_u'(u) = -1$, $W_u''(u) = 0$. The index u from $W_u(\cdot)$ shows an explicit dependence on the constant u. From Lemma 5.1, $W_u(x)$ is a concave function for all $x < u$ and decreasing in the neighborhood of u. We can therefore choose u_0 such that $W_{u_0}(x^*) = 0$. Since $W_u(u) = \frac{u\kappa(\alpha\sigma-\kappa)}{c} > 0$, $W_u'(u) = -1$ and $W_u(x)$ is concave for $x < u$, we have

$$W_u(x) \leq x - u + \frac{u\kappa(\alpha\sigma - \kappa)}{c}, \qquad \forall x < u. \tag{5.41}$$

If we insert x^* into (5.41), we get $W_u(x_0) < 0$ for $u > \frac{x^*c}{c+\kappa(\kappa-\alpha\sigma)}$. So the existence of u_0 such that $W_{u_0}(x_0) = 0$ follows.

Let $W_{A_2}(x)$ denote a function given by (5.35). For any u we can choose $A_2 = A_2(u)$ such that $W_{A_2(u)}(x^*) = W_u(x^*)$, in particular $A_2(u_0) = 0$. Let $f(u) = W_u'(x^*)$, then f is a decreasing function of u with $f(x^*) = -1$. Also let $g(A_2) =$

$W'_{A_2}(x^*)$, then g is a decreasing function of A_2. Due to the fact that $A_2(u)$ is a decreasing function of u, $g(A_2(u))$ is a decreasing function of u. To find $g(A_2(x^*))$ we insert x^* and $W'_{A_2(x^*)}(x^*) = \frac{x^*\kappa(\alpha\sigma-\kappa)}{c}$ in (5.34) and get

$$g(A_2(x^*)) = W_{A_2(x^*)}(x^*) = \frac{\kappa - \alpha\sigma}{\frac{\alpha\sigma}{2} + \frac{(x^*\kappa)^2\alpha\sigma}{2} - \kappa} \geq -1.$$

Therefore as u decreases from u_0 to x^*, the function $f(u)$ will continuously increase from -1 to a value greater than -1, whereas the function $g(A_2(x^*))$ will continuously decrease to -1. Hence, there exists a $u_1 \in (x^*, u_0)$ such that $g(A_2(u_1)) = f(u_1)$. This choice of u_1 and $A_2 = A_2(u_1)$ ensures that the value and the first derivative of W are equal at x^*. The equality of the second derivatives follows from the differential equation that is satisfied on both sides of x^* and the fact that $a(x^*) = 1$. □

Remark 5.1. Let C, u_1, $W(x)$ be as in Lemma 5.2, then for any h, we have that $N(x) = hW(x)$ satisfies (5.38) with $N'(u_1) = h$ and $N''(u_1) = 0$. We thus have from the arbitrary choice of h that there exists $u_1 > x^*$ and a twice continuously differentiable function $W(x)$ given by (5.35) for $x \leq x^*$ and satisfies (5.38) for $x > x^*$ with $W''(u_1) = 0$.

The solution to the differential equation (5.38) is given by

$$W_2(x) = C_1 m(x) + C_2 n(x), \tag{5.42}$$

where C_1, C_2 are free constants, $m(x)$ and $n(x)$ are defined by

$$m(x) = (x^2\kappa^2 - 1)^{\frac{2\kappa-\alpha\sigma}{2\kappa}} U\left(\frac{-\kappa + \sqrt{(3\kappa - 2\alpha\sigma)^2 - 4c}}{2\kappa}, \frac{\alpha\sigma - 2\kappa}{\kappa}, x\kappa\right), \tag{5.43}$$

$$n(x) = (x^2\kappa^2 - 1)^{\frac{2\kappa-\alpha\sigma}{2\kappa}} F\left(\frac{-\kappa + \sqrt{(3\kappa - 2\alpha\sigma)^2 - 4c}}{2\kappa}, \frac{\alpha\sigma - 2\kappa}{\kappa}, x\kappa\right), \tag{5.44}$$

and U, F are associated Legendre polynomials of the first and second kind respectively.

The continuity and the differentiability of the function $W(x)$ at the point x^* implies that

$$\begin{aligned} C_1 &= Ak_1, \\ C_2 &= Ak_2, \end{aligned}$$

where

$$k_1 = e^{rx^*} \frac{n'(x^*) - rn(x^*)}{n'(x^*)m(x^*) - n(x^*)m'(x^*)}, \tag{5.45}$$

$$k_2 = e^{rx^*} \frac{rm(x^*) - m'(x^*)}{n'(x^*)m(x^*) - n(x^*)m'(x^*)}. \tag{5.46}$$

Thus the solution to (5.32) is given by

$$W(x) = \begin{cases} Ae^{rx} & \text{if} \quad 0 < x < x^*, \\ A(k_1 m(x) + k_2 n(x)) & \text{if} \quad x \geq x^*, \end{cases} \tag{5.47}$$

and the corresponding optimal reinsurance control is given by

$$\hat{a}(x) = \begin{cases} \frac{\kappa \alpha \hat{R}_t}{r} & \text{if} \quad 0 \leq \hat{R}_t \leq x^*, \\ 1 & \text{if} \quad x^* < \hat{R}_t. \end{cases} \tag{5.48}$$

5.6 The Conjectured Impulse Control

Following the method used by Cadenillas and Zapatero [11] which is also employed in Zhang and Song [69], we conjecture an impulse control $\hat{\pi} = (\hat{a}, \hat{\mathcal{J}}, \hat{\xi})$ and verify that the conjectured control is optimal and that it solves the QVI, so that by theorem 5.1 $V(x)$ is the value function of the problem.

Conjecture: The optimal impulse control is characterized by two parameters β and L with $0 < \beta < L < \infty$, such that it is optimal not to intervene while the reserve process R stays inside the interval $(0, L)$. If on the other hand the reserve process reaches the boundary L, then the control should be exercised to push it instantaneously to level β. That is,

$$\tau_1 = \inf\{t \geq 0 : \hat{R}_t \notin (0, L)\}, \tag{5.49}$$

$$\tau_i = \inf\{t > \tau_{i-1} : \hat{R}_t \notin (0, L)\}, \quad i = 2, 3, \ldots \tag{5.50}$$

$$\hat{R}_{\hat{\tau}_i} = \hat{R}_{\hat{\tau}_i^-} - \xi_i = \beta I_{\{\hat{R}_{\hat{\tau}_i = L}\}}. \tag{5.51}$$

If we initially have $x > L$, then optimal control would jump to β. The value function will satisfy

$$V(x) = V(\beta) + g(x - \beta) = V(\beta) + k(x - \beta) - K \qquad \forall x \in [L, \infty). \tag{5.52}$$

If V were differentiable at L, then from (5.52), we will get

$$\begin{aligned} V'(L) &= k, \\ V'(\beta) &= k. \end{aligned}$$

We also note that in the no intervention region (continuation region) $x \in (0, L)$, the value function $V(x)$ should satisfy,

$$\max_{a \in [0,1]} \left\{ \frac{1}{2}(a^2 - x^2\kappa^2)V''(x) + x\kappa(\kappa - \alpha\sigma a)V'(x) - cV(x) \right\} = 0. \tag{5.53}$$

From section 5.5, a solution of (5.53) is given by

$$W(x) = \begin{cases} Ae^{rx} & \text{if} \quad 0 < x < x^*, \\ A(k_1 m(x) + k_2 n(x)) & \text{if} \quad x \geq x^*, \end{cases} \tag{5.54}$$

where $m(x)$, $n(x)$, k_1 and k_2 are given by (5.43), (5.44), (5.45) and (5.46) respectively. We summarize this by conjecturing that the solution is described by (5.49)-(5.51), (5.47)-(5.48) and that the three unknown quantities β, L, A

constitute the solution of the system of three equations

$$W'(\beta) = k, \tag{5.55}$$

$$W'(L) = k, \tag{5.56}$$

$$W(L) = W(\beta) + k(L - \beta) - K. \tag{5.57}$$

Proposition 5.1. *Suppose that $W(x)$ is given by (5.47), then there exists a $u_1 > x^*$ such that $W''(u_1) = 0$ and*

$$W''(x) > 0 \qquad \textit{for} \quad x > u_1.$$

Proof. The existence of u_1 follows from Lemma 5.2 and Remark 5.1. Since $W(x)$ is a solution of equation (5.38) for $x > x^*$, we differentiate 5.38 to get

$$(1 - x^2\kappa^2)W'''(x) - 2x\kappa\alpha\sigma W''(x) + 2(\kappa^2 - \alpha\sigma\kappa - c)W'(x) = 0. \tag{5.58}$$

Letting $x = u_1$ we get

$$W'''(u_1) = \frac{2(c + \alpha\sigma\kappa - \kappa^2)}{u_1^2\kappa^2 - 1}. \tag{5.59}$$

Since we are only interested in the positive values of u_1, we therefore have $W'''(u_1) > 0$ for every $u_1 > \frac{1}{\kappa}$ if $\kappa \in \left(0, \frac{\sigma\alpha + \sqrt{(\alpha\sigma)^2 + 4c}}{2}\right)$ or $W'''(u_1) > 0$ for every $u_1 \in (0, \frac{1}{\kappa})$ if $\kappa > \frac{\sigma\alpha + \sqrt{(\alpha\sigma)^2 + 4c}}{2}$. Thus $W''(x) > 0$ in the right neighbourhood of u_1 if $u_1 > \frac{1}{\kappa}$ or $u_1 \in (0, \frac{1}{\kappa})$. If there exists a constant $\bar{x}$ such that $W''(\bar{x}) \leq 0$, then $\tilde{x} = \inf\{x : W''(x) \leq 0\} < \infty$. For $x = \tilde{x}$ in equation (5.59), we get that $W'''(\tilde{x}) > 0$. Therefore $W''(x) < 0$ in the left neighborhood of $\tilde{x}$ which is a contradiction to the definition of $\tilde{x}$. We must thus have $\tilde{x} = \infty$ and then get the desired result. $\square$

From Lemma 5.1, 5.2, Remark 5.1 and Proposition 5.1, we obtain

$$W''(x) = \begin{cases} < 0 & \text{if} \quad x < u_1, \\ > 0 & \text{if} \quad x > u_1. \end{cases} \tag{5.60}$$

We therefore see that $W'(x)$ is strictly decreasing on the interval $(0, u_1)$ and is strictly increasing on the interval (u_1, ∞). So the existence of L and β satisfying (5.55)-(5.57) implies that

$$\beta < u_1 < L. \tag{5.61}$$

Proposition 5.2. *Let*

$$\hat{x} = \inf\{x \geq 0 : W(x) = \mathbb{M}W(x)\}, \tag{5.62}$$

then $\hat{x} = L$.

Proof. Let $G : [0, x] \mapsto \mathbf{R}$ be defined by

$$G(y) = W(y) + k(x - y) - K.$$

Clearly

$$G'(y) = W'(y) - k.$$

From (5.60), (5.61), (5.55) and (5.56) it is easy to see that

$$W'(x) = \begin{cases} > k & \text{if} \quad 0 \leq x < \beta, \\ < k & \text{if} \quad \beta < x < L. \end{cases} \tag{5.63}$$

We therefore have that $G(y)$ is increasing in $[0, x]$ whenever $x \in (0, \beta]$, also increasing in $[0, \beta]$ whenever $x \in (\beta, L)$ and decreasing in $[\beta, x]$.
By the definition of the maximum utility $\mathbb{M}$ and the above analysis, we get

$$\mathbb{M}W(x) = \begin{cases} W(x) - K & \text{if} \quad 0 \leq x < \beta, \\ W(\beta) + k(x - \beta) - K & \text{if} \quad \beta \leq x \leq L. \end{cases} \tag{5.64}$$

We therefore have

$$W(x) - \mathbb{M}W(x) = K > 0. \quad \text{if} \quad 0 \leq x < \beta, \tag{5.65}$$

and if $\beta \leq x \leq L$,

$$W(x) - \mathbb{M}W(x) = W(x) - W(\beta) - k(x - \beta) + K.$$

Let $Z(x) = W(x) - \mathbb{M}W(x)$ so that $Z'(x) = W'(x) - k$. From (5.63) it is clear that $Z(x)$ is decreasing in $[\beta, L]$. However, from (5.64) and (5.57) we have

$$Z(L) = W(L) - \mathbb{M}W(L) = 0. \tag{5.66}$$

Therefore

$$Z(x) = W(x) - \mathbb{M}W(x) > 0 \quad \text{if} \quad \beta \leq x < L. \tag{5.67}$$

From (5.65), (5.66) and (5.67) we obtain that L is the smallest point that makes $W(x) = \mathbb{M}W(x)$, i.e. $L = \inf\{x \geq 0 : W(x) = \mathbb{M}W(x)\}$. So $\hat{x} = L$.

□

Proposition 5.1 and 5.2 above provides a conjectured structure of the solution v to the QVI (5.19)-(5.22), that is

$$v(x) = \begin{cases} W(x) & \text{if} \quad 0 \leq x \leq L, \\ W(\beta) + k(x - \beta) - K & \text{if} \quad x \geq L, \end{cases} \tag{5.68}$$

where $W(x)$ is given by (5.47) and the constant A, L and β are determined by (5.55)-(5.57).

Proposition 5.3. *There exists constants A, L and β which satisfies equations (5.55)-(5.57).*

Proof. Define the function $F(x)$ for all $x > 0$ by

$$F(x) = \begin{cases} re^{rx} & \text{if} \quad 0 \leq x \leq x^*, \\ k_1 m'(x) + k_2 n'(x) & \text{if} \quad x \geq x^*, \end{cases} \tag{5.69}$$

where $m(x)$ and $n(x)$ are given by (5.43) and (5.44) respectively while x^* is given by equation (5.37). Noting Proposition 5.1, it is easy to see that

$$\lim_{x \to \infty} F(x) = +\infty,$$

$$\lim_{x \to 0} F(x) = +\infty.$$

Let u_1 be given as in Proposition 5.1 and let $\lambda = F(u_1)$. From equation (5.60) and the fact that $AF(x) = W'(x)$, we can see that $F(x)$ is strictly decreasing on $[0, u_1]$ and is strictly increasing on $[u_1, +\infty)$. We have that if $A \in (0, \frac{k}{\lambda})$, then there must exist two points β^A and L^A with $u_1 \in (\beta^A, L^A)$ such that $AF(\beta^A) = AF(L^A) = k$. It is quit obvious that $\beta^A = L^A = u_1$ if $A = \frac{k}{\lambda}$. It is also easy to see that β^A is an increasing function of A, while L^A is a decreasing function of A for $A \in (0, \frac{k}{\lambda}]$.

Define

$$I(A) := \int_{\beta^A}^{L^A} (k - AF(y))dy.$$

Due to the fact that the limits in the integral and the integrand are continuous functions of A, the function $I(A)$ is also a continuous function of A. The fact that both the integrand and the interval $[\beta^A, L^A]$ are decreasing with respect to A, implies that $I(A)$ is a decreasing function of A. Since $AF \to 0$ uniformly on the compact set $(0, +\infty)$, we can see that $L^A \to +\infty$ and $(k - AF(y)) \to k$ as $A \to 0$. We therefore have that $I(A) \to +\infty$ as $A \to 0$. Since $I(\frac{k}{\lambda}) = 0$, there exists an $A < \frac{k}{\lambda}$ such that

$$I(\tilde{A}) = \int_{\beta^{\tilde{A}}}^{L^{\tilde{A}}} (k - \tilde{A}F(y))dy = K.$$

Noting that $AF(x) = W'(x)$, we therefore get

$$W(L^{\tilde{A}}) = W(\beta^{\tilde{A}}) + k(L^{\tilde{A}} - \beta^{\tilde{A}}) - K.$$

Thus equations (5.55)-(5.57) are satisfied with the choice of $A = \tilde{A}$, $\beta = \tilde{\beta}$ and $L = L^{\tilde{A}}$. $\square$

5.6.1 Verification Theorem

We want to show that the function defined by (5.68) is a solution to the QVI and that it coincides with the value function.

Theorem 5.2. *The function v given by equation (5.68) with $W(x)$ given by (5.47) is continuously differentiable on $(0,\infty)$ and is twice continuously differentiable on*
$(0,L)\cup(L,\infty)$. *This function is a solution to (5.19)-(5.22), subject to $W'(x^*)=0$ $u_1>\frac{1}{\kappa}$ if $\kappa\in\left(0,\frac{\sigma\alpha+\sqrt{(\alpha\sigma)^2+4c}}{2}\right)$ or $u_1\in(0,\frac{1}{\kappa})$ if $\kappa>\frac{\sigma\alpha+\sqrt{(\alpha\sigma)^2+4c}}{2}$, where u_1 is given in Proposition 5.1.*

Proof. From the construction of $W(x)$ on $[0,x^*]$ equation (5.47) is satisfied. We also know from equation (5.60) that $W''(x)\le 0$ for $x\in[x^*,u_1]$. Therefore, to see that $W(x)$ satisfies equation (5.62), it is sufficient to show that

$$\frac{x\kappa\alpha\sigma W'(x)}{W''(x)}\ge 1,$$

for all $x\in[x^*,u_1]$. Let the point $x=x^*$ be the optimal value of $W(x)$ for $x\in(0,u_1)$, with $u_1>\frac{1}{\kappa}$ if $\kappa\in\left(0,\frac{\sigma\alpha+\sqrt{(\alpha\sigma)^2+4c}}{2}\right)$ or $u_1\in(0,\frac{1}{\kappa})$ if $\kappa>\frac{\sigma\alpha+\sqrt{(\alpha\sigma)^2+4c}}{2}$, then $W'(x^*)=0$. From equation (5.60) we know that $W'(x)$ is strictly decreasing for $x\in(0,u_1)$, thus $W'(x)$ must be strictly decreasing for $x\in[x^*,u_1)$. But $W'(x^*)=0$, we therefore have $W'(x)<W(x^*)=0$ for $x\in(x^*,u_1)$ with $W'(u_1)=-1$ for $x=u_1$ satisfying the condition of Proposition 5.1 and the continuity condition of $W(x)$. For $x\in(u_1,L)$, we have according to equation (5.60) that $W'(x)$ is strictly increasing and $W''(x)>0$. Therefore $a=1$ is a maximizer of the left-hand side of (5.32). Thus $W(x)$ satisfies equation (5.32) 0n $[u_1,L]$ since it is a solution to equation (5.47) on this interval.
From Proposition 5.2 and the definition of $v(x)$, we have

$$\mathbb{M}v(x)<v(x)\qquad \text{for } x\in[0,L).$$

We will now show that

$$\mathbb{M}v(x) = v(x) \qquad \text{for } x \in [L, \infty). \tag{5.70}$$

Let $Z(\eta) = v(x-\eta) + g(\eta)$ for $x \geq L$, then

$$Z(\eta) = \begin{cases} W(x-\eta) + k\eta - K & \text{if} \quad 0 \leq x-\eta < L, \\ W(\beta) + k(x-\beta) - 2K & \text{if} \quad x-\eta \geq L, \end{cases} \tag{5.71}$$

From (5.63), it is easy to see that $Z(\eta)$ attain it maximum in the interval $[0, L)$ when $x - \eta = \beta$. We therefore have that

$$\begin{aligned} \mathbb{M}v(x) &= \sup_{\eta \in (0,x)} Z(\eta) \quad \text{if} \quad x \geq L \\ &= W(\beta) + k(x-\beta) - K \\ &= v(x). \end{aligned}$$

On the other hand, for $x \geq L$

$$\begin{aligned} & \frac{1}{2}(1 - x^2\kappa^2)v''(x) + x\kappa(\kappa - \alpha\sigma)v'(x) - cv(x) \qquad (5.72) \\ = \;& x\kappa(\kappa - \alpha\sigma)k - cv(x) \\ \leq \;& x\kappa(\kappa - \alpha\sigma)k - cW(L) \\ = \;& -\Big(x\kappa(\alpha\sigma - \kappa)k + cW(L)\Big) \\ < \;& 0. \end{aligned}$$

Which shows that

$$\mathbb{L}^a v(x) < 0 \quad \text{and} \quad \mathbb{M}v(x) = v(x) \quad \text{if} \quad x \geq L.$$

Thus $v(x)$ given by equation (5.68) with $W(x)$ given by (5.47) satisfies (5.19)-(5.22). $\square$

5.7 Conclusion

We have presented a solution to the impulse control and optimal stopping problem for insurance reserves under stochastic volatility. Our choice of risk neutral measure for the proposed model gives a closed form solution for both the optimal control and optimal Value Function. We also presented and proved the verification theorem of our problem.

Chapter 6

Optimal Dividend and Reinsurance Policies for Uncertain Stochastic Processes.

6.1 Introduction

In this chapter, we present and investigate the dividend optimization problem under uncertainty theory. The risk of the insurance firm is controlled by using a proportional reinsurance policy. It is considered that the evolution of the cash reserves of the firm is driven by hybrid processes. The surplus cash reserves earn interest at a constant rate. The objective of the firm is to maximize the total expected discounted dividends paid out to shareholders. The situation is modeled as an impulse-classical control problem. We define an impulse control for hybrid processes and solve the optimal control problem when insurance reserves are modeled by hybrid processes.

To provide better specifications and forecasting of the evolution of reserves in an insurance company, we extend the model by X. Zhang and M. Song [69] to uncertain stochastic model by including a one-dimensional canonical process into their model. The canonical process is an uncertainty processes representing incurred but not reported reserves and the belief degree at which uncertain events will occur.

6.2 Model Formulation

Let $(\Gamma \times \Omega, \mathcal{L} \otimes \mathcal{F}, \{\mathcal{L}_t \otimes \mathcal{F}_t\}_{t\in[0,T]}, \mathcal{M} \times P)$ be a filtered hybrid space satisfying the usual conditions endowed with a standard Brownian motion $B = \{B_t; t \in [0,T]\}$ adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ and a standard canonical process $C = \{C_t; t \in [0,T]\}$ adapted to the filtration $\{\mathcal{L}_t\}_{t\in[0,T]}$. The filtration $\{\mathcal{L}_t \otimes \mathcal{F}_t\}$ represent the information available at time t for the hybrid process $X_t = (B_t, C_t)$ and any control made is based on this information. Our state variable is the reserve process $R = \{R_t; t \in [0,T]\}$ where $T \in [0,\infty)$ is a time horizon. The reserve process R_t represents the liquid assets of an insurance company at time t.

Definition 6.1. (i)An uncertain random variable τ is called an uncertain random time if it is non-negative and can also take the value ∞ on $(\Gamma \times \Omega, \mathcal{L}_T \otimes \mathcal{F}_T)$. (ii) Suppose that a filtration $\{\mathcal{L}_t \otimes \mathcal{F}_t\}$ is given, τ is called a stopping time with respect to this filtration if for each $t \in [0,T]$ the event

$$\{\tau \le t\} \in \mathcal{L}_t \otimes \mathcal{F}_t.$$

Definition 6.2. A classical control $a = \{a(t)\}_{t\in[0,T]}$ is an $\{\mathcal{L}_t \otimes \mathcal{F}_t\}$-adapted hybrid process such that $a(t) \in [0,1]$.

We let $a(t) \in [0,1]$ be a classical control representing the retention rate at time t for the reserve processes R_t. The distribution of dividends is described by a sequence of increasing stopping times $\{\tau_i; i = 1,2,\ldots\}$ and a sequence of uncertain random variables $\{\xi_i; i = 1,2,\ldots\}$, which are associated with the times and amount of dividends paid out to shareholders. We also assume that the reserves earn interest at a constant force $\rho > 0$. The dynamics of our controlled process is thus given by,

$$dR_t = [a(t)\mu + \rho R_t]dt + \sigma_1 a(t)dB_t + \sigma_2 a(t)dC_t - dL(t), \tag{6.1}$$

$$R_0 = x \qquad \text{is the initial reserves,}$$

with $\mu > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$ being constant and $L(t) = \sum_{k=1}^{\infty} I_{\{\tau_k<t\}}\xi_k$ being the amount of dividends received by shareholders on the time interval $[0,T]$.

The uncertainty term C_t represents incurred but not reported reserves and the belief degree at which uncertain event will occur while the indeterminate term B_t represent the randomness of the reserve process.

Definition 6.3. An impulse control for a hybrid process is a double sequence

$$\vartheta = (\mathcal{T}, \xi) = (\tau_1, \tau_2, \ldots, \tau_n, \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots)$$

where $0 \leq \tau_1 < \tau_2 < \tau_3 < \ldots$ is an $\{\mathcal{L}_t \otimes \mathcal{F}_t\}$-adapted sequence of increasing stopping times and $\xi_1, \xi_2, \ldots$ are $\{\mathcal{L}_t \otimes \mathcal{F}_t\}$ measurable uncertain random variables with $\xi_i \in [0, R_{\tau_i^-}]$; for $i = 1, 2, 3, \ldots$.

In this chapter, an impulse control is used to describe the judgement on dividend distribution to shareholders. The insurance company pays ξ_i as the i-th dividend at some stopping time τ_i, which implies that $R(\tau_i) = R(\tau_i^-) - \xi_i$. On the other hand decisions on how much needs to be reinsured are made by classical control $a(t)$.

Definition 6.4. The combined classical control a and impulse control ϑ given by the triple

$$\pi := (a, \vartheta) = (a, \mathcal{T}, \xi)$$

is called an admissible control. The class of all admissible controls is denoted by $\mathcal{A}(x)$.

We define the stopping (bankruptcy) time by

$$\tau^\pi := \inf\{t \geq 0 : R(t) = 0\}.$$

We assume that $R(t)$ vanishes for $t \geq \tau$ as we are only dealing with the optimization problem during the time interval $[0, \tau)$. At time $t \in [0, T]$, the controlled reserve process R_t^π is given by

$$R_t^\pi = \begin{cases} x + \int_0^t (\mu a + \rho R_s^\pi) ds + \int_0^t \sigma a dB_s + \int_0^t a\sigma_2 dC_s - \sum_{k=1}^\infty I_{\{\tau_k < t\}} \xi_k, & \text{if } t < \tau \\ 0, & \text{if } t \geq \tau. \end{cases}$$

Define the performance functional $J(x,\pi)$ by

$$J(x,\pi) := E\left[\sum_{k=1}^{\infty} e^{-c\tau_k} g(\xi_k) I_{\{\tau_k<\tau\}} \Big/ R_0^{\pi} = x\right] \tag{6.2}$$

where $c > \rho$ and the function $g : [0,\infty) \mapsto (-\infty,\infty)$ is given by

$$g(\eta) = k\eta - K, \tag{6.3}$$

where $k \in (0,1)$ and $K \in (0,\infty)$ are constants, with $1-k$ being interpreted as tax rate and K as a fixed cost when dividends are paid.

Problem 6.2.1. *The problem is to determine the value function $V(x)$ and the optimal control $\pi^* = (a^*, \mathcal{T}^*, \xi^*) \in \mathcal{A}$ such that*

$$V(x) = \sup_{\pi\in\mathcal{A}} J(x,\pi) = J(x,\pi^*). \tag{6.4}$$

6.3 The Value Function

For every $x \in (0,\infty)$ denote the value function by $V(x)$, where

$$V(x) = \sup\{J(x,\pi); \pi \in \mathcal{A}(x)\} = \sup_{\pi\in\mathcal{A}(x)} E\left[\sum_{k=1}^{\infty} e^{-c\tau_k} g(\xi_k) I_{\{\tau_k<\tau\}}/R_0^{\pi} = x\right]. \tag{6.5}$$

Theorem 6.1. *(Principle of Optimality)*
For any $(t,x) \in [0,T]\times\mathbb{R}$ and $0 < t < t^* < T$ we have

$$V(t,x) = \sup_{\pi\in\mathcal{A}} E\left[\int_t^{t^*} f(R_s,\pi,s)ds + V(t^*, R_{t^*})\right] \tag{6.6}$$

where the arbitrary uncertain stochastic process $\{f|[0,T]\times\mathbb{R}\times\mathcal{A}\to\mathbb{R}\}$. Hence, for $f(R_t,\pi,t) = 0$ we get

$$V(t,x) = \sup_{\pi\in\mathcal{A}} E\left[V(t^*, R_{t^*})\right] \tag{6.7}$$

Proof. We denote the right hand side of (6.6) by $\tilde{V}(t,x)$. It follows from the definition of $V(t,x)$ that, for any $\pi \in \mathcal{A}$

$$V(t,x) \geq E\left[\int_t^{t^*} f(R_s,\pi,s)ds + \int_{t^*}^T f(R_s,\pi,s)ds + G(R_T)\right],$$

Since the uncertain processes $dC_s(s \in [t,t^*])$ and $dC_s(s \in [t^*,T])$ are independent, we know that

$$\int_t^{t^*} f(R_s,\pi,s)ds \qquad \text{and} \qquad \int_{t^*}^T f(R_s,\pi,s)ds$$

are independent. Thus by Theorem 2.2 we have

$$V(t,x) \geq E\left\{\int_t^{t^*} f(R_s,\pi,s)ds + E\Big[\int_{t^*}^T f(R_s,\pi,s)ds + G(R_T)\Big]\right\}.$$

Taking the supremum for the above inequality with respect to $\pi \in \mathcal{A}$, we get $V(t,x) \geq \tilde{V}(t,x)$. On the other hand, for every $\pi \in \mathcal{A}$, we have

$$\begin{aligned}
& E\left[\int_t^T f(R_s,\pi,s)ds + G(R_T)\right] \\
= \;& E\left\{\int_t^{t^*} f(R_s,\pi,s)ds + E\Big[\int_{t^*}^T f(R_s,\pi,s)ds + G(R_T)\Big]\right\} \\
\leq \;& E\left[\int_t^{t^*} f(R_s,\pi,s)ds + V(t^*,R_{t^*})\right] \\
\leq \;& \tilde{V}(t,x).
\end{aligned}$$

Hence $V(t,x) \leq \tilde{V}(t,x)$, thus $V(t,x) = \tilde{V}(t,x)$ and (6.7) follows when $f(R_t,\pi,t) = 0$. □

Now define the differential operator $\mathbb{L}^a$ by

$$\mathbb{L}^a V(t,x) = \frac{a^2\sigma_1^2}{2}V_{xx} + [\mu a + \rho x]V_x + V_t - cV.$$

Theorem 6.2. *(Equation of Optimality)*

The value function V is a solution of the following terminal problem of the Equation of Optimality,

$$\sup_{\pi \in \mathcal{A}} \Big\{ \mathbb{L}^a V(t,x) \Big\} = 0 \tag{6.8}$$

with terminal condition that $V(T,x) = G(R_T) = \sum_{k=1}^{\infty} e^{-c\tau_k} g(\xi_k) I_{\{\tau_k < \tau\}}$, where $\xi_i = R(\tau_i^-) - R(\tau_i)$.

Proof. Let $R_t < \beta < +\infty$ for $t \in [0,T]$ and define $\tau_j = \inf\{t \geq 0 : R_t = \beta\}$ for $j = 0, 1, 2, \ldots$

Define $\tau_t^* = \max\{\tau_j : \tau_j < t\}$ noting that $\tau_t^* \to \infty$ as $t \to \infty$ $a.s.$. Consider an arbitrarily chosen impulse control

$$\vartheta = (\tau_1, \tau_2, \ldots, \tau_n, \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots)$$

and let $\tau_0 = 0$. Choose $a \in [0,1]$ and put $R_t^\pi = R_t$. Since V' is bounded on $(0,\infty)$ implies that

$$E\Bigg[\int_0^\infty (V'(R_t))^2 \sigma_1^2 a^2 dt \Bigg] < \infty.$$

For $n = 1, 2, 3, \ldots$ and for every $t, t^* \in [0,T]$ from Theorem 2.3 we obtain

$$\begin{aligned} & V(t^* \wedge \tau^* \wedge n, R_{t^* \wedge \tau^* \wedge n}) - V(t,x) \qquad (6.9) \\ = & \int_t^{t^* \wedge \tau^* \wedge n} \mathbb{L}^a V(s, R_s) ds + \int_t^{t^* \wedge \tau^* \wedge n} \sigma_1 a V'(s, R_s) dB_s + \int_t^{t^* \wedge \tau^* \wedge n} \sigma_2 a V'(s, R_s) dC_s. \end{aligned}$$

Clearly

$$E\Bigg[\int_t^{t^* \wedge \tau^* \wedge n} \sigma_1 a V'(s, R_s) dB_s \Bigg] = E\Bigg[\int_t^{t^* \wedge \tau^* \wedge n} \sigma_2 a V'(s, R_s) dC_s \Bigg] = 0.$$

Thus taking expectation of both sides of (6.9) and letting $n \to \infty$ and $\beta \uparrow \infty$, we obtain $\tau^* \to T$. Thus

$$E\Big[V(t^*, R_{t^*}) \Big] = V(t,x) + E\Bigg[\int_t^{t^*} \mathbb{L}^a V(s, R_s) ds \Bigg]. \tag{6.10}$$

By using the principle of optimality, Theorem 6.1 and (6.10) we obtain

$$0 = \sup_{\pi \in \mathcal{A}} E\left[\int_t^{t^*} \mathbb{L}^a V(s, R_s)ds\right]. \tag{6.11}$$

Dividing (6.11) by $t^* - t$, and letting $t^* \to t$ we get the required results. The terminal condition holds obviously. $\square$

6.4 Solution of the Optimal Control Problem

We easily deduce from the equation of optimality, Theorem 6.2 that the value function V satisfies the HJB equation as follows

$$\max_{a \in [0,1]} \left\{ \frac{a^2\sigma_1^2}{2} V_{xx} + [\mu a + \rho x]V_x + V_t - cV \right\} = 0. \tag{6.12}$$

Letting $a(x)$ be the maximizer of (6.12) over all $a \in (-\infty, \infty)$ we obtain,

$$a(x) = -\frac{\mu^2 V_x^2}{2\sigma_1^2 V_{xx}}. \tag{6.13}$$

For those x for which the argmaximum of the left-hand side of (6.12) does not coincide with the end points of the interval $[0,1]$, we can substitute the expression (6.13) into (6.12) to get

$$-\frac{\mu^2 V_x^2}{2\sigma_1^2 V_{xx}} + \rho x V_x + V_t - cV = 0. \tag{6.14}$$

A general solution of this equation subject to the terminal condition that $V(T,x) = G(R_T) = \sum_{k=1}^{\infty} e^{-c\tau_k} g(\xi_k) I_{\{\tau_k < \tau\}}$ is given by

$$V(t,x) = A(t)x^\lambda, \tag{6.15}$$

where $A(T) = 1$ and $\lambda \in (0,1)$. This implies that $V_t = A'x^\lambda$, $V_x = A\lambda x^{\lambda-1}$ and $V_{xx} = A\lambda(\lambda-1)x^{\lambda-2}$ which after substituting into (6.14) we get

$$-\frac{\mu^2}{2\sigma_1^2}\frac{\lambda}{\lambda-1} + \rho\lambda + \frac{A'}{A} - c = 0. \tag{6.16}$$

A solution to (6.16) is given by

$$A(t) = e^{-(c+\frac{\mu^2}{2\sigma_1^2}\frac{\lambda}{\lambda-1}-\rho\lambda)(T-t)}. \tag{6.17}$$

Hence

$$V(t,x) = x^{\lambda} e^{-(c+\frac{\mu^2}{2\sigma_1^2}\frac{\lambda}{\lambda-1}-\rho\lambda)(T-t)}, \tag{6.18}$$

with

$$\hat{a}(x) = \frac{\mu x}{\sigma_1^2(1-\lambda)}. \tag{6.19}$$

This solution is only valid when the increasing function $a(x) \leq 1$, that is if and only if $x \leq x^*$, where

$$x^* := \frac{\sigma_1^2}{\mu}(1-\lambda). \tag{6.20}$$

Thus for $x > x^*$ equation (6.18) is not valid, in this case we have $a(x) > 1$. However the maximization in (6.12) must be taken over $a(x) \in [0,1]$. Hence for $x > x^*$ we must have $a(x) = 1$, and (6.12) becomes

$$\frac{\sigma_1^2}{2}V_{xx} + [\mu + \rho x]V_x + V_t - cV = 0. \tag{6.21}$$

We will make use of the Feynman-Kac formulae to connect the PDE (6.21) to uncertainty theory to solve for the value function V.

Definition 6.5. Let X_t be an uncertain process, then the generator A of X_t is

$$Af(X) = \lim_{t \to 0^-} \frac{E_x\big[f(X_t)\big] - f(X_t)}{t} \quad \text{for all} \quad x \in \mathbb{R}. \tag{6.22}$$

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the limit exists at x is denoted by $\mathcal{D}_A(x)$ on the other hand $\mathcal{D}_A$ denote the set of functions for which the limit exists for all $x \in \mathbb{R}^n$.

Theorem 6.3. *(The Feynman-Kac Formula)*
Let $f \in \mathcal{C}^2(\mathbb{R}^n)$ and let $q \in \mathcal{C}(\mathbb{R}^n)$. Assume that q is lower bounded and put

$$v(t,x) = E_x\left[e^{-\int_0^t q(X_s)ds} f(X_t)\right]. \tag{6.23}$$

Then

$$\frac{\partial v}{\partial t} = Av - qv, \qquad t > 0 \quad x \in \mathbb{R}^n$$
$$v(0,x) = f(x) \qquad x \in \mathbb{R}^n$$

Proposition 6.1. *The value function V for the controlled process in the intervention region region $x^* < R_t$ subject to the initial condition $V(0,x) = V(x)$ and terminal condition $V(T,x) = G(R_T) = \sum_{k=1}^{\infty} e^{-c\tau_k} g(\xi_k) I_{\{\tau_k < \tau\}}$ is given by*

$$V(t,x) = e^{-ct} \int_0^1 V\Big(-\frac{\mu}{\rho} + e^{\rho t}\big(x + \frac{\mu}{\rho}\big) + \frac{\sigma_1 \sqrt{3}}{\rho\pi}(e^{\rho t} - 1) \ln \frac{\alpha}{1-\alpha}\Big) d\alpha \tag{6.24}$$

where $\alpha \in (0,1)$ is the α-path of the uncertain differential equation given by

$$dR_t = (\mu + \rho R_t)dt + \sigma_1 dX_t \tag{6.25}$$

Proof. In connection with the optimal impulse control in the intervention region we have PDE (6.21) which by the Feynman-Kac formula can be expressed as

$$\frac{\sigma_1^2}{2} V_{xx} + [\mu + \rho x]V_x - cV = AV - qV.$$

When $q = c$ we get

$$\frac{\sigma_1^2}{2} V_{xx} + [\mu + \rho x]V_x = AV.$$

An application of the generator for uncertain process gives

$$dR_t = (\mu + \rho R_t)dt + \sigma_1 dX_t \tag{6.26}$$

where $X_t = (B_t, C_t)$. We can solve (6.26) by using Theorem 14.3 of Liu [41] and get

$$R_t = e^{\rho t}\left(x + \int_0^t \mu e^{-\rho s} ds + \int_0^t \sigma_1 e^{-\rho s} dX_s \right).$$

A simplification of the above solution gives

$$R_t = -\frac{\mu}{\rho} + e^{\rho t}\left(x + \frac{\mu}{\rho} \right) + \sigma_1 \int_0^t e^{\rho(t-s)} dX_s, \tag{6.27}$$

with R_t being an uncertain normal variable, i.e.,

$$R_t \sim \mathcal{N}\bigg(-\frac{\mu}{\rho} + e^{\rho t}\Big(x + \frac{\mu}{\rho}\Big), (e^{\rho t} - 1)\frac{\sigma_1}{\rho}\bigg).$$

The Feynman-Kac formula gives

$$\begin{aligned} V(t,x) &= E_x\big[e^{-\int_0^t q(R_s)ds} V(R_s)\big] \\ &= E_x\big[e^{-ct} V(R_s)\big]. \end{aligned}$$

If we apply Yao and Chen [66] Theorem on expected value of solution, which state that for a monotone function V,

$$E\big[V(X_t)\big] = \int_0^1 V(X_t^\alpha)d\alpha = \int_0^1 \Upsilon_t^{-1}(\alpha)d\alpha, \tag{6.28}$$

where $\Upsilon_t^{-1}(\alpha) = V(X_t^\alpha)$ is the inverse uncertainty distribution of $V(X_t)$, we get

$$V(t,x) = e^{-ct}\int_0^1 V\bigg(-\frac{\mu}{\rho} + e^{\rho t}\big(x + \frac{\mu}{\rho}\big) + \frac{\sigma_1\sqrt{3}}{\rho\pi}(e^{\rho t} - 1)\ln\frac{\alpha}{1-\alpha}\bigg)d\alpha.$$

□

Note that the inverse function for the reserve process is given by

$$\begin{aligned} \Upsilon_t^{-1}(\alpha) &= -\frac{\mu}{\rho} + e^{\rho t}\Big(x + \frac{\mu}{\rho}\Big) + \sigma_1\int_0^t e^{\rho(t-s)}\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}ds \\ &= -\frac{\mu}{\rho} + e^{\rho t}\Big(x + \frac{\mu}{\rho}\Big) + \frac{\sigma_1\sqrt{3}}{\rho\pi}(e^{\rho(t)} - 1)\ln\frac{\alpha}{1-\alpha}. \end{aligned}$$

Theorem 6.4. *The value function V for the proposed optimal impulse control problem, Problem 6.2.1, is given by*

$$V(t,x) = \begin{cases} x^\lambda e^{-(c+\frac{\mu^2}{2\sigma_1^2}\frac{\lambda}{\lambda-1}-\rho\lambda)(T-t)}, & \textit{if} \quad 0 < x < x^*, \\ e^{-ct}\int_0^1 V\Big(-\frac{\mu}{\rho} + e^{\rho t}\big(x + \frac{\mu}{\rho}\big) + \frac{\sigma_1\sqrt{3}}{\rho\pi}(e^{\rho t} - 1)\ln\frac{\alpha}{1-\alpha}\Big)d\alpha & \textit{if} \quad x \geq x^*, \end{cases} \tag{6.29}$$

and its corresponding optimal reinsurance control is given by

$$\hat{a}(t) = \begin{cases} \frac{\mu\hat{R}_t}{\sigma_1^2(1-\lambda)} & \textit{if} \quad 0 < \hat{R}_t < x^*, \\ 1 & \textit{if} \quad x^* < \hat{R}_t. \end{cases} \tag{6.30}$$

Proof. The result follows by combining the value function V in the continuation region from equation (6.18) with the value function V in the intervention region from equation (6.24) of Proposition 6.1. The optimal control value follows from equation (6.19). $\square$

6.5 Conclusion

In our study of combined optimal impulse control and classical control of insurance reserves, modeled by hybrid processes, we have managed to present and prove the theorem, principal of optimality and equation of optimality. The two theorems, Theorem 6.1 and Theorem 6.2, assisted us in solving the proposed problem, Problem 6.2.1, of determining the optimal value function V and the optimal control $a(t)$. We managed to find a closed form solution for the optimal control and a closed form solution for the value function.

Chapter 7

Optimal Proportional Reinsurance Policies Under Interest Rates in Itô-Liu Markets with Jump

7.1 Introduction

In this chapter, we present and investigate the optimal control problem for insurance firms when reserves are modeled by uncertain stochastic processes with an uncertain V jump. The principle of optimality and the equation of optimality are obtained for uncertain stochastic processes with a V jump. The risk of the insurance firm is controlled by using a proportional reinsurance policy. We assume that the surplus cash reserves earn interest at a constant rate $\rho > 0$. The optimal consumption rate and the optimal proportional reinsurance policy for insurance reserves under uncertain stochastic model are derived.

This chapter is motivated by the need to invite some domain experts to evaluate the belief degree when samples are not available to estimate a probability distribution. We consider the extension of the optimal control problem by proposing a model driven by Brownian motion, canonical process and V jump. The Brownian motion term measures random indeterminacy while the canonical process and the V jump measure uncertainty and uncertain jump, respectively. An uncertain stochastic process with jumps is a more realistic description of real world phenomena. For example, in many cases the stock price may jump at scheduled or unscheduled times due to a sudden shift in policy by a central bank, economic crisis, war, or any other natural disaster. These factors need to be incorporated into the modeling of uncertain events.

7.2 Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a filtered complete probability space satisfying the usual conditions endowed with a one-dimensional standard Brownian motion $B = \{B_t; t \in [0, T]\}$ adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ where $T \in [0, \infty)$ is a time horizon. The Brownian filtration $(\{\mathcal{F}_t\}_{t\in[0,T]})$ is generalized by $\sigma(B_s : s \leq t)$ and the P-null sets of $\mathcal{F}$. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and define a filtered uncertainty space $(\Gamma, \mathcal{L}, \{\mathcal{L}_t\}_{t\in[0,T]}, \mathcal{M})$ endowed with a one-dimensional standard canonical process $C = \{C_t; t \in [0, T]\}$ adapted to the filtration $\{\mathcal{L}_t\}_{t\in[0,T]}$. The canonical process filtration $\{\mathcal{L}_t\}_{t\in[0,T]}$ is generalized by $\sigma(C_s : s \leq t)$ and $\mathcal{M}$-null sets of $\mathcal{L}$ where the canonical process C_t is defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. For related properties of the canonical process filtration, the reader is referred to Fei [21]. We will make use of the following result about uncertain variables in our analysis of the optimal control problem.

Theorem 7.1. *(Zhu [70]) Let Λ be a normally distributed uncertain variable with the uncertain distribution*

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R}.$$

Then for any real number a,

$$\frac{\sigma^2}{2} \leq E[a\Lambda + \Lambda^2] \leq \sigma^2.$$

Proof. We only need to consider the case $a > 0$ as a similar method is employed when $a \leq 0$. Let

$$x_1 = \frac{-a - \sqrt{a^2 + 4r}}{2}, \qquad x_2 = \frac{-a + \sqrt{a^2 + 4r}}{2},$$

which is derived from the solution of the equation $ax + x^2 = r$ for any real number r. Denote the minimum value of $ax + x^2$ by $y_0 = -\frac{a^2}{4}$. Then

$$\begin{aligned}
& E[a\Lambda + \Lambda^2] \\
= & \int_0^{\infty} \mathcal{M}\{a\Lambda + \Lambda^2 \geq r\}dr - \int_{y_0}^{0} \mathcal{M}\{a\Lambda + \Lambda^2 \leq r\}dr \\
= & \int_0^{\infty} \mathcal{M}\{(\Lambda \leq x_1) \cup (\Lambda \geq x_2)\}dr - \int_{y_0}^{0} \mathcal{M}\{(\Lambda \geq x_1) \cap (\Lambda \leq x_2)\}dr. \quad (7.1)
\end{aligned}$$

Because

$$\begin{aligned}
\mathcal{M}\{\Lambda \leq x_2\} &= \mathcal{M}\{[(\Lambda \geq x_1) \cap (\Lambda \leq x_2)] \cup (\Lambda \leq x_1)\} \\
&\leq \mathcal{M}\{(\Lambda \geq x_1) \cap (\Lambda \leq x_2)\} + \mathcal{M}\{(\Lambda \leq x_1)\},
\end{aligned}$$

we have

$$\mathcal{M}\{(\Lambda \geq x_1) \cap (\Lambda \leq x_2)\} \geq \mathcal{M}\{\Lambda \leq x_2\} - \mathcal{M}\{\Lambda \leq x_1\} = \Phi(x_2) - \Phi(x_1).$$

We note that

$$\mathcal{M}\{(\Lambda \leq x_1) \cup (\Lambda \geq x_2)\} \leq \mathcal{M}\{\Lambda \leq x_1\} + \mathcal{M}\{\Lambda \geq x_2\} = \Phi(x_1) + 1 - \Phi(x_2).$$

Hence, it follows from (7.1) that

$$
\begin{aligned}
E[a\Lambda+\Lambda^2] &\le \int_0^{\infty}\Phi(x_1)dr+\int_0^{\infty}[1-\Phi(x_2)]dr-\int_{y_0}^{0}\big[\Phi(x_2)-\Phi(x_1)\big]dr\\
&= \int_0^{\infty}\frac{1}{1+\exp\left(-\frac{\pi x_1}{\sqrt{3}\sigma}\right)}dr+\int_0^{\infty}\frac{1}{1+\exp\left(\frac{\pi x_2}{\sqrt{3}\sigma}\right)}dr\\
&\quad -\int_{y_0}^{0}\frac{1}{1+\exp\left(-\frac{\pi x_2}{\sqrt{3}\sigma}\right)}dr+\int_{y_0}^{0}\frac{1}{1+\exp\left(-\frac{\pi x_1}{\sqrt{3}\sigma}\right)}dr\\
&= \int_{-a}^{-\infty}\frac{a+2x}{1+\exp\left(-\frac{\pi x}{\sqrt{3}\sigma}\right)}dx+\int_0^{\infty}\frac{a+2x}{1+\exp\left(\frac{\pi x}{\sqrt{3}\sigma}\right)}dx\\
&\quad -\int_{-\frac{a}{2}}^{0}\frac{a+2x}{1+\exp\left(-\frac{\pi x}{\sqrt{3}\sigma}\right)}dx+\int_{-\frac{a}{2}}^{-a}\frac{a+2x}{1+\exp\left(-\frac{\pi x}{\sqrt{3}\sigma}\right)}dx\\
&= a\int_0^{a}\frac{1}{1+\exp\left(\frac{\pi x}{\sqrt{3}\sigma}\right)}dx+2\int_a^{\infty}\frac{x}{1+\exp\left(\frac{\pi x}{\sqrt{3}\sigma}\right)}dx\\
&\quad +2\int_0^{\infty}\frac{x}{1+\exp\left(\frac{\pi x}{\sqrt{3}\sigma}\right)}dx-\int_0^{a}\frac{a-2x}{1+\exp\left(\frac{\pi x}{\sqrt{3}\sigma}\right)}dx\\
&= 4\int_0^{\infty}\frac{x}{1+\exp\left(\frac{\pi x}{\sqrt{3}\sigma}\right)}dx\\
&= \sigma^2. \qquad (7.2)
\end{aligned}
$$

On the other hand, because

$$\mathcal{M}\{(\Lambda\le x_1)\cup(\Lambda\ge x_2)\}\ge\mathcal{M}\{\Lambda\ge x_2\}=1-\Phi(x_2),$$

and

$$\mathcal{M}\{(\Lambda\ge x_1)\cap(\Lambda\le x_2)\}\le\mathcal{M}\{\Lambda\le x_2\}=\Phi(x_2),$$

we have from (7.1) that

$$
\begin{aligned}
E[a\Lambda+\Lambda^2] &\geq \int_0^{\infty}[1-\Phi(x_2)]dr-\int_{y_0}^{0}\Phi(x_2)dr \\
&= \int_0^{\infty}\frac{1}{1+\exp\left(\frac{\pi x_2}{\sqrt{3}\sigma}\right)}dr-\int_{y_0}^{0}\frac{1}{1+\exp\left(-\frac{\pi x_2}{\sqrt{3}\sigma}\right)}dr \\
&= \int_0^{\infty}\frac{a+2x}{1+\exp\left(\frac{\pi x_2}{\sqrt{3}\sigma}\right)}dx-\int_{-\frac{a}{2}}^{0}\frac{a+2x}{1+\exp\left(-\frac{\pi x_2}{\sqrt{3}\sigma}\right)}dx \\
&= 2\int_0^{\infty}\frac{x}{1+\exp\left(\frac{\pi x_2}{\sqrt{3}\sigma}\right)}dx+\int_0^{\frac{a}{2}}\frac{x}{1+\exp\left(-\frac{\pi x_2}{\sqrt{3}\sigma}\right)}dx \\
&\quad +a\int_{\frac{a}{2}}^{\infty}\frac{1}{1+\exp\left(-\frac{\pi x_2}{\sqrt{3}\sigma}\right)}dx \\
&= \frac{\sigma^2}{2}+\frac{6\sigma^2}{\pi^2}\int_0^{\frac{a\pi}{2\sqrt{3}\sigma}}\frac{z}{1+e^z}dz+\frac{\sqrt{3}a\sigma}{\pi}\int_{\frac{a\pi}{2\sqrt{3}\sigma}}^{\infty}\frac{1}{1+e^z}dz \\
&\geq \frac{\sigma^2}{2}. \qquad (7.3)
\end{aligned}
$$

Combining (7.2) and(7.3) yields the result. □

Theorem 7.2. *(Liu [42]) Let f be a convex function on $[a,b]$, and let Λ be an uncertain variable that takes values in $[a,b]$ and has expected value e. Then*

$$E[f(\Lambda)]\leq\frac{b-e}{b-a}f(a)+\frac{e-a}{b-a}f(b). \qquad (7.4)$$

Proof. For each $\gamma\in\Gamma$, we have

$$a\leq\Gamma(\gamma)\leq b \qquad \text{and}$$

$$\Gamma=\frac{b-\Gamma}{b-a}a+\frac{\Gamma-a}{b-a}b.$$

It follows from the convexity of f that

$$\Gamma=\frac{b-\Gamma}{b-a}f(a)+\frac{\Gamma-a}{b-a}f(b).$$

Taking expected values on both sides, we obtain the inequality. □

7.3 Jump Uncertainty

Definition 7.1. (Deng and Zhu [19]) An uncertain variable $Z(r_1, r_2, t)$ is said to be a jump uncertain variable with parameters r_1 and r_2, for $(0 < r_1 < r_2 < 1)$ and for $t > 0$ if it has a jump uncertain distribution

$$\Phi(x) = \begin{cases} 0, & \text{if} \quad x < 0, \\ \frac{2r_1}{t}x, & \text{if} \quad 0 \leq x < \frac{t}{2}, \\ r_2 + \frac{2(1-r_2)}{t}\left(x - \frac{t}{2}\right), & \text{if} \quad \frac{t}{2} \leq x < t, \\ 1 & \text{if} \quad x \geq t, \end{cases} \tag{7.5}$$

The uncertain distribution Φ of a Z jump uncertain variable has a discontinuous point at which the value of Φ has a jump with step $r_2 - r_1$.
A jump uncertain process is defined by a Z jump uncertain variable as follows.

Definition 7.2. (Deng and Zhu [19]) An uncertain process V_t is said to be a V jump process with parameters r_1 and r_2, for $(0 < r_1 < r_2 < 1)$ and for $t \geq 0$ if:

(i) $V_0 = 0$

(ii) V_t has stationary and independent increments,

(iii) every increment $V_{s+t} - V_s$ is a Z jump uncertain variable $Z(r_1, r_2, t)$.

Lemma 7.1. *Let V_t be a V jump uncertain process, and $\Delta V_t = V_{t+\Delta t} - V_t$. Then*

$$E[\Delta V_t] = \frac{3 - r_1 - r_2}{4}\Delta t. \tag{7.6}$$

Proof.

$$\begin{aligned} E[\Delta V_t] &= \int_0^{\infty} (1 - \Phi(x))dx \\ &= \int_0^{\Delta t} (1 - \Phi(x))dx \\ &= \int_0^{\frac{\Delta t}{2}} \left(1 - \frac{2r_1 x}{\Delta t}\right)dx + \int_{\frac{\Delta t}{2}}^{\Delta t} \left(1 - r_2 - \frac{2(1-r_2)}{\Delta t}\left(x - \frac{\Delta t}{2}\right)\right)dx \\ &= \frac{3 - r_1 - r_2}{4}\Delta t. \end{aligned}$$

□

The following theorem, which is presented for the first time for uncertain stochastic processes with uncertain jump, will play a critical role in the formulation of the equation of optimality.

Theorem 7.3. *Let B_t be a Brownian motion, C_t be an uncertain canonical process, and V_t be a V jump uncertain process. Denote $\Lambda = b\theta + d\eta + n\zeta$, where $\theta = \Delta B_t$, $\eta = \Delta C_t$, $\zeta = \Delta V_t$, $b, d, n \in \mathbb{R}$. Also let θ, η and ζ be independent. Then for any real numbers a and m,*

$$E[a\Lambda + m\Lambda^2] = \frac{an(3 - r_1 - r_2)}{4}\Delta t + 3mb^2\Delta t + o(\Delta t). \tag{7.7}$$

Proof. We note that

$$\begin{aligned} E[a\Lambda + m\Lambda^2] &\geq aE[\Lambda] + mb^2\Delta t \\ &= aE[b\theta + d\eta + n\zeta] + mb^2\Delta t \\ &= \frac{an(3 - r_1 - r_2)}{4}\Delta t + mb^2\Delta t. \end{aligned} \tag{7.8}$$

On the other hand, noting that $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ we get that

$$\begin{aligned} a\Lambda + m\Lambda^2 &= a(b\theta + d\eta + n\zeta) + m(b\theta + d\eta + n\zeta)^2 \\ &\leq a(b\theta + d\eta + n\zeta) + 3m(b^2\theta^2 + d^2\eta^2 + n^2\zeta^2) \\ &= (ab\theta + 3mb^2\theta^2) + (ad\eta + 3md^2\eta^2) + (an\zeta + 3mn^2\zeta^2). \end{aligned}$$

Since $(ab\theta + 3mb^2\theta^2)$, $(ad\eta + 3md^2\eta^2)$ and $(an\zeta + 3mn^2\zeta^2)$ are independent, we have

$$E[a\Lambda + m\Lambda^2] \leq E[ab\theta + 3mb^2\theta^2] + E[ad\eta + 3md^2\eta^2] + E[an\zeta + 3mn^2\zeta^2].$$

From Theorem 7.1 we have that

$$E[ad\eta + 3md^2\eta^2] = o(\Delta t).$$

Clearly

$$E[ab\theta + 3mb^2\theta^2] = 3mb^2\Delta t.$$

By Theorem 7.2 we have that

$$\begin{aligned} E[an\zeta + 3mn^2\zeta^2] &\leq \frac{E[\zeta]}{\Delta t}\Big(an\Delta t + 3mn^2(\Delta t)^2\Big) \\ &= \frac{3 - r_1 - r_2}{4}\Big(an\Delta t + 3mn^2(\Delta t)^2\Big) \\ &= \frac{an(3 - r_1 - r_2)}{4}\Delta t + o(\Delta t) \end{aligned}$$

Hence

$$E[a\Lambda + m\Lambda^2] \leq \frac{an(3 - r_1 - r_2)}{4}\Delta t + 3mb^2\Delta t + o(\Delta t). \quad (7.9)$$

Combining the inequality (7.8) and inequality (7.9), we can obtain

$$E[a\Lambda + m\Lambda^2] = \frac{an(3 - r_1 - r_2)}{4}\Delta t + 3mb^2\Delta t + o(\Delta t).$$

□

7.4 Uncertain Stochastic Optimal Control with Uncertain Jumps

In this section, we adopt the expected value-based method as in [70] for the uncertain stochastic optimal control problem with jumps. The uncertain stochastic optimal control problem with jumps is to find the optimal decision such that some objective functions subjected to uncertain stochastic process with jumps provided by uncertain stochastic differential equation with jumps is optimized. Unless stated otherwise, we assume that B_t is a one-dimensional standard Brownian motion, C_t is a one-dimensional standard canonical process and V_t a V jump uncertain process with parameters r_1 and r_2 $(0 < r_1 < r_2 < 1)$, and ΔB_t, ΔC_t and ΔV_t are independent. Suppose that if we apply the control process $u = u(t) \in \mathcal{U}$ the state of a system at time t is described by a controlled uncertain stochastic process X_t^u, with a jump of the form

$$\begin{aligned} dX_t^u &= \mu(t, X_t^u, u)ds + \sigma(t, X_t^u, u)dB_t + \gamma(t, X_t^u, u)dC_t + \chi(t, X_t^u, u)dV_t \\ X_0^u &= x_0 \end{aligned} \quad (7.10)$$

where $X_t \in \mathbb{R}$, $\mu : [0,T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$, $\gamma : [0,T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ and $\chi : [0,T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$. It is considered that μ, σ, γ and χ satisfy the conditions for the existence and uniqueness of a solution for (7.10). The function $u \in \mathcal{U} \subset \mathbb{R}$ is the decision (control) variable whose value can be chosen in the given set $\mathcal{U}$ at any time t in order to control the process X_t.

For any time $t \in (0,T)$, let $J(t,x)$ denote the expected optimal reward obtainable in $[t,T]$ with the condition that at time t we are in state $X_t = x$. We consider the following hybrid optimization problem.

Problem 7.4.1. *The problem is to find the value* $J(t,x)$ *and the optimal control* u^* *such that*

$$J(t,x) = \sup_u E\left[\int_t^T f(s,X_s,u)ds + G(X(T),T)\right] = E\left[\int_t^T f(s,X_s,u^*)ds + G(X(T),T)\right] \tag{7.11}$$

subject to

$$\begin{aligned} dX_t &= \mu(t,X_t,u)ds + \sigma(t,X_t,u)dB_t + \gamma(t,X_t,u)dC_t + \chi(t,X_t,u)dV_t \\ X_t &= x \end{aligned} \tag{7.12}$$

where without loss of generality we have let $X_t^u = X_t$ *for every* $t \in [0,T]$.

To solve problem 7.4.1 the following principle of optimality for uncertain stochastic optimal control is considered.

Theorem 7.4. *(Principle of Optimality)*
For any $(t,x) \in [0,T] \times \mathbb{R}$ *and every* $\Delta t > 0$, *with* $t + \Delta t < T$ *we have*

$$J(t,x) = \sup_u E\left[\int_t^{t+\Delta t} f(s,X_s,u)ds + J(t+\Delta t, x+\Delta X_t)\right] \tag{7.13}$$

where $x + \Delta X_t = X_{t+\Delta t}$.

Proof. We denote the right hand side of (7.13) by $\tilde{J}(t,x)$. It follows from the definition of $J(t,x)$ that,

$$J(t,x) \geq E\left[\int_t^{t+\Delta t} f(s,X_s,u|_{[t,t+\Delta t)})ds + \int_{t+\Delta t}^T f(s,X_s,u|_{[t+\Delta t,T]})ds + G(X(T),T)\right],$$

where $u|_{[t,t+\Delta t)}$ and $u|_{[t+\Delta t,T]}$ are the values of decision u restricted on $[t, t+\Delta t)$ and $[t+\Delta t, T]$ respectively. Since the uncertain processes $dC_s(s \in [t, t+\Delta t])$ and $dC_s(s \in [t+\Delta t, T])$ are independent, we know that

$$\int_t^{t+\Delta t} f(s, X_s, u|_{[t,t+\Delta t)})ds \qquad \text{and} \qquad \int_{t+\Delta t}^{T} f(s, X_s, u|_{[t+\Delta t,T]})ds$$

are independent. Thus by Theorem 2.2 we have

$$J(t,x) \geq E\left\{ \int_t^{t+\Delta t} f(s, X_s, u|_{[t,t+\Delta t)})ds + E\left[\int_{t+\Delta t}^{T} f(s, X_s, u|_{[t+\Delta t,T]})ds + G(X(T), T) \right] \right\}.$$

Taking the supremum for the above inequality with respect to $u|_{[t,t+\Delta t)}$ followed by supremum with respect to $u|_{[t+\Delta t,T]}$, we get $J(t,x) \geq \tilde{J}(t,x)$. On the other hand, for every u, we have

$$\begin{aligned}
& E\left[\int_t^T f(s, X_s, u)ds + G(X(T), T) \right] \\
= \quad & E\left\{ \int_t^{t+\Delta t} f(s, X_s, u|_{[t,t+\Delta t)})ds + E\left[\int_{t+\Delta t}^{T} f(s, X_s, u|_{[t+\Delta t,T]})ds + G(X(T), T) \right] \right\} \\
\leq \quad & E\left[\int_t^{t+\Delta t} f(s, X_s, u)ds + J(t+\Delta t, x+\Delta X_t) \right] \\
\leq \quad & \tilde{J}(t,x).
\end{aligned}$$

Hence $J(t,x) \leq \tilde{J}(t,x)$, thus $J(t,x) = \tilde{J}(t,x)$ as required. □

7.5 Equation of Optimality for Uncertain Stochastic Processes with Jump.

We now give a fundamental result called the equation of optimality for hybrid processes with uncertain V jump.

Theorem 7.5. *(Equation of optimality) Let $J(t,x)$ be twice differentiable on*

$[0, T] \times \mathbb{R}$. *Then we have*

$$-J_t = \sup_u \left\{ f(t,x,u) + J_x \mu(t,x,u) + \frac{3}{2} J_{xx} \sigma^2(t,x,u) + \frac{3 - r_1 - r_2}{4} \chi(t,x,u) J_x \right\} \tag{7.14}$$

where $J_t = J_t(t,x)$, $J_x = J_x(t,x)$ *and* $J_{xx} = J_{xx}(t,x)$ *are the partial derivatives of the function* $J(t,x)$ *with respect to* t *and* x.

Proof. For any $\Delta t > 0$, we have

$$\int_t^{t+\Delta t} f(s, X_s, u) ds = f(t,x,u)\Delta t + o(\Delta t). \tag{7.15}$$

By Taylor series expansion, we have

$$\begin{aligned} J(t+\Delta t, x + \Delta X_t) &= J(t,x) + J_t(t,x)\Delta t + J_x(t,x)\Delta X_t + \frac{1}{2} J_{tt}(t,x)\Delta t^2 \\ &\quad + \frac{1}{2} J_{xx}(t,x)\Delta X_t^2 + J_{tx}(t,x)\Delta t \Delta X_t + o(\Delta t) \end{aligned} \tag{7.16}$$

Substituting Equation (7.15) and (7.16) into Equation (7.13) yield

$$\begin{aligned} 0 &= \sup_u \Big\{ f(t,x,u)\Delta t + J_t(t,x)\Delta t + E\big[J_x(t,x)\Delta X_t + \frac{1}{2} J_{tt}(t,x)\Delta t^2 \\ &\quad + \frac{1}{2} J_{xx}(t,x)\Delta X_t^2 + J_{tx}(t,x)\Delta t \Delta X_t\big] + o(\Delta t) \Big\} \end{aligned} \tag{7.17}$$

Let Λ be a hybrid variable such that $\Delta X_t = \Lambda + \mu(t,x,u)\Delta t$. It follows from (7.17) that

$$\begin{aligned} 0 &= \sup_u \Big\{ f(t,x,u)\Delta t + J_t(t,x)\Delta t + J_x(t,x)\mu(t,x,u)\Delta t \\ &\quad + E\Big[\big(J_x(t,x) + J_{xx}(t,x)\mu(t,x,u)\Delta t + J_{tx}(t,x)\Delta t\big)\Lambda + \frac{1}{2} J_{xx}(t,x)\Lambda^2\Big] + o(\Delta t) \Big\} \\ 0 &= \sup_u \Big\{ f(t,x,u)\Delta t + J_t(t,x)\Delta t + J_x(t,x)\mu(t,x,u)\Delta t \\ &\quad + E\big[a\Delta + m\Delta^2\big] + o(\Delta t) \Big\} \end{aligned} \tag{7.18}$$

where $a = J_x(t,x) + J_{xx}(t,x)\mu(t,x,u)\Delta t + J_{tx}(t,x)\Delta t$ and $m = \frac{1}{2} J_{xx}(t,x)$. It follows from the hybrid differential equation that

$\Lambda = \Delta X_t - \mu(t,x,u)\Delta t = \sigma(t,x,u)\Delta B_t + \gamma(t,x,u)\Delta C_t + \chi(t,x,u)\Delta V_t$ is a hybrid variable. Theorem 7.3 implies that

$$\begin{aligned} E\big[a\Delta + m\Delta^2\big] &= \frac{an(3-r_1-r_2)}{4}\Delta t + 3mb^2\Delta t + o(\Delta t) \\ &= \frac{(3-r_1-r_2)}{4}\chi(t,x,u)J_x(t,x)\Delta t \\ &\quad + \frac{3}{2}J_{xx}(t,x)\sigma^2(t,x,u)\Delta t + o(\Delta t). \end{aligned} \tag{7.19}$$

Substituting Equation (7.19) into Equation (7.18) yields

$$\begin{aligned} -J_t(t,x,u) &= \sup_u \Big\{ f(t,x,u)\Delta t + J_x(t,x)\mu(t,x,u)\Delta t + \frac{3}{2}J_{xx}(t,x)\sigma^2(t,x,u)\Delta t \\ &\quad + \frac{3-r_1-r_2}{4}\chi(t,x,u)J_x(t,x)\Delta t + o(\Delta t)\Big\} \end{aligned} \tag{7.20}$$

Dividing Equation (7.20) by Δt, and letting $\Delta t \to 0$, we obtain the result (7.14) which completes the proof of the theorem. □

In the next section we apply the equation of optimality in investigating the problem of optimal control of insurance reserves.

7.6 Optimal Control of Insurance Reserves

Let $(\Gamma \times \Omega, \mathcal{L} \otimes \mathcal{F}, \{\mathcal{L}_t \otimes \mathcal{F}_t\}_{t\in[0,T]}, \mathcal{M} \times P)$ be a filtered complete uncertain space satisfying the usual conditions endowed with a standard Brownian motion $B = \{B_t; t \in [0,T]\}$ adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, a standard canonical process $C = \{C_t; t \in [0,T]\}$ adapted to the filtration $\{\mathcal{L}_t\}_{t\in[0,T]}$ and V_t a V jump uncertain process with parameters r_1 and r_2 $(0 < r_1 < r_2 < 1)$. The filtration $\{\mathcal{L}_t \otimes \mathcal{F}_t\}$ represent the information available at time t for the uncertain stochastic process $X_t = (B_t, C_t)$ and any control made is based on this information. Our state variable is the reserve process $R = \{R_t; t \in [0,T]\}$ where $T \in [0,\infty)$ is a time horizon. The reserve process R_t represents the liquid assets of an insurance company. We let $a(t) = a_t \in [0,1]$ be a classical control which represents the retention rate at time t and let $q(t) = q_t \in [0,\infty)$ represent the consumption rate of the reserve process. We assume that the reserves earn interest at a constant force $\rho > 0$. The dynamics of the controlled process is given by,

$$dR_t^{a,q} = [a_t\mu + \rho R_t - q_t]dt + \sigma_1 a_t dB_t + \sigma_2 a_t dC_t + \sigma_3 a_t dV_t, \qquad (7.21)$$
$$R_t = x,$$

where $R_t^{a,q}$ is the controlled reserve process, $\mu > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_3 > 0$ being constant. The uncertainty term C_t represents incurred but not reported reserves and the belief degree at which uncertain events will occur, the indeterminate term B_t represents the randomness of the reserve process while the uncertain jump term V_t represents a jump in insurance reserves caused by economic crises, war, announcements of monetary policy, and so on.

Definition 7.3. A classical control $a = \{a(t)\}_{t\in[0,T]}$ is an $\{\mathcal{L}_t \otimes \mathcal{F}_t\}$-adapted uncertain stochastic process such that $a(t) \in [0, 1]$.

The following optimal control problem for insurance reserves is considered:

$$\begin{cases} J(t,x) & = \sup_{a_t\in[0,1]} E\left[\int_0^T e^{-ct}\frac{q(t)^\lambda}{\lambda}dt + (R_T - b)\right] \\ & \text{subject to} \\ dR_t & = [a_t\mu + \rho R_t - q_t]dt + \sigma_1 a_t dB_t + \sigma_2 a_t dC_t + \sigma_3 a_t dV_t \end{cases} \qquad (7.22)$$

where $R_t = R_t^{a,q}$, the constant $b > 0$ represents transactional costs, c and λ are given constants such that $c > 0$ and $\lambda \in (0, 1)$.

Theorem 7.6. *The optimal consumption rate and the optimal retention rate for insurance reserves under uncertain stochastic model with uncertain jump is given by,*

$$q(x) = x\big(A\lambda\big)^{\frac{1}{\lambda-1}} \qquad \textit{and} \qquad a(x) = \frac{x}{3\sigma_1^2(1-\lambda)}\left[1 + \frac{3 - r_1 - r_2}{4}\sigma_3\right],$$

where

$$A\lambda = \left\{\frac{\rho\lambda - c + \frac{2\lambda}{3\sigma_1^2(\lambda-1)}\left[1 + \frac{3-r_1-r_2}{4}\sigma_3\right]}{\lambda - 1}\right\}^{\lambda-1}. \qquad (7.23)$$

Proof. It follows from the equation of optimality (7.14) that

$$\begin{aligned} -J_t &= \max_{q,a}\left\{e^{-ct}\frac{q^\lambda}{\lambda} + (\rho x - q)J_x + kaJ_x + \frac{3a^2\sigma_1^2}{2}J_{xx}\right\} \\ &= \max_{q,a} L(q,a) \end{aligned} \tag{7.24}$$

where $L(q,a)$ represent the term enclosed by the braces and $k = (1 + \frac{3-r_1-r_2}{4}\sigma_3)$. The optimal (q,a) satisfies

$$\begin{aligned} \frac{\partial L(q,a)}{\partial q} &= e^{-ct}q^{\lambda-1} - J_x = 0 \qquad \Rightarrow q = [e^{ct}J_x]^{\frac{1}{\lambda-1}} \\ \frac{\partial L(q,a)}{\partial a} &= kJ_x + 3a\sigma_1^2 J_{xx} = 0 \qquad \Rightarrow a = -\frac{kJ_x}{3\sigma_1^2 J_{xx}}. \end{aligned}$$

Substituting the preceding results into Equation (7.24) yields

$$-J_t = e^{-ct}\frac{\left(e^{ct}J_x\right)^{\frac{\lambda}{\lambda-1}}}{\lambda} + \left(\rho x - \left(e^{ct}J_x\right)^{\frac{1}{\lambda-1}}\right)J_x - \frac{k^2J_x^2}{3\sigma_1^2 J_{xx}} + \frac{k^2J_x^2}{6\sigma_1^2 J_{xx}}$$

or

$$-J_t e^{ct} = \frac{\left(e^{ct}J_x\right)^{\frac{\lambda}{\lambda-1}}}{\lambda} + \left(\rho x - \left(e^{ct}J_x\right)^{\frac{1}{\lambda-1}}\right)J_x e^{ct} + \frac{2k^2J_x^2}{3\sigma_1^2 J_{xx}}e^{ct}. \tag{7.25}$$

We conjecture that the solution to the above nonlinear second-order-differential equation (7.25) is of the form $J(t,x) = Ax^\lambda e^{-ct}$ with the terminal condition that $J(T,x) = R_T - b$. It follows that $J_t = -Acx^\lambda e^{-ct}$, $J_x = A\lambda x^{\lambda-1}e^{-ct}$ and $J_{xx} = A\lambda(\lambda-1)x^{\lambda-2}e^{-ct}$. Substituting these identities into Equation (7.25), we get

$$Acx^\lambda = \frac{(A\lambda)^{\frac{\lambda}{\lambda-1}}}{\lambda}x^\lambda + \left(\rho A\lambda - (A\lambda)^{\frac{\lambda}{\lambda-1}}\right)x^\lambda + \frac{2A\lambda k^2}{3\sigma_1^2(\lambda-1)}x^\lambda.$$

Dividing both sides by x^λ, and rearranging we find

$$(A\lambda)^{\frac{1}{\lambda-1}} = \frac{\rho\lambda - c + \frac{2\lambda k^2}{3\sigma_1^2(\lambda-1)}}{\lambda - 1}.$$

We thus conclude that

$$A\lambda = \left\{\frac{\rho\lambda - c + \frac{2\lambda}{3\sigma_1^2(\lambda-1)}\left[1 + \frac{3-r_1-r_2}{4}\sigma_3\right]}{\lambda-1}\right\}^{\lambda-1}. \tag{7.26}$$

Using Equation (7.27) we deduce that the optimal consumption rate and the optimal retention rate is found respectively by

$$q(x) = x\big(A\lambda\big)^{\frac{1}{\lambda-1}} \qquad \text{and} \qquad a(x) = \frac{x}{3\sigma_1^2(1-\lambda)}\left[1 + \frac{3-r_1-r_2}{4}\sigma_3\right].$$

□

Theorem 7.7. *The value function* J *for the optimal control problem, Problem 7.4.1, is given by*

$$J(t,x) = \frac{x^{\lambda}e^{-ct}}{\lambda}\left\{\frac{\rho\lambda - c + \frac{2\lambda}{3\sigma_1^2(\lambda-1)}\left[1+\frac{3-r_1-r_2}{4}\sigma_3\right]}{\lambda-1}\right\}^{\lambda-1}. \tag{7.27}$$

Proof. The result follow immediately from Theorem 7.6. □

7.7 Conclusion

We have managed to present and prove the theorem, principal of optimality and equation of optimality for uncertain stochastic processes with uncertain jump size known as V jump uncertain process. As the application of the equation of optimality, a controlled insurance reserve model was discussed and the optimal policies were analytically derived.

Chapter 8

A Maximum Principle for Partial Information Forward-Backward Uncertain Stochastic Control with Application to Insurance and Finance

8.1 Introduction

In this chapter we present and investigate the maximum principle for optimal control of forward-backward uncertain stochastic differential equations (FBUSDE). The results are applied to partial information combined classical and impulse control problem for insurance reserves. The sufficient and necessary optimal condition for the local critical points of the combined classical and impulse control problem are given.

The problem of managing the operating cash to meet demand is called the cash-balance or cash management problem. This problem in insurance gives rise to backwards stochastic differential equations (BSDE) and in most cases forward backward stochastic differential equations (FBSDE) when indeterminacy is measured by randomness. An extensive study of (BSDE) and (FBSDE) can be found in Ma and Yong [63] and references therein.

This chapter of the thesis is motivated by recent publications on optimal control of (FBSDE), for example Bahlali et. al. [5], Øksendal and Sulem [50] and Wang et. al. [63] and the need to invite a domain expert to measure the belief degree. The paper by Wang et. al studies the partial information classical and impulse control problem of forward-backward systems driven by Lévy processes. They derive a maximum principle to give the sufficient and necessary optimal conditions for local critical points under random indeterminacy. This chapter on the other hand gives the sufficient and necessary optimal conditions for local critical points under uncertain random indeterminacy.

The objective of this chapter of the thesis are to add to the existing body of information on uncertain stochastic systems. We therefore introduce for the first time forward-backward uncertain stochastic differential equation and their application to the optimal control problem of insurance reserves.

8.2 Forward-Backward Uncertain Stochastic Differential Equation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a filtered probability space endowed with a m-dimensional standard Brownian motion $B = \{B_t; t \in [0, T]\}$ adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ where $T \in [0, \infty)$ is a time horizon. The Brownian filtration $(\{\mathcal{F}_t\}_{t\in[0,T]})$ is generalized by $\sigma(B_s : s \leq t)$ and the P-null sets of $\mathcal{F}$. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and define a filtered uncertainty space $(\Gamma, \mathcal{L}, \{\mathcal{L}_t\}_{t\in[0,T]}, \mathcal{M})$ endowed with a n-dimensional standard canonical process $C = \{C_t; t \in [0, T]\}$ adapted to the filtration $\{\mathcal{L}_t\}_{t\in[0,T]}$. The canonical process filtration $\{\mathcal{L}_t\}_{t\in[0,T]}$ is generalized by $\sigma(C_s : s \leq t)$ and $\mathcal{M}$-null sets of $\mathcal{L}$ where the canonical process C_t is defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. In what follows, we consider the following general forward-backward uncertain stochastic differential equation (FBUSDE), for $t \in [0, T]$,

$$\begin{aligned}
dX_t &= b(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t, Z_t)dB_t + \zeta(t, X_t, Y_t, Z_t)dC_t \\
dY_t &= g(t, X_t, Y_t, Z_t)dt + Z_t dB_t + K_t dC_t, \qquad t \in [0, T], \qquad (8.1) \\
& X_0 = x, \qquad Y(T) = h(X_T),
\end{aligned}$$

where

$$\begin{aligned}
b &= (b_1, \dots, b_p)^T : \Gamma \times \Omega \times [0, T] \times \mathbb{R}^p \times \mathbb{R}^d \times \mathbb{R}^{p\times m} \to \mathbb{R}^p \\
& \text{is} \quad \mathcal{P} \otimes \mathcal{B}_p \otimes \mathcal{B}_d \otimes \mathcal{B}_{p\times m}/\mathcal{B}_p \quad \text{measurable} \\
\sigma &= (\sigma_{kl})_{p\times a} : \Gamma \times \Omega \times [0, T] \times \mathbb{R}^p \times \mathbb{R}^d \times \mathbb{R}^{p\times m} \to \mathbb{R}^{p\times a} \\
& \text{is} \quad \mathcal{P} \otimes \mathcal{B}_p \otimes \mathcal{B}_d \otimes \mathcal{B}_{p\times m}/\mathcal{B}_{p\times a} \quad \text{measurable} \\
\zeta &= (\zeta_{kl})_{p\times m} : \Gamma \times \Omega \times [0, T] \times \mathbb{R}^p \times \mathbb{R}^d \times \mathbb{R}^{p\times m} \to \mathbb{R}^{P\times m} \qquad (8.2) \\
& \text{is} \quad \mathcal{P} \otimes \mathcal{B}_p \otimes \mathcal{B}_d \otimes \mathcal{B}_{p\times m}/\mathcal{B}_{p\times m} \quad \text{measurable} \\
g &= (g_1, \dots, g_d)^T : \Gamma \times \Omega \times [0, T] \times \mathbb{R}^p \times \mathbb{R}^d \times \mathbb{R}^{d\times m} \to \mathbb{R}^d \\
& \text{is} \quad \mathcal{P} \otimes \mathcal{B}_p \otimes \mathcal{B}_d \otimes \mathcal{B}_{d\times m}/\mathcal{B}_d \quad \text{measurable.}
\end{aligned}$$

Here, $\mathcal{P}$ denotes the σ-algebra of progressively measurable subsets of $\Gamma \times \Omega \times [0, T]$.

We define

$$\mathfrak{M}[0, T] \triangleq \mathbf{M}^2(0, T, \mathbb{R}^p) \times \mathbf{M}^2(0, T, \mathbb{R}^d) \times \mathbf{M}^2(0, T, \mathbb{R}^{d\times m}) \times \mathbf{M}^2(0, T, \mathbb{R}^{p\times m}),$$

where $\mathbf{M}^2(0, T, \mathbb{R}^p)$ $\big($rep. $\mathbf{M}^2(0, T, \mathbb{R}^{p\times m})\big)$ is defined in Definition 2.15. The norm of this space is defined by

$$||(X, Y, Z, K)|| = \Big\{E \sup_{t\in[0,T]} |X(t)|^2 + E \sup_{t\in[0,T]} |Y(t)|^2 + E \int_0^T |Z(t)|^2 dt + E \int_0^T |K(t)|^2 dt\Big\}^{\frac{1}{2}}$$

for all $(X, Y, Z, K) \in \mathfrak{M}[0, T]$.

We also denote

$$\mathfrak{N}[0, T] \triangleq \mathbf{M}^2(0, T, \mathbb{R}^d) \times \mathbf{M}^2(0, T, \mathbb{R}^{d\times m})$$

and

$$||(Y, Z)|| = \Big\{E \sup_{t\in[0,T]} |Y(t)|^2 + E \int_0^T |Z(t)|^2 dt\Big\}^{\frac{1}{2}}.$$

The following definition of solvability of FBUSDE is presented for the first time.

Definition 8.1. A process $(X(\cdot), Y(\cdot), Z(\cdot), K(\cdot)) \in \mathfrak{M}[0,T]$ is called an adapted solution of (8.1) if the following holds for any $t \in [0,T]$, $a.s.$,

$$\begin{aligned}
X(t) &= x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)dB_s + \int_0^t \zeta(s, X_s, Y_s, Z_s)dC_s \\
Y(t) &= h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s)ds + \int_t^T Z_s dB_s + \int_t^T K_s dC_s.
\end{aligned}$$

Furthermore, we say that a FBUSDE (8.1) is solvable if it has an adapted solution and non-solvable if it is not solvable.
For the remainder of this chapter, we will use the fact that

$$E\Big[\int_a^b X(t)dC(t)\Big] = E\Big[\int_a^b Y(t)dB(t)\Big] = 0,$$

for every $a, b \in [0,T]$, $X \in \mathbf{M}^2(0,T,\mathbb{R}^p)$, and $Y \in \mathbf{M}^2(0,T,\mathbb{R}^d)$.

Theorem 8.1. *Let b, σ, ζ, g and h satisfy (8.2). Moreover, assume that*

$$\begin{aligned}
&|g(t,y,z) - g(t,y,z)| \le L\Big[|y-\bar{y}| + |z-\bar{z}|\Big], \\
&|\sigma(t,x,y,z) - \sigma(t,x,y,\bar{z})| \le L_0|z-\bar{z}|, \\
&|h(x) - h(\bar{x}| \le L_1|x-\bar{x}| \\
&\textit{for all } x,\bar{x} \in \mathbb{R}^p,\ y,\bar{y} \in \mathbb{R}^d,\ z,\bar{z} \in \mathbb{R}^{d\times m},\ \textit{and},\ (\gamma,\omega,t)-a.e.,
\end{aligned} \tag{8.3}$$

with

$$L_0 L_1 < 1. \tag{8.4}$$

Then there exists a $T_0 > 0$, such that for any $T \in (0, T_0]$ and any $x \in \mathbb{R}^p$, (8.1) admits a unique adapted solution $(X,Y,Z,K) \in \mathfrak{M}[0,T]$.

Proof. Let $0 < T_0 \le 1$, be undetermined and $T \in (0,T_0]$. Let $x \in \mathbb{R}^p$ be fixed. We introduce the following norm:

$$||(Y,Z)||_{\bar{\mathfrak{N}}[0,T]} = \sup_{t\in[0,T]} \Big\{E|Y(t)|^2 + E\int_0^T |Z(t)|^2 dt\Big\}^{\frac{1}{2}}, \quad t \in [0,T] \tag{8.5}$$

for all $(Y, Z) \in \mathfrak{N}[0, T]$. We let $\bar{\mathfrak{N}}[0, T]$ be the completion of $\mathfrak{N}[0, T]$ in $\mathbf{M}^2(0, T, \mathbb{R}^d) \times \mathbf{M}^2(0, T, \mathbb{R}^{d\times m})$ under the norm (8.5). For simplicity we assume that $d = m = 1$. Take any $(Y_i, Z_i) \in \mathfrak{N}[0, T]$, $i = 1, 2$. We solve the following Forward Uncertain Stochastic Differential Equation (FUSDE) for X_i:

$$dX_i = b(t, X_i, Y_i, Z_i)dt + \sigma(t, X_i, Y_i, Z_i)dB(t) + \zeta(t, X_i, Y_i, Z_i)dC(t),$$
$$X_i(0) = x. \tag{8.6}$$

From the definition of hybrid differential equations, Definition 2.17, (8.6) admit a unique solution $X_i \in \mathbf{M}^2(0, T, \mathbb{R}^p)$ under the conditions of (8.4). By the Itô-Liu formula 2.3 we have

$$X^2(t) = X^2(0) + 2\int_0^t X(s)dX(s) + \int_0^t (dX(s))^2 \tag{8.7}$$

$$X^2(0) = X^2(T) - 2\int_0^T X(s)dX(s) - \int_0^T (dX(s))^2. \tag{8.8}$$

Putting (8.7) into (8.8), we get

$$X^2(t) = X^2(T) - 2\int_t^T X(s)dX(s) - \int_t^T (dX(s))^2.$$

Letting $X(t) = |X_1(t) - X_2(t)|$ and noting that $dB(t)dC(t) = dtdB(t) = dtdC(t) = 0$, we have

$$\begin{aligned} E|X_1(t) - X_2(t)|^2 &= 2E\int_0^t |X_1 - X_2|\big(b(s, X_1, Y_1, Z_1) - b(s, X_2, Y_2, Z_2)\big)ds \\ &\quad + E\int_0^t \big(\sigma(s, X_1, Y_1, Z_1) - \sigma(s, X_2, Y_2, Z_2)\big)^2 ds, \end{aligned} \tag{8.9}$$

and by the Lipschitz continuity of b and σ, (8.4), we obtain

$$\begin{aligned} E|X_1(t)-X_2(t)|^2 &\leq E\int_0^t 2L|X_1-X_2|\big(|X_1-X_2|+|Y_1-Y_2|+|Z_1-Z_2|\big)ds \\ &+E\int_0^t \big(L(|X_1-X_2|+|Y_1-Y2|)+L_0|Z_1-Z_2|\big)^2 ds, \\ &\leq E\int_0^t \Big[(2\varepsilon L^2+\frac{3}{2\varepsilon}+4L^2)|X_1-X_2|^2+(\frac{3}{2\varepsilon}+4L^2)|Y_1-Y_2|^2 \\ &+(2L_0^2+\frac{3}{2\varepsilon})|Z_1-Z_2|^2\Big]ds. \end{aligned} \tag{8.10}$$

By the Gronwall's inequality, we obtain

$$\begin{aligned} & E|X_1(t)-X_2(t)|^2 \\ \leq\ & e^{T(2\varepsilon L^2+C_\varepsilon)}E\int_0^T \Big[(C_\varepsilon|Y_1-Y_2|^2)+(2L_0^2+\frac{3}{2\varepsilon})|Z_1-Z_2|^2\Big]dt \end{aligned} \tag{8.11}$$

where $C_\varepsilon=\frac{3}{2\varepsilon}+4L^2$ and $\varepsilon>0$.

We now solve the following Backwards Uncertain Stochastic Differential Equation (BUSDE)

$$\begin{aligned} & d\bar{Y}_i = g(t,X_i,Y_i,Z_i)dt+\bar{Z}_i dB(t)+\bar{K}_i dC(t), \quad t\in[0,T] \\ & \bar{Y}_i(T)=h(X_i(T)) \qquad i=1,2. \end{aligned} \tag{8.12}$$

From the existence and uniqueness theorem of solutions to uncertain backward stochastic differential equations [21], (8.12) admits a unique adapted solution $(\bar{Y}_i,\bar{Z}_i)\in\mathfrak{N}[0,T]\subseteq\bar{\mathfrak{N}}[0,T]$. We have thus defined a map $\theta:\bar{\mathfrak{N}}[0,T]\to\bar{\mathfrak{N}}[0,T]$ by $(Y_i,Z_i)\to(\bar{Y}_i,\bar{Z}_i)$. Applying Itô-Liu formula 2.3 to $|\bar{Y}_1(t)-\bar{Y}_2(t)|^2$ and Hölders inequality, we get

$$
\begin{aligned}
& E|\bar{Y}_1(t)-\bar{Y}_2(t)|^2+E\int_t^T|\bar{Z}_1-\bar{Z}_2|^2ds \\
= \;& E|\bar{Y}_1(T)-\bar{Y}_2(T)|^2+2E\int_t^T|\bar{Y}_1(t)-\bar{Y}_2(t)||g(s,X_1,Y_1,Z_1)-g(s,X_2,Y_2,Z_2|ds \\
\leq \;& L_1^2E|X_1(T)-X_2(T)|^2 \\
& +E\Bigg\{\Big(\int_t^T|\bar{Y}_1(s)-\bar{Y}_2(s)|^2ds\Big)^{\frac{1}{2}}\Big(\int_t^T|g(s,X_1,Y_1,Z_1)-g(s,X_2,Y_2,Z_2)|^2ds\Big)^{\frac{1}{2}}\Bigg\}.
\end{aligned}
$$

Using the fact that $2ab\leq 2\varepsilon a^2+\frac{1}{2\varepsilon}b^2$, g is lipschitz and letting $L^*=L_1^2E|X_1(T)-X_2(T)|^2$, we get

$$
\begin{aligned}
& E|\bar{Y}_1(t)-\bar{Y}_2(t)|^2+E\int_t^T|\bar{Z}_1-\bar{Z}_2|^2ds \\
\leq \;& L^*+2\varepsilon E\int_t^T|\bar{Y}_1(s)-\bar{Y}_2(s)|^2ds+\frac{1}{2\varepsilon}E\int_t^T|g(s,X_1,Y_1,Z_1)-g(s,X_2,Y_2,Z_2)|^2ds \\
\leq \;& L^*+2\varepsilon E\int_t^T|\bar{Y}_1(s)-\bar{Y}_2(s)|^2ds \\
& +\frac{L^2}{2\varepsilon}E\int_t^T\Big(|X_1-X_2|+|Y_1-Y_2|+|Z_1-Z_2|\Big)^2ds \\
\leq \;& L^*+2\varepsilon E\int_t^T|\bar{Y}_1(s)-\bar{Y}_2(s)|^2ds \\
& +\frac{3L^2}{2\varepsilon}E\int_t^T|X_1-X_2|^2+|Y_1-Y_2|^2+|Z_1-Z_2|^2ds
\end{aligned}
$$

Letting $\bar{C}_\varepsilon=2\varepsilon$ and using (8.11) and (8.4), we get

$$
\begin{aligned}
& E|\bar{Y}_1(t)-\bar{Y}_2(t)|^2+E\int_t^T|\bar{Z}_1-\bar{Z}_2|^2ds \\
\leq \;& e^{T(\bar{C}_\varepsilon L^2+C_\varepsilon)}E\int_0^T\Big[(C_\varepsilon|Y_1-Y_2|^2)+(2L_0^2+\frac{3}{2\varepsilon})|Z_1-Z_2|^2\Big]ds \\
& +\frac{3L^2}{2\varepsilon}E\int_0^T|Y_1-Y_2|^2+|Z_1-Z_2|^2ds+\bar{C}_\varepsilon E\int_t^T|\bar{Y}_1(s)-\bar{Y}_2(s)|^2ds
\end{aligned}
$$

Using Gronwall's inequality gives

$$\begin{aligned}
& E|\bar{Y}_1(t)-\bar{Y}_2(t)|^2+E\int_t^T|\bar{Z}_1-\bar{Z}_2|^2ds \\
\leq\;& e^{T(\bar{C}_\varepsilon L^2+\bar{C}_\varepsilon+C_\varepsilon)}E\int_0^T\Big[(C_\varepsilon|Y_1-Y_2|^2)+(2L_0^2+\frac{3}{2\varepsilon})|Z_1-Z_2|^2\Big]ds \\
& +\frac{3L^2e^{\bar{C}_\varepsilon}}{2\varepsilon}E\int_0^T|Y_1-Y_2|^2+|Z_1-Z_2|^2ds \\
\leq\;& \left(e^{T(\bar{C}_\varepsilon L^2+\bar{C}_\varepsilon+C_\varepsilon)}C_\varepsilon+\frac{3L^2e^{\bar{C}_\varepsilon}}{2\varepsilon}\right)E\int_0^T|Y_1-Y_2|^2ds \\
& +\left(\big(e^{T(\bar{C}_\varepsilon L^2+\bar{C}_\varepsilon+C_\varepsilon)}\big)\big(2L_0^2+\frac{3}{2\varepsilon}\big)+\frac{3L^2e^{\bar{C}_\varepsilon}}{2\varepsilon}\right)E\int_0^T|Z_1-Z_2|^2ds \\
\leq\;& \left(e^{T(\bar{C}_\varepsilon L^2+\bar{C}_\varepsilon+C_\varepsilon)}C_\varepsilon+\big(e^{T(\bar{C}_\varepsilon L^2+\bar{C}_\varepsilon+C_\varepsilon)}\big)\big(2L_0^2+\frac{3}{2\varepsilon}\big)+\frac{3L^2e^{\bar{C}_\varepsilon}}{\varepsilon}\right) \\
& \times||(Y_1,Z_1)-(Y_2,Z_2)||^2_{\bar{\mathfrak{N}}[0,T]}, \qquad (8.13)
\end{aligned}$$

where $\bar{C}_\varepsilon$ and C_ε are independent of $T>0$. The last inequality (8.13) follows from the fact that, for any $(Y,Z)\in\bar{\mathfrak{N}}[0,T]$,

$$E|Y(t)|^2\leq||(Y,Z)||^2_{\bar{\mathfrak{N}}[0,T]},\qquad t\in[0,T]$$

$$\int_0^T E|Z(t)|^2dt\leq||(Y,Z)||^2_{\bar{\mathfrak{N}}[0,T]}$$

Since (8.4) holds, by choosing $\varepsilon>0$ small enough and choosing $T>0$ small enough, we obtain

$$||(\bar{Y}_1,\bar{Z}_1)-(\bar{Y}_2,\bar{Z}_2)||_{\bar{\mathfrak{N}}[0,T]}\leq\alpha||(Y_1,Z_1)-(Y_2,Z_2)||_{\bar{\mathfrak{N}}[0,T]},$$

for some $0<\alpha<1$. This means that the map $\theta:\bar{\mathfrak{N}}[0,T]\to\bar{\mathfrak{N}}[0,T]$ is contractive. Hence, the result follows by the Contraction Mapping Theorem for all small enough $T>0$. □

The results of this section are useful in the formulation of the optimal control problem for forward backward uncertain differential equation which is the main results of this chapter.

8.3 Problem Formulation

Let $(\Gamma \times \Omega, \mathcal{L} \otimes \mathcal{F}, (\mathcal{L}_t \otimes \mathcal{F}_t)_{t\in[0,T]}, \mathcal{M} \times P)$ be a filtered uncertain probability space. Suppose that we are given a subfiltration $\mathcal{G}_t \subseteq \mathcal{F}_t$ representing information available to the controller at time t, $t \in [0,T]$. Let $\vartheta = (\mathcal{T}, \xi) = (\tau_1, \tau_2, \ldots, \tau_n, \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots)$ be an impulse control described in Definition 6.3, where $0 \leq \tau_1 < \tau_2 < \tau_3 < \ldots$ is an $\{\mathcal{L}_t \otimes \mathcal{G}_t\}$-adapted sequence of increasing stopping times and $\xi_1, \xi_2, \ldots$ are $\{\mathcal{L}_t \otimes \mathcal{G}_t\}$ measurable uncertain random variables. We define the impulse process $L(t)$ by

$$L(t) = \sum_{i=1}^{\infty} I_{\{\tau_i < t\}} \xi_i, \qquad t \leq T. \tag{8.14}$$

Suppose the controlled forward backwards uncertain stochastic systems involving classical and impulse control, in the unknown processes $X(t)$, $Y(t)$, $Z(t)$ and $K(t)$ is described by the following FBUSDE:

$$\begin{aligned} dX_t &= b(t, X_t, a_t)dt + \sigma(t, X_t, a_t)dB_t + \zeta(t, X_t, a_t)dC_t + A_t dL_t \\ dY_t &= -g(t, X_t, Y_t, Z_t, a_t)dt + Z_t dB_t + K_t dC_t - D_t dL_t, \qquad t \in [0,T], \\ & X_0 = x, \qquad Y(T) = \mu X(T), \end{aligned} \tag{8.15}$$

where

$$b : \Gamma \times \Omega \times [0,T] \times \mathbb{R} \times \mathcal{A} \to \mathbb{R}$$

$$\sigma : \Gamma \times \Omega \times [0,T] \times \mathbb{R} \times \mathcal{A} \to \mathbb{R}$$

$$\zeta : \Gamma \times \Omega \times [0,T] \times \mathbb{R} \times \mathcal{A} \to \mathbb{R}$$

$$g : \Gamma \times \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{A} \to \mathbb{R}$$

are measurable. The result of giving the impulse ξ_i is that the state jumps from $(X(\tau_i^-), Y(\tau_i^-))$ to $(X(\tau_i), Y(\tau_i)) = (X(\tau_i^-) + A(\tau_i)\xi_i, Y(\tau_i^-) - D(\tau_i)\xi_i)$. Let $\mathcal{A}_{\{\mathcal{L}_t \otimes \mathcal{G}_t\}}$ denote the class of all $\{\mathcal{L}_t \otimes \mathcal{G}_t\}$-measurable controls, such that the uncertain stochastic system (8.15) admit a unique strong solution.

Define the performance functional $J(x,\pi)$ by

$$\begin{aligned} J(x,\pi): \; &= \; E\Bigg[\int_0^T f(t,X(t),Y(t),Z(t),K(t),a(t))dt\Bigg] \\ &+E\Bigg[h_1(Y(0))+h_2(X(T))+\sum_{i=1}^{\infty} n(\tau_i,\xi_i)\Bigg] \end{aligned} \tag{8.16}$$

where f, h_1, h_2 and g are functions such that

$$\begin{aligned} &E\Bigg[\int_0^T |f(t,X(t),Y(t),Z(t),K(t),a(t))|dt\Bigg] \\ &+E\Bigg[|h_1(Y(0))|+|h_2(X(T))|+\sum_{i=1}^{\infty} |n(\tau_i,\xi_i)|\Bigg] < \infty. \end{aligned} \tag{8.17}$$

Problem 8.3.1. *The problem is to determine the value function $V_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}(x)$ and the optimal control $\pi^*=(a^*,\mathcal{T}^*,\xi^*)\in\mathcal{A}_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}$ such that*

$$V_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}(x) = \sup_{\pi\in\mathcal{A}_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}} J(x,\pi) = J(x,\pi^*). \tag{8.18}$$

8.4 Optimal Control of Uncertain Stochastic Processes with Partial Information

In this section we establish a maximum principle for the proposed problem, Problem 8.3.1. The necessary and sufficient conditions for the local critical points $(a^*,\mathcal{T}^*,\xi^*)$ is given.

Let $\pi(t)=(a(t),\vartheta(t))\in\mathcal{A}_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}$, where $\vartheta(t)=L(t)$. We need to first make the following assumptions.

Assumption (1) For all $s\in[0,T]$ and bounded $\{\mathcal{L}_s\otimes\mathcal{G}_s\}$-measurable uncertain random variables $\theta(\gamma,\omega)$, the control

$$\bar{a}_s(t)=\theta(\gamma,\omega)I_{(s,T]}, \qquad s\in[0,T] \tag{8.19}$$

belong to $\mathcal{A}_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}$.

Assumption (2) For all $\bar{\pi}=(\bar{a},\bar{\vartheta})\in\mathcal{A}_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}$ where $\bar{\pi}$ is bounded, there exists

$\delta > 0$ such that the control

$$(a(t) + z\bar{a}(t), \vartheta(t) + z\bar{\vartheta}(t)) \in \mathcal{A}_{\{\mathcal{L}_t \otimes \mathcal{G}_t\}} \qquad (8.20)$$
$$\forall z \in (-\delta, \delta), \qquad t \in [0, T].$$

Definition 8.2. The Hamiltonian

$$H : \Gamma \times \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{A} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \qquad (8.21)$$

is defined by

$$\begin{aligned} H(t,x,y,z,k,a,\lambda,p,q,r) &= f(t,x,y,z,k,a) + \lambda(t) g(t,x,y,z,k,a) + b(t,x,a)p(t) \\ &\quad + \sigma(t,x,a)q(t) + \zeta(t,x,a)r(t). \end{aligned} \qquad (8.22)$$

We assume that H is Fréchet differentiable in the variable x, y and k.

Remark 8.1. Let V be an open subset of a Banach space M and let $F : V \to \mathbb{R}$.
(i) We say that F has a directional derivative (Gateaux derivative) at $x \in V$ in the direction $y \in M$ if

$$D_y F(x) := \lim_{\epsilon \to 0} \frac{F(x + \epsilon y) - F(x)}{\epsilon} \qquad \text{exists.}$$

(ii) We say that F is Fréchet differentiable at $x \in V$ if there exists a linear map $L : M \to \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{|F(x+h) - F(x) - L(h)|}{||h||} = 0 \qquad \forall h \in M.$$

In this case we call L the Fréchet derivative (Strong derivative) of F at x, and we write

$$L = \bigtriangledown_x F.$$

(iii) If F is Fréchet differentiable, then F has a directional derivative in all directions $y \in M$ and

$$D_y F(x) = \bigtriangledown_x F(y).$$

To Problem 8.3.1, we associate a pair of FBUSDE's in the adjoint processes $\lambda(t)$, $p(t)$, $q(t)$ and $r(t)$ as follows.

(i) Forward system in the unknown process $\lambda(t)$:

$$\begin{aligned} d\lambda(t) &= \frac{\partial H}{\partial y}\big(t, X_t, Y_t, Z_t, K_t, a_t, \lambda_t, p_t, q_t, r_t\big)dt + \frac{\partial H}{\partial z}\big(t, X_t, Y_t, Z_t, K_t, a_t, \lambda_t, p_t, q_t, r_t\big)dB_t \\ & \frac{\partial H}{\partial k}\big(t, X_t, Y_t, Z_t, K_t, a_t, \lambda_t, p_t, q_t, r_t\big)dC_t \\ & \lambda(0) = h_1'(Y(0)). \end{aligned} \tag{8.23}$$

(ii) Backward system in the unknown processes $p(t)$, $q(t)$, and $r(t)$:

$$\begin{aligned} dp(t) &= -\frac{\partial H}{\partial x}\big(t, X_t, Y_t, Z_t, K_t, a_t, \lambda_t, p_t, q_t, r_t\big)dt + q_t dB_t + r_t dC_t \\ & p(T) = \mu\lambda(T) + h_2'(X(T)). \end{aligned} \tag{8.24}$$

For notational convenience we will use the notation

$$\frac{\partial H}{\partial y}\big(t, X_t, Y_t, Z_t, K_t, a_t, \lambda_t, p_t, q_t, r_t\big) = \frac{\partial H}{\partial y}(t),$$

similarly with other partial derivatives.

Theorem 8.2. *(Maximum Principle)*

Let $\pi \in \mathcal{A}_{\{\mathcal{L}_t \otimes \mathcal{G}_t\}}$ with corresponding solutions $X(t)$, $Y(t)$, $Z(t)$, $K(t)$, $\lambda(T)$, $p(t)$, $q(t)$ and $r(t)$ of (8.15), (8.23) and (8.24). Assume that for all $\pi \in \mathcal{A}_{\{\mathcal{L}_t \otimes \mathcal{G}_t\}}$ the following growth conditions hold:

$$E\bigg[\int_0^T \Big\{Y^2(t)\Big(\frac{\partial H}{\partial y}^2(t) + \frac{\partial H}{\partial k}^2(t)\Big) + \lambda^2(t)\big(Z^2(t) + K^2(t)\big)\Big\}dt\bigg] < \infty$$

$$E\bigg[\int_0^T \Big\{X^2(t)\big(q^2(t) + r^2(t)\big) + p^2(t)\big(\sigma^2(t) + \zeta^2(t)\big)\Big\}dt\bigg] < \infty. \tag{8.25}$$

The following are equivalent.

If

$$\bar{a}(t)\frac{\partial f}{\partial a}(t) + \bar{a}(t)q(t)\frac{\partial \sigma}{\partial a}(t) + \frac{\partial f}{\partial z}(t)\hat{Z}(t, \bar{a}, \bar{\vartheta}) + \frac{\partial f}{\partial k}(t)\hat{K}(t, \bar{a}, \bar{\vartheta}) = r(t)\hat{X}(t, \bar{a}, \bar{\vartheta})\frac{\partial \zeta}{\partial x}(t), \tag{8.26}$$

then

(1) (a,ϑ) *is a critical point for* $J(a,\vartheta)$*, in the sense that*

$$\frac{d}{dz}J(a+z\bar{a},\vartheta+z\bar{\vartheta})|_{z=0}=0 \qquad \forall(\bar{a},\bar{\vartheta})\in\mathcal{A}_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}, \tag{8.27}$$

(2) *and*

$$E\left[\sum_{i=1}^{\infty}\Big(p(\tau_i)A(\tau_i)+\frac{\partial n}{\partial\vartheta}(\tau_i)-\lambda(\tau_i)D(\tau_i)\Big)\bar{\vartheta}_i\right]=0, \tag{8.28}$$

where the expectation E *is conditional to the partial information* $\{\mathcal{L}_t\otimes\mathcal{G}_t\}$.

Proof. We define

$$\begin{aligned}
\hat{X}(t,\bar{a},\bar{\vartheta}) &= \frac{d}{dz}X(t,a+z\bar{a},\vartheta+z\bar{\vartheta})|_{z=0},\\
\hat{Y}(t,\bar{a},\bar{\vartheta}) &= \frac{d}{dz}Y(t,a+z\bar{a},\vartheta+z\bar{\vartheta})|_{z=0},\\
\hat{Z}(t,\bar{a},\bar{\vartheta}) &= \frac{d}{dz}Z(t,a+z\bar{a},\vartheta+z\bar{\vartheta})|_{z=0},\\
\hat{K}(t,\bar{a},\bar{\vartheta}) &= \frac{d}{dz}K(t,a+z\bar{a},\vartheta+z\bar{\vartheta})|_{z=0}.
\end{aligned} \tag{8.29}$$

We therefore have

$$\begin{aligned}
\hat{X}(0,\bar{a},\bar{\vartheta}) &= \frac{d}{dz}X(0,a+z\bar{a},\vartheta+z\bar{\vartheta})|_{z=0}=0,\\
\hat{X}(T,\bar{a},\bar{\vartheta}) &= \frac{d}{dz}X(T,a+z\bar{a},\vartheta+z\bar{\vartheta})|_{z=0}=\frac{\hat{Y}(T,\bar{a},\bar{\vartheta})}{\mu},\\
d\hat{X}(t,\bar{a},\bar{\vartheta}) &= \left[\frac{\partial b}{\partial x}(t)\hat{X}(t,\bar{a},\bar{\vartheta})+\frac{\partial b}{\partial a}(t)\bar{a}(t)\right]dt+\int_0^t\left[\frac{\partial\sigma}{\partial x}(s)\hat{X}(s,\bar{a},\bar{\vartheta})+\frac{\partial\sigma}{\partial a}(s)\bar{a}(s)\right]dB_s\\
&\quad+\int_0^t\left[\frac{\partial\zeta}{\partial x}(s)\hat{X}(s,\bar{a},\bar{\vartheta})+\frac{\partial\zeta}{\partial a}(s)\bar{a}(s)\right]dC_s.\\
d\hat{Y}(t,\bar{a},\bar{\vartheta}) &= -\Big[\frac{\partial g}{\partial x}(t)\hat{X}(t,\bar{a},\bar{\vartheta})+\frac{\partial g}{\partial y}(t)\hat{Y}(t,\bar{a},\bar{\vartheta})+\frac{\partial g}{\partial z}(t)\hat{Z}(t,\bar{a},\bar{\vartheta})+\frac{\partial g}{\partial a}(t)\bar{a}\Big]dt\\
&\quad+\hat{Z}(t,\bar{a},\bar{\vartheta})dB_t+\hat{K}(t,\bar{a},\bar{\vartheta})dC_t+D(t)d\bar{\vartheta}_t.
\end{aligned} \tag{8.30}$$

We need to first prove that $(1) \Rightarrow (2)$. If we assume that (1) holds, we then have

$$\begin{aligned}
0 &= \frac{d}{dz}J(a+z\bar{a},\vartheta+z\bar{\vartheta})|_{z=0} \\
&= E\Big[\int_0^T \Big(\frac{\partial f}{\partial x}(t)\hat{X}(t,\bar{a},\bar{\vartheta})+\frac{\partial f}{\partial y}(t)\hat{Y}(t,\bar{a},\bar{\vartheta})+\frac{\partial f}{\partial z}(t)\hat{Z}(t,\bar{a},\bar{\vartheta})\Big)dt\Big] \\
&\quad +E\Big[\int_0^T \Big(\frac{\partial f}{\partial k}(t)\hat{K}(t,\bar{a},\bar{\vartheta})+\frac{\partial f}{\partial a}(t)\bar{a}\Big)dt\Big] \\
&\quad +E\Big[h_1^{'}(Y(0))\hat{Y}(0,\bar{a},\bar{\vartheta})+h_2^{'}(X(T))\hat{X}(T,\bar{a},\bar{\vartheta})+\sum_{\tau_i\le T}\frac{\partial n}{\partial\vartheta}(\tau_i)\bar{\vartheta}_i\Big]. \quad (8.31)
\end{aligned}$$

By the Itô-Liu formula 2.3, we get

$$\begin{aligned}
& E\Big[h_1^{'}(Y(0))\hat{Y}(0,\bar{a},\bar{\vartheta})\Big] \\
&= E\Big[\lambda(0)\hat{Y}(0,\bar{a},\bar{\vartheta})\Big] \\
&= E\Big[\lambda(T)\hat{Y}(T,\bar{a},\bar{\vartheta})-\sum_{i=1}^{\infty}\lambda(\tau_i)D(\tau_i)\bar{\vartheta}_i-\int_0^T\Big(\frac{\partial H}{\partial y}(t)\hat{Y}(t,\bar{a},\bar{\vartheta})+\frac{\partial H}{\partial z}(t)\hat{Z}(t,\bar{a},\bar{\vartheta}))\Big)dt\Big] \\
&\quad +E\Big[\int_0^T\lambda(t)\Big(\frac{\partial g}{\partial x}(t)\hat{X}(t,\bar{a},\bar{\vartheta})+\frac{\partial g}{\partial y}(t)\hat{Y}(t,\bar{a},\bar{\vartheta})+\frac{\partial g}{\partial z}(t)\hat{Z}(t,\bar{a},\bar{\vartheta})\Big)dt\Big], \quad (8.32)
\end{aligned}$$

where $0 \le \tau_i \le T$. We also have that

$$\begin{aligned}
& E\Big[h_2^{'}(X(T))\hat{X}(T,\bar{a},\bar{\vartheta})\Big] \\
&= E\Big[\Big(p(T)-\mu\lambda(T)\Big)\hat{X}(T,\bar{a},\bar{\vartheta})\Big] \\
&= E\Big[p(T)\hat{X}(T,\bar{a},\bar{\vartheta})\Big]-E\Big[\lambda(T)\hat{Y}(T,\bar{a},\bar{\vartheta})\Big]. \quad (8.33)
\end{aligned}$$

An application of the Itô-Liu formula 2.3 to $E\Big[p(T)\hat{X}(T,\bar{a},\bar{\vartheta})\Big]$, yields

$$
\begin{aligned}
& E\Big[p(T)\hat{X}(T,\bar{a},\bar{\vartheta})\Big] \\
= \;& E\Big[p(0)\hat{X}(0,\bar{a},\bar{\vartheta}) + \sum_{i=1}^{\infty} p(\tau_i)A(\tau_i)\bar{\vartheta}_i \; + \int_0^T p(t)\Big(\frac{\partial b}{\partial x}(t)\hat{X}(t,\bar{a},\bar{\vartheta}) + \frac{\partial b}{\partial a}(t)\bar{a}(t)\Big)dt\Big] \\
& -E\Big[\int_0^T \frac{\partial H}{\partial x}(t)\hat{X}(t,\bar{a},\bar{\vartheta})dt - \int_0^T q(t)\Big(\frac{\partial \sigma}{\partial x}(t)\hat{X}(t,\bar{a},\bar{\vartheta}) + \frac{\partial \sigma}{\partial a}(t)\bar{a}(t)\Big)dt\Big] \qquad (8.34)
\end{aligned}
$$

where $0 \leq \tau_i \leq T$. If we substitute (8.32), (8.33) and (8.34) into (8.31), we get

$$
\begin{aligned}
0 \;=\;& E\Big[\sum_{i=1}^{\infty}\Big(p(\tau_i)A(\tau_i) + \frac{\partial n}{\partial \vartheta}(\tau_i) - \lambda(\tau_i)D(\tau_i)\Big)\bar{\vartheta}_i\Big] \\
& +E\Big[\int_0^T \Big(\frac{\partial f}{\partial x}(t) + \lambda(t)\frac{\partial g}{\partial x}(t) + p(t)\frac{\partial b}{\partial x}(t) - \frac{\partial H}{\partial x}(t) + q(t)\frac{\partial \sigma}{\partial x}(t)\Big)\hat{X}(t,\bar{a},\bar{\vartheta})dt\Big] \\
& +E\Big[\int_0^T \Big(\frac{\partial f}{\partial y}(t) - \frac{\partial H}{\partial y}(t) + \lambda(t)\frac{\partial g}{\partial y}(t)\Big)\hat{Y}(t,\bar{a},\bar{\vartheta})dt\Big] \\
& +E\Big[\int_0^T \Big(2\frac{\partial f}{\partial z}(t) - \frac{\partial H}{\partial z}(t) + \lambda(t)\frac{\partial g}{\partial z}(t)\Big)\hat{Z}(t,\bar{a},\bar{\vartheta})dt\Big] \\
& +E\Big[\int_0^T \Big(\frac{\partial f}{\partial a}(t) + q(t)\frac{\partial \sigma}{\partial a}(t)\Big)\bar{a}(t) + \frac{\partial f}{\partial k}(t)\hat{K}(t,\bar{a},\bar{\vartheta})dt\Big]. \qquad (8.35)
\end{aligned}
$$

From our definition of the Hamiltonian process (8.22), we have

$$
\begin{aligned}
\frac{\partial H}{\partial x}(t) &= \frac{\partial f}{\partial x}(t) + \lambda(t)\frac{\partial g}{\partial x}(t) + p(t)\frac{\partial b}{\partial x}(t) + q(t)\frac{\partial \sigma}{\partial x}(t) + r(t)\frac{\partial \zeta}{\partial x}(t) \\
\frac{\partial H}{\partial y}(t) &= \frac{\partial f}{\partial y}(t) + \lambda(t)\frac{\partial g}{\partial y}(t) \\
\frac{\partial H}{\partial z}(t) &= \frac{\partial f}{\partial z}(t) + \lambda(t)\frac{\partial g}{\partial z}(t) \\
\frac{\partial H}{\partial k}(t) &= \frac{\partial f}{\partial k}(t) + \lambda(t)\frac{\partial g}{\partial k}(t) \\
\frac{\partial H}{\partial a}(t) &= \frac{\partial f}{\partial a}(t) + \lambda(t)\frac{\partial g}{\partial a}(t) + q(t)\frac{\partial \sigma}{\partial a}(t) + r(t)\frac{\partial \zeta}{\partial a}(t). \qquad (8.36)
\end{aligned}
$$

Substituting (8.36) into (8.35) gives the required results, that is

$$E\left[\sum_{i=1}^{\infty}\Big(p(\tau_i)A(\tau_i)+\frac{\partial n}{\partial \vartheta}(\tau_i)-\lambda(\tau_i)D(\tau_i)\Big)\bar{\vartheta}_i\right]=0 \tag{8.37}$$

whenever

$$\bar{a}(t)\frac{\partial f}{\partial a}(t)+\bar{a}(t)q(t)\frac{\partial \sigma}{\partial a}(t)+\frac{\partial f}{\partial z}(t)\hat{Z}(t,\bar{a},\bar{\vartheta})+\frac{\partial f}{\partial k}(t)\hat{K}(t,\bar{a},\bar{\vartheta})=r(t)\hat{X}(t,\bar{a},\bar{\vartheta})\frac{\partial \zeta}{\partial x}(t). \tag{8.38}$$

The proof of $(2)\Rightarrow(1)$ can be done by reversing the above argument. □

8.5 Optimal Dividend Distribution with Partial Information for Uncertain Stochastic Reserves.

In this section we study the optimal control problem of an insurance company which can adjust its reinsurance policy rate to obtain optimal profit and optimal dividend distribution policy. The objective of the firm is to find optimal dividend distribution policy which will minimize the total deviation of its reserves to some pre-set target.

Suppose the state of the reserve process for an insurance firm is described by the following controlled forward-backward uncertain stochastic differential equation

$$\begin{aligned} dX_t &= (\mu a(t)+\rho X_t)dt+\sigma_1 a(t)dB_t+\sigma_2 a(t)dC_t-dL(t)\\ dY_t &= -g(t,Y_t)dt+Z_tdB_t+K_tdC_t+dL(t)\\ X(0) &= x, \qquad Y(T)=\alpha X(T) \end{aligned} \tag{8.39}$$

where $L(t)=\sum_{i=1}^{\infty} I_{\{\tau_i<t\}}\xi_i$ is the dividend distribution policy and $\alpha\in\mathbb{R}-\{0\}$is a given constant.

Suppose in addition we are given a subfiltration

$$\{\mathcal{L}_t\otimes\mathcal{G}_t\}\subseteq\{\mathcal{L}_t\otimes\mathcal{F}_t\}$$

representing the information available to the controller at time t.

Define the performance functional $J(x,\pi)$ by

$$J(x,\pi) := E\left[\int_0^T \frac{a^\nu(t)}{\nu}dt + Y_g^{-X_a(T)}(0) + \sum_{i=1}^{\infty} n^2(\xi_i)I_{\{\tau_i<\mathcal{T}\}}\Big/\{\mathcal{L}_t\otimes\mathcal{G}_t\}\right] \quad (8.40)$$

where the function $n:[0,\infty)\mapsto(-\infty,\infty)$ is given by

$$n(\eta) = \tilde{k}\eta - \tilde{K}, \quad (8.41)$$

where $\tilde{k}\in(0,1)$ and $\tilde{K}\in(0,\infty)$ are constants, with $1-\tilde{k}$ being interpreted as tax rate and $\tilde{K}$ as a fixed cost when dividends are paid.

Problem 8.5.1. *The problem is to determine the value function $V_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}(x)$ and the optimal control $\pi^* = (a^*,\mathcal{T}^*,\xi^*)\in\mathcal{A}_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}$ such that*

$$\begin{aligned} V_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}(x) &= \sup_{\pi\in\mathcal{A}_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}} J(x,\pi) \\ &= \sup_{\pi\in\mathcal{A}_{\{\mathcal{L}_t\otimes\mathcal{G}_t\}}} E\left[\int_0^T \frac{a^\nu(t)}{\nu}dt + Y_g^{-X_a(T)}(0) + \sum_{i=1}^{\infty} n^2(\xi_i)I_{\{\tau_i<\mathcal{T}\}}\Big/\{\mathcal{L}_t\otimes\mathcal{G}_t\}\right] \end{aligned}$$

The above problem, Problem 8.5.1, is a combined classical and impulse control problem of FBUSDE under partial information $\{\mathcal{L}_t\otimes\mathcal{G}_t\}$. The solution to Problem 8.5.1 is obtained by making use of Theorem 8.2. From Theorem 8.2 we note that

$$\begin{aligned} &f(t,x,y,z,k,a) = \tfrac{a^\nu}{\nu}; && h_1(y) = y; \quad h_2(s) = 0; \\ &n(\tau_i,\xi_i) = n^2(\xi_i); && b(t,x,a) = (\mu a+\rho x); \quad \sigma(t,x,a) = a\sigma_1; \\ &\zeta(t,x,a) = a\sigma_2; && A(t) = -1; \quad D(t) = 1. \end{aligned} \quad (8.42)$$

The Hamiltonian (8.22) is thus given by

$$H(t,x,y,z,k,a,\lambda,p,q,r) = \frac{a^\nu}{\nu} + \lambda(t)g(t,y) + (\mu a+\rho x)p(t) + \sigma_1 a q(t) + \sigma_2 a r(t) \quad (8.43)$$

where

$$d\lambda(t) = \lambda(t)\frac{\partial g(t,y)}{\partial y}dt, \qquad \lambda(0) = 1, \quad (8.44)$$

and

$$\begin{aligned} dp(t) &= \rho p_t dt + q_t dB_t + r_t dC_t \\ p(T) &= \alpha\lambda(T). \end{aligned} \tag{8.45}$$

The solution to (8.44) is given by

$$\lambda(t) = \exp\left\{ \int_0^t \frac{\partial g(s,y)}{\partial y} ds \right\}, \qquad t \in [0,T]. \tag{8.46}$$

If (8.26) hold, then then

$$\begin{aligned} &\bar{a}(a^{\nu-1} + q(t)\sigma_1) = 0 \\ &a^* = \big(-q(t)\sigma_1\big)^{\frac{1}{\nu-1}}, \end{aligned} \tag{8.47}$$

and by sufficient and necessary optimality condition (8.28) we have

$$\sum_{\tau_i \leq T} E\Big[2(\tilde{k}\xi_i + \tilde{K})\tilde{k} - p(\tau_i) - \lambda(\tau_i) \Big/ \mathcal{L}_{\tau_i} \otimes \mathcal{G}_{\tau_i}\Big] = 0. \tag{8.48}$$

That is, for each $\tau_i \leq T$, we have

$$2\tilde{k}^2 E\Big[\xi_i \Big/ \{\mathcal{L}_{\tau_i} \otimes \mathcal{G}_{\tau_i}\}\Big] = E\Big[p(\tau_i) + \lambda(\tau_i) - 2\tilde{k}\tilde{K} \Big/ \mathcal{L}_{\tau_i} \otimes \mathcal{G}_{\tau_i}\Big]. \tag{8.49}$$

Since ξ_i is an $\{\mathcal{L}_{\tau_i} \otimes \mathcal{G}_{\tau_i}\}$-measurable uncertain random variable, we have

$$\xi_i^* = \frac{1}{2\tilde{k}^2} E\Big[p(\tau_i) + \lambda(\tau_i) - 2\tilde{k}\tilde{K} \Big/ \mathcal{L}_{\tau_i} \otimes \mathcal{G}_{\tau_i}\Big], \tag{8.50}$$

where $\lambda(t)$ is given by (8.46) and $p(t)$ is given by (8.45). We summarize the above results with the following theorem.

Theorem 8.3. *Let $p(t)$, $q(t)$ and $r(t)$ be solutions of (8.45) and let $\lambda(t)$ be given by (8.46). Then the optimal reinsurance policy and the optimal dividend distribution policy for an insurance firm is given by*

$$\begin{aligned} a^* &= \big(-q(t)\sigma_1\big)^{\frac{1}{\nu-1}}, \\ \xi_i^* &= \frac{1}{2\tilde{k}^2} E\Big[p(\tau_i) + \lambda(\tau_i) - 2\tilde{k}\tilde{K} \Big/ \mathcal{L}_{\tau_i} \otimes \mathcal{G}_{\tau_i}\Big] \end{aligned} \tag{8.51}$$

respectively, where the pair $(a^*(t), \vartheta^*(t))$ are local critical point of the combined classical and impulse control Problem 8.5.1.

8.6 Conclusion

In this chapter we managed to prove the existence and uniqueness theorem of solutions to FBUSDE under Lipschitzian conditions. The derived FBUSDE can be applied in other areas of research, such as engineering, biology, economics and physics. We on the other hand applied FBUSDE in the classical and impulse control problem of forward-backward systems. We considered the partial information problem for insurance firm driven by uncertain stochastic processes. Due to the non-Markovian nature of the proposed problem, dynamic programming principle could not be applied. As a result, we derived a maximum principle for FBUSDE when there is partial information available to the controller. We relaxed the concavity conditions of the utility function and the Hamiltonian and gave a sufficient and necessary optimality conditions for the local critical points of the control problem. We then solved the dividend distribution for an insurance firm which can reinsure its reserves. We also gave explicit solutions for the proposed insurance problem.

Chapter 9

Summary and Future Directions

In this thesis, we studied the optimal control problem for insurance reserves under different forms of indeterminacy. The first part of the thesis studied the optimal control and optimal stopping problem under random indeterminacy. In chapter 3 and 4, a Levy jump optimal stopping and impulse control problem was investigated. Closed form solutions for the stopping problem and the combined classical and impulse control problem were presented under the smooth pasting condition. Future research for both chapter 3 and 4 could look into the case when the value function is not continuously differentiable at the boundary. In chapter 5 a stochastic volatility model is presented under random indeterminacy. Under the risk neutral assumptions for the proposed model, we explicitly solve the problem and construct its value function with the optimal policy. In an incomplete market, there are infinitely many equivalent martingale measures. Common examples of these measures are the minimal martingale measure, the relative entropy minimizer, and the Esscher measure. Further research could develop the findings in chapter 5 by studying the optimal control problem under these measures when volatility is also assumed to be random.

The second part of the thesis studied the optimal control problem for an insurance company in the present of both uncertain and random indeterminacy. In chapter 6, the dividend optimization problem and the reinsurance policy for an insurance company under uncertain random indeterminacy are investigated. Closed form solutions for the value function and reinsurance policy are presented. In chapter 7, the control problem for insurance firms under uncertain random indeterminacy with an uncertain V jump was investigated. Future research could study the optimal control problem of insurance firms when jumps are modeled by uncertain random renewal processes instead of a V jump process. Unlike V jump processes, uncertain random renewal processes measure both uncertain and random indeterminacy and measures multiple jumps. The last chapter, chapter 8, present and investigate the maximum principle for optimal control of forward-backward uncertain stochastic differential equations (FBUSDE). The objectives of chapter 8 are to add to the existing body of information on uncertain stochastic systems and their application to insurance.

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