PRICING AND HEDGING OF DEFAULTABLE CLAIMS IN DISCONTINUOUS MARKET

A PROJECT SUBMITTED IN FULFILMENT OF THE ACADEMIC REQUIREMENTS FOR THE DEGREE OF MASTERS OF SCIENCE IN THE FACULTY OF SCIENCE AND AGRICULTURE

By

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Dedication

This project is dedicated to my family.

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Declaration

I hereby declare that this project was carried out originally by me under the direct supervision of Dr. J. M. Ngnotchouye in the School of Mathematical Sciences, University of KwaZulu-Natal Pietermaritzburg and Dr. O. M. Pamen in the department of Mathematical Science, University of Liverpool.

It has not been submitted in any form to any University or institution for any degree or diploma. Acknowledgement has been duly ascribed where the work of others were used.

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Abstract

Credit risk has become one of the highest-profile risk facing participants in the financial markets. In this dissertation, we study the pricing and hedging of defaultable claim in a discontinuous market. Here, we present the pricing of credit default swap under stochastic intensity within the set up of a generic reduced form credit risk model. In this context, we present different approaches to pricing and hedging of defaultable claim in a discontinuous market and then proffer results concerning the trading of credit default swap. We first assume that the default intensity is deterministic and the rate of interest is equal to zero. We derive a closed-form solution for replicating strategy for an arbitrary non-dividend paying defaultable claim. We then extend the established results under deterministic intensity to the case of stochastic intensity, where the objective is to hedge both default (jump) risk and the spread (volatility) risk.

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Chapter 1

INTRODUCTION

Over the last thirty years, mathematical finance has been rapidly an expanding field of science. The main reason is the success of sophisticated quantitative methodologies in helping professional to manage financial risk. Hence, it may be reasonable that newly developed credit derivatives industry will also benefit from the use of advanced mathematics. This helps to handle credit risk, which is one of the fundamental factors of financial risk.

Indeed, a great interest has grown in the development of advanced mathematical models for finance and at the same time, we can note a tremendous acceleration in research efforts aimed at a better understanding, modeling and hedging of credit risk. This is the risk caused by the possibility that a company will have financial troubles and will have to default on payments which it owes to its lenders.

In a financial market, the default of one firm in paying its bond usually has important influences on the other ones. This has been shown clearly by several recent default events during the credit crisis [9]. Defaultable instruments, or credit-linked derivatives, are financial securities that pay their holders amounts that are contingent on the occurrence of a default event such as the bankruptcy of a firm or non-repayment of a loan. The market in credit-linked derivative products has grown astonishingly, from \$631.5 billion global volume in the first half of 2001, to above $12 trillion through the first half of 2005 [16] [(ISDA data reported at http://www.credit-deriv.com/globalmarket.htm)]. The growth from mid-2004 through mid-2005 alone was 128% percent. They now account for approximately 10% of the total Over The Counter(OTC) derivatives market.

This fact certainly raises the question whether the credit derivatives market (specifically Credit Default Swaps (CDSs)) still has a future and whether it is still worth putting effort into their pricing. Since the original purpose of CDSs was to hedge credit risk, and since there will still be a need for this in the future, it is safe to say that both questions can be answered with "yes". However, it is also almost certain that products will be held simple and will be subject to more regulation than in the past. Furthermore, the market for credit derivatives will probably not be as liquid as it used to be close to its peak. Credit derivatives market is based primarily on credit or default risk. In order to protect investors from this risk, the credit derivatives market emerged with various products whose sole purpose is to hedge credit risk. A credit derivative is a contract between a protection buyer and a protection seller to transfer the credit risk of an asset without the actual transfer of the asset.

A credit default swap is the most straightforward type of a credit derivative. It is an agreement between two counter-parties that allows one counter-party to be long a third-party credit risk, and the other counter-party to be short the credit risk. Explained another way, one counter-party is selling insurance and the other counter-party is buying insurance against the default of the third party. In a credit default swap, the protection buyer makes periodic premium payments to the protection seller in exchange for the promise that if a default occurs, the protection seller will receive the defaulted security and repay the protection buyer a percentage of what was owed. The premiums of the credit default swap contract are determined by the market's view of how likely it is that default will occur before the credit swap matures.

Pricing the credit default swap involves determining the fixed payments from the market-maker to the investor. In this case, it is sufficient to extract the price from the bond market. One does not need to model default or any other complicated credit risk process. To apply risk-neutral pricing theory, one needs to construct a hedge for the credit default swap. For example, suppose that two counter-parties, a market maker, and an investor, enter into a two-year credit default swap. They specify what is called the reference asset, which is a particular credit risky bond issued by a third-party corporation or sovereign. For simplicity, if it is assumed that the bond has exactly two years remaining to mature and is currently trading at par value, the market maker agrees to make regular fixed payments (with the same frequency as the reference bond) for two years to the investor. In exchange the market maker has the following right: If the third party defaults at any time in that two years, the market maker makes his regular fixed payment to the investor and puts the bond to the investor in exchange for the bond's par value plus interest. The credit default swap is thus a contingent put - the third party must default before the put is activated.

In this simple example, it is sufficient to construct a static hedge. This means the cash instruments are purchased once, and once only, for the life of the credit default swap; they will not have to be sold until the termination of the credit default swap.

The aim of this project is the pricing and hedging of defaultable claim in a discontinuous market. Market discontinuity is a shift in any of the market forces that can be predicted and affect the performance of the company. The project is organized as as follows; Chapter 2 describes the mathematical preliminaries such as probability and stochastic processes, Poisson processes, basic stochastic calculus and financial market. In Chapter 3 we introduce various approaches to pricing and hedging of defaultable claim in a discontinuous market. The pricing of a defaultable claim under deterministic intensity and under stochastic intensity is discussed in Chapter 4, while Chapter 5 is devoted to the conclusion.

Chapter 2

MATHEMATICAL PRELIMINARIES

This chapter deals with a review of some mathematical results that are important for the study of pricing and hedging of credit default swap in a discontinuous market. These results comprise the basic concept of stochastic processes and their properties, Lévy processes which are stochastic processes with jumps. We present stochastic calculus with jump culminating in the solution of stochastic differential equation with jump.

More details on the basic concepts of stochastic processes can be found in [1], [10] and [12]. Lévy processes are discussed extensively in the excellent books [13] and [17].

2.1 PROBABILITY AND STOCHASTIC PROCESS

This section deals with the definitions of probability and stochastic processes.

2.1.1 Probability as a measure

Definition 2.1 (Measurable space). *Given a non empty set Ω, a σ-algebra $\mathcal{F}$ on Ω is a collection of subsets of Ω satisfying the following three conditions:*

(i) $\Omega \in \mathcal{F}$;

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, *where* $A^c = \Omega \setminus A$;

(iii) $A_1, A_2, A_3, \dots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

Definition 2.2 (Measurable function). *Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be two measurable spaces. A function $f : E \to F$ is measurable if for all $A \in \mathcal{F}$, $f^{-1}(A) \in \mathcal{E}$.*

Definition 2.3 (Generated σ-algebra). *Given a measurable space $(\Omega, \mathcal{F})$ and $\mathcal{A}$, a set of subsets of Ω, we define the σ-algebra generated by $\mathcal{A}$ (denoted by $\sigma(\mathcal{A})$) by*

$$\sigma(\mathcal{A}) := \bigcap \big\{\mathcal{H} : \mathcal{H} \ is \ a \ \sigma - algebra \ and \ \mathcal{A} \subseteq \mathcal{H}\big\},$$

that is, $\sigma(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$.

An important example of a generated σ-algebra is $\mathcal{B}(\mathbb{R})$ which is generated by the open subsets of $\mathbb{R}$ and referred to as the Borel σ-algebra.

Definition 2.4 (Standard product space). *Given measurable spaces $(S_1, \mathcal{S}_1)$ and $(S_2, \mathcal{S}_2)$, we define the direct product of $\mathcal{S}_1$ and $\mathcal{S}_2$, denoted by $\mathcal{S}_1 \otimes \mathcal{S}_2$, to be the σ-algebra generated by sets of the form $\mathbf{B}_1 \times \mathbf{B}_2$, where $\mathbf{B}_1 \in \mathcal{S}_1$ and $\mathbf{B}_2 \in \mathcal{S}_2$. Then $(S_1 \times S_2, \mathcal{S}_1 \otimes \mathcal{S}_2)$ forms a measurable space which we call the standard product space.*

Definition 2.5 (Probability space). *Given a measurable space $(\Omega, \mathcal{F})$. A probability measure is a mapping $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfying the following conditions:*

(i) $\mathbb{P}(A) \geq 0$, $\forall A \in \mathcal{F}$*;*

(ii) $\mathbb{P}(\Omega) = 1, \quad \mathbb{P}(\emptyset) = 0$*;*

(iii) If $A_1, A_2, A_3, \cdots \in \mathcal{F}$ *with* $A_i \cap A_j = \emptyset$, $\forall i \neq j$*, then* $\mathbb{P}\Big(\bigcup\limits_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ *is called probability space.*

Note that a mapping $\mu : \mathcal{F} \to \mathbb{R}$ that satisfies property (i) and (iii) is called a measure, and the triplet $(\Omega, \mathcal{F}, \mu)$ is called measure space.

Definition 2.6 (Random Measure)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $(\Omega, \mathcal{F})$ *a measurable space. Then* $M : \Omega \times \mathcal{F} \to \mathbb{R}$ *is a random measure if*

- *For every* $\omega \in \Omega$*,* $M(\omega, \cdot)$ *is a measure on* $\mathcal{F}$
- *For every* $A \in \mathcal{F}$*,* $M(\cdot, A)$ *is measurable.*

Definition 2.7 (Absolute continuity and equivalence of probabilities)**.** *If* $\mathbb{P}$ *and* $\mathbb{Q}$ *are probability measures on the same measurable space* $(\Omega, \mathcal{F})$ *then we say that* $\mathbb{P}$ *is absolutely continuous with respect to* $\mathbb{Q}$*, denoted by* $\mathbb{P} \prec \mathbb{Q}$*, if* $\mathbb{Q}(A) = 0 \Rightarrow \mathbb{P}(A) = 0$, $\forall A \in \mathcal{F}$*. We say that the measures are equivalent, denoted by* $\mathbb{P} \sim \mathbb{Q}$ *if for all* $A \in \mathcal{F}$, $\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0$.

Definition 2.8 (Filtration or Information flow)**.** *Given a measurable space* $(\Omega, \mathcal{F})$*, a filtration* $\mathbb{F}$ *is a set of* σ*-algebras* $\{\mathcal{F}_t\}_{t \in I}$*, indexed by a set* $I \subset \mathbb{R}$*, with* $\mathcal{F}_t \subset \mathcal{F}$ *for each* $t \in I$ *and*

$$t_1 \leq t_2 \Rightarrow \mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \text{ for any } t_1, t_2 \in I.$$

The collection $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a filtered probability space.

Definition 2.9 (Completeness filtered probability spaces)**.** *A probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is said to be complete if* $B \subset A$, $\mathbb{P}(A) = 0 \Rightarrow B \in \mathcal{F}$*. A filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *is complete if* $(\Omega, \mathcal{F}, \mathbb{P})$ *is complete in the previous sense and* $\mathcal{F}_0$ *contains all sets* $A \in \mathcal{F}$ *such that* $\mathbb{P}(A) = 0$.

Definition 2.10. *Given a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *we say that* $A \in \mathcal{F}$ *holds almost surely a.s. if* $\mathbb{P}(A) = 1$.

2.1.2 Random variables and stochastic processes

Definition 2.11 (Random variables)**.** *Given a measurable space* $(\Omega, \mathcal{F})$, *a function* $X : \Omega \times \mathbb{R}$ *is said to be a random variable (r.v.) if, for any open set* $B \subset \mathbb{R}$,

$$X^{-1}(B) := \{\omega \in : X(\omega) \in B\} \in \mathcal{F}.$$

Note that X is alternatively referred to as $\mathcal{F}$-measurable.

Definition 2.12 ((σ-algebras generated by r.v.'s)**.** *For a r.v.* X, *on a measurable space* $(\Omega, \mathcal{F})$, *we define* $\sigma(X)$, *the* σ*-algebra generated by the r.v.* X, *by*

$$\sigma(X) := \{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}.$$

Equivalently, we could define $\sigma(X)$ to be the smallest σ-algebra on such that X is $\sigma(X)$-measurable.

Definition 2.13 (Probability density function)**.** *The probability density function (pdf) of a continuous random variable* X *with support* $\mathbb{R}$ *is an integrable function* $f(x)$ *satisfying the following:*

- $f(x)$ *is positive everywhere in the support* $\mathbb{R}$, *that is,* $f(x) > 0$, *for all* x *in* $\mathbb{R}$*;*
- *The area under the curve* $f(x)$ *in the support* $\mathbb{R}$ *is 1, that is:*

$$\int_{\mathbb{R}} f(x)dx = 1,$$

- *If* $f(x)$ *is the pdf of* x, *then the probability that* x *belongs to* A, *where* A *is some interval, is given by the integral of* $f(x)$ *over that interval, that is:*

$$\mathbb{P}(X \in A) = \int_A f(x)dx.$$

Note that in probability theory, a probability density function, is a function that describes the relative likelihood for this random variable to take on a given value.

Example 2.14 (Exponential random variable)**.** *A positive random variable Y is said to follow an exponential distribution with parameter $\lambda > 0$ if it has a probability density function of the form*

$$\lambda e^{-\lambda y} 1_{y \geq 0}.$$

Definition 2.15 (Stochastic process)**.** *Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a stochastic process $X = \{X_t\}_{t \in I}$ is a collection of random variables indexed by a set I which is often referred to as time. For each realization of the randomness ω, the trajectory $X(\cdot, \omega) : t \to X_t(\omega)$ defines a function of time, called the sample path of the process.*

Thus stochastic processes can also be a random functions which are random variables taking values in function spaces.

Definition 2.16 (Càdlàg processes)**.** *A stochastic process $X = \{X_t\}_{t \in I}$ is càdlàg if its trajectories are right continuous with finite left limits a.s. at any time $t \in I$.*

Definition 2.17 (Càdlàg function)**.** *A function $f : [0, T] \to \mathbb{R}$ is said to be càdlàg if it is right-continuous with left limits: for each $t \in [0, T]$ the limits*

$$f_{t-} = \lim_{s \to t, s < t} f_s, \quad f_{t+} = \lim_{s \to t, s > t} f_s \tag{2.1}$$

exist and $f_t = f_{t+}$.

This set of functions is known as discontinuous function. In most of the literature, it is denoted by 'RCLL' which simply mean Right Continuous and Left Limit.

If t is a discontinuity point, we denote the jump of f at t by

$$\Delta f_t = f_t - f_{t-}.$$

A càdlàg function f can have a countable number of discontinuities (i.e. $\{t \in [0,T], f_t \neq f_{t-}\}$ is finite or countable).

Definition 2.18 (Adapted processes). *Given a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$*, a stochastic process* $X = \{X_t\}_{t \in I}$ *is said to be* $\mathbb{F}$*-adapted if* X_t *is* $\mathcal{F}_t$*-measurable for all* t *in* I*.*

Definition 2.19 (Predictable processes). *Given a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$*, then a continuous-time stochastic process* $(X_t)_{t \geq 0}$ *is predictable if* X*, considered as a mapping from* $\Omega \times \mathbb{R}_+$*, is measurable with respect to the* σ*-algebra generated by all left-continuous adapted processes.*

Definition 2.20 (Stopping times). *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a filtered probability space. A random variable* $\tau : \Omega \to [0, \infty]$ *is a stopping time if* $\{\tau \leq t\} \in \mathcal{F}_t$ *for all* $t \geq 0$*.*

Definition 2.21 (Stopped processes). *For* $X = \{X_t\}_{t \in I}$ *a stochastic process and* τ *a stopping time, we define the process* X *stopped at* τ *by*

$$X_t^\tau := X_{t \wedge \tau},$$

where $a \wedge b = \min(a, b)$ *for* $a, b \in \mathbb{R}$*.*

Definition 2.22 (Brownian motion). *The Brownian motion or Wiener process* $W = \{W_t\}_{t \geq 0}$ *is a stochastic process satisfying the following three properties:*

(i) $W_0 = 0$,

(ii) The trajectories W_t *are continuous a.s.,*

(iii) W *has independent increments with*

$$W_t - W_s \sim \mathcal{N}(0, t - s) \text{ for } 0 \leq s \leq t,$$

where $\mathcal{N}(\mu, \sigma^2)$ *denotes the normal distribution with expected value* μ *and variance* σ^2*.*

Definition 2.23 (Lévy Process)**.** *A càdlàg stochastic process* $(X_t)_{t\geq 0}$ *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* $\mathbb{R}$ *such that* $X_0 = 0$ *is called a Lévy process if it possesses the following properties:*

- *Independent increments: for every increasing sequence of times* $0 < t_0 < t_1 \cdots t_n$*, the random variables* $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ *are independent,*
- *Stationary increments: the law of* $X_{t+h} - X_t$ *does not depend on* t*,*
- *Stochastic continuity:* $\forall \varepsilon > 0$, $\lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$*,*
- *At any fixed time, the probability of having a jump is zero:* $\forall t, \mathbb{P}[X_{t-} = X_t] = 1$*.*

Proposition 2.1. *Let* X *be a continuous Lévy process. Then there exist* $\gamma \in \mathbb{R}$ *and a symmetric positive definite matrix* A *such that*

$$X_t = \gamma_t + W_t,$$

where W *is the Brownian motion with covariance matrix* A*.*

The proof can be found in [18], page 4.

2.1.3 Expectations

Definition 2.24 (Simple random variable)**.** *Given a measurable space* $(\Omega, \mathcal{F})$*,* X *is a simple random variable on it if it can be written in the form*

$$X(\omega) = \sum_{k=1}^{m} a_k 1_{A_k}(\omega),$$

where $a_k \in \mathbb{R}$ *and* $A_k \in \mathcal{F}$ *for all* k*.*

Definition 2.25 (Expectation of simple random variable)**.** *For a simple random variable* $X = \sum_{k=1}^{m} a_k 1_{A_k}$ *defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *its expectation is defined by*

$$\mathbb{E}(X) := \int_{\Omega} X d\mathbb{P} := \sum_{k=1}^{m} a_k \mathbb{P}(A_k).$$

Definition 2.26 (General expectations)**.** *The expectation of a non-negative random variable* X *defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is defined as*

$$\mathbb{E}(X) := \sup \left\{ \int_{\Omega} Y d\mathbb{P} : Y \geq 0 \textit{ is a simple r.v. and } Y \leq X a.s. \right\}.$$

This can be extended to a general r.v. X by introducing random variables X^+ and X^-:

- $X^+(\omega) := \max(0, X(\omega))$,
- $X^-(\omega) := -\min(0, X(\omega))$,

If $\mathbb{E}(X^+)$ and $\mathbb{E}(X^-)$ are both finite, then X is integrable and we define the expectation of X by:

$$\mathbb{E}(X) := \mathbb{E}(X^+) - \mathbb{E}(X^-).$$

Definition 2.27 (Conditional expectation)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* X *an integrable random variable on it and* $\mathcal{A}$ *a sub* σ*-algebra of* $\mathcal{F}$*. The conditional expectation of* X *given* $\mathcal{A}$ *is an* $\mathcal{A}$*-measurable function* $\mathbb{E}(X|\mathcal{A}) : \Omega \times \mathbb{R}$ *which satisfies*

$$\int_A \mathbb{E}(X|\mathcal{A}) d\mathbb{P} = \int_A X d\mathbb{P}, \quad \forall A \in \mathcal{A}.$$

We can also define the conditional expectation of a random variable X with respect to another random variable Y as

$$\mathbb{E}(X|Y) := \mathbb{E}(X|\sigma(Y)),$$

defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 2.1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let X and Y be random variables defined on it and let $\mathcal{A}$ and $\mathcal{G}$ be sub σ-algebras. The following properties of conditional expectation holds:*

(i) $\mathbb{E}(X|\mathcal{A}) = \mathbb{E}(X)$ *if X is independent of $\mathcal{A}$;*

(ii) $\mathbb{E}\big(\mathbb{E}(X|\mathcal{A})\big) = \mathbb{E}(X)$*;*

(iii) $\mathbb{E}\big(\mathbb{E}(X|\mathcal{A})|\mathcal{G}\big) = \mathbb{E}(X|\mathcal{G})$ *if $\mathcal{G} \subset \mathcal{A}$;*

(iv) $\mathbb{E}\big(\mathbb{E}(XY|\mathcal{A})\big) = X\mathbb{E}(Y|\mathcal{A})$ *if X is $\mathcal{A}$-measurable.*

2.1.4 Martingales

Definition 2.28 (Martingales)**.** *Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, a stochastic process $X = \{X_t\}_{0 \le t < 1}$ is a martingale relative to the filtration $\mathbb{F}$ or an $\mathbb{F}$-martingale if*

(i) *X is adapted to $\mathbb{F}$,*

(ii) $\mathbb{E}|X_t| < \infty$ *for all $0 \le t < \infty$,*

(iii) $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ *a.s. for all $0 \le s \le t$.*

The process X is a super-martingale if in place of *(iii)* we have

$$\mathbb{E}(X_t|\mathcal{F}_s) \le X_s.$$

The process X is a sub-martingale if in place of *(iii)* we have

$$\mathbb{E}(X_t|\mathcal{F}_s) \ge X_s.$$

A martingale can be constructed given a random variable Y revealed at T (i.e., $\mathcal{F}_T$ -measurable) with $E|Y| < \infty$, the process $(\mathcal{M}_t)_{t\in[0,T]}$ defined by $M_t = E[Y|\mathcal{F}_t]$ is a martingale.

Definition 2.29 (Local martingales). *An adapted stochastic process* $\{X_t\}_{t\leq 1}$ *defined on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *is a local martingale if there is an increasing (to infinity) sequence of stopping times* $\{\tau_n\}$ *such that the stopped processes*

$$X_{\tau_n} = \{X_{\tau_n \wedge t}\}$$

are $\mathbb{F}$*-martingales for each* n.

Note that every martingale is necessarily a local martingale.

Definition 2.30 (Semi-martingale). *A process* X *defined on the filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *is called a semi-martingale if it can be decomposed as*

$$X_t = M_t + A_t,$$

where M *is a local martingale and* A *is a cadlag adapted process.*

Definition 2.31 (Markov property). *Let* $X = \{X_t\}_{t\in I}$ *be a stochastic process on filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. X *possesses the Markov property if for all* $t,\ s \in I,\ s < t$, *expectation*

$$\mathbb{P}\big(X_t \in A|\mathcal{F}_s\big) = \mathbb{P}\big(X_t \in A|X_s\big).$$

The Markov property is an important property of Lévy processes and it states that the conditional probability distribution of future state of the process depends only on the present state. So for every random variable Y depending on the history $\mathcal{F}_s$ of X_s one must have

$$E[Y|\mathcal{F}_s] = E[Y|X_s].$$

Lévy processes satisfy a stronger version of the Markov property, namely, for all t, the process $(X_{t+s} - X_t)_{s\geq 0}$ has the same law as the process $(X_s)_{s\geq 0}$ and is independent of $(X_s)_{0\leq s\leq t}$.

2.2 POISSON PROCESSES

Poisson process provides a useful tool for the model of discontinuous random variable. In finance, in can be used to model a discontinuous jumps on assets prices or jumps in stock prices. A brief introduction of Poisson process on general measurable space will be given in this section.

Definition 2.32. *Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with intensity λ and for each $n \in N$, $T_n = \sum_{i=1}^{n} \tau_i$. The process $(N_t, t \geq 0)$ defined by*

$$N_t := \sum_{n\geq 1} 1_{t\geq T_n} \tag{2.2}$$

is called a Poisson process with intensity (or parameter) λ.

Proposition 2.2.1. *Let $(N_t)_{t\geq 0}$ is a Poisson process.*

(i) For any $t > 0$, the infinite sum in equation (2.2) is almost surely finite.

(ii) For any ω, the sample path (or trajectories) $t \to N_t(\omega)$ is piecewise constant with only jumps of size 1.

(iii) The sample paths (or trajectories) $t \mapsto N_t$ are càdlàg function.

(iv) $\forall t > 0$, $N_{t-} = N_t$ with probability 1.

(v) $\forall t > 0$, N_t follows the Poisson law with parameter λt:

$$\mathbb{P}[N_t = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

(vi) (N_t) is continuous in probability:

$$\forall t > 0, N_s \underset{s\to t}{\overset{\mathbb{P}}{\to}} N_t. \tag{2.3}$$

(vii) The characteristic function of N_t is given by

$$E[e^{iu.N_t}] = \exp\{\lambda t(e^{iu} - 1)\}, \forall u \in \mathbb{R}. \tag{2.4}$$

(viii) (N_t) *has independent increments: for any* $t_1 < \cdots < t_n$*,* $N_{t_n} - N_{t_{n-1}}, \cdots, N_{t_2} - N_{t_1}, N_{t_1}$ *are independent random variables.*

(ix) The increments of N *are homogeneous: for any* $t > s$*,* $N_t - N_s$ *has the same distribution as* N_{t-s}*.*

(x) (N_t) *has the Markov property:*

$$\forall t > s,\ E[f(N_t)|N_u, u \leq s] = E[f(N_t)|N_s],$$

where f *is a bounded continuous function.*

(xi) The Poisson process is a Lévy process.

The Poisson process N_t counts the number of random times $\{T_n, n \geq 1\}$ occurring in $[0, t]$, where the random times T_n are partial sums of a sequence of independent and identity distributed(i.i.d.) exponential random variables.

Definition 2.33 (Counting process). *Let* (T_n) *be a sequence of positive time with* $T_n \to \infty$ *a.s. then*

$$N_t = \sum_{n \geq 1} 1_{t \geq T_n}$$

is called a counting process

Put in an another way, a counting process is an increasing piecewise constant process with jumps of size 1 only and almost surely finite.

The characterization of Lévy processes is done by first characterizing Lévy processes which are counting processes.

Proposition 2.2. *Let* N_t *be a Lévy process and a counting process. Then* N_t *is a Poisson process.*

This proposition uses the characterisation of the exponential distribution and its detailed proof will be found in [18], page 6.

Definition 2.34 (Poisson distribution)**.** *The Poisson distribution with parameter or intensity λ is the distribution of a r.v. X which has probabilities*

$$\mathbb{P}(X = x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & \textit{if } x = 0, 1, 2, \cdots, \\ 0, & \textit{otherwise} \end{cases}$$

Note that the mean and variance of X are both equal to λ.

Definition 2.35 (Compound Poisson process)**.** *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution η is a stochastic process $(X_t)_{t\geq 0}$ defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i \tag{2.5}$$

where jumps sizes $(Y_i)_{i\geq 1}$ are i.i.d. with distribution η and (N_t) is a Poisson process with intensity λ, independent from $(Y_i)_{i\geq 0}$.

In other words, a compound Poisson process is a piecewise constant process which jumps at jump times of a standard Poisson process and whose jump sizes are i.i.d. random variables with a given law.

Definition 2.36 (Poisson random measure)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\Omega, \mathcal{F})$ a measurable space and μ a measure on $(\Omega, \mathcal{F})$. Then $M : \Omega \times \mathcal{F} \to \mathbb{R}$ is a Poisson random measure with intensity μ if*

- *$\forall A \in \mathcal{F}$ with $\mu(A) < \infty$, $M(A)$ follows the Poisson law with parameter $E[M(A)] = \mu(A)$.*
- *For any disjoint sets $A_1, \cdots, A_n, M(A_1), \cdots, M(A_n)$ are independent.*

A Poisson random measure is a positive integer-valued random measure. It can be constructed as the counting measure of randomly scattered points.

Definition 2.37 (Jump measure). *Let X be an $\mathbb{R}$-valued cadlag process. The jump measure of X is a random measure on $\mathcal{B}\Big([0;1) \times \mathbb{R}^d\Big)$ defined by*

$$J_X(A) = \#\{t : \Delta X_t \neq 0 \quad and \quad (t, \Delta X_t) \in A\},$$

where $\mathcal{B}$ is a Borelian σ-algebra.

The jump measure of a set of the form $[s,t] \times A$ counts the number of jumps of X between s and t such that their sizes fall into A. For a counting process, since the jump size is always equal to 1, the jump measure can be seen as a random measure on $[0;\infty)$.

Proposition 2.3. *Let X be a Poisson process with intensity λ. Then J_X is a Poisson random measure on $[0;\infty)$ with intensity $\lambda \times dt$.*

It can be said that a crucial result of the theory of Lévy processes is that the jump measure of a general Lévy process is also a Poisson random measure.

2.2.1 Path structure of a Lévy process

Definition 2.38 (Lévy measure). *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$. The measure ν on $\mathbb{R}$ defined by:*

$$\nu(A) = E\Big[\#\Big\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\Big\}\Big], \quad A \in \mathcal{B}(\mathbb{R}) \tag{2.6}$$

is called the Lévy measure of X.

Theorem 2.39 (Lévy-Itô decomposition). *Let $(X_t)_{t\geq 0}$ be $\mathbb{R}$-valued Lévy process and ν its Lévy measure. Then* *(i)* ν *is a Lévy measure on* $\mathbb{R}\backslash\{0\}$ *and satisfies:*

$$\int_{\mathbb{R}} (|x|^2 \wedge 1)\nu(dx) < \infty.$$

(ii) The jump measure J_X*, is a Poisson random measure on* $[0, \infty[\times\mathbb{R}$ *with intensity measure* $\nu \times dt$*.*

(iii) There exist a vector $\gamma \in \mathbb{R}$ *and a d-dimensional Brownian motion* $(B_t)_{t\geq 0}$ *with covariance matrix A such that*

$$X_t = \gamma t + B_t + N_t + M_t, \tag{2.7}$$

where

$$N_t = \int_{|x|\geq 1, s\in[0,t]} xN(ds \times dx)$$

and

$$M_t = \int_{\epsilon\leq|x|<1, s\in[0,t]} x\{N(ds \times dx) - \nu(dx)ds\}.$$

The first three terms in equation (2.7) are independent and the convergence in the last term is almost sure and uniform in t on the set $[0, T]$.

Consider a triplet (A, ν, γ) which is called characteristic triplet or Lévy triplet of the process X_t, the Lévy-Itô decomposition says that for every Lévy process there exist a vector γ, a positive definite matrix A and a positive measure ν that uniquely determine its distribution.

A proof of the **Theorem 2.39** can be found in page 96 of [17].

Proposition 2.4 (Lévy-Khinchin representation)**.** *Let* $(X_t)_{t\geq 0}$ *be a Lévy process on* $\mathbb{R}$ *with characteristic triplet* (A, ν, γ)*. Then its characteristics function is given by*

$$E[e^{iuX_t}] = \exp\left\{t(i\gamma u - \frac{Au^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{|x|\leq 1})\nu(dx))\right\}. \tag{2.8}$$

Proposition 2.5. *Let* $(X_t)_{t\geq 0}$ *be a Lévy process on* $\mathbb{R}$ *with characteristic triplet* (A, ν, γ)

(i) (X_t) *is a martingale if and only if* $\int_{|x|\geq 1} |x|\nu(dx) < \infty$ *and*

$$\gamma + \int_{|x|\geq 1} x\nu(dx) = 0.$$

(ii) *exp*(X_t) *is a martingale if and only if* $\int_{|x|\geq 1} e^x\nu(dx) < \infty$ *and*

$$\frac{A}{2} + \gamma + \int_{-\infty}^{\infty} (e^x - 1 - x1_{|x|\leq 1})\nu(dx) = 0.$$

This proposition is a consequence of the Lévy-Khinchin formula.

2.3 BASIC STOCHASTIC CALCULUS FOR JUMP PROCESSES

This section presents a brief overview of stochastic integration and stochastic differential equation. The interested reader is referred to [12],[10] and [13] for a rigorous treatment of the material.

Definition 2.40 (Stochastic integral)**.** *Let* X *be a semi-martingale and* H *a locally bounded, adapted càglàd process given on a common filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$*. Let* π_n *be a sequence of partitions of* $[0,t]$ *with* $\lim_{n\to\infty} \|\pi_n\| \to 0$*. Then we define the stochastic integral (or Itô integral) of* H *with respect to* X *by*

$$\int_0^t H_u dX_u := \lim_{n\to\infty} \sum_{t_{i-1}, t_i \in \pi_n} H_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}),$$

where this limit is defined in terms of convergence in probability [15] and [1].

The stochastic integral defined in this manner is itself a semi-martingale. This definition can be extended to allow for predictable and locally bounded integrands.

Definition 2.41 (Quadratic variation/covariation)**.** *Let* X *and* Y *be stochastic processes defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. Let* π_n *be a sequence of partitions of* $[0,t]$ *with*

$\lim_{n\to\infty}\|\pi_n\| \to 0$. *Then we define the quadratic covariation of X and Y by*

$$[X,Y]_t := \lim_{n\to\infty} \sum_{t_{k-1},t_k\in\pi_n} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}),$$

where this limit is defined in terms of convergence in probability.

Quadratic variation of a single process X is given by

$$[X]_t = [X,X]_t.$$

Definition 2.42 (Stochastic differentials). *Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, a semi-martingale S and a stochastic process X expressible in the form*

$$X_t = X_0 + \int_0^t A_u du + \int_0^t B_u dS_u$$

for some $\mathbb{F}$-progressively measurable process A and B, then process X has stochastic differential dX_t given by

$$dX_t = A_t dt + B_t dS_t.$$

Lemma 2.3.1. *Let X be a semi-martingale and H a process of finite variation. Then, we have $[X,H]_t = 0$ for all t (and thus also $d[X,H]_t = 0$).*

The following theorem gives some conditions under which the process of stochastic integration relative to a local martingale preserves the local martingale property. The proof is omitted and the reader is referred to [14] for further details.

Theorem 2.3.2. *Let M be a local martingale and let H be a predictable, locally bounded stochastic process. Then the stochastic integral $\int_0^t H_u dM_u$ is a local martingale.*

The following quotes without proof are two results proved by K. Itô [15] for stochastic integrals relative to the Brownian motion process and extended to general stochastic integrals with respect to semi-martingales.

Theorem 2.3.3 (Itô's product Rule). *For semi-martingales X and Y, the stochastic differential of XY is given by*

$$d(X_tY_t) = X_{t-}dY_t + Y_{t-}dX_t + dX_tdY_t,$$

where $dX_tdY_t = d[X,Y]_t$, the differential of the quadratic covariation process.

Theorem 2.3.4 (Itô's formula). *Given a stochastic process X and a function $f(t,x)$, continuously differentiable in t and twice differentiable in x, the stochastic differential of the process $Y_t := f(t,X_t)$ is given by*

$$dY_t = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(dX_t)^2.$$

2.3.1 Change of variable formula for Lévy-Itô processes

With respect to a Poisson random measure, the stochastic integral allows us to define a new process known as Lévy-Itô process which extends the notion of the Lévy process. Note that a Lévy process statisfies

$$X_t = \mu t + \sigma W_t + \int_0^t \int_{|x|>1} xN(ds\times dx) + \int_0^t \int_{|x|\leq 1} x\bar{N}(ds\times dx),$$

where M is a Poisson random measure with intensity $dt \times \nu$. So a Lévy-Itô process can have a non-constant coefficient, that is,

$$X_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{|x|>1} \gamma_s(x)N(ds\times dx) + \int_0^t \int_{|x|\leq 1} \gamma_s(x)\bar{N}(ds\times dx),$$

where μ and σ are adapted locally bounded processes and $\gamma_t(x)$ is an adapted random function, left-continuous in t, measurable in x, such that the process

$$\int_{|x|>1} \gamma_t^2(x)\nu(dx)$$

is locally bounded.

In the absence of jumps, the change of variable formula (Itô formula) for a function $f \in C^2$ takes the form

$$f(X_T) = f(X_0) + \int_0^T f'(X_t)dX_t + \frac{1}{2}\int_0^T f''(X_t)\sigma_t^2 dt.$$

When the process has a finite number of jumps on $[0,T]$, one can write $X_t := X_t^c + \sum_{s\le t}\Delta X_s$ where X_t^c is the continuous part of X_t and ΔX_s are jumps in X_t and apply the same formula between the jump times:

$$f(X_T) = f(X_0) + \int_0^T f'(X_t)dX_t^c + \frac{1}{2}\int_0^T f''(X_t)\sigma_t^2 dt + \sum_{t\le T:\Delta X_t\neq 0}\{f(X_t) - f(X_{t-})\}.$$

When the number of jumps is infinite, the later sum may diverge, but we still have

$$f(X_T) = f(X_0)+\int_0^T f'(X_{t-})dX_t+\frac{1}{2}\int_0^T f''(X_t)\sigma_t^2 dt+\sum_{t\le T:\Delta X_t\neq 0}\{f(X_t)-f(X_{t-})-f'(X_{t-})\Delta X_t\} \tag{2.9}$$

To make the decomposition appear and show that the class of Lévy-Itô processes is stable with respect to transformations with C^2 functions, we rewrite the above expression as follows:

$$\begin{aligned} f(X_T) \quad = \quad & f(X_0) + \int_0^T \Big\{\mu_t f'(X_{t-}) + \frac{1}{2}\int_0^T f''(X_t)\sigma_t^2 dt \\ & + \int_{|x|\le 1} (f(X_t+\gamma_t(x)) - f(X_t) - \gamma_t(x)f'(X_t))\nu(dx)\Big\}dt \\ & + \int_0^T f'(X_t)\sigma_t dW_t + \int_0^T\int_{|x|\le 1} (f(X_{t-}+\gamma_t(x)) - f(X_{t-}))\tilde{M}(dt\times dx) \\ & + \int_0^T\int_{|x|\le 1} (f(X_{t-}+\gamma_t(x)) - f(X_{t-}))M(dt\times dx). \end{aligned}$$

Proposition 2.6 (Stochastic exponential)**.** *Let $(X_t)_{t\le 0}$ be a Lévy-Itô process with volatility coefficient σ. There exists a unique cadlag process $(Z)_{t\le 0}$ such that*

$$dZ_t = Z_{t-}dX_t \quad Z_0 = 1. \tag{2.10}$$

Z is given by:

$$Z_t = e^{X_t - \frac{1}{2}\int_0^T \sigma_s^2 ds}\prod_{0\le s\le T}(1+\Delta X_s)e^{-\Delta X_s}. \tag{2.11}$$

Z is called the stochastic exponential of X and is denoted by $Z = \mathcal{E}(X)$.

2.3.2 Stochastic Differential Equations

Geometric Lévy process is a solution to Stochastic differential equation driven by Lévy process.

The Geometric Lévy process: Consider the stochastic differential equation

$$dX_t = X_{t-}\left[\alpha dt + \beta dB_t + \int_{\mathbb{R}} \gamma(t,z)\bar{N}(dt,dx)\right], \tag{2.12}$$

where α, β are constants, $\gamma(t,z) \geq 1$ and $\bar{N}(dt,dx)$ is given as

$$\bar{N}(dt,dx) = \begin{cases} N(dt,dx) - \nu(dx)dt, & \text{if } |x| < R, \\ N(dt,dx), & \text{if } |x| \geq R, \end{cases}$$

for some $R \in [0,\infty]$. The solution of X_t will be found but we will first rewrite equation (2.12) as follows:

$$\frac{dX_t}{X_{t-}} = \alpha dt + \beta dB_t + \int_{\mathbb{R}} \gamma(t,x)\bar{N}(dt,dx).$$

Now, define

$$Y_t = \ln X_t.$$

Then by Itô formula

$$\begin{aligned} Y_t &= \frac{X_t}{X_{t-}}\Big[\alpha dt + \beta dB_t\Big] + \int_{|x|<R} \Big\{ \ln(X_{t-} + \gamma(t,x)X_{t-}) - \ln(X_{t-}) \\ &\quad -X^{-1}(t^{-}\gamma(t,z)X_{t-})\Big\}\nu(dx)dt \\ &\quad \int_{\mathbb{R}} \Big\{ \ln(X_{t-} + \gamma(t,x)X_{t-}) - \ln(X_{t-})\bar{N}(dt,dx)\Big\} \\ &= (\alpha - \tfrac{1}{2}\beta^2)dt + \beta dB_t + \int_{|x|<R} \{\ln(1+\gamma(t,x)) - \gamma(t,x)\}\nu(dx)dt \\ &\quad + \int_{\mathbb{R}} \{\ln(1+\gamma(t,x))\bar{N}(dt,dx)\Big\}. \end{aligned}$$

Hence

$$\begin{aligned} Y_t &= Y_0 + (\alpha - \tfrac{1}{2}\beta^2)t + \beta B_t + \int_0^t \int_{|z|<R} \{\ln(1+\gamma(s,x)) - \gamma(s,x)\}\nu(dx)ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \{\ln(1+\gamma(s,x))\bar{N}(ds,dx)\} \end{aligned}$$

and this gives the solution

$$\begin{aligned} X_t &= X_0 \exp\Big\{ + (\alpha - \tfrac{1}{2}\beta^2)t + \beta B_t \\ &\quad + \int_0^t \int_{|x|<R} \{\ln(1+\gamma(s,x)) - \gamma(s,x)\}\nu(dx)ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \{\ln(1+\gamma(s,x))\bar{N}(ds,dx)\Big\}. \end{aligned}$$

We call the process X_t a geometric Lévy process.

Theorem 2.43 (**Existence and Uniqueness of Solutions of Lévy SDEs**). *Consider the following Lévy SDEs in* $\mathbb{R} : X_0 = x_0 \in \mathbb{R}$ *and*

$$dX_t = \alpha(t, X_t)dt + \sigma(t, X_t)dB_t + \int_{\mathbb{R}} \gamma(t, X_{t-}, z)\bar{N}(ds, dz)$$

where $\alpha : [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ *and* $\gamma : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfy the following conditions*

- *There exist a constant* $C_1 < \infty$ *such that*

$$||\sigma(t,x)||^2 + |\alpha(t,x)|^2 + \int_{\mathbb{R}} \sum_{k=1}^{l} |\gamma_k(t,x,z)|^2 \nu_k(dz_k) \leq C_1(1+|x|^2)$$

  for all $x \in \mathbb{R}$

- *There exist a constant* $C_2 < \infty$ *such that*

$$||\sigma(t,x) - \alpha(t,y)||^2 + |\alpha(t,x) - \alpha(t,y)|^2 + \sum_{k=1}^{l} \int_{\mathbb{R}} \gamma^{(k)}(t,x,z) - \gamma^{(k)}(t,y,z_k)|^2 \nu_k(dz_k) \leq C_2|x-y|^2;$$

  for all $x, y \in \mathbb{R}$.

Then there exists a unique cadlag adapted solution X_t *such that*

$$E[|X_t|^2] < \infty \quad \textit{for all } t.$$

Solutions of Lévy SDEs in the time homogeneous case are called Lévy diffusions.

2.4 FINANCIAL MARKET

The fundamental principle in pricing theory in an ideal financial market is that there are no arbitrage opportunities. In real world, arbitrage opportunities do exist but only for very short time periods. If every claim can be replicated perfectly, i.e. at time 0 the investor can set up a portfolio and has an adapted trading strategy which replicates the payoff of the claim perfectly at maturity, then the market is called complete. In a complete market under the absence of arbitrage, the price of any claim is uniquely determined as the value of its replicating portfolio.

It is assumed that we are operating in a discontinuous market, for example when asset prices are observed over small time scales, in particular in the case of high frequency data. In such a context the price trajectories are typically piecewise constant and jump only at random discrete points in time in reaction to trading or significant new information. While in such a context one observes frequent jumps, discontinuous models with less frequent jumps may arise whenever small changes in prices are neglected and only major price movements are registered as a jump. It will equally be assumed that all securities are perfectly divisible, i.e. we can purchase and sell portions of a single unit of an asset.

In constructing the market model, we first assume an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfying the conditions of completeness and right-continuity. We assume that $\mathcal{F}_T = \mathcal{F}$ and that $\mathcal{F}_0$ is trivial in the sense that, for every $A \in \mathcal{F}_0$, either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$. Filtration is modeling the flow of information available to traders in the market. The probability measure $\mathbb{P}$ is called the real world probability measure. It is assumed that there are $d+1$ primary traded assets (stocks, bonds or options), whose prices are given by stochastic processes $S_0, \cdots, S_d$ which are adapted to the filtration and are continuous and strictly positive semi-martingale. This technical assumption were made so that stochastic integrals can be interpreted in relation to the stock prices in accordance with the general theory set out.

The price processes are implicitly measured in units relative to some common measure of value, known as a numéraire.

Definition 2.44 (Numéraire)**.** *A numéraire is a price process X_t that is strictly positive a.s. for each $t \in [0, T]$.*

Assume $S_0(t)$ to be the price process of a non-dividend paying asset, which is strictly positive a.s. and so can be used as our numéraire. Traditionally the bank account B_t is used as a numéraire.

Definition 2.45 (Bank account)**.** *The bank account B_t specifies the value at time $t \in [0, T]$ of 1 unit invested at time 0. It is usually specified by*

$$B_t = e^{R_t},$$

where R_t is a positive process and $R_0 = 0$.

This model is used to value contingent claims which is interpreted in the economic sense as being financial contracts whose value is determined exactly by the price of an underlying financial asset.

Definition 2.46 (Contingent claim)**.** *A contingent claim X with maturity date T is an arbitrary $\mathcal{F}_T$-measurable random variable.*

The concept of trading strategy is a key tool in the set up of no-arbitrage type of argument.

Definition 2.47 (Trading strategy)**.** *A trading strategy is an $\mathbb{R}^{d+1}$-valued predictable locally bounded process*

$$\phi(t) = (\phi_0(t), \cdots, \phi_d(t)), \quad t \in [0, T],$$

satisfying $\int_0^T \mathbb{E}(\phi_0(t))dt < \infty$, $\sum_d^{i=0} \int_0^T \mathbb{E}(\phi_i^2(t))dt < \infty$, so that the stochastic integral $\int_0^t \phi(u)dS_u$ exists.

Here $\phi_i(t)$ is the number of shares of asset i held in the portfolio at time t. The predictability of ϕ means that the composition of the portfolio at time t is entirely determined by information available before time t, i.e. the investor determines how many units of each stock to hold at time t based on the stock prices S_{t-}. A negative value of a component of ϕ indicates that the particular stock has been short sold. The ability for the components of ϕ to take non-integer values represents the assumption that the traded stocks are perfectly divisible, i.e. we can purchase/sell a portion of 1 stock unit.

Definition 2.48 (Value process)**.** *The value process of the trading strategy , denoted by $V_\phi(t)$, is given by*

$$V_\phi(t) := \sum_{i=0}^{d} \phi_i(t) S_i(t), \quad t \in [0, T],$$

which, as its name suggests, is simply the value of the portfolio at time t.

Definition 2.49 (Gain process)**.** *Gain process $G_\phi(t)$ is define as*

$$G_\phi(t) := \sum_{i=0}^{d} \int_0^t \phi_i(u) dS_i(u),$$

which represents the capital gains generated by the portfolio.

Definition 2.50 (Self-financing trading strategies)**.** *A trading strategy is called self-financing if $V_\phi(t)$ satisfies*

$$V_\phi(t) = V_\phi(0) + G_\phi(t) \text{ for all } t \in [0, T],$$

which says that changes in the value of our portfolio come only from capital gains and not from injections or withdrawals of funds.

Our process can now be expressed in terms of the designated numéraire $S_0(t)$ which can be discounting by the bank account B_t

Definition 2.51 (Discounted price process)**.** *Discounted price process can be defined by*

$$\tilde{S}(t) := \frac{S_t}{S_0(t)} = (1, \tilde{S}_1(t), \cdots, \tilde{S}_d(t)),$$

where $\tilde{S}_i(t) = S_i(t)/S_0(t)$, $i = 1, 2, \cdots, d$.

Definition 2.52 (Discounted wealth process)**.** *Discounted wealth process is defined by*

$$\tilde{V}_\phi(t) := \frac{V_\phi(t)}{S_0(t)} = \phi_0(t) + \sum_{i=1}^{d} \phi_i(t)\tilde{S}_i(t).$$

Definition 2.53 (Discounted Gain process)**.** *Discounted gain process is defined by*

$$\tilde{G}_\phi(t) := \sum_{i=1}^{d} \int_0^t \phi_i(u) d\tilde{S}_i(u).$$

As was mentioned earlier, the presence of arbitrage gives investors the ability to generate riskless profits from no initial outlay and thus represents a market failure. The aim of no-arbitrage pricing is to establish conditions on this market model that eliminate potential arbitrage opportunities. To do this we require a formal definition of what arbitrage means in this market model:

Definition 2.54 (Arbitrage Opportunity)**.** *A self-financing trading strategy ϕ is an arbitrage opportunity if V_ϕ satisfies the conditions:*

(i) $V_\phi(0) = 0$ *(Zero initial net-investment),*

(ii) $\mathbb{P}(V_\phi(T) \geq 0) = 1$ *(No chance of loss),*

(iii) $\mathbb{P}(V_\phi(T) > 0) > 0$ *(positive probability of gain).*

A key tool in no-arbitrage pricing is the concept of equivalent martingale measures.

Definition 2.55 (Equivalent martingale measure, EMM.)**.** *Given a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, we say that $\mathbb{P}^*$ is an equivalent martingale measure (EMM) if:*

(i) $\mathbb{P}^* \sim \mathbb{P}$*,*

(ii) The discounted price process $\tilde{S}$ is $\mathbb{P}^*$-martingale.

This market model, time is treated as a continuous variable. In the case of discrete time it can also be shown that an arbitrage-free market model must admit an EMM yielding what is known as the fundamental theorem of asset pricing which states that for a market model, the No Arbitrage (NA) condition is equivalent to the existence of an EMM. In continuous time models, a stronger condition than NA is needed. At this point we assume that there exists an EMM $\mathbb{P}^*$ for this market model (and thus no arbitrage opportunities) and consider a subclass of trading strategies.

Definition 2.56 (Admissible trading strategy)**.** *A self-financing trading strategy ϕ is called* ($\mathbb{P}^*$)*-admissible if the discounted gains process $\tilde{G}_\phi(t)$ is a $\mathbb{P}^*$-martingale.*

The link between our model and the valuation of contingent claims is the concept of a replicating strategy, which we define as follows.

Definition 2.57 (Replicating strategy)**.** *An admissible trading strategy ϕ such that*

$$V_\phi(t) = X$$

is a replicating startegy for a contingent claim X.

Definition 2.58 (Attainable Claim)**.** *We say that a contingent claim X is attainable if a replicating strategy for X exists.*

Thus, for an attainable contingent claim, a portfolio which produces the same cash flow at maturity can be constructed and is thus equivalent to holding the claim itself. So the price of the contingent claim X at time t, denoted by $P(t)$, should satisfy

$$P(t) = V_\phi(t) \text{ for all } t \in [0, T]$$

for there to be no arbitrage opportunities.

Chapter 3

PRICING AND HEDGING IN DISCONTINUOUS MARKET

In many stochastic models, the exclusion of the simultaneous buying and selling of securities in different markets or in derivative form in order to take advantage of differences in prices for the same asset is an essential property. Market completeness is not realistic financially and in theory, it is a robust property. Indeed, it can be seen that, given a complete market model, the addition of even a small jump risk breaks down market completeness. Thus, in models with jumps, market completeness is an exception rather than the rule [17]. An option can be valued only in one arbitrage-free way in a complete market. And this is done by defining the value of the option as the cost of replicating the option. To perfectly hedge in a real market is not possible which makes pricing by replication meaningless. This is because even in continuous time trading, there are risks that one cannot perfectly hedge. Thus we have to reconsider hedging in the more realistic sense by approximating a target payoff with a trading strategy. Different ways to measure risk thus lead to different approaches to hedging which consists of two parts: the cost of the hedging strategy and a risk premium required by the option seller to cover her residual (unhedgeable) risk. The various approaches to pricing and hedging options in discontinuous markets will be discussed as follows: Merton's approach presented in Section **3.1**, ignores the extra risk jumps. The notion of superhedging, discussed in Section **3.2** leads to bound of prices and it is a preference-free approach to the hedging problem in discontinuous markets. Choosing an optimal hedge by minimizing some measures of hedging error is the idea of the combination of utility maximization and dynamic trading. This leads to the notion of utility indifference price, discussed in Section **3.3**.

3.1 MERTON'S APPROACH

Robert Merton first introduced the application of jump process in option pricing. He considered the jump-diffusion model

$$S_t = S_0 \exp\Big[\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\Big], \tag{3.1}$$

where W_t is a Brownian Motion, N_t is a Poisson process with intensity λ independent of W_t and $Y_i \sim N(m, \delta^2)$ are i.i.d. random variables independent from W, N. Merton assigns a choice as in the Black-Scholes model by changing the drift of the Brownian motion but leaving the other ingredients unchanged:

$$\mathbb{P}_M : \quad S_t = S_0 \exp\Big[\mu^M t + \sigma W_t^M + \sum_{i=1}^{N_t} Y_i\Big], \tag{3.2}$$

where W_t^M is a standard Brownian Motion, N_t, Y_i are as in (3.1), independent from W^M and μ^M is chosen such that $\hat{S}_t = S_t e^{-rt}$ is a martingale under $\mathbb{P}$:

$$\mu^M = r - \frac{\sigma^2}{2} - \lambda E\big[e^{Y_i} - 1\big] = r - \frac{\sigma^2}{2} - \lambda\Big[\exp\big(m + \frac{\delta^2}{2}\big) - 1\Big]. \tag{3.3}$$

Equation (3.2) is an equivalent martingale measure obtained by shifting the drift in (3.1) while the jumps are left unchanged. Merton justified that in equation (3.2), the risk-neutral properties of the jump component of S_t are supposed to be the same as its statistical properties. He applied this in the price of a European option with a payoff $H(S_T)$ which is given by

$$\Pi_t^M = e^{-r(T-t)} E^{\mathbb{P}_M}\big[H(S_T)|\mathcal{F}_t\big]. \tag{3.4}$$

Furthermore, since S_t is a Markov process under $\mathbb{P}_M$, so $\mathcal{F}_t$ contains as much information as S_t , thus:

$$\Pi_t^M = \Pi^M(t, S_t) = e^{-r(T-t)} E^{\mathbb{P}_M}\big[(S_T - K)^+ | S_t = S\big]. \tag{3.5}$$

Then by conditioning on the number of jumps N_t , we can express Π_t^M as a weighted sum of Black-Scholes prices, setting $\tau = T - t$:

$$\begin{aligned} \Pi^M(t,S) &= e^{-r\tau} \sum_{n\geq 0} \mathbb{P}_M(N_t = n) E^{\mathbb{P}_M}\left[H\left(S \exp\left[\mu^M \tau + \sigma W_\tau^M + \sum_{i=1}^n Y_i\right]\right)\right] \\ &= e^{-r\tau} \sum_{n\geq 0} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} E^{\mathbb{P}_M}\left[H\left(S e^{nm + \frac{n\delta^2}{2} - \lambda \exp(m + \frac{\delta^2}{2}) + \lambda\tau} e^{r\tau - \frac{\sigma_n^2}{2} + \sigma_n W_\tau}\right)\right] \\ &= e^{-r\tau} \sum_{n\geq 0} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \Pi(\tau, S_n, \sigma_n), \end{aligned}$$

where $\sigma_n^2 = \sigma^2 + \frac{n\delta^2}{\tau}$,

$S_n = S e^{nm + \frac{n\delta^2}{2} - \lambda\tau \exp(m + \frac{\delta^2}{2}) + \lambda\tau}$ and

$$\Pi(\tau, S; \sigma) = e^{-r\tau} E\big[H(S e^{r - \frac{\sigma^2}{2}\tau + \sigma W_\tau})\big] \tag{3.6}$$

is the value of a European option with time to maturity τ and payoff H in Black-Scholes model with volatility σ. Since

$$\hat{\Pi}_t^M = e^{-rt}\Pi_t^M = E^{\mathbb{P}_M}\big[e^{-rT}(S_T - K)^+ | \mathcal{F}_t\big],$$

the discounted value $\hat{\Pi}_t^M$ is a martingale under $\mathbb{P}_M$, so

$$\hat{\Pi}_T^M - \hat{\Pi}_0^M = \hat{H}(S_T) - E^{\mathbb{P}_M}[H(S_T)]. \tag{3.7}$$

Merton derives the hedging portfolio which is self-financing strategy (ϕ_t^0, ϕ_t) by

$$\phi_t = \frac{\partial \Pi^M}{\partial S}(t, S_{t-}), \quad \phi_t^0 = \phi_t S_t - \int_0^t \phi dS. \tag{3.8}$$

The risk from the diffusion part is hedged from this self-financing strategy, but the discounted hedging error is:

$$\hat{H} - e^{-rT} R_\phi(T) = \hat{\Pi}_T^M - \hat{\Pi}_0^M - \int_0^T \frac{\partial \Pi^M}{\partial S}(u, S_{u-}) d\hat{S}_u, \tag{3.9}$$

where $R_\phi(T) = E^{\mathbb{P}_M}\big[(S_T - K)^+ | S_t = S\big]$. From Merton's rational, jump risk can be hedged if the jumps across the stocks are independent.

3.2 SUPERHEDGING

A conventional way to hedging is to find a self-financing strategy ϕ such that

$$\mathbb{P}(V_\phi(T) = V_0 + \int_0^T \phi dS \geq H) = 1. \tag{3.10}$$

Here ϕ superhedge against the claim H. The cost of cheapest superhedging strategy is the cost of superhedging, given as;

$$\Pi^{\sup}(H) = \inf \left\{ V_0, \mathbb{P}(V_0 + \int_0^T \phi dS \geq H) = 1) \right\}.$$

When some option seller is willing to take the risk at some certain price, it means he can at least partially hedge this option with a cheaper cost, thus this price represents an upper bound for the option. Similarly, the cost of superhedging a short position in H, that is $-\Pi^{\sup}(-H)$ gives a lower bound on the price. Therefore, we pin down an interval

$$\big[-\Pi^{\sup}(-H), \Pi^{\sup}(H) \big].$$

Proposition 3.1 (Cost of superhedging)**.** *Consider a European option with a positive payoff H on an underlying asset described by a semimartingale $(S_t)_{t\in[0,T]}$ and assume that*

$$\sup_{\mathbb{P}\in M(S)} E^{\mathbb{P}}[H] < \infty, \tag{3.11}$$

where $M(S)$ is the set of probability measures.

Then the following duality relation holds:

$$\inf_{\phi\in S} \{\hat{V}_t(\phi), \mathbb{P}(\phi \geq H) = 1\} = \operatorname*{ess\,sup}_{\mathbb{P}\in M(S)} E^{\mathbb{P}}[H|\mathcal{F}_t]. \tag{3.12}$$

In particular, the cost of the cheapest superhedging strategy for H is given by

$$\Pi^{\sup}(H) = \operatorname*{ess\,sup}_{\mathbb{P}\in M_a(S)} E^{\mathbb{P}}[\hat{H}], \tag{3.13}$$

where $M_a(S)$ is the set of martingale measure absolutely continuous with respect to $\mathbb{P}$.

More details on the above result can be found in page 74 of [7].

With respect to equivalent martingale measures, superhedging cost corresponds to the value of the option under the least favorable martingale measure.

Proposition 3.2 (Application of superhedging in exponential-Lévy model). *Consider $S_t = S_0 \exp X_t$ where (X_t) is a Levy process,*

- *if X has infinite variation, no Brownian component, negative jumps of arbitrary size and Levy measure ν: $\int_0^1 \nu(dy) = +\infty$ and $\int_{-1}^0 \nu(dy) = +\infty$ then the range of prices*

$$\left[\inf_{\mathbb{P}\in M(S)} E^{\mathbb{P}}[(S_T - K)^+], \sup_{\mathbb{P}\in M(S)} E^{\mathbb{P}}[(S_T - K)^+]\right]$$

  for a call option is given by

$$[(S_0 e^{rT} - K)^+, S_0]$$

- *if X is a jump-diffusion process with diffusion coefficient σ and compound Poisson jumps, then the price range for the call option is*

$$[C^{BS}(0, S_0; T, K; \sigma), S_0],$$

  where $C^{BS}(0, S, T, K, \sigma)$ denote the value of a call option in a Black-Scholes model with volatility σ.

From the above, the superhedging cost is too high. More details on the above result can be found in page 40 of [6] and page 215 of [2] respectively.

3.3 UTILITY MAXIMIZATION

The unrealistic results of the superhedging approach stem from the fact that it gives equal importance to hedging in all scenarios which can occur with nonzero probability, regardless of the actual loss in a given scenario. But a more flexible approach involves weighting scenarios according to the losses incurred and minimizing this weighted average loss. We formalized this idea using the notion of expected utility. Expected utility has a long tradition in the theory of choice under uncertainty. An agent pick some strategy to maximize utility level:

$$\max_{Z} E^{\mathbb{P}}[U(Z)], \tag{3.14}$$

where $U : \mathbb{R} \to \mathbb{R}$ is concave, increasing, and $\mathbb{P}$ is seen as a probability distribution objectively describing the future events. The concavity of U is related to the risk aversion of the agent. A typical example is the logarithmic utility function $U(x) = \ln \alpha x$. Another example is the exponential utility function $U^{\alpha}(x) = 1 - \exp(-\alpha x)$ where $\alpha > 0$ determines the degree of risk aversion: a large α corresponds to a high degree of risk aversion.

3.3.1 Certainty equivalent

A classical concept to measure risk aversion for an uncertain payoff H is the notion of certainty equivalent $c(x, H)$ defined as the sum of cash which, added to the initial wealth, results in the same level of expected utility:

$$U(x + c(x, H)) = E[U(x + H)] \Rightarrow c(x, H) = U^{-1}(E[U(x + H)]) - x.$$

At the same level x, faced with the same H, the higher compensation you require, the more you averse the risk. An investor who uses expected utility as a criterion is then indifferent between receiving the random payoff H or the lump sum $c(x, H)$.

The certainty equivalent is an example of a nonlinear valuation. In general, the certainty equivalent of $\lambda > 0$ units of the contract H is not obtained by multiplying by λ the value of one unit given by $c(x, \lambda H) \neq \lambda c(x, H)$. Also, $c(x, H)$ depends on the initial wealth x held by the investor.

3.3.2 Utility indifference pricing

If the investor follows a self-financing strategy $(\phi_t)_{t\in[0,T]}$ during $[0,T]$ to maximize her final wealth, then her final wealth is given by

$$V_T = x + \int_0^T \phi_t dS_t. \tag{3.15}$$

A utility maximizing investor will therefore, attempt to choose a trading strategy ϕ to optimize the utility of her final wealth:

$$u(x,0) = \sup_{\phi\in S} E^{\mathbb{P}}[U(x + \int_0^T \phi_t dS_t)]. \tag{3.16}$$

Suppose now the agent buys an option, with terminal payoff H, at price p, then

$$u(x-p,H) = \sup_{\phi\in S} E^{\mathbb{P}}[U(x - p + H + \int_0^T \phi_t dS_t)]. \tag{3.17}$$

Utility indifference price is therefore, define as the price $\pi_U(x,H)$

$$u(x,0) = u(x - \pi_U(x,H), H). \tag{3.18}$$

Equation (3.18) is means that an investor with initial wealth x and utility function U, trading in the underlying, will be indifferent between buying or not buying the option at price $\pi_U(x,H)$. The notion of certainty equivalent is extended by the notion of utility indifference pricing to a setting where uncertainty is taken into account.

Notice firstly that indifference pricing in not linear:

$$\pi_U(x,\lambda H) \neq \lambda\pi_U(x,H) \text{ and } \pi_U(x,H_1+H_2) \neq \pi_U(x,H_1) + \pi_U(x,H_2)$$

Second, the utility indifference price depends in general on the initial wealth of the investor except for special utility functions such as $U^\alpha(x) = 1 - e^{-\alpha x}$.

Third, buying and selling are not symmetric operations since the utility function weighs gains and losses in an asymmetric way. The utility indifference selling price defined as the price p

solves:

$$u(x,0) = u(x + p, -H). \tag{3.19}$$

which means that the selling price is given by $-\pi_U(x, -H)$, in general, it is different from the buying price $\pi_U(x, H)$. This approach naturally leads to a pair of prices $\{\pi_U(x, H), -\pi_U(x, -H)\}$.

Note that there are special cases of the expected utility maximization, where the loss function is quadratic. Here the agent choose to minimize the hedging error in the mean square sense. Different criterion to be minimized in the least squares sense can be:

- hedging error at maturity which is Mean-variance hedging and
- hedging error measure locally in time which is local risk minimization.

The two approaches are equivalent if the discounted price is a martingale measure. More details on the above approaches can be found in page 336 of [17].

Chapter 4

PRICING AND HEDGING OF DEFAULTABLE CLAIM

The objective of this chapter is a detailed study of pricing and hedging defaultable claim within the framework of generic reduced form credit risk model. It will be more suitable to deal with a generic dividend paying asset, since most basic properties of prices of defaultable assets and related trading strategies are already apparent in a general set up. The risk-neutral valuation of defaultable claim is supported by the desire to produce an arbitrage-free model of default-free and defaultable assets [4, 19]. The replication of defaultable claims in the structural approach, which was initiated by Merton and Black and Cox, is entirely different, since the value of the firm is usually postulated to be a tradeable underlying asset. Bielecki et al [3] worked within the reduced-form framework, where they focused on the possibility of an exact replication of a given defaultable claim through a trading strategy based on defaultable and default-free securities. According to Ramin Okhrati et al [11], the locally risk minimizing approach is carried out when the underlying process has jumps and the derivative linked to a default event and the probability measure is not necessarily risk-neutral. Robert A. Jarrow et al [8] generalizes existing reduced-form models to include default intensities dependent on the default of a counterparty. In their model, firms have correlated defaults not only to an exposure to common risk factors, but also to firm-specific risks that are termed 'counterparty risks'. The rest of this chapter is arranged as follows; section **4.1** gives a brief summary of general result concerning the valuation of the defaultable claim. In section **4.2**, pricing of Credit Default Swap(CDS) under deterministic intensity was discussed. Section **4.3** focuses on pricing of CDS under stochastic intensity.

4.1 PRICING DEFAULTABLE CLAIMS

A company defaults when it fails to fulfill some important obligations arising from a debt contract. A default risk is a probability that a counter-party in a financial contract will not fulfill its commitments to meet her obligations stated in the contract. If this happens, a defaultable event has occurred. According to [5], bankruptcy, failure to pay, restructuring, repudiation or moratorium, obligation and accelerated defaults are the six types of credit events.

4.1.1 Defaultable claim

A random or default time is a strictly positive random variable τ, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In order to exclude trivial cases, $\mathbb{P}\{\tau > 0\} = 1$ and $\mathbb{P}\{\tau \leq T\} \leq 1$. The jump process $H_t = 1_{\{\tau \leq t\}}$ associated with τ is introduced and $\mathbb{H}$ is the filtration generated by this process. The process H has right continuous sample path which is equal to zero before random time τ and is equal to 1 for $\tau \leq t$. If the filtration generated by H is given by $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ for any $t \in \mathbb{R}_+$, then $\mathcal{H}_t = \sigma(H_u : u \leq t)$. And if in addition, some auxiliary filtration $\mathbb{F}$ is given such that if $\mathcal{G}_t \subseteq \mathcal{F}_t$ for every $t \in [0, T]$, then $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, meaning that $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$. The information generated by the occurrence of τ up to t is represented by $\mathcal{H}_t$

The following are basic properties of the filtration $\mathbb{H}$.

(H.1) $\mathcal{H}_t = \sigma(\{\tau \leq u\} : u \leq t)$

(H.2) $\mathcal{H}_t = \sigma(\sigma(\tau) \cap \{\tau \leq t\})$

(H.3) $\mathcal{H}_t = \sigma(\tau \wedge t) \vee (\{\tau > t\})$

(H.4) $\mathcal{H}_t = \mathcal{H}_{t+}$

(H.5) $\mathcal{H}_\infty = \sigma(\tau)$

(H.6) For any $A \in \mathcal{H}_\infty$, $A \cap \{\tau \leq t\} \in \mathcal{H}_t$.

In order to establish (H.6), we consider an arbitrary event A of the form $A = \{\tau \leq s\}$ for some $s \in \mathbb{R}_+$

Lemma 4.1.1. *let Y be a $\mathcal{G}$-measurable random variable, therefore*

$$1_{\{\tau\leq t\}}\mathbb{E}_{\mathbb{P}^*}(Y|\mathcal{H}_t) = \mathbb{E}_{\mathbb{P}^*}(1_{\{\tau\leq t\}}Y|\mathcal{H}_\infty) = 1_{\{\tau\leq t\}}\mathbb{E}_{\mathbb{P}^*}(Y|\tau), \tag{4.1}$$

and

$$1_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}(Y|\mathcal{H}_t) = 1_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}^*}(1_{\{\tau>t\}}Y)}{\mathbb{P}^*\{\tau > t\}}$$

Proof. Let us check that

$$\mathbb{E}_{\mathbb{P}^*}(1_{\{\tau\leq t\}}Y|\mathcal{H}_\infty) = 1_{\{\tau\leq t\}}\mathbb{E}_{\mathbb{P}^*}(Y|\tau).$$

From the basic properties of the filtration (H.6), we have that for any $A \in \mathcal{H}_\infty$, $A \cap \{\tau \leq t\} \in \mathcal{H}_t$. Therefore

$$\begin{aligned}
\int_A \mathbb{E}_{\mathbb{P}^*}(1_{\{\tau\leq t\}}Y|\mathcal{H}_\infty)d\mathbb{P}^* &= \int_A 1_{\{\tau\leq t\}}Y d\mathbb{P}^* = \int_{A\cap\{\tau\leq t\}} Y d\mathbb{P}^* \\
&= \int_{A\cap\{\tau\leq t\}} \mathbb{E}_{\mathbb{P}^*}(Y|\mathcal{H}_t)d\mathbb{P}^* \\
&= \int_A 1_{\{\tau\leq t\}}\mathbb{E}_{\mathbb{P}^*}(Y|\mathcal{H}_t)d\mathbb{P}^* \\
&= \int_A \mathbb{E}_{\mathbb{P}^*}(1_{\{\tau\leq t\}}Y|\mathcal{H}_t)d\mathbb{P}^*,
\end{aligned}$$

since the event $\{\tau \leq t\}$ is in $\mathcal{H}_t$. To prove the second formula, we need to establish that

$$1_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}(Y|\mathcal{H}_t) = 1_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}^*}(1_{\{\tau>t\}}Y)}{\mathbb{P}^*\{\tau > t\}}.$$

We equally need to ascertain that for $A \in \mathcal{H}_t$

$$\int_A \mathbb{E}_{\mathbb{P}^*}(1_{\{\tau>t\}}Y|\mathcal{H}_t)d\mathbb{P}^* = \int_A 1_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}^*}(1_{\{\tau>t\}}Y)}{\mathbb{P}^*\{\tau > t\}}d\mathbb{P}^*.$$

If we consider events of the form $A = \{\tau \leq s\}$ for $s \leq t$, then both sides of the last equality becomes zero. But if we consider the event $A = \{\tau > t\} \in \mathcal{H}_t$, then we will obtain

$$\int_A \mathbb{E}_{\mathbb{P}^*}(1_A Y|\mathcal{H}_t)d\mathbb{P}^* = \int_A 1_A Y d\mathbb{P}^* = \int_\Omega 1_A Y d\mathbb{P}^* = \frac{\mathbb{E}_{\mathbb{P}^*}(1_{\{\tau>t\}}Y)}{\mathbb{P}^*\{\tau > t\}}\mathbb{P}\{A\}+\int_A 1_A\frac{\mathbb{E}_{\mathbb{P}^*}(1_{\{\tau>t\}}Y)}{\mathbb{P}^*\{\tau > t\}}d\mathbb{P}^*.$$

□

Definition 4.1 (Defaultable claim)**.** *A defaultable claim maturing at T is the quadruple $(X, (C_t)_{t\in[0,T]}, (Z_t)_{t\in[0,T]}, \tau)$, where X is an $\mathcal{F}_T$-measurable random variable called promised contingent claim, $(C_t)_{t\in[0,T]}$ is an $\mathbb{F}$-adapted process of finite variation called promised dividend, $(Z_t)_{t\in[0,T]}$ is $\mathbb{F}$-predictable process and τ is the default time.*

A dividend process h describe all cash flows associated with a defaultable claim over the lifespan $]0, T]$, that is, after the contract was initiated at time 0. The choice of 0 as the date of inception is arbitrary.

Definition 4.2 (Dividend process)**.** *Let $(X, (C_t)_{t\in[0,T]}, (Z_t)_{t\in[0,T]}, \tau)$ be a defaultable claim maturing at T. The dividend process h of a defaultable claim is a stochastic process defined as*

$$h_t = X1_{\{\tau>t\}}1_{[T,\infty[}(t) + \int_{]0,t]}(1 - H_u)dC_u + \int_{]0,t]} Z_u dH_u, \tag{4.2}$$

where Z is the recovery process which specifies the recovery payoff at default. We should note that the premium at time 0 is not included in the dividend process h associated with a defaultable claim.

The premium paid in installments up to dafault or maturity date is denoted by the process C. The 'price' of a defaultable claim is denoted by a constant, which is known as a constantly paid premium or credit default rate, i.e $C_t = kt$, for some constant $k > 0$.

If the payoffs X and Z of the contracts are known, then finding the level of k that makes the swap valueless at the beginning is the valuation of a swap. Most often, in a defaultable claim, $X = 0$, and Z is known in reference to the recovery rate of the reference credit-risky entity. Though the process C is discontinuous in a more practical approach with jumps occurring at the premium payments dates.

Since

$$\int_{]0,t]} (1 - H_u) dC_u = \int_{]0,t]} 1_{\{\tau > u\}} dC_u = C_{t-} 1_{\{\tau \leq u\}} + C_t 1_{\{\tau > u\}},$$

it implies that the dividend process h follows a process of finite variation on $[0, T]$ which means that if default occurs at some date t, the promised dividend $C_t - C_{t-}$ that is due to be collected at this date will be ignored.

$$\int_{]0,t]} Z_u dH_u = Z_{\tau \wedge t} 1_{\{\tau > t\}} = Z_\tau 1_{\{\tau > t\}},$$

if we denote $\tau \wedge t = \min(\tau, t)$.

The process $h_u - h_t$, $u \in [t, T]$ may depend on the past behavior of the claim prior to t wish denote all cash flows from the defaultable claim received by an investor who buys it at time$t$.

If there is a spot martingale measure $\mathbb{P}^*$, which means that $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, and when discounted by the savings account B, it is given as

$$B_t = \exp\left(\int_{]0,t]} r_u du \right).$$

4.1.2 Buy and hold strategy

Let $S^i, i = 1, \cdots, k-1$ denotes the price processes of k primary securities in an arbitrage-free financial model. All processes are assumed to be given on a filtered probability space

$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where where $\mathbb{P}$ is known as the real life probability measure. Let it equally be assumed that the processes $S^i, i = 1, \cdots, k$ follow semi-martingales and we introduce the discounted price processes $S_t^{i*} = \dfrac{S_t^i}{B_t}$.

Consider an additionally traded security that pays dividends in the time interval $[0, T]$ in line with the process of finite variation h, with $h_0 = 0$. Let S denote a yet unspecified price process of this security and let a $\mathbb{G}$-predictable, $\mathbb{R}^{k+1}$-valued process $\phi = (1, 0, \cdots, 0, \phi^k)$ be a generic trading strategy, where ϕ_t^j is the number of shares of the j^{th} asset held at time t. S^0 is identified here with S so that S is the 0^{th} asset.

If we consider a buy-and-hold strategy $\psi = (1, 0, \cdots, \psi^k)$, where ψ^k is a $\mathbb{G}$-predictable process, the associated wealth process $V(\psi)$ satisfies

$$V_t(\psi) = S_t + \psi_t^k B_t, \quad \forall t \in [0, T], \tag{4.3}$$

so that its initial value is $V_0(\psi) = S_0 + \psi_0^k$ if one unit of the 0^{th} asset was purchased at time 0, at the initial price S_0, and it was held until time T and that all the proceeds from dividends were re-invested in the savings account B using a buy-and-hold strategy $\psi = (1, 0, \cdots, 0, \psi^k)$, where ψ^k is a $\mathbb{G}$-predictable process.

Definition 4.3 (Self financing strategy). *A strategy $\psi = (1, 0, \cdots, 0, \psi^k)$ is said to be self financing if its value process satisfies the SDE*

$$dV_t(\psi) = dS_t + dh_t + \psi_t^k B_t,$$

or for every $t \in [0, T]$

$$V_t(\psi) - V_0(\psi) = S_t - S_0 + h_t + \int_{]0,t]} \psi_u^k dB_u \tag{4.4}$$

The process ψ^k, with respect to S, h and B will be represented in a way that ψ is self-financing and the random variable $X = \int_{]0,T]} B_u^{-1} dh_u$ is $\mathbb{P}^*$-integrable.

Lemma 4.1.2. *For every $t \in [0,T]$. the discounted wealth $\hat{V}_t(\psi) = B_u^{-1}V_t(\psi)$ of any self-financing buy-and-hold trading strategy ψ satisfies*

$$\hat{V}_t(\psi) = \hat{V}_0(\psi) + \hat{S}_t - \hat{S}_0 + \int_{]0,t]} B_u^{-1} dh_u. \tag{4.5}$$

Hence,

$$\hat{V}_T(\psi) - \hat{V}_t(\psi) = \hat{S}_T - \hat{S}_t + \int_{]t,T]} B_u^{-1} dh_u, \tag{4.6}$$

for every $t \in [0,T]$.

Proof. For all $t \in [0,T]$, let $\tilde{V}(\psi) := V_t(\psi) - S_t = \psi_t^k B_t$ be an auxiliary process and substituting it in equation (4.4), the following result will be obtained

$$\tilde{V}_t(\psi) = \tilde{V}_0(\psi) + h_t + \int_{]0,t]} \psi_u^k dB_u,$$

where the process $\tilde{V}(\psi)$ follows a semi-martingale. Applying the Itô product rule gives

$$\begin{aligned} d(B_t^{-1}\tilde{V}(\psi)) &= B_t^{-1} d\tilde{V}(\psi) + \tilde{V}(\psi) dB_t^{-1} \\ &= B_t^{-1} dh_t + \psi_t^k B_t^{-1} dB_t + \psi_t^k B_t dB_t^{-1} \\ &= B_t^{-1} dh_t, \end{aligned}$$

where the identity $B_t^{-1} dB_t + dB_t^{-1} = 0$. So integrating $B_t^{-1} dh_t$, it will give

$$B_t^{-1}(V_t(\psi) - S_t) = B_0^{-1}(V_0(\psi) - S_0) + \int_{]0,t]} B_u^{-1} dh_u$$

which is the same with equation (4.5). □

Lemma 4.1.2 holds if the assumption that the savings account B represented by S^k is relaxed. The price S^k is a strictly positive continuous semi-martingale. Therefore $\psi = (1, 0, \cdots, 0, \psi^k)$ is self-financing if the wealth process

$$V_t(\psi) = S_t + \psi_t^k S_t^k, \quad \forall t \in [0,T],$$

satisfies the equation

$$V_t(\psi) - V_0(\psi) = S_t - S_0 + h_t + \int_{]0,t]} \psi_u^k S_u^k,$$

for every $t \in [0,T]$.

4.1.3 Spot Martingale Measure

Consider an arbitrage-free market which accepts a martingale measure $\mathbb{P}^* \equiv \mathbb{P}$ where $\mathbb{P}$ is associated with the choice of B which is strictly positive continuous semi-martingale.

Definition 4.4 (Spot martingale measure)**.** $\mathbb{P}^*$ *is a spot martingale measure if the discounted price* $\hat{S}^{i*}$ *of any non-dividend paying traded security follows a* $\mathbb{P}^*$*-martingale with respect to* $\mathbb{G}$*.*

Remember that in any self-financing trading strategy $\phi = (0, \phi^1, \phi^2, \cdots, \phi^k)$, the discounted wealth process $V^*(\phi)$ is a local martingale under $\mathbb{P}^*$. Now, we will consider an admissible strategy for which the discounted wealth process $V^*(\phi)$ is a martingale under $\mathbb{P}^*$ and deduce that the trading strategy ψ is also admissible, so that its discounted wealth process $V^*(\psi)$ follows a martingale under $\mathbb{P}^*$ with respect to $\mathbb{G}$.

Making a natural assumption that the market value at time t of the 0^{th} security comes from the cash flow occurring in the open interval $]t, T]$, one can derive a pricing formula for the defaultable claim since $S \in [0, T]$ which implies that $S_T = \hat{S}_T = 0$. S will be referred to as the ex-dividend price of the 0^{th} asset.

Definition 4.5 (Ex-dividend price of the 0^{th} asset)**.** *A process S with $S_T = 0$ is the ex-dividend price of the 0^{th} asset if the discounted wealth process $\hat{V}(\psi)$ of any buy and hold strategy ψ follows a $\mathbb{G}$-martingale under $\mathbb{P}^*$.*

Proposition 4.1. *For every $t \in [0, T]$, the ex-dividend price process S associated with the dividend process h fulfills,*

$$S_t = B_t \mathbb{E}_{\mathbb{P}^*}\left(\int_{]t,T]} B_u^{-1} dh_u \middle| \mathcal{G}_t \right). \tag{4.7}$$

Proof. The stated martingale property (4.11) of the discounted wealth process $\hat{V}(\psi)$ gives, for every $t \in [0, T]$,

$$\mathbb{E}_{\mathbb{P}^*}\left(\hat{V}_T(\psi) - \hat{V}_t(\psi) | \mathcal{G}_t \right) = 0.$$

Considering the condition of integrability to ensure the existence of S_t in equation (4.6), we obtain

$$\hat{S}_t = \mathbb{E}_{\mathbb{P}^*}\left(\hat{S}_T + \int_{]t,T]} B_u^{-1} dh_u \middle| \mathcal{G}_t\right).$$

By the definition of ex-dividend price, $S_T = \hat{S}_T = 0$, the equation above gives equation (4.7). □

The ex-dividend price S satisfies

$$S_t = 1_{\{t<\tau\}} \hat{S}_t, \tag{4.8}$$

for every $t \in [0, T]$ and the process $\tilde{S}$ represents the ex-dividend pre-default price of the defaultable claim.

For every $t \in [0, T]$, the cumulative dividend price process $\bar{S}$ connected with the dividend process h is a $\mathbb{G}$-martingale under $\mathbb{P}^*$. It is denoted by the formula,

$$\bar{S}_t = B_t \mathbb{E}_{\mathbb{P}^*}\left(\int_{]t,T]} B_u^{-1} dh_u \middle| \mathcal{G}_t\right). \tag{4.9}$$

Replacing the saving account B by S^k, the corresponding valuation formula becomes

$$S_t = S_t^k \mathbb{E}_{\mathbb{P}^{S^k}}\left(\int_{]t,T]} (S^k)_u^{-1} dh_u \middle| \mathcal{G}_t\right) \tag{4.10}$$

where $\mathbb{P}^{S^k}$ is a martingale measure on $(\Omega, \mathcal{G}_T)$ associated with S^k, which is a probability measure on $(\Omega, \mathcal{G}_T)$ and it is denoted by the formula

$$\frac{d\mathbb{P}^{S^k}}{d\mathbb{P}^*} = \frac{S_T^k}{S_0^k B_T}, \quad \mathbb{P}^* \text{a.s.}$$

4.1.4 Self Financing Trading Strategies

Given a general trading strategy $\phi = (\phi^0, \phi^1, \cdots, \phi^k)$ with $\mathbb{G}$-predictable components, $V_t(\phi) = \sum_{i=0}^k \phi_t^i S_t^i$ is the associated wealth process $V(\phi)$ where $S^0 = S$. For all $t \in [0, t]$, $V_t(\phi) = V_0(\phi) + G_t(\phi)$ is self-financing strategy, where the gain process is given as

$$G_t(\phi) = \int_{]0,t]} \phi_u^0 dhu + \sum_{i=0}^k \int_{]0,t]} \phi_i^u dS_u^i.$$

4.1.5 Martingale properties of Prices of Defaultable Claim

The discounted cumulative dividend price $\hat{S}_t$, $t \in [0,T]$, of a defaultable claim, is a $\mathbb{P}^*$-martingale with respect to $\mathbb{G}$. The discounted ex-dividend price S_t^*, $t \in [0,T]$, satisfies

$$S_t^* = \hat{S}_t + \int_{]t,T]} B_u^{-1} dh_u, \quad \forall t \in [0,T] \tag{4.11}$$

and thus it follows a supermartingale under $\mathbb{P}^*$ if and only if the dividend process h is increasing.

In application to be considered in the next section, the finite variation process $(C_t)_{t\in[0,T]}$ is interpreted as the positive premium paid in installments by the claim holder to the counterparty in exchange for a positive recovery. It will be assumed that $(C_t)_{t\in[0,T]}$ is a decreasing process but $X \geq 0$ and $(Z_t)_{t\in[0,T]} \geq 0$.

Assuming now that $(C_t)_{t\in[0,T]} \equiv 0$, then the premium for a defaultable claim is paid in advance at time 0. In this case, the dividend process h is manifestly increasing, and thus the discounted ex-dividend price S^* is a supermartingale under $\mathbb{P}^*$. In general, the martingale properties of the price of a defaultable claim depends on the specification of a claim and conventions due to the prices.

4.2 PRICING A CREDIT DEFAULT SWAP UNDER DETERMINISTIC INTENSITY

This section deals with the pricing of CDS under deterministic intensity. Throughout this section, the spot martingale measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}_T)$ is used. Assuming that the auxiliary filtration $\mathbb{F}$ is trival, then $\mathbb{G} = \mathbb{H}$ and the interest rate r is zero.

4.2.1 Valuation of a CDS

Definition 4.6. *A CDS with a constant rate k and recovery at default is a defaultable claim $(0, C, Z, \tau)$, where $Z_t \equiv \delta_t$ and $C_t = -kt$ for every $t \in [0, T]$. The cadlag function $\delta : [0, T] \to \mathbb{R}$ denotes the default protection and the constant $k \in \mathbb{R}$ represents the CDS premium.*

4.2.2 Ex-dividend price of a Credit Default Swap

Consider a Credit Default Swap with the rate k, which was commenced at time $t = 0$. Its market value at time t depends on the level of the rate k. Assume that k is an arbitrary constant and that the default protection payment is received at the time of default, which is δ_t if the default occurs before maturity or at maturity date T.

With respect to equation (4.7), the ex-dividend price of credit default swap maturing at T with rate k is represented by the formula

$$S_t(k) = \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{t<\tau\leq T\}}\delta_\tau|\mathcal{H}_t\Big) - \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{t<\tau\}}k((\tau \wedge T) - t)|\mathcal{H}_t\Big), \tag{4.12}$$

where default protection stream is represented by the first conditional statement and the survival annuity stream is represented by the second conditional statement.

Lemma 4.2.1. *The ex-dividend price at time $t \in [s, T]$ of a credit default swap started at s, with rate k and protection payment δ_τ at default, equals*

$$S_t(k) = 1_{\{t<\tau\}}\frac{1}{G_t}\Big(-\int_t^T \delta_u dG_u - k\int_t^T G_u du\Big). \tag{4.13}$$

Proof. : From the set $\{t < \tau\}$ and in view of **Lemma 4.1.2**

$$\begin{aligned} S_t(k) &= -\frac{\int_t^T \delta_u dG_u}{G_t} - k\Big(\frac{TG_T + \int_t^T u dG_u}{G_t} - t\Big) \\ &= \frac{1}{G_t}\Big(-\int_t^T \delta_u dG_u - k\Big(TG_T - tG_t - \int_t^T u dG_u\Big)\Big). \end{aligned}$$

Since

$$\int_t^T G_u du = TG_T - tG_t - \int_t^T u dG_u, \tag{4.14}$$

therefore, equation (4.13) holds.

□

4.2.3 Market Credit Default Swap Rate

Let us assume now that the recovery function δ is given and that a credit default swap was initiated at some date $s \leq t$ and its initial price was equal to zero.

Definition 4.7. *A market credit default swap started at s is a credit default swap initiated at time s whose initial value is equal to zero. A T-maturity market credit default swap rate at time s is the level of the rate $k = k(s,T)$ that makes a T-maturity credit default swap started at s valueless at its inception. A market credit default swap rate at time s is thus determined by the equation $S_s(k(s,T)) = 0$, where S is defined by equation* (4.12).

Given Lemma 4.2.1, for all $s \in [0,T]$, the T-maturity market credit default swap rate $k(s,T)$ solves the following equation

$$\int_s^T \delta_u dG_u + k(s,T) \int_s^T G_u du = 0,$$

hence,

$$k(s,T) = -\frac{\int_s^T \delta_u dG_u}{\int_s^T G_u du}. \tag{4.15}$$

Assuming that at time $t = 0$, the market gives the premium of a credit default swap for any maturity T. From this, $k(0,T)$ is the T-maturity market credit default swap rate for a given recovery function δ written as

$$k(0,T) = -\frac{\int_0^T \delta_u dG_u}{\int_0^T G_u du}.$$

Let the maturity date T be fixed, $k(s,T)$ written as k_s and all credit default swaps have a common recovery function δ. Note that the ex-dividend pre-default value at time $t \in [0,T]$ of a credit default swap with any fixed rate k can be easily related to the market rate k_t. The following result, in which the quantity $\nu(t,s) = k_t - k_s$ represents the calendar credit default swap market rate is obtained.

Proposition 4.2. *The ex-dividend price of a market credit default swap started at s with recovery δ at default and maturity T equals, for every $t \in [s,T]$,*

$$S_t(k_s) = 1_{\{t<\tau\}}(k_t - k_s)\frac{\int_t^T G_u du}{G_t} = 1_{\{t<\tau\}}\nu(t,s)\frac{\int_t^T G_u du}{G_t}, \tag{4.16}$$

or more explicitly

$$S_t(k_s) = 1_{\{t<\tau\}}\frac{\int_t^T G_u du}{G_t}\left(\frac{\int_s^T \delta_u dG_u}{\int_s^T G_u du} - \frac{\int_t^T \delta_u dG_u}{\int_t^T G_u du}\right). \tag{4.17}$$

Proof. Observe that $S_t(k_s) = S_t(k_s) - S_t(k_t)$. From equation (4.13),

$$k(t,T) = -\frac{\int_t^T \delta_u dG_u}{\int_t^T G_u du,}$$

If $S_t(k) = 0$. Substituting k_t and k_s in equation (4.16) yields equation (4.17). □

4.2.4 Forward Start Credit Default Swap

Here, we will consider a forward start credit default swap initiated at time $s \in [0,U]$ with default protection over the future time interval $[U,T]$. Now, if the reference entity defaults before the start date U, there will be no payment and the contract is terminated. Then the price of this contract at any date $t \in [s,U]$ equals

$$S_t(k) = \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{U<\tau\leq T\}}\delta_\tau|\mathcal{H}_t\Big) - \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{U<\tau\}}k((\tau\wedge T) - U)|\mathcal{H}_t\Big). \tag{4.18}$$

The price $S_t(k)$, $t \in [s,U]$, can be considered as either the ex-dividend price or the cumulative dividend price. This is because a forward start credit default swap does not pay any dividends prior to the start date U. Note that since G is continuous, the probability of default occurs at time $U = 0$ and thus for $t = U$, equation (4.18) becomes

$$S_t(k) = \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{t<\tau\leq T\}}\delta_\tau\Big|\mathcal{H}_t\Big) - \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{t<\tau\}}k((\tau\wedge T) - t)\Big|\mathcal{H}_t\Big),$$

which is the same as equation (4.12), since a forward start credit default swap becomes a standard credit default swap at time T .

Equation (4.18) can be rewritten explicitly as

$$S_t(k) = 1_{\{t<\tau\}} \frac{1}{G_t} \Big(- \int_U^T \delta_u dG_u - k \int_U^T G_u du \Big),$$

if G is continuous.

A forward credit default swap in which k is chosen at time t in such a way that the contract is valueless at time t is known as a forward start market credit default swap at time $t \in [0, U]$. The following equation determines the corresponding pre-default forward credit default swap rate $k(t, U, T)$.

$$S_t(k(t,U,T)) = \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{U<\tau\leq T\}}\delta_\tau \Big| \mathcal{H}_t\Big) - \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{U<\tau\}} k(t,U,T)((\tau \wedge T) - U) \Big| \mathcal{H}_t\Big) = 0,$$

which gives,

$$k(t,U,T) = -\frac{\int_U^T \delta_u dG_u}{\int_U^T G_u du}$$

for every $t \in [0, U]$. We can express the price of an arbitrary credit default swap in terms of k and $k(t, U, T)$ as;

$$S_t(k) = S_t(k) - S_t(k(t,U,T)) = (k(t,U,T) - k)\mathbb{E}_{\mathbb{P}^*}\Big(1_{\{U<\tau\}}((\tau \wedge T) - U) \Big| \mathcal{H}_t\Big),$$

or more explicitly,

$$S_t(k) = 1_{\{t<\tau\}}(k(t,U,T) - k)\frac{\int_U^T G_u du}{G_t}.$$

for every $t \in [0, U]$.

Similar representation of the formula above are also valid in the case of stochastic default intensity where they are used to price options on a forward credit default swap.

4.2.5 Case of a Constant Default Intensity

Let us assume here that $F_t = 1 - e^{-\gamma t}$ for a constant default intensity $\gamma > 0$ under $\mathbb{P}^*$ and $\delta_t = \delta$ is independent of t. With regards to Lemma 4.2.1, the valuation formula for a credit default swap can be further simplified and the ex-dividend price of a credit default swap with rate k equal

$$S_t(k) = 1_{\{t<\tau\}}(\delta\gamma - k)\gamma^{-1}\Big(1 - e^{-\gamma(T-t)}\Big),$$

for every $t \in [0, T]$.

Equation (4.15) gives that $k_s = \delta\gamma$, so that the market rate k_s is independent of s for every $s < T$. This process follows a trivial martingale under $\mathbb{P}^*$. It can be observed that the ex-dividend price of a market credit default swap will not hold if default intensity is not constant.

4.2.6 Price dynamics of a Credit Default Swap

Consider a credit default swap and assume that

$$G_t = \mathbb{P}^*(\tau > t) = \exp\Big(-\int_{]0,t]} \gamma_u du\Big), \tag{4.19}$$

where the default intensity γ_t under $\mathbb{P}^*$ is a non-negative deterministic function. Let us first focus will be on the dynamics of the ex-dividend price of a credit default swap with rate k started at some date $s < T$.

Lemma 4.2.2. *The dynamics of the ex-dividend price $S_t(k)$ on $[s, T]$ are*

$$dS_t(k) = -S_{t-}(k)dM_t + (1 - H_t)(k - \delta_t\gamma_t)dt, \tag{4.20}$$

where the $\mathbb{H}$-martingale M under $\mathbb{P}^*$ is given by the formula

$$M_t = H_t - \int_{]0,t]} (1 - H_u)\gamma_u du, \quad \forall t \in \mathbb{R}_+. \tag{4.21}$$

Hence, the process $\bar{S}_t(k)$, $t \in [s,T]$, given by the expression

$$\bar{S}_t(k) = S_t(k) + \int_s^t \delta_u dH_u - \int_s^t (1-H_u)du \tag{4.22}$$

is a martingale for $t \in [s,T]$.

Proof. Recall that

$$S_t(k) = 1_{\{t<\tau\}}\tilde{S}_t(k) = (1-H_t)\tilde{S}_t(k)$$

so that

$$dS_t(k) = (1-H_t)d\tilde{S}_t(k) - \tilde{S}_{t-}(k)dH_t.$$

With equation (4.13), we obtained

$$d\tilde{S}_t(k) = \gamma_t\tilde{S}_t(k)dt + (k-\delta_t\gamma_t)dt.$$

Proof of equation (4.20) is complete given the expression of M_t in equation (4.21). To prove the second statement, it is pertinent to know that the process N given by

$$N_t = S_t(k) - \int_s^t (1-H_u)(k-\delta_u\gamma_u)du = -\int_s^t S_{u-}(k)dM_u$$

is an $\mathbb{H}$-martingale under $\mathbb{P}^*$. Though for all $t \in [s,T]$

$$\bar{S}_t(k) = N_t + \int_s^t \delta_u M_u,$$

so that $\bar{S}(k)$ is also a $\mathbb{H}$-martingale under $\mathbb{P}^*$. Observe that the cumulative dividend price of a credit default swap is represented br the process $\bar{S}(k)$ given in (4.22), so that we will expect the martingale property $\bar{S}(k)$. □

It can equally be represented as;

$$dS_t(k) = -\tilde{S}_{t-}(k)dM_t + (1-H_t)(k-\delta_t\gamma_t)dt. \tag{4.23}$$

In some cases, it can be useful to reformulate the dynamics of a market credit default swap in terms of market observables, such as credit default swap spreads. The dynamics of the ex-dividend price $S_t(k_s)$ on $[s,T]$ can also be written as

$$dS_t(k) = -S_{t-}(k)dM_t + (1-H_t)\left(\frac{\int_t^T G_u du}{G_t}d_t\nu(t,s) - \nu(t,s)dt\right). \tag{4.24}$$

4.2.7 Replication of a Defaultable Claim

A strategy $\phi_t = (\phi_t^0, \phi_t^1)$, $t \in [0,T]$ is self-financing if the wealth process $V(\phi)$, defined as

$$V_t(\phi) = \phi_t^0 S_t(k) + \phi_t^1, \tag{4.25}$$

satisfies

$$dV_t(\phi) = \phi_t^0 (dS_t(k) + dh_t), \tag{4.26}$$

where $S(k)$ is the ex-dividend price of a CDS with the dividend stream h.

Definition 4.8 (Self financing trading strategy). *A self financing trading strategy ϕ is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$ if and only if the following holds*

(i) $V_t(\phi)$ equals pre-default value of the claim $(X, 0, Z, \tau)$ on the random interval $[0, \tau \wedge T[$

(ii) $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \leq T\}$,

(iii) $V_T(\phi) = X$ on the set $\{\tau > T\}$.

If a self-financing trading strategy satisfies condition (ii) and (iii) of Definition 4.8, then (i) holds as well.

A strategy ϕ replicates a contingent claim Y if $V_T(\phi) = Y$. On the set $\{\tau \leq t \leq T\}$ the ex-dividend price $S(k) = 0$ and thus the total wealth is necessarily invested in B, so that it is constant. This means that ϕ replicates Y if and only if $V_{\tau \wedge T}(\phi) = Y$ where Y is an arbitrage contingent claim settling at T.

Lemma 4.2.3. *For any self-financing strategy ϕ on the set $t \in [0,T]$, the following equation holds for the total wealth process;*

$$\Delta_\tau V(\phi) := V_\tau(\phi) - V_{\tau-}(\phi) = \phi_\tau^0 (\delta_\tau - \tilde{S}_\tau(k)). \tag{4.27}$$

Proof. Let us assume that ϕ^0 is a cadlag function and $\mathbb{G}$-predictable. Recall that the ex-dividend price $S(k)$ drops to zero at default time. So on the set $\{\tau < T\}$, the jump of the wealth process $V(\phi)$ at time τ equals,

$$\Delta_\tau V(\phi) = \phi^0_\tau \Delta_\tau S + \Delta_\tau h,$$

where $\Delta_\tau S(k) = S_\tau(k) - S_{\tau-}(k) = -\tilde{S}_\tau(k)$ and $\Delta_\tau h = \delta_\tau$. □

In hedging of a defaultable claim, Let Y an $\mathcal{H}_T$-measurable random variable admit the following representation

$$Y = 1_{\{\tau \leq T\}} q_\tau + 1_{\{\tau > T\}} c_T, \tag{4.28}$$

where $q : [0, T] \to \mathbb{R}$ is a Borel measurable function, and c_T is a constant.

Proposition 4.3. *Assume that G is continuous and $\hat{q}$ is an cadlag function such that the random variable $\hat{q}_\tau$ is $\mathbb{P}^*$-integrable. Then the $\mathbb{H}$-martingale $\hat{M}$ is represented as*

$$\hat{M}_t = \hat{M}_0 + \int_{]0,t]} (\hat{q}_u - \hat{g}_u) dM_u \tag{4.29}$$

where the continuous function $\hat{g} : \mathbb{R}_+ \to \mathbb{R}$ is given by the formula

$$\hat{g}_t = \frac{1}{G_t} \mathbb{E}_{\mathbb{P}^*}(1_{\{t<\tau\}} \hat{q}_\tau) = \frac{1}{G_t} \int_t^\infty \hat{q}_u dG_u. \tag{4.30}$$

On the set $\{t \leq \tau\}$, $\hat{g}_t = \hat{M}_{t-}$. Then equation (4.29) can be rewritten as

$$\hat{M}_t = \hat{M}_0 + \int_{]0,t]} (\hat{q}_u - \hat{M}_{u-}) dM_u.$$

Let a contingent claim settling at T be given as a random variable Y represented in equation (4.28). Consider a defaultable claim of the form $(X, 0, Z, \tau)$, where $X = c_T$ and $Z_t = q_t$. Apply **Proposition 4.3** to the function $\hat{q}$, where $\hat{q}_t = q_t$ for $t < T$ and $\hat{q}_t = c_T$ for $t \geq T$, then

$$\hat{g}_t = \frac{1}{G_t}\Big(-\int_t^T q_u dG_u + c_T G_T \Big), \tag{4.31}$$

and thus for the process $\hat{M}_t = \mathbb{E}_{\mathbb{P}^*}(Y|\mathcal{H}_t)$, $t \in [0,T]$ satisfies

$$\hat{M}_t = \mathbb{E}_{\mathbb{P}^*}(Y) + \int_{]0,t]} (q_u - \hat{g}_u) dM_u, \tag{4.32}$$

with $\hat{g}$ represented by equation (4.31). Note that $\tilde{S}(k)$ is the pre-default ex-dividend price process of a credit default swap with rate k and maturity T, and also that $\tilde{S}(k)$ is a continuous function of t if G is continuous.

Proposition 4.4. *Assume that the inequality $\tilde{S}_t(k) \neq \delta_t$ holds for every $t \in [0,T]$. Let ϕ^0 be cadlag function given by the formula*

$$\phi_t^0 = \frac{q_t - \hat{g}_t}{\delta_t - \tilde{S}_t(k)}, \tag{4.33}$$

and let $\phi_t^1 = V_t(\phi) - \phi_t^0 S_t(k)$, where the process $V(\phi)$ is given by (4.26) *with the initial condition $V_0(\phi) = \mathbb{E}_{\mathbb{P}^*}(Y)$, where Y is given by* (4.28)*. Then the self-financing trading strategy $\phi = (\phi^0, \phi^1)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$, where $X = c_T$ and $Z_t = q_t$.*

Proof. With respect to Lemma 4.2.1, the dynamics of the price $S(k)$ is

$$dS_t = -S_{t-}(k) dM_t + (1 - H_t)(k - \delta_t \gamma_t) dt,$$

and on the set$\{\tau > t\}$ is equally

$$dS_t = d\tilde{S}_t(k) = (\gamma_t \tilde{S}_t(k) + k - \delta_t \gamma_t) dt. \tag{4.34}$$

Recall that the wealth $V(\phi)$ of any admissible self-financing strategy is an $\mathbb{H}$-martingale under $\mathbb{P}^*$. Since under the present assumptions $dB_t = 0$. The wealth process $V(\phi)$ on the set $\{\tau > t\}$, gives

$$dV_t(\phi) = \phi_t^0 (d\tilde{S}_t(k) - k dt) = -\phi_t^0 \gamma_t (\delta_t - \tilde{S}_t(k)) dt, \tag{4.35}$$

For the martingale $\hat{M} = \mathbb{E}_{\mathbb{P}^*}(Y|\mathcal{H}_t)$ associated with Y, and with regards to equation (4.32), on the set $\{\tau > t\}$,

$$d\hat{M}_t = -\gamma_t q_t - \hat{g}_t dt. \tag{4.36}$$

For every $t \in [0,T]$, it will be good to find ϕ^0 such that $V_t(\phi) = \hat{M}_t$. Focusing on the equality $1_{\{t<\tau\}} V_t(\phi) = 1_{\{t<\tau\}} \hat{M}_t$ for pre-default values, a comparison of (4.35) with (4.36) gives

$$\phi_t^0 = \frac{q_t - \hat{g}_t}{\delta_t - \tilde{S}_t(k)}, \quad \forall t \in [0,T]. \tag{4.37}$$

It can be seen that if $V_0(\phi) = \hat{M}_0$ then also for every $t \in [0,T]$, $1_{\{t<\tau\}} V_t(\phi) = 1_{\{t<\tau\}} \hat{M}_t$. The second component of a self-financing strategy ϕ is given by $\phi_t^1 = V_t(\phi) - \phi_t^0 S_t(k)$, where $V(\phi)$ is given by (4.26) with the initial condition $V_0(\phi) = \mathbb{E}_{\mathbb{P}^*}(Y)$, so that $\phi_0^1 = \mathbb{E}_{\mathbb{P}^*}(Y) - \phi_0^0 S_0(k)$.

To show that $V_t(\phi) = \hat{M}_t$ for every $t \in [0,T]$, we compare the jumps of both processes at time τ. From equation (4.32), the jump of $\hat{M} = \Delta_\tau \hat{M} = q_\tau - \hat{g}_\tau$. Using (4.27), it will result that the jump of the wealth process satisfies

$$\Delta_\tau V(\phi) = \phi_\tau^0(\delta_\tau - \tilde{S}_\tau(k)) = q_\tau - \hat{g}_\tau,$$

and thus in conclusion, $V_t(\phi) = \hat{M}_t$ for every $t \in [0,T]$. ϕ is admissible and $V_T(\phi) = V_{\tau\wedge T}(\phi) = q(\tau \wedge T) = Y$, so that ϕ replicates a claim Y. □

4.3 PRICING OF CREDIT DEFAULT SWAP UNDER STOCHASTIC INTENSITY

This section deals with hedging both default (jump) risk and spread (Volatility) risk.

4.3.1 Hazard Process

Consider that some reference filtration $\mathbb{F}$ such that $\mathcal{F}_t \subseteq \mathcal{G}$ is given. For every $t \in \mathbb{R}_+$, $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t)$. The filtration $\mathbb{G}$ is the full filtration which includes the observation of default events. Assume also that any $\mathbb{G}$-martingale is also a $\mathbb{F}$-martingale. This assumption is sometimes referred to as H hypothesis. Note that τ is an

$\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time but not necessarily an $\mathbb{F}$-stopping time. The hazard process of a random time τ is closely related to the process F defined through the formula

$$F_t = \mathbb{P}^*\{\tau \leq t | \mathcal{F}_t\}, \quad \forall t \in \mathbb{R}_+.$$

If the survival process G_t is denoted by $G_t = 1 - F_t = \mathbb{P}^*\{\tau > t | \mathcal{F}_t\}$ and for every $t \in \mathbb{R}_+$, $G_t > 0$, then the process $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$, given by the formula

$$\Gamma_t = -\ln(1 - F_t) = -\ln G_t, \quad \forall t \in \mathbb{R}_+,$$

is termed the hazard process of a random time τ with respect to the reference filtration $\mathbb{F}$ or the $\mathbb{F}$-hazard process of τ. Note that Γ follows an $\mathbb{F}$-submartingale and the hazard process becomes evident from the following equation

$$\mathbb{E}_{\mathbb{P}^*}(1_{\{T<\tau\}} | \mathcal{G}_t) = 1_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{P}^*}(G_T | \mathcal{F}_t) = 1_{\{T<\tau\}} \mathbb{E}_{\mathbb{P}^*}(e^{\Gamma_t - \Gamma_T} | \mathcal{F}_t) \tag{4.38}$$

which holds for any two dates $0 \leq t \leq T$.

In addition, the hypothesis that any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale holds. In this case, the hazard process Γ is known to be an increasing process. An additional assumption will help more, in that G is an absolutely continuous, decreasing process given by equation (4.19). Under this assumption, equality (4.38) can be rewritten as follows

$$\mathbb{E}_{\mathbb{P}^*}(1_{\{T<\tau\}} | \mathcal{G}_t) = 1_{\{t<\tau\}} \mathbb{E}_{\mathbb{P}^*}\bigg(\exp\Big(\int_t^T \gamma_u du \Big) | \mathcal{F}_t \bigg). \tag{4.39}$$

We maintain that the assumption that the interest rate risk is negligible, specifically, $r = 0$ so that $B_t = 1$ for every $t \in \mathbb{R}_+$. Finally, it is assumed that the filtration $\mathbb{F}$ is generated by a Brownian motion W under $\mathbb{P}^*$. Remember that all (local) martingales with respect to a Brownian filtration are continuous.

4.3.2 Market Credit Default Swap Rate

Here, we value a credit default swap and derive a general formula for market credit default swap rate which means that the default protection stream is now represented by an $\mathbb{F}$-predictable process δ. As before, it is assumed that the default protection payment is received at the time of default, and it is equal to δ_t if default occurs at time t, prior to or at maturity date T. To simplify certain pricing formula, one may be willing to assume instead that the default protection is given by a constant δ.

Here, the ex-dividend price of a credit default swap with rate k maturing at T is given by the formula

$$S_t(k) = \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{t<\tau\leq T\}}\delta_t|\mathcal{G}_t\Big) - \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{t<\tau\}}k((\tau\wedge T)-t)|\mathcal{G}_t\Big) \tag{4.40}$$

where the two conditional expectations represent the current values of two legs of a credit default swap: the default protection stream and the survival annuity stream.

Making the standard assumption that $\mathbb{E}_{\mathbb{P}^*}|\delta_\tau| < \infty$, the following result is a counterpart of Lemma 4.2.1 which shows that pricing formula in equation (4.13) extends to the case of stochastic default intensity.

Lemma 4.3.1. *The ex-dividend price at time $t \in [s,T]$ of a credit default swap started at s, with rate k and protection payment δ_t at default, equals*

$$S_t(K) = 1_{\{t<\tau\}}\frac{1}{G_t}\mathbb{E}_{\mathbb{P}^*}\bigg(-\int_t^T \delta_u dG_u - k\int_t^T G_u du|\mathcal{F}_t\bigg). \tag{4.41}$$

Proof. The proof of **Lemma 4.3.1** follows the same argument as the proof of **Lemma 4.2.1** in addition with the following formula

$$\mathbb{E}_{\mathbb{P}^*}(1_{\{t<\tau\leq T\}}Z_t|\mathcal{G}_t) = 1_{\{t<\tau\}}\frac{1}{G_t}\mathbb{E}_{\mathbb{P}^*}\bigg(\int_t^T Z_u dG_u|\mathcal{F}_t\bigg), \tag{4.42}$$

which holds for any $\mathbb{F}$-predictable process such that $\mathbb{E}_{\mathbb{P}^*}|Z_\tau| < \infty$.

Applying equation (4.42) to the process $Z_u = \delta_u 1_{[0,T]}(u) - k(u-t)$ for $u \in [t,T]$ makes it easier to derive (4.41) from (4.40) which gives

$$1_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{P}^*}\bigg(\int_t^T (\delta_u 1_{[0,T]}(u) - k(u-t)) dG_u | \mathcal{F}_t \bigg).$$

Using the formula (4.38) to compute the expectation $\mathbb{E}_{\mathbb{P}^*}(1_{\{T<\tau\}} k(T-t)|\mathcal{G}_t)$ gives

$$S_t(K) = 1_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{P}^*}\bigg(- \int_t^T \delta_u dG_u - k\Big(TG_T - tG_t - \int_t^T u dG_u\Big) \bigg| \mathcal{F}_t \bigg).$$

To conclude the proof, G is assumed to be a continuous increasing process and

$$\int_t^T G_u du = TG_T - tG_t - \int_t^T u dG_u.$$

$\square$

For all $s \in [0,T]$, the T-maturity credit default swap market rate $k(s,T)$ admits a generic representation analogous to (4.15), namely,

$$k(s,T) = - \frac{\mathbb{E}_{\mathbb{P}^*}\Big(\int_s^T \delta_u dG_u \Big| \mathcal{F}_t \Big)}{\mathbb{E}_{\mathbb{P}^*}\Big(\int_s^T G_u du \Big| \mathcal{F}_t \Big)} . \tag{4.43}$$

4.3.3 Price dynamics of a Credit Default Swap

Under stochastic intensity, the dynamics of a credit default swap will have an additional continuous martingale term, related to an uncertain behavior of the credit spread before default.

Proposition 4.5. *The dynamics of the ex-dividend price $S_t(k)$ on $[s,T]$ are*

$$dS_t(k) = -S_{t-}(k) dM_t + \frac{1-H_u}{G_u} d\hat{n}_t + (1-H_t)(k - \delta_t \gamma_t) dt, \tag{4.44}$$

where the $\mathbb{G}$-martingale M under $\mathbb{P}^*$ equals

$$M_t = H_t - \int_{]0,t]} (1-H_u)\gamma_u du, \quad \forall t \in \mathbb{R}_+, \tag{4.45}$$

and the continuous $\mathbb{F}$-martingale (and $\mathbb{G}$-martingale) $\hat{n}$ under $\mathbb{P}^*$ is given by the formula

$$\hat{n}_t = \mathbb{E}_{\mathbb{P}^*}\Big(- \_int_0^T \delta_u dG_u + k \int_0^T u dG_u - kTG_T | \mathcal{F}_t \Big). \tag{4.46}$$

The proof of Proposition 4.5 is base on the following predictable representation theorem.

Proposition 4.6. *Let $\hat{M}_t = \mathbb{E}_{\mathbb{P}^*}(Z_\tau | \mathcal{G}_t)$ where Z is an arbitrary $\mathbb{F}$-predictable process such that $\mathbb{E}_{\mathbb{P}^*}|Z_\tau| < 1$. Then we have, for every $t \in \mathbb{R}_+$,*

$$\hat{M}_t = \hat{M}_0 + \int_{]0,t]} (Z_u - \hat{g}_u) dM_u + \int_{]0,t]} \frac{1 - H_u}{G_u} d\hat{n}_u$$

where the continuous $\mathbb{F}$-martingale (and $\mathbb{G}$-martingale) $\hat{n}$ is given by the formula

$$\hat{n}_t = \mathbb{E}_{\mathbb{P}^*}\Big(\int_0^\infty Z_u dF_u | \mathcal{F}_t \Big) = -\mathbb{E}_{\mathbb{P}^*}\Big(\int_0^\infty Z_u dG_u | \mathcal{F}_t \Big)$$

and the continuous, $\mathbb{F}$-adapted process $\hat{g}$ is given by

$$\hat{g}_t = e^{\Gamma_t}\Big(\hat{n}_t - \int_{]0,t]} Z_u dF_u \Big) = -\frac{1}{G_t} \mathbb{E}_{\mathbb{P}^*}\Big(\int_0^\infty Z_u dG_u | \mathcal{F}_t \Big).$$

Moreover, $\hat{M}_t = \hat{g}_t$ on the set $\{t < \tau\}$.

Proof of ***Proposition 4.5***. . To establish formula equation (4.44), **Proposition 4.6** will be applied to the process $Z_t = \delta_t 1_{[0,T]}(t) - k(t \wedge T)$. Note that for every $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}^*}(Z_\tau | \mathcal{G}_t) &= \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{t<\tau\leq T\}} \delta_t - k(\tau \wedge T) | \mathcal{G}_t \Big) \\ &= \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{t<\tau\leq T\}} \delta_t - 1_{\{t<\tau\}} k(\tau \wedge T) | \mathcal{G}_t \Big) + 1_{\{t<\tau\}} \delta_\tau - 1_{\{t<\tau\}} k\tau. \end{aligned}$$

In view of equation (4.40),

$$S_t(k) = \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{t<\tau\leq T\}} \delta_t - 1_{\{t<\tau\}} k(\tau \wedge T) | \mathcal{G}_t \Big) + 1_{\{t<\tau\}} kt,$$

so that

$$S_t(k) = \mathbb{E}_{\mathbb{P}^*}(Z_\tau | \mathcal{G}_t) - 1_{\{t<\tau\}} \delta_t + 1_{\{t<\tau\}} kt + 1_{\{t<\tau\}} kt. \tag{4.47}$$

From **Proposition 4.6**, it follows that the martingale $\hat{M}_t = \mathbb{E}_{\mathbb{P}^*}(Z_\tau|\mathcal{G}_t)$ satisfies, for every $t \in [0,T]$,

$$\hat{M}_t = \hat{M}_0 + \int_{]0,t]} (Z_u - \hat{g}_u) dM_u + \int_{]0,t]} \frac{1-H_u}{G_u} d\hat{n}_u, \tag{4.48}$$

where the continuous $\mathbb{F}$-martingale $\hat{n}$ is given by the formula

$$\hat{n}_t = \mathbb{E}_{\mathbb{P}^*}\Big(- \int_t^T \delta_u dG_u + k \int_t^T u dG_u - kTG_T|\mathcal{F}_t\Big), \tag{4.49}$$

and the process $\hat{g}$ is given by

$$\hat{g}_t = \frac{1}{G_t}\mathbb{E}_{\mathbb{P}^*}\Big(- \int_t^T \delta_u dG_u - k \int_t^T u dG_u + kTG_T|\mathcal{F}_t\Big). \tag{4.50}$$

Since $1_{\{t<\tau\}}\hat{M}_t = 1_{\{t<\tau\}}\hat{g}_t$ and thus, in view of equation (4.47), $S_t(k) = 1_{\{t<\tau\}}(\hat{g}_t + kt)$. It is useful to observe that $\hat{g}$ is a continuous process, so that $\tilde{S}_{t-}(k) = 1_{\{t<\tau\}}(\hat{g}_t + kt)$ for every $t \in [0,T]$. From equation (4.47),

$$\begin{aligned}
S_t(k) &= \hat{M}_t - 1_{\{t<\tau\}}\delta_t + 1_{\{t<\tau\}}kt + 1_{\{t<\tau\}}kt \\
&= \hat{M}_t - \int_{]0,t]} (\delta_u - ku) dH_u + 1_{\{t<\tau\}}kt \\
&= \hat{M}_t - \int_{]0,t]} (\delta_u - ku) dM_u - \int_{]0,t]} (1-H_u)(\delta_u - ku)\gamma_u du + 1_{\{t<\tau\}}kt
\end{aligned}$$

Consequently, using equation (4.48) and noting that $Z_t = \delta_t - kt$ for all $t \in [0,T]$,

$$\begin{aligned}
S_t(k) &= \hat{M}_0 - \int_{]0,t]} \hat{g}_u dM_u + \int_{]0,t]} \frac{1-H_u}{G_u} d\hat{n}_u - \int_{]0,t]} (1-H_u)(\delta_u - ku)\gamma_u du + 1_{\{t<\tau\}}kt \\
&= \hat{S}_0(k) - \int_{]0,t]} S_{u-}(k) dM_u + \int_{]0,t]} ku dM_u + \int_{]0,t]} \frac{1-H_u}{G_u} d\hat{n}_u \\
&\quad - \int_{]0,t]} (1-H_u)(\delta_u - ku)\gamma_u du + 1_{\{t<\tau\}}kt \\
&= \hat{S}_0(k) - \int_{]0,t]} S_{u-}(k) dM_u + \int_{]0,t]} ku dH_u + \int_{]0,t]} \frac{1-H_u}{G_u} d\hat{n}_u \\
&\quad - \int_{]0,t]} (1-H_u)\delta_u\gamma_u du + 1_{\{t<\tau\}}kt \\
&= \hat{S}_0(k) - \int_{]0,t]} S_{u-}(k) dM_u + \int_{]0,t]} \frac{1-H_u}{G_u} d\hat{n}_u + \int_{]0,t]} (1-H_u)(k - \delta_u\gamma_u) du,
\end{aligned}$$

where $\hat{M}_0 = \hat{S}_0(k)$ and

$$\int_{]0,t]} S_{u-}(k) dH_u + 1_{\{t<\tau\}}kt = \int_{]0,t]} S_{u-}(k) dH_u + (1-H_t)kt = \int_{]0,t]} (1-H_t)k du.$$

$\square$

Using the dynamics of the process $\tilde{S}(k)$ for all $t \in [0,T]$, recall that $\tilde{S}(k)$ is the pre-default ex-dividend price of a credit default swap, so that $S_t(k) = 1_{\{t<\tau\}}\tilde{S}_t(k)$. Therefore, prior to default on the set $\{t < \tau\}$,

$$dS_t(k) = d\tilde{S}_t(k) = (\gamma_t\tilde{S}_t(k) + k - \delta_t\gamma_t)dt + \frac{1}{G_u}d\hat{n}_t \tag{4.51}$$

and it resulted that $\tilde{S}_t(0) = S_0(k)$. The formula above is an extension of equation (4.34) which shows in particular, that the pre-default ex-dividend price $\tilde{S}_t(k)$ is a continuous, $\mathbb{F}$-adapted process, since $\tilde{S}_t(k) = \hat{g}_t + kt$, where the continuous $\mathbb{F}$-adapted process $\hat{g}$ is given by equation (4.50).

4.3.4 Replicating Strategies with Credit Default Swaps

Assume now that protection payments δ^i for $i = 0, \cdots, k-1$ with maturities $T^i \geq T$, rates k^i and $k \geq 1$ credit default swaps are traded. The kth asset is the constant savings account $B_t = 1$. Consider hedging a defaultable claim $(X, 0, Z, \tau)$ such that $\mathbb{E}_{\mathbb{P}^*}|Z_\tau| < \infty$.

Definition 4.9. *A self-financing strategy $\phi = (\phi^0, \cdots, \phi^k)$ replicates a defaultable claim $(X, 0, Z, \tau)$ if its wealth process $V(\phi)$ satisfies the following equalities:*

$$V_T(\phi)1_{\{T<\tau\}} = X1_{\{T<\tau\}}$$

and

$$V_\tau(\phi)1_{\{T<\tau\}} = Z_\tau 1_{\{T<\tau\}}.$$

In dealing with replicating strategies, with regards to the definition above, assume that the components of the process ϕ are $\mathbb{F}$-predictable processes. A self-financing trading strategy ϕ is admissible if the stopped wealth process $V_{t\wedge\tau}(\phi)$, $t \in [0,T]$, is a $\mathbb{P}^*$-martingale.

Proposition 4.7. *Assume that there exist $\mathbb{F}$-predictable processes $\phi_0, \cdots, \phi^{k-1}$ such that*

$$\sum_{i=0}^{k-1}\phi_t^i(\delta_t^i - \tilde{S}_t^i(k^i)) = Z_t - \hat{g}_t, \quad \sum_{i=0}^{k-1}\phi_t^i\iota_t^i = \iota_t, \tag{4.52}$$

where the $\mathbb{F}$-predictable processes $\iota^i, i = 0, \cdots, k-1$ and ι are given by the equation in equation (4.55), and the continuous, $\mathbb{F}$-adapted process $\hat{g}$ is given by (4.58). Let $\phi_t^k = V_t(\phi) - \sum_{i=0}^{k-1} \phi_t^i S_t^i(k^i)$, where the process $V(\phi)$ is given by

$$dV_t(\phi) = \sum_{i=0}^{k-1} \phi_t^i (S_t^i(k^i) + dh_t^i) \tag{4.53}$$

with the initial condition $V_0(\phi) = \mathbb{E}_{\mathbb{P}^*}(Y)$ and Y is given by

$$Y = 1_{\{T \geq \tau\}} Z_\tau + 1_{\{T<\tau\}} X. \tag{4.54}$$

Then the self-financing trading strategy $\phi = (\phi^0, \cdots, \phi^k)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$.

Proof. Since $dB_t = 0$, for the wealth process $V(\phi)$ we obtain, on the set $\{\tau > t\}$,

$$dV_t(\phi) = \sum_{i=0}^{k-1} \phi_t^i (d\tilde{S}_t^i(k^i) - k^i dt) = \sum_{i=0}^{k-1} \phi_t^i \Big(\gamma_t (\tilde{S}_t^i(k^i) - \delta_t^i) dt + \frac{1}{G_t} d\hat{n}_t^i \Big),$$

where δ and k replaced by δ^i and k^i and the second equality follows from (4.51) with $\hat{n}^i, i = 0, \cdots, k-1$ given by (4.46). With respect to the predictable representation property of a Brownian motion,

$$dV_t(\phi) = \sum_{i=0}^{k-1} \phi_t^i \Big(\gamma_t (\tilde{S}_t^i(k^i) - \delta_t^i) dt + \frac{1}{G_t} \iota_t^i dW_t \Big) \tag{4.55}$$

for some $\mathbb{F}$-predictable processes $\iota^i; i = 0, \cdots, k-1$ such that $d\hat{n}_t^i = \iota_t^i dW_t$.

In dealing with a defaultable claim $(X, 0, Z, \tau)$, apply Proposition 4.6 to the process $\bar{Z}$ given by the formula $\bar{Z}_t = Z_t 1_{[0,T[}(t) + X 1_{[T,\infty[}(t)$ to get

$$\hat{M}_t = \hat{M}_0 + \int_{]0,t]} (Z_u - \hat{g}_u) dM_u + \int_{]0,t]} \frac{1 - H_u}{G_t} d\hat{n}_u, \tag{4.56}$$

where the continuous $\mathbb{F}$-martingale $\hat{n}$ is given by the formula

$$\hat{n}_t = \mathbb{E}_{\mathbb{P}^*} \Big(- \int_0^T Z_u dG_u + G_T X | \mathcal{F}_t \Big), \tag{4.57}$$

and the process $\hat{g}$ equals

$$\hat{g}_t = \frac{1}{G_t} \mathbb{E}_{\mathbb{P}^*} \Big(- \int_t^T Z_u dG_u + G_T X | \mathcal{F}_t \Big). \tag{4.58}$$

Recall that $\mathbb{P}^*(\tau = T) = 0$ Following from the set $\{t < \tau\}$,

$$d\hat{M}_t = -\gamma_t(Z_t - \hat{g}_t)dt + \frac{1}{G_u}d\hat{n}_u = -\gamma_t(Z_t - \hat{g}_t)dt + \frac{1}{G_u}\iota_t dW_t, \tag{4.59}$$

for some $\mathbb{F}$-predictable processes ι such that $d\hat{n}_t = \iota_t dW_t$. The existence of ι follows from the predictable representation property of W.

The strategy $\phi = (\phi^0, \cdots, \phi^k)$ replicates a claim $(X, 0, Z, \tau)$ prior to default, provided that its initial value $V_0(\phi)$ is equal to $\mathbb{E}_{\mathbb{P}^*}(Y)$, and the components $(\phi^0, \cdots, \phi^{k-1})$ are judiciously chosen so that the equality $dV_t(\phi) = d\hat{M}_t$ holds on $\{t < \tau\}$. More explicitly, the $\mathbb{F}$-predictable processes $(\phi^0, \cdots, \phi^{k-1})$ are bound to satisfy

$$\sum_{i=0}^{k-1} \phi_t^i(\delta_t^i - \tilde{S}_t^i(k^i)) = Z_t - \hat{g}_t, \quad \sum_{i=0}^{k-1} \phi_t^i \iota_t^i = \iota_t \quad \forall t \in [0, T], \tag{4.60}$$

where the first condition is essential only for those values of $t \in [0, T]$ for which $\gamma_t \neq 0$.

It will be good to compare the jumps of $\hat{M}$ and $V(\phi)$ at time τ to complete the proof. Observe that $\Delta_\tau \hat{M} = Z_\tau - \hat{g}_\tau$ and that the wealth process of ϕ,

$$\Delta_\tau V(\phi) = \sum_{i=0}^{k-1} \phi_t^i(\delta_t^i - \tilde{S}_t^i(k^i)) = Z_t - \hat{g}_t,$$

where the last equality follows from (4.60). In conclusion, $V_{t\wedge\tau}(\phi) = \hat{M}_{t\wedge\tau}$ for every $t \in [0, T]$. In particular, ϕ is admissible in the sense that the stopped wealth process $V_{t\wedge\tau}(\phi)$, $t \in [0, T]$, is a $\mathbb{P}^*$-martingale, and $V_{t\wedge\tau}(\phi) = Y$, where Y is given in (4.54).

This means that ϕ replicates a defaultable claim $(X, 0, Z, \tau)$. Hence, the stopped $\mathbb{P}^*$-martingale $\hat{M}_{t\wedge\tau}$, represents the arbitrage price of this claim on $[0, \tau \wedge T]$ where $\hat{M}$ is given by equation (4.56). □

4.3.5 Forward Start Credit Default Swap

A forward start credit default swap initiated at some date $s \in [0, U]$ gives the default protection over the future time interval $[U, T]$. The price of this contract at any date $t \in [s, U]$ equals equation (4.18) or more explicitly,

$$S_t(k) = 1_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{P}^*}\Big(- \int_U^T \delta_u dG_u - k \int_U^T G_u du | \mathcal{F}_t \Big). \tag{4.61}$$

A forward start market credit default swap at time $t \in [0, U]$ is a forward credit default swap, which is valueless at time t. The corresponding forward credit default swap rate $k(t, U, T)$ is thus an $\mathcal{F}_t$-measurable random variable implicitly determined by the equation

$$S_t(k(t, U, T)) = \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{U<\tau\leq T\}}\delta_\tau | \mathcal{G}_t\Big) - \mathbb{E}_{\mathbb{P}^*}\Big(1_{\{U<\tau\}} k(t, U, T)((\tau \wedge T) - U) | \mathcal{G}_t\Big) = 0$$

and for all $t \in [0, U]$,

$$k(t, U, T) = -\frac{\mathbb{E}_{\mathbb{P}^*}\Big(\int_U^T \delta_u dG_u | \mathcal{F}_t\Big)}{\mathbb{E}_{\mathbb{P}^*}\Big(\int_U^T G_u du | \mathcal{F}_t\Big)}. \tag{4.62}$$

The difference between equation (4.62) and the rate of forward start CDS underdeterministic intensity is that it is $\mathcal{F}_t$-measurable random variable.

For an arbitrary forward credit default swap with rate k we have, for every $t \in [0, U]$,

$$S_t(k) = S_t(k) - S_t(k(t, U, T)) = (k(t, U, T) - k)\mathbb{E}_{\mathbb{P}^*}\Big(1_{\{U<\tau\}}((\tau \wedge T) - U) | \mathcal{G}_t\Big) \tag{4.63}$$

or more explicitly

$$S_t(k) = 1_{\{t<\tau\}}(k(t, U, T) - k) \frac{1}{G_t} \mathbb{E}_{\mathbb{P}^*}\Big(\int_U^T G_u du | \mathcal{F}_t \Big). \tag{4.64}$$

The above representation is useful in the valuation and hedging of options on a forward start credit default swap.

Chapter 5

CONCLUDING REMARKS

The purpose of this thesis was to review the general framework for the pricing and hedging of defaultable claim and to extend the established result under deterministic intensity to the case of stochastic intensity. We focused on reduced form models and directed our attention towards the pricing framework for defaultable bonds.

In the financial literature, the risk of trading a defaultable claim is divided into two components which are the jump risk linked with the default event and the jump risk associated with the volatile character of the pre-default price of a defaultable claim. Dealing with both kind of risks simultaneously in an efficient way becomes the problem. But in our Chapter 4, the **Proposition 4.7** shows that it is possible to deal with both kind of risks in a generic intensity based model.

The default risk was perfectly hedged in the first equality in equation (4.52) and the spread risk was hedged effectively in the second equality. We can conclude from these formulae that keeping unexpected jumps that may occur prior to maturity under control will hedge the default risk. So with more standard methods related to the volatilities and correlations of underlying stochastic processes, the spread risk was hedged.

Chapter 5 – CONCLUDING REMARKS

In Chapter 4, the assumption was that the interest rate is equal to zero, though it was not an important restriction. The condition was imposed only for a comprehensive description and explanation for which it will not be difficult to extend all the results in Chapter 4 to the case of a deterministic short-term rate r_t. Though, in the case of stochastic intensity, this case will not be easy to analyze. So we can therefore either make the assumption that Brownian motion drives both the default intensity and the short-term rate or that the default intensity and short-term rate are driven by two correlated Brownian motions.

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