PUPILS' NEEDS FOR CONVICTION AND EXPLANATION WITHIN THE CONTEXT OF DYNAMIC GEOMETRY.

BY

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ABSTRACT

Recent literature on mathematics education, and more especially on the teaching and learning of geometry, indicates a need for further investigations into the possibility of devising new strategies, or even developing present methods, in order to avert what might seem to be a "problem" in mathematics education. Most educators and textbooks, it would seem, do not address the need (function and meaning) of proof at all, or those that do, only address it from the limited perspective that the only function of proof is verification. The theoretical part of this study, therefore, analyzed the various functions of proof, in order to identify possible alternate ways of presenting proof meaningfully to pupils.

This work further attempted to build on existing research and tested these ideas in a teaching environment. This was done in order to evaluate the feasibility of introducing "proof" as a means of explanation rather than only verification, within the context of dynamic geometry. Pupils, who had not been exposed to proof as yet, were interviewed and their responses were analyzed. The research focused on a few aspects. It attempted to determine whether pupils were convinced about explored geometric statements and their level of conviction. It also attempted to establish whether pupils exhibited an independent desire for why the result, they obtained, is true and if they did, could they construct an explanation, albeit a guided one, on their own.

Several useful implications have evolved from this work and may be able to influence, both the teaching and learning, of geometry in school. Perhaps the suggestions may be useful to pre-service and in-service educators.

DEDICATION

To

My spiritual masters, Swami Sivananda and Swami Sahajananda.

My wife, Ronicka and my children, Shavani and Deveshnie.

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I, Vimolan Mudaly, declare that the research involved in my dissertation submitted in partial fulfillment of the M. Ed. Degree in Mathematics Education, entitled **Pupils' needs for conviction and explanation within the context of dynamic geometry**, represents my own and original work.

Vimolan Mudaly

CONTENTS

INTRODUCTION AND OVERVIEW

Proof and proving in mathematics and mathematics education has been the subject of much discussion and debate, as will be seen in the chapters that follow. This research, whilst attempting to investigate whether pupils' display any need for conviction and explanation in dynamic geometry, also takes into account the different views and ideas that surrounds the concept of proof. Kline (1982 : 317) and Bell (1945 : 4), along with many other mathematicians, expressed a clear need for proof, but recently certain mathematicians, like Zeilberger (1994 : 11), have questioned the need for proof in mathematics. Some authors, like Horgan (1993 : 74-82), have also argued that verification by computer is making proof obsolete.

This research also explores the need for proof from an educational perspective. Several teachers of mathematics have argued that the proving of riders in examinations has been a major factor in pupils' poor results. Most teachers and textbooks do not address the need (function and meaning) of proof at all, or those that do, only address it from the limited perspective that the only function of proof is that of verification.

The theoretical part of this study is contained in chapters one and two. Chapter One analyses the various functions of proof within mathematics, in order to identify possible alternate ways of presenting proof meaningfully to pupils. In this chapter the need for proof in mathematics is dealt with, and it takes into account the arguments for and against the advent of computer assisted proofs. Many definitions of mathematical proof have been recorded in the various texts and articles that are available and this chapter reflects on some of them.

Chapter Two considers proof in the mathematics curriculum at schools. It examines the need for pupils to prove in mathematics. It also takes cognisance of the levels expounded by the Van Hieles and also previous research conducted by other authors and researchers, including that of De Villiers (1990;1991) and Zack (1997 : 291-297).

The empirical part of this investigation will build on previous research, related to pupils' cognitive needs for conviction and explanation within the context of geometry. In particular, a main aspect of the study will be to check the findings of De Villiers (1990;1991), that pupils' exhibit a need for explanation, independent from their need for conviction, using the pencil and paper method, whereas this study is based in the context of dynamic geometry. This research will further try to establish whether a guided explanation provides a deeper understanding of that which the pupil is already convinced of. Therefore the purpose of the research is to evaluate the feasibility of introducing "proof" as a means of explanation rather than only verification, within the context of dynamic geometry.

Chapter Three briefly summarizes the research methodology that was used in the collation of the data. A brief overview of the process that was involved is discussed. Chapter Four deals explicitly with the analysis of the data that was collected. The significant questions that were asked and the responses obtained are discussed in this chapter.

Finally, Chapter 5, deals with the conclusions that were drawn from the pupils' responses and some recommendations are made. These recommendations are highly relevant and would hopefully serve as a springboard for further research.

CHAPTER ONE

PROOF IN MATHEMATICS

1.1 Introduction

Understanding the role that proving has played in mathematics must be considered important. Although proof should be seen as serving many functions, it would seem that establishing certainty in a statement has been its main function. According to Davis and Hersh (1984 : 249) this can be traced back to the Greek mathematicians who saw the proof process as that of validation and certification. Of course, this meant that once a statement had been proved then that statement was true beyond any shadow of doubt. Even the first theorem, which was proved by Thales of Miletus (600B.C.), showed that a diameter divides a circle into two equal parts. This may seem quite obvious to the reader, but it did show that proof was necessary and possible (Davis and Hersh, 1984 : 248). According to Jones (1996 : 142) it is the most important aspect of mathematics and it is that which distinguishes it from other disciplines.

The significance of a mathematical proof is not only contained in the end result, but also in the process of proving. Mathematicians are often interested in finding new types of arguments, in breaking new grounds, so that new and existing statements can be proved using these new found links. This is why even incorrect proofs are not discarded, but rather they are scrutinized carefully for ideas that may be used to prove new and existing problems.

Recent developments in proof strategies (namely, computer generated proofs), has threatened to alter the existing concept of proof. It was never envisaged that certainty of a statement could be based entirely on the results of a computer, whose language and operating mechanisms is difficult to understand. The existing concept of mathematical proof is clear in that it allowed the reader to follow the argument in a logical way, which if understood, would give much insight to the problem.

It must be acknowledged though, that different standards and types of proof (Tall, 1996 : 28) exist at a formal level, and therefore different forms of proof might be appropriate in different contexts. Despite the method and context within which proof is done, proof has remained the main tool which mathematicians use for verifying, communicating, explaining, systematizing and discovering. It is therefore important to continue to closely examine (and re-examine) the teaching of proof at school. Jones (1996 : 144) has, for example, pointed out that *"we need to continue to look for ways of laying the foundation for a deeper appreciation of the role of proof"*. Hanna and Jahnke, (1993 : 329) have made similar comments when they stated that : *"The last fifteen years has seen a remarkable reassessment of the role and meaning of proof, one which has influenced the attitude and practice of mathematicians, the theory of mathematics education, and the curriculum"*.

Much has been said and written about proof in mathematics and many salient issues have been raised. Despite the vast amount of literature available, it would seem that *which vanishes in the distance, and conjectures that it leads to a peak in the clouds or below the horizon. But when he sees a peak, he believes that it is there simply because he sees it. If he wishes someone else to see it, he points to it, either directly or through the chain of summits, which led him to recognise it himself. When his pupil also sees it, the research, the argument, the proof is finished."*

This extract of Hardy's intimates that proof, to an extent, is informal. The chain of summits merely refers to a chain of statements in a proof. But proof has yielded far more discussion than what has been stated above.

1.2 **The need for proof in mathematics**

The need for proof has been the source of much deliberation in mathematics circles. It has evoked many responses from various authors of mathematics texts. Some of these responses have been inspired by innovations in computer generated proofs, (for example, the Appel and Haken proof of the Four colour theorem) and to some extent, by the work of Lakatos, who claimed that proof and refutation are the essential driving forces behind mathematical discovery. In fact, many people have narrowly misinterpreted proof as that of simply serving the function of verification alone. Many authors (Hanna:1996; De Villiers:1995) have recognised that this idea of proof is inadequate and they have been stating that there should be a balance between the different functions of proof. This can easily be seen by the number of conferences that specifically deal with proofs and proving and journal articles that are being published at these conferences, like the conference in London in 1995 (Justifying and proving in school mathematics) and the Topic Group on Proof at the ICME conference in Sevilla in 1996.

Steiner (1975 : 93) had also noted the explicit need for proof when he had stated that : *"In principle, it is held, mathematical proof is essential to mathematical knowledge"*. Bell, E.T. (1945 : 4) similarly said that : *"Without the strictest deductive proof from admitted assumptions, explicitly stated as such, mathematics does not exist"*. The main reason for this is that mathematical certainty cannot be attained from mere empirical evidence. The following simple example (Peterson, 1990 : 153) illustrates the point that a mathematician cannot rely on empirical evidence alone. Consider this sequence of integers: 31, 331, 3331, 33331, 333331 and 3333331. What is it that is so special about these numbers? They are all prime numbers. Is 33333331 a prime as well? Yes it is. Is the next integer in this sequence a prime? Unfortunately, no! This number turns out to be the product of two numbers, 17 and 19 607 843. Thus it can be seen that although empirical evidence may appear to be convincing, only a formal proof provides absolute certainty. Empirical results may ensure a high level of conviction but it cannot guarantee the truth of the statement because it provides no grounds on which we can accept the evidence. Although it can be said that strong empirical evidence may provide belief in the truth of a result, thus motivating the search for its logical explanation.

Proofs give mathematician an assurance that a statement is true, if it has been proved using sound statements that were previously obtained and proved. Slomson (1996 : 12) states that proofs *"give us the justification for the mathematical methods we use, and good proofs also help us to understand the mathematics"*. In a lighter vein he states that *"mathematics without proof is like brandy without alcohol : the spirit of the subject is missing"* (Slomson, 1996 : 12).

Barbara Ball (1996:34) also conveys the need for proof when she said that *"proof can bring understanding of why methods work and , consequently of how those methods might be adapted to cope with new and altered circumstances*" and "*we do them (pupils) a great disservice if we exclude it (proof) from the curriculum*". Schoenfeld (as quoted by Hanna, 1996 : 1) similarly replied to the question "Do we need proof in mathematics education ?" as follows: *"Absolutely. Need I say more? Absolutely."*. Otte (1994 : 312) believes that proof is essential : *"... formal proof is required, which however connects not empirical facts but formal propositions"*.

Putnam (1986 : 63) also states that *"proof will continue to be the primary method of mathematical verification"*. She (1986 : 52) envisages that if Martians were to communicate with man then they would say : *"We recognize proof, and we value proof as highly as you do when we can get it. What we don't understand is why you restrict yourself to **proof** – why you refuse to accept **confirmation**"*. Here she argues that it is this fact (not accepting quasi-empirical methods as proof) that prevents us from using certain profound discoveries in mathematics. She believes that it is often quite convincing from quasi-empirical methods that statements may be true but the lack of a formal proof prevents us from using the result to discover other truths in mathematics.

Kitcher (1984 : 180), like many other authors, also emphasizes the need for proof as follows: "proof provides optimal support for the conclusion, in that other ways of obtaining the conclusion from those premises would be *more* vulnerable to challenge". In essence, Kitcher is not saying that steps in a proof are invulnerable to challenge, but rather that, proofs would fare much better as compared to other forms of argument. Kitcher (1984 : 181) further argues that proofs not only increases understanding but also generates new knowledge. It is well known that many theorems in mathematics have more than one proof. This discovering of new proofs for old theorems, increases understanding and provides a greater insight into the different relationships that exist between theorems and these various proofs.

Ultimately it must be said of proof that "*anyone who has looked into the contributions of mathematics to human thought would not sacrifice the concept of proof*" (Kline, 1982 : 317).

1.3 **Proof is seen mainly as verification**

There is little doubt that the traditional role of proof has been seen mainly in terms of verification of the correctness of mathematical statements. Proof, it would seem, served the explicit function of convincing skeptics about the truth of a statement. Coe and Ruthven (1994 : 42) summarized this by claiming that "the most salient function of proof is that it provides grounds for belief". In fact, a survey in 1984 by De Villiers (1990 : 18) revealed that more than 50 % Higher Education Diploma students in mathematics education agreed that the only function of proof was that of "making sure", that is, the verification of the truth of the results.

Despite being the dominant view for a long time, several authors have cautioned against such stereotyped views. Bell (1976 : 24) stated that *"conviction is normally reached by quite other means than following a logical proof; proof is essentially a public activity of validation which follows the reaching of conviction, though it may be conducted internally"*. In fact, Hersh (1993 : 390) echoes a similar sentiment when he says that : *"... more than whether a conjecture is correct, mathematicians must know **why** it is correct"*. Reid (1996 : 185) echoed similar sentiments when stating: *"I would like to question the common assumption that the role of deductive reasoning or proving in mathematics is the verification of conjectures"*.

The fact that proof is more than just verification of conjectures is emphasized by John Searl (1996 : 21) when he observed that: *"Of course, it is usually more important to understand the meaning and implications of a theorem than its proof. Sometimes the method of proof offers some insight into the meaning of the result but sometimes the proof has been so refined by successive generations of mathematicians that insight is lost. So the nature of mathematical proof is not as well defined as some writers appear to believe. Further it has not been demonstrated beyond reasonable doubt that the rote learning of mathematical proofs inculcates, either logical thought, technical fluency or mathematical insight. There is, however, evidence that good investigational work does"*.

1.4 The role of computer generated proofs

Computers are fast becoming a useful tool in the proving process. Many complex and very long proofs have been presented, by mathematicians, which has made use of computers. This has inevitably evoked some debate as to whether these 'proofs' can really be classified as proofs. Of greater interest is the fact that computers can check the truth of mathematical conjectures, using millions of possible values in a few seconds.

But computer generated proof of mathematical statements has created a fair amount of controversy amongst some mathematicians. Horgan (1993 : 74-82) pronounces the "death" of proof because, with computer assisted proofs, he argues that mathematicians will move away from deductive proofs. The invention of high-speed computers has certainly created a new awareness that proofs in mathematics may never be the same. In fact, some mathematicians are ill at ease about computer generated proofs (Kleiner and Hadar, 1997 : 22). A good example of this is Appel and Haken's proof of the Four Colour Theorem. The proof required 1200 hours of computing on three different computers (Mackernan, 1996 : 18). As Hersh (1993 : 393) states, "not everyone was overjoyed" with this proof. He quotes Halmos as follows: "*I do not find it easy to say what we learned from all that. We are still far from having a good proof of the Four-Colour Theorem. I hope as an article of faith that the computer missed the right concept and the right approach. 100 years from now the map theorem will be, I think, an exercise in a first –year graduate course, provable in a couple of pages by means of the appropriate concepts, which will be completely familiar by then*" (Hersh,1993:393).

Zeilberger (1994 : 11), a mathematician of some note, also states that : "*The writing is on the wall that, now that the silicon saviour has arrived, a new testament is going to be written ... . The computer has already started doing to mathematics what the* *telescope and microscope did to astronomy and biology. In the future, not all mathematicians will care about absolute certainty, since there will be so many exciting new facts to discover ...".*

A basic and very important aspect that needs to be considered is the fact that computers can perform large calculations and can provide us with very high levels of conviction, but it does not give explanations. Otte (1994 : 310) expanded on this idea when he said *"that a proof which does nothing but prove in the sense of mere verification must be unsatisfactory"*. Why? Because computer proofs cannot explain, generalize, nor can it "*enrich our intuition*" (Otte, 1994 : 310). Undoubtedly, computer proofs might establish a high level of certainty in a mathematical statement, but from previous computer proofs (like the Appel and Haken proof) it would seem that it does not remove the skepticism that you would find in a formal but traditional mathematical proof.

What is it about the computer generated proof that causes concern? Kleiner and Hadar (1997 : 22), have noted a comprehensive list of concerns which are listed below:

- "control over the subject must be shared with a foreign agent – machine".
- "mathematics seems to resemble an experimental science". For example in order to satisfy oneself about the proof, one needs to repeat the procedure with another computer. Often, verifying results on another computer seems impractical (Peterson, 1990 : 276).

- "computer proofs are not 'surveyable'" The actual computer programs are not published, so it would be difficult to verify the procedures it employs. Besides, we do not fully understand the physical processes by which the computer works (Hersh, 1993 : 393). Of greater significance, maybe, is that most computer generated proofs are far too long.
- "both computer hardware and software are subject to error".
- "while the accepting of a traditional mathematical proof is a social process, that of a computer proof is not". This argument is based on the fact that computer proofs cannot be read generally, unless of course, the reader understands the language of the computer. These proofs cannot be verified by the normal processes, understood and internalised, and used by other mathematicians.
- "the function of proof is – or should be – to enlighten the reader, in addition to validating the result which it purports to prove." Kleiner and Hadar argue that computer proofs fail in this respect. In other words, computer generated proofs do not serve the important function of explanation.

In addition to the concerns mentioned above by Kleiner and Hadar, the following concerns were expressed by Peterson (1990 : 276) :

- In general, it would seem that several people have a hand in writing out these computer programs. This would mean that there is greater chance of human error. In fact, in some cases, programs were written by people in different countries at different times. It might just be possible that one member of the team writing out these programs might have left and replaced by a new member, which would imply further that the continuous train of thought might be broken.

- The computations are generally done in bits and pieces, over a number of years, and requires large amounts of time. Again the possibilities of errors do exist, especially when these bits and pieces are brought together.

Another relevant and important argument against those that are suggesting that computer proof will make human proof obsolete is that computers have serious limitations with regard to general verification because they can only test for a finite number of cases.

Yehuda Rav (In Press : 1) argues that even if we had a supercomputer which would give us an answer to any mathematical question that we put to it, we would still need proof. His argument goes more or less as follows. Suppose there is such a computer, called PYTHIAGORA, which would not only answer any mathematical question but it would do so at the speed of light. This would mean that we could test the validity of any statement by simply checking whether it is true by asking PYTHIAGORA. The Riemann Hypothesis and the Fermat Theorem would have been deemed to be true or false a long time ago. Rav's argues that whether a statement is true or false is not as important as the actual process involved in searching for the truth or falsity. He states that "*proofs rather than the statement-form of theorems are the bearers of mathematical knowledge*" (Rav, In press : 18). In other words, greater knowledge is derived from the process of proving and the underlying meaning that is attributed to the proof, instead of simply knowing whether the statement is true or false. Rav (In press : 18), further emphasizes this when he stated that we should "think of proofs as a network of roads in a public transportation system, and regard statements of theorems as bus stops; the site of the stops is just a matter of convenience". The point, according to Rav, is that knowing whether a conjecture is true or not (as PYTHAGORIA might tell us) is not as worthwhile as knowing how to get there (why is it true or not?).

Some people believe that the computer should be regarded as a legitimate means of conducting proof, and it would seem that these computer proofs are here to stay. Moreover, since it is believed that computer proofs are no different from the traditional proof, they may be freer from man-made errors and liable to contain new machine errors.

1.5 Some important functions of proof

The following are some of the thoughts on proof that have already been expressed by different authors:

- "... proof is a discourse designed to convince or persuade or it is an argument or presentation of evidence that convinces or produces belief." (Exner and Rosskopf, 1970 : 197)
- " ... proof is ritual, and a celebration of the power of pure reason. Such an exercise in reassurance may be necessary in view of all the messes that clear thinking gets us into." (Davis and Hersh, 1981 : 151)

- "Proof is no more than the testing of the products of our intuition ... The sensible thing to do seems to be to admit that there is no such thing, generally, as absolute truth (proof) in mathematics, whatever the public may think." (Raymond L. Wilder(1944) as quoted by Kline, 1982 : 314)
- "Thus mathematical argument (proof) becomes a tool in the dialectic between what the mathematician suspects to be true and what the mathematician knows to be true." (Schoenfeld, 1985 : 173)
- "...proof is proof from premises". (Steiner, 1975 : 96-97)

A more precise definition of proof was given by Kitcher (1984 : 36) when he said that "*a proof in a system is a sequence of sentences in the language of the system such that each member of the sequence is either an axiom of the system or a sentence which results from previous members of the sequence in accordance with some rule of the system*". According to Lakatos (1986 : 155) also admitted that most students of the modern philosophy of mathematics would tend to instinctively define proof according to their formalist conception of mathematics, that is : "*Proof is a finite sequence of formulae of some given system, where each formula of the sequence is either an axiom of the system or a formula derived by a rule of the system from some of the preceding formula*".

With respect to the above definitions of proof, it is therefore necessary to analyse the different functions of proof. The model expounded by De Villiers (1990 : 18) expands on Bell's model (Bell, 1979 : 366) and it is therefore presented here in no particular order. According to this model, proof has the following functions :

- Verification
- Explanation or illumination
- Systematization
- Discovery
- Communication
- Self realization

1.5.1 **Proof as a means of verification / justification**

Earlier (refer to page 8), some aspects of verification was discussed. The following are a few examples of definitions of proof which are inclined towards the verification function of proof:

- Proof is " a logically sound piece of reasoning by which one mathematician could convince another of the truth of some assertion". (Devlin, 1988 : 148)
- Proof is that which attests the veracity or authenticity, the guarantee, the evidence, the process of verification of the accuracy of operations and reasonings ..." (Garnica, 1996 : 257)

This function of proof is concerned with the *truth* of a statement or proposition and is indeed a very important function of proof. The reason for emphasising this function is that sometimes intuition or strong experimental evidence can turn out to be misleading or wrong. Any evidence based on a finite number of test cases or examples is not good enough, because the one exception to the rule may be just outside the range of experimentation or computation (Peterson, 1990 : 14). An example of a well-known theorem, which was believed to be true for a long time, was Merten's Conjecture, which was first proposed by Thomas Stieltjies in 1885 and again by F. Merten in 1897. This conjecture "concerns the function *M(n)* defined on positive whole numbers *n* by setting *M(n)* to be the difference between the number of numbers less than *n* which are products of an even number of distinct primes and the number of numbers which are products of an odd number of distinct primes" (Devlin, 1985 : 30). This conjecture seemed to be correct according to the computations of Merten himself, von Sterneck and, Cohen and Dress. But in October 1983, using complex computer methods, Hermann te Riele and Andrew Odlyzko, finally disproved the conjecture that existed for about 86 years (Devlin, 1988 : 218).

Another useful example which shows that verification is indeed an important process is given by Rotman (1998 : 2). It involves perfect squares, which can be defined as an integer of the form a^2. He considers the statements $S(n)$, for $n > 1$ (n can also be equal to 1) : $S(n) : 991\, n^2 + 1$ is not a perfect square. This statement is true for many n values. In fact the smallest value of n for which $S(n)$ is false is

$$n = 12\ 055\ 735\ 790\ 331\ 359\ 447\ 442\ 538\ 767$$

which is approximately 1.2×10^{29}. Rotman (1998 : 3) goes further when he quotes a special case of Pell's equation (given a prime p, when there are integers m and n with $m^2 = pn^2 + 1$). A rather spectacular result is obtained when $p = 1\ 000\ 099$. The smallest n for which $1\ 000\ 099n^2 + 1$ is a perfect square has 1115 digits. On a lighter note, Rotman (1998 : 3) makes a very pertinent comment which is reproduced here : *"The latest scientific estimate of the age of the earth is 20 billion (20, 000, 000, 000) years, or about 7.3×10^{12} days, a number much smaller than 1.2×10^{29}, let alone*

10^{1115}. If, starting on the very first day, mankind had verified statement S(n) on the nth day, then there would be, today, as much evidence of the general truth of these statements as there is that the sun will rise tomorrow morning. And yet some statements S(n) are false!".

Rotman was, in the above quotation, attesting to the fact that empirical evidence alone is insufficient and verification is a vital function of proof.

Another very simple example, which conveys the idea, that verification is an essential function of proof is the expression $n^2 - n + 41$ which produces prime numbers when the numbers 1, 2, 3, ... is substituted for n. This is true for n up to 40, but when n = 41, the answer is not prime anymore. Similarly, the Chinese conjectured in 500 BC, that if $2^n - 2$, is divisible by n, then n must be prime. Substituting values for n will reveal that this conjecture seems to be true, because $2^3 - 2$ is divisible by 3, $2^4 - 2$ is not divisible by 4 and $2^5 - 2$ is divisible by 5. In fact, this conjecture holds true for all values up to $n = 340$, but was in 1819 discovered to be false for $n = 341$ (which is a composite number). This meant that this conjecture was not true in general!

Tymoczko (1986 : 126) argued that "the practice of mathematics is essentially the verification of rigorous proofs". Schoenfeld (Coe and Ruthven, 1994 : 42) argued that "mathematical argument becomes a tool in the dialectic between what the mathematician suspects to be true and what the mathematician knows to be true". This view of proof functions "as the last judgement, the final word before a problem is put to bed" (Hersh,1993 : 390). This really implies that proof is the only way to obtain conviction. De Villiers (1990 : 18) argues that this claim is distorted because "proof is not necessarily a prerequisite for conviction – conviction is far more frequently a prerequisite for proof". This explains why mathematicians spend many months attempting to find a formal proof for a specific conjecture – they must already be convinced of the truth of the conjecture. This means that mathematicians generally believe, that behind every conjecture there must be a sequence of comprehensible, irrefutable, logical arguments, which move from a hypothesis to a conclusion.

Almeida (1996 : 86) also states that "proof is about *reaching* rather than *following* conviction and where conviction may come from intuition or by weight of evidence".

The major argument that faith in a particular statement or proposition can **only** be attained by a rigorously formulated proof is false. In fact, a very high level of conviction can be reached even in the absence of proof. The Riemann Hypothesis is a good example. This hypothesis by Riemann is amongst the greatest unsolved problems in mathematics. The hypothesis concerns the roots of the zeta function – the complex numbers z at which the zeta function equals zero. Riemann conjectured that these roots lie on the line parallel to the imaginary axis and half a unit to the right of it (Davis and Hersh, 1990 : 363-364). The evidence supporting this conjecture is so strong that it has compelled belief despite the absence of a formal, rigorous proof. It has indeed been verified by calculations that the first 70 000 000 complex zeros do, in fact, satisfy Riemann's Hypothesis. More importantly, many profound theorems on the representation of numbers as sums of primes or in other interesting forms, which have been independently proved, can be deduced from the Riemann Hypothesis and other similar assumptions (Bell, 1945 : 315). This strongly suggests the truth of the Riemann Hypothesis.

De Villiers (1995 : 155) has argued that in "mathematical research , proof is not an *absolute prerequisite* for conviction. To the contrary, it can be argued that some form of *priori* conviction is probably far more frequently a prerequisite for the finding of proof, than the other way around". It therefore seems clear that conviction is not attained by proof alone.

Perhaps, it should be stated that "empirical evidence" within dynamic geometry tends to be more convincing than the "empirical evidence" within number theory. This stems from the simple notion that it is impossible to test a conjecture for every possible number in the latter. A conjecture may be true for a very large number, but no guarantee exists that there is no other larger number, which disproves the theory. Whilst, on the other hand, the former allows for the dragging of a diagram to almost any position within a few minutes. Anything contrary to the conjecture could be quite easily detected. This is due to the fact that the variables used in the latter are discrete, whilst the variables used in the former are continuous (De Villiers, 1998 : 373).

1.5.2 <u>**Proof as a means of explanation**</u>

The following is a definition, which conveys the meaning of proof as a function of explanation:

- "By proof is meant a deductively valid, rationally compelling argument which shows why this *must* be so ...". (Mary Tiles, 1991 : 7)

This function of proof helps the individual make sense of a mathematical result and to satisfy the individual's curiosity as to *why* it may be true. This function of proof has been neglected because proof has been seen as performing only the function of verification. Coe and Ruthven (1994 : 42) claim that less emphasis has been placed on explanation because much writing about proof "has been from a philosophical rather than a pedagogical perspective". But Hanna (1996 : 16) states that "with today's stress on 'meaningful' mathematics, teachers are being encouraged to focus on the explanation of mathematical concepts ...". Wittmann (1996 : 16) quotes David Gale as saying that "*the main goal of all science is to first observe and then to explain. In mathematics the explanation* **is** *the proof*"(bold added). Schoenfeld (1985 : 172) demonstrated this important function of proof quite succinctly when he states that: " *'Prove it to me' comes to mean 'explain to me why it is true', and argumentation* (proof) *becomes a form of explanation, a means of conveying understanding*".

Although it is quite possible to achieve a high level of conviction that a conjecture holds true by experimentation, this does not provide a deeper understanding as to *why* the conjecture may be true (De Villiers,1990 : 19). Experimentation, especially if it is computer driven, may provide a large degree of certainty but it does not necessarily provide the insight or understanding of how the result may be true as a consequence of other already established results. Hersh (1993 : 396) states that "what proof should do for the student is provide insight into why the theorem is true" and at the high-school level "the primary role of proof is explanation"(1993 : 398).

Johnston Anderson (1996 : 32-38) appropriately summarises the explanatory role of proof in order to establish a deeper understanding of why certain results always hold true : "*Proof should be seen as being about explaining, albeit carefully and precisely. It is where instrumental understanding gives way to relational understanding. It should be seen as the essence of mathematics and all pupils who study mathematics should meet it at some time, at some level.*" Slomson also (1996 : 12) expressed the idea that "*good proofs not only convince us of the truth of mathematical statements, but also helps us to understand what is going on*".

The following two quotations also emphasize that proof as a means of explanation plays an important role in mathematics:

- "The mathematician's reaction shows quite clearly that a proof which does nothing but prove in the sense of mere verification must be unsatisfactory. A proof is also expected to generalize, to enrich our intuition, to conquer new objects, on which our mind may subsist". (Otte (1994 : 310))
- "the functions of proof are to generate knew knowledge and to *advance mathematical understanding*"(my emphasis).(Kitcher (1984 : 189))

This function of proof deserves greater emphasis than it seems to have been given at this point in time. Hanna (1996 : 135) supports this as follows: "*The best proof, even in the eyes of practicing mathematicians, is one that not only establishes the truth of a theorem but also helps understand it. Such a proof is also more persuasive and more likely to be accepted*".

1.5.3 **Proof as a means of systematization**

These are some of the ideas that have been expressed about systematization as a function of proof:

- "*All mathematical proofs must be deductive.* Each proof is a chain of deductive arguments, each of which has its premises and conclusion." (Kline , 1962 : 42)
- Proof is "the logical organisation of a body of mathematical knowledge". (Kline, 1968 : 2)
- "A proof is a directed tree of statements, connected by implications, whose end point is the conclusion and whose starting points are either in the data or are generally agreed facts or principles." (Bell, 1976 : 26)

This function of proof is concerned with the logical organisation of propositions into a deductive system. In fact this aspect deals with the logical structure of the actual reasoning involved in the formal proof – the writing down of ideas in a logical sequence. The focus is on making logical connections between statements; statements that may already be known to be true (by local proof) or assumed to be true. Lakatos (1986 : 167) stated that "certain statements are derivable from other statements by means of 'pure reason', and a corpus of connected material can be built from a few fundamental statements known as axioms".

Often when we prove we assume categorically the existence of some previously defined knowledge, namely definitions, axioms and theorems. Definitions are "generalizations found in mathematical systems" (Travers, et al, 1977 : 80). These generalizations are related to other concepts in a particular hierarchy. Skemp cited in Floyd (1981 : 82) states that: "Definitions can thus be seen as a way of adding precision to the boundaries of a concept, once formed; and of stating explicitly its relation to other concepts". Axioms are statements, the results of which are accepted without proof. Axiomatization plays an important role in proving. In descriptive ("a posteriori") axiomatization (De Villiers, 1986 : 3), certain axioms, which are already arranged in some hierarchical system, is selected for the process of writing down a proof. The axioms are arranged in a particular, logical way which taking into account the relatedness of all the statements. Theorems are also generalizations in mathematical systems. Theorems are general statements which can be shown to be logical consequences of the axioms, definitions, and previously proven theorems in a mathematical system (Travers, et al, 1977 : 81).

Systematizing, according to Coe and Ruthven (1994:42), "*lies in the fact that logical structure is concerned with formal, explicit arguments, publicly agreed and conforming to standard conventions*". Bell (1979 : 366) also believes that proof serving the function of systematizing is the process of collecting a set of known results and then organising it into a hierarchical deductive sequence involving the choice of suitable starting points as axioms. Mary Tiles also emphasizes the idea of "natural order". She takes this idea a step further by saying "the point of axiomatization and the provision of proofs either directly from axioms or by reference to previously proved theorems is that a rational order is imposed" (Tiles, 1991 : 17).

Clearly, the order that she wrote of, should reflect the way in which things are learnt and understood. Otte (1994 : 314) described mathematical proof as "deriving new theorems from those already known …".

According to De Villiers (1990 : 20) proof is an essential tool in the systematization of known facts into a deductive system of axioms, definitions and theorems. But the production of conviction is not the prime reason for this formal structuring of proof (Exner and Rosskopf, 1970 : 197). De Villiers (1990:20) provides an extensive list of important functions of a deductive systematization, which are now reproduced here:

- It helps with the identification of inconsistencies, circular arguments and hidden or not explicitly stated assumptions
- It unifies and simplifies mathematical theories by integrating unrelated statements, theorems and concepts with one another, thus leading to an economical presentation of results
- It provides a useful global perspective or bird's eye view of a topic by exposing the underlying axiomatic structure of that topic from which all the other properties may be derived
- It is helpful for applications both within and outside mathematics, since it aids checking for applicability of a whole complex structure or theory simply by evaluating the suitability of its axioms and definitions
- It often leads to alternate deductive systems which provide new perspectives and/or are more economical, elegant and powerful than existing ones

Michael Otte (1994 : 299) sums up the systematization function of proof by stating that proofs "... connect propositions into longer chains of reasoning. Proofing (sic) may then become an exercise in the correct arrangements of propositions ...". So proving, it seems, achieves a coherent organisation although the value of this logical pattern is grossly underrated. As Kitcher (1984 : 218) states that "*to be a proof is to be a member of a system of reasonings serving two functions : (a) providing optimal generation of new knowledge from old and (b) providing increased understanding of statements previously accepted*".

1.5.4 **Proof as a means of discovery**

In criticizing formalism in mathematics, the claim is often made that new mathematical results are always discovered by means of intuition and/or quasi-empirical methods before they are verified by the production of proofs. For example, Hanna (as quoted in De Villiers, 1990 : 21) claims that "mathematical concepts and propositions are ... conceived and formulated before proofs are put in place". Even Steiner (1975 : 100) echoed a similar sentiment when he stated that : " On the spot he (the mathematician) may discover important premises as yet unproved and prove them". Paul Halmos (quoted in De Villiers (34 : 1996)) similarly emphasizes the precedence of discovery over proof as follows: "The mathematician at work ... arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof."

Peterson (1990 : 16), however, makes it clear as follows that discoveries are often not first made empirically, but can occur quite unintentionally during proof : "*Often, the* *obsession with proof is itself an important source of new ideas and mathematical methods. Efforts to prove that closed curves divide space into an outside and an inside led to the new mathematical field of algebraic topology, a central topic in modern mathematics. It's unlikely that any attack on a particular practical problem would have led to such novel abstract ideas".* Even Alexander Graham Bell did not set out to invent the telephone. All he was trying to do was invent something to help the hard of hearing. Similarly, when Thomas Edison invented the phonograph, all he was trying to do was develop something that might record telephone conversations. So, it can be seen that even in general life discoveries are made without any explicit intention to do so.

Yehuda Rav (In press) demonstrates the idea of discovery by taking the example of Christian Goldbach's conjecture which states that every even integer greater than 6 can be represented as the sum of two distinct odd primes. Rav (3) states that whether the conjecture turns out to be true or not is of no "theoretical or practical importance", because of the immense discoveries the attempts at solving this conjecture has yielded. Jean Merlin in 1911 thought he had proved Goldbach's conjecture and another famous problem, the twin-prime conjecture. He, in fact, outlined a sieve method, which generalised the sieve of Eratosthenes. His proofs turned out to be invalid, but it led to the invention of the Merlin sieve method, which is today used in number theory. This method has developed to such an extent that it has become a subject in its own right (Rav : In press, 4). Besides the sieve methods that were developed, other mathematicians, like L. Schnirelmann in 1930, achieved new and important discoveries while attempting to prove it. The Goldbach conjecture has in fact acted like a catalyst for new discoveries. It can be conjectured that, in the absence of the Goldbach conjecture, many relevant theories in mathematics might not have surfaced.

Another way proof can often lead to new discoveries is that when proving a result, one discovers that certain conditions were not necessary, thus leading to an immediate generalization. An example of this is given in De Villiers (1997) with reference to the following result:

"If ABCD is an equilic quadrilateral and equilateral triangles are drawn on AC, DC and DB, away from AB, then the three new vertices, P, Q and R are collinear" (refer to the Figure 1). An equilic quadrilateral is a quadrilateral ABCD having AD = BC and where these sides are inclined at 60^0 to each other. Through further experimentation, De Villiers discovered that the result was also true for any quadrilateral ABCD with AD = BC, if similar triangles were constructed on AC, DC and DB and angle APC = angle ASB (refer to Figure 2). After proving his result, De Villiers noticed that he had never used the property that AD = BC in the proof. This meant that the result could be further generalized to any quadrilateral.

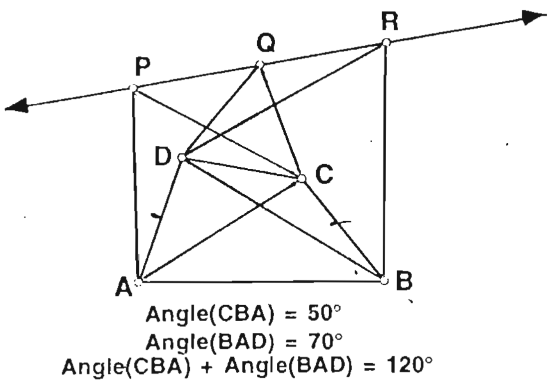

Figure 1

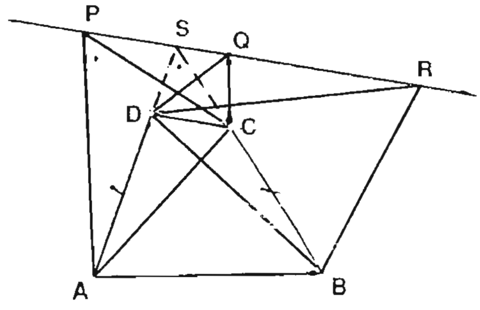

Figure 2

It is also true that purely "logical analysis" can lead to simultaneous "discovery" and "proof". Figure 3 represents a quadrilateral, which has its sides tangential to a circle centre A. KC = KE = x, EH = HG = n, GI = IF = m, and JF = JC = y. It can easily be seen that the sum of the opposite pairs of sides are equal, that is, JK + HI = KH + JI, because JK + HI = x + y + n + m and KH + JI = x + y + n + m. Furthermore, the above serves as an explanation (proof) for the observation.

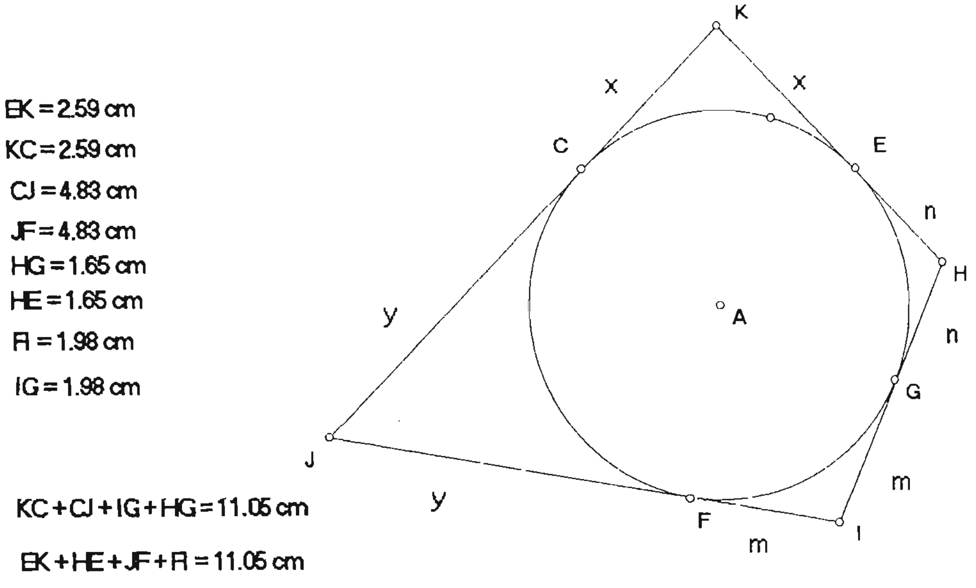

Figure 3

1.5.5 **Proof as communication**

Mathematical proof has afforded mathematicians ample opportunities to communicate with each other. In fact, it might be the aspect of communicating with other people that makes proof a human activity. Many books relate the letters written by great mathematicians (Poincare; Wittgenstein; Hardy and many more) to their colleagues communicating a new proof, or even the inability to prove a statement. Without doubt proof creates an ideal forum for healthy, critical debate because it has become a source of communication between the broader mathematical community. This is what inspired Tymoczko (1986 : 127), in an editorial, to say that verification of proofs is a public affair, an elaborate social process that proceeds by the canons and paradigms of a particular community of experts". He goes on further to say that the "verification of proofs would involve such factors as the dissemination of results through a community".

De Villiers (1990 : 22) also relates a view which indicates that proof "is a unique way of communicating mathematical results between professional mathematicians, between lecturers and students, between teachers and pupils, and among students and pupils themselves". So it is clear that proof is a social activity which involves gathering, reporting and disseminating knowledge.

Of greater significance though, must be the tremendous communication that exists between many people that are working on the same proof (example Appel and Haken and associates). In the case of the Four-Colour Theorem, which Appel and Haken together with a team of mathematicians proved using computers, it was clear that a vast communication network existed between them because of the huge amount of work that had to be done in order to achieve the proof.

There is also a similar type of communication that exists between professional mathematicians, resulting in the discovery of proofs. Often mathematicians themselves, in attempting to prove a theorem, unwittingly set up a communication between themselves and other mathematicians, which results in a proof. The Bieberbach Conjecture (as quoted by Devlin, 1985 : 31) is a good example. In 1916, Ludwig Bieberbach, wrote a paper in which he proved that if a complex function of the form $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ is such that no two values of z of absolute values less than 1 give the same value of $f(z)$, then the absolute value of a_2 is at most 2. In a footnote Bieberbach conjectured that for all values of n, the coefficient of a_n of z^n will be at most n in absolute value. Many mathematicians worked on this conjecture. The following mathematicians (Devlin, 1985 : 31) proved certain aspects of the proof : Jean Dieudonne in 1931, C. Loewner in 1923, Z. Charzynski and M. Schiffer in 1960, M. Ozawa in 1969, R. Pederson and M. Schiffer in 1972 and finally Louis De Branges in 1984.

It appears that proof is a form of social interaction, which involves communication between mathematicians, either directly or independently. The value of proof as a means of communication is emphasized as follows by De Villiers (1990 : 22) : "... such a social filtration of a proof in various communications contributes to its refinement and the identification of errors, as well as sometimes to its rejection by the discovery of a counter-example".

1.5.6 **Proof as a means of self-realization**

This is an aesthetic function of proof and is very important because it deals with exactly what the human mind feels satisfied with. Although most mathematicians know that a proof will benefit many others, the inner joy and personal satisfaction at discovering a proof is the main intrinsic motivating factor. As Davis and Hersh (1990 : 369) state : *"Perhaps, though, there is another purpose to proof – as a testing ground for the stamina and ingenuity of the mathematician. We admire the conqueror* *of Everest, not because the top of Everest is a place we want to be, but just because it is so hard to get there*". This may just explain why a mathematician may work with a problem for many years – the thought of doing what no one else has done before. The following extract may serve to further emphasize this point. Klaus Barner (1997 : 1294) asked Andrew Wiles (Wiles is accredited with the proof of the Fermat conjecture) what was it about the Fermat conjecture that fascinated him ? Wiles initially responded that that it was the "romantic history" of the problem, which drew his attention (at the age of 11!). When probed further he responded that *"because Fermat said he had a proof, but none was found"*. It is exactly this intellectual challenge which drives mathematicians to find new frontiers in mathematical proofs.

De Villiers (1997) experienced this intellectual challenge whilst working with Van Aubel's theorem, which states that the centres of squares on the sides of any quadrilateral *ABCD* form a quadrilateral *EFGH* with equal and perpendicular diagonals (refer to figure 4 below).

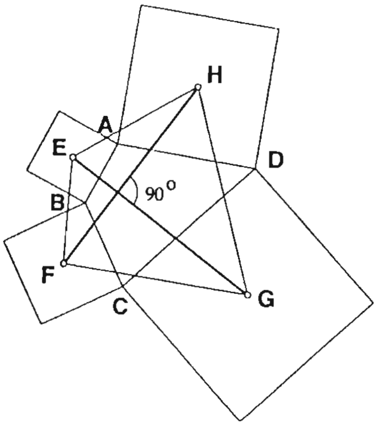

Figure 4

He discovered two more generalizations of Van Aubel's theorem, using the dynamic geometry program Cabri, namely:

1. If similar *rectangles* are constructed on the sides of any quadrilateral as shown in figure 5 below, then the centres of these rectangles form a quadrilateral with *perpendicular* diagonals.
2. If similar *rhombi* are constructed on the sides of any quadrilateral as shown in figure 6 below, then the centres of these rhombi form a quadrilateral with *equal* diagonals.

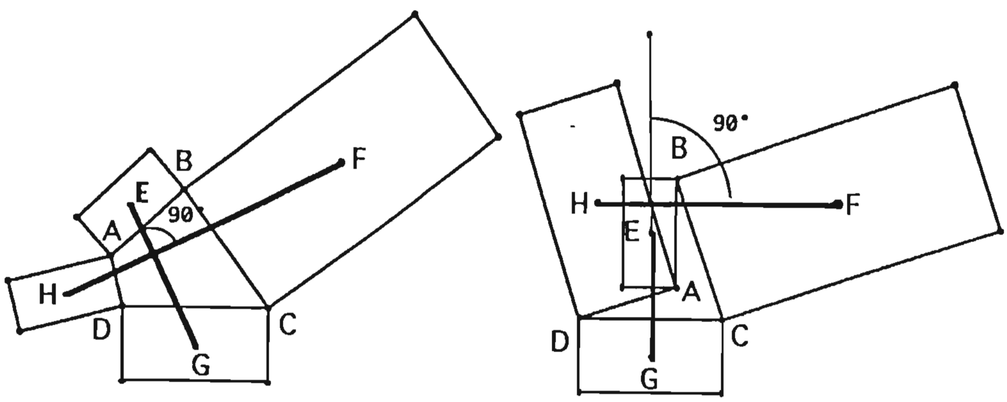

Figure 5

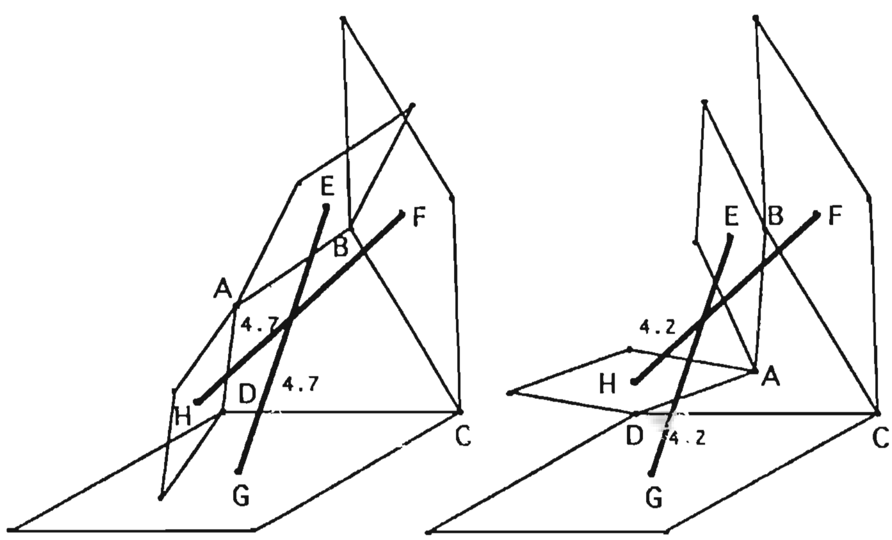

Figure 6

De Villiers was quite convinced that the above statements were correct and further convinced himself by using the property checker of *Cabri* to confirm his results. This property checker, tests the statement to check whether it would always be correct – if the statement is correct then it would state that it is "*true in general*". If it is false, then the property checker produces a counter-example. After receiving confirmation of the truth of both of these generalizations, De Villiers (1997 : 17) described his feelings as follows: "*As was the case previously, I did not really experience a need for further certainty, but rather of* ***explanation*** *(why were they true?) and of* ***intellectual challenge*** *(can I prove them?).*"

It is this very same intellectual challenge and aesthetic emotion which Poincare spoke about when he said that constructing proofs is a "satisfaction of our needs" (Kline : 17). In fact, Poincare went on to state that the person lacking this "aesthetic sensibility will never be a real creator". Very often the satisfaction of this aesthetic need may only come after many attempts to find a solution, and it is then even more satisfying. Alfred Adler (1984 : 9-10) emphasizes this point when he wrote : "*A new mathematical result, entirely new, never before conjectured or understood by anyone, nursed from the first tentative hypothesis through labyrinths of false attempted proofs, wrong approaches, unpromising directions, and months or years of difficult and delicate work - there is nothing, or almost nothing, in the world that can bring a joy and a sense of power and tranquility to equal those of its creator*".

Davis and Hersh (1984 : 250) described this beauty and power of mathematical proof quite clearly when they wrote : "*A shudder might even run down our spines if we believe that with a few magic lines of proof we have compelled all the right triangles* *in the universe to behave in a regular Pythagorean fashion*". This is the beauty of a mathematical proof.

This aesthetic desire is so pervasive that it often compels mathematicians to carefully re-examine already proven results, particularly if they do not have elegant proofs. Kline (as quoted by Peterson, 1990 : 288) states in this regard that "*much research for new proofs of theorems already correctly established is undertaken simply because the existing proofs have no aesthetic appeal*".

Mathematicians often persevere for many years in attempting to prove new or existing theorems. However, there are very few material benefits for discovering a proof. There is no, or very little monetary gain, there is no Nobel prize for mathematics[1], and very little public recognition (mainly amongst other serious mathematicians) is accorded to great mathematical achievers. So that which spurs them on to find satisfactory mathematical arguments can only be attributed to the intellectual challenge that they experience and the tremendous beauty they discover in a proof. This is also true for those who discover new proofs for old theorems.

[1] The highest award for research achievement in mathematics is the Field's Prize, which comprises a relatively small sum of money.

CHAPTER TWO

PROOF IN SCHOOL MATHEMATICS

2.1 <u>The general response</u>

With the present school mathematics curricula set out as it is, pupils are required to begin proving at Grade 10 (in some schools it begins at the Grade 9 level). A quick perusal of literature based on proof at the secondary school level will reveal that pupils experience many problems with regard to the need for proof. De Villiers (1990 : 17) states that *"the problems that pupils have with perceiving a need for proof is well-known to all high school teachers and is identified without exception in all educational research as a major problem in the teaching of proof"*. In fact many teachers are asked the same question by pupils all the time: *Why do I have to prove ?*

Gonobolin (1954 : 61), for example, found that pupils do not recognise the need for proof of geometric theorems especially if the proofs are visually obvious or it can be established empirically. Senk (1989 : 309) also refers to a study which revealed that *"although teaching students to write proofs has been an important goal of the geometry curriculum for the college bound in the United States for more than a century, contemporary students rank doing proofs in geometry among the least important , most disliked, and most difficult of school mathematics topics"*. Data collected by the Cognitive Development and Achievement in Secondary School Geometry project confirmed that writing proofs was indeed a difficult task for most students (Senk, 1989 : 309).

Driscoll (1988 : 21) emphasized that proof is difficult, since it requires a high level of cognitive activity: "*A formal mathematics proof is a complex cognitive task. ... These are taxing, if not futile, demands on pre-formal students and there are many such students in the ninth and tenth grades, where formal proof is a frequent objective in the curriculum*". Due to the cognitive complexity of proof, many pupils simply resort to memorization, as explained by Jones (1996 : 235): "*For many children it* (proof) *became an exercise of memory; facts and processes to be learnt and reproduced*".

There is a growing body of knowledge which indicates that many students do not understand proof, for example:

- A study by Suydam (1985 : 483) showed that about 50% of pupils saw no need to prove what they considered obvious.
- Senk (1985 : 454) found that only 30% of pupils attained 70% mastery on six geometry problems involving proofs.
- Usiskin (1982) also determined that although 50% of secondary school graduates completed a year of geometry less than 15 % mastered proof writing.
- Bell (1976 : 23) carried out an investigation of 160 grammar school girls and discovered that only 10 % of them could give an acceptably complete, deductive argument (proof).
- Reynolds (Bell, 1979 : 370) studied the "proof concepts of grammar school pupils and concluded that, in general, formal axiomatic proof was not understood even by 17-year old pupils specialising in mathematical and scientific subjects".

- Williams (Driscoll, 1988 :156) surveyed eleventh grade pupils and found that less than 30 % exhibited any understanding of the meaning of proof, and that almost 60 % were unwilling to argue, for the sake of argument, from any hypothesis they considered false.

These statistics reveal that the current instruction processes of mathematics proof are inadequate and are quite concisely summed up by Retzer (1996 : 60) when he stated that *"proof making is one of the most dreaded activities in American mathematics. Students often ask for assurance that this will not appear on an upcoming examination".*

From an educational perspective, pupils' poor performance in geometry examinations could be attributed to the cognitive difficulty in proving the geometry riders and theorems in those papers. This poor performance of pupils in geometry riders and theorems might also be attributed to the following underlying reasons:

- pupils' may be at the inappropriate Van Hiele level (De Villiers, 1996:6) in order to attempt proof. Van Dormolen (1977 : 32) writes as follows in regard to the Van Hiele theory (see page 48) : *"Now, in mathematics education it is essential that the teacher bears in mind that it is impossible to operate on the higher level if the lower one has not yet been reached".*
- inadequate traditional teaching strategies that are employed when proof is taught. It would seem that insufficient emphasis is placed on proof heuristics with most teachers preferring the direct presentation of proof.

- Negative attitude, brought about by the pupils not understanding the role or meaning of proof when the concept of proof was initially introduced.

Some authors like Mackernan (1996 : 18) believe that proof should not feature in the mathematics curriculum at all since pupils have such difficulty with it. He believes that anything taught at school must be both enjoyable and useful, and proof (the kind that requires a certain amount of rigor) according to him, is neither useful nor enjoyable. His argument is based on the fact that discovering of patterns is enjoyable but he questions the attempt to deduce a formal proof for this pattern. John Searle (1996:21) similarly questioned the traditional approach to teaching of proof at school, because "it has not been demonstrated beyond reasonable doubt that the rote learning of mathematical proof inculcates either logical thought, technical fluency or mathematical insight".

Dave Hewitt (1996:27) counters this by insisting that proof should be in the curriculum, mainly because students learn about properties and that "proof is not only about properties but is also about an awareness of properties". This means that if a pupil has had the opportunity to explore the relationship between the angles at the centre and the angles at the circumference of a circle, via measurement in a number of cases, then the pupil is aware that the angle at the centre is twice the angle at the circumference. In fact the pupil might be quite convinced that the result will hold true for all circles. According to Mackernan, this should be sufficient for the pupil, and there is no need for further proof. However, Hewitt speaks about bringing about an **understanding** of the problem. Learning proof through rigor means that it must be accepted on trust, without an "awareness" of the properties through experimentation and exploration, whilst learning proof with an "awareness" of the properties gives insight and therefore leads to a better understanding of the problem. According to Hewitt (1996:31) : "*Attending to proof can force a student to examine what they do know and what they do not know about something. This forces attention onto properties, and it is not unusual for someone to learn a number of new things on the way to a proof*". In other words, in order to understand a result (and its proof) one needs to first understand the properties involved. If, for example, the teacher presents a proof based on properties which the pupil is not aware of then the pupil may have a difficulty in understanding it. So, it is quite obvious that every pupil must be aware of all properties involved in a proof.

The concern with the present state of affairs with regard to proof was expressed by Greeno (as quoted by Hanna,1996:1) : "*Regarding educational practice, I am alarmed by what appears to be a trend towards making proof disappear from pre-college mathematics education, and I believe that this could be remedied by a more adequate theoretical account of the epistemological significance of proof in mathematics*". Greeno criticizes viewpoints, like that of Mackernan, and he pleads for a better theoretical account of the role of proof.

Although proof has been relegated to a less prominent role (Hanna,1996 : 1) in the secondary school mathematics curricula in some countries, like the United Kingdom during the past 10 years or so, it is being re-established again. However, the role of proof appears to be no longer seen simply as a means of simple verification, but rather has far greater value in mathematics education (for example, involving explanation, discovery, and so on). It would be appropriate to look at proof, which is motivated by prior personal conviction due to experimentation by the pupils themselves. Thus, proof becomes a means of helping the child make sense of a mathematical result and to satisfy the child's curiosity as to *why* it may be true. Hanna (1996 : 16) stated that "with today's stress on 'meaningful' mathematics, teachers are being encouraged to focus on the explanation of mathematical concepts........". Hersh (1993 : 396) states that "what proof should do for the student is provide insight into why the theorem is true" and at the high-school level "the primary role of proof is explanation"(1993 : 398).

In an experiment carried out by Zack (1997 : 291-297), fifth grade children were given a task of counting squares of ANY size, where the squares resembled a chessboard pattern. The first was a 5 by 5 grid and was quite easily done by them. Further questions were posed, where the children were asked to count the squares in a 10 by 10 grid and eventually they were asked to count the squares in a 60 by 60 grid. It was quite evident that some of the children were able to establish patterns, make conjectures and then test these conjectures. This enabled Zack (1997:4-296) to conclude that even with grade five children (ten and eleven year old children) "there is evidence of conviction prior to proving; their arguments are based upon their conviction *that* their pattern works in all instances." She finally concluded that the children displayed strong evidence of the need for an explanation. In fact, despite finding one pupil's result very good, the majority of the pupils insisted on finding out *why* the result worked as it did.

Slomson (1996 : 13) made the following useful points in regard to the teaching of proof:

- "There has never been a golden age when all pupils left school knowing about mathematical proofs". This statement is in line with the generally held notion that very few pupils are in fact successful with mathematical proofs.
- "The logical structure of mathematics is one of its most attractive features, and it might be that by playing down the role of proof we are failing to attract potentially talented pupils to becoming mathematicians".
- "Pupils can reasonably be expected to investigate problems and conjecture results, when it would be unreasonable to expect proofs. (It is the nature of good proofs, like good music or good novels, that far fewer people are capable of creating them than of appreciating them.) Unfortunately, this can lead to two wrong ideas. First, that once you have spotted the pattern, the problem is solved. Second, the way pupils are encouraged to investigate problems often gives a wrong idea about how mathematicians think".
- "It is unreasonable to expect anyone to come up with proofs at the end of investigations unless they have been shown in a didactic fashion lots of examples of proofs which they can use as models for their own attempts".

Perhaps the following quote by Hanna (1996 : 12) adequately summarises the need for pupils to prove in mathematics : "*With today's stress on teaching 'meaningful' mathematics, teachers are being encouraged to focus on the explanation of mathematical concepts and students are being asked to justify their findings and assertions. This would seem to be the right climate to make the most of proof as an explanatory tool, as well as to exercise it in its role as the ultimate form of mathematical justification. But for this to succeed, students must be made familiar* *with the standards of mathematical argumentation; in other words,* ***they must be taught proof***" (my emphasis). Similarly Anderson (1996 : 34) expressed his feelings for the need for proof: "*This places proof at the very heart of the mathematical experience and therefore, if we wish to convey to pupils something of what mathematics is really about, then we do them (and ourselves) a disservice if we exclude it from the curriculum*".

Hoyles (1996 : 59) also acknowledges that proof is essential at the secondary school stage, but points out that: "*... if formal proof is presented only as a way to demonstrate something that students are already convinced is true, it is likely to remain a meaningless activity. The challenge to mathematics educators is to widen the notion of proof and to build connections between its diverse aspects*". A similar view is expressed by Goldenberg (1996 : 184) when stating that "*we often hear of negative associations that students have with proof : the game seems to be played with a distrustful or unwarrantedly skeptical nature, or requires one to engage in an empty, post-hoc, proof ritual even though one is already fully convinced of the truth of the statement*".

2.2 Previous research based on pupils' need for conviction and explanation within the context of geometry.

Relevant research was carried out by De Villiers in 1990 and 1991. As part of the initial study, in 1990, high school pupils were asked to judge 42 geometry theorems according to the following set of codes :

Code 1 : Believe it is true from own conviction;

Code 2 : Believe it is true because it appears in the textbook or because the teacher said so;

Code 3 : Do not know whether it is true or not;

Code 4 : Do not think it is true;

Code 0 : Unanswered;

This study revealed that between 50% and 70% of the pupils based their conviction on authoritarian grounds, that is, Code 2, rather than on personal conviction. In order to verify these findings, De Villiers conducted further research in township schools in the Durban area in 1991.

The aims of the investigation were to try to establish :

- which geometric statements the children were convinced about and the reasons for that conviction
- which geometric statements the children found doubtful or false, and the reasons for their views in this respect

The investigation was based on the hypotheses that :

- the majority of pupils would base their conviction of the truth of the given statements on the authority of the teacher and / or textbook rather than personal conviction.

- the majority of children would not easily distinguish false statements on their own, but will be dependent on the authority of the teacher and / or textbook for this distinction.

The investigation yielded some significant results. Code 2 responses were in virtually all 42 cases, on average two or three times greater than Code 1 responses. This allowed De Villiers (1991 : 22) to draw the following conclusion: *"The certainty or conviction of the majority of pupils with respect to prescribed statements presently seem to be based on authoritarian grounds rather than on personal conviction.*

De Villiers (1991:22) attributes this to the fact that there is a dominance of the traditional approach which really involves the imposition of the teacher's ideas instead of an investigative approach, which involves conjecturing, testing of the conjecture, refining of the conjecture, understanding and, finally, justification. In similar but separate studies (De Villiers, 1991:22), Smith in 1986 and De Villiers & Njisane in 1987, it was shown that of the 1959 standard 7 to standard 10 pupils who were interviewed, 88% were certain of the truth of the statements that were presented to them, yet only 7% indicated that they were certain because the statements could be proved. This motivated De Villiers (1991:22) to draw another conclusion: *"Only for a minority of the pupils, proofs seems to have the function of conviction / justification"*.

The following results were also obtained by De Villiers (1991 : 23) in a teaching experiment in 1987 in which standard 7 pupils were involved as well as further interviews with standard 6 to standard 10 pupils

1. After initially discovering the conjecture, *"If the midpoints of the adjacent sides of a quadrilateral ABCD are consecutively connected, then a parallelogram EFGH is formed", by construction and measurement,* 94 % of the standard 7 pupils spontaneously indicated that further quasi-empirical testing would satisfy their need for certainty.
2. 8 out of 11 standard 6 to standard 10 pupils interviewed, also spontaneously obtained certainty with respect to the above conjecture by means of construction and measurement of a number of different quadrilaterals.

This motivated De Villiers (1991 : 24-25) to make the following conclusion:

- *"the majority of pupils spontaneously choose to satisfy their need for personal conviction in new and unknown situations by quasi-empirical testing"*

A significant result that was obtained was that "despite displaying no further need for deductive verification, the 3 pupils who had used construction and measurement with respect to the given isosceles trapezium, still exhibited a need for explanation which had not been satisfied by their quasi-empirical approach" (De Villiers, 1991 : 24). This lead to the following conclusion:

- *"pupils who have convinced themselves by quasi-empirical testing still exhibit a need for explanation, which seems to be satisfied by some sort of informal or formal logico-deductive arguments".*

Another important aspect was that the given logico-deductive explanation appeared to increase their confidence/certainty in the statement (De Villiers, 1991 : 25). This teaching experiment showed that although pupils were already quite satisfied about the truth of the statements after experimentation, they still displayed a need for further explanation.

Other significant research was done by Zack (1997 : 291-298) in her fifth grade class (refer to details on pages 42 and 43). She finally concluded (1996 : 296) that "there is evidence of conviction prior to proving; their arguments are based upon their conviction *that* their pattern works in all instances". Her findings (1996 : 296) further suggests "that the students who succeed in convincing their peers are those whose justifications are based upon the generalizations". Of great importance is the fact that her students emphasized that their criteria for proof included (1996 : 297) :

1. a need for evidence,
2. that the proof must make sense, and
3. the person presenting must say why it works.

For example, her students responded as follows to Johnston Anderson's (1996 : 296) rule ([n(n+1)(2n+1)/6]):

Ross stated : "brilliant, but he should state why it works".

Lew stated : "I think that if the Johnston Anderson's rule had evidence, if Johnston himself explained why it worked it would be more convincing".

Rina stated : Johnston's expression was " a great way to figure out the problem but it doesn't make sense ... I think a mathematical proof is when you say it works and if it works for everything show why".

From the above, it is clear that her students displayed a need to know *why*, in other words, a need for *explanation*, despite being convinced of the validity of the rule.

2.3 **The Van Hiele Model of geometric thought**

Most pupils will never encounter geometry proof before they reach secondary school. Strangely enough, it seems that even when they reach the secondary school phase they still have problems with understanding and writing proof. This, according to Dina van Hiele-Geldof and her husband, Pierre Marie van Hiele, is due to the pupils' level of geometric maturity. Through extensive research they were able to posit the existence of five distinct levels of geometric maturity. All these levels describe the thinking process, which when assisted by the appropriate instructional strategies, allows the learner to move sequentially from the most basic stage (visualization or recognition) to the final stage of rigor (Crowley, 1987 : 1). According to the theory, pupils pass through these levels in consecutive order, but not all pupils pass through these levels at the same rate. The levels as proposed by the van Hieles are as follows:

Level 1 : (Basic level) Recognition or Visualization

At this basic level the pupil is aware of space and has a knowledge of a certain basic vocabulary. The pupil can recognize, for example, a square but will not be able to list any properties of the square. So, in other words, the child recognizes specified shapes holistically, but not by its properties.

Level 2 : Analysis

At this level pupils begin to understand that the shapes that they are working with, through observation and playing around with it, has certain properties. The child can now see that the square is made up of all equal sides, or the rectangle has opposite sides equal. But the child still cannot find the relevant links between the different figures, for example, the relationship between a square and a rectangle.

Level 3 : Ordering or Informal deduction

At this level, pupils discern the relationships between and within geometric figures. For example, pupils can conclude that if the opposite sides of a quadrilateral are parallel then the quadrilateral is a parallelogram or that a square is a rectangle. So, at this level they can determine the characteristics of an entire class of figures, for example, quadrilaterals. At this level pupils cannot employ deductive strategies to solve geometric problems. They may be able to follow a proof but may not themselves be able to prove.

Level 4 : Deduction

At this level, the pupil understands the significance of deduction as a means of solving geometric problems. They also understand the role of axioms, postulates, definitions and theorems. They can use their knowledge to construct a proof of a statement. In fact, the relationship between a statement and its converse is also understood.

Level 5 : Rigor

The least amount of research has been done concerning this level, but it suffices to state that pupils at school rarely reach this level. Here the pupil is supposed to work in a variety of axiomatic systems, that is, non-Euclidean geometries can be studied, and different systems can be compared.

An important point to remember is that not all pupils in the same class attain the same levels at the same time. Research done by Usiskin and Senk regarding pupils and their Van Hiele levels (Driscoll, 1988 : 160) determined the following:

- During the year of geometry, more than 50 % of the students at the lowest level moved to Levels 2 and 3, but about a third of them remained in Level 1.
- After a full year of a geometry course with proof, only about 50 % of the students could do more than simple proofs.
- As a predictor of how well students would do with proof after a year-long geometry course, the van Hiele model proved to be successful. More importantly, it seems that if the student enters geometry at Level 1 or below, there is very little chance of success with proof. Entry at Level 2 will guarantee the child a better

than even chance of success, whilst entry at Level 3 implies a very good chance of success.

According to the Van Hiele model two people reasoning at different levels will not understand each other; Further to this, in other words, a student who has attained only level n will not understand the thinking of level $n + 1$ or higher (compare Senk, 1989 : 310). Level 3 becomes the transitional stage from informal to formal geometry. At this level students are able to derive short chains of reasoning whilst at level 4, students can write formal proofs.

Research conducted by Senk (1989) confirmed the positive link between the Van Hiele levels and successful proof writing. The fact that students might not be at the correct van Hiele level in order to attempt a proof will explain why these students attain very little success at proof writing.

A serious limitation of the van Hiele theory is that proof is only assigned to van Hiele Level 3. Here proof serves the function of systematization. It now appears that the other functions of proof, like explanation, could be addressed at earlier levels. At van Hiele Level 1 pupils could be provided with diagrams which they can manipulate themselves and visually draw conclusions. Thereafter they can measure and deduce properties of the different figures and at the same time, new terminology can be learnt. This would then be van Hiele Level 2. Only then would it be appropriate to challenge them to construct such dynamic quadrilaterals themselves, thus assisting the transition to Level 3.

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CHAPTER THREE

RESEARCH METHODOLOGY AND OVERVIEW

The present study builds on research by De Villiers (refer to pages 45-49) but it contextualizes it specifically within dynamic geometry. The purpose of this study is to determine whether pupils have any need for conviction and explanation within the context of dynamic geometry. Furthermore, this study will test curriculum material that was developed as a result of previous empirical and theoretical research. The material allows the child to discover a solution to a problem by guiding the child through stages that are easy and practical. As the child progresses through the worksheets, the child is allowed to record his/her conclusions and conjectures and is led to an explanation (proof).

The empirical part of this research has focussed on the following major research questions:

Given a self-exploration opportunity within dynamic geometry:

- are pupils convinced about the truth of the explored geometric statements and what is their level of conviction ? Do they require further conviction?
- do they exhibit a desire for an explanation for why the result is true?
- can they construct a logical explanation for themselves with guidance?
- do they find the guided, logical explanation meaningful?

As a result of these questions it was decided to use the method of qualitative analysis by means of one-to-one task based interviews. This method makes it easier to document the high level of information that individual children reveal about their sense making of situations and contexts. Furthermore, this method would allow the researcher greater control to observe and take note of, how each pupil went through the task sheet. As Novak and Gowin (1984 : 12) stated: "For this reason most psychologists prefer to do research in the laboratory, where variation in events can be rigidly prescribed or controlled. This approach clearly increases the chances for observing regularities in events and hence for creating new concepts". The researcher acknowledges that the classroom situation is dynamic, due to the interaction of pupils with each other, the teacher, the subject content and the environment. By reducing the number of external variables, one narrows the focus, giving generalizations based on findings during task-based interviews greater credibility. Such findings, however, might be able to dictate future classroom practice.

The tasks to be used in the interviews have been conceptualized within an action research paradigm. The tasks are based on curriculum materials that have been conceptualized within a theoretical framework of the different functions of proof, as well as empirical research on pupils' cognitive needs with respect to conviction and explanation. This research will determine how well they cope with the tasks provided and whether they construct meaning as it has been conceptualized. Based on these results, the material may have to be reviewed or redesigned. This explains the need for action research. As Cohen and Manion (1986 : 208) stated : "Action research is a small-scale intervention in the functioning of the real world and a close examination of the effects of such intervention". This in effect summarises the purpose of this experiment. The strategic plan that would be implemented involves the pupil interacting with the developed curriculum material, careful observation and thereafter reflection.

The researcher chose to work with pupils from Glenover Secondary School due to the convenience of having easy access to the computer laboratory and arrangements could easily be made to interview the pupils. Seventeen pupils, about 14 years old, were interviewed from grade 9 (standard 7). These pupils were selected randomly by their computer studies class teacher, who chose every ninth pupil appearing in the attendance register. They were selected from a group of 153 pupils in February 1997. At this stage, the pupils had not written any examinations for the year and therefore their individual academic performances could not be commented on. Grade 9 pupils were ideal for this study because the questions are suited to their level, and since they were just beginning with proofs in geometry.

The school at which this experiment was carried out, was previously administered by the ex-House of Delegates. There were a larger number of Indian pupils at this school in 1997, as compared to pupils from other race groups. All of the pupils selected for these interviews were Indian. The school itself is situated in Westcliff, in Chatsworth, which is a predominantly Indian suburb south of Durban. The residents of Westcliff are generally those of the middle to lower income group. The mathematics results at this school, has been below average over the previous years, and this was apparent from the matric results and the fact that below 10% of the pupils doing mathematics at matric level offer it at the higher grade level. On this basis the average mathematics achievement of pupils at this school could be considered to be below average.

Pupils were not exposed to this particular problem at the school before and therefore did not know the solution or what to expect beforehand. However, pupils needed to have some knowledge of equilateral triangles, the formula to determine the area of a triangle and basic factorising skills. This was well within the capabilities of the grade 9 (standard 7) pupils.

A brief synopsis of the interviewing process is necessary in order to place the responses obtained in perspective. Although the pupils did not know exactly what to expect, they initially displayed an unwillingness to participate in this experiment. They feared failure and felt that they were incapable of working with mathematics in a computer environment because they never experienced any such thing before. The interviews were conducted in the school computer laboratory over a period of one and a half months. The interviews were subject to the availability of the laboratory and the pupils, because the school only allowed the use of its laboratory during school hours. The laboratory was adequately equipped for the purposes of the interviews. It was initially envisaged that all pupils involved would be brought together for a short period in order to familiarize them with the general use and application of the computer software – *Sketchpad*. This was not possible for two reasons:

- permission was not granted for the use of all pupils at any one given time, and
- only five computers could be used for the purposes of an orientation, which meant that 17 pupils would not have been able to satisfactorily enjoy and learn the basics about the software.

In any event, this did not affect the experiment because minimal knowledge was expected from the pupils about the software. Each pupil was made to feel at ease before the interview commenced, in order to ensure that they would respond in a way that would reflect their understanding of the task provided.

The task that the pupils had to work through was based on an equilateral triangle. The sketch of the equilateral triangle was presented ready-made to the pupils, although the task of constructing it for themselves might have been an interesting task on its own. The decision to present the equilateral triangle to them was based on the following reasons:

- it would take each pupil a long time to figure out how to construct a dynamic equilateral triangle because they were not familiar with *Sketchpad*.
- The construction of the triangle was not one of the objectives of this experiment. So presenting the construction to them did not affect the essence of the experiment.

All measurements were clearly visible on the screen of the computer, so that pupils could easily view any changes that might have taken place. This is an example of what pupils would have seen on the screen (refer to Figure 7).

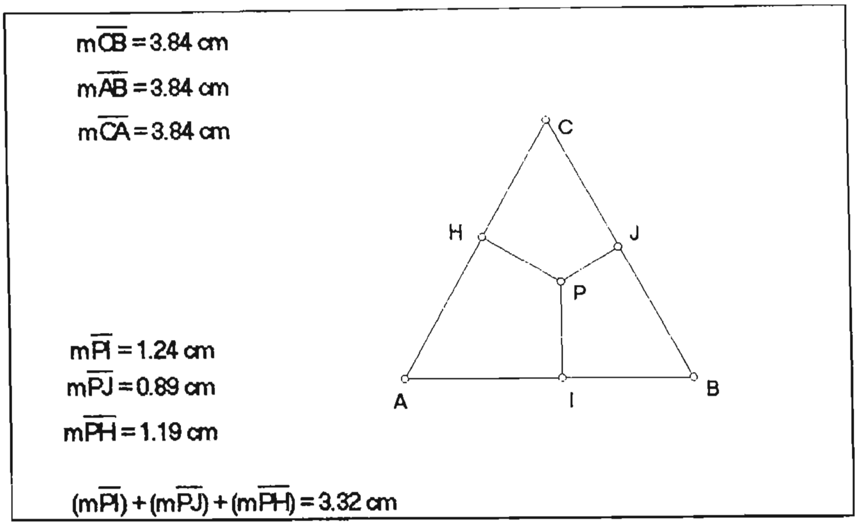


Figure 7

On being seated pupils were given the task sheet (refer to APPENDIX 2) and were asked to read through it. At the commencement of the interview (when the tape recording began), pupils were asked whether they understood the question posed to them. This was done for the following reasons:

- to break the ice and make them feel at ease during the course of the interview, and
- to ensure they knew exactly what they had to determine.

The schedule of questions that followed, was designed and redesigned after three trial runs. This is what it finally looked like :

THE CHILD AFTER SUFFICIENT TIME,
WILL BE ASKED TO WRITE A CONJECTURE.

↓

THE CHILD WILL BE ASKED WHETHER
S/HE IS SURE OF THE CONJECTURE.

↓

DETERMINE THE LEVEL OF CONVICTION.
(50 %, 80 %, 90 %, 100%) ?

↓

ASK THE CHILD : IF THE ISLAND (TRIANGLE)
WAS BIGGER WILL THE RESULT BE ANY DIFFERENT?

DETERMINE WHETHER THE CHILD FINDS THE RESULT SURPRISING.

IF NO – WHY ?

IF YES – WHY ?

ESTABLISH WHETHER THE CHILD HAS A DESIRE
TO KNOW WHY THE RESULT IS TRUE ?
(WOULD YOU LIKE TO KNOW WHY THIS IS TRUE ?)

ASK THE CHILD: CAN YOU EXPLAIN WHY THIS IS TRUE?

IF A CHILD SAY YES BUT GIVES
A SIMPLISTIC REASON EG. I
CAN DRAG IT AROUND AND
SHOW THAT IT IS TRUE.

IF THE CHILD SAYS NO

A SHEET CONTAINING SEVERAL GUIDELINES
WILL BE HANDED TO THE CHILD. ASK THE
CHILD TO READ IT.

RESEARCH QUESTION : CAN THE CHILD NOW GIVE
AN EXPLANATION ?
(GUIDE IF NECESSARY)

↓

ESTABLISH LEVEL OF UNDERSTANDING
DID THE EXPLANATION INCREASE THE
CHILD'S UNDERSTANDING ?

Each interview was approximately twenty minutes long and each was audio taped. Although these questions were structured around the critical questions, it also allowed for variation in expected responses from the pupils, and further probing was done in particular cases.

Finally, the data analysis amounted to systematically grouping and summarizing the responses, and providing a coherent organising framework that encapsulated and explained the way each pupil produced meaning whilst working through the tasks provided.

CHAPTER FOUR

DATA ANALYSIS

4.1 INTRODUCTION

The question chosen for this investigation was as follows (refer to Figure 8):

Investigation: Distances

Sarah, a shipwreck survivor manages to swim to a desert island. As it happens, the island closely approximates the shape of an equilateral triangle. She soon discovers that the surfing is outstanding on all three of the island's coasts and crafts a surfboard from a fallen tree and surfs every day. Where should she build her house *P* so that the total sum of the distances from *P* to all three beaches is a minimum? (She visits them with equal frequency). Before you proceed further, first write down your intuitive guess in the space below where you think *P* should be placed for the total sum of the distances to be a minimum.

Figure 8

The above question is a replica of that which was given to the pupils. The schedule of questions on pages 60 to 61, was based on this task-sheet and was merely a guideline to important questions and was not strictly adhered to, in order to allow for individual differences and further probing.

Pupils were asked these questions to ensure that the researcher understood exactly what they were saying : "Are you sure (certain)?", "Do you desire an explanation ?

Do you want to know why the result is always true?". Other questions were also asked. For example: " Where do you think Sarah should build her house? " or "Are you surprised with the results?".

During the latter part of the interview, pupils were expected to write out an explanation. They were asked to do so at the back of their task sheet. Also, much time was needed for the pupils to carry out their testing process (using the mouse to drag the point P around) and for the calculation. During these processes, the tape recorder was stopped and restarted when they were ready to continue. The shortest interview took 17 minutes and the longest one took 26 minutes. No time constraints were placed on the pupils – they worked at their own pace.

When the pupil arrived for the interview s/he was given the question to read. After they had completed the question they were asked whether they understood the question. At this point the interview began. Very few understood the question after the first reading as can be seen from the following responses:

- Kovilan stated that : "Sarah wants to build her house so that the distance to the beaches will be equal".
- Floyd said that : "Sarah wanted to find the minimum distance from the three sides of the triangle".
- Emily also stated that : "...... it had to be the shortest distance to all the beaches".
- Higashnee said that : ".............. the sum of all the beaches must be the smallest".

- Rhyam's response was : "............she wants to get a way so that she can go to three beaches at the same time".

From the pupils' it was also clear that none of them had seen the question before. To enable the pupils to correctly understand the problem, the researcher then asked each pupil to re-read the question carefully, and providing some guidance where necessary.

After it was clear that the pupils understood the question, they were asked to make an initial intuitive guess (refer to Figure 9 which appeared in the task sheet).

Investigation: Distances

Sarah, a shipwreck survivor manages to swim to a desert island. As it happens, the island closely approximates the shape of an equilateral triangle. She soon discovers that the surfing is outstanding on all three of the island's coasts and crafts a surfboard from a fallen tree and surfs every day. Where should she build her house *P* so that the total sum of the distances from *P* to all three beaches is a minimum? (She visits them with equal frequency). Before you proceed further, first write down your intuitive guess in the space below where you think *P* should be placed for the total sum of the distances to be a minimum.

Figure 9

The question "Where do you think that Sarah should build her house?" elicited a common response among most pupils, namely that Sarah should build her house at the centre. The pupils were asked why they felt that the house should be built at the centre. Kumarasen, for example, responded by saying that: *"...if you build anything in the centre then there is always a short distance around it"*. Kumarasen seemed quite convinced of his conjecture and so was Manivasan, whose reason was *" ...because everything will be equal"*. Rowan believed that it should be at the centre because *"it will be close … it will be the same distance to all the beaches"* and therefore the sum will be a minimum. Karishma felt that the sum would be a minimum if the point P was at the centre because *"it will be closer to all three beaches"*. Ansuya's reason was similar when she said *"because it seems the easiest way to get to any of the three beaches"*. On the other hand Kerushnee wasn't so sure. She said that *"…maybe here the sum will be the shortest"*. She was totally non-committal. The only person who answered differently was Nicolas, who believed that Sarah should build her house close to one apex of the triangular island. When he was asked why he felt that way, his response was that he just felt that way.

The next task required the pupils to investigate the problem with a ready-made sketch on *Sketchpad* where, the distances to the sides and their sum, was already provided. Pupils were then asked to move the point around in the triangle and to observe whether any changes were taking place (refer to Figure 10 from task sheet).

Sketch

Step 1: Construct a dynamic equilateral triangle *ABC*.

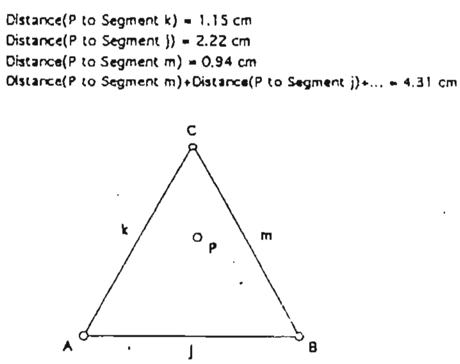
Step 2: Construct a point P in the triangle.

Step 3: Measure the distances from P to the three sides.

Step 4: Select the three distances and choose Calculate to add the three distances.

Investigate

Drag point *P* around the interior of the triangle. What do you notice regarding the total sum of the distances? Drag *A*, *B* or *C* to change the size of the equilateral triangle and again drag point *P* around the interior of the triangle. What do you notice now? What happens if *P* is dragged outside the triangle?



Conjecture

In the space below, write a conjecture regarding your observations above.

Figure 10

After the pupils had moved the point around, they had to make a conjecture regarding their observations. All the pupils found that when moving the point around, the distances to the three sides of the triangle changed, but their sum did not. All of them seemed quite surprised at the result, and when asked whether they found the result surprising, the following responses were obtained:

Kerushnee : (emphatically) *Yes I find the result very surprising.*

Ansuya : (confidently) *Yes. I thought it would change.*

Kumarasen : *Yes, because at first you think it should be at the centre and the sum will be small. But now it can be anywhere.*

Floyd : (emphatically) *I didn't expect it. It is surprising !*

This could probably be attributed to the result so clearly contradicting their initial expectation. It was also noticed that the majority of them began to smile after they came up with their conjecture, which indicated that the discovery was not an unpleasant, but a pleasant surprise.

4.2 Pupils' need and level of conviction regarding the truth of the discovered conjecture.

The main purpose of this section was to establish how pupils convinced themselves of the truth of the conjecture, as well as the level of their certainty. It took only a few minutes for pupils to convince themselves about the truth of this conjecture. The researcher was surprised to find that most pupils (14) stopped within a few minutes of experimentation because they felt that there was no need to conduct further computer testing of the conjecture.

When Nicholas stopped after a few minutes of experimentation, his response was as follows:

Researcher : *You've stopped Nicholas. What's wrong ?*

Nicholas : *No matter where I move the point P the total remains constant.*

Researcher : *You mean the sum remains constant?*

Nicholas : *Yes.*

Researcher : *Are you convinced ?*

Nicholas : *Yes.*

Nicholas was asked whether he wanted to try some more, and he reluctantly tried again and confidently stated :

Nicholas : *Okay, I'm convinced that the sum is always constant.*

When asked whether he was really completely sure, he responded rather irritably :

Nicholas : *I'm convinced that it wouldn't change at all no matter how long I try.*

Nicholas went further by saying that he was a 100 % convinced, and that he didn't need to conduct further tests in order to convince himself. Floyd similarly showed a very high level of conviction, for example:

Floyd : *No matter where I put the 'centre' point in the triangle its always going to be the same.*

Researcher : *When you say the centre point, do you mean point P ?*

Floyd : *Yes.*

Researcher : *And what happens ?*

Floyd : *The sum of all the distances from point P to the sides is always the same. It's not changing.*

Researcher : *Are you convinced ? Do you want to move it around some more ?*

Floyd : *No.*

Researcher : *But are you convinced already ?*

Floyd : *(emphatically) Yes, I'm convinced !*

Researcher : *If I asked you how many percent convinced are you, what would you say ?*

Floyd : *(emphatically) 100%*

Researcher : *You are 100% convinced that no matter where point P is* ...(pausing)

Floyd : *(completing the statement) .... the sum of the distances will still be the same !*

To probe his level of conviction further regarding the generality of the statement, for any equilateral triangle, the researcher continued as follows:

Researcher : *I want to know what would happen if I grabbed this point of the triangle and made it bigger or smaller ? Do you think the result will change ?*

At this point the researcher increased the size of the equilateral triangle which resulted in an increase in the sum of the distances to the sides.

Floyd : *Yes, the result will change.*

Researcher : *When you say the result will change, are you saying that the sum will change ?*

Floyd : *It will change because the distance from the house to the beaches will be different.*

Researcher : *Are you speaking about the different triangles ?*

Floyd : *For the different triangles.*

Researcher : *No, I'm asking, if you made the triangle bigger or smaller, will the result in that triangle change ?*

Floyd : *No it will not change .... it will be the same.*

Researcher : *What do you mean 'the same' ?*

Floyd : *Wherever I move the house within the triangle, the sum of all the distances will be the same.*

Researcher : *So you're saying irrespective of the size of the triangle ....*

Floyd : *Irrespective of the size, the distance will be the same. You can build the house anywhere.*

Researcher : *Are you sure ?*

Floyd : *Yes, sir, I'm positive.* (sounding satisfied)

It was interesting to note that Floyd did not even try to move the point around within the new triangle. He remained steadfast in his belief that it will remain the same. The researcher then moved the point around just to reinforce what Floyd was saying.

Clearly, both Nicholas and Floyd had obtained high levels of conviction, any attempt to get both of them to carry out further tests seemed pointless. Vinolia showed similar levels of conviction. She was confident of her conjecture and didn't *want* to continue any further. Nirvana also displayed a very high level of confidence in her conjecture as follows:

Researcher : *Okay Nirvana, you seemed* to *have moved it to a number of points. What is your observation ?*

Nirvana : *The distances are changing and the sum ……………..*

Researcher : *Which distances are changing ?*

Nirvana : *All of them and the sum remains the same.*

Researcher : *Do you think that this is the same throughout the triangle ?*

Nirvana : *Throughout the triangle.*

Researcher : *Do you think that if I moved the point to the corner there* (pointing with the finger) *then the sum will remain the same ?*

Nirvana : *Yes !*

Researcher : *Are you convinced ?*

Nirvana : *Yes !*

Researcher : *You don't want to try ?* (The researcher was attempting to establish whether she was simply saying 'yes' to satisfy the researcher or did she really mean it ?)

Nirvana : *I'll try………………… (after a while) yes it remains the same*

Researcher : *So, no matter where you moved it in the triangle, it will be the same ?*

Nirvana : *Yes.*

Researcher : *If I asked you how many percent convinced are you, what would you*

Researcher : *I noticed that you didn't move to many points……… are you saying that what you observed will be the case anywhere in the triangle ?*

Kerushnee : *Maybe.*

Researcher : *If we made the triangle bigger or smaller, do you think now if we moved the point around, will the sum change ?*

(The researcher changed the size of the triangle.)

Kerushnee : *I don't think so.*

Researcher : *Try it and see.*

(Because she was not so confident in the way she said it, the researcher requested that she move the point on the inside just to check whether her statement was correct.)

Kerushnee : (after a while of actual testing) *The sum is not changing.*

Researcher : *So what can you say irrespective of the size ?*

(In other words, will the size of the equilateral triangle affect the result obtained?)

Kerushnee : *No matter where you move the point the sum will still be the same.*

Researcher : *Do you feel that you are convinced that that will be the case always ? Are you sure that if I moved it to a corner point, it will not change ?*

Kerushnee : *I don't think it will change.*

Despite some initial lack of confidence, Kerushnee later in the interview, showed that she was now quite convinced.

Researcher : *Kerushnee, I want <u>you</u> to convince yourself that what you are saying will always be true.* (The emphasis was on her convincing herself.)

Kerushnee : *But when I'm moving it everywhere it still remains the same.*

Researcher : *So if I asked you if you were 60% convinced, what would you say ?*

Kerushnee : *I think I'm more than 60% convinced !*

Researcher : *How many percent convinced are you ?*

Kerushnee : *(confidently) 100%*

Researcher : *So you don't have any doubt that it would always hold ?*

Kerushnee : (emphatically) *I did try and I don't think there is such a point !*

The need for sufficient empirical exploration before the attainment of certainty was also evident in the interview with Kumarasen.

Kumarasen : *I noticed that the distances from the house to the beaches always changes but the sum is always constant.*

Researcher : *Are you saying that no matter where P is the sum of the distances is always the same ?*

Kumarasen : *Yes.*

Researcher : *What if I moved P to a corner will the sum change ?*

Kumarasen : *No !*

Researcher : *How many percent convinced are you ? ……… Would you say about 60% ?*

Although he seemed quite convinced that the result will always hold, his reply was surprising.

Kumarasen : *About 55 %.*

It might be that his expressed lack of certainty at this point was in regard to the truth of the statement for ANY equilateral triangle (generalization 2), rather than whether it was true for ANY point within the GIVEN equilateral triangle (generalization 1). This seemed to have been the case, as is borne out by the rest of the interview.

Researcher : *What do you think would happen if we changed the size of the triangle ?*

(The researcher did not change the size of the triangle.)

Kumarasen : *Then the sum of all the distances will change.*

Researcher : *What do you mean that the sum of all the distances will change ?*

Kumarasen : *When you make the triangle bigger then the sum will change.*

Researcher : *But within the same triangle will the sum change ?*

Kumarasen : *Yes !*

Researcher : *Okay, then let us make the triangle bigger* (the researcher increased the size of the triangle)...... *now test the conjecture.*

Kumarasen : (after a while of testing) *The sum never changes even if the triangle is made bigger or smaller. The sum will always stay the same.*

Now it seemed that he was quite convinced.

Researcher : *You said before that you were 55% convinced, now how many percent convinced are you ?*

Kumarasen : *About 100%, because no matter how big or small you make the triangle the sum will always be the same.*

Researcher : *You don't have any doubts ?*

Kumarasen : *No !*

Emily also needed extensive empirical exploration before gaining a high level of conviction in generalization 2.

Researcher : *So you are saying that no matter which triangle you have the principle will be the same ? (This was in the context of a smaller or larger equilateral triangle.)*

Emily : *(emphatically) Yes.*

Researcher : *If I had to ask you how convinced you are of this and you had to give it to me in the form of a percentage what would you say ?*

Emily : *About 70%.*

Researcher : *So you still have some doubt ?*

Emily : *Yes.*

Researcher : *Do you want to try some more to further convince yourself ?*

Emily : *Yes.*

Researcher : *(after a while) I see you've stopped. What does that mean ?*

Emily : *I'm .... I'm ....*

Researcher : *Are you a little bit more convinced ?*

Emily : *Yes.*

Researcher : *What percentage do you think ?*

Emily : *About 90% now.*

Natasha was also initially unsure about generalization 1, but through further investigation Natasha became absolutely convinced of the conjecture.

Natasha : *No, it won't change !*

Researcher : *Are you sure ? Are you confident of your answer ? You're looking unsure ?*

Natasha : *No, I'm quite confident that it won't change.*

Researcher : *Do you want to try it again ? You moved it around the centre only, you did not move it around the corners.*

Natasha : *(after a while) It remains the same.*

When asked whether the result would still hold for a larger or a smaller equilateral triangle (generalization 2), she responded positively, stating that she was sure that the result will be the same. Yet when she was asked how convinced she was about this, she replied that she was only 70% convinced. However, after some further investigation, she became 100 % convinced about generalization 2. The following provides a summary of the findings in this section.

After the initial experimental exploration, the following levels of initial conviction were displayed :

- 12 (70.5 %) were 100 % convinced
- 2 (11.8 %) were 98 % to 99 % convinced
- 2 (11.8 %) were 70 % convinced
- 1 (5.9 %) were 55 % convinced

After further exploration, the following levels of conviction were displayed:

- 14 (82.3 %) were 100 % convinced
- 2 (11. 8 %) were 98 % to 99 % convinced
- 1 (5.9 %) was 90 % convinced

It was evident that the more pupils experimented the more convinced they became. The level of conviction of all pupils was very high, and for many pupils just a few minutes of experimentation on the computer was sufficient to achieve this level of conviction.

- From the pupils interviewed, it was clear that pupils could achieve a very high level of conviction about the truth of a geometric statement by exploration on computer. In fact, their level of conviction was much higher than that expected by the researcher. It is also possible that pupils might not have achieved this level of conviction, so quickly and easily, if they had used only the pencil and paper construction and discovery method.

The following finding was made with respect to the question that was asked of the pupils:

<u>FINDING 1</u>: **The pupils developed very high levels of conviction in relation to this conjecture within a dynamic geometry environment.**

4.3 <u>Pupils' need for explanation (or understanding of why the result is true).</u>

The purpose of this section was to try and establish whether pupils exhibited a need for explanation of the conjecture they had made. Do they want to know why the conjecture is true? Do they display a desire for a deeper understanding, independent from their conviction? The researcher found that the majority of pupils expressed a desire for an explanation. In fact 94 % (16) of the pupils said that they would want an explanation and only one pupil (6%) took a while before saying that she would also like an explanation. Some extracts from the interviews are now given.

Researcher : *Do you think then, now that you are a 100% convinced, that there is a need for an explanation ?*

Manivasan : *Yes.*

Researcher : *Would you want an explanation ?*

Manivasan : *Yes.*

Researcher : *Why ?*

Manivasan : *So I can <u>understand</u> it.* (emphasis by child)

It seemed clear that despite being convinced, Manivasan also wanted to understand why the result is true. He seemed to want something more than just being able to observe and accept the validity of the statement. Rowan responded in a similar way.

Researcher : *Do you think that there is a need for an explanation ? Do you want to know why this is true ?*

Rowan : *Yeah, there is a need for explanation.*

Researcher : *Why do you think there is a need ? ... Why ?*

Rowan : *So we will be able to understand more clearly that diagram.*

The researcher acknowledges that by asking the question "Do you want to know why this is true?" the pupil may have been led to answering in the affirmative. However, Rowan's body language and response seemed to indicate that he had made up his mind on his own and that he truly wanted to know why. Although Rowan was absolutely sure (100%) of the statement, he nevertheless seemed to express a need for further understanding. Similar responses were given by the following students, all of whom seemed to express some curiousity regarding the result.

Researcher : *Do you think, now that you are very convinced, ... is it necessary to know why this is the case ?*

Rodney : *Yes.*

Researcher : *Why do you want an explanation for this ?*

Rodney : *To satisfy my curiosity.*

Researcher : *Why do you think there is a need for an explanation ?*

Karishma : *Because I'm curious and I'd like to know what's going on.*

Researcher : *Why do you think there is a need for an explanation ?*

Debashnee : *Because I'm a curious person and I would like to find a solution for things. I would like to do the same for this.*

Researcher : *Do you desire an explanation for what is going on ?*

Ryham : *Yes.*

Researcher : *You really would want to know why ?*

Rhyam : *Yes.*

Researcher : *Why ? Why would you want know why ?*

Rhyam : *I like to find out why things are taking place.*

Higashnee's response was very similar when she said : " *I would like to find out about it myself and know more about it than finding out from the computer*".

From the above, it was clear that she also expressed a desire to satisfy her curiousity herself rather than just be given an explanation. Natasha expressed the same desire to know why as follows.

Researcher : *Do you think now that you are 100% convinced……… do you think that it is necessary to have some sort of explanation as to why the result is true ?*

Natasha : *Yes.*

Researcher : *Why do you think there should be an explanation ?*

Natasha : *Out of interest I would want to know <u>why</u>.*

Again it would seem that there was a deeper urge to find an explanation rather than to check whether the result was really true. Indeed, Natasha had earlier already stated that she was 100% convinced about the truth of the result, and did not express any doubt regarding the validity of the statement. However, some pupils did not explicitly state a need for an explanation or curiousity, for example :

Researcher : *Do you think there is a need to explain why this is the case ? Do you have a desire to know why ?*

Vinolia : *No, I'd like to …………*

Researcher : *You'd like to know ………*

Vinolia : ……… *to go more ahead.*

Although Vinolia's did not clearly state that she desired some explanation, but it appeared as if her statement to "*go more ahead*" implied that she desired some further understanding beyond the experiences she had whilst she carried out her task. The transcription does not adequately carry the entire emotion that she showed during her interview.

It further seems clear that the pupils desire for further explanation or a deeper understanding had not been satisfied by the empirical exploration on computer. This exploration only seemed to convince them, but did not appear to have satisfied some deeper need for explanation and understanding.

Leann was the only pupil who did not immediately indicate a need for explanation.

Researcher : *Leann, now that you are a 100% convinced, do you think that there is a need for explanation ?*

Leann : *I don't think so.*

Researcher : *Why would you not want an explanation ?*

Leann : *I'm quite convinced it is true.*

Researcher : *But then, wouldn't you want to know why the result is true ?* (researcher's emphasis)

Leann : *Yes, it might be necessary.*

It might be that she had interpreted the first question as one which was enquiring about her need for conviction rather than explanation. Her response later on ("I'm quite convinced it is true") seems to indicate that this was indeed the case. It was for this reason that the researcher in the next question attempted to enquire more explicitly whether she wanted to know why the result was true (by emphasising the word "why"). Although this question might be viewed as a somewhat leading question, it seemed necessary to make sure that the pupil understood that it was not about further conviction, but about understanding why the result was true. This question then resulted in the pupil finally stating that it might be necessary. It was not entirely clear, however, whether she was merely responding to satisfy the researcher.

Eventually all the pupils seemed to express some desire to have an explanation. This desire clearly did not emanate from a need to further verify the result as they already had very high levels of conviction. They seemed to want to 'understand' the problem which 'interested' them and which made them 'curious'. De Villiers (1991 : 25) similarly found that: *"Pupils who have convinced themselves by quasi-empirical testing still exhibit a need for explanation, which seems to be satisfied by some sort of informal or formal logico-deductive argument"*. It seems that the distinction between experimental conviction and explanation is important. Although pupils could not make this distinction, they perhaps intuitively felt that that which was being observed required an explanation, that is, a deeper understanding of why it was true. There must be something more than an ordinary, simplistic answer. That is what pupils appeared to have needed.

Pupils were also asked why they wanted an explanation, but very few could give a clear reason for their need. It seemed that they had difficulty verbalising this desire for an explanation. In fact, the majority simply said 'I don't know why', when asked 'why?'. The pupils were further asked whether they would like to attempt an explanation on their own. However, only 6 (36 %) were willing to try, and then only came up with empirical arguments based on their earlier exploration (for example, if we move the point around we can see that the sum of the distances is constant). This was not unexpected as these pupils had not yet been exposed to proof in geometry.

Even with their limited knowledge the 6 pupils were eager to attempt an *explanation*, although they gave very simplistic explanations, for example, they referred to the observed empirical evidence. The interview with Kumurasen was a good example of this.

Researcher : *Now that you are so* convinced, *do you like to know why that is true ?* (referring to his observation)

Kumarasen : *(confidently) Yes I would.*

Researcher : *Can you give me an explanation for why that is true ?*

Kumarasen : *Well by changing the size of the triangle you are changing the distance of the house from the beach.*

Researcher : *Okay. What you are saying is not really an explanation. What we want are logical reasons. Do you think that you can come up with logical reasoning ?*

Kumarasen : *No.*

FINDING 2 : The pupils appeared to display an intrinsic desire for an explanation, that is, a need for understanding the conjecture, independent of its verification.

4.4 Pupils' ability to construct a logical explanation with guidance.

The basic research question, investigated in this section, was whether pupils could construct their own logical explanations with some guidance. Pupils were first asked to read the introduction (refer to Figure 11) which clearly illustrates the difference between observation / experimentation and explanation. The purpose of this was to 'break the ice' and to attempt to illustrate to pupils that the activity, which they had engaged in up to that point, was in fact, only observation and experimentation.

Distances: Explaining

You are no doubt at this stage quite convinced that the total sum of the distances from a point P to all three sides of a given equilateral triangle is always constant. But can you **explain** why it is true?

Although further exploration on *Sketchpad* may succeed in convincing you even more of the truth of your conjecture, it really provides no explanation; it only confirms its truth. For example, the regular observation that the sun rises every morning clearly does not constitute an explanation; it only reconfirms the validity of the observation. To explain something, one therefore has to try and explain it in terms of something else, e.g. the rotation of the earth around the polar axis.

Recently a mathematician named Feigenbaum made some new experimental discoveries in fractal geometry using a computer just as you have used *Sketchpad* earlier to discover your conjecture. These discoveries were then later explained by Lanford and other mathematicians. Carefully read and comment on the following quotation in this respect:

> "*Lanford and other mathematicians were not trying to validate Feigenbaum's results any more than, say, Newton was trying to validate the discoveries of Kepler on the planetary orbits. In both cases the validity of the results was never in question. What was missing was the explanation. Why were the orbits ellipses? Why did they satisfy these particular relations? ... there's a world of difference between validating and explaining.*"
>
> - D. Gale (1990) in The **Mathematical Intelligencer**, 12(1), 4.

Figure 11

The pupils were then asked to complete the next section (see Figure 12) which provided guidelines for the finding of explanation to the problem, in a logical, sequential way.

Explain

Here are some hints for planning a possible explanation. Read and work through it if you want, or try to construct your own explanation.

E1. Label all three sides as a and the distances from *P* to the sides *AB*, *BC* and *CA* respectively as h_1, h_2 and h_3.

E2. Write expressions for the areas of triangles *PAB*, *PBC* and *PCA* in terms of the above distances.

E3. Add the three areas and simplify your expression by taking out a common factor.

E4. How does the sum in P3 relate to the total area of triangle ABC? What can you conclude from this?

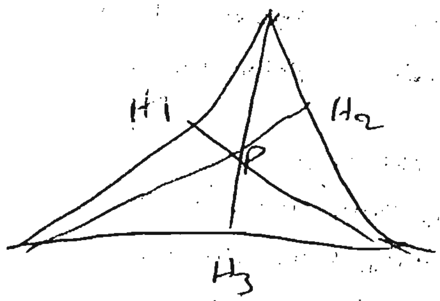
E5. Which property therefore explains why this result is true?

E6. Discuss your explanation with your partner or group.

Figure 12

It should again be noted that these pupils had not yet been exposed to the writing of proofs (explanations) for geometric statements. The guided explanation given to them required them to follow six steps in determining a possible solution. They were comfortable with the ease of the instructions because they could understand what was required. Refer to Figure 13 for Nicholas's written work.

$AREA = \frac{1}{2} b \times h$



$AREA \text{ OF } \Delta_1 = \frac{1}{2} b \times h_1 = A_1$

$AREA \text{ OF } \Delta_2 = \frac{1}{2} b \times h_2 = A_2$

$AREA \text{ OF } \Delta_3 = \frac{1}{2} b \times h_3 = A_3$

$\text{Total } AREA \text{ OF } \Delta = \frac{1}{2} b \times h = \frac{1}{2} \cancel{2} a h$

$\cancel{\frac{1}{2} ah}$ $A_1 + A_2 + A_3 = A$ of the ~~...~~ big Δ

$Sum = \frac{1}{2} a h_1 + \frac{1}{2} a h_2 + \frac{1}{2} a h_3$ = Area of whole Δ

$= \frac{1}{2} a (h_1 + h_2 + h_3) = \frac{1}{2} a H$

$= h_1 + h_2 + h_3 = H$

Figure 13

Nicholas went through the sheet with considerable ease.

Researcher : *Okay, I can see that you have done that* (referring to the writing down of expressions for the areas of the three small triangles). *The next step asks you to add all three up. Do you know what to do ?*

Nicholas : *Yes.*

Researcher : (after a while) *You've got A_1, A_2 and A_3 and you've got expressions for them. Now add these expressions.*

(after a while) *Have you done that Nicholas ?*

Nicholas : *Yes.*

Researcher : *Now simplify it. ……………………………… Have you done that ?*

Nicholas : *Yes.*

Researcher : *I've noticed that you removed half 'a' as a common factor.*

Nicholas : *Yes.*

Researcher : *Describe what you have done.*

Nicholas : *I've removed half a as a common factor and I've got half a into $h_1 + h_2 + h_3$.*

Researcher : *Nicholas can you tell me how these three triangles relate to the area of the large triangle ?*

Nicholas : *The area of the three triangles when you add it up, will give you the area of the big triangle.*

Researcher : *If that is the case and we found the sum of the areas of the three triangles, then what can we conclude ?*

Nicholas : *(silence)*

Researcher : *That the areas of these triangles equal to …………… ?*

Nicholas : *area of the big triangle.*

Researcher : *Now look at E4. I want you to write down this expression.*

Nicholas : (after a while) *I noticed that the big triangle also had half a in it. So I cancelled off the half a from the big triangle and half a from the three small triangles.*

Researcher : *And what have we arrived at ?*

Nicholas : *The height of the three triangles ………… when you add it up it gives you the height of the big triangle.*

Researcher : *What does this mean to you ?*

Nicholas : *No matter what the heights of the three smaller triangles are, it will always equal the height of the big triangle.*

Researcher : *So what does it mean in terms of Sarah's house ?*

Nicholas : *It means that no matter where she puts her house the total distances will always be constant.*

Floyd also worked through the worksheet quite easily. Very little was required of the researcher in terms of "leading" the pupils to a solution. The interview with Floyd is now presented followed by his written work (see Figure 14):

Researcher : *Do you have the three expressions ?*

Floyd : *Yes.*

Researcher : *Can I have a look at them ………………. That's okay. The next step E3 asks you to add the three areas and simplify them by taking out a common factor. Can you do that ?*

Floyd : *Yes…………..* (after a while) *Okay.*

Researcher : *Now tell me this Floyd, instead of writing it. How does the sum in E3 relate to the total area of the triangle ?*

Floyd : *I've divided the triangle into three different parts and I've found the area, ...I mean the height of the triangle ....*

Researcher : *But my question is: how is the sum of the three triangles you've got, related to the entire triangle.*

Floyd : *If you add the whole three triangles then it will give you the sum of the whole thing.*

Researcher : *So you're saying that the sum of the three triangles ....*

Floyd : .... *is equal to the big triangle.*

Researcher : *Now I want you to use that and come up with some form of explanation.*

Floyd : .... *(after some time) Okay, I've found the height of each triangle and I added them together and I've taken out a common factor and I found that* $h_1 + h_2 + h_3 = H$*, which is the height of the whole triangle.*

Researcher : *But what does it mean ?* $h_1 + h_2 + h_3 - H$ .... *What does it mean ?*

Floyd : *I found the areas of each of the three triangles and then found the sum* $h_1 + h_2 + h_3 = H$............

Researcher : *What does it mean to you if it is equal to H ?*

Floyd : *Yes, when I move it around it does not change* h_1, h_2, h_3 *no matter how much I move it around............*

Researcher : *Are you saying that* h_1, h_2 *and* h_3 *will not change ?*

Floyd : h_1, h_2 *and* h_3 *will change, but when you add all three up, it will remain the same.*

Researcher : *So you're saying that* h_1, h_2 *and* h_3 *changes but when you add them up the sum is staying the same.*

Floyd : *Yes, the sum of every height in the triangle is still the same.*

Researcher : *What does this mean with respect to Sarah ?*

Floyd : *No matter where she builds her house on the island the distance from her house to the beaches will still be the same.*

Researcher : *When you say distance you are referring to the ?* ……………

Floyd : *The sum.*

FLOYD.

71 cm : 1.74

97 cm : 41

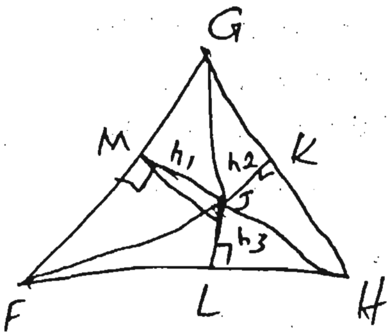
97 cm : 51

Sum = 2.66 cm   Sum = 2.66 cm

Area of GFJ $= \frac{1}{2} Bh.$

$= \frac{1}{2} GH . MJ$

$= \frac{1}{2} a h_1$



area of GJH $= \frac{1}{2}Bh.$

$= \frac{1}{2}ah_2$

area of FJH $= \frac{1}{2}Bh$

$= \frac{1}{2}ah_3.$

Sum $= \frac{1}{2}ah_1 + \frac{1}{2}ah_2 + \frac{1}{2}ah_3 =$ Area of FGH

$\cancel{\frac{1}{2}}a(h_1+h_2+h_3) = \cancel{\frac{1}{2}}aH =$

$h_1 + h_2 + h_3 = H$

Figure 14

Floyd took slightly longer than Nicholas but he understood exactly what he was doing. In fact, Floyd showed a lot of satisfaction as he worked through the explanation stage.

Some pupils had more difficulty arriving at a logical explanation, for example, Kerushnee. Besides taking longer to arrive at the explanation, she required more guidance. Like the other pupils, she was asked to go through the sheet and attempt to follow the guidelines set out, but after a while the researcher noticed that she had not written anything down. This is how the interview proceeded from there.

Researcher : *What is the area of the large triangle ?*

Kerushnee : *Half base times height.*

Researcher : *Okay. Then what is the base of the large triangle ?*

Kerushnee : *a*

Researcher : *So you should write that down. That b that you wrote represents the base. ... What is the base ?*

Kerushnee : *a*

Researcher : *Then maybe you should write that. So you would write half a.*h_1.

Kerushnee : *I've got it here. Half base times height – half . a .* h_1.

Researcher : *You should do it separately. What does small h represent ?*

Kerushnee : *The height of the big triangle.*

Researcher : *So capital H does not represent that ? ……… Somewhere in the sheet you are asked to find a relationship between the large triangle and the small triangles. What relationship do you think exists ?*

Kerushnee : *The total …………the areas of the small triangles = the area of the big triangle.*

Researcher : *Maybe that is what you ought to write. ……………… What can you conclude from this ?*

Kerushnee : *The total area of the big triangle is equal to the sum of the small ones* *inside.*

Researcher : *That we know already, okay. Can you simplify your expression further?*

Kerushnee : *Can I take out the h ?*

Researcher : *Why h ?*

Kerushnee : (silence)

Researcher : *Okay, look at both sides, what can you do to both sides ?*

Kerushnee : (silence)

Researcher : *Okay Kerushnee, let us look at it again. On this side we have half a.H and on this side we have half a.*$(h_1 + h_2 + h_3)$*. What can we do to simplify that ?*

Kerushnee : *I only know that* $H = h_1 + h_2 + h_3$.

Researcher : *But why would you do that ?*

Kerushnee : *Because they are going to be equal to the same thing.*

These statements were encouraging. She seemed to know that $H - h_1 + h_2 + h_3$, but she could not tell why.

Researcher : *Okay, you're saying that all of these (* $h_1 + h_2 + h_3$*) are going to equal to this (H). Why ?*

Kerushnee : *(silence)*

Researcher : *You are telling me the right thing, but look at your equation and tell me why ?*

Kerushnee : *(silence)*

Researcher : *Do you agree that this equation is like a scale ?*

Kerushnee : *Yes.*

Researcher : *The left hand side equals the right hand side ?*

Kerushnee : *Yes.*

Researcher : *Think of the scale, what you do on the left you do on the right.*

Kerushnee : *Maybe I should cancel* ………

Researcher : *What would you cancel ?*

Kerushnee : *The ………………h's.*

Researcher : *Why the h's ?*

Kerushnee : *(silence)*

Researcher : *Okay in any equation what can be cancelled from both sides ?*

Kerushnee : *The common factor.*

Researcher : *What is the common factor ?*

Kerushnee : *The h's.*

Researcher : *Are you saying that the h here is the same as the* $h_1 + h_2 + h_3$ *?*

Kerushnee : *No.*

Researcher : *There's a half here, is there a half there?*

Kerushnee : *Oh, yes.*

This sudden insight made Kerushnee feel excited. From here onwards she found that to arrive at the explanation was much easier.

Researcher : *What else is common ?*

Kerushnee : *The a.*

Researcher : *Okay cancel off what you think should be cancelled off.*

………(after she had done that) *What does this mean ?*

Kerushnee : *It means that if you add up all the heights of the small triangles it will* *give you the height of the big triangle.*

Researcher : *What can we conclude from that ? What does it really mean to us ?*

Kerushnee : *It means that the triangle can be any ....the area can be any amount but the heights will still be the same when you add them together.*

Researcher : *Why ? Why would they be the same ?*

Kerushnee : *Because they belong to an equilateral triangle.*

Researcher : *You are saying that these (h_1, h_2, h_3) can change but their sum will be the same. Why ?*

Kerushnee : *(silence)*

Researcher : *Each of the individual values can change. Do you agree ?*

Kerushnee : *Yes.*

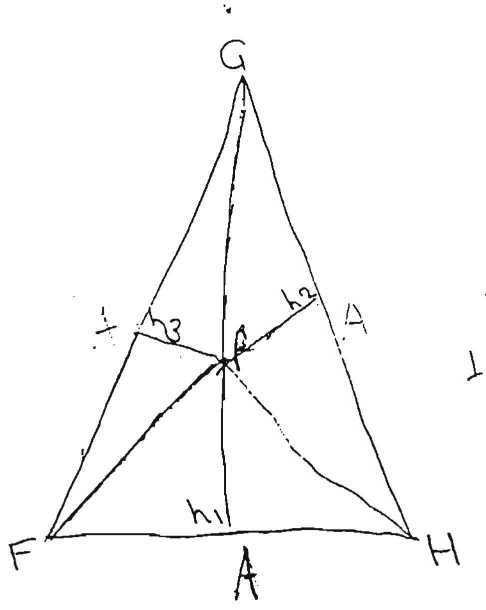
Researcher : *But what can you say about the sum ?*

Kerushnee : *The sum is always the total height of the big triangle.*

Researcher : *Which means that the height will be fixed and therefore ... .?*

Kerushnee : .... *the sum will always remain the same.*

Although Kerushnee required much more guidance, and it took her much longer, she seemed at the end to have understood the explanation (refer to her written work in Figure 15).



$$\tfrac{1}{2}B \times H = \tfrac{1}{2}A \times h_1 + \tfrac{1}{2}A \times h_2 + \tfrac{1}{2}A \times h_3$$

$$A = \tfrac{1}{2}A \times (h_1 + h_2 + h_3)$$

$$=$$

$$\tfrac{1}{2}B \times H = \tfrac{1}{2}A \times h$$

$$\tfrac{1}{2}B \times H = (\tfrac{1}{2}B \times h_1) + (\tfrac{1}{2}B \times h_2) + (\tfrac{1}{2}B \times h_3)$$

$$\tfrac{1}{2}A \times H = (\tfrac{1}{2}A \times h_1) + (\tfrac{1}{2}A \times h_2) + (\tfrac{1}{2}A \times h_3)$$

$$= \tfrac{1}{2}A \times (h_1 + h_2 + h_3)$$

$$= \tfrac{1}{2}A$$

$$H = h_1 + h_2 + h_3$$

Figure 15

Rowan came up with a slightly different explanation, as can be seen in the following excerpt.

Researcher : *Now Rowan I've noticed that you've completed……… What have you done there ? I can see that you found the sum. What have you finally arrived at ?*

Rowan : *half a into h_1 plus h_2 plus h_3.*

Researcher : *Thereafter you equated the sum of the areas of the triangles to that of the area of the large triangle. Why have you equated them ?*

Rowan : *The sum of the areas does not change.*

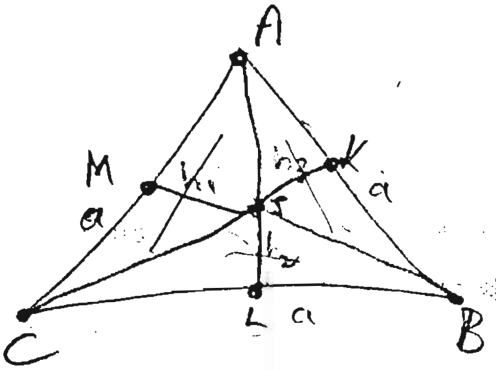
Researcher : ……… *because ?* ……………*why do you think the sum of the areas does not change ?*

Rowan : *Because the large triangle does not change.*

Unlike the other pupils, he immediately noticed that the area of the large triangle was constant, and therefore the sum of the areas of the small triangles also had to be constant (implying $h_1 + h_2 + h_3$ is constant).

Refer to Figure 16 to Rowan's written work.

ROWAN



Area of Δ AJC
$= \frac{1}{2} B \times h$
$= \frac{1}{2} a h_1$

Area of Δ CJB
$= \frac{1}{2} B \times h$
$= \frac{1}{2} a h_2$

Area of Δ AJB.
$= \frac{1}{2} B \times h$
$= \frac{1}{2} a h_3$

$\frac{1}{2} a h_1 + \frac{1}{2} a h_2 + \frac{1}{2} a h_3$
$= \frac{1}{2} a (h_1 + h_2 + h_3)$

Area of $\Delta ABC = \frac{1}{2} a h$

$\cancel{\frac{1}{2}} \cancel{a} (h_1 + h_2 + h_3) = \cancel{\frac{1}{2}} a H$

$h_1 + h_2 + h_3 = H$

Figure 16

From the interviews it was apparent that the pupils were able to construct logical explanations, although some required more guidance than others.

<u>FINDING 3</u> : If given proper guidance, pupils' were able to construct a logical explanation for the conjecture.

4.5 <u>Pupils' interpretation of the guided, logical explanation.</u>

The basic research question in this section was to try and establish whether pupils had experienced the guided, logical explanation as meaningful. More specifically, did it satisfy their earlier expressed needs for explanation or understanding? To attempt to establish this, pupils were asked at the end of the interview whether they found the explanation satisfying or good.

The following responses from Rowan and Manivasan were typical.

Researcher : *Do you find this explanation insightful ?*

Rowan : *Yes.*

Researcher : *Do you think it is a good explanation ?*

Rowan : *Yes.*

Researcher : *You enjoyed working with it ?*

Rowan : *Yes.*

Researcher : *Do you think that this was a good explanation ?*

Manivasan : *Yes.*

Researcher : *Did you find it insightful ?*

Manivasan : *Yes.*

Researcher : *Did you understand it well ?*

Manivasan : *Yes.*

Researcher : *You think it was ?*

Manivasan : *Yes.*

Given that these pupils had not been exposed to such a problem and explanation before, Rhyam's response, in particular, seemed to indicate that it was a satisfying experience for him. His statement was perhaps even surprising in the light of the generally poor performance of these pupils in mathematics tests and examinations..

Researcher : *Do you think that this explanation we gave you there is a good one ?*

Rhyam : *Yes.*

Researcher : *Did you find it insightful ?*

Rhyam : *Yes.*

Researcher : *Did you enjoy working with it ?*

Rhyam : *I wish I could do it again !*

Every pupil answered positively, and seemed to be satisfied with the guided explanation they had worked through. However, the researcher acknowledges that from the results obtained, it is very difficult to conclusively state that the pupils found the explanation insightful. The extracts above do not indicate their exact thoughts. Although it can be argued that they were simply agreeing with the researcher to please him, their smiles gave some indication that they appeared to find the explanation good. Perhaps more care should have been taken to probe further in order to extract their true feelings.

This sense of satisfaction seemed to have arisen from their own participation in establishing the explanation, as well as that it appeared to have satisfied their earlier expressed need for an understanding of why it was true. More research in this area, in particular the development of good diagnostic techniques, is however necesssary.

Finding 4: The guided, logical explanation appeared to satisfy their earlier expressed need for explanation and understanding.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

This research has yielded some valuable results in terms of the teaching and learning of geometry theorems and problems. Given the fundamental importance of proof within mathematics as a discipline, as outlined in the theoretical part of this study, proof should remain an essential part of the secondary school curriculum. Moreover, the teaching (and learning) approach used in the empirical research seemed to provide pupils a greater, and more meaningful, understanding of the role of proof. This study concentrated mainly on the introduction of proof to pupils as a means of explanation rather than as verification.

The following statistics reveal the significant level of conviction that pupils experience whilst exploring geometric statements using *Sketchpad*:

- After initially experimenting, 12 (70,5%) were 100% convinced, 2 (11.8%) were 98% to 99% convinced, 2 (11.8%) were 70% convinced, 1 (5.9%) were 55% convinced.
- Asking them to explore the conjecture further yielded the following results: 14 (82.3%) were 100% convinced, 2 (11.8%) were 98% to 99% convinced, and 1 (5.9%) were 90% convinced.

The research also indicated that pupils had a need for an explanation (deeper understanding) which was independent of their need for conviction. In fact, almost all the pupils stated that they wanted an explanation for why the statement was true. Only one pupil initially felt that there was no need for any further explanation. So it would seem that pupils do exhibit an intrinsic desire for an explanation. Although they had a high level of conviction with respect to their conjecture, it did not seem to satisfy their need for an explanation. They showed signs of being very convinced of what they experienced. Such conviction often reduces a problem to that of the obvious, in other words 'I can see that it is true so why do I need an explanation for it?'. If they were so sure of the result then it should have made no difference to them whether there was some logical explanation for it or not. Yet they expressed a strong desire for an explanation. It seemed that they had recognised the fact that they had merely observed the result through experimentation. Perhaps it could be stated that they were aware of the difference that existed between observation, through experimentation, and *knowing* why it was really true. They undoubtedly wanted to know *why* the result was true and not *whether* the result was true. From the pupils' responses it seemed that the explanation provided insight into the reason why it was true.

More significantly, this research found that given proper guidance, pupils can construct reasonable explanations for their conjectures. Although the different pupils were able to do this at their own pace, they were nonetheless able to do it. It has been stated that these pupils had not constructed a proof before and like most teachers the researcher believed that they could not. From talking to other colleagues, the researcher established that this perception is quite rife. The pupils involved in the experiment, showed that, with guidance, they could construct a proof. In a sense, the act of moving points on a screen and seeing the results displayed on the screen, is a type of proof in itself. Constructing a logical argument hereafter becomes much easier, because seeing the images on the screen, allowed them to see the generalization in the particular diagrams they were constructing. Even an attempt to confuse them (by changing the sizes of the equilateral triangle, although this may be viewed as testing their level of conviction) did not work. This clearly indicates that through active participation pupils can achieve greater understanding of geometrical concepts. The researcher asked the pupils whether they would change their conjecture if the equilateral triangle was made bigger or smaller and, as was stated earlier, they were convinced that the same result would hold, irrespective of the size of the triangle as long as it was an equilateral triangle. Their conviction was of a very high degree, which enabled them to state that they did not need more time to explore. Some of the pupils were showing signs of impatience due to my constantly asking them whether or not they were convinced. This method was therefore a powerful tool for pupils to explore and make their own conjecture, which they can test and then prove. The researcher was convinced that pupils at this point did not display a need for further proof (that is to logically validate their conjecture), but rather to understand why their conjecture was always true. This is how mathematics is experienced as compared to merely learning it. Even Grade Nine pupils saw the need for proof as an explanation (within the context of the problem they worked on), which was, for the researcher, an astounding fact. Thus if proof is going to feature in the curriculum, then it must be presented in such a way that pupils do it for themselves and not simply learn what the teacher or text-book says.

Although it appeared that the pupils found the guided, logical explanation meaningful, no conclusive results can be obtained from the evidence of this research.

This is mainly due to the fact that the pupils answered all questions in the affirmative, thus giving the impression that they were simply agreeing with the researcher. Further probing might have been useful in drawing firmer conclusions.

Although the researcher anticipated that the novelty of 'doing' mathematics might produce some important information, the nature of the responses obtained has certainly been overwhelming. The pupils that were interviewed enjoyed working through the mathematics, although the circumstances seemed quite formal, which resulted in them being somewhat anxious. Unlike the paper and pencil investigations, prior to a proof, the dynamic nature of the presentation surprised them. Looking at this problem presented to them, their surprise did not take long to manifest itself. There was no way that pupils could have correctly guessed that Sarah could build her house anywhere on the island, unless they had seen the result previously. But the results displayed on the screen convinced them that the result did not change for as long as Sarah built her house within the triangle. Pupils, who dragged the point outside, noticed that the sum of the distances were no longer constant. Within the context of the problem this made sense as it would have meant that she would have to build her house in the sea.[2]

Another significant point that ought to be mentioned is that pupils never distrusted the results displayed on the computer. They never gave any indication that they felt that the researcher had 'doctored' the results. This was essential because they were quick to formulate their conjecture that Sarah could build her house anywhere on the island

[2] The result can actually be generalised to points outside the equilateral triangle, but then use must be made of directed line-segments to represent the distances to the sides.

and the sum of the distances would be constant. As it was stated before, none of the pupils had explored mathematics, in this experimental way before, and subsequently, the researcher has been inundated with requests that we continue with similar problems. This interest was encouraging. This researcher was convinced that it was possible to achieve similar, if not better, results with other problems. An important factor, which contributed to this, must have been the notion that in the traditional sense, the theorem and its proof is directly presented to pupils whilst in this experiment pupils were able to first make a conjecture and thereafter, build an explanation, albeit a guided one, themselves. Their experiences, in a short period of time, was far greater than that which they would have gained using traditional methods. In a matter of minutes pupils were able to drag the point inside to many different positions and at the same time they observed the sum of their distances. No matter how accurately a pupil worked, similar results through any other medium might have been very difficult to attain.

Perhaps, it must also be stated that the way the problem was presented was also a novel experience to the pupils. Rephrased the question could have read : Do you think the sum of the distances from any point within an equilateral triangle to its sides will be the same ? Besides being ordinary and boring, it does not allow pupils to become creative and imaginative. The problem itself allowed pupils to imagine an island which was in the shape of an equilateral triangle, and they could relate to the fact that Sarah wanted to surf. Besides using their creativity, they were able to use their imagination in this activity.

Although the debate about the relative emphasis that should be placed on proof might continue, pupils in this experiment showed that they actually desired a proof as a means of explanation. Quite obviously, this experiment focused on one problem and therefore a generalisation may not be appropriate. Nevertheless it seems likely that similar responses could be evoked in other appropriate contexts.

The idea here is that the move should be away from the *traditional method of teaching proof* rather than moving away from proof itself. Instead of the teacher simply verifying the truth of a statement by directly providing a proof, pupils should be conjecturing and seeking explanations for their own observations. Instead of learning the proof that somebody else has already written out, pupils should be encouraged to observe, conjecture, test and seek explanations. This experiment showed that pupils are capable of doing just that. Of course, it might be necessary for the teacher to guide pupils. It is clear that the processes of formulating proof has been hidden from pupils, because of the way proof is presented to them. Pupils believe that the proofs of statements are just written out by someone without the person experiencing any difficulties whatsoever. Pupils here were able to see that proving is a process, whereby empirical testing played a vital role in making a conjecture. The process continued with the conjecture being refined and finally constructing a proof. Also significant was the fact that in some instances pupils had to examine and re-examine the logical statements that they had written.

Not only did pupils learn some of the processes involved in proving, they also learned about the properties of the figure that they worked with. In this case they were able to see that the length of the sides of an equilateral triangle were equal. If time permitted they could have also measured the angles of the triangle.

An important concluding idea that must be mentioned is that the pupils really began writing their proof towards the end of the interview. Most of the work was done by visually observing the changes to the data, if there were any. All that the explorations did, which the pupils engaged in, was increase their conviction. They were able to mentally collate and analyse the information they observed.

Reconstruction of the mathematics curriculum needs to take place where much thought must be given to introducing dynamic geometry at Grade 7 level (or earlier). Recognition must be given to the tremendous power of the computer in creating powerful contexts for the teaching and learning of mathematics, in general and, more especially, mathematical proof. Surely, the "curriculum is to be thought of in terms of activity and experience rather than of knowledge to be acquired and stored" (McIntosh as cited in Floyd, 1981 : 9).

It would be interesting to continue the experiment on a larger scale where group work would be assessed and whether larger groups cope with this type of approach.

A primary aim of future mathematics education courses should be to create an awareness of proving techniques which allow pupils to generate conjectures and develop proofs on their own. Mathematics educators often display techniques that are merely mechanical routines, more especially in the teaching of proof. Attitudes must be changed, by teaching mathematics educators themselves, about proving techniques. An argument that most educators might present is that the method presented here may not be financially viable. Despite this problem, learning about these techniques is essential because the future might bring about changes that would allow pupils to work with computers. Thus teacher educators need to begin focusing on this aspect of proving techniques in their in-service training (INSET) programs. This will enable teachers to gain insight into the many and varied functions of proof.

For the mathematics educator, the following recommendations can be made:

- Problems (or theorems) should be presented to pupils in a way that would ensure greater understanding, as compared to simply directly presenting the theorem and its proofs. Instead of pupils learning proofs for re-writing them for examinations and tests only, they must be taught to prove, with guidance if necessary, on their own. The researcher is confident that as pupils are exposed to more examples, less guidance from the teacher would be necessary.
- Pupils must be allowed to attempt to construct their own proof. If worksheets are well planned then pupils can be guided to a proof that may satisfy their desire for a proof as an explanation.
- Although this experiment concentrated on a one-to-one interview, it could be hypothesised that the results would be just as significant in groups as well.
- Proof writing must be made to be an enjoyable task rather than a session of boring facts.

With respect to curriculum planners and officials of education departments :

- The idea is to ensure that the child has every opportunity to do well in mathematics. Proof writing is an important part of mathematics examinations. Results in examinations and the mathematical literacy of pupils could improve, mainly because of the interest they will show in the subject.
- Curriculum planners should include material, which uses this method for teaching proof, as this approach of exploring, conjecturing and explaining could also be carried without the use of a computer. The computer and software used only makes it the process easier.
- Teacher education institutions should also focus on proof and proving.

This research also presents a good platform for further related research in the following areas:

I. An investigation, which would chart the progress of pupils through the Van Hiele levels in a dynamic geometry environment. Although the Van Hiele theory does not foresee the possibility of pupils' understanding proof before the attainment of Level 3, a dynamic geometry environment might stimulate proof (particularly as a means of explanation) on Level 2, or perhaps even Level 1.

II. Research which would indicate whether similar results can be obtained with a classroom of pupils, instead of a one-to-one interview.

III. Further research, which would investigate a greater variety of problems, making use of the different functions of proof.

IV. Perhaps, more relevant to the present classroom situation in many South African schools, an investigation can be carried out to ascertain whether these results are also true for non-dynamic geometry environments.

V. Further research needs to be carried out in order to determine whether examination and test results improve if pupils are exposed to these types of environments.

VI. A substantive, longitudinal study should be carried out in order to determine whether pupils have acquired an understanding of the different functions of proof.

R : RESEARCHER K = KUMARASEN

R	Ok, Kumarasen, now that you understand the question, I want you to make a conjecture as to where you think Sarah should build her house?
K	In the centre.
R	Why?
K	Well if you build anything in the centre then there's always a short distance all around it.
R	What I want you to do now is to move point P around and observe carefully these distances. Do you know what these represent? (Pointing to the distances on the screen.) Okay. Further, observe this sum. (After a while.) Why have you stopped?
K	I noticed that the distances from the house to the beach always changes but the sum is always constant.
R	Are you saying that no matter where P is the sum is always constant?
K	Yes.
R	What if I move P to a corner, would the sum change?
K	No. (Emphatically.)
R	Are you convinced?
K	Yes.
R	You are not looking so sure. How many percent convinced are you? (silence.) Would you say about 60%?...
K	About 55%.
R	Let's see if you can convince yourself further? ... How would you convince yourself further?
K	By measuring from her house to the beach at each point on the triangle.
R	Do you want to try to do that?
K	Yes. No matter where P is the sum of the distances will always be the same.
R	What do you think would happen if we changed the size of the equilateral triangle?
K	Then the sum of the distances will change.
R	What do you mean when you say that the sum of the distances will change?
K	When you make the triangle bigger then the sum will change.
R	But within that triangle will the sum change?
K	Yes.
R	Okay, let's make this triangle bigger. ... now test the conjecture.
K	The sum never changes even if the triangle is bigger or smaller. The sum will always stay the same.
R	Are you saying that as long as it is an equilateral triangle, the sum within the triangle will be the same?
K	Yes
R	Are you sure?
K	Yes.
R	You said before that you were 55% convinced, now how many percent convinced are you?

K	About 100%. Because no matter how big or small you make the triangle the sum of the distances will always remain the same.
R	Do you have any doubts?
K	No.
R	Now that you are so convinced, would you like to know why it is true?
K	Yes, I would.
R	Can you give me an explanation as to why that is true?
K	Well by changing the size of the triangle you are changing the distances from the house to the beaches.
R	Ok. What you are saying is not really giving us an explanation. What we want are logical reasons. Do you think that you can come up with some form of logical reasoning?
K	No.
R	Would you like to have such a logical explanation?
K	Yes.
R	Initially you told me that she should build her house at the centre, now you've concluded that, really, it can be anywhere. Do you find the result surprising?
K	Yes, because at first you think that it should be in the centre and the sum will be small but now it can be anywhere.
R	I'm now going to give you a sheet. I want you to read through it and when you are finished we'll take it from there.
R	(After a while.) With regard to what you have read, can you tell in your own words what that says?
K	I can't explain it.
R	What distinction is made between … what two things?
K	(Silence.)
R	There's a distinction made between two kinds of activities …
K	(Silence.)
R	Would you agree that there is a distinction between experimentation and explanation?
K	Yes.
R	What do you think you were doing?
K	Experimenting where she should build her house so that she can have a small total distance from her house to the beaches.
R	Do you agree that you have been experimenting and you have not come up with an explanation?
K	Yes.
R	So now we want to come up with some logical explanation as to why this is true. Now read through this … (handing pupil a sheet.) … (After a while.) I see you have written something down. What have you got?
K	I've written down the areas of the three triangles.
R	And what have you done here? (Pointing to the sheet.)
K	I've equated the sum to the area of the large triangle.
R	Why?

K	Because the 3 triangles add up to the area of the big triangle.
R	I notice that you wrote $h = h_1 + h_2 + h_3$. What does this mean?
K	It means that if you add the areas up you will get the area of the big triangle.
R	But that is terms of the area. I'm referring to this part here. You have finally arrived at something which says that $h_1 + h_2 + h_3 = h$. What does that mean?
K	It means that if you add the area of the 3 triangles ....
R	Hoes h represent the area of the triangle?
K	Yes.
R	Look at your diagram and tell me what h_1, h_2 and h_3 represent?
K	h_1 represents this triangle, and ...
R	Does it really represent the triangle?
K	It represents the distance from the house to the beach.
R	In the context of a triangle what does it mean?
K	The area.
R	Alright you've got a formula for area, what is it?
K	½ base times height.
R	So what does h_1 represent then?
K	The distance from the house to the beach.
R	In terms of the triangle?
K	The height.
R	So what does h_1, h_2 and h_3 mean now?
K	The 3 different heights. If you add them up you get the big height of the whole triangle.
R	What did you notice about h_1, h_2 and h_3 when you moved point P around?
K	They always changed but the sum remained constant.
R	What do you know about the height of the large triangle?
K	But the height always stays constant.
R	S what does it really mean?
K	She can build her house anywhere within the island the sum of the distances will always be the same.
R	Does this satisfy your need for an explanation? Do you find this explanation satisfactory?
K	Yes.
R	Does it explain to you why this result is true?
K	Yes, no matter where you build the house the constant sum is there.
R	So are you satisfied with this kind of explanation?
K	Yes.
R	Did you really understand it?
K	Yes.
R	Thank you.

M = RESEARCHER R = RHYAM

M	Ok, Rhyam where do you think that Sarah should build her house?
R	In the centre.
M	Why do you think in the centre?
R	Because the centre seems to be most suitable to go to any beach.
M	That's fine. What I want you to do now is move the point P around and whilst you're doing that I want you to observe the distances PM, PC and PK. At the same time observe the sum. Move the point around as much as you want to and when you are satisfied that you have moved it around enough you can tell me.
R	(After a while.) Ok. The sum remains constant and PM, PL and PK always change.
M	Are you saying that it's true throughout or only at the points you moved it to?
R	Throughout.
M	Are you quite sure that if I moved the point P to this corner here, that it won't change?
R	Yes.
M	It won't change?
R	No.
M	Are you convinced?
R	Yes.
M	If I asked you how many percent convinced are you, what would you say?
R	100 %
M	100 %! If I made this a larger equilateral triangle, do you think the result will change?
R	No.
M	You're quite sure.
R	Yes.
M	Wouldn't you like to just check?
R	Ok. (After a while.) Yes it remains the same.
M	Do you find this result quite surprising? You did say that P should be at the centre before, now you indicate that it can be anywhere in the triangle. Do you find the result surprising?
R	Yes.
M	I'm going to give you a sheet which I want you to read. Read just the initial part first. (After a while.) Can you in your own words tell me what you have read?
R	(Silence.)
M	What are the two things that are being differentiated between?
R	Exploration and explanation.
M	Can you give me another word for exploration?
R	Explaining.
M	Are you saying that exploration is another word for explanation?
R	Finding out further…
M	Ok, exploration is similar to experimenting, whilst explanation is finding logical reasons for why

	something is true. We have been experimenting all this while, but now we want to find an explanation. Do you think that there is a need for an explanation?
R	Yes.
M	You really would want to know why?
R	Yes.
M	Why, why would you want to know why?
R	I like to find out why things are taking place.
M	You simply want to know why? Good. Rhyam, at the bottom of the page you would notice that there are a few points. I want you to work through each point and ask questions if you want to. (After a while.) It seems you are finished. Can you tell me what you have done?
R	I've added all the areas of the small triangles.
M	And what have you got?
R	The most common is ½.
M	Is only ½ common?
R	½ a.
M	Thereafter what did you do?
R	½ a into $h_1 + h_2 + h_3$.
M	Why did you equate the area of the large triangle to the sum of the areas of the small triangles?
R	They're equal.
M	Why would you say that?
R	Because the small ones add up to the big one.
M	What does your last statement mean to you? (Referring to $h = h_1 + h_2 + h_3$.)
R	The small triangles are changing.
M	So what can we say about h_1, h_2 and h_3?
R	They are also changing.
M	What can we say about h?
R	It doesn't change.
M	What is another word for "it doesn't change"?
R	Stays constant.
M	How would you summarise what you have just said?
R	$h_1 + h_2 + h_3$ remains constant although h_1, h_2 and h_3 are changing because h is the same.
M	So where should Sarah build her house?
R	She can build it anywhere on the island.
M	Do you think this explanation we gave you there is a good one?
R	Yes.
M	Did you find it insightful?
R	Yes.
M	Did you enjoy working through it.
R	Yes. I wish I could do it again.
M	Thank you.

M = RESEARCHER R = ROWAN

M	Rowan, where do you think that Sarah should build her house?
R	In the centre.
M	So it will be close … it will be the same distance to all three beaches.
M	Do you think that if it the same distance to all three beaches then the sum will be a minimum?
R	Yes.
M	I want you to grab this point P and move it around. Observe at the same time the distances PM, PL and PK and at the same time observe their sum. As soon as you're satisfied you may stop. (After a while.) Now tell me what did you observe.
R	The sum of the distances does not change while the distances changed.
M	These distances here (pointing) changed? Is that what you're saying?
R	Yes.
M	Are you sure? Do you think that if I had to move P right to the bottom here (pointing to the triangle), the sum won't change?
R	It won't change.
M	Are you positive?
R	Yes.
M	Would you say that you are quite convinced?
R	Yes.
M	If I made this a larger equilateral triangle, do you think the sum will change?
R	No.
M	A smaller equilateral triangle?
R	No.
M	Wouldn't you want to just check?
R	Okay. (After a while.) Yes it is the same.
M	Alright if I asked you how many percent convinced are you, what would you say?
R	100%
M	A 100%?
R	Yes.
M	Rowan do you think that there is a need for an explanation? Do you want to know why this is true?
R	Yeah, there is a need for an explanation.
M	Why do you think there is such a need? … Why?
R	So we will be able to understand more clearly the diagram.
M	So you actually want to understand. Do you find this result surprising?
R	Yes.
M	Okay. Take this sheet and read through it and when you're finished I'll ask you some questions. (After a while.) Can you briefly tell me what you read in the preamble?
R	(Silence.)

M	Would you be able to?
R	Yes.
M	What have you read in the preamble?
R	(Silence.)
M	What did it say?
R	(Silence.)
M	Okay there is distinction being made between two things, what are these two things?
R	Exploration and explanation.
M	Is there a difference between these two?
R	Yes.
M	What does exploration entail? … What is another word for exploration? … What were we doing all this while?
R	Experimenting.
M	So we were experimenting and exploring. Now we want to find a possible explanation. Do you think that you can give us an explanation?
R	No.
M	Don't you want to try?
R	No.
M	At the bottom of the same sheet you will find a list of instructions. Read through that and see if you can come up with an explanation. (After a while.) Now Rowan I noticed you've completed. What have you got there?
R	$\frac{1}{2}(h_1 + h_2 + h_3)$
M	Why did you eventually equate the sum of the areas of the triangles to that of the big triangle?
R	The sum of the areas does not change.
M	Why does it not change?
R	Because the large triangle does not change.
M	So you are saying that the areas of the small triangles will change but their sum won't because the area of the large one does not change. That is correct. What did you finally arrive at?
R	$h = h_1 + h_2 + h_3$
M	What does it mean to us?
R	The sum of the distances will not change, but if you move P then the distances will change.
M	What does it mean in terms of Sarah's house?
R	She could build her house anywhere on the island.
M	Do you find this explanation insightful?
R	Ycs
M	Do you think that it is a good explanation?
R	Yes.
M	Did you enjoy working with it?
R	Yes.
M	Thank you.

M = RESEARCHER K = KERUSHNEE

M	Kerushnee, you seem to understand the question. Where do you think that Sarah should build her house?
K	I think that Sarah should build her house in the centre.
M	Why?
K	Maybe here the sum of the distances will be the smallest.
M	This computer program will allow you to take the point and move it around. I want you to look at the diagram on the screen. Move this internal point around and observe the distances MJ, JL and JK. At the same time observe the sum of the distances. (After a while.) Why did you stop?
K	The sum of the distances, they don't change.
M	But what is changing?
K	The distances from Sarah's house to the beaches.
M	I noticed that you didn't move to many points ……are you saying that what you observed will be the case anywhere in the triangle?
K	Maybe.
M	If we made the triangle bigger or smaller, do you think now if we moved the point around, will the sum change?
K	I don't think so.
M	Try it, let us see.
K	(After a while.) The sum is not changing.
M	So what can we say irrespective of the size?
K	No matter where you move the point the sum will still remain the same.
M	Do you feel that you are convinced that that will be the case always? Are you sure that if I moved it to that corner point there, it will not change?
K	I don't think it will change.
M	At every point, you are fairly convinced that it will be true?
K	No. Not that convinced.
M	Let's try it again. How do you think you can convince yourself some more?
K	If I could measure all the sides and find the sum.
M	How would you do that? … What does the sides have to do with those distances?
K	I will measure all the distances from Sarah's house.
M	Are you saying that you will physically measure it with a ruler or the computer?
K	With the computer.
M	But they have been measured already and they have been added here. That has already been done.
K	I would quickly draw one.
M	So you would like to quickly draw one and measure it yourself?
K	Yes.
M	Do you think that your equilateral triangle will be different from yours?
K	No.

M	Kerushnee, I want **you** to convince yourself that what you are saying will always be true.
K	(After a while.) When I'm moving it everywhere it still remains the same.
M	So if I asked you if you were 60 % convinced what would you say?
K	I think I'm more that 60 % convinced.
M	How many percent convinced are you?
K	100 %.
M	So, you don't have any doubts that it will not hold true at some point?
K	I did try and I don't think there is such a point.
M	You seem to be fairly convinced. Keeping in mind the conjecture you made, do you find this result surprising?
K	Yes, I find the result very surprising.
M	Kerushnee, now that you told me that you are 100 % convinced, do you think that you need to know why the result is true?
K	Yes.
M	Do you want to attempt an explanation on your own?
K	Yes. I think it is because it is an equilateral triangle. Anywhere she builds her house it would be the same.
M	You are basically repeating the statement. You have read the question and you have seen the result. What we mean by explanation is that you must explain in terms of something else. What you are giving is a simplistic answer. What we need is a logical reasoning – we want you to logically explain why it is the case. Do you think you can do that?
K	No, I don't think so.
M	Would you like to see such an explanation?
K	Yes.
M	I'm going to give you a sheet now, which has an explanation on it. Note that there are 6 points here. What you would need to do is read through the page and work through each point. Would you want to try?
K	Yes.
	TAPE FAULTY
K	They say that nobody ever questions his discovering and that, er, …he wanted to know how that was so.
M	Would you agree that he is making a distinction between an explanation and experimenting?
K	Yes.
M	In some sense what have you done here? What would you describe that work as?
K	Explanation.
M	Is that an explanation?
M	Did you give us an explanation for what was going on?
K	Experimentation.
M	That was experimentation, but what you don't yet have is the explanation.
M	Do you know what they're saying here?
K	Yes.

M	For example, if the sun rises every morning and somebody asks you why does the sun rise, you can't just say it rises. There must be some explanation for that and that is the reason for what is going on now. I want you to go through E1 to E6. If there is something you don't understand please ask.
K	Okay.
M	(After a while.) What is the area of the large triangle?
K	½ base times height.
M	Ok. Then what is the base of the large triangle.
K	a
M	So you should write that down. That b that you wrote represents the base. What is the base?
K	a
M	Then maybe you should write that. So you would write ½ ah.
K	I've got it here ½ base x height = ½ ah.
M	Maybe you should do it separately.
M	What does small h represent?
K	The height.
M	Off?
K	The big triangle.
M	So capital H does not represent that? Some where in the sheet they ask you to find a relationship between the large triangle and the small triangle. What relationship you think exists?
K	The total … the area of the small triangles = to area of the big triangle.
M	Maybe that's what you ought to write now. (After a while.) What can you conclude from this?
K	The total area of the big triangle = three small areas inside.
M	That we know already. Can you simplify your expression further. ($\frac{1}{2}ah = h_1 + h_2 + h_3$). Can you simplify further?
K	Can I take out the h?
M	Why h?
K	(Silence).
M	Okay look at both sides. What can you do to both sides?
K	(Silence).
M	Okay Kerushnee, let's look at it again. On this side we have $\frac{1}{2}a(h_1 + h_2 + h_3)$. What can we do to simplify that?
K	I only know that $h = h_1 + h_2 + h_3$.
M	But why would you do that?
K	Because they are going to equal to the same thing.
M	You're saying that all of this ($h_1 + h_2 + h_3$) are going to equal this (h). Why?
K	(Silence). Because this is an equilateral triangle. No matter how many triangles you get inside, the height of the big triangle is the same.
M	You are saying the correct thing, but look at your equation and tell me why $h = h_1 + h_2 + h_3$
K	(Silence.)

M	Do you agree that this equation is like a scale?
K	Yes.
M	The left hand side = the right hand side.
K	Yes.
M	Think of the scale. What you do on the left you must do on the right.
K	Maybe I should cancel …
M	What would you cancel?
K	The ….. h's.
M	Why the h's?
K	(Silence.)
M	Okay in any equation what can we cancel from both sides?
K	The common factor.
M	What is the common factor?
K	The h's.
M	Are you saying that the h here is the same as the $h_1 + h_2 + h_3$ here?
K	No.
M	There is a ½ here. Is there a ½ there?
K	Oh yes.
M	What else is common?
K	The a.
M	Now cancel off, what you think should be cancelled off. (After a while.) What does this mean?
K	It means that if you add up all the heights of the small triangles, it will give you the height of the big triangle.
M	What can we conclude from that? What does it really mean to us?
K	It means that the triangle can be …. The area can be any amount but the heights will still be the same when you add them together.
M	Why? Why would they be the same?
K	Because they belong to an equilateral triangle.
M	You said that these (h_1 , h_2 , h_3) can change but the sum will be the same. Why?
K	(Silence.)
M	Each of the individual values can change. Do you agree?
K	Yes.
M	But what can you say about the sum?
K	Their sum is always the total height of the big triangle.
M	Which means that the height will be fixed and therefore …?
K	… the sum will always remain the same.
M	Does that explain to you what you have observed?
K	Yes.
M	I just want to make sure you understood it. Can you repeat it to me?
K	No matter where Sarah builds her house the total sum will always be the same.

M	Why would she be able to do that?
K	Because the three distances, no matter how they change the height of the big triangle will always remain the same.
M	So you understand it fairly well?
K	Yes.
M	Does the explanation really satisfy your curiosity?
K	Yes.
M	Thank you.

	to do the same for this.
M	Really, that is good. Do you think that you would be able to come up with an explanation on your own?
D	No.
M	Wouldn't you want to try?
D	No.
M	What we have here is a sheet, which gives a possible explanation. I want you to read through it and let me know when you're finished? ………….. Okay Debashnee, you seem to have read through. What is being said in the preamble?
D	(Silence.)
M	Would you be able to say anything?
D	No.
M	Do you find that there is a distinction being made between two things? Do you know what these two things are?
D	Exploration and explanation.
M	You can see this distinction, but what is another word for exploration?
D	Experimentation.
M	Yes experimentation. What have you been doing? Were you exploring or were you experimenting?
D	Experimenting.
M	What we need to do now is find an explanation for our observation, is that not so?
D	Yes.
M	At the bottom of this sheet, there are points E1 to E6. I want you to go through all these points. Draw a diagram by copying exactly what you see on the screen and attempt to come up with an explanation. Would you be able to do that?
D	Yes.
M	(After a while.) I notice that you are finished. Very briefly tell me what you've done.
D	I've found the area of the three sides.
M	Sides? Areas of the sides?
D	No. The area of the triangles. I then added up the areas.
M	Which triangle did you work with? The big one or the small ones?
D	The small ones.
M	And then what did you do?
D	I added it up.
M	You added them up here (pointing) and what did you finally get?
D	I've got $\frac{1}{2}a(h_1, h_2, h_3)$.
M	Is it h_1, h_2, h_3? Or is it $h_1 + h_2 + h_3$?
D	$h_1 + h_2 + h_3$.
M	Okay, so I noticed here, you equated the area of the large triangle, to the sum of the areas of the small triangles. Why?
D	Because it's equal.

M	What's equal?
D	The area of the large triangle and the sum of the areas of the small triangle.
M	Thereafter, what have you done?
D	Thereafter I cancelled both the ½'s...... (silence).
M	What else have you cancelled off?
D	The a.
M	So ½ a was cancelled off and what have you got?
D	I've got $h = h_1 + h_2 + h_3$.
M	What does this really mean?
D	(Silence.)
M	What can you say about this statement with respect to what you observed?
D	(Silence.)
M	What can you say about h_1 , h_2 and h_3 as we moved point P around?
D	It changes.
M	So they changed. What did we notice about $h_1 + h_2 + h_3$?
D	It remained the same.
M	And what can we say about this h here (pointing) ?
D	It's the same height.
M	When you say that it is the same height I presume you mean it is constant?
D	Yes.
M	So you're saying that h_1 , h_2 , h_3 changed, but $h_1 + h_2 + h_3$ remained the same?
D	Yes.
M	But then you also stated that h is constant?
D	Yes.
M	So what does this imply?
D	Although h_1 , h_2 and h_3 changes their sum will stay the same.
M	Where then should Sarah build her house?
D	Anywhere.
M	Now that you've gone through this explanation, do you think that it was insightful?
D	Yes.
M	Do you think that you would be able to work with something like this again?
D	Yes.
M	Thank you.

M = RESEARCHER E = EMILY

M	Emily can you tell what does the question require?
E	Sarah wants to find … er … er…
M	… a spot where she can build her house. What are the conditions for that spot?
E	It has to be the shortest distance to all the beaches.
M	Is it the shortest distance? Is that what she really wants?
E	Yes.
M	The question requires you to find the point within the island where Sarah would be able to build a house such that the sum of all distances from the house to the three beaches will be a minimum. Do you understand exactly what I'm talking about?
E	Yes.
M	Emily, tell me where you think that Sarah should build her house?
E	In the middle?
M	I want you to check whether what you're saying is correct? Move point P around and observe these three distances listed on the left hand side PM, PL and PK. Below is the sum of these three distances. Observe both simultaneously and tell me what you notice.
E	(After a while.) They change.
M	What is changing?
E	The distances from P to M, P to L and P to K.
M	What else did you observe?
E	The sum of the distances remain the same.
M	What does this mean to you?
E	No matter where she builds her house the sum of the distances will always be the same.
M	Just so that I know that you understand, can you say what you just said in other words?
E	(Silence)
M	You just said that no matter where she build her house the sum of the distances will be the same. Considering what you said initially that she should build her house in the middle, what do you say now?
E	I think that she can build her house anywhere.
M	If I had to make this triangle bigger, do you think the result will change?
E	It will change if the triangle is made bigger.
M	Yes if the triangle is made bigger then the sum will change if it is compared to the other triangle. But within this bigger triangle will it change?
E	No.
M	Would you want to just check?
E	Yes.
E	(After a while). Yes it remains the same within the bigger triangle.
M	So you are saying that no matter which equilateral triangle you use the principle will be the same?
E	Yes.

M	If I ask you how many percent convinced are you, what would you say?
E	About 70 %.
M	So you do have some doubt?
E	Yes.
M	Do you want to try again to further convince yourself?
E	Yes.
M	(After a while). I see you've stopped. What does that mean?
E	I'm … … I'm … …
M	Are you a little bit more convinced?
E	Yes.
M	What percentage do you think?
E	About 90% now.
M	Do you find the result here surprising?
E	A little.
M	Compared to what you had guessed?
E	Yes.
M	Would you like to know why this is the case?
E	Yes.
M	Would you want to attempt an explanation on your own?
E	Yes.
M	Okay I'll give you some time. (After a while.) I've noticed that you have not written anything down. What does that mean?
E	I can't …. (silence).
M	You can't find an explanation?
E	No.
M	I'm giving you a sheet now, which has points E1 to E6. These points help you with determining a possible explanation. If you look at E1 it says that you must label all the sides 'a' and the distances from P to the sides AB, BC and AC respectively as h_1, h_2 and h_3.Please do that now.
E	(After a while). I'm finished with that.
M	Now E2 says, write expressions for the areas of these triangles.
E	(After a while). Okay.
M	The next step requires the removal of a common factor. (After a while). Emily, you seem to have completed that. Can you tell me your simplified answer?
E	$\frac{1}{2} a (h_1 + h_2 + h_3)$
M	The next step asks: How does the sum of the areas of the small triangles relate to the area of the large triangle?
E	The sum of the small triangles add up to the big triangle.
M	You mean the areas?
E	Yes.
M	Equate them and see what you get.

E	(After a while). Okay.
M	Explain to me what you've arrived at.
E	I've got ½ a which is common on both sides, so I've taken them out and I've come up with $h_1 + h_2 + h_3 = h$.
	TAPE FAULTY.
E	As I move the point around $h_1 + h_2 + h_3$ changes.
M	Does $h_1 + h_2 + h_3$ change?
E	No, h_1, h_2 and h_3 changes.
M	What can you say about the sum?
E	The sum remains the same.
M	What does this really mean?
E	Sarah can build her house anywhere and the distance will always be the same.
M	Did you enjoy this?
E	Yes.
M	Thank you.

M = RESEARCHER H = HIGASHNIE

M	Higashnie, now that you understand the question, where do you think Sarah should build her house?
H	In the centre.
M	Why do you say so?
H	Because it will be closer to all the beaches and the distances will be the same.
M	What I want you to do now is grab the point P, like this, and move it around and I want you to observe what would happen on the left hand side here. Observe these distances PM, PL and PK and at the same time observe the sum of those distances at the bottom. I'll give you some time to do that. (After a while). Okay you've stopped. What can you tell me about what you've observed?
H	The distances change but the sum remains the same.
M	Do you think it will be the same throughout this triangle?
H	Yes, throughout the triangle.
M	What if I made this triangle slightly bigger, an equilateral triangle nonetheless ... I made it bigger, what do you think would happen?
H	It won't change. It will remain the same.
M	If I made it smaller, would it change?
H	No.
M	Why don't you test it?
H	Okay. Yes it remains the same.
M	So you seem quite convinced that the result will be the same. If I asked you how many percent convinced are you, what would you say?
H	100 %.
M	Arc you surc?
H	Yes.
M	Do you find the result surprising?
H	Yes.
M	Now that you are a 100 % convinced, do you think that there is a need to find an explanation for what you've discovered?
H	Yes.
M	Why? Why, do you think there should be an explanation?
H	I would like to find out more about it myself and know more about it than just finding out from the computer.
M	So you what to really know why it is true?
H	Yes.
M	Do you think that you would be able to come up with an explanation yourself?
H	No.

M	Wouldn't you like to try?
H	No.
M	Higashnie, this sheet contains a possible explanation. First read through the preamble, and I'll ask you some questions. (After a while). Can you give me a brief summary of what you read? (Silence). Would you be able to?
H	No.
M	Higashnie, there is a distinction being made between two things. What are they?
H	Explanation and exploration.
M	What is another word for exploration?
H	Experimenting.
M	What have we been doing all this while?
H	Experimenting.
M	Yes, we were experimenting. What we need to do now is find an explanation. Read the sheet at the bottom. There are six points which I would like you to work through. I'll give you some time. (After a while). Higashnie, now that you've completed that task, can you briefly tell me what you've done?
H	I added the areas of the three triangles.
M	Yes.
H	I then equated it to the area of the big triangle.
M	What have you got?
H	I came up with a common factor ½, I cancelled it and I've got $h_1 + h_2 + h_3$.
M	What is that equal to?
H	½.
M	I can see you cancelled ½ and a, but what have you arrived at?
H	$h_1 + h_2 + h_3$.
M	What is that equal to?
H	h.
M	So you've got $h_1 + h_2 + h_3 = h$. What does that mean?
H	The sum is the same throughout and Sarah can build her house anywhere on the island.
M	Did you easily understand the explanation?
H	Yes.
M	Did you find it insightful? Did you enjoy working with it?
H	Yes.
M	Thank you.

M = RESEARCHER K = KOVILAN

M	Kovilan, now that you understand the question, where do you think that Sarah should build her house?
K	In the centre.
M	Why do you say that it should be in the centre?
K	(Silence.)
M	Did you just guess?
K	Yes.
M	Kovilan I want you to check whether what you're saying is correct. Grab this point P and move it around. As you move it around I want you to observe the measurements PM, PL and PK, which are the distances from the house to the beaches. At the same time I want you to observe the sum. Then I want you to tell me what you observed.
K	(After a while). The sum of the total – the three distances change but the sum does not.
M	Can we go through that again? What changes?
K	The distances.
M	Which distances?
K	The three – PK, PL and PM.
M	In other words the distances from P to the three sides. But what else did you notice?
K	PL and PK and PM remain constant.
M	But you just said that they change.
K	The total – PM + PL + PK remains constant.
M	So that doesn't change?
K	Yes.
M	Are you convinced that that is true?
K	Yes.
M	How many percent convinced are you?
K	100%.
M	You are a 100% convinced?
K	Yes.
M	What would happen if I made this a slightly bigger or smaller? I want to know whether the result you were a hundred percent sure of …… will it still be true?
K	Yes.
M	Are you convinced that that would be true? Do you want to try again? (After a while.) Alright, are you convinced that what you said is correct?
K	Yes.
M	Do you think that there should be an explanation for this? Do you want to know why this is true?
K	Yes.
M	Do you think that you would be able to come up with an explanation on your own?
K	Yes, I'll try.

M	I've noticed Kovilan that you have not written anything, what's going on? Do you know how to explain it?
K	No.
M	I've got a sheet with a possible explanation in a step wise way. Read through it. (After a while). Have you done the first part?
K	Yes.
M	Now look at E2. Do that now. (After a while.) Have you completed that?
K	Yes.
M	Now add them and simplify your answer. (After a while). What have you got?
K	I've added ½ ah_1 + ½ ah_2 + ½ a h_3 and I've noticed a common factor.
M	What is your final answer?
K	½ a ($h_1 + h_2 + h_3$).
M	The next step requires you to find the relationship between the areas of the three triangles and the area of the large triangle. What is that relationship?
K	(Silence).
M	Is there any relationship?
K	The areas add up.
M	The areas of which one adds up?
K	The small ones.
M	To which triangle do they add up?
K	The big one.
M	So the areas of the small triangles adds up to the area of the large one?
K	Yes.
M	Now show this relationship – write it down.
K	I wrote ½ a ($h_1 + h_2 + h_3$) = ½ ah and I cancelled off ½ a.
M	And what have you arrived at?
K	$h_1 + h_2 + h_3 = h$.
M	What does it mean?
K	In the smaller triangles the heights change but the sum of the large one does not.
M	What do you mean by the sum of the large one?
K	The height of the large triangle.
M	What does this mean then? If the small ones change and the large one does not, what can we say about the sum?
K	It won't change. It remains constant.
M	Did you understand the explanation?
K	Yes.
M	Thank you.

M = RESEARCHER N = NICHOLAS

M	Nicholas, now that you understand the question, where do you really think that Sarah should build her house?
N	I thought towards any one of the corners of the … of the …
M	Island?
N	Yes.
M	Why?
N	(Silence).
M	Nicholas I want you to check whether you are right. Grab point P and move it around within the triangle. Observe the distances PL, PM and PK and at the same time observe the sum of these distances. (After a while). You've stopped Nicholas. What's wrong?
N	No matter where I move point P their total remains constant.
M	You mean their sum remains constant?
N	Yes.
M	Are you convinced?
N	Yes.
M	Do you want to try some more?
N	(After a while). Okay, I'm convinced that the sum will always remain constant.
M	That changes from what you thought it would be?
N	Yes.
M	Are you surprised?
N	Yes.
M	How convinced are you that this will not change?
N	I'm convinced that it will not change no matter how long I try.
M	How many percent convinced would you say?
N	100%.
M	You're actually a 100% convinced?
N	Yes.
M	Nicholas, do you want to know why this is true?
N	Yes.
M	Do you think you would be able to come up with an explanation?
N	I can try.
M	I'll give you a moment. (After a while). I've noticed you didn't write anything at all.
N	I didn't come up with an explanation.
M	You can't?
N	No.
M	Nicholas, I'm going to give you a sheet which has a possible explanation on it. Let us go through E1 to E6. Can you do that now. (After a while). Okay I can see that you've done that. The next step asks for you to add all three up. Do you know what to do?
	Yes.

	I notice that you've got $A_1 + A_2 + A_3$. Simplify that expression. (After a while). Describe what you've done.
N	I've removed ½ a as a common factor and I've got ½ a $(h_1 + h_2 + h_3)$.
M	Nicholas can you tell me how these areas of the three triangles relate to the area of the large triangle?
N	The area of the three triangles when you add it up, will give you the area of the big triangle.
M	If that is the case and we found the sum of the three triangles then what can we conclude? …(Nicholas is silent). That the sum of the area of these three triangles equal to …
N	The area of the big triangle.
M	Now can you look at E4. I want you to write down this expression.
N	I noticed that the big triangle also had ½ a in it. So I cancelled off the ½ a from the big triangle and ½ a from the 3 other triangles.
M	And what have we arrived at?
N	The height of the three triangles … when you add it up it gives you the height of the big triangle.
M	What does this mean to you?
N	No matter what the heights of the three smaller triangles it will always equal the height of the big triangle.
M	So what does it mean in terms of Sarah's house now?
N	It means that no matter where she puts the house the total distances will always be constant.
M	So do you think that this is a good explanation?
N	Yes.
M	Thank you, Nicholas.

M = RESEARCHER R = RODNEY

M	If you understand the question Rodney, I would like for you to tell me where do you think Sarah should build her house on the island?
R	In the centre.
M	Why centre? Do you have any particular reason for saying centre?
R	It would be most appropriate.
M	Then Rodney I want you to check. I want you to grab point P and move it within the triangle. Observe the distances PM, PL and PK and at the same time observe the sum of these distances. Do that now. (After a while). What did you observe?
R	Wherever she goes …
M	No, no. I'm talking about your moving the point. Alright carry on.
R	Its cqual to thc samc distancc.
M	What is the same distance? Are those there (pointing to PM, PL and PK) the same?
R	No, they all changed.
M	So you're saying that the distances from the house to the beaches changed. Now what remain the same?
R	The sum of the distances.
M	Do you think that this will be the case for all points within the triangle?
R	Ycs.
M	You're sure?
R	Yes.
M	Do you want to convince yourself some more or do you think that you are fairly convinced?
R	Convinced.
M	If I made this a larger or a smaller equilateral triangle do you think the result will be the same?
R	Yes.
M	Now I'd like to know from you how many percent convinced are you?
R	100%.
M	You have no doubt at all?
R	No doubt.
M	Do you find the results surprising? Initially you told me that she should build her house at the centre but now you've changed your mind, do you find the results surprising?
R	Yes.
M	Do you think, now that you are very convinced, … …is it necessary to know why this is the case?
R	Yes.
M	Why do you want an explanation for this?
R	To satisfy my curiosity.
M	Do you think you would be able to come up with some explanation for this on your own?
R	Yes … no.

M	If you think you can, you must try.
R	Yes, I'll try.
M	I've given you some time now, Rodney. I see that you've not written anything at all. Why?
R	'cos I couldn't find any solution.
M	I'm going to give you a sheet that contains a possible explanation.
	TAPE FAULTY AND INAUDIBLE FOR APPROXIMATELY & REMAINING MINUTES OF INTERVIEW.

M = RESEARCHER F = FLOYD

M	Floyd, now that you understand the question, where do you think that Sarah should build her house?
F	At the centre.
M	Do you want to check your results? Grab point P and move it around within the triangle. Observe carefully the distances we have on the left here. They would indicate he distances from the centre to the beaches and also observe the sum of the distances and then let me know what you think is happening.
F	(After a while). No matter where I put the centre point in the triangle it's always going to be the same.
M	So when you say "the centre point" you really mean the point P?
F	Yes.
M	And what happens?
F	The sum of all the distances from point P to the sides is always the same. It's not changing.
M	Are you convinced? Do you want to move it around some more?
F	No.
M	But are you convinced already?
F	Yes, I'm convinced (emphatically).
M	If I had to ask you how many percent convinced are you, what would you say?
F	100% (emphatically).
M	You are a 100% convinced that no matter where you take the point P …
F	The sum of the distances will still be the same.
M	Do you find this result surprising?
F	I didn't expect it. It is surprising
M	Now Floyd, you've established that you can place the point anywhere within the triangle and the distance is going to be a minimum. I want you to tell me Floyd, whether you want to know why this is true?
F	Yes, I'd like to.
M	Do you think you would be able to give me a possible explanation?
F	Maybe.
M	Do you want to try?
F	Yes.
M	(After a while) Floyd, I've noticed that you've tried for quite a while now – have you come up with some form of explanation?
F	No. I've tried and I can't get an explanation.
M	Floyd, I'm going to give you a sheet which has an explanation. I want you to go through this sheet and see whether you can understand it firstly and then come up with an explanation. I would suggest that you go through E1 first. (After a while) Have you got E1 Floyd?
F	Yes.

M	E2 requires you to write expressions for the areas of the different triangles.
F	(After a while) Okay, that's done.
M	Do you have the 3 expressions.
F	Yes.
M	Can I have a look at them? … That's okay. The next point E3 asks you to add the 3 areas and simplify them by taking out a common factor.
F	Yes …. (After a while).Okay.
M	(After a while) So you've taken out your common factor? Now can you tell me, Floyd, instead of writing it. How does the sum of the areas in E3 relate to the total area of the triangle?
F	I've divided the triangle into 3 different parts and I've found the area, … I mean the height of each triangle …
M	But my question is "how is the sum of the areas of the 3 triangles you've got there relate to the entire triangle"?
F	If you add the whole 3 triangles it will give you the sum of the whole thing.
M	So you're saying that the sum of the areas of the 3 triangles …
F	… is equal to the area of the big triangle.
M	Now I want you to use that and come up with some form of explanation.
F	(After a while) Okay, I've found the height of each of the triangles and I added them together and I've taken out a common factor and I found that $h_1 + h_2 + h_3 = h$ which is the height of the whole triangle.
M	But what does it mean? $h_1 + h_2 + h_3 = h$. hat does it mean?
F	I found the area of each of the 3 triangles and found the sum $h_1 + h_2 + h_3 = h$.
M	What does it mean to you if it is equal to h?
F	Yes, when I move it around it does not change - h_1, h_2 and h_3, no matter how much I move it around ….
M	Are you saying that h_1, h_2 and h_3 will not change?
F	h_1, h_2 and h_3 will change, but when you add all three up, it will remain the same.
M	So you're saying that h_1, h_2 and h_3 changes, but when you add them up …… the sum is staying the same.
F	Yes.
M	And what is the value there? (moving the point around)
F	The sum of every height in the triangle is still the same.
M	What does this mean with respect to Sarah?
F	No matter where she builds her house on the island, the distance from her house to the beaches will still be the same.
M	When you say the distance you are referring to ….
F	The sum.
M	Once more question before you go. I want to know if I grab this point of the triangle here and I make the triangle bigger or smaller, do you think the result will change?
F	Yes, the result will change.
M	When you say the result will change, are you saying that the sum will change? What do you

	mean?
F	It will change because the distance from the house to the beach will be different …
M	Are to saying – from the different triangles …?
F	For the different triangles.
M	No, I'm asking if you made the triangle bigger or smaller … I want to know whether the result in this triangle …. the result in this particular triangle?
F	No it will not change … it will be the same.
M	What do you mean "the same"?
F	Wherever I move the house or hut within this triangle the sum will be the same.
M	So you're saying, irrespective of the size of the triangle …
F	Irrespective of the size of the triangle, the sum of the distances will be the same. You can build the house or hut anywhere.
M	Are you sure?
F	Yes, sir, I'm positive.
M	Wouldn't you want to check?
F	No.
M	Thank you.

M = RESEARCHER K = KARISHMA

M	Karishma, you seem to understand the question. Before we begin, could you quickly tell me where you think Sarah should build her house?
K	In the centre.
M	Why in the centre?
K	It will be closer to all three beaches.
M	What I want you to do now is grab the point P and move it around within the triangle. Please observe what happens to PM, PL and PK. At the same time I want you to observe the sum of those three distances, here. You may continue and stop when you are satisfied.
K	I'll stop now.
M	Are you satisfied?
K	Yes.
M	What's your observation?
K	PM, PL and PK changes but the sum remains the same.
M	So you're saying that wherever point P was moved in the triangle, the distances changed but the sum did not.
K	Yes.
M	Do you think that if I moved point P to this apex, the sum will remain the same?
K	Yes.
M	Throughout the triangle? Are you quite convinced?
K	Yes.
M	If I had to ask you how many percent convinced are you, what would you say?
K	100 %
M	You have no doubt at all?
K	No.
M	Alright, what if I made this a smaller equilateral triangle. Do you think the result will hold true for that triangle as well?
K	Yes.
M	Are you positive?
K	Yes.
M	Why don't you just check?
K	(After a while). Yes it is still the same.
M	Do you desire an explanation for what is going on?
K	Yes.
M	Why do you think that there is a need?
K	Because I'm curious and I'd like to know what is going on.
M	So, just out of curiosity you'd like to know what's going on? Do you think that you would be able to come up with an explanation yourself?
K	No.
M	You don't want to try?

K	No.
M	Okay, I've got a sheet which has a possible explanation. Please read through it. I'm going to ask you a few questions just now. (After a while). Did you understand what was written here? Would you like to briefly explain to me what you read?
K	No.
M	Did you notice that the entire preamble is making a distinction between two things?
K	Yes.
M	What two things are they distinguishing between?
K	Explanation and exploration.
M	What do you think we were doing here?
K	We were exploring.
M	We were exploring, isn't that so? We were experimenting. Now what we want to do is find an explanation. At the bottom of the sheet you find steps E1 to E6. Work through each step and then let us see what you come up with.
K	After a while). I added the areas of the three small triangles and equaled it to the area of the big triangle.
M	What did you actually get?
K	½ a as a common factor.
M	What have you finally arrived at? What is the last statement that you've got?
K	$h = h_1 + h_2 + h_3$.
M	So what can we say? … I mean what does that result tell us?
K	(Silence).
M	You've got $h = h_1 + h_2 + h_3$, what does it mean to you?
K	(Silence).
M	Okay, what do you know about h_1, h_2 and h_3 when you moved the point around?
K	The distances changed.
M	What did we notice about the sum of the distances?
K	It remained the same.
M	That is fine. What do we know about this h here, the height of the triangle?
K	It's constant.
M	So we are saying that no matter what the distances are, the sum will always be equal to h? what does that mean?
K	It means that that would also be …. The sum would also be constant.
M	Now what does that mean in terms of Sarah's house?
K	She can build her house anywhere.
M	Tell me do you think that this is a good explanation?
K	Yes.
M	Did you enjoy working with it? Did you find it insightful?
K	Yes.
M	Thank you.

M = RESEARCHER N = NATASHA

M	Natasha, you seem to understand the question. Before we begin would you like to tell me where you think Sarah should build he house?
N	In the middle.
M	Why in the middle?
N	It will be equal for her to come from C and B.
M	We are not looking at equal distances to the beaches, we're looking at the sum of those distances. So you think the middle will be most appropriate. What I want you to do now is grab this point and move it around within the triangle. At the same time observe these distances on the left hand side. Also observe the sum of those distances. Do that now and when you stop I will know that you are finished. (After a while) What do you observe?
N	It's changing everytime … everytime I move it.
M	What changes?
N	These values here (pointing to the distances) that is, the distances are changing.
M	What happened to the sum?
N	It remained the same.
M	I noticed that you moved it for a fairly long time. So are you convinced that this is true? Are you saying, that you would not be able to find a spot within this triangle where the sum might change?
N	No it won't change.
M	Are you sure? Are you confident of your answer? … You're looking unsure.
N	No, I'm quite confident that it won't change.
M	Do you want to try it again? You moved it around the centre only, you did not move it around the corners?
N	(After a while) It remains the same.
M	So you're fairly convinced? What would happen now if I made the triangle bigger, I drew a bigger equilateral triangle? I want to know, will the results you obtained for this triangle be the same for the bigger one? Will it work for the larger triangle?
N	Yes.
M	Why don't you check?
N	(After a while) Yes, it is the same.
M	You're fairly convinced? What I want to know then is how many percent convinced are you?
N	70 %
M	This indicates that you're not so convinced about it. Do you want to try again to convince yourself some more?
N	Yes. (After a while) I'm a 100 % convinced now.
M	Are you sure? What would have convinced you to a 100 % now?
N	Everytime I am moving this mouse only the distances are changing and the total is not changing.
M	Are you surprised with this result?

N	Yes.
M	Now that you are 100 % convinced do you think it's necessary to have some sort of explanation as to why that result is true?
N	Yes.
M	Why do you think that there should be an explanation?
N	Out of interest, I would want to know why.
M	Do you think you would be able to come up with an explanation on your own?
N	No.
M	What I do have here is a sheet which contains a possible explanation. At the bottom of the sheet you will find steps E1 to E6. Now E1 says you must label, if you look at the diagram that is appearing on the sheet that I have given you, it says label the sides. I want you to draw a diagram on a page and then work through these steps. Do that now. (After a while) Can you read out to me what you have written there?
N	I've got $\frac{1}{2}a(h_1 + h_2 + h_3)$
M	The next step required you to find a relationship between the sum of the areas of the small triangles to the area of the large triangle.
N	It will make up the big triangle.
M	When you say "make up the big triangle" are you saying the sum of the areas of the 3 small triangles is equal to the area of the big triangle?
N	Yes.
M	Now I want you to write an expression for the area of the big triangle.
N	(After a while) I got ½ a as a common factor. I've removed ½ a and I got $h = h_1 + h_2 + h_3$.
M	What does this mean to you? What can you conclude from that?
N	No matter how much h_1, h_2 and h_3 changes h will remain the same.
M	h_1, h_2 and h_3 are changing, but what can we say about the sum?
N	The sum doesn't change.
M	So you're saying that no matter how much h_1, h_2 and h_3 change, the sum does not change. Why do you think that is the case?
N	Because h is the sum and h is constant.
M	How does it relate to Sarah's house? Where do you think she should build her house?
N	In the centre … anywhere.
M	Is it the centre or anywhere?
N	Anywhere because no matter where she builds it the sum will always be the same.
M	Looking at the work we have done, do you think the explanation was good? Did it enlighten you?
N	Yes.
M	Did it improve your understanding?
N	Yes.
M	Thank you.

M = RESEARCHER A = ANSUYA

M	Okay Ansuya, now that you understand the question, where do you think that Sarah should build her house?
A	In the middle of the equilateral triangle.
M	You seem quite convinced that it is the middle, why?
A	Because it seems the easiest way to get to any of the beaches.
M	I want you to test that. Grab the point P and move it around within the triangle. I want you to observe as you move the point around the values I have on the left here – these are the distances from Sarah's house to the beaches – and at the same time observe the sum of the distances here. (After a while) You've stopped. What did you observe?
A	The distances of the smaller lines in it are changing but the total distance is the same.
M	When you say "smaller line" are you actually talking about PL, PK, PM which represents the distances from the house to the beaches. They don't change, you say?
A	Well they change but the total doesn't change.
M	But I noticed you only moved it around the centre, are you convinced? Do you want to try again?
A	Yes. (After a while) It's the same.
M	Are you convinced?
A	Yes.
M	If you are so convinced then I just want to know what would happen if I made this a larger equilateral triangle. Do you think your result will still hold?
A	Yes.
M	Are you convinced?
A	Yes.
M	Do you want to check?
A	Yes. (After a while) It's the same.
M	If I asked you how many percent convinced are you, what would you say?
A	98 % - 99 %
M	So you're fairly convinced. Do you find the result surprising?
A	Yes. I thought it would change.
M	Did you enjoy that?
A	Yes.
M	The next thing that I'd like to know is whether you have any desire for an explanation? Do you think there is a need to find an explanation for this result?
A	Yes It was surprising.
M	Why do you think there should be an explanation?
A	I thought like there would be a specific point but it's anywhere.
M	Do you think you would be able to come up with an explanation on your own?
A	No, I don't think so.

M	I'm giving you a sheet now that has an explanation. I want you to read through that now. There are steps E1 to E6. Work through that and when you stop I'll ask you some questions. (After a while) Can you describe to me what you have done?
A	I've added the areas of the smaller triangles up and I've decided that ½ a is common so I put ½ a (h_1 , h_2 , h_3).
M	When you say h_1 , h_2 , h_3 I presume you're saying $h_1 + h_2 + h_3$ as it is written in your sheet.
A	Yes.
M	Okay, then the next thing here says, how does this sum that you have established relate to the area of the large triangle?
A	The sum of the 3 small areas is equal to the sum of the area of the large triangle.
M	So you really mean the sum of the areas of the smaller triangles is equal to the area of the large triangle? Is that what you're saying?
A	Yes.
M	I want you to find the relationship between the area of the large triangle and the sum of the areas of the 3 smaller ones. (After a while) How did you get $h = h_1 + h_2 + h_3$?
A	½ was common.
M	Was only ½ common?
A	½ a was common.
M	Yes, you've got it written down correctly. What does this mean to us?
A	When you add the heights of the smaller triangles you get the height of the larger triangle.
M	What can we conclude about h of the larger triangle?
A	It's the same.
M	When you say "same" do you mean constant?
A	Yes.
M	Then what can we say about h_1 , h_2 and h_3?
A	They change when you move point P.
M	So what conclusion can we draw from that?
A	While h stays the same the heights of the smaller triangles change.
M	In terms of Sarah's house, what does it mean?
A	Wherever she builds her house the sum of the distances will be the same.
M	Do you find this explanation enlightening?
A	Yes.
M	Thank you.

M = RESEARCHER N = NIRVANA

M	Now that you understand the question where do you think that Sarah should build her house?
N	In the centre.
M	Why?
N	Eh…eh…
M	Are you just saying the centre or do you have a reason for saying so?
N	I'm just guessing.
M	Okay I want you to grab point P and move it around. Observe the distances PM, PL and PK and at the same time observe the sum of these distances. (After a while). Okay Nirvana, you seemed to have moved it to a number of points. What is your observation?
N	The distances are changing and the sum …
M	Which distances are changing?
N	All of them and the sum remains the same.
M	Do you think that this is the case throughout the triangle?
N	Throughout the triangle.
M	Do you think that if I moved the point P to the corner there the sum will remain the same?
N	Yes!
M	Are you convinced?
N	Yes!
M	You don't want to try?
N	I'll try ….. yes it remains the same.
M	So no matter where you moved it in the triangle the result will be the same?
N	Yes.
M	If I asked you how many percent convinced are you, what would you say?
N	100 %.
M	That means that you are highly convinced that there is no point within the triangle for which the sum will change?
N	(nodding her head)
M	You are shaking your head – is it yes or no?
N	No.
M	There is no point?
N	No.
M	What if I changed this to a larger equilateral triangle, do you think the result will still hold?
N	Yes.
M	Do you think if I added up all the distances from any point within a larger equilateral triangle the sum will be the same?
N	It will, yes. (Hardly audible).
M	Do you find the result surprising?
N	Yes.
M	Now that you are so convinced, do you think that there is a need for an explanation?

N	Yes.
M	Why is there such a need?
N	(Silence).
M	Why? ....
N	I don't know.
M	But you still think that there should be an explanation?
N	Yes.
M	Do you think that you would be able to come up with an explanation?
N	No.
M	On this sheet there's a possible explanation. Read through it.
	TAPE FAULTY
M	Nirvana I notice that you've written out the entire explanation. Can you tell me what you've done in this part here?
N	I've added the areas of these small triangles and I got ½ a h_1 + ½ a h_2 + ½ a h_3, and I found the common factor and I got ½ a ($h_1 + h_2 + h_3$).
M	What is the relationship between the areas of the three triangles and the area of the large triangle?
N	They are smaller.
M	If you take the areas of the small triangles and you add it up will it be smaller, equal or larger than the area of the big triangle?
N	Equal.
M	Are you sure?
N	Yes.
M	Tell me what you did thereafter.
N	I equated them. I got ½ a h = ½ a ($h_1 + h_2 + h_3$).
M	I can see all of that but what have you got in the last part?
N	I cancelled them.
M	And what have you got?
N	I've got $h = h_1 + h_2 + h_3$.
M	What does this mean?
N	That Sarah can build her house anywhere.
M	What makes you say that from here? ... what do we know about h_1 , h_2 , h_3 when you moved it around?
N	It changed.
M	What can you say about the height of the large triangle?
N	It remained the same.
M	So what does it mean?
N	No matter where you move the point in the triangle the sum is a constant.
M	Do you think that this explanation helped with your understanding?
N	Yes.
M	Thank you.

R = RESEARCHER M = MANIVASAN

R	Okay Manivasan, where do you think that Sarah should build her house?
M	In the centre
R	Why the centre?
M	Because everything will be equal.
R	What I want you to do right now is grab that point P and move it around. Please observe what happens to the distances here on the left and at the same time the sum of these distances at the bottom. Okay when you've stopped I'll know that you've finished. (After a while). Tell me what you observed.
M	PM, PL and PK is changing but the sum is not.
R	Do you think that this is true throughout the triangle?
M	Yes.
R	Are you convinced?
M	Yes.
R	If I asked you how many percent convinced are you, what would you say?
M	70 %.
R	So you are not so convinced? Do you want to try some more? Do you want to try or do you think it is not necessary?
M	I'll try.
R	(After a while). What can you say?
M	I've tried all over and it still remains the same.
R	Do you think that you are more convinced now?
M	Yes.
R	How many percent convinced are you?
M	I'm sure a 100 % (emphatically).
R	If I made this a larger or smaller equilateral triangle, do you think the result will still hold?
M	Yes.
R	Are you sure about that?
M	Yes.
R	Why don't you check?
M	(After a while). It is the same.
R	Do you find the result surprising?
M	Yes.

R	Do you think that now that you are a 100 % convinced, that there is a need for an explanation?
M	Yes.
R	Would you want an explanation?
M	Yes.
R	Why?
M	So I can understand it.
R	Do you think that you would be able to come up with an explanation on your own?
M	No.
R	You don't want to try?
M	No.
R	Alright then, I've got a sheet which has a possible explanation on it. Go through the sheet and then we'll look at it together. (After a while). I can see what you've got, but just for the tape can you tell me what you have?
M	½ a $(h_1 + h_2 + h_3)$.
R	Now look at the triangle you've drawn. How does the areas of the three small triangles relate to the area of the big triangle?
M	The three small ones are equal to the large one.
R	Are you saying that each small one is equal to the large one?
M	No, it makes up the large one.
R	Write down the area of the large one and then find the relationship with the areas of the small ones. (After a while). What have you got?
M	½ ah is common in both.
R	½ ah is common to both? Is h also common on both sides?
M	No, only ½ a.
R	Thereafter what have you done?
M	(Silence).
R	Okay, I can see that you removed it, and what have you arrived at?
M	$h = h_1 + h_2 + h_3$.
R	But what does that mean Manivasan?
M	It means that h_1 , h_2 and h_3 is …. (silence).
R	Continue.
M	…. is equal to the large triangle.
R	h_1 , h_2 and h_3 are equal to? Lets examine that. What can you say about h?
M	h is constant.
R	Does h_1 , h_2 and h_3 change as you move the point around?
M	Yes.
R	What can we say about the sum then?
M	The sum remains the same.
R	So what can you conclude?
M	You can change h_1 , h_2 and h_3, but the sum will not change.

R	So what can you say about Sarah's house?
M	It can be built anywhere on the island.
R	Do you think that this was a good explanation?
M	Yes.
R	Did you find it insightful?
M	Yes.
R	Did you understood it well?
M	Yes.
R	You think so?
M	Yes.
R	Thank you.

M – RESEARCHER V = VINOLIA

M	Okay Vinolia, I can see that you understand the question now. Where do you think that Sarah should build her house?
V	In the centre.
M	Why?
V	Because I think it is appropriate … to attend all beaches.
M	Vinolia, I want you to test that. But before you do, I want you to note that on the left hand side the distances from Sarah's house to the beaches are written and at the bottom we have the sum of these distances. As you move P around observe these distances and their sum. Try that now. (After a while). What have you got?
V	The total sum of the distances is the same but each distance on its own changes.
M	So you are saying that the things on the left are changing but this is not?
V	Yes.
M	Are you sure that this is true throughout because I noticed that you only moved it around the centre?
V	Okay I'll try again. (After a while). It still remains the same.
M	Are you convinced that it remains the same?
V	Yes.
M	Okay Vinolia, what would happen if I made this a larger or smaller equilateral triangle?Do you think that the result will be the same?
V	Yes.
M	You are convinced?
V	Yes.
M	Why don't you check?
V	Okay … it remains the same.
M	If I asked you how many percent convinced are you what would you say?
V	99 %.
M	So you are fairly convinced. Do you find the result surprising?
V	Yes, it is.
M	Do you think that there is a need to know why this is true? Do you have any desire for an explanation?
V	Yes.
M	Why?
V	I'd like to …. go more ahead.
M	Do you think that you would be able to come up with an explanation on your own?
V	No.
M	You wouldn't want to try?
V	No.
M	Okay Vinolia, I'm going to give you a sheet which has a possible explanation on it. There is

	a lot to read. Go through each step and when you are ready we'll discuss it.
M	(After a while). I can see that you have finished the first step. Can you briefly tell me what you've got for the three triangles?
V	Area of APC = ½ BH = ½ ah_1, area of APB = ½ BH = ½ ah_2 and area of BPC = ½ BH = ½ ah_3.
M	Now I want you to add the expressions and simplify it. (After a while). Okay what have you got?
V	I wrote out the expression and I noticed that ½ a is common, and so I got ½ a and in brackets I got h_1 , h_2 and h_3.
M	The next step requires you to establish some relationship which exists between the sum of the areas of the small triangles and the area of the large triangle.
V	The three small triangles make up the large equilateral triangle.
M	So what can we say about the sum of the areas of three triangles as compared to the area of the large triangle?
V	They are equal.
M	Write that down now and simplify it. (After a while). Can you describe what you've got there?
V	I've equalled the sum of the three triangles and I've related it to the big triangle and I noticed that ½ a is common and it leaves me with $h = h_1 + h_2 + h_3$.
M	Your calculations are correct but what does it mean?
V	The height of the three small triangles equals the height of the large triangle.
M	That is true. But from our observations, looking at the big triangle, what can we say about h?
V	It does change because the triangle is the same.
M	What can we say about h_1 , h_2 and h_3 when we point P around?
V	It does change.
M	So what can we say about the sum?
V	The sum is equal to the height of the … (silence).
M	If the height of the large triangle does not change , what can we say about $h_1 + h_2 + h_3$?
V	They don't change.
M	Do you find that this explanation increased your understanding of the result obtained?
V	Yes.
M	Did you enjoy it?
V	Yes.
M	Do you think that the explanation we have here was done in a proper step-wise fashion? Were you able to follow it?
V	Yes, it is understandable.
M	Thank you.

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APPENDIX A

Investigation: Distances

Sarah, a shipwreck survivor manages to swim to a desert island. As it happens, the island closely approximates the shape of an equilateral triangle. She soon discovers that the surfing is outstanding on all three of the island's coasts and crafts a surfboard from a fallen tree and surfs every day. Where should she build her house *P* so that the total sum of the distances from *P* to all three beaches is a minimum? (She visits them with equal frequency). Before you proceed further, first write down your intuitive guess in the space below where you think *P* should be placed for the total sum of the distances to be a minimum.

....

Sketch

Step 1: Construct a dynamic equilateral triangle*ABC*.

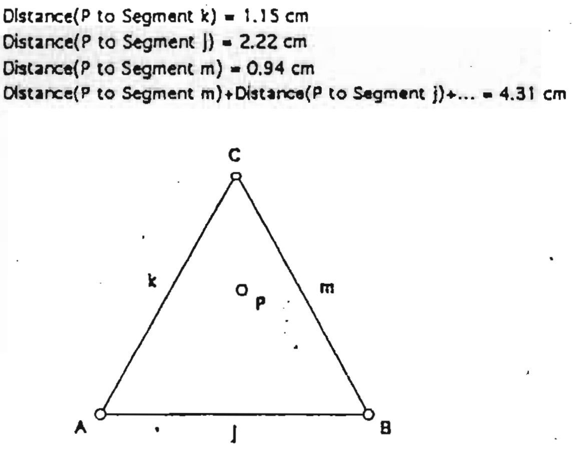
Step 2: Construct a point P in the triangle.

Step 3: Measure the distances from P to the three sides.

Step 4: Select the three distances and choose Calculate to add the three distances.

Investigate

Drag point *P* around the interior of the triangle. What do you notice regarding the total sum of the distances? Drag *A*, *B* or *C* to change the size of the equilateral triangle and again drag point *P* around the interior of the triangle. What do you notice now? What happens if *P* is dragged outside the triangle?



Conjecture

In the space below, write a conjecture regarding your observations above.

....

Explore more

Construct any triangle *ABC* and an arbritrary point *P* inside. Does your conjecture above still hold if you have, say, an isosceles, right, scalene or obtuse triangle?

....

Present Your Findings

Discuss your results with your partner or group. To present your findings print some sketches with measures and captions to illustrate your conjecture and findings.

Distances: Explaining

You are no doubt at this stage quite convinced that the total sum of the distances from a point P to all three sides of a given equilateral triangle is always constant. But can you **explain** why it is true?

Although further exploration on *Sketchpad* may succeed in convincing you even more of the truth of your conjecture, it really provides no explanation; it only confirms its truth. For example, the regular observation that the sun rises every morning clearly does not constitute an explanation; it only reconfirms the validity of the observation. To explain something, one therefore has to try and explain it in terms of something else, e.g. the rotation of the earth around the polar axis.

Recently a mathematician named Feigenbaum made some new experimental discoveries in fractal geometry using a computer just as you have used *Sketchpad* earlier to discover your conjecture. These discoveries were then later explained by Lanford and other mathematicians. Carefully read and comment on the following quotation in this respect:

> "*Lanford and other mathematicians were not trying to validate Feigenbaum's results any more than, say, Newton was trying to validate the discoveries of Kepler on the planetary orbits. In both cases the validity of the results was never in question. What was missing was the explanation. Why were the orbits ellipses? Why did they satisfy these particular relations? ... there's a world of difference between validating and explaining.*"
>
> - D. Gale (1990) in The Mathematical Intelligencer, 12(1), 4.

Explain

Here are some hints for planning a possible explanation. Read and work through it if you want, or try to construct your own explanation.

E1. Label all three sides as a and the distances from P to the sides AB, BC and CA respectively as h_1, h_2 and h_3.

E2. Write expressions for the areas of triangles PAB, PBC and PCA in terms of the above distances.

E3. Add the three areas and simplify your expression by taking out a common factor.

E4. How does the sum in P3 relate to the total area of triangle ABC? What can you conclude from this?

E5. Which property therefore explains why this result is true?

E6. Discuss your explanation with your partner or group.