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A DISCRETE FOURIER TRANSFORM BASED ON SIMPSON'S RULE

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A DISCRETE FOURIER TRANSFORM BASED ON SIMPSON'S RULE

by

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As the candidate's supervisors we have approved this thesis for submission.

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Preface

The work described in this dissertation was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban, from February 2013 to December 2013, under the supervision of Dr P. Singh and co-supervised by Dr V. Singh. This study represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

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Dan Behlule Sibandze

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I, Dan Behlule Sibandze, affirm that

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Dedication

This thesis is dedicated to both my parents, the late Mr Shadrack Sibandze and Mrs Sellinah Sibandze, who did not only raise and nurture me but also taxed themselves dearly over the years for my education and intellectual development.

Acknowledgment

I would like to thank all the people who contributed in some way to the work described in this thesis. First and foremost, I thank my supervisor Dr Pravin Singh and co-supervisor Dr Virath Singh for their guidance from the commencement of this work. They tirelessly and with much enthusiasm provided a friendly and cooperative atmosphere at work and also useful feedback and insightful comments on my work. They contributed to a rewarding experience by engaging me in new ideas and demanding a high quality of work in all my endeavours. Thank you very much.

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Notation

$\mathbf{A}, \mathbf{A_N}$	Symmetric matrix of order N
$C(k), C(k,l)$	Fourier coefficients
$\mathbb{C}^N$	Complex space of n-tuples
$\mathbf{D}, \mathbf{D_N}$	Diagonal matrix of order N
$\mathbf{f}$	Vector or image matrix
$\mathbf{f_0}$	Even indexed components of a signal
$\mathbf{f_1}$	Odd indexed components of a signal
$\mathbf{F}$	Transform of $\mathbf{f}$
$\mathbf{F_0}$	Transform of $\mathbf{f_0}$
$\mathbf{F_1}$	Transform of $\mathbf{f_1}$
$\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}_N}$	DFT matrix of order N
$\mathbf{I}, \mathbf{I_N}$	Identity matrix of order N
$\mathbf{P}, \mathbf{P_N}$	Permutation matrix of order N
$\mathbb{R}^N$	Real space of n-tuples
$\mathbf{U}, \mathbf{U_N}$	SDFT matrix of order N
ω, ω_N	N^{th} root of unity
*	Hermitian conjugate
$\cdot *$	Element-wise multiplication
$\otimes$	Convolution
$\Vert \cdot \Vert$	Euclidean norm
$.^T$	Transpose of a matrix
$\langle \cdot, \cdot \rangle$	Inner product

Abstract

Fourier transforms are mathematical operations which play a vital role in the analysis of mathematical models for problems originating from a broad spectrum of fields. In this thesis, we formulate a discrete transform based on Simpson's quadrature for $4m + 2$ quadrature nodes and analyse its various properties and provide detailed proofs thereof. In addition, we make applications to encryption and watermarking in the frequency domain.

Contents

Introduction

In chapter 1 we provide a brief literature survey of the Discrete Fourier Transform and show how it arises from trapezoidal quadrature. In addition, we list some of the important properties satisfied by the transform.

In chapter 2 we derive the Simpson Discrete Fourier Transform from Simpson's quadrature. We provide an expression for the transformation matrix as well as an expression for its inverse. We investigate the effect of the transform on a real signal.

In chapter 3 we provide detailed proofs for several properties satisfied by the Simpson Discrete Fourier Transform. We use the duality property to obtain an expression for the minimal polynomial. From the minimal polynomial we obtain eight distinct eigenvalues for the transformation matrix.

In chapter 4 we derive a two dimensional Simpson Discrete Fourier Transform. We use this transform to show the effect of the duality property in two dimensions. In addition, we apply the transform to encryption and watermarking in the frequency domain.

Chapter 1

Discrete Fourier Transform

1.1 Introduction

Fourier analysis is a family of mathematical techniques, based on decomposing functions into sinusoids. It is named after Jean Baptiste Joseph Fourier (1768-1830), a French mathematician and physicist who claimed that any continuous, periodic signal could be represented as the sum of properly chosen sinusoidal waves [24]. By a signal, we mean any variable that contains some kind of information about the behaviour or attributes of some phenomenon. Fourier analysis has many scientific applications in physics, partial differential equations [10], image and signal processing, numerical analysis and other areas [12]. The input signal is said to be in the time domain while the transformed signal is said to be in the frequency domain. The whole applicability stems from the ease of manipulation in the frequency domain. The properties establish relationships between the time and frequency domains.

The Discrete Fourier Transform(DFT) is the most important discrete transform used to perform Fourier analysis in many practical applications [24]. The representation of signals and systems in the frequency domain is an important tool in the study of signal processing, communication, and other fields. In many applications, discrete-time signals and systems are characterized by finite-length sequences. A widely used frequency domain representation of such a sequence is its DFT. It deals with signals that are discrete and periodic. When the DFT contains only real numbers from an even function, the sine component of the the DFT is zero, and the DFT becomes a Discrete Cosine Transform (DCT). The DCT is used to compress data and this is particularly important for the storage and transmission of images, since each image involves a large amount of data. Data compression can be achieved in speech and video transmissions and for recording biomedical signals such as Electroencephalography (EEGs) and Electrocardiography (ECGS) [13].

The continuous Fourier transform is the source of all discretized Fourier transforms, such as the DCT, DST (Discrete Sine Transform), DFRFT (Discrete Fractional Fourier Transform) and the DFT. The DFT can calculate a signal's frequency spectrum. This is a direct examination of information encoded in the frequency, phase and amplitude of the component sinusoids. The DFT can find a system's frequency response from the system's impulse response, and vice versa. This allows systems to be analyzed in the frequency domain just as convolution allows systems to be analyzed in the time domain. The DFT can be used as an intermediate step in more elaborate signal processing techniques. The DFT is efficiently implemented using the Fast Fourier Transform (FFT), an algorithm for convolving signals that are hundred times faster than conventional methods [8]. The discrete nature of the DFT results in spectral leakage which refers to the misrepresentation of the Fourier coefficients as a result of sampling and windowing applied to signals which often result in blurring effect [11].

In recent developments, a fractional DFT has been defined by discretizing the continuous fractional Fourier transform [7]. Another popular definition stems from the use of projection operators [2].

1.2 Derivation of the DFT

In this section, we introduce the Discrete Fourier Transform (DFT) and show how it arises from numerical quadrature. Recall that the coefficients in a Fourier series expansion for a continuous, 2π-periodic function $f(t)$ have the form

$$C(k) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt. \tag{1.1}$$

To obtain an approximation of the coefficients, we divide the interval $[0, 2\pi]$ into N equally spaced subintervals of length $h = \frac{2\pi}{N}$ and label the nodes $t_j = jh$, $j = 0, 1, \cdots, N$. Using the trapezoidal rule for numerical integration [4] in (1.1) yields

$$\begin{aligned} C(k) &\approx \frac{h}{4\pi}\left[e^{-ikt_0} f(t_0) + e^{-ikt_N} f(t_N) + 2\sum_{j=1}^{N-1} e^{-ikt_j} f(t_j)\right] && (1.2) \\ &= \frac{1}{N} \sum_{j=0}^{N-1} e^{-ikt_j} f(t_j) && (1.3) \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \omega^{kj} f(j) && (1.4) \end{aligned}$$

where from (1.2) to (1.3) we have used $f(t_0) = f(t_N)$ and $e^{-ikt_0} = e^{-ikt_N}$. Furthermore, we adopt the notation $f(j) = f(t_j)$ and $\omega = e^{-\frac{2\pi i}{N}}$, where the latter is the N^{th} root of unity.

Given these approximations to $C(k)$ for $k = 0, 1, \cdots, N-1$, however, one may recover $f(j)$, $j = 0, 1, \cdots, N-1$. To see this, let

$$F(k) = \sum_{j=0}^{N-1} \omega^{kj} f(j), \tag{1.5}$$

$k = 0, 1, \cdots, N-1$ so that $C(k) \approx \frac{F(k)}{N}$.

Multiply both sides of (1.5) by ω^{-kl}, where $l \in \{0, 1, \cdots, N-1\}$, and sum over k

to get

$$\begin{aligned}
\sum_{k=0}^{N-1} \omega^{-kl} F(k) &= \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} f(j)\omega^{(j-l)k} \\
&= \sum_{j=0}^{N-1} f(j) \sum_{k=0}^{N-1} \omega^{mk} \qquad (1.6)
\end{aligned}$$

where $m = j - l$ is at most $N - 1$.

We require the following lemma:

Lemma 1.1. *Let $S = \sum_{k=0}^{N-1} \omega^{mk}$, then*

$$S = \begin{cases} 0 & \textit{if } \ m \neq 0 \\ N & \textit{if } \ m = 0 \end{cases}$$

Proof. When $m = 0$, the result is trivial.

When $m \neq 0$, then consider

$$\begin{aligned}
S &= \sum_{k=0}^{N-1} \omega^{mk} \\
\omega^m S &= \sum_{k=0}^{N-1} \omega^{m(k+1)} \\
S(1-\omega^m) &= 1 - \omega^{mN} \\
S &= \frac{1-(\omega^N)^m}{1-\omega^m} \\
&= 0
\end{aligned}$$

The last step follows since $\omega^N = 1$ and $\omega^m \neq 1$. $\square$

Applying lemma 1.1 to (1.6), we obtain

$$\sum_{k=0}^{N-1} \omega^{-kl} F(k) = f(l)N$$

from which it follows that

$$f(j) = \frac{1}{N}\sum_{k=0}^{N-1} \omega^{-kj} F(k), \tag{1.7}$$

$j = 0, 1, \cdots, N-1$.

Equations (1.5) and (1.7) represents the DFT and its inverse transformation, respectively. The system (1.5) can be written in matrix vector form $\mathbf{F} = \boldsymbol{\mathcal{F}}\mathbf{f}$, where $\mathbf{F} = [F(0), F(1), \cdots, F(N-1)]^T$, $\mathbf{f} = [f(0), f(1), \cdots, f(N-1)]^T$ and

$$\boldsymbol{\mathcal{F}} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{(N-1)} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(N-1)} & \omega^{2(N-1)} & \dots & \omega^{(N-1)(N-1)} \end{bmatrix}. \tag{1.8}$$

Since $\boldsymbol{\mathcal{F}}^*\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{F}}^* = N\mathbf{I}$, this allows for the inversion $\mathbf{f} = \frac{1}{N}\boldsymbol{\mathcal{F}}^*\mathbf{F}$ which is equivalent to (1.7), where * denotes the Hermitian conjugate. One can easily see from (1.8) that $\boldsymbol{\mathcal{F}}$ is symmetric.

1.3 Properties of the Discrete Fourier Transform

In this section we summarize some of the properties of the DFT [5].

Time Reversal

By this operation, we mean that a reversal in the time domain results in a corresponding reversal in the frequency domain. That is, $\boldsymbol{\mathcal{F}}\mathbf{f}(-k)$ is the transform of $f(-j)$, $j = 0, 1, \cdots, N-1$, where $f(-j) = f(N-j)$ by the periodicity N of the signal.

Conjugation

The transform of the conjugate signal $\overline{\mathbf{f}}$ implies a reversal and conjugation in the frequency domain, that is,

$$\boldsymbol{\mathcal{F}}\overline{\mathbf{f}}(k) = \overline{\boldsymbol{\mathcal{F}}\mathbf{f}(-k)} \tag{1.9}$$

$k = 0, 1, \cdots, N-1$.

Time Shift

The transform of a shifted signal $f(j-n)$ in the time domain results in the multiplication by a complex exponent ω^{kn} in the frequency domain, that is

$$\sum_{j=0}^{N-1} \omega^{kj} f(j-n) = \omega^{kn} \boldsymbol{\mathcal{F}}\mathbf{f}(k) \tag{1.10}$$

Frequency Shift

A frequency shift, k_0, in the frequency domain is equivalent to the transformation of the signal $\omega^{-k_0 j} f(j)$ in the time domain, that is

$$\boldsymbol{\mathcal{F}}\mathbf{f}(k-k_0) = \sum_{j=0}^{N-1} \omega^{kj} \left[\omega^{-k_0 j} f(j)\right]. \tag{1.11}$$

Definition 1.3.1. *For two periodic signals* $\mathbf{f}$ *and* $\mathbf{g}$ *of same length* N*, the circular convolution* $\mathbf{f} \otimes \mathbf{g}$ *is defined by*

$$(\mathbf{f} \otimes \mathbf{g})(j) = \sum_{m=0}^{N-1} f(m) g(j-m)$$

for $j = 0, 1, \cdots, N-1$.

Convolution in Time Domain

The convolution operation in the time domain is related to a multiplication in the frequency domain in the following manner

$$\boldsymbol{\mathcal{F}}(\mathbf{f} \otimes \mathbf{g}) = \boldsymbol{\mathcal{F}}\mathbf{f} \cdot * \boldsymbol{\mathcal{F}}\mathbf{g} \tag{1.12}$$

where $\cdot *$ indicates element by element product of components.

Convolution in Frequency Domain

The convolution operation in the frequency domain is related to a multiplication in the time domain in the following manner

$$(\boldsymbol{\mathcal{F}}\mathbf{f} \otimes \boldsymbol{\mathcal{F}}\mathbf{g}) = N\boldsymbol{\mathcal{F}}(\mathbf{f} \cdot * \mathbf{g}). \tag{1.13}$$

Plancherel's Theorem

Let $\mathbf{f}$ and $\mathbf{g}$ be periodic discrete-time signals with period N. Then

$$\begin{aligned} \langle \boldsymbol{\mathcal{F}}\mathbf{f}, \boldsymbol{\mathcal{F}}\mathbf{g} \rangle &= N \sum_{j=0}^{N-1} f(j)\overline{g(j)} \\ &= N\langle \mathbf{f}, \mathbf{g} \rangle \end{aligned} \tag{1.14}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{C}^N$. For $\mathbf{f} = \mathbf{g}$ in (1.14), we obtain Parseval's identity

$$\|\boldsymbol{\mathcal{F}}\mathbf{f}\|^2 = N\|\mathbf{f}\|^2. \tag{1.15}$$

Duality

If the signal $\mathbf{f}$ is transformed, it results in a signal $\boldsymbol{\mathcal{F}}\mathbf{f}$ in the frequency domain. If the latter signal is transformed, it results in the reversal of the original signal $\mathbf{f}$ in the time domain, namely

$$\boldsymbol{\mathcal{F}}^2\mathbf{f}(j) = Nf(-j). \tag{1.16}$$

1.4 Minimal Polynomial

Applying the duality property to (1.16) results in recovering the original signal, that is $\boldsymbol{\mathcal{F}}^4\mathbf{f} = N^2\mathbf{f}$ from which it follows that the DFT matrix satisfies the property

$$\boldsymbol{\mathcal{F}}^4 = N^2\mathbf{I}. \tag{1.17}$$

The minimal polynomial is given by

$$\phi(\lambda) = \lambda^4 - N^2 \tag{1.18}$$

Hence, the eigenvalues, obtained by setting $\phi(\lambda) = 0$, are given by

$$\sqrt{N}e^{i\frac{\pi}{2}l} \tag{1.19}$$

$l = 0, 1, 2, 3$, which are either real or purely imaginary.

For $N > 4$, the eigenvalues are therefore repeated. If the minimal polynomial has linear elementary divisors, then the matrix is diagonalizable [15], hence, there exists a linear independent set of eigenvectors of $\boldsymbol{\mathcal{F}}$ that span $\mathbb{C}^N$. The eigenvalues of a matrix play an important role in the spectral resolution of functions of the matrix into their constituent components and their multiplicities gives the number of linearly independent eigenvectors corresponding to each eigenvalue [15]. McClellan and Parks [16] studied the eigenstructure of the DFT matrix in detail and provided a means of numerically generating a linearly independent set of eigenvectors corresponding to the related eigenspaces. Much research has focussed on generating an orthogonal set of eigenvectors by using matrices that commute with the DFT matrix [9] and more recently a method based on complete generalized Legendre sequence has been proposed [19].

Chapter 2

Simpson Discrete Fourier Transform

2.1 Derivation of the SDFT

We now introduce [22] a transformation arising from the Simpson's rule of numerical quadrature [4]. Simpson's rule applied to (1.1) to approximate $C(k)$ with even N gives

$$\begin{aligned} C(k) &\approx \frac{h}{6\pi}\left[e^{-ikt_0}f(t_0)+e^{-ikt_N}f(t_N)+2\sum_{j=1}^{\frac{N}{2}-1}e^{-ikt_{2j}}f(t_{2j})\right. \\ &\quad \left.+\,4\sum_{j=0}^{\frac{N}{2}-1}e^{-ikt_{2j+1}}f(t_{2j+1})\right] \\ &= \frac{1}{3N}\left[2\sum_{j=0}^{\frac{N}{2}-1}e^{-ikj\frac{4\pi}{N}}f(t_{2j})+4\sum_{j=0}^{\frac{N}{2}-1}e^{-ik(2j+1)\frac{2\pi}{N}}f(t_{2j+1})\right] \\ &= \frac{2}{3N}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k2j}f(2j)+\frac{4}{3N}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k(2j+1)}f(2j+1) \end{aligned} \tag{2.1}$$

where we have used the periodicity of f, that is, $f(t_0)=f(t_N)$.

Define

$$F_0(k) = \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \omega^{k2j} f(2j) \tag{2.2}$$

and

$$F_1(k) = \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \omega^{k(2j+1)} f(2j+1), \tag{2.3}$$

then

$$F(k) = F_0(k) + F_1(k). \tag{2.4}$$

For any vector $\mathbf{f} = [f(0), f(1), \ldots, f(N-1)]^T$ of length N, define $\mathbf{f_0} = [f(0), f(2), \cdots, f(N-2)]^T$, $\mathbf{f_1} = [f(1), f(3), \cdots, f(N-1)]^T$ comprising of the even and odd indexed components of $\mathbf{f}$ respectively and the discrete transforms $\mathbf{L_0 f_0}$ by

$$\begin{aligned} \mathbf{L_0 f_0}(k) &= F_0(k) \\ &= \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \omega^{k2j} f_0(j) \end{aligned} \tag{2.5}$$

and $\mathbf{L_1 f_1}$ by

$$\begin{aligned} \mathbf{L_1 f_1}(k) &= F_1(k) \\ &= \frac{4}{3} \sum_{j=0}^{\frac{N}{2}-1} \omega^{k(2j+1)} f_1(j) \end{aligned} \tag{2.6}$$

with

$$F(k) = F_0(k) + F_1(k)$$

$k = 0, 1, \cdots, \frac{N}{2} - 1$.

Define $\mathbf{F_0} = \left[F_0(0), F_0(1), \cdots, F_0\left(\frac{N}{2} - 1\right)\right]^T$ and

$\mathbf{F_1} = \left[F_1(0), F_1(1), \cdots, F_1\left(\frac{N}{2}-1\right)\right]^T$.

Theorem 2.1. *The even indexed components $f(2j)$ can be retrieved from the inverse transform $\mathbf{L_0^{-1}F_0}(j)$, where*

$$\mathbf{L_0^{-1}F_0}(j) = \frac{3}{N}\sum_{k=0}^{\frac{N}{2}-1}\omega^{-k2j}F_0(k).$$

Proof.

$$F_0(k) = \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k2j}f(2j) \tag{2.7}$$

Multiply both sides of (2.7) by ω^{-k2p} and sum over k to get

$$\begin{aligned}
\sum_{k=0}^{\frac{N}{2}-1}\omega^{-k2p}F_0(k) &= \frac{2}{3}\sum_{k=0}^{\frac{N}{2}-1}\sum_{j=0}^{\frac{N}{2}-1}\omega^{2k(j-p)}f(2j) &(2.8)\\
&= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\sum_{k=0}^{\frac{N}{2}-1}\omega^{2k(j-p)}f(2j) &(2.9)\\
&= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}f(2j)\sum_{k=0}^{\frac{N}{2}-1}\omega^{2k(j-p)}. &(2.10)
\end{aligned}$$

If $m = j - p$, then

$$\begin{aligned}
\sum_{k=0}^{\frac{N}{2}-1}\omega^{-k2p}F_0(k) &= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}f(2j)\sum_{k=0}^{\frac{N}{2}-1}\omega^{2km} &(2.11)\\
&= \frac{2}{3}f(2p)\left(\frac{N}{2}\right) &(2.12)
\end{aligned}$$

where from (2.11) to (2.12) we have used lemma 1.1. Solving for the even components $f(2p)$ we get

$$f(2p) = \frac{3}{N}\sum_{k=0}^{\frac{N}{2}-1}\omega^{-k2p}F_0(k)$$

or equivalently

$$f(2j) = \frac{3}{N}\sum_{k=0}^{\frac{N}{2}-1}\omega^{-k2j}F_0(k)$$

for $j = 0, 1, \ldots, \frac{N}{2} - 1$. □

Theorem 2.2. *The odd indexed components $f(2j+1)$ can be retrieved from the inverse transform $\mathbf{L_1^{-1}F_1}(j)$, where*

$$\mathbf{L_1^{-1}F_1}(j) = \frac{3}{2N} \sum_{k=0}^{\frac{N}{2}-1} \omega^{-k(2j+1)} F_1(k)$$

Proof. The proof for this theorem is similar to the proof for theorem 2.1. □

Hence, we note that the transformations in (2.5) and (2.6) are related to two separate transforms arising from the trapezoidal rule, namely, one for the even indexed components and one for the odd indexed components. These transformations satisfy a number of mathematical relationships that can be used to reduce computational effort.

2.2 Relationships in the Frequency Domain

Theorem 2.3. *The transform $\mathbf{F_0}$ has periodicity of $\frac{N}{2}$.*

Proof. For $k \in \mathbb{Z}$

$$\begin{aligned} F_0\left(k + \tfrac{N}{2}\right) &= \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \omega^{(k+\frac{N}{2})2j} f(2j) \\ &= \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \omega^{k2j} \omega^{Nj} f(2j) \\ &= \frac{2}{3} \sum_{j=0}^{\frac{N}{2}-1} \omega^{k2j} f(2j) \\ &= F_0(k) \end{aligned}$$

□

Theorem 2.4. *The transform* $\mathbf{F_1}$ *has anti-periodicity of* $\frac{N}{2}$.

Proof.

$$\begin{aligned} F_1\left(k+\tfrac{N}{2}\right) &= \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{\left(k+\frac{N}{2}\right)(2j+1)}f(2j+1) \\ &= \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k(2j+1)}\omega^{\frac{N}{2}(2j+1)}f(2j+1) \\ &= \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k(2j+1}(-1)^{2j+1}f(2j+1) \\ &= -\frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k(2j+1)}f(2j+1) \\ &= -F_1(k) \end{aligned}$$

□

It follows from theorem 2.3, theorem 2.4 and equation (2.4) that

$$F\left(k+\tfrac{N}{2}\right) = F_0(k) - F_1(k) \tag{2.13}$$

Theorem 2.5. *For a real signal* $\mathbf{f}$,

$$\overline{F\left(\tfrac{N}{2}-k\right)} = F\left(\tfrac{N}{2}+k\right)$$

Proof.

$$F\left(\tfrac{N}{2}-k\right) = F_0\left(\tfrac{N}{2}-k\right) + F_1\left(\tfrac{N}{2}-k\right)$$

From equation (2.5) we obtain

$$F_0\left(\tfrac{N}{2}-k\right) = \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{\left(\frac{N}{2}-k\right)2j}f(2j). \tag{2.14}$$

Conjugating (2.14) results in

$$
\begin{aligned}
\overline{F_0\left(\frac{N}{2}-k\right)} &= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1} \overline{\omega}^{\left(\frac{N}{2}-k\right)2j} f(2j) \\
&= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1} \omega^{-\left(\frac{N}{2}-k\right)2j} f(2j) \\
&= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1} \omega^{k2j} f(2j) \\
&= F_0(k) \\
&= F_0\left(\frac{N}{2}+k\right)
\end{aligned}
$$

and the last step follows from theorem 2.3. We can similarly prove from (2.6) and using theorem 2.4 that

$$
\overline{F_1\left(\frac{N}{2}-k\right)} = -F_1(k) = F_1\left(\frac{N}{2}+k\right).
$$

Hence, it follows that

$$
\begin{aligned}
\overline{F\left(\frac{N}{2}-k\right)} &= \overline{F_0\left(\frac{N}{2}-k\right)} + \overline{F_1\left(\frac{N}{2}-k\right)} \\
&= F_0\left(\frac{N}{2}+k\right) + F_1\left(\frac{N}{2}+k\right) \\
&= F\left(\frac{N}{2}+k\right)
\end{aligned}
$$

□

Theorem 2.6. *The transform* **F** *has periodicity* N.

Proof.

$$
\begin{aligned}
F(k+N) &= F_0(k+N) + F_1(k+N) \\
&= F_0\left(k+\frac{N}{2}+\frac{N}{2}\right) + F_1\left(k+\frac{N}{2}+\frac{N}{2}\right) \\
&= F_0\left(k+\frac{N}{2}\right) - F_1\left(k+\frac{N}{2}\right)
\end{aligned}
$$

Using the periodicity $\frac{N}{2}$ of $\mathbf{F_0}$ and antiperiodicity $\frac{N}{2}$ of $\mathbf{F_1}$ we now obtain

$$\begin{aligned} F(k+N) &= F_0(k) - [-F_1(k)] \\ &= F_0(k) + F_1(k) \\ &= F(k) \end{aligned}$$

$\square$

As theorem 2.5 holds for a vector $\mathbf{f}$ with real components, setting $k = 0$ implies that $F(\frac{N}{2}) = \overline{F(\frac{N}{2})}$ or $F(\frac{N}{2})$ is real. Also setting $k = \frac{N}{2}$ implies $F(N) = \overline{F(0)}$, but $F(N) = F(0)$ from theorem 2.6, hence $F(0)$ is also real.

These conclusions are illustrated for the real signal $\mathbf{f} = [2, 6, 8, 2, 5, 1, 3, 2, 7, 10]^T$ in table 2.1 where $\mathbf{F}$ is computed using MATLAB and tabulated.

Table 2.1: Transform of a real signal

k	$F(k)$
0	44.6667
1	14.3864 + 1.7171i
2	-1.4978 + 5.9485i
3	-8.7197 + 4.1961i
4	-14.1689 + 4.5526i
5	-11.3333
6	-14.1689 - 4.5526i
7	-8.7197 - 4.1961i
8	-1.4978 - 5.9485i
9	14.3864 - 1.7171i

2.3 Matrix Representation of the Transform

Using theorem 2.3 and theorem 2.4, the transformation (2.4) can be written in block matrix vector notation as

$$\mathbf{F} = \begin{bmatrix} \mathbf{F_0} + \mathbf{F_1} \\ \mathbf{F_0} - \mathbf{F_1} \end{bmatrix} = \frac{2}{3} \begin{bmatrix} \mathbf{A} & 2\mathbf{DA} \\ \mathbf{A} & -2\mathbf{DA} \end{bmatrix} \begin{bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \end{bmatrix}, \tag{2.15}$$

where the $\left(\frac{N}{2} \times \frac{N}{2}\right)$ symmetric matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{(N-2)} \\ 1 & \omega^4 & \omega^8 & \dots & \omega^{2(N-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(N-2)} & \omega^{2(N-2)} & \dots & \omega^{(\frac{N}{2}-1)(N-2)} \end{bmatrix} \tag{2.16}$$

and $\mathbf{D} = \text{diag}[1, \omega, \dots, \omega^{(\frac{N}{2}-1)}]$. It follows that equation (2.15) can be written as

$$\mathbf{F} = \frac{2}{3} \begin{bmatrix} \mathbf{A} & 2\mathbf{DA} \\ \mathbf{A} & -2\mathbf{DA} \end{bmatrix} \mathbf{Pf} \tag{2.17}$$

since

$$\begin{bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \end{bmatrix} = \mathbf{Pf}, \tag{2.18}$$

where $\mathbf{P}$ is the $(N \times N)$ permutation matrix defined by its components $P_{j+1,2j+1} = 1$ and $P_{\frac{N}{2}+j+1,2j} = 1$ for $j = 0, 1, \cdots, \frac{N}{2} - 1$. The effect of $\mathbf{P}$ is to group the even indexed components and odd indexed components of $\mathbf{f}$ together. Hence,

$$\mathbf{F} = \mathbf{Uf},$$

where the transformation matrix $\mathbf{U}$ is given by

$$\mathbf{U} = \frac{2}{3}\begin{bmatrix} \mathbf{A} & 2\mathbf{DA} \\ \mathbf{A} & -2\mathbf{DA} \end{bmatrix}\mathbf{P}. \tag{2.19}$$

Theorem 2.7. *The matrix product $\mathbf{AA}^*$ is diagonal and given by*

$$\mathbf{AA}^* = \tfrac{N}{2}\mathbf{I}_{\frac{N}{2}}$$

Proof.

$$(\mathbf{AA}^*)_{pq} = \sum_{l=1}^{\frac{N}{2}} A_{pl}A^*_{lq} \tag{2.20}$$

where $\mathbf{A}_{pq} = \omega^{2(p-1)(q-1)}$. Hence,

$$\begin{aligned}
(\mathbf{AA}^*)_{pq} &= \sum_{l=1}^{\frac{N}{2}} \omega^{2(p-1)(l-1)}\overline{\omega}^{2(q-1)(l-1)} \\
&= \sum_{l=1}^{\frac{N}{2}} \omega^{2(p-1)(l-1)}\omega^{-2(q-1)(l-1)} \\
&= \sum_{l=1}^{\frac{N}{2}} \omega^{2(l-1)(p-q)} \\
&= \sum_{l=1}^{\frac{N}{2}} \omega^{2(l-1)m},
\end{aligned}$$

where $m = p - q$.

If $m \neq 0$, that is $p \neq q$, then

$$(\mathbf{AA}^*)_{pq} = 0$$

follows from lemma 1.1.

If $m = 0$, that is, $p = q$,

$$(\mathbf{AA}^*)_{pp} = \frac{N}{2}.$$

□

Corollary 2.3.1. *The matrix* $\mathbf{A}$ *is normal.*

Proof. The proof is similar to that of theorem 2.7 by exploiting the symmetry of $\mathbf{A}$. □

The diagonal matrix $\mathbf{D}$ is unitary, that is, $\mathbf{D}\mathbf{D}^* = \mathbf{D}^*\mathbf{D} = \mathbf{I}_{\frac{N}{2}}$, which follows from the fact that $\omega\overline{\omega} = \overline{\omega}\omega = 1$.

Theorem 2.8. *The inverse transformation matrix* $\mathbf{U^{-1}}$ *is given by*

$$\mathbf{U}^{-1} = \frac{3}{2N}\mathbf{P}^T \begin{bmatrix} \mathbf{A}^* & \mathbf{A}^* \\ \frac{1}{2}\mathbf{A}^*\mathbf{D}^* & -\frac{1}{2}\mathbf{A}^*\mathbf{D}^* \end{bmatrix}.$$

Proof.

$$\begin{aligned}
\mathbf{U}\mathbf{U}^{-1} &= \frac{2}{3}\begin{bmatrix} \mathbf{A} & 2\mathbf{D}\mathbf{A} \\ \mathbf{A} & -2\mathbf{D}\mathbf{A} \end{bmatrix} \mathbf{P}\frac{3}{2N}\mathbf{P}^T \begin{bmatrix} \mathbf{A}^* & \mathbf{A}^* \\ \frac{1}{2}\mathbf{A}^*\mathbf{D}^* & -\frac{1}{2}\mathbf{A}^*\mathbf{D}^* \end{bmatrix} \\
&= \frac{1}{N}\begin{bmatrix} \mathbf{A} & 2\mathbf{D}\mathbf{A} \\ \mathbf{A} & -2\mathbf{D}\mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{A}^* & \mathbf{A}^* \\ \frac{1}{2}\mathbf{A}^*\mathbf{D}^* & -\frac{1}{2}\mathbf{A}^*\mathbf{D}^* \end{bmatrix} \\
&= \frac{1}{N}\begin{bmatrix} \mathbf{A}\mathbf{A}^* + \mathbf{D}\mathbf{A}\mathbf{A}^*\mathbf{D}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{A}\mathbf{A}^* + \mathbf{D}\mathbf{A}\mathbf{A}^*\mathbf{D}^* \end{bmatrix} \\
&= \frac{1}{N}\begin{bmatrix} N\mathbf{I}_{\frac{N}{2}} & \mathbf{0} \\ \mathbf{0} & N\mathbf{I}_{\frac{N}{2}} \end{bmatrix} \\
&= \mathbf{I}
\end{aligned}$$

□

Chapter 3

Properties of the Simpson Discrete Fourier Transform

The Simpson Discrete Fourier Transform (SDFT) and its inverse are transformations relating the time and frequency domains. We state and prove the important properties of time reversal, conjugation, even time shift, frequency shift, convolution in time domain, convolution in frequency domain and Plancherel's theorem. By discarding $\frac{1}{N}$ in (2.1), the Simpson Discrete Fourier Transformation may be defined by

$$\mathbf{Uf}(k) = \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1} \omega^{k2j} f(2j) + \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1} \omega^{k(2j+1)} f(2j+1), \tag{3.1}$$

for $k = 0, 1, \ldots, N-1$.

The transpose $\mathbf{U}^T$ of the transformation matrix $\mathbf{U}$ in (2.19) is represented by

$$\mathbf{U}^T = \frac{2}{3}\mathbf{P}^T \begin{bmatrix} \mathbf{A} & \mathbf{A} \\ 2\mathbf{AD} & -2\mathbf{AD} \end{bmatrix}. \tag{3.2}$$

The effect of $\mathbf{U}^T$ on a signal $\mathbf{f}$ is given by

$$\mathbf{U}^T\mathbf{f}(2k) = \frac{2}{3}\sum_{j=0}^{N-1} \omega^{k2j} f(j), \tag{3.3}$$

$$\mathbf{U}^T\mathbf{f}(2k+1) = \frac{4}{3}\sum_{j=0}^{N-1}\omega^{(2k+1)j}f(j), \tag{3.4}$$

where $k = 0, 1, \ldots, \frac{N}{2} - 1$.

3.1 Time Reversal

By this operation, it is meant that a reversal in the time domain results in a corresponding reversal in the frequency domain, that is $\mathbf{Uf}(-k)$ is the transform of $f(-j)$, $j = 0, 1, \ldots, N-1$.

Proof.

$$\begin{aligned}
\mathbf{Uf}(-k) &= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{-k2j}f(2j) + \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{-k(2j+1)}f(2j+1) && (3.5)\\
&= \frac{2}{3}\sum_{j=-\frac{N}{2}+1}^{0}\omega^{k2j}f(-2j) + \frac{4}{3}\sum_{j=-\frac{N}{2}+1}^{0}\omega^{k(2j-1)}f(-2j+1) && (3.6)\\
&= \frac{2}{3}\sum_{j=1}^{\frac{N}{2}}\omega^{k2j}f(N-2j) + \frac{4}{3}\sum_{j=1}^{\frac{N}{2}}\omega^{k(2j-1)}f(N-2j+1) && (3.7)\\
&= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k2j}f(N-2j) + \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k(2j+1)}f(N-2j-1) && (3.8)
\end{aligned}$$

We have shifted the index in (3.8) since the arguments of both summations in (3.7) have periodicity of $\frac{N}{2}$ [5]. Further, using the periodicity N of $\mathbf{f}$ we obtain

$$\mathbf{Uf}(-k) = \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k2j}f(-2j) + \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k(2j+1)}f(-2j-1). \tag{3.9}$$

□

From equation (3.9) it follows that if $\mathbf{f}$ is an even signal, that is $f(-j) = f(j)$, $j = 0, 1, \ldots, N-1$ then $\mathbf{Uf}(-k) = \mathbf{Uf}(k)$ and if $\mathbf{f}$ is an odd signal, that is $f(j) = -f(j)$, $j = 0, 1, \ldots, N-1$ then $\mathbf{Uf}(-k) = -\mathbf{Uf}(k)$. Hence the transform of an even signal is even and that of an odd signal is odd.

3.2 Conjugation

The transform of the conjugate signal $\overline{f(j)}$ implies a reversal and conjugation in the frequency domain, that is $\overline{\mathbf{Uf}(-k)}$.

Proof. From (3.1) we obtain

$$\begin{aligned}
\overline{\mathbf{Uf}(-k)} &= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1} \overline{\omega}^{-k2j}\overline{f(2j)} + \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1} \overline{\omega}^{-k(2j+1)}\overline{f(2j+1)} \\
&= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1} \omega^{k2j}\overline{f(2j)} + \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1} \omega^{k(2j+1)}\overline{f(2j+1)}. \qquad (3.10)
\end{aligned}$$

□

A signal $\mathbf{f}$ may be decomposed into an even and an odd part in the following manner:

$$f(j) = \frac{f(j)+f(-j)}{2} + \frac{f(j)-f(-j)}{2} \qquad (3.11)$$

Define the even part $\mathbf{f}_e$ by $f_e(j) = \frac{f(j)+f(-j)}{2}$ and the odd part $\mathbf{f}_o$ by $f_o(j) = \frac{f(j)-f(-j)}{2}$, then

$$\mathbf{f} = \mathbf{f}_e + \mathbf{f}_o.$$

Transforming the signal in (3.11) and using the time reversal property we get

$$\mathbf{Uf}(k) = \frac{\mathbf{Uf}(k)+\mathbf{Uf}(-k)}{2} + \frac{\mathbf{Uf}(k)-\mathbf{Uf}(-k)}{2}. \qquad (3.12)$$

As a special case, consider a signal $\mathbf{f} \in \mathbb{R}^N$. Equation (3.10) reduces to

$$\overline{\mathbf{Uf}(-k)} = \mathbf{Uf}(k) \qquad (3.13)$$

From (3.12) and (3.13), this reduces to

$$\mathbf{Uf}(k) = \mathbf{Uf}_e(k) + \mathbf{Uf}_o(k)$$

$$
\begin{aligned}
&= \frac{\mathbf{Uf}(k)+\overline{\mathbf{Uf}(k)}}{2}+\frac{\mathbf{Uf}(k)-\overline{\mathbf{Uf}(k)}}{2} \\
&= Re\left(\mathbf{Uf}(k)\right)+Im\left(\mathbf{Uf}(k)\right) \qquad (3.14)
\end{aligned}
$$

Theorem 3.1. *For any real even signal* $\mathbf{f}_e$*, the transformation* $\mathbf{Uf}_e$ *is real, that is*

$$\overline{\mathbf{Uf}_e(k)} = \mathbf{Uf}_e(k)$$

Proof.

$$
\begin{aligned}
\overline{\mathbf{Uf}_e(k)} &= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\overline{\omega}^{k2j}f_e(2j)+\frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\overline{\omega}^{k(2j+1)}f_e(2j+1) \\
&= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{-k2j}f_e(2j)+\frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{-k(2j+1)}f_e(2j+1) \\
&= \frac{2}{3}\sum_{j=-\frac{N}{2}+1}^{0}\omega^{k2j}f_e(-2j)+\frac{4}{3}\sum_{j=-\frac{N}{2}+1}^{0}\omega^{-k(-2j+1)}f_e(-2j+1) \qquad (3.15)
\end{aligned}
$$

Replace j by $j+\frac{N}{2}+1$ in the second summation in (3.15) to get

$$\overline{\mathbf{Uf}_e(k)} = \frac{2}{3}\sum_{j=-\frac{N}{2}+1}^{0}\omega^{k2j}f_e(-2j)+\frac{4}{3}\sum_{j=-N}^{-\frac{N}{2}-1}\omega^{-k(-2j-N-1)}f_e(-2j-N-1)$$

since $\mathbf{f}_e$ is even and periodic in N, it follows that

$$
\begin{aligned}
\overline{\mathbf{Uf}_e(k)} &= \frac{2}{3}\sum_{j=-\frac{N}{2}+1}^{0}\omega^{k2j}f_e(-2j)+\frac{4}{3}\sum_{j=-N}^{-\frac{N}{2}-1}\omega^{k(2j+1)}f_e(-2j-1) \\
&= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k2j}f_e(2j)+\frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k(2j+1)}f_e(2j+1) \\
&= \mathbf{Uf}_e(k) \qquad (3.16)
\end{aligned}
$$

$\square$

Theorem 3.2. *For any real odd signal* $\mathbf{f}_o$*, the transformation* $\mathbf{Uf}_o$ *is purely imaginary, that is*

$$\overline{\mathbf{Uf}_o(k)} = -\mathbf{Uf}_o(k).$$

Proof. The proof for this theorem is similar to the proof for theorem 3.1. □

Hence from theorem 3.1, theorem 3.2 and (3.14) we conclude that for $\mathbf{f} \in \mathbb{R}^N$, the transformation of the even part of $\mathbf{f}$ is the real part of the transformation of $\mathbf{f}$, that is, $\mathbf{Uf}_e = Re(\mathbf{Uf})$ and the transformation of the odd part of $\mathbf{f}$ is the imaginary part of the transformation of $\mathbf{f}$, that is $\mathbf{Uf}_o = Im(\mathbf{Uf})$.

3.3 Even Time Shift

The transform of an even shifted signal $f(j-2n)$ in the time domain results in the multiplication by a complex exponent ω^{2kn} in the frequency domain.

Proof.

$$\begin{aligned}
\omega^{2kn}\mathbf{Uf}(k) &= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k2(j+n)}f(2j) + \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k(2j+1+2n)}f(2j+1) \\
&= \frac{2}{3}\sum_{j=-n}^{\frac{N}{2}-1-n}\omega^{k2(j+n)}f(2j) + \frac{4}{3}\sum_{j=-n}^{\frac{N}{2}-1-n}\omega^{k(2j+1+2n)}f(2j+1) \\
&= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k2j}f(2j-2n) + \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k(2j+1)}f(2j+1-2n)
\end{aligned}$$

□

It must be stressed that it is only possible to effect an even time shift since this shifts even indexed quadrature nodes to even indexed nodes and odd indexed quadrature nodes to odd indexed nodes, being a characteristic of Simpson's quadrature. An odd time shift will not preserve this property.

3.4 Frequency Shift

A frequency shift in the frequency domain is equivalent to the transformation of the signal $\omega^{-k_0 j} f(j)$ in the time domain.

Proof.

$$\begin{aligned}\mathbf{Uf}(k-k_0) &= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{(k-k_0)2j}f(2j)+\frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{(k-k_0)(2j+1)}f(2j+1)\\ &= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{2kj}\left[\omega^{-2k_0 j}f(2j)\right]+\frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k(2j+1)}\left[\omega^{-k_0(2j+1)}f(2j+1)\right]\end{aligned} \tag{3.17}$$

□

3.5 Convolution in Time Domain

The convolution operation in the time domain is related to a multiplication in the frequency domain in the following manner

$$\mathbf{U}^T(\mathbf{f}\otimes\mathbf{g})(2k) = \frac{3}{2}\mathbf{U}^T\mathbf{f}(2k)\mathbf{U}^T\mathbf{g}(2k), \tag{3.18}$$

$$\mathbf{U}^T(\mathbf{f}\otimes\mathbf{g})(2k+1) = \frac{3}{4}\mathbf{U}^T\mathbf{f}(2k+1)\mathbf{U}^T\mathbf{g}(2k+1), \tag{3.19}$$

where $\mathbf{f}$ and $\mathbf{g}$ are periodic signals of length N.

Proof. We prove the result (3.19). Using (3.4) we obtain

$$\begin{aligned}\frac{3}{4}\mathbf{U}^T\mathbf{f}(2k+1)\mathbf{U}^T\mathbf{g}(2k+1) &= \frac{4}{3}\left(\sum_{j=0}^{N-1}\omega^{(2k+1)j}f(j)\right)\left(\sum_{p=0}^{N-1}\omega^{(2k+1)p}g(p)\right)\\ &= \frac{4}{3}\sum_{j=0}^{N-1}\sum_{p=0}^{N-1}\omega^{(2k+1)(j+p)}f(j)g(p)\end{aligned} \tag{3.20}$$

Changing the index of the inner summation in (3.20) to $l = j + p$, we get

$$\begin{aligned}
\frac{3}{4}\mathbf{U}^T\mathbf{f}(2k+1)\mathbf{U}^T\mathbf{g}(2k+1) &= \frac{4}{3}\sum_{j=0}^{N-1}\sum_{l=j}^{N-1+j}\omega^{(2k+1)l}f(j)g(l-j) \\
&= \frac{4}{3}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1}\omega^{(2k+1)l}f(j)g(l-j) \\
&= \frac{4}{3}\sum_{l=0}^{N-1}\omega^{(2k+1)l}\sum_{j=0}^{N-1}f(j)g(l-j) \\
&= \frac{4}{3}\sum_{l=0}^{N-1}\omega^{(2k+1)l}(f\otimes g)(l) \\
&= \mathbf{U}^T(\mathbf{f}\otimes\mathbf{g})(2k+1). \qquad (3.21)
\end{aligned}$$

□

Similarly, we can prove (3.18).

3.6 Convolution in Frequency Domain

The convolution operation in the frequency domain is related to a multiplication in the time domain in the following manner

$$\mathbf{U}\mathbf{f}\otimes\mathbf{U}\mathbf{g} = N\mathbf{U}(\hat{\mathbf{f}}\cdot *\hat{\mathbf{g}}), \qquad (3.22)$$

where $\cdot *$ denotes componentwise product of vectors and $\hat{f}(2m) = \sqrt{\frac{2}{3}}f(2m)$ and $\hat{f}(2m+1) = \sqrt{\frac{4}{3}}f(2m+1)$ and $\hat{\mathbf{g}}$ is similarly defined.

Proof. We prove the result in (3.22) for the odd indexed components.

$$\begin{aligned}
&(\mathbf{U}\mathbf{f}\otimes\mathbf{U}\mathbf{g})(2k+1) \\
&= \sum_{j=0}^{N-1}\mathbf{U}\mathbf{f}(j)\mathbf{U}\mathbf{g}(2k+1-j)
\end{aligned}$$

$$= \sum_{j=0}^{N-1} \left[\frac{2}{3} \sum_{p=0}^{\frac{N}{2}-1} \omega^{j2p} f(2p) + \frac{4}{3} \sum_{p=0}^{\frac{N}{2}-1} \omega^{j(2p+1)} f(2p+1) \right] \tag{3.23}$$

$$\times \left[\frac{2}{3} \sum_{m=0}^{\frac{N}{2}-1} \omega^{(2k+1-j)2m} g(2m) + \frac{4}{3} \sum_{m=0}^{\frac{N}{2}-1} \omega^{(2k+1-j)(2m+1)} g(2m+1) \right] \tag{3.24}$$

Equation (3.24) reduces to

$$\frac{4}{9} \sum_{j=0}^{N-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \omega^{j2p} f(2p) \omega^{(2k+1-j)2m} g(2m)$$

$$+ \frac{16}{9} \sum_{j=0}^{N-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \omega^{j(2p+1)} f(2p+1) \omega^{(2k+1-j)(2m+1)} g(2m+1) \tag{3.25}$$

since the omitted products can be shown to be zero. The first term in (3.25) can be written as

$$\frac{4}{9} \sum_{j=0}^{N-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \omega^{j2p} f(2p) \omega^{(2k+1)2m} \omega^{-j2m} g(2m)$$

$$= \frac{2}{3} \sum_{p=0}^{\frac{N}{2}-1} \sum_{m=0}^{\frac{N}{2}-1} \omega^{(2k+1)2m} \frac{2}{3} f(2p) g(2m) \sum_{j=0}^{N-1} \omega^{2j(p-m)}. \tag{3.26}$$

The last summation in (3.26) is zero unless $p = m$ (from lemma 1.1) and equation (3.26) simplifies to

$$\frac{2}{3} N \sum_{m=0}^{\frac{N}{2}-1} \omega^{(2k+1)2m} \sqrt{\frac{2}{3}} f(2m) \sqrt{\frac{2}{3}} g(2m) = \frac{2}{3} N \sum_{m=0}^{\frac{N}{2}-1} \omega^{(2k+1)2m} \hat{f}(2m) \hat{g}(2m). \tag{3.27}$$

Similarly, the second term in (3.25) may be shown to equal

$$\frac{4}{3} N \sum_{m=0}^{\frac{N}{2}-1} \omega^{(2k+1)(2m+1)} \hat{f}(2m+1) \hat{g}(2m+1). \tag{3.28}$$

Adding (3.27) and (3.28), we finally obtain

$$
\begin{aligned}
&(\mathbf{Uf} \otimes \mathbf{Ug})(2k+1) \\
&= N\left[\frac{2}{3}\sum_{m=0}^{\frac{N}{2}-1}\omega^{(2k+1)2m}\hat{f}(2m)\hat{g}(2m) + \frac{4}{3}\sum_{m=0}^{\frac{N}{2}-1}\omega^{(2k+1)(2m+1)}\hat{f}(2m+1)\hat{g}(2m+1)\right] \\
&= N\left[\frac{2}{3}\sum_{m=0}^{\frac{N}{2}-1}\omega^{(2k+1)2m}(\hat{\mathbf{f}}\cdot *\hat{\mathbf{g}})(2m) + \frac{4}{3}\sum_{m=0}^{\frac{N}{2}-1}\omega^{(2k+1)(2m+1)}(\hat{\mathbf{f}}\cdot *\hat{\mathbf{g}})(2m+1)\right] \\
&= N\mathbf{U}(\hat{\mathbf{f}}\cdot *\hat{\mathbf{g}})(2k+1).
\end{aligned} \tag{3.29}
$$

Likewise, it can be shown that (3.22) is true for the even indexed components. □

3.7 Plancherel's Theorem

For $\mathbf{f}, \mathbf{g} \in \mathbb{C}^N$

$$
\langle \mathbf{Uf}, \mathbf{Ug}\rangle = \frac{4N}{9}\left[\sum_{j=0}^{\frac{N}{2}-1} f(2j)\overline{g(2j)} + 4\sum_{j=0}^{\frac{N}{2}-1} f(2j+1)\overline{g(2j+1)}\right]. \tag{3.30}
$$

Proof.

$$
\langle \mathbf{Uf}, \mathbf{Ug}\rangle = \sum_{k=0}^{\frac{N}{2}-1}\mathbf{Uf}(2k)\overline{\mathbf{Ug}(2k)} + \sum_{k=0}^{\frac{N}{2}-1}\mathbf{Uf}(2k+1)\overline{\mathbf{Ug}(2k+1)}. \tag{3.31}
$$

The second summation in (3.31) is by definition

$$
\begin{aligned}
&\sum_{k=0}^{\frac{N}{2}-1}\mathbf{Uf}(2k+1)\overline{\mathbf{Ug}(2k+1)} \\
&= \sum_{k=0}^{\frac{N}{2}-1}\left[\frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{(2k+1)2j}f(2j) + \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{(2k+1)(2j+1)}f(2j+1)\right] \\
&\quad\times\left[\frac{2}{3}\sum_{p=0}^{\frac{N}{2}-1}\omega^{-(2k+1)2p}\overline{g(2p)} + \frac{4}{3}\sum_{p=0}^{\frac{N}{2}-1}\omega^{-(2k+1)(2p+1)}\overline{g(2p+1)}\right].
\end{aligned} \tag{3.32}
$$

Multiplying out (3.32), we get

$$\begin{aligned}
&\sum_{k=0}^{\frac{N}{2}-1} \mathbf{Uf}(2k+1)\overline{\mathbf{Ug}(2k+1)} \\
&= \frac{4}{9}\sum_{j=0}^{\frac{N}{2}-1}\sum_{p=0}^{\frac{N}{2}-1}\sum_{k=0}^{\frac{N}{2}-1} \omega^{2(2k+1)(j-p)} f(2j)\overline{g(2p)} \\
&\quad + \frac{16}{9}\sum_{j=0}^{\frac{N}{2}-1}\sum_{p=0}^{\frac{N}{2}-1}\sum_{k=0}^{\frac{N}{2}-1} \omega^{2(2k+1)(j-p)} f(2j+1)\overline{g(2p+1)} \\
&\quad + \frac{8}{9}\sum_{j=0}^{\frac{N}{2}-1}\sum_{p=0}^{\frac{N}{2}-1}\sum_{k=0}^{\frac{N}{2}-1} \omega^{(2k+1)(2j-2p-1)} f(2j)\overline{g(2p+1)} \\
&\quad + \frac{8}{9}\sum_{j=0}^{\frac{N}{2}-1}\sum_{p=0}^{\frac{N}{2}-1}\sum_{k=0}^{\frac{N}{2}-1} \omega^{(2k+1)(2j-2p+1)} f(2j+1)\overline{g(2p)}. \qquad (3.33)
\end{aligned}$$

In the last term of (3.33), let $m = 2j - 2p + 1$, $m \in \mathbb{Z}$ to obtain

$$\frac{8}{9}\sum_{j=0}^{\frac{N}{2}-1}\sum_{p=0}^{\frac{N}{2}-1} f(2j+1)\overline{g(2p)} \sum_{k=0}^{\frac{N}{2}-1} \omega^{(2k+1)m}. \qquad (3.34)$$

Since $m \neq 0$, it follows that (3.34) is zero from lemma 1.1. Likewise, it can be shown that the second last term in (3.33) is also zero. Hence (3.33) simplifies to

$$\begin{aligned}
\sum_{k=0}^{\frac{N}{2}-1} \mathbf{Uf}(2k+1)\overline{\mathbf{Ug}(2k+1)} &= \frac{4}{9}\sum_{j=0}^{\frac{N}{2}-1}\sum_{p=0}^{\frac{N}{2}-1}\sum_{k=0}^{\frac{N}{2}-1} \omega^{2(2k+1)(j-p)} f(2j)\overline{g(2p)} + \\
&\quad \frac{16}{9}\sum_{j=0}^{\frac{N}{2}-1}\sum_{p=0}^{\frac{N}{2}-1}\sum_{k=0}^{\frac{N}{2}-1} \omega^{2(2k+1)(j-p)} f(2j+1)\overline{g(2p+1)}
\end{aligned} \qquad (3.35)$$

If $j \neq p$, (3.35) vanishes. If $j = p$, (3.35) reduces to

$$
\begin{aligned}
&\sum_{k=0}^{\frac{N}{2}-1} \mathbf{Uf}(2k+1)\overline{\mathbf{Ug}(2k+1)} \\
&\quad = \frac{4}{9}\sum_{j=0}^{\frac{N}{2}-1} f(2j)\overline{g(2j)}\left(\tfrac{N}{2}\right) + \frac{16}{9}\sum_{j=0}^{\frac{N}{2}-1} f(2j+1)\overline{g(2j+1)}\left(\tfrac{N}{2}\right) \\
&\quad = \frac{2N}{9}\left[\sum_{j=0}^{\frac{N}{2}-1} f(2j)\overline{g(2j)} + 4\sum_{j=0}^{\frac{N}{2}-1} f(2j+1)\overline{g(2j+1)}\right]
\end{aligned} \tag{3.36}
$$

Similarly, the first sum in (3.31) can also be shown to simplify to (3.36) from which (3.30) follows. □

For $\mathbf{f} = \mathbf{g}$ in (3.30), we obtain Parseval's identity

$$
\|\mathbf{Uf}\|^2 = \frac{4N}{9}\left[\sum_{j=0}^{\frac{N}{2}-1} |f(2j)|^2 + 4\sum_{j=0}^{\frac{N}{2}-1} |f(2j+1)|^2\right]. \tag{3.37}
$$

3.8 Duality

The SDFT satisfies the duality property

$$
\mathbf{U}^2\mathbf{f}(2k) = \frac{2}{3}Nf(-2k) - \frac{4}{9}Nf\left(-2k - \tfrac{N}{2}\right) \tag{3.38}
$$

$$
\mathbf{U}^2\mathbf{f}(2k+1) = \frac{4}{3}Nf(-2k-1) - \frac{2}{9}Nf\left(-2k-1-\tfrac{N}{2}\right), \tag{3.39}
$$

which has been proven in [23].

For the signal $\mathbf{f}$ defined by $f(j) = j + 1$ for $j = 0, 1, \cdots, 9$ and shown in figure 3.1 as a stem plot, the effect of the duality property on signal $\mathbf{f}$ for the DFT as well as for the SDFT is shown in figure 3.2 and figure 3.3, respectively.

The DFT duality (1.16) reverses the signal from the beginning only, whilst the SDFT duality reverses the signal from the beginning as well as from midway and involves a linear combination of both according to equations (3.38) and (3.39).

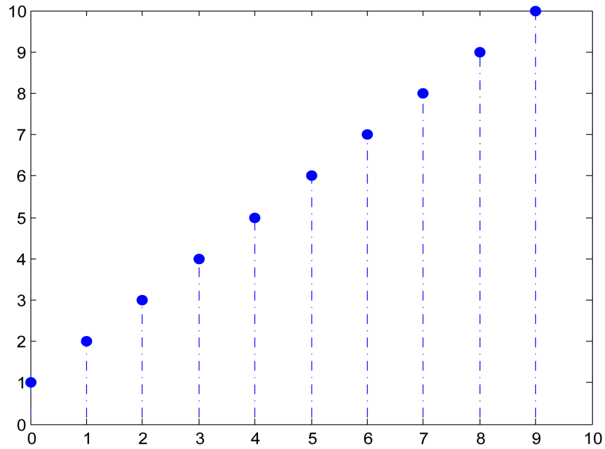

Figure 3.1: Original Signal

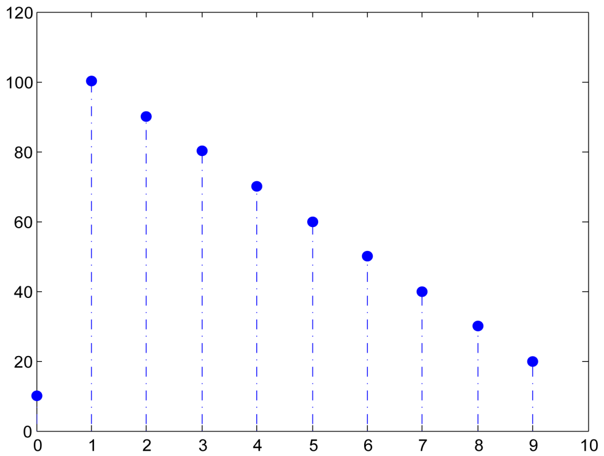

Figure 3.2: DFT Duality

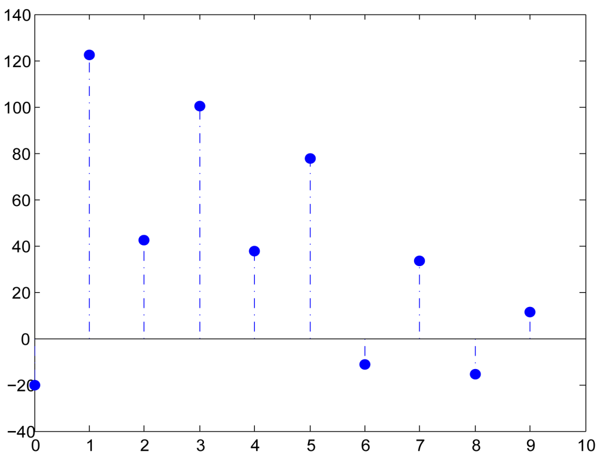

Figure 3.3: SDFT Duality

3.9 The Minimal Polynomial

In order to derive the minimal polynomial for the SDFT matrix, we require an expression for $\mathbf{U}^4\mathbf{f}$ in component form. To obtain $\mathbf{U}^4\mathbf{f}$ we apply the duality property to $\mathbf{g}$ where

$$\mathbf{g}(2k) = \mathbf{U}^2\mathbf{f}(2k) \tag{3.40}$$

and

$$\mathbf{g}(2k+1) = \mathbf{U}^2\mathbf{f}(2k+1). \tag{3.41}$$

From (3.38) and (3.39) we get

$$\mathbf{U}^2\mathbf{g}(2k) = \frac{2}{3}Ng(-2k) - \frac{4}{9}Ng\left(-2k - \tfrac{N}{2}\right) \tag{3.42}$$

In order to obtain $\mathbf{g}(-2k)$, we replace k by $-k$ in (3.40) and (3.38) to obtain

$$\mathbf{g}(-2k) = \frac{2}{3}Nf(2k) - \frac{4}{9}Nf\left(2k - \tfrac{N}{2}\right). \tag{3.43}$$

In order to obtain $g(-2k - \frac{N}{2})$, we first observe that $-2k - \frac{N}{2}$ is odd. Replace k by $-k - \frac{N}{4} - \frac{1}{2}$ in (3.41) and (3.39) to obtain

$$\mathbf{g}(-2k - \tfrac{N}{2}) = \frac{4}{3}Nf(2k + \tfrac{N}{2}) - \frac{2}{9}Nf(2k). \tag{3.44}$$

Substituting (3.43) and (3.44) in (3.42) we get

$$\mathbf{U}^2\mathbf{g}(2k) = N^2\left[\frac{44}{81}f(2k) - \frac{8}{27}f\left(2k - \tfrac{N}{2}\right) - \frac{16}{27}f\left(2k + \tfrac{N}{2}\right)\right]. \tag{3.45}$$

Similarly, it may be shown that

$$\begin{aligned}
\mathbf{U}^2\mathbf{g}(2k+1) &= -\frac{2}{9}Ng(-2k-1-\tfrac{N}{2}) + \frac{4}{3}Ng(-2k-1) \\
&= -\frac{2}{9}N\left[-\frac{4}{9}Nf(2k+1) + \frac{2}{3}Nf\left(2k+1+\tfrac{N}{2}\right)\right] \\
&\quad + \frac{4}{3}N\left[-\frac{2}{9}Nf\left(2k+1-\tfrac{N}{2}\right) + \frac{4}{3}Nf(2k+1)\right] \\
&= N^2\left[\frac{152}{81}f(2k+1) - \frac{8}{27}f\left(2k+1-\tfrac{N}{2}\right)\right. \\
&\quad \left. - \frac{4}{27}f\left(2k+1+\tfrac{N}{2}\right)\right]. \qquad (3.46)
\end{aligned}$$

Using the periodicity N of $\mathbf{f}$ in (3.45) and (3.46) we get

$$\begin{aligned}
\mathbf{U}^2\mathbf{g}(2k) &= \mathbf{U}^4\mathbf{f}(2k) \\
&= N^2\left[\frac{44}{81}f(2k) - \frac{8}{9}f\left(2k+\tfrac{N}{2}\right)\right] \qquad (3.47)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{U}^2\mathbf{g}(2k+1) &= \mathbf{U}^4\mathbf{f}(2k+1) \\
&= N^2\left[\frac{152}{81}f(2k+1) - \frac{4}{9}f\left(2k+1+\tfrac{N}{2}\right)\right] \qquad (3.48)
\end{aligned}$$

for $k = 0, 1, \cdots, \frac{N}{2} - 1$.

Applying the duality property to (3.47) and (3.48) we obtain

$$\mathbf{U}^6\mathbf{f}(2k) = N^3\left[\frac{136}{243}f(-2k) - \frac{1040}{729}f\left(-2k-\tfrac{N}{2}\right)\right] \qquad (3.49)$$

$$\mathbf{U}^6\mathbf{f}(2k+1) = N^3\left[\frac{656}{243}f(-2k-1) - \frac{520}{729}f\left(-2k-1-\tfrac{N}{2}\right)\right] \qquad (3.50)$$

A final application of the duality property to (3.49) and (3.50) yields

$$\mathbf{U}^8\mathbf{f}(2k) = N^4\left[\frac{4528}{6561}f(2k) - \frac{1568}{729}f\left(2k+\tfrac{N}{2}\right)\right] \qquad (3.51)$$

$$\mathbf{U}^8\mathbf{f}(2k+1) = N^4\left[\frac{25696}{6561}f(2k+1) - \frac{784}{729}f\left(2k+1+\tfrac{N}{2}\right)\right]. \tag{3.52}$$

Let $r_1 = \frac{\sqrt{N}}{3}\sqrt{9+\sqrt{17}}$ and $r_2 = \frac{\sqrt{N}}{3}\sqrt{9-\sqrt{17}}$, then from (3.47) and (3.51) it can be shown that the even components of the vector

$$[\mathbf{U}^8 - (r_1^4+r_2^4)\mathbf{U}^4 + r_1^4r_2^4\mathbf{I}]\mathbf{f}$$

satisfy the equation

$$[\mathbf{U}^8 - (r_1^4+r_2^4)\mathbf{U}^4 + r_1^4r_2^4\mathbf{I}]\mathbf{f}(2k) = 0. \tag{3.53}$$

Likewise, it can be shown from (3.48) and (3.52) that the odd components satisfy

$$[\mathbf{U}^8 - (r_1^4+r_2^4)\mathbf{U}^4 + r_1^4r_2^4\mathbf{I}]\mathbf{f}(2k+1) = 0 \tag{3.54}$$

for $k = 0, 1, \cdots, \frac{N}{2}-1$.

Hence,

$$\begin{aligned}\mathbf{U}^8 - (r_1^4+r_2^4)\mathbf{U}^4 + r_1^4r_2^4\mathbf{I} &= (\mathbf{U}^4 - r_1^4\mathbf{I})(\mathbf{U}^4 - r_2^4\mathbf{I})\\ &= \mathbf{0}\end{aligned} \tag{3.55}$$

From (3.55) we get the minimal polynomial

$$\phi(\lambda) = (\lambda^4 - r_1^4)(\lambda^4 - r_2^4), \tag{3.56}$$

from which we deduce that the spectrum $\sigma(\mathbf{U})$ is given by

$$\sigma(\mathbf{U}) = \left\{r_1e^{il\frac{\pi}{2}},\ r_2e^{il\frac{\pi}{2}}\ :\ l = 0,1,2,3\right\}. \tag{3.57}$$

Thus we notice that the eigenvalues are either real or purely imaginary. In addition, the minimal polynomial has linear elementary divisors and hence, $\mathbf{U}$ is diagonalizable. Hence, the eigenvectors of $\mathbf{U}$ form a basis for $\mathbb{C}^N$.

For $N = 14$, the eigenvalues of the SDFT matrix $\mathbf{U}$, computed in MATLAB, are summarized in table 3.2 and agree with those calculated from (3.57).

Table 3.2: Eigenvalues of $\mathbf{U}$ (14×14)

-4.518154	
-4.518154	
4.518254	
4.518254	
	-4.518154i
	4.518254i
	4.518254i
	-2.754320i
	-2.754320i
-2.754320	
-2.754320	
2.754320	
2.754320	
	2.754320i

We note from (1.19) that the eigenvalues of the DFT matrix normalized with respect to $\sqrt{N}$, lie on a unit circle. However, from (3.57), the normalized eigenvalues of the SDFT matrix lie on the circumference of concentric circles of radii less than unity and greater than unity as depicted in figure 3.4 in the complex plane.

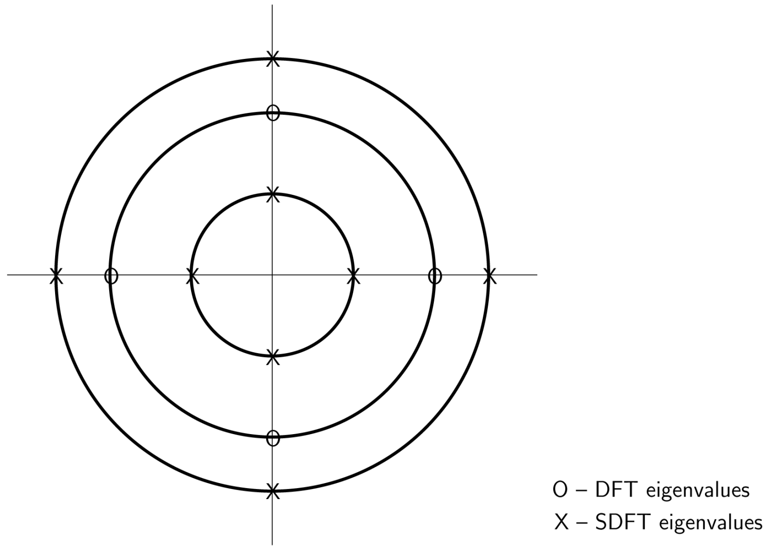

Figure 3.4: Normalized eigenvalues

Chapter 4

2-D Simpson Discrete Fourier Transform

Digital images are of vital importance in the application to ultrasound imaging, magnetic resonance imaging, computer tomography, geophysics and astronomy [13]. Images that are sinusoidally contaminated by noise are most easily cleaned by denoising in the frequency domain by the application of suitable filters [14]. Image compression in the frequency domain essentially ignores Fourier coefficients of small magnitude thus resulting in a reduction of data storage without a significant loss of image quality [18]. As a two dimensional (2-D) discrete Fourier transform is obtained by an application of the trapezoidal rule in 2-D, here we similarly derive a 2-D Simpson discrete Fourier transform using Simpson's rule. A 2-D transform is necessary for application to image processing. In this chapter we briefly apply the 2-D SDFT to encryption and watermarking in the frequency domain [21].

4.1 Derivation of the 2-D SDFT

In order to formulate a 2-D SDFT we adopt the notation $\mathbf{U_N}$, where N denotes the order of the square SDFT matrix $\mathbf{U}$ defined in (2.19). Then $\mathbf{U_N}$ is defined by

$$\mathbf{U_N}\mathbf{f}(k) = \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1} \omega_N^{k2j} f(2j) + \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1} \omega_N^{k(2j+1)} f(2j+1), \tag{4.1}$$

where $\omega_N = e^{-\frac{2\pi i}{N}}$ and $k = 0, 1, \cdots, N-1$.

The transformation is represented by the matrix

$$\mathbf{U_N} = \frac{2}{3}\begin{bmatrix} \mathbf{A_N} & \mathbf{2D_N A_N} \\ \mathbf{A_N} & -\mathbf{2D_N A_N} \end{bmatrix} \mathbf{P_N}, \tag{4.2}$$

where the $\frac{N}{2} \times \frac{N}{2}$ matrix $\mathbf{A_N}$ is defined by

$$\mathbf{A_N} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{(N-2)} \\ 1 & \omega_N^4 & \omega_N^8 & \cdots & \omega_N^{2(N-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{(N-2)} & \omega_N^{2(N-2)} & \cdots & \omega_N^{(\frac{N}{2}-1)(N-2)} \end{bmatrix}, \tag{4.3}$$

the matrix $\mathbf{D_N} = \mathrm{diag}\left(1, \omega_N, \cdots, \omega_N^{\frac{N}{2}-1}\right)$, the $N \times N$ permutation matrix $\mathbf{P_N}$ is defined by its components $P_{N_{j+1,2j+1}} = 1$ and $P_{N_{\frac{N}{2}+j+1,2j}} = 1$ for $j = 0, 1, \cdots, \frac{N}{2}-1$ with the inversion matrix defined by

$$\mathbf{U_N^{-1}} = \frac{3}{2N}\mathbf{P}^T \begin{bmatrix} \mathbf{A_N^*} & \mathbf{A_N^*} \\ \frac{1}{2}\mathbf{A_N^* D_N^*} & -\frac{1}{2}\mathbf{A_N^* D_N^*} \end{bmatrix}. \tag{4.4}$$

The two dimensional Fourier series of a function $f(x, y)$ periodic in the variables x and y of period 2π is given by [6]

$$f(x, y) = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} C(k, l) e^{i(kx+ly)} \tag{4.5}$$

where the Fourier coefficients

$$C(k, l) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x, y) e^{-i(kx+ly)} dxdy \tag{4.6}$$

Approximating the double integral in (4.6) above by Simpson's quadrature in the

x-direction with the aid of a grid $x_n = \frac{2\pi n}{N}$, $n = 0, 1, \cdots, N$ gives

$$\begin{aligned}
&C(k,l)\\
&= \frac{1}{4\pi^2}\int_0^{2\pi}\left[\int_0^{2\pi} f(x,y)e^{-ikx}dx\right]e^{-ily}dy\\
&\approx \frac{1}{4\pi^2}\int_0^{2\pi}\frac{h_x}{3}\left[f(x_0,y)e^{-ikx_0} + f(x_N,y)e^{-ikx_N} + 2\sum_{n=1}^{\frac{N}{2}-1} f(x_{2n},y)e^{-ikx_{2n}}\right.\\
&\quad\left. + 4\sum_{n=0}^{\frac{N}{2}-1} f(x_{2n+1},y)e^{-ikx_{2n+1}}\right]e^{-ily}dy\\
&= \frac{h_x}{12\pi^2}\left[\int_0^{2\pi} f(x_0,y)e^{-ikx_0}e^{-ily}dy + \int_0^{2\pi} f(x_N,y)e^{-ikx_N}e^{-ily}dy\right.\\
&\quad\left. + 2\sum_{n=1}^{\frac{N}{2}-1}\int_0^{2\pi} f(x_{2n},y)e^{-ikx_{2n}}e^{-ily}dy + 4\sum_{n=0}^{\frac{N}{2}-1}\int_0^{2\pi} f(x_{2n+1},y)e^{-ikx_{2n+1}}e^{-ily}dy\right]
\end{aligned} \tag{4.7}$$

where $h_x = \frac{2\pi}{N}$.

Combining the first three terms in (4.7) by using the fact that $f(x_0,y)e^{-ikx_0} = f(x_N,y)e^{-ikx_N}$ gives

$$\begin{aligned}
C(k,l) \quad &\approx \frac{h_x}{12\pi^2}\left[2\sum_{n=0}^{\frac{N}{2}-1}\int_0^{2\pi} f(x_{2n},y)e^{-ikx_{2n}}e^{-ily}dy\right.\\
&\quad\left. + 4\sum_{n=0}^{\frac{N}{2}-1}\int_0^{2\pi} f(x_{2n+1},y)e^{-ikx_{2n+1}}e^{-ily}dy\right]\\
&= \frac{h_x}{6\pi^2}\left[\sum_{n=0}^{\frac{N}{2}-1}\int_0^{2\pi} f(x_{2n},y)e^{-ikx_{2n}}e^{-ily}dy\right. && (4.8)\\
&\quad\left. + 2\sum_{n=0}^{\frac{N}{2}-1}\int_0^{2\pi} f(x_{2n+1},y)e^{-ikx_{2n+1}}e^{-ily}dy\right] && (4.9)
\end{aligned}$$

Consider (4.8) and define $G(x_{2n},y) = f(x_{2n},y)e^{-ikx_{2n}}e^{-ily}$. Applying the Simpson's quadrature rule in the y-direction with the aid of a grid $y_m = \frac{2\pi m}{M}$, $m =$

$0, 1, \cdots, M$ and with $h_y = \frac{2\pi}{M}$ yields

$$\begin{aligned}
\int_0^{2\pi} G(x_{2n}, y)dy &= \frac{h_y}{3}\left[G(x_{2n}, y_0) + G(x_{2n}, y_M) + 2\sum_{m=1}^{\frac{M}{2}-1} G(x_{2n}, y_{2m})\right.\\
&\quad \left. + 4\sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m+1})\right]\\
&= \frac{h_y}{3}\left[2\sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m}) + 4\sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m+1})\right]\\
&= \frac{2h_y}{3}\sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m}) + \frac{4h_y}{3}\sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m+1}) \qquad (4.10)
\end{aligned}$$

substituting (4.10) in (4.8) gives

$$\begin{aligned}
&\frac{h_x}{6\pi^2}\sum_{n=0}^{\frac{N}{2}-1}\int_0^{2\pi} G(x_{2n}, y)dy\\
&= \frac{h_x}{6\pi^2}\sum_{n=0}^{\frac{N}{2}-1}\left[\frac{2h_y}{3}\sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m}) + \frac{4h_y}{3}\sum_{m=0}^{\frac{M}{2}-1} G(x_{2n}, y_{2m+1})\right]\\
&= \frac{h_x h_y}{9\pi^2}\sum_{n=0}^{\frac{N}{2}-1}\sum_{m=0}^{\frac{M}{2}-1} f(x_{2n}, y_{2m})e^{-ikx_{2n}}e^{-ily_{2m}}\\
&\quad + \frac{2h_x h_y}{9\pi^2}\sum_{n=0}^{\frac{N}{2}-1}\sum_{m=0}^{\frac{M}{2}-1} f(x_{2n}y_{2m+1})e^{-ikx_{2n}}e^{-ily_{2m+1}}. \qquad (4.11)
\end{aligned}$$

The first term of equation (4.11) simplifies to

$$\begin{aligned}
&\frac{h_x h_y}{9\pi^2}\sum_{n=0}^{\frac{N}{2}-1}\sum_{m=0}^{\frac{M}{2}-1} f(x_{2n}, y_{2m})e^{-ikx_{2n}}e^{-ily_{2m}}\\
&= \frac{4}{9NM}\sum_{n=0}^{\frac{N}{2}-1}\sum_{m=0}^{\frac{M}{2}-1} f(2n, 2m)e^{-ik\frac{2\pi}{N}(2n)}e^{-il\frac{2\pi}{M}(2m)}\\
&= \frac{4}{9NM}\sum_{n=0}^{\frac{N}{2}-1}\sum_{m=0}^{\frac{M}{2}-1} f(2n, 2m)\omega_N^{2nk}\omega_M^{2ml} \qquad (4.12)
\end{aligned}$$

where $\omega_M = e^{-\frac{2\pi i}{M}}$ and we have adopted the notation $f(x_n, y_m) = f(n, m)$.

Define

$$S_1(k,l) = \frac{4}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n,2m)\omega_N^{2nk}\omega_M^{2ml}. \tag{4.13}$$

Consider the second term in (4.11). This simplifies to

$$\begin{aligned}
&\frac{2h_x h_y}{9\pi^2}\sum_{n=0}^{\frac{N}{2}-1}\sum_{m=0}^{\frac{M}{2}-1} f(x_{2n},y_{2m+1})e^{-ikx_{2n}}e^{-ily_{2m+1}} \\
&= \frac{8}{9NM}\sum_{n=0}^{\frac{N}{2}-1}\sum_{m=0}^{\frac{M}{2}-1} f(2n,2m+1)\omega_N^{2nk}\omega_M^{(2m+1)l} \\
&= \frac{8}{9NM}\sum_{n=0}^{\frac{N}{2}-1}\sum_{m=0}^{\frac{M}{2}-1} f(2n,2m+1)\omega_N^{2nk}\omega_M^{(2m+1)l} \qquad (4.14)
\end{aligned}$$

and define

$$S_3(k,l) = \frac{8}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n,2m+1)\omega_N^{2nk}\omega_M^{(2m+1)l}. \tag{4.15}$$

A similar approach is applied to (4.9) to obtain

$$S_2(k,l) = \frac{8}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m)\omega_N^{(2n+1)k}\omega_M^{2ml} \tag{4.16}$$

$$S_4(k,l) = \frac{16}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m+1)\omega_N^{(2n+1)k}\omega_M^{(2m+1)l} \tag{4.17}$$

where $k = 0,1,\cdots,N-1;\ l = 0,1,\cdots,M-1$.

Define the Simpson Discrete Fourier Transformation in 2-D by

$$F(k,l) = S_1(k,l) + S_2(k,l) + S_3(k,l) + S_4(k,l), \tag{4.18}$$

then the approximation for the Fourier coefficients are given by

$$C(k,l) \approx \frac{F(k,l)}{NM}.$$

Summing (4.13) and (4.16) we get

$$
\begin{aligned}
&S_1(k,l) + S_2(k,l) \\
&= \frac{2}{3}\sum_{m=0}^{\frac{M}{2}-1}\left[\frac{2}{3}\sum_{n=0}^{\frac{N}{2}-1} f(2n,2m)\omega_N^{2nk} + \frac{4}{3}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m)\omega_N^{(2n+1)k}\right]\omega_M^{2ml} \\
&= \frac{2}{3}\sum_{m=0}^{\frac{M}{2}-1} \hat{F}(k,2m)\omega_M^{2ml} \qquad (4.19)
\end{aligned}
$$

where

$$
\hat{F}(k,2m) = \frac{2}{3}\sum_{n=0}^{\frac{N}{2}-1} f(2n,2m)\omega_N^{2nk} + \frac{4}{3}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m)\omega_N^{(2n+1)k} \qquad (4.20)
$$

Similarly

$$
\begin{aligned}
&S_3(k,l) + S_4(k,l) \\
&= \frac{4}{3}\sum_{m=0}^{\frac{M}{2}-1}\left[\frac{2}{3}\sum_{n=0}^{\frac{N}{2}-1} f(2n,2m+1)\omega_N^{2nk} + \frac{4}{3}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m+1)\omega_N^{(2n+1)k}\right]\omega_M^{(2m+1)l} \\
&= \frac{4}{3}\sum_{m=0}^{\frac{M}{2}-1} \hat{F}(k,2m+1)\omega_M^{(2m+1)l} \qquad (4.21)
\end{aligned}
$$

where

$$
\hat{F}(k,2m+1) = \frac{2}{3}\sum_{n=0}^{\frac{N}{2}-1} f(2n,2m+1)\omega_N^{2nk} + \frac{4}{3}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m+1)\omega_N^{(2n+1)k}, \qquad (4.22)
$$

then equation (4.18) may be re-written as

$$
F(k,l) = \frac{2}{3}\sum_{m=0}^{\frac{M}{2}-1} \hat{F}(k,2m)\omega_M^{2ml} + \frac{4}{3}\sum_{m=0}^{\frac{M}{2}-1} \hat{F}(k,2m+1)\omega_M^{(2m+1)l}. \qquad (4.23)
$$

Equations (4.20) and (4.22) are equivalent to the 1-D SDFT on the columns of the $N \times M$ matrix $\mathbf{f}$, which consists of elements $f(n,m)$ and may be represented by $\hat{\mathbf{F}} = \mathbf{U_N f}$ whilst equation (4.23) represents the 1-D SDFT on the rows of $\hat{\mathbf{F}}$. We need to transform rows of $\hat{\mathbf{F}}$. To do this, let us transform columns of $\hat{\mathbf{F}}^T$ to obtain

$$\mathbf{U_M}\hat{\mathbf{F}}^T = \mathbf{U_M}\mathbf{f}^T\mathbf{U_N}^T.$$

Transposing to the original form, we get

$$\begin{aligned} \mathbf{F} &= \hat{\mathbf{F}}\mathbf{U_M}^T \\ &= \mathbf{U_N}\mathbf{f}\mathbf{U_M}^T \end{aligned} \tag{4.24}$$

which is the matrix form representing the 2-D SDFT of $\mathbf{f}$.

From (4.18),

$$\begin{aligned} F(k,l) &= \frac{4}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n,2m)\omega_N^{2nk}\omega_M^{2ml} + \frac{8}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m)\omega_N^{(2n+1)k}\omega_M^{2ml} \\ &+ \frac{8}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n,2m+1)\omega_N^{2nk}\omega_M^{(2m+1)l} \\ &+ \frac{16}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m+1)\omega_N^{(2n+1)k}\omega_M^{(2m+1)l} \end{aligned} \tag{4.25}$$

for $k = 0, 1, \cdots, N-1$; $l = 0, 1, \cdots, M-1$.

Equation (4.25) essentially represents Simpson's quadrature over a 2-D mesh for a periodic function. For an image $\mathbf{f}$ of size 6×10, the mesh (x_n, y_m) is represented in figure 4.1 for $n = 0, 1, \cdots, N-1$; $m = 0, 1, \cdots, M-1$.

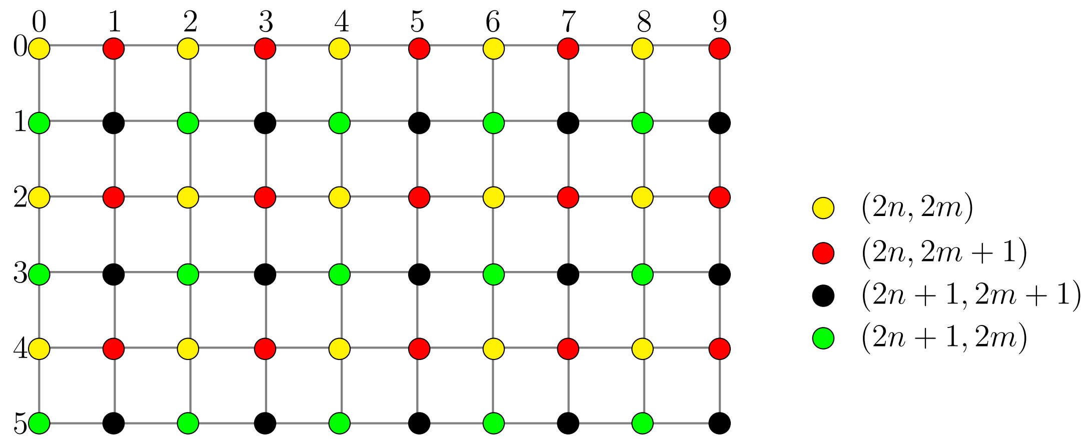

Figure 4.1:

In order to recover $f(2p, 2q)$, we multiply both sides of (4.25) by $\omega_N^{-2pk}\omega_M^{-2ql}$ and summing over k and l, it can be shown from lemma 1.1 that the last three terms are zero, leaving us with

$$\begin{aligned}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1} F(k,l)\omega_N^{-2pk}\omega_M^{-2ql} &= \frac{4}{9}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}\left[\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n,2m)\omega_N^{(2n-2p)k}\omega_M^{(2m-2q)l}\right] \\ &= \frac{4NM}{9} f(2p,2q) \qquad (4.26)\end{aligned}$$

Thus the inversion formula is

$$f(2p,2q) = \frac{9}{4NM}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1} F(k,l)\omega_N^{-2pk}\omega_M^{-2ql}. \qquad (4.27)$$

Similarly, it can be shown that

$$f(2p+1,2q) = \frac{9}{8NM}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1} F(k,l)\omega_N^{-(2p+1)k}\omega_M^{-2ql} \qquad (4.28)$$

$$f(2p,2q+1) = \frac{9}{8NM}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1} F(k,l)\omega_N^{-2pk}\omega_M^{-(2q+1)l} \qquad (4.29)$$

$$f(2p+1,2q+1) = \frac{9}{16NM}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1} F(k,l)\omega_N^{-(2p+1)k}\omega_M^{-(2q+1)l} \qquad (4.30)$$

Let $\mathbf{S_1}$ be the $\frac{N}{2} \times \frac{M}{2}$ matrix consisting of elements $S_1(k,l), k = 0, 1, \cdots, \frac{N}{2} - 1;$ $l = 0, 1, \cdots, \frac{M}{2} - 1.$

Theorem 4.1. $\mathbf{S_1}$ *has periodicity of* $\frac{N}{2}$ *in the first variable and* $\frac{M}{2}$ *in the second variable.*

Proof. From equation (4.13)

$$\begin{aligned}
S_1(k,l) &= \frac{4}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n,2m)\omega_N^{2nk}\omega_M^{2ml} \\
S_1\left(k+\tfrac{N}{2}, l+\tfrac{N}{2}\right) &= \frac{4}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n,2m)\omega_N^{2n\left(k+\frac{N}{2}\right)}\omega_M^{2m\left(l+\frac{M}{2}\right)} \\
&= \frac{4}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n,2m)\omega_N^{2nk}\omega_M^{2ml} \\
&= S_1(k,l)
\end{aligned}$$

□

Let $\mathbf{S_4}$ be the $\frac{N}{2}\times\frac{M}{2}$ matrix consisting of elements $S_4(k,l)$, $k=0,1,\cdots,\frac{N}{2}-1$; $l=0,1,\cdots,\frac{M}{2}-1$ then similarly it can be shown that $S_4\left(k+\frac{N}{2},l\right)=-S_4(k,l)$ and $S_4\left(k,l+\frac{M}{2}\right)=-S_4(k,l)$.

Let $\mathbf{S_2}$ be the $\frac{N}{2}\times\frac{M}{2}$ matrix consisting of elements $S_2(k,l)$, $k=0,1,\cdots,\frac{N}{2}-1$; $l=0,1,\cdots,\frac{M}{2}-1$.

Theorem 4.2. $\mathbf{S_2}$ *is anti-periodic in the first variable and periodic in the second variable.*

Proof. From equation (4.16)

$$S_2(k,l) = \frac{8}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m)\omega_N^{(2n+1)k}\omega_M^{2ml}$$

$$\begin{aligned}
S_2\left(k+\tfrac{N}{2},l\right) &= \frac{8}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m)\omega_N^{(2n+1)\left(k+\frac{N}{2}\right)}\omega_M^{2ml} \\
&= \frac{8}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m)\omega_N^{(2n+1)k}\omega_N^{(2n+1)\frac{N}{2}}\omega_M^{2ml}
\end{aligned}$$

where $\omega_N^{(2n+1)\frac{N}{2}} = (-1)^{2n+1}$ thus,

$$\begin{aligned} S_2\left(k+\tfrac{N}{2},l\right) &= -\frac{8}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m)\omega_N^{(2n+1)k}\omega_M^{2ml}. \\ &= -S_2(k,l) \end{aligned} \tag{4.31}$$

Also

$$\begin{aligned} S_2\left(k,l+\tfrac{M}{2}\right) &= \frac{8}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m)\omega_N^{(2n+1)k}\omega_M^{2m(l+\frac{M}{2})}. \\ &= \frac{8}{9}\sum_{m=0}^{\frac{M}{2}-1}\sum_{n=0}^{\frac{N}{2}-1} f(2n+1,2m)\omega_N^{(2n+1)k}\omega_M^{2ml} \\ &= S_2(k,l) \end{aligned} \tag{4.32}$$

□

Let $\mathbf{S_3}$ be the $\frac{N}{2}\times\frac{M}{2}$ matrix consisting of elements $S_3(k,l)$ for $k=0,1,\cdots,\frac{N}{2}-1$; $l=0,1,\cdots,\frac{M}{2}-1$ then similarly it can be shown that $S_3\left(k+\frac{N}{2},l\right)=S_3(k,l)$ and $S_3\left(k,l+\frac{M}{2}\right)=-S_3(k,l)$. Hence $S_1(k,l)$ is periodic in the first and second variables, $S_2(k,l)$ is antiperiodic in the first variable and periodic in the second variable, $S_3(k,l)$ is periodic in the first variable and antiperiodic in the second variable, whilst $S_4(k,l)$ is antiperiodic in the first and second variables. Then (4.25) can be written in the block matrix form

$$\mathbf{F} = \begin{bmatrix} \mathbf{S_1}+\mathbf{S_2}+\mathbf{S_3}+\mathbf{S_4} & \mathbf{S_1}+\mathbf{S_2}-\mathbf{S_3}-\mathbf{S_4} \\ \mathbf{S_1}-\mathbf{S_2}+\mathbf{S_3}-\mathbf{S_4} & \mathbf{S_1}-\mathbf{S_2}-\mathbf{S_3}+\mathbf{S_4} \end{bmatrix}. \tag{4.33}$$

For real $\mathbf{f}$ it is seen from (4.25) that the component $F(0,0)$ is real and is the weighted sum of the elements of matrix $\mathbf{f}$. This is referred to as the dc component. Similarly, the frequency components $F\left(0,\frac{M}{2}\right)$, $F\left(\frac{N}{2},0\right)$ and $F\left(\frac{N}{2},\frac{M}{2}\right)$ are real. From (4.18) we obtain

$$F(0,0) = S_1(0,0)+S_2(0,0)+S_3(0,0)+S_4(0,0) \tag{4.34}$$

$$\begin{aligned} F\left(0,\tfrac{M}{2}\right) &= S_1\left(0,\tfrac{M}{2}\right)+S_2\left(0,\tfrac{M}{2}\right)+S_3\left(0,\tfrac{M}{2}\right)+S_4\left(0,\tfrac{M}{2}\right) \\ &= S_1(0,0)+S_2(0,0)-S_3(0,0)-S_4(0,0) \end{aligned} \tag{4.35}$$

$$F\left(\tfrac{N}{2},0\right) = S_1(0,0)-S_2(0,0)+S_3(0,0)-S_4(0,0) \tag{4.36}$$

$$F\left(\tfrac{N}{2},\tfrac{M}{2}\right) = S_1(0,0)-S_2(0,0)-S_3(0,0)+S_4(0,0). \tag{4.37}$$

Adding equations (4.34) - (4.37) gives

$$S_1(0,0) = \frac{1}{4}\left[F(0,0)+F\left(0,\tfrac{M}{2}\right)+F\left(\tfrac{N}{2},0\right)+F\left(\tfrac{N}{2},\tfrac{M}{2}\right)\right] \tag{4.38}$$

Adding (4.34) and (4.35) and then subtracting (4.36) and (4.37) gives

$$S_2(0,0) = \frac{1}{4}\left[F(0,0)+F\left(0,\tfrac{M}{2}\right)-F\left(\tfrac{N}{2},0\right)-F\left(\tfrac{N}{2},\tfrac{M}{2}\right)\right] \tag{4.39}$$

Adding (4.34) and (4.36) and then subtracting (4.35) and (4.37) gives

$$S_3(0,0) = \frac{1}{4}\left[F(0,0)-F\left(0,\tfrac{M}{2}\right)+F\left(\tfrac{N}{2},0\right)-F\left(\tfrac{N}{2},\tfrac{M}{2}\right)\right] \tag{4.40}$$

Adding (4.34) and (4.37) and then subtracting (4.35) and (4.36) gives

$$S_4(0,0) = \frac{1}{4}\left[F(0,0)-F\left(0,\tfrac{M}{2}\right)-F\left(\tfrac{N}{2},0\right)+F\left(\tfrac{N}{2},\tfrac{M}{2}\right)\right] \tag{4.41}$$

These are some of the relationships that aid in understanding the frequency spectrum.

Duality

We will illustrate the effect of the duality property on an image. Using MATLAB, a three-leaved rose was generated. One leaf was generated from $3\cos 3\theta$ for $\frac{7\pi}{6} \leq \theta \leq \frac{3\pi}{2}$ and is shown in blue. Another leaf was generated from $2\cos 3\theta$ for $\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$ and is shown in green. The last leaf is shown in red and is generated by $\cos 3\theta$ for $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$. This picture is shown in figure 4.2.

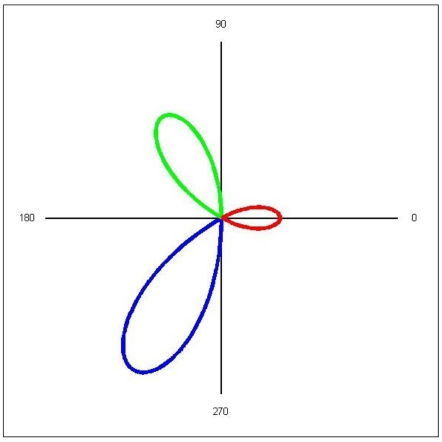

Figure 4.2: Original rose

The duality property for the 2-D DFT is applied in the following manner, namely

$$\boldsymbol{\mathcal{F}}_{\boldsymbol{N}} \left(\boldsymbol{\mathcal{F}}_{\boldsymbol{N}} \mathbf{f} \boldsymbol{\mathcal{F}}_{\boldsymbol{M}}^{T}\right) \boldsymbol{\mathcal{F}}_{\boldsymbol{M}}^{T} \tag{4.42}$$

This is done on each layer separately, namely, the R (red), G (green) and B (blue) layers. The effect of the duality is to reflect the petals in the x and y axis which is equivalent to a rotation of 180 $^\circ$ about the origin. This is illustrated in figure 4.3.

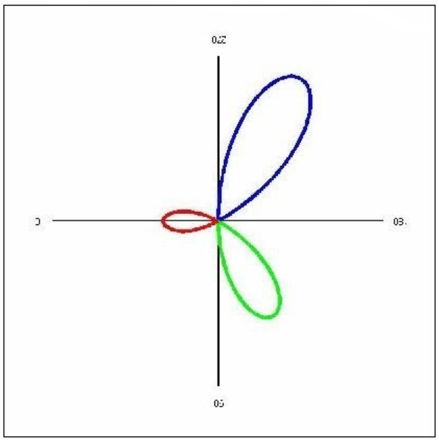

Figure 4.3: DFT duality

The duality property for the 2-D SDFT is applied in the following manner, namely

$$\mathbf{U_N}\left(\mathbf{U_N}\mathbf{f}\mathbf{U_M^T}\right)\mathbf{U_M^T} \tag{4.43}$$

This is done on each layer separately, namely, the R, G and B layers. The effect of the duality in this case is to rotate the image by $180\,^\circ$ about the origin as well as leave residual images translated by $\frac{N}{2}$ in the y direction and $\frac{M}{2}$ in the x direction. The background is as a result of the SDFT treating the pixel elements $(2n, 2m)$, $(2n, 2m+1)$, $(2n+1, 2m)$ and $(2n+1, 2m+1)$ in the ratio $1:2:2:4$ according to (4.25). This is illustrated in figure 4.4

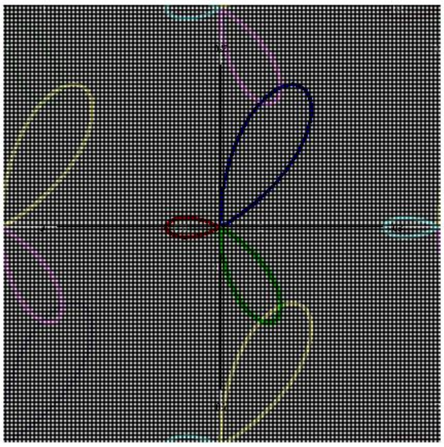

Figure 4.4: SDFT duality

4.2 Applications

Encryption in the Frequency Domain

Encryption is a process of encoding information in a way that an adversary cannot read it. In an encryption scheme, the information referred to as a plain image, is encrypted using an encryption algorithm turning it into an unreadable cipher image. Image encryption is classified into two catergories: spatial domain methods and frequency domain methods. The spatial domain method is based on direct manipulation of pixels in an image that usually destroys the correlation among pixels and makes the encrypted image incompressible [1].

Frequency domain processing techniques are based on modification of the Fourier transform of an image. Encryption in the frequency domain has the advantage that the plain image can be recovered without loss of information.

Consider the plain image in figure 4.5 and the key image in figure 4.6.

Figure 4.5: Plain image

Figure 4.6: Key image

The plain image is transformed using the 2-D SDFT and a fraction of it is added to the 2-D SDFT of the key image resulting in the transform of the cipher image. This transform is inverted to give the cipher image in figure 4.7.

Figure 4.7:

By adjusting the fraction, the plain image can be completely embedded in the key image. This is illustrated in figure 4.8 and figure 4.9, which shows partial embedding.

Figure 4.8:

Figure 4.9:

Watermarking in the Frequency Domain

Watermarking is the method of embedding information like personal data or a logo into digital contents like images , audio or video. It can be detected to verify the authenticity of the digital media. The watermarking is essentially used in in copyright protection and ownership proving. The watermark can be embedded into the image in the spatial domain or frequency domain. In spatial domain techniques, the watermark is embedded directly into the pixel data [21]. In frequency domain techniques, the image data is first converted to frequency domain using Fourier transforms such as the DFT. The watermark is embedded in a particular manner to the frequency domain coefficients and then the inverse transform is performed to restore the watermarked image.

Consider the image $\mathbf{f}$ in figure 4.10. Its DFT amplitude spectrum $\left(\left|\boldsymbol{\mathcal{F}_N}\mathbf{f}\boldsymbol{\mathcal{F}}_{\boldsymbol{M}}^T\right|\right.$ where $|\cdot|$ denotes the absolute value of the elements) is shown in figure 4.11. In contrast, the SDFT amplitude spectrum $|\mathbf{F}|$ (see equation 4.24) is plotted in figure 4.12. The top left hand corners of both figures denote the dc component which is always real and hence bright which in the SDFT case corresponds to $F(0,0)$ from (4.25). The remaining three brighter corners in figure 4.11 and figure 4.12 is due to the periodicity of N and M in both transforms. The additional bright spots in figure 4.12 corresponds to the region near $F\left(0,\frac{M}{2}\right)$, $F\left(\frac{N}{2},M\right)$, $F\left(N,\frac{M}{2}\right)$ and $F\left(\frac{N}{2},0\right)$. The bright region at the center in figure 4.12 is located around $F\left(\frac{N}{2},\frac{M}{2}\right)$. These extra five bright regions in the SDFT spectrum arise because of the periodicity and anti-periodicity in $\frac{N}{2}$ and $\frac{M}{2}$. These nine bright regions are characteristic of the SDFT amplitude spectrum.

We have extracted a 7×7 submatrix from the amplitude spectrum shown in figure 4.12 (scaled by 10^{-6}) and centred about $|F\left(\frac{N}{2},\frac{M}{2}\right)|$. This is shown in (4.44) where the bold face entry corresponds to $|F\left(\frac{N}{2},\frac{M}{2}\right)|$.

$$\begin{pmatrix} 0.1110 & 0.0739 & 0.0646 & 0.0602 & 0.1138 & 0.0614 & 0.0194 \\ 0.1035 & 0.0888 & 0.0328 & 0.0521 & 0.0876 & 0.0831 & 0.1109 \\ 0.2132 & 0.1238 & 0.2506 & 0.1333 & 0.1075 & 0.0631 & 0.1146 \\ 0.0734 & 0.2127 & 0.2669 & \mathbf{3.5896} & 0.2669 & 0.2127 & 0.0734 \\ 0.1146 & 0.0631 & 0.1075 & 0.1333 & 0.2506 & 0.1238 & 0.2132 \\ 0.1109 & 0.0831 & 0.0876 & 0.0521 & 0.0328 & 0.0888 & 0.1035 \\ 0.0194 & 0.0614 & 0.1138 & 0.0602 & 0.0646 & 0.0739 & 0.1110 \end{pmatrix} \tag{4.44}$$

This matrix is centro-symmetric. This is true for all submatrices extracted in a similar manner up to order $(N-1)\times(M-1)$. This property must be conserved in watermarking in order to obtain a real image in the spatial domain. This restricts the positioning as well as the types of watermarks that may be used.

Figure 4.10:

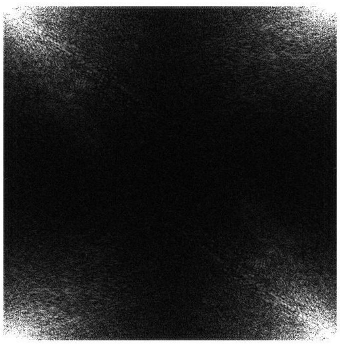

Figure 4.11:

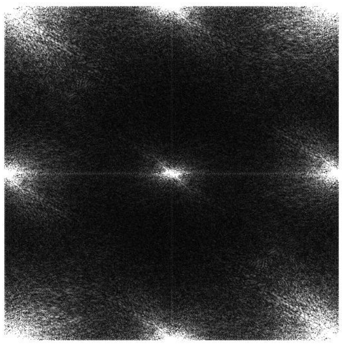

Figure 4.12:

We have inserted a square watermark in $\mathbf{F}$, which is visible in the amplitude spectrum $|\mathbf{F}|$ in figure 4.13 . After inverting $\mathbf{F}$ we obtain the watermarked (in the frequency domain) image in figure 4.14. This difference between the original image in figure 4.10 and the watermarked image in figure 4.14 is visibly not discernible to the naked eye.

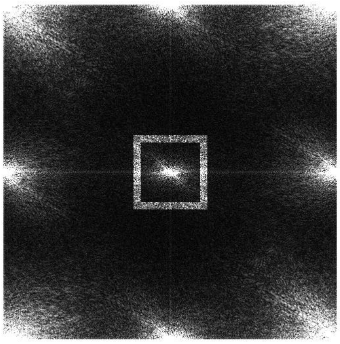

Figure 4.13:

Figure 4.14:

Phase Effect

The Fourier coefficients $F(k, l)$ represent the contribution due to different frequencies. In order to illustrate the effect of the phase, we normalize each Fourier coefficient by $|F(k, l)|$. This, in effect, gives each frequency the same contribution, thus removing the effect of the amplitudes. Upon inversion, we obtain a feint image which is basically an outline of the original image. This is illustrated in figure 4.15 below. This also illustrates the importance of the magnitude of the Fourier coefficients as their relative contributions have been removed.

Figure 4.15:

Conclusion

Our aim in this thesis has been to put forth a relatively new transform, examine its mathematical properties and show some areas of applications. This is by no means exhaustive as wherever the DFT is applied so can the SDFT be applied. We however, did not aim to pursue the efficiency of each of the transforms. It is left as a future task to fully investigate the eigenbasis corresponding to the SDFT transform. It is also possible to define a fractional SDFT by means of projection operators using spectral theory [2]. In addition, it would be interesting to define and examine new transforms based on different closed form Newton-Cotes quadrature formulae. The 0(mod)4 case would be interesting to investigate and compare to the 2(mod)4 case presented in this project.

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