

AN ANALYSIS OF PRE-SERVICE MATHEMATICS TEACHERS' GEOMETRIC
THINKING AND CLASSROOM DISCOURSE USING A COMMUNICATIVE LENS

BY

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DECLARATION

I hereby declare that this thesis is the result of my own original work and that no part of it has been presented for another degree at this university or elsewhere.

Candidate's signature



Date:

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As the candidate's supervisor, I approve the submission of this thesis.

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DEDICATION

This thesis is dedicated to the Almighty God for His continued guidance and protection, making it possible for me to accomplish this qualification in His own time.

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ABSTRACT

Learning geometry equips learners with cognitive skills such as visualisation, critical thinking, spatial reasoning and problem-solving abilities, that are necessary for learning mathematics in general. However, geometry is noted to be difficult for learning as well as teaching. An investigation of this difficulty, especially with teachers, will help address its teaching and learning. The purpose of this study was to analyse pre-service teachers' geometric thinking and classroom discourse using the commognitive lens. The study was guided by three objectives, which were to analyse the pre-service teachers' discursive thinking in geometry; the nature of their routine thinking in solving the geometric tasks, and how these informed their classroom geometric discourse. The study aligned itself to the qualitative approach and was underpinned by the interpretivist research paradigm. Eight pre-service teachers who were second-year university students and had taken geometry as part of their programme modules, participated in the study. The study site was conveniently selected, whilst the participants were selected on purposively. Geometry worksheet (test), interview and classroom observation, were used to generate written, verbal (oral) response, and visual data in relation to the study objectives. The data was analysed using the themes of the commognitive framework. The results show that both literate and colloquial word use were found in the discourses of the pre-service teachers. Many participants in Group A used more literate words to define and explain geometric concepts and how they solved the geometry problems, than the participants in Group B, who used both literate and colloquial words. Also, the routine solution strategies of many in Group A showed more of an explorative way of thinking compared to those in Group B, who demonstrated more of a ritualised way of thinking. In addition, multiple solutions to tasks were found by many of Group A participants than those in Group B. Generally, many of the study participants demonstrated limited geometric thinking. Misconceptions were evident in the discourses of some pre-service teachers in both groups. Other key findings from the classroom observation were that, many participants in Group A demonstrated an explorative instruction that is characterised by developing learner understanding and using different kinds of visual mediators as compared to participants in Group B, whose classroom geometric discourse was ritualised in nature. In other words, their teaching was more procedure-driven than conceptual. **The study concludes that many of the PSTs possess limited geometric thinking. In addition, those who possessed good geometric thinking were more capable of engaging learners in explorative instruction compared to those with limited geometric**

thinking. These findings may have an influence on mathematics teacher educators' efforts to develop teaching competence among pre-service teachers.

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CHAPTER ONE

INTRODUCTION

1.1 Background to the study

Several research findings have revealed learners' poor performance in mathematics throughout the globe (Abreh et al., 2018; Bruce, 2016; Kiwanuka et al., 2015; Musa & Dauda, 2014; Ngirishi & Bansilal, 2019; Wonu & Zalmon, 2017). This poor performance has attracted the attention of mathematics research communities and mathematics educators who are constantly engaged in rigorous investigations to determine how to improve the quality of teaching and learning in the subject, for improved performance. The reason for this special attention to mathematics is that, it has been considered the backbone of all scientific, technological and industrial advancement. Knowledge of mathematics is key to all technological advancement and economic growth of a nation. People can better manage their lives and information around them when they have some fundamental knowledge of mathematics (Kiwanuka et al., 2015). Because mathematics is so important, many countries have placed much focus on improving the quality of teaching and learning of the subject (Aslan-Tutak, 2015; Robichaux-Davis & Gaurino, 2016). Researchers believe that the study of geometry has a central place in the mathematical curriculum. The learning of geometry enables people to better understand their world, as well as to improve the teaching and learning of mathematics (Luneta, 2015; Riastuti et al., 2017). Despite the significant role of geometry in mathematics education, a considerable number of research results show that learners, worldwide, have difficulty in learning geometric concepts (Aslan-Tutak, 2015; Ozerem, 2012; Sulistiowati et al., 2019).

Geometry is known to play a significant role in learning mathematics (National Council of Teachers of Mathematics [NCTM], 2000; Zuya & Kwalat, 2015). It is a branch of mathematics that is concerned with the study of points, shapes, straight lines, spatial figures, space, including the properties and relations among them (Bassarear, 2012; Biber, et al., 2013; Luneta, 2015). The study of geometry develops learners' critical thinking skills and problem-solving abilities (Maulana & Yuniawati, 2018). It is considered a reality-based topic in the mathematics curriculum (Ness & Farenga, 2007).

Geometry has a long history and its inclusion in the development of mathematics continues to receive significant attention in (mathematics education) its teaching and learning due to its potential to develop and improve mathematics understanding among learners (Luneta, 2015; Robichaux-Davis & Guarino, 2016). Learners' knowledge of geometry enables them to explore, conjecture, deduce and improve their development of mathematical ideas and reasoning skills (Maulana & Yuniawati, 2018; Sulistiowati et al., 2019). Learning geometry aims to develop learners' spatial reasoning and abilities for learning geometric shapes and objects encountered in daily life (Ozdemir & Goktepe Yildiz, 2015). According to NCTM (2000), geometry equips learners with reasoning abilities necessary for developing their mathematical thinking. NCTM (2000, p. 41) expects that learners will be able to “analyse characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships”. In addition, learners should be able to develop the competence to identify and describe the attributes of shapes using the appropriate vocabulary, to be capable of defining geometric shapes, to develop spatial thinking, to have visual ability and to develop deductive thinking among other skills (Jones, 2000). Thus, there are high expectations for learning geometry. According to Sulistiowati et al. (2019) and Yurmalia and Herman (2021), learners' development of geometric habits of mind, such as spatial reasoning and visualisation, are critical tools for learning mathematics. Jones (2000) maintains that much of the learner's thinking necessary for learning mathematics is spatial in nature. There is a significant relationship between geometric understanding and mathematical competencies (Robichaux-Davis & Guarino, 2016).

1.1.1 Learners' performance in geometry

Due to the importance and contribution of geometry to our daily life and the development of critical thinking and problem-solving abilities, it is included in the mathematics curricula of many countries (Jones 2000; Luneta, 2015). However, a growing body of research shows that learners find geometry difficulty to learn (Ngirishi & Bansilal, 2019; Sulistiowati et al., 2019). Many learners find geometry a difficult subject matter and regard it as the most difficult and confusing aspect in the mathematics curriculum, and hence find it irrelevant to their daily lives (Luneta, 2015; Ngirishi & Bansilal, 2015). Luneta (2015) asserts that many learners find geometry concepts more complex and difficult than other topics of mathematics. As a result of these and other factors,

learners are not particularly enthused about learning geometry, which leads to poor mathematics performance in both national and international level examinations (Abreh et al., 2018; Bora & Ahmed, 2018; Bruce, 2016). Despite the effort of mathematics stakeholders and suggestions made in the mathematics curriculum document (Ministry of Education [MoE], 2010) regarding the teaching of geometry to enable learners to realise its importance and to appreciate it in their lives, learners' poor performance in mathematics and geometry persists, and it seems to be a global problem (Bora & Ahmed, 2018).

In Ghana, like many other countries, many learners find geometry a challenging subject matter to learn. As a way of providing feedback on previous examinations, mathematics examiners engage in diagnostic analysis of learners' results to determine their strengths and weaknesses in the various content areas of mathematics, which is reported by the West African Examination Council (WAEC). In such reports, geometry has, for many years, been among the content areas that have been repeatedly highlighted as learners' major weaknesses WAEC (2015, 2017, 2018). The reports have continually indicated that candidates have difficulty in solving tasks in geometry. According to the reports, most candidates do not attempt questions on geometry. The few who attempt such questions only display errors that shows their misunderstanding of the subject matter. Some difficulties involve inability to recall and apply geometric properties and circle theorems to solve related problems in plane geometry (WAEC, 2015). Other areas are **those of solving** problems on cyclic quadrilaterals, tangent and chord theorems (WAEC, 2017). Research has shown similar geometric misconceptions and learning difficulties among learners around the globe (Aslan-Tutak, 2015; Ling et al., 2016; Mulungye et al., 2016; Robichaux-Davis & Gaurino, 2016). To improve learners' difficulties in learning geometry and to make them appreciate the subject, teachers need to possess deep and flexible content knowledge of the subject matter to enhance its teaching and learning activities in the classroom (Robichau-Davis & Gaurino, 2016).

1.1.2 The role of the teacher in geometry education

Research shows that the quality of instruction learners receive significantly influences their learning gains (Jentsch & Schlesinger, 2018; Yi et al., 2022). To enable learners to attain the expected competencies in learning geometry, teachers must organise learning experiences that are critical to the development of geometric ideas. In the educational system of every nation, teachers

are assigned the responsibility to interpret the curriculum for learners. Thus, the implementation of every education system depends on teachers, making their function vital. The teacher has the role of transforming and representing the subject matter knowledge of the curriculum to learners in an understandable way (Shulman, 1986). Jentsch and Schlesinger (2018) outline three dimensions of instructional quality in mathematics education, which are: classroom management, personal learning support and cognitive activation. Even though all are important, what is worth mentioning, as a critical responsibility of mathematics teachers, is the support they provide for individual learners in terms of guidance and constructive feedback, and how teachers use problem-solving tasks to activate and promote the learning process. Thus, in the context of geometric education, teachers and pre-service teachers need to learn how to design and implement teaching strategies necessary for supporting instructional quality in geometric discourse (Llinares, 2021). Learning, understanding and application of knowledge in solving problems all emanate from instructional quality (Yi et al., 2022).

1.1.3 Teachers' content knowledge

Central to instructional quality is the content knowledge possessed by the teacher. Teachers need to possess adequate and functional content knowledge of geometric concepts and ideas to teach the subject. The teacher's content knowledge of geometry serves as a knowledge-base for teaching, and facilitating learners' geometric knowledge construction (Aslan-Tutak, 2015). Learners' geometric thinking can be adequately developed if teachers have, in their knowledge repertoire, adequate, comprehensive and flexible geometric knowledge for teaching (Robichaux-Davis & Guarino, 2016). High learning gains for mathematics learners can be attributed to the strength of the mathematics teaching force. The continued focus on teachers' knowledge is because learners tend to gain more when taught by excellent teachers than when taught by underperforming ones (Maruli, 2014). Teachers' knowledge of the content is the most influential variable in successful teaching. The depth of the teacher's understanding of the subject matter serves as an underlying factor for the quality of classroom instruction. It also serves as a predictor of learners' achievement (Hill et al., 2005; Robichaux-Davis & Gaurino, 2016). Almost all classroom activities are centred on the knowledge-base of the teacher. According to Krauss et al. (2008), the teacher's content knowledge about the subject matter forms a pivotal point on which teacher competency is built. This means that the teacher's ability to navigate his or her way through quality and effectual

instruction depends on how well his or her knowledge is rooted in the subject matter. Aslan-Tutak and Adams (2015) assert that teachers' subject matter knowledge has a strong effect on their teaching practices. The authors add that teachers should possess adequate knowledge of what they are supposed to teach before determining how to develop that knowledge among learners. Teachers who have adequate and in-depth mathematical knowledge for teaching have the potential to guide and facilitate learners' understanding of the subject matter (Danisman & Tanisli, 2017).

According to Mudaly (2015), teachers' evidence-based knowledge can be achieved by examining their thinking about mathematical concepts. He maintains that how a teacher thinks informs his or her teaching actions, and the subsequent learning experiences learners are offered. With the focus on geometry, both teachers and pre-service teachers need to possess deep geometric thinking to successfully teach all content areas in geometry and most importantly, to support learning in geometric discourse through guidance and constructive feedback (Ball et al., 2008; Danisman & Tanisli, 2017; Yi et al., 2022).

1.1.4 Geometry as a discourse

According to Sfard (2008), a discourse is a special type of communication activity that depends on its admissible actions as defined in a particular community of learners. The discourses learners are engaged in largely depend on teachers' thinking. They are the kinds of communication activities teachers make available for learning. Discourses as communication, take the form of spoken language, written text, artefacts and physical objects used in learning. According to Sfard (2008, p. 93), discourses are "different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus define different communities of communicating actors". An inference from Sfard's (2008) definition means the keywords that are identified and used in communication (spoken or textual), the visual tools (mediators) that are used to communicate concepts and the procedures that are followed in communication make a discourse distinct, especially in geometry, which makes use of some specified words to communicate the concept. As a result, engaging learners in an effective communication of geometric concepts requires high level of teacher competence.

To improve performance, it is necessary to reconsider the discourse in mathematics education, especially in geometry. The quality of mathematics education depends, to a large extent, on the strengths of the teaching force. According to Meli (2020), a significant approach to improving the teaching of mathematics is based on the professional development of teachers, particularly, pre-service teachers who enter the teaching profession every year.

1.2 Statement of the problem

Research studies on teachers' mathematics content knowledge and learners' learning gains show that these two variables are positively correlated (Baumert et al., 2010; Hill et al., 2005). In other words, the teacher's content knowledge significantly relates to the learners' achievement. Robichaux-Davis and Guarino (2016) assert that for teachers to help learners learning, they must have a broad range of knowledge and a deep understanding of the subject matter.

Literature supports an investigation of teachers' and pre-service teachers' content knowledge of geometry. The concern about poor performance in mathematics has initiated several research inquiries into the factors underlying this problem. The findings of such research show geometry as the main area many learners find difficult to learn (Luneta, 2015; Sulistiowati et al., 2019). Similar learning difficulties in geometry are found among Senior High School (SHS) learners in Ghana, many of whom, do not answer questions in this area, when there is an alternative (Fletcher & Anderson, 2012). However, geometry has been noted to be a reality-based topic in the mathematics curriculum for which research shows that learners' geometry competencies also correlate positively with their mathematics performance (Ness & Farenga, 2007). This could mean that a way to improve learners' mathematics performance is to pay critical attention to the teaching and learning of geometry.

The effectiveness and quality of geometric instruction rests on the teachers' geometric content knowledge, especially pre-service teachers who join the teaching service each year. Mudaly (2015) claims that a realistic part of a teacher's knowledge is how they think, which affects all the activities they use in the classroom to guide learning. This makes teachers' (both in-service and pre-service) geometric thinking a significant area of research in response to the pursuit of finding ways to improve learners' performance in geometry in particular and in mathematics in general. According

to Luneta (2015), it is not only learners who find geometry a difficult topic to learn, teachers also find it difficult to teach, and in most cases, some teachers even skip the teaching of geometry content within the mathematics curriculum (Clement & Battista, 1992). Anecdotal evidence shows that some teachers in our present-day classrooms also skip the teaching of geometry. Research shows that many pre-service teachers are not prepared to teach geometry because they do not have the required geometry content knowledge, which affects their classroom activities (Duartepe Aksu, 2013; Martinovic & Manizade, 2018). Jones and Tzekaki (2016) claim that pre-service teachers' lack of in-depth geometric subject matter knowledge, impacts how they teach in the classroom. This suggests that a research inquiry that investigates pre-service teachers' learning complexities can make a significant contribution to teachers' preparatory programmes, as well as the teaching and learning of geometry in schools, a notion that forms the basis for this study. Pre-service teachers should be able to demonstrate a flexible and adequate understanding of the geometric topics that they will teach. Ball et al. (2008) assert that pre-service teachers need to have an in-depth content knowledge of mathematics to be able to teach it well.

If mathematics stakeholders wish to see significant improvement in learners' performance in mathematics, then it will be desirable to focus attention on the teaching and learning of geometry, which research has shown, equips learners with important mathematics learning abilities such as visualisation, spatial reasoning abilities, critical reasoning and problem-solving skills (Maulana & Yuniawati, 2018; NCTM, 2000; Sulistiowati et al., 2019). These mathematics learning qualities can be experienced when teachers have deep content knowledge of geometry and can use the knowledge to design quality classroom instructional activities, for learners' engagement. A teacher's thinking is his or her realistic part of mathematical knowledge, which can be accessed in detail if given the opportunity to explain, justify, or substantiate his or her mathematical ideas (Hufferd-Ackles et al., 2004; Mudaly, 2015; Sfard, 2007, 2008). **Even though several studies have been conducted on pre-service teachers' geometric thinking (Arslan et al., 2016; Baktemur et al., 2021; Biber et al., 2013; Ndlovu, 2015; Unlu, 2022; Zuya & Kwalat, 2015), those that connect the pre-service teachers' geometric thinking and pedagogical practices are rarely explored.** This study, therefore, **filled this gap by** analysing pre-service mathematics teachers' geometric thinking and their classroom discourse using Sfard's (2008) commognitive lens. The participants will generally be referred to as pre-service teachers (PSTs).

1.3 Critical research questions and objectives

The purpose of the study was to analyse pre-service teachers' geometric thinking and how it informs their classroom geometric discourse. The following objectives were outlined to guide the study:

1. To analyse the nature of pre-service mathematics teachers' discursive thinking in geometry.
2. To analyse the nature of pre-service mathematics teachers' routine thinking in geometry.
3. To examine how pre-service mathematics teachers' geometric thinking influences their classroom discourse.

As guided by the objectives stated above, the study intended to answer the following critical research questions:

1. What is the nature of pre-service mathematics teachers' discursive thinking in geometry?
2. What is the nature of pre-service mathematics teachers' routine thinking in geometry?
3. How does pre-service mathematics teachers' geometric thinking influence their classroom discourse?

1.4 Significance of the study

According to Schoenfeld (2000), two main purposes drive research in mathematics education. The first deals with identifying the nature of teaching and learning in schools, and the mathematical thinking of learners. The second is concerned with the use of the insight gained to improve mathematics discourse. The two purposes aligned with the focus of this study, which analysed pre-service teachers' geometric thinking and their classroom geometric discourse. The aim was to identify the nature of the pre-service teachers' geometric thinking and to use the findings to help improve the teaching and learning of geometry through teacher support.

Available literature in Ghana on investigating teachers' knowledge shows that their geometry skills is one of the least researched. This study on pre-service teachers' geometric thinking reveals broader views of the nature of their geometric thinking. The study draws stakeholders' attention to the nature of the geometric thinking the pre-service teachers possess before entering the teaching

profession. This is necessary because how a teacher organises his or her teaching and the kind of learning experiences offered to learners depends on his or her understanding of the subject matter. A teacher needs to possess a deep and flexible understanding of the subject matter to be successful in teaching and learning (Shulman, 1986). This study contributes to the understanding of the participated pre-service teachers' geometric thinking for teaching geometry, a content area that continues to pose learning challenges to learners in Ghana.

The use of the commognitive framework makes a significant contribution to the existing theoretical frameworks that have been used to study teachers' knowledge. Its significance is based on its elements that provide a clear path for analysing the discursive thinking of both teachers and learners, which can be used to analyse classroom teaching and learning situations. Thus, the use of this framework in this investigation draws researchers' (especially Ghanaians) attention to its use in studying various phenomena related to teaching and learning in mathematics education.

The study makes another useful contribution to teacher preparatory programmes. Pre-service teachers undergo training to develop teaching competencies to communicate the knowledge to learners. Naidoo (2011) asserts that teachers ought to be given the needed support to succeed in teaching, and this support must be given before they are fully assigned a duty or begin teaching. Findings from her study, draw attention to the need for mathematics teacher educators to offer the needed support to pre-service teachers with special attention to the curriculum components they will teach.

In addition to teacher support, the study brings into focus the framework used in analysing pre-service teachers' geometric thinking in a broader perspective of the constructs of the framework. As part of developing the pre-service teachers' content knowledge for teaching, particularly in geometry, the use of the constructs of the framework can cause a discursive turn in teaching and in assessing learners' understanding and progress with instruction (formative, and even summative assessment) in teacher education institutions. In this case, minimal focus will be paid to solutions to questions, to include justification of the informed choices. Integrating the constructs of Sfard's (2008) commognitive framework into classroom teaching and, assessing learning and understanding, can form a solid foundation for developing the pre-service teachers' thought

processes for teaching. Thus, in communicating their thought processes, conscious effort could be placed on the words they use in communication, the use of visual mediators to communicate and explain geometric and mathematical concepts, what narratives they use to substantiate their strategies and how these inform their routines. Evidence from the framework used to investigate and analyse the participants' classroom geometric discourse shows that the framework has promising use in mathematics educational research (Presmeg, 2016), both in teaching and learning, and to develop mathematics teachers' teaching proficiencies and communicative competence for effective mathematics teaching.

1.5 Defintion of terms

Learners – is used in the context of lifelong learners. This includes anybody studying any point in time to acquire knowledge.

Pre-service teachsers – refers to students/learners studying to become teachers. In this study, the term learners and pre-service teachers will used interchangeably.

1.6 Overview of the study

The study has been organised into the following chapters.

Chapter one provides the background to the study. It presents the perspective of the problem under investigation and delimit it to the area of the study. It also outlines the problem statement and the motivation for the study. The research objectives and critical questions that guided the study are provided. The contribution the study makes to mathematics education are discussed and ends with an overview of the study.

Chapter two provides a review of the related literature on the study. Among the areas reviewed are geometry and its place in the mathematics curriculum, mathematics education in Ghana, the acquisitionist and participationist views to learning, and cognitive abilities for learning geometry. The chapter also presents a review of the literature on pre-service teachers' geometry thinking in the content areas of the study, including their spatial abilities.

Chapter three provides the theoretical framework as the lens to conduct the study and a means to gain a deeper understanding of the data generated. The use of the framework and its usefulness to the current study are also discussed in the chapter.

Chapter four focuses on the research paradigm, the research design and the methods used. It describes the development of the instrument for generating data, the piloting of the instrument and the processes involved in data generation. It explains the approaches used to gain access to conduct the study and, the data analysis procedure.

Chapter five answers the first critical research question. It presents an analysis of the participants' geometric thinking in terms of the discursive constructs of the framework. It also presents the findings and relates it to the literature.

Chapter six concentrates on the second critical question and presents findings on the participants' routine thinking in solving the geometry tasks assigned. The chapter focuses on the dominance of ritual, or exploratory ways of thinking in their routine strategies.

Chapter seven builds on chapters five and six. It focuses on how the participants' geometric thinking and their solution strategies influence their classroom discourse.

The final chapter eight, presents a summary of the study and its findings. It also discusses the limitations of the study and offers recommendations and suggestions for further research.

1.7 Conclusion

This chapter started with the background to the study and the statement of the problem. It presented the guiding study objectives as well as the critical research questions. The chapter concluded with the preview of the chapters to follow. The next chapter, discusses the literature review.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

The previous chapter focused on the background to the study and the underlying rationale that necessitates this study. This chapter presents a review of related literature regarding mathematics education and geometry in particular. This places the study within the existing literature of mathematical studies on geometry and related issues in teaching and learning. The review begins with the concept of geometry, a historical account of geometry, the place of geometry in the mathematics curriculum, and relevant issues in the teaching and learning of geometry. Further attention is given to some key abilities for learning geometry such as visualisation, spatial reasoning abilities, and other related factors.

The review proceeds with teachers' geometric thinking and its influence on teaching in the classroom context. It continues with the teachers' role in teaching geometry and ways of facilitating the development of geometric thinking among learners. The last part of the chapter describes what the literature review means for this study.

2.2 Definition of geometry

Several authors have attempted to define geometry based on its historical account. A general idea based on this account shows that geometry is a branch of mathematics that deals with the study of shapes, their measurement and dimensions. For this study to have a basis and direction, some definitions of geometry are examined.

According to Luneta (2015), geometry is the branch of mathematics that focuses on the study of shapes and their properties and the relationships among them. It is the science of study that deals with the **properties of shapes** of the world. This definition relates well to this study. As part of analysing pre-service teachers' thinking, attention was paid to their thinking about geometric shapes and their properties. The thinking of the pre-service teachers about geometry, shows their knowledge and understanding as a basis of their competency for the teaching profession.

Previously, Clement and Battista (1992, p.420) said that, geometry is the “study of spatial objects, relationships and transformations that have been formalised (or mathematised) and the axiomatic mathematical systems that have been constructed to represent them”. Geometric objects relate to space as a physical dimension, and since space forms the basis of our living environment, spatial thinking is deemed an important tool for teaching and learning geometry. Spatial thinking deals with the formation of ideas or thoughts using spatial relationships in learning geometry (Lowrie et al., 2018; Ozdemir & Goktepe Yildiz, 2015). Clement and Battista (1992) add that spatial reasoning is an intellectual process of engaging in mental representation, construction and manipulation of spatial objects, and the relationships among their properties. Since spatial reasoning is crucial to learning geometry, as well as to developing problem solving skills, it becomes an important part of pre-service teachers' geometric thinking and an essential ability to develop among learners.

Also, according to Jones (2000, p. 124), a useful and probably modernised definition of geometry was attributed to the British mathematician, Sir Christopher Zeeman who said, "geometry comprises those branches of mathematics that exploit visual intuition (the most dominant of our senses) to remember theories, understand proof, inspire conjecture, perceive reality and give global insight”.. This definition, shows that visual intuition is a critical tool for learning geometry. What the eyes perceive aids the thought processes in learning. Visual intuition (senses) and geometry can be termed as two sides of a coin for which visual intuition is essential for the exploration of geometrical concepts. The visual senses therefore are examined in this study.

2.3 The place of geometry in a school mathematics curriculum

Geometry plays an important role in the school mathematics curriculum and forms one of the basic components of learning mathematics NCTM (2000). A major goal of teaching mathematics is to enable learners to develop reasoning abilities and skills to solve challenging problems in their daily lives. The mathematics curriculum aims to equip learners with problem-solving abilities to understand and devise appropriate strategies to solve challenging tasks (MoE, 2010; Sulistiowati et al., 2018). For example, the Ghanaian mathematics curriculum document identifies as one of its aims of teaching mathematics, the ability to “use mathematics in daily life by recognising and applying appropriate mathematical problem-solving strategies” (MoE, 2010, p. ii). This suggests

that problems encountered in life may remain unresolved without basic knowledge of mathematics. Geometry has notably contributed significantly to the central goal of teaching mathematics (NCTM, 2000; Safrina et al., 2021; Sulistiowati et al., 2018). Geometry plays a major role in developing learners' spatial reasoning, visual abilities, and critical thinking skills that generally enhance mathematics learning and develop problem-solving abilities (Safrina et al., 2021; Suarsana, 2019). Learning geometry develops learners' spatial and visual skills, which are necessary to solve problems and helps to understand the world we live in (Ndlovu, 2014). Acquisition of spatial and visual skills in learning geometry has the potential to develop problem-solving abilities among learners. Hence, developing learners' proficiency in learning geometry is given greater attention in research as a basis for realising improved performance in mathematics. Most researchers and mathematics educators believe that geometry is the central component of mathematics education (Cheng & Mix, 2014; Safrina et al., 2021).

Major reforms in mathematics education have focused on geometry content. For example, the 1989 reform (NCTM, 1989) in mathematics included two-and three-dimensional geometry which aimed to enable learners to:

1. describe, model, draw, and classify shapes,
2. investigate and predict the results of combining, subdividing, and changing shapes,
3. develop a spatial sense,
4. relate geometric ideas to number and measurement ideas, and
5. recognise and appreciate geometry in the world (NCTM, 1989).

Although the entire mathematics curriculum may have broader expectations for learners' engagement, it may not be that much different from the five goals outlined above. The learning of geometry focuses on concept development and relates to the learning of other content in the mathematics curriculum. Geometric exploration enables the development of the learners' visual sense, which is a tool for learning mathematics. For example, geometric shapes and other diagrammatic representations of mathematical concepts can be well analysed when the learner has a high sense of visualisation and spatial reasoning. Since the mathematics curriculum aims to provide learners with problem-solving abilities, teaching mathematics requires teachers to engage learners in purposeful activities in geometry lessons to help develop their visualisation and spatial

reasoning skills, as critical tools necessary for learning geometry and mathematics as a whole (Cheng & Mix, 2014; Winarti, 2018).

Mathematics is considered the backbone of all scientific exploration, and infers that no development in the human and scientific world can be realised, if mathematics is left out of the school curricula. This can be attributed to the centrality of geometry in the mathematics curriculum. Jones (2000, p. 3) **claims** that, tasks requiring geometric knowledge emerge in all science courses and programmes, like “chemistry (computational chemistry and the shapes of molecules), material physics (modeling various forms of glass and aggregate materials), biology (modeling of proteins, docking of drugs on other molecules, etc.), Geographic Information Systems (GIS), and most fields of engineering”. Zhang (2017) and Luneta (2015) **add that**, knowledge of geometry is fundamental for learning various subjects such as physics, geography, astronomy, art, geology, engineering, technology, chemistry, biology, and many others. The importance of geometry in scientific development gives it a central role in mathematics education. There are numerous applications of geometry in various areas of study such as computer-aided designs and geometric modelling, robotics, medical imaging, computer animation and visual representations (Jones, 2000). **Jones** (2000) observes that there has been phenomenal growth in mathematics, which is largely geometrical. His observation has shown that most of this geometrical growth in mathematics occurs in dynamic systems, mathematical visualisation, and geometric algebra. This observation suggests that geometry continues to grow in mathematics education, which requires great attention to its teaching and learning in order to develop functional geometric thinking among learners.

2.4 Geometry education internationally

The teaching and learning of geometry continue to gain attention from researchers and stakeholders of education due to its central role in learning mathematics (Jones, 2000). Research shows that most of the cognitive abilities required for learning mathematics, are obtained from geometric discourse (Atanasova-Pachemska, et al., 2016). Examples are visualisation, spatial reasoning, and problem-solving abilities. This suggests that critical emphasis placed on geometry education has the potential to improve general mathematics performance among learners. According to Jones (2000), geometry, which is among the earlier branches of mathematics, has

undergone significant growth from the earliest times to the 21st century, and is still well recognised for its coherence and richness in content. However, it seems that geometry education in the classroom has not undergone changes in parallel to the growth in content and structure (Kuzle & Gracin, 2020). This could mean that, despite the phenomenal growth in geometric content, not enough has been realised in terms of its teaching and learning in the classroom.

Learners are supposed to be exposed to and have an in-depth knowledge of geometry because it deals with most of the concepts in our daily lives (Luneta, 2015). Learners are to be engaged in quality teaching experiences to enable them to gain proficient geometric thinking. To develop quality geometric thinking among learners, they must be guided to construct geometric meaning from the learning experiences (Manizade & Orrill, 2020). This means that learners need to be encouraged to take ownership of learning by actively participating in the lesson and learn by doing. Assisting and developing learners to individualise the learning process enables them to be independent and creative thinkers (Sfard, 2007, 2008). For example, learners are expected to be engaged in learning experiences that will enable them to observe and compare observations, hypothesise, prove and justify the truth or otherwise of mathematical claims (Jones, 2000). This makes geometric discourse important in school mathematics. Research shows that geometry is important in many ways. Learners' engagement in geometric discourse develops their cognitive skills and mathematical way of thinking (Kuzle, 2022). Kuzle (2022) asserts that geometric thinking permeates all the mathematics we do through visualisation, and through visual thinking and analysis of issues. Despite the importance of geometric discourse, the teaching and learning of geometry seem concealed in classroom mathematics lessons (Kuzle, 2022).

Several studies show that learners across the globe find geometric concepts difficult to learn (Adolphus, 2011; Carlin, 2009; Luneta, 2015), which reflects in their performance in both national and international examinations. For example, Carlin (2009), who conducted a comparative study of geometry curricula, reported that despite the effort to expand geometry education across all grades from pre-school through to Grade 12, the Trends in International Mathematics and Science Study [TIMSS] results showed that learners in the United States (US) demonstrated difficulties with geometric concepts compared to learners of Asian countries such as Singapore and China. Martinovic and Manizade (2018) claim that geometry is seen in the mathematics curriculum in two ways. It is either taken as a separate strand (high school and the colleges) or it is integrated

with other strands in mathematics (elementary and some high schools). The authors add that it is a requirement for high school graduation in most states in the US, to take a number of geometry strands.

This raises a continued concern with regard to the learners' low performance in geometry and measurement strands, content areas that are strongly related to STEM programmes or courses (Steele, 2013). According to Zhang and Bergstrom (2016), Grade 10–12 teachers in the United States find the teaching of geometry a major challenge. The authors add that American learners have been performing poorly in geometry over the past two decades in both national and international assessments. This shows that not only do learners find it difficult to learn geometry, but teachers who are trained to teach geometry for the conceptual understanding of learners, also find it difficult. Thus, teachers may not have the depth of content knowledge required to develop learners' geometric thinking. This observed knowledge gap among learners and teachers in the United States is in line with Luneta's (2015) claim from research that "geometry is difficult to teach as well as to learn" (p.1). According to Provasnik et al. (2012), both the National Assessment of Educational Progress (NAEP) and Trends in International Mathematics and Science Study produce similar results, showing that the United States Grade 8 learners received the weakest performance in geometry among the mathematics content areas, which are numbers, algebra, data and chance, and geometry. Dobbins et al. (2014) claim that learners' difficulty in geometry begins during early education and becomes worse as they get to high school.

Research shows that learners in South Africa experience similar learning difficulties in geometry. They do not understand most of the basic geometric concepts (Alex & Mammen, 2018; Luneta, 2015; Ngirishi & Bansilal, 2019). According to Ngirishi and Bansilal (2019), learning outcomes in mathematics among South African learners have been very low. This has received research attention, especially within the scope of geometry. The authors add that many learners see geometry as the most complicated mathematics strand and think it has no bearing on their lives. Hence, they do not find it motivating enough to learn.

Perhaps the difficulty of geometry prompted educational authorities in South Africa to revise their curriculum in 2006 so that learners in Grades 10-12 were no longer required to take geometry as

compulsory in their final examinations. In 2011, geometry being compulsory in final examinations was restored since many learners chose not to study it because it was optional. When it was restored, many teachers were quite uncomfortable to teach the subject (Ngrishi & Bansilal, 2019). This concern motivated the study conducted by the Ngrishi and Bansilal (2019) to explore high school teachers' understanding of geometric concepts. Grades 10 and 11 learners' understanding was explored in terms of van Hiele's levels of geometric thinking. The participants comprised 147 learners selected from three high schools to participate in a study. The study was conducted at a time when the participants had completed studying geometry. Data was collected using a questionnaire schedule and interviews. The questionnaire contained 15 adapted multiple-choice questions and a worksheet containing six open-ended questions. The authors found that learners demonstrated difficulties in defining geometric terms and concepts, interrelations of properties and shapes, as well as class inclusion of shapes. It was also found that most of the learners operated at the visual and analytical levels of van Hiele's thinking model. Thus, the participants performed below expectations. The authors recommended that teachers should use appropriate language and engage them in meaningful learning to improve their performance. In a previous study by Luneta (2014) on foundation phase teachers' knowledge of geometry, the author found that the student teachers (pre-service) demonstrated limited knowledge of basic geometry. Similarly, Couto (2014), who studied pre-service teachers' knowledge of elementary geometry concepts, found a weak performance by the pre-service teachers on a test that addressed elementary geometry concepts.

Knowledge of geometry is important to learners in every part of the world, to understand their environment and to benefit from careers related to geometry (Jones, 2000; Luneta, 2015). A proficient understanding of geometry and mathematics would enable learners to be successful in their chosen careers and endeavours in life (Jones, 2000; Luneta, 2015). For example, the building engineer needs geometry to set up a square and an upright building. The carpenter will also need knowledge of geometry to roof the building. All other careers, such as STEM related ones, need geometry. This means that geometry and mathematics in general can be a gateway to brightening one's future. According to Naidoo (2011), learning mathematics paves the way to a better future. Thus, geometry is included in the mathematics curriculum to prepare teachers for teaching STEM-related courses as well as prepare learners for its jobs (Jones, 2000).

It seems that attempts made by many countries to improve mathematics education, focuses on the geometry content of the mathematics curriculum. Like many countries, stakeholders of education in Nigeria acknowledge the importance of geometry in the life of people and the development of the country (Fabiya, 2017). According to Fabiya (2017), learning geometry is critical to learning both primary and secondary mathematics curricula in Nigeria, since it provides a valuable cognitive tool such as visualisation, for understanding the subject. According to Fabiya (2017), reports from the Chief Examiner on Nigerian learners' difficulties in mathematics, highlight geometry as an area where their performance is very low. In response to this, Fabiya (2017) investigated the geometry concepts that senior secondary school learners find difficult to learn. The study sample consisted of 500 learners selected from 30 co-educational schools in Ekiti State, Nigeria. A questionnaire comprising 23 items on geometry concepts was used to collect data. The study was guided by three research questions and a hypothesis. Results showed that two among the eight topics the learners perceived to be difficult, were coordinate geometry and construction. The author recommended to teachers to teach these topics or concepts using a participatory method with the use of instructional materials. According to the author, other studies in Nigeria show that mass failure in mathematics results from difficulties associated with the teaching and learning of geometry content in mathematics. Adegun and Adegun (2013), who looked at the difficult parts of the mathematics curriculum from the learners' and teachers' points of view, said that SHS learners' poor mathematics grades are caused by the difficulty they have in learning geometry.

When Fabiya's (2017), study results showed learners' difficulty with some geometric concepts, he recommended the use of instructional materials in geometric instruction. These materials are concrete objects that learners can see, hold, touch, and move around to learn mathematical concepts. Similarly, Sfard (2008) asserts that visual mediators (also called instructional materials) are objects that control communication in a discourse. When learners can touch, manipulate, and see the object of communication, it enhances their thinking processes and narratives about the object (Sfard, 2008). This facilitates meaning-making in a discourse between both the teachers and the learners, and enhances the individualisation of the learned concept. Viewing learning as a social construction means that learners learn with lead discussants or knowledgeable others in settings that allow knowledge to be constructed and reconstructed to achieve a shared meaning (Sfard,

2007, 2008). The use of instructional materials promotes a learner-centred approach to learning where learners are motivated to think of ideas, explore and experiment with new ways of doing things, and to communicate with others. In this situation, the teacher or lead discussant supports and facilitates the learners' process (Keiler, 2018). The view of teaching with materials (visual mediators) is supported in the sense that learners are not made to passively follow the teacher's instruction but to share his or her view of the learning process. This form of teaching and learning is endorsed by the NCTM (2000) due to its potential to enhance understanding of mathematical concepts and of geometry in particular.

Malaysia, like other countries, places a major focus on the teaching and learning of geometry to provide its learners with visual and analytical reasoning abilities. This is done in connection with their mathematical teaching aim, which is to “provide learners with a deep understanding of mathematical concepts so that they can relate, explain, and apply the mathematical knowledge to solve daily problems more innovatively” (Boo & Leong, 2016, p. 3). To achieve this aim, a new mathematics curriculum introduced in 2011 brought some changes, particularly in geometry, to foster the achievement of their mathematical aim (Boo & Leong, 2016). However, research shows that learning geometry at the primary and secondary school levels has not been easy for learners since many of them are unable to develop an understanding of the concepts, reasoning abilities, and problem-solving skills related to geometry. Research shows that Malaysian learners’ inability to understand those geometric concepts contributed to their poor performance in the subject area, which is reflected in their performance in mathematics (Boo & Leong, 2016).

2.5 Mathematics education in Ghana

Despite the suggested activities in the Ghanaian mathematics curriculum, to guide learners’ mathematics and geometric thinking, learners' performance in national examinations seems not to reflect the desired result. Good mathematics performance is required for learners’ admission to the institutions of higher learning in Ghana. It also determines one’s eligibility to study desired programmes especially in science and technology, in which success in learning depends on a good foundational knowledge of mathematics and geometry in particular (Luneta, 2015; Zhang, 2017).

Despite more contact hours devoted to high school mathematics instruction, results of learners' performance have been disappointing. The majority of learners' passes range from 45% to 55%. Concerned with such performances, many researchers have investigated this issue to identify the possible causes and provide the necessary remedies. Many of these studies have been founded on mathematical curriculum specifications and other learner-related factors, teaching processes, and the environment of the mathematics discourse (Watson & Harel, 2013). Research reveals that the teacher-centred teaching is the common method of instruction used at the various levels of pre-tertiary institutions in Ghana. For example, a study by Mereku (2003) conducted at the primary school level, investigated the alignment between the suggested teaching methods in the official mathematics curriculum (textbooks and syllabi) and teachers' classroom practices. He described the classroom instructional pattern as "teacher-led class discussion using situations and examples, followed by pupils' examples in exercises" (Mereku, 2003, p. 63). This finding contradicts the suggested participatory teaching methods in the mathematics curriculum to engage both teachers and learners in an interactive learning situation (MoE, 2010). The participatory approach is to allow learners to actively participate in the discourse, to become acquainted with and gain insight into mathematical concepts. According to Sfard (2008), active participation in a discourse is necessary to develop an exploratory way of learning and reasoning.

Learning through facts and procedures is what Sfard (2008) terms the acquisitionist approach to learning (teacher-centred instruction) where learners receive only verbal explanations for everything they must learn. This is considered a mechanistic approach to learning and is often suitable for the most basic forms of learning mathematics, such as terminology. This mechanistic approach has been criticised as "behaviour without mind" (Sfard, 2015 p. 130). This could also mean learning with little or no thinking. De Lina and Tall (2007) assert that the formation of mathematics concepts results from doing mathematics. Learning is said to be more effective when the learner is involved in developing and constructing concepts and then processing or thinking (reflection) about the knowledge constructed.

Mereku's (2003) findings show that teachers seem not to follow the suggested activities for mathematics discourses. This puts difficulty on the learners' learning habits in terms of forcibly committing to memory something they may not understand. This could be a reflection of the

learners' continuous low mathematics performance in national examinations. Learners might struggle to pass, sometimes, not because they truly understand, but because they rely on rules coupled with some cues to enhance recall. Learners who are taught through rule-based instructions generally learn by memorisation. According to Fletcher (2000), despite all efforts to encourage mathematics educators in Ghana to adopt the constructivist approach to teaching and learning mathematics, the reality of teaching at the basic and senior high school levels has been to place a strong emphasis on memorisation and imitation at the expense of understanding, resulting in low achievement in national and international examinations (Agyei, 2012). Agyei (2012) also attributes such poor performance in mathematics to poor presentation or delivery of mathematical concepts. Such a poor presentation occurs when the lead discussant assumes a central role in the discourse.

According to Arakaza and Mugabo (2022), Ottevanger, et al. (2007) and Swan (2005), the teacher-centred method of teaching is characterised by the teacher doing most of the talking in the mathematics discourse. The teacher supplies all the information to the learners. The learners sit back and follow what the teacher does with little or no contribution on their part. The learners' role in the discourse is to write notes, imitate the teacher's solution procedure to worked examples and apply them to other similar tasks. Learners in such discourses become passive recipients of information and are often treated as if they have nothing to contribute to the development of the discourse (Fletcher, 2009). This teaching approach limits learners' cognitive processes as their explorative thinking abilities are curtailed by engaging them in routines with little or no understanding. These examples within Sfard's (2008) framework, are characteristics of ritualised (acquisitionist) instruction.

One of the aims of teaching mathematics, however, is to enable learners to apply the knowledge gained to solve their daily problems (MoE, 2010). This is made possible through teacher guidance and fostering learners take responsibility for their own learning by self-construction of knowledge and making meaning from the learning activities. Thus, learners' engagement in mathematics discourse, should enable them to provide narratives about the objects of mathematics (Sfard, 2008). Learners must ideally, modify and expand their knowledge through the learning of mathematics (Ben-Zvi & Sfard 2007). This does not seem to be the case for learners in Ghana, where they mostly learn by imitating what the teacher does. In the commognitive perspective,

learners only develop ‘how’ to routinely complete mathematics tasks, which leads to rigidity in thinking. Thus, mathematics discourse in Ghanaian mathematics education can be described as one that follows the acquisitionist perspective, in which learners are made to follow procedures in solving mathematical tasks.

In a recent study, Mensah and Agyei (2019) desired to understand the philosophical stance of Ghanaian teachers, their teaching styles and the use of ICT in mathematics discourse. The study, which employed an exploratory case study design, involved six high school mathematics teachers selected from six schools. The background of the teachers indicated that all of them had taught for a minimum period of about seven years and were all considered professional teachers. The study found that their philosophical stand in teaching, followed the absolutist philosophical view in which they adopted the teacher-centered approach in mathematics discourse.

2.6 Geometry education in Ghana

Ghana's mathematics curriculum includes a significant amount of geometry, which could be viewed as a core skill in our daily activities. Our physical environment is composed of geometric shapes and hence learners need to develop geometry thinking and skills to explore and understand this environment (Jones, 2000; Luneta, 2015). In response to the ever-growing geometric knowledge in human life, the Ghana Education Service has a prepared mathematics curriculum with needed content to develop functional geometric thinking among learners (MoE, 2010). Unlike in the United States, where geometry is a separate course for high school learners, geometry is integrated within the Ghanaian mathematics curriculum with other topics, and is taught from pre-school to senior high school stages (Grade 12). Through these levels, bits of geometry are learned, and the geometry content has been designed such that at every level, the understanding of geometry concepts aims to match the learners' abilities. According to Bora and Ahmed (2018), exposing learners to geometric discourse from the early stages of education through to higher institutions, contributes to the development of critical thinking and problem-solving abilities, which help in learning mathematics.

Despite the early exposure of Ghanaian learners to the geometry content of the mathematics curriculum to enable them to develop the required cognitive abilities such as visualisation, spatial

reasoning, and problem-solving abilities associated with geometry discourse and to facilitate their understanding of mathematics, research and available records show that the performances of learners at various levels within the education system, falls below expectations and has remained low for many years (Abreh, et al., 2018, Bruce, 2016; Fletcher, 2018). Ghana's Grade 8 learners participated in TIMSS in 2003 to ascertain how they compared with learners from other countries in the various mathematics strands. Results showed that geometry was among the content areas where the learners performed poorly (Anamuah-Mensah et al., 2014). Also, in TIMSS (2011), a similar report on low performance of Ghanaian learners was obtained, and further analysis showed that geometry was among the poorly performed content areas (Mullis et al., 2011). The Ghanaian Grade 8 learners who participated in TIMSS, could not demonstrate the cognitive abilities needed to respond to the tasks on geometry concepts they were tested on. Those reports from international examinations are the similar as the analysis of learners' performance in national examinations, as mentioned previously.

The state of geometry performance by junior high school learners (Grade 8), is a worrying problem because the geometry taught from pre-school to this level is to meant prepare them for higher geometry content in the SHS (Grades 10–12). The SHS mathematics “builds on the knowledge and competencies developed at the junior high school level” (MoE, 2010, p.ii). Learners' inability to grasp the requisite knowledge and competencies before entering SHS shows that they enter this level with knowledge gaps, which will pose learning challenges to them. According to Hailikari et al. (2008), a major difficulty that instructors face in teaching, is the learners' lack of previous knowledge needed to understand the higher-level content of the curriculum.

With this concern, Baffoe and Mereku (2010) studied Ghanaian learners' (entering the SHS) understanding of geometry. The study intended to measure the learners' geometric thinking at SHS entry level. The sample comprised 188 first-year learners (Grade 10), selected from two schools to participate in the study. The data collection instrument was an adapted van Hiele's geometry test. This test and an aptitude test were administered to the learners in the first month on campus. It was found that a little above half (59%) of the participants attained van Hiele's level 1. 11% attained level 2 and 1% attained level 3. This shows that 1% of the participants who had just entered SHS, demonstrated the requisite knowledge needed to successfully learn geometry.

Hailikari et al. (2008) assert that learners' previous knowledge influences their knowledge acquisition and ability to apply that knowledge in problem-solving. The authors claim that learning is actively constructing new knowledge based on previous knowledge. They go on to say that learning may be hampered if previous knowledge is inadequate or fragmented. Learning new material without adequate previous knowledge or having misconceptions, may result in rote memorisation with little understanding. Hailikari et al. (2008) assert that this kind of surface learning happens when the learner cannot connect new knowledge to what is already known.

Similar concerns about geometry education probably motivated Asemani et al. (2017) to find the geometric thinking level of SHS 3 (Grade 12) learners, who at this level may have been exposed to almost all the geometry content in the SHS mathematics curriculum and are expected to attain the van Hiele's level 4. Their study used a quantitative approach with 200 participants who were randomly selected from three schools. Data was collected using the adapted van Hiele geometry test based on the first four levels. Learners were tested on their ability to identify shapes by their appearance, recognise geometric shapes by their properties, analyse properties of geometric shapes, show an understanding of the relations among axioms, definitions, theorems, proofs, and postulates. The instrument contained 20 multiple-choice items. The results showed that 42.5% of the learners could not attain any level. 33% attained level 1, 22.5% attained level 2, 1.5% attained level 3, and 0.5% attained level 4. This indicates that the learners' geometric thinking was very low, as only 0.5% could demonstrate adequate understanding of relationships among properties, axioms, definitions, theorems, proofs, and postulates. Thus, results of Asemani et al. (2017) showed that learners' geometric knowledge at SHS three (Grade 12) was below expectation.

The increasingly low performance in mathematics and geometry in particular, by Ghanaian learners at various levels raise several concerns, especially the knowledge mathematics teachers possess for teaching. According to Hourigan and Leavy (2017), a critical focus of research in education is teachers' knowledge, due to its significance on learners' learning gains. As a result, attention has been drawn to pre-service mathematics teachers' knowledge for teaching geometry.

Due to several concerns regarding Ghanaian learners' poor performance in geometry (Baffoe & Mereku, 2010, Asemani et al. 2017, Yalley et al., 2021), and the fact that teachers' competency in

geometry is critical to effective teaching of the subject, Armah et al. (2017) investigated the geometric thinking of Ghanaian pre-service teachers in the colleges of education. The van Hiele geometric criterion was used as a yardstick for assessment. The sample included 300 pre-service teachers conveniently selected from four colleges of education. The participants had received instruction in both content (including geometry) and methodology, and were about to begin their teaching practice (internship), specifically at the basic level (Grades 7 to 9). The authors used an adapted van Hiele Geometric Test to collect data. Results showed that 16.33% of the participants were at level 0, 27% attained level 1, 32% attained level 2, 17.67% attained level 3, 6% attained level 4, and 1% attained van Hiele level 5. This shows that most of the pre-service teachers could not attain the minimum level of geometric knowledge they needed to teach effectively at the basic school level. In other words, most of the pre-service teachers demonstrated geometric thinking below the level at which they were expected to teach. This could be the major reason why Ghanaian learners' performance in mathematics, particularly geometry, remains under par. The authors confirmed this point of view by saying that most of the pre-service teachers performed worse than what is expected of learners in junior high school.

In a similar study, driven by the fact that teachers' geometric thinking influences their teaching efficacy of geometry, Bonyah and Larbi (2021) assessed Ghanaian pre-service mathematics teachers' geometric thinking levels within the first three van Hiele levels, the exact content they would be teaching. The study was quantitative in nature and used a descriptive survey design. A sample of 217 pre-service teachers who had taken geometry as one of their modules, were randomly selected from four colleges of education.

The development of the data collection instrument was informed by the van Hiele Geometry Test (VHGT) and the objectives for learning geometry at the junior high schools in Ghana (MoE, 2010). Some of the van Hiele items were modified to make them suitable for the context of the study. The 40 multiple-choice tests used to collect data covered the first three levels of van Hiele's geometric thinking, and the application of geometric properties to solve related tasks in geometry. Results showed that even though all the pre-service teachers achieved level one, more than half could not demonstrate geometric thinking beyond level two. This means that less than half attained level 3, the minimum level required by pre-service teachers to develop their geometric teaching

competence. Not only do research results show pre-service teachers' weak geometric thinking, evidence from the Chief Examiners' reports on the examinations they write at the college level, also show their poor geometric content knowledge (Armah et al. 2017). It can be conjectured that when teachers do not have deep geometrical thinking, the only approach to teaching may be to embark on procedures. Thus, research shows that both learners and teachers find geometric concepts a difficult content area of the mathematics curricula (Luneta, 2015).

2.7 The acquisitionist and the participationist approaches to learning

Several teaching strategies are used in mathematics discourse to achieve various outcomes. According to Sfard (2008), these strategies can be classified as acquisitionist or participationist. Research has shown that the most frequently used strategy in mathematics instruction has been the acquisitionistic approach. Whilst change as a product of learning is not doubted, the concern is what that change is (Sfard, 2015) and probably, what causes that change. Most of the changes are acquired along the path of the acquisitionist metaphor (Presmeg, 2016). Since this approach, seemed not to produce the independence and the autonomy required for mathematical discourse, a contrasting view known as the participationist view emerged. The participationist view builds on the fact that learning occurs when people participate in different kinds of activities (Sfard, 2008). The participationists view learning as actively participating in a “patterned collective way of doing” (Sfard 2008, p.78). This definition is also supported by Bruner (1966), who asserts that learning is an active process where the learner constructs knowledge based on his present or previous experiences. It could be inferred from the above that learning results from doing, or engaging in, interactive activities that are culturally specific and make use of appropriate resources (tool-mediated) to facilitate concept development (Sfard, 2015). Active participation of learners in mathematics discourse forms the basis through which mathematical thinking is developed (Essack, 2015). Participation in a discourse also enhances learners' development of intellectual autonomy, through which they can make use of “their own intellectual capacities to reason about mathematical ideas” and self-thinking (Presmeg, 2016, p. 2). Perhaps, this is the focus of mathematics education. Learners who are to engage in independent reasoning have the potential to produce narratives that are acceptable in the mathematics learning community (Sfard, 2008).

Proponents of the participationist perspective on learning, are of the view that learning comes about through social and cultural interaction (Sfard, 2015). Communication plays a major role in teaching and learning. As a result, learners must interact with peers, teachers and available resources and actively engage in communication of mathematical ideas to enhance their sense making. According to Essack (2015), learning realisation occurs when the learner can communicate or talk about mathematical objects. This enables facilitators of mathematical discourse to determine what goes on in learners' minds through what they say and do as they participate in the discourse (Sfard, 2015). It also enables facilitators to see how well learners are understanding what they are being taught and to identify problems or misunderstandings for remedy.

Sfard (2008) considers mathematical concepts as abstract entities. Therefore, mathematics educators are concerned about how to bring these abstract concepts into reality-based learning. The acquisitionist approach to teaching curtails learners' ability to understand mathematics. Within the acquisitionist view, mathematics is more of teaching learners how to solve a task since they focus much on operationality and rigour. To a large extent, this approach does not consider individual differences in learning. A facilitator who is more in tune with this approach, teaches and expects learners to sit back and follow what he or she does with little or no contribution. This makes learning mathematics seem like information processing instead self constructed knowledge.

2.8 Learning Geometry

Learning, in the participationist perspective, occurs when there is growth in discourse through active participation (doing), either individually or with others. Learning is said to be an initiation into a patterned activity (Ben-Zvi & Sfard, 2007; Sfard, 2008). It is the ability to produce narratives about an object of study or about one's world. Learning takes place through social interaction or in collaboration with other participants. Learning, according to Sfard (2008), is a permanent change in discourse. She situates her communication framework on Vygotsky's view of knowledge creation through social interaction with others. According to Vygotsky as cited in Kivkovich (2015), knowledge creation results from cognitive changes that rely on social-cultural interaction. Learning results from the processes of interpersonal relationships among learners, through which they develop their cognitive abilities. This translates into Sfard's (2008)

commognitive perspectives on learning, in which the learner plays an active role in the patterned activity. Learner engagement in the discourse serves as a key point for developing conscious knowledge about the object of study. Like mathematics, geometry is considered a discourse with its own specialised form of communication which has words or terms that guide and improve communication in the discourse (Sfard, 2008).

In the discourse of mathematics, there are specialised words or terms (vocabulary) that are used for communication. As indicated previously, geometry has its own created vocabularies that keep geometric discourse in focus. As we focus and reflect on the vocabularies of geometric objects and communicate about them, it leads to an enhanced understanding of the subject matter. Communication plays a critical role in learning. According to Kivkovich (2015), a learner's active participation in an interaction with others is fundamental to a high level of learning. Communication or dialogue enables learners to think and talk about their learning perspectives. Facilitating learner communication in a discourse is an important strategy to make learners individualise learning.

We live with geometry and talk about its concepts almost every day, utilising geometric knowledge in our world of living (Luneta, 2015; Ndlovu, 2014). Learning geometry in a meaningful way enables learners to realise the immediate importance of geometry and mathematics in general. Developing learners' ability to learn and practice, or utilise the knowledge in a flexible manner, requires teachers to facilitate learners' construction of ideas (Kivkovich, 2015). This grounds the concept of learning as self-knowledge creation and not as something imposed. Such facilitators know how knowledge develops among learners and how to organise instruction around their learning perspectives.

2.9 Definitions in mathematics

According to Brunheira and da Ponte (2016), without definitions, mathematics and its communication could not be possible. This suggests that definitions play an important role in mathematics discourse. To define is to use words to express or describe mathematical concepts succinctly (McCammon, 2018). Thus, a mathematical definition should draw attention to certain

features or point to the exact intended concepts. Zaslavsky and Shir (2005) outline the following four features of mathematical definitions:

1. Introducing the objects of a theory and capturing the essence of a concept by conveying its characteristic properties,
2. Constituting fundamental components for concept formation,
3. Establishing the foundation for proofs and problem-solving and
4. Creating uniformity in the meaning of concepts, which allows us to communicate mathematical ideas more freely (Zaslavsky & Shir, 2005).

Winicki-Landman and Leikin (2000, p. 17) add that the definition must portray the following mathematical characteristics:

1. Defining is giving a name. The name of the new concept is presented in the statement used as a definition and appears only once in the statement.
2. For defining the new concept, only previously defined concepts may be used.
3. A definition establishes necessary and sufficient conditions for the concept.
4. The set of conditions should be minimal.
5. A definition is arbitrary.

In this regard, definitions should communicate mathematical concepts clearly and in a meaningful way. According to McCammon (2018), a statement that contains necessary and sufficient conditions of a concept, qualifies it to be its definition. The author adds that the properties of the concept could be presented as the necessary conditions. Herbst et al. (2005, p.17) also consider mathematical definitions as a “statement of the necessary and sufficient conditions that an object must meet to be labeled by a certain word or expression”. According to Brunheira and da Ponte (2016), mathematical definitions play a fundamental role in learning the structures of mathematics, and learners’ success in the subject depends on how they understand the role of mathematical definitions in learning. They are a vehicle for deepening one’s understanding of mathematics (Brunheira & da Ponte, 2016; Ndlovu, 2014).

2.10 Cognitive abilities and attributes for learning geometry

2.10.1 Definitions of geometric concepts

Learning to define geometric figures and concepts can be a useful starting point of engagement. Therefore, mathematics teachers' definitions of geometric concepts, becomes critical for enhanced teaching of the subject matter. They should be able to show an understanding of the definitions of geometric shapes and their related properties in a conceptual manner, which plays an important role in learning. According to Ndlovu (2014), definitions play a critical role in the Euclidean axiomatic system, and they are a starting point to think geometrically.

Geometry possibly requires more definitions in learning than any other topic in the mathematics curriculum. The learning of geometric shapes and figures depends on one's understanding of what the shape is. According to Ndlovu (2014), the definition of geometric shapes and figures is crucial because it involves their fundamental properties. For example, a triangle, which is often defined as a polygon with three sides and angles, communicates important features of the shape and its associated properties, which is needed to solve related problems. Researchers have shown the key role that definitions play in the learning of geometry (Ndlovu, 2014; Usiskin & Griffin, 2008). Ndlovu (2014, p. 6642) claims that “definitions of terms form the basis from which properties of the terms are logically defined, and the means by which the user can name and classify geometric objects”. This could mean that definitions form the starting point of learning geometry. Mathematics teachers' knowledge and competencies in defining geometric concepts, are necessary for effective teaching of the subject.

2.10.2 Terminologies, language and geometric discourse

Effective communication between teachers and learners is essential for teaching and learning mathematics (Mulwa, 2015). Both teachers and learners need to hold a common meaning of words and terminologies used in a discourse. Analysis of research findings shows that learners who are even more proficient in teaching language, (say English), sometimes are unable to follow certain discipline-based discourse due to terminologies that may not be in their learning vocabulary (Oyoo, 2009). This suggests that in a discourse, teachers often use words and terms that are more aligned to the subject however, those words may not be fully understood by learners. According to Atebe and Schafer (2010, p. 53), geometric discourse “stresses the use of language

(terminology) more than any other topic of the mathematics curriculum”. Hence, to learn geometry and acquire a good sense of geometric thinking, there is a need to be equipped with the meaning of, and be associated with the basic geometric terms often used in the discourse. The main purpose of language in mathematics instruction is to make it possible for both the instructor and the learner to precisely communicate their understanding of mathematics (Mulwa, 2015).

Alex and Mammen (2018) assert that learners’ inability to understand geometric concepts and the appropriate terminologies, is the underlying reasons for not achieving success in learning. Terminology forms the most fundamental knowledge in any field of study. It is the set of terms that are associated with and unique to a field of study (Sfard, 2008). Understanding technical terminologies, is key to successful learning of mathematics, particularly geometry (Alex & Mammen, 2018; Atebe & Schafer, 2010). In learning geometry, it is important that learners grasp an understanding of the appropriate technical terms and develop the ability to use these terms in communicating geometric ideas in a related discourse. Research shows that lack of language competency in a subject domain, such as geometry, hinders its meaningful learning (Atebe & Schafer, 2010).

Most of the basic words used in our daily communication contain geometric terms. Some of these words are; line, point, plane, angles, triangles, squares, rectangles, parallel, perpendicular, circles, and many others. They help us to communicate and understand the world around us, hence the need to learn geometry (Luneta, 2015). According to Clements and Battista (1992, p. 420), there is a need to understand geometry because we live in a world that is “inherently geometric”. We live in a world of shapes and lines, plane figures, etc. To be able to analyse, understand and interpret our world of living, it is important to study geometry, which deals with such figures and their properties (Biber et al., 2013; Riastuti et al., 2017), and understand its specialised terms and language used in communication.

Several studies have shown learners’ misconceptions in solving a wide range of tasks in geometry. Those tasks range from word use (terminology), through classification of shapes, to application of geometric properties to solve related tasks (Luneta, 2015; Ngirishi & Bansilal, 2019; Robichaux-Davis & Guarino, 2016; Sulistiowati, et al., 2019). Learners' geometric misconceptions can be

linked to their non-exposure to appropriate geometric vocabulary in the discourse (Alex & Mammen, 2018; Atebe & Schafer, 2010; Mulwa, 2015). To remediate learner errors that result from inappropriate terminology, it is important that early discourse in geometry focus on the use of precise terminology, both in communication and in written work (Alex & Mammen, 2018; Atebe & Schafer, 2010).

The concept of precise terminology is one of the subcomponents of the elements of Sfard's (2008) commognitive framework, under word use. The framework holds the fundamental tenet that mathematics is a form of discourse due to the distinctive pattern of activities associated with it. Sfard (2008) categorises the discourse of words into colloquial and literate. Literate word use has a shared and specific meaning within the mathematics learning community. Teaching mathematics, which is considered a form of discourse, characterised by distinctive use of vocabulary that is grounded in the subject, must adhere to vocabulary-driven instruction in the discourse. For instance, we have words that signify numbers and operations, as well as words that refer to geometric shapes. Words that are associated with mathematical objects are referred to as literate words. This signifies that lead discussants in mathematics discourse must make careful use of literate words and draw learners' attention to such words (terms) with their appropriate meaning explained. Hence, colloquial words (words with different meanings in day-to-day use) must be avoided in a discourse. When such words are used in a discourse, their subject-specific meaning should be emphasised. For example, 'angle', which has a literate meaning viz 'an amount of opening formed by two rays at a point', can colloquially mean 'one's viewpoint in ordinary discussion'. The word 'similar' which has a literate meaning as 'same shape but not the same size', can also be colloquially interpreted as 'resembling but not exact'. Sfard (2008) cautions teachers to be mindful of the words they choose to explain mathematical concepts and those of geometry in particular.

Teaching geometry aims to enable learners to produce narratives about the field of study. The goal is to help learners develop a firm understanding of the basic, fundamental terminology required to compose or produce narratives that are accepted in the discourse. Human minds process information that is self-constructed, better than when it is imposed by lead discussants. Hence, to ensure learners' proficiency in understanding and use of terms in a discourse, learners must be

guided to construct narratives of the field of study (geometry) to show their understanding of the terms in the discourse. Thus, until learners can make meaning of the terminologies in geometry and use them correctly in communication (oral or written), they may continue to have challenges in learning advanced topics in geometry (Alex & Mammen, 2018; Atebe & Schafer, 2010).

Alex and Mammen (2018) studied 126 volunteer, university students' understanding of geometric terminology using van Hiele's theory as a framework. A questionnaire with 60 multiple-choice items (30 verbal and 30 visuals on geometric terminology) was used to collect data. The authors wanted to obtain data on how **the students** could identify the names and terminology of some geometric concepts (verbal description) and associate them with their visual representation. The study found that majority of the participants demonstrated a fairly good understanding of geometric terminology. According to the authors, the participants performed better on the visual representation of geometric concepts than on the verbal presentation (identifying names), which showed a lower level of geometric thinking within van Hiele's model. The author concluded, that a combination of both visual (diagrams) and verbal description should be integrated into instructional approaches to develop learners' conceptual understanding of geometry. They added that complementing both verbal description and visuals (pictorial) in an instruction, has the potential to support and fill learners' (pre-service teachers') content gaps found in the study. The interpretation drawn from the finding is that the participants were able to identify the geometric concepts in diagram form rather than the names or geometric terms for those concepts on lines, circles, triangles and quadrilaterals.

Similarly, Atebe and Schafer (2010) studied learners' knowledge of the vocabulary in geometric discourse. The study was based on the fact that learners' acquisition and use of correct geometric terminology are critical for successful learning of the subject. The study participants were drawn from both Nigeria and South Africa. The study used the stratified sampling technique to select 144 learners. The data collection instrument was a questionnaire of sixty multiple-choice items. Analysis of the data showed that the learners' knowledge of basic geometric terminology was limited. This suggests that the learners' success in learning geometry can be impeded since the basic geometric terms that form the foundation for learning, are not firmly grasped. The authors also found that the learners' verbal geometric terminology correlated with their ability in visual

tasks in geometry. The acquisition of basic terminology is necessary for learning and communicating ideas in geometric discourse. According to Atebe and Schafer (2010), terminology forms a critical tool for communication. Therefore, teachers must possess adequate knowledge of terminology and vocabulary to communicate ideas and engage learners in geometric discourse. Zayyadi et al. (2020) and Khan et al. (2017) assert that a teacher's communication competence with regard to terminology and word use is a significant component in achieving and developing learners' learning outcomes. Learning will be affected when inappropriate words are used in communication (Khan et al., 2017, Sfard, 2008, Zayyadi et al., 2020).

2.10.3 Geometric properties

Geometric properties serve as an important tool for learning the subject matter. According to Bassarear (2012) and Luneta (2015), geometry deals with the study of shapes and figures, their properties, and the relationships among them. The study of geometry goes beyond the study of shapes around us. A learner's ability to identify and classify shapes depends on some characteristic features of their properties. According to Ndlovu (2014), learners' proficient understanding of geometry is built on sound knowledge of the properties of geometric shapes. From the structural design of a geometric shape, emerges associated properties that must be learned and understood. Research shows that high school learners often have difficulty identifying the properties of geometric shapes (Atebe, 2008). This hinders their ability to classify shapes, as well as to apply their properties to solve related tasks in geometry. Knowledge of geometric properties helps in devising strategies to solve a task. It also serves as a foundation for learning more complex or advanced concepts in geometry (Ngirishi & Bansilal, 2019; Ndlovu, 2014). Ndlovu (2014) claims that the properties of geometric shapes and their relations need to be understood by both teachers and learners to facilitate classroom communication. Sabey (2009) adds that learning about geometric concepts, properties, and their relationships, deepens learners' understanding of the subject matter and offers them the opportunity to communicate ideas in the discourse. As part of the purpose of this study, the PSTs' thinking about geometric properties was analysed.

Chigonga et al. (2017) explored exiting Grade 9 learners' knowledge and understanding of the properties of quadrilaterals. The study with 84 exiting 9th graders, was a qualitative case study. The instrument for collecting data was an adapted questionnaire from van Hiele's test for assessing learners' geometric thinking, and contained 25 multiple-choice items. The participants were given

an hour to respond to the test items. The data, analysed in terms of correct or incorrect responses, revealed that the participants had limited knowledge of quadrilateral properties. According to the authors, a majority of the participants provided wrong answers. They added that except for the first item, the remaining items were answered incorrectly by more than half of the participants. For example, a little more than half of the participants gave the wrong response to the item “The diagonals of a rhombus are perpendicular” (Chigonga et al., 2017, p. 3). The authors advocated for secondary school learners to be taught quadrilateral properties through a discovery approach.

In a similar study, Alex and Mammen (2014), investigated Grade 10 learners’ knowledge of the properties of quadrilaterals and triangles. The authors found that the learners were unable to identify geometric shapes by their properties. In addition, the learners showed difficulty in identifying the properties associated with quadrilaterals and triangles. The authors claimed that the participants who had limited knowledge of geometry is a knowledge gap that could affect their understanding of advanced topics in geometry. The authors called for immediate attention to be paid to the learners’ learning needs to bring them to the expected levels which required teachers to possess in-depth knowledge of the properties of geometric shapes to facilitate learners’ classroom learning and also to diagnose learners’ misconceptions and errors, for correction.

2.10.4 Spatial visualisation

Visualisation basically has a long tradition in mathematics discourse. Mudaly (2016) asserts that visualisation in mathematics is not a new discovery. Rosken and Rolka (2006) gave an example where the blind Euler, despite his blindness, was able to produce about 355 pages of work due to his “visual imagination as well as his phenomenal memory” (p. 1). In addition to the historical nature of visualisation, Stylianou, as cited in Mudaly (2016) posits that visual thinking in mathematics dates back to the work of Euclid’s book on geometry, titled ‘The Elements’. In the book, geometrical ideas were arranged into definitions, axioms and theorems. Most of his arguments in the book were based on geometric figures and shapes, which indicate that the concept of visualisation has been used in mathematics for many years. The role of visualisation in mathematics discourse has received much attention by researchers (Arcavi, 2003; Lowrie et al., 2018; Winarti, 2018) and has numerous definitions by several authors. A notable definition of visualisation by Arcavi (2003, p. 217) states that:

Visualisation is the ability, process and product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understanding.

The definition shows that visual ability is an indispensable tool for mathematical exploration through which meaning is attached to mathematical concepts in learning. Atanasova-Pachemska et al. (2016, p. 1) define visualisation as "the creation of a mental image of a given concept". Its utility in a discourse has the potential to enhance learners' thinking and understanding of several concepts in many fields of study. Employing visual abilities in exploring mathematical or geometric concepts, builds one's capacity to develop a global and intuitive sense of understanding in learning. In the view of Mudaly and Rampersad (2010, p. 25), "visualisation can be a physical or mental process". In other words, visualisation occurs when we see something with our physical eye or with the eye of the mind (imagination). As put forth by the cognitive scientist, "we learn to see; we create what we see", visual reasoning, or 'seeing to think', can be developed through learning and training (Atanasova-Pachemska, et al., 2016, p. 2). If pre-service teachers have developed a good sense of visual reasoning, they can guide and develop the learners' visual sense of reasoning in classroom learning.

Research in spatial reasoning is receiving attention because of its use in learning Science, Technology, Engineering and Mathematics (STEM) courses and in understanding the natural spatial nature of the world (Jones, 2000; Lowrie et al., 2018). Several studies have been conducted to analyse the effect of visual abilities on problem-solving as well as how learners' visual abilities and spatial reasoning can be developed. Concerns have been raised that learners should start receiving instruction that develops their visualisation to enhance their subsequent mathematics discourse (Atanasova-Pachemska et al., 2016; Riastuti et al., 2017). In response, Atanasova-Pachemska et al. (2016) studied the visualisation of geometry problems with a focus on primary school education. They worked with learners whose teachers believed in visualisation in mathematics discourse and integrated visual tools into their lessons or otherwise. The study was experimental in nature and comprised an experimental group of learners whose teachers taught geometry using visuals and encouraged visualisation problems (using I.C.T). The control group

however comprised learners whose teachers indicated that learners get confused with visualisation problems and left it in the hands of the learners to resolve those problems. The learners in the two groups were tested on their competence in solving geometric problems with or without visualisation of geometric tasks. The test aimed to determine if visualisation influences the successful learning of geometry. The results showed that the experimental group received a higher average mean score on the geometry test than their control group counterparts. Further analysis showed that there was a significant difference in favour of the experimental group. This means that teachers who taught through visual means and encouraged their use in problemsolving, produced learners who performed significantly better on visualisation and problem-solving tests than learners whose teachers were not so enthused about teaching with visual means. Also, those in the experimental group had their visual and spatial reasoning well developed compared to their counterparts in the control group. The authors concluded that the use of visualisation in the process of teaching problem-solving, influences success in learning and recommended that mathematics teachers should implement this teaching and learning strategy in geometric discourse.

2.10.5 Spatial orientation

According to Lowrie et al. (2018, p. 10), spatial orientation requires a “transformation of imagining a change in one’s own perspective”. This ability enables a learner to identify geometric shapes in different positions. In other words, it deals with the ability to view and recognise geometric shapes when displayed in different locations or positions. Clement and Battista (1992) add that spatial orientation deals with the understanding of how spatial objects are related when viewed from different positions. Learners who have developed their visual processing ability to manipulate and transform visual imagery and information are better trained to acquire an important mathematical skill for learning (Lowrie, et al., 2018). This is an important ability for learning geometry. For example, a learner’s ability to identify a shape in its non-standard form depends on his/her spatial orientation.

2.10.6 Spatial ability

Several abilities enhance the learning of geometry. Learners’ visualisation is supported by their spatial ability or spatial intelligence. Spatial intelligence, according to Riastuti et al. (2017), is the

learners' ability to perceive the world as a spatial entity and transform its spatial perception in many ways. It is the learners' ability to mentally manipulate the spatial relations among objects. Spatial ability is one of the special cognitive abilities learners need to solve problems in geometry. Acquisition of such cognitive ability enables learners to develop geometric thinking in both concrete and abstract senses. Riastuti et al. (2017) claim that most of the problems learners have with learning geometry, can be traced back to their inability to think spatially.

Similarly, Lowrie et al. (2018) investigated the impact of an intervention on learners' spatial reasoning. The study, which was experimental in nature, included 337 participants selected from six schools. These participants were grouped into eight (8) experimental classes and seven (7) control classes. The intervention was conducted by teachers who participated in a 5-day professional workshop on how to develop spatial reasoning abilities. The teachers were also exposed to the constructs of spatial reasoning, which was the lessons' pedagogical framework. They were then encouraged to use such lesson activities to strengthen their classroom pedagogy. The intervention with the learners lasted for ten weeks (20 lessons), and the learners were engaged in learning activities that developed their spatial thinking. The delivery of the intervention programme followed the Experience-Language-Pictorial-Symbolic-Application (ELPSA) framework, whilst the control group was taught through standard transitional mathematics instruction. A Spatial Reasoning Instrument (SRI) used to collect the data focused on three concepts: spatial orientation, spatial rotation, and spatial visualisation. Results showed that learners in the experimental group performed better in spatial reasoning abilities than those in the control group. According to the authors, there was evidence of high participation and spatial reasoning development, based on the stages of the ELPSA model. The authors concluded, based on the evidence of the effectiveness of the intervention, that it should be applied when teaching STEM programmes.

Research shows that learners who have high spatial reasoning often make fewer errors in solving geometry and mathematics tasks. For example, Riastuti et al. (2017) used a qualitative approach to analyse learners' errors in geometry from a spatial intelligence perspective. Purposive sampling was used to select the participants. Data was collected through tests and interviews. Data on spatial intelligence was obtained from the scores on spatial intelligence tests. Of the 35 participants, 11

were classified as high spatial learners, 17 with medium spatial intelligence and 7 with low spatial intelligence. The results showed that even though there were different errors in solving the geometric tasks in each category of spatial intelligence, learners with low spatial intelligence often made errors in their solutions due to deficiencies in their visual abilities. This means that learners with low spatial intelligence were unable to see unseen features embedded in geometric tasks, limiting their ability to draw relational structures and connections about the tasks, to aid their understanding (Arcavi, 2003; Matlen et al., 2018).

A deep reflection on the above findings shows that learning experiences which promote spatial intelligence, seems to be missing in our traditional mathematics classroom, and needs special and immediate attention. A conscious effort must be made to develop this cognitive skill among school learners to improve their performance in learning geometry and mathematics in general. Learners need to be trained to utilise their visual abilities in learning. From the teaching point of view, Atanasova-Pachemska et al. (2016) claim that visual ability is a powerful teaching and learning tool that needs to be utilised to develop learners' understanding of mathematical concepts and disciplines associated with STEM. The authors add that many reasons support the teaching and learning of mathematics through visualisation, at all levels of education. Seeing differently is not inborn but needs to be created, developed and learned and must be given special attention in mathematics discourse. Arcavi (2003) asserts that we learn about the world through our sense of vision.

2.10.7 The use of diagrams in geometric discourse

Diagrams play a significant role in the teaching and learning of geometry. According to Samkoff et al. (2012), diagrams are visual forms used to present information in mathematics. Diagrams make it possible to identify relationships between mathematical concepts. Watson et al. (2013) and Yahya et al. (2022) claim that geometry is one of the content areas in mathematics where diagrams are used to convey concepts, as well as in its teaching and learning. Jones (2013, p. 38) defines a geometric diagram as “a figure composed of lines, serving to illustrate a definition or statement, or to aid in the proof of a proposition”. Sfard (2008) considers mathematical objects as abstract in nature, and the only possible way to reduce the abstractness of these objects is the use of diagrams

or visual data, for example, a point, an arc, or a line. These geometric concepts can be well conceptualised through representation in visual form.

Jones (2013) asserts that the use of diagrams in mathematics education is common for a number of reasons. A notable one is the expression “a picture is worth a thousand words” or its equivalent, “hearing a hundred times is not as good as seeing once” (p. 37). This suggests that information conveyed through diagrams is better processed and retained in the mind than that which is heard. This could also mean that the eyes are receptive or sensitive to visual data. Learners' geometric thinking can be well developed if diagrams are made an integral component of the geometric discourse. Samkoff et al. (2012) remark that mathematics education and mathematicians view diagrams as “an integral component of doing and understanding mathematics” (p. 49); also “drawing diagrams is commonly cited as a heuristic for mathematics problem-solving that students should engage in” (p. 50). This shows that diagrams have the potential to enhance learning mathematics and geometric understanding as well as to develop problem-solving abilities, which are central to the teaching and learning of mathematics.

Geometric concepts must be conveyed to learners using diagrams, and they must also be made to draw them to show their understanding of those concepts. According to Thom and McGarvey (2015), drawing is a tool used in geometric education to discover how learners represent geometric ideas. It allows learners to gain an in-depth understanding of geometric concepts. Drawing gives learners the chance to become familiar with geometric principles, concepts, and their relationships. Thom and McGarvey (2015) assert that learners' acts of drawing and the processes attached to drawing have a significant impact on how well they understand geometric concepts. Learners' ability to draw is an external representation of the internal thinking process. Thus, the act of drawing (visual representation) helps learners to reason before their visual production. In other words, the use of diagrams in learning can be regarded as a powerful thinking tool. Brizuela and Gravel (2013) claim that visual representation (drawing) is the process one engages in to make meaning of a phenomenon. An opportunity to support one's thinking and information processing is through visual representation (Lowrie, 2020). Hence, the use of diagrams and drawing can be a powerful tool for developing learners' geometric thinking.

Diagrams enhance the thinking and reasoning abilities of learners (Yahya et al., 2022). Rizwan et al. (2018) investigated how learners' geometric thinking can be developed and deepened with diagrams as an instructional teaching aid. The study intended to identify high school learners' misconceptions about geometric concepts and how teachers can use diagrams to help develop the learners' conceptual understanding of angles. The study used an experimental design that comprised both control and experimental groups. Sixty high school learners participated in the study. The control group was taught using the traditional method (chalk and talk) of instruction, whilst the experimental group was taught using diagrams to explain concepts of angles. A 25-item test with multiple-choice and short-answer questions was used to collect data. The study found that the experimental group benefited significantly as the learners used diagrams to clearly explain their geometric ideas about angles. The use of diagrams engages both teachers and learners in a discourse. According to Rizwan et al. (2018), actively engaging learners in a discourse enables them to acquire an in-depth understanding of the concepts that arise from doing mathematics, instead of being passive learners who only watch what the teacher does. The use of diagrams in teaching and learning can provide an avenue for learners to learn geometric concepts in a less abstract context. Another important benefit to be obtained from learning with diagrams is the development of learners' visual abilities, which is important for learning geometry as well as mathematics and for developing problem-solving abilities (Hasanah et al., 2019).

2.11 Pre-service teachers' geometric thinking

2.11.1 Pre-service teachers' understanding of angles and parallel lines

Understanding the concept of angles is critical for learning geometry (Arslan et al., 2016; Yigit, 2014). Biber et al. (2013) assert that angles form one of the basic geometric concepts that permeate almost all other topics in geometry. This could mean that teachers and learners would benefit greatly if their geometric thinking is based on a good conceptual grasp of angles. Angles are formed when two straight lines meet or intersect at a point called the vertex. Musser et al. (2014) define angles as the union of two straight lines at a common point. Some concepts that need to be learned about angles are their notations, representation, types. For example, angle ABC denoted as $\angle ABC$ could be interpreted as the angle formed at the vertex B of line segments AB and BC . According to Kontorovich and Zazkis (2016), Smith et al. (2014) and Mullins (2020), angles can

be viewed as static or dynamic. The static deals with angles formed by two arms or sides, whilst the angle as dynamic, involves movement such as a turn. Several angles are formed when two straight lines intersect or when a transversal intersects two straight lines. The intersection of these straight lines produces several angle properties that need to be learned.

Studies reveal that learners have difficulty in understanding basic geometric concepts necessary for learning the subject matter and important aspects of mathematics (Biber et al, 2013; Clement & Battista, 1992; Crompton, 2014; Ngirishi & Bansilal, 2019; Zuya & Kwalat, 2015). These researchers also show that learners have several misconceptions about angles in geometry. For learners' difficulties and misconceptions to be addressed and remedied, teachers need to possess good geometric thinking to provide effective learning support to learners (Crompton, 2014). Research reveals that teachers and pre-service teachers also possess similar learning difficulties with geometric concepts (Biber et al., 2019; Zuya and Kwalat, 2015).

A study by Zuya and Kwalat (2015) may have been informed by the work of Biber et al. (2013), who found that the participants focused attention to the appearance of the geometric task instead of the geometric properties. They could not relate to or apply the known properties to devise solutions for other tasks. According to Biber et al. (2013), the learners' main learning difficulties were the over-generalisation of properties of specific cases to others, and difficulties with the concept of parallelism. Zuya and Kwalat (2015) based their study on these findings to investigate teachers' geometric thinking. The goal was for the teachers to identify the knowledge learners lacked and to determine strategies to help address the problem. The study showed that most of the teachers could not identify the knowledge that learners lacked to solve the task on angles and parallel lines. Also, the teachers were unable to suggest specific strategies to address or remediate the learners' learning difficulties because the teachers themselves had a limited understanding of angles and parallel lines. This causes concern because of the teachers' role to guide and facilitate learning in the classroom and to provide learning support based on identified learning difficulties.

Other studies with pre-service teachers as participants, show similar results (Yigit, 2014). Yigit (2014) studied four pre-service mathematics teachers' understanding and mental construction of angles. Data was collected through clinical interviews conducted in one-on-one sessions. The four

participants were those who were willing to explain their thought processes about their mental construction of angles. From the results, all the participants showed adequate schema on 2-line angles, but they were less flexible in constructing 0-line and 1-line angles. It meant that these pre-service teachers had limited knowledge on angles.

2.11.2 Pre-service teachers' understanding of triangles

Triangles have a central role in the study of geometry, which forms the basis of the study of polygons. A triangle is a polygon with three sides (Smith, 2014) with a narrative that the interior angles sum up to 180° (Sfard, 2008; Smith, 2014). Triangles can be classified by sides or by angles. For classification by sides, they are equilateral, isosceles and scalene, and by angles, they are right triangle, acute triangle and obtuse triangle. The various types of triangles have their own associated properties that must be taught to give learners a firm foundation for learning higher concepts in the discipline. Types of triangles are represented visually using marks to design the diagrams. These marks in the communication framework are termed 'iconic mediators'. Sfard (2008) asserts that the iconic mediators are used to communicate certain important features of geometric shapes to learners. For example, an equal number of marks placed on any two sides of a triangle communicates that the two sides are equal, hence, the name isosceles. It also connects to the property that 'angles opposite the equal sides are also equal'. Learners' abilities to identify and interpret iconic mediators are useful to learning and solving tasks in the discourse (Sfard, 2008).

Ndlovu (2015) conducted a study to investigate pre-service teachers' understanding of geometric definitions and class inclusion of triangles and quadrilaterals. For this section, only findings about triangles are discussed. The focus was to determine pre-service teachers' ability to define a triangle in an economical or non-economical way. The survey design was used and 16 pre-service teachers participated in the study. In the pre-test, the pre-service teachers were asked to define some types of triangles in their own words. Analysis of data showed that they were able to identify the geometric shapes and their properties. Based on the study focus on the economy of definitions, the author remarked that even though they demonstrated good knowledge of definitions, they generally lacked the economy of definitions. The author concluded that teachers' geometric knowledge was weaker than expected.

The study of the definitions of triangles comes with several properties required for teaching and learning proofs, showing the relationships among many figures in geometry as well as their properties (Atebe, 2008). For example, in teaching the analytical proof of the circle theorem, the properties of triangles serve as the fundamental knowledge. Other properties of the shape can be the exterior angle formed when a side of the triangle is extended, known as the exterior angle theorem, which states that the exterior angle formed is equal to the sum of the two opposite interior angles. The properties of geometric figures and shapes are critical tools for solving problems in the discipline (Ndlovu, 2014; Ngirishi & Bansilal, 2019).

2.11.3 Pre-service teachers' understanding of quadrilaterals

Several studies report that pre-service teachers' have difficulty understanding quadrilaterals, definitions, and inclusion criteria of classifying quadrilaterals (Baktemur et al., 2021; McCammon, 2018; Ndlovu, 2014; Pickreign, 2007; Rianasari et al, 2016; Ulger & Broutin, 2017; Wang, 2013; Zilkova, 2014). According to Rianasari et al. (2016), quadrilateral classification is one of the topics taught in secondary school mathematics, so it is expected that teachers and pre-service teachers understand the definitions and inclusion relations among them.

For example, in a study conducted by Ulger and Broutin (2017) to investigate pre-service mathematics teachers' understanding of quadrilaterals and their relationships, open-ended questions were posed to 27 participants through clinical interviews. Analysis showed that there were more personal definitions (colloquial word use) in their responses, than formal definitions (literate word use). According to the authors, the PSTs defined the shapes based on their experiences with them, which showed more prototype figures. Adding some details, the authors reported that although some of them defined a parallelogram as “a quadrilateral whose opposite sides are parallel to each other”, the majority used incorrect properties in their definitions. Others defined a parallelogram from the prototype point of view, saying that “a parallelogram does not have right angles; a parallelogram is an oblique quadrilateral”. The authors explained that those definitions resulted from a prototype figure. According to Fujita (2012), prototype figures are those formed during the learners' first [and subsequent] encounter with such geometric objects. The authors of the study claimed that many participants showed difficulties with inclusion relations

within the quadrilaterals. Literature shows similar findings as reported by Ulger and Broutin (2017).

Baktemur et al. (2021) investigated pre-service teachers' conceptions and misconceptions of definition, classification and inclusion in which 20 purposely selected participants were involved. The case study design was used. The participants were to take a module in Methods of Teaching Mathematics, after they had taken various content modules, including Analytic Geometry and Elementary Geometry. For the two weeks devoted to the teaching of geometry, instruction focused on van Hiele's levels of geometric thinking on two- and three-dimensional shapes and related properties. The learners were asked to design tasks that would support learning and understanding in the classroom. The instrument used to generate data was a convex quadrilateral test developed by the researchers. The test items were open-ended, true-false type questions, definitions and inclusion relations among quadrilaterals, as well as identification of shapes and the drawing of alternative quadrilaterals. Data was generated before and after taking the module. The study revealed that the PSTs, before taking the module, showed inadequate understanding of hierarchical classification of quadrilaterals and experienced difficulties with definitions. It was also found that the participants described the quadrilaterals with no indication of inclusion relations which means that the PSTs did not show any knowledge that 'a rectangle could also be called a parallelogram' or 'a square could be called a rectangle'.

Learners with such thinking within the Sfard (2008) commognitive theory, may think that a geometric shape cannot have two names. Often, learners will give a 'no' answer when they are asked 'if a square can also be called a rectangle or a parallelogram'. This is a demonstration of a ritualised way of thinking, and in most cases, it happens due to strict rules of definition (Sfard, 2008), or, how the object has been experienced or encountered over the past learning years (Fujita, 2012; Ulger & Broutin, 2017).

In another study by Ndlovu (2014), 16 pre-service mathematics teachers participated. Analysis of data generated about the participants' thinking on definitions and inclusion relations revealed that, though they showed some understanding of definitions, their thinking on inclusion relations was limited. Ndlovu (2014) concluded that the participants' understanding of quadrilaterals was lower

than expected. Most of the pre-service teachers held the view that a rectangle is not a parallelogram. Also, some did not believe that a square is a rhombus (Ndlovu, 2014). According to Rianasari et al. (2016), pre-service teachers frequently prefer partition classification to hierarchical classification because in partition, they do not consider shapes that share properties with others. Studies show that pre-service teachers' geometric thinking on quadrilaterals is below expectations, as some are even unable to identify or recognise geometric shapes in non-standard positions (Contay & Paksu, 2012; Duatepe-Paksu et al., 2012; Rianasari, 2016; Zilkova, 2014).

2.11.4 Pre-service teachers' understanding of circles and circle theorems

Unlu (2022) studied pre-service mathematics teachers' concept definitions of a circle, a circular region, and a sphere. The participants comprised 56 pre-service teachers enrolled in a teacher education programme at the university. The instrument used to collect data was a test consisting of three items on circles, circular regions, and spheres. The data was analysed in terms of correctness and generalisability criteria. After the analysis of the initial data, the author conducted a semi-structured interview with nine of the participants. The author found that most of the pre-service teachers' responses or answers to questions on circles were either insufficient or incorrect. Also, a few of the participants provided appropriate examples that related to the concepts examined.

Similarly, Aksu (2019) conducted a study to investigate pre-service mathematics teachers' ability to respond to mistakes learners make, in learning about circles. The participants were expected to identify the learners' source of mistakes and suggest a correct solution. The study, which employed a qualitative inquiry approach, used open-ended questions to obtain data from 30 participants. The author found that the participants could not identify the source of the learners' mistakes. In addition, they could not suggest or propose any correct solutions to the challenges learners faced. According to the author, there is a need for pre-service teachers to be aware of the demands of the professional career and to prepare themselves for the desired expectations.

Research shows that learners at the various levels of study, from elementary, through high school to teacher education levels, also show difficulties in learning concepts about circles and circular regions (Aksu & Kul, 2018; Cantimer & Sengul, 2017). Research indicates that secondary school

learners have difficulties in remembering basic concepts in circles. In addition, they are unable to express their ideas and understanding of the relationship between the concepts, even though they can recognise them (Cantimer & Sengul, 2017). It could be inferred that it is these unresolved difficulties that manifest themselves, even at the teacher education level. PSTs carry several unresolved misconceptions from previous learning, which accounts for their difficulties in learning at higher level education. For example, a study conducted by Mudaly (2021) involved 51 undergraduate mathematics students. Figure 2.1 was one of the tasks used in the study, and the people who took part were asked to write down the value of the angle formed at A.

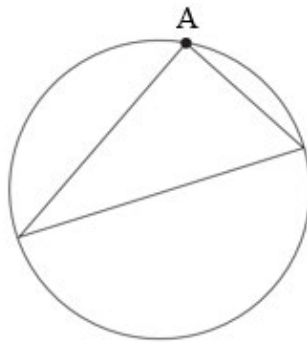


Figure 2.1: Task on circle theorem (adapted from Mudaly, 2021, p. 4)

It was found that 34 of the 51 participants indicated the value as 90° , and substantiated that ‘angle in a semi-circle is 90° ’. Thus, these participants took the chord to be a diameter without first examining whether or not it passed through the centre of the circle. This could result from weak visual abilities as well as a misconception in the application of a learned theorem. According to Yahya et al. (2022), most learners have difficulty in learning geometry because of their poor visualisation skills.

The difficulties of Ghanaian learners regarding circles, have been documented both by researchers (Fletcher & Anderson, 2012) and by the Chief Examiner’s report on analysis of candidates’ solutions to questions. It is known that learners only answer questions on geometry, particularly circles and circle theorems, when that is the only alternative, and in such cases, their solutions clearly demonstrate weak knowledge (Fletcher & Anderson, 2012).

2.11.5 Pre-service teachers' spatial abilities

Sections 2.8.4 to 2.8.6 show the importance of spatial skills that learners need to successfully engage with learning geometry and other mathematics content areas. This is an indication that teachers' instructional experiences designed for classroom discourse, must take into consideration important cognitive learning abilities such as spatial skills. This presupposes that teachers must have their spatial abilities developed and be able to develop such skills among learners in geometric discourse. According to Ozdemir and Goktepe Yildiz (2015), spatial abilities are highly connected with geometric thinking. Hence, pre-service teachers being prepared to join the teaching profession, should be able to demonstrate this knowledge as part of their thought processes for effective teaching. In view of this, Ozdemir and Goktepe Yildiz (2015) conducted a study to examine pre-service teachers' spatial abilities in problem-solving. The study used the qualitative approach, making use of clinical interviews to generate data. According to the authors, these interviews enabled them to connect with the participants' thought processes in an interactive manner. Three pre-service teachers willingly participated in the study. Sixteen (16) questions comprising open-ended, multiple-choice types, and those requiring drawings, were used for data generation. The qualitative data analysis showed a wide range of spatial abilities among the pre-service teachers. According to the authors, one of the pre-service teachers was found to have poor visualisation ability, and his attempts to answer almost all of the questions were based on superficial analysis rather than deep thought about the tasks, and hence he failed. This pre-service teacher was then found to be performing at the unstructured level. The second pre-service teacher was found to be operating at the multi-structural level and, as a result of his ability to employ some level of spatial visualisation skills, he was able to progress in his responses to a medium level. The last pre-service teacher operated at a higher level and was found to possess high spatial thinking skills. According to the authors, this pre-service teacher was able to relate information with others, regarding the given problem, and hence was able to satisfactorily solve the given tasks. It can be inferred from this study that the development of spatial abilities is not automatic among teachers, and therefore, an effort must be made for its development through several specific activities. According to Ozdemir and Goktepe Yildiz (2015), the development of spatial abilities can only be realised by engagement in various purposeful activities in a discourse.

2.12 Research on pre-service teachers' definition ability

Despite the importance of definition in learning, research has noted that many pre-service teachers have difficulty with mathematics definitions (Kemp & Vidakovic, 2021). According to Ndlovu (2014, p. 195), a major difficulty for pre-service teachers in learning geometry is “defining 2D shapes and their properties”. Kemp and Vidakovic (2021) report that the major difficulty that students enrolled in higher-level mathematics face, is completing tasks on definitions. They lack the ability to provide a statement that conveys the meaning of mathematical objects and figures. According to Kemp and Vidakovic (2021), this difficulty with definitions occurs among students, despite their successful learning of advanced mathematics modules in colleges and universities.

Speer et al. (2015) backed by research evidence, claim that that prospective high school mathematics teachers lack sufficient understanding of the high school mathematics curriculum. For teachers to be able to rely on good content knowledge necessary for all manner of classroom teaching activities, including definitions, it is important that they first acquire a good level of understanding of definitions in mathematics (Chesler, 2012). Definitions are important for assessing learners' conceptual understanding of geometric concepts (Usiskin et al., 2008; Zazkis & Leikin, 2008), and teachers need to demonstrate this skill in teaching mathematics to enhance their teaching competency (Erdogan & Dur, 2014).

2.13 Pre-service teachers' conceptual and procedural knowledge of geometry

Teachers' geometric thinking for teaching requires knowledge of both concepts and procedures. Understanding of concepts and procedures is necessary for mathematical proficiency. For mathematics teachers to be competent and effective in teaching, they must possess a deep and flexible knowledge of the subject matter with understanding of both concepts and procedures in the subject (Alex & Mammen, 2018; Erdogan, 2020; Zuya et al., 2017). This suggests that teachers' proficient geometric thinking should entail an understanding of both the concepts and procedures of the subject matter and how to blend them, to foster effective teaching. Sfard (2008) supports this view when she explains that understanding both concepts and procedures are equally important in the teaching and learning of mathematics, even though conceptual understanding (explorative discourse) is the focus of school mathematics. Rittle-Johnson and Schneider (2015)

consider this type of knowledge to be bidirectional, hence, teachers need to demonstrate adequate knowledge of both for teaching.

Even though one may ask which one comes first, the **NCTM** (2000) claims that procedural knowledge should not be taught, in the absence of conceptual understanding. Also, Sabey (2009) remarks that a deep conceptual understanding of the subject matter can prevent learners from using incorrect procedures. Thus, even though both conceptual and procedural understanding are important, conceptual understanding could be more valuable for teaching so that teaching is not limited to rules and procedures. According to Alex and Mammen (2018), there has been special attention to learning with understanding, because it is the main aim of teaching mathematics and a significant approach in preparing learners for the 21st century.

Conceptual knowledge is an understanding of mathematical concepts that enables one to make connections between them and engage in meaningful learning (Rittle-Johnson & Schneider, 2015). Knowing ‘how’, ‘when’, and ‘why’ mathematical concepts apply in various contexts is associated with conceptual understanding. Learners with conceptual understanding, learn mathematics by connecting the new ideas to what they already possess in their existing schema and also relate these to new situations (Erdogan, 2017; Nahdi & Jatisunda, 2020; Sabey, 2009). Conceptual understanding draws on the ability to make connections and generalise from abstractions, and the ability to produce principles and properties about an object (Rittle-Johnson et al., 2016). Nahdi and Jatisunda (2020) add that conceptual understanding goes beyond knowing information to an understanding and interpretation of information in a meaningful way. It helps represent ideas in many ways and devise multiple solutions to mathematical tasks, which is an attribute of problem-solving abilities (Maulana & Yuniawati, 2018; Nur & Nurvitasari, 2017). Conceptual understanding in mathematics helps learners become more flexible and develops their ability to think creatively (Kivkovich, 2015; Ortiz, 2016).

Procedural knowledge, on the other hand, refers to knowledge of rules and procedures. It deals with how to follow rules and routine procedures to solve mathematical tasks and the skills necessary to perform them, accurately, efficiently and with flexibility. It is characterised by the ability to follow and use step-by-step procedures, also known as algorithms to solve a task (Hurrel,

2021; Rittle-Johnson, 2017). A learner who relies on algorithms can solve a given task but may have little or no understanding of why a certain procedure works in other situations. Learners need to know how to do basic operations quickly and correctly, both in their heads and on paper (Erdogan, 2017; Nahdi & Jatisunda, 2020; Sabey, 2009).

According to Erdogan (2017), incorporating conceptual and procedural knowledge into instruction enables learners to develop and demonstrate high-order mathematical thinking skills. Whilst these two kinds of knowledge are important for teaching, research shows that teachers and pre-service teachers seem to demonstrate more procedural knowledge than conceptual knowledge. For example, Sabey (2009) studied 15 pre-service teachers' understanding of Euclidean geometry. The PSTs were taking secondary mathematics education as their study programme at the university. The study used mixed methods and collected data using both paper-and-pencil tests and interviews. The author selected the top three PSTs who answered all the 15 questions correctly and the bottom three PSTs who correctly answered only eight (about half) of the 15 questions. Two from the high performing group, two from the low group, and one from the middle group were invited for follow-up interviews. Results showed that the PSTs in the high group demonstrated conceptual and procedural knowledge, and the PSTs in the low group showed deficiencies in knowledge of all the strands, including conceptual and procedural. The author claims that there was evidence of most of the PSTs' use of procedural knowledge in their solution strategies. According to Sfard (2008), learners stick to procedures and rules because they consider them the easiest way to solve mathematical tasks. Sfard (2008) asserts that classroom instruction mostly focuses on rules rather than teaching for understanding.

Bryan (2002) studied pre-service mathematics teachers' subject matter knowledge on the procedural-conceptual dimension to find out the depth of their knowledge of mathematical topics they would teach. The study found that the majority of pre-service teachers could not justify or demonstrate any conceptual understanding of the answers they provided to the tasks involved. According to the author, only a fourth of the participants successfully explained the rationale or conceptual basis of the solution. Most of the questions were solved using rules and procedures, which may result from familiarity with those approaches. Researchers (Mann & Enderson, 2017;

Rittle-Johnson & Schneider, 2015) claim that learners follow procedures in solving mathematical tasks due to familiarity and ease of recall.

Literature shows that the use of a set of rules in solving mathematical tasks, is common among learners because they think it leads to the expected results. This affects understanding and limits their ability to apply the knowledge in new situations (Akhter & Akhter, 2018; Al-Mutawah et al., 2019). Sabey (2009) reported Cooney's findings that pre-service teachers had difficulty understanding topics in the high school mathematics curriculum, irrespective of their successful study of advanced mathematics modules at university. According to Sabey (2009), several studies show that most pre-service teachers can solve mathematics tasks by following procedures, but demonstrate weak conceptual understanding of their solutions.

Erdogan (2017) investigated pre-service teachers' conceptual structures about geometry using a qualitative approach. The study that included 60 pre-service teachers, 44 females and 16 males, used the Free Word Association Test to collect data. Content analysis of the test responses showed that the pre-service mathematics teachers demonstrated inadequate or insufficient knowledge of the conceptual structures of the geometric concepts because most of their responses focused on the basic concepts. The author claimed, with support from literature that analysis of learners' knowledge, skills and geometric thinking at all levels of education, shows that they have an insufficient conceptual understanding of geometry. The author adds that knowledge of isolated basic concepts in geometry makes no meaning unless those concepts are associated with other concepts in mathematics, which form a basis for meaning-making and conceptual understanding to develop.

Similarly, Yurniwati and Soleh's (2021) analysis of pre-service teachers' conceptual and procedural geometric knowledge showed that the pre-service teachers tend to demonstrate surface-level or basic knowledge of geometry, in which deep learning is not supported. The authors assert that teachers need to acquire an understanding of a substantial body of geometric content knowledge to scaffold learners' thinking in the discourse. NCTM (2000) supports this by saying that teaching for understanding helps improve learners' learning outcomes and hence the main aim of the mathematics teacher education programme.

On the other hand, Zuya et al. (2017) studied pre-service mathematics teachers' conceptual and procedural knowledge of geometry. The study, with 28 participants, investigated the relationship between the participants' conceptual and procedural knowledge. A test consisting of 15 items on concepts and 15 items on procedures, was used to collect data. The items were open-ended and required the participants to compose their responses. Analysis of the data showed that the pre-service mathematics teachers demonstrated both conceptual and procedural knowledge of geometry. The study also found a significant relationship between the participants' conceptual and procedural knowledge, even though they performed better on conceptual items than on procedural items. This showed that the participants knew how to solve the tasks and also demonstrated an understanding of the solution processes. In other words, they knew more than just the rules for learning geometry. According to the authors, conceptual knowledge draws on the ability to justify and substantiate why one uses a particular approach to solve a task. It is a demonstration of understanding of what works and why. Teachers can be effective in teaching when they have an in-depth understanding of the concepts, principles and relations between content in the mathematics curriculum (Schneider & Stern, 2010).

Marchionda (2006) and Svensson and Molmqvist (2021) claim that, not only in geometry do pre-service teachers demonstrate insufficient conceptual understanding, but also in topics such as fractions, integers, and statistics. In addition, when pre-service teachers are asked to explain their solutions, they often try to identify rules they could use instead of the underlying meaning related to the mathematical tasks (Sabey, 2009). This shows a lack of conceptual knowledge.

Mann and Enderson (2017) investigated learners' approaches to problem-solving based on their preference for conceptual or procedural strategies. The study also focused on the preferred solution strategies among those whose mathematics performance was above average, and those who performed below average. The study used a single-factor, between-subjects experiment on conceptual vs. procedural approaches to instruction. The learners watched short videos that presented a series of instructions to solve problems in two ways; one using a conceptual approach and the other using a procedural approach. A convenient sampling technique was used to select 65 learners who had also taken an introductory mathematics module at the university. Another

criterion that guided the selection of the participants, was based on those who had completed a pre-requisite modules and general education mathematics. Each instructional approach was evaluated by the learners. The results showed that learners' preference for the procedural approach was higher than that of the conceptual approach. The author remarked that procedural and conceptual strategies develop differently among learners and hence recommended that faculty modules should be taught by integrating both procedural and conceptual strategies, to facilitate the development of learners' depth of mathematics understanding. The author also states that using conceptual strategies in teaching will make learners to think critically, learn deeply, and be able to use what they have learned in new situations.

2.14 Fostering classroom geometric discourse

Teachers' mathematical knowledge is pivotal to all classroom activities (Ball et al., 2008). These activities are meant to develop and enhance learning. Research shows that the teachers' abilities to design and deliver quality classroom instruction, is informed by the depth of the content knowledge possessed (Kivkovich, 2015). Similarly, a teacher needs to possess deep and flexible geometric content knowledge to be able to provide better learning support in the classroom. In-depth and flexible geometric knowledge deals with the structures that requires one's ability to understand geometric concepts, facts, definitions, properties, proofs, and theorems (Bassarar, 2012; Erdogan, 2017). Teachers need to have adequate thinking ability about these geometric structures and their relationships, in order to enhance their classroom delivery.

The teacher should select an appropriate teaching technique that engages learners in learning through doing. According to Nissim et al. (2016), teaching techniques are the approaches teachers use to achieve their lesson aims. The technique used by the teacher generally contributes to the quality of teaching and learning. The authors maintain that the more supportive and adaptable a strategy is, the more effective it is in the teaching and learning process. Sfard (2008) maintains that learning takes place in an interactive setting. Thus, participatory teaching methods are suitable for teaching and facilitates the learning of geometric concepts. The use of participatory teaching methods places learners in the central position of the instructional process (Fletcher, 2009). Kivkovich (2015) asserts that better learning takes place among learners if they are active participants in a discourse with knowledgeable people. According to Sfard (2008), mathematical

objects are abstract, which may require a teaching approach that lessens the abstract nature of the concept being taught. This could mean that if abstract concepts are communicated in an abstract setting through only verbal explanation or the traditional method, no meaningful learning can take place since such a learning approach, lays emphasis on memorisation (Kivkovic, 2015).

The use of functional words in communication is significant in learning. According to Ajayi and Lawani (2015), the language of discourse is important for learning effectiveness. Sfard (2008) adds that a discourse is characterised by keywords (terminology and vocabulary) to communicate specific concepts, particularly in geometry, which makes use of mathematical terminology (Atebe & Schafer, 2010). This means that teachers' use of formal and functional words to designate specific geometric concepts, is crucial for learning and understanding. Teachers who have acquired good terminologies (vocabularies) in a discourse, are able to express ideas in definite and concise ways (Ajayi & Lawani, 2015; Oyoo, 2009; Robert, 2010). Hence, teachers' ability to use formal and functional words in their discourse, could make a significant contribution to learners' understanding and application of learned concepts. The use of appropriate words in a discourse enables learners to express their ideas confidently. According to Alex and Mammen (2018) and Robert (2010), when learners understand geometric terminologies, they can describe their ideas in ways that specify the particular spatial concept under discussion. Language forms a critical tool for communication in a discourse (Sfard, 2008), hence, there is a need to use and guide learners' communication with appropriate use of functional keywords.

Learners' geometric thinking can be facilitated through tool-mediated instruction. Sfard (2008) asserts that visual mediators are objects that learners use to form the centre of their communication. The cognitive system is responsive to visual stimuli. Hence, visual mediated instructions have the potential to develop learners' knowledge construction and knowledge retention. The visual mediators can be diagrams, graphs, sketches, etc. The categories of visual mediators are iconic, symbolic, and concrete (Sfard, 2008). These tools have a significant role to play in mathematics and they bring abstract concepts into reality.

Concrete mediators, or manipulatives, are key to learners' understanding in a discourse and enable them to learn in an interactive setting (Horan & Carr, 2018). As learners handle and play with the

concrete materials, they engage in deep thinking and make some discoveries. This comes from the view of Rondina (2019) that hands-on learning contributes to learners' cognitive development and helps them become constructive thinkers. Learning geometry has been noted for its potential to develop learners' reasoning abilities and problem-solving skills (Aksu, 2019). These skills can be developed and enhanced if learners are not just provided with verbal information to assimilate, but are taught through instruction that is designed for concrete or pictorial experience before the abstract (Mudaly and Naidoo, 2015). Tool mediated instruction enables learners to become active learners and to individualise learning (Kontas, 2016; Sfard, 2008). It also gives them confidence to express their ideas and take responsibility for their learning (McDonough, 2016). Research shows that the use of concrete mediators in instruction has the potential to meet the learners' preferred ways of learning (Kablan, 2016). These views suggest that learners' geometric thinking can be greatly fostered if they are taught with concrete mediators or manipulatives. The benefit is that learners' own constructed knowledge is understood, retained, and applied to new learning situations, especially in problem-solving environments (Cope, 2015; Kontas, 2016). They can also learn in groups and share ideas when they use concrete mediators in the classroom (Chan & Idris, 2017).

Bassarear (2012) notes that geometry deals with the study of shapes, their properties and their relationships, which means that knowledge of geometric properties is crucial for developing geometric thinking. Luneta (2015) asserts that learning geometry at high school emphasises the use of properties to guide learning as well as fostering competence for writing mathematical and geometric proofs. Thus, learners' weak knowledge of geometric properties can impede the learning proficiency that is expected of them. All geometric shapes have their own associated properties which need to be taught to learners for them to attain proficient thinking in the discourse. Consider the question in Figure 2.2, which is one of the tasks used in this study. Figure $ABCD$ is a rhombus and equilateral triangle ABX lie on side AB . If angle $BCD = 82^\circ$, calculate angle ADX and angle BDX .

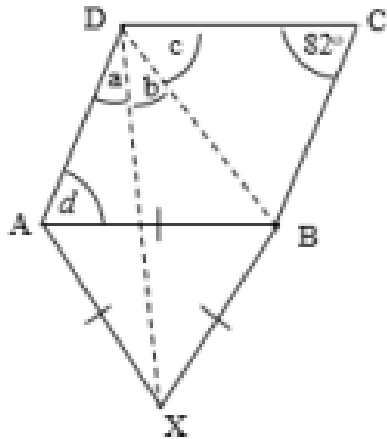


Figure 2.2: Task 3.4 on worksheet

From Figure 2.2, the name of the shape $ABCD$ is given. The rest of the information needed relies on one's ability to interpret the iconic mediators used to design the triangle ABX , to identify the name, think of its properties, and relate these properties to the entire design of the task, before it can be solved. Hence, teaching must focus attention on the properties and all other guiding principles such as definitions and theorems, to enhance learning. Mathematics discourse requires learners to have a proficient understanding of definitions, theorems, and the ability to write proofs (Luneta, 2015; Ndlovu, 2014; Sfard, 2008). Another way to develop learners' geometric thinking is through their spatial abilities, which has been discussed in the literature.

2.15 Conclusion

This chapter began with an overview of some definitions of geometry and how they relate to the current study. It also looked at the place of geometry in the mathematics curriculum and the state of mathematics education in Ghana. Some cognitive processes needed to study geometry were discussed. Further discussion was based on research studies on pre-service teachers' geometric thinking in the various content areas used in plane geometry. The chapter concluded with some concepts on how to foster learners' geometric thinking. Chapter three, will focus on the theoretical framework that underpinned this study.

CHAPTER THREE

THEORETICAL FRAMEWORK

3.1 Introduction

This study analysed pre-service mathematics teachers' geometric thinking and their classroom discourse. The previous chapter presented a review of related literature which served as a basis for situating the study in an academic investigation. This chapter discusses the theoretical framework that served as a lens for conducting this study. It explains the potentiality of the framework for studying geometric thinking as a discourse. The chapter starts by presenting a discussion of the theoretical basis of the commognitive framework and its suitability for this study. It then focuses on the various constructs of the framework as well as a review of previous research that has been conducted using the commognitive framework. The chapter ends with a discussion on how the constructs of the framework will serve as a basis of analysis for this study.

3.2 The commognition theory

Sfard's (2008) commognition theory served as a theoretical lens for this study. The theory emanates from the idea of thinking as a form of communication. Commognition stems from the theory that a person engages in self-communication by thinking, when engaged in an activity. The term commognition is derived from two keywords, viz., cognition (thinking) and communication. According to Sfard (2008), thinking is a form of communication and is individualised in nature. This means that one communicates with oneself whilst thinking. She stresses that the processes of communication and individual cognitive processes (thinking) are different occurrences of the same phenomenon. Thus, in commognitive perspectives, there is no difference between thinking and communication. Thinking is an individualised kind of communication (Nardi, et al., 2014; Sfard, 2007, 2008). Communication, on the other hand, is "a collectively performed rule-driven activity" that directs the activities of people or self (Sfard, 2008, p. 118). The commognition theory is a term coined to show the interrelationship between the processes of thinking and communication. The theory provides a discursive framework for studying and interpreting an individual's activity to gain insights into "intricacies of learning" (Sfard, 2008, p. 566).

The commognition stance of cognitive processes (thinking) and communication is used here as learning and teaching activities. Sfard (2008, p. 570) asserts that learning is an individual development in a “patterned collective activity” often guided by teachers. In the commognitive theory, communication types include expressions in the form of text, spoken, artefacts and use of tangible objects. Communication activities that assemble people together are regarded as discourses. In Sfard’s (2008, p. 93) expression, “different types of communication set apart by their objects, the kinds of mediators used, and the rules followed” that bring participants together in a community of communication, are called discourses. Discourses are therefore tools of communication such as keywords (spoken and textual) and their uses, and visual mediating tools used in communicating concepts to others in regulated activities. This shows a link between discourse and communication in the commognitive theory. Teaching can also be considered as a form of a discourse that follows patterned activities through which concepts are communicated to learners. Mathematics lessons are characterised by collective activities that are organised for learners to participate purposefully in. Teaching that aims at engaging learners in a purposeful discourse place them in an active stance for self-knowledge construction rather than being made to follow teachers’ directives passively. Sfard (2008) maintains that mathematics is a type of special discourse that is characterised by many forms of communication tools which are recognised by members of the mathematics community. Mathematics discourse takes into consideration all forms of communication, namely spoken, textual, and visual tools that govern the study of mathematical objects. Mathematical discourse consists of communication of types of ideas such as concepts, proofs, laws and theorems that are generally restricted to mathematics. This shows that mathematics discourse is characterised by communication tools that are recognised by fellows of a unified community.

In commognition, thinking is linked with learning. Thinking is a personalised form of communication, when associated with school learning, deals with the process of restructuring and extending one’s discourse on an object of study with the assistance of teachers (Ben-Zvi & Sfard, 2007; Sfard, 2008). Communication is the mediating tool between teachers and learners in a classroom discourse, which makes teaching a social interaction through which learners are engaged in purposefully organised activities, in order to construct knowledge. Learning takes place

when learners modify their existing knowledge or develop new knowledge, based on the current activities (Sfard, 2008).

With evidence of learners' low performance in mathematics, which has been extensively documented through research activities and reports from Chief examiners on national examinations, literature falls short of studies examining teachers' knowledge by analysing their thinking, which is the source of all informed actions and inactions, and interactions with learners in a discourse. Since teachers are tasked with the role of interpreting the curriculum to learners, there is an impetus for a study that focuses on what teachers know, by analysing their thinking through their speech and what they do in mathematical discourse, instead of focusing only on their solutions to tasks. Such studies will generate a discussion of what teachers know and what they do in mathematical discourse (Ben-Yehuda et al., 2005). This will provide the opportunity to understand and identify teachers' deficiencies and the needed measures to develop their competency. It is worthy to note that an educational outcome (success or failure) is a product of collective doing (Ben-Yehuda et al, 2005). Teachers engage learners in a discourse to develop their autonomy in learning, based on experiences with mathematical objects. Learners learn with understanding and gain much autonomy when they actively participate in a discourse instead of being passive recipients (Nachlieli & Tabach, 2012; Sfard, 2008). If learners are not performing as expected, then there is a need to examine the possible source of disconnection between teachers' and learners' interaction in a discourse.

3.3 Elements of mathematical discourse

Sfard (2008) identifies the elements of mathematics discourse as word use, visual mediators, routines and narratives. The following sections provide a description of these elements.

3.3.1 Word use

According to Sfard (2008), a discourse is characterised by the kinds of keywords used. It refers to the vocabularies or terminologies of communication. Communication forms a major tool through which teachers convey knowledge and ideas to learners. The kind of words used in a discourse gives it a unique feature. In a mathematics discourse, the words, terms, or vocabularies used must have specific mathematical meaning. For example, the use of words such as differentiate, circumference, perimeter, vertically opposite angles and topology in communication, indicate what

is known as mathematical discourse (Kim et al., 2017; Nardi et al., 2014; Sfard, 2008). There are certain words that are used in everyday discourse with a loosely attached meaning but have an exact or definite meaning in mathematics discourse. Consider the word ‘half’ in ‘I ate half of an orange’, which could mean ‘approximately half’ but has a specific mathematical meaning of “exact” (Berger, 2013, p. 3). Whilst a word used in literate mathematical discourse has an exact and definite meaning, this is not the same in colloquial discourse. Colloquial discourse emerges spontaneously from daily talk and may have different meanings to different people. Sfard (2008) asserts that word use plays a significant role in a discourse because the meaning of the word communicates the exact intent of the user.

In a mathematics classroom, learners make sense of the mathematical objects when the words of the discourse have a common meaning to all members in the mathematical community (Sfard, 2007). Mathematical words used as a signifier to mathematical concepts in an African context, have the same meaning in the European context. For example, the words ‘co-interior angles’ have a common meaning and understanding around the globe and are associated with two-line segments which are crossed by another line segment called a transversal. The meaning or understanding associated with the ‘co-interior angles’, is the same among members of the mathematics community throughout the world. This kind of discourse is what Sfard (2007) referred to as literate discourse. An important aspect of teaching is the kind of words that are used to explain mathematical concepts. Sfard (2008) argues that learners use words to construct mathematical ideas in a discourse. According to her, mathematical objects are abstract and as a result, cannot be assessed by our senses. Word use therefore is an important component of mathematical discourse.

The four categories of words use are; passive, routine, phrase, and object driven (Sfard, 2008). Passive word use is a stage where learners are exposed to words about the object. At this stage, learners cannot make any contribution to the discourse (Gcasamba, 2014). For example, in learning geometry, learners will be able to give the names of shapes according to appearance. At this level, learners can only associate the shapes to their names but they cannot mention their properties. The learner cannot give any reason why the shape is a rectangle. Learners’ responses are often based on what is perceived and not on reasoning.

The routine-driven stage is characterised by learner development of word use in an action-based activity. Development of words among learners is based on routine techniques (Sfard, 2008). At this stage, word use by learners is based on a distinct discursive routine, where the learner matches tasks to new words. Learners at the routine-driven stage are able to add a collection of properties of the object to the shape identified. In geometry discourse, a learner operates at routine driven when the kind of word use is supported by certain visual properties of the object. Thus, a learner's use of words in describing a geometric figure in a discourse, is informed by some geometric properties. In addition, a learner can explain his/her course of action. A learner who is asked to explain why a figure is a 'rectangle' for example, is likely to base his/her reasoning on the fact that 'the opposite sides are equal' and 'each interior angle measures 90° '.

According to Sfard (2008, p. 207), the development of words which are associated with objects of the discourse and form the "basic building blocks" of learners' discourse, is the phase-driven stage. Learners at this stage are able to describe geometric shapes using more malleable words. They can provide an accurate description of geometric figures. The learner can also give a formal description of shapes (Sfard, 2008).

Object-driven word use in mathematics discourse is more oriented to objectification. The development of object-driven word use is through reification and alienation, the two related parts of objectification (Sfard, 2008). Reification is the transformation among learners where they move from talking about processes to talking about objects. In other words, learners' word use is more oriented to objects than to processes. It is an evolution from an operational to a structured sense of reasoning that is more of "object-like entities" (Sfard, 1992, p. 60). Alienation, according to Sfard (2008, p. 295), is the use of "discursive forms to present phenomena in an impersonal way". Word use that is object-driven can be matched with object-level learning, which is the discourse that "expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines and producing new endorsed narratives" (Sfard, 2008, p. 255). Learner development in the two related parts of the objectification in mathematical objects, such as geometry, enables him or her to acquire and develop more flexibility in talk about the objects and greater ability to engage in effective communication in geometric discourse. Learners

communicate mathematically and in a more efficient and concise manner when they are helped to make sense of the objectification process (Ben-Yehuda, et al., 2005).

3.3.2 Visual mediator

Visual mediators are pictorial materials that are used to communicate geometric and mathematical concepts in general. They are visual materials that provide the participants of a discourse, a tool to talk about in constructing geometric ideas. Visual mediators, according to Sfard (2008, p. 147), are imagery materials that “discussants identify [as] the object of their talk” and steer communication among them. These materials personify geometric objects that learners perceive through their visual senses. The uses of visual mediators afford participants of a discourse the opportunity to communicate their thinking about geometry and the various objects being learned. Learners make meaning of geometric objects through their visual senses. Visual senses can therefore be considered as a medium that aid in a natural way of reasoning. Visual senses enable learners to create mental imagery of geometric objects on their minds. Sfard (2008) maintains that visual mediators provide a tool to be operated upon in a discourse. Such tools aid thinking by providing learners with some discursive prompts. Examples of visual mediators are variables (algebraic symbols), graphs, sketches (drawings), diagrams, numbers, formulae, manipulatives, etc. Visual tools play a critical role in geometric discourse. The medium through which learners acquire knowledge about geometric objects is through their visual senses. This makes learners’ visualisation ability an important tool for learning geometry, and mathematics in general. Sfard (2008) categorises three visual mediators which are, iconic, symbolic and concrete. A brief description of these categories follows.

3.3.2.1 Symbolic Mediator

Communication of geometric concepts is enhanced, many times, by using symbols that are realised through human visual senses. These symbols are created artefacts, such as written symbols or numerals, purposely used to provide additional communication (Sfard, 2008). Although geometric objects may seem abstract (Hidayah et al., 2018; Sfard, 2008), they have symbols that are meant to provide visual prompts to facilitate learning. These visual prompts, together with numeric relations, are known as symbolic mediators. Symbolic mediators communicate several important cues in geometric discourse. For example, if two lines AB and CD are parallel, this is often

represented symbolically as $AB \parallel CD$. When the two lines are equal in length, it is often represented as $|AB|=|CD|$ or $AB = CD$. Similarly, a triangular figure can be represented by the symbols Δ . Learners' knowledge of these symbols enables them to develop geometric competence in the discourse.

3.3.2.2 *Iconic mediator*

Iconic mediators, in this study, include visual tools in the form of signs or marks used to design geometric figures, more especially, two-dimensional figures. These visual tools provide participants in geometric discourse with an opportunity to create imagery for mental manipulation and reasoning (Sfard, 2008). The use of iconic mediators communicates certain important features of geometric objects and makes them more meaningful to interpret. Thus, the use of iconic mediators in geometric discourse, helps learners to create mental images through their visual senses to facilitate their reasoning. For example, two parallel lines, which may be difficult to represent concretely, can be presented in the form of diagrams, or sketches, by drawing two-line segments with arrows on them. In this way, the visual tools provide learners with materials to think with, during a geometric discourse (Yurmalia & Herman, 2021). According to Hidayah et al. (2018), the use of visual tools enables learners' thinking abilities to be developed and enhanced.

The iconic mediators can be used together with the symbolic mediators. For example, Figure 3.1 denotes triangle ABC (symbolically represented as ΔABC). Apart from the figure being a triangle, there is additional information indicated by the 'marks' placed on the sides of the triangle. The two marks placed on the sides of the triangle communicate that the length AB is equal to length AC , which is mathematically written as $|AB|=|AC|$ or $AB = AC$. Further information, drawn from these symbols, is that the size of the angles opposite the equal sides are also equal.

Thus, $\angle ABC = \angle ACB$

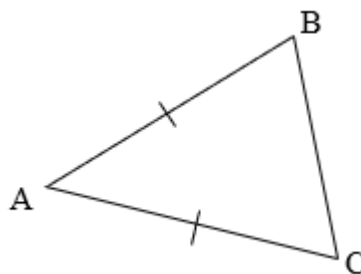


Figure 3.1: A triangle (isosceles) designed with iconic mediators

Consider the diagram in the Figure 3.2.

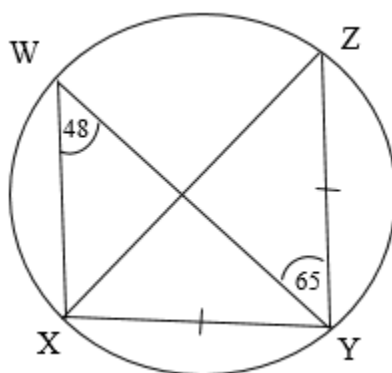


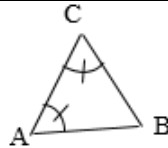
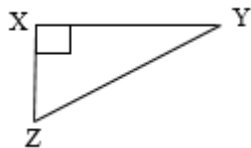
Figure 3.2: A question on circles designed with iconic artefacts.

In the diagram, $|ZY| = |XY|$, $\angle WYZ = 65^\circ$ and $\angle XWY = 48^\circ$, find $\angle WYX$.

These artefacts marked on geometric objects communicate additional information to the reader in order to obtain a full understanding of the geometric tasks. Learner understanding of the symbolic and iconic mediators is a pre-requisite to achieving success in learning geometry and increases the ability to solve related problems. Drawing the learners' attention to such cues enables them to be aware of and to process their meaning in geometric discourse. Understanding of such tasks would be incomplete if these visual prompts were not understood and could lead to difficulties in problem-solving abilities in geometry. Table 3.1 includes examples of mediators of symbolic and iconic representational meanings of some geometric concepts.

Table 3.1: Symbolic and iconic representation of some geometric concepts

Line	Word use	Symbolic mediator	Iconic mediator
1	Line-segments AB and CD are parallel	$AB \parallel CD$	
2	Perpendicular line segments and right angles	$AB \perp BC$	
3	Isosceles triangle ABC with equal sides AB and BC	In $\triangle ABC$, $ AB = BC $ or $AB = BC$	

	Isosceles triangle ABC with equal angles at (vertices) A and C	In $\triangle ABC$, $\angle A = \angle C$ or $\angle CAB = \angle ACB$	
4	Triangle XYZ such that XZ is perpendicular to XY	$\triangle XYZ$, $XZ \perp XY$	

3.3.2.3 Concrete mediator

According to Sfard (2008, p. 112), concrete objects are “tangible, material objects that exist independently of communication”. The use of concrete objects in mathematics helps in the formation of purely abstract concepts. According to Sfard (2008), concrete objects in a discourse facilitates effective communication. She adds that things that are seen create images that “help interlocutors in making discursive decisions” (p. 173). This suggests that a person’s thinking abilities, decision making skills, and communication with self and others, is greatly enhanced when we can see the object of our talk. Concrete mediators are visual tools that can be seen and handled. Examples are rulers, set squares, counters, abacuses, geoboards, and the like (Sfard, 2008). Concrete mediators are interactive materials that are used in geometric discourse. The use of these mediators in learning geometry enables learners to take active participation in learning through a hands-on approach. According to Rondina (2019), learners’ involvement in hands-on activities greatly influences their cognitive development. Research shows that interactive learning through the use of concrete mediators encourages learner involvement, facilitates understanding and recall, and leads to improved performance (Hidayah, et al., 2018; Pullen & Lane, 2016; Rondina, 2019). Sfard (2008) asserts that concrete mediators have the potential to allow learners to produce new, endorsed narratives in a discourse. She adds that when contradictory comments are made, once the object of the communication is seen, our sense of understanding evolves. The key notion of concrete mediators is that they can be seen and manipulated, or moved about, whilst learning (Sfard, 2008). She adds that “communication is mediated by images [...] and this is true even if all these concrete objects are seen and operated upon, only with the interlocutors’ mind’s eye” (p. 174).

3.3.3 Routines

Routines can be described as following a procedure or distinctive discursive pattern in a geometric discourse. They are repetitive patterns (Berger, 2013) which are characterised by a given discourse. Sfard (2008, p. 208) defines routines as “a set of meta-rules that describe a repetitive discursive action”. She distinguishes between the ‘how’ and ‘when’ of a routine. She explains the ‘how’ of a routine as the set of meta-rules that show the course of action. It determines how to deal with a geometric task. It can be described as a step-by-step guide to an action.

The ‘when’ of a routine describes the applicability or its appropriateness. Sfard (2008, p. 208) clarifies that the ‘when’ of a routine is a “collection of meta-rules that determines those situations in which the discussants would deem the performance as appropriate”. Classroom discourse is regulated by rules that guide action. Classroom discourses follow distinctive, patterned ways. These discourses show that mathematical words and visual mediators can be used to produce and substantiate narratives. The narratives produced or created can be new or modifications of an existing one. Through routines, we gain insight about what pre-service teachers do and say, and their patterns of substantiation in a geometric discourse. It unveils distinctive patterned ways pre-service teachers communicate about geometry. The focus on pre-service teachers’ routines reveals valuable information about their creativity, competencies and strategies used in working with geometry.

Routines used by pre-service teachers could provide valuable information about the kind of discourse or pattern they will organise in their classroom for learners. Sfard (2007, 2008) argues that several studies assign value to the processes of a discourse and hence, lay emphasis on process as a basis for developing learner autonomy or independence in learning. Pre-service teachers’ routines of ‘how’ and ‘when’ play an important role in teaching. They are mandated to know both ‘how’ to perform a procedure and ‘when’ that procedure works, for teaching purposes. Their acquisition of such knowledge will enable them to develop learner competencies in geometric discourse instead of limiting their teaching to only the ‘how’ of a routine, that is currently predominant in school mathematics discourse (Gcasamba, 2014).

Sfard (2008) provides categories of routines based on their goals. These are explorations, deeds and rituals. Explorations are routines that aim at creating endorsed narratives of mathematical objects. Deeds cause changes in the environment, and rituals aim at producing the ‘how’ that attracts social reward. In this study of analysing pre-service teachers’ geometric thinking, attention was paid to their rituals and explorative routines in the discourse. Pre-service teachers’ state of thinking about these categories, provides an insight into their competencies and capabilities when they engage with geometric objects. The following section presents a detailed discussion of rituals and exploration routines.

3.3.3.1 *Ritualised and explorative routines*

In this section, review on rituals and exploration of routines in mathematics discourse, is put together, even though each is different from the other. Sfard (2008) asserts that rituals are the distinctive rules that guide an action and are pre-determined by people in authority, such as authors, teachers or lead discussants. Rituals are basic units of a discourse and deal with ‘how’ to get something done. It can be understood as the process of an algorithm which uses a step-by-step approach to perform an activity. In other words, rituals are limited to the ‘how’ of routines. A person working within rituals can demonstrate high knowledge of procedures but may be limited about the ‘when’ or the ‘why’ underling such action. Ritualised routines are prone to imitating the knowledgeable person (teacher) or colleague in a discourse. They are considered as a basis of transformation to explorative discourse (Sfard, 2008).

Explorative discourse is the learner’s ability to explain ‘when’ to use a routine and ‘why’ the routine works. Exploration is the implicit or explicit understanding of geometric objects of study. Learner development of explorative routines enables him/her to apply knowledge in several situations. Learners, in this case, exhibit meta-thinking and are capable of devising multiple approaches to a task and, can justify their routines (Sfard 2008; Essack, 2015). Explorative routines have a great range of applicability to learning mathematical objects. Whilst rituals deal with the rules that determine the course of action which, in most cases, place constraints on the learners’ thinking/reasoning, exploration enables diversity of thinking, leading to a demonstration of high order thinking in learning mathematics.

Both ritualised and explorative routines are central to geometric discourse. Rituals are considered as an initial phase for learners (Sfard, 2008), and are a necessary foundation on which the learner builds upon, to develop the ‘when’ and ‘why’ criteria for transition into an explorative routine. Mathematics discourse requires that learners be equipped with both rituals and exploration of routines, but with much emphasis on the latter (Sfard, 2008). Learners need to develop fundamental concepts and accurate procedures of routines in mathematical discourse but these must serve as a basis to develop explorative competencies. Rituals are structured and often require some form of acceptance or approval from the lead discussant. Such discourse, due to its limitation on learners’ diversity in reasoning, often affect their range of applicability (Essack, 2015).

Learners develop autonomy in learning when there is phenomenal growth in moving from ritual to explorative routines. Learners who have developed an exploratory routine in a discourse have the capacity to produce endorsed narratives about learned geometric objects. They can use knowledge gained to determine the ‘when’ and ‘why’ of routines concerning the ‘how’, to create a new endorsed narrative which satisfies the aim of school mathematics. Learners’ ability to demonstrate such endorsed narratives is said to result from a wide range of connected routines. This enables learners to develop a high sense of flexibility about geometric objects and the ability to substantiate them (Sfard, 2008).

3.3.4 Narratives

Narratives, according to Sfard (2008, p. 134), are “any sequence of utterances framed as a description of objects, that is subject to endorsement or rejection with the help of discourse-specific substantiation procedures”. These are mathematical statements, either spoken (verbal) or written (text), and can be considered by the mathematical community as being true or false. The truth or falsity of a narrative is evaluated through mathematical procedures and reasoning. In commognition, a narrative is endorsed when it is true, and rejected when its truth cannot be established. An example of endorsed narratives are theorems, definitions, axioms, properties, theories and the like. Narratives include a sequence of utterances that have been proven or accepted for use in mathematical discourse. The narrative ‘the sum of the interior angles of a triangle is 180° ’ is endorsed to describe the interior angles of a geometric shape bounded by three straight lines.

Narratives are produced using words, mediators and routines. According to Berger (2013), there is a difference between the rules of endorsement in school mathematics and that of formal mathematics. The author asserts that ‘the sum of angles in a triangle is always 180° , holds only for school mathematics discourse which is backed by the axioms of Euclidean geometry. However, the statement is not necessarily true in a more formal mathematics discourse, such as hyperbolic geometry and spherical geometry.

3.4 Mathematics as a discourse

As mentioned earlier, a discourse is a form of communication that engages people in a specialised activity. A discourse is mathematical if the object of the talk, the processes and the tools of communication focus on mathematical objects. Mathematical objects, according to Sfard (2008), are abstract, discursive constructs that have specialised mathematical signifiers and form part of a discourse. The commognitive theory explains that objects of mathematical discourse include signifiers, discursive objects, primary objects realisations. A signifier is a primary object with its associated procedures. Commognitive capacities develop as a result of the human ability to acquire higher discursive levels. The distinct components of commognitive capacities are commognitive objects that involve abstracting, reasoning and objectifying. The other commognitive subjects deal with subjectification and consciousness (Sfard, 2008).

Growth in mathematical discourse is a result of learning. According to Sfard (2008), learning is a communication with self and others in a social interaction to produce modification and change in what one knows. Ben-Zvi and Sfard (2007, p. 81) maintain that “school learning is a process of modifying and extending one’s discourse”. From a constructivist perspective, learners construct their own knowledge by active participation in a discourse. Learner participation is key to enabling him/her to individualise learning. Active participation in learning is a process of developing autonomy in mathematics discourse. When learning has been effective, learners demonstrate discursive growth in mathematics discourse and can approach mathematical objects and tasks on their own with little or no support from the knowledgeable other (peer, teacher or lead discussant). Learner development as a result of change in discourse is evident when the learner has developed the intellectual capacity to modify his/her mathematical thinking through individualisation.

According to Sfard (2008), development is characterised by the learner's ability to "use the same mathematical signifiers in different ways to perform the same mathematical task", using different procedures (p. 161).

Sfard (2000) describes two types of learning in commognitive theory which are object-level learning and meta-level learning. She describes object-level learning as an "expansion of exploratory discourse attained through extending a vocabulary in constructing new routines and producing new endorsed narratives" (p.253). In other words, it is the learners' ability to produce or generate new narratives in a discourse that is based on previously endorsed ones. In object-level learning, the discourse of learning expansion is based on objects. Meta-level learning involves an explanation about object-level rules. It involves changes in rules of discourse. The changes deal with moving from familiar tasks, such as one's ability to define a word or identify geometric figures, to doing it differently in an unfamiliar way, using words with different uses/contexts (Sfard, 2008). Learner development of meta-level learning is when he/she can explain other participants' actions but not about the object itself (Essack, 2015). An example of learning at the meta-level is moving from properties of a rectangle to that of a square. Knowledge of new concepts may be conflicting with the properties of a rectangle that the learner knows already, for example, taking a decision on what properties of a rectangle apply to the properties of a square. Learners need to know the distinguishing features of properties of various geometric figures and individualise them in learning. There are three classifications of meta-rules which are; procedural application of a routine, results of a procedure, and applicability of a routine (Sfard, 2008). When learners have developed these three classifications of meta-rules, they are able to show some signs of creativity and higher cognitive abilities in a discourse. Their routines show an appropriate application of 'when' and 'why' of routines which probably result from extending the discourse beyond the 'how' of the routine. A learner expands his/her existing discursive layers as a result of changes of 'when' and 'why', an element of developing competence in mathematical discourse.

3.5 Studies informed by the commognitive framework

Despite the recency of Sfard's (2008) commognitive theory, it is gradually gaining much attention for investigating several aspects of the teaching and learning of mathematics. Perhaps it has been

developed to solve the ‘quandaries’ of mathematics discourse. As Presmeg (2016, p. 423) puts it, the “teaching and learning of mathematics that has posed a lot of quandaries to many, has the timely proponent” of Sfard’s (2008) theory. The theory shows ‘dualistic aspects of mathematics education and its research underlying the quandaries. Presmeg (2016) adds that, the commognitive framework has emerged as a theoretical view that provides insight, not only into the teaching and learning of mathematics, but also into the entire fabric of human development and what it means to be human. Thus, Presmeg (2016) sees commognition as a lens for research which has the potential for unveiling human development. It also serves as a tool to investigate the processes of learning to become a mathematics educator, and the teaching activities in a discourse. Hence, the theory provides a framework to study mathematics discourse at all levels of learning, such as elementary (Ben-Yehuda et al., 2005; Gcasamba, 2014, Roberts, 2016, Essack, 2015), and pre-service teachers’ development as they undergo training (Nachlieli & Katz, 2017; Tuset, 2018; Wang, 2013). There is also research that used the theory to investigate teaching and learning of mathematics at a tertiary level (Tasara, 2017; Tabach & Nachlieli, 2011). Although these studies have investigated various areas in mathematics, such as arithmetic, algebra and function, calculus and geometry, etc, focus is first laid on a review of studies conducted specifically in geometry, since they are of direct relevance to this study. Further attention is paid to other mathematical objects in which the theory has been applied in learning, other than geometry, such as algebra and arithmetic. To gain further insights that will help to decide what needs to be done to enhance and refine this study, a review of the existing research literature in mathematics education is conducted.

Wang (2013) conducted a study that focused on understanding prospective teachers’ levels of geometric thought through discursive analysis. The study used Sfard’s (2008) discursive framework to investigate the geometric thinking of prospective teachers who had enrolled in a measurement and geometry course, in their undergraduate programme. Data was collected using a pencil and paper test and interviews. A week after they participated in a pre and post-test using van Hiele’s geometry test, they engaged in another pre and post interview for 90 minutes. According to the author, the purpose of the interview was to gain insight into how the participants’ geometric thought aligned with the test taken. The author took video recording of the interviews and transcribed them for discursive analysis. Written responses of two participants on the test were used to gain initial information about their geometric thoughts. Further insights were gained

through the interview, from what they said and did as the interview unfolded. The study found that one of the participants showed tremendous improvement in geometric discourse in the use of words and routines. Sam (pseudo name) was able to use definitions to classify polygons, which showed a higher level of competency. Sam could engage in informed deductive reasoning, and as a result, could reason at level 3 of van Hiele's geometric thought. Sam's competencies developed as she could identify polygons using definitions, with the ability to substantiate her routines at a meta-level. According to the author, there was a substantial improvement by Lulu, the other participant, who attained level 4 of van Hiele's geometric thinking, in the post-test. Lulu showed application of knowledge of quadrilaterals to do mathematical proofs. Lulu's growth in geometric discourse was found to be a continuous progression as she had acquired the needed language, using axioms, definitions and symbols to do mathematical proof. There was also evidence of the participants' use of both literate and colloquial words in the discourses. The author concluded by confirming the usefulness of the framework in such a study. The diversity of the units of analyses enabled the author to attest to the potential utility of the framework in gaining greater insight into the pre-service teachers' thinking. Through the discursive framework, greater insight into the pre-service teachers' thinking at the various van Hiele's levels was gained, as revealed by the author. As per the constructs of the framework, what the participants said about word use and narratives of the object, parallelograms and their properties, were analysed in detail, in addition to what they did. Knowing the kind of words pre-service teachers used in a discourse, and their state of appropriateness, has significant implications on effectiveness of communication in teaching. Word use is central to teachers' communication activities in a discourse. Teachers' word use in communication can impede learners' understanding if words used do not signify the exact, intended meaning (Gharbavi & Iravani, 2014), or are used inconsistently. The quality of teachers' discourse largely depends on the suitability of words that best explain the concepts under consideration. Teachers, therefore, need to choose words that are aligned with literate mathematical discourse and can be substantiated due to their use in the community of scholars in that discourse. If pre-service teachers have not developed the correct use of words in a discourse, they are less likely to organise and provide learners with the most effective opportunities for learning.

In another study, Berger (2013) investigated in-service teachers' mathematical discourse. The participants were two practicing teachers who participated in a programme to enhance their knowledge for teaching mathematics. According to the author, the commognition theory which has its basis in Vygotsky and Wittgentien's view of mathematics, resonated with his experiences of teaching mathematics (Berger, 2013). He was further informed of its use due to the extensive analytic constructs of the theory for examining and interpreting activities in mathematics discourse. The author's desire was to use the constructs to examine how 'word use' and 'change in discourse' affect mathematical activities. The mathematical object of focus in the study, was function. The study showed that learners' word use was very important as it served as a basis for change in discourse, which manifested in one of the participants' understanding. The study also showed how the other participant approached the task using a ritualised routine. In all, the author attested to the usefulness of the framework in enabling him to gain an understanding of certain phenomena on function.

The commognitive theory also serves as a framework for analysing the quality of teaching opportunities teachers make available, for learning mathematics. Tuset (2018) used the framework to investigate pre-service teachers' mathematics discourse and as a tool for developing ambitious teaching. According to the author, ambitious teaching is a kind of teaching that enables learners to demonstrate an authority in communicating mathematical ideas thereby developing their explorative sense for learning mathematics. This kind of teaching required learners to actively participate in a discourse. The study reported the analysis of only one participant's teaching. The teaching was analysed as either being ritualised or explorative. The author reported high ability of the participant to demonstrate a good level of explorative instruction. The participant's geometric discourse was characterised by asking learners to provide an explanation to substantiate their narratives about geometric figures. The learners were also prompted for further explanations by asking the 'why' questions, to extend their reasoning about the object being learned. According to the author, the participant's teaching further revealed opportunities for the learner's ability to explore, which supported their transition to explorative discourse. The author concluded that the framework enabled him to gain insights into teaching opportunities that pre-service teachers provided for learners' participation in a discourse.

It is worth mentioning that the potentiality of the commognitive framework to investigate mathematical learning based on its constructs, also enables teaching and its quality to be explored. The constructs of the theory, enables pre-service teachers to develop good teaching skills that facilitate learning. Pre-service teachers' awareness and use of the construct develops their communicative competence and builds their capacity to teach to expectation. In Tuset's (2018) study, the pre-service teachers were required to justify their discourses in a lesson. Asking learners to substantiate their narratives gives them the opportunity to extend their reasoning about the learned object. This provides learners with the opportunity to claim responsibility for their own learning instead of being trained to see mathematics as following rules determined by teachers or knowledgeable others. Learners are mostly not motivated to learn mathematics when they are made to excessively follow strict rules of discourse, and they think of the subject as a mere recall of rules of the game, which never considers how one reasons to construct ones' own knowledge. According to Sfard (2017), the memorise-symbolic-manipulation in learning is what she termed ritualised participation in a discourse which often results from ritualised instruction. This kind of instruction, following rules, enables the learner to please the teacher but not to develop his/her creative thinking in learning mathematics (Heyd-Metzuyanim & Graven, 2016). Thus, the framework serves a multi-purpose function as it provides a clear direction to investigate learner and teacher thinking, as well as to examine the teaching potential of pre-service and in-service teachers.

Pre-service teachers' ability to engage learners in mathematics discourse depends on what they themselves have experienced. The generalisation is that teachers teach the way they were taught. Hence, how a pre-service teacher has experienced and been engaged in mathematics discourse, predicts the kind of opportunities he/she would organise for classroom practice. With the quest to determine the learning opportunities made available to pre-services teachers during their training programme, Nachlieli and Katz (2017) conducted a study to investigate the learning processes of pre-service teachers in the continuum of rituals to exploration. The authors designed a course that aimed to enhance mathematical thinking of these students by providing an opportunity for explorative participation in the discourse. The participants in the study were 18 pre-service teachers who had enrolled in a course (module) that sought to develop their mathematical reasoning abilities. For the purpose of the study, the discourse was designed to give the pre-service

teachers the opportunity to engage in cognitive demand tasks that have multiple solution paths that required them to draw connections between their solutions and mathematical ideas. Data for the study included participants' written examinations results and lesson plans for promoting their students' mathematical thinking. Data was analysed in terms of ritual to explorative continuum. Results showed that participants focused on explorative participation in their reasoning about the tasks that required them to suggest multiple solution paths. This prompt enabled participants' actions to be guided by some explorative questions, to enhance their transition. The author however indicated that their first reactions to the tasks were more ritual in nature. The participants made an effort to recall previously learned formulas and procedures in solving the tasks until they were given further suggestions.

Although teaching mathematics aims to develop learner exploratory participation in the discourse to enhance conceptual understanding, ritual participation seems to be inevitable in learning. Learners thought processes are developed from ritual to explorative by engaging in appropriate explorative teaching strategies. Such approaches enable learners to extend their reasoning to answer questions by producing new endorsed narratives (Sfard & Lavie, 2005). According to Nachlieli and Katz (2017, p. 2), "exploration is an act of production" of new endorsed narratives from existing ones or from the current situation. It is worthy to note that pre-service teachers, even at the college of education, initially expressed rigidity in their thinking by solving the task in a ritualised manner until they were prompted to exercise further reasoning to produce new and endorsed narratives about the task. This challenges pre-service teachers to also develop and exercise the skill of guiding and facilitating their learners' sound knowledge construction that goes beyond following rules determined by lead discussants.

Extending the use of the framework in investigating teaching of calculus, Tasara (2017) conducted a study to analyse teachers' mathematics discourse on derivatives. The study was concerned with how teachers teach derivatives component of calculus at the secondary school level. The study was based on suggestions from research that calculus has remained difficult for many learners to grasp. Of the nine mathematics teachers who participated in the study, the author reported on data obtained from one participant. Data was generated using interviews and lesson observations, which were recorded, using audio and video respectively. Two interviews were conducted, one before

and one after the observation of the teacher's classroom lesson. After transcribing the audio data from the interview and video data from the classroom observation, an analysis was made using the commognitive constructs to determine how the teacher used words, together with the supporting narratives, in a discourse, with critical consideration to the specialised language or vocabularies related to derivatives. Other attention was focused on the use of visual mediators and routines in the discourse. The author found that there were inconsistencies of word use in the discourse. The author noted that the words used in the calculus discourse were not consistent with those of literate mathematics discourse. Thus, both literate and colloquial words were used in the discourse. The words "the gradient of a curve", were used when the teacher actually meant "gradient of a curve at a point" (p. 4). According to the author, "inconsistent use of words can hinder learning and understanding" (p. 4). The author draws teachers' attention to word use in mathematics discourse. As the author rightly cautioned, word use forms the central point of a mathematics discourse. It is invariably agreed that teachers clarify almost everything in a discourse, and their word use serves as a medium through which concepts are communicated and clarified to learners. Word use in a discourse should therefore signify the specific meaning that the teacher intends to convey, particularly in mathematics, where word use provides precise and concise communication in a discourse. Learner word use in a discourse is supposed to be shaped by teachers or knowledgeable others. The use of the right mathematical words is crucial to learning success in the discourse. Teachers need to choose appropriate words related to specific mathematical objects and to be consistent when using them in their communication with learners. According to Zayyadi et al. (2020), learners' ability to learn with understanding results from teachers' communication ability, since the words used to explain ideas play a significant role in learning. Learners get confused when teachers are not consistent in their word use, in communicating ideas and concepts, resulting in learners' learning difficulty. As rightly noted by Tasara (2017), teachers' inconsistency in use of words in a discourse can create difficulty in learning and hinder understanding, in a discourse.

3.6 The commognitive framework used in this study

All the four commognitive constructs were used as units of analysis in this study. The first construct, 'word use' focused on the kind of words pre-service teachers identify and process as part of the cognitive process of learning geometry. Success in learning stems from making meaning of words used in communication, either in text or verbal. Geometric words are highly

specialised or have unique meaning in the mathematics community. It is for this reason that mathematics, and its various objects, are considered a universal subject. Learner understanding of geometric words is supposed to be concise and precise. Imprecise meaning of geometric words causes difficulty in learning the subject matter and related objects in the mathematics curriculum. When learners develop imprecise, or vague meanings of geometric words, it is an indication that they have developed and brought words commonly used in everyday communication, into learning geometry. Learners may make errors in mathematical communication when they use non-specialised words in geometric discourse (Gcasamba, 2014). The situation worsens when people who are trained to teach others, do not themselves have a precise understanding or meaning of the words of geometric discourse.

Pre-service teachers need to possess a well-developed visual sense before they can meaningfully guide their learners in geometric discourse. In this respect, one may ask, about the visual ability of pre-service teachers in identifying the ‘objects of their talk’ to enhance the communication about these objects? Visual sense is a critical tool for learning geometry and mathematics in general. These elements of the framework guided the study to analyse how the pre-service teachers are able to interpret visual objects that they used in geometric discourse. The pre-service teachers’ visual sense plays a crucial role in their spatial reasoning and enables them to create ideas and manipulate them mentally. Since pre-service teachers are supposed to facilitate the knowledge construction of learners, they need to understand geometry and develop its essential learning tools, such as visualisation and spatial reasoning, in order to teach it well. Teacher competence in visual senses enables them to use visual tools such as diagrams, drawings, pictures and physical materials in a discourse.

Participants’ ability to teach geometry effectively, largely depends on ritualised and explorative cognitive processes as part of their routines of geometric discourse. Pre-service teachers must develop flexible and fluent reasoning beyond ritual, to enhance their competency and confidence in teaching geometry. Such teaching develops the learners’ potential and ability to produce endorsed narratives in mathematics and in geometric discourses in particular. Learner ability to produce new endorsed narratives is a criterion for accepting learners in the mathematics community. This ability emerges from the know-how (knowledge) of teachers themselves. A

teacher with such knowledge is able to guide learners to produce utterances, oral or textual, to describe their thinking about mathematics objects and activities (Sfard, 2008).

3.7 Conclusion

The chapter began by describing Sfard's (2008) commognitive framework and its constructs, taking into consideration how it is related to the current study. This was followed by a review of previous research conducted through the lens of the commognitive theory. The commognitive theory provides an exhaustive path of analysing the pre-service teachers' geometric thinking and their classroom discourse, with the support of the qualitative approach. Considering how the constructs relate to this study, the importance of the words used in a discourse are discussed. Learning the keywords (terminology or vocabulary) of a discourse, forms its basic building blocks. Hence, difficulties with keywords or geometric terminologies may pose learning challenges and can affect learning outcomes in the discourse. Attention to the keywords was in light of whether they were used mathematically or colloquially.

Next was the discussion on visual mediators used in communicating geometric concepts. Understanding how pre-service teachers' pay attention to how certain visual mediators are used in geometric discourse, was deemed to be of great importance. A further discussion on routines of the framework was dealt with. How the participants solved geometric tasks paved the way for solution strategies to be classified as ritual or explorative. The routines showed whether the participants were prone to using a set of meta-rules in solving geometric tasks or whether they were inclined to producing objective properties of the task, based on which tasks were solved.

Finally, a discussion of narratives and how they could help understand the way participants substantiate or justify their discursive moves in the discourse, was presented. The kind of narrative produced was of great concern, whether they are endorsed in the mathematics community of learning or not. Almost all discursive moves in geometric discourse are supported by an appropriate narrative. The chapter concludes and paves way for the next one, which discusses the research design and the methodology of the study.

CHAPTER FOUR

RESEARCH DESIGN AND METHODOLOGY

4.1 Introduction

The commognitive lens, the theory that underpins this study, was discussed in the previous chapter. This chapter describes how the research process was carried out. The primary aim was to analyse pre-service mathematics teachers' geometric thinking and their classroom geometric discourse. The study was divided into two parts. The first part conducted a detailed analysis of the participants' geometric thinking using constructs of Sfard's (2008) commognitive framework. The second part looked at how pre-service teachers talked about geometry in the classroom and how that related to their thought processes about geometry. The goal was to figure out what kinds of learning opportunities they offered, as a way to improve learner geometry thinking.

The chapter also describes the philosophical view underlying the study which offers an approach through which the study was conducted. It further describes the research methodology, the study participants and criteria for their selection, the data generating instruments, and the measures taken to refine these instruments. It then describes the stages involved in generating data and the approach used to analyse the data.

Describing a study approach and its underpin is built on the fact that all research follows a philosophical view or paradigm, which is "a basic set of beliefs that guides an action" (Guba, as cited in Creswell, 2014, p.35). Conducting a study from a philosophical viewpoint and following its informed methodology, is a necessary approach that contributes to better research practices and has the potential to produce more valid results and findings (Ritchie et al., 2014). This section of the research work is grounded by the critical research questions that the study sought to answer, and the lens through which knowledge was produced. An account of this section follows.

4.2 The critical research questions

This study analysed pre-service mathematics teachers' geometric thinking and classroom discourse. It determined pre-service teachers' thinking about keywords they identify and process in learning and understanding of geometry, the meaning and interpretation they attach to visual

mediators and their visualisation ability, the routines they use to solve problems in geometry, and how they justify and substantiate, or make mathematical arguments, when working with geometry. In addition, effort was made to determine their routines on the continuum of ritual to explorative ways of thinking. The next step assessed the quality of interaction regarding the participants' engagement with learners in communicating geometric concepts, based on the constructs of the framework. These foci guided the formulation of the following critical questions in this study to guide the study:

1. What is the nature of pre-service mathematics teachers' discursive thinking in geometry?
2. What is the nature of pre-service mathematics teachers' routine thinking in geometry?
3. How does pre-service mathematics teachers' geometric thinking influence their classroom discourse?

4.3 The interpretive research paradigm

The interpretive paradigm seeks to understand a phenomenon through subjective human experiences (Mohamed, 2017). In this paradigm, researchers focus on understanding how individuals interpret what they have experienced in their world of living. The interpretive paradigm, according to Cohen et al. (2011), is driven by concern for people, and is used to comprehend the lived experience of people. The belief of the interpretive researchers is that what is considered to be real, is based on people's subjective experiences. Individuals then become the central focus for interpretive researchers, who seek to gain an understanding of how they interpret the environment around them (Creswell, 2014). The interpretivist, seeking to understand the interpretation humans make of their experiences, often relies on a written text, verbal (spoken words) and nonverbal forms of communication.

This approach was found to be appropriate for this study, since it dealt with written, verbal and facial cues as forms of communication. In addition, the interpretive paradigm uses approaches that rely on people's participation in a natural life setting. Considering the focus that sought to analyse pre-service mathematics teachers' thinking within the context of their experiences with geometry, the interpretive paradigm was considered the best fit within which human thinking could be analysed, interpreted and described in order to enhance its validity. Pre-service mathematics teachers' geometric thinking was a result of how each individual had experienced and interpreted

geometry in the mathematics curriculum. Part of the data was generated from the pre-service teachers' written text and verbal explanations of their solutions to the geometry tasks. The focus on the written text and the verbal explanations, was to gain insight and an in-depth understanding of their views, and to interpret the thinking of the participants, based on the analysis of the data generated. Human thinking can be analysed by creating contact with their minds through conversation and can afford researchers the opportunity to understand their subjective experiences. Interpretive researchers believe that language and shared meaning are the only ways to understand reality, whether it is given or made by people (Creswell, 2007; Lapan et al., 2012).

Another tool, used by the interpretivist to collect data, is observation, that enables them to observe (watch) and collect data about events (Ary et al., 2010). Other data was generated through observation within the classroom because the classroom is the most natural setting to witness how the pre-service teachers interact with colleagues in a discourse. According to Denzin (2010), natural environments are used to examine and explain the experiences that people have created. When using the interpretive paradigm, the researcher gains experiences as they emerge, and gathers multiple accounts when grouping stories. Examining the pre-service teachers' teaching behaviour, actions and thoughts on geometric discourse would provide an insight, such that a high degree of interpretation of the participants' teaching could be analysed and described.

According to Cohen et al. (2011), the interpretive research paradigm allows for the study of individual attributes, such as opinions, behaviour, and attitudes. The interpretive paradigm helps to obtain data by examining the world in multiple ways. Cohen et al. (2011) assert that in interpretive research, theory should be based on the evidence of multiple data and be created directly from experience. Multiple data collection is therefore the hallmark of interpretive researchers. The interpretive paradigm is thus characterised by relying on naturalistic techniques (text analysis, interviews and observations). These techniques ensure a comprehensive and adequate interaction between the researcher and those being interacted with, to construct a collaborative and meaningful reality. It is further characterised by the tendency for meaning making to emerge naturally from the research process and from qualitative methods (Neuman, 2014). The use of the paradigm provided an in-depth understanding and insight into meanings and actions of the pre-service teachers when engaged with geometry.

4.3.1 Situating the study in the interpretivist's viewpoint

The aim of the interpretive paradigm is to understand the social environment in which one lives and to discover the meaning people create in their natural life setting (Creswell, 2014). The use of the interpretive paradigm helped to interact with the pre-service mathematics teachers as they interpreted their geometric experiences (Tracy, 2013). The interpretivist shares the view that there are varied meanings to experiences that one encounters. People give subjective interpretations of their experiences which can be known by relying on the views and experiences of the participants in the situation researched (Creswell, 2014). Getting to know the meaning of situations is through discussions and interactions with people, to make sense of the meanings they have constructed about the places where they live and work. The interpretive paradigm is applicable in this study, because it helped to analyse each participant's geometric thinking based on how they had experienced it in their learning pursuits.

Reality, in the interpretive paradigm, is the subjective meaning people attach to their experiences that results from the way objects or things have been perceived (Astin & Long, 2014). The interpretivist gains insight into the reasons why individuals behave the way they do through their social interaction. People create subjective meanings that result from their experiences with the objects they encounter. Such experiences result in varied and multiple meanings that are of interest to researchers who also seek to understand, by relying on participants' views (Denzin 2010; Creswell, 2007). The interpretive paradigm enables researchers to collect data in a context that will promote one's understanding of an individual's worldview (Creswell, 2014; Tracy, 2013). It lays emphasis on "subjective understanding or interpretation of human action" (Babbie et al., 2010, p.30). This study analysed the thinking underlying a particular action. It was therefore crucial that pre-service teachers were first given some tasks to complete, then to engage them in a dialogue and probe further to better understand individual's thought processes, and the meaning they construct in learning geometry. People do things in different ways which are best understood if they are given the opportunity to explain why. Interacting with individuals offers the opportunity to understand the reasons underlying their experiences with geometry.

As asserted by Creswell (2007), the subjective meanings of participants' behaviour are best negotiated through social interaction with the participants. Thus, the interpretivist relies heavily on social interactions to explore the subjective meanings that individuals create, based on their experiences within situations. According to Creswell (2014), interpretive researchers believe that knowledge is an act of social construction, and reality is only assessed through the meanings that are assigned by people.

In Sfard's (2008) commognitive framework, thinking is a form of communication that may include verbal, non-verbal and textual communication. Thinking is a phenomenon that is personalised in nature and therefore is considered as subjective, which can be understood only in ways that have been experienced by the individual participants. In this study, the interpretive paradigm was found to be more suitable because of the belief that participants' thinking cannot be reduced to an objective search, but rather through social engagement to access their views about their geometric experiences through written texts, verbal, and facial cues. Such engagement with participants normally takes the form of an interview or participant observation, which allows an in-depth study of the phenomenon being investigated. The framework, coupled with the interpretive paradigm, offered an exhaustive approach to study the participants' thinking through the subjective meanings they had created, resulting from their experiences with geometry.

The interpretive paradigm enables researchers to study several features of an individual, some of which are opinions, attitudes and behaviour (Cohen et al., 2011). In this study, the paradigm offered many opportunities to gain insight into an understanding of the geometric thinking of the participants from different perspectives through the multiple ways of data collection. Data was generated using a test to obtain textual solutions to geometric tasks, followed by engaging the participants in communication to understand their thought processes, underlying how they solved the tasks. Their geometric discourse in the classroom was then observed to find out how they interacted with their peers as a learning process of communicating knowledge in the teaching and learning situation. The choice of this philosophical viewpoint was guided by the three basic informed decisions on ontology, epistemology and methodology which provide information about reality, what knowledge is, and the appropriate ways to gain knowledge (Neuman, 2014).

4.3.2 Ontological assumptions in the interpretive paradigm

Ontology deals with the beliefs one holds concerning the nature of reality. It considers the question, ‘what is the nature of reality’ and what is to be known regarding its features or characteristics (Lincoln & Guba, as cited in Mohamed, 2017). It relies on the tenet of social interaction and the contributions one makes to social phenomena. The research paradigm was used to analyse, understand and describe the PSTs geometric thinking based on how they have experienced it. In addition, the interpretive paradigm aspires to understand the relevance of social events which pave the way for a profound understanding of human behaviour and actions, and also analysed how such thinking exhibited by the up-coming teachers, informed their teaching behaviour within the classroom context. Neuman (2014) claims that ontology focuses on what exists or the quest to know about the nature of reality. To make an assumption underlying the phenomenon being studied based on what was considered to be real and in existence, the two fundamental positions of ontology, namely the realist and the nominalist, were considered (Neuman, 2014). Realists view the world as something out there, that is pre-existing and needs to be discovered and that “exists independent of humans and their perception” (Neuman, 2014, p. 94). The belief is that inquiry into reality gets contaminated when it is searched for using our existing ideas containing subjective and cultural interpretations.

On the other hand, the nominalist takes the view that reality is not directly experienced or discovered. Humans come to know the world based on their experiences which occur through subjective means and interpretations. What is believed to be real, and hence exists, depends on what has been experienced. According to Neuman (2014), people’s experiences are organised into categories based on their personal and cultural worldview, which sometimes happens without their realisation. For the nominalist, no matter how subjective human interpretations may be, they cannot be removed from their sense of reality. From the above discussion, the realist-nominalist ontological assumptions can be placed on a continuum. The nominalist view of reality, holds that people’s understanding of the world is dependent on interpretative-cultural factors and not on objective knowledge as far as human experiences are concerned.

Considering the focus of this study, which is concerned with PSTs’ thinking, consideration was given to the nominalist’s ontological perspective in which that thinking is relative, especially in

mathematics tasks, where the same solution can be obtained with different thinking. This study relies on the belief that learners engage in personalised thinking in solving problems in geometry. Learner thinking evolves from his or her experiences, which makes thinking personalised and subjective in nature (Sfard, 2008). The PSTs' thinking can then be studied and interpreted based on what they have experienced, hence it is that they are given the opportunity to explain their thinking. Thus, the phenomenon analysed in this study was believed to be a reality that is not objective, but formed by human subjective experiences. This phenomenon therefore, is said to exist and can be studied through social interaction.

4.3.3 Epistemological assumptions in interpretive paradigm

Epistemology focuses on what knowledge is, and how it can be known, with major emphasis on the relationship between the inquirer and the object of inquiry. Its major assumption is how knowledge is acquired and communicated in ways that are acceptable (Rehman & Alharthi, 2016). The authors add that epistemologists are more interested in the kind of relationship that exists between the knower and what is to be known. In other words, it is how researchers acquire knowledge about what is being studied, that is relevant. It emphasises the kind of relationship that exists between the researcher and the participant from whom data is obtained.

To the nominalist,

the best knowledge about the world that we can produce is to offer carefully considered interpretations of specific people in specific settings. We can offer interpretation of what we think other people are doing and what we believe to be their reasons in a specific setting. To produce social science knowledge, we must inductively observe, interpret, and reflect on what other people are saying and doing in specific social contexts whilst we simultaneously reflect on our own experiences and interpretations (Neuman, 2014, p. 95).

Guided by the question of how the researcher acquires knowledge about the phenomenon being studied, a close relationship was established with the PSTs, which paved the way to observe, study, interpret and reflect on what they said and did, and to report based on their viewpoints (Neuman, 2014).

4.3.4 Methodological assumptions in interpretive paradigm

In selecting the methodology, critical consideration was paid to ontology and epistemology. According to Sarantakos (2013), ontological and epistemological stances have great influence on methodological consideration and further guide the choices made on the research design and study instruments. Methodology, which deals with how knowledge is produced about the phenomenon of interest, results from what one has assumed to exist and, the right approach to creating knowledge about it. It is the process that governs how research is conducted and how data is collected, analysed and interpreted (Creswell, 2014).

Cohen et al. (2011) suggest that in any attempt to understand the subjective world of human experiences, there must be an effort to get ‘inside’ the participant so that there could be understanding from within. In a similar domain under the interpretative paradigm, the researcher goes on a journey with the subjects to learn about how they see the world. The suggestion was adopted in this study. The PSTs’ geometric thinking and their classroom discourses were understood from ‘within’ the participants.

Diverse research parameters guide how research is conducted. Methodology assumes a central role in the research process. It is a vehicle through which ontological and epistemological principles are converted into strategies that show how a study is to be carried out (Sarantakos, 2013). These strategies result in how data is being generated so that the processes surrounding the data generation will help the researcher to analyse, understand and interpret the behaviour of the participants. Based on the discussions above, coupled with the purpose of the study to analyse humans’ thinking, this study situated itself in the interpretivist paradigm.

4.4 Research methodology

Research methodology deals with the design, sampling techniques and approaches used in the study to analyse pre-service mathematics teachers’ geometric thinking. Pre-service teachers’ thinking was chosen as a variable of the study, due to its significant contribution to teachers’ professional development. According to Mudaly (2015), thinking forms an evidence-base component of teachers’ knowledge. The tendency to study thinking is because it involves complex and demanding mental functions such as creativity, innovation and originality in reasoning. Thus,

thinking goes beyond one's ability to just solve a task. If teachers are made aware of the elements of mathematical thinking, which are conjecturing, justifying, proving, visualising and exploring (Tiwari, et al., 2021), they would most probably integrate these elements into their mathematics discourse.

The purpose of this research demanded that multiple forms of data be obtained about pre-service teachers' thinking. The need for such data was to understand, interpret and succinctly describe the participants' thinking as part of their possessed knowledge for teaching. In this regard, the pre-services teachers' thinking was investigated utilising multiple data sources to gain an in-depth understanding and insight into this variable of the study. It was concluded that using the qualitative case study method would be a helpful approach for this study.

4.4.1 Commognitive research

According to Sfard (2008), any interpretive research conducted along the commognitive perspective should focus attention on both the 'what' and the 'how' of human thinking. Commognitive research is rooted in the interpretive research perspective and supports the view that human thinking can be assessed through social interactions via communication, which is central in a discourse (Ryve, 2006; Sfard, 2008). It continues down the route of creating narratives about a person's thought processes. By utilizing stories about the world, the discursive activity of cognition seeks to mediate and establish optimal practices (Sfard, 2012).

In the evolution of the commognitive standpoint on research, Presmeg (2016) asserts that it was created as a theoretical perspective to address, not only the challenges in mathematics discourse, but also the whole dimension of human development and what it takes to be human. Sfard (2008) points out that it is evident that the proposed concept of thinking entails a wide range of data-gathering processes, therefore a large and diverse range of analytical approaches can be envisaged. In addition to the discourse and conventional analyses that have already been done, **those ascribe to** the communicational approach to cognition, have yet to create and test their own data-handling algorithms that are tailored to their individual needs. This view of Sfard (2008) places the commognitive method of research to be both compatible and evolving with an interpretative and

a qualitative method. This gives the impetus to place commognitive research in the interpretive, qualitative perspective and be guided by its principles of investigation.

4.4.2 Qualitative approach

According to Creswell (2014), qualitative research provides a means for studying and understanding the meaning people ascribe to a social or human problem. It is a research approach that is mostly used within the interpretive paradigm. Creswell (2007) asserts that individuals seek knowledge about their world of living through subjective meanings based on how objects and things have been experienced. This study analysed the PSTs' geometric thinking and their classroom discourse. Thinking is an individualised phenomenon; therefore, the qualitative approach was found useful to study the PSTs' geometric thinking through their individualised, subjective meaning on how they have experienced geometry. The aim was to understand the PSTs' experiences with geometry from their own perspective by interacting with them (Astin & Long, 2014).

In the qualitative process of inquiry, researchers build knowledge in multiple ways, as experienced by individuals. In other words, qualitative researchers rely on meanings constructed by individuals in several ways (Creswell, 2014). As a result, any attempt to understand or explore an individual's world of living or experiences, should be carried out by looking for their varied views. The qualitative researchers believe that the phenomenon being studied has several dimensions and hence the need "to portray it in its multifaceted form" (Leedy & Ormrod, 2015, p. 269). In this study, multiple views were sought using the PSTs' solutions to worksheets, their responses to interviews and their practices in the classroom setting.

Also, thinking is a personal phenomenon and can only be assessed through the subjective responses (spoken or written) that are offered by the participants, when socially engaged by researchers. The qualitative inquiry was found to be a useful approach since it relies mostly on spoken or written texts and is open to the subjective view of the researcher in the discussion and findings of the study (Cohen et al., 2011; Creswell, 2014).

Creswell (2014) emphasised that asking more open-ended questions gives the researcher the opportunity to “listen carefully to what people say or do in their life setting” (p. 21). This makes the process of interaction between the inquirer and the participants, a critical tool for data collection in a qualitative study. Ary et al. (2010) explain that the qualitative approach allows for intensive interactions between inquirers and participants through face-to-face interaction, discussion and observation of what occurs in the real-life context.

This approach assisted in generating more open-ended questions during the interview session to check for consistency with what they say that they do in the process. This is because what people do may differ from what they say they do, for which observation provides a reality check (Cohen et al., 2011). The PSTs’ classroom geometric discourse helped to observe what they said and did in the process of teaching geometry.

The use of the qualitative approach to conduct this study in a naturalistic context helped to observe the PSTs’ geometric discourse and to report the actual and detailed views of the respondents’ (Creswell, 2014). In addition, employing a qualitative approach helped to interact with the study participants in their natural setting and allowed the generation of detailed data that was responsive to the study focus (Creswell, 2014).

4.5.3 Case study

According to Creswell (2007), case study of qualitative inquiry is used to explore a phenomenon over an extended time period. A case study involves the study of a single or more cases in which the focus can be on an individual or a group, a process or an activity, extending for some time, to obtain detailed information (Creswell, 2014). Neuman (2014) concurs that a case study explores several characteristics of cases. Cases can be individuals, organisations, groups or units of events. It examines in detail numerous and extensive cases over a period of time. The case study approach was suitable for this study to analyse the PSTs’ geometric thinking and classroom discourse over a period of time.

The goal of a case study is to obtain and report on a detailed description of experiences through an in-depth understanding of the cases being studied, by using multiple methods of data collection

(Ary et al., 2010; Neuman, 2014). It proceeds with the beliefs of the interpretivist that seek to conduct an in-depth study and draw descriptions about participants' interactions with others in a context specific situation, and obtain meaning about the phenomenon being studied (Maree, 2016). The use of the case study was based on its multiple tools such as interviews, observation and focus group discussions, and, is supported with audio and visual materials for data collection (Creswell, 2007).

Neuman (2014) argues that a case study seeks to study many details of the cases. Hence, it was applicable to this study, in which the focus was to study several details of the PSTs' geometric thinking. The case study method, coupled with the study framework, which provided an extensive construct for exploring human thinking, paved the way to gain insight into several dimensions of each PST's thought processes about geometry, which included word use and how words are processed in learning geometry. Other areas included visual mediators, routines and narratives. Further insight was obtained by observing how the PSTs' geometric thinking influenced their classroom geometric discourse. This allowed for a detailed description of the variables studied (Yin, 2014).

4.5.4 Selecting the institution

The first consideration in selecting the participating institution, related to institutions that offer teacher education programmes with a specialty in mathematics education. The choice was based on working with PST's who could provide rich data for the study. The next consideration was to work with a nearby institution.

Convenience sampling is a non-random sampling technique used to select the places of study because of proximity (Ary et al, 2010; Sarantakos, 2013). Creswell (2014) asserts that in a qualitative study, the researcher has the choice of selecting an area of study and the study participants, if they are found to be capable of providing the needed information to meet the study aims. Cohen et al. (2011) go on to say that convenience sampling is the appropriate approach used to select a study area due its proximity and easy accessibility to facilitate the research process. The study required a prolonged engagement with the study participants to generate extensive and

detailed data from several sources, in line with the study purpose. Hence, the two most important criteria considered in choosing the institution were proximity and accessibility.

Cohen et al. (2011) assert that the non-probability sampling technique is often used by researchers whose aim of study is not to generalise its finding to any larger population. This is a qualitative study that sought to describe the geometric thinking of participating PSTs, in depth. Hence, working with many institutions was not a priority. In addition, the study did not seek to generalise its findings. As a result, only one institution was used, which made it unrepresentative of the existing educational institutions in Ghana.

4.5.5 University's profile

The mandate of the university, one of the public tertiary institutions within the study area, is to provide quality education to citizens who wish to join the educational sector after course completion. This is part of the government's commitment to provide education for all its citizens and those aspiring to be teachers. Prospective teachers undergo training to gain knowledge and expertise required for teaching purposes. Teacher preparatory institutions are tasked with equipping prospective teachers with the knowledge necessary to be successful in the teaching profession. Such knowledge encompasses content, pedagogy and the school curriculum (Shulman, 1986).

The university has several departments that offer various programmes in teacher education, including mathematics, which is responsible for mathematics teacher preparatory programmes. The department has the basic role of preparing prospective teachers for teaching at the Senior High School (SHS) level. This puts the SHS mathematics curriculum into focus. Teaching mathematics at the SHS level aims "to enable all Ghanaian young people to acquire the mathematics skills, insights, attitudes and values that they will need to be successful in their chosen career and daily lives" (MoE, 2010, p. ii). It is believed that all learners must learn mathematics as a school subject, as it is a subject underlying all technological development and a tool for a nation's economic growth. Learners at this level are expected to develop sound mathematical competencies for two purposes. It is expected that the SHS leavers would be able to gain admission into tertiary institutions and be successful in the study of "associated vocations in mathematics, science,

business, industry and a variety of career options” (MoE, 2010, p.ii). Also, the knowledge gained should enable those who are unable to further their education, to successfully engage in trading activities. Education programmes, guided by such goals, prepares prospective mathematics educators to foster the intended curriculum goals.

The department’s varied courses, classified into content, pedagogy, and enrichment, are geared toward holistic development of prospective teachers for effective teaching. There is a pre-internship programme (on-campus teaching practice) that enables prospective teachers to put the knowledge gained into practice, for further development for teaching. The pre-internship programme enables prospective teachers to design and execute a planned lesson to experiment with certain teaching behaviours and teaching skills (Otsupius, 2014). The department, like any other mathematics education department, has several content courses in the mathematics education programme which are structured to equip the pre-service teachers with required knowledge, to enable them to teach effectively at the SHS level and to prepare them for further studies in related disciplines in mathematics.

4.5.6 Gaining institutional access

To gain institutional access, the head of the Department of Mathematics Education was visited, where the purpose of the study and the processes for data generation were discussed. The Head was also informed that, participation was voluntary, each participant had the right to withdraw from the study and, confidentiality of the data collected was assured. A written consent from him was required to conduct the study.

4.5.7 Informed consent

Researchers believe that participating in research activity should be voluntary and based on one’s clear understanding of the purpose of the study and what information will be required after his/her agreement. One must understand the research processes, to make a decision to participate. A letter, which described the purpose of the study and how data was to be generated and recorded, was given to the prospective participants to seek their consent and voluntary participation. Willing participants were required to append their signatures on the written consent. Contact details (phone

numbers and e-mails) of the researcher, the study supervisor and personnel at the University of KwaZulu-Natal, were provided. A copy of the letter is attached (see Appendix B).

4.5.8 The study participants

Eight second year PSTs were selected, based on two factors. The first was those who had learned geometry as part of their study programme, and had also been exposed to courses in principles and methods of teaching. The choice of the participants was based on Cohen et al's. (2011) claim that that purposive sampling can be used to access data from people who have in-depth knowledge about the phenomenon under study. Neuman (2014) support this view and add that choosing participants for a study can be done purposively, provided they can provide the researcher with the necessary information to gain a rich and in-depth understanding of the phenomenon being studied.

The phenomenon investigated in this study, was thinking. The participants' geometric thinking was investigated to gain insight into the *when* and the *why* of their written geometric discourses on the *how*. According to Sfard (2008), thinking is personalised and can be assessed in depth by engaging the individual in communication. Hence, the participants were interviewed to generate data about their thought processes. After explaining the purpose of the study, they were again reminded of their voluntary participation and their free will to withdraw their participation, if they wished. People should not be compelled to participate in a study when they are not willing (Neuman, 2014).

About 90 PSTs from a class of 97 showed interest in taking a test, but they were made aware that taking the test did not guarantee participation in the actual study. Participation was entirely voluntary. The intention for the test was to gain an initial insight into their solution strategies and select the PSTs based on performance. Consideration was also given to those who solved the tasks using different approaches (various solution strategies). Since the 16 test items were based on SHS mathematics content, those who correctly solved 12 or more items were considered to have performed well and were classified as Group A, whilst those who correctly solved less than 12 items were considered not to have performed well (not so good) and were classified as Group B. Of the ten who were invited to participate in the study, eight (four from each group) showed their

willingness to participate in the interview and the classroom lesson observation. The eight were considered adequate since in-depth data was to be generated.

4.5.9 Piloting the study

The test was pilot tested to obtain further possible inputs to improve upon its quality in terms of clarity and meaningfulness of the items. This step, according to Leedy and Ormrod (2015), is to confirm that the questions are clear and capable of eliciting the information required. Piloting study instruments at an early stage is a step taken by researchers to detect, or uncover, any difficulties in understanding the items (test or interview), and to highlight areas that may require modification, to help refine the instrument (Dikko, 2016; Gani et al., 2020). The pilot test helped to identify and modify detected ambiguities, misunderstood words, and other inadequacies in the test questions.

Maree (2016) advises that the population from which the sample is drawn for the pilot test, should have the same characteristics as those who take part in the main study. Cohen et al. (2011) concur and add that piloting the instrument to a different group with similar characteristics, will enable the researcher to analyse the possible trends seen during the piloting stage in case such trends occur in the actual study. Based on these, the test was piloted on students who were also PSTs from different institutions and had similar characteristics. Responses obtained in the piloting exercises helped to improve the quality of the test instrument. The piloting also helped to determine which questions were more capable of eliciting the needed data, with regard to geometric thinking. Thus, the pilot study helped to decide whether the questions on the instrument triggered or inspired geometric thinking, and to check if the questions were adequate for the study. Another important benefit of the piloting was that it helped to gain insights into the kinds of responses to be obtained in the actual study, and to determine their suitability to answer the critical research questions. During the piloting, two PSTs were selected to practice the interview process before the main study. The practicing interview, helped in refining questions that enabled the participants to freely express their thoughts in depth.

4.6 Stages of data generation

Data generation, according to Cohen et al. (2011) is the practice of creating, or eliciting information from research participants to gain an understanding of the phenomenon under study,

to answer the research questions. The instruments and processes involved in data generation play a significant role in research and using inappropriate instruments and procedures may affect the credibility of the study. Lapan et al. (2012) and Yin (2014) assert that a qualitative case study is characterised by collecting data from multiple sources to facilitate understanding of the phenomenon being studied. In this research, data was generated from three multiple sources, viz worksheets (test), semi-structured interviews (task-base), and an observation guide. The sequence of data generation is shown in Figure 4.1, with details in the sections that follow.

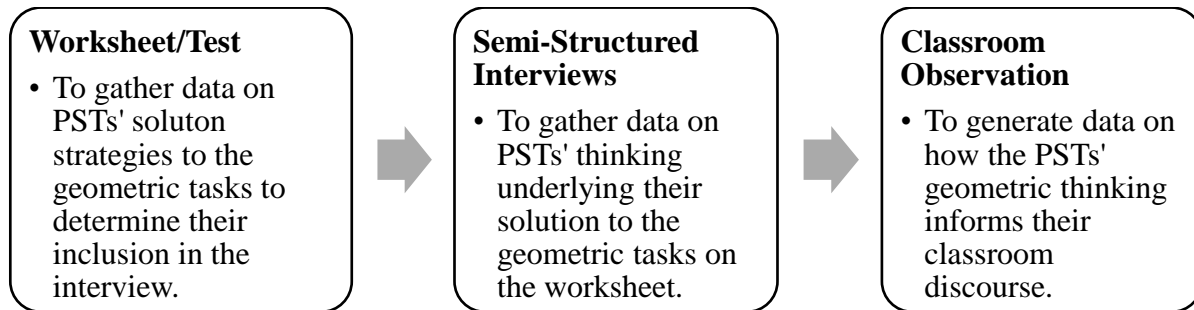


Figure 4.1. Steps involved in data generation.

Three research instruments, namely worksheets, interviews and observation, were used in relation to the critical research questions. The table below outlines the critical questions and how the data was obtained.

Table 4.2: Alignment of critical research questions with research instruments

Critical Research Questions	Research Instruments
1. What is the nature of pre-service mathematics teachers' discursive thinking in geometry?	The worksheet and interview
2. What is the nature of pre-service mathematics teachers' routine thinking in geometry?	The worksheet and interview
3. How does pre-service mathematics teachers' geometric thinking influence their classroom discourse?	Classroom observation

4.6.1 Stage 1: Structured worksheet

Although this study was conducted through a qualitative approach, a worksheet was administered to the study participants, not for the purpose of quantification, but to provide written responses to the items on the worksheet (test). The main rationale was to facilitate the selection of two groups, with distinct characteristics. Those who performed well in the testing phase were labeled as Group A and those who performed not so good, were the Group B participants.

A worksheet (test) consisting of 16 questions, was used to obtain first hand data about the participants' solution strategies. These tests normally consist of tasks or a series of tasks used to elicit the information required. According to Nwadinigwe (2002), it is a data collection tool which is administered to study participants by subjecting them to testing conditions, to enable the researcher to obtain information about their performance concerning the phenomenon of study. It is often used to generate data on people's competence on the objective of the study.

The development of the test was based on several factors. The first was to select items that could measure the participants' geometric ideas in relation to the study's purpose. Maree (2016) and Cohen et al. (2011) maintain that worksheets should be designed to purposefully obtain the right information for achieving the study purpose.

The worksheet items were based on angles and properties of parallel lines, triangles, quadrilaterals and circles in the plane geometry content, of the SHS mathematics syllabus for Ghanaian schools (MoE, 2010). The literature was also reviewed on what has been documented on learners' difficulties and misconceptions in plane geometry (Luneta, 2015; Mirna, 2018; Ngrishi & Bansilal, 2019). Insight gained from this review provided insightful ideas for constructing the test items. The test items were open-ended and the purpose was to gain insight into the PSTs' solution strategies for those tasks.

The worksheet was administered to the PSTs on the appointed date and time, and they were given an hour so that each of them could devise a solution to the test items. The participants were selected based on their performance and solution strategies to participate in the interview.

4.6.2 Stage 2: Semi-structured interviews

Interviews, being one of the most commonly used instruments for data generation by qualitative researchers, were used to generate data. It is a tool used to explore several traits of individuals including their views, feelings and beliefs concerning issues under investigation. Data generated by interviews are often in the form of words uttered by the study participants (Gani et al., 2020; Ary et al., 2010; Dikko, 2016; Majid et al., 2017). According to Nwandinigwe (2002) and Babbie et al. (2010), it is the process of data collection where the researcher dialogues with the study participants to elicit data. Interviews proceed through oral questioning to which the respondents respond orally (Babbie et al., 2010), providing constant interaction between the researcher (interviewer) and the respondent (interviewee).

With the aim to analyse, interpret and describe pre-service teachers' geometric thinking, the interviews helped to communicate with the PSTs, to gain an in-depth account of their thinking. This form of communication involves talking and listening to each other as the process of data generation unfolds (Cohen, et al., 2011; Sarantakos, 2013). Best and Kahn (2006) contend that the main purpose of an interview as a data generation method, is to gain access to what is on a person's mind. It is a tool that enables a participant to share his or her opinion on issues, experiences, what informs particular actions, or, the basis under which something is done in response to questions related to the study purpose (Wahyuni, 2012). Thus, the interview was a suitable way to create contact with the minds of the PSTs to access their thought processes governing the solutions provided.

The choice of an interview was further informed by Sfard's (2008) commognitive framework which considered thinking as communication. Engaging the PSTs in communication helped to gain insight into their thinking processes on geometry, through the explanations they provided about the strategies underlying their solutions to the geometry tasks. Thinking is human behaviour that is not seen, heard or touched. Hence, the most appropriate way to investigate such human behaviour in detail is to allow the person to express him/herself in their own words (Sfard, 2008). Data generation through interviews provides the opportunity to ask questions that demand explanations and justifications of one's actions (Lapan et al., 2012).

According to Sarantakos (2013), an interview can be structured, unstructured or semi-structured. The structured is formal and is often restricted to a predetermined set of questions, whilst the unstructured takes the form of questions that arise as the interview unfolds. In the case of the semi-structured interviews, a laid down set of questions are followed but the researcher could formulate new ones in the course of the interview, for probing or for clarification (Maree, 2016; Sarantakos, 2013). In this study, the semi-structured interview was used due to its flexible nature.

The semi-structured interview was used due to its flexibility in allowing for follow-up questions (DeJonckheere & Vaughn, 2019). Due to the different ways they solved the tasks, it was not possible to ask all of the PSTs the same questions. Although, some of the questions were the ‘why’ and the ‘when’, that required participants to substantiate their solutions, most of the questions depended on individualised solutions (written responses) to the geometry tasks. This made the interview a semi-structured type since it contained questions that were planned before the exercise and others that arose from the participants’ responses (Ary et al., 2010; Sarantakos, 2013). Thus, some general questions were constructed in advance whilst others were formulated based on participants’ solutions. Appendix E shows a copy of the semi-structured questions.

In line with the purpose of the study, each PST was expected to explain how he or she had experienced and understood geometry. The semi-structured interview enabled them to express themselves freely in an individualised manner. Gani et al. (2020) maintain that the semi-structured interview is characterised by its potential to address ‘how’, ‘when’ and ‘why’ questions about how an individual has experienced an object and the meaning and interpretation assigned to that experience.

The place to conduct the interview was chosen because it was conducive for conversation and quiet enough to support the recording. Also, a good rapport was created with interviewees, allowing them to communicate freely and willingly. There was no indication of judgement by the researcher either in expression or facial looks to PSTs responses, based on the suggestion for conducting the interviews (Best & Kahn, 2006; Gani et al., 2020; Jacob & Furgerson, 2012).

4.6.3 Stage 3: Classroom Observations

The PSTs' classroom geometric discourses were observed to understand how their geometric thinking influenced their teaching. Cohen et al. (2011) assert that classroom observation allows the researcher to obtain live data from a naturally occurring setting. Observing an activity as it proceeds is characterised by immediate awareness, and has a high probability of producing valid data. Classroom observation was used as the tool for obtaining such data in its natural setting to help gain a detailed understanding of how the PSTs communicated their geometric thinking, in line with commognitive constructs (Sfard, 2008). Maree (2016) claims that observation takes place in our daily activities where our visual senses are used to assemble data. The observation assisted in obtaining original information in connection with the study aims. Classroom observation helped to gather valid and extensive data about the PSTs' teaching of related issues in connection with the data generated during the interviews. Ary et al. (2010) explain that observation is a basic tool to generate data to gain insights into issues being studied. The use of observation helped to understand and describe their teaching behaviour with regard to how the PSTs' geometric thinking informed their geometric discourse.

Teaching is an interactive process where teachers communicate concepts, facts, attitudes and knowledge to learners. Teaching is said to be effective when the objectives of instruction have been achieved. Classroom interaction is geared towards facilitating learners' achievement of the stated instructional objectives. The desire to generate data as teaching unfolded, was to understand how prospective teachers presented geometric concepts in ways that enhanced learners' sense making of the content being taught. Geometry, like mathematics, is a discourse and needs to be communicated in a patterned, collective way (Sfard, 2007). The intent was to observe how the prospective teachers drew attention to certain keywords in geometric discourse, as the fundamental process of guiding learners' understanding (Atebe & Schafer, 2010; Mulwa, 2015). Learning geometry, just as any other mathematical subject, requires identifying and processing basic words of the subject (Sfard, 2008). Understanding these basic words enables learners to construct knowledge on their own, which requires that teachers facilitate learners' knowledge creation through appropriate use of words and other commognitive constructs. Deeper insights and understanding of classroom activities can be known through classroom observation (Maree, 2016; Cohen et al., 2011).

According to Cohen et al. (2011) and Creswell (2014), observation can be classified into four types, namely: complete participant, observer as a participant, participant as an observer, and complete observer. The observation type used in this study was the complete observer, where the researcher only observes without participating or interfering in the activities (Creswell, 2014). Before the observational schedule, the PSTs were visited on several occasions during the pre-internship programme (on-campus teaching practice) as a way of gaining and strengthening familiarity. According to Babbie et al. (2010), researchers must be close to study participants in order to gain a high degree of credibility and trustworthiness. All the observed lessons were video recorded in order to obtain rich data and also to cater for any information that may have missed. The video recording was supported by taking field notes.

Observation can be classified as structured, semi-structured or unstructured, depending on the extent to which it is organised (Cohen et al., 2011; Sarantakos, 2013). In structured observation, the researcher follows an organised procedure, planned in advance, on what is to be observed, before starting the observation process. There is a high level of standardisation in what is observed. The structured observation makes use of planned and well-defined traits to be observed. In semi-structured observation, the observer outlines the desired traits to be observed but also focuses attention on certain equally important issues that may be noticed in the course of the observation. An unstructured observation does not employ any strict procedures for data generation but takes note and records any traits that may be of interest as far as they relate to the study (Kuranchie, 2021).

Structured observation was used in this study. Placing the study in a commognitive perspective and guided by its elements, formed one of the main rationales for the classroom observation. The structured observation helped to gain insight into how the commognitive constructs could be seen or followed in the PSTs classroom discourse (Sfard, 2008). In agreement with Sarantakos (2013), Cohen et al. (2011) claim that structured observation makes use of organised procedures to be followed. In a similar way, the traits to be observed were planned in advance before the observation schedule. These traits related to the constructs of the commognitive framework, which were how the PSTs used, keywords, visual mediators, narratives, and routines to provide learning

opportunities for developing geometric thinking. Other traits observed were how they delivered instruction in ritualised or explorative ways of concept development. It was conjectured that the constructs of Sfard's (2008) commognitive framework has the potential to develop teachers' communicative competence in a discourse.

During the observation, notes were taken in a field diary and all observed lessons were videotaped. The use of the field diary was to take note of issues such as facial cues or expressions, gestures, PSTs' movement, and others. Being mindful of the ethics of the practice, participants were assured that there would be no pictures showing their faces and that all written text and verbal responses would be reported using pseudonyms.

4.7 Data analysis plan

Data analysis is the process of examining and deriving meaning from data to understand, interpret and explain the phenomena being investigated (Cohen, et al., 2011). Data analysis is the process of transforming data into findings. Data analysis in qualitative study can be inductive or deductive. In this study, the deductive analysis approach was used based on the guiding research questions and the informed theoretical framework employed. Deductive analysis is informed by the themes predetermined within a theoretical framework. It considers the proposed themes within a framework that guides the research. The researcher is usually aware of the themes in the framework before he or she starts collecting data.

In this study, the data analysis was guided by the themes of the commognitive constructs, which are keywords (literate or colloquial), visual mediators, narratives and routines. All the research questions were analysed through a deductive approach. The rationale was to conduct a search for the presence of Sfard's (2008) commognitive constructs (traits) in the geometric discourses of the PSTs. In other words, the search was to analyse and understand the PSTs' planning processes in devising solutions to geometric tasks, as well as the learning opportunities they provided for classroom learning of geometry.

Research questions one and two were answered through the analysis of text and interviews, whilst research question three was answered through the analysis of the classroom observation data. The

analysis of the interviews helped to understand the rationale behind the solution strategies of the PSTs in order to report accurately. According to Ary et al. (2010), even though there may be slight changes in qualitative data analysis, it can be categorised into three: namely organisation and familiarity, coding and reducing, and interpreting and representing.

The data collected in the study took the form of text (solutions to tasks), audio-taped material, video recordings, and field notes. Based on Ary et al.'s (2010) suggestion, familiarity with the data started by listening and re-listening to the audio-taped data, successive reading of the field notes, examining the PST's solutions to the geometry tasks, and viewing the video tapes many times. The audio-taped data and video recordings were transcribed into text, whilst the field notes were typed. Care was taken to transcribe the exact words obtained from the participants to avoid any potential bias that may occur in the analysis process. The self-transcribing of data by the researcher, helped to gain a depth of familiarity (King et al., 2018). To avoid several pages of transcribed data for each of the eight participants, the transcribed data was organised in terms of responses to tasks, and questions asked, to enhance the actual analysis. During the transcription, nonverbal information from the data collection process was added.

The data was arranged as described and then began the coding. According to Tracy (2013), coding is the process of labelling and systematising the data obtained. Leavy (2017) adds that it is the process whereby words or phrases are assigned to segments of data. The discourses of the PSTs were coded based on the themes and characteristics of the informed commognitive constructs. The coding was done by identifying the words used in communicating geometric concepts and ideas, and how further geometric concepts were communicated using visual mediators, narratives and routines. The analysed geometric thinking of the PSTs was presented and discussed on the themes within the chosen framework. In addition, the analysis was interpreted by reflecting on the participants' word usage to gain greater understanding from them.

4.8 Trustworthiness of the study

Trustworthiness, or rigor, means the extent to which one can establish the authenticity or quality of study findings based on confidence in data obtained, interpretation, and approaches used (Connelly, 2016). Trustworthiness has its roots in validity and reliability, which is often used to

ensure the quality of quantitative studies. Due to the protest of the use of the term ‘trustworthiness’, in qualitative inquiry Ary et al. (2010) and Mohamed (2017) outlined criteria for ensuring the quality of qualitative inquiry which is accepted by most qualitative researchers, regarding trustworthiness. These are credibility, dependability, tranferability and confirmability.

4.8.1 Credibility

Credibility, which is analogous to internal validity of a quantitative inquiry, deals with the confidence in the truth of research findings (Connelly, 2016; Anney, 2014; Mohamed, 2017). It seeks to answer whether the information obtained is what the researcher intended, and whether the participants can agree to the study findings, based on the information provided (Lapan et al., 2012; Mohamed, 2017). In other words, its purpose is to determine whether the research findings are plausible representations of the information drawn from the actual data provided by the study participants, as well as to ascertain whether it is correctly interpreted in the views of the participants (Korstjens & Moser, 2018). Korstjens and Moser (2018) add that credibility of research findings is supported by prolonged engagement with research participants, persistent observation, triangulation, peer debriefing and member checks.

To ensure credibility of the findings of this study, time was spent with the PSTs to ensure that a good relationship was established with them. This was to develop familiarity with the PSTs in order to be recognised as a partial member of their community. Such association helped to gain insight into the phenomena studied. Prolonged engagement increases rapport and enables participants to volunteer detailed information that they would sometimes provide at the beginning of the study Connelly (2016). In this sense, it allowed the participants to communicate freely and act naturally, without fear or apprehension about releasing certain vital information.

Another step taken to ensure credibility of the study results, was to generate data through triangulation, to obtain rich data from varied or multiple sources to enhance comprehensive understanding of the phenomena studied. Data sources used in the study included testing (worksheets), interviews and classroom observation, to gain detailed understanding in order to enhance the accuracy and credibility of the study findings and to minimize bias. Triangulation aids in minimising bias and double-checks the reliability of participant responses (Anney, 2014).

There was also peer debriefing apart from discussion with my supervisor. According to Lincoln and Guba as cited in Creswell (2014) peer debriefing is the practice of opening up to a neutral peer in a way that resembles an analytical session, in order to explore parts of the inquiry that could otherwise just be latent in the inquirer's mind. In this situation, the study was shared with disinterested peers (people who are knowledgeable sources on the topic, but may not be a stakeholder in the study outcome) such as colleagues in doctoral studies, and experienced mathematics educators (both at secondary and tertiary levels), for possible scholarly guidance to improve the study findings.

In order to prevent the chance of incorrectly interpreting the data acquired, Yin (2014) describes member checking as the process of allowing the research participants to review and verify the data collected via interviews, participant observations, and documentation. Based on this, the participants were visited on several occasions, for them to verify the transcribed data to ensure it corresponded with what they said.

4.8.2 Dependability

Dependability, which is similar to reliability in quantitative studies, is the extent to which the data would be stable over a period of time, in different situations but similar conditions (Anney, 2014; Connelly, 2016; Mohamed, 2017). It concerns itself with the consistency of the study findings. Dependability in qualitative research is ensured by an 'audit trail' and offers a rich description of the study process (Anney, 2014; Korstjens & Moser, 2018; Mohamed, 2017). Audit trail is the process of frequent engagement with the participants to assess the findings, interpretations and recommendations of the study, to determine if all that is reported in the study could be supported by the data that was provided (Connelly, 2016; Korstjens & Moser, 2018).

To ensure consistency of the study findings, an effort was made to provide a detailed description of the methodology which dealt with choosing the study site and sampling the study participants. Also, the exact data generation procedure is described in detail, such that it can be followed to assess the dependability criterion.

4.8.3 Transferability

Transferability of qualitative research is a quantitative form of generalisability. It is the extent to which the findings, or methods of a qualitative study, can be transferred to other participants in another context (Anney, 2014; Mohamed, 2017; Korstjens & Moser, 2018). According to Anney (2014), transferability is the extent to which one can determine the applicability of the findings of a particular inquiry to other contexts, or with other participants. This can be achieved by providing a complete and detailed description of the research context, that would enable one to judge the applicability of the approach to another context. In this study, a complete, detailed and thoughtful description of the research processes has been provided to enable its conclusion to be applicable to other situations. There is also an explicitly detailed description of the study purpose and aims, including approaches used in data generation, to enable a high sense of transferability of the study findings.

4.8.4 Confirmability

Confirmability in qualitative research can be associated with objectivity in a quantitative inquiry. This can be achieved if effort is made to establish credibility, dependability and transferability (Mohamed, 2017). Anney (2014) posits that confirmability is achieved if other researchers are able to corroborate the results of the study. In this case, the data and interpretation of the findings are considered original and not from a researcher's own imagination. According to Cope (2014), research confirmability can be attained when the researcher can show that the data is an accurate reflection of the participants' viewpoints and that the participants' responses are devoid of the researcher's biases. Issues of confirmability can be addressed through Ary et al.'s (2010) suggestions of using audit trail, reflexivity and triangulation.

Efforts were made to achieve credibility, dependability and transferability, which, to a large extent, guarantees the establishment of confirmability of the findings. A reflexive journal was kept to document issues or events that occurred in course of the study, for reflection.

4.9 Conclusion

This chapter provided a detailed account of how the study was carried out using the informed philosophical views, its method and procedures. The chapter started with a brief introduction of

the purpose of the study and the elements of the theoretical framework that served as a lens for conducting the study. Next, was the outline of the critical research questions, followed by a discussion of the interpretive paradigm used. The chapter also included a discussion of major issues of concern in research, namely, research design, methodology and the methods used. The research instruments used to generate data from the study participants and the processes of data generation were also discussed in detail. The data generated was then transcribed and prepared for analysis.

CHAPTER FIVE

PRE-SERVICE TEACHERS' DISCURSIVE GEOMETRIC THINKING

5.1 Introduction

In the previous chapter, the philosophical basis underling the study, the research design, and the methodology were discussed. It also provided a detailed description of the stages followed, in data generation. Data was generated using worksheets, task-based interviews and classroom lesson observations. This chapter presents the analysis of the data obtained through tests and task-based interviews on the pre-service teachers' (PSTs') geometric thinking of/on plane geometry topics in the senior high school mathematics curriculum. The analysis, which was informed by the literature review and the research questions, is presented in three chapters. This chapter focusses on the nature of the PSTs' discursive geometric thinking in geometric discourse, whilst the other research questions, (viz., two and three) are presented and discussed in chapters six and seven respectively.

In the write-up of this chapter and throughout the thesis, italicised texts are used to present the pre-service teachers' actual words used in communication. All other expressions are paraphrasing of their words, the observations made, the textual solutions they provided, and the interviews held with them.

5.2 The results of the study

The analysis of the results of the PSTs' geometric thinking was further informed by the constructs of the commognitive framework, which are, word use, visual mediators, narratives, and routines. In this respect, the analysis was conducted deductively. The presentation of the analysis has been organised in the following manner. It begins with a focus on how the PSTs used appropriate words to define and describe geometric concepts and properties in a concise and precise way, as well as the word use in explaining their solution strategies. Next, it looks at the PSTs' consciousness of centering their talk about the visual mediators governing the tasks, and are discussed in sub-themes within the framework. Further analysis focused on their ability to substantiate their discursive actions by using appropriate sources of narratives within the geometric discourse. The final

analysis deals with the routine patterns used by the PSTs, to devise solutions to the geometry tasks. These are briefly discussed in the next section.

5.3 Path of discourse analysis in this study

This section explains the discourse analysis conducted in the study as guided by constructs of Sfard's (2008) commognitive framework mentioned in section 5.2. These constructs have been captured and explained in detail in Chapter 3, section 3.3. Table 5.1 explains how these constructs are employed, or associated, in this current study. The constructs of the framework are highly connected in discourse that sometimes seem inseparable. However, there is an attempt to put them into groups since they can have a significant effect on how to teach.

Table 5.1: Description of discourse analysis in this study

Commognitive construct	Indicators
Word use	Word use may appear in many forms. The goal of this study was to examine how participants used words to define geometric concepts and figures such as angles, acute angles, triangles, and their related properties. The ability to identify names of geometric shapes and concepts and other terminologies, forms part of word use. In addition, attention was given to how words were used to explain solutions to geometric tasks. The wording of the geometric properties was also of great interest, for example, angles (adjacent angles) on a straight line, vertically opposite angles, exterior angles, opposite interior angles, etc.
Visual mediator	Visual mediators come in many forms as a tool for learning geometry. Analysis was based on the participants' interpretations of diagrams (visual abilities or visual thinking), their ability to draw geometric figures and related concepts, using visual abilities to interpret the diagrams, symbolic mediators, iconic mediators, or concrete mediators.
Narrative	Attention was given to the geometric concepts and ideas that participants used to solve the geometric tasks. For example, if a participant models the equation

	for interior angles of triangles by equating their sum to 180° , he/she needs to justify by the response, ‘the interior angles of a triangle add up to 180° ’. Similarly, if two straight lines intersect, and a participant equates the two opposite angles, his/her action must be informed by the narrative that ‘vertically opposite angles are equal’. In short, the intention was to find out which geometric concepts the participants used to support their discursive actions.
Routines	The focus was on the procedure for solving a given task. The major concern was placed on the participants’ ability to model equations connecting the angles in the tasks. According to Sfard (2008), this action needs to be informed by keywords, visual mediators, and narratives.

The indicators outlined in Table 5.1 draw connections between the constructs of the framework and geometric discourse. The constructs were employed to analyse the PSTs’ geometric thinking since they were found to be the best fit for discourse analysis.

5.4 Pre-service teachers’ word use in geometric discourse

This section presents the findings and discussion on the PSTs’ word use in describing geometric concepts, terminologies, geometric properties, and word use in explaining their solution strategies to the geometric tasks used in the study. The discussion is based on the four content areas, namely, angles and straight lines, triangles, quadrilaterals and circles.

5.4.1 Word use in defining /describing geometric concepts

According to Sfard (2008), the kind of word use (terms of vocabulary) gives a discourse its distinctive features. Word use plays a significant role in mathematics discourse. It provides the avenue to describe specific ideas in a discourse especially definitions of geometric terms, concepts and theorems. Sfard (2007, 2008) claims that mathematical words have a shared or a specialised meaning among participants within the discourse, in which case it is literately used, whilst non specialised words are colloquial. Word use plays a key role in defining mathematical concepts. Concept definitions are the kinds of words used to designate or specify a given concept (Fujita & Jones, 2006). Concept definitions play a critical role in the teaching and learning of mathematics

(Baktemur, et al., 2021; Cunningham & Roberts, 2010). In this section, the PSTs' word uses in defining geometric concepts and properties, are analysed and described. The PSTs were asked to share their thinking and understanding of angles and straight lines, and other geometric figures such as triangles, quadrilaterals and circles, as well as their properties. The purpose was to gain insight into how they used words, whether literately or colloquially, in defining geometric concepts.

All the PSTs shared their understanding of the definition of the concept of an angle. The eight participants stated that angles are formed when two straight lines, or rays, meet at a point. The definitions of Stephen and Jones from Group A, are shown in the following excerpts:

... an angle is formed when two lines or two rays meet. When they [rays] meet the space between the vertex of the two lines is what we call the angle ... (Stephen).

... an angle is the figure formed when two rays meet at a common point. I can say that an angle is the space between two lines when they meet at a common point (Jones).

Similarly, the Group B participants provided definitions that showed their understanding of the concept of angles. Examples of these definitions are shown in the extracts below:

... angles are formed when two straight lines meet at a point ... (Alex).

... when two lines meet at a point, they form an angle. The point at which the lines meet is called a vertex (Cynthia).

It is evident from the preceding excerpts that a key attribute that emerged from their definitions was that angles are produced when two straight lines meet. The participants' discourses showed their knowledge of some terminologies associated with the definitions provided. They stated that the point at which the rays meet is a vertex, whilst the rays are called the arms of the angle formed. This shows that the participants had developed an understanding of the concept of angles and could use acceptable words to express their thoughts about the object.

To investigate further, the participants were asked if they could define, or describe, the concept of an angle in a different way. Their responses, except for Stephen's, were either 'no' or a repetition

of the same central idea using different wording. Clement (A) and Albert (B) reaffirmed the same thinking using planes. For example, Albert said:

... I can say that when you have a cardboard, or any straight edge and ... join them together ... you ...can equally get an angle.

Stephen (A), on the other hand, explained his understanding of angles as the amount of ‘opening’ formed by a ray when it turns from one direction to another. His response was as follows.

... when we take a ray or a line segment [draws on paper] when it is rotated from its starting point to the terminal point, the space found between the rays is called an angle.

Stephen defined the concept of an angle from different perspectives. The word ‘rotate’ is synonymous with the word ‘turn’. Describing an angle from this perspective shows that Stephen has developed good thinking about an angle and was able to describe it with appropriate use of words. Stephen’s description can be summarised to mean an angle is a measure of a turn.

Analysis of the participants’ responses showed that seven of them defined an angle in a static form. According to Kontorovich and Zazkis (2016) and Smith et al. (2014), learners view angles in a static form when they see them as geometric figures in which the focus is on the position of the sides or the arms. Mullins (2020, p. 5) asserts that considering angles as static forms is “simply pictorial or figurative representation”, which makes it difficult for learners to identify angles in various positions, such as 0° and 180° (Smith et al., 2014). Only Clement (A) demonstrated adequate thinking of angles in a dynamic form involving movement (Clement & Burns as cited in Mullins, 2020). Learners understand angles in various forms when they conceptualise angles as dynamic, which makes them develop schemas that can be used in later learning (Mullins, 2020; Smith et al., 2014). Thus, the PSTs showed limited thinking about the concept of angles. Nevertheless, findings show that their understanding of the concept of an angle was articulated with appropriate word use. They used endorsed words to express their thoughts.

Maxwell (A) drew two straight lines to intersect and indicated all the four angles formed, whilst Stephen drew two rays that shared a common endpoint and described the two angles around the

vertex. The remaining six participants' discourses (two in Group A and the rest in Group B), in addition, their drawings seemed to imply that their idea of defining an angle is linked to one angle formed when two straight lines meet. All the participants were able to mention the types of angles and described them based on the opening or measure (Smith, 2012), with acceptable word use.

Albert's (B) word use in defining both complementary and supplementary angles was inaccurate. He defined supplementary angles as "*angles that sum up to 180°* ". What needs to be noted is the missing word, signifying the number of angles involved. When he was asked about the number of angles involved, he said, "*it could be two or more*". The other participants described and indicated the number of angles involved. For example, Alex (B) defined complementary angles as "*when two angles sum up to 90°* " and supplementary angles as "*when the two angles sum up to 180°* ".

The Ghanaian mathematics curriculum may be classified as a spiral curriculum, where concepts are introduced to learners in portions from lower levels to higher levels of study (MoE, 2010). It is a system that ensures that the learners' knowledge-base is well grounded in the basic concepts before advancing to higher ones. In other words, it is an educational system that enables learners to absorb the most basic concepts, so that understanding can be developed by relating new knowledge to existing knowledge. For example, one's understanding of an acute angle is necessary for learning triangle classification by angle, as 'acute triangle' and a 'right angle', leading to learning 'right-angled triangle' or 'right triangle', and others. The PSTs' performance was expected because of their long exposure to the concept of angles in their previous lesson in mathematics.

Generally, the results show that many of the PSTs possessed good understanding of angles and their types, and held sufficient ability to describe them. Pre-service teachers need to possess in-depth geometric thinking about angles to effectively teach geometry to their future learners. According to Biber et al. (2013), almost every topic in geometry requires good knowledge of angles, as one of its basic concepts. This implies that anyone who has not grasped these basic concepts may encounter learning difficulties as he/she progresses in learning higher content areas of geometry, particularly with the PSTs who are being trained to teach these concepts. Placing this in the commognitive theory, the participants are said to use mathematically literate words to

describe their thinking and understanding of angles. Sfard (2008) asserts that learners should be able to use appropriate words to communicate clear intent. This quality was noticed to be present in many of the PSTs' discourses about angles. The participants' thinking on the concepts of parallel lines was analysed, whilst seeking an explanation of the solution to the tasks involved in the study. The next paragraph presents the participants' talk about triangles.

Data on the participants' discourses on triangles show that many of them had a good ability to define triangles and related concepts using mathematically literate words, except for a few who demonstrated difficulty in choosing appropriate words in such a discourse. All the participants in Group A described triangles and their related concepts using functional words rooted in geometric discourse. They used acceptable words in the community of mathematics learners to describe triangles. For example, Stephen (A) defined a triangle as "*a three-sided figure which is enclosed with straight lines and has three interior angles*". Similarly, Clement (A) defined triangles as "*a triangle is a figure abounded by three straight lines*". Their definitions with the overarching words 'three-sided figure' show that they have conceptualised a triangle as a polygon with three sides. The definition of a triangle from the Group B participants was similar to those in Group A.

The participants were asked to name and describe types of triangles. Stephen's (A) responses showed that he had adequate knowledge of the types of triangles, and he classified them into two categories, namely, by side and by angle. The response from one participant in group A suggested that he had little knowledge about triangle classification by angles and by sides. The most common categories mentioned by all, were equilateral, isosceles, scalene, and right-angled triangle (right triangle). A right-angled triangle was mentioned by some participants (in both groups) during further probing. For example, Jones (A) mentioned the third one by using the phrase ... "*and the last one is an equilateral triangle*". The use of such a phrase could indicate that the three mentioned were the only triangles in geometry.

An interview segment with Maxwell (A) is as follows:

Researcher: Can the types of triangles be classified into two categories?

Maxwell: *Classification of triangles? Please, I have no idea.*

He was then asked to draw the types mentioned. He drew and described them in a mathematically acceptable way, with literate word use. After drawing those mentioned (triangles by sides), he was probed further.

Researcher: Can you draw any other types of triangles?

Maxwell: ... *I have remembered one triangle. The right-angle triangle.*

He drew and said, "*It has this side [pointing to the 90° angle] to be 90°*".

Researcher: Why did you say it is a right-angled triangle?

Maxwell: ... *because one of side [moving the pen around the right-angle space] is 90°.*

It was observed from Maxwell's discourse that he used inaccurate words to designate certain geometric concepts. For example, he constantly referred to the word 'angle' as 'side' as in his discourse, eg. "*it has this side to be 90°*". It is worth mentioning here that asking him to draw the ones mentioned played a significant role. Seeing the ones drawn seemed to create some internal connections in his mind, which probably reminded him of another one, even though not all were mentioned.

The participants' responses may mean that the three types of triangles (by side) are the common ones that they may have used frequently in their geometric discourse, which could have influenced their recall, even though one may claim the right-angled triangle is also common due to its association with the Pythagoras theorem. It was observed in both Groups that the most common types of triangles in their discourse, were the triangles by sides and the right-angled triangle.

Apart from Maxwell, the concepts on triangles described by the participants were good, and they used accepted words in their discourse. The correctness of their discourses was tied to triangle descriptions by sides, which were easily articulated. Differences in word use started to emerge when they were describing the properties of triangles, and the position of the angles, particularly with isosceles triangles. Jones' (A) responses during the interview are shown in the following excerpt:

Researcher: What can you say about the angle properties in an isosceles triangle?

Jones: *For isosceles, we can say two of the angles are the same.*

Researcher: Which of the angles?

Jones: *Angles facing [opposite] the equal sides.*

Researcher: What about the scalene triangle?

Jones: *For scalene, since all the sides are different... all the angles are also different.*

Researcher: What is a line of symmetry?

Jones: *It is a line which divides a shape into two equal parts.*

Researcher: What can you say about the lines of symmetry of the shapes drawn/mentioned?

Jones: *An equilateral triangle has three lines of symmetry.*

Researcher: Okay.

Jones: *For isosceles, it will have emm... it will have only one, since ... two sides are the same, we can only have it through the point where the two [equal] lines meet. Scalene has no line of symmetry.*

Researcher: What can you say about the line of symmetry and the angle formed at the vertex by the equal sides of an isosceles triangle?

Jones: *Since the sides are the same, the line of symmetry divides the angle at the top [vertex].*

Jones' substantiating narratives concerning triangles could be attributed to his knowledge of the object. His narratives are endorsed, and match well with the content knowledge of triangles. Analysis of his response indicates that he has adequate thinking about the properties of triangles, especially isosceles triangles. The responses of Stephen and Clement were similar to that of Jones' discourse above. Data showed that three of the PSTs in Group A often made use of mathematically literate words in their discourses about triangles.

On the other hand, Maxwell (A) described the position of the angles in isosceles triangles using colloquial discourse. He described an isosceles triangle as "*a triangle in which the two base angles are equal*". The words 'base angles' are often heard in traditional geometric discourse, where in most cases, isosceles triangles are drawn with equal angles at the bottom and are commonly described as 'an isosceles triangle has base angles equal'. Many teachers sometimes place

emphasis on the ‘base angles’ in their discourse, probably because of the way the triangle is mostly drawn. Thus, the phrase ‘base angles’ is matched with the prototypic sketch of the figure and seems to be one of the most remembered properties of an isosceles triangle.

All the Group B participants also used such colloquial words as used by Maxwell (A) to describe the position of the equal angles in an isosceles triangle. Even when the isosceles triangle was presented in a different orientation from the prototype (equal angles at the base), they still referred to the equal angles as the ‘base angles’. Thus, their words used in describing the position of the equal angles in an isosceles triangle were not object driven, and hence, colloquial in nature. Such a description is not acceptable in the geometry learning community since it could possibly lead to errors in the discourse, as found in Nsiah’s (B) solution to task 3.3 (Appendix D) The discourses of the Group B participants are exemplified in the excerpt below:

Researcher: Is there any other properties of an isosceles triangle you can talk of?

Alex: *It has one line of symmetry and two angles are equal.*

Researcher: Which of the angles?

Alex: *The base angles.*

Researcher: Show me the base of this triangle.

Alex: *This angle* [points to the angles opposite the equal sides]

Similarly, Cynthia said, “*In an isosceles triangle, the base angles are equal*”, which is a representation of the discourses of Albert and Nsiah.

The Group A participants, except for Maxwell, described the exterior angle theorem of a triangle using mathematically literate and colloquial words. Those who expressed the theorem correctly used appropriate words, which are endorsed in school mathematics. This was evident in the way they expressed their thinking about the exterior angle theorem. The following excerpts show the responses of the three Group A and one Group B participants when they were asked about their understanding of the exterior angle theorem.

... the exterior angle will be equal to the sum of the two opposite interior angles (Stephen, A).

... for the exterior angle, the sum of two non-adjacent angles is equal to the exterior angle
(Nsiah, B).

Based on their responses, it is evident that they used functional and literate words to describe their thinking about the exterior angle theorem of a triangle through the word use of ‘sum’ to show the relationship between the angles.

On the other hand, Maxwell’s description was incorrect because it lacked the operation that connects the two opposite interior angles in the theorem. Maxwell stated that:

What I know is [that] if you have a triangle like this [draws] this is the exterior angle [points to the exterior angle]. So, the exterior angle is equal to two interior angles.

Mathematics is about patterns and relationships between variables; failing to specify the exact operation can be confusing and lead to errors. Connecting the angles with the word ‘and’ is not endorsed and is said to be used in a colloquial way. This omission of the operation was seen in the discourses of three of the Group B participants, except for Albert, who used an incorrect name (co-interior) for the interior angles, even though he stated the operation in his discourse. He said “*For the exterior angle theorem ... the sum of the two co-interior angles is equal to the exterior angle of a triangle*”. Cynthia and Alex provided a similar response. For example, Cynthia said [with the support of drawing] “*I know that the two opposite angles are equal to the ... exterior angle*”.

It is obvious from these participants’ responses that their word use was colloquial in nature, since they could not use explicit words to succinctly state the exterior angle theorem of a triangle. For example, Albert termed the ‘two opposite interior angles in a triangle as ‘co-interior angles’. Placing the statement of the theorem within the commognitive framework, it is claimed that word use by the Group B participants and Maxwell in Group A, in substantiating their narratives about the theorem, is not endorsed in geometric discourse and hence is an error (Sfard, 2008).

On quadrilaterals, the participants’ geometric discourse gave more insight about the words they used in describing quadrilaterals together with their properties. Further insight was gained into how the participants used words to substantiate their narratives at an object level or a meta-level.

All the participants shared their understanding of a quadrilateral, which they defined using appropriate words. Their definitions showed that a quadrilateral is a four-sided figure. A representative definition by Clement and Maxwell, (Group A), is shown in the following excerpts:

... quadrilaterals are plane figures which are bounded by four straight lines (Clement).

... quadrilaterals are four sided figures (Maxwell).

The Group B participants defined a quadrilateral in a similar manner, shown in the following excerpts:

... a quadrilateral is a four-sided figure (Cynthia).

... any plane figure that is bounded by four sides is called a quadrilateral (Nsiah).

... a quadrilateral is a four-sided figure (Albert).

The preceding excerpts show how all the participants described their understanding of a quadrilateral using mathematically literate words. The participants gave an exhaustive list of types of a quadrilateral, which they were then asked to define.

All the participants defined a parallelogram using endorsed words. They concisely defined the term parallelogram, with either the necessary and sufficient conditions (Zilkova, 2014) or with inferred properties. Of the eight, Clement, Jones and Stephen, all in Group A and Cynthia (B) defined a parallelogram as the four-sided figure which has two pairs of parallel sides. This definition is shown in the following excerpts:

... it is a quadrilateral which has two pairs of sides parallel (Jones, A).

... it is a four-sided plane figure formed from two pairs of parallel lines (Clement, A).

... a parallelogram is a polygon with opposite sides parallel to each other (Cynthia, B).

The rest of the participants defined it as follows:

... a parallelogram is a four-sided figure which has two sides and two opposite angles equal (Maxwell, A).

... a parallelogram is a type of a quadrilateral whose opposite sides are equal (Albert, B).

Varied thinking, and hence varied word use, emerged when the participants were asked to define the types of quadrilaterals. Some of the participants used mathematically literate words in their definitions whilst others used colloquial words. Two PSTs from Group A (Stephen and Clement) and Cynthia (B) provided a literate definition of a rectangle and a square. The rest of the participants provided incomplete definitions, or the definition did not show the exact description of the figure under discussion. For example, Jones (A) defined a rectangle as “... *a quadrilateral ... with opposite sides parallel.*” He also defined a square as “a quadrilateral having all sides equal.” What needs to be noted is that the definition of a rectangle matches that of a parallelogram, and that of a ‘square’ matches that of a rhombus, hence the need to mention that their angles are equal or right angles. The rest of the five participants defined these figures in a similar way. This finding supports that of Fujita and Jones (2006), who also found that most of the pre-service teachers could not define these two shapes correctly. They could not exercise any logical reasoning to exclude the shape from a more general one.

In defining the quadrilaterals, only Stephen defined a rectangle from the perspective of an inclusion in a parallelogram. Stephen’s (A) definition shows that he may have considered that some concepts or properties of a rectangle also apply to a parallelogram (de Villiers, as cited in Rianasari et al., 2016), which informed his definition along the path of class inclusion.

Not all the PSTs were able to define a rhombus with an acceptable use of words. Stephen (A) said, “*A rhombus is a parallelogram with all sides being equal*”. Cynthia provided a similar understanding when she said, “*a rhombus too is a quadrilateral with all equal sides*”. A deep reflection on their responses shows that Stephen (A) and Cynthia (B) demonstrated some knowledge of inclusion criterion in their definitions. Stephen stated that “*a rhombus is a parallelogram with ...*”, whilst Cynthia excluded a rhombus from her definition of a square when she defined it as “*a quadrilateral with all equal sides*”, but carefully defined a square as “*a square has all the sides and angles equal*”. She may have linked this definition to her previous definition of a rhombus.

The rest of the participants could not define a rhombus with concise word use. Jones (A) defined a rhombus with no restriction on membership. He said, “*a rhombus is a quadrilateral which has two pairs of sides being parallel*”. A careful consideration of Jones’ definition is more suitable for a parallelogram, for which a rhombus shares this property in definition. Hence, it should have come with the extra word-phrase ‘with all sides equal’ to make it unique or to exclude it from the larger shapes.

The results show that the PSTs have difficulty in using appropriate words to define some quadrilaterals such as rhombus, trapezium and the like. In addition, it was found that, except for Stephen, almost all the participants’ attempts to define the shapes, made no effort to consider the inclusion relation among the quadrilaterals. This finding supports that of Baktemur et al. (2021), who also found, in their study, that the participating teachers described the types of quadrilaterals without any attention to the inclusion criteria among the shapes.

Maxwell’s (A) word use in defining several types of quadrilaterals could not be fully analysed because, at some point in time, he said he had forgotten the definitions of some of the shapes that were central to our discussion. Maxwell was able to express some properties of a parallelogram using appropriate words but could not transfer such thinking to other related types. For example, Maxwell previously described a parallelogram as having opposite sides of equal length and the diagonals bisect each other. When asked to define a rhombus, he responded, “*I know the figure but I have forgotten the definition. I know it is like a square. All the sides are equal, but I have forgotten the definition*”. This suggests that even though he may have seen the shape in his mind, he may not have fully internalised the characteristic features of it, in which case, he was unable to use appropriate words to describe it in an object-driven way.

Understanding circles is concerned, not only with the ability to solve complex tasks, but also with the ability to understand and define their fundamental concepts and related properties. Knowledge of such concepts gives learners a solid foundation upon which higher content knowledge is developed (Alex & Mammen, 2018; Atebe & Shafer, 2010). As part of analysing the participants’ geometric thinking, they were asked to share their understanding of a circle and parts of a circle as

fundamental concepts to learning circle theorems. The following paragraphs show how the participants used words to describe different ideas and some of a circle's properties.

All the participants in Group A and two in Group B demonstrated good thinking about the concepts of circles, based on their use of acceptable words to describe the basic concepts of circles. The definitions contained the essential features of a circle, in which all the points on the circle are equidistant from a fixed point known as the centre. The definitions of these six participants are given in the following excerpts.

... a circle is defined as a locus of points that is equidistant from a given fixed point, which is the centre (Clement, A).

... a circle is a set of all points in a plane that have a fixed distance, called the radius, from the centre of the circle (Jones, A).

... it is a round plane figure in which all points on the boundary are equidistant from a fixed point (Maxwell, A).

... a circle is a locus of points equidistant from a fixed point called the centre (Stephen, A).

... a circle is a plane figure in which all the points on the boundary are of equal distance from the centre of the circle (Nsiah, B).

... a circle is a set of points in a plane where all the points on the circle have an equal distance from the centre (Alex, B).

Analysis of the above excerpts in defining a circle shows that their word uses are mathematically literate. This suggests that they have developed the competencies to use appropriate words to describe the salient features in defining a circle, which reads “... *set of all points ... fixed distance ... from the centre*”. The use of such words is accepted in the mathematics curriculum, which endorses their word use and is also object-driven. The results show that six of the participants demonstrated the competence to define a circle with appropriate word use. A similar finding was reported in a study conducted by Wright (2013) in which the majority (79%) of the participating pre-service teachers, provided a formal definition of a circle. Within the commognitive theory, these are mathematically literate ways to use words that are correct and accepted (Sfard, 2008).

The rest of the two Group B participants' words were not object-driven. Their definitions lacked some key words that express the features of a circle. Albert (B) had an idea of a circle but struggled with words to express his thoughts. Having realised that his first definition was not clear, he said, *"If I should come again, it is a point or any point at a rounded shape ... a point found at a rounded shape to a fixed point at the centre"*. Cynthia (B) also said *"A circle is a closed, two-dimensional curve"*. When she was asked to repeat her definition, she said, *"I can define it as a round object with no corners"*. Describing a circle as a closed two-dimensional curve is incorrect (Wright, 2013). The words used by these two Group B participants are not endorsed in geometric discourse, and hence are colloquial in nature (Sfard, 2008).

The participants were further engaged about the various parts of a circle, which forms the basis for successfully learning a circle theorem (Jamhari & Wongkia, 2018; Ntow & Hissan, 2021). Mathematics, particularly geometry, deals with some special basic terms that are essential for communicating one's understanding of the object. Understanding arises in a situation where both the learner and the teacher share a common meaning of basic words in the discourse. Learners' understanding of the language (terms) of mathematics is necessary for their sense-making in the discourse (Anthony & Walshaw, 2009).

Analysis of the participants' word use in defining parts of the circle shows that they used endorsed mathematical words in their discourse, except for a few responses which were questionable. Among the parts of a circle that were defined by the participants with appropriate word use, were circumference, arc, and radius. All the participants defined a circumference as the distance around a circle. For example, in defining a circumference, Cynthia stated, *"A circumference is the distance around the circle, or the perimeter of a circle"*. Similarly, all the participants used appropriate words to express the radius of a circle. What is significant to all the definitions provided by the participants, was the use of the words 'centre and circumference' in their definitions.

Analysis of the participants' discourses on circumference, radius and arcs, show that they have used the appropriate words to communicate their thinking about these geometric concepts. Atebe and Schafer (2010) assert that language (terms, word use, and terminologies) is an important tool in communication. This means that the PSTs could define and explain these parts of a circle to the

learners. Thus, literate words were used to define some parts of a circle. Definitions form the basic unit of a discourse (Leikin & Zazkis, 2010), and it is important that learners' geometric thinking is developed on their ability to define some key basic concepts of the discourse, to enhance later learning. Within the context of commognitive theory, they are said to express their thoughts on these concepts using mathematically literate words. These words used when learning circles, contribute to a distinct discourse (Sfard, 2008).

The word diameter, received different definitions from the participants. Seven of them defined it using appropriate words that indicate the specific spatial concept accepted in geometric discourse. Two representative definitions of the diameter from the Group A participants are exemplified in the following excerpts:

... it is a line that is being drawn to pass through the centre of the circle to touch two parts of the circumference (Clement).

... a diameter is a line that has its two ends touching the circumference of the circle and passes through the centre of the circle (Stephen).

Similar representative definitions of the Group B participants are also shown below.

... a diameter is a line drawn through the centre ... to the circumference. The endpoints of the line touch the circumference (Albert).

... a diameter is the line which divides the circle into two equal parts, and it is drawn through the centre of the circle (Cynthia).

It is evident from the preceding excerpts that identifying a straight line drawn through the centre with its endpoints on the circumference of the circle, qualifies it to be a diameter. Thus, these representative definitions of the seven participants show endorsed word use in their discourse. It means that their word use in the communication framework is said to be mathematically literate (Sfard, 2008).

On the other hand, Maxwell's (A) word use was open to different interpretations. He used the word 'divide' in his definition of a diameter and connected it to the words 'equal parts'. He said, "*a diameter is a line that divides a circle into equal parts*". Not being convinced about his definition,

he was asked to draw it. There was no indication in the drawing that the straight line passes through the center of the circle. This shows Maxwell's understanding of diameter is not linked with the centre of the circle. A similar finding was reported by Mudaly (2021) in which a chord with no indication of it passing through the centre, was considered as a diameter by the participants. The use of the word 'divide' to define a diameter is a word used in a colloquial way.

The participants' discourses on chord, the centre of a circle, and tangent, were also analysed. Analysis of their responses showed that the Group A participants, except for Maxwell, used appropriate words to define the chord as a straight line whose endpoints lie on the circumference of a circle.

Four of the participants, two from each Group, used words in the definition of a tangent that showed that they clearly understood the concept. For example, when talking about the tangent, Jones said, "*a tangent is a straight line which touches a circle at only one point*". They added the accepted condition 'touches at only one point' as found in the literature (Musser, et al., 2014; Smith, 2012). This definition is accepted and contains a literate use of words. On the other hand, three participants defined a tangent with similar words but lacked the condition of 'touching at only one point'. For example, Clement (A) defined a tangent as "*a line that is being drawn to touch the circumference of a circle*". Also, Albert (B), whose definition is similar to that of Cynthia, stated that "*a tangent is a line, or a straight line drawn to touch a circle*". They used the word 'touch' and connected it to a literate meaning. Analysis shows that the three used insufficient words to define a tangent. Within the commognitive framework, these three participants are said to use colloquial words in their discourse (Sfard, 2007, 2008).

As shown in the following transcript, Maxwell (A) was able to identify a tangent from a diagram but could not describe it in his own words.

Researcher: What is the name of the line FG?

Maxwell: *It is a tangent to a circle.*

Researcher: Ok, what is a tangent of a circle?

Maxwell: *These definitions ... [pause] I know the name, but the definition I have forgotten.*

Maxwell's word use in describing diameter was found to contain a colloquial discourse. It is therefore not surprising that in the question posed to him on what a tangent is, he started with "*these definitions ...*". This suggests that he may have some geometric terminologies in his knowledge repertoire but may lack the competence to define them using appropriate words. According to Kemp and Vidakovic (2021), many expectations of learners, who enrol in higher-level courses in mathematics at a college or university, are not met due to their difficulties in completing tasks involving definitions of mathematical concepts. This is happening even though many of them have been successful in taking advanced mathematics courses in their programmes of study. This difficulty was seen in the discourses of some of the PSTs, especially Maxwell (A), who performed quite well on the geometry tasks, but seemed to lack the ability to define certain geometric concepts.

The participants were asked to talk about their understanding of a cyclic quadrilateral. Stephen and Jones described the cyclic quadrilateral using appropriate words that are endorsed in geometric discourse. In this case, they used mathematically literate words in their discourse. The response on cyclic quadrilaterals by Jones and Stephen is as follows:

... a cyclic quadrilateral is an inscription of a quadrilateral in a circle where the vertices of the quadrilateral touch the circumference of the circle (Jones, A).

... a cyclic quadrilateral is a four-sided figure within a circle in which it has four vertices touching the circumference of the circle (Stephen, A).

Maxwell's (A) words in defining quadrilaterals could not be analysed because he could not use any appropriate words to define a cyclic quadrilateral. Zilkova (2014) asserts that many future (pre-service) teachers have insufficient knowledge of geometry and continues to say that their major difficulties are defining 2D shapes. Maxwell was one of the participants in Group A who had demonstrated some difficulties in defining many of the 2D shapes and other geometric concepts used in this study. A previous question posed to him to define a line he correctly named as a tangent, yielded the response "*these definitions ...*". Then, after some silence, he said, "*I know*

the name, but the definition I have forgotten”. A similar response was obtained when he was asked to define a cyclic quadrilateral shown in the following transcript.

Researcher: What is a cyclic quadrilateral?

Maxwell: *A cyclic quadrilateral hmm ... I can draw it but the definition no ...*

Maxwell’s definitions of geometric concepts suggest that he is deficient in defining geometric concepts. Definitions form a critical component of Euclidean geometry and a starting point for teachers (and learners) to develop sound knowledge for teaching related aspects of the secondary school mathematics curriculum (Guner & Gutlen, 2016; Speer et al., 2015). Definitions serve as the foundation for logically determining geometric properties, as well as a means of guiding learners in the identification and classification of geometric objects (Ndlovu, 2014). This implies that knowledge of defining geometric objects forms a basis on which other concepts can be learned. PSTs’ acquisition of the ability to define geometric concepts and terminologies, is critical for the development of their geometric competencies necessary for effective teaching. Being good at geometry, one needs to know its concept definitions and have extensive knowledge and understanding of their properties (Maier & Benz, 2014). Generally, the discourses of the Group A participants contained more literate words than those in Group B.

5.4.2 Word use in describing solution to the tasks

Sfard’s (2008) notion of word use suggests that learners, especially PSTs, should be able to use words with literate meaning to describe specific ideas. If words are not used in a literate context, it allows for several interpretations by learners during communication. Learners can communicate their ideas in geometric discourse only when they have acquired the correct words and terms in the discourse (Atebe & Schafer, 2010). This notion forms one of the general aims of teaching mathematics in Ghana, which states that the mathematics syllabus is designed to enable learners to “communicate effectively using mathematical terms” (MoE, 2010, p. ii). This means that teachers should be able to acquire and use appropriate terms (keywords) in teaching mathematics as well as geometry, which is noted to use more technical terms than any other discipline in the mathematics curriculum (Ashfiel & Prestage, 2006). This formed the rationale for analysing the PSTs’ word use in describing their solutions to the geometric tasks through the commognitive lens.

This section describes the word use by the PSTs in communicating their thinking and solution strategies to the geometric tasks.

The first task that the PSTs had to solve was to find a missing angle in the set of adjacent angles, formed on a straight line. The focus was to analyse the words used in stating the properties that informed their solutions, and the procedures followed.

All the eight PSTs correctly solved the task and explained it in a similar way. Analysis of their discourses showed that they used both literate and colloquial words. For example, Stephen (A) said:

Looking at the diagram ... we have three angles all on the straight line. All the three angles at this (pointed to it) vertex ... on a straight line ... sum up to 180° .

Similarly, Jones (A) explained that:

... angle x and this angle 90° and ... 64° ... all lie on a straight line. So, the three angles add up to 180° .

A reflection of the preceding excerpts shows that they have used words that convey the intended meaning and are acceptable in geometric discourse. For example, Stephen used the words, “*all on the straight line*”. He also indicated that all the angles share a common vertex. Jones also substantiated his routine with similar word use. This could also mean that when a straight line is filled with angles, then their sum could be taken as 180° . Similar word use was found in six of the PSTs’ discourses. For example, Nsiah (B) said, “... *the sum of all these angles on a straight line is 180°* ”. Five participants (3 in Group A and 2 in Group B) fall under this category. Within the communicative framework, their word use is said to be mathematically literate because they depict the exact intent of the property (Sfard, 2008).

Three of the participants substantiated their routines or approaches by using incomplete words in stating the property. For example, Maxwell (A) used the words “... *angles on a straight line add up to 180°* ”. This description contains incomplete information in geometric discourse and is classified as colloquial word use. Maxwell connected his colloquial word use with literate discourse in explaining his thinking. What needs to be highlighted is that the omission of the word

‘adjacent’ leaves the property open to different interpretations by learners, hence, could lead to an error in devising solutions to related tasks, as was found by Ngirishi and Bansilal (2019). The authors noticed that one of the study participants added two non-adjacent angles on a straight line and equated it to 180° , with the reason that s/he remembers one property which states “angles on a straight line add up to 180° ” (Ngirishi & Bansilal, 2019, p. 8). This misconception could be attributed to the omission of the word ‘adjacent’ which sends a signal of a common ray or same vertex of the angles. This suggests that the property should be stated by emphasising the word ‘... adjacent angles ...’ (Greer, 1979). Cynthia said, “*all straight angles sum up to 180°* ”. In the same way, Albert said “*a straight line angle ... add up to 180°* ”. Maxwell, Albert and Cynthia used words in a colloquial way as guided by Sfard’s (2008) theory.

Tasks 1.2 and 1.3 were similar in design and required the PSTs to reflect on and devise solutions to them. Task 1.2 was on angles associated with the intersection of two straight lines, and task 1.3 contained two parallel lines intersected by a transversal, both of which required similar approaches, and hence similar thinking and word use in their routine.

In explaining their solutions, six of them, all from Group A and 2 from Group B, produced words on the contextual structure of the task, using the appropriate words in naming the property associated with the task. Their word use was based on object-level narratives. Those who produced the properties associated with the task design made use of acceptable keywords in their discourses. The following excerpts show how Stephen expressed his thinking in task 1.3.

Researcher: Explain how you devised a plan to solve task 1.3.

Stephen: ... *in the diagram, we know that we have two parallel lines and a transversal.*

Researcher: Why did you say the lines are parallel?

Stephen: ... *lines are parallel because of the symbols* [point to the arrows on the line].

Stephen: ... *with the transversal line drawn through the two parallel lines ... we have new angle properties introduced. I also realised that the same variables were used in the angles that are opposite to each other. So, with this vertically opposite angle...*

Stephen's use of the geometric properties associated with the parallel lines in planning to devise a solution to the task, was similar to the discourses of the remainder in Group A and the two in Group B. Their ability to use these words to describe the geometric properties, is evidence of connective thinking in geometric discourse. For example, Jones said " $3(x - 20)$ is vertically opposite to $2x$ ". Because they substantiated their interpretation with appropriate words to describe the properties involved in solving the task, they are mathematically literate. Lefrida et al. (2021) claim that the identification of properties related to geometric tasks, falls under the summarising category of keywords in a discourse. Thus, the PSTs should be able to identify the appropriate names of the properties they wish to use in their preferred solution approach.

Cynthia (B) used both mathematical literate and colloquial words to name the geometric properties connecting the two identified angles. She also named the straight line drawn to intersect the two parallel lines, as the diagonal. Her solution routine however, yielded the correct result. This could mean that Cynthia may have developed the routine of solving the geometric task but may be deficient in word use, in naming and describing some geometric concepts. This deficiency is exemplified in the excerpt below.

Cynthia: *We are having direct opposite angles, which are $3(x - 20)$ and $2x$.*

Researcher: You mentioned alternate angles in your plans. What are they?

Cynthia: *When angles alternate, that means you see the position of the angles with a line that looks like a Z.*

Researcher: Z?

Cynthia: *Yeah, when you have the Z symbol, it means the angle here and this... alternate.* [points to the angles in the corners of the Z symbol].

Researcher: What are co-interior angles?

Cynthia: *Co-interior angles are angles... [pause]. They are angles within the lines. That means this angle and this angle, they are within, that is interior.* [draws to explain what interior angles are].

Researcher: What is the name of the line drawn across the two parallel lines?

Cynthia: *It can be a diagonal.* [pause] *... I always call it a diagonal.*

The preceding excerpts show that Cynthia had difficulty in using the correct words to describe the spatial geometric ideas explored. This difficulty in describing the angle properties of parallel lines was also found in Albert's discourse, where the properties were described with colloquial words.

Researcher: What are co-interior angles?

Albert: *They... [pause]...they are interior angles whose sum add up to 180°.*

Researcher: What is your understanding of alternate angles?

Albert: *... are angle you realise that one angle is exterior and one angle will be interior but they are marked at one transversal.*

These responses show that Albert and Cynthia, both in Group B, have some deficiencies in the use of appropriate words to describe these geometric concepts. This difficulty was also shown in Albert's discourse on alternate angles. He used exterior and interior to mean angles on opposite sides of a transversal. He could also not describe what co-interior angles are. These participants were able to produce associated properties to solve the task. However, analysing their thinking about word use in describing these geometric concepts showed that they only know the alternate angles by their position and not by definition or description. This could mean that Albert and Cynthia, being trained as future teachers, have difficulty in using acceptable words to explain these geometric properties and concepts. In a study by Ngirishi and Bansilal (2019, p. 8), a participant recounted that when they were taught geometry (viz parallel lines), "the terms were never explained" to them, and that they were just told which angles were equal, with their appropriate names. Placing this within the commognitive theory, they are said to use words in a colloquial way.

When analysing the participants' thought processes about triangles, attention was devoted to their word use, together with their discursive actions. Almost all the tasks on triangles had different solution approaches for which the PSTs were required to demonstrate knowledge of, by identifying the name of the geometric shapes and their related properties, needed to devise a solution on a preferred approach. The focus is to determine how the PSTs use and process words in their discourse.

The following sections include the participants' discourses on triangles. They all solved and explained task 2.1, making use of appropriate words governing the task, in the preferred routine. An interview with Maxwell is shown below.

Researcher: Tell me about how you organised your thinking when solving for m and n in task 2.1?

Maxwell: ... *know that these are interior angles of a triangle* ...

Researcher: Why did you say a triangle?

Maxwell: *I can see the figure here [points to $\triangle ABE$] is bounded by three straight lines.*

Maxwell: ... *and here [pointing to 62°] is equal to angle x , vertically opposite angles.... I applied the exterior angle theorem.*

Stephen also used words in a more literate way with endorsed narratives that matched with visual properties and associated narratives. Stephen's explanation in shown is the following extract:

... I first considered my triangle AEB ... since I know the interior angles of a triangle add up to 180° . To find m , I take my triangle BFC, I realise ... m is exterior angle two opposite interior angles for my angle m are 75° and B . To find B , I used the property of vertically opposite to angle n ... I can call my angle, say t [referring to the angle found at B].

Maxwell and Stephen's discourses showed that they used words that are acceptable in geometry. They first identified the name of the geometric figure to determine the suitable properties required to devise the solution. For example, Maxwell named the figure as a triangle and substantiated that "...the figure ... is bounded by three straight lines". Stephen, upon identifying the name of the shape, said, "*I know the interior angles of a triangle add up to 180°* ". All the participants demonstrated knowledge of word use and properties, as shown in the discourses of Maxwell and Stephen, both in Group A. They all used similar words in explaining how they solved for n in the task. For example, Cynthia (B) explained that:

... the diagram here is a triangle...in all triangles, the interior angles sum up to 180° ... we come to the other side of the triangle too. Let's assume we have 'a' here as the angle... n and a are vertically opposite so they are the same. ...

Differences in word use emerged depending on the approach one preferred, to solve for the value of m . Words used to describe their solutions were found to be both literate and colloquial. Those whose word use was based on the structure of the task, used them in an object-driven way. The PSTs used similar wording and approaches to solve the remaining triangle tasks. Some stated the exterior angle theorem of a triangle, without the arithmetic operation that connects the two interior angles, as the theorem demands. The omission of the arithmetic operation makes the statement of the theorem incomplete and hence colloquially expressed.

Task 2.3 required thinking that involves the use of any two of the following: (1) the exterior angle theorem of a triangle, (2) the interior angle sum of a triangle, or (3) the straight line angle, to devise a preferred solution. All the participants, except Clement (A), used words governing the task structure based on their visual, informed thinking, governing the design of the task. The words used by each of the PST's depended on the internalisation process that he or she engaged in. These internal processes are what commognition is all about. Engaging in thinking (individual cognition) to determine which way to go (Sfard, 2008). Those whose thinking was guided by the visual analysis and inclusion of the exterior angle, probably first thought of using words associated with the exterior angle theorem before thinking of any other properties useful for the task.

On the other hand, Clement (A) used equally accepted words associated with the use of the two properties, which are the interior angle properties of triangles and adjacent angles on straight lines. His individualised collective patterned way of thinking (Sfard, 2008), resulted in the use of two simultaneous approaches in which he demonstrated good algebraic thinking in terms of the words used in explaining his approach. Clement's discourse is shown in the following extract:

... the sum of the interior angles in that triangle is 180° , so ... (1). Then I look at the straight line on which we can find the angle $2x$ and m being adjacent angles on it. From (1) which

is 68° plus m plus x equals 180° , when I subtract 68° from both sides, I get ... I then solved to get the answers ...

The preceding excerpt shows that Clement used appropriate words in his discourse. In a similar previous task on the straight line, he correctly stated the straight line angle property by using the endorsed words ‘adjacent angles’. This is also seen in the preceding extract when he said, “*angles $2x$ and m [are] adjacent ...*” He showed knowledge of the sequential algebraic steps needed to solve the task, with correct use of words. Literature shows that teachers must possess functional knowledge of appropriate words about mathematical objects, to engage future learners in mathematical discourse using literate words (Atebe & Schafer, 2010; Berger, 2013; Roberts & LE Roux, 2019). From a commognitive perspective, Clement is also said to use literate words in explaining his thinking about solving the task (Sfard, 2008).

Regarding quadrilaterals, task 3.1 required the participants to use appropriate wording in the figure identification and to use its associated properties in devising a solution to the task. Task 3.1 showed no indication of parallelism on any of its opposite sides. Six participants (all Group A’s and two from Group B) noticed this and hence treated the intersections of the side lines separately. Their word use was centred on adjacent angles on a straight line and the emerging angles concerned with the intersection of two straight lines.

A demonstration of representative thinking among the six participants is from the discourse of Clement. His word use in expressing his thinking about how he planned to solve the task, is shown in the excerpt below.

Researcher: Okay, talk to me about how you organised your thought processes to devise a strategy to solve the task.

Clement: *First of all ... angle QRM should be equal to angle GRS and is vertically opposite angles. I further find angle QLM ... adjacent angles. So, this side is a quadrilateral $QLMR$. I add all ... angles ... it sums up to 360° .*

Clement’s (A) solution plan, which reflects **that of those in** Group A participants, and two in Group B, illustrates the functional use of words in describing their thinking processes. This shows that

the identification of geometric figures in a task, and thinking about their associated properties, cannot be left out when developing geometric thinking. The structure of the task, which is a quadrilateral with the sides (lines) extended at the vertices, informed the participants' thinking about the use of properties of angles on a straight line and vertically opposite angles. As indicated in his discourse, "*angle QRM should be equal to angle GRS ... [because] of vertically opposite angles*". Analysis of the word use of the six participants shows that they have used words that are endorsed about the properties... used in their discourse. Their word use indicate that they have developed good thinking and can express such ideas in meaningful ways. They have used words in a more mathematical way in explaining the solution of the task. Alex and Mammen (2018) assert that the use of appropriate words and terminologies in geometric discourse, is necessary to avoid any possible misconceptions. According to Sfard (2008), learners' interest in participating in mathematics discourse would be enhanced when literate words form a specialised role in communication and they are also guided and corrected in their usage. Guiding learners' word use in a discourse plays a significant role in developing their autonomy in mathematising (Ben-Zvi & Sfard, 2007; Sfard, 2008). Thus, the PSTs who frequently used endorsed words have developed the competence to engage learners in mathematics discourse using literate words.

Albert (B), on the other hand, used a combination of colloquial and mathematical words. He initially started by using literate mathematical words based on correct use of geometric properties identified between the angles. He however made an incorrect assumption about a property connecting the angle to be found, and another one in its position that appeared similar without any mathematical support. Albert's discourse is exemplified in the excerpt below.

Researcher: Can you explain your solution to me?

Albert: *The angle R [referring to QRM] is 103° . That is vertically opposite angles. Angle U [referring to angle QLM] is ... 57° . So having known angle U, then I can say that angle x is corresponding angle to angle KLF.*

Researcher: Can you explain it again with the informing properties?

Albert: *The x, I used the property of corresponding angles ... I realised that where the x is placed and where I am having my 123° , they are corresponding angles.*

Albert started well with the step-by-step analysis of the geometric figure by assigning literate words to geometric properties of related angles. He then made a wrong assumption that the placement of the unknown angle x and angle 123° were in relative positions of the transversal to the straight lines. This wrong assumption, leading to wrong word use, was made even though there was no indication that the two lines are parallel. He repeated this wrong assumption with confidence when a question was posed to go over the informing properties.

Alex (B) also made that same wrong assumption and used the wrong word use in geometric property. On his worksheet, he just stated that x is equal to 123° . A question was asked, to justify his decision. He responded, “... *that one is corresponding to 123° here* [pointing to angle 123°]”.

Task 3.2, also on quadrilaterals with arrows on both sides, required the participants to identify the geometric construction of the task, assign it the appropriate name (parallelogram) and use it to identify the properties in devising strategies to solve for the variables. All the angles involved were represented by variables and hence, required critical thinking to devise the solution. This task was found to pose some challenges to two of the Group B participants, Alex and Nsiah. Alex was able to identify the figure as a parallelogram, but he was unable produce appropriate properties to guide his solution. This will be discussed in detail in the next chapter.

The Group A participants and two from Group B engaged in diverse thinking, which resulted in different properties emerging to guide their solution. All the properties raised were accepted and hence, endorsed in geometric discourse. These properties were raised because of interpreting the salient properties of the task. The following excerpt shows the discourse of Jones and Maxwell in their interpretation and word use.

Jones: ... *m and y are alternate angles.*

Researcher: Why did you say that?

Jones: ... *lie on opposite sides of the transversal and are within the parallel lines. Also, PT and QR are parallel ... considered $3x$ plus m ... two consecutive angles in quadrilateral sum up to 180° $2x$ and m are opposite angles ... $2x$ corresponds to the y .*

Maxwell: ... *this is parallel to this* [point to the longer side] *and this is also parallel to that*

[pointing to the shorter sides] *and in parallelograms, opposite angles are equal*

Those who were able to solve the task used appropriate words about the governing properties of the parallelogram. Success and proficient geometric thinking rely on the ability to identify the properties related to the task's structure, through which geometric properties are connected to the visual properties of the tasks (Matlen et al., 2018; Ndlovu, 2014). The successful participants seemed to produce properties based on how they individualised and interpreted the task.

Tasks on circles provided an opportunity to analyse the participants' word use in explaining their solutions. All the participants, at the stage of devising a solution to the task, used appropriate words governing the construction of the task.

Analysis of the excerpt below shows that Stephen's (A) understanding of devising strategies to solve the task, was based on his knowledge of the words used in identifying the names of the parts of the circle and their usage in stating the circle theorems governing task 4.1. An interview with Stephen is described in the following extract:

Researcher: Can you explain how you solved the task to me?

Stephen: *When I look at the diagram, I have an arc PR.*

Researcher: Ok.

Stephen: *This arc is subtending an angle at the circumference and another at the centre. If I should know the angle measure the arc PR subtend at the centre ... I can find the angle measure for the angle it subtends at the circumference.*

Researcher: Ok.

Stephen: *... when an arc subtends an angle at the centre and also at the circumference ...*

Researcher: Yes, proceed.

Stephen: *And looking at the quadrilateral PQRS, it is a cyclic quadrilateral ... Because the vertices of figure PQRS touches the circumference of the circle.*

It could be seen from the above excerpts that Stephen made use of the appropriate words to describe the task. He used the word 'arc' and connected it to the angle subtended at the centre of the circle and the angle at the circumference. In his discourse, he said, "*this arc is subtending an*

angle at the circumference and another at the centre”. He then used appropriate words to state the circle theorem governing the relationship between the two angles. Stephen used the precise geometric terms in his discourse about the task.

Jones demonstrated similar thinking with words used in a mathematical way. His use of words in devising a solution to the task was endorsed in geometric discourses. In such discourses, he has used words that are mathematically literate. An extract that follows shows Jones’ responses in an interview session.

Researcher: Can you help me understand your solution?

Jones: ... *an angle is subtended by an arc PR at the centre ... also subtended at the circumference.*

Researcher: Ok.

Jones: ... *this one ... is a quadrilateral because it has four sides and is cyclic.*

Researcher: Why?

Jones: *All the vertices touch the circle. So, I can apply cyclic quadrilateral theorem on x and y which states that opposite angles are supplementary.*

The discourses of Stephen and Jones are representative of the discourses of all the participants about task 4.1. The tasks on circles that received appropriate word use were tasks 4.1 and 4.3 (see Appendix D). They used appropriate words to name the parts of a circle and state the theorem and properties used in their solution strategies. These were the only two tasks where all the participants were able to express their solution strategies with correct use. Details of their solution are discussed in the next chapter. Analysis of their word use is classified as mathematically literate within the Sfard’s (2008) framework. Their words were specific and were communicated in a clearer and more understandable way in describing the specific spatial ideas and relationships they intended.

The more capable ones used the same geometric words (names of circle parts) in their discourses about the other circle tasks. For example, in task 4.2, all the Group A participants and Albert in Group B, used associated and accepted words in their discourse. This was evident in the discourses of the five participants. For example, an interview with Maxwell is shown in the extract below.

Maxwell: *This is a triangle. That is LOM.*

Researcher: Ok.

Maxwell: *And this is the centre which means that this triangle is isosceles triangle.*

Researcher: Why is it an isosceles triangle?

Maxwell: *Because, any line from the centre to the circumference is a radius.*

Researcher: Ok proceed.

Maxwell: *So, from O to L is a radius and from O to M is also a radius, and the two lines are the same. It means we are talking about an isosceles triangle.*

Similar to Maxwell's discourse, the participants expressed their thinking using literate mathematical words to describe the spatial concepts involved. The successful participants had developed a good plan about how to solve the task. The identification of the shapes and their connections probably enhanced their ability to identify the associated properties necessary to devise an appropriate plan to solve the task. Developing a high sense of problem-solving ability in geometry, and mathematics in general, requires a connection between the unknown and how it relates to the given data (Ortiz, 2016). Although this connection is seen in the participants' discourses, it is mostly found in the discourses of Stephen and Jones. As PSTs are being trained to develop expertise on how to guide learners' knowledge construction and develop good thinking processes about the subject matter, it is important that they themselves have developed deep and flexible thinking about topics in the mathematics curriculum to enhance their teaching effectiveness, particularly, in providing a reason for action (Sfard, 2008). In general, the PSTs used both literate and colloquial words in their discourses. There were more literate words in the discourses of those in Group A compared to their counterparts in Group B. This finding supports that of Berger (2013) and Wang (2013) who also found that learners' discourses showed evidence of words used in both literate and colloquial ways.

5.5 Visual mediator

Along with words, many tasks in geometry come with some visual mediators, or prompts, that communicate certain important, distinctive features to learners. These visual mediators help learners to make meaning of such tasks. The visual mediators are often in the form of graphs,

tables, diagrams, concrete materials, symbols, icons, etc. (Sfard, 2007, 2008). In this study, focus was placed on those that are associated with geometric tasks. Hence, this study is interested in analysing how PSTs identify and process these mediators associated with geometric concepts. In other words, it was deemed necessary to analyse how the participants' meaning-making of geometric tasks was informed by their visual abilities, in addition to their use of diagrams in their discourse. The next section discusses the PST's use of diagrams to explain their thinking about some geometric concepts.

5.5.1 Drawing of geometric concepts/shapes

Drawing in geometric education serves as a medium to understand learners' representation of geometric concepts. It is a means to gain insights into learners' geometric thinking ability (Thom & McGarvey, 2015). The act of drawing provides learners with the opportunity to become aware of geometric ideas, concepts, and their relationships. According to Thom and McGarvey (2015), learners' acts of drawing and their processes of attending to drawing, contribute immensely to the ways they understand geometric concepts, and communicate their thinking to others (Sfard, 2008).

It was observed that some of the participants drew diagrams of the concepts being explored before trying to define them. This shows that diagrams play a significant role in the participants' thinking processes. In this way, the participants seemed to have attached their defining abilities to what their eyes saw and perceived. For example, when Stephen (A) was asked if he could define or describe an angle in another way, he was seen making a sketch (see Figure 5.1) on the paper to explain the idea.

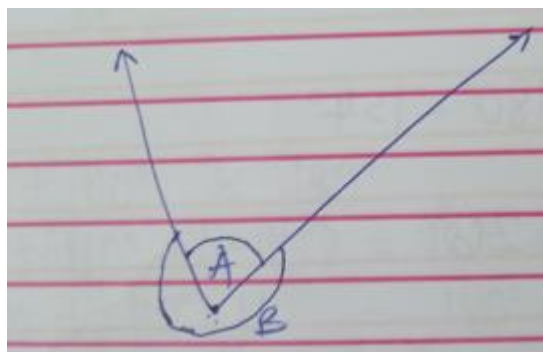


Figure 5.1: Stephen's sketch of an angle as a turn.

After the drawing, he explained that:

When we take a ray ... when it is rotated from its starting point to the terminal point, the space found ... between the rays is called an angle.

Based on the sketch, he connected two angles formed at the vertex of the rays, which he labelled with the letters A and B.

This result showed that the diagram served as a visual representation of what he had in mind. The diagram probably enabled him to observe the important information, he needed to express with words. In this regard, the diagram could be considered a powerful tool for thinking. This supports the finding of Rizwan et al. (2018) that those who learned through diagrams explained their geometric ideas in a precise way. It also supports the claim by Brizuela and Gravel (2013) that visual representation (drawing or diagram) is a way to capture and make meaning as well as to interpret an idea or a phenomenon. Visual representation provides an opportunity to support one's thinking and increases the ability to process information (Lowrie, 2020). This was evident in Stephen's action of drawing before explaining, which shows that his thinking was supported by the representation (diagram drawn).

Similarly, Cynthia (B), Jones (A) and Maxwell (A) explained their understanding of the exterior angle theorem of a triangle with the support of drawings, even when they had not been asked to do so. In their attempt to define, they drew diagrams to guide their definition. For example, Maxwell stated that:

What I know is [that] if you have a triangle like this [draws] this is the exterior angle [point at the angle]. So, the exterior angle is equal to [the sum of] two [opposite] interior angles.

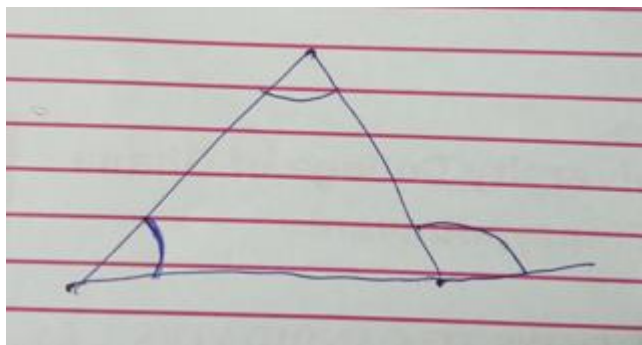


Figure 5.2: Maxwell's diagram to help define the exterior angle theorem of a triangle.

Maxwell (A), upon making the drawing, pointed to the corners marked by the arcs to explain which ones are the exterior, and the interior angles, in the theorem.

In the same way, Jones (A) provided a representation of the theorem as shown in Figure 5.3

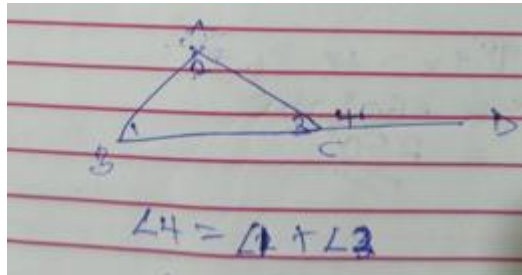


Figure 5.3: Jones' representation of the exterior angle theorem of a triangle

In defining the theorem with the aid of the diagram, he said:

... the exterior angle theorem states that when [draws] you sum two opposite interior angles in a triangle, it is equal to its exterior angle (Jones).

Figures 5.2 and 5.3 show how some of the PSTs used diagrams to support what they wanted to say about the theorem. According to Hasanah et al. (2019), the goal of teaching mathematics is to equip learners with thinking and problem-solving skills and to develop their ability to communicate mathematical ideas or convey information through [...] diagrams.

Another observation was that the PSTs used diagrams to emphasise what they had said. They emphasised what they had communicated orally using visual representation to convey the intended information. It could also mean that these PSTs may have thought they could not do much with words, hence the attempt to draw to put emphasis on some aspects of key concepts they mentioned in their communication. For example, Alex (B) first defined a cyclic quadrilateral as “a four-sided figure within a circle”. When he was asked to redefine, he drew it as shown in Figure 5.4 and said, “This is what I mean”. He pointed to where the vertices of the quadrilateral touched the circumference of the circle.

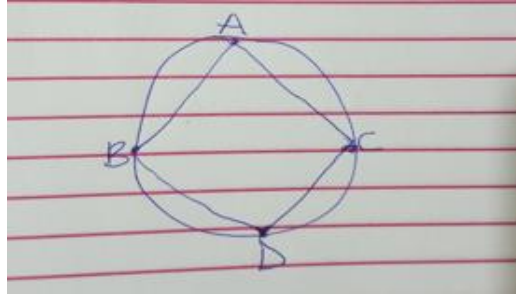


Figure 5.4: A drawing produced by Alex to define a cyclic quadrilateral.

He then redefined it as “... a four-sided figure which has its vertices touching the circle”. He may have embraced the notion that a diagram is worth a thousand words (Jones, 2013). It is also seen that the use of the diagram helped him to use the appropriate geometric terms in his discourse. According to Sfard (2008), the use of drawings helps learners to keep their discourse focused.

Attention is paid to the use of diagrams for diverse reasons. Some of the participants, who found it difficult to provide word-descriptions of some geometric figures and concepts, resorted to drawing as a means of explanation. For example, Maxwell (A), when asked to define a cyclic quadrilateral, responded that “... I can draw it but the definition noo ...” Maxwell’s sketch is shown in Figure 5.5.

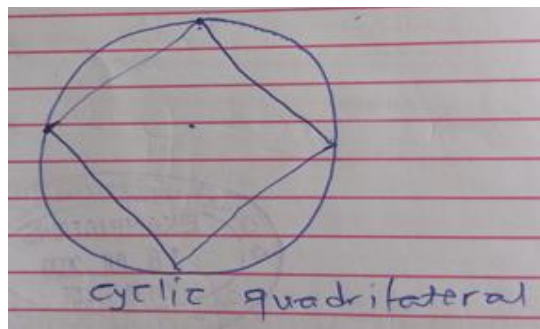


Figure 5.5: Maxwell’s sketch for a cyclic quadrilateral

This shows that Maxwell had a good mental image but found it difficult to offer verbal or word descriptions for this geometric concept. It suggests that Maxwell has difficulty in defining geometric concepts. He could not translate the image in his mind to verbal representation, even though he was able to correctly represent it in a visual form. Just as learners should be guided to

use diagrams to convey mathematical ideas (Hasanah et al., 2019), they should be guided to make verbal descriptions of diagrams used in learning. Mudaly and Naidoo (2015) assert that creating diagrams contributes to understanding. If Maxwell had reflected on the diagram created, he could have made some meaning from it and expressed it in words.

The PSTs' act of attaching their defining abilities to what they had visualised or seen, suggests that diagrams play a significant role in geometric discourse, both in learning and recalling abilities. Those who drew diagrams to support their defining and describing abilities might have felt incapable of defining those concepts alone, without sketches. Thus, they used the diagrams to support their verbal representation of concepts. Gagatsis (as cited in Mudaly & Reddy, 2016) asserts that visual representation provides learning support in the reflection and communication of mathematical ideas. Sfard (2008) remarks that most mathematicians often resort to visual imagery for abstract discourse. These visual images are drawn to help keep the discourse focused when communicating with others.

The questions to solicit their thoughts about sketching certain types of angles, were favourably answered by all the participants. They were able to sketch and indicate the opening of the arms as related to the questions asked. Appropriate thinking and the ability to sketch the various angle types are necessary, because the angle types also formed the basis for learning the classification of triangles by angles. All the participants at some point in time, drew diagrams to indicate their understanding of the various concepts explored in the study and also to support their explanations. Sfard (2008) asserts that the use of diagrams helps learners to express their ideas in a most enhanced way. She adds that learners' understanding and use of visual cues is important for learning mathematics.

5.5.2 Visual ability (Visual interpretation)

Visual ability is one of the important tools for learning geometry, and mathematics in general (Attanasova-Pachemska et al., 2016; Sfard, 2008). Most tasks in plane geometry are presented in diagrams, which requires the ability to interpret, or make sense of diagrams through our visual senses. When the participants were asked to explain how they organized their thoughts to solve

the tasks, they were seen to base their stories by relying on how the structure of the diagrams looked to them.

Task 1.1 contained a number of angles formed on a straight line, which required the PSTs to solve for the unknown angle involved. This was a task that required minimal visual ability as a tool for devising a solution. All the PSTs interpreted the task correctly and demonstrated knowledge of the value of the visual cue (90°) which they used to devise the correct solution. Data showed that, comparably, all the participants had understood the concept of adjacent angles on a straight line and could apply it to solve related tasks. They solved the task correctly because they were able to interpret the visual cues used in geometric communication to aid understanding and, to guide their thinking processes to design an appropriate plan to solve it. Stephen's and Maxwell's explanations are noted below:

... Looking at the diagram that is given, we have three angles all on a straight line
(Stephen, A).

... These are angles on a straight line (Maxwell, A).

In trying to understand their interpretation of the visual cue, they all mentioned that the sign shows an angle value of 90° which they used in their calculation. Such thinking of the PSTs was expected because of their long exposure to learning geometry up to their current level of study. They may have been exposed to such symbols in geometric instruction in their previous studies.

The PSTs exercised varying degrees of visual ability to understand and solve the rest of the tasks in the study. The way they understood the tasks resulted from the extent to which they visualised and interpreted them. Visualisation is the process of creating mental images and manipulating these images to make discoveries and understand mathematical problems (Attanasova-Pachemska, et al., 2016). Arcavi (2003) puts visualisation in a figurative form by considering it as “seeing the unseen”. This notion is supported by Mudaly (2021, p. 2) when he asserts that “visualisation involves more than just seeing”. In this sense, seeing beyond the ordinary, could be termed as an act of visual exploration, which involves mental processes.

The participants analysed the tasks (1.2 and 1.3) from different perspectives when they were asked to talk about them. Task 1.2 is an intersection of two straight lines and task 1.3 is intersection of two parallel lines with a transversal. By talking about the mental process involved in understanding the tasks, they were to describe their internal representation of the task's structure. There were two visual thinking approaches, each of which led to the correct answer. On task 1.2, Jones (A) demonstrated his thinking on the normal straight line concept in geometry. He stated that *"this is a straight line ... now I can see if I put a here [adjacent to 130°] ... I say, adjacent angles on a straight line sum up to 180° "*. Jones demonstrated his visually informed thinking around the straight line and solved the task in two steps (to be discussed in detail on routine thinking). Albert and Nsiah, who were both in Group B, also showed this way of thinking.

The other three Group A participants and two from Group B, demonstrated visually informed thinking in the structure of the task design. According to Matlen et al. (2018, p. 2), "the ability to perceive the relational structure of a mathematical [task] allows the problem solver to more easily draw connections across problems or mathematical ideas, and to think more conceptually about mathematics". These participants were seen to be drawing connections that may have been informed by their visual recognition and interpretation of the task's structure. These connections probably led to the use of the emerging property of 'vertically opposite angles' in planning. In an interview with Stephen, he explained that:

... looking at the diagram given, I can say that angle 130° is vertically opposite to the two angles (m and 60°) ...

A similar property guided, thinking solution was seen in Alex's (B) discourse. In an interview, he responded that:

... I can see that angles (m and 60°) are vertically opposite to the angle 130° . So, m plus 60° is equal to 130° .

Those who thought this way, informed by such visual processes, and in line with their associated properties, solved the task in a few steps (the simplest).

A similar property guided solution, was used to solve task 1.2, and was applied to task 1.3. This time, six of the PSTs, three from each Group, recognised the relational structure of the task based on visual interpretation, hence they used the emerging vertically opposite angle property, in an explorative routine solution. In a probing dialogue with Maxwell (A) on why he used that solution strategy, he said:

*I realised that ... I will have to first find the value of x ... to make the work simpler
... finding the values of x , I will be able to apply ... the properties to find the rest*

Delving into his thinking, he was asked to explain why he said solving for x first will make the work simpler. He responded that:

... because this is [pointing at $3(x - 20)$] and this [pointing at $2x$], the two angles facing each other ...are vertically opposite.

In Maxwell's discourse, he claimed that "*I will have to first find the value of x ... to make the work simpler*". This has support from Matlen et al.'s. (2018) claim that when learners have perceived the relational structure of a task, they are able to easily draw connections among the mathematical ideas, which reflects the ways a solution is devised. Results show that those who demonstrated high visual abilities were able to devise solutions informed by properties associated with the task's structure. Several studies (Anwar & Juandi, 2020; Lowrie et al., 2018; Riastuti et al., 2017) have shown that learners with high visual abilities often possess high problem-solving abilities and are able to devise solutions in the most the efficient ways.

Participants' visual sense of thinking is informed by their exploration of geometric concepts. Attanasova-Pachemska, et al. (2016) assert that several reasons support and substantiate visualisation abilities for teaching and learning mathematics. They add that seeing things differently is not inborn, but something that can be created and learned. Many of the participants in task 1.3, substantiated their narratives about geometric concepts and theorems relying on their visual abilities. Visualisation plays a significant role in teaching and learning geometry, where most of the tasks are presented in diagrams. Learning geometry is enhanced when one has high visualisation ability. Mudaly and Reddy (2016) assert that to visualise is to see in your mind, either from an internal or external perspective. It is a tool for learning mathematics and geometric

concepts by interpreting important features of diagrams and geometric figures or shapes, to enhance conceptualisation of geometric ideas. Even though all participants provided the correct solution to the tasks, it was observed that those who devised the most enhanced and easiest approaches, were those who utilised a high sense of visual thinking in the contextual structure of the task design. This ability was found more in the discourses of the Group A participants compared to their counterparts in Group B.

According to cognitive science, “we learn to see, we create what we see; visual reasoning or ‘seeing to think’ is learned, it can also be taught, and it is important to teach it” (Attanasova-Pachemska et al., 2016, p. 33). The authors assert that learning to see and creating what to see are among the reasons that make visualisation a powerful tool for enhancing learners’ understanding of concepts in many disciplines, including mathematics. According to them, effective implementation of the visualisation process is useful for making mathematical discoveries and understanding mathematical problems. As mentioned in the previous submission, those who utilised skillful visual abilities demonstrated enhanced strategies for solving the task beyond the more basic ones of using adjacent angles on a straight line. The phrase in Attanasova-Pachemska et al.’s (2016) assertion, ‘seeing to think’ is shown in the discourses of the participants. Clement’s (A) planning about how to solve the task was based on his ability to interpret certain key portions of the task through visual mediators as declared in his expression, “*I observed from the diagram that ...*” Thus, Clement verified his approach based on what he observed. According to Mudaly and Reddy (2016), visualisation plays a significant role as a tool used to verify a solution.

The participants’ sense of visual thinking was the most important tool for exploring and understanding all aspects of tasks on plane geometry. When they were asked to find the unknown angles in the given triangles, they were all observed to be interpreting the task construction through the sketches of the diagrams. In the process, they were observed to be using their pens to trace some portions of the task.

For example, in tasks 2.1 and 2.2, the solutions and discourses of the PSTs’ showed that more participants in Group A demonstrated a visual ability to see beyond the ordinary sketches. Of the eight participants, the five who used the exterior angle theorem of a triangle as demanded by task

2.1, comprised all the Group A participants and one from Group B. It must be added that whilst all the PSTs solved the task correctly, the approaches used showed the degree of iteration between their visual ability and thinking, which deals with how they connected the external representation of the task to their internal processes.

One characteristic feature of visualisation is the ability to solve complex tasks, which is linked to mental manipulation of visual information in connection with related, or existing ideas, to discover new emergent patterns (Anwar & Juandi, 2020). Task 2.2 seemed complex for many of the PSTs (see Figure 5.6).

In the figure, PQRS and PUT are straight lines. $\angle PQU = 120^\circ$, If $|PQ| = |QU|$, find $\angle URS$.

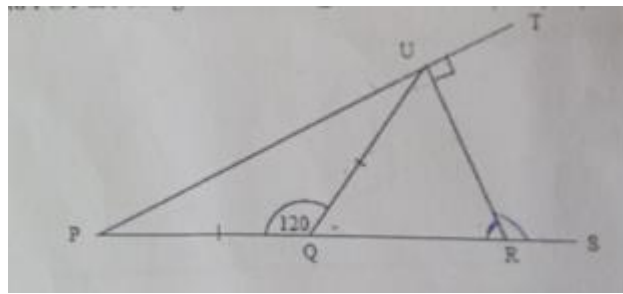


Figure 5.6: A task with combinations of triangles.

Three of the Group A participants were able to apply the exterior angle theorem, but only to the immediate triangle QRU . Of the three, two demonstrated knowledge of applying the theorem to the bigger triangle PRU . Stephen, one of the two, showed knowledge of using the bigger triangle PRU from the start but preferred applying the theorem to the immediate triangle QRU . In the interview, he said:

to find angle URS, emm... I can choose to use the exterior angle theorem either for this triangle [pointing at triangle QRU] or for the greater triangle [PRU] but I chose this [triangle QRU] triangle.

Similarly, Clement, when asked if he could solve the task with an alternative approach, said:

Yeah, this time around I will still use the exterior angle property, but this time it is a little bit short. Ok, I will find angle QPU, which I got 30° . Since angle RUT is 90° by indication, I know that ... this place [pointing at angle PUR] can also be 90° ... the exterior angle

theorem, I will ... add the 30° to the 90° ... [to] be equal to angle at this place [pointing at angle URS].

The rest of the participants (five), who may have found the task to be a bit complex, preferred using the straight line algorithm to solve the exterior angle. In their study, Atanasova-Pachemska et al. (2016) reported on an explanation given by teachers, that learners get confused in applying their visual abilities to complex geometry tasks. Hence, they prefer to use known formulae and procedures to obtain the solution.

Communication on geometric concepts with oneself, and even with others, may be ineffective if the object of talk cannot be seen or imagined (Sfard, 2008). Objects seen “help interlocutors in making discursive decisions and in sustaining the sense of mutual understanding” (Sfard, 2008, p. 147). This feature was observed in the discursive actions of the participants. At some point in time, they were seen murmuring to themselves (although not loud enough to be heard) as part of devising strategies to understand and solve the task. This action could also be referred to as internal communication, or talking with oneself, either heard or not (Sfard, 2008).

Similar visual thinking, guided by the structure of the task design, was used to solve the tasks on quadrilaterals and circles. For example, the participants were observed to be making visual interpretations of the angles subtended at the various parts of the circle, in relation to the related arcs.

Tasks on quadrilaterals and circles, similarly required thinking based on visual interpretation about an angle subtended at the circumference and at the center of a circle by the same arc, and possibly, a cyclic quadrilateral. All the participants described their thought processes in mathematical ways, informed by their visual recognition. All the participants were observed to be interpreting some important features of the task design. They were seen to be moving the pen around the angles formed by the arc *PR*. Analysis of the participants’ discourses show that their actions were informed by their ability to recognise and interpret what triggered their visual abilities.

Thinking is a complex activity in problem-solving processes, in which information is processed in the mind (Anwar & Juandi, 2020). The participants processed the visual information to aid their understanding in devising solutions to the variables involved. It appeared that their line of visual reasoning about task 4.1 (see Appendix D) was a demonstration of what they have been exposed to in most of their instruction and learning activities. Their discursive actions of finding the unknown angle at the centre before finding the angle x at the circumference, could be linked to a prototypic diagram found in many books, as well as what happens in traditional mathematics. It has been a normal practice that in most cases, the angle at the circumference is always drawn on top of the centre angle subtended by the same arc. Therefore, their thinking of finding the angle y before x , was not triggered by their visual senses. Analysis of the solution path showed that, their visual interpretation was more linked to a prototype form of the task, which seemed to dominate their mental image. A similar finding was reported by Cunningham and Roberts (2010) when they stated that pre-service teachers lacked the understanding to interpret prototypical geometry tasks seen in many textbooks. This often leads learners to rely on recall of previously endorsed narratives, to substantiate their new narratives (Sfard, 2008). Even though that task was solved by all the participants, the approach used showed some evidence of rigid thinking.

5.5.3 Symbolic Mediator

Symbolic mediator, for the purpose of this study, is used to mean visual cues or mathematical notations that used to denote mathematical concepts. Sfard (2008) asserts that symbols are artefacts that need to be operated upon or interpreted to produce endorsed narratives. They are visual objects created purposely for communication in a discourse. Many forms of symbols are used in communicating concepts in geometric discourse. The advantages of paying attention to such symbols in solving problems in geometry, were discussed in the literature (see Chapter 3, section 3.3.2.1).

With task 2.2, which required the participants to find the value of the unknown angle, which was communicated with a symbolic mediator, they were observed to be drawing some connections between the symbol and the task design. This observation was made at the initial stages of engaging with the task that reads, ‘In the figure, $PQRS$ and PUT are straight lines, $\angle PQU = 120^\circ$. If $|PQ| = |QU|$, find $\angle URS$ ’. Some tried to understand the worded instruction governing the task

and were observed to be associating, or connecting, the symbolic information $|PQ| = |QU|$, to the diagram. They were seen to be identifying the line segment \overline{PQ} and \overline{QU} . In the process of identifying the $\angle URS$, they were observed to be moving their pens along the line segments UR and RS to locate the space between them, relative to the vertex.

This action was taken because, as with the other geometric tasks, the angle to be found was not labelled by any variable. This meant that the participants' inability to identify, interpret and connect symbolic mediators with the diagram, could be a hindrance to solving the task successfully.

All the PSTs were able to identify the required angle using a symbolic mediator. This showed their understanding and interpretation of the symbol used to represent the angle. For example, $\angle URS$ means an angle with a vertex R with sides \overline{UR} and \overline{RS} (Smith, 2012). Pre-service teachers should be able to acquire and demonstrate adequate knowledge of angles and the various symbols used in the discourse, to teach those concepts of geometry in the mathematics curriculum (Duartepe Aksu, 2013). Sfard (2008) asserts that the production of narratives is informed by the symbolic realisation governing that discourse. The participants' narratives about the symbolic mediator $\angle URS$ have general endorsement in the learning community of geometric discourse.

Even though all the participants were able to interpret the angles to be found in task 3.4 (see Figure 5.7), the task seemed to pose some difficulties for the participants' visual interpretation. Of the eight participants, only Stephen (A) could successfully interpret the visual cues of the task, leading to a successful solution. Many did not attempt it at the testing stage, whilst those who attempted it demonstrated inadequate thinking which may have resulted from weak visual interpretation. For example, when Cynthia was asked to try it, her response was that "*this task looks confusing*".

Maxwell's explanation based on visual interpretation of the geometric task 3.4, shows an error in thinking according to his visual recognition. Maxwell (A) made a wrong visual inference about a line he interpreted to be perpendicular even though there was no symbolic indication of perpendicular angle relations between the two intersecting straight lines (see Figure 5.7).

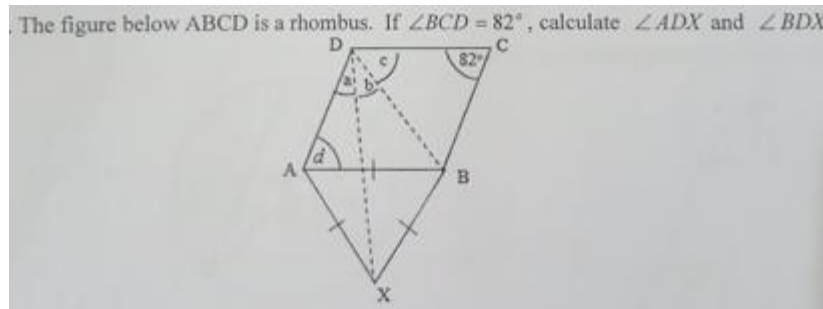


Figure 5.7: A task that seems complex for the participants' interpretation.

Based on his interpretation, he used a value of 90° by assuming that line DX is a vertical to line AB . When he was asked for justification, he said “*that is how I see it*”. An interview with Maxwell is as follows:

Maxwell: *As I said the properties of parallelogram, so, since angle this ... is the same as this angle ... angle and is 82° .*

Researcher: Ok.

Maxwell: *We are looking for this angle [referring to a] and because of the line [pointing to line DX] I used the right-angled triangle. [see Figure 5.7]*

Researcher: Which side did you say is 90° ?

Maxwell: *This [pointing to the angle line DX makes to the left of AB].*

Researcher: Why did you say it is 90° ?

Maxwell: *It looks like a perpendicular line.*

Analysis of Maxwell's (A) discourse reveals that he was misled by what he saw regarding relationships between the line segments. Geometric discourse, together with words, use symbols to provide the intended meaning to learners. In addition to symbols, diagrams are often designed with what Sfard (2008) terms as iconic mediators (to be discussed in detail in the next section), to communicate intended meaning, to aid full understanding of the task. This is what, Maxwell possibly did not take time to observe, to clarify his thinking.

In a similar manner, task 4.5 on circles, used a symbol to communicate the angle to be calculated. Four of the Group A and one Group B participant devised strategies by interpreting the geometric task, by matching it with the symbolic mediator $\angle XQY$, the angle to be found. Those who provided

the correct answer were observed to rely on visual cues, to identify the part of the diagram that had been represented by $\angle XQY$. This was evident through their hand movements of tracing line segment XQ (\overline{XQ}) and line segment QY (\overline{QY}) to determine the opening at the point where the two straight lines intersect (vertex). These participants claimed to have used their visual thinking abilities to understand the task which enabled them to devise appropriate strategies to solve it.

5.5.4 Iconic Mediator

Iconic mediators are visual representations (marks) used to design diagrams (Sfard, 2008). These iconic realisations are essential in mathematics discourse, particularly geometry, where most tasks are presented in diagram form. Sfard (2008) claims that learners need to identify and interpret visual representations to enhance their successful learning of school mathematics. These iconic mediators serve as a warrant to mathematicians' production of narratives about geometric objects. As part of analysing the participants' geometric thinking, some of the diagrams were designed with icons to communicate the intended meaning of certain geometric ideas governing the tasks. The participants were required to utilise their visual abilities to identify and interpret, to fully understand the question and its demands. The intention of such tasks design, was to see how well the participants could pay attention to and interpret the iconic realisations to understand and solve the task as required.

All the participants interpreted these mediators during the interview sessions. They substantiated their narratives by relying on the visual cues of the sides containing the hatch marks. This was evident in their discursive actions. In an interview with Jones (A) on task 2.3, he explained that:

... I first considered triangle PQU which these marks [pointing at the marks made on sides PQ and QU] indicates it is an isosceles triangle. ... So, it means that the angles that is facing each side [pointing to the equal sides] they are the same.

Analysis of Jones' discourse in substantiating his actions in every step, shows that he has gained a good level of fluency in interpreting symbolic artefacts used in geometric diagrams. This interpretation was evident in all the participant's' discourses, which informed their routines of representing the angles opposite the equal sides of the triangle, with the same variable to indicate

that both angles had the same measure or were equal in value. This shows that their mathematical discourse aligns with literate geometric discourse. According to Sfard (2008, p. 148), “artefacts include icons such as individually designed diagrams ...”. She goes on to say that learners’ fluency in such discourse remains the ultimate goal in learning mathematics in school. Choosing a solution path to complete such tasks depends on one’s interpretation.

The highlight of Jones’ discourse is that he substantiated all that he said, even when he had not been asked to justify. He indicated the location of the angles in the diagram and provided names for the geometric shapes based on his visual recognition of the iconic mediators on the task. This is seen in his response, “*I first considered triangle PQU which the marks [pointing to the marks made on sides PQ and QU] which indicates it is an isosceles triangle*”. Thus, the marks (iconic mediators) on the geometric shape were used as a visual cue, or a visual trigger, that informed his thinking (Arcavi, 2003; Sfard, 2008). What is also worth mention, is his geometric fluency in interpreting the meaning of the iconic mediators in relation to devising strategies to solve the task, as expressed in his discourse, “*so it means that the angles that are facing each side are the same*”. According to Sfard (2008), narratives can be substantiated based on memory recall that is linked to previously endorsed narratives. This assertion can be associated with Jones’ discourse of linking the visual information to other endorsed narratives of equal angles. Jones’ initial discourse is a representation of what was observed about the discourses of all the participants. Different approaches were used to find the required angle *URS*.

In all the tasks designed with the iconic mediators, the PSTs relied on their visual abilities to interpret them, to inform their discursive actions (Sfard, 2007, 2008). Analysis of their discourse shows that their actions were based on what they saw on the diagram. The participants’ relied on things seen to inform their thinking of the names of the geometric shapes and the properties associated with them. This is evident in Stephen’s discourse when he said, “*when I look at the diagram, I realise that I have line PQ to be parallel to SR and QR to be parallel to PT*”. This diagram was a quadrilateral with arrows on both of its opposite sides. This figure received the same interpretation from all the PSTs, even though not all of them could identify the appropriate properties to solve it. Their inability to solve the task could be a result of their weak internal processes.

The results show that many of the PSTs' successful attempts to solve the tasks, were as a result of their understanding of the iconic realisations about the tasks. According to Sfard (2008), learners' problem-solving abilities are supported by the realisation and meaning making of iconic mediators about tasks. She adds that frequent use of such realisations in solution procedures is necessary for individualising learning. It could mean that these PSTs may have frequently been exposed to, and used these iconic realisations in their previous learning.

5.5.5 Concrete Mediators

Concrete mediators are materials in the real world that are associated with, or used, in the teaching of mathematical concepts (Sfard, 2008). Concrete mediators, commonly known as physical manipulatives, are the available materials in one's environment that are normally used to teach mathematical concepts. It was observed that two of the participants, one in each Group, used concrete mediators in explaining their thinking about geometric concepts, such as the formation of an angle. Albert (B) crossed his arms to form an X shape. Also, Clement referred to playing materials in our environment to explain the concept of an angle, when he was asked whether he could explain an angle in another way. He said, *"angle again can be defined using the normal shapes that we have been playing with in the house. I know that whenever there ... are shapes or boards... when they are joined together, an angle is formed"*. This shows that these two individuals have acquired the knowledge of using materials around us to explain the concept of an angle. The use of materials to teach mathematical concepts enables learners to take an active position in the teaching and learning process and also enables them to verbalise their thinking (Jones & Tiller, 2017; Mudaly & Naidoo, 2015). Within the commognitive theory, Sfard (2008) argues that learning with concrete mediators helps learners produce endorsed narratives and also to express ideas with relatively few words.

5.6 Narratives

A narrative is any spoken or written text, usually "framed as a description of objects, of relations between objects, or of activities with or by objects" (Sfard, 2008, p. 134). According to the author, these narratives can be labelled as true if they are known and well accepted mathematical facts.

Berger (2013, p. 3) adds that in a formal mathematics learning community, the narratives are approved and are called “mathematical theories”, which can be axioms, definitions, or theorems.

The PSTs were required to produce narratives to substantiate or justify their discursive actions. In other words, they were expected to justify their actions based on an accepted discourse on the objects under consideration. The participants’ responses are presented in the following sections. This would be brief as some have been discussed in the previous sections, in the context of their word use. Issues of certain definitions have been discussed with particular focus on the wording of such discourse.

In the content of angles and straight lines, all the participants demonstrated adequate knowledge of mathematical statements (endorsed narratives) required to substantiate new narratives produced about the tasks. In answering task 1.1, they added the angles and equated them to 180° . This action was substantiated by the narrative that ‘the sum of adjacent angles on a straight line is equal to 180° ’. Similarly, in tasks 1.2 and 1.3, which dealt with the intersection of two straight lines and two parallel lines crossed by a transversal, respectively, those who preferred solving for the angles along the straight line(s), justified their approach with the same mathematical statement for task 1.1. Others who observed the emerging properties of these tasks used endorsed properties such as those of vertically opposite angles, corresponding angles, and alternate angles, to equally substantiate the properties associated with the intersection of two parallel lines with a transversal. Analysis of their responses shows that many participants in Group A produced object-level narratives compared to their counterparts in Group B. Whilst all the participants could produce an endorsed narrative of the concept of parallelism in solving task 1.4, Alex (B) could not think of any, and hence could not solve the task.

Data on triangles show that their arguments were substantiated by their knowledge of angle properties related to the object. They substantiated their action of modelling or developing an algebraic equation connecting the angles of a triangle by its related narrative. All participants substantiated the equation modelled by the narrative that ‘the sum of interior angles in a triangle is 180° ’. Learners’ geometric discourse on angle sum in triangles should be backed by the statement that the sum of angles in a triangle is 180° (Smith, 2012). Similarly, they produced an

associated narrative to justify their discursive actions on tasks in quadrilaterals and, those involving circles. Throughout the study, it was observed that many discourses of Group A participants were found to contain substantiation compared to their Group B counterparts, who provided upon request for justification. Yehuda et al. (2005) assert that a person's **competence** in finding solutions to tasks is often shown by their ability to substantiate. The habits of the PSTs in Group A support this claim.

A narrative is endorsed at any level of scholarly mathematical discourse if it can be derived from other endorsed narratives in accordance with universally accepted principles (Sfard, 2008). It was observed that before the participants produced the governing properties of the shapes, they focused on the iconic mediators to identify the shape, to produce the correct associated properties.

5.7 Routines involved in the participants' solutions

According to Sfard (2008, p. 234), a routine is “a set of metarules that describe a repetitive discursive action”. In other words, they are the steps that are followed to solve mathematical tasks in which each step of the solution is produced based on the previous action (Roberts & Le Roux, 2018). In geometry, particularly in non-worded tasks, which are often in diagrams, angle relations are modelled based on one's understanding of angle properties or theorems related to the task.

The repetitive patterns were associated with the approach used in modelling the angle relations (algebraic equations) governing the geometric figure, to solve for the angles involved. The participants engaged in several repetitive routines to solve the tasks. The routines involved the use of sequential steps in simplifying the algebraic equations modelled to the tasks. In effect, varied routines were demonstrated by the participants across and within the geometric tasks. Most of the modelled equations were linear, which gave some insight into how the PSTs solve linear equations. It is worth mention that though the focus of this study was more on geometric thinking than algebraic thinking, how they solved the equations was equally important in analysing the mathematics knowledge held by pre-service teachers for teaching (Ferrer, 2020; Strand & Mills, 2014).

In all the equations modelled by the PSTs, it was observed that they solved for the unknown in an objectified way. They produced discourse that showed their knowledge of maintaining the horizontal equivalence between the left-hand and right-hand of the equations. They used each equation as a signifier to generate the next equation, with features of objectified discourse maintained (Sfard, 2008). The discursive sequence of operations executed by the PSTs, included the use of distributive property to clear or remove bracket(s) from the equation, and the use of the additive and multiplicative inverse. For example, when Jones formulated the equation $a + 130^\circ = 180^\circ$, he said, “*I can say I will subtract 130° from both sides*”. Cynthia also said, “*so you subtract 80° from both sides ...*” This shows that the PSTs (with some evidence in solutions) possess knowledge to solve linear equations with guiding discourses that are endorsed in the mathematics learning community (Sfard, 2008). Details of the PSTs’ routines, used in their solution strategies along the path of ritual and exploratory discourse, are presented in chapter six.

5.8 Conclusion

The chapter started with a discourse analysis that showed how the participants’ discursive thinking was analysed. Representative discourses of the participants have been discussed based on their competencies and deficiencies in geometric thinking. The chapter presented and discussed participants’ discourses with regard to the constructs of the framework, with support from the literature.

The focus of this chapter was to introduce the participants’ discursive thinking in geometry. The next chapter deals with the detailed discussion of their routines along ritual and explorative ways of thinking in solving the geometric tasks.

CHAPTER SIX

ROUTINE GEOMETRIC THINKING

6.1 Introduction

In Chapter Five, the PSTs' solutions and their explanations about their thinking processes regarding the given geometry tasks were analysed in terms of the constructs of the commognitive framework. This chapter deals with the analysis of the participants' geometric routines, in search for the presence of ritual and explorative discourse in the strategies they used to solve the geometry tasks.

Placing geometry in the commognitive theory assumes that geometric communication exists between signifiers. These signifiers support communication in different representations. These can be in the form of spoken language, diagrams, symbols and icons. Such signifiers offer a broader understanding of, and concept development in geometric discourse. Learners' competence in geometric discourse is marked by their knowledge of these signifiers in their various forms.

The results of the PSTs' routine solutions to the tasks, were analysed based on the classifications and characteristics of ritualised and explorative routines within Sfard's (2008) commognitive framework. According to Sfard (2008), both ritual and explorative routines are important in mathematics education. They complement each other in developing learner competence in mathematics. In other words, 'how' to solve a particular task, and 'when' and 'why' such an approach works, are central to the teaching and learning of mathematics, even though the focus of school mathematics is to produce learners with explorative thinking, which focusses on the 'when' and 'why'. Sfard (2008) asserts that rituals are acceptable at the initial phase of learning but should be a basis for the learner to understand the 'when' and the 'why', which shows a learner's transition from ritual discourse to explorative discourse. In the following sub-sections, the participants' ritualised routines are described in an attempt to draw attention to the window of explorative routine.

6.2 Ritual discourse

In Sfard's (2008) commognitive framework, a discourse is ritual if it contains talk and actions on entities in a dis-objectified manner. The author associates ritual discourse with performing the 'how' of a routine, which uses a set or meta-rule to solve a task. Therefore, in trying to analyse the PSTs' ritual routines, evidence in the ways and manner in which they demonstrated their thinking and skills for solving the geometric tasks presented to them, was sought for. The participants' ritual routine is discussed under four (4) sub-themes:

1. The use of step – by – step solution strategy
2. Series of (computational) steps in solving a task
3. Devising a solution using a simultaneous equation
4. Narrow range of applicability/less attention to important features of the tasks

Apart from task 1.1, which required the application of the property of adjacent angles on a straight line, which is often termed the 'straight line algorithm', in the write up, it was used quite frequently, even when the task design necessitated a simpler approach by its objectified property.

6.2.1 The use of a step-by-step solution strategy

Rituals, according to Sfard (2008), are the unique guidelines that govern an action and are pre-established by people in positions of authority, such as authors, professors, or lead discussants. Rituals are the fundamental building blocks of a discourse and address "how" to carry out an action. It can be viewed as the operation of an algorithm that carries out a task in a step-by-step approach. Sfard (2008) asserts that **an initial step** in acquiring a new notion, is to imitate an authoritative figure by adhering to a set of rules. This is regarded as first-hand knowledge, upon which additional content is developed.

Task 1.2 required the participants to find the value of the unknown angle represented by the letter *m*. Generally, the value could be found by using the angle property on a straight line, or the application of vertically opposite angles (vertical angles), depending on one's understanding. All the eight participants solved these tasks correctly using different approaches. Of the eight, four (one in Group A and three in Group B) used the angle property on a straight line to solve the task on a step-by-step basis, whilst the remaining four used the vertical angle property. Those who used

the step-by-step approach first calculated the angle, labelled p (Nsiah's plan), which is adjacent to 130° on a straight line CD , before considering the angle p together with angles m and 60° on another straight line AB . This process is evident in Nsiah's (B) solution to task 1.2 as shown in Figure 6.1.

Task 1.2: Find the size of the angle marked by m .

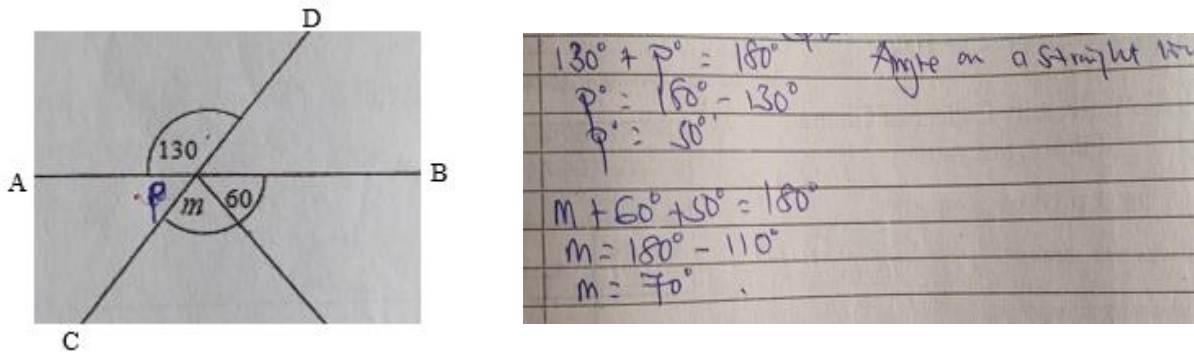


Figure 6.1 Nsiah's step-by-step solution to task 1.2.

In trying to understand their thought processes governing the solutions provided, the participants were interviewed as follows:

Researcher: Could you help me to understand how you solved for the value of m ?

Nsiah (B): ... I used angles on a straight line to first find the value of p . With this, I added p and 130° and equated to 180° , and subtracting 130° from 180° , I got the p to be 50° . Now, using the horizontal straight line, I added the three angles and equated them to 180° . I then solved for m which is 70° .

In a similar manner, Albert (B) explained that:

... I used the approach of angle on a straight line. So, I have 130° plus the ' a ' so their sum is equal to 180° . So, I calculated for a . Equally, ... I have my angles a and b , and I can say they are vertically opposite angles and are equal. ... considering this straight line [referring to straight line AB] b plus m plus 60° gives me 180° . I make m the subject and solved it to get 70° .

It is evident from the preceding excerpts, which are similar to the explanations of the other two, that they used the step-by-step approach to work with different straight lines (or the same) to obtain their answers. The participants' ability to devise solutions to the task could result from several processes. These participants, upon visualising the task, may have recognised a straight line which they internalised, to produce the preferred solution. They seemed to have their a priori knowledge more linked to straight lines.

Albert's explanation shows that he knew of vertically opposite angles, but he chose to work on the observed straight line. His knowledge of vertical angles is shown in his solution below:

<p>Step 1</p> $130^\circ + a^\circ = 180^\circ$ $a^\circ = 180^\circ - 130^\circ$ $a^\circ = 50^\circ$ <p>Step 2</p> $\angle a^\circ = \angle b^\circ$ $50^\circ = \angle b^\circ$	<p>Step 3</p> $b^\circ + m^\circ + 60^\circ = 180^\circ$ <p>but $b^\circ = 50^\circ$</p> $50^\circ + m^\circ + 60^\circ = 180^\circ$ $m^\circ + 110^\circ = 180^\circ$ $m^\circ = 180^\circ - 110^\circ$ $m^\circ = 70^\circ$
--	--

Step 1: Angle on a straight line add up to 180°

Step 2: Vertically opposite angles are equal.

Step 3: Again

Step 3: Angle on a straight line add up to 180°

Figure 6.2: Albert's solution to task 1.2 with substantiating narratives.

He explained that "I have my angles a and b , and I can say they are vertically opposite angles and are equal" [Refer to Figure 6.1, Task 1.2]. He labelled the angle adjacent to 130° on the straight line AB as a . He then related the angle value of a (50°) to that of b (the angle space adjacent to [m and 60°]) also on straight line AB and supported by the narrative 'vertical angles' (Step 2, in Fig.

6.2). He used the straight line AB to solve for the value of m . It is evident in Albert's discourse that he preferred using the straight line in a horizontal appearance or prototype view. His preference for the use of the straight line could be linked to his internal representation of the task that resulted from his visual ability. Albert's mention of vertical angles, associated with the intersection of the two straight lines AB and CD , could also mean that he did not exercise his visual ability to see beyond the ordinary, that the angle 130° is also vertical to the sum of (m and 60°) and hence, equal in value. Thus, the task was mentally manipulated at a shallow level by Albert.

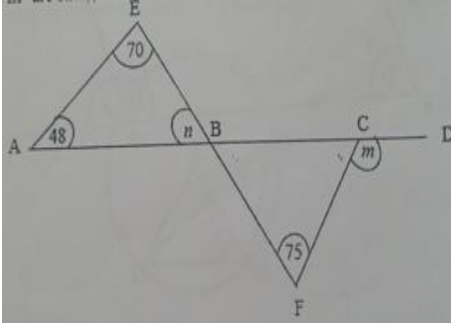
What was different from Jones' solution was that his conception of a straight line seemed to be more open compared to that of Albert. When he first calculated the adjacent angle to 130° on the straight line AB , and then used the value obtained (50°) on the straight line CD to calculate the value of m . In all the solutions devised using the step-by-step approach, these four participants showed adequate knowledge of the procedures needed to solve the task and explained their thinking, which showed fluency with the procedure.

The participants' ability to solve the task resulted from an internalisation process that took place upon recognition of the straight lines. Internally, they may have connected the straight line seen to a picture in their minds, which possibly informed their thinking, and formulated the equation on the straight line angle value.

Some of the participants were more capable of using the straight line algorithm and other fundamental concepts in geometry to find answers to some of the tasks on triangles and quadrilaterals. For example, on triangles, the purpose of most of the tasks was to gain insight into the participants' preferred approach of using either the 'exterior angle theorem' or the 'straight line algorithm'.

In task 2.1, the exterior angle could be solved for, using the exterior angle theorem, or otherwise. It was found that three Group B participants calculated the adjacent angle to the exterior angle so that they could easily apply the straight line algorithm to find the correct answer. They all explained their solutions with a clear sense of knowledge and substantiated each step with an appropriate geometry property. Cynthia (B) solution is shown in Figure 6.3.

Task 2.1: Find the values of n and m in the figure below.



$$\begin{aligned}
 & 70^\circ + 48^\circ + n = 180^\circ \quad (\text{Angle in a triangle sum up to } 180^\circ) \\
 & 118^\circ + n = 180^\circ \\
 & n = 180^\circ - 118^\circ \\
 & n = 62^\circ \\
 \\
 & \triangle BCF: a^\circ + 75^\circ + b^\circ = 180^\circ \quad (\text{Angle in a triangle sum up to } 180^\circ) \\
 & \text{but } a^\circ = 62^\circ \quad (\text{direct opposite angles}) \\
 & \therefore 62^\circ + 75^\circ + b^\circ = 180^\circ \\
 & 137^\circ + b^\circ = 180^\circ \\
 & b^\circ = 180^\circ - 137^\circ \\
 & b^\circ = 43^\circ \\
 \\
 & m = 75^\circ + 62^\circ \quad (\text{exterior angle is equal to two opposite}) \\
 & m + b^\circ = 180^\circ \quad (\text{straight angle}) \\
 & m + 43^\circ = 180^\circ \\
 & m = 180^\circ - 43^\circ \\
 & m = 137^\circ \\
 \\
 & \therefore n = 62^\circ \text{ and } m = 137^\circ
 \end{aligned}$$

Figure 6.3: Cynthia's step-by-step solution to task 2.1.

Cynthia's approach is a representational solution of the remaining two in Group B, who solved for m using the straight line approach. All the Group A participants, together with one from Group B, explained that they could add angles 62° and 75° , as the two interior opposite angles to get the value of the exterior angle m as 137° .

The solution strategies devised by the participants were based on how they thought about the task. For example, when Cynthia was asked why she solved the task that way, she said, "*I used this approach because of the properties I see within this diagram*". This means that Cynthia's solution was an externalisation of her internalisation process about the task. Her word use of "*properties I see within the diagram*" could mean that her action of solving the task was connected to her visualisation process that stimulated her mental action. Mudaly (2021) asserts that a person's physical action of devising a solution to a task is related to his/her visualisation process. He further claims that "it would be difficult to conceive of a process where no mental action or image has occurred" (p. 2). This relation between physical action and mental action was seen in Cynthia's

discourse when she said, *“I used this approach because of the properties I see within this diagram”*.

These three Group B participants probably wanted to work with a clearer focus. This could be an approach they may be comfortable with and knew it leads to the correct answer, hence, their familiarity with it. Mann and Enderson (2017) partly attribute learners’ preference for a procedure-driven approach for solving mathematical tasks, to their ease of remembering and their familiarity with such an approach.

This finding is in line with the literature which shows that learners often prefer solving tasks using a set of procedures (Akhter & Akhter, 2018; Zuya et al., 2017). According to the authors, most learners rely on the use of algorithms to solve tasks in mathematics because it never fails. For example, a study conducted by Mann and Enderson (2017) to assess learners’ preference for a procedure (rule or formula-driven) rather than concept-driven, found that learners preferred the use of a rule-based approach (procedure) to the conceptual approach.

Placing this within the commognitive framework, the use of a set of procedures to solve a task is characteristic of a ritual routine (Sfard, 2008). Sfard (2007, 2008) asserts that ritual routines are characterised by following strict rules, mostly determined by the teacher or an authority. It is concerned with ‘how’ to get something done with no focus on ‘when’ or ‘why’ the approach works.

According to Sfard (2008), learning a new concept begins with imitating an authority by following a set of procedures. Among the basic concepts of teaching plane geometry are ‘the sum of adjacent angles on a straight line’ and others, such as the ‘sum of interior angles in a triangle’, for learning polygons. These concepts serve as a foundation for learning geometry. For example, these two basic concepts are used to prove the exterior angle theorem of a triangle, for which learners (and teachers) are supposed to demonstrate implicit and explicit understanding, and apply them to solve related tasks. Thus, both fundamental thinking and high concept thinking are necessary for holistic understanding and use in solving problems (Al-Mutawah et al., 2019; Zuya et al., 2017). According to Zuya et al. (2017), knowledge of both fundamental and enhanced concepts in geometry, and other topics in mathematics, is needed for teachers to be effective in teaching. Sfard (2008) advises

that as learners are taught procedures as a way of imitating the knowledgeable other (teacher), the procedures must also serve as a basis to transit to explorative thinking. Even though both ritual and explorative routines are good for learning, Sfard (2008) maintains that school mathematics aims at producing learners with explorative way of reasoning.

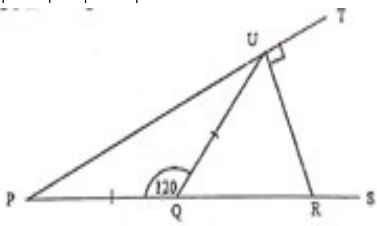
These procedure-driven solutions were discussed in the literature where researchers have noticed that most classroom instructions often focus on how to solve problems by showing learners methods and algorithms, as a way of creating familiarity with such questions (Sfard, 2008). This practice sometimes enables learners to memorise such procedures, with little or no understanding of their underlying concepts. This practice often results in difficulty in using knowledge in new situations.

6.2.2 Series of (computational) steps in solving a task

Sfard (2008) asserts that a person functioning within rituals can have strong procedural knowledge but may have inadequate understanding of the ‘when’ or ‘why’ behind such activity. These people typically limit themselves to using first-hand information to accomplish new tasks. Such a routine mode of thinking frequently results in the development of a set of computational solution steps to problems (Sfard, 2007, 2008).

A greater number of participants used a series of steps in solving task 2.2, compared to their solution to task 2.1. The number increased from three, to five (one in Group A and all four in Group B). Maxwell’s (A) solution is representative of those who used a series of computational steps. See Figure 6.4.

Task 2.2: In the figure, PQRS and PUT are straight lines and $\angle PQU = 120^\circ$. If $|PQ| = |QU|$, find $\angle URS$.



$$\begin{aligned} \angle P + \angle U + 120 &= 180 \\ \angle P &= \angle U = x \\ x + x + 120 &= 180 \\ 2x &= 180 - 120 \\ x &= \frac{60}{2} \\ x &= 30^\circ \\ \angle QUR + 30 + 90 &= 180 \\ \angle QUR &= 180 - 120 \\ \angle QUR &= 60^\circ \\ 120 + \angle UQR &= 180 \\ \angle UQR &= 180 - 120 \\ \angle UQR &= 60^\circ \\ 60 + 60 + \angle R &= 180 \\ \angle R &= 180 - 120 \\ \angle R &= 60^\circ \\ \angle R + \angle URS &= 180 \\ \angle URS &= 180 - \angle R \\ \angle URS &= 180 - 60 \\ \angle URS &= 120 \end{aligned}$$

Figure 6.4: Maxwell's solution to task 2.2 using a series of steps.

Maxwell explained his thinking and substantiated his routines using the appropriate narrative. It was noted that, even when several angles were formed at a point by different line segments, he used only one letter to name the angle. Some of the angles were named correctly, whilst others were incorrect. The following extract shows his responses governing the solution.

Maxwell: *This [pointing to triangle PQU] is an isosceles triangle*

Researcher: Why do you say it is an isosceles triangle?

Maxwell: ... *because of this sign here [pointing to the marks (iconic mediators) on sides PQ and QU of the triangle]. ... the sides [pointing to sides? PQ and QU] are equal ... definitely their angles are also equal.*

Researcher: Could you show me which of the angles are equal?

Maxwell: *This and this angle P [pointing to angle UPQ] and angle U [referring to PUQ].*

He continued that

So, since this is 120° and the sum of the interior angles of a triangle adds up to 180°, I represented the equal angles by x and added them and solved to get x to be 30°. Now we

are looking for this [pointing to angle URS] but we have to get this angle [pointing to angle UQR] and this angle [pointing to angle QUR]. ... is an angle on a straight line [referring to line segment PT on which U is located]. So, you sum the angle and equate it to 180° , and then you get this [angle QUR] to be 60° this [line PS] is also a straight line so 120° plus angle UQR you equate it to 180° and you solve to get 60° . Now I added this [angle UQR] and this [angle QUR] then I find for the angle here [pointing to angle URQ] and this angle R [angle URQ] is 60° then I used the angle on a straight line to find for angle URS.

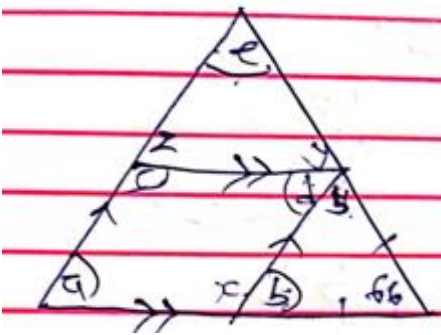
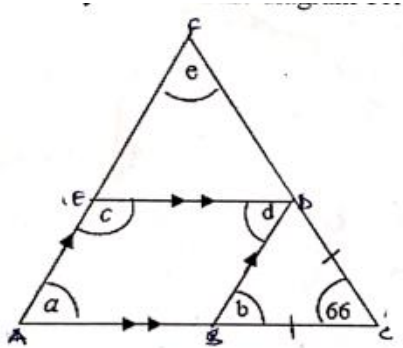
Maxwell's responses indicate his fluency in thinking and skill in devising a plan, using a series of computational steps to solve the task. Those who solved the task using this approach also demonstrated similar thinking by using a series of computations that they were comfortable with to find $\angle URS$. They preferred to first find the adjacent angle URQ so that they could use the straight line algorithm to calculate the answer. They demonstrated proficient skills in their solutions. All the participants demonstrated these computational solutions, except for Stephen and Jones, who skipped finding the angle URQ and instead, applied the exterior angle theorem to triangle QRU . It could be seen from the task in Figure 6.3, that the exterior angle theorem could also be applied to triangle PUR .

The results of this study indicate that the PSTs' solution strategy for the task was based on over-dependence on procedures. Similar studies in the literature report this finding (Mann & Enderson, 2017). Learners' preference for, and over-dependence on algorithms often results from familiarity with such mathematical procedures (Mann & Enderson, 2017). According to Mann and Enderson (2017), learners are so proficient in using algorithms such that they often lack the confidence to explore new ways of solving tasks. This was observed in the PSTs' solution to task 2.2. Even though some participants applied the exterior angle theorem to task 2.1, task 2.2 may have seemed new to them. The task depicted a combination of two triangles (or partitioning a larger triangle into two triangles), for which an application of the exterior angle theorem was equally applicable to the larger triangle PUR . This could mean they lacked confidence to apply the exterior angle theorem to the larger triangle. Those who used the theorem applied it to the immediate triangle QRU . Mann and Enderson (2017) assert that lack of confidence to apply learned concepts to new

situations, is a challenge that needs to be addressed to prepare them for the tasks ahead (especially teachers). This demonstration of following a series of steps to solve the task within the commognitive framework, is characterised as ritual (Sfard, 2008). Sfard (2008) states that, despite the aim of school mathematics, which seeks to develop learners' explorative thinking (producing new narratives), the most common teacher routine in the classroom has been the 'how' of a routine rather than the 'when' and 'why'. This practice often limits learners' understanding and, as a result, restricts their thinking to strict rules, as was found in this study. Sfard (2008) asserts that mathematical procedures are considered as ritual when they are not objectified to the task.

A similar series of computational steps was found in two of the participants' solutions to task 3.3. In the task (see Figure 6.5), only two of the angles marked by letters, b and c , required some form of calculation. The rest of the angles could be obtained by the application of objectified properties about the shapes. Almost all the participants calculated for b and c , and applied properties to find a and d . For angle e , Maxwell (A) and Albert (B) further used the 'straight line algorithm' along the straight sides of the triangle to first solve for angles marked z and y , as shown in Maxwell's (A) plan in Figure 6.5, even though the values of those angles could be obtained by properties. In addition, interpreting the iconic mediators shows that angle e corresponds with angle b and are equal. These two participants used the straight lines along the sides of the triangle to find the angles z and y (see Figure 6.5) so they could solve the angle marked e by applying the sum of the interior angles of a triangle. Maxwell's plan to solve the task is shown below.

Task 3.3: Find the value of angles marked by letters in the figure below.



$$\begin{aligned}
 b + b + 66 &= 180 \\
 2b &= 180 - 66 \\
 2b &= 114 \\
 b &= \frac{114}{2} \\
 b &= 57^\circ \\
 b + x &= 180 \\
 x &= 180 - b \\
 b + b &= 57 \\
 x &= 180 - 57 \\
 x &= 123 \\
 x &= c = 123^\circ
 \end{aligned}$$

$$\begin{aligned}
 b &= d = 57^\circ \\
 \angle d &= a = 57^\circ
 \end{aligned}$$

$$\begin{aligned}
 \angle d + \angle b + y &= 180 \\
 57 + 57 + y &= 180 \\
 y &= 180 - 114 \\
 y &= 66 \\
 \angle c + \angle z &= 180 \\
 123 + \angle z &= 180 \\
 \angle z &= 180 - 123 \\
 &= 57 \\
 \angle y + \angle z + \angle e &= 180 \\
 66 + 57 + \angle e &= 180 \\
 \angle e &= 180 - 123 \\
 \angle e &= 57
 \end{aligned}$$

Figure 6.5: Maxwell's solution to task 3.3 using a series of steps.

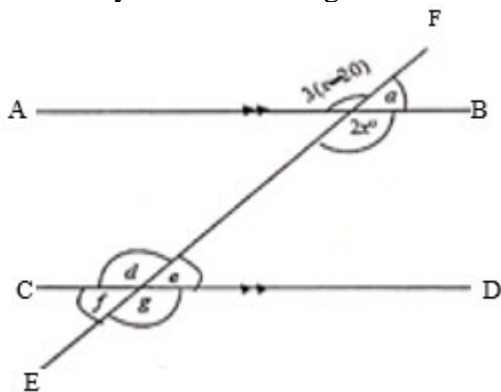
The routines used by Maxwell and Albert seemed to indicate what they were familiar with. This led to the approach of using a well-established routine to obtain those angles. Their routine seemed to be a result of over-relying on the use of a straight line algorithm to devise solutions to the tasks.

6.2.3 Devising solutions using simultaneous equations

According to Sfard (2008, p. 267), many learners cling to the ritual of "doing exactly" the same thing they do or might have been doing with others to sustain a bond or reward. They frequently

use the fundamental concept or standard method of doing a task. The use of a straight line algorithm to solve a problem is a fundamental concept in learning geometry. Adhering to such a fundamental concept is one of the characteristics of a ritualised way of thinking (Sfard, 2008). Task 1.3 (see Figure 6.6) has two parallel lines crossed by a transversal. Angles $3(x - 20)$ and $2x$ are vertically opposite, and a is adjacent to all the angles on either of the straight lines. Thus, the angles marked by letters could be solved by using a preferred approach of either the objectified property or a straight line approach. Clement preferred to work with straight lines and formulated two equations in two variables as a solution strategy. These two equations were solved simultaneously as shown in Figure 6.6.

Task 1.3: Find the value of the angles marked by letters in the figure.



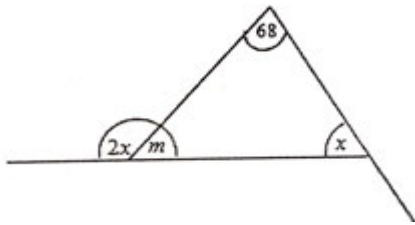
$$\begin{aligned}
 &3(x-20) + a = 180^\circ \text{ (Adj. angles)} \\
 \Rightarrow &3x - 60 + a = 180^\circ \\
 &3x + a = 240^\circ \dots \text{--- (1)} \\
 &\text{Also} \\
 &3(x-20) + 2a + 2x = 360^\circ \text{ (Angles)} \\
 &2x - 60 + 2a + 2x = 360^\circ \\
 &5x + 2a = 420^\circ \dots \text{--- (2)} \\
 &\text{Solving eqn (1) and (2) Simultaneously} \\
 &3x + a = 240^\circ \dots \text{--- (1)} \\
 &5x + 2a = 420^\circ \dots \text{--- (2)} \\
 &\text{By substitution method } a = 240^\circ \\
 &\text{Hence:} \\
 &x = 60^\circ \\
 &a = 60^\circ
 \end{aligned}$$

Figure 6.6: Clement's (A) solution to task 1.3.

He explained how he devised his solution in a flexible manner. When he was probed about the missing steps that led to the answers, he explained that after formulating the two equations, he used a calculator. This approach was also used by Albert (B). Clement's (A) and Albert's (B) solutions indicate that they know how to use other mathematical ideas to solve problems on parallel lines.

Similarly, Clement (A) used a simultaneous equation to solve task 2.3, whilst the remaining seven participants applied the exterior angle theorem of a triangle, in their solutions. Clement's solution is shown in Figure 6.7.

Task 2.3: Find the angle value of x and m in the figure below.



$$\begin{aligned}
 &68^\circ + m + x = 180 \quad (\text{Sum of interior angles in triangle}) \\
 &68 + m + x = 180 \quad \text{--- ①} \\
 &m + x = 112 \quad \text{--- ①} \\
 &2x + m = 180 \quad \text{--- ② (Sum of Adj angles on a straight line)} \\
 &\text{Solve eqn ① \& ② simultaneously}
 \end{aligned}$$

Figure 6.7: Clement's solution to task 2.3, using simultaneous equations.

Clement (A) described his thought processes in devising a solution to the tasks 2.3 as follows.

This time, I used the simultaneous equation. ... I found the sum of interior angles in that triangle. ... and wrote 68° plus m plus x is equal to 180° ... sum of interior angles in triangle. Then I took the straight line... $2x$ and m being adjacent angles on it. ... $2x$ and m is equal to 180° . So, if I have these two equations ... so, I solved both simultaneously and my x is 68° and m is 44° .

The use of simultaneous equations to solve the tasks in geometry (parallel lines) can be viewed in two ways. It can have both positive and negative impacts on learning geometry, taking into consideration the level at which these geometry concepts are taught in Ghanaian schools. In this case, it can be used to validate and deepen learners' understanding of how different approaches can be used to solve geometry tasks. This can help them appreciate and make connections about how they can apply such knowledge within and across mathematics topics. It can also boost their confidence in learning mathematics by the exposure to various strategies used in problem-solving. These qualities can be achieved when the learners are good at algebra and know how to solve simultaneous equations.

On the other hand, a learner who has difficulty learning simultaneous equations may find this approach to solving tasks in geometry more challenging, and difficult to understand. A common difficulty found among senior high school learners is the ability to devise an effective method to solve simultaneous equations (Ugboduma, 2012). Simultaneous equations are perceived to be difficult, and as a result, most learners have little or no interest in studying or attempting such questions during a test or examination (Johari & Shahrill, 2020; Ugboduma, 2012). Learners try it

in a test or examination, not because they understand it, but purposely to pass (Shahrill, 2018; Shahrill & Clarke, 2019). This perceived difficulty, often prevents learners from utilising such ideas to any related topics in the mathematics curriculum (Johari & Shahrill, 2020). Thus, if learning geometry, which has been noted to present difficulty to learners (Ngirishi & Bansilal, 2019; Robichaux-Davis & Guarino, 2016), is met with another perceived difficult concept, it can greatly reduce learners' interest in learning. Interest is a crucial and important learning component. Ugboduma (2012) considers interest as a crucial variable for learning, and is of the view that learners get deeply involved in a lesson when they have an interest in it. It would therefore be important that where possible, learners must first be exposed to the most precise and simplest solutions to tasks, as an initial scaffold to learning. Alternative approaches can probably be used to extend their way of thinking, to deepen their understanding, and to enable them to appreciate different ways of solving mathematics problems.

6.2.4 A narrow range of applicability/less attention to the features of the tasks

Sfard (2008) asserts that the over-reliance on the use of procedures in learning, often hinder learners' ability to apply their knowledge in new situations. She adds that the ritual practices often limit the ability to engage in complex reasoning in mathematics. They did not take time to explore and interpret other information the task may contain.

Certain tasks on quadrilaterals received ritualised practices from some of the participants. For task 3.1, which used a quadrilateral with no parallel sides, two of the participants in Group B applied equality to the angle properties associated with parallel lines. This can be described as a ritual way of thinking by over-generalising the properties of parallel lines to any task with a similar structure. The appearance of the lines in the task looked parallel. However, concepts in mathematics or geometry are communicated by the use of symbols or icons to design tasks. According to Sfard (2008), they are signifiers used in geometric discourse to communicate or describe certain important features of the tasks to learners. In this case, Alex wrongly applied the parallel line property, hence arrived at the wrong answer $x = 123^\circ$, and substantiated that they are corresponding angles. Upon subsequent probing, he still could not describe any visually informed thinking of identifying whether the lines were parallel or not. When he was asked why he used that approach, he said, *“that is properties of angles ... and that one [pointing to x] is corresponding to*

123° here [pointing to the value]”. Alex could not offer a satisfactory response when he was asked to indicate the condition under which we apply the equality of two corresponding angles. He responded that:

... when one angle is given and it corresponds to what you are supposed to find, then you use the corresponding angles.

Alex’s response shows that his line of thinking seemed not to be on identifying any signifier or any iconic mediator about the task, but focused on the appearance of the lines. His responses suggest that he knows of the corresponding angle property but does not know when it is applied. Even though his answer was wrong, he demonstrated his thinking about ‘how’ to obtain the answer but could not tell ‘when’ to apply the property.

Albert also explained that:

angle R that is angle GRS is equal to 103° that is vertically opposite angles are equal. And angle U [referring to angle QLM] ... is 57°. So, having known angle U then I can say that angle x is a corresponding angle to angle FLM. So, angle x is equal to 123°. The reason is that they are corresponding angles.

Albert was on the correct solution path when he solved for the interior angles, for which the remaining step was to equate the sum of the interior angles to 360° in order to find for x . Suddenly, he changed his decision, which was quite odd. His responses to further probing questions were a confirmation of his answer, with the statement that “*I realise that where the x is placed and where I am having my 123°, they are corresponding angles*”.

Thus, no effort was made to ascertain the parallelism of the two straight lines. This could be interpreted as a weak iteration between his thinking and visualisation. According to Bruce et al. (2015), Clement and Battista (1992) and Sinclair et al. (2016), good visual abilities are required to develop good geometric thinking. Within the commognitive theory, a mathematical discourse that puts emphasis on the ‘how’ at the expense of the ‘when’ is classified as a ritual. Thus, by Sfard’s (2008) classification, their routine is ritual since their focus was more on the ‘how’ without

considering ‘when’ that approach works. This could also mean that, either they had forgotten, or they knew the properties but did not know when they are equal.

Further, Alex could not make any relational geometric thinking to task 3.2, which was a parallelogram (see Figure 6.8). Analyses of the routines used by two of the PSTs in Group B, suggest that angles formed on a straight line seemed to attract their attention the most. After formulating angle relations of the variables informed by the straight line PT , Alex got stuck, and could not proceed. Several questions were asked to draw his attention to the signifiers of the task, so that he could draw connections between the angles. When asked how he could put ideas together to solve the task, he mentioned the straight line and the opposite angles properties, but became silent for a while and said he had forgotten. Finally, he said the angles marked by variables could not be solved for. An interview to understand Alex’s (B) thinking processes, is shown below:

Alex: *You will use angles on a straight line to solve for them. In this case, the figure attached to ... [long pause]*

Researcher: What again?

Alex: *You use eemm angles which are opposite to each other.*

Researcher: Ok. Now explain how you will solve for the various variables.

Alex: [Silence for some time], ... *Ahm I have forgotten.*

Researcher: You just said you would use angles on a straight line and opposite angles.

Alex: *Yeah*

Researcher: Can you make any connections between any of the angles?

Alex: *The angles on a straight line. That is when you use $3x$ plus y , then you equate to 180° because angles on a straight line will be equal to 180° . But in this case, we have x and y which ... are not the same ... [pause for a while]*

Researcher: In your opinion, can the variables be solved for or not?

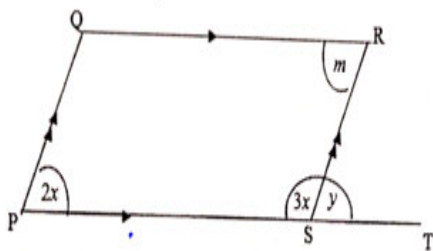
Alex: *In my opinion no.*

Alex, mentioning that x and y are not the same, could have drawn his attention to exploring the possible relationship between the angles. This would have been facilitated if he had observed the iconic mediators used to design the task, which indicated that opposite sides are parallel, and hence

figure $PSRQ$ was a parallelogram. Thus, he paid little or no attention to the important features of the task that could have informed his decision. He mentioned opposite angles but got stuck since the angles were indicated with different variables, as he rightly commented about x and y that “...in this case, we have x and y which ... are not the same ...” Thus, the angles were considered in isolation due to a weak connection between thinking and visualisation. He ended up saying that the value of the letters in the task, could not be determined.

Nsiah’s (B) solution is a replica of that of Alex’s (B). Nsiah could not continue with the solution after he formulated the equation relating the angles on the straight line and labelled equation (1) as in Figure 6.8.

Task 3.2: Find the values of the angle marked by letters in the figure.

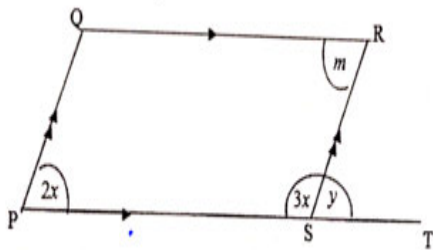


$$\begin{aligned} \cancel{2x} + 3x & \quad 3x + y = 180^\circ \quad \text{--- (1)} \\ y & = 180^\circ - 3x \\ \therefore 3x + 180^\circ - 3x & = 180^\circ \end{aligned}$$

Figure 6.8: Nsiah’s first attempt to task 3.2

The many cancellations in Nsiah’s work show some uncertainty in the way he planned to solve the task. He began by writing $2x + 3x$ and struck it off. He may have considered the two angles as co-interior angles but changed his mind, probably due to the way they appeared. Therefore, not engaging in clear visual thinking about the two angles, viz $2x$ and $3x$, might have caused his change of mind, which resulted in cancelling what he initially wrote. This cancellation could have resulted from a commognitive process of thinking with himself about the solution (Mudaly, 2015). The interview process, which focused on definitions and properties of geometric concepts, probably reminded him of something that he wanted to try, but he also gave up in the process. The second trial is shown in Figure 6.9.

Task 3.2. Find the value of the angles marked by letters in the figure.



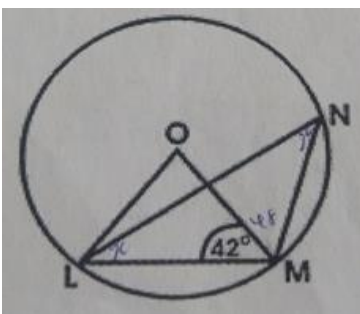
$$\begin{aligned} 3x + y &= 180^\circ \\ m + 3x &= 180^\circ \\ 3x + m + y &= 2 \end{aligned}$$

Figure 6.9: Nsiah's second attempt to solve task 3.2

He was able to state that $m + 3x = 180^\circ$ (co-interior angles) but was still a challenged problem to solve. When asked a further question to see if he could demonstrate any kind of informed thinking, he said, "let's skip for now".

In the same way, not much attention was paid to some important features of the tasks on circles. Nsiah and Cynthia (both in Group B) had used the properties of an isosceles triangle to solve related tasks in this study. They failed however, to connect this knowledge to a similar on circles probably because it was not designed with any iconic mediator. These participants failed to identify and interpret that joining the ends of a chord to the centre of a circle, forms an isosceles triangle. Nsiah made a wrong assumption which may have resulted from his internalisation and externalisation processes. There seemed to be a weak iteration between his visual and thinking processes (Mudaly, 2021), which resulted in his wrong solution task 4.2 shown in Figure 6.10.

Task 4.2. In the figure, LMN are points on a circle with centre O . If angle $LMO = 42^\circ$, find angle LMN .



$$\begin{aligned} 2x + 90^\circ &= 180^\circ \\ 2x &= 180^\circ - 90^\circ \\ 2x &= 90^\circ \\ x &= 45^\circ \\ \therefore \angle LMN &= 45^\circ \end{aligned}$$

Figure 6.10: Nsiah's solution to task 4.2

He considered line segment LN to be a diameter, leading to his assumption that angle LMN is 90° (see Figure 6.10). Another wrong assumption is considering the sides LM and MN to be equal and hence equating the angles opposite these sides to x . In trying to understanding his thinking processes underlying why he solved it that way, he said, “... *eemm that is the way I understood it*”. Cynthia (B), who did not attempt task 4.2, claimed, “*that because the angle at the centre of the circle was not given, it would be difficult to solve such a task*”.

Generally, circle theorems have been a major problem for high school learners in Ghana. The Chief Examiner’s report on learners’ performances, released every year, mostly touches on the learners’ knowledge deficiency in solving questions on the circle theorems (WAEC, 2015, 2017, 2018). According to these reports, most learners perform poorly on questions in this area of geometry. Learners prefer to solve questions on other topics rather than the circle theorem, and the solutions of the few who attempt these questions show that they don’t know much about the subject matter (Fletcher & Anderson, 2012). Findings in this study also showed similar difficulties among the participants. Almost all the questions on the circle theorem were solved by participants in Group A, compared to those in Group B. Question (4.4) was partially solved by Maxwell (A), and Cynthia, Albert and Nsiah, in Group B. Question 4.2 was incorrectly solved by Nsiah (B) and questions 4.2 & 4.5 were not attempted at all by Cynthia (B). This analysis is shown in Table 6.1.

Table 6.1: Details of PSTs’ performance on circle theorem

Q. No.	Correct	Partial (Incomplete)	Incorrect	Not Attempted
4.1	All participants			
4.2	5		Nsiah (B)	Alex (B) Cynthia (B)
4.3	All participants			
4.4	3	Maxwell(A); Albert, Alex Cynthia and Nsiah (all in B)		
4.5	6			Cynthia (B) Alex (B)

In addition, data suggests that participants in Group B attempted questions on other content areas they were tested on, compared to questions on circle theorems.

As mentioned previously, the discussion of the participants' ritual discourse serves as the first layer to provide insight and attention into the window of exploratory discourse. These are described and discussed in the subsections that follow.

6.3 Explorative discourse

This section describes and analyses the PSTs' solution strategies devised to solve the tasks and provides explanations on the perspective of explorative discourse within the Sfard's (2008) commognitive theory. According to Sfard (2008), an explorative discourse is the production of an endorsed narrative about the properties of geometric objects, as well as their relationships. Although evidence of explorative discourse was found in both groups, it was dominant in the discourses of the group A participants. The PSTs whose routines were a manifestation of explorative discourse, showed in their explanation that they have deep and flexible geometric thinking about the objects involved. From the data, evidence of explorative discourse was identified and explained in the sub-themes:

- (1) Frequent expressions of associated geometric properties of objects,
- (2) Engagement in multiple solution path and
- (3) Demonstration of high visual ability.

6.3.1 Frequent expressions about the geometric properties of the objects

Sfard (2008) notes that a key feature of explorative discourse is the ability to produce an endorsed narrative about the properties of an object. Findings from the study showed that some of the PSTs (from both Groups) focused their attention on the properties associated with the task design. That meant, their discourses about the solution strategies were more of justifying with the objectified properties that necessitated the preferred routine. This solution strategy was mostly found in the discourses of the PSTs in Group A as compared to those from Group B. The Group A participants substantiated their discursive actions by producing endorsed geometric properties about the task. It was found that the participants' plan for devising solutions was based on identifying the most associated properties of the task.

Their ability to identify the associated properties could be attributed to critical visual engagement and mental manipulation with the task(s). Mudaly (2021, p.2) maintains that we manipulate objects in our minds in order to “create a deeper understanding”. This process resulted in their discursive actions, which aligns with the explorative way of thinking (Sfard, 2008). They were less prompted to justify why they chose a particular solution path. Their solution path, was probably based on how the task registered in their minds (internalisation). This is evident in the extract of Stephen’s (A) response that follows:

... the diagram given I can say that the angle 130° is vertically opposite to the two angles (m and 60°). Therefore, knowing that vertically opposite angles are equal, I can write it as

...

Similarly, Jones (A) explained that:

... from the diagram, we can see that $3(x - 20)$ is vertically opposite to $2x$ which are equal. So, I equate the two angles ...

This property-guided explanation was also found in Cynthia’s (B) discourse who said:

... we are having direct opposite [vertical] angles here which are $3(x - 20)$ and $2x$ [points to the angles] which are equal.

Their responses show that they have developed the competence to engage in a critical exploration of the task design and to identify its objectified properties. Their responses show that their solution strategies are informed by identifying the most objectified properties of the task. Participants (in both Groups), who were unable to produce the objectified properties of the geometric shape (s), preferred working with the straight line algorithm.

The participants’ thinking, to produce objectified properties about the geometric tasks, was also frequent in their discourse on triangles. The tasks on triangles were designed so that variables in the exterior angle position would draw attention to the exterior angle theorem of a triangle. Explorative thinking was evident in the comments of Group A participants about their discourses on triangles, compared to their counterparts in Group B. For example, in task 2.1, all the participants in Group A, together with Alex in Group B, recognised the angle to be found as an exterior angle and hence, they used the exterior angle theorem in their solution to the task. Some

of them gave an explanation that indicated their understanding of how the theorem was developed, using the ‘sum of angles in a triangle’ and ‘the adjacent angles on a straight line’. Hence, they felt confident in applying the theorem to the task. The following extracts depict their explanation.

... this is an exterior angle [pointing to m] ... [the sum of the] two ... opposite [interior] angles will be equal to the exterior angle. ... you know 62° we add to 75° and you get 137° that will be equal to the exterior angle ... m (Alex, B).

... I should find ... m . For m , ... is an exterior angle... the sum of two opposite interior angles of the triangle [pointing to angle 75° and angle CBF], labeled n . So, this n and 75° , when you sum them, it is going to give me m ... the exterior angle. So, then I can say the m is equal to 75° plus 62° . Then adding, m is 137° (Jones, A).

... having found the two opposite interior angles for my m , ... m my exterior angle is equal to 75° plus 62° which gave me 137° (Stephen, A).

Similar property-guided thinking was evident in Jones’ (A) discourse on task 2.2. His discursive actions, as seen in the extract below, were based on identifying properties associated with the task. He may have frequently engaged with geometric tasks and understood that the easiest way of solving geometric tasks is by applying associated properties. There was, however, a decrease in the number (from five to three) of participants who focused on the identification of the exterior angle in task 2.2. The rest of the participants’ responses focused on the use of angle properties in a triangle and the straight line algorithm. Jones’ response is shown below.

Researcher: Can you explain how you organised your thinking to solve the task?

Jones: *...to find or calculate for angle URS ... the angle here [moving the pen along the arms UR and RS]. I planned to use the exterior angle theorem because of its position. So that if I get here [pointing to angle QUR] and this angle [pointing at angle UQR], then I sum them and use the exterior angle theorem.*

This solution strategy, was evident in the discourses of three participants in Group A. They demonstrated high explorative thinking due to their ability to produce associated properties to the task (Sfard, 2008). The rest used an equally good approach of the straight line algorithm, possibly

because they felt their answers would be guaranteed through the rule-based discourse. Group A participants' demonstration of such thinking means that they have a good understanding and the mental ability to interpret information in a meaningful way, as noted by Nahdi & Jatisunda (2020). In explorative discourse, discursive actions are based on understanding mathematical principles underlying one's routine and the ability to explain and substantiate why such an action is taken (Sfard, 2008). Such discourse is the "product of a process that connects prior knowledge with new knowledge" (Nahdi & Jatisunda, 2020, p. 2). The Group A participants' competence in engaging in explorative discourse, demonstrates that they have developed rich connections between the properties of geometric shapes used in the study. Although they talked about the exterior angle in relation to triangle QUR , the theorem could be applied equally to triangle PRU .

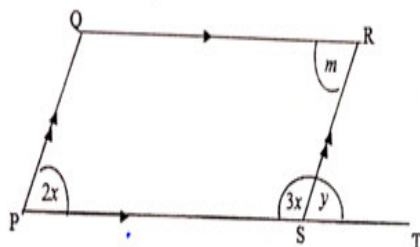
According to Dewi and Asnawati (2019), teachers should show a clear understanding of geometric concepts to inform their knowledge of applying these concepts in solving related tasks. In this study, the PSTs who produced associated properties, performed beyond the use of strict rules (straight line algorithm), which was dominant in the Group B participants. The Group A participants showed competence in devising solutions using geometric concepts and properties of geometric figures and shapes. This geometric competence was evident in their solution processes. During the interview, the participants in Group A substantiated their discursive actions with endorsed narratives, which is a feature of explorative discourse (Sfard, 2008). These competencies enabled them to devise objectified solutions to the tasks. The PSTs' ability to develop such competence in thinking, was an indication that they have the potential to plan and work with the associated geometric properties based on the design of the task, to enhance learners' understanding (Dewi & Asnawati, 2019; Sugeng & Nurhanurawati, 2018).

A similar explorative routine was evident in some of the participants' solutions for tasks on triangles and quadrilaterals. The characteristic feature of the explorative routine is the application of the exterior angle theorem which was needed to find the angle measure of m in task 2.1 on triangles. All Group A participants, together with Alex, in Group B, were able to show their knowledge and competence in applying the theorem to the task. This routine, guided by the theorem, led to the use of fewer steps in solving for m compared with those whose solutions were devised along the straight line concept using a series of steps. The participants who produced an

explorative discourse, showed that they were capable of engaging in mathematical thinking by producing objectified properties as a guide to solve geometric tasks (Sfard, 2008). This was evident in their word use and attention to some visual cues in solving the task. It was evident in the planning phase that they focused attention on the position of the angle to be found. Participants were often found to be moving their pens, showing the two angles they needed to complete the task using the exterior angle theorem of a triangle.

Explorative discourse is characterised by the ability to apply the concepts of an object, in a practical solution, to its related task. In this case, the learner is said to have developed the competence to analyse, discern and solve problems related to the learned task (Hurrel, 2021; Sfard, 2008). According to Hurrel (2021), explorative thinking is the ability to link the knowledge gained to other mathematical objects. This knowledge competence was seen in the routines of the Group A participants and one from Group B.

Task 3.2. Find the value of the angles marked by letters in the figure below



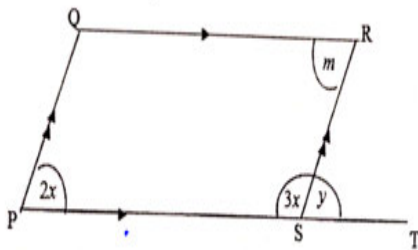
$$\begin{aligned} 2x + 3x &= 180^\circ \\ 5x &= 180 \\ x &= \frac{180}{5} \\ x &= 36 \\ \text{Reason: co-interior angles add up to } 180^\circ \end{aligned}$$

Figure 6.11: Albert's objectified solution to task 3.2.

Stephen (A), Albert (B), and Cynthia (B), when engaged with task 3.2 on the parallelogram in Figure 6.11, analysed it based on the iconic mediators on the line segments. Their approach may have been enhanced by their internalisation processes. As a result, they each formulated the equation $2x + 3x = 180^\circ$ and substantiated it as the co-interior angles. The ability to discern this property of the task, is a demonstration of high visually informed thinking. Dewi and Asnawati (2019) assert that teachers or learners with good visual abilities can explore and apply the most appropriate strategies to solve mathematical tasks. Maxwell also solved the task by applying the objectified properties of a parallelogram, using the equality of opposite angles and the sum of

interior angles of the generic shape [quadrilaterals], which is 360° . He formulated the equation $2x + 3x + 2x + 3x = 360^\circ$ and solved by demonstrating proficient algebraic skills when explaining what he did. Clement made some mathematical arguments, informed by the observed properties, to arrive at the same initial strategy, devised by Stephen and Albert. These arguments were based on the properties of corresponding angles and the angle sum on a straight line. This is shown in his solution below:

3.2. Find the value of the angles marked by letters in the figure below



$$\begin{aligned}
 2x &= y \quad (\text{Corresponding Angles}) \\
 3x + y &= 180^\circ \quad (\text{Sum of Adj. Angles on a straight line}) \\
 \Rightarrow 3x + 2x &= 180^\circ \\
 \therefore 5x &= 180^\circ \\
 x &= 36^\circ \\
 3x + m &= 180^\circ \quad (\text{Co-interior angles}) \parallel
 \end{aligned}$$

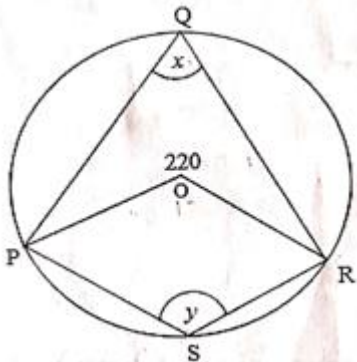
Figure 6.12: Clement's solution through critical reasoning

Clement's (A) solution shows evidence of engaging in critical thinking to devise a possible solution. Not being able to readily identify that $2x$ and $3x$ were co-interior angles, he resorted to reasoning by making mathematical arguments to arrive at the same initial equation that Stephen (A) and Albert (B) started with. Dewi and Asnawati (2019) assert that learners' ability to learn geometry is based on their tendency to make mathematical arguments using geometric concepts and properties. It forms the basis of building learners' critical reasoning abilities.

The ability to solve the tasks on circles, was seen more in Group A participants, than those in Group B. Group A participants demonstrated high competencies in applying theorems and properties to solve the tasks on circles. The most answered task on circles was task 4.1, which is governed by two theorems. These are (1) the theorem governing the angle subtended at the centre and at the circumference of a circle by the same arc and (2), the theorem governing a cyclic quadrilateral. It must be noted that any of the variables involved in the task, could have been found using knowledge of the first theorem. Even though this is being discussed under explorative discourse, where they all identified objectified properties of the task design (Sfard, 2008), their discourse showed that they were familiar with the theorem in its prototypic sketch. As noted

previously, in most traditional mathematics classrooms, the diagram for theorem (1) above, is drawn such that the angle at the circumference is on top of the angle at the centre. Hence, even when it would have been easier to first solve for y in the task, the participants in both Groups preferred to solve for x before y . This is evident in Stephen's solution in Figure 6.13.

4.1 Find the values of x and y in the diagram below.



$$\begin{aligned}
 &\angle POR + 220^\circ = 360^\circ && \text{Angles at a point} \\
 &\angle POR = 360^\circ - 220^\circ \\
 &\angle POR = 140^\circ \\
 &x = \frac{1}{2} \angle POR \\
 &x = \frac{1}{2} (140^\circ) \\
 &x = 70^\circ \\
 &x + y = 180^\circ && \text{Cyclic quadrilateral} \\
 &70^\circ + y = 180^\circ \\
 &y = 180^\circ - 70^\circ \\
 &y = 110^\circ
 \end{aligned}$$

Figure 6.13: Stephen's solution to task 4.1.

6.3.2 Engagement in multiple solution paths (multiple strategies)

According to Sfard (2008), learners' ability to create an accepted narrative regarding the characteristics of geometric objects as well as their interactions, enables them to engage in complex mathematical reasoning and as a result, they are able to devise multiple solutions to tasks. One important way of doing, or learning mathematics, is the ability to engage in cognitive strategies to devise multiple ways of solving tasks through the use of objectified properties (Sfard, 2008).

The ability to devise multiple solutions to tasks was found in the discourses of less than half of the participants, when they were asked if the task could be solved in any other way. It was found that multiple solution ability was demonstrated more among those in Group A compared to those in Group B.

For example, Stephen (A) demonstrated his competence to solve most of the tasks using multiple strategies. In tasks 1.2 and 1.3, which he first solved using an associated property, he also solved

them using an alternative approach. He demonstrated good knowledge of applying the objectified property to solve the tasks and of using the straight line algorithm.

Stephen's use of multiple strategies indicates that he has acquired the geometric thinking needed to solve tasks in multiple ways and can teach learners how to solve tasks using alternative approaches. Such instruction has support from the NCTM's (2000) assertion that instructional strategies should develop learners' capabilities, to apply multiple and appropriate strategies to solve problems. Exposing learners to this kind of instruction develops their flexibility in thinking and their ability to choose the best strategies to solve a task. Klerlein and Hervey (2020) add that it fulfils the goal of equipping learners with a range of strategies needed to solve problems and to understand that some tasks may be solved by applying more than one strategy. Within the commognitive framework, the competence of devising multiple solutions to tasks is a demonstration of an explorative way of thinking and an indication of higher order thinking abilities (Sfard, 2008). Ability to devise multiple solutions to a given task is a characteristic associated with the demonstration of high problem-solving skills, and the ability to be flexible and creative in thinking (Kivkovich, 2015; Maulana & Yuniawati, 2018; Nur & Nurvitasari, 2017; Ortiz, 2016; Robichaux-Davis & Guarino, 2016).

Stephen solved task 1.4 using multiple approaches. In recall, we find that Alex (B) could not think of any possible strategy to solve it, whilst Clement (A) said the task was a bit complex and that he needed to include some line segments. But Stephen and Maxwell, Group A, showed that the task could be solved in multiple ways. Strategy one was based on extending the straight line perpendicular from the top parallel line to the bottom parallel line to form a triangle in order to apply the emerging angle properties. In strategy two, he explained his thinking on the concept of parallelism, by introducing a new line segment parallel to the existing ones to produce some associated properties that could be applied. These two solutions are shown below.

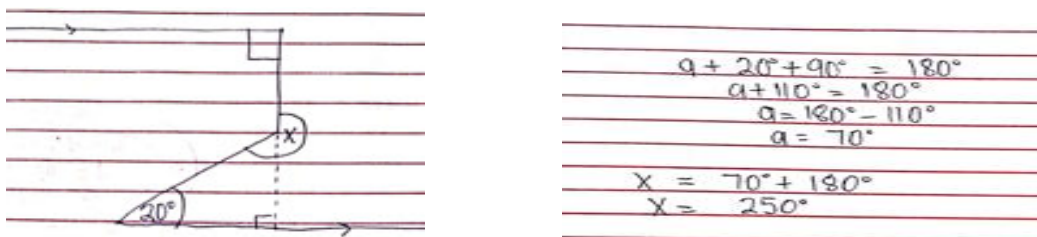


Figure 6.14: Stephen's first solution to task 1.4.

Stephen explained his thinking about the solution in Figure 6.14 by saying that:

... when I extend this line to meet this parallel line at 90° , I realise that the angle for x has been divided into two. One inside the triangle I have formed using the line that I have extended and the other over here [pointing on the extended straight line].

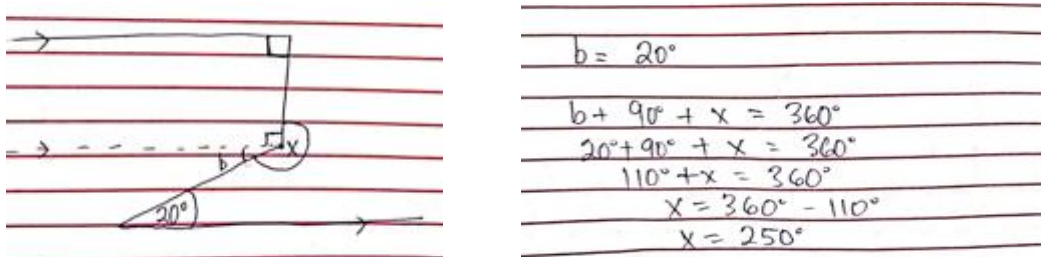


Figure 6.15: Stephen's alternative solution to task 1.4

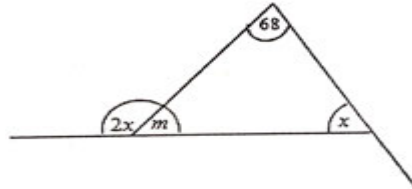
In his plan for the alternative solution, he said:

I decided to introduce another parallel line [the dotted line] here. ... I introduce a third parallel to meet the vertex at this point at 90° . The reasoning why I did that was because I want to find the angle x with the method using angle at a point.

The multiple solutions demonstrated by these participants show that they have acquired good knowledge to enable them to teach learners to solve tasks using multiple strategies. Teaching several strategies for solving problems can help learners adjust to the one they understand most. Teachers, where possible, need to explain how to solve a task in multiple ways to dispel learners' notions that mathematical tasks are solved in only one way. Knowledge of multiple solutions enables teachers to teach with confidence. Teachers can teach with confidence and enthusiasm when they can answer questions from different perspectives. Its importance is also felt in developing learners' autonomy in learning. In the view of multiple approaches to solving mathematical tasks, Leikin (as cited in Bingolbali, 2011) argues that multiple approaches to tasks is a meta-mathematical habit that enhances advanced mathematical thinking, which contributes to the learners' critical thinking, creativity and problem-solving skills.

Some of the participants who solved the tasks on triangles, by applying the exterior angle theorem, also demonstrated the knowledge of solving the same task using different methods. Jones' multiple solutions to the tasks are shown below.

Task 2.3. Find the value of the angle marked by letters in the figure below



$2x = 68^\circ + x$	$2x + m = 180^\circ$
$2x - x = 68^\circ$	$m = 180^\circ - 2x$
$x = 68^\circ$	$(180 - 2x) + 68^\circ + x = 180^\circ$
	$x - 2x + 248^\circ = 180^\circ$
$m + 2x = 180^\circ$	$-x = -68$
$m + 2(68^\circ) = 180^\circ$	$x = 68$
$m + 136^\circ = 180^\circ$	$m + 68^\circ + 68^\circ = 180^\circ$
$m = 180^\circ - 136^\circ$	$m = 180^\circ - 136^\circ$
$m = 44$	$m = 44^\circ$

Figure 6.16: Jones' multiple solutions to task 2.3.

Evidence from research shows that teachers who can demonstrate knowledge of multiple solution strategies to tasks, are better able to provide a differentiated discourse in class to enhance learners' explorative mathematical thinking, and to provide quality instructional sessions to aid learning (Bingolbali, 2011).

Some of the PSTs seemed uncomfortable when asked to devise another solution to the tasks using a different method. They probably felt that trying to produce one solution to the task was enough. The most common response to the question about possible alternative solutions was 'no' or 'no idea'. For example, Jones responded, "*Different? No idea*", when asked to devise an alternative solution to task 1.4. On the same task, Clement laughed and said, "*no other method comes to*

mind". This negative reply was also found in the response of the Group B participants to the tasks they could not solve using multiple approaches.

This could mean that these participants possessed little or no idea of different ways to solve a task in geometry. A few PSTs (mostly from Group A) demonstrated competence in completing the tasks using multiple approaches. This **could** mean that **many of the** participants who are upcoming teachers, have limited knowledge of multiple solution strategies, which is an important approach for developing learners' problem-solving abilities. According to Semanisinova (2021), one of the most important aims of PSTs' programmes is to build their competence, to guide and provide relevant and constructive feedback to learners. The mathematics teacher should be able to provide feedback to learners on the solutions and strategies they employ, as well as to help them to think about other possible methods to solve the same task. Teachers should be able to detect learners' solution paths as right or wrong, from multiple perspectives. The teacher should possess the knowledge capacity to provide the solution method that is specific, or appropriate to a given task, and also use concepts and properties to devise solutions for learners. The quality of this feedback and teacher guidance depends on the depth of knowledge the teacher possesses (Semanisinova, 2021).

When PSTs have limited knowledge about solving geometric tasks, it can seriously affect the quality of instruction and guidance they would provide for their future learners. Thus, a learner may be marked wrong, even if his or her solution process is correct, which may happen because the learner did not present the solution in the way the teacher is familiar with, or used to. In this study, more of the Group B participants showed limited knowledge of solving some tasks that had different solution approaches. Indeed, many of the tasks used in the study could be solved in more than one way and the participants were required to demonstrate their thinking abilities by devising multiple solution paths. The study findings showed that many of the Group B participants could not solve most of the tasks in multiple ways. This finding supports that of Bingolbali (2011) who found that the study participants who were teachers, were not open to the multiple solutions that learners provided for the tasks assigned. In addition, the teachers demonstrated difficulty in evaluating (grading) different solutions that were presented to them. Their difficulty was that they could not tell if the solutions to the open-ended tasks were correct or not.

In summary, the discourses of a many participants in Group A were noted to be more objectified in their geometric thinking. Their talks and solution for the tasks were more efficient and object-driven in nature. The discursive actions of these participants showed more substantiation with geometric properties, in an unprompted manner. Some of the Group B participants' responses involved more procedure-based explanations, which shows the 'how' of the solution. In most cases, they were prompted to substantiate their narratives. The Group B participants' talk on the geometric task showed how to execute the task, compared to the Group A, who in most cases, were more concerned with thinking about the appropriate property to use in devising strategies to solve the tasks.

The discourses of three participants in Group A revealed that they think of using properties to provide a more objectified discourse about an object. In addition, signifiers used in communicating geometric concepts seemed to attract more attention of the Group A participants than their counterparts in Group B. These actions of the Group As, seemed to be informed by their ability to recognise and interpret certain important features of the tasks. The habits of recognising, analysing and interpreting the features of the task were more evident in the discursive actions of members of Group A than their counterparts in Group B. There was evidence of discursive fluency in the discourses of these participants in Group A due to a greater ability to substantiate their narratives. Some of the Group B participants made lesser attempts to understand the structural design of the tasks. This contributed to their dominant strategy of solving most of the tasks using a straight line algorithm. Thus, the Group B participants engaged more in processual thinking than object-oriented thinking (Sfard, 2008). Some of these participants seemed to connect their processes with recall instead of engaging in explorative thinking about the tasks. Whilst the Group A participants made an attempt to individualise the discourse, by drawing connections among narratives to produce an endorsed one, more of the Group B participants were noted to rely on the recall of procedures showing a lack in objectifying their discourse.

6.3.3 Demonstration of high visual abilities

Visual mediators among the elements of Sfard's (2008) commognitive constructs, make visual abilities an important tool for learning mathematics and geometry. According to Sfard (2008), the

practices of connecting our thinking to scanned (visualised) mediators enable us to remember more, and have visual abilities to activate our thinking in response to discursive cues. Visual mediators could be diagrams, sketches, graphs, pictures, and the like. According to Mudaly (2021), diagrams stimulate learners' thinking about concepts and ideas associated with mathematical objects. Good visual interpretation largely depends on drawing a connection between thinking and visualisation in relation to previous knowledge. Dewi and Asnawati (2019) claim that learners' geometric abilities are well developed when they have good visual abilities. In this study, those who demonstrated an explorative discourse in their routine, utilised or relied on their visual abilities to gain a clearer understanding of the task, before devising the solution.

Considering participants' engagement with the tasks, they were often found to rely on their visual **abilites** to analyse and interpret some visual cues, based on the structure of the task. This is apparent in their discourses about planning to solve task 1.3, as in the excerpts that follow:

... looking at the diagram given, I can say that the angle 130° is vertically opposite to the two angles (m and 60°) ... (Stephen, A).

... from the diagram, we can see that $3(x - 20)$ is vertically opposite to $2x$ which are equal. So, I equate the two angles (Jones, A).

... I used this approach because of the properties I see within this diagram (Cynthia, B).

... in geometry, when you see these short lines or dashes, it means ... (Clement, A).

It is evident from these excerpts that these PSTs' solution plans were centred on what they saw and interpreted about the task. There were instances where the participants used their pens to trace or move along the line segments. These findings about the participants' responses and their actions made it obvious that visual abilities formed an integral part of their thinking processes, needed to devise solutions to the tasks. Atanasova-Pachemska et al. (2016) consider visual abilities as mental processing of visually obtained information that enables mathematical discovery and understanding of mathematical tasks. Some of the study participants were noted to utilise their visual abilities of 'seeing to think', or think with what was seen; and in most cases, it was evident in their actions when they were observed to be tracing some parts of the task. These actions might

have aided their ability to recognise certain important features of the task and to think about associated properties.

Although all participants utilised their visual abilities in thinking about the tasks, it was observed in some of their utterances that their action plans were based on what they saw and analysed (visual senses). They were all given the same tasks, but they analysed and interpreted them differently, leading to different approaches in their solutions. This could mean that those who faced challenges in solving some of the tasks had weak visual abilities, among other factors. Riastuti et al. (2017) assert that learners' difficulty in understanding geometric concepts, and devising the most appropriate solutions to tasks, is due to their weak visual abilities. This weakness in their visual abilities makes it difficult for them to recognise, analyse and make meaning of what they see to gain deeper understanding of the task (Dewi & Asnawati, 2019; Mudaly, 2021).

Visual ability can therefore be considered as one of the major tools for learning mathematics and plane geometry, where much information is communicated by using diagrams and other signifiers to communicate important features, for clearer understanding (Sfard, 2008). The utterances of the participants in the study seemed to show that their efforts to devise solutions to the tasks were based on their visual senses.

Wai (2018) adds to the debate on visuals by saying that learning geometry greatly relies on visual reasoning. This could mean that visual reasoning abilities contribute significantly to developing high geometric thinking. Research has documented that those teachers who have high visuals, spatial skills and reasoning abilities positively influence their classroom teaching and learning activities (Dewi & Asnawati, 2019; Ozdemir & Goktepe Yildiz, 2015). According to Ozdemir and Goktepe Yildiz (2015), there is an interrelation between spatial skills and geometric thinking. It can therefore be conjectured that those participants who demonstrated their geometric thinking through high visual reasoning, have developed a good level of visual ability necessary for developing and enhancing learners' visual thinking abilities. Thus, teachers who have been observed to possess a good level of content knowledge, as well as good visual abilities, would be able to provide supportive experiences to guide and shape learning in the classroom (Dewi & Asnawati, 2019; Wai, 2018).

Those who explored geometric tasks with a high visual sense were able to solve the tasks in a well-organized way because the development of geometric skills is highly dependent on one's visual abilities. Task 3.3 was a combination of different geometric shapes, which required the participants to employ their visual skills first, to identify the geometric shapes to guide their thinking about the related properties needed to solve it. Though all the participants obtained the correct answers, diverse approaches were used in the solution, resulting from their differences in visual abilities. Whilst those with high visual abilities were able to notice and use the task's emerging properties, others with low or weak visuals, used straight line algorithms to solve the tasks.

6.4 Routine thinking on the inclusion relation of quadrilaterals

According to Sfard (2008), when an object has a set of endorsed narratives which are subset of another object, the objects are said to be highly similar and can be called by the same name. This can be applied to the inclusion relations of classifying quadrilaterals. Within the framework, the process of classifying quadrilaterals can be classified as either a ritualised or an explorative way of thinking.

In a ritualised discourse, a shape such as a 'square' is considered as a distinct object and cannot be called by another name. Hence, it does not have any relationship with other shapes. However, in explorative discourse, a shape can have several names, such as 'a square also being called a rectangle or a parallelogram' when they have similar endorsed narratives (Sfard, 2008). According to Tuset (2019), defining and classifying quadrilaterals, promotes learners' development of geometric thinking and their ability to make mathematical argument.

In this study, PSTs were required to define the various quadrilaterals, to draw them and show their class inclusion. Although definition, drawing and some the properties have been considered in the previous sections, any reappearance is meant for emphasis and clarity of description. The following sub-sections describe the participants' thinking about inclusion relations of quadrilaterals.

6.4.1 Relation between rectangle and parallelogram

Analysis shows that the PSTs have difficulty with inclusion relations among quadrilaterals. Of the eight participants, only two were able to identify that a rectangle is a parallelogram, using the necessary conditions whilst the other used reasoning about the properties. Their justification was based on the necessary condition of defining a parallelogram as a figure that has two pairs of parallel sides or a quadrilateral with two opposite sides being parallel (Rianasari et al., 2016; Ulger & Broutin, 2017). For example, Jones (A) said, “... *because its two pairs of opposite sides are parallel*”. Clement (A) also said that “*it [rectangle] has two pairs of opposite sides parallel*”. Stephen (A), on the other hand, showed his understanding of the inclusion of a rectangle in a parallelogram, although he did not mention the concept of parallelism of both sides. He compared some properties of the two geometric figures as a way of justifying the inclusion of a rectangle in a parallelogram. Stephen’s discourse is shown in the excerpt below.

A rectangle ... has two pairs of opposite and equal sides ... a parallelogram ... a figure with two pairs of opposite and equal sides. ... has opposite interior angles of equal measure. ... looking at a rectangle, all interior angles are 90°, meaning the opposite angles are also of equal measure.

According to Rianasari et al. (2016), showing relationships among quadrilaterals is built on exercising one’s deductive reasoning. These responses from the three Group A participants indicate that they have a good grasp on the idea that rectangles are a subset of parallelograms.

Cynthia (B), on the other hand, said that a rectangle is a parallelogram, but justified it from a quadrilateral point of view. Her justification was that “... *because it is also a four-sided figure*”. This wrong justification for inclusion criteria indicates that Cynthia (B) did not understand, or found it difficult to engage in relational thinking about the inclusion of rectangles in parallelograms.

In addition, Maxwell (A) and the rest of the participants in Group B responded that a rectangle is not a parallelogram. Maxwell gave the reason that “*for the properties of a parallelogram, two opposite sides and angles are equal but for a rectangle, I know that their opposite sides are the*

same but the angles are 90°, so a rectangle cannot be a parallelogram”, which shows Maxwell’s ritual thinking. His response shows that he could not reason that all the angles being equal is the same or similar to opposite angles being equal. Thus, he considered the properties as isolated facts, which is classified as ritual thinking within the commognitive framework (Sfard, 2008). Alex also said no and explained that *“a rectangle has two of its opposite sides equal and parallel and a parallelogram has two of its opposite angles equal”*. It should be noted that even the comparison is wrong because the equality and parallelism of the sides of the rectangle were compared with the opposite angles of the parallelogram. Through further prompts to draw his attention to the comparison, he responded, *“even the way they look is not equal”*. This could mean that his thinking about the inclusion relations of the shapes is based on their appearance and not their governing properties. This finding supports that of other researchers such as Ngirishi and Bansilal (2019) and Baktemur et al. (2021) who also found that learners struggle with the class inclusion of rectangles in a parallelogram. For example, many participants in the study of Baktemur et al. (2021) did not consider a rectangle as a parallelogram at the beginning of an intervention. This, within Sfard’s (2008) commognitive theory, is characterised by considering shapes and their properties in isolation, and is classified as a ritual discourse.

6.4.2 Relationship between a square and a parallelogram

It was observed, on the inclusion of squares in parallelograms, that the PSTs found it difficult to accept this shape inclusion. Of the eight participants, three in Group A demonstrated acceptable thinking about the inclusion of squares in a parallelogram. Jones’ (A) and Clement’s (A) responses showed knowledge of inclusion among the two, with the justification of parallelism of opposite sides. Stephen’s justification was still based on reasoning about the properties of the two shapes.

The five others either said ‘no’ or ‘yes’ with the wrong justification. For example, Cynthia still justified her ‘yes’ response with the concept of both shapes being quadrilaterals. She said, *“a square is a parallelogram because a square also is a four-sided figure as a parallelogram”*. She was probed further to see if she could give a more acceptable reason, but her response showed no clear reasoning. She stated that *“... because it [square] also has four sides and with the square, all the sides are equal”*.

Cynthia's response shows that she may be referring to, or thinking about partition classification, instead of hierarchical classification, hence referring to the quadrilaterals. The rest of the participants said that a square is not a parallelogram, giving the reason for example, that "*... for a square, it is a type of a quadrilateral which all the sides are equal but that of a parallelogram it is not like that*" (Albert).

Other responses included:

Alex (B): *... for parallelogram two of its sides are equal but for square all the sides are equal and the angles are also equal. But for a parallelogram only two of its opposite angles are equal.*

Maxwell (A): *For the square all the sides are equal but for the parallelogram all the sides are not equal. Only opposite sides are equal.*

The participants' responses are an indication of their difficulty in accepting that a square is a parallelogram, which is a demonstration of ritual thinking because of their inability to draw any relationship between the shapes (Sfard, 2008). Their persistent unsatisfactory response could mean that they may not have been exposed to a hierarchical classification, in which emphasis is laid on particular concepts that are a subset of a more general concept (Baktemur et al., 2021).

6.4.3 Relationship between a rhombus and a parallelogram

The inclusion of the rhombus in the parallelogram was favourably responded to by almost all the participants, compared to previous questions. Seven of the participants provided favourably, even though some gave a wrong justification. Cynthia (B), as in previous responses, stated that both figures have four sides (quadrilateral), whilst Jones and Clement (both in A) claimed that both figures have opposite sides parallel. Stephen responded that, "*a rhombus is also a special type of parallelogram. It satisfies having two opposite and equal sides and angles*".

In probing further to understand what may have informed their thinking, it was observed that some of them drew the two shapes to support their reasoning. This could mean that their thinking may have been informed by the appearance of the two shapes, but not by the parallel nature of both sides. For example, Albert said, "*both figures may have common properties because they look similar*". When Maxwell drew the figures, he said, "*I ... opposite angles of a rhombus are equal,*

and opposite angles of the parallelogram are also equal". When probed further, he responded, *"that is how I see it"*.

6.4.4. Relationship between a square and a rectangle

The participants' thinking on shape inclusion seemed to have ended with comparing the shapes to parallelograms. When we moved to comparison within shapes, particularly squares and rectangles, only Stephen (A) responded favourably, and said, *"a square is a rectangle which has all sides to be equal. Yes, it is a special type of a rectangle"*. This could mean that he has a good understanding of the inclusion relation between a square and a rectangle. In other words, Stephen understands that squares form a subset of rectangles. Findings concerning Stephen's justification, supports that of Rianasari et al. (2016, p. 4) who also found their study participants using the phrase "... is a special form of ..." to justify the inclusion of squares in rectangles. Cynthia, wrongly supported her claim from a quadrilateral point of view.

The remaining six participants said that a square is not a rectangle, and supported it with reasons as follows:

Maxwell (A): *... for a square all the sides are equal and for rectangle opposite sides are equal.*

Albert (B): *the square is a type of a quadrilateral in which all the sides and the angles are the same but with that of a rectangle, their opposite sides are equal.*

Clement (A): [smiled, sat quietly, and said] *A square cannot be a rectangle. A rectangle has its opposite sides measuring equal but a square has all sides equal.*

Alex (B): *... for square, all the sides are equal but, in a rectangle, only two of its sides are equal.*

The participants' responses show that their patterns of thinking are almost the same. They considered the shapes in isolation not considering any possible relationship that could be drawn from the properties (Sfard, 2008). These responses suggest a ritual routine. Thus, these participants did not accept that a square is a subset of a rectangle. Only two (in Group A) of the eight participants showed that a square is a rectangle. This finding supports that of Rianasari et al. (2016)

who also found that few of the study participants, who were pre-service teachers, showed good thinking about the inclusion relations of squares in rectangles.

6.4.5 Relationship between a square and a rhombus

Only Clement (A) responded favourably that a square is a rhombus and justified that, *“a square is a rhombus. Their characteristics are almost the same. Four sides equal ..., both have opposite sides being parallel, diagonals bisecting at 90° , a type of rhombus”*.

For the rest of the participants, it was difficult to think that a square is a subset of a rhombus. Maxwell justified his ‘no’ response by saying that *“for the square all the sides are equal. Even though for the rhombus, all the sides are equal, the angles in the square [each] is 90° but for the rhombus the angles can be different. Opposite angles are the same”*.

This shows the ritual thinking of Maxwell and the rest of the participants. Maxwell thought that the fact that each angle in a square is 90° was different from the fact that opposite angles in a rhombus are the same.

Stephen and Jones had performed quite well on the shape inclusion when compared with a parallelogram. However, data from the study shows that their understanding of the relationship among the other shapes was not well developed, or they had no knowledge about it. On square and rhombus, Stephen said, *“a square is not a rhombus because in a square, all interior angles measure 90° , but in rhombus, we do not have that property”*.

This response is similar to that of Maxwell. Jones said ‘no’, but he could not give any justification. Thus, the participants still considered the properties of shapes learned as isolated facts, with no effort to think about the relationships among them. This shows that their thinking about the inclusion of shapes is purely ritual (Sfard, 2008).

6.4.6 Relations between a rhombus and a rectangle

The responses from the participants to the inclusion of a rhombus in a rectangle, show that they have little or no knowledge of such a concept in geometry. The responses showed their ritual

thinking since they could not draw any relations concerning the sides of these figures, which give the necessary and sufficient conditions together with their properties (Rianasari et al., 2016). All the participants initially said ‘no’, even though Cynthia, after considering some properties, changed her decision with some level of uncertainty. She had been wrongly supporting her thinking of inclusion relations from a quadrilateral viewpoint. During the interview, seeking to understand her thinking on the relationship between a rhombus and a rectangle, she said, *“I don’t think so even though they are all four-sided figures”*. She paused for a short while and said *“Yes, it can be a rectangle because with rhombus we have two opposite angles equal like the rectangle”*. Even though her response showed some level of uncertainty, it is possible she may have begun to draw some connections between the properties of the two shapes.

The responses from those who said ‘no’ but used some related properties to justify their argument, are evidence of ritual thinking since they could not make any logical deduction from the properties they raised (Sfard, 2008). Almost all the participants justified their discourse by using the same properties, except for Alex, who said he had no idea. For example, Jones said, *“for a rectangle two opposite sides are equal in length but for a rhombus, all the sides are equal”*. This probably shows that the properties of geometric figures are learned by rote with no understanding of the relationship among them (Hurrell, 2021), which is a demonstration of ritualised thinking within the framework (Sfard, 2008). Perhaps, what they could think of, was to make a logical deduction from the properties apart from the concept of parallelism. For example, consider the properties of the following quadrilaterals, A and B.

A: has opposite sides equal

B: has all sides equal

Question: Which property is a subset of the other?

It can be noted that answering the above question demands critical thinking. It may be conjectured that it is for these critical thinking, reasoning and problem-solving abilities, that the teaching and learning of geometry is given more attention in the mathematics curriculum (Erdogan & Dur, 2014; NCTM, 2000). Possibly, if the participants had engaged in such a logical deduction from the properties, their thinking about inclusion relations may have improved.

6.6 Conclusion

This chapter commenced with a description and analysis of the PSTs' solution strategies for the geometry tasks that were used. The data used here were their solutions to the geometry tasks and the explanations of those solution strategies. The solutions provided for the test gave an initial understanding of the routines used. The explanations provided during the interviews allowed for further insight into their thought process.

Based on the analyses conducted on the solutions and their explanations, the PSTs' routine thinking was classified as either ritual discourse or explorative discourse, as guided by the characteristics of the commognitive theory. The analyses of the PSTs' routines were presented and supported with appropriate evidence from the solutions and the explanations. Their routine thinking was analysed and categorised into four sub-themes under ritualised discourse, and three sub-themes under explorative discourse. The ritualised sub-themes were: the use of step-by-step approaches; a series of computational steps; the use of simultaneous equations; and a low range of applicability/less attention to important features of the tasks. The explorative sub-themes were: frequent utterances of geometric properties; engaging in multiple solution paths; and high visual abilities.

Further analysis of the PSTs' routines was conducted on their thinking about inclusion relations of quadrilaterals and some basic proofs in geometry. These were discussed in the light of their difficulties in believing that some quadrilaterals can have different names based on the necessary and sufficient conditions.

This chapter and the previous one provided an insight into the PSTs' discursive geometric thinking, as well as the nature of their routine strategies in solving geometric tasks, and their thinking about the class inclusion of quadrilaterals. The next chapter deals with how this geometric thinking informs their classroom discourse.

CHAPTER SEVEN

CLASSROOM GEOMETRIC DISCOURSE

7.1 Introduction

The previous chapter analysed the pre-service teachers' routine geometric thinking in relation to ritual and explorative ways. Based on the insight gained about the participants' thinking with respect to the commognitive framework, coupled with their ritualised or explorative ways of thinking, this chapter analyses their classroom geometric discourse. In reporting and discussing the classroom discourse in this chapter, the terms 'colleagues', 'learners' or 'colleague learners' will be used synonymously.

7.2 The PSTs' classroom geometric discourse

7.2.1 Stephen's (A) classroom geometric discourse

Stephen (A) performed well by solving all 16 geometry tasks in the test during the first phase of data generation in the study. He explained his thought processes governing the definitions of geometric concepts and the solutions of the tasks using mathematical words. He defined some quadrilateral shapes with knowledge of class inclusion and solved most of the tasks using multiple approaches. He also demonstrated adequate reasoning on some fundamental geometric proofs and showed evidence of an explorative way of thinking in his discourse.

Stephen taught the properties of the rhombus using paper folding and measurement. He started the lesson on the assumption that the class had learned about parallelograms and reviewed the learners' previous knowledge on parallelograms. The interaction during the review process is shown below.

Stephen: *What is a parallelogram?*

Colleague: *A parallelogram is a plane figure bounded by four sides and has four angles.*

When Stephen was not satisfied with the response, he asked for another colleague's view. One said that: *"A parallelogram has two [pairs of] parallel sides, two opposite sides to be equal and two opposite angles to be equal"*.

Stephen refined the definition, drew the parallelogram on the whiteboard and explained the properties that the colleague used in stating the definition. He drew the learners' attention to one important and necessary condition for defining or considering a four-sided figure as a parallelogram. He said:

If you have a figure of this sort [pointing at the figure on the whiteboard] ... all you need to do is to look and see if the two pairs of opposite sides are parallel, then it is a parallelogram.

After a review of some properties of the parallelogram, he moved to the day's topic and said:

Today we will be talking about a rhombus, and in fact, a rhombus is a special parallelogram.

He explained what he meant by 'special parallelogram' by saying that a rhombus has all the properties that a parallelogram has. The unique feature is that all the sides are equal.

Stephen gave samples of paper cut-outs of a rhombus, with varying dimensions, to the colleague learners in groups and guided the folding activities with some instructions. The extract below shows the classroom dialogue that accompanied the teaching activities.

Stephen: *What do you realise after the folding?*

Colleague: *They [one part] fall on each the other perfectly.*

Stephen: *Ok first, what is the name of the shape formed?*

Colleague: *Triangle.*

Stephen: *So, we have established that when the rhombus is folded along the diagonal, one part falls on the other perfectly. In the same way, when it is folded along the other diagonal [shows], one part falls exactly on the other.*

Based on Stephen's first demonstration of folding a parallelogram (during the review), in which the shapes formed overlapped, he questioned the colleague learners about what accounted for the overlapping shapes in the parallelogram, and perfect fitting shapes in the rhombus when folded along the diagonal. After a short time (probably examining the critical features of the two shapes), a colleague in Group A said, *"I think because in the rhombus, all the sides are equal"*. He asked

if the whole class agreed with him. He further asked for any different opinion as a way of giving everyone the opportunity to share any possible different ideas.

The colleague learners within the various class grouping were found to share ideas governing the activities of the object. Cooperative engagement was evident among them. Chan and Idris (2017) describe cooperative learning as a teaching technique in which learners are placed in small groups where they assist one another in the learning process.

Stephen continued to ask questions throughout the lesson. When he had written two of the answers from the learners' measurements of angles on the whiteboard, he asked for the learners' observation about the angles measured. A colleague (in B) answered, "*the opposite angles are equal*". He consistently asked questions to ensure that the learners, although engaged in the activity, followed the key concepts he was teaching. Through the paper folding, cutting, and measurement activities, he directed the class to observe the following properties of the rhombus:

1. All sides are equal.
2. There are two lines of symmetry.
3. The diagonals bisect the angles at the vertices (corner angles).
4. The diagonals bisect at right angles.

Stephen paid particular attention to the learners' use of the correct and appropriate geometric terms in communicating their ideas, to ensure that they developed and used those terms in geometric discourse. An example of Stephen's guidance of literate word use in geometric discourse is shown below:

Stephen: *What do you notice about the diagonal and the angle at the vertex (corner angles)?*

Colleague: *The diagonals divide the angle into two equal parts.*

Stephen: *When something is divided into two equal parts, what term (word) do we use for it in mathematics or geometry?*

Stephen: *It starts with "b"*

A colleague learner raised the hand and said, 'bisect'.

Stephen asked them to use that word to describe their observation as “*the diagonals bisect [the] corner angles*”.

As a way of ensuring the colleague learners’ understanding and application of the concepts learned, he solved examples with them. The example required the learners to make use of the property that each diagonal bisects the angle at the vertex. Stephen, crafted a task to test their understanding and relational thinking gained in the lesson. Overall, the materials chosen for teaching the properties of a rhombus, the activities performed, questions used within the discourse, the kind of explanation given, with critical concern for word use, the diagrams used to mediate the concrete and abstract concepts, all indicate that Stephen demonstrated a high level of in-depth geometric thinking for teaching the selected topic.

7.2.2 Jones’ (A) geometric discourse

Jones (A) performed quite well in the test. He scored correctly on fifteen of the sixteen items. He used mathematically literate words in defining almost all the geometric concepts and described his thought processes about the solutions devised to solve the tasks. He solved one of the tasks in a ritualised way and the rest of the tasks were solved with some evidence of explorative thinking, during the task-based interview. He also solved most of the tasks using multiple approaches and showed knowledge of the proofs involved.

Jones prepared to teach the first two circle theorems. As a way of firmly developing the colleague learners’ understanding of the sketch of the theorem, he reviewed their previous knowledge on the definition and meaning, of parts of a circle. Among the parts mentioned by the colleagues in response to his question were diameter, circumference, sector, chord, and the like. Jones assessed his colleagues’ ability to define a circle including its various parts. He started the review by asking the colleagues to define a circle, as shown in the following dialogue:

Jones: *What is a circle?*

Colleague: *A circle is the locus of points equidistant from a fixed point.*

He repeated this definition, drew a circle on the board and added that the fixed point is called the centre of the circle, which he labeled with the letter O . He continued with the review.

Jones: *What is a diameter?*

Colleague: *A diameter is a line that divides a circle into two equal halves.*

Considering this definition to be colloquial (Sfard, 2008), he asked them to produce a more mathematical and acceptable definition, as seen in the following dialogue.

Jones: *It is ok but I want a more geometrical definition.*

Colleague: *A diameter is a line segment drawn through the centre of the circle with its endpoints lying on the circle.*

Jones accepted this definition, drew a circle (using free a hand or not drawn to scale) and a chord passing through the centre on the whiteboard and laid emphasis on the words ‘through the centre’ and ‘endpoints on the circle’. He pointed at these for the learners to see, and labelled the endpoints of the line segment as AB and the centre with the letter O .



Figure 7.1 Jones' sketch of a diameter

He continued to talk about what the terms meant, and showed his colleagues visual representation of these terms by drawing them on the whiteboard.

Jones: *What is a chord?*

Colleague: *It is a line drawn from one point on the circle to another point on the circle.*

Jones (the teacher) seemed to be confused about the relationship between a chord and a diameter. A colleague asked, “*can a diameter also be called a chord*”? He said “*no, a diameter is not a*

chord”, and gave the reason that a chord does not pass through the centre of the circle. This generated some intellectual debate among the peers. Then, another colleague contributed that:

I think you told us that a chord is a line drawn from one part of a circle to touch the other part. So, looking at the diagram, we can see that the line is drawn from one part of the circle to touch the other part of the circle. So, I think per your definition, the diameter is also a chord.

Jones listened to him and responded that it is a chord which passes through the centre of a circle. This means that teachers should be careful in answering learners’ questions that they may not be too conversant with, to prevent learners from losing confidence in them. When all doubts were cleared, the lesson continued.

Jones: *What is a tangent?*

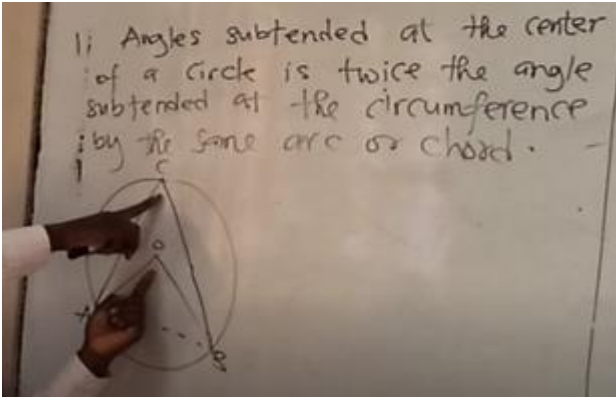
Colleague: *It is a line drawn to touch the circumference of a circle.*

Jones: *At how many points does a tangent touch the circle?*

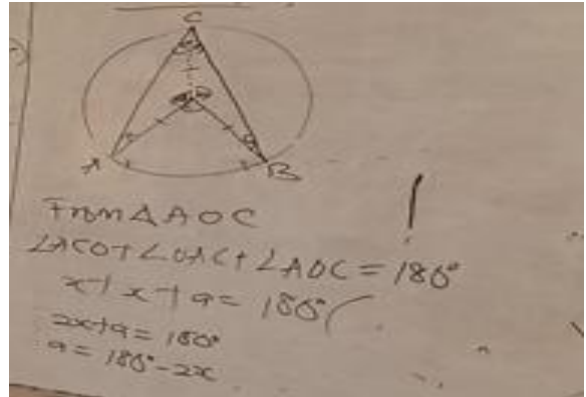
Colleague: *at one point.*

All these parts of a circle were drawn for the learners to create a mental image of those spatial ideas. For example, when the colleague answered that a tangent touches at one point, he drew, and added that the point is called the point of tangency. After this review, Jones introduced the day’s lesson, which was ‘circle theorem’.

Jones wrote the theorem governing the angle subtended at the centre and at the circumference by the same arc. He explained these two key words, ‘angle at the centre’ and ‘angle at the circumference’, in the theorem to the class, as seen in Figure 7.2 (a). In demonstrating the proof, he drew a radius from the centre, *O*, of the circle to the point *C* on the circumference to form two smaller triangles. He explained that the two triangles were isosceles because they were bounded by two radii, and he demonstrated the equality of the sides with an iconic mediator (see Figure 7.2b).



(a)



(b)

Figure 7.2. Jones' definition and the initial proof of the theorem

He described the properties of an isosceles triangle and used the same variables to represent the angles opposite the equal sides. He also labelled the angles formed at the centre of the circle by the three radii as a , b , and c (see Figure 7.2b). When he modeled the angle for the interior angles of the isosceles triangles, he asked the learners for the justification for it.

Jones: *What informed this equation?*

Colleague: *The sum of angles in a triangle is equal to 180° .*

Jones: *How about the angle at the circle?*

Colleagues (chorus): *the sum of angles at a point is equal to 360° .*

He explained the proof process to the learners in a comprehensive way. Finally, the results obtained showed that $c = 2(x + y)$, where c , is the angle at the centre, and $(x + y)$ is the angle at the circumference. He solved examples and ended the class.

7.2.3 Maxwell's (A) classroom geometric discourse

Maxwell solved 15 of the 16 geometry items. He explained his thoughts using both colloquial and mathematically literate words. He seemed to have a little difficulty in using appropriate words to define some of the geometric concepts. He solved some of the tasks using multiple approaches. In his discourse, he demonstrated both ritual and an explorative way of thinking.

Maxwell reviewed the learners' knowledge on the previous lesson, on types of angles, before introducing the day's topic. The review is shown in the following dialogue:

- Maxwell: *What is a straight angle?*
Colleague *It is an angle that measures 180° .*
Maxwell: *What is an acute angle?*
Colleague *... an angle that measures less than 90° .*

The colleagues demonstrated good understanding in the rest of the types of angles reviewed. Maxwell made no attempt to draw the types of angles. He covered the topic ‘Angle properties of parallel lines’. He mentioned railway tracks as a real example of parallel lines. He drew two parallel lines and a transversal (in a slant position) across the parallel lines and said:

Whenever you have parallel lines and you draw a transversal line, it makes eight angles with the parallel lines. We are going to learn the properties of these lines [parallel lines], deal with some proofs, and then solve some questions.

He labelled the angles formed as follows:



Figure 7.3. Maxwell’s sketch for teaching properties of parallel lines.

Maxwell had some misconceptions about a transversal line. Before he could continue, after drawing in Figure 7.3, a colleague asked:

Colleague: *If I choose to draw a perpendicular line across the parallel lines, is it still a transversal?*

Maxwell said:

A transversal ... is a line that goes through two straight lines in a plane at two distinct points. ... so this [pointing to the perpendicular line drawn across the two parallel lines] is not a transversal.

Maxwell's response seemed to show that he only associated transversals with those drawn in a slant position [prototype sketch]. It seems he did not know that a transversal could intersect two straight lines at any angle. Hence, a transversal could be perpendicular if it intersects the two parallel lines at right angles.

The colleagues were not convinced (noise in the class). One colleague stated that the line that crosses any two or more parallel lines could intersect at right angles or any angle. Maxwell made no comment but asked the colleagues to clap for him and continued with the lesson.

He talked about the properties of the angles formed (see Figure 7.3) and said:

'Angle a and angle f', and 'angle c and angle h', [wrote on the whiteboard] they are vertically opposite angles. Now when we talk about vertically opposite angles ... they are equal. So whenever you are solving a question or I give you a question to solve and you see these angles [pointing to the angles opposite each other, a and f] ... they are equal. Please, do you understand?

He continued by posing the following question:

Maxwell: *What other ... vertically opposite angles do you see here?*

Colleague: *angle e and angle b are also vertically opposite angles.*

The remaining properties were taught as described above. No attempt was made by Maxwell to explain or describe the properties in words. The properties were taught by just telling them those that are equal, together with their names. For example, he said,

Corresponding angles are always equal, just like vertically opposite angles. So, angles g and e, angles f and h, angles a and c, angles b and d, they are all corresponding angles.

There was low participation among the colleagues. They were seen as passive learners (participants) just following what Maxwell was saying.

A colleague asked, “*Sir I once came across the word ‘adjacent’ on these properties in a book. Please, what are they?*” Maxwell waited for a second and said, “*anybody who can help?*” One colleague responded that, two or more angles are said to be adjacent when they are next to each other.

Maxwell continued to teach the properties of these angles, with a questionable approach. There was evidence that he knew what he wanted to convey, but he presented it in a way that could be confusing to learners, even though his colleagues understood him. The proof process is shown in Figure 7.4b.

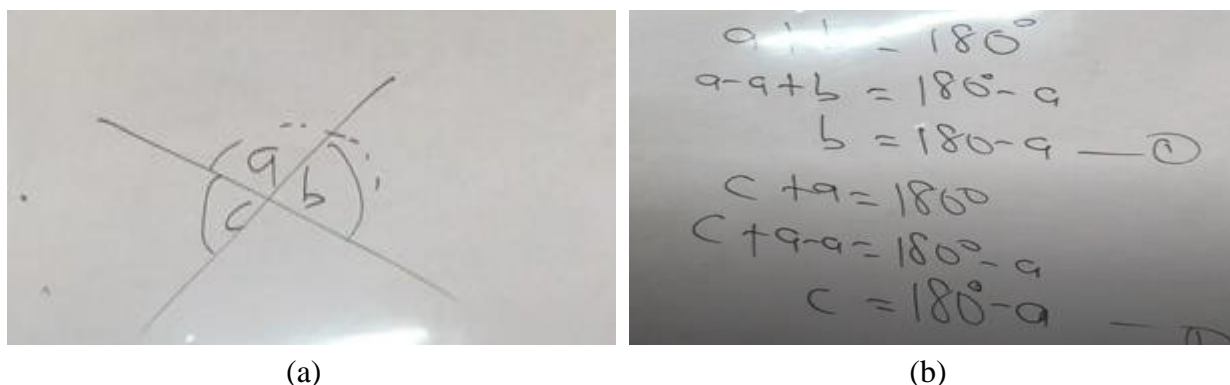


Figure 7.4. Maxwell's proof process of the equality of vertically opposite angles

He explained it as follows:

From the property of angles on a straight line, we know that these [pointing to angles a and b] are [adjacent] angles on a straight line. If we sum a and b , we should get 180° . I want to make b the subject, so I will subtract a from both sides of the equation [see Figure 7.4 (b)] to get $b = 180 - a \dots (1)$.

The second equation was obtained through a similar explanation. After obtaining these equations, he said, *from equations one and two, we can confidently say that $b = c$* . His claim was that b and c are both equal to $180^\circ - a$.

He further explained why alternate angles are equal and solved examples with the colleagues, which brought the lesson to an end. He rushed through the concluding part of the lesson. For example, in the figure given, which showed no sign of parallel lines, he went ahead to apply the equality of two corresponding angles, until he was prompted by a member of the class, before drawing the arrows.

7.2.4 Clement's (A) geometric discourse

Clement solved 15 of the 16 items. He attempted task 3.4 but could not provide an appropriate solution. He defined the geometric concepts using mathematically literate words. Clement demonstrated both a ritual and an explorative way of thinking in devising solutions to the tasks, and solved some of the tasks using multiple approaches. He was able to prove some of the fundamental geometric concepts explored in the study.

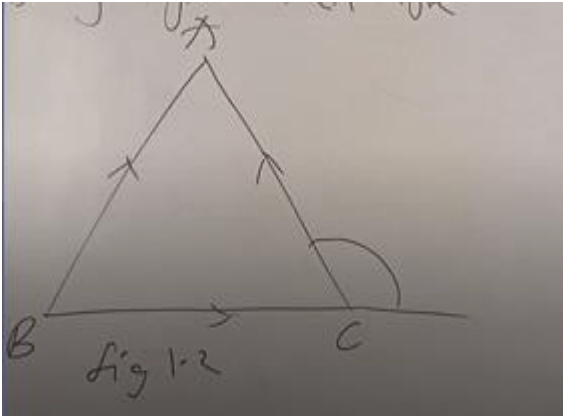
Clement presented a lesson on the exterior angle theorem of a triangle. He began by reviewing the colleagues' knowledge on the properties of parallel lines, by calling one of the colleagues to draw parallel lines on the board. Among the properties reviewed, were alternate, vertically opposite angles, corresponding angles and co-interior angles.

Clement introduced the topic and indicated that the properties they would be using are of the alternate and corresponding angles. He said:

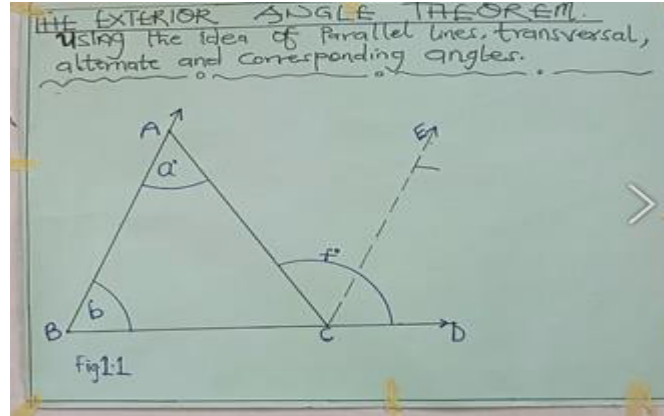
We are going to use these two (alternate and corresponding angles) to come up with or verify the exterior angle theorem.

Clement shared with his colleagues the lesson objectives, which required of them to prove the exterior angle theorem and use it to solve problems.

He explained that an exterior angle is formed when the side of a triangle is extended and drew it on the whiteboard for them to visualise and internalise (see Figure 7.5a). He pasted a cardboard of a figure to explain the exterior angle theorem as is shown in Figure 7.5b.



(a)



(b)

Figure 7.5. Clement's proof of the exterior angle theorem.

He engaged the colleagues in a discourse based on the figure pasted on the whiteboard. The discourse is shown in the following excerpt:

Clement: *What do you see from the two diagrams?*

Colleague: *There is a line drawn through point C in Fig. 1.1 (see Figure 7.5b)*

Clement: *Ok. Any different observation?*

Colleague: *There is a line that is parallel to line BA*

Clement: *How do you know they are parallel lines?*

Colleague: *There are arrows on lines BA and CE.*

Clement: *Ok. I like your answer.*

Clement asked any of the colleagues to come to the board and show why the lines are parallel.

Colleague: *Sir, we have this line BA and also ... line CE. So, this line is parallel to this line* [moving the two hands along the lines to indicate they are parallel]

Clement: *Why?*

Colleague: *Because they move with a constant ...* [still moving the two hands along the lines]

One colleague said, *Sir I want to help him.*

Clement: *Yes. Come and help your brother* [Laughter in the class].

Colleague: *They are parallel because there is an arrow here* [pointing to the arrow on line BA] *showing that this line is going in the same direction as this* [pointing to the arrow on line CE] *and they can never meet.*

Clement commented that “the arrows are very clear that ... line BA and then line CE are parallel”. When he said this, somebody whispered, “Ahhh there is an arrow at the top. I didn’t see oo”. Clement jokingly said, “Now all those without eyes can see clear [laughter in the class]. Clement asked, “what would you call line AC?” After some silence in the class, he said, “or do you want me to extend this?” When he extended the line AC beyond the parallel lines, there was a chorus response that the line AC is a transversal.

Now he asked:

What can you see or say about the angles ‘a’ and ‘b’ and the angle formed at ‘f’ here [pointing at the whole of the exterior angle formed]?

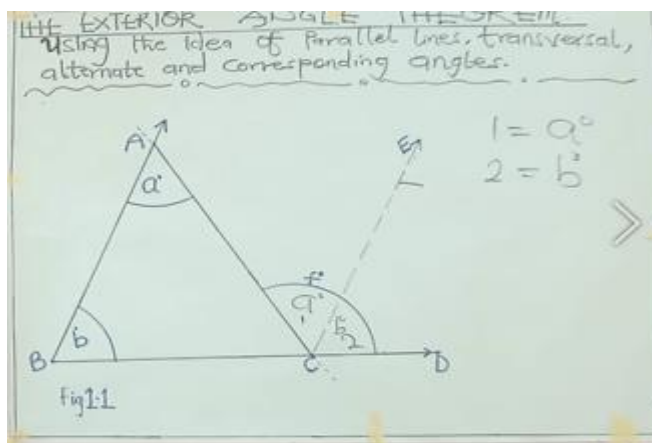
Colleague: *angle a alternate with [pointed to angle ACE].*

When Clement asked who agreed with what the colleague said, the whole class was quiet. When he realised the figure seemed unclear (probably due to weak visualisation) to the class, he labelled the partition of the angle f as ‘1’ and ‘2’. He asked a colleague to go to the board and explain. He said “angle a is equal to 1”. This time around, when Clement asked how many agreed with him, the class gave a favourable response. The lesson continued.

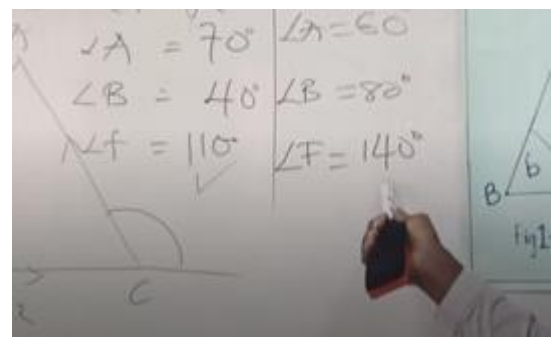
Clement: *Ok, what can you say about angle b ... and either 1 or 2?*

Colleague: *I can say that angle b corresponds with 2 [was invited to show on the board].*

This time it seemed clear to the class. See Figure 7.6.



(a) exterior theorem by properties



(b) exterior theorem by measurement

Figure 7.6. The exterior theorem of a triangle.

Clement asked, “*What is the relationship between a and b and the whole of angle f ?*”

Colleague: *I see the summation of angles a and b is equal to angle f .*

Clement helped the class to realise that the sum of angles a and b is equal to angle f .

He further supported the learners’ understanding of the theorem by using the measurement of the exterior angle and the two non-adjacent interior angles. He shared some cardboard cut-outs of triangles of different dimensions and asked them to measure the labelled interior angles a and b , and the exterior angle f . He guided the class to use the values obtained from the measurement to establish the validity of the exterior angle theorem through the properties.

He asked the colleagues if they could describe the theorem in their own words. This was an opportunity to verbalise their thinking about the theorem and to produce a narrative. Sfard (2008) asserts that narratives can be accepted or rejected.

Colleague: *The two opposite interior angles is equal to the exterior angle.*

Clement said ‘ok’, but asked for another response.

Second colleague: *The sum of the two opposite interior angles is equal to the exterior angles.*

He lauded them for their contribution but laid emphasis on the use of the word “sum” as was found in the second colleague’s submission. He said:

All that Alex is trying to say is that, mathematically, the sum of these angles [pointing to the two opposite interior angles] should be equal to the exterior angle.

Clement seemed to be teaching too many things within a 30-minute lesson. He solved one example and also taught the proof of the theorem. He supported every step with the appropriate narrative. Based on the classroom observation, Clement did not assume the central role of the proof process. There was evidence of interaction between himself and the colleagues in every step of the routine.

7.2.5 Albert's (B) geometric discourse

Albert solved 13 of the 16 items. He explained his thought processes in solving the tasks using both colloquial and mathematically literate words. Almost all the triangle questions were solved in the same ritual type way of thinking. He demonstrated weak thinking in the basic geometric proofs.

Albert taught 'parts of a circle', as a new topic. He reviewed the colleagues' previous knowledge of shapes, based on the previous lesson. Among the parts mentioned were squares, triangles, rectangles and circles. He drew the shapes and focused on the circle. The extract below shows the classroom interaction:

Albert: *In your own words, what can you say a circle is?*

Colleague: *A circle is a round shape.*

Albert: *You have tried.* [invited another person's view]

Colleague: *A plane figure bounded by curves.*

Another colleague defined a circle as "*a locus of points equidistant from a fixed point*". He accepted this definition and explained that a circle is derived from the Greek word 'kirkos' which means a hoop or ring. He then defined a circle similar to that of the last person and wrote it on the whiteboard (see Figure 7.7).

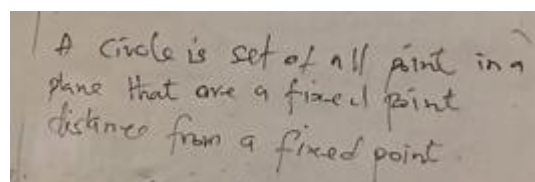


Figure 7.7: Albert's written definition of a circle.

He explained the definition with the help of a drawing. He used the word 'equidistant' to explain the fixed-point distances from the centre, as in the definition given. He defined all the concepts or terminologies used in a circle, with the help of a chart and drew them on the whiteboard. Some of these definitions are as follows:

A chord is a straight line segment whose endpoints lie on the circle.

An arc is any portion or section of the circumference of a circle.

A diameter is a line through the centre that touches two points on the circle.

He emphasised that a chord must pass through the centre of a circle before it could be considered as a diameter.

However, Albert confirmed a colloquial definition put forth by one of the colleagues who asked if he could define a diameter as “*a line that divides a circle into two equal parts*”, for which Albert answered ‘yes’, it is true.

Albert was noted to be reading from a sheet, moving from the board to the table where he kept the sheets. The colleagues were placed entirely in a passive position. Not even one colleague came to the board to do something. The lesson comprised Albert telling the colleagues what to learn. It was observed that the colleagues’ attention was quite low, and often, Albert had to shout ‘hellooo’ to attract their attention. During the lesson, the only time colleagues voices were heard, apart from the noise they were making in the class, was their response of ‘yes sir’ to Albert’s question ‘do you understand?’

He handled students’ questions by confirming both errors and correct thinking. When he had explained a segment as “*a segment is a region of a circle which is cut from the rest of the circle by a chord*”, he stated that any time you use a chord to cut a circle, that place can be termed as a segment. One colleague asked, “*if the line passes through the centre, [is it] a segment?*” Albert said yes, drew it, and explained that it is a special case known as a semi-circle.

He also gave a colloquial definition of a sector as “*any two radii that touch at the circumference of a circle*”, but drew it correctly. He concluded the lesson without assessing the colleagues’ understanding of what they had learned. He only told them what they would be learning in the next lesson.

7.2.6 Alex’s (B) geometric discourse

Alex solved nine of the sixteen items. This means that he obtained no score for five of the geometric tasks. He did not attempt task 3.4. He explained his thinking processes governing the

solution of the tasks using colloquial, and literate words. Alex demonstrated both ritual and explorative ways of thinking in his geometric discourse.

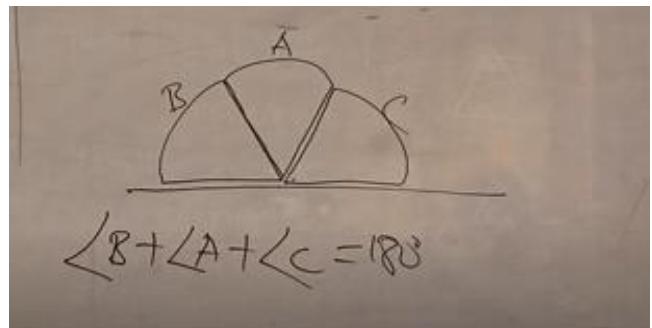
Alex taught the topic ‘the sum of the interior angles in a triangle’. At the beginning of the lesson, he did not engage the learners in any review of related previous knowledge. He said:

Today we are going to talk about the sum of the interior angles of a triangle, and I know you know the properties of a triangle, right? So, we are going to do a simple thing to indicate that the interior angle of a triangle is 180° .

He shared paper cut-outs of triangles with the colleagues in groups and gave them some instructions about the activity to perform. He asked the colleagues to open their compasses to any reasonable radius and make an arc at the vertices of the triangles and cut them out. He asked them to arrange the cut-out angles on a horizontal line, with their vertices together (see Figure 7.8 a and 7.8b).



(a) arrangement for a semi-circle



(b) sum of the angles is 180°

Figure 7.8: An arrangement of the cut-out corner angles of a triangle.

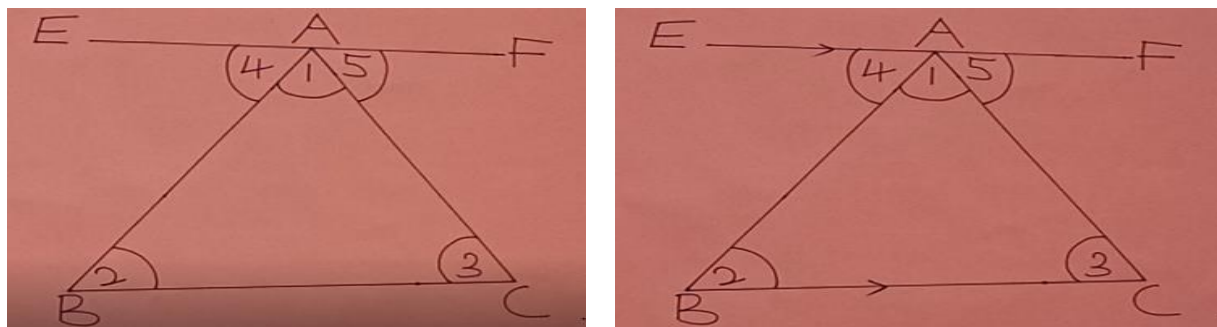
He engaged them in a discourse.

Alex: What do you notice from your arrangement?

Colleague: *It has formed a semi-circle.*

Alex explained that since it had formed a semi-circle, it meant the arrangement forms a straight line. He asked the class to use their protractors to measure the angle along the straight edge of the arrangement. Based on the angle obtained, he explained that the sum of the angles in the triangle is 180° (see Figure 7.8b).

He elaborated by using properties associated with parallel lines. Alex pasted a chart prepared for this lesson, with no indication that the lines were parallel, until he was prompted by a colleague during the lesson.



(a) diagram with no indication of parallelism (b) diagram showing the icons when prompted
Figure 7.9. Relating the sum of interior angles of a triangle, to angle on a straight line.

He explained that the adjacent angles 4, 1, and 5 are on a straight line. This means that their sum is 180° . He needed to find the sum of angles 1, 2 and 3, the interior angles in the triangle. He explained using the alternate angle properties that angle 2 is alternative to angle 4 and are equal. He then asked:

Alex: *Do you see any other alternate angles in the figure?*

Chorus response: *Angle 3 and angle 5.*

He affirmed the response and said they are equal. He continued that in the sum $\angle 1 + \angle 2 + \angle 3 = 180^\circ$, angle 2 and angle 3 can be replaced with angle 4 and angle 5 respectively. Thus, $\angle 1 + \angle 4 + \angle 5 = 180^\circ$. This means that the sum of the interior angles in a triangle is equal to 180° . He concluded the lesson by inviting one colleague to solve a question he wrote on the board.

7.2.7 Cynthia's (B) geometric discourse

Cynthia was the only lady among the participants. She solved eleven of the sixteen geometry items, and received a partial mark for one of the items. She did not attempt some of the questions. She explained her thought processes using both colloquial and literate words. She demonstrated ritual ways of thinking in solving three of the tasks. She could not use appropriate words to define or describe some geometric concepts, even though she knew the names of those angle properties.

Cynthia prepared to teach the exterior angle theorem of a triangle. She used a test to review the learners' knowledge and understanding of the previous lesson, which was about properties of a triangle and adjacent angles on a straight line. In the introduction of her lesson, she said:

Last week we learned about the properties of a triangle and a [straight] line. So, I want to test your knowledge and understanding with this question.

She drew a triangle with one missing interior angle to be determined, and designed another task on finding one unknown angle on a straight line. She first asked the colleagues to try the task individually, and then called one of them to solve it on the whiteboard and explain the method to the class. She said, *“today we are going to combine these properties to get one thing”*.

In developing the learners' thinking on the exterior angle theorem, she drew a triangle and extended one side to form an angle outside the closed shape. She explained what she meant by an exterior angle, saying, *“an exterior angle is formed when the line of one side of a triangle is extended”*. She pointed on the diagram, to show to the colleagues the place termed the exterior angle. She added that, *“It is formed outside the triangle ...”* After the explanation, she built on previous lessons and gave them a task to find the unknown interior and exterior angles.

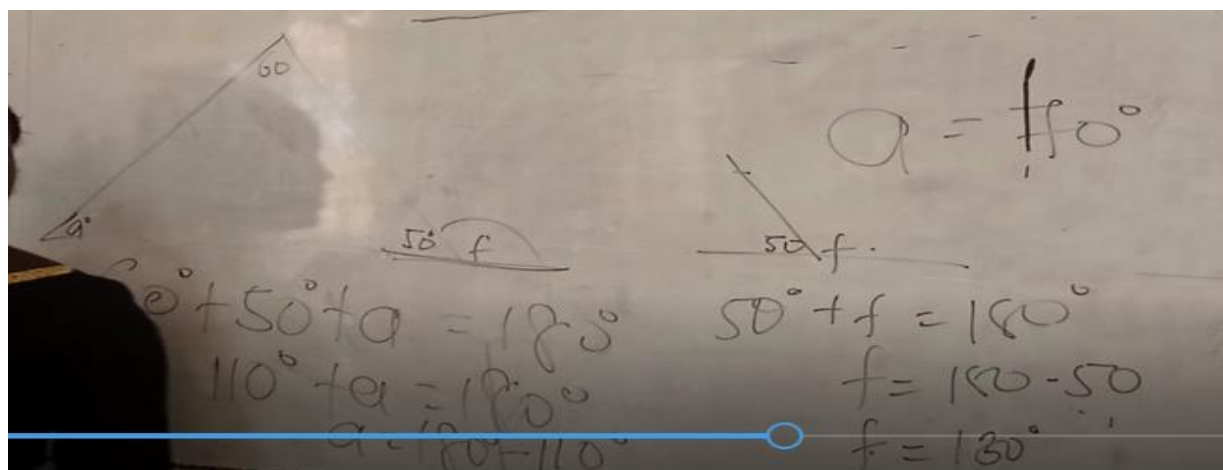


Figure 7.10: Cynthia's example solved in class.

She said to the class, *“Solve for the interior angle as we did last week”*. Cynthia introduced a new concept based on their previous knowledge.

It was evident that she understood what she was teaching but her pedagogical approach was not clear, even though the colleagues observed that the exterior angle is equal to the sum of the two opposite interior angles. She guided the learners to solve for a in the triangle, and the exterior angle f .

Then she said:

... now we have our f ... 130° , and our a , ... 60° , we are coming to make some comparing here so please pay attention ...

She modeled the equations of the angles in the triangle, and on the straight line, and said:

... both the triangle and the line have the same properties [angle sum] which add up to 180° .

She then compared it without using the transitive property to equate or combine the two equations. She asked the learners to identify what was common in the two equations. She cancelled the 50° and the 180° from the two equations as shown below.

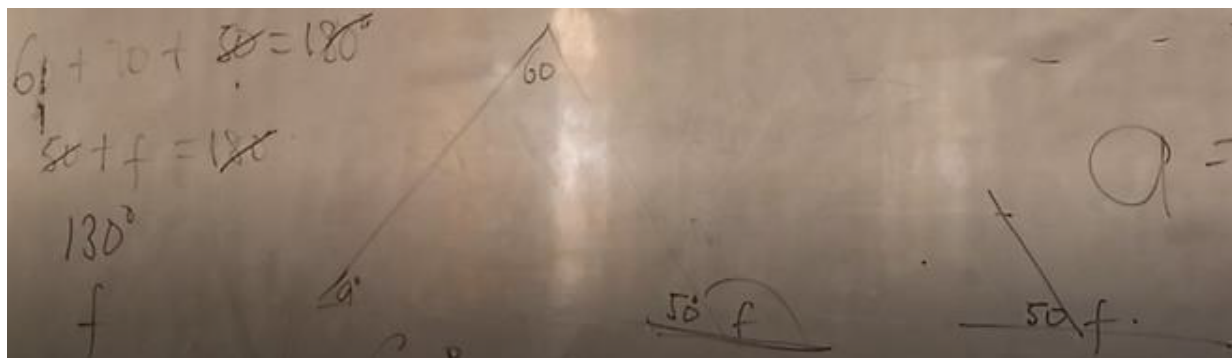


Figure 7.11: Cynthia's proof of the exterior angle theorem.

She explained the process as follows:

Cynthia: *... looking at these equations, what is common?*

One colleague responded 50° .

Cynthia: *Very good. We have 50° here and 50° there, [which] means we are taking off 50° here and 50° there because they are common [cancelled the 50° from each of the two equations].*

Cynthia: *Now what again is common here?* [pointing to the two equations].

One responded 180° .

Cynthia: [Repeated after him] *Very good* [cancelled out the 180° from both equations] *now it is left with what? '60° plus 70°' and 'f' here* [pointing to the f].

She further explained that:

Now with this ... as we combine 60°, 70°, 50° we got 180°, and when you combine the 50° and f we got 180°. So, what is the relationship between this f and 50° plus 70°. Let us combine 50° and 70°. What will you get?

One responded: 130°

Cynthia: *So, what is the relationship between this f and 130, or what can you say about them?*

One responded: *When you combine 60° to 70°, it is the same as the f.*

Cynthia: *Good, clap for him.*

After this, she said, *“Let’s work more examples for you to understand it well”*.

Before the examples, she drew the colleagues’ attention to the fact that when they solved for the value of f using the straight line, they got 130° . Also, looking at the triangle, they got a to be 70° from their solution. When you add this 60° and 70° you get 130° , it is the same as the f which is the exterior angle. She then generalised that:

It means whenever we are finding the exterior angle, it is the same as or is equal to two opposite interior angles. So, is that clear? Are you okay? Any question?

Cynthia’s approach used for teaching the concept of the exterior angle theorem was quite odd. She knew what she wanted to teach but had a problem with the pedagogical approach. She dominated the teaching and learning process. She said almost everything, except answers to the few questions that she posed to her colleagues.

She gave the following trial work and asked them to solve the unknown angles marked by, a , b and c .

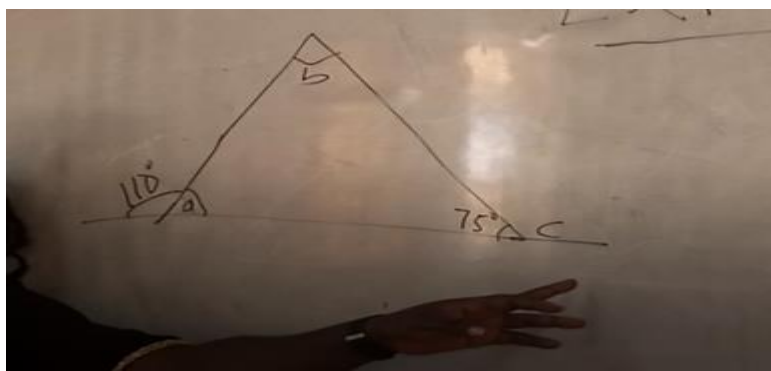


Figure 7. 12: Cynthia's task used to assess her colleagues' understanding

She called on three of her colleagues in turn, to solve for the angle value for each variable. When they finished, Cynthia explained their solutions. She assessed the lesson by asking the colleagues about what they had learned. A colleague said, "*we have learned that two opposite interior angle is equal to one exterior angle*". Based on this question, she ended the class.

7.2.8 Nsiah's (B) geometric discourse

Of the 16 items in the geometric test, Nsiah answered ten of them correctly and obtained a partial score for one task on circles. He did not attempt task 4.2 on circles. He used both literate and colloquial words in his discourse and showed both ritual and explorative ways of thinking in devising solutions to the geometric tasks.

He prepared to teach one property of the circle theorem based on the angle subtended at the circumference of a circle by a diameter. He briefly reviewed the previous lesson. He said:

In our previous lesson, we looked at parts of a circle, I want you to mention the parts that you learned to me.

Among the types mentioned were circumference, radius, centre, diameter, and chord, all of which he wrote on the whiteboard. He said, "*Having known these, we have to look at their meaning*". He explored the learners' understanding of the parts mentioned, as shown in the following excerpt:

Nsiah: *What is a circumference?*

Colleague: *It is the distance around a circle.*

Nsiah: *What is a diameter?*

Colleague: *The line that passes through the centre with its endpoints on the circle.*

The above excerpts show how Nsiah refreshed the colleagues' knowledge on their previously learned concepts, in relation to the topic to be taught.

Nsiah told the colleague learners to sit in groups and shared the mathematical set to each group. He gave the following instructions:

- (1) *Take your compass and open it to any convenient radius and draw a circle.*
- (2) *Locate your centre and label it as O .*
- (3) *Draw a line from one part through the centre to the other part of the circle [diameter] and label the endpoints A and B .*
- (4) *Draw a chord from point A to any point C on the circumference of the circle drawn.*
- (5) *Draw a line from the point B to meet point C on the circumference.*
- (6) *Use the protractor to measure your angle ACB .*

The group followed the instructions given by Nsiah as seen in the activity being performed below (see Figure 7.13).

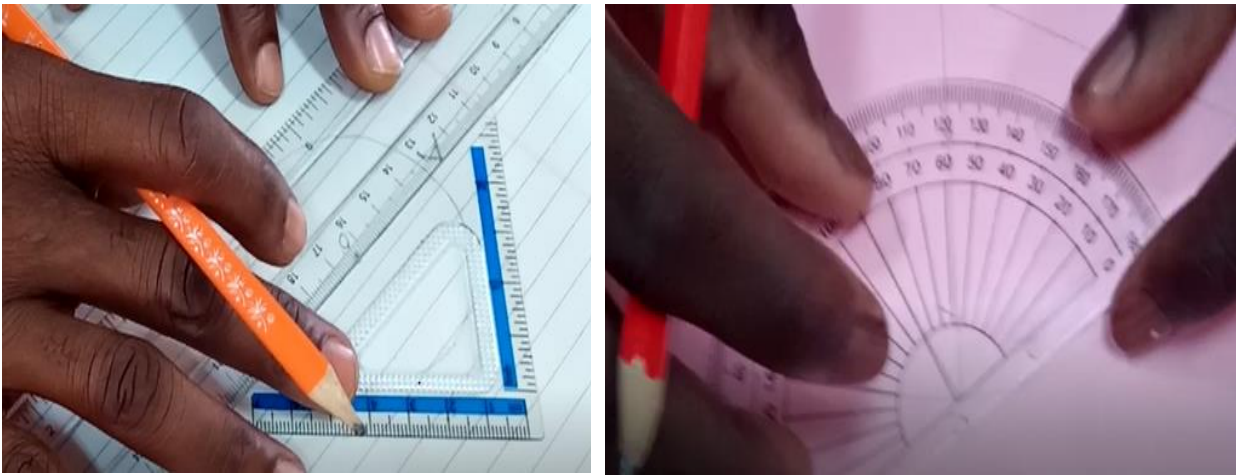


Figure 7.13: Different groups drawing and measuring as instructed

Nsiah asked them to use the protractor to measure the angle subtended at the circumference by the diameter. Throughout the instruction, he moved from one group to the other to observe what each group was doing, and was found to be interacting with some members of the groups. He also drew

the diagram as per the instruction provided for the class engagement. He asked each group to give the value of the angle measured. Each group determined the angle to be 90° .

He said:

This is an angle obtained by measurement. So, let's use the analytical approach to see [prove] that the angle subtended at the circumference of circle by a diameter will give us 90° .

He gave further instruction on the proof process by asking the class to draw a line from the centre O to the point C on the circumference of the circle. He explained that each of the straight lines drawn from the centre to the points on the circumference, is a radius. He made a hatch mark and labeled each r (radius). He engaged the class with the following question:

Nsiah: *How many triangles do we have?*

Colleague: *Two triangles.*

He explained that because the smaller $\triangle AOC$ and $\triangle BOC$ are each bonded by two radii, the triangles are called isosceles triangles. He added the property that “*one property of an isosceles triangle is that the base angles are equal*”.

Based on this property, he labelled $\angle OAC$ and $\angle OCA$ with the same variable x and $\angle AOC$ to be y . Using the same analogy, he labelled $\angle OBC$ and $\angle OCB$ with the same variable z and $\angle BOC$ as w .

He asked the class “*What is the total sum of the interior angles of a triangle?*” One colleague responded that it is 180° . He generated three equations using the two triangles, and the straight line AOB as shown below.

$$2x + y = 180^\circ \quad \dots \quad \dots \quad (1)$$

$$2z + w = 180^\circ \quad \dots \quad \dots \quad (2)$$

$$y + w = 180^\circ \quad \dots \quad \dots \quad (3)$$

from eqn ① and eqn ②
 Add eqn ① and eqn ②
 $2x + y + 2z + w = 360^\circ$
 $2x + 2z + y + w = 360^\circ$
 $2(x+z) + 180^\circ = 360^\circ$
 $2(x+z) + 180^\circ - 180^\circ = 360^\circ - 180^\circ$
 $2(x+z) = 180^\circ$

$\frac{1}{2} \times 2(x+z) = \frac{1}{2} \times 180^\circ$
 $x+z = \frac{180^\circ}{2}$
 $x+z = \frac{2 \times 90^\circ}{2}$
 $x+z = 90^\circ$

Figure 7.14: Proof of angle subtended in a semi-circle.

He associated the proof with the angle obtained from measurement in the first activity. He said, “From the analytical proof, we can see that angle $x + z = 90^\circ$, as obtained from measurement”.

He stated the theorem, based on the activity, that:

Any angle subtended at the circumference by a diameter is equal to 90° .

He concluded the lesson by telling the class captain to see him for a take home assignment.

7.3 Analysis of the PSTs’ classroom geometric discourse

7.3.1 Word use in classroom geometric discourse

The category of analysis of word use draws on PSTs’ use of functional words in their classroom geometric discourse. According to Sfard (2008), language use plays a significant role in mathematics discourse. The analysis focused on the use of words and their related meanings in the context of the geometric shape. Sfard (2008) maintains that a discourse is characterised by keywords, and mathematics, particularly geometry, includes some specific keywords used to designate specific concepts in the subject (Atebe & Schafer, 2010). Word use in a discourse can be classified as mathematical or colloquial (explained in Chapter three section 3.3.1). Both literate and colloquial words were found in the discourses of the participants. The use of words differed across and within groups.

Stephen used functional words in his classroom discourse. He explained the various terms by using accepted words in learning geometry. He redefined a parallelogram, as stated by one of the colleagues, as “*a four-sided figure with two pairs of parallel lines*”. He termed the straight line that connects the two non-adjacent vertices as the ‘diagonal’ and also preferred that the colleagues use the word ‘bisect’ to describe the observation that ‘the diagonals bisect at right angles’. Stephen used words that are endorsed in geometric discourse. Stephen’s understanding of these technical terms enabled him to communicate with his colleagues in a concise manner. This finding supports the inference from other researchers’ views on terminologies in geometry, that pre-service teachers who possess adequate knowledge of mathematical terms in a discourse, are able to communicate ideas in a concise and acceptable way to enhance learners’ acquisition for and use of appropriate terms in communication (Atebe & Schafer, 2010; Oyoo, 2009; Roberts, 2010).

Stephen also showed concern about his colleagues’ use of literate words in their discourse. Eg. When he wanted to draw their attention to one of the intended properties, he asked about their observations regarding the folded rhombus. He did accept the response “*that diagonals divide the angles into two equal parts*”, but encouraged them to use the word ‘bisect’ instead. Stephen knew how to use literate words to communicate ideas and how to shape or correct the way other people used words in the discourse.

Similarly, Jones (A) did not accept colloquial word use in defining some of the concepts taught. For example, one of the colleagues defined a diameter as “*a line that divides a circle into two equal parts*”. Finding it unsuitable, he called for another definition, which yielded the response “*a diameter is a line segment drawn through the centre of the circle with its endpoints on the circle*”. Jones accepted this definition, drew it on the whiteboard and emphasised the phrase, ‘through the centre’. He corrected his colleagues’ definition of a tangent by encouraging them to add the phrase, ‘touches at one point’. It was also found in Clement’s (A) lesson that when one of his colleagues stated the exterior angle theorem of a triangle without the word ‘sum’, he corrected him and emphasised the use of the word ‘sum’ in stating the theorem.

This guidance provided by Stephen and Jones shows that they have developed the competence to guide teaching and learning of geometry, with appropriate words in an objectified way. Sford

(2008) maintains that meaningful learning occurs when learners' word use is guided and shaped. Whilst Clement, Jones and Stephen (all in Group A) were correcting their colleagues' word use in a discourse, two others in Group B were approving of colloquial word use. For example, Albert confirmed a colleague learner's question to find out if a diameter could be defined as "*a line that divides a circle into two equal parts*". He replied in the affirmative way.

The rest of the participants used both colloquial and literate words in their discourses. According to Sfard (2008), colloquial word use is informal and could lead to several interpretations. It is not often used in discourses practised in schools. Based on the observed lessons, some of the PSTs used or endorsed the use of colloquial words in their discourses. For example, in Albert's lesson on parts of a circle, where he read most of the definitions from a paper, he endorsed a colloquial definition given by one of the colleagues. A colleague asked if a diameter could be defined a diameter as "*a line that divides a circle into two equal parts?*" Albert endorsed this definition, which was stated with colloquial words and is not used in school mathematics (Sfard, 2008). Albert also defined a segment as "*a region of a circle which is cut from the rest of the circle by a chord*". He demonstrated this by drawing and shading a portion bounded by a chord and an arc.

Similarly, Alex (B) described a property of a straight line by using colloquial words. He said, "*the sum of angles on a straight line is 180°* ". This narrative was used as a guide to model equations involving the adjacent angles formed on a straight line.

Cynthia also used the colloquial word 'combine' to mean 'add' in her discourse. She also stated the theorem that "*the exterior angle is equal to the two opposite interior angles*". What needs to be noted is the omission of the words 'sum of' in stating the theorem. Nsiah (B) also used the words 'base angles' to refer to angles opposite the equal sides, in an isosceles triangle. According to Sfard (2008), language or word use performs specific functions in mathematics discourse with literate meaning in the context of its use. For example, in stating the theorem, the words 'sum of' cannot be omitted.

Geometry, like any other topics in the mathematics curriculum, has unique terminologies that learners need to comprehend, to guide their understanding in learning advanced geometric

concepts (Atebe & Schafer, 2010). In this study, some of the PSTs developed their colleague learners' thinking with word use in defining key geometric concepts or terms, required to learn higher geometric content.

Word use, in defining and describing geometrical terms plays an important role in the discourse. It is an essential tool through which geometric concepts are communicated. According to Atebe & Schafer (2010), learners' acquisition and understanding of the basic geometric terms, forms a sound basis to enable them to clearly communicate their ideas. This suggests that certain words and terminologies in geometric discourse should not be assumed to be known by the learners, but need to be taught. In this study, some of the Group A participants were found to develop colleagues' understanding of certain words, that were prerequisite to understanding the main topic to be taught. Some researchers claim that learners' understanding of geometric words enables them to describe specific spatial ideas clearly and to show possible relationships among them (Alex & Mammen, 2018; Roberts, 2010). Teachers need to develop learners' understanding of basic geometric concepts and terminologies (word use). Sfard (2008) asserts that what gives a discourse its special features are the keywords used. Learners could be proficient in learning geometry when certain keywords are learned, understood and used in communication. This supports Atebe and Schafer's (2010) view that keywords (concepts and terminology) form the building blocks, or the basic knowledge, on which further content is built. Understanding of these keywords is fundamental for learning geometry. The authors contend that learners' proficient communication of geometric ideas depends on how well the basic terminologies have been understood.

According to Sfard (2008), language forms the main tool for communicating ideas in a discourse. This makes the kind of word use in explaining solution processes to learners, very important in every discipline. Learners' understanding can be developed and deepened when words are used in a literate context. On the other hand, learners' understanding may be affected if colloquial words are used in a discourse. In the classroom context, communication forms the means through which teachers present ideas and content of the curriculum to learners. In teaching properties of geometry, teachers need to explain using certain words that would help learners to make relationships and draw relevant connections to guide their understanding.

7.3.2 Visual mediators

According to Sfard (2008), visual mediators are artefacts (pictorial materials) that are used to communicate mathematical concepts to learners. These can be diagrams, symbols, sketches and the like. Visual mediators assist learners to create imagery of the concepts learned. Learners can communicate their thoughts better, when the object of discussion is seen (Sfard, 2008). She classifies these mediators into iconic, symbolic and concrete mediators. The PSTs used several visual mediators to communicate the desired geometric concepts in their lessons. These are discussed in the following paragraphs.

7.3.2.1 *Diagrams and sketches*

Diagrams are used to convey, or communicate geometric concepts, which serve as a realisation that forms a valuable source of a person's information in a discourse (Sfard, 2008). The use of diagrams is central to learning because they offer great prospects for enhancing learners' understanding (Sfard, 2008). By using diagrams to explain ideas, learners can picture the ideas in their minds (mental imagery) and remember them later.

All the PSTs incorporated diagrams in teaching the geometric concepts in their lessons. The diagrams aroused and sustained their colleagues' attention in the lessons, compared to other lessons which were dominated by the teacher's verbal explanations. The diagrams enabled the colleague learners to create mental images of spatial ideas in the geometric discourse. For example, Jones (A), Albert (B) and Nsiah (B), who taught topics on circles, supported their verbal explanations of the various concepts by using diagrams. Sfard (2008) asserts that diagrams make objects accessible, for learners to produce and substantiate mathematical narratives. This notion was embraced by the PSTs. After a concept had been defined, it was visually represented on the whiteboard. For example, Jones made the definition of a diameter more accessible to colleague learners, by drawing it and emphasising that the straight line (the chord) must pass through the centre of the circle before it is considered a diameter. This is in line with Samkoff et al.'s (2012) view that the use of diagrams provides a more accessible explanation for learning mathematics. The authors state that diagrams enable learners (problem solvers) to view and integrate pieces of information with "less cognitive effort" (p. 49).

A great advantage of diagrams is that they show relationships between variables or concepts (Jones, 2013; Sfard, 2008). When Jones stated the theorem that ‘an angle subtended by an arc at the centre is twice the angle subtended at the circumference’, he supported this with a visual representation to show the relationship between the angles (see Figure 7.2).

It was observed that, most of the colleagues followed the diagram-supported instruction with rapt attention. Most learners benefit from visual and verbal representation of mathematical concepts (Jones, 2013; Mudaly, 2012; Sfard, 2008). All learners learn in different ways. For mathematics instruction to be effective and meet the various learning styles of learners, concepts need to be presented using multiple visual instructional strategies (Mudaly & Naidoo, 2015). Such visual aids developed the colleague learners’ ability to make meaning from Jones’ lesson.

The PSTs embraced the use of diagrams, to ensure that their learners not only processed the verbal explanation of geometric concepts but also visualised and developed a deeper understanding of the geometric ideas (Jones, 2013;). After Stephen had guided the colleagues through the paper folding activity, he sketched and explained the properties of a rhombus. In Matlen et al.’s (2018) view, diagrammatic representation of concepts enables learners to make sense of ideas and draw relational meaning among the concepts learned. Jones (2013, p. 1) adds that diagrams are an integral aspect of doing and making meaning of mathematics, and are often used in the teaching and learning of geometry, not only because of the nature of the geometric objects, but because diagrams offer an “effective problem representation that enables complex geometric processes and structures to be represented holistically”.

Geometric concepts and properties are easier to learn and understand when they are presented in visual forms such as diagrams (Jones, 2013). The colleague learners welcomed the use of the diagrams in learning, as was evident by their participation and contributions in the lesson. This pedagogical perspective, adopted by Stephen, aroused and sustained the colleagues’ interest throughout the lesson, and Stephen did not have to struggle with lengthy verbal explanations of the properties. According to Jones (2013, p. 1), the reason for the widespread use of diagrams in teaching and learning is that “a picture is worth a thousand words” or “hearing a thousand times is not as good as seeing once”. Possibly, the participants’ interest and active participation in the

lesson were heightened as they were not made to process verbal information, but were shown the visual meaning of the properties, which may be easier to integrate into their way of learning. Stephen's attempts to assess the colleagues' progress with the instruction, yielded responses and articulation of the properties learned based on the diagrams used (on the whiteboard). In line with the commognitive theory, "communication is mediated by images", which develops learners' fluency in a discourse that being the goal of mathematics learning (Sfard, 2008, p. 148). This suggests that learners' thinking can be highly enhanced when diagrams are used in teaching and learning.

Results showed that the use of the diagrams by the PSTs helped to communicate the intended geometric ideas, so that the colleagues could follow the lesson in a meaningful way. This finding corroborates with other researchers (Dundar & Otten, 2022; Poch et al., 2015) who also found that the use of visual resources such as diagrams, enables learners to make meaning of the learning activities. The use of diagrams, as instructional aids in geometry, make it easier to present geometric concepts and also make the concepts clearer for the learners. Diagrams form an integral part of learning and understanding mathematics (Jones, 2013; Matlen et al., 2018).

7.3.2.2 *Iconic mediators*

According to Sfard (2008), icons are artefacts used to design diagrams, drawings and graphs. They are marks used to design mathematical tasks to communicate important features. Learners produce a factual narrative in a discourse based on iconic realisations (Sfard, 2008). This means that learners' proficiency in mathematics depends on their ability to identify and interpret such iconic mediators to aid understanding. To develop learners' understanding, teachers should draw learners' attention to such visual cues to facilitate learning. In geometry, where most tasks are presented with diagrams, making learners aware of any additional visual prompts can be useful in their discourse (Sfard, 2007; 2008).

The PSTs used iconic mediators to design the diagrams in the lessons. These iconic mediators communicated some features of the diagrams. For example, narratives about right angles, types of triangles, types of quadrilaterals and related properties, were produced by the colleague learners in the discourse when they had observed the various icons used to design the diagrams (Sfard,

2008). When one colleague described a triangle as isosceles, he substantiated by stating that the two sides of the triangle have an equal number of icons (hatch marks).

Some of the PSTs ensured that narratives produced by their colleague learners, were based on an observed iconic mediator and not a mere assumption. In Clement's (A) lesson on the exterior angle of a triangle (see Figure 7.5), he asked about the relationship between the side BA and the straight line drawn through the vertex C . One of the colleagues said they are parallel. When Clement asked 'why', he referred to the arrow on the side BA and to the straight line CE . This supports Sfard's (2008) notion that iconic realisations help learners to produce factual narratives in a discourse. Clement's probe was an effort to ensure that narratives are produced by observing and interpreting the icons used in the diagram. During the lesson, he commented that:

Sometimes you may see these straight lines [pointing to the parallel lines] as parallel, but do not make a wrong assumption when these marks or signs [iconic mediators] are not there. Always check first before you consider the lines as parallel.

This shows that Clement has good geometrical thinking as a basis to guide learners in the discourse. Also, Stephen drew attention to the fact that, in the quadrilateral drawn, the opposite sides are parallel and equal, which is a rhombus. Similarly, in Jones' proof of one of the circle theorems (see Figure 7.2b), he drew a straight line from C to the centre, O , of the circle and said:

From here [pointing on the diagram], this is radius OC and OA is also a radius. So, it means that they are equal in measure.

When he said this, he designed the two radii with icons indicating that they are equal. Then, he labelled the angles opposite the equal sides with the same variable x in the proof process.

In Nsiah's (B) lesson on circles, he informed the colleagues that any line segment drawn from the centre to any point on the circumference, is of equal length and made an equal number of hatch marks on them. He also indicated the angle the diameter formed at the circumference of a circle, by using a 'square mark' iconic mediator to show that the angle is 90° .

Many of the Group A participants were more conscious of guiding the development of geometric thinking using the iconic mediators, compared to the Group B participants. All the PSTs used

iconic mediators in the context of the content taught. However, three of the PSTs in Group A based their narratives on the icons and deliberately drew their colleagues' attention, whilst the rest of the participants in the study assumed that their colleagues knew them and were silent on them. Within the commognitive framework (Sfard, 2008), learners' fluency and interpretation of visual cues in a discourse is an important goal of teaching mathematics, which is achieved when iconic mediators are explained to learners.

7.3.2.3 *Symbolic mediators*

Symbols are artefacts used in mathematical communication (Sfard, 2008). Many geometric concepts are communicated using symbols as signifiers. Knowledge of these artefacts enhances one's understanding in the discourse and improves the communication of ideas. According to Sfard (2008, p. 184), learners' "ability to create an endorsed narrative about geometric shapes" starts by visualising and interpreting their associated symbols.

The only symbol used by the PSTs, was for angle, ' \angle '. For example, in teaching properties of parallel lines, although letters were used to label the angles, on the chart, they were accompanied by the angle symbols during explanations in the lessons of Alex (B), Clement (A) and Cynthia (B). Maxwell labelled the spaces between the straight lines related to the vertex, with letters, and continued to use these letters in explaining the various related properties of angles as shown below.

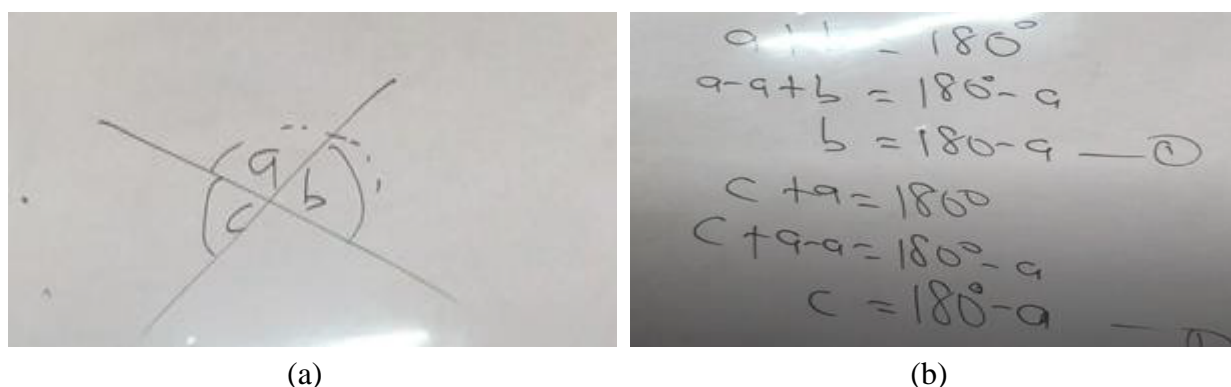


Figure 7.15: Maxwell's representation of concepts of angles with only variables.

Even when he used the word 'angle' in his explanation, he constantly used a variable (algebraic symbol) to represent the concept of an angle. Although the PSTs taught some concepts on parallel

and perpendicular lines (right angles), they made no attempt to develop the learners' thinking on related symbolic artefacts.

7.3.2.4 *The PSTs' use of concrete mediators in classroom geometric discourse*

Sfard (2007, 2008) explains that concrete mediators are visual objects that can be seen, handled and manipulated for the purpose of learning mathematics, and help in the production of factual narratives about the object. Examples are drawing tools, paper cut-outs and gestures. They allow for interactive engagement between the teacher and the learner, and bring about learner participation in learning through hands-on activities (Horan & Carr, 2018). Learning through hands-on activities enhances learners' cognitive development and helps them to become constructive thinkers (Rondina, 2019).

Stephen and Clement (Group A) and Alex and Nsiah (Group B), used concrete mediators in their classroom discourse. Stephen used paper folding to develop his colleague learners' geometric thinking on the properties of a rhombus. He gave samples of varying dimensions of cut-out rhombi to the colleagues in small groups and engaged them in folding activities. The following extract shows the interactive dialogue between Stephen and his colleagues.

Stephen: *What did you realise after the folding?*

Colleague: *They [one part] falls? on each other perfectly.*

Stephen: *If one part falls exactly on the other, what do we call this line [points to the one line connecting the two vertices]? If they are falling on each other perfectly, what do we call that line?*

Colleague: *Line of symmetry.*

Stephen: *Ok, first, what is the name of the shape formed?*

Alex: *Triangle.*

According to Sfard (2008), the use of concrete mediators enables learners to produce factual narratives in the discourse. Stephen used leading questions to engage the colleagues, in producing narratives governing the activities performed. One colleague said that the folding of the rhombus along one of its diagonals forms a triangle. The colleagues followed the lesson critically and were

able to provide the answers expected by Stephen, in relation to the activities being performed. This finding is in line with Duatepe-Paksu's (2017) observation that as learners engage in the activities of folding and unfolding, they begin to examine certain critical features of the objects more closely and derive meaning, which often creates stronger memory and enhances retention.

Stephen engaged the colleagues in critical thinking by asking for their observations about the activity performed. When the parallelogram he folded during the review session formed an overlapping shape, he asked the colleagues if they could provide a reason for that, by comparing the folded rhombus to that of the parallelogram. Maxwell (A) said, "*I think because in the rhombus, all the sides are equal*". Maxwell reasoned this way to support the non-overlapping sides of the rhombus, when he folded it. Stephen asked if the class agreed with Maxwell, and probed for any different opinions, as a way of giving everyone the opportunity to share a different line of thinking.

Observation showed that the use of paper folding provided the colleagues with the opportunity to express their mathematical thinking. Sfard (2008) asserts that when learners are provided with a physical object that forms the centre of their discussion, they can communicate their thinking in a rich and meaningful way among themselves and with the knowledgeable other. This notion is supported by Wares (2016), that concrete materials help in communicating ideas in learning. Researchers (Rondina, 2019; Uribe & Wilkins, 2017) also found that when concrete materials are used in a lesson, learners have the chance to verbalise their thinking about the observed object.

Duatepe-Paksu (2017) suggests that organising the learners in groups, in paper folding activities, enables them to share ideas and learn from one another. The colleagues who were put into groups were found to share ideas during the activities. There was evidence of cooperative engagement among the learners. Chan and Idris (2017) describe cooperative learning as a teaching technique in which learners are put into small groups and learn from one another in the discourse. The colleagues were sharing ideas with each other about the folding activities they were engaged in.

He used questions to guide his colleague learners' observations regarding the desired lesson objectives. Through the paper folding, cutting and measurement, he showed the following properties of the rhombus:

1. All sides being equal
2. Two lines of symmetry
3. The diagonals bisect the angles at the vertices (corner angles).
4. The diagonals bisect at right angles.

The use of concrete mediators enhanced the colleagues' understanding and discoveries in learning. This was evidenced by processes from which the properties emerged or were discovered. In most cases, the properties were mentioned in response to questions posed by Stephen, to guide their discoveries in the activities being performed. Several studies reported similar findings, that the use of concrete objects in teaching mathematics enables learners to explore, make meaning of and understand the concepts being taught (Cockett & Kilgour, 2015; Moyer & Westenskow, 2013; Rondina, 2019). Situating this finding in the Sfard's (2008) commognitive framework, is based on her belief that the use of concrete mediators in a discourse, enables learners to express ideas in a meaningful way, which facilitates their meaning making in the discourse.

Sfard (2008) asserts that a discourse is characterised by its distinctive word use. Stephen paid particular attention to the colleague learners' use of the appropriate geometric terms in communicating ideas to make sure that they learned and used these terms, when talking about geometry. Stephen's guidance of literate word use is shown below:

Stephen: *What do you notice about the diagonal and the angle at the vertex (corner angles)?*

Nsiah (B): *The diagonals divide the angle into two equal parts.*

Stephen: *When something is divided into two equal parts, what term (word) do we use for it in mathematics/geometry?*

Stephen: *It starts with "b"*

A colleague said, 'bisect'. Stephen asked the learners to use this preferred word to describe their observation as "*the diagonals bisect [the] corner angles*". Guiding learners' meaningful use of

words in a discourse, is consistent with Sfard's (2008) commognitive framework. Learners' ability to acquire and use geometric terms meaningfully is fundamental in learning geometry (Alex & Mammen, 2018). The authors add that learners must learn technical terms to communicate their ideas correctly.

Nsiah (B) taught one of the circle theorems in a reality-based instructional approach using a concrete mediator. When he had provided the visual representation of the theorem on the whiteboard and had guided the colleagues to draw, but with varying radii, he asked them to measure the angle the diameter subtended at the circumference, to which each group gave the answer of 90° . Similar to other hands-on-activities, all the colleagues participated in this lesson. Nsiah encouraged other group members to measure it to verify for themselves. This corroborates Kontas' (2016) claim that teachers do not only use concrete objects to guide learners' understanding in learning, but most importantly, to actively engage them in the learning process. This engagement enables learners to individualise learning (Sfard, 2008). When learners have been actively involved in doing and have individualised the concept, it gives them confidence and makes them take ownership of their own learning (McDonough, 2016). Learners' understanding of concepts is deepened, retention is enhanced, and they can apply the understanding in a problem-solving situation (Al-Mutawah et al., 2019). After the 'doing' phase of the instruction, Nsiah deepened the colleague learners' understanding, by using geometric properties and concepts to prove the theorem analytically. All the lessons designed by the PSTs using concrete mediators, shared some characteristics with the concrete-representation-abstract model of instruction.

Throughout the lessons that incorporated the use of visual mediators, learners' active participation was observed, which showed that the approach actually met their style of learning. Kablan (2016) asserts that the use of concrete mediators in teaching accommodates for the various learning styles of learners. This kind of instruction aligns with learning through doing. It has support from NCTM (2000) that knowing mathematics is doing mathematics and constructing knowledge. In addition, research shows that knowledge constructed by learners themselves is understood, retained and applied to new situations (Cope, 2015; Kontas, 2016). From the lesson observation, learners developed an interest in these lessons due to their active participation. Learners' class contribution was observed to be high since many of them often raised their hands, wanting to answer questions

posed by the PSTs. Researchers agree with this notion that learners see mathematics as fun, and become engaged in the lesson when they are guided to do their own exploration (Cockett & Kilgour, 2015; Cope, 2015; Kontas, 2016). Within Sfard's (2008) theory, learning takes place when one works with a more knowledgeable person to receive direction and guidance so that one can gain autonomy in learning. When appropriate support and guidance are given to learners, their understanding can be deepened more, than when they learn on their own. This could mean that a learner may learn more when receiving support from a knowledgeable person (Naidoo, 2011; Sfard, 2008).

7.3.3 Narratives

Sfard (2008, p. 134) explains that a narrative is a sequence of words that are “framed as a description of objects, of relationships between objects, or of processes with or by objects”. Narratives can be endorsed or rejected in the context of “discourse-specific substantiation procedures” about the object. Narratives are endorsed and labelled as ‘true’ when they are accepted in a community of learning, otherwise, they are rejected (Sfard, 2008). A narrative can be any text, written or spoken, framed to describe an object (Sfard, 2007).

Sfard (2007, p. 574) asserts that in school mathematical discourse, the endorsed narratives are called the mathematical theories, which include discursive construction such as “definitions, proofs, and theorems”. Geometric discourse requires narratives to be produced as a description of the various definitions, theorems, proofs and properties governing geometric concepts and figures.

The PSTs produced various narratives in their geometric discourse, which ranged from definitions to properties, to theorems governing geometric shapes. Some of the PSTs developed their colleagues' geometric thinking on definitions of terms within the content area. For example, in Jones' lesson on the circle theorem, he guided the colleagues to define all the basic terms necessary for their understanding and meaningful engagement in the lesson. This finding supports (Kemp & Vidakovic, 2021; Leikin & Zazkis', 2010) assertions that definitions form the basic building blocks of any content knowledge in mathematics. In addition to Jones defining these geometric terms, he represented them in visual forms on the whiteboard. Visual representation helps in producing narratives about geometric terms and figures (Sfard, 2008). Cunningham and Roberts (2010, p. 3)

maintain that to define geometric concepts, it is not the definition that comes to the learners' minds, but their "prior experiences with diagrams and their attributes" associated with those concepts.

According to Sfard (2008), narratives can be accepted or rejected in the context of the content under discussion. It was observed that not all the definitions produced by the colleague learners were accepted by Jones. For example, Jones did not accept or endorse the definition put forth by a colleague that "*a diameter is a line that divides a circle into two equal halves*", but accepted another one that "*a diameter is a line segment drawn through the centre of the circle with its endpoints lying on the circle*". Jones rejected the previous definition, probably to avoid any misconception that learners may associate with it, as was found in the work of Mudaly (2021) where learners claimed a line to be a diameter even though there was no indication that it passed through the centre of the circle. Alex and Mammen (2018) affirm that producing appropriate narratives about geometric concepts, reduces the incidents of misconceptions. The three participants in Group A were more particular about how their colleagues produced endorsed narratives compared to their counterparts in Group B.

The use of endorsed narratives was seen in the PSTs' explanations in many forms, such as descriptions of geometric figures, identifying and substantiating some geometric properties that govern particular solutions, and the use of appropriate theorems in solving the various tasks in their lessons. Some of the narratives included; definitions, producing names of geometric figures, properties associated with parallel lines and geometric figures as well as the properties and theorems associated with circles. For example, a polygon with three sides was named a triangle, and the narrative about its interior angles raised by Alex was that "*the sum of the interior angles of a triangle is 180°* ", an object-level type of narrative associated with a triangle, as explained by Sfard (2008). Thus, the study focused attention on the kinds of endorsed narratives, such as the properties and the narratives that the PSTs used to substantiate or justify their routine actions in geometric discourse. For example, angles that are formed on a straight line can be summed and equated to 180° , with the supporting narrative that 'the sum of adjacent angles on a straight line added up to 180° '.

7.3.4 Routines

Sfard (2008) defines routines as repetitive patterns that characterise a particular discourse. They are mathematical regularities one follows in creating and substantiating narratives in a discourse. In other words, they are metarules that regulate learners' actions in a discourse. Routines are classified as ritual or explorative. In the following sections, the different patterns that the PSTs followed in developing geometric concepts, are discussed in line with the characteristics of ritualised and explorative discourse.

7.3.4.1 *Ritualised discourse of teaching.*

Ritualised discourse, according to Sfard (2008), is characterised by using rules in solving mathematical tasks. She associates ritualised discourse with the use of meta-rules to show 'how' to solve a mathematical task.

Based on the classroom observation, the lessons of three of the PSTs, one from Group A and two from Group B, were found to exhibit some characteristics of ritualised instruction. They provided strict rules of how to solve some of the examples they used, to develop the learners' geometric thinking. In Cynthia's (B) lesson on 'the exterior theorem of a triangle', she solely determined for the learners the steps to follow, in solving for the missing angles in the task. She demonstrated nearly all the patterned routines they were supposed to learn. Sfard (2008) contends that the most common type of instruction in our classrooms, is of the ritualised type. She adds that learners begin the learning of mathematical routines in a ritual way by following a sequence of instructions for the purpose of "creating and sustaining a bond with other people", particularly the knowledgeable other (p. 241). This sequence of instructions focuses on 'how' to get something done. When Cynthia had written a task on the whiteboard, she said, "*Solve for the interior angle as we did last week*". This could be an order for the colleague learners to follow (or recall) the same steps for solving a problem that was taught previously.

Another ritualised routine pattern in her discourse emerged from the statement that "*so it means whenever we are finding the exterior angle, it is the same as [adding] the two opposite interior angles*". According to Sfard (2008), ritualised discourse, in most cases, is associated with prompts and is extremely restricting. Cynthia seemed to be limiting what they could do, forgetting that

tasks may not always appear as she had presented them. This finding is in line with the claim of Nahdi and Jastisunda (2020) that classroom mathematics teaching mostly focuses on developing learners' knowledge of procedures for solving mathematical tasks.

This was followed by a series of questions that seemed to be in a command form. The questions were '*Is that clear?*', '*Are you okay?*', '*Any questions?*' Little attention was given to the colleague learners' participation, except for one task where she called a colleague to solve on the board.

The colleague learners were made to imitate the solution process as taught by Cynthia, as it is an acceptable way in the interim phase of a discourse (Sfard, 2008). When one learner was asked to solve a question on the board, Cynthia said, "*Do it as I taught you*". Sfard (2008, p. 267) acknowledges that the learning process often starts with "loosely related rituals" by depending on situational clues? (imitation). Thus, teachers also try to guide learning by imitation. According to Sfard (2008), ritualisation makes learners dependent on learning clues, which has restricted applicability but is unavoidable in developing learners' new mathematics discourses. Even though teaching by imitation may be unavoidable, the goal of school mathematics teaching is to develop an explorative way of thinking (Sfard, 2008).

Ritualised instruction can be associated with Sfard's (2008) term acquisitionist approach to learning, which is characterised by receiving and processing verbal information from teachers, on how to solve a task. She associates this type of learning as "behavior without mind" (Sfard, 2008, p. 92). In other words, it is learning with little or no thinking, or learning by memorisation.

In Maxwell's (A) lesson of teaching properties of parallel lines, he provided verbal information for the learners to process. Emphasis was laid on the rules of the discourse. The properties were just stated for them to learn and apply, in solving related tasks. Even though he started the lesson with a brief review, the actual concept development showed more of the 'how' throughout his discourse. He drew a transversal to the two straight lines and labelled the angles formed at the corners. He said,

So, we are going to talk about the properties of angles. Now angle a and angle f, angle c and angle h [wrote on the board] are vertically opposite angles [asked the colleagues to repeat after him].

He continued that:

Now when we talk about vertically opposite angles, they are equal. They are what? [the learners responded ‘they are equal’]. ... do you understand? [colleagues responded ‘yes sir’].

After that he asked the learners to identify the rest of the vertically opposite angles in the figure shown (Figure 7.16)



Figure 7.16. Maxwell's drawing for teaching parallel lines.

Maxwell: *Can anybody identify another type of vertically opposite angle?*

Colleagues: *Angle d and angle g.*

Next, he asked the learners to identify angles in a similar relative position. Learners responded angles *g* and *e*, and angles *h* and *f*. Maxwell named them as ‘corresponding angles’ and applauded the learners. He added that corresponding angles are equal. He also emphasized that ‘angle *a* and angle *b*’ are on a straight line and that they add up to 180° . He solved examples to demonstrate the application of the properties when attempting mathematics tasks.

Sfard (2008, p. 267) maintains that “whenever the conversion of the full-fledged explorative discourse fails to occur and what was supposed to be a transitory stage gains permanence, teaching methods are the immediate suspect”. We see from Maxwell’s (A) teaching, that he presented almost everything the learners had to learn about the properties of parallel lines. He probably, felt that the learners knew nothing about the topic and hence needed to fill their minds with information. According to Ardeleanu (2019), teachers often regard learners as “having gaps in knowledge” that need to be filled, by giving them lots of information. Learners are therefore not

engaged in the teaching and learning process and are treated as passive learners (Fletcher, 2009; Swan, 2005). Thus, the teacher plays a dominant role as an instructor. This was evident in Maxwell's (A) discourse. After he drew the parallel lines with the transversal, he pointed at some angles and gave their associated names. As a characteristic of acquisitionist learning, learners only played the role of passively listening to the verbal information that Maxwell offered (Ardeleanu, 2019; Bishara, 2015; Fletcher, 2009; Sfard, 2008; Swan, 2005). Learners are often provided with procedures and facts to learn, which is termed a mechanistic approach to learning (Sfard, 2008). Swan (2005) also calls this teaching a transmission approach where the teacher (in this case Maxwell), transmitted or told the learners which of the angles are equal, and their names, for the learners to commit to memory.

After the facts given by Maxwell, the only time the learners' voices were heard, was when they repeated (in chorus) the name 'vertically opposite angles' after him. The next chorus answer was a response of 'Yes sir' to Maxwell's question "*please, do you understand?*" Ardeleanu (2019), Fletcher (2009) and Swan (2005), remark that teachers, in mechanistic approaches to teaching, only question the learners to check if they are following the lesson, or to direct them in a particular way. This was found in Maxwell's discourse, when on several occasions, he asked the learners to repeat some words after him, or to check their understanding, not individually, but the whole class.

Maxwell emphasised which angles were equal based on the position of the intersection of the straight lines, in the form of a rule. He said, "*so whenever you are solving a question... and you see these angles* [pointing to directly opposing angles], *they are equal*". Thus, the learners were only told to recognise the angles that were equal. A similar finding was reported by (Ngirishi & Bansilal, 2019, p. 89) in which one of their study participants explained his learning experiences on the properties of parallel lines, that in a geometry lesson, "the terms were never explained, they were just told which of the angles were equal".

Maxwell's discourse was characterised by his assumed central role of giving the learners facts and procedures to guide their learning. This supports similar findings reported in the literature, where most traditional classrooms portray teacher dominance in the discourse of teaching procedures (Ardeleanu, 2019; Bishara, 2015; Sfard, 2008). It was also observed that he did not use any

teaching activity such as measuring to facilitate knowledge construction. It was purely a ‘chalk and talk’ approach to teaching. Similar findings about Cynthia’s and Maxwell’s lessons were observed about Albert’s lesson on parts of a circle.

7.3.4.2 *Evidence of explorative routine*

Sfard (2008) maintains that explorative routine is the learner’s ability to produce associated properties of mathematics objects. It is characterised by knowing the ‘when’ and the ‘why’ underlying a particular routine. It is the implicit and explicit understanding of mathematics and geometry in particular. Sfard (2008) claims that producing endorsed narratives about mathematical objects remains the ultimate goal of school mathematics. This means that learners should be engaged in a lesson that enables them to construct and re-construct ideas as a way of producing endorsed narratives, a learning habit and ability that align with Sfard’s (2008) thinking and learning.

There was evidence that some of the PSTs engaged the colleague learners in knowledge construction in their discourses. These were mostly found in lessons that made use of different visual mediators, to ensure the learners’ active involvement. For example, in Stephen’s (A) lesson, he used paper folding (origami) to guide the colleague learners’ exploration of the properties of a rhombus. This lesson design engaged the colleague learners to assume a central role in finding out things for themselves, which contributed to producing narratives through the construction and re-construction of ideas (Sfard, 2008). Sfard (2007, p. 609) asserts that “agreeing about the discourse to follow and the readiness to shape one’s own discourse in its image are the important factors in learning”, and this can “only be achieved through [learner] participation”. Based on the colleague learners’ participation and the questions posed by Stephen, the colleagues were able to produce narratives, some true (endorsed) and others false (rejected). Stephen guided them to reconstruct the rejected narratives by suggesting some word use in their discourses, as conforms to literature. Thus, the colleague learners were able to produce endorsed and objectified properties of the rhombus. Other researchers (Rondina, 2019; Uribe & Wilkins, 2017) have reported similar findings that instructional design that puts learners at the centre of the teaching and learning, often helps them to produce appropriate verbal representation of what has been observed.

The effectiveness of teaching and learning was realised in closing the lesson, when Stephen assessed what they had learned. They produced endorsed narratives about the properties of the rhombus as per the objectives of the lesson, because the object of discourse was seen and manipulated in diverse ways. Sfard (2008, p. 146) asserts that irrespective of the “intangibility” of mathematical objects and concepts, mathematical communication depends on what has been seen, which reduces the “abstract type of talk”. The colleague learners were taught the properties of a rhombus, not in an abstract way, but in a concrete way of experiencing the properties.

Sfard’s (2008) advocacy for the use of visual mediators in mathematics discourse, suggests that learners’ visual senses play a significant role in developing their reasoning abilities. Many concepts in geometry are presented in diagrams, such as the use of icons in designing diagrams and concrete materials, that are signifiers of geometric concepts. In Clement’s (A) lesson on the proof of the exterior angle theorem of a triangle (see Figure 7.5b), the concept of parallel lines in that task was presented in a different way from how it has been experienced in many traditional mathematics classrooms (horizontal appearance). He developed the learners’ visual reasoning in the lesson. This is exemplified in the following extracts:

Clement: *What do you see from the two diagrams?*

Colleague: *There is a line drawn through point C in Fig. 1.1 [see Figure 7.5b].*

Clement: *Ok. Any different observation?*

Colleague: *There is a line that is parallel to line BA.*

Clement: *How do you know they are parallel lines?*

Colleague: *There is an arrow on the lines BA and CE.*

Clement: *Ok. I like your answer.*

Clement invited another colleague to come to the board and explain why the straight lines are parallel.

Colleague: *Sir, we have this line BA and also ... line CE. So, this line is parallel to this line*
[moving the two hands along the lines to indicate that they are parallel].

Clement: *Why?*

Colleague: *Because they move with a constant ...* [still moving the two hands along the straight lines].

It can be deduced from the preceding excerpts that the colleague thought of parallelism of the two straight lines on mere appearance rather than evidence in concept communication in geometry, as seen in his utterance “*Because they move with a constant ...*” which he demonstrated by moving his two figures along the straight lines.

One colleague said, *Sir I want to help him.*

Clement: *Yes. Come and help your brother* [Laughter in the class].

Colleague: *They are parallel because there is an arrow here* [pointing to the arrow on line BA] *showing that this line is going in the same direction as this* [pointing to the arrow on line CE] *and they can never meet.*

Clement commented that “*The arrows are very clear that ... line BA and then line CE are parallel.* When he said this, someone whispered, “*Ahhh there is an arrow at the top. I didn’t see oo*”.

Clement jokingly said, “*Now all those without eyes can see clearly* [colleagues laugh upon this utterance].

Clement developed the learners’ ability to produce narratives based on the ‘when’. Thus, even though some of the learners may have pre-conceived that the lines are parallel (as answered), not all of them may have realised that the straight lines were designed with iconic mediators that substantiated their notion of parallelism (Sfard, 2008). Clement’s comment that “*now all those without eyes can see clearly*”, could be a prompt to learners to critically examine features of figures to aid interpretation. Clement asked, “*What would you call line AC?*” (see Figure 7.6a). After some silence, he said, “*or do you want me to extend this*”? When he extended the line AC beyond the parallel lines, there was a chorus response that the line AC is a transversal. This could be a good effort to develop the learners’ visual reasoning. This mode of presentation exercised the learners’ levels of spatial reasoning.

Sfard (2008) maintains that explorative routine is the ability to identify properties of mathematical objects. Even though all the PSTs developed their colleague learners’ abilities to solve geometric tasks through their related properties, the medium in which the properties were developed differed between and within the two Groups. The discourses of three of the PSTs in Group A and two in Group B showed more of the explorative instructional discourse. Many of the PSTs in Group A

designed instruction that had the potential to develop the colleague learners' explorative ways of thinking compared to their Group B counterparts. The instructional design of Group A members, shares more of the characteristics of learning by doing, from the constructivist perspective on teaching, compared to that of Group B participants, whose instructional design required the colleague learners to process the information given to them in the traditional way of teaching and learning (Fletcher, 2009; Sfard, 2008; Swan, 2005).

7.4 Conclusion

This Chapter focused on the participants' geometric discourse. In the first place, learning opportunities that the participants in both groups offered, for learning geometry were presented. Secondly, the presented geometric discourses were analysed based on the characteristics of Sfard's (2008) commognitive constructs. Also, the geometric discourses were also analysed based on the group characteristics. The discussion of the findings was situated in the related literature.

CHAPTER EIGHT

SUMMARY, RECOMMENDATIONS AND LIMITATIONS

8.1 Introduction

Chapter seven presented a discussion of the PSTs' geometric discourse in themes that highlighted the constructs of the commognitive framework. This chapter, which concludes the investigation presents the summary, recommendations, and limitations of the study.

8.2.1 The nature of pre-service mathematics teachers' discursive thinking in geometry

The first critical question sought to analyse the PSTs' discursive thinking in geometry through the constructs of the commognitive framework. Sfard (2008) maintains that thinking is a form of communication which takes place with oneself or with others and is individualised in nature. She talks about four key ways to understand how a person thinks, which serves as the basis of analysis and discussion in the study.

8.2.1.1 PSTs' word use in geometric discourse

The PSTs' word use in geometric discourse was analysed in two ways, namely: word use in the definition of geometric terms and word use in explaining their solution strategies. It was found that they used both literate and colloquial words in their discourse. This was evident in the kind of word use describing their thinking processes, about the objects on which the interview was centred. This finding is in line with other researchers' view that learners' discourses are a combination of both literate and colloquial words (Berger, 2013; Tasara, 2017; Wang, 2013).

Analysis of the PSTs' word use in defining geometric terms showed differences between and within the two participating groups. Many discourses of the Group A participants showed literate word use in defining the geometric terms and properties, compared with their counterparts in Group B. Many of the Group A participants were capable of using endorsed words in their discourses. Endorsed words are those that have a shared meaning in the community of mathematics learning (Sfard, 2008). Three of the Group A participants' discourses demonstrated that they had acquired the appropriate words for comprehensively describing their geometric thinking. It is

important to use the correct words to define mathematics concepts, since they play a critical role in the teaching and learning of mathematics (Baktemur et al., 2021; Cunningham & Roberts, 2010; Fujita & Jones, 2006). The PSTs' ability to use literate words to define geometric concepts and properties would enable them to teach definitions of geometric concepts well, as a fundamental aspect of developing learners' geometric thinking in learning. It would also enable them to explain geometric properties in their exact spatial sense.

Ndlovu (2014) asserts that definitions of geometric concepts and shapes are critical in learning geometry, because they form the basis for learning the properties of geometric shapes. Leikin and Zazkis (2010) claim that definitions form the basic unit of discourse, and it is important that learners' geometric thinking is developed on their ability to define some key terms and concepts of the geometric object. These competencies were demonstrated by three of the Group A participants. Stephen, Jones, and Clement used literate words to define and describe geometric terms and properties, taking into consideration certain important diagrammatic features. For example, Stephen defined the exterior angle theorem of a triangle as follows: “... *the exterior angle is equal to the sum of the two opposite interior angles*”. Also, when Jones said that the isosceles triangle has two equal angles, he specified these angles by saying “*angles facing [opposite] the equal sides*”. Jones also described another property of the isosceles triangle, viz that the line of symmetry divides the angle formed at the point where the two equal sides meet. Sfard (2008) asserts that word use is very important because it is responsible for portraying the exact meaning the user wants to convey. Analysis of the above description provided by these PSTs, shows that their word use communicates the exact intent, from which meaning could be derived. The discourses of these PSTs show that they have used words in functional perspectives of geometry to define the concepts.

These participants demonstrated the ability to describe geometric concepts with literate word use, which is central to teaching and learning. Literate word use is necessary for developing classroom mathematical communication because thinking manifests itself through communication. Literate word use in mathematical communication is important in ensuring successful learning in geometric discourse (Sfard, 2007, 2008). Primarily, word use in mathematics communication helps teachers and learners to engage in mathematical discourse, with a high level of precision (Mulwa, 2015).

Also, the results show that five of the PSTs used words in both literate and colloquial ways to define and describe the objects under discussion. By colloquial, they either used words that are open to several interpretations or used incomplete words or terms in their definitions. The use of colloquial words in mathematics discourse was found among all the PSTs in Group B and one in Group A. For example, three participants in Group B and one in Group A, similarly described the exterior angle property as follows: “... *the exterior angle is equal to the two opposite interior angles*” (see sections 5.4.1 and 5.4.2).

The theorem was colloquially stated. What is worth noting about stating the theorem is the omission of the word ‘sum’ signifying the operation between the two interior angles. The omission of the word renders the theorem incomplete. According to Sfard (2008), the use of words in mathematics discourse should draw attention to the exact meaning, as used in school mathematics. When words are not used to specify an exact idea, they allow for several interpretations and lead to errors in discourse (Atebe & Schafer, 2010; Sfard, 2008).

Other geometric concepts which were defined or described using colloquial words were ‘diameter’, ‘rhombus’, etc. For example, some of the participants defined a diameter as “*a line that divides a circle into two equal parts*”. Sfard (2008) asserts that words that lead to multiple interpretations are characteristic of colloquial word use. This means that such words do not support the development of accurate and specific spatial reasoning among learners. This finding supports that of Mudaly (2021) who observed that the study participants considered a straight line (that seemed to pass through the centre of a circle) as a diameter without carefully examining its spatial condition, as to whether it passed through the centre or not.

Further analysis showed that some of the PSTs seemed to have developed their geometric thinking on weak fundamental concepts. These are people who have passed tertiary courses in mathematics and geometry, yet demonstrated difficulties with the definitions of some basic geometric terms. For example, Maxwell (A) solved many of the geometric tasks and demonstrated both ritual and explorative ways of thinking, but was found to be deficient in wording the definitions. He had already defined some of the geometric terms with colloquial words. When he was asked to define

a tangent, he commented, “*these definitions*” and said, “*I know the name, but the definition I have forgotten*”. This shows that Maxwell could not find any words to express his thoughts on what a tangent is. Also, when he was asked to define or describe a cyclic quadrilateral, he responded, “*a cyclic quadrilateral, hmmm ... I can draw it but the definition no*”. This finding supports Kemp and Vidakovic’s (2021) assertion that many expectations of college students are not met because, in spite of passing advanced courses taken in their programme of study, they are unable to complete definitions of terms in the same content domain. According to Guner and Gutlen (2016) and Speer et al. (2015), definitions form part of the fundamental component of Euclidean geometry and also serve as a starting point for teachers (and learners) to develop sound and adequate knowledge of content in the mathematics curriculum.

The data showed that the PSTs further used both literate and colloquial words to explain their solution processes. As indicated earlier, Sfard (2008) asserts that words use in a discourse can be literate or colloquial in nature. Those who used literate words were found to specify the intended spatial concept, by using appropriate words. For example, some participants used the word ‘adjacent’ in stating the angle property of a straight line. They also used endorsed words to state the theorem of the exterior angle of a triangle (see section 5.4.2). It was found that many of the Group A participants described their thoughts about their solutions with functional use of words, as well as by stating the properties and theorems to substantiate their routine solutions. Thus, those who performed well in solving the geometric tasks, also showed that they had a good understanding of the basic ideas in the content area.

Some of the participants used colloquial words to substantiate the narratives that informed their solution strategies. In the analysis of their word use, some said, “*angles on a straight line is 180°*”. It should be observed that the property was stated with incomplete word use. Inappropriate word use or an incomplete stated narrative, often leads to misconceptions and errors. Similar findings are reported in the literature, that learners’ errors in their solution processes were a result of being exposed to incomplete word use in stating a property. As a result, the learners applied these properties which resulted in an error solution (Ngirishi & Bansilal, 2019).

Also, it was found that participants' wrong use of words in stating the property of an isosceles triangle, related to the interior angle and the equal sides, led to a wrong solution by Nsiah's (B) to task 3.3. In most cases, the property was stated in a traditional classroom discourse that '*the base angles are equal*'. This word used in stating the property could emanate from the reliance on the visual appearance of the isosceles triangle, as often found in traditional mathematics classrooms, where isosceles triangles are drawn with the base containing the equal angles (prototypic figure). This suggest that the word is used in a passive-driven form. Sfard (2008) explains that passive-driven words are those acquired during the early stages of word development in learning, where learners associate words with visual recognition of the appearance of geometric shapes (prototypic appearance). Nsiah may have been familiar with the term 'base angles' because that is what he may have been taught in his geometry lessons. According to Sfard (2008), this may result from blending word use with objectified talk about the object.

In conclusion, from the analysis of their word use, it could be said that there were many colloquial words used in the discourses of the PSTs in Group B compared to their colleagues in Group A. Thus, many of the PSTs who performed well in the test were able to use mathematically literate words in their geometric discourse. They used words that are endorsed and communicate the meaning of the exact geometric concept. On the other hand, some PSTs in Group B used words colloquially. Thus, most of the words used in substantiating their solution processes, were prone to several interpretations, which could be a major hindrance to understanding and developing geometric thinking.

8.2.1.2 Visual mediator

The participants seemed to rely on the various visual mediators in communicating their ideas about the objects of study. Sfard (2008) asserts that visual mediators are objects that can be seen and operated upon, that form part of our discursive processes. Many of the participants linked the way they thought to the things they saw, which then influenced how they used words. The visual mediators seemed to coordinate the participants thinking strategies as well as their communication abilities. For example, the participants' ability to talk about vertically opposite angles depended on the interpretation made about the structure of the task designed. When Stephen was asked to explain his solution plan to task 1.3, he said,

... with the transversal line drawn through the two parallel lines ... we have new angle properties introduced. I also realised that the same variables are used in the angles that are opposite to each other. So, with this vertically opposite angle...

The preceding excerpt shows that his thinking about the emerging properties, was based on the diagram interpretation. Thus, the participants' communication was informed by the visual mediators and associated word use. Ryve et al. (2013) reported similar findings in which visual mediators and associated words, formed a central part upon which learners built their communication.

It was observed that the various visual mediators formed the central part of the participants' communicational abilities. These were diagrams, iconic, symbolic and concrete mediators. The most used ones during the interview were the diagrams, iconic and symbolic visual mediators. Some of the participants demonstrated mediational flexibility by using a combination of them. That is, some of the participants demonstrated the ability to use different visual mediators to make meaning of geometric shapes and concepts (Sfard, 2008).

Thus, the participants in Group A were conscious of interpreting the structure of the task together with its iconic mediators, before devising a solution strategy. This was evident from the way they connected their talk to the visual signifiers governing the task. It was also noticed that no decision was made without figuring out the iconic mediators and what they meant.

Some of the participants in Group B were not too conscious of the iconic mediators use in designing geometric task(s). They mentioned properties associated with parallel lines and claimed equality without first looking out for evidence of the parallelism of the two straight lines. This made them apply the property wrongly and led to a wrong solution.

8.2.1.3 Narratives

The participants' discourses contained endorsed narratives about the geometric objects. The participants focused on existing narratives about geometric figures and shapes to produce new narratives, some being accepted, and others rejected. For example, the participants produced

narratives about a straight line, as found in Clement's discourse that "*the sum of the adjacent angles on a straight line is 180°* ". They also produced narratives on triangles as found in the discourse of Jones' that "*the sum of the interior angles in a triangle is 180°* ". To some extent, the participants were able to produce endorsed narratives such as definitions, theories, theorems, and proofs related to geometric objects, as used in scholarly mathematics. This finding aligns with Sfard's (2008, p. 200) assertion that communication is a "rule-regulated activity". She adds that discourses result from "rule-governed processes", expressed as object-level rules about mathematical objects. These governed rules can be regarded as the "properties of the objects of this discourse and take the form of narratives about these objects" (Sfard, 2008, p. 201).

8.2.1.4 Routines

The PSTs demonstrated adequate thinking of the required metarules needed to devise solutions to the tasks involved in the study. They determined appropriate strategies to solve the tasks, even though some wrong calculations were observed in certain solutions processes. They used endorsed narratives of the properties and theorems, as a basis to formulate the various linear equations of the angles involved in a particular geometric object. There were more substantiating narratives in the discourses of the participants in Group A, compared to their counterparts in Group B. In most cases, the participants in Group A produced narratives in their planning stages of devising solutions. This was also observed in a few discourses of the Group B participants, even though some of them stated the property when they were asked to justify their routine strategies.

Thus, many of the PSTs in Group A demonstrated adequate routine thinking about when to use a particular solution strategy and the skills involved in performing the procedures. This finding is supports that of Supardi et al. (2021), who made similar observations when they studied students' problem-solving skills through the commognitive perspective.

8.2.2 The nature of the pre-service mathematics teachers' routines thinking when solving tasks in geometry

8.2.2.1 Ritualised routine

Many of the tasks used in this study could be solved in more than one way. The approaches could be the most basic (such as the straight line algorithm) or more advanced thinking, indicating an

awareness of the associated properties governing the task design. Analysis of the PSTs' solutions shows that many of them preferred solving the tasks in a ritual way that uses a step-by-step strategy to devise solutions to the tasks. These solution strategies are discussed in section 6.2.1. This finding resonates with that of other researchers, Zuya et al. (2017), that learners often prefer solving tasks using a set of procedures. Most learners rely on the use of algorithms to solve tasks in mathematics. In a study by Mann and Enderson (2017) to investigate learners' preference for rule or formula-driven (procedures) or concept-driven approaches to learning, they found that learners preferred the use of a rule-based approach (procedure) to the conceptual approach. Within Sfard's (2008) commognitive framework, the use of a set of procedures to solve a task is a manifestation of a ritual routine. Ritual routines are characterised by the use of strict rules, mostly determined by the teacher or authority. The focus of ritual routine is 'how' to get something done, with little or no attention to 'when' or 'why' the approach works.

Another finding is that the participants (especially those in Group B), over-reliance on procedural approaches to solving tasks, led them to devise a series of computational steps in their solutions to the tasks (see section 6.2.2, Figures 6.3 and 6.4). It could be seen in Task 3.3 (Figure 6.4), that only two of the angles marked by letters needed some form of calculation. However, Maxwell (A) and Albert (B) preferred using a series of computational steps to find the answers. This finding supports that of Mann and Enderson (2017), who also reported similar findings and claimed that learners often prefer the use of algorithms to solve mathematical tasks because they find those methods to be familiar. The authors add that learners' proficient use of algorithms often obstructs their confidence in doing things in a new way. Sfard (2008) asserts that learners stick to the ritual way of doing exactly the same thing as they might have been doing, to obtain a reward. She adds that the dominant classroom teaching of 'how' to solve tasks in the absence of the 'when' and 'why' mostly restricts their thinking to rules.

Also, two participants solved two of the tasks by generating two linear equations in two variables, and solved them simultaneously. Although it could be an alternative approach to solving the tasks, one participant explained his solution devised along the straight line algorithm, and shared some characteristics of ritualised way of reasoning. knowledge. He could not show any other approach to solve the task.

There was the narrow range of applicability among those who followed the set of procedures to solve the tasks. Two of the participants, in Group B, could not think of when to apply certain properties that govern the geometric figures. They applied the ‘equality of corresponding angles’ to solve task 3.1, even when there was no indication that the two straight lines were parallel. The task was probably not examined to check for possible signifiers, often used to communicate or describe some important features of tasks (Sfard, 2008). Also, Nsiah’s (B) numerous but unsuccessful attempts to solve task 3.2 (see Figures 6.7 and 6.8) could be interpreted as an inability to apply the properties of a parallelogram to the task. He only formulated an equation on the observed straight line, but could not continue. This failure may have resulted from his inability to interpret the icons used to design the task. Placing this within Sfard’s (2008) commognitive framework, he seemed to be deficient in ‘when’ to apply the best strategies to solve tasks.

8.2.2.2 Explorative routine

Even though there was evidence of explorative routine among the PSTs, it was mostly found in the discourses of the participants in Group A as compared to those in Group B. A key finding about their discourse (Group A), is the frequent mention of geometric properties in relation to how the tasks were solved. This is evident in Stephen’s (A) response in the extract that follows:

... in the diagram given I can say that the angle 130° is vertically opposite to the two angles (m and 60°). Therefore, knowing that vertically opposite angles are equal, I can write it as ...

Similarly, Jones (A) explained that:

... from the diagram, we can see that $3(x - 20)$ is vertically opposite to $2x$ which are equal. So, I equate the two angles ...

This property-guided explanation was also found in Cynthia’s (B) discourse, as follows.

... we are having direct opposite [vertical] angles here which are $3(x - 20)$ and $2x$ [points to the angles] which are equal.

The preceding excerpts show that these PSTs have developed the competence to engage in a critical exploration of identifying the objectified properties, that govern the design of a task. Sfard (2008) asserts that a characteristic feature of an explorative routine is the ability to produce

narratives about the properties of the mathematical object. Even though some participants in Group B also showed proficient use of properties, those in Group A were less prompted for justification of the property that informed a particular routine. The participants in Group A demonstrated an explorative thinking, which means they had a good understanding of the geometric concepts and were able to interpret information in a meaningful way. A similar observation was made by Nahdi and Jatisunda (2020) that learners who have a good grasp of geometric concepts are able to communicate ideas meaningfully. This result show that some participants, especially those in Group A have a good level of understanding of geometric concepts and properties and were able to apply those concepts to solve a variety of tasks in geometry (see Dewi & Asnawati, 2019). Sfard (2008) asserts that learners' ability to substantiate their discursive actions with endorsed narratives, is characteristic of an explorative discourse which helps in developing one's **competence** in devising objectified solutions to tasks. Many Group A participants demonstrated these competencies of thinking about the associated geometric properties, devise solutions to tasks, with clear evidence of understanding (Dewi & Asnawati, 2019; Sugeng & Nurhanurawati, 2018).

Also, it was found that some Group A participants solved some tasks using multiple strategies. This was a demonstration of critical thinking and problem-solving abilities. According to Sfard (2008), one of the characteristics of the explorative routine is flexibility in thinking, which is also an indication of higher order thinking, and the ability to devise several ways of solving mathematical tasks. In task 1.4, in which Alex (B) could not devise a single solution, Stephen (A) solved this task in multiple ways, as shown in section 6.3.1, (Figures 6.13 and 6.14). This could mean that his thinking was not limited to one way of solving a task, but as far as his explorative mind could search. Also, Figure 6.15 shows Jones' (A) multiple approaches to solving task 2.3. Kivkovich (2015) and Ortiz (2015) claim that the ability to devise multiple ways of solving a task, is a demonstration of good problem-solving skills; flexibility and a creative way of thinking. Semanisinova (2021) asserts that the aim of teacher education programme is to develop future teachers' content knowledge to be able to provide the appropriate guidance to enhance learning in the classroom. Teachers' ability to offer quality guidance in classroom learning, depends on the depth of their content knowledge (Semanisinova, 2021). It can be said that the participants in Group A, who demonstrated multiple ways of solving some of the tasks, have the competence to guide learners' geometric discourse in an explorative way. Klerlein and Hervey (2020) remark

that classroom teaching and learning needsto equip learners with a range of relevant strategies to solve problems in mathematics. The Group A participants are equipped with a range of strategies to assist learners in geometric problem-solving.

Another key finding within the explorative routine, was a demonstration of high visual skills or abilities among the participants. Sfard's (2008) inclusion of visual mediators as one of the constructs of the commognitive framework, signifies that there is a strong connection between thinking and visualisation. The PSTs who showed an explorative way of thinking seemed to see beyond the ordinary. This is evident in how they explained their plans, noted in the following excerpts:

... looking at the diagram given, I can say that ... (Stephen, A).

... from the diagram, we can see that ... (Jones, A).

... I used this approach because of the properties I see within this diagram (Cynthia, B).

... in geometry, when you see these short lines or dashes, it means ... (Clement, A).

It is evident from the preceding excerpts that these PSTs used words that signify their reliance on what they saw, in interpreting the tasks presented in diagrams. These results show that some participants relied on their visual skills a major tool to exercise their thinking. These findings support the view of Dewi and Asnawati (2019), who claim that learners who have good visual abilities, can develop their geometric thinking in an explorative way. Good visual abilities support the way one recognises, analyses, and interprets geometric figures, in learning. It was found in the study that those who demonstrated an explorative discourse in their routine, utilised or relied on their visual abilities to gain a clearer understanding of the tasks before devising the solutions. The PSTs exercised varying degrees of visually informed analysis of the tasks, which resulted in different approaches to their solutions.

Also, the participants' responses and their actions made it clear that visual abilities formed an integral part of their thinking processes. According to Atanasova-Pachemska et al. (2016), visual ability is the mental processing of visually obtained information, that enables mathematical discovery and understanding of mathematical tasks. Some of the participants were found to use what they saw to help them think, or to think with what they saw. In most cases, this was evident when they were seen tracing parts of the task.

Learners' inability to connect thinking with visualisation can impede their understanding of learning mathematics, particularly, plane geometry, in which most tasks are presented in diagrams. In this study, those who could not solve some of the tasks, could be said to have weak visual abilities. Riastuti et al. (2017) reported similar finding from a study where learners who made errors in solving the tasks, were those with low spatial **intelligence** due to weak visual abilities. According to Dewi and Asnawati (2019) and Mudaly (2021), learners' weak visual abilities hinder their ability to recognise, analyse and make meaning of what they see, preventing a deeper understanding of the task. However, the participants in Group A demonstrated high levels of visual abilities and can be said to have held a deeper understanding of the geometric tasks presented to them. Hence, their competence in recognising, analysing, and making meaning from the diagrams. Many of the PSTs in Group B were deficient in this visual reasoning competence.

8.2.3 The pre-service mathematics teachers' geometric thinking and their classroom geometric discourse

8.2.3.1 Word use in classroom geometric discourse

A distinctive feature of a discourse is the kind of keywords used (Sfard, 2008). Atebe and Schafer (2010) add, that geometry includes the use of language and keywords to designate specific concepts in the subject. This could mean that communicating geometric concepts can be effective, if attention is given to the kind of keywords used. The words used in a discourse can be classified as mathematically literate or colloquial.

Observation of the PSTs' classroom geometric discourses, show that they used both literate and colloquial words in their lessons. Some PSTs in Group A were found to use and also accept endorsed words that related to the geometric object being discussed. These Group A participants were found to use functional words in their discourse. In Stephen's (A) lesson, when one of his colleague learners defined a parallelogram in a colloquial way, he refined it as "*a four-sided figure with two pairs of parallel lines*". He used words to mean specific ideas he wanted to put across and was found to correct his colleagues' inappropriate use of words. Also, Jones and Clement (both in Group A) were found to emphasise literate words in their definitions. This finding supports the inference from other researchers' views on terminologies in geometry, that pre-service teachers

who possess adequate knowledge of mathematical terms are able to communicate ideas in a concise and acceptable way, to enhance learners' acquisition and use of appropriate terms in communication (Atebe & Schafer, 2010; Roberts, 2010).

It was also found that the rest of the five participants were not mindful of their use of words, as well as of their colleagues' word use in the discourse. They used both literate and colloquial words in their discourse and also accepted the same from their colleagues. For example, Albert (B) accepted his colleagues' definition of a diameter as "*a line that divides a circle into two equal parts*". Similar colloquial words were noticed in the discourses of the rest of the five participants. According to Sfard (2008), language or word use performs specific functions in mathematics discourse, with literate meaning in the context of its use. She asserts that the use of colloquial words affects the development of literate mathematics discourse. Learners' understanding may be affected if colloquial words are used in a discourse.

The PSTs must possess basic terminologies of geometric objects in their knowledge repertoire, in order to communicate ideas in mathematics discourse. According to Sfard (2008), language forms the main tool for communicating ideas in a discourse. As a result, the type of words used, or language used becomes an important tool in explaining solutions in all disciplines. Learners' understanding can be developed and deepened when words are used in a literate way (Sfard, 2008). Atebe and Schafer (2010) and Oyoo (2009) add that learners could be proficient in learning geometry, and have their conceptual understanding well developed naturally, if certain keywords are learned, understood, and used in communication. However, it was observed that many of the PSTs in this study, were not mindful of their use of words in their classroom discourses, hence, they could not provide appropriate guidance toward their learners' word use and accepted the learners' use of colloquial words in defining geometric terms.

8.2.3.2 Use of diagrams in geometry lesson

All the PSTs used diagrams in their lessons, in teaching geometric concepts. The diagrams they used to convey or communicate the geometric concepts taught, served as a valuable source of information in the discourse, as they became the objects that coordinated their communication. This is in line with Sfard's (2008) assertion that the use of diagrams to explain ideas, enables

learners to picture the ideas in their minds and offers great prospects for enhancing their understanding. The PSTs may have held a similar belief as Sfard's (2008) that a diagram makes objects accessible for learners to operate upon, in order to produce and substantiate mathematical narratives. This finding supports the view of Samkoff et al.'s (2012) that the use of diagrams makes mathematical explanations easily understood and also enables learners (problem solvers) to view and integrate pieces of information with less cognitive effort.

Also, the use of the diagrams by the PSTs helped in communicating the intended geometric ideas and also helped their colleagues' meaningful learning. This finding resonates with the finding of Dundar and Otten (2022), who also found that the diagrams that served as visual aids enabled the learners to make meaning from the learning activities. The finding also supports the belief of Jones (2013) and Matlen et al. (2018) that diagrams form an integral part of learning and understanding in mathematics. Within the commognitive theory, "communication is mediated by images, which develops learners' fluency in discourse and the goal of mathematics learning" (Sfard, 2008, p. 148).

8.2.3.3 Use of iconic mediators in geometric lesson

It was observed that the PSTs used iconic mediators to design the diagrams drawn in their lessons, to communicate and bring attention to some important features of the diagram, and also to help learners' production of endorsed narratives. Results showed that the colleague learners made meaning from the visual icons used to design the diagrams to produce narratives about the concepts of right angles, types of triangles, types of quadrilaterals, and related geometric properties (Sfard, 2008). For example, one of the colleagues substantiated his classification of a triangle as isosceles, by saying that the equal number of icons (marks) placed on the two sides of the triangle indicated that the sides were equal, hence the given name. Also, arrows were used by the PSTs designing two straight lines to communicate the concept of parallelism to learners. This action is in line with Sfard's (2008) view that icons are artefacts used to design diagrams and graphs in order to communicate certain important features in discourse and also help in learners' production of endorsed narratives.

It is worthy to note that even though all the PSTs used iconic mediators in their lessons, those in Group A seemed to demonstrate consciousness in guiding the development of geometric ideas using the iconic mediators, as compared to their colleagues in Group B, who seemed to assume the understanding of these icons by their colleagues. For example, in a diagram with two straight lines drawn by Alex (B) for a lesson, he claimed the equality of the properties associated with parallel lines, until he was prompted by his colleagues to indicate that the lines were in fact parallel (see Figure 7.9). According to Sfard (2008), learners' fluency in a discourse can be attained only when visual cues are used in discourses, and interpreted for them.

8.2.3.4 Use of symbolic mediators

Although the PSTs taught some concepts on parallel lines, perpendicular lines (right angles) and triangles, it was observed that no attempt was made to develop the learners' thinking on related symbolic artefacts. The only symbol used by the PSTs (Alex, A; Clement, A; Cynthia, B) was ' \angle ' to represent the concept of an angle, and was used in their explanation but with no awareness creation to learners. Maxwell (A) labelled the angles formed at the intersection of two straight lines with letters and continued to use these letters in his proof, even though they signified angles (see Figure 7.15). This violates Sfard's (2008) suggestion that learners need to be taught the use of artefacts in mathematics communication. She adds that learners' ability to produce an endorsed narrative about geometric shapes starts by visualising and interpreting their associated symbols.

8.2.3.5 The PSTs' use of concrete mediators in geometric discourse

Concrete mediators have been explained by Sfard (2008) as visual objects that can be seen, handled, and manipulated for learning mathematics. Four of the PSTs, two from each group, used concrete mediators in their classroom geometric discourses. The use of the various mediators by these PSTs in their lessons, allowed for interactive engagement between the teacher and his/her colleague learners. The colleague learners were found to take an active part in the lesson. This finding supports the view of Horan and Car (2018) that teachers' use of concrete materials in their lessons enable learners to become active participants in learning. Rondina (2019) adds that the use of concrete materials in a lesson, engages learners in active learning and develops their cognitive skills in mathematics.

According to Rondina (2019), learning through hands-on approach develops the cognitive abilities of learners and enables them to become constructive thinkers. In Sfard's (2008) view, the use of concrete mediators enables learners to produce factual and endorsed narratives in a discourse. In Stephen's lesson, in which he used paper folding (origami) to develop the colleague learners' geometric thinking about the properties of a rhombus, he was found to use questioning techniques to engage his colleagues' production of narratives governing the activities performed. Most of the responses given by his colleagues were found to meet his expectations, with the exception of a few where he corrected their word use. His colleague learners were noted to engage in critical reasoning through the hands-on learning approach. Duatepe-Paksu (2017) has recorded similar findings and claims that as learners engage in the activities of folding and unfolding, they begin to examine certain critical features of the objects more closely and derive meaning, which often creates stronger memory and enhances retention.

Another key finding is that the use of paper folding in the lesson provided the colleagues with the opportunity to express their mathematical thinking. This supports Sfard's (2008) notion that learners are able to communicate their thinking in a meaningful way, when physical objects are used and form the centre of their discussion. Wares (2016) reports a similar finding that the use of physical materials in the teaching and learning of mathematics enables learners to express their views about those objects. Researchers have also found that learners are able to verbalise their thinking about the manipulatives (concrete mediators) when those materials are used in mathematics lessons (Rondina, 2019; Uribe & Wilkins, 2017).

It was also found that lessons that made use of visual mediators, kept the colleague learners actively involved in doing and learning, which could mean that the instructional design may have met their style of learning. Research shows that the use of concrete mediators in teaching, accommodates the various learning styles of learners (Kablan, 2016). Within such an instructional design, learners learn by doing, and it fosters their ability to construct their knowledge. In addition, the literature shows that knowledge constructed by learners themselves is well understood, retained, and applied to new situations (Cope, 2015; Kontas, 2016). It was observed from the object-mediated lesson that the colleague learners appeared to be interested in those lessons, probably due to their active participation. Learners' class contribution was seen to be high, since

many of them often raised their hands to respond to the questions posed by the leading PST. Within Sfard's (2008) theory, learning takes place when one engages with a more knowledgeable person in order to receive direction and guidance, so that one can gain autonomy in learning. When learners receive appropriate support and guidance from knowledgeable others, it deepens their understanding (Naidoo, 2011; Sfard, 2008).

8.2.3.6 Narratives about geometric figures and shapes

Sfard (2007) asserts that "in school mathematical discourse, the endorsed narratives are called the mathematical theories, which include discursive constructions such as definitions, proofs, and theorems" (p. 574). Geometric discourse requires narratives to be produced as a description of the various definitions, theorems, proofs and properties governing geometric concepts and figures.

It was found that the PSTs were able to produce narratives governing the various geometric concepts and figures used in the study, even though some were not explicitly stated. Among the narratives produced were definitions, properties and theorems governing geometric shapes. It was observed that some of the PSTs developed their colleagues' geometric thinking on definitions of terms within the content area. Some of the PSTs in Group A guided their colleagues' use of basic terms and vocabulary in specific contexts in their discourse. Some of the PSTs used definitions to explain certain properties of geometric figures, as seen in the discourse of Stephen (A) in his lesson on the rhombus. This finding supports other researchers' views that definitions form the basic building blocks of any content knowledge in mathematics (Kemp & Vidakkovic, 2021; Leikin & Zazkis, 2010). It was observed that Jones defined certain geometric terms and further drew them (visual representation) on the whiteboard for his colleagues. The diagrams that he drew also helped to produce narratives about geometric concepts and figures, as noted by (Sfard, 2008). Cunningham and Roberts (2010, p. 3) maintain that when learners try to define (produce narratives) geometric concepts, it is not the definition that comes to the learners' mind but the learners' "previous experiences with diagrams and attributes" that are connected to those concepts.

Narratives can be accepted or rejected depending on the context of the subject matter (Sfard, 2008). In a similar manner, Jones did not accept all of the definitions stated by the colleague learners. For example, he accepted the definition that "*a diameter is a line segment drawn through the centre*

of the circle with its endpoints located on the circle”, but rejected the definition put forth by another colleague that *“a diameter is a line that divides a circle into two equal halves”*. Jones rejected the definition, probably to avoid any misconception that learners may associate with it, as was found in the work of Mudaly (2021) where learners claimed a line to be a diameter even though there was no indication that it passed through the centre of the circle. The ability to produce an appropriate narrative about geometric concepts is important, to avoid misconceptions (Alex & Mammen, 2018). When compared to their peers in Group B, three people in Group A were more particular about how their colleagues made endorsed narratives.

8.2.3.7 Evidence of ritualised routine in classroom instruction

Evidence from the classroom observation showed that three of the PSTs delivered instruction that shared some characteristics of ritualised routine, as classified by Sfard (2008). In their lesson, they provided strict rules that served as a guide for their colleague learners in learning. For example, it was observed in Cynthia’s (B) lesson on the ‘exterior angle theorem of a triangle’, that she dominated the teaching and learning process by giving them steps to follow, in solving a task. Sfard (2008) claims that when classroom discourse is dominated by ritualised instruction, learners start learning mathematics by following the sequence of instructions, to create a bond with others. A similar observation was made in Cynthia’s instruction when she gave the task to the class and said, *“solve for the interior angle as we did last week”*. This instruction could have been a request to colleagues to repeat the same steps they took, to complete a previous task.

Also, there were some elements of ritualised routines in the statement *“so it means whenever we are finding the exterior angle, it is the same as [adding] the two opposite interior angles”*. Sfard (2008) claims that ritualised routine is associated with strict procedures and is extremely restricting. Cynthia appeared to be limiting what they could do, forgetting that tasks may not always appear as she had presented them. This is in line with Mann & Enderson (2017) and Sfard (2008), who have reported that learners’ knowledge of procedures is the common instructional approach used by teachers. A general guideline can probably be given, but it needs to be done only after a series of examples have been considered. After this alert was given by Cynthia, she followed up with several questions that seemed a bit commanding. These were *“is that clear?”* *“Are you*

okay?” “Any questions?” It was observed that little attention was given to the colleague learners’ participation, with the exception of one task, where she asked a colleague to solve on the board.

There was evidence of Cynthia making the colleague learners learn through imitation. When one learner was asked to solve a task on the board, Cynthia said, “*do it as I taught you*”. According to Sfard (2008’ p. 267), the process of learning begins with “loosely related rituals” depending on situational clues (imitation), however, it is only an acceptable way of learning at the initial stages of discourse. This finding resonates with Sfard’s (2008) observation that the most common approach to mathematics teaching is developing learners’ knowledge of rules and procedures, often determined by the teachers.

Another finding about the ritualised routine of these three participants’ lesson delivery, was the use of traditional methods of instruction. In Maxwell’s (A) lesson on ‘properties of parallel lines’, he only drew the parallel lines with a transversal and labelled the various angles (see Chapter 7 sections 7.2.3 and 7.3.4.1 for detailed discussion). After that, he mentioned which of the angles were equal and provided their associated names. This finding supports the notion of Ardeleanu (2019) that teachers often regard learners as having knowledge gaps that need to be filled, by giving them lots of information to learn. Learners are therefore not engaged in the teaching and learning process and are treated as passive learners (Fletcher, 2009; Swan, 2005). Thus, the teacher plays a dominant role as an instructor, as was evident in Maxwell’s (A) discourse. This kind of instruction is termed by Sfard (2008) as the acquisitionist approach to learning, in which the learner only receives and processes verbal information from teachers. Albert’s (B) lesson was also a demonstration of teacher dominance in the lesson delivery (see Chapter seven, section 7.2.5).

8.2.3.8 Evidence of explorative routine in classroom instruction

The instructional delivery of five of the PSTs (3 in Group A and 2 in Group B) exhibited the potential to develop learners’ learning ability and retention, by guiding them to construct and re-construct new ideas as a way of producing endorsed narratives in a discourse. Sfard (2008) maintains that producing endorsed narratives remains the ultimate goal of school mathematising, and this can take place on the foundations of discursive learning. According to Sfard (2015, p. 131), discursive learning refers to “the activity of becoming able to tell and produce ever new

stories about the world”, or object of learning, which aligns well with the participationist view of learning.

There was evidence of some of the PSTs engaging the colleague learners in knowledge construction, during the discourse. This was mostly found in discourse where all kinds of visual mediators were used. **Utilising those materials, was** found to enable the learners to produce objectified narratives about the object of talk. According to Sfard (2008), visual-mediated instruction contributes to learners’ production of narratives. In Stephen’s lesson, in which paper folding was used to investigate the properties of a rhombus, it was found that the colleagues’ responses matched the intended lesson objectives, which is evidence of effective lesson (achieved lesson objectives). A characteristic feature of an explorative routine is the ability to produce identified properties of mathematical objects (Sfard, 2008). Other researchers like Rondina (2019) and Uribe and Wilkins (2017) report similar findings, where instructional design that put learners at the centre of the teaching and learning, often helped them to produce verbal representations of what had been observed.

Three of the PSTs in Group A and two in Group B, designed instruction that shared more of the characteristics of learning by doing, as characterised in the constructivist perspective on teaching and learning (Bruner, 1966). This shows that the instructional design of Group A members, was more aligned with the characteristics of explorative discourse (Fletcher, 2009; Sfard, 2008; Swan, 2005).

8.3 Interconnective thinking within the commognitive constructs

Based on the findings from this study, the commognitive theory has indeed served as a potential lens for the analysis of the PSTs’ geometric thinking and classroom discourse. Presmeg (2016) claims that the commognitive theory has emerged as a theoretical lens to provide an insight into teaching and learning of mathematics, and the entire fabric of human development. It was found that the PSTs, particularly those in Group A, who performed well in the test (worksheet), demonstrated an interconnected thinking within the commognitive constructs. Three of the PSTs used words in a more literate way to define and describe the geometric concepts that were investigated. Their word use was based on the visual mediators employed to design the tasks.

These PSTs could not have carefully used literate words to describe their thinking, if they had not seen and interpreted the diagrams, and associated visual mediators. The reliance on their visual abilities was seen in the way they chose the underlying properties (word use) of the geometric tasks, which influenced their routine strategies of ‘how’ to solve the tasks, and the ‘why’ and the ‘when’ involved, in substantiating and explaining their thinking processes. Thus, their discourses were mostly mediated by their visual interpretational abilities (Sfard, 2007, 2008).

These competencies were also seen in their geometric discourses in the classroom. The PSTs in Group A were mostly found to guide their colleague learners’ word use with regard to description, definition of terms and identification of geometric properties governing the tasks (Kim et al., 2017; Lefrida et al., 2021). Explanation of their solution routines to the tasks was characterised by drawing colleague learners’ attention to what informed a particular strategy, as well as by substantiating with an appropriate narrative governing the object. Many of the Group A participants also engaged their colleague learners in active learning through the use of visual mediators, and in most cases, their instructional discourses were characterised by developing learners’ explorative way of thinking. These competencies demonstrated by the Group A participants were less evident among those in Group B.

8.4 Researcher’s thoughts

Most learners see geometry content to be difficult, in the mathematics curriculum. This seems to affect learners’ interest in taking mathematics-related courses or programmes of study at the secondary school or at the university level. The effect has negative implications for economic growth and technological advancement in this era of the 21st century, where technology permeates our daily activities. Literature shows that learning competence and proficient geometric thinking have a positive effect on learners’ mathematics performance. However, literature has documented that lots of research indicate learners’ poor performance in mathematics and geometry in particular. The impetus for this study emerged from concerns about learners continued poor performance in geometry.

One significant tool for learning geometry is the properties of geometric shapes. It was observed that some of the participants demonstrated inadequate thinking about the geometric properties

needed, to devise solutions to the tasks. This implies that mathematics teacher educators should make a conscious effort to develop and deepen the PSTs' thinking on the properties of geometric objects. Proficient thinking of geometric properties, serves as the foundational competence in learning geometry. In most cases, the failure of some of the PSTs to devise the appropriate solutions to the tasks, was a result of weak knowledge of geometric properties. On the contrary, some of the participants who performed well, demonstrated good thinking about the properties that govern the tasks. This was evident in their discourses, where they made mention of the informed property needed to devise the solution. This means that mathematics teacher educators must focus on teaching for understanding and application of these properties in problem-solving, or learners will continue to view geometry as a difficult subject to learn.

Also, attention needs to be drawn to visualisation and spatial reasoning as critical cognitive tools in geometry, which seem to be neglected in its teaching and learning. It is worth mentioning that tasks in mathematics are presented in three ways, namely; numeric, worded and diagrams. This means that in the same way as we need number sense to learn numeric tasks and algebraic thinking to learn worded tasks, we also need to demonstrate a high visual sense to interpret and solve tasks presented in diagrams. Thus, mathematics teacher educators need to design lessons that help learners to exercise their visual thinking abilities. Such lessons should take into consideration visual rotation, visual orientation, and visualisation. For example, it was observed that one of the participants, although having some ideas about the task, could not engage in the needed visual and spatial reasoning of manipulating the task in the mind, which resulted in a wrong solution.

The PSTs use of the kinds of visual mediators in their lessons served as a medium that helped them to communicate their geometric ideas. In such instructions, it was observed that the colleague learners' participation was high, and they showed interest in what they were doing. The visual tools made the geometric properties and concepts more accessible to the colleague learners, by reducing their abstract nature and led them to discover the properties of the rhombus. The teacher only corrected the use of words in their discourses. In those lessons, teachers were found to facilitate colleague learners' construction of knowledge rather than being knowledge distributors. In practice therefore, mathematics teacher educators should endeavour to use more visual tools to mediate communication between themselves and the learners, and among the learners, to enhance

knowledge construction as advocated by the constructivists. These kinds of instructions could be seen as teaching for retention against teaching for memorisation, which is mostly through the processing of verbal information and is teacher-centred.

8.5 Recommendations

The following recommendations are made based on the study findings:

It is recommended that as part of PSTs' development of geometric thinking, they have to acquire the basic terminologies, language, and keywords needed to enable them to communicate their knowledge of geometric content in the mathematics curriculum.

Also, the PSTs should develop their geometric thinking by identifying and making meaning of the various visual mediators that are used in communicating certain important geometric concepts in the discourse. Attention should be focused on the use of diagrams in teaching and learning of geometry, the use of icons to design diagrams, and the related properties that emerge from such use. Other visual tools used in the discourse, such as symbols and concrete mediators, could also be enforced.

In addition, the PSTs must develop their geometric thinking on the narratives associated with the various geometric objects in the mathematics curriculum, some of which are the definitions, properties, theories, theorems, etc. These narratives, when they have been well connected with their a priori knowledge, can form a strong basis for them to produce new and endorsed narratives about geometric objects.

It is further recommended that the PSTs focus attention on multiple strategies needed to solve tasks in geometry. The PSTs should be mindful of the repetitive patterns that govern how to devise appropriate solutions to geometric tasks, in the various content areas. In doing so, critical consideration should be given to the narratives such as the properties, theories definitions, theorems, and other concepts that serve as the foundational knowledge for learning geometry.

Concerning the second research question, the PSTs should focus attention on developing and enhancing their explorative way of thinking in geometry. Even though discourses constitute both

ritual and explorative approaches, school mathematics aims at developing learners' explorative ways of thinking (Sfard, 2008). Ritualised discourse, even though it is accepted in learning, is supposed to serve as a foundation to develop explorative discourse, which provides several benefits to learning, as was found with the participants who devised multiple solutions to tasks, demonstrating explorative thinking. Thus, the PSTs must work hard to develop their problem-solving abilities by moving from adhering to strict rules of learning, to exploring several ways of doing mathematics.

Based on the findings from the third research question, it is recommended that the PSTs develop deep and flexible geometric thinking in order to enhance their competence in teaching geometry content in the senior high school mathematics curriculum. Teaching is the act of blending content with pedagogical principles in order to create enhanced learning opportunities for learners. This is based on the findings that many of the PSTs in Group A, who showed an explorative way of thinking, designed an instruction that has the potential to develop the learners' conceptual grasp of geometry.

8.6 Limitations

First, the sample size used in the study consisted of only eight participants, all of whom were selected from one university and from a year group. Regardless of the detailed data obtained for the purpose of the study, which makes the findings insightful, the study should be extended to other PSTs of the same programme in institutions of higher learning in Ghana for a broader understanding of the variables of study.

Secondly, even though the study focused on a wider content area on plane geometry in the Ghanaian mathematics curriculum, there may be other areas that could have been investigated.

Thirdly, since the study was interested in all aspects of classroom learning behaviour, two cameras could have been used, one for the teaching activities and the other for the coverage of learners' interactions. Notwithstanding, the video recording was supplemented by taking field notes.

8.7 Suggestions for further research

Research over the years has focused on investigating PSTs' knowledge for teaching various content areas in the mathematics curriculum, using several theories as the study lens. The commognitive framework served as a useful approach to investigating PSTs' geometric thinking. Based on the findings of the study, the following suggestions are made for further research under the commognitive framework, to:

1. analyse pre-service mathematics teachers' sources of errors made in geometric discourse,
2. explore pre-service mathematics teachers' responses to learners' errors in geometry,
3. explore pre-service mathematics teachers' difficulties in transiting from ritual to explorative discourse in geometry,
4. analyse mathematics teacher educators', and pre-service teachers' participation in geometric discourse.

8.8 Conclusion

This chapter presented findings of the study in relation to the critical research questions. The study analysed pre-service teachers' geometric thinking and classroom discourse. The findings revealed that many participants in Group A demonstrated good discursive geometry thinking compared to those in Group B. In addition, many in Group A demonstrated an explorative way of thinking with regard to geometric discourse and the strategies used in solving the tasks, compared to their counterparts in Group B, who mostly displayed a ritualised way of thinking. It was further found during the classroom observation, that many in Group A designed an instruction that showed the potential to develop the learners' explorative ways of thinking and to facilitate conceptual grasp of geometry. Some Group B participants produced an instructional design that was ritualised in nature. The chapter concluded with recommendations, limitations, and suggestions for further research.

REFERENCES

- Abreh, M. K., Owusu, K. A., & Amedahe, F. K. (2018). Trends in performance of WASSCE candidates in the science and mathematics in Ghana: Perceived contributing factors and the way forward. *Journal of Education*, 198(1), 113-123. doi:10.1177/0022057418800950
- Adegun, I. K., & Adegun, B. O. (2013). Students and teachers' view of difficult areas in mathematics syllabus: Basic requirement for science and engineering. *Journal of Education and Practice*, 4(2), 235-243.
- Adolphus, T. (2011). Problems of teaching and learning of geometry in secondary schools. *International Journal for Emerging Sciences*, 1(2), 143–152. 2011
- Agyei, D. D. (2012). *Preparation of pre-service teachers in Ghana in integrate information and communication technology in teaching of mathematics*. Unpublished Doctoral Dissertation, University of Twente.
- Ajayi, K. O., & Lawani, A. O. (2015). Speaking mathematically: The role of language and communication in teaching and learning of mathematics. *The International Journal of Technologies in Learning*, 21(1), 1-7.
- Akhter, N., & Akhter, N. (2018). Learning in mathematics: Difficulties and perceptions of students. *Journal of Educational Research, Dept. of Education*, 21(1), 147-163.
- Aksu, Z. (2019). Pre-service mathematics teachers' pedagogical content knowledge regarding students' mistakes on the subject of circle. *International Journal of Evaluation and Research in Educaion*, 8(3), 440-445.
- Aksu, Z., & Kul, U. (2018). Conceptual knowledge of middle school students about circle: The case of Artvin province (in Turkey). *International Artvin Symposium, Artvin, October 2018*.
- Alex, J. K., & Mammen, K.J. (2014). An assessment of the readiness of grade 10 learners for geometry in the context of curriculum and assessment policy statement (caps) expectation. *International Journal Education Science*, 7(1), 29-39.
- Alex, J. K., & Mammen, K. J. (2018). Students' understanding of geometry terminology through the lens of Van Hiele theory. *Pythagoras*, 39(1), 1-8.
- Al-Mutawah, M. A., Thomas, R., Eid, A., Mahmoud, E. Y., & Fateel, M. J. (2019). Conceptual understanding, procedural knowledge and problem-solving skills in mathematics: High school graduates work analysis and standpoints. *International journal of education and practice*, 7(3), 258-273.
- Anamuah-Mensah, J., Mereku, D. K., & Asabere-Ameyaw, A. (2004). *Ghanaian Junior Secondary School Students' Achievement in Mathematics and Science: Results from*

Ghana's participation in the 2003 Trends in International Mathematics and Science study.
Accra: Ministry of Education Youth and Sports

- Anney, V. N. (2014). Ensuring the quality of the findings of qualitative research: Looking at trustworthiness criteria. *Journal of Emerging Trends in Educational Research and Policy Studies (JETERAPS)*, 5(2), 272-281.
- Anthony, G., & Walshaw, M. (2009). Characteristics of effective teaching of mathematics: A view from the West. *Journal of Mathematics Education*, 2(2), 147-164.
- Anwar, A., & Juandi, D. (2020). Studies of level of visual thinking in geometry. *Journal of Physics: Conference Series*, 1470, 1-5. (012095). Doi:1088/1742-6596/1470/1/012095
- Anwar, A., Turmudi, T., Juandi, D., Wahyuni, R., & Muntazhimah, M. (2022). Visual thinking skills in solving geometry problems based on learning style: A grounded theory study. *European Online Journal of Natural and Social Sciences*, 11(3), 635-642.
- Arakaza, A., & Mugabo, L. R. (2021). Teachers' practices in teaching mathematics in selected schools of Bujumbura Mairie Province in Burundi. *Rwandan Journal of Education*, 5(2), 140-152.
- Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52(3), 215-241.
- Ardeleanu, R. (2019). Traditional and modern teaching methods in mathematics. *Journal of Innovation in Psychology, Education and Didactics*, 23(2), 133-140.
- Armah, R. B., Cofie, P. O., & Okpoti, C. A. (2017). The geometric thinking levels of pre-service teachers in Ghana. *Higher Education Research*, 2(3), 98-106.
doi: 10.11648/j.her.20170203.14
- Arslan, C., Erbay, H. N., & Guner, P. (2016). Prospective mathematics teachers' ability to identify mistakes related to angle concept of sixth grade students. *European Journal of Education Studies*, 2(12), 190-204.
- Ary, D., Jacobs, L. C., Sorensen, C., & Razavieh, A. (2010). *Introduction to Research in Education* (8th ed). Belmont, CA: Cengage Learning.
- Asemani, E., Aseidu-Addo, S., & Oppong, R. A. (2017). The geometric thinking levels of senior high school students in Ghana. *International Journal of Mathematics and Statistics Studies*, 5(3), 1-8.
- Aslan-Tutak, F., & Adams, T. L. (2015). A study of geometry content knowledge of elementary preservice teachers. *International Electronic Journal of Elementary Education*, 7(3), 301-318.

- Astin, F., & Long, A. (2014). Characteristics of qualitative research and its application. *British Journal of Cardiac Nursing*, 9(2), 93-98.
- Atanasova-Pachemska, T., Gunova, V., Lazarova, L. K., & Pachemska, S. (2016). Visualization of geometry problems in primary math education- needs and challenges. *Istraživanje Matematičkog Obrazovanja*, 15(8), 33-37.
- Atebe, H. U. (2008). *Student's van Hiele levels of geometric thought and conception in plane geometry: A collective case study of Nigeria and South Africa* (Doctoral dissertation). Rhodes University.
- Atebe, H. U., & Schafer, M. (2010). Beyond teaching language: Towards terminological primacy in learners' geometric conceptualisation. *Pythagoras*, 71, 53-64.
- Babbie, E., Mouton, J., Vorster, P. & Prozesky, B. (2010). *The practice of social research*. Cape Town: Oxford University Press.
- Baffoe, E., & Mereku, D. K. (2010). The van Hiele levels of understanding of students entering senior high school in Ghana. *African Journal of Educational Studies in Mathematics and Sciences*, 8, 51-61.
- Baktemur, G., Ayan Civak, R., & Isiksal Bostan, M. (2021). Pre-service middle school teachers' (mis)conception of definitions, classifications, and inclusion relations of quadrilaterals. *European Journal of Science and Mathematics Education*, 9(4), 183-198. <https://doi.org/10.30935/scimat/11206>
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59, 397-407.
- Bassarear, T. (2012). *Mathematics for elementary school teachers*. (5th edn.). London: Brooks/Cole.
- Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., . . . Tsai, Y. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, 47(1), 133-180. doi:10.3102/0002831209345157
- Ben-Yehuda, M., Lavy, L., Linchevski, L., & Sfard, A. (2005). Doing wrong with words: What bars students' access to arithmetic discourses. *Journal for Research in Mathematics Education*, 36(3), 176-247.
- Ben-Zvi, D., & Sfard, A. (2007). Ariadne's thread, daedalus' wings and the learners' autonomy. *Education Et Didactique*, 1(3), 117-134.
- Berger, M. (2013). Examining mathematical discourse to understand in-service teachers' mathematical activities. *Pythagoras*, 34(1), <https://doi.org/10.4102/pythagoras.v34i1.197>

- Best, J. W., & Kahn, J. V. (2006). *Research in Education* (10th ed.). New York, NY: Pearson Education Inc.
- Biber, C., Tuna A., & Korkmaz, S. (2013). The mistakes and the misconceptions of the eighth-grade students on the subject of angles. *European Journal of Science and Mathematics Education*, 1(2), 50- 61.
- Bingolbali, E. (2011). Multiple solutions to problems in mathematics teaching: Do teachers really value them? *Australian Journal of Teacher Education*, 36(1), 18-31.
- Bishara, S. (2015). Active and traditional teaching mathematics in special education. *Creative Education*, 6, 2313-2324. <https://doi.org/10.4236/ce.2015.622238>
- Bonyah, E., & Larbi, E. (2021). Assessing van Hiele's geometric thinking levels among elementary pre-service mathematics teachers. *African Educational Research Journal*, 9 (4), 844 – 851.
- Boo, J. Y., & Leong, K. E. (2016). *Teaching and learning of geometry in primary school using GeoGebra*. Retrieved from researchgate.net/publication/311912319
- Bora, A., & Ahmed S. (2018). Secondary school students' attitudes towards their learning geometry: A survey of Diphu town secondary school. *International Journal of Research and Analytical Review*, 5(3), 265-267.
- Brizuela, B. M., & Gravel, B. (2013). *Show me what you know: Exploring representations across STEM disciplines*. New York: Teachers College Press.
- Bruce, D. (2016). Mathematics anxiety among Ghanaian students: A case study of students of Kinbu senior high/technical school, Accra and Hermann-Gmeiner SOS junior high school, Tema. *Journal of Education and Practice*, 75-83.
- Bruner, J. S. (1966). *Towards a theory of instruction*. Cambridge, Massachusetts: Belknap Press.
- Brunheira, L. & da Ponte, J. P. (2016). Prospective teachers work on defining quadrilaterals through an exploratory approach. *Didactica Mathematicae*, 38, 33-56.
- Bryan, T. J. (2002). *The conceptual knowledge of preservice secondary mathematics teachers: How well do they know the subject matter they will teach?* (Unpublished Doctoral Dissertation). The University of Texas at Austin, Austin Tx.
- Cantimer, G. G., & Sengul, S. (2017). Misconceptions and mistakes of middle school 7th and 8th grade students about circle (in Turkey). *Gazi Eğitim Bilimleri Dergisi*, 3(1), 17-27.
- Carlin, R. W. (2009). *A comparative study of geometry curricula*. LSU Master's Thesis. 1348. https://digitalcommons.lsu.edu/gradschool_theses/1348 Louisiana State University.
- Chan, L. L. & Idris, N. (2017). Cooperative learning in mathematics education. *International Journal of Academic Research in Business and Social Sciences*, 7(3), 539-553.

- Cheng, Y., & Mix, K. S. (2014). Spatial training improves children's mathematics ability. *Journal of Cognition and Development*, 15(1), 2-11.
- Chesler, J. (2012). Pre-service secondary mathematics teachers making sense of definitions of functions. *Mathematics Teacher Education and Development*, 14(1), 27-40.
- Chigonga, B., Kahle, G., & Chuene, K. (2017). *Exploring grade 9 learners' knowledge of properties of quadrilaterals*. Retrieved from <https://uir.unisa.ac.za>
- Clements, D. H. & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed), *Handbook on Mathematics Teaching and Learning*. (pp. 420-464). New York: Macmillan.
- Cockett, A., & Kilgour, P. W. (2015). Mathematical manipulatives: Creating an environment for understanding, efficiency, engagement, and enjoyment. *Teach Collection of Christian Education*, 1(1), 47-54.
- Cohen, L., Manion, L., & Morrison, K. (2011). *Research Methods in Education*. London: Routledge.
- Connelly, L. M. (2016). Trustworthiness in qualitative research. *Medsurg Nursing*, 25(6), 435.
- Cooney, T. J. (1995). Conceptualizing teachers' ways of knowing. *Educational Studies in Mathematics*, 38,163-187.
- Cope, D. G. (2014). Methods and meanings: credibility and trustworthiness of qualitative research. *Oncology Nursing Forum*, 41(1), 89 – 91. Doi: 10.1188/14.ONF.89-91.
- Cope, L. (2015). Math manipulatives: Making the abstract tangible. *Delta Journal of Education*, 5(1), 10-19.
- Contay, E. G., & Paksu, A. D. (2012). Preservice mathematics teachers' understandings of the class inclusion between kite and square. *Procedia-Social and Behavioral Sciences*, 55, 782-788.
- Couto, A. (2014). Pre-service teachers' knowledge on elementary geometry concepts. *Journal of the European Teacher Education Network*, 9, 57-73.
- Creswell, J. W. (2007). *Qualitative Inquiry & Research Design: Choosing among Five Approaches* (2nd ed.). Thousand Oaks, CA: Sage.
- Creswell, J. W. (2014). *Research design: Qualitative, quantitative and mixed methods approaches*. (4th ed). London: Sage Publications, Inc.
- Crompton, H. (2014, March). using mobile learning to support students' understanding in geometry: A design-based research curriculum study. In *Society for Information*

- Technology & Teacher Education International Conference* (pp. 3024-3032). Association for the Advancement of Computing in Education (AACE).
- Cunningham, R. F., & Roberts, A. (2010). Reducing the mismatch of geometry concepts definitions and concept images held by pre-service teachers. *The Journal*, 1, 1-17.
- Danisman, S., & Tanisli, D. (2017). Examination of mathematics teachers' pedagogical content knowledge of probability. *Malaysian Online Journal of Educational Sciences*, 5(2), 16-34.
- De Lina, R., & Tall, D. (2007). Procedural and embodiment and magic in linear equations. *Educational Studies in Mathematics*, 6(1), 3-18.
- DeJonckheere, M., & Vaughn, L. M. (2019). Semi structured interviewing in primary care research: A balance of relationship and rigour. *Fam Med Com Health*, 1-8. Doi: 10.1136/fmch-2018-000057
- Denzin, N. K. (2010). Moments, mixed methods, and paradigm dialogs. *Qualitative Inquiry*, 16(6), 419-427.
- Dewi, I. L. K., & Asnawati, S. (2019). Relationship among pre-service teacher conception of geometry and teaching skills. *Journal of Physics: Conference Series*, 1280, 1 – 5. doi: 10.1088/1742-6596/1280/4/042042
- Dikko, M. (2016). Establishing construct validity and reliability: Pilot testing of a qualitative interview for research in Takaful (Islamic insurance). *The Qualitative Report*, 21(3), 521-529.
- Dobbins, A., Gagnon, J. C., & Ulrich, T. (2014). Teaching geometry to students with math difficulties using graduated and peer-mediated instruction in a response-to-intervention model. *Preventing School Failure: Alternative Education for Children and Youth*, 58(1), 17-25.
- Duatepe Aksu, A. (2013). Predicting the geometry knowledge of pre-service elementary teachers. *Cumhuriyet International Journal of Education*, 2(3), 15-27.
- Duatepe-Paksu, A. (2017). Constructing a rhombus through paper folding. *International Journal of Mathematics Education in Science and Technology*, 48 (5), 763-767. doi: 10.1080/0020739X.2017.1282048
- Duatepe-Paksu, A., Pakmak, G. S., & Iymen, E. (2012). Preservice elementary teachers' identification of necessary and sufficient conditions for a rhombus. *Procedia - Social and Behavioral Sciences*, 46, 3249 – 3253.
- Dundar, R. K., & Otten, S. (2022). High school students' use of diagrams in geometric proofs. *International Journal of Science and Mathematics Education*. <https://doi.org/10.1007/s10763-022-10286-2>

- Erdogan, A. (2017). Examining pre-service mathematics teachers' conceptual structures about geometry. *Journal of Education and Practice*, 8(27), 65-74.
- Erdogan, O., & Dur, Z. (2014). Pre-service mathematics teachers' personal figural concepts and classification about quadrilaterals. *Australian Journal of Teacher Education*, 39(6), 107-133. <http://dx.doi.org/1014221/ajte2014v39n6.1>
- Essack, R. M. (2015). *Exploring grade 11 learner routines on functions from a commognitive perspective* (Unpublished doctoral thesis). University of Witwatersrand, Johannesburg, South Africa.
- Fabiyi, T. R. (2017). Geometry concepts in mathematics perceived difficult to learn by senior secondary school students in Ekiti state, Nigeria. *Journal of Research & Method in Education*, 7(1), 83-90.
- Ferrer, G. A. (2020). Error analysis in the operations of algebraic expressions of grade 8 students at Kalinga State University laboratory high school. *International Journal of English Literature and Social Sciences*, 5(6), 2494-2504.
- Fletcher, J. A. (2000). Constructivism and mathematics education in Ghana. *Journal of the Mathematical Association of Ghana*, 12, 27-33.
- Fletcher, J. A. (2009). Learning algebraic concepts through group discussion. *Journal of Science and Mathematics Education*, 4, 31-47.
- Fletcher, J. A. (2018). Performance in mathematics and science in basic schools in Ghana. *Academic Discourse: An International Journal*, 10(1), 1-18.
- Fletcher, J. A., & Anderson, S. (2012). Improving students' performance in measurement at the senior high school level using the Geometer's Sketchpad. *Journal of Science and Mathematics Education*, 6(1), 63-79.
- Fujita, T. (2012). Learners' level of understanding of the inclusion relations of quadrilaterals and prototype phenomenon. *The Journal of Mathematical Behavior*, 31, 60-72. <http://dx.doi.org/10.1016/j.jmathb.2011.08.003>
- Fujita, T., & Jones, K. (2006). Primary trainee teachers' understanding of basic geometrical figures in Scotland. *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education. Prague: PME. Vol. 3*, 129-136.
- Gani, A., Imtiaz, N., Rathakrishnan, M., & Krishnasamy, H. N. (2020). A pilot test for establishing validity and reliability of qualitative interview in the blended learning English proficiency course. *Journal of Critical Reviews*, 7(5), 140-143.
- Gcasamba, L. C. (2014). *A discursive analysis of learners' mathematical thinking: the case of functions* (Unpublished master's thesis). University of Witwatersrand. South Africa.
- Gharbavi, A., & Iravani, H. (2014). Is teacher talk pernicious to students? A discourse analysis of teacher talk. *Procedia-Social and Behavioral Sciences*, 98, 552-561.

- Greer, A. (1984). *New comprehensive mathematics for 'O' level* (2nd ed.). England: Stanley Thornes (Publishers) Ltd.
- Guner, P., & Gulten, D.Ç. (2016). Pre-service primary mathematic teachers' skills of using the language of mathematics in the context of quadrilaterals. *International Journal on New Trends in Education & Their Implications*, 7(1), 13–27.
- Hailikari, T., Katajavuori, N., & Lindblom-Ylänne, S. (2008). The relevance of prior knowledge in learning and instructional design. *American Journal of Pharmaceutical Education*, 72(5), 1-9.
- Hasanah, S. I., Tafrilyanto, C. F., & Aini, Y. (2019). Mathematical reasoning: The characteristics of students' mathematical abilities in problem-solving. *Journal of Physics: Conference Series*, 1188, 012057. IOP Publishing.
- Herbst, P., Gonzalez, G., & Macke, M. (2005). How can geometry students understand what it means to define in mathematics? *The Mathematics Educator*, 15(2), 17-24.
- Heyd-Metzuyanim, E., & Graven, M. (2016). Between people-pleasing and mathematizing: South African learners' struggle for numeracy. *Educational Studies in Mathematics*, 91(3), 349-373.
- Hidayah, I., Dwijanto, D., & Istiandaru, A. (2018). Manipulatives and question series for elementary school mathematics teaching on solid geometry. *International Journal of Instruction*, 11(3), 649-662. <https://doi.org/10.12973/iji.2018.11344a>
- Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42(4), 371-401.
- Horan, & Carr, (2018). How much guidance do students need? An intervention study on kindergarten mathematics with manipulatives. *International Journal of Educational Psychology*, 7(3), 286-316.
- Hourigan, M., & Leavy, A. M. (2017). Preservice primary teachers' geometric thinking: Is pre-tertiary mathematics education building sufficient strong foundations? *The Teacher Educator*, 52(4), 346-364. <https://doi.org/10.1080/08878730.2017.1349226>
- Hufferd-Ackles, K., Fuson, K. C., & Gamoran Sherrin, M. (2004). Describing levels and components of math-talk learning community. *Journal for research in mathematics education*, 35(2), 1-10.
- Hurrell, D. P. (2021). Conceptual knowledge or procedural knowledge or conceptual knowledge and procedural knowledge: Why the conjunction is important for teachers. *Australian Journal of Teacher Education*, 46, (2), 57 – 71. <http://dx.doi.org/10.14221/ajte.2021v46n2.4>

- Jacob, S. A., & Furgerson, S. P. (2012). Writing interview protocols and conducting interviews: Tips for students new to the field of qualitative research. *The Qualitative Reports*, 17(42), 1-10.
- Jamhari, J., & Wongkia, W. (2018). Experiencing the angle properties of a circle. *AMT*, 74(3), 24-33.
- Jentsch, A., & Schlesinger, L. (2018). Measuring instructional quality in mathematics education. *CERME*. 10, 1-8.
- Pg. Johari, P. M. A. R., & Shahrill, M. (2020). The common error in the learning of the simultaneous equations. *Infinity*, 9(2), 263-274.
- Jones, K. (2000). Critical issues in the design of the geometry curriculum. In B. Barton (Ed.), *Readings in Mathematics Education* (pp. 75-90). Auckland, New Zealand: University of Auckland.
- Jones, K. (2013). Diagrams in the teaching and learning of geometry: Some results and ideas for future research. *Proceedings of the British Society for Research into Learning Mathematics*, 33(2), 37-42.
- Jones, J. P., & Tiller, M. (2017). Using concrete manipulatives in mathematical instruction. *Dimensions of Early Childhood*, 45(1), 18-23.
- Jones, K., & Tzekaki, M. (2016). Research on the teaching and learning of geometry. In A. Gutierrez, G. Leder, & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education: The Journey Continues* (pp. 109-149). Rotterdam: Sense Publishers. http://dx.doi.org/10.1007/978-94-6300-561-6_4
- Juandi, D. (2020). Studies of level visual thinking in geometry. *Journal of Physics: Conference Series*, 1470(1), 012095. IOP Publishing.
- Kablan, Z. (2016). The effect of manipulatives on mathematics achievement across different learning styles. *Educational Psychology*, 36(2), 277-296, DOI: 10.1080/01443410.2014.946889
- Keiler, L.S. (2018). Teachers' roles and identities in student-centered classrooms. *IJ STEM Ed*, 5, 34. <https://edtechbooks.org/-TrXd>
- Kemp, A., & Vidakovic, D. (2021). Ways secondary mathematical teachers apply definitions in Taxicab geometry for a real-life situation: Midset. *Journal of Mathematics Behaviour*, 62, 1-18.
- Khan, A., Khan, S., Zia-ul-islam, S., & Khan, M. (2017). Communication skills of a teacher and its role in the development of the students. *Journal of Education and Practice*, 8(1), 18-21.

- Kim, D. J., Choi, S., & Lim, W. (2017). Sfard's commognitive framework as a method of discourse analysis in mathematics. *International Journal of Cognitive and Language Sciences*, 11(11), 481- 485.
- King, N., Horrocks, C., & Brooks, J. (2018). *Interviews in qualitative research*. London: SAGE Publications.
- Kiwanuka, H. N., Noortgate, W. V., Anumendem, D. N., & Namusisi, S. (2015). Factors affecting mathematics achievement of first-year secondary school students in Central Uganda. *South African Journal of Education*, 35(3), 1-16.
- Kivkovich, N. (2015). A tool for solving geometric problems using mediated mathematical discourse (for teachers and pupils). *Procedia-Social and Behavioural Sciences*, 209, 519-525.
- Klerlein, J., & Hervey, S. (2020). *Mathematics as a complex problem-solving activity: Promoting students' thinking through problem-solving*. From <https://www.generationready.com/wp-content/uploads/2020/12/Mathematics-as-a-Complex-Problem-Solving-Activity.pdf>
- Kontas, H. (2016). The effect of manipulatives on mathematics achievement and attitudes of secondary school students. *Journal of Education and Learning*, 5(3), 10-20.
- Kontorovich, I., & Zazkis, R. (2016). Turn vs. shape: teachers cope with incompatible perspectives on angle. *Educational Studies in Mathematics*, 93(2), 223-243.
- Korstjens, I., & Moser, A. (2018). Series: Practical guidance to qualitative research. part 4- trustworthiness and publishing. *The European Journal of General Practice*, 24(1), 120–124. <https://doi.org/10.1080/13814788.2017.1375092>
- Kotsopoulos, Lee, J., Heide, D., & Schell, A. (2009). *Discursive routines and endorsed narratives as instances of mathematical cognition* [Paper presentation]. Proceedings of the 31st Annual meeting of North American Chapter of the International Group for the Psychology of Mathematics Education, Atlanta, GA.
- Krauss, S., Burnner, M., Kunter, M., Baumert, J., & Neubrand, M. (2008). Pedagogical content knowledge and content knowledge of secondary mathematics teachers. *Journal of Educational Psychology*, 100(3), 716-725. doi:10.1037/0022-0663.100.3.716
- Kuranchie, A. (2021). *Research made easy*. Kumasi: Bookworm Publications.
- Kuzle, A. (2022). The teaching of geometry in primary education: Is geometry still neglected in school mathematics? *Twelfth Congress of the European Society for Research in Mathematics Education*, Feb 2022, Bozen-Bolzano, Italy. ffh1-03751494 (1-9).

- Kuzle, A., & Gracin, D. G. (2020). Making sense of geometry education through the lens of fundamental ideas: an analysis of children's drawings. *The Mathematics Educator*, 29(1), 7-52.
- Lapan, S., Quartaroli, M. T., & Riemer, R. J. (Eds.). (2012). *Qualitative research: An introduction to methods and designs*. San Francisco: John Wiley & Sons, Inc.
- Leavy, P. (2017). *Research design: Quantitative, qualitative, mixed methods, arts-based, and community-based participatory research approaches*. New York, NY: The Guilford Press.
- Leedy, P. D., & Ormrod, J. E. (2015). *Practical research: Planning and design* (11th ed.). England: Pearson Education Limited.
- Lefrida, R., Siswono, T. Y. E., & Lukito, A. (2021). A commognitive study on field-dependent students' understanding of derivative. *Journal of Physics: Conference Series*, 1747(1), 012025. IOP Publishing.
- Leikin, R., & Zazkis, R. (2010). On the content-dependence of prospective teachers' knowledge: A case of exemplifying definitions. *International Journal of Mathematical Education in Science and Technology*, 41(4), 451–466.
- Lesseig, K., Hine, G., & Na, G. S. (2019). Perceptions on proof and the teaching of proof: A comparison across preservice secondary teachers in Australia, USA and Korea. *Mathematics Education Research Journal, Early View Online First*. doi: 10.1007/s13394- 019-00260-7
- Ling, G. C. L., Shahrill, M., & Tan, A. (2016). Common misconceptions of algebraic problems: Identifying trends and proposing possible remedial measures. *Advanced Science Letters*, 22(5-6), 1547-1550.
- Llinares, S. (2021). Instructional quality of mathematics teaching and mathematics education. *Journal of Mathematics Teacher Education*, 24, 1-3. <https://doi.org/10.1007/s10857-021-09488-2>
- Low, J. (2013). Unstructured and semi-structured interviews in health research. *Researching health: Qualitative, quantitative and mixed methods*, 2, 87-105.
- Lowrie, T. (2020). The utility of diagrams in elementary problem-solving. *Cognitive Development*, 55. <https://doi.org/10.1016/j.cogdev.2020.100921>
- Lowrie, T., Logan, T., Harris, D., & Hegarty, M. (2018). The impact of an intervention program on students' spatial reasoning: Student engagement through mathematics-enhanced learning activities. *Cognitive Research: Principles and Implications*, 3 (50), 1-10. <https://doi.org/10.1186/s41235-018-1047-y>
- Luneta, K. (2014). Foundation phase teachers' (limited) knowledge of geometry. *South African Journal of Childhood Education*, 4(3), 71-86.

- Luneta, K. (2015). Understanding students' misconception: An analysis of final grade 12 examination questions in geometry. *Pythagoras*, 36(1), 1-11. Doi: 10.4102/pythagoras.v36i1.261
- Maier, A. S., & Benz C. (2014). Children's conceptual knowledge of triangles manifested in their drawings. In P. Liljedahl, S. Oesterle, C. Nicol & D. Allan (Eds.) *Proceedings of the Joint Meeting of PME 38 and PME-NA 36*(4), pp.153-160. Vancouver, Canada: PME
- Majid, M. A. A., Othman, M., Mohamad, S. F., Lim, S. A. H., & Yusof, A. (2017). Piloting for interviews in qualitative research: Operationalization and lessons learnt. *International Journal of Academic Research in Business and Social Sciences*, 7(4), 1073-1080.
- Manero, V., & Arnal-Bailera, A. (2021). Understanding proof practices of pre-service mathematics teachers in geometry. *Mathematics Teaching Research Journal*, 13(3), 99-130.
- Manizade, A., & Orrill, C. H. (2020). International perspectives on frameworks for mathematics teachers' knowing and quality of mathematics instruction. *Research in Mathematics Education*, 22(2), 93-97.
- Mann M., & Enderson, M. C. (2017). Give me a formula not the concept! Student preference to mathematical problem-solving. *Journal for Advancement of Marketing Education*, 25, 15-24.
- Marchionda, H. (2006). *Preservice teachers' procedural and conceptual understanding of fractions and the effects of inquiry-based learning on this understanding*. (Doctoral thesis) Clemson University. Retrieved from https://tigerprints.clemson.edu/all_dissertations/37
- Maree, K. (2016). *First Steps in Research*. Pretoria: Van Schaik.
- Martinovic, D., & Manizade, A. G. (2017). Using grounded theory to extend PCK framework at the secondary level. *Journal for Research in Mathematics Education*, 7(60), 1-17.
- Martinovic, D., & Manizade, A. G. (2018). The challenges in the assessment of knowledge for teaching geometry. *ZDM*, 50, 613-629. doi:10.1007/s11858-018-0934-4
- Maruli, S. (2014). Quality in Teaching: A review of literature. *International Journal of Education and Research*, 2(12), 193-200.
- Matlen B.J., Richland L.E., Klostermann E.C., Lyons E. (2018). Impact and prevalence of diagrammatic supports in mathematics classrooms. In P. Chapman, G. Stapleton, A. Moktefi, S. Perez-Kriz, F. Bellucci (Eds). *Diagrammatic Representation and Inference*. Lecture Notes in Computer Science, 110871. Springer, Cham
- Maulana, F., & Yuniawati, N. T. (2018). Students' problem solving ability in non-routine geometry problem. *International Journal of Information and Education Technology*, 8(9), 661-667. doi: 10.18178/ijiet.2018.8.91118

- McCammon, J. E. (2018). *Preservice teachers' understanding of geometric definitions and their use in the concept of special quadrilaterals*. Unpublished Doctoral Dissertation, Georgia State University.
- McDonough, A. (2016). Good concrete activity is a good mental activity. *Australian Primary Mathematics Classroom*, 21(1), 3-7.
- Meli, B. M. (2020). Qualities of an ideal effective teaching for primary and secondary school teachers in Tanzania. *Journal of Education and Practice*, 11(9), 237-244.
- Mensah, F. S., & Agyei, D. D. (2019). Philosophy of teaching, teaching style and ICT use: A qualitative study of the perspectives of high school mathematics teachers. *African Journal of Educational Studies in Mathematics and Sciences*, 15(1), 1-13.
- Ministry of Education. (2010). *Teaching syllabus for senior high school core mathematics*. Accra: Curriculum Research and Development Division.
- Mereku, (2003). Methods in Ghanaian primary mathematics textbooks and teachers' classroom practice. In J. Williams (Ed.) *Proceedings of the British Society for Research in Learning Mathematics*, 23 (2), 61-65.
- Mirna, M. (2018). Error analysis of students in mathematics department to learn plane geometry. *Conference Series: Materials Science and Engineering*, 335, 1-4.
doi:10.1088/1757-899X/335/1/012116
- Mohamed, H. (2017). Qualitative research approach in LIS education: Comparative study. *Journal of Research & Method in Education*, 1(2), 83-89.
- Morgan, C., & Sfard, A. (2016). Investigating changes in high-stakes mathematics examinations: A discursive approach. *Research in Mathematics Education*, 18(2), 92-119.
- Moyer, P. & Westenskow, A. (2013). Effects of virtual manipulatives on student achievement and mathematics learning. *International Journal of Virtual and Personal Learning Environment*, 4(3), 35-40.
- Mudaly, V. (2015). Pre-service teacher knowledge: Thinking about conceptual understanding. *Journal of Social Sciences*, 1-9.
- Mudaly, V. (2016). Pre-service teachers' understanding of the concept of proof. *International Scientific Researches Journal*, 72 (5), 137- 158.
- Mudaly, V. (2021). Constructing mental diagrams during problem-solving in mathematics. *Pythagoras*, 42(1), 1-8.
- Mudaly, V., & Naidoo, J. (2015). The concrete-representational-abstract sequence of instruction in mathematics classroom. *Perspectives in Education*, 33(1), 42-56.

- Mudaly, V., & Rampersad, R. (2010). The role of visualisation in learners' conceptual understanding of graphical functional relationships. *African Journal of Research in MST Education*, 14(1), 36-48.
- Mudaly, V., & Reddy, L. (2016). The role of visualisation in the proving process of Euclidean geometry problems. *International Scientific Researches Journal*, 72 (8), 178-204.
- Mullins, S. B. (2020). Angling for the right results: Students' conceptualizations of angles. *Journal of Research in Education*, 29(1), 1-47.
- Mullis, V. S., Martin, M. O., Foy, P. & Aron, A. (2011). *Trends in international mathematics and science study*. International Results in Mathematics TIMSS & PIRLS, International Study Centre, Lynch School of Education; Boston College.
- Mulungye, M. M., O'Connor, M., & Ndethiu, S. (2016). Sources of student errors and misconceptions in algebra and effectiveness of classroom practice remediation in Machakos County--Kenya. *Journal of Education and Practice*, 7(10), 31-33.
- Mulwa, E. C. (2015). Difficulties encountered by students in the learning and usage of mathematical terminology: A critical literature review. *Journal of Education and Practice*, 6(13), 27-37.
- Musa, M. & Duada, E. S. (2014). Trends analysis of students' mathematics performance in west african senior secondary certificate examination: Implication for Nigeria's Vision 20: 2020. *British Journal of Education*, 2(7), 50-54.
- Musser, G. L., Peterson, B. E., & Burger W. F. (2014). *Mathematics for elementary teachers*. Hoboken, NY: John Wiley & Sons, Inc.
- Nachlieli, T., & Katz, Y. (2017). Ritual towards explorative classroom participation of pre-service elementary school mathematics teachers. *Proceedings of CERME 10*, 2964-2971.
- Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom – The case of function. *International Journal of Educational Research*, 51, 10-27.
- Nahdi, D. S., & Jatisunda, M. G. (2020). Conceptual understanding and procedural knowledge: a case study on learning mathematics of fractional material in elementary school. In *Journal of Physics: Conference Series*, 1477(4), 1-6.
- Naidoo, J. (2011). *Exploring master teachers' use of visuals as tools in mathematical classrooms* (Unpublished doctoral thesis). University of KwaZulu-Natal, Durban, South Africa.
- Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of calculus. *Research in Mathematics Education*, 16(2), 182-198.
- National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for mathematics teachers*. Reston, VA NCTM.

- National Council of Teachers of Mathematics. (2000). *Principles and standards for school Mathematics*. Reston, VA: NCTM.
- Ndlovu, M. (2014). Preservice teachers' understanding of geometrical definitions and class inclusion: An analysis using the van Hiele model. *8th International Technology, Education and Development Conferences*. Valencia: Spain.
- Ness, D., & Farenga, S. J. (2007). *Knowledge under construction: The importance of play in developing children's spatial and geometric thinking*. Rowman & Littlefield Publishers.
- Neuman, W. L. (2014). *Social research methods: Qualitative and quantitative approaches* (7th ed.). Harlow: Pearson Education Limited.
- Ngirishi, H., & Bansilil, S. (2019). An exploration of high school learners' understanding of geometric concepts. *Problems of Education in the 21st Century*, 77(1), 1-14.
- Nissim, Y., Weissblueth, E., Scott-Webber, L., & Amar, S. (2016). The effect of a stimulating learning environment on pre-service teachers' motivation and 21st century skills. *Journal of Education and Learning*, 5(3), 29-39.
- Ntow, F. D., & Hissan, Y. (2021). The impact of concept-based instruction on senior high school students' achievement in circle theorems. *African Journal of Educational Studies in Mathematics and Sciences*, 17(1), 113-132.
- Nur, A. S., & Nurvitasari, E. (2017). Geometry skills analysis in problem solving reviewed from the difference of cognitive style students junior high school. *Journal of Educational Science and Technology*, 3(3), 205-210.
- Nwadinigwe, I. P. (2000). *Fundamental of research methods and statistics* (2nd ed.). Lagos: Sibon Books Limited.
- Ortiz, E. (2016). The problem-solving process in a mathematics classroom. *Transformation*, 1(1), 4-12. Retrieved from <https://nsuworks.nova.edu/transformations/vol1/iss1/1>
- Otsupius, I. A. (2014). Micro-teaching: A technique for effective teaching. *African Research Review*, 8(4), 183 – 197. DOI: <http://dx.doi.org/10.4314/afrrrev.v8i4.15>
- Ottevanger, W., Van den Akker, J. J. H., & De Feiter, L. (2007). *Developing science, mathematics, and ICT education in sub-Saharan Africa: Patterns and promising practices* (No. 101). World Bank Publications.
- Oyoo, S. O. (2009). Beyond general proficiency in language of instruction: Towards the appropriate perspective on language for effective learning in African science classrooms. In M. Schäfer & C. McNamara (Eds.), *Proceedings of the 17th Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology Education* (pp. 197-212). Grahamstown: SAARMSTE.
- Ozdemir, A. S., & Goktepe Yildiz, S. (2015). The examination of elementary mathematics pre-service teachers' spatial abilities. *Procedia-Social and Behavioral Sciences*, 174, 594-601.

- Ozerem, A. (2012). Misconceptions in geometry and suggested solutions for seventh grade students. *Procedia-Social and Behavioral Sciences*, 55, 720-729.
- Pickreign, J. (2007). Rectangles and rhombi: How well do preservice teachers know them? *The Journal*, 1, 1-6.
- Poch, A. L., van Garderen, D., & Scheuermann, A. M. (2015). Students' understanding of diagrams for solving word problems: A framework for assessing diagram proficiency. *Teaching Exceptional Children*, 153-162.
- Presmeg, N. (2016). Commognition as a lens for research. *Educational Studies in Mathematics*, 91(3), 423- 430. <https://doi.org/10.1007/s10649-015-9676-1>
- Provasnik, S., Kastberg, D., Ferraro, D., Lemanski, N., Roey, S., and Jenkins, F. (2012). *Highlights from TIMSS 2011: Mathematics and science achievement of U.S. fourth- and eighth-grade students in an international context (NCES 2013-009)*. National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education. Washington, DC.
- Pullen, P. C., & Lane, H. B. (2016). Hands-on decoding: Guidelines for using manipulative letters. *Learning Disabilities: A Multidisciplinary Journal*, 21(1), 27-37.
- Rehman, A. A., & Alharthi, K. (2016). An introduction to research paradigms. *International Journal of Educational Investigations*, 3(8), 51-59.
- Rianasari, V. F., Julie, H., & Sulistyani, N. (2016). Primary teachers' understanding of the inclusion relation of quadrilaterals. *Proceedings of the 2nd International Conference on Education and Training-Improving The Quality of Education and Training Through Strengthening Networking*. Pp. 263-266.
- Riastuti, N., Mardiyana, M., & Pramudya, I. (2017). Students' errors in geometry viewed from spatial intelligence. *International Conference on Mathematics and Science Education*, 1-5. Doi: 10.1088/1742-6596/895/1/012029
- Ritchie, J., Lewis, J., Nichols, C. M., & Ormston, R. (2014). *Qualitative research in practice*. London: Sage
- Rittle-Johnson, B. (2017). Developing mathematics knowledge. *Child Development Perspectives*, 11(3), 184-190. <https://doi.org/10.1111/cdep.12229>
- Rittle-Johnson, B., & Schneider, M. (2015). Developing conceptual and procedural knowledge in mathematics. In R. Cohen Kadosh & A. Dowker (Eds.), *Oxford Handbook of Numerical Cognition* (pp. 1102-1118). Oxford, UK: Oxford University Press. <https://doi.org/10.1093/oxfordhb/9780199642342.013.014>
- Rittle-Johnson, B. Fyfe, E., & Loehr, A. (2016). The content of instruction within a mathematics lesson: Implications for conceptual and procedural knowledge development. *British Journal of Educational Psychology*, 86, 576 - 591. <https://doi.org/10.1111/bjep.12124>

- Rizwan, M., Naseem, S., & Raza, S. W. (2018). Use of diagrams as instructional aids in teaching of geometrical concepts at secondary school level. *International Journal on Language, Research and Education Studies*, 2(1), 1-18. Doi: 10.30575/2017/IJLRES-2018010401
- Roberts, A. (2010). Language in mathematics classroom – catalyst or impediment to learning? In V. Mudaly (Ed.), *Proceedings of the 18th Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology Education* (Vol. 1, pp. 15-24), Pinetown: SAARMSTE.
- Roberts, A. (2016). *A study of grade 8 and 9 learner thinking about linear equations, from a commognitive perspective* (Unpublished masters' thesis). University of Cape Town.
- Roberts, A., & Le Roux, K. (2019). A commognitive perspective on grade 8 and grade 9 learner thinking about linear equations. *Pythagoras*, 40(1), 1-15.
- Robichaux-Davis, R., & Guarino, A. (2016). Assessing elementary pre-service teachers' knowledge for teaching geometry. *International Journal of Mathematics and Statistics Invention*, 4(1), 12-20.
- Rondina, J. Q. (2019). Hand's on mind's on approach using localized manipulative enhanced students' achievement in mathematics word problems. *Science International*, 31(2), 327-329.
- Rosken, B., & Rolka, K. (2006). A picture is worth a 1000 words—the role of visualization in mathematics learning. In *Proceedings 30th conference of the International Group for the Psychology of mathematics education* (Vol. 4, pp. 457-464). Prague: Charles University.
- Ryve, A. (2006). *Approaching mathematical discourse: Two analytical frameworks and their relation to problem-solving interactions*. Sweden: Malardalen University, Sweden.
- Ryve, A., Nilsson, P., & Pettersson, K. (2013). Analysing effective communication in mathematics group work: The role of visual mediators and technical terms. *Educ Stud Math*, 82, 497-514.
- Sabey, K. (2009). *Secondary preservice teachers' understanding of Euclidean geometry*. Unpublished Doctoral Thesis submitted to the University of Northern Iowa. Retrieved from <https://scholarworks.uni.edu/etd/691>
- Safrina, Y., Ikhsan, M., & Zubainur, C. M. (2021). Improving student geometry problem-solving skills through spatial training. *Advances in Social Science, Education and Humanities Research*, 627, 30-35.
- Samkoff, A., Lai, Y., & Weber, K. (2012). On the different ways that mathematicians use diagrams in proof construction. *Research in Mathematics Education*, 14(1), 49-67.
- Sarantakos, S. (2013). *Social Research* (4th ed). Palgrave Macmillan.

- Schneider, M., & Stern, E. (2010). The developmental relations between conceptual and procedural knowledge: A multimethod approach. *Developmental Psychology*, 46(1), 178-192.
- Schoenfeld, A. H. (2000). Purposes and methods of research in mathematics education. *Notices of the AMS*. (June/July), 641-649.
- Semanisinova, I. (2021). Multiple-solution tasks in pre-service teachers' course on combinatorics. *Mathematics*, 9, 1-18. <https://doi.org/10.3390/math9182286>
- Sfard, A. (2000). Steering (dis)course between metaphors and rigor: Using focal analysis to investigate an emergence of mathematical objects. *Journal for Research in Mathematics Education*, 31(3), 296-327.
- Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification: The case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy*. Washington DC: Mathematical association of America
- Sfard, A. (2012). Introduction: Developing mathematical discourse-some insights from communicational research. *International Journal of Educational Research*, 51(3), 1-9.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York, NY: Cambridge University Press.
- Sfard, A. (2015). Learning, commognition and mathematics. In D. Scott & E. Hargreaves (Eds.), *The Sage Handbook of Learning* (pp. 129-138). London Sage. <https://doi.org/10.4135/9781473915213.n12>
- Sfard, A. (2007). When the Rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The Journal of the Learning Sciences*, 16(4), 567-615.
- Sfard, A., & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different? Early numerical thinking revisited. *Cognition and Instruction*, 23, 237-309
- Shahrill, M. (2018). Teachers' devotion to review lessons: Insights to the mathematics lessons in Brunei Darussalam. *Journal of Physics: Conference Series*, 1028(1), 012158 <https://doi.org/10.1088/1742-6596/1028/1/012158>
- Shahrill, M., & Clarke, D. J. (2019). Pedagogical features that influences mathematics classroom practices: A Bruneian perspective. *Kasetsart Journal of Social Sciences*, 40,(2), 341-348.
- Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Sinclair, N., Bartolini Bussi, M. G., de Villiers, M., Jones, K., Kortenkamp U., Leung, A. & Owens, K. (2016). Recent research on geometry education. *ZDM*, 48(5), 691-719.

- Smith, K. J. (2012). *The nature of mathematics* (12th ed.). Belmont: Brooks/Cole, Cengage Learning.
- Smith, C. P., King, B., & Hoyte, J. (2014). Learning angles through movement: Critical actions for developing understanding in an embodied activity. *Journal of Mathematical Behavior*, 36, 95–108.
- Somayajulu, R.B. (2012). *Building pre-service teacher's mathematical knowledge for teaching of high school geometry* (Unpublished doctoral thesis). Ohio State University, Columbus, Ohio.
- Speer, N. M., King, K. D., & Howell, H. (2015). Definitions of mathematical knowledge for teaching: Using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education*, 18(2), 105-122.
- Steele, M. D. (2013). Exploring the mathematical knowledge for teaching geometry and measurement through the design and use of rich assessment tasks. *Journal of Mathematics Teacher Education*, 16(4), 245-268.
- Strand, K., & Mills, B. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on algebra. *The Mathematics Enthusiast*, 11(2), 385-432.
- Suarsana, I., Lestari, I. A. P. D., & Mertasari, N. M. S. (2019). The effect of online problem posing on students' problem-solving ability in mathematics. *International Journal of Instruction*, 12(1), 809-820.
- Sugeng, S., & Nurhanurawati, M. C. (2018). The effect of various media scaffolding on increasing understanding of students' geometry concepts. *Journal on Mathematics Education*, 9(1), 95 – 102.
- Sulistiowati, D. L., Herman, T., & Jupri, A. (2019). Student difficulties in solving geometry problem based on van Hiele thinking level. *Journal of Physics: Conference Series*, 1-7. doi:10.1088/1742-6596/1157/4/042118
- Supardi, L., Zayyadi, M., Lanya H., Hasanah, S. I., & Hidayati, S. N. (2021). Commognitive analysis of students' errors in solving high order thinking skills problems. *Turkish Journal of Computer and Mathematics Education*, 12(6), 950-961.
- Surif, J., Ibrahim, N. H., & Mokhtar, M. (2012). Conceptual and procedural knowledge in problem solving. *Procedia-Social and Behavioural Sciences*, 56, 416-425.
- Svensson, C., & Holmqvist, M. (2021). Pre-service teachers' procedural and conceptual understanding of pupils' mean value knowledge in grade 6. *International Electronic Journal of Mathematics Education*, 16(3), 1-12. <https://doi.org/10.29333/iejme/11067>
- Swan, M. (2005). *Improving learning in mathematics: Challenges and strategies*. Retrieved from <http://www.successforall.gov.uk/>

- Tabach, M., & Nachlieli, T. (2011). Combining theories to analyse classroom discourse: A method to study learning process. *Paper presented at the Seventh Congress of the European Society for Research in Mathematics Education*, University of Rzeszów, Poland.
- Tasara, I. (2017). Commognitive analysis of a teacher's mathematical discourse on the derivative. *Proceedings of the British Society for Research into Learning Mathematics*, 37(3), 1-6.
- Thom, J. S., & McGarvey, L. M. (2015). The act and artifact of drawing(s): Observing geometric thinking with, in, and out through children's drawing. *ZDM Mathematics Education*, 47, 465-481.
- Tiwari, S., Obradovaic, D., Rathour, L., Mishra, L. N., & Mishra, V. N. (2021). Visualization in mathematics teaching. *Journal of Advances in Mathematics*, 21, 431-439.
- Tracy, S. J. (2013). *Qualitative research methods: Collecting evidence, crafting analysis, communicating impact*. Malden: John Wiley & Sons Ltd. Publications
- Tuset, G. A. (2018). A commognitive lens to study pre-service teachers' teaching in the context of achieving a goal of ambitious mathematics teaching. *Proceeding, Congress of the European Society for Research in Mathematics Education*, 1-8.
- Tuset, G. A. (2019). *Preservice teachers' geometrical discourse when leading classroom discussion about defining and classifying quadrilaterals*. Retrieved from http://hal.archives.ouvertes.fr/hal-024222763/file/TWG18_53.pdf
- Ugboduma, S. O. (2012). Students' preference of method of solving simultaneous equations. *Global Journal of Educational Research*, 11(2), 129-136.
- Ulger, T. K., & Broutin M. S. T. (2017). Pre-service mathematics teachers' understanding of quadrilaterals and the internal relationships between quadrilaterals: The case of Parallelograms. *European Journals of Educational Research*, 6(3), 331-345.
- Unlu, M. (2022). Investigation of preservice mathematics teachers' concept definitions of circle, circular region, and sphere. *International Journal of Mathematical Education in Science and Technology*, 53(7), 1787-1814.
- Uribe, L., & Wilkins, J. (2017). Manipulative use and elementary school students' mathematics learning. *Journal of Science and Mathematics Education*, 15(8), 1541-1557.
- Usiskin, Z., & Griffin, J. (2008). *The classification of quadrilaterals: A study of definition*. Charlotte, NC: Information Age Publishing.
- Usiskin, Z., Griffin, J., Witonsky, D., & Willmore, E. (2008). *The classification of quadrilaterals: A study in definition*. Charlote, NC: Information Age Publishing.
- Wahyuni, D. (2012). The research design maze: Understanding paradigms, cases, methods and methodologies. *Journal of Applied Management Accounting Research*, 10(1), 69-80.

- Wai, J. (2018). Why spatial reasoning matters for education policy. *American Enterprise Institute*, 1-8. doi:<https://eric.ed.gov/?q=Why+Spatial+Reasonin+Matters+For+Education+Policy.&id=ED592011>
- Wang, S. (2013). Understanding prospective teachers' levels of geometric thoughts: insights from a discursive analysis. *Proceedings of the 35th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education by Psychology of Mathematics Education - North American Chapter*.
- Wares, A. (2016). Mathematical thinking and origami. *International Journal of Mathematical Education in Science and Technology*, 47(1), 155-163.
- Watson, A., & Harel, G. (2013). The role of teachers' knowledge of functions in their teaching: A conceptual approach with examples from two cases. *Canadian Journal of Science, Mathematics and Technology Education*, 13(2), 154-168.
- Watson, A., Jones, K., & Pratt, D. (2013). *Key ideas in teaching mathematics: Research-based guidance for ages 9-19* (1st ed.). Oxford: Oxford University Press.
- West African Examinations Council (2015). *Chief Examiners' Report for Senior School Certificate Examination*. Accra: WAEC Press.
- West African Examinations Council (2017). *Chief Examiners' Report for Senior School Certificate Examination*. Accra: WAEC Press.
- West African Examinations Council (2018). *Chief Examiners' Report for Senior School Certificate Examination*. Accra: WAEC Press.
- Winarti, D. W. (2018). Developing spatial reasoning activities within geometry learning. *Journal of Physics: Conference Series* 1088. Doi.10.1088/1742-6596/1088/1/012004
- Winicki-Landman, G., & Leikin, R. (2000). On equivalent and non-equivalent definitions: Part 1. *For the learning of Mathematics*, 20(1), 17-21.
- Wonu, N., & Zalmon, I. G. (2017). Diagnosis and remediation of senior secondary students' common learning difficulties in mathematics from Chief examiners' report. *European Journal of Research and Reflection in Educational Sciences*, 5(1), 7-23.
- Wright, V. (2013). Pre-service teachers' concept image for circle and ellipse. In V. Steinle, L. Ball & C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow. Proceedings of the 36th Annual Conference on the Mathematics Education Research Group of Australia* (pp. 682-689). VIC: MERGA.
- Yahya, F. H., Kurniawan, H., Shamsuddin, N. R., & Zain, R. M. (2022). Effects of visual learning geometry kits (VLGeoKits) on students' level of geometrical thinking. *Advances in Social Science, Education and Humanities Research*, 630, 1-6.
- Yalley, E., Armah, G., & Ansah, R. K. (2021). Effect of the van Hiele instructional model on students' achievement in geometry. *Education Research International*, 1-10.

<https://doi.org/10.1155/2021/6993668>

- Yi, M., Wang, J., Flores, R., & Lee, J. (2022). Measuring pre-service elementary teachers' geometry knowledge for teaching 2-dimensional shapes. *EURASIA Journal of Mathematics, Science and Technology Education*, 18(8), 1-15. doi.org/10.29333/ejmste/12220
- Yigit, M. (2014). An Examination of pre-service secondary mathematics teachers' conceptions of angles. *The Mathematics Enthusiast*, 11(3), 707-736. DOI: <https://doi.org/10.54870/1551-3440.1322>
- Yin, R. K. (2014). *Case study research: Design and methods* (5th ed.). London: SAGE Publications.
- Yurmalia, D., & Herman, T. (2021). Student visualization in solving geometry problem: the case of reflection. *Journal of Physics: Conference Series*, 1806, 1-6. doi:10.1088/1742-6596/1806/1/012052
- Yurniwati, Y., & Soleh, D. A. (2021). Analyzing conceptual and procedural knowledge of geometry among prospective teachers: Indonesian perspective. *Journal of Physics: Conference Series*, 1752, 1-7. Doi:10.1088/1742-6596/1752/1/012067
- Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, 36(4), 317-346.
- Zayyadi, M., Nusantara, T., Hidayanto, E., Sulandra, I. M., & Sadijah, C. (2020). Content and pedagogical knowledge of prospective teachers in mathematics learning: Commognitive framework. *Journal for the Education of Gifted Young Scientists*, 8(1), 515-532. DOI: <https://dx.doi.org/10.17478/jegys.642131>
- Zazkis, R., & Leikin, R. (2008). Exemplifying definitions: a case of a square. *Educational Studies in Mathematics*, 69(2), 131-148.
- Zhang, D., & Bergstrom, C. (2016). Geometry intervention for K-12 students with and without disabilities: A research synthesis. *International Journal of Education Research*, 80, 134-154.
- Zhang, D. (2017). Effects of visual working memory training and direct instruction on geometry problem solving in students with geometry difficulties. *A Contemporary Journal*, 15(1), 117-138.
- Zilkova, K. (2014). Testing pre-service primary education teachers in quadrilaterals. *Human and Social Sciences*, 2(1), 195-197.
- Zuya, H. E., Matawal, D. B., & Kwalat, K. S. (2017). Conceptual and procedural knowledge of pre-service teachers in geometry. *International Journal of Innovative Education Research*, 5(1), 30-38.

Zuya, H. E., & Kwalat, S. K. (2015). Teacher's knowledge of students about geometry. *International Journal of Learning, Teaching and Educational Research*, 13(3), 100-114.

APPENDICES

Appendix A: Ethical Clearance Certificate from UKZN



15 October 2021

Ernest Larbi (219095877)
School Of Education
Edgewood Campus

Dear E Larbi,

Protocol reference number: HSSREC/00001799/2020

Project title: An analysis of pre-service mathematics teachers' geometric thinking and classroom discourse using commognitive lens

Approval Notification – Amendment Application

This letter serves to notify you that your application and request for an amendment received on 06 October 2021 has now been approved as follows:

- Change in research instruments

Any alterations to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form; Title of the Project, Location of the Study must be reviewed and approved through an amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number.

PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

All research conducted during the COVID-19 period must adhere to the national and UKZN guidelines.

Best wishes for the successful completion of your research protocol.

Yours faithfully



Professor Dipane Hlalele (Chair)

/dd

Humanities & Social Sciences Research Ethics Committee
UKZN Research Ethics Office Westville Campus, Govan Mbeki Building
Postal Address: Private Bag X54001, Durban 4000
Tel: +27 31 260 8350 / 4557 / 3587
Website: <http://research.ukzn.ac.za/Research-Ethics/>

Founding Campuses: Edgewood Howard College Medical School Pietermaritzburg Westville

INSPIRING GREATNESS

Appendix B: Permission letter to conduct the study



Informed Consent Letter to the Head of Mathematics Department

1st September, 2020

The Head Department of Mathematics Education
College of
University of
Ashanti Region Ghana

Dear Sir

PERMISSION TO CONDUCT A RESEARCH STUDY IN YOUR DEPARTMENT

I am writing to request your permission to conduct a research study in your school. This research study is entitled:

An analysis of pre-service mathematics teachers' geometric thinking and classroom discourse using a commognitive lens

My name is Ernest Larbi, and I am currently studying towards a Doctor of Philosophy Degree at the University of KwaZulu-Natal (UKZN), Durban. My student number is 219095877. As part of the requirements of this degree, I am required to complete a research thesis. This study focuses on pre-service mathematics teachers' geometric thinking.

I require 10 pre-service mathematics teachers to participate in this research. I would be very grateful if you would consent to these pre-service teachers' participation in this study. They will be selected from your department.

If you agree to this, they will be invited to take part in a test, which will be used as a basis for their inclusion in the main study. After the test, there will be an individual interview with the selected pre-service teachers. Their teaching will also be observed during on-campus teaching practice.

All discussions, interviews and dialogues with participants will be audio recorded using an infinix phone, and the classroom observation will be video recorded. Thereafter, both the audio and video recordings will be transcribed verbatim to produce transcriptions. This research information (data) is required for the analysis of data and completion of the actual write up of the thesis. Collecting research information for this study will take approximately 60 minutes with each individual in the interview. The video recording of the classroom observation will be based on the time allowed for their teaching. However, it will not go beyond 40 minutes. The test and the in-depth individual interviews will take place after their lecture hours but on the school premises, with your permission. Times and dates will be discussed and arranged with you and the participants at a later stage. I will try to ensure that the testing and the interview takes place during their free periods, in an attempt to avoid any disruptions during lessons. The lesson observation will however take place during on-campus teaching practice with your permission. I will not deprive them of their right or opportunities, especially since I intend to use some of their free time in order to collect sufficient data for my study.

As indicated earlier, data generation activities will also take place after lectures hours and on school premises with your consent. If I am unable to collect my data during their free hours (after lectures), I will make arrangements with your consent and with the participants on an appropriate time including weekends, if possible.

Please note:

* Times and dates of this data generation process will be at your sole discretion. I have merely presented you with an outline of what I intend to do, however you are free to make any changes and suggestions, if necessary.

* Participation is completely voluntary and participants have the right to withdraw from this study at any time. They will not be penalised if they choose to do so.

* Confidentiality and anonymity will be maintained at all times. The identity of your school and all participants will not be revealed at any time, as pseudonyms (different names) will be used to protect everyone's right to privacy.

* Any information provided by the participants will not be used against them, or against the school, and will be used for purposes of this research only.

* Participation in this study will not result in any cost to your school or the participants.

* Neither the participants nor your school will receive financial remuneration. However, costs incurred by participants as a result of their involvement in this project will be covered.

* This study does not intend to harm the participants in any way.

* Both parents/guardians as well as participants will be handed letters of consent which they will have to carefully read and sign, before I begin data collection.

I may be contacted at:

Email address: ertlarbi@gmail.com

Tel: +233 24 481 2402

My supervisor, Prof. Vimolan Mudaly, with contact details:

Email address: mudalyv@ukzn.ac.za

Tel: 0027 839 770 577

You may also contact the College of Humanities, Research Office through:

HSSREC@ukzn.ac.za

Tel: 0027 031 260 8350

If you would like any further information or if you are unclear about anything, please feel free to contact me at any time. Your co-operation and consent will be greatly appreciated.

If you grant permission to conduct this research at your school, please complete the form below and return to me.

Warm regards

.....

Ernest Larbi.

DECLARATION

I (full name/s of school principal) of (name of school) hereby confirm that I understand the contents of this document and the nature of this research project, and I consent to the pre-service teachers' participating in this research project. I also grant permission for my school to be used as the research site.

Additional consent

I understand that interviews will be audio-recorded and I grant permission for this.

YES/NO

I understand that lesson observation will be video-recorded and I grant permission for this.

YES/NO

I understand that the pre-service teachers and the department are free to withdraw from the research project at any time.

YES/NO

SIGNATURE OF HEAD OF DEPARTMENT

DATE

.....

.....

Appendix C: Letter requesting pre-service teacher for participation



1st September, 2020

Dear Student,

REQUEST FOR PARTICIPATION IN A STUDY

I am Ernest Larbi, a doctoral student of the school of education, University of KwaZulu-Natal, Edgewood Campus with student number 219095877. As part of the program, I am required to undertake a research work in my discipline, which is mathematics.

My research topic is “**An analysis of pre-service mathematics teachers’ geometric thinking and classroom discourse using a commognitive lens**”.

As a result, I write to make a request for your voluntary participation in the study which will involve;

1. Answering some test questions on geometry
2. Engaging you in an interview to enable you talk about what you have done
3. A classroom observation of your geometry lesson with students/peers (teaching practice)

The interview and lesson observation will be audio and video recorded respectively.

I wish to state that this study will not in any way disrupt teaching and learning and the interview will be conducted after contact hours.

This exercise is mainly for research purposes and any information provided to me will be treated with utmost confidentiality. Also, your name or the name of the school will not be mentioned in any part of the study report. You are at liberty to withdraw your participation from the study should you feel uncomfortable.

I am convinced that this study will contribute to effective teaching and learning of geometry and mathematics in general in our educational setting.

Should you require any further information about this study, never hesitate to contact my supervisor by the address:

Prof. Vimolan Mudaly

Email: mudalyv@ukzn.ac.za

Or College of Humanities research office by the e-mail: HSSREC@ukzn.ac.za

Yours sincerely

.....

Ernest Larbi

e-mail: ertlarbi@gmail.com

Contact: +233 24 481 2402

DECLARATION

I,, (full name of pre-service teacher)

having read and understood the content of the letter, agree to participate in your research activity.

I am also aware that the interview will be audio-recorded and the classroom observation will be video-recorded.

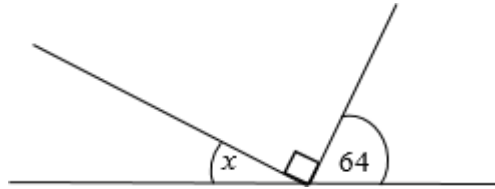
Signature:.....

Date:.....

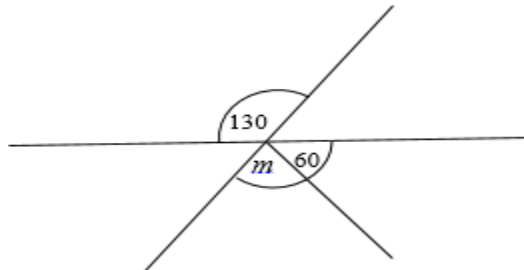
Appendix D: Geometry test

1. Angles and parallel lines

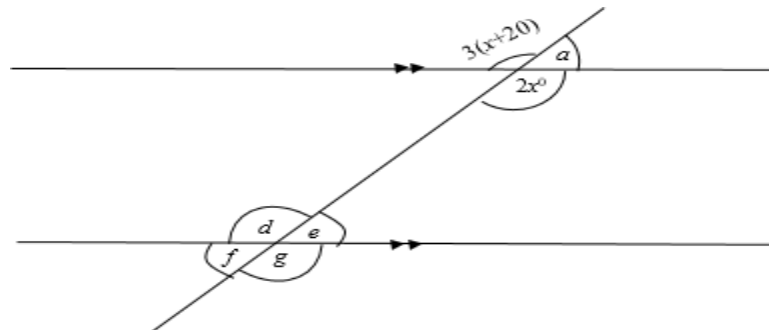
1.1 Find the value of x in the figures below:



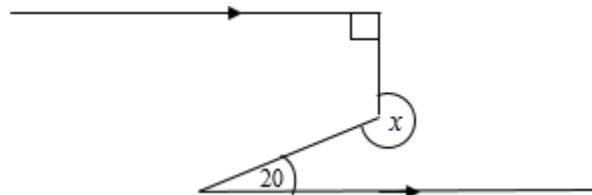
1.2 Find the value of m in the figure below



1.3 Find the values marked by letters

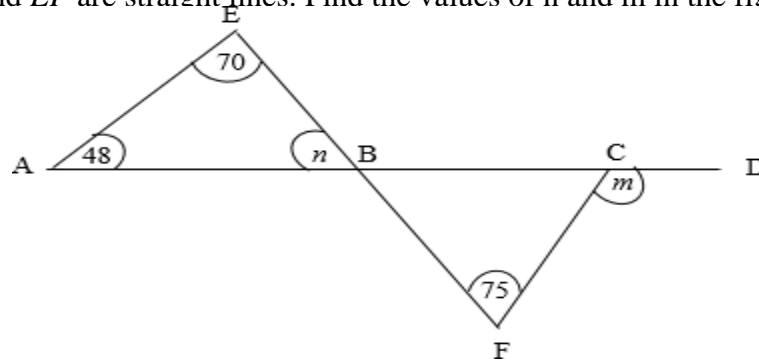


1.4 Find the value of x in the figure

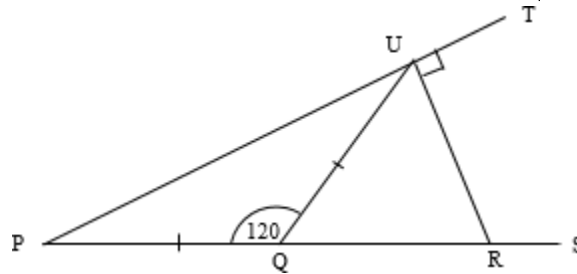


2. Triangles

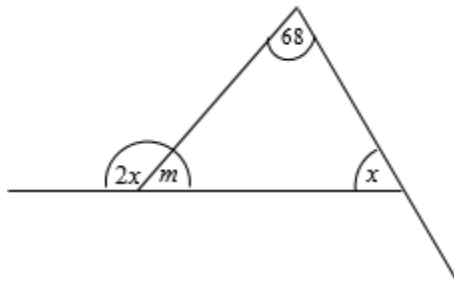
2.1 In Fig. 5, AD and EF are straight lines. Find the values of n and m in the figure below.



2.2 In the fig. PQRS and PUT are straight lines. $\angle PQU = 120^\circ$. If $|PQ| = |QU|$, find $\angle URS$.

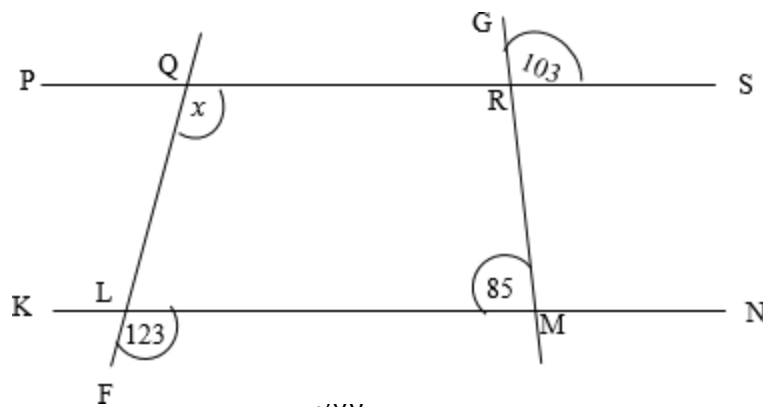


2.3 Solve for x and m in the figure below.

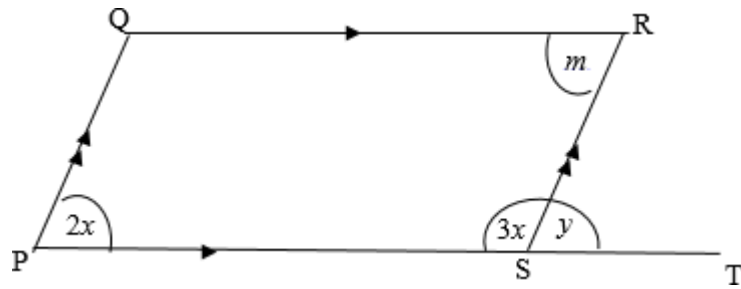


3. Quadrilaterals/Combination of shapes

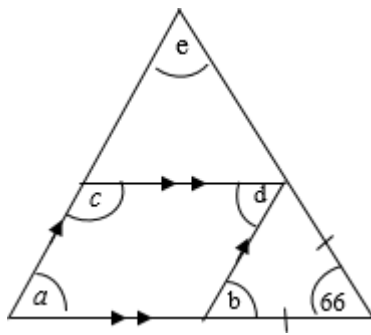
3.1 Find the value of x in the diagram below



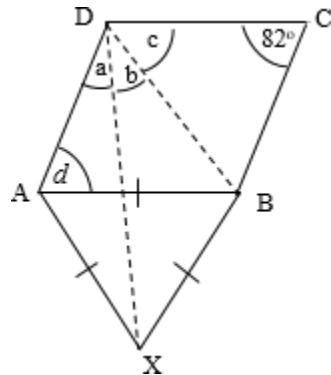
3.2 Find the values of the angles marked by letters



3.3 Find the values of angles marked by letters in the diagram below

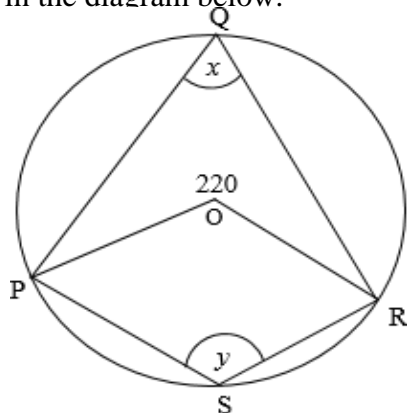


3.4 The figure below ABCD is a rhombus. If $\angle BCD = 82^\circ$, calculate $\angle ADX$ and $\angle BDX$

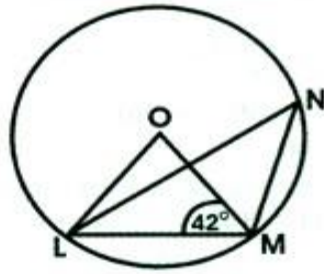


4. Circles

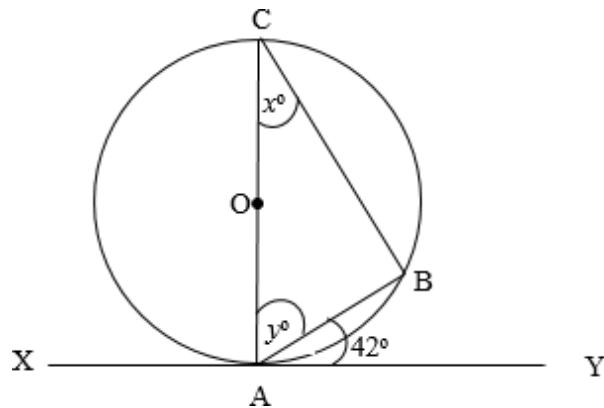
4.1 Find the values of x and y in the diagram below.



4.2 In Fig. 12, LMN are points on a circle with centre O . If angle $LMN = 42^\circ$, find angle LMN

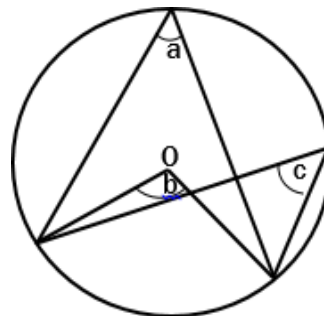


4.3 Find the values of x and y in the figure below.

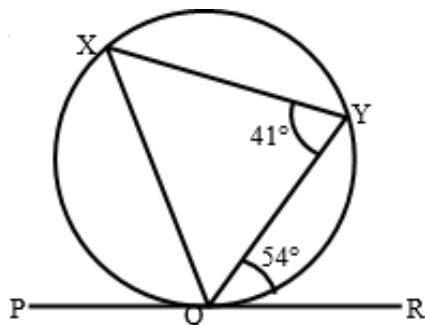


4.4 In the diagram, find the relationship between

- (i) b and a
- (ii) a and c
- (iii) b , a and c



4.5 In Fig. 14, PQR is a tangent to the circle at Q . $\angle YQR = 54^\circ$ and $\angle XYQ = 41^\circ$. Find the size of $\angle XQY$.



Appendix E: Semi-structured interview protocol

Interview protocol

1. Angles and parallel lines

What is your understanding of the concept of an angle? Can you demonstrate it on paper?

Can you explain the concept of an angle in any other way?

Can you show this using a diagram?

Can you give some examples of angles?

Task 1.1

How did you find the value of x in Task 1.1?

Can you explain why you employed this method or approach in answering the question?

Can you find the value of x in any other way?

Task 1.2

How will you find the value of x in task 1.2?

Why did you use that approach?

Is there any other method that can be used in solving for x in task 1.2?

Task 1.3

How will you find the values of the angles marked by letters in task 1.3?

What informed your reasoning for using that or this method?

What geometric ideas/properties did you use to find the angles in the question?

What other approach can you use to solve the task?

What is the relationship between angle d and angle g in task 1.3?

What is the name of the line drawn across the two parallel lines in task 1.3?

What are corresponding angles?

What are alternate angles?

Task 1.4

Can you explain how you solved for x in task 1.4?

Why did you use this approach?

What geometric ideas or properties informed this approach?

What other method can you use to find x ?

2. PSTs' conception of triangles

What is your understanding of a triangle?

What are the types of triangles?

Probing: Can you mention any other types of triangles?

What properties define these triangles?

Can the types of triangles be put into two groups?

What is the sum of the interior angles in a triangle?

Can you show a simple proof for this?

Task 2.1

How did you solve for the values of n and m in task 2.1?

Can you explain why you used this approach?

What geometric properties did you use in this approach?

Probe: There will be a follow up question depending on the approach used to find m .

Task 2.2

What geometric properties will you need to solve for the angle in task 2.2

What is the name of triangle PQU ?

Why did you say that? / What informed your answer?

What other way can you use to identify the name of the triangle in the task?

Can you find angle URS using another method different from what you used earlier?

Task 2.3

How will you solve for x and m in task 2.3?

What geometric ideas did you use to solve for the values of x and m ?

Can you show any other method that you can use to find the values of x and m ?

What is your understanding of the exterior angle theorem?

Can you demonstrate the proof of this theorem?

3. Quadrilaterals

What is your understanding of a quadrilateral?

- What are the types of quadrilaterals?

Can you define them?

What is a parallelogram?

Can you draw it?

- What can you say about the (1) sides, (2) angles and (3) diagonals of the parallelogram?
- There will be follow up questions based on what the participant draws.

Can you draw a parallelogram different from the one you drew earlier?

- What can you say about the sides, angles and diagonals of the parallelogram?
- There will be follow up questions based on what the participant draws.

Task 3.1

How will you solve for x in the diagram above?

Why did you solve it using that method?

Task 3.2

Can you explain how you solve for x , m and y in task 3.2?

What geometric ideas do you need to find the values of x , m and y in the diagram?

Can you describe the approach used?

Can you solve for the variables in another way?

Task 3.3

How will you find the angles marked by letters in task 3.3?

What geometric ideas did you use to find the values of the angles marked by letters?

Explain the reasoning behind your solution, indicating the properties.

Task 3.4

What geometric property informed your solution approach?

Explain how you solved the question.

Classifications of Quadrilaterals/Parallelograms

You are kindly requested to classify parallelograms by responding to the questions below and justify your response.

Is a rectangle a parallelogram?

Yes

☐

No

☐

Justify/Explain

.....

.....

.....

Is a square a parallelogram?

Yes

☐

No

☐

Justify/Explain

.....

.....

.....

Is rhombus a parallelogram?

Yes

☐

No

☐

Justify/Explain

.....

.....

.....

Is a square a rectangle?

Yes

☐

No

☐

Justify/Explain

.....

.....

.....

Is a square a rhombus?

Yes

☐

No

☐

Justify/Explain

.....
.....
.....

Is a rhombus a rectangle?

Yes

☐

No

☐

Justify/Explain

.....
.....

4. Circles

What is a circle?

Can you draw a circle and name the parts?

What is a cyclic quadrilateral?

Probe: If answers are not exhaustive, a circle will be drawn for them to answer questions that will be asked.

On the work sheet provided, provide the names of the parts of the circle as indicated.

Probe: Participants will be asked to describe the various parts of the circle.

In each of the questions on circle the participants were asked to:

Explain how they solved the task and the governing geometric properties that they used.

They were probed based on their solution strategy.

For example, can you state the theorem required to solve this problem?

Appendix F: Lesson observation protocol

Classroom observation was guided by the constructs of the commognitive framework.

Variables for Observation	Expectation(s) / Indication(s)
Pre-Observation exercise	
Review of relevant previous knowledge	To help colleagues link or connect new knowledge to existing ones to facilitate understanding.
Main- Lesson Observation	
Word use	Determine the type of words that pre-service teachers used in geometric discourse. Attention was paid to determining whether the kinds of words or terminologies used in developing geometric thinking were colloquial or literate. That is, the language employed in a geometric discourse.
Visual mediators	Observe the kinds of visual objects (paper cutouts, symbols, diagrams, etc.) or resources the pre-service teachers used to help communicate geometric concepts or develop geometric thinking in a discourse.
Routines	Observe the patterns the pre-service teachers used to regulate colleagues' actions in a discourse. For example, guiding them to determine when to apply a theorem or when it is suitable to use a routine in discourse.
Narratives	Determine the kinds of written or spoken statements used by pre-service teachers to describe and justify geometric discourse. For instance, axioms, theorems, definitions, etc.
Teaching methods (Partitionist or Acquisitionist teaching)	Find out whether the lesson is teacher dominance (lectures, teacher-led demonstrations) or learner dominance (activities, discussions, cooperative learning approaches, etc.).
Ritual of Explorative routine (Will be linked to teaching methods)	To assess whether the instruction seeks to establish social approval- ritual, or to produce or verify an endorsed narrative among learners- explorative (constructing alternative solution to tasks)
Learner involvement in the discourse	The degree of learner involvement in the discourse. Will the learners be treated as acquisitionist (passive) or participationist (active) members of the discourse?

To whom it may concern

This is to confirm that the doctoral thesis submitted by

ERNEST LARBI

has been language edited

TOPIC: An analysis of pre-service mathematics
teachers' geometric thinking and classroom discourse
using a commognitive lens

M. Govender

Date: 21/11/2022

monica.govender@outlook.com

084 4646898

Appendix H: Turnitin report

AN ANALYSIS OF PRE-SERVICE MATHEMATICS TEACHERS' GEOMETRIC THINKING AND CLASSROOM DISCOURSE USING A COMMOGNITIVE LENS

ORIGINALITY REPORT

7 %	5 %	4 %	1 %
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	hdl.handle.net Internet Source	1 %
2	researchspace.ukzn.ac.za Internet Source	1 %
3	vital.seals.ac.za:8080 Internet Source	<1 %
4	files.eric.ed.gov Internet Source	<1 %
5	Discourse Perspective of Geometric Thoughts, 2016. Publication	<1 %
6	Submitted to University of KwaZulu-Natal Student Paper	<1 %
7	www.pmena.org Internet Source	<1 %
8	etd.lib.metu.edu.tr Internet Source	<1 %