FISCHER-CLIFFORD THEORY AND CHARACTER TABLES OF GROUP EXTENSIONS

by

ZWELETHEMBA EUGENE MPONO

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Abstract

The smallest Fischer sporadic simple group Fi_{22} is generated by a conjugacy class D of 3510 involutions called *3-transpositions* such that the product of any noncommuting pair is an element of order 3. In Fi_{22} there are exactly three conjugacy classes of involutions denoted by D, T and N and represented in the ATLAS [26] by $2A$, $2B$ and $2C$, containing 3510, 1216215 and 36486450 elements with corresponding centralizers $2{\cdot}U(6,2)$, $(2 \times 2_+^{1+8}{:}U(4,2)){:}2$ and $2^{5+8}{:}(S_3 \times 3^2{:}4)$ respectively. In Fi_{22}, we have $N_{Fi_{22}}(2^6) = 2^6{:}SP(6,2)$, where 2^6 is a $2B$-pure group, and thus the maximal subgroup $2^6{:}SP(6,2)$ of Fi_{22} is a 2-local subgroup.

The full automorphism group of Fi_{22} is denoted by $\overline{Fi}_{22}$. In $\overline{Fi}_{22}$, there are three involutory outer automorphisms of Fi_{22} which are denoted by e, f and θ and represented in the ATLAS [26] by $2D$, $2F$ and $2E$ respectively. We obtain that $\overline{Fi}_{22} = Fi_{22}{:}\langle e\rangle$ and it can be easily shown that $\overline{Fi}_{22} = Fi_{22}{:}\langle e\rangle = Fi_{22}{:}\langle f\rangle = Fi_{22}{:}\langle \theta\rangle$. As e, f and θ act on Fi_{22}, then we obtain the subgroups $C_{Fi_{22}}(e) \cong O^+(8,2){:}S_3$, $C_{Fi_{22}}(f) \cong SP(6,2) \times 2$ and $C_{Fi_{22}}(\theta) \cong 2^6{:}O^-(6,2)$ of Fi_{22} which are generated by $C_D(e)$, $C_D(f)$ and $C_D(\theta)$ respectively.

In this thesis we are concerned with the construction of the character tables of certain groups which are associated with Fi_{22} and its automorphism group $\overline{Fi}_{22}$. We use the technique of the Fischer-Clifford matrices to construct the character tables of these groups, which are split extensions. These groups are $2^6{:}SP(6,2)$, $2^6{:}O^-(6,2)$ and $2^7{:}SP(6,2)$. The study of the group $2^6{:}SP(6,2)$ is essential, as the other groups studied in this thesis are related to it. The groups $SP(6,2)$ and $O^-(6,2)$ of 6×6 matrices over $GF(2)$, played crucial roles in our construction of the group $SP(6,2)$ as a group of 7×7 matrices over $GF(2)$ which would act on 2^7. Also the character table of $2^5{:}S_6$, the affine subgroup of $SP(6,2)$ fixing a nonzero vector in 2^6, is constructed by using the technique of the Fischer-Clifford matrices. This character table is used

in the construction of the character table $2^6{:}SP(6,2)$.

The character tables computed in this thesis have been accepted for incorporation into GAP and will be available in the latest version of GAP.

Preface

The work described in this thesis was carried out under the supervision and direction of Professor Jamshid Moori, Department of Mathematics and Applied Mathematics, University of Natal, Pietermaritzburg, from February 1994 to December 1997.

The thesis represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

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I am grateful to Professor Doctor Herbert Pahlings and the GAP team at Aachen for verifying the consistency and accuracy of the character tables and for incorporating these character tables into the GAP.

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Notation and conventions

Throughout this thesis all groups will be assumed to be finite, unless otherwise stated. We will use the notation and terminology from the ATLAS [26] and [68].

$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
G, N, H, K	groups
1_G	the identity element of G
$H \leq G$	H is a subgroup of G
$H \cong G$	H is isomorphic to G
F	a field
F^*	$F - \{0\}$
$\langle x, y \rangle$	the subgroup generated by x and y
$N{\cdot}G$	an extention of N by G
$N{:}G$	a split extention of N by G
h^g	conjugation of h by g
nX	a general conjugacy class of G with representatives of order n
$g_1 \sim g_2$	g_1 is conjugate to g_2
$o(g)$	order of $g \in G$
$C_G(g)$	the centralizer of g in G

$[g]$	a conjugacy class of G with representative g
$N_G(H)$	the normalizer of the subgroup H in G
Hg	the right coset of H in G
X, Y, Ω	sets
$\|\Omega\|$	the cardinality of the set Ω
$1^\alpha 2^\beta 3^\gamma \ldots$	cycle structure of a permutation
$Irr(G)$	the set of irreducible characters of G
I_G	the identity character of G
$\chi(G\|H)$	the permutation character of G on H
χ_H	the restriction of the character χ of G to the subgroup H
ψ^G	the induction of the character ψ of subgroup H to G
na, nb, $\ldots$	an irreducible character of G of degree n
$\langle \chi_i, \chi_j \rangle$	the inner product of the characters χ_i and χ_j
$dim(V)$	the dimension of a vector space V
D_n	diheral group of order $2n$
S_n	the symmetric group on n symbols
$GF(q)$	the Galois field of q elements
$V(n,q)$	a vector space of dimension n over $GF(q)$
$SP(2n,q)$	symplectic group of dimension $2n$ over $GF(q)$
$O^+(2n,q)$	the full orthogonal group leaving the form f^+ on $V = V(2n,q)$ invariant
$O^-(2n,q)$	the full orthogonal group leaving the form f^- on $V = V(2n,q)$ invariant
$O^+(8,2)$	the full orthogonal group (simple) of dimension 8 over $GF(2)$, $\|O^+(8,2)\| = 2^{12} \times 3^5 \times 5^2 \times 7$
$O^-(6,2)$	the full orthogonal group of dimension 6 over $GF(2)$, $\|O^-(6,2)\| = 2^7 \times 3^4 \times 5$, ATLAS [26]: $U(4,2){:}2$
2^n	an elementary abelian group of order 2^n

Contents

Chapter 1

Introduction

Let D be a conjugacy class of involutions such that the product of any noncommuting pair of elements of D has order 3. Elements of D are called *3-transpositions*. A group which is generated by the conjugacy class D of 3-transpositions is called a *3-transposition* group and subgroups generated by elements of D are called *D-subgroups*. B. Fischer in [39] introduced and studied the 3-transposition groups. Fischer classified all finite 3-transposition groups with no non-trivial normal soluble subgroups. In the process of classifying the 3-transposition groups, Fischer discovered three new groups Fi_{22}, Fi_{23} and Fi_{24}. The conjugacy class D is unique in each of the three groups and these groups act as rank-3 permutation groups on D by conjugation. In each of these groups there is a maximal set L of pairwise commuting elements of D with lengths 22, 23 and 24 respectively. The elements of L are said to form a basic set of transpositions. The subgroup generated by the basic set of transpositions is an elementary abelian group. For more information on 3-transposition groups and D-subgroups, readers are referred to [3], [28], [34], [39], [40], [84], [87] and many other relevant sources.

In Fi_{22} there are exactly three conjugacy classes of involutions denoted by D, T and N and represented in the ATLAS [26] by $2A$, $2B$ and $2C$, containing 3510, 1216215 and 36486450 elements respectively. The centralizers of elements corresponding to these conjugacy classes are $2{\cdot}U(6,2)$, $(2 \times 2_{+}^{1+8}{:}U(4,2)){:}2$ and $2^{5+8}{:}(S_3 \times 3^2{:}4)$ respectively. The 3510 involutions in $D = 2A$ are the 3-transpositions of Fi_{22}. The 22 basic transpositions of D in Fi_{22} generate an elementary abelian group $\langle L\rangle$ of order 2^{10} whose normalizer in Fi_{22} is $2^{10}{:}M_{22}$. Under the action of $2^{10}{:}M_{22}$ on D, we have three orbits D_1, D_2 and D_3 such that

(i) $D_1 = L$ contains the 22 basic transpositions which generate 2^{10}.

(ii) D_2 contains $2^5 \times 77 = 2464$ transpositions each commuting with just one hexad of the basic transpositions.

(iii) D_3 contains $2^{10} = 1024$ transpositions which commute with none of the basic transpositions.

The conjugacy class D of 3510 involutions in Fi_{22} generates Fi_{22}. The group $N_{Fi_{22}}(\langle L\rangle) = 2^{10}{:}M_{22}$ is a maximal subgroup of Fi_{22} and its automorphism group is $2^{10}{:}\overline{M}_{22} = 2^{10}{:}M_{22}{:}2$ which is a maximal subgroup of $\overline{F}i_{22}$. The character tables of $2^{10}{:}M_{22}$ and $2^{10}{:}\overline{M}_{22}$ were constructed by Moori in [80] and [81]. For more information on Fi_{22}, see [3], [25], [33], [71], [66], [83], [85], [86], [88], [100], [118] and many other relevant sources.

Theorem 1.0.1 *The simple group Fi_{22} has exactly 14 conjugacy classes of maximal subgroups as follows:*

$2{\cdot}U(6,2)$	$O(7,3)$ *(two classes)*
$O^+(8,2){:}S_3$	$2^{10}{:}M_{22}$
$2^6{:}SP(6,2)$	$(2 \times 2_+^{1+8}{:}U(4,2)){:}2$
$S_3 \times U(4,3){:}2$	${}^2F_4(2)'$
$2^{5+8}{:}(S_3 \times A_6)$	$3_+^{1+6}{:}2^{3+4}{:}3^2{:}2$
S_{10} *(two classes)*	M_{12}

Proof. This is part(i) in the Main Theorem of [71]. □

From the work of [119], we obtain that Fi_{22} has an outer automorphism group of order 2. The full automorphism group of Fi_{22} is denoted by $\overline{F}i_{22}$. In $\overline{F}i_{22}$, there are three involutory outer automorphisms of Fi_{22} which are denoted by e, f and θ and represented in the ATLAS [26] by $2D$, $2F$ and $2E$ respectively. We obtain that $\overline{F}i_{22} = Fi_{22}{:}\langle e\rangle$ and it can be easily shown that

$$\overline{F}i_{22} = Fi_{22}{:}\langle e\rangle = Fi_{22}{:}\langle f\rangle = Fi_{22}{:}\langle \theta\rangle \quad .$$

As e, f and θ act on Fi_{22}, then we obtain the subgroups $C_{Fi_{22}}(e) \cong O^{+}(8,2){:}S_3$, $C_{Fi_{22}}(f) \cong SP(6,2) \times 2$ and $C_{Fi_{22}}(\theta) \cong 2^6{:}O^{-}(6,2)$ of Fi_{22} which are generated by $C_D(e)$, $C_D(f)$ and $C_D(\theta)$ respectively. The character table of $O^{+}(8,2){:}S_3$ was calculated by Moori in [82]. For more information on the automorphism groups of simple groups, readers are referred to [119].

Theorem 1.0.2 *$\overline{Fi}_{22}$ has exactly 13 conjugacy classes of maximal subgroups as follows:*

Fi_{22}	$2{\cdot}U(6,2){:}2$
$G_2(3){:}2$	$3^5{:}(2 \times U(4,2){:}2)$
$O^{+}(8,2){:}S_3 \times 2$	$2^{10}{:}M_{22}{:}2$
$2^7{:}SP(6,2)$	$(2 \times 2_{+}^{1+8}{:}U(4,2){:}2){:}2$
$S_3 \times U(4,3){:}2^2$	${}^2F_4(2)$
$2^{5+8}{:}(S_3 \times S_6)$	$3_{+}^{1+6}{:}2^{3+4}{:}(S_3 \times S_3)$
$M_{12}{:}2$	

Proof. This is part(ii) in the Main Theorem of [71]. □

Most of the maximal subgroups of the sporadic simple groups are of extension type. With the classification of finite simple groups being complete, more recent work in group theory involves the study of other aspects of finite groups. The structures and character tables of group extensions play important roles in these studies. Character tables of finite groups can be constructed using various techniques. However B. Fischer studied a technique which can be used to construct character tables of group extensions. This technique, which is known as the technique of the *Fischer-Clifford* matrices, derives its fundamentals from the Clifford Theory and provides very powerful information for constructing character tables. In this thesis, we use this technique to construct the character tables of cetain subgroups of Fi_{22} and its automorphism group $\overline{Fi}_{22}$ which are split extensions.

In Chapter 2, we discuss the general theory of group extensions. Since every group extension is a short exact sequence of groups and homomorphisms, in Section 2.1 we discuss the background theory of exact sequences, build up to short exact sequences, and discuss the general theory of group extensions. In Section 2.2 we discuss the theory of semidirect products and give a proof from [108] that every split extension $\overline{G}$ of N by G is equivalent to a semidirect product of N by G. We also study a result from [105] that every semidirect product $\overline{G}$ of N by G realizes a homomorphism $\theta : G \longrightarrow Aut(N)$. In Section 2.3, we discuss the conjugacy classes of the elements of group extensions. We also give some general results involving conjugacy classes in finite groups. We then go on to discuss the technique of *coset analysis* for computing conjugacy classes of group extensions $\overline{G}$ of N by G where N is an abelian normal subgroup of $\overline{G}$. This technique which works for both split and nonsplit extensions was first developed and used by Moori in [80] and [81] and has since been widely used for computing the conjugacy classes of group extensions in all cases where it is applicable. For example it has also been used by Salleh in [106], Whitley in [116]. We also developed two programmes in CAYLEY [17] which we call Programmes A and B. These programmes can be used to compute the conjugacy classes and the orders of the class representatives for split extensions $\overline{G} = N{:}G$ where N is an elementary abelian p-group, (for prime p) on which a linear group G acts. Programme A is based on the coset analysis technique. These programmes have been applied to the groups that have been studied in this thesis, for example the group $2^6{:}SP(6,2)$. For further information and reading on group extensions, we encourage readers to consult Hall [55] and Humphreys [57] and other relevant books on group theory.

In Chapter 3, we present some results on group characters which are used in the later chapters. We mostly concentrate on those results which would be useful for the technique of the Fischer-Clifford matrices that is fully discussed in Chapter 5. In this thesis, we construct character tables of certain groups associated with the smallest Fischer sporadic simple group Fi_{22} and its automorphism group $\overline{F}i_{22}$. We start by discussing the general theory of representations and characters, and go on to discuss the restricted, induced and permutation characters, which will be used in the later chapters for constructing the character tables of the groups that are studied in this thesis. The characters being studied are ordinary complex characters. We give a proof that the permutation character of any group G acting on the cosets of its subgroup H is the character induced from the identity character of H. We use the notation $\chi(G|H)$ to denote this permutation character and we use I_G to denote the identity character of any group G. So with this notation we have $\chi(G|H) = (I_H)^G$. We also give a proof from [60] of the Frobenius Reciprocity Theorem, which gives a relationship between restricted and induced characters and their constituents. For a finite group G and $H \leq G$, then the relationship between the characters of G and those of H is of fundamental importance. For further reading on representations and characters, readers are encouraged to consult [2], [4], [7], [12], [23], [24], [35], [61], [63], [64], [67], [72], [74], [90], [101], [109], [110], [114] and many other relevant sources.

In Chapter 4, we shall concentrate on symplectic groups. We discuss the general theory of symplectic groups and their affine subgroups. One particular affine subgroup $2^5{:}S_6$ of the symplectic group $SP(6,2)$ has been studied in this thesis and is discussed in Chapter 6. The symplectic groups are constructed by defining some bilinear form on the underlying vector space and then taking all the form-preserving automorphisms of the space. Two of the groups studied in this thesis are split extensions of elementary abelian 2-groups by the symplectic group $SP(6,2)$ and are maximal subgroups of the smallest Fischer sporadic simple group Fi_{22} and its automorphism group $\overline{F}i_{22}$ respectively. The other group studied in this thesis is a split extension of an elementary abelian 2-group by the orthogonal group $O^-(6,2)$, where $O^-(6,2)$ is a maximal subgroup of $SP(6,2)$ of index 28. For further reading and information on symplectic groups, readers are encouraged to consult [10], [19], [29], [32], [51], [58], [57], [59] and [115].

In Chapter 5, we shall discuss the theory behind the technique of the Fischer-Clifford matrices. We shall however begin by discussing the Clifford Theory and then go on to discuss the theory of the Fischer-Clifford matrices. Given a group extension $\overline{G} = N{\cdot}G$ such that every irreducible character of N can be extended to its inertia group then for each class representative $g \in G$, we are able to construct a matrix $M(g)$ called the Fischer-Clifford matrix. By using these matrices together with the fusion maps and character tables of some subgroups of G which are inertia factors of the inertia groups in $\overline{G}$, we are able to construct the complete character table of $\overline{G}$. The technique of the Fischer-Clifford matrices has also been discussed and used in [30], [31], [41], [42], [43], [75], [76], [98], [106] and [116]. In the subsequent chapters, we will use this technique and other group theoretic and character theoretic information that have been discussed in the previous chapters to construct the character tables of the groups which have been studied in this thesis. For the Fischer-Clifford matrices, we shall follow the work of Whitley [116] very closely. Sometimes additional information given in the introduction of Chapter 6, together with other methods such as the character restrictions, have to be used to compute the entries of $M(g)$.

In Chapter 6 we study the group $2^6{:}SP(6,2)$ which is a maximal subgroup of the smallest Fischer simple group Fi_{22} of index 694980. Let $\overline{G} = 2^6{:}SP(6,2)$ be the split extension of $N = 2^6$ by $G = SP(6,2)$, where N is the vector space of dimension 6 over $GF(2)$ on which G acts naturally. We construct the character table of $\overline{G}$ using the technique of the Fischer-Clifford matrices. This character table will be divided row-wise into blocks where each block corresponds to an inertia group $\overline{H}_i = N{:}H_i$, where the H_i's are the inertia factors. The character table of $\overline{G}$ can be constructed by finding the Fischer-Clifford matrix $M(g)$ for each class representative g of G and using the character tables of the inertia factors. We use the properties of the Fischer-Clifford matrices which are discussed in Section 5.2.2 of Chapter 5 to compute their entries. In some cases we need to use the following additional information to compute these entries:

(i) For χ a character of any group H and $h \in H$, we have $|\chi(h)| \leq \chi(1_H)$, where 1_H is the identity element of H.

(ii) For χ a character of any group H and h a p-singular element of H, where p is a prime, then we have $\chi(h) \equiv \chi(h^p) mod(p)$.

(iii) For any irreducible character χ of a group H and for $h_i \in C_i$ then $d_i = \frac{b_i\chi(h_i)}{\chi(1_H)}$ is an algebraic integer, where C_i is the i-th conjugacy class of H and $b_i = |C_i| = [H : C_H(h_i)]$. Obviously if $d_i \in \mathbb{Q}$, then $d_i \in \mathbf{Z}$.

We also study a group of the form $2^5{:}S_6$ which is maximal and affine in $SP(6,2)$ of index 63. We construct the character table of this affine subgroup using the technique of the Fischer-Clifford matrices. This character table is necessary since it will be used to construct the character table of $\overline{G}$. In the process we also construct the character table of $3^2{:}D_4$ which is maximal in S_6 of index 10. This character table is used in the construction of the character table of $2^5{:}S_6$. The Fischer-Clifford matrices and the character table of $2^6{:}SP(6,2)$ are given in Section 6.4. In Sections 6.5 and 6.6 we deal with the fusion of $2^6{:}SP(6,2)$ into Fi_{22} and the permutation character of Fi_{22} on $2^6{:}SP(6,2)$ respectively.

In Chapter 7, we study the group $C_{Fi_{22}}(\theta) \cong 2^6{:}O^-(6,2)$ which is a maximal subgroup of $2^6{:}SP(6,2)$ of index 28. We determine its Fischer-Clifford matrices and hence construct its character table. We use the properties of the Fischer-Clifford matrices which are discussed in Chapter 5 (Section 5.2.2) and in some cases we also use the additional information discussed in the introduction of Chapter 6, to compute the entries of the Fischer-Clifford matrices. Motivation for this problem came from Moori's papers [83] and [85]. Moori in [83] obtained the generators for the groups $C_{Fi_{22}}(e)$, $C_{Fi_{22}}(f)$ and $C_{Fi_{22}}(\theta)$, where

$$C_{Fi_{22}}(e) \cong O^+(8,2){:}S_3,\ C_{Fi_{22}}(f) \cong SP(6,2)\times 2 \quad \text{and} \quad C_{Fi_{22}}(\theta) \cong 2^6{:}O^-(6,2) \quad .$$

From [83] we obtain that the above groups are D-subgroups of Fi_{22} generated by $C_D(e)$, $C_D(f)$ and $C_D(\theta)$ respectively. The complete fusion map of $2^6{:}O^-(6,2)$ into $2^6{:}SP(6,2)$ will be fully determined. Our computations by using GAP [107], show that the group $2^6{:}O^-(6,2)$ does not sit inside any other maximal subgroup of Fi_{22}.

In Chapter 8, we study the group $2^7{:}SP(6,2)$ which is a maximal subgroup of $\overline{F}i_{22}$ of index 694980. The maximal subgroup $2^6{:}SP(6,2)$ of Fi_{22}, where 2^6 is a $2B$-pure group and that $N_{Fi_{22}}(2^6) = 2^6{:}SP(6,2)$, is a 2-local subgroup of Fi_{22}. We have $2^6{:}SP(6,2) \leq N_{Fi_{22}}(2^6{:}SP(6,2))$ and since Fi_{22} is simple, the maximality of $2^6{:}SP(6,2)$ in Fi_{22} implies that $N_{Fi_{22}}(2^6{:}SP(6,2)) = 2^6{:}SP(6,2)$. In $\overline{F}i_{22}$, we obtain that $2^6{:}SP(6,2) \leq N_{\overline{F}i_{22}}(2^6{:}SP(6,2))$, but $N_{\overline{F}i_{22}}(2^6{:}SP(6,2)) \neq \overline{F}i_{22}, Fi_{22}$. By Theorem C in [118] and the results of [71], we deduce that $N_{\overline{F}i_{22}}(2^6{:}SP(6,2)) = 2^7{:}SP(6,2)$ and hence $2^7{:}SP(6,2) = (2^6{:}SP(6,2)){:}\langle e\rangle$. In Chapter 6, the conjugacy classes and the Fischer-Clifford matrices of the group $2^6{:}SP(6,2)$ have been computed. In this chapter, the conjugacy classes and the Fischer-Clifford matrices of the group $2^7{:}SP(6,2)$ will be computed. We shall use the technique of the Fischer-Clifford matrices to construct the character table of $2^7{:}SP(6,2)$. We shall use the properties of the Fischer-Clifford matrices which are discussed in Chapter 5 (Section 5.2.2) and in some cases we shall also use additional information discussed in the introduction of Chapter 6, to compute their entries. For example the Fischer-Clifford matrix $M(2D)$ in $2^7{:}SP(6,2)$ had 70 possible candidates of which we had to eliminate 69. This elimination was achieved by using the additional information and methods. The fusion map of this group into $\overline{F}i_{22}$ will be fully determined. However the fusion map of $2^6{:}SP(6,2)$ into $2^7{:}SP(6,2)$ will be crucial in determining the fusion map of $2^7{:}SP(6,2)$ into $\overline{F}i_{22}$. This will help to determine those classes of elements of $2^7{:}SP(6,2)$ that fuse into Fi_{22}. Those conjugacy classes of elements of $2^7{:}SP(6,2)$ which contain classes of $2^6{:}SP(6,2)$ will fuse into Fi_{22} and the others will fuse into $\overline{F}i_{22} - Fi_{22}$. Using the permutation character of Fi_{22} on $2^6{:}SP(6,2)$, which was determined in Chapter 6, we will be able to identify those irreducible characters of $\overline{F}i_{22}$ that are involved in the permutation character of $\overline{F}i_{22}$ on $2^7{:}SP(6,2)$. Hence this permutation character will be completely determined.

All the computations were carried out with the aid of CAYLEY [17] and GAP [107] running on a SUN GX2 computer. For notation on the conjugacy classes of elements and permutation characters, we follow the notation used in the ATLAS [26] and the ATLAS of BRAUER CHARACTERS [68]. All our groups and sets are finite unless otherwise specified. Programmes A for $2^5{:}S_6$, $3^2{:}D_4$, $2^6{:}O^-(6,2)$ and $2^7{:}SP(6,2)$ that have been used to compute the conjugacy classes of of these groups will be given in the Appendix A, just before the Bibliography. The character tables computed in this thesis have been accepted for incorporation into GAP and will be available in the latest version of GAP. The consistency and accuracy of the character tables have been verified by the GAP team at Aachen.

Chapter 2

Group Extensions

Most of the maximal subgroups of the sporadic simple groups are of extension type. The groups studied in this thesis are all split extensions and hence in this chapter we discuss the general theory of the group extensions. Since every group extension is a short exact sequence of groups and homomorphisms, in Section 2.1 we discuss the background theory of exact sequences, build up to short exact sequences, and discuss the general theory of group extensions. In Section 2.2 we discuss the theory of semidirect products and give a proof from [108] that every split extension $\overline{G}$ of N by G is equivalent to a semidirect product of N by G. We also study a result from [105] that every semidirect product $\overline{G}$ of N by G realizes a homomorphism $\theta : G \longrightarrow Aut(N)$. In Section 2.3, we discuss the conjugacy classes of the elements of group extensions. We also give some general results involving conjugacy classes in finite groups. We then go on to discuss the technique of coset analysis for computing conjugacy classes of group extensions $\overline{G}$ of N by G where N is an abelian normal subgroup of $\overline{G}$. This technique which works for both split and nonsplit extensions was first developed and used by Moori in [80] and [81] and has since been widely used for computing the conjugacy classes of group extensions in all cases where it is applicable. For example it has also been used by Salleh in [106], Whitley in [116]. We also developed two CAYLEY Programmes A and B. These programmes can be used to compute the conjuagacy classes and the orders of the class representatives for split extensions $\overline{G} = N{:}G$ where N is an elementary abelian p-group, (for prime p) on which a linear group G acts. Programme A is based on the coset analysis technique.

These programmes have been applied to the groups that have been studied in this thesis, for example the group $2^6{:}SP(6,2)$. For further information and reading on group extensions, we encourage readers to consult Hall [55] and Humphreys [57] and other relevant books on group theory.

2.1 Exact Sequences and Group Extensions

Definition 2.1.1 *Let* $\{\ldots, A_{n-1}, A_n, A_{n+1}, \ldots\}$ *and* $\{\ldots, \alpha_{n-1}, \alpha_n, \alpha_{n+1} \ldots\}$ *be sets of groups and homomorphisms respectively. Then we call*

$$\cdots \xrightarrow{\alpha_{n-1}} A_{n-1} \xrightarrow{\alpha_n} A_n \xrightarrow{\alpha_{n+1}} A_{n+1} \rightarrow \cdots \qquad (*)$$

a sequence of groups and homomorphisms. We say that the sequence $(*)$ *is exact if* $ker(\alpha_n) = Im(\alpha_{n-1})$ *for each successive pair* (α_{n-1}, α_n).

Theorem 2.1.2 *Let* A *and* B *be groups,* α_1, α_2 *and* α_3 *be homomorphisms. Then*

(i) The homomorphism $A \xrightarrow{\alpha_2} B$ *is one-to-one iff the sequence* $\{1\} \xrightarrow{\alpha_1} A \xrightarrow{\alpha_2} B$ *is exact.*

(ii) The homomorphism $A \xrightarrow{\alpha_2} B$ *is onto iff the sequence* $A \xrightarrow{\alpha_2} B \xrightarrow{\alpha_3} \{1\}$ *is exact.*

(iii) The homomorphism $A \xrightarrow{\alpha_2} B$ *is an isomorphism iff the sequence* $\{1\} \xrightarrow{\alpha_1} A \xrightarrow{\alpha_2} B \xrightarrow{\alpha_3} \{1\}$ *is exact.*

Proof. (i) Suppose that the sequence $\{1\} \xrightarrow{\alpha_1} A \xrightarrow{\alpha_2} B$ is exact. Then $ker(\alpha_2) = Im(\alpha_1)$. However $Im(\alpha_1) = \{1\}$. Thus $ker(\alpha_2) = \{1\}$ and hence α_2 is one-to-one.
Conversely suppose that $A \xrightarrow{\alpha_2} B$ is one-to-one. Then $ker(\alpha_2) = \{1\}$. However from the sequence $\{1\} \xrightarrow{\alpha_1} A \xrightarrow{\alpha_2} B$ we have that $Im(\alpha_1) = \{1\} = ker(\alpha_2)$ and hence sequence is exact.
(ii) Suppose that $A \xrightarrow{\alpha_2} B \xrightarrow{\alpha_3} \{1\}$ is exact. Then $ker(\alpha_3) = Im(\alpha_2)$. However $ker(\alpha_3) = B$ and thus $Im(\alpha_2) = B$ and hence α_2 is onto.
Conversely suppose that $A \xrightarrow{\alpha_2} B$ is onto. Then we have that $Im(\alpha_2) = B$. However from $A \xrightarrow{\alpha_2} B \xrightarrow{\alpha_3} \{1\}$, we obtain that $ker(\alpha_3) = B = Im(\alpha_2)$. Hence the sequence is

exact.
(iii) Suppose that $\{1\} \xrightarrow{\alpha_1} A \xrightarrow{\alpha_2} B \xrightarrow{\alpha_3} \{1\}$ is exact. Then $ker(\alpha_2) = Im(\alpha_1) = \{1\}$. Thus α_2 is one-to-one. Also from the exactness of sequence we have that $ker(\alpha_3) = B = Im(\alpha_2)$. Hence α_2 is onto and hence an isomorphism.
Conversely suppose that α_2 is an isomorphism. Then we obtain that $ker(\alpha_2) = \{1\}$ and $Im(\alpha_2) = B$. Thus from the sequence $\{1\} \xrightarrow{\alpha_1} A \xrightarrow{\alpha_2} B \xrightarrow{\alpha_3} \{1\}$ we obtain that $ker(\alpha_2) = \{1\} = Im(\alpha_1)$ and $Im(\alpha_2) = B = Ker(\alpha_3)$ and hence the sequence is exact. $\square$

Definition 2.1.3 *A short exact sequence of groups and homomorhisms is an exact sequence of the form* $\{1\} \to N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \to \{1\}$.

Definition 2.1.4 *If* $\{1\} \to N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \to \{1\}$ *is a short exact sequence, then we say that* $\overline{G}$ *is an extension of* N *by* G.

Remark 2.1.5 If $\overline{G}$ is an extension of N by G given by the short exact sequence $\{1\} \to N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \to \{1\}$, then

$$\overline{G}/\delta(N) = \overline{G}/ker(\sigma) \cong G \quad \text{and} \quad \delta(N) \cong N \ .$$

Definition 2.1.6 *An extension* $\{1\} \to N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \to \{1\}$ *is said to be equivalent to the extension* $\{1\} \to N \xrightarrow{\delta_1} \overline{G}_1 \xrightarrow{\sigma_1} G \to \{1\}$ *if there exists a homomorphism* $\phi : \overline{G} \longrightarrow \overline{G}_1$ *such that the diagram*

$$\begin{tikzcd}
N \arrow[r, "\delta"] \arrow[dr, "\delta_1"'] & \overline{G} \arrow[r, "\sigma"] \arrow[d, "\phi"] & G \\
& \overline{G}_1 \arrow[ur, "\sigma_1"'] &
\end{tikzcd}$$

commutes.

Using the *five lemma* it can be shown that ϕ is an isomorphism between $\overline{G}$ and $\overline{G}_1$. It is also easy to prove that the equivalence of group extensions defined above is an equivalence relation.

Remark 2.1.7 If $\{1\} \to N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G$ is an exact sequence and θ is the homomorphism of $\overline{G}$ into $Aut(\delta(N))$ given by $\theta(g) = \theta_g \downarrow_{\delta(N)}$, where θ_g is the inner automorphism of G induced by g, then the map $\tau : \overline{G} \longrightarrow Aut(N)$ given by the rule $\tau(g) = (\delta \downarrow_N)^{-1}\theta_g\delta$ is a homomorphism. Moreover if $\alpha : N \longrightarrow Aut(N)$ is the homomorphism given by $\alpha(n) = \alpha_n$, where α_n is the inner automorphism of N induced by n, then

$$\begin{tikzcd}
N \arrow[rr, "\delta"] \arrow[dr, "\alpha"'] & & \overline{G} \arrow[dl, "\tau"] \\
& Aut(N) &
\end{tikzcd}$$

commutes.

Definition 2.1.8 *Let G and N be groups and $\alpha : N \longrightarrow Aut(N)$ as in Remark 2.1.7. A factor system of N by G is a pair (θ, β) of functions, where $\theta : G \longrightarrow Aut(N)$ and $\beta : G \times G \longrightarrow N$ such that if we let $\theta(i) = i'$, $\beta(i,j) = c_{i,j}$, then we obtain that*

$$i'j' = (ij)'\alpha(c_{i,j}) \quad \textit{and} \quad c_{i,jk}c_{j,k} = c_{ij,k}(c_{i,j}k') \quad .$$

Definition 2.1.9 *Suppose that (θ, β) is a factor system of N by G. We say that (θ, β) belongs to an extension $\{1\} \to N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \to \{1\}$ and a function $\lambda : G \longrightarrow \overline{G}$ such that $\sigma\lambda = I_G$ if and only if $\theta = \tau\lambda$, where τ is defined in Remark 2.1.7 and $(\lambda(i))(\lambda(j)) = (\lambda(ij))(\delta(c_{i,j}))\ \forall\ i,j \in G$. The factor system (θ, β) of N by G is equivalent to the factor system (θ^*, β^*) of N by G if there is a function $\gamma : G \longrightarrow N$ such that $\forall\ i,j \in G$,*

$$i^* = \alpha(\gamma(i))i' \quad \textit{and} \quad c^*_{i,j} = \gamma^{-1}(ij)c_{i,j}(j'(\gamma(i)))\gamma(j) \quad .$$

The function γ is an equivalence of (θ, β) with (θ^*, β^*).

It can be shown that for an extension $\{1\} \to N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \to \{1\}$ and the map $\lambda : G \longrightarrow \overline{G}$ such that $\sigma\lambda = I_G$ there is a unique factor system (θ, β) belonging to the extension and λ.

Theorem 2.1.10 *Let N and G be groups and (θ, β) be a factor system of N by G. Then there is a group $\overline{G}$ and a homomorphism $\lambda : G \longrightarrow \overline{G}$ such that $\{1\} \to N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \to \{1\}$ is an extension, $\sigma\lambda = I_G$ and the factor system (θ, β) belongs to the extension and λ.*

Proof. See Theorem 9.4.5 of [108]. $\square$

Theorem 2.1.11 *Let the factor system (θ, β) belong to the extension $\{1\} \to N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \to \{1\}$ and the map $\lambda : G \longrightarrow \overline{G}$ and let the factor system (θ^*, β^*) belong to the extension $\{1\} \to N \xrightarrow{\delta_1} \overline{G}_1 \xrightarrow{\sigma_1} G \to \{1\}$ and the map $\lambda_1 : G \longrightarrow \overline{G}_1$. Then the extensions are equivalent iff the corresponding factor systems are equivalent.*

Proof. See Theorem 9.4.6 of [108]. $\square$

2.2 Semidirect Products and Split Extensions

Definition 2.2.1 *Let $\overline{G}$ be a group and N, G be subgroups of $\overline{G}$ such that*

(i) N is normal in $\overline{G}$

(ii) $\overline{G} = NG$

(iii) $N \cap G = \{1\}$

then $\overline{G}$ is called a semidirect product of N by G. The subgroups N and G are said to be complementary.

Remark 2.2.2 If both subgroups N and G are normal in $\overline{G}$, then $\overline{G}$ is a direct product of N and G.

For $\overline{G}$ a semidirect product of N by G, then every element in $\overline{G}$ can be expressed uniquely in the form ng, where $n \in N$ and $g \in G$ and the multiplication of elements of $\overline{G}$ is given by

$$(n_1g_1)(n_2g_2) = n_1n_2^{g_1}g_1g_2 \quad ,$$

where $n^g = gng^{-1}$. Also there is a homomorphism $\theta : G \longrightarrow Aut(N)$ given by $\theta(g) = \theta_g$, where $g \in G$, $\theta_g : N \longrightarrow N$ is defined by $\theta_g(n) = gng^{-1}$ and θ_g is an automorphism of N. Hence G acts on N.

Definition 2.2.3 *Let $\overline{G}$, N and G be as defined above and $\theta : G \longrightarrow Aut(N)$. Then the semidirect product $\overline{G}$ of N by G is said to realize θ if $\theta_g(n) = n^g\ \forall\ n \in N, g \in G$.*

Remark 2.2.4 For $\overline{G}$ a semidirect product of N by G, then $\overline{G}$ is isomorphic to a semidirect product of N by G that realizes θ for some $\theta : G \longrightarrow Aut(N)$.

Theorem 2.2.5 *Let N and G be groups, $\theta \in Hom(G, Aut(N)), \overline{G} = N \times G$ as a set with multiplication defined by $(n_1, g_1)(n_2, g_2) = (n_1\theta_{g_1}(n_2), g_1g_2)$. Let δ, σ and λ be functions given by $\delta(n) = (n, 1_G), \sigma(n, g) = g$ and $\lambda(g) = (1_N, g)$. Then*

(i) $\{1\} \rightarrow N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \rightarrow \{1\}$ *is an extension of N by G*

(ii) δ *is an isomorphism of N with a subgroup N_1 of $\overline{G}$*

(iii) λ *is an isomorphism of G with a subgroup G_1 of $\overline{G}$*

(iv) $\overline{G}$ *is a semidirect product of N_1 by G_1 that realizes a homomorphism ψ satisfying $[\psi(\lambda(g))](\delta(n)) = \delta(\theta_g(n))$, for all $n \in N, g \in G$*

(v) $\sigma\lambda = I_G$.

Proof. See Theorem 9.2.1 of [108]. $\square$

Definition 2.2.6 *An extension $\{1\} \rightarrow N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \rightarrow \{1\}$ is called*

(i) abelian if $\overline{G}$ is abelian

(ii) central if $Im(\delta) = \delta(N) \subset Z(\overline{G})$

(iii) cyclic if G is cyclic

(iv) split if there is a monomorphism $\lambda : G \longrightarrow \overline{G}$ such that $\sigma\lambda = I_G$.

Remark 2.2.7 If an extension is abelian, central or cyclic, then so is every equivalent extension.

Theorem 2.2.8 *[108] If an extension splits, then so does any equivalent extension.*

Proof. Let $\{1\} \to N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \to \{1\}$ be a split extension such that it is equivalent to the extension $\{1\} \to N \xrightarrow{\delta_1} \overline{G}_1 \xrightarrow{\sigma_1} G \to \{1\}$. Let ϕ be the homomorphism that gives the equivalence. Then there is a monomorphism $\lambda : G \longrightarrow \overline{G}$ such that $\sigma\lambda = I_G$. Let $\lambda_1 = \phi\lambda$, then $\lambda_1 : G \longrightarrow \overline{G}_1$ is a monomorphism such that $\sigma_1\lambda_1 = \sigma_1\phi\lambda = \sigma\lambda = I_G$. □

We say that a factor system (θ, β) splits if a corresponding extension splits. Hence the above Theorem 2.2.8 implies that a factor system equivalent to a factor system which splits also splits. We also obtain the following theorem.

Theorem 2.2.9 *Let (θ, β) be a factor system of N by G. Then the following statements are equivalent*

(i) *(θ, β) splits*

(ii) *(θ, β) is equivalent to another factor system (θ^*, β^*) such that $\theta^* \in Hom(G, Aut(N))$ and $c^*_{i,j} = 1_N \quad \forall\, i, j \in G$.*

Proof. See Theorem 9.5.3. of [108]. □

Theorem 2.2.10 *[108] Every split extension of N by G is equivalent to a semidirect product of N by G.*

Proof. Let $\overline{G}$ be a split extension of N by G. Then by Theorem 2.2.9, there is an equivalent split extension $\overline{G}_1$, a map λ_1 and a factor system (θ^*, β^*) belonging to the pair $(\overline{G}_1, \lambda_1)$ such that $\theta^* \in Hom(G, Aut(N))$ and $c^*_{i,j} = 1_N \ \forall\, i, j \in G$. By Theorem 2.1.10, there is an extension $\overline{G}_2$ and a map λ_2 such that (θ^*, β^*) belongs to $(\overline{G}_2, \lambda_2)$. By Theorem 2.2.5 $\overline{G}_2$ is a semidirect product of N by G which realizes the homomorphism θ^*. Hence by Theorem 2.1.11 $\overline{G}$ and $\overline{G}_2$ are equivalent. Hence the result. □

From the above theorem, we have that every split extension $\overline{G}$ of N by G is equivalent to a semidirect product of N by G. Hence the terms split extension and semidirect product can be used interchangeably to mean one and the same entity. From now on by an extension $\overline{G}$ of N by G we mean that N is a normal subgroup of $\overline{G}$ and $\overline{G}/N \cong G$. Thus an extension $\overline{G}$ of N by G is a short exact sequence of the form

$$\{1\} \to N \to \overline{G} \xrightarrow{\pi} G \to \{1\}$$

such that $ker(\pi) = N$ and $Im(\pi) = G$. If $\overline{G}$ is an extension of N by G, we simply write $\overline{G} = N{\cdot}G$. In the case where $\overline{G}$ is a split extension we use the notation $\overline{G} = N{:}G$.

Theorem 2.2.11 *Let N be a group, G_1 and G_2 be subgroups of $Aut(N)$. Then there is an isomorphism α from $N{:}G_1$ onto $N{:}G_2$ such that $\alpha(N) = N$ and $\alpha(G_1) = G_2$ if and only if G_1 and G_2 are conjugate in $Aut(N)$.*

Proof. See [56]. $\square$

If N is a finite abelian group, G_1 and G_2 are cyclic subgroups of $Aut(N)$, then Holmes in Theorems 2 and 3 of [56] gives conditions on N, G_1 and G_2 for which $N{:}G_1$ and $N{:}G_2$ will be isomorphic.

Definition 2.2.12 *Let $\overline{G} = N{\cdot}G$ and $\{1\} \to N \to \overline{G} \xrightarrow{\pi} G \to \{1\}$ be the corresponding short exact sequence. Let $g \in G$ and $\overline{g} \in \overline{G}$ such that $\pi(\overline{g}) = g$. Then $\overline{g}$ is called a lifting of g in $\overline{G}$.*

Lemma 2.2.13 *([105],[116]) Let $\overline{G}$ be an extension of N by G where N is abelian. Then there is a homomorphism $\theta : G \longrightarrow Aut(N)$ such that $\theta_g(n) = \overline{g}n(\overline{g})^{-1}, n \in N$ and θ is independent of the choice of liftings $\{\overline{g} \mid g \in G\}$.*

Proof. Let $a \in \overline{G}$ and γ_a denote conjugation by a. Since N is a normal subgroup of $\overline{G}$, $\gamma_a \downarrow_N \in Aut(N)$ and the function $\mu : \overline{G} \longrightarrow Aut(N)$ defined by $\mu(a) = \gamma_a \downarrow_N$ is a homomorphism. If $a \in N$, then since N is abelian we have $\mu(a) = I_N$. Thus there is a homomorphism $\mu^* : \overline{G}/N \longrightarrow Aut(N)$ which is given by $\mu^*(Na) = \mu(a)$. However $G \cong \overline{G}/N$ and for any lifting $\{\overline{g} \mid g \in G\}$, the function $\phi : G \longrightarrow \overline{G}/N$ defined by $\phi(g) = N\overline{g}$ is an isomorphism. If $\{\overline{g}_1 \mid g \in G\}$ is another choice of liftings, then

$\overline{g}\,\overline{g}_1^{-1} \in N$ for every $g \in G$ and thus $N\overline{g} = N\overline{g}_1$. Therefore the isomorphism ϕ is independent of the choice of liftings. Let $\theta : G \longrightarrow Aut(N)$ be the composition $\mu^* \circ \phi$. For $g \in G$ and $\overline{g}$ a lifting of g, then $\theta(g) = \mu^*(\phi(g)) = \mu^*(N\overline{g}) = \mu(\overline{g}) \in Aut(N)$ and thus for $n \in N$, we have $\theta_g(n) = \mu(\overline{g})(n) = \overline{g}n(\overline{g})^{-1}$. Hence the result. □

Remark 2.2.14 *[116]* Let $\overline{G}$ be an extension of N by G where N is abelian and for each $g \in G$ let $\overline{g}$ be a lifting of g. We identify G with $\overline{G}/N$ under the isomorphism $g \longmapsto N\overline{g}$. Thus $\{\overline{g} \mid g \in G\}$ is a right transversal for N in $\overline{G}$ and thus every $x \in \overline{G}$ has a unique expression of the form $x = n\overline{g}$ where $n \in N$ and $g \in G$.

Definition 2.2.15 *Let $\overline{G}$ be an extension of N by G, where N is abelian and let $\theta : G \longrightarrow Aut(N)$ be a homomorphism. Then $\overline{G}$ is said to realize θ if $\theta_g(n) = n^{\overline{g}}$ for every $n \in N$ and every lifting $\overline{g}$ of g in $\overline{G}$.*

Lemma 2.2.13 asserts that every extension $\overline{G}$ of N by G, where N is abelian determines a homomorphism θ which is realized by $\overline{G}$ and θ describes the normality of N in $\overline{G}$.

Let $\overline{G}$ be an extension of N by G. Then we obtain the short exact sequence $\{1\} \to N \to \overline{G} \xrightarrow{\pi} G \to \{1\}$. A choice of liftings $\overline{g}$ for each $g \in G$ defines a function $\lambda : G \longrightarrow \overline{G}$, where λ is not necessarily a homomorphism, such that $\pi\lambda = I_G$. The range of λ is called a *transversal* for N in $\overline{G}$ or a *complete set of coset representatives* for N in $\overline{G}$ since it contains exactly one representative from each coset of N.

Definition 2.2.16 *Let $\overline{G}$ be an extension of N by G and λ be as defined above. Define a function $\beta : G \times G \longrightarrow N$ by*

$$\beta(x, y) = \lambda(x)\lambda(y)[\lambda(xy)]^{-1} \quad .$$

Then β is called a factor set or a cocycle of $\overline{G}$.

Remark 2.2.17 A factor set or cocycle depends on the choice of a transversal for N in $\overline{G}$. If $\overline{G}$ is a semidirect product of N by G, then the map λ given by $\lambda(x) = x$ for all $x \in G$ is a transversal making $\beta(x, y) = xy(xy)^{-1} = 1_N$, for all $x, y \in G$. In general, by using Definition 2.2.16 we can deduce that λ is a homomorphism if and only if $\beta(x, y) = 1_N$ for all $x, y \in G$. Hence we can regard a factor set as a measure of the extent of deviation of $\overline{G}$ from being a semidirect product.

2.3 The conjugacy classes of group extensions

The conjugacy classes of elements of a group provide vital information about the structure of the group. Butler in [16] states that given a group G and considering each prime p dividing $|G|$, the classes of elements of order p^r, for all possible values of r are determined by computing a Sylow p-subgroup, analyzing its classes of elements and then determining their fusion into G. Furthermore the classes of composite order $p^r t$, where $(p,t)=1$ are determined by computing the centralizer $C_G(g)$ for each class representative g, where $o(g)=p^r$ and analyzing the classes of the centralizer or the classes of the centralizer modulo a normal p-subgroup such as $\langle g \rangle$. The computation of the centralizers plays an important role in the determination of the conjugacy classes of elements of a finite group. Recently several authors studied the properties of conjugacy classes and parallelism between results on characters and results on conjugacy classes. For example the following result of Fulman [46] deals with the number of conjugacy classes of elements of order n in a finite group G.

Theorem 2.3.1 *Let G be a finite group and $p_1, p_2, \ldots, p_m$ be the distinct primes dividing some $n \in \mathbf{Z}$. Then the number of conjugacy classes in G of elements of order n is a multiple of*

$$\prod_{i=1}^{m} \frac{p_i - 1}{gcd(|G|, p_i - 1)} \quad .$$

Proof. See [46]. $\square$

Let G be a finite group, and $g \in G$. We denote by $\sigma_G(g)$ the set of all prime divisors of $[G : C_G(g)]$, the length of the conjugacy class of g. We define $\sigma^*(G) = max\{|\sigma_G(g)| : g \in G\}$. Let $\pi(G)$ be the set of all primes dividing the order of G and for $p \in \pi(G)$ we denote by G_p a Sylow p-subgroup of G and define

$$n_p(G) = [N_G(G_p) : C_G(G_p)] \quad \text{and} \quad \Delta(G) = \{p \in \pi(G) \mid n_p(G) > 1\} \quad .$$

If G is nonabelian, then by [20] we have

$$\sigma^*(G) > \sum_{p \in \Delta(G)} \frac{n_p(G) - 1}{n_p(G)} \quad \text{and} \quad 2\sigma^*(G) > |\Delta(G)| \quad .$$

Chillag and Herzog [22] described the groups with $\sigma^*(G) = 1$. They showed that $\sigma^*(G) = 1$ if and only if $G = A \times H$, where A is abelian and

(i) H is a nonabelian p-group (for some prime p), or

(ii) $H = O_q(H)P$ with P a p-group (p and q are distinct primes), $O_q(H)$ and P abelian and $H/O_q(H)$ a Frobenius group.

Casolo in [21] studies finite groups with $\sigma^*(G) \leq 3$. In the following we list some elementary properties of $\sigma_G(g)$ and $\sigma^*(G)$. For proofs, see [22].

(i) Let $x, y \in G$ such that $xy = yx$ and $(o(x), o(y)) = 1$. Then

$$\sigma_G(xy) \supseteq \sigma_G(x) \cup \sigma_G(y)$$

(ii) Let H be a normal subgroup of G, $x \in H$ and $y \in G$. Then

(a) $\sigma_H(x) \subseteq \sigma_G(x)$ and $\sigma^*(H) \leq \sigma^*(G)$.

(b) $\sigma_{G/H}(Hy) \subseteq \sigma_G(y)$ and $\sigma^*(G/H) \leq \sigma^*(G)$.

Before going into the detailed discussion of the coset analysis technique, which is the main part of this section, we would like to state the following relevant results.

Theorem 2.3.2 *Let G be a finite group*

(i) Suppose that C_1 and C_2 are two conjugacy classes of G such that $C_1 \neq [1_G]$ and $C_1^n = C_2$ for some integer $n \geq 2$, where

$$C_1^n = \{x_1 x_2 \cdots x_n \mid x_i \in C_1,\ 1 \leq i \leq n\} \ .$$

Then there exists some normal subgroup N of G and $g \in G - N$ such that C_1 is the coset Ng and the map $x \mapsto x^n$ is a bijection from C_1 onto C_2.

(ii) If G has a normal subgroup N and $g \in G - N$ such that the coset Ng is a single conjugacy class of G, and such that for some $n \in \mathbf{Z}$ the map $x \longmapsto x^n$ for $x \in Ng$ is a monomorphism, then Ng^n is a conjugacy class of G and $(Ng)^n = Ng^n$.

Proof. See [11]. $\square$

Proposition 2.3.3 *Let $\overline{G} = N{\cdot}G$, $\overline{g} \in \overline{G}$ a lifting of $g \in G$, C be the centralizer of $N\overline{g}$ in G and $\overline{C}$ be the complete preimage in $\overline{G}$ of C. Then*

(i) the union of the cosets $N\overline{x}$ which are conjugate in G to $N\overline{g}$, is the union of the conjugacy classes $L_1, L_2, \ldots, L_r$ of $\overline{G}$

(ii) $\overline{C}$ acts on the coset $N\overline{g}$ by conjugation

(iii) $\overline{C}$ has r orbits in its action on $N\overline{g}$ and the orbit representatives $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_r$ are representatives of the conjugacy classes $L_1, L_2, \ldots, L_r$ of $\overline{G}$

(iv) the centralizer $C_{\overline{G}}(\overline{g}_i)$ for $1 \leq i \leq r$ is the stabilizer of $\overline{g}_i$ in $\overline{C}$ in its action on $N\overline{g}$.

Proof. See [15]. □

We now discuss the technique of *coset analysis* which is used for determining the conjugacy classes of elements of group extensions $\overline{G} = N{\cdot}G$ where N is an abelian normal subgroup of $\overline{G}$. The technique works for both split and nonsplit extensions and was developed and first used by Moori in [80]. For each conjugacy class $[g]$ in G with representative $g \in G$, we analyse the coset $N\overline{g}$, where $\overline{g}$ is a lifting of g in $\overline{G}$ and

$$\overline{G} = \bigcup_{g \in G} N\overline{g} \quad .$$

To each class representative $g \in G$ with lifting $\overline{g} \in \overline{G}$, we define

$$C_{\overline{g}} = \{x \in \overline{G} \mid x(N\overline{g}) = (N\overline{g})x\} \quad .$$

Then $C_{\overline{g}}$ is the stabilizer of $N\overline{g}$ in $\overline{G}$ under the action by conjugation of $\overline{G}$ on $N\overline{g}$, and hence $C_{\overline{g}}$ is a subgroup of $\overline{G}$.

Remark 2.3.4 It is not difficult to see that N is a normal subgroup of $C_{\overline{g}}$.

Lemma 2.3.5 *[116] $C_{\overline{g}}/N = C_{\overline{G}/N}(N\overline{g})$.*

Proof. Consider Nk, where $k \in \overline{G}$. Then

$$\begin{aligned} Nk \in C_{\overline{G}/N}(N\overline{g}) &\Leftrightarrow Nk(N\overline{g})(Nk)^{-1} = N\overline{g} \\ &\Leftrightarrow NkN\overline{g}Nk^{-1} = N\overline{g} \\ &\Leftrightarrow NkN\overline{g}k^{-1} = N\overline{g} \\ &\Leftrightarrow NkNn\overline{g}k^{-1} = N\overline{g} \ \forall n \in N \\ &\Leftrightarrow Nkn\overline{g}k^{-1} = N\overline{g} \ , \ \forall n \in N \\ &\Leftrightarrow kn\overline{g}k^{-1} \in N\overline{g} \ , \ \forall n \in N \\ &\Leftrightarrow k \in C_{\overline{g}} \\ &\Leftrightarrow Nk \in C_{\overline{g}}/N \ . \end{aligned}$$

Thus we obtain that $C_{\overline{g}}/N = C_{\overline{G}/N}(N\overline{g})$. □

Remark 2.3.6 Using Remark 2.3.4 and Lemma 2.3.5 we deduce that $C_{\overline{g}} = N{\cdot}C_{\overline{G}/N}(N\overline{g})$. For $\overline{g}$ a lifting of $g \in G$ in $\overline{G}$, we can identify $C_{\overline{G}/N}(N\overline{g})$ with $C_G(g)$ and write $C_{\overline{g}} = N{\cdot}C_G(g)$ in general. If $\overline{G} = N{:}G$ then we can identify $C_{\overline{g}}$ with $C_g = \{x \in \overline{G} \mid x(Ng) = (Ng)x\}$, where the lifting of g in $\overline{G}$ is g itself since $G \leq \overline{G}$ in the case of a split extension.

Corollary 2.3.7 *If $\overline{G} = N{:}G$, then $C_g = N{:}C_G(g)$.*

Proof. We have that N is a normal subgroup of C_g. Now we show that $C_G(g) \leq C_g$ and that $N \cap C_G(g) = \{1\}$. Let $x \in C_G(g)$. Then we obtain $(Ng)^x = x(Ng)x^{-1} = xNgx^{-1} = Nxgx^{-1} = Ng$. Thus $x \in C_g$ and hence $C_G(g) \leq C_g$. Since $N \cap C_G(g) \leq N \cap G = \{1_G\}$, then we have that $N \cap C_G(g) = \{1_G\}$. Hence the result. □

The conjugacy classes of $\overline{G}$ (recall that $\overline{G} = N{\cdot}G$ where N is abelian) will be determined by the action by conjugation of $C_{\overline{g}}$, for each conjugacy class $[g]$ of G, on the elements of $N\overline{g}$. To act $C_{\overline{g}}$ on the elements of $N\overline{g}$, we first act N and then act $\{\overline{h} \mid h \in C_G(g)\}$, where $\overline{h}$ is a lifting of h in $\overline{G}$. We outline this action in two steps as follows:

STEP 1: *The action of N on $N\overline{g}$*: Let $C_N(\overline{g})$ be the stabilizer of $\overline{g}$ in N. Then for any $n \in N$ we have

$$x \in C_N(n\overline{g}) \quad \Leftrightarrow \quad x(n\overline{g})x^{-1} = n\overline{g}$$

$$\begin{aligned}
&\Leftrightarrow\quad xnx^{-1}x\overline{g}x^{-1} = n\overline{g} \\
&\Leftrightarrow\quad n(x\overline{g}x^{-1}) = n\overline{g}\ , \quad \text{since } N \text{ is abelian} \\
&\Leftrightarrow\quad x\overline{g}x^{-1} = \overline{g} \\
&\Leftrightarrow\quad x \in C_N(\overline{g}).
\end{aligned}$$

Thus $C_N(\overline{g})$ fixes every element of $N\overline{g}$. Now let $|C_N(\overline{g})| = k$. Then under the action of N, $N\overline{g}$ splits into k orbits $Q_1, Q_2, \ldots, Q_k$, where

$$|Q_i| = [N : C_N(\overline{g})] = \frac{|N|}{k}\ ,$$

for $i \in \{1, 2, \ldots, k\}$.

STEP 2: *The action of* $\{\overline{h} \mid h \in C_G(g)\}$ *on* $N\overline{g}$: Since the elements of $N\overline{g}$ are now in the orbits $Q_1, Q_2, \ldots, Q_k$ from Step 1 above, we need only act $\{\overline{h} \mid h \in C_G(g)\}$ on these k orbits. Suppose that under this action f_j of these orbits $Q_1, Q_2, \ldots, Q_k$ fuse together to form one orbit Δ_j, then the f_j's obtained this way must satisfy

$$\sum_j f_j = k$$

and we have

$$|\Delta_j| = f_j \times \frac{|N|}{k}\ .$$

Thus for $x = d_j\overline{g} \in \Delta_j$, we obtain that

$$\begin{aligned}
|[x]_{\overline{G}}| &= |\Delta_j| \times |[g]_G| \\
&= f_j \times \frac{|N|}{k} \times \frac{|G|}{|C_G(g)|} \\
&= f_j \times \frac{|\overline{G}|}{k|C_G(g)|}
\end{aligned}$$

and thus we obtain that

$$|C_{\overline{G}}(x)| = \frac{|\overline{G}|}{|[x]_{\overline{G}}|} = |\overline{G}| \times \frac{k|C_G(g)|}{f_j|\overline{G}|} = \frac{k|C_G(g)|}{f_j}\ .$$

Thus to calculate the conjugacy classes of $\overline{G} = N{\cdot}G$, we need to find the values of k and the f_j's for each class representative $g \in G$.

Remark 2.3.8 However in the case of $\overline{G} = N{:}G$ a split extension, we analyse the coset Ng instead of $N\overline{g}$ since in this case $G \leq \overline{G}$. Under the action of N on Ng, we always assume that $g \in Q_1$. Also instead of acting $\{\overline{h} \mid h \in C_G(g)\}$ on the k orbits $Q_1, Q_2, \ldots, Q_k$ we just act $C_G(g)$ on these orbits. Since $g \in Q_1$, then $C_G(g)$ always fixes Q_1 and thus we will always have $f_1 = 1$. Hence

$$k = \sum_j f_j = 1 + \sum_m f_m \quad ,$$

where the sum is taken over all m such that $g \notin Q_m$.

In the following we prove and discuss techniques that are useful in the determination of the orders of the elements of $\overline{G} = N{:}G$.

Theorem 2.3.9 *Let $\overline{G} = N{:}G$ and $dg \in \overline{G}$ where $d \in N$ and $g \in G$ such that $o(g) = m$ and $o(dg) = k$. Then m divides k.*

Proof. We have that

$$1_{\overline{G}} = (dg)^k = dd^g d^{g^2} d^{g^3} \ldots d^{g^{k-1}} g^k \quad .$$

Since G acts on N and $d \in N$, we have $d, d^g, d^{g^2}, \ldots, d^{g^{k-1}} \in N$. Hence $dd^g d^{g^2} \ldots d^{g^{k-1}} \in N$. Thus we must have that $dd^g d^{g^2} \ldots d^{g^{k-1}} = 1_N$ and $g^k = 1_G$. Hence m divides k. $\square$

Theorem 2.3.10 *Let $\overline{G} = N{:}G$ such that N is an elementary abelian p-group, where p is prime. Let $dg \in \overline{G}$ where $d \in N$ and $g \in G$ such that $o(g) = m$ and $o(dg) = k$. Then either $k = m$ or $k = pm$.*

Proof. Since N is an elementary abelian p-group and $d \in N$, then we have that $o(d) = 1$ or $o(d) = p$. Suppose that $d \neq 1_N$, then $o(d) = p$. Now we observe that

$$(dg)^m = dd^g d^{g^2} d^{g^3} \ldots d^{g^{m-1}} g^m \quad .$$

Since $g^m = 1_G$, we deduce that $(dg)^m \in N$. If $(dg)^m = 1_N$, then k must divide m and Theorem 2.3.9 implies that $k = m$. If $(dg)^m \neq 1_N$, then $o((dg)^m) = p$ and hence

$(dg)^{pm} = 1_N$. Thus we obtain that $k \mid pm$ and hence $pm = kx$ for some positive integer x. However from Theorem 2.3.9 we have $k = mb$ for some positive integer b. Since $o(dg) = k$ and we assumed $(dg)^m \neq 1_N$, we must have $m \neq k$ and hence $b \neq 1$. Now $pm = kx$ and $k = mb$ imply that $pm = mbx$, so that $p = bx$. Since p is a prime and $b \neq 1$, we must have $p = b$ and $x = 1$. Therefore we obtain that $k = pm$. Hence the result. $\square$

Remark 2.3.11 Let $\overline{G} = N{:}G$, where N is an elementary abelian p-group. Let $dg \in \overline{G}$ with $d \in N$, $g \in G$ such that $o(g) = m$ and $o(dg) = k$, then we observe that

$$(dg)^m = d.d^g.d^{g^2}.\dots.d^{g^{m-1}}g^m \quad .$$

Since $g^m = 1_G$, we obtain that $(dg)^m = w$, where $w \in N$ and it is given by

$$w = d.d^g.\dots.d^{g^{m-1}} \quad .$$

By Theorem 2.3.10 above, we have that if $w = 1_N$ then $k = m$ and if $w \neq 1_N$ then $k = pm$.

We have used the method of coset analysis discussed above (outlined in Steps 1 and 2) together with Theorems 2.3.9 and 2.3.10 and Remark 2.3.11 in developing Programmes A and B in CAYLEY which are applied for the computation of conjugacy classes and the orders of the class representatives of the extension $\overline{G} = N{:}G$ where N is an elementary abelian p-group for prime p on which a linear group G acts.

For example consider $\overline{G} = N{:}S$ where S is a matrix group, with generators $A_1, A_2, \dots, A_t$ acting on the vector space $N = V(n,q)$ with orbits $O_1, O_2, \dots, O_{k'}$ on $V^*(n,q)$. Then the first part of Programme A computes the orbits $Q_1, Q_2, \dots, Q_k$ for each conjugacy class of S while the second part acts the centralizers of elements of S on $\{Q_1, Q_2, \dots, Q_k\}$ to determine the f_j's, Δ_j's and the corresponding d_j's, where d_jg is a representative of the Δ_j, as described in Step 2. The Programme B computes the elements $w \in N$ which are used in determining the orders of $d_jg \in \overline{G}$, as required by Remark 2.3.11.

PROGRAMME A

$V : vector\ space(n, GF(q));$
$S : matrix\ group(V);$
$S.generators : A_1, A_2, \ldots, A_t;$
$c : classes(S);$
$O_1 : matrix\ orbit(S, vec(\alpha_1, \ldots, \alpha_n), false);$
$O_2 : matrix\ orbit(S, vec(\beta_1, \ldots, \beta_n), false);$
$\vdots$
$O_{k'} : matrix\ orbit(S, vec(\delta_1, \ldots, \delta_n), false);$
$O : O_1\ join\ O_2\ join \cdots join\ O_{k'};$
$for\ i = 1\ to\ n(c)\ do;$
$print\ c[i],\ '\$N';$
$e = null;$
$w = vec(0)\ of\ V;$
$while\ O - e\ ne\ [\,]\ do;$
$d = null;$
$for\ each\ x\ in\ O\ do;$
$y = [x + w + (x * c[i])];$
$d = d\ join\ y;$
$end;$
$print\ d,\ '\$N';$
$print\ ' * * * * * *';$
$e = d\ join\ e;$
$if\ O - e\ ne\ [\,]\ then;$
$w = setrep(O - e);$
$end;$
$end;$
$r = null;$
$u = vec(0)\ of\ V;$
$while\ O - r\ ne\ [\,]\ do;$
$m = null;$
$for\ each\ g\ in\ centralizer(S, c[i])\ do;$

$l = [u * g];$
$m = m\ join\ l;$
$end;$
$print\ 'A\ block\ for\ the\ vectors\ under\ the\ action\ of\ centralizer\ :';$
$print\ m;$
$r = m\ join\ r;$
$if\ O - r\ ne\ [\ ]\ then;$
$u = setrep(O - r);$
$end;$
$end;$
$print\ '*******************************';$
$end;$

PROGRAMME B

$V : vector\ space(n, GF(q));$
$S : matrix\ group(V);$
$S.generators : A_1, A_2, \ldots, A_t;$
$c : classes(S);$
$g = c[i];$
$d = vec(\alpha_1, \ldots, \alpha_n);$
$w = d + d * g + d * (g^2) + d * (g^3) + \ldots + d * (g^{m-1});$
$print\ w;$

In Programme B we have $o(g) = m$ and $g \in S$ is a class representative, for $1 \leq j \leq n, \alpha_j \in GF(q)$, $d * g = d^g$, and $+$ signifies the operation in V and $dg \in \overline{G}$ is a class representative from the coset Ng.

In [15] and [16], Butler gives various algorithms which can be used for computing conjugacy classes in finite groups and in permutation groups respectively. In [16], Butler gives the inductive schema for computing the conjugacy classes in permutation groups. This schema is given as Algorithm 1 in this paper.

Chapter 3

Group Characters

In this chapter, we present some results on group characters which are used in the later chapters. We mostly concentrate on those results which would be useful for the technique of the Fischer-Clifford matrices that is fully discussed in Chapter 5. In this thesis, we construct character tables of certain groups associated with the smallest Fischer sporadic simple group Fi_{22} and its automorphism group $\overline{F}i_{22}$. We start by discussing the general theory of representations and characters, and go on to discuss the restricted, induced and permutation characters, which will be used in the later chapters for constructing the character tables of the groups that are studied in this thesis. The characters being studied are ordinary complex characters. We give a proof that the permutation character of any group G acting on the cosets of its subgroup H is the character induced from the identity character of H. We use the notation $\chi(G|H)$ to denote this permutation character and we use I_G to denote the identity character of any group G. So with this notation we have $\chi(G|H) = (I_H)^G$. We also give a proof from [60] of the Frobenius Reciprocity Theorem, which gives a relationship between restricted and induced characters and their constituents. For a finite group G and $H \leq G$, then the relationship between the characters of G and those of H is of fundamental importance. For further reading on representations and characters, readers are encouraged to consult [2], [4], [7], [12], [23], [24], [35], [63], [61], [64], [67], [72], [74], [90], [101], [109], [110] and many other relevant sources.

3.1 Representations and Characters

Definition 3.1.1 *Let G be a group, F a field and $GL(n, F)$ the general linear group which is the multiplicative group of all nonsingular $n \times n$ matrices over F for some integer n. Then a homomorphism $\rho : G \longrightarrow GL(n, F)$ is called a representation of G over F or simply an F-representation. The representation ρ is said to have degree n. The function $\chi : G \longrightarrow F$ given by $\chi(g) = trace(\rho(g))$ is called the F-character of G afforded by the F-representation ρ. The degree of χ is the same as that of ρ.*

Two F-representations ρ_1 and ρ_2 of G are said to be *equivalent* if there exists $P \in GL(n, F)$ such that $\rho_1(g) = P\rho_2(g)P^{-1}$ for all $g \in G$. An F-representation ρ of G is said to be *reducible* if it is equivalent to a representation α which is given by

$$\alpha(g) = \begin{pmatrix} \beta(g) & \gamma(g) \\ 0 & \delta(g) \end{pmatrix}$$

for all $g \in G$, where β, γ, δ are F-representations of G. If ρ is not reducible, then it is said to be *irreducible*. Since similar matrices have the same trace, then it follows that equivalent representations afford the same character. The character afforded by an irreducible representation is called an *irreducible* character. Sums and products of characters are themselves characters.

Theorem 3.1.2 (**Schur's Lemma**) *Let $\rho_1 : G \longrightarrow GL(n, F)$ and $\rho_2 : G \longrightarrow GL(m, F)$ be two irreducible representations of a group G over a field F. Assume that there exists a matrix P such that $P\rho_1(g) = \rho_2(g)P$ for all $g \in G$. Then either P is the zero matrix or P is nonsingular so that $\rho_1(g) = P^{-1}\rho_2(g)P$.*

Proof. See Theorem 1.8 of [89]. □

Corollary 3.1.3 *[89] If $\rho : G \longrightarrow GL(n, F)$ is an irreducible representation of a group G over an algebraically closed field F, then the only matrices which commute with all matrices $\rho(g)$, $g \in G$ are scalar matrices aI_n, where $a \in F$ and I_n is the $n \times n$ identity matrix.*

Proof. Let P be an $n \times n$ matrix such that $P\rho(g) = \rho(g)P$ for all $g \in G$. Then for any $a \in F$ we have that

$$(aI_n - P)\cdot\rho(g) = \rho(g)\cdot(aI_n - P),\ \forall g \in G\ . \quad (1)$$

Let $m(x) = det(xI_n - P)$ be the characteristic polynomial of P. Since $m(x)$ is a polynomial over F and F is algebraically closed, then there exists $a_1 \in F$ such that $m(a_1) = 0_F$. Hence $det(a_1I_n - P) = 0_F$ and thus $a_1I_n - P$ is singular. Then from relation (1) above and Schur's Lemma, we obtain that $a_1I_n - P = 0$ and hence $a_1I_n = P$. $\square$

Definition 3.1.4 *Let G be a group, F a field and $\phi : G \longrightarrow F$ be a function which is constant on conjugacy classes. Then ϕ is called a class function of G.*

From the above definition, we observe that every character is a class function. We shall use the notation $Irr(G)$ to denote the set of all irreducible characters of the group G.

From now on, we will consider representations and characters of a finite group G over the complex field $\mathbb{C}$.

We can show that every class function ϕ of G can be uniquely expressed in the form $\phi = \sum_{\chi\in Irr(G)} b_\chi\chi$, where $b_\chi \in \mathbb{C}$. Moreover ϕ is a character if and only if all $b_\chi \in \mathbb{N} \cup \{0\}$ and $\phi \neq 0$. We can also show that the following properties hold:

(i) Two representations of G have the same character if and only if they are equivalent.

(ii) The number of irreducible characters of G is equal to the number of conjugacy classes of elements of G.

(iii) Any character of G can be written as a sum of irreducible characters.

Definition 3.1.5 *Let G be a group, χ be a character of G and $Irr(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$ such that $\chi = \sum_{i=1}^{r} n_i\chi_i$, where $n_i \in \mathbb{N} \cup \{0\}$. Then those χ_i for which $n_i > 0$ are called the irreducible constituents of χ. In general, if ψ is a character of G such that $\chi - \psi$ is a character or is zero, then ψ is a constituent of χ.*

Theorem 3.1.6 (**Generalized Orthogonality Relation**) *Let G be a group and $Irr(G) = \{\chi_1, \chi_2, \dots, \chi_r\}$. Then the following holds for every $h \in G$.*

$$\frac{1}{|G|}\sum_{g\in G}\chi_i(gh)\chi_j(g^{-1}) = \delta_{ij}\frac{\chi_i(h)}{\chi_i(1_G)} \quad .$$

Proof. See Theorem 2.13 of [60]. □

Let G be a group, ρ be a representation of G which affords the character χ. Let $g \in G$ such that $o(g) = n$. Then the following conditions hold

(i) $\rho(g)$ is similar to a diagonal matrix $\mathrm{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$

(ii) $\varepsilon_i^n = 1$

(iii) $\chi(g) = \sum_i \varepsilon_i$

(iv) $|\chi(g)| \leq \chi(1_G) =$ degree of χ

(v) $\chi(g^{-1}) = \overline{\chi(g)}$, where $\overline{\chi(g)}$ is the complex conjugation of $\chi(g)$.

The above conditions are proved as Lemma 2.15 in [60].

Definition 3.1.7 *Let χ and ψ be class functions of a group G. Then the inner product of χ and ψ is defined by*

$$\langle\chi, \psi\rangle = \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\psi(g)} \quad .$$

The following theorems are derived from the generalized orthogonality relation and are called the first and second orthogonality relations respectively.

Theorem 3.1.8 *[60]*(**First Orthogonality Relation**) *Let G be a group and $Irr(G) = \{\chi_1, \chi_2, \dots, \chi_r\}$. Then*

$$\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)} = \delta_{ij} = \langle\chi_i, \chi_j\rangle \quad .$$

Proof. Using the generalized orthogonality relation and taking $h = 1_G$, then the result follows immediately. □

Theorem 3.1.9 *[60](***Second Orthogonality Relation***) Let G be a group and $Irr(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$ and $\{g_1, g_2, \ldots, g_r\}$ be a set of representatives of the conjugacy classes of elements of G. Then*

$$\sum_{\chi \in Irr(G)} \chi(g_i)\overline{\chi(g_j)} = \delta_{ij}|C_G(g_i)| \quad .$$

Proof. Let X be the character table of G. Then viewed as a matrix, X is an $r \times r$ matrix whose (i,j)-th entry is given by $\chi_i(g_j)$. Let C_i be the conjugacy class which contains g_i and D be the diagonal matrix with entries $\delta_{ij}|C_i|$. Then by the first orthogonality relation, we obtain that

$$|G|\delta_{ij} = \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \sum_{t=1}^{r} |C_t|\chi_i(g_t)\overline{\chi_j(g_t)} \quad .$$

Then we obtain a system of r^2 equations which can be written as a single matrix equation as follows

$$|G|I = XD\overline{X}^T \quad ,$$

where I is the identity $r \times r$ matrix and $\overline{X}^T$ is the transpose of $\overline{X}$. Since X is a nonsingular matrix, then we obtain that

$$|G|I = D\overline{X}^T X \quad .$$

Rewriting the above matrix system as a system of equations yields

$$|G|\delta_{ij} = \sum_{t=1}^{r} |C_i|\overline{\chi_t(g_i)}\chi_t(g_j) \quad .$$

Hence we obtain that

$$\sum_{\chi \in Irr(G)} \chi(g_j)\overline{\chi(g_i)} = |C_G(g_i)|\delta_{ij} \quad .$$

□

3.2 Normal Subgroups

Let G be a group and χ be a character of G afforded by a representation ρ. Then we define

$$ker(\chi) = \{g \in G \mid \chi(g) = \chi(1_G)\} \quad .$$

It can be shown (for example see Whitley [116]) that $ker(\chi) = ker(\rho)$ and hence $ker(\chi)$ is a normal subgroup of G. If $Irr(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$, then every normal subgroup of G is the intersection of some of the $ker(\chi_i)$.

Theorem 3.2.1 *Let G be a group and N be a normal subgroup of G. Then*

(a) If χ is a character of G and $N \subseteq ker(\chi)$, then χ is constant on the cosets of N in G and the function $\hat{\chi}$ defined on G/N by $\hat{\chi}(Ng) = \chi(g)$ is a character of G/N.

(b) If $\hat{\chi}$ is a character of G/N, then the function χ defined by $\chi(g) = \hat{\chi}(Ng)$ is a character of G.

(c) In both (a) and (b) above, $\chi \in Irr(G)$ if and only if $\hat{\chi} \in Irr(G/N)$.

Proof. See Theorem 2.2.2. of [116]. $\square$

If N is a normal subgroup of G and ρ is representation of G such that $N \subseteq ker(\rho)$, then there exists a unique representation $\hat{\rho}$ of G/N defined by $\hat{\rho}(Ng) = \rho(g)$. Thus knowing ρ, we can obtain $\hat{\rho}$ and vice versa. We also obtain that ρ is irreducible if and only if $\hat{\rho}$ is irreducible. Hence ρ and $\hat{\rho}$ can be identified. If ρ affords a character χ of G, then $\hat{\rho}$ affords a character $\hat{\chi}$ of G/N and also χ and $\hat{\chi}$ can be identified. Under this identification, we obtain that

$$Irr(G/N) = \{\chi \in Irr(G) \mid N \subseteq ker(\chi)\} \quad .$$

Thus the irreducible characters of G/N are precisely those irreducible characters of G which contain N in their kernels.

Definition 3.2.2 *Let G be a group, N a normal subgroup of G and $\hat{\chi}$ be a character of G/N. Then the character χ of G defined by*

$$\chi(g) = \hat{\chi}(Ng)$$

is called a lifting of $\hat{\chi}$ *to* G.

Thus given characters of G/N, we can obtain some characters of G by the lifting process. The character $\hat{\chi}$ and its lifting χ have the same degree.

3.3 Restriction of Characters

Definition 3.3.1 *Let G be a finite group and $H \leq G$. If ρ is a representation of G, then the restriction of ρ to H is a representation of H. This representation is denoted by ρ_H. If χ is a character of G afforded by ρ, then the restriction of χ to H is denoted by χ_H and is a character of H afforded by the representation ρ_H such that*

$$\chi_H = \sum_{\psi \in Irr(H)} k_\psi \psi \quad ,$$

where $k_\psi \in \mathbb{N} \cup \{0\}$.

The characters χ_H and χ take on the same values on the elements of H. If χ_H is irreducible, then χ is irreducible in G but the converse is not true in general. Karpilovsky in [70] proves a theorem (Theorem 23.1.4) due to Gallagher(1966) that if $H \leq G$, $\chi \in Irr(G)$ such that $\chi(g) \neq 0 \ \forall \ g \in G - H$, then χ_H is irreducible and for any $g \in G - H$, $\chi(g)$ is a root of unity. We also observe that (see [67]) every irreducible character of H is a constituent of some irreducible character of G restricted to H.

Theorem 3.3.2 *[67] Let G be a group, $H \leq G$, $\chi \in Irr(G)$ and $Irr(H) = \{\psi_1, \psi_2, \ldots, \psi_r\}$. Then*

$$\chi_H = \sum_{i=1}^{r} k_i \psi_i \quad ,$$

where $k_i \in \mathbb{N} \cup \{0\}$ satisfy the following relation

$$\sum_{i=1}^{r} k_i^2 \leq [G : H] \quad .$$

Moreover, equality in the above relation holds if and only if $\chi(g) = 0$ for all $g \in G - H$.

Proof. We obtain that

$$\sum_{i=1}^{r} k_i^2 = \langle \chi_H, \chi_H \rangle = \frac{1}{|H|} \sum_{h \in H} \chi(h)\overline{\chi(h)} \quad .$$

Since χ is irreducible, then we have that

$$\begin{aligned} 1 &= \langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} \\ &= \frac{1}{|G|} \sum_{h \in H} \chi(h)\overline{\chi(h)} + \frac{1}{|G|} \sum_{g \notin H} \chi(g)\overline{\chi(g)} \\ &= \frac{|H|}{|G|} \sum_{i=1}^{r} k_i^2 + K \end{aligned}$$

where

$$K = \frac{1}{|G|} \sum_{g \notin H} \chi(g)\overline{\chi(g)} \quad .$$

Since $K \geq 0$ we have that

$$\frac{|H|}{|G|} \sum_{i=1}^{r} k_i^2 = 1 - K \leq 1 \quad .$$

Hence

$$\sum_{i=1}^{r} k_i^2 \leq [G : H] \quad .$$

The equality holds if and only if $\chi(g) = 0$ for all $g \in G - H$. □

Theorem 3.3.3 *Let G be a group, H be a normal subgroup of G and $\chi \in Irr(G)$. Then all the constituents of χ_H have the same degree.*

Proof. See Proposition 20.7 of [67]. □

3.4 Induced Characters

Let G be a group and $H \leq G$ such that the set $\{x_1, x_2, \ldots, x_r\}$ is a transversal for H in G. Let ϕ be a representation of H of degree n. Then we define ϕ^* on G as follows:

$$\phi^*(g) = \begin{pmatrix} \phi(x_1 g x_1^{-1}), \phi(x_1 g x_2^{-1}), \ldots, \phi(x_1 g x_r^{-1}) \\ \phi(x_2 g x_1^{-1}), \phi(x_2 g x_2^{-1}), \ldots, \phi(x_2 g x_r^{-1}) \\ \vdots \\ \phi(x_n g x_1^{-1}), \phi(x_n g x_2^{-1}), \ldots, \phi(x_n g x_r^{-1}) \end{pmatrix}$$

where $\phi(x_i g x_j^{-1})$ are $n \times n$ submatrices of $\phi^*(g)$ satisfying the property that

$$\phi(x_i g x_j^{-1}) = 0_{n\times n} \ \forall \ x_i g x_j^{-1} \notin H \quad .$$

Then we can show that ϕ^* is a representation of G of degree n.

Definition 3.4.1 *Let G, H, ϕ and ϕ^* be as above. Then the representation ϕ^* is called the representation of G induced from the representation ϕ of H and we denote this by writing $\phi^* = \phi^G$.*

If ψ is a representation of H which is equivalent to ϕ, then it can be shown that ψ^G is equivalent to ϕ^G. Thus the induction process preserves equivalence between representations.

Definition 3.4.2 *Let G be a group and $H \leq G$. Let χ be a class function of H. Then we define χ^G as follows:*

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \chi^\circ(xgx^{-1}) \quad ,$$

where

$$\chi^\circ(h) = \begin{cases} \chi(h) & \text{if } h \in H \\ 0 & \text{otherwise} \end{cases} \quad .$$

Then χ^G is a class function of G, called the induced class function of G induced from χ. Also we have that $deg(\chi^G) = [G : H] deg(\chi)$.

Theorem 3.4.3 *[60](***Frobenius Reciprocity Theorem***) Let G be a group, $H \leq G$ and suppose that χ is a class function of H and ϕ is a class function of G. Then*

$$\langle \chi, \phi_H \rangle = \langle \chi^G, \phi \rangle$$

Proof. We obtain that

$$\langle \chi^G, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi^G(g)\overline{\phi(g)} = \frac{1}{|G|}\frac{1}{|H|} \sum_{g \in G} \sum_{x \in G} \chi^\circ(xgx^{-1})\overline{\phi(g)} \quad .$$

Putting $y = xgx^{-1}$ and since ϕ is a class function, then we obtain that $\phi(y) = \phi(g)$. Hence we have

$$\begin{aligned} \langle \chi^G, \phi \rangle &= \frac{1}{|G|}\frac{1}{|H|} \sum_{g \in G} \sum_{x \in G} \chi^\circ(xgx^{-1})\overline{\phi(g)} = \frac{1}{|G|}\frac{1}{|H|} \sum_{y \in G} \sum_{x \in G} \chi^\circ(y)\overline{\phi(y)} \\ &= \frac{1}{|H|} \sum_{y \in H} \chi(y)\overline{\phi(y)} = \langle \chi, \phi_H \rangle. \end{aligned}$$

Hence the result. □

Let $H \leq G$ and ϕ be a representation of H that affords a character χ of H. Then χ^G is a character of G afforded by the induced representation ϕ^G of G. The character χ^G is called the *induced character* of G. The induction and restriction processes do not necessarily preserve irreducibility of characters. For further reading on induced characters, readers are encouraged to consult [5], [6], [64], [91] and many other relevant sources.

Theorem 3.4.4 *Let G be a group and $H \leq G$. Let χ be a character of H, $g \in G$ and $\{x_1, x_2, \ldots, x_m\}$ be a set of representatives of the conjugacy classes of elements of H which fuse into $[g]$ in G. Then we obtain that*

$$\chi^G(g) = |C_G(g)| \sum_{i=1}^{m} \frac{\chi(x_i)}{|C_H(x_i)|} \quad ,$$

where we have that $\chi^G(g) = 0$ whenever $H \cap [g] = \emptyset$.

Proof. We have that

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \chi^\circ(xgx^{-1}) \quad .$$

If $H \cap [g] = \emptyset$, then $xgx^{-1} \notin H$ and thus $\chi^{\circ}(xgx^{-1}) = 0 \quad \forall\, x \in G$ and hence $\chi^G(g) = 0$. Now if $H \cap [g] \neq \emptyset$, then let $h \in H \cap [g]$. Then as x runs over G, then $xgx^{-1} = h$ for exactly $|C_G(g)|$ values of x. Hence we obatin that

$$\chi^G(g) = \frac{1}{|H|}\sum_{x \in G}\chi(xgx^{-1}) = \frac{|C_G(g)|}{|H|}\sum_{h \in H \cap [g]}\chi(h) = |C_G(g)|\sum_{i=1}^{m}\frac{\chi(x_i)}{|C_H(x_i)|} \quad .$$

Hence the result. $\square$

Definition 3.4.5 *Let G be a group $H \leq G$ and χ be a character of G such that $\chi = \lambda^G$ for some linear character λ of H. Then χ is called monomial. If G is such that every $\chi \in Irr(G)$ is monomial, then G is called an M-group or a monomial group.*

It can be shown that every nilpotent group is an M-group (see [94]). According to the Taketa Theorem (see Theorem 52.5 in [27]), every M-group is solvable. For further results on M-groups we encourage the readers to consult [60]. For a group-theoretic characterization of M-groups, see Parks [99].

Theorem 3.4.6 *Let G be a group, H and K be subgroups of G such that $H \leq K$. Let χ be a character of H. Then we obtain that $\chi^G = (\chi^K)^G$.*

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be a complete set of representatives of the conjugacy classes of H which fuse into $[g]$, $g \in G$ and let $\{y_1, y_2, \ldots, y_m\}$ be the representatives of the conjugacy classes of K which fuse into $[g]$ in G. For any $[z]$ in K which fuses into $[g]$ in G for which there is no $x_i \in H$ such that $[x_i]$ fuses into $[z]$, then we set $\chi^K(z) = 0$. Thus since $H \leq K$, suppose without loss of generality that $\{y_1, y_2, \ldots, y_m\}$ is a complete set of representatives of the conjugacy classes of K which fuse into $[g]$ in G for which $\exists\, x_i \in H$ such that $[x_i]$ fuses into $[y_j]$ in K. Then $n \geq m$ and we obtain that

$$\chi^K(y_j) = |C_K(y_j)|\sum_i \frac{\chi(x_i)}{|C_H(x_i)|} \quad ,$$

where the summation is taken over all i for which $[x_i]$ fuses into $[y_j]$ in K. Then we obtain that

$$\begin{aligned}(\chi^K)^G(g) &= |C_G(g)|\sum_{j=1}^{m}\frac{\chi^K(y_j)}{|C_K(y_j)|} = |C_G(g)|\sum_{j=1}^{m}\sum_{i}\frac{\chi(x_i)}{|C_H(x_i)|}\\ &= |C_G(g)|\sum_{i=1}^{n}\frac{\chi(x_i)}{|C_H(x_i)|} = \chi^G(g).\end{aligned}$$

Hence the result. $\square$

Theorem 3.4.7 *Let G be a group, $H \leq G$ and $\{\chi_1, \chi_2, \ldots, \chi_n\}$ be a set of characters of H. Then*

$$(\sum_{i=1}^{n}\chi_i)^G = \sum_{i=1}^{n}\chi_i^G \quad .$$

Proof. Let $\{x_1, x_2, \ldots, x_m\}$ be a set of representatives of the conjugacy classes of H which fuse into $[g]$ of G. Then we obtain that

$$\begin{aligned}(\sum_{i=1}^{n}\chi_i)^G(g) &= |C_G(g)|\sum_{j=1}^{m}\frac{(\sum_{i=1}^{n}\chi_i)(x_j)}{|C_H(x_j)|} = |C_G(g)|\sum_{j=1}^{m}\frac{\sum_{i=1}^{n}\chi_i(x_j)}{|C_H(x_j)|}\\ &= |C_G(g)|\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{\chi_i(x_j)}{|C_H(x_j)|} = \sum_{i=1}^{n}|C_G(g)|\sum_{j=1}^{m}\frac{\chi_i(x_j)}{|C_H(x_j)|}\\ &= \sum_{i=1}^{n}\chi_i^G(g)\end{aligned}$$

Hence the result. $\square$

The above theorem asserts that the induction process of characters of a subgroup to the parent group is an additive operation. If $H \leq G$, χ a character of H and $\{g_1, g_2, \ldots, g_n\}$ is a transversal for H in G, then for any $g \in G$, we obtain that

$$\chi^G(g) = \sum_{i=1}^{n}\chi^\circ(g_i g g_i^{-1}) \quad .$$

Definition 3.4.8 *Let G be a group, $g \in G$ and p be a prime. If $p \not| \, o(g)$, then g is called p-regular. If $p \mid o(g)$, then g is called p-singular.*

Theorem 3.4.9 *[27] Let G be a group and p be a prime. Then every $g \in G$ can be uniquely expressed as $g = g_1g_2$, where $g_1, g_2 \in G$, g_1 and g_2 commute, g_1 is p-regular and g_2 is p-singular of order a power of p.*

Proof. Suppose that $o(g) = p^n q$ for some $n \in \mathbb{N} \cup \{0\}$ and $q \in \mathbb{N}$ such that $(p, q) = 1$. Let $a, b \in \mathbb{Z}$ such that

$$ap^n + bq = 1$$

and that $g_1 = g^{ap^n}$ and $g_2 = g^{bq}$. Then we have that $g_1, g_2 \in G$, $g_1 g_2 = g^{ap^n + bq} = g$ and that g_1 and g_2 commute since they are both powers of g. Moreover we obtain that $o(g_1) = q$ and $o(g_2) = p^n$. Hence g_1 is p-regular and g_2 is p-singular. Thus we have established at least one decomposition of g. Now suppose that $g = g_3 g_4$, where $g_3, g_4 \in G$, g_3 and g_4 commute and that g_3 is p-regular and g_4 is p-singular of order a power of p. Then $(o(g_3), o(g_4)) = 1$ and thus we obtain that $o(g_3 g_4) = o(g_3).o(g_4)$. Hence $o(g_3) = q$ and $o(g_4) = p^n$. However we have that

$$g_3 = g_3^{ap^n + bq} = g_3^{ap^n}.g_3^{bq} = g_3^{ap^n} = (gg_4^{-1})^{ap^n} = g^{ap^n}.g_4^{-ap^n} \quad ,$$

since g_4 and g_4^{-1} commute with g. Hence we obtain that

$$g_3 = g^{ap^n}.g_4^{-ap^n} = g^{ap^n} = g_1 \quad .$$

Similarly we obtain that

$$g_4 = g_4^{ap^n + bq} = g_4^{bq} = (g_3^{-1}g)^{bq} = g_3^{-bq}.g^{bq} = g^{bq} = g_2 \quad .$$

This establishes the uniqueness of the decomposition of $g \in G$. Hence the result. □

Definition 3.4.10 *Let G be a group and p be a prime. Let $H \leq G$ such that $H = A \times B$, where $A = \langle a \rangle$ and a is a p-regular element of G, and B is a p-subgroup of G. Then H is called a p-elementary subgroup of G.*

Lemma 3.4.11 *[27] Let G be a group and p be a prime. Then every cyclic subgroup of G is a p-elementary subgroup.*

Proof. Let $H = \langle g \rangle$, $g \in G$. Since we have that $g = g_1 g_2$, where g_1, g_2 are the p-regular and p-singular components of g respectively as given by Theorem 3.4.9, then we can write $H = \langle g \rangle = \langle g_1 \rangle \times \langle g_2 \rangle$. Hence result. □

From the above lemma and Definition 3.4.10, we can deduce that if H is a subgroup of G such that $H = A \times B$, where A is an arbitrary cyclic group and B is a p-group, then we have that

$$H = A \times B = \langle g \rangle \times B = \langle g_1 \rangle \times (\langle g_2 \rangle \times B) \quad ,$$

where g_1, g_2 are the p-regular and p-singular components of g respectively, such that $(\langle g_2 \rangle \times B)$ is a p-group. Hence H is a p-elementary subgroup of G.

Theorem 3.4.12 (**Brauer's Theorem on Induced Characters**) *Every complex character of a group G is a $\mathbf{Z}$-linear combination of characters induced from linear characters of p-elementary subgroups of G, for some prime p.*

Proof. See [27]. □

Brauer's theorem on induced characters asserts that every complex character χ of a group G satisfies the following relation

$$\chi = \sum k_i \psi_i^G \quad ,$$

where $k_i \in \mathbf{Z}$ and ψ_i are linear complex characters of p-elementary subgroups of G. Thus χ is a $\mathbf{Z}$-linear combination of monomial characters induced from p-elementary subgroups of G. In [113], Van Der Waall proved that every nonidentity irreducible character of a finite group G is a $\mathbf{Z}$-linear combination of monomial characters of G none of which contains the identity character of G as an irreducible constituent.

3.5 Permutation Characters

We say that a group G *acts* on a set X if there is a homomorphism $\phi : G \longrightarrow S_X$, where S_X is the symmetric group on X. We say that G acts faithfully on X if ϕ is a monomorphism. In this case G can be identified with a subgroup of S_X and G becomes a *permutation group* on X. In this section we assume that X is a finite set.

Definition 3.5.1 *Let G be a group acting on a set X such that for any two k-tuples $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ of k distinct elements of X, there exists $g \in G$ for which $x_i^g = y_i$ for $i = 1, 2, \ldots, k$. Then we say that G is k-transitive on X.*

If G is 1-transitive on X, then we say that G is transitive. In this case G has only one orbit on X.

If G acts on X, we define a representation $\pi : G \longrightarrow GL(n, \mathbb{C})$, where $n = |X|$. Let $X = \{x_1, x_2, \ldots, x_n\}$. For each $g \in G$ we define $\pi_g = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } x_i^g = x_j \\ 0 & \text{otherwise} \end{cases} .$$

Then π_g is a permutation matrix of the action of g. The representation π defined above is called the *permutation representation* of G obtained from the action of G on X.

Definition 3.5.2 *Let G be a group and X be a set such that G acts on X. Then we denote the character afforded by the permutation representation π by $\chi(G|X)$. This character is called the* permutation character *of G associated with the action of G on X. It is not difficult to show that for $g \in G$ we have*

$$\chi(G|X)(g) = |\{x \in X \mid x^g = x\} = \quad \textit{the number of points of } X \textit{ fixed by } g \quad .$$

Suppose that G acts transitively on X and G_x is the stabilizer of $x \in X$. Then the action of G on X is the same as the action of G on the cosets of $H = G_x$. Hence $\forall\ g \in G$, $\chi(G|X)(g)$ also gives the number of cosets of $H = G_x$ which are fixed by $g \in G$ and in this case we denote this number by $\chi(G|H)(g)$. Due to the fact that the action of G on X is the same as the action of G on the cosets of H, then we can write $\chi(G|H) = \chi(G|X)$.

Theorem 3.5.3 *Let G be a group acting transitively on a set X. Let $\alpha \in X$, $H = G_\alpha$ and $\chi(G|H)$ be the permutation character of this action. Then*

$$\chi(G|H) = (I_H)^G \quad .$$

Proof. We have that

$$(I_H)^G(g) = \frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} I_H(xgx^{-1}) = \frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} 1 \quad .$$

Now if $xgx^{-1} \in H$, then $xg \in Hx$. Thus $Hxg = Hx$ and hence Hx is fixed by $g \in G$. However the summation is taken over all $x \in G$ such that $xgx^{-1} \in H$. Hence the summation is taken over all $x \in G$ for which the coset Hx is fixed by $g \in G$. But $\forall\, y \in Hx$, $Hx = Hy$ and thus we obtain that

$$\sum_{x \in G, xgx^{-1} \in H} 1 = |H||\{Hx \mid Hxg = Hx\}|$$

and hence we obtain that

$$(I_H)^G(g) = \frac{1}{|H|}|H||\{Hx \mid Hxg = Hx\}| = |\{Hx \mid Hxg = Hx\}| = \chi(G|H)(g) \quad .$$

Hence the result. □

From the above theorem, we deduce that the permutation character of a group acting on the cosets of its subgroup is monomial since it is induced from the identity character of that subgroup. Thus a permutation character provides an example of a monomial character. Let $\chi(G|H)$ be a permutation character of G. Then we obtain that $\chi(G|H) = \sum \lambda_i \chi_i$, where $\lambda_i \in \mathbb{N} \cup \{0\}$ and $\chi_i \in Irr(G)$. If $\lambda_i \in \{0, 1\}$, then we say that $\chi(G|H)$ is *multiplicity-free*. Breuer and Lux in [13] classified all the multiplicity-free permutation characters of the sporadic simple groups and their automorphism groups. As we will see in the later chapters, the permutation characters of Fi_{22} on $2^6{:}SP(6,2)$ and $\overline{F}i_{22}$ on $2^7{:}SP(6,2)$ are multiplicity-free.

The following result will be used in later calculations to determine the conjugacy class fusions of subgroups of G.

Corollary 3.5.4 *Let $H \leq G$. Let $g \in G$ and let $x_1, x_2, \ldots, x_m$ be representatives of the conjugacy classes of H that fuse to $[g]$. Then*

$$\chi(G|H)(g) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_H(x_i)|} \quad .$$

Proof. This follows from Theorem 3.4.4. □

Corollary 3.5.5 *Let G act on X and $\chi(G|X)$ be the permutation character of the action. Let $X_1, X_2, \ldots, X_k$ be the orbits of G on X and $H_i = G_{x_i}$ be the stabilizer of $x_i \in X_i$ and $\chi_i(G|H_i)$ be the permutation character of G on the cosets of H_i. Then*

$$\chi(G|X) = \sum_{i=1}^{k} \chi_i(G|H_i) \quad .$$

Proof. From the stabilizers $H_1, \ldots, H_k$, we obtain that

$$\begin{aligned}
\chi(G|X)(g) &= \sum_{i=1}^{k} |\{H_i x \mid H_i x g = H_i x\}| \\
&= \sum_{i=1}^{k} |\{H_i x \mid H_i x g = H_i x,\ x \in G,\ x g x^{-1} \in H_i\}| \\
&= \sum_{i=1}^{k} \chi_i(G|H_i)(g)
\end{aligned}$$

Hence the result. □

Theorem 3.5.6 *[60] Let G be a group acting on a set X with $\chi(G|X)$ as the permutation character of the action. If X splits into k orbits under the action of G, then*

$$\langle \chi(G|X), I_G \rangle = k \quad .$$

Proof. Suppose that the k orbits of X under the action of G are $\{X_1, \ldots, X_k\}$. Then we obtain that

$$X = \bigcup_{i=1}^{k} X_i \quad .$$

Let $x_i \in X_i$ and H_i be the stabilizer of $x_i \in X_i$. Also let $\chi_i(G|H_i)$ be the permutation character of G on the cosets of H_i. Then we obtain that

$$\chi(G|X) = \sum_{i=1}^{k} \chi_i(G|H_i) \quad \text{where} \quad \chi_i(G|H_i) = (I_{H_i})^G \quad .$$

By the Frobenius reciprocity theorem, we obtain that

$$\langle \chi_i(G|H_i), I_G \rangle = \langle (I_{H_i})^G, I_G \rangle = \langle I_{H_i}, I_{H_i} \rangle = 1 \quad .$$

Hence we obtain that

$$\langle \chi(G|X), I_G \rangle = \sum_{i=1}^{k} \langle \chi_i(G|H_i), I_G \rangle = \sum_{i=1}^{k} 1 = k \quad .$$

Hence the result. □

From the above theorem, we observe that if G is a group acting on a finite set X and $\chi(G|X)$ is the permutation character associated with this action, then G is transitive on X if and only if $\langle \chi(G|X), I_G \rangle = 1$.

Let G act transitively on X. Then all subgroups G_x of G, for $x \in X$ are conjugate in G. If r is the number of orbits of G_x on X, then we say that the *rank* of G is r. It is clear that G is 2-transitive if and only if the rank of G is equal to 2.

Corollary 3.5.7 *Let G act transitively on X and $\chi(G|X)$ be the permutation character of the action. Let $x \in X$, $H = G_x$ be the stabilizer of x and r be the number of orbits of H on X. Then we obtain that*

$$\langle \chi(G|H), \chi(G|H) \rangle = r \quad .$$

Proof. By the Frobenius reciprocity, we obtain that

$$\langle \chi(G|H), \chi(G|H) \rangle = \langle \chi(G|H), (I_H)^G \rangle = \langle \chi(G|H)_H, I_H \rangle = r \quad .$$

Hence the result. $\square$

In the Corollary 3.5.7 if we let $\chi(G|X) = I_G + \sum_i \lambda_i \chi_i$, where $\chi_i \in Irr(G)$, then we have

$$r = \quad \text{rank of G} \quad = 1 + \sum_i \lambda_i^2 \quad .$$

In particular G is 2-transitive on X if and only if $\chi(G|X) = I_G + \chi$ for some irreducible character $\chi \neq I_G$.

In the following, we present without proof, some properties of permutation characters. These properties have been proved as Theorem 2.5.6 in [116]. Let G be a group, $H \leq G$ and $\chi = \chi(G|H)$. Then the following properties hold

(i) $deg(\chi)$ divides $|G|$.

(ii) $\langle \chi, \psi \rangle \leq deg(\psi)$ for all $\psi \in Irr(G)$.

(iii) $\langle \chi, I_G \rangle = 1$.

(iv) $\chi(g) \in \mathbb{N} \cup \{0\}$ for all $g \in G$.

(v) $\chi(g) \leq \chi(g^m)$ for all $g \in G$ and $m \in \mathbb{N} \cup \{0\}$.

(vi) $\chi(g) = 0$ if $o(g)$ does not divide $|G|/deg(\chi)$.

(vii) $\chi(g)\frac{|[g]|}{deg(\chi)}$ is an integer for all $g \in G$.

Theorem 3.5.8 *Let K be a proper subgroup of H where H is a proper subgroup of G. The set of all conjugates of K in G which are also subgroups of H splits into r conjugacy classes of subgroups of H. Let $K_1, K_2, \ldots, K_r$ be representatives of these r conjugacy classes of subgroups of H. Then the number of conjugates of H in G which contain K is given by*

$$\frac{1}{[N_G(H):H]}\sum_{i=1}^{r}[N_G(K):N_H(K_i)] \quad .$$

Proof. See [38] and [50]. □

Corollary 3.5.9 *[50] Let G be a finite group and H be a subgroup of G containing a fixed element x. Then the number h of conjugates of H in G which contain x is given by*

$$h = \frac{1}{[N_G(H):H]}\sum_{i=1}^{m}\frac{|C_G(x)|}{|C_H(x_i)|} \quad ,$$

where $x_1, x_2, \ldots, x_m$ are representatives of the conjugacy classes of elements of H which fuse into $[x]$ in G.

Proof. The number of conjugates of x in G and H are $[G : C_G(x)]$ and $[H : C_H(x)]$ respectively. However H contains $\sum_{i=1}^{m}[H : C_H(x_i)]$ conjugates of x in G. Then the result follows immediately by the previous theorem. □

Theorem 3.5.10 *[50] Let G be a finite group and H be a subgroup of G containing a fixed element x such that $(o(x), [N_G(H) : H]) = 1$. Then the number h of conjugates of H in G which contain x is $\chi(G|N_G(H))(x)$. In particular*

$$h = \sum_{i=1}^{m}\frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|} \quad .$$

where $x_1, x_2, \ldots, x_m$ are representatives of the conjugacy classes of elements of $N_G(H)$ which fuse to $[x]$ in G.

Proof. Let Ω be the set of all conjugates of the subgroup H in G. Then G acts transitively by conjugation on Ω and the point stablizer $G_H = N_G(H)$. Thus the permutation character of G acting on Ω is given by $\chi(G|N_G(H)) = (I_{N_G(H)})^G$. By definition

$$\chi(G|N_G(H))(x) = |\{H^g \mid (H^g)^x = H^g\}| = |\{H^g \mid x \in N_G(H^g)\}|$$

gives the number of fixed points of x in Ω. Let $\overline{x}$ be the image of x under the natural homomorphism $N_G(H^g) \longmapsto N_G(H^g)/H^g$. Since $(o(x), [N_G(H^g) : H^g]) = 1$, it follows that $o(\overline{x}) = 1$ and hence $x \in H^g$. Therefore $\chi(G|N_G(H))(x) = |\{H^g \mid x \in H^g\}|$. We also have that

$$\chi(G|N_G(H))(x) = (I_{N_G(H)})^G(x) = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|},$$

where $[x]_G \bigcap N_G(H) = \bigcup_{i=1}^{m} [x_i]_{N_G(H)}$. $\square$

Given a group G, then G acts on the cosets of all its subgroups H such that the permutation character associated with that particular action is given by $\chi(G|H) = (I_H)^G$. In view of this fact, the most natural questions to ask are as follows:

(i) Given two subgroups $H, K \leq G$, is it possible to have $(I_H)^G = (I_K)^G$?

(ii) If the answer to question (i) is in the affirmative, then if $H, K \leq G$ such that $(I_H)^G = (I_K)^G$, is H conjugate to K in G?

Indeed, the answer to question (i) is in the affirmative. However two subgroups H and K of a group G inducing the same permutation character does not necessarily guarantee that the two subgroups are conjugate in G. It can however happen under certain circumstances that H and K would be conjugate in G but there is no guarantee in general. The work of Caranti, Gavioli and Mattarei in [18] addresses question (ii) for finite p-groups, where p is prime. Feit in [36] and Guralnick in [54] established that if $H, K \leq G$ satisfy $(I_H)^G = (I_K)^G$ and have index a prime or the square of a prime, then they are conjugate with exceptions that can be described quite satisfactorily. Based on these results it follows that if G is a p-group for p prime and $[G : H] \leq p^2$, then H and K are conjugate. Guralnick in [54] provides an example of a finite p-group of order p^5 with two subgroups of index p^3 that are not conjugate but induce the same permutation character. In [18], the authors give a construction that for an odd prime p, there exists a p-group G, $|G| = p^7$ with $H, K \leq G$ of index p^3 where H is nonabelian and K is abelian such that they induce the same permutation character in G.

Chapter 4

Symplectic Groups

Classical linear groups are divided into three main categories, namely the symplectic, orthogonal and unitary groups. In this chapter, we shall concentrate on symplectic groups. We discuss the general theory of symplectic groups and their affine subgroups. One particular affine subgroup $2^5{:}S_6$ of the symplectic group $SP(6,2)$ has been studied in this thesis and is discussed in Chapter 6. The symplectic groups are constructed by defining some bilinear form on the underlying vector space and then taking all the form-preserving automorphisms of the space. Two of the groups studied in this thesis are split extensions of elementary abelian 2-groups by the symplectic group $SP(6,2)$ and are maximal subgroups of the smallest Fischer sporadic simple group Fi_{22} and its automorphism group $\overline{F}i_{22}$ respectively. The other group studied in this thesis is a split extension of an elementary abelian 2-group by the orthogonal group $O^-(6,2)$, where $O^-(6,2)$ is a maximal subgroup of $SP(6,2)$ of index 28. For further reading and information on symplectic groups, readers are encouraged to consult [10], [19], [29], [32], [51], [58], [57], [59] and [115].

4.1 Symplectic Forms

Definition 4.1.1 *Let V be a vector space over a field F and let $f : V \times V \longrightarrow F$ be a function such that for all $u, v, w \in V$ and all $\alpha, \beta \in F$ we have* *(i)* $f(\alpha u + \beta v, w) = \alpha f(u,w) + \beta f(v,w)$

(ii) $f(w, \alpha u + \beta v) = \alpha f(w,u) + \beta f(w,v)$

Then f is called a bilinear form on V. If f is a bilinear form on V such that for all $u \in V$ we have $f(u,u) = 0$, then f is called an alternating (symplectic) form on V. If f is a symplectic form on V such that for all $u \in V$, $u \neq 0$, there exists $v \in V$ for which $f(u,v) \neq 0$, then f is said to be non-degenerate.

Let V be a vector space and f be a symplectic form on V. Then we obtain that for all $u, v \in V$

$$f(u+v, u+v) = f(u+v, u) + f(u+v, v) = f(u,u) + f(v,u) + f(u,v) + f(v,v) \quad .$$

However we have that

$$f(u+v, u+v) = f(u,u) = f(v,v) = 0$$

and thus we obtain that $f(u,v) = -f(v,u)$.

4.2 Symplectic Spaces

Definition 4.2.1 *Let V be a vector space over a field F and f be a bilinear form on V such that*

(i) $f(u,u) = 0 \quad \forall\, u \in V$

(ii) $f(u,v) = -f(v,u) \quad \forall\, u, v \in V$

Then the pair (V, f) is called a symplectic space over the field F.

Remark 4.2.2 If $char(F) \neq 2$, then the properties (i) and (ii) in the above definition are equivalent. Moreover the symplectic space (V, f) becomes non-degenerate if f is non-degenerate.

Let (V, f) and (W, g) be symplectic spaces over the same field F, then we say that $V \cong W$ if and only if there exists $T \in L(V, W)$ an isomorphism such that $\forall\, u, v \in V$

$$f(u, v) = g(T(u), T(v)) \quad .$$

If $T \in L(V, V)$ is an isomorphism such that $\forall\, u, v \in V$

$$f(u, v) = f(T(u), T(v))$$

then T is called an *isometry* on (V, f).

Definition 4.2.3 *Let (V, f) be a symplectic space and U be a subspace of V. Then we define*

$$U^{\perp} = \{v \in V \mid f(u, v) = 0,\ \forall\, u \in U\} \quad .$$

Then $U^{\perp}$ is called the perpendicular space of U.

Note that for all $u \in U$ we have

$$f(0, u) = f(u - u, u) = f(u, u) - f(u, u) = 0 - 0 = 0 \quad ,$$

so that $0 \in U^{\perp}$. It is not difficult to show that $U^{\perp}$ is a subspace of V.

Let (V, f) be a symplectic space and define $R(V)$ by $R(V) = V^{\perp}$. Then we call $R(V)$ the *radical* of V.

Theorem 4.2.4 *Let (V, f) be a symplectic space. Then $R(V) = 0$ iff f is non-degenerate.*

Proof. Suppose that $R(V) = 0$. Let $u \in V,\ u \neq 0$. Then $u \notin R(V)$ and hence there is $v \in V$, such that $f(u, v) \neq 0$. Hence f is non-degenerate.
Conversely suppose that f is non-degenerate. Then for $u \in V,\ u \neq 0$, there is $v \in V$ such that $f(u, v) \neq 0$. Hence $u \notin R(V)$, for all $u \in V, u \neq 0$. Thus we obtain that $R(V) = 0$. $\square$

Let (V, f) be a symplectic space and U be a subspace of V. Then we obtain that

$$U \cap U^{\perp} = R(U) \quad .$$

Definition 4.2.5 *Let V be a vector space over a field F and f be a bilinear form on V such that for $u, v \in V$*

$$f(u,v) = f(v,u) \quad .$$

Then f is called an orthogonal form.

Let (V, f) be a symplectic space and $\{V_1, V_2, \ldots, V_n\}$ be subspaces of V such that

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

and that $f(v_i, v_j) = 0$ for all $v_i \in V_i$, $v_j \in V_j$ for which $i \neq j$. Then we say that V is an *orthogonal sum* of the subspaces $\{V_1, V_2, \ldots, V_n\}$ and we denote this by writing

$$V = V_1 \perp V_2 \perp \ldots \perp V_n \quad .$$

Theorem 4.2.6 *Let (V, f) be a symplectic space and $\{V_1, V_2, \ldots, V_n\}$ be subspaces of V such that*

$$V = V_1 \perp V_2 \perp \ldots \perp V_n \quad .$$

Then

$$R(V) = R(V_1) \perp R(V_2) \perp \ldots \perp R(V_n) \quad .$$

Proof. We have that $R(V)$ is a subspace of V. Now let $v \in R(V)$, then we obtain that $\forall\, u \in V$, $f(u,v) = 0$. Since V is an orthogonal sum of $\{V_1, V_2, \ldots, V_n\}$ and $v \in V$, then we obtain that

$$v = v_1 + v_2 + \cdots + v_n \quad ,$$

where $v_i \in V_i \; \forall\, 1 \leq i \leq n$. Then we obtain that for all $u \in V$

$$0 = f(u,v) = f(u, v_1 + v_2 + \cdots + v_n) = f(u, v_1) + f(u, v_2) + \cdots + f(u, v_n) \quad .$$

Now for all $u \in V_i$, we have $f(u, v_j) = 0 \; \forall\, i \neq j$ and hence

$$0 = 0 + 0 + \cdots + f(u, v_i) + 0 + \cdots + 0 \quad .$$

So that $v_i \in R(V_i)$. Hence we obtain that $v \in R(V_1) + R(V_2) + \cdots + R(V_n)$. Hence $R(V) = R(V_1) + R(V_2) + \cdots + R(V_n)$. However since $R(V_i)$ is a subspace of V_i and V is a direct sum of the V_i's, $1 \leq i \leq n$, then we obtain that

$$V_i \cap \sum_{j \neq i} V_j = 0$$

and hence we obtain that

$$R(V_i) \cap \sum_{j \neq i} R(V_j) = 0 \quad .$$

Therefore

$$R(V) = R(V_1) \oplus R(V_2) \oplus \cdots \oplus R(V_n) \quad .$$

If $v_i \in R(V_i)$ and $v_j \in R(V_j)$, then $v_i \in V_i$ and $v_j \in V_j$. Thus we obtain that

$$R(V) = R(V_1) \perp R(V_2) \perp \ldots \perp R(V_n) \quad .$$

Hence the result $\square$

Remark 4.2.7 Let (V, f) be a symplectic space and U be a subspace of V. Then we can also show that (see [58])

1. $dim(U^{\perp}) \geq dim(V) - dim(U)$.
2. if V is non-degenerate, then $dim(U^{\perp}) = dim(V) - dim(U)$.
3. if V is non-degenerate, there exists a linear isomorphism $\theta : V \longrightarrow \hat{V}$ given by $x^{\theta}(y) = f(x, y)$, where $\hat{V}$ is the dual space of V.
4. if U is non-degenerate, then $V = U \perp U^{\perp}$.

Definition 4.2.8 *Let (V, f) be a symplectic space and $u, v \in V$ such that $f(u, v) = 1$. Then the vectors $u, v \in V$ are called a hyperbolic pair and the 2-dimensional subspace of V generated by $\{u, v\}$ is called a hyperbolic plane.*

Remark 4.2.9 It is not difficult to see that every hyperbolic plane is non-degenerate.

Theorem 4.2.10 *[32] Let (V, f) be a non-degenerate finite dimensional symplectic space over a field F. If U is a subspace of V such that $U \cap U^{\perp} = 0$, then $V = U \oplus U^{\perp}$.*

Proof. Since V is finite dimensional, then U is finite dimensional. Let $\{u_1, u_2, \ldots, u_k\}$ be a basis for U. Then $U^{\perp}$ will be the collection of all vectors $v \in V$ for which $f(u_i, v) = 0, \ 1 \leq i \leq k$. Since V is non-degenerate, then $dim(U^{\perp}) = dim(V) -$

$dim(U)$ and thus we obtain that $dim(V) = dim(U^{\perp}) + dim(U)$. Since $U \cap U^{\perp} = 0$, then we obtain that

$$V = U \oplus U^{\perp} \quad .$$

□

Theorem 4.2.11 *Let (V, f) be a symplectic space over a field F such that $dim(V) = n$ and $dim(R(V)) = r$, then we obtain that*

$$V = H_1 \perp H_2 \perp \ldots \perp H_m \perp R(V) \quad ,$$

where $H_i, 1 \leq i \leq m$ are hyperbolic planes and we further have that $n - r = 2m$.

Proof. We have that $R(V)$ is a subspace of V. However if $R(V) = V$, then $m = 0$ and thus $n - r = 0 = 2 \times 0$ and the proof is complete. Thus W. L. O. G. suppose that $R(V) \neq V$. Then let $u \in V - R(V)$. Since $u \notin R(V)$, the $\exists\ w \in V$ such that $f(u, w) \neq 0$. Thus we can choose $v \in V$ such that $f(u, v) = 1$ and thus $\{u, v\}$ is a hyperbolic pair. This is true, because if $f(u, w) = k \neq 0$, then for $v = \frac{1}{k}w$ we have $f(u, v) = 1$. Now suppose that H_1 is the hyperbolic plane generated by $\{u, v\}$ and that $H_1^{\perp} = V_1$. Since H_1 is a hyperbolic plane, then it is non-degenerate and thus we obtain that $V = H_1 \perp V_1$ and we also have that $R(H_1) = 0$ and hence

$$R(V) = R(H_1) \perp R(V_1) = R(V_1) \quad .$$

We now apply induction on $dim(V) = n$. Since $H_1 = \langle u, v \rangle$, then $dim(H_1) = 2$ and thus we obtain that $dim(V) = n = dim(H_1) + dim(V_1)$. Thus we obtain that $dim(V_1) = n - 2 < dim(V)$ and thus by induction hypothesis we obtain that

$$V_1 = H_2 \perp H_3 \perp \ldots \perp H_m \perp R(V_1) \quad ,$$

where H_i, $2 \leq i \leq m$ are hyperbolic planes and that $2(m-1) = n - 2 - r$, thus we get $2m = n - r$. Since $R(V_1) = R(V)$ and $V = H_1 \perp V_1$, then we obtain that

$$V = H_1 \perp H_2 \perp H_3 \perp \ldots \perp H_m \perp R(V)$$

and that $n - r = 2m$. Hence the result. □

The following result shows that the dimension of a non-degenerate symplectic space must be even.

Corollary 4.2.12 *Let (V, f) be a non-degenerate symplectic space of dimension n over a field F. Then*

$$V = H_1 \perp H_2 \perp \ldots \perp H_m \quad ,$$

where H_i, $1 \leq i \leq m$ are hyperbolic planes and $n = 2m$.

Proof. By above theorem, we obtain that

$$V = H_1 \perp H_2 \perp \ldots \perp H_m \perp R(V) \quad .$$

However V is non-degenerate and thus $R(V) = 0$. Hence we obtain that

$$V = H_1 \perp H_2 \perp \ldots \perp H_m$$

and $dim(V) = n = 2m$. □

Let (V, f) be a symplectic space over a field F with $dim(V) = 2m$ and let the set $B = \{u_1, v_1, u_2, v_2, \ldots, u_m, v_m\}$ be a basis for V such that $\{u_i, v_i\}$ is a hyperbolic pair for all $1 \leq i \leq m$ and also such that for all $i \neq j$, $i, j \in \{1, 2, \ldots, m\}$ we have

$$f(u_i, u_j) = f(u_i, v_j) = f(v_i, v_j) = 0 \quad .$$

Then we call the set B a *hyperbolic basis* for V and we have that every non-degenerate symplectic space has a hyperbolic basis.

Theorem 4.2.13 *Let (V, f) be a non-degenerate symplectic space and $\{x_1, \ldots, x_r\}$ be a linearly independent set of elements of V such that $f(x_i, x_j) = 0 \ \forall\ i, j$. Then there is a linearly independent set $\{y_1, y_2, \ldots, y_r\}$ of elements of V such that*

$$V = H_1 \perp H_2 \perp \ldots \perp H_r \perp V_1 \quad ,$$

where V_1 is a subspace of V and H_i, $1 \leq i \leq r$ are hyperbolic planes and $2r \leq dim(V)$.

Proof. Since V is non-degenerate, $R(V) = 0$. Hence $x_1 \notin R(V)$ and as in Theorem 4.2.10 there is $y_1 \in V$ such that $f(x_1, y_1) = 1$. Let $H_1 = \langle x_1, y_1 \rangle$. Then H_1 is a hyperbolic plane. Since H_1 is non-degenerate, $V = H_1 \perp H_1^{\perp}$. Now $0 = R(V) = R(H_1) \perp R(H_1^{\perp})$ implies that $R(H_1^{\perp}) = 0$. Hence $H_1^{\perp}$ is non-degenerate. Since

$f(x_1, x_i) = 0$ for $2 \leq i \leq r$, $x_i \in H_1^{\perp}$. Hence $\{x_2, x_3, \ldots, x_r\} \subseteq H_1^{\perp}$. Since $dim(H_1^{\perp}) = dim(V) - 2 \leq dim(V)$, by induction there exists $\{y_2, y_3, \ldots, y_r\} \subseteq H_1^{\perp}$ such that

$$H_1^{\perp} = H_2 \perp H_3 \perp \ldots \perp H_r \perp V_1 \quad ,$$

where $H_i = \langle x_i, y_i \rangle, 2 \leq i \leq r$. Since $V = H_1 \perp H_1^{\perp}$, we have

$$V = H_1 \perp H_2 \perp H_3 \perp \ldots \perp H_r \perp V_1 \quad .$$

Therefore $dim(V) = 2r + dim(V_1)$ and hence $2r \leq dim(V)$. $\square$

Theorem 4.2.14 *[58](***Witt's Theorem***) Let (V, f) be a non-degenerate symplectic space and U_1, U_2 be two subspaces of V and $T : U_1 \longrightarrow U_2$ be an isometry. Then there exists an isometry $S : V \longrightarrow V$ such that $S \downarrow_{U_1} = T$.*

Proof. We have that

$$U_1 = H_1 \perp H_2 \perp \ldots \perp H_m \perp R(U_1) \quad ,$$

where H_i, $1 \leq i \leq m$ are hyperbolic planes. Thus we obtain that

$$T(U_1) = T(H_1) \perp T(H_2) \perp \ldots \perp T(H_m) \perp T(R(U_1)) = U_2 \quad .$$

If $H_i = \langle u_i, v_i \rangle$, then we obtain that

$$T(H_i) = T(\langle u_i, v_i \rangle) = \langle T(u_i), T(v_i) \rangle = H_i' \quad ,$$

and H_i' is a hyperbolic plane in U_2. We also obtain that $T(R(U_1)) = R(T(U_1)) = R(U_2)$. Thus we obtain that

$$U_2 = H_1' \perp H_2' \perp \ldots \perp H_m' \perp R(U_2) \quad .$$

Suppose that

$$H = H_1 \perp H_2 \perp \ldots \perp H_m \quad \text{and} \quad H' = H_1' \perp H_2' \perp \ldots \perp H_m' \quad ,$$

then we obtain that

$$U_1 = H \perp R(U_1) \quad \text{and} \quad U_2 = H' \perp R(U_2) \quad .$$

Since H_i, H_i', $1 \leq i \leq m$ are hyperbolic planes, then they are non-degenerate and hence $R(H_i) = R(H_i') = 0 \ \ \forall\, 1 \leq i \leq m$. Thus we obtain that $R(H) = R(H') = 0$ and hence H, H' are non-degenerate. Therefore we obtain that

$$V = H \perp H^{\perp} = H' \perp (H')^{\perp} \quad .$$

However since V is non-degenerate, then $R(V) = 0$ and thus

$$R(V) = R(H) \perp R(H^{\perp}) = R(H') \perp R((H')^{\perp})$$

and hence we obtain that $R(H^{\perp}) = R((H')^{\perp}) = 0$ and thus $H^{\perp}, (H')^{\perp}$ are non-degenerate. Since $H \subseteq U_1$ and $H' \subseteq U_2$, then $R(U_1) \subseteq H^{\perp}$ and $R(U_2) \subseteq (H')^{\perp}$. Let $\{x_1, x_2, \ldots, x_k\}$ be a basis for $R(U_1)$. Then $f(x_i, x_j) = 0 \ \ \forall\, i, j$. Thus by Theorem 4.2.12 there exists a linearly independent set $\{y_1, y_2, \ldots, y_k\}$ such that

$$H^{\perp} = K_1 \perp K_2 \perp \ldots \perp K_k \perp L \quad ,$$

where L is a subspace of $H^{\perp}$ and $K_i = \langle x_i, y_i \rangle$ such that $f(x_i, y_i) = 1$. Since T is an isometry, then $\{T(x_1), T(x_2), \ldots, T(x_k)\}$ is a linearly independent set in $T(R(U_1)) = R(U_2)$ and we also obtain that $0 = f(x_i, x_j) = f(T(x_i), T(x_j))$. Again by Theorem 4.2.12 there exists a linearly independent set $\{y_1', y_2' \ldots y_k'\}$ such that

$$(H')^{\perp} = K_1' \perp K_2' \perp \ldots \perp K_k' \perp L' \quad ,$$

where $K_i' = \langle T(x_i), y_i' \rangle$ such that $f(T(x_i), y_i') = 1$. Thus we obtain that

$$\begin{aligned} V &= H_1 \perp H_2 \ldots \perp H_m \perp K_1 \perp K_2 \ldots \perp K_k \perp L \\ &= H_1' \perp H_2' \ldots \perp H_m' \perp K_1' \perp K_2' \ldots \perp K_k' \perp L' \end{aligned}$$

and thus we obtain that $R(L) = R(L') = 0$ and

$$dim(V) = 2(m+k) + dim(L) = 2(m+k) + dim(L') \quad .$$

Thus we obtain that $dim(L) = dim(L')$. Hence there exists an isometry $M : L \longrightarrow L'$. Define a linear transformation $S : V \longrightarrow V$ by

$$S(h) = T(h) \ \forall\, h \in H, S(x_i) = T(x_i) \ \ 1 \leq i \leq k$$

$$S(y_i) = y_i' \ \ 1 \leq i \leq k, S(\ell) = M(\ell) \ \forall\, \ell \in L \quad .$$

Then S is an isometry on V and $S \downarrow_{U_1} = T$, where

$$U_1 = H_1 \perp H_2 \perp \ldots \perp H_m \perp \langle x_1, x_2, \ldots, x_k \rangle \quad .$$

□

4.3 Symplectic Groups

Let (V, f) be a non-degenerate symplectic space of dimension $2n$ over a field F. Then the set of all isometies of V forms a group which is called a *symplectic group* and is denoted by $SP(2n, F)$. If $F = GF(q)$ is a Galois field of q elements, where $q = p^r$ for some r and p is a prime, then we denote $SP(2n, F)$ by $SP(2n, q)$. We further obtain that $SP(2n, F) \leq GL(2n, F)$.

Theorem 4.3.1 *$SP(2n, F)$ is a transitive permutation group on the set of all hyperbolic pairs.*

Proof. $SP(2n, F)$ has a permutation representation on the set of all hyperbolic pairs $\{u, v\}$ given by $T \mapsto T_1$, where $T \in SP(2n, F)$ and

$$T_1 = \begin{pmatrix} \{u, v\} \\ \{T(u), T(v)\} \end{pmatrix} \quad .$$

Let $\{u_1, v_1\}$, $\{u_2, v_2\}$ be two hyperbolic pairs. Then we have that

$$f(u_1, v_1) = f(u_2, v_2) = 1 \quad .$$

Thus there is a linear automorphism T such that $T(u_1) = u_2$ and $T(v_1) = v_2$. Let

$$H_1 = \langle u_1, v_1 \rangle \quad \text{and} \quad H_2 = \langle u_2, v_2 \rangle \quad .$$

Then we observe that $T : H_1 \longrightarrow H_2$ is an isometry. By Witt's Theorem, there is an isometry $S \in SP(2n, F)$ such that $S \downarrow_{H_1} = T$. Hence $SP(2n, F)$ is transitive on the set of all hyperbolic pairs. $\square$

Theorem 4.3.2 *[58] Let (V, f) be a non-degenerate symplectic space of dimension $2n$ over $GF(q)$. Then the number of hyperbolic pairs of V is $q^{2n-1}.(q^{2n} - 1)$.*

Proof. We observe that $|V| = q^{2n}$. Let $\{u, v\}$ be a hyperbolic pair. Then we have that $f(u, v) = 1$ and hence $u \in V^*$. Thus we have that $dim(\langle u \rangle) = 1$ and hence we obtain that $dim(\langle u \rangle^{\perp}) = 2n - 1$. Therefore the number of elements of V which are not in $\langle u\rangle^{\perp}$ is $q^{2n}-q^{2n-1}$. Since $|F^{*}|=q-1$, the number of elements $v\in V$ for which $f(u,v)=1$ is given by $\frac{q^{2n}-q^{2n-1}}{q-1}$. Thus the number of hyperbolic pairs is given by

$$(q^{2n}-1).(\frac{q^{2n}-q^{2n-1}}{q-1})=(q^{2n}-1).q^{2n-1}(\frac{q-1}{q-1})=q^{2n-1}.(q^{2n}-1) \quad .$$

□

Theorem 4.3.3 *[32] Let (V,f) be a non-degenerate symplectic space of dimension $2n$ over $GF(q)$. Then*

$$|SP(2n,q)|=q^{n^2}\prod_{i=1}^{n}(q^{2i}-1) \quad .$$

Proof. Since V is non-degenerate, then there is a hyperbolic basis for V. Let $\{u_1,v_1,u_2,v_2,\ldots,u_n,v_n\}$ be a fixed hyperbolic basis for V. Let $T\in SP(2n,q)$ and since T is an isometry, we have $\{T(u_1),T(v_1),T(u_2),T(v_2),\ldots,T(u_n),T(v_n)\}$ is a hyperbolic basis for V. Thus we obtain that $|SP(2n,q)|$ is the number of hyperbolic bases for V. Then we apply induction on n, to count the number of ways of choosing hyoerbolic bases for V. There are $q^{2n-1}.(q^{2n}-1)$ ways of choosing a hyperbolic pair $u_1,v_1\in V$. Let $H_1=\langle u_1,v_1\rangle$, then the restriction $\overline{f}$ of f to $H_1^{\perp}$ is non-degenerate and thus making $(H_1^{\perp},\overline{f})$ into a non-degenerate symplectic space. Thus the remaining vectors of the hyperbolic basis for V may be chosen as a hyperbolic basis for $(H_1^{\perp},\overline{f})$. Since $dim(H_1^{\perp})=2n-2$, the number of hyperbolic bases for $(H_1^{\perp},\overline{f})$ is equal to $|SP(2n-2,q)|$. Hence we obtain that

$$\begin{aligned}|SP(2n,q)| &= q^{2n-1}.(q^{2n}-1).|SP(2n-2,q)|\\ &= q^{2n-1}.(q^{2n}-1).q^{(n-1)^2}\prod_{i=1}^{n-1}(q^{2i}-1)\\ &= q^{n^2}\prod_{i=1}^{n}(q^{2i}-1).\end{aligned}$$

Hence the result. □

If V is a $2n$-dimensional non-degenerate symplectic space over a field F, and $SP(2n,F)$ the symplectic group of isometries of V, then the centre $Z(SP(2n,F)$ of $SP(2n,F)$ consists of the transformations $T=kI$, where $k=\pm 1$. This is true since a symplectic transformation necessarily has determinant 1. The factor group $SP(2n, F)/Z(SP(2n, F))$ is called the *projective symplectic group* and is denoted by $PSP(2n, F)$. The projective symplectic groups are generally simple. In fact they are all simple except for $PSP(2,2) = PSL(2,2)$, $PSP(2,3) = PSL(2,3)$ and $PSP(4,2)$. If $F = GF(q)$, then $SP(2n, F)$ and $PSP(2n, F)$ are denoted by $SP(2n, q)$ and $PSP(2n, q)$ respectively. In this case $Z(SP(2n, q)) = \{I\}$ if $char(F) = 2$ and $Z(SP(2n, q)) = \{I, -I\}$ if $char(F) \neq 2$. Thus

$$|PSP(2n, q)| = \frac{1}{(2, q-1)} \times |SP(2n, q)| = \frac{q^{n^2}}{(2, q-1)} \prod_{i=1}^{n} (q^{2i} - 1) \quad .$$

If V is a vector space of dimension n and H is a subspace of V of dimension $n-1$, then we say that H is a *hyperplane* of V. If $F = GF(q)$ and H is a hyperplane in V, then H contains q^{n-1} points.

Definition 4.3.4 *Let V be a non-degenerate symplectic space over a field F and $T \in SP(2n, F)$, $T \neq I$ such that for some hyperplane H of V, we have*

(i) $T(h) = h \quad \forall\, h \in H$

(ii) $T(x) - x \in H \quad \forall\, x \in V - H$

Then T is called a symplectic transvection of V.

Theorem 4.3.5 *[58] Let T be a symplectic transvection with hyperplane H. Then there is a non-zero $w \in V$ such that $H = \langle w\rangle^{\perp}$ and for all $v \in V$ we have $T(v) = v + cf(w, v)w$ for $c \in F$. Conversely for $w \neq 0, w \in V$ and $0 \neq c \in F$ define $T : V \longrightarrow V$ by $T(v) = v + cf(w, v)w$ for all $v \in V$. Then T is a symplectic transvection with hyperplane $\langle w\rangle^{\perp}$.*

Proof. Let $x \in V - H$. Since T is nonidentity, $T(x) - x \neq 0$. Let $y \in H$ such that $y \neq 0$ and $T(x) - x = y$. Since H is a hyperplane and $x \notin H$, $V = \langle x\rangle \oplus H$. Then $dim(H^{\perp}) = 1$ and hence $H^{\perp} = \langle w\rangle$ for some $w \neq 0, w \in V$. Define $\phi : V \longrightarrow F$ by $\phi(v) = \phi(\lambda x + h) = \lambda$ and it can be shown that ϕ is a linear functional, so there is $z \in V$ such that $z^\theta = \phi$, where $\theta : V \longrightarrow \hat{V}$ is the linear isomorphism given in Remark 4.2.7 part 3. For all $v \in V$, $\phi(v) = z^\theta(v) = f(z, v)$,

$$\begin{aligned} T(v) &= T(\lambda x + h) = \lambda T(x) + T(h) = \lambda T(x) + h = \lambda(y + x) + h \\ &= \lambda y + \lambda x + h = v + \lambda y = v + \phi(v)y = v + f(z, v)y. \end{aligned}$$

Now for all $h \in H$,

$$f(h, x) = f(T(h), T(x)) = f(h, x + y) = f(h, x) + f(h, y) \quad .$$

So $f(h, y) = 0$ for all $h \in H$, that is $y \in H^\perp$. Since $H = \langle w\rangle^\perp$ then $y \in \langle w\rangle$, so that $y = c_1 w$ for some $c_1 \in F$. Since $y \neq 0$, then $c_1 \neq 0$. Therefore $T(v) = v + c_1 f(z, v)w$. Since $0 = \phi(h) = f(z, h)$ for all $h \in H$, $z \in H^\perp$ and thus $z = c_2 w$ for some $c_2 \in F$. Hence $T(v) = v + c_1 c_2 f(w, v)w$.

Conversely for $0 \neq c \in F$ and $0 \neq w \in V$, define $T : V \longrightarrow V$ by $T(v) = v + cf(w, v)w$ for all $v \in V$. It can be shown that $T \in SP(2n, F)$. Let $H = \langle w\rangle^\perp$ then for $h \in H$, $T(h) = h + cf(w, h)w = h + 0 = h$ and if $v \in V$ then $T(v) - v = cf(w, v)w = kw$ for some $k \in F$. Since $\langle w\rangle \subseteq \langle w\rangle^\perp$, then $T(v) - v \in \langle w\rangle^\perp = H$. Therefore T is a symplectic transvection with the hyperplane $H = \langle w\rangle^\perp$. □

If T is a transvection, then by theorem 4.3.5 there exists $c \in F^*$ and $w \in V^*$ such that $T = T_{c,w}$. For $T = T_{c,w}$ we say that T is a transvection in direction w. Let X be the set of all symplectic transvections of V. Then it can be shown that $\langle X\rangle$ is transitive on V^* and on hyperbolic pairs. (See [58])

Theorem 4.3.6 *[58] $SP(2n, F)$ is generated by the set of all symplectic transvections.*

Proof. For $n = 1$, we obtain that $SP(2, F) \cong SL(2, F)$ and that $SL(V) = \langle X\rangle$ by Proposition 2.4.6 of [10] and the proof is complete. Suppose that $n > 1$ and let $\{x, y\}$ be a hyperbolic pair and $S \in SP(2n, F)$. Then $\{S(x), S(y)\}$ is also a hyperbolic pair. Since $\langle X\rangle$ is transitive on hyperbolic pairs, then there exists $T \in \langle X\rangle$ such that

$$T(x) = S(x) \quad \text{and} \quad T(y) = S(y) \quad .$$

Let $P = T^{-1}S : \{x, y\} \longrightarrow \{x, y\}$ and $H = \langle x, y\rangle$. Then $V = H \perp H^\perp$. Since P fixes H, then $P(H^\perp) = H^\perp$ and thus P also fixes $H^\perp$. Thus we obtain that

$P \downarrow_{H^\perp} = P'$ is an isometry on $H^\perp$. Now suppose the result is true for all symplectic spaces whose dimensions are less than $2n$. Since $dim(H^\perp) = 2n-2$, then by the induction hypothesis

$$P' = \prod_i T_i' \quad ,$$

where T_i''s are symplectic transvections of $H^\perp$. Now we define $T_i : V \longrightarrow V$ by $T_i(h+h') = h + T_i'(h')\ \forall\ h \in H,\ h' \in H^\perp$ and all indices i. If T_i' is a transvection with hyperplane $\langle h_i' \rangle^\perp \cap H^\perp$, where $h_i' \in H^\perp$, then T_i will also be a transvection with hyperplane $H^\perp(\langle h_i' \rangle^\perp \cap H^\perp)$. Since

$$P' = \prod_i T_i' \quad \text{and} \quad P = T^{-1}S \quad ,$$

then we obtain that

$$S = \prod_i TT_i$$

and thus $S \in \langle X \rangle$. Hence the result. □

Corollary 4.3.7 *$SP(2n, F)$ is transitive on V^*.*

Proof. The result follows immediately since $SP(2n, F)$ is generated by the set of all symplectic transvections of V. □

All elements of $SP(2n, F)$ have determinant 1. We can also show that $SP(2n, F)$ is perfect except for the cases $SP(2,2), SP(2,3)$ and $SP(4,2) \cong S_6$. The isomorphism between $SP(4,2)$ and S_6 has been discussed in some detail in [58].

Theorem 4.3.8 *Let q be a power of an odd prime p. Then $SP(2n, q)$ has irreducible characters ψ_1 and ψ_2 of degrees $(q^n+1)/2$ and $(q^n-1)/2$ respectively. Moreover*

$$|\psi_1(x) + \psi_2(x)|^2 = |C_V(x)|$$

for all $x \in SP(2n, q)$ and $V = V(2n, q)$ is the natural module of $SP(2n, q)$.

Proof. See Theorem 4.8 of [59]. □

4.4 The Affine Subgroups of Symplectic Groups

Let V be a vector space of dimension $2n$ over $GF(q)$, where q is a power of a prime p. Then $SP(2n,q)$ is transitive on the nonzero points of V. Then we consider the subgroup of $SP(2n,q)$ which is a stabilizer of a nonzero vector of V and study the structure of this subgroup.

Definition 4.4.1 *Let $\{e_1, e_2, \ldots, e_{2n}\}$ be a basis for V and f be a non-degenerate symplectic form $f : V \times V \longrightarrow F$ defined by $f(e_i, e_j) = \delta(i, 2n+1-j)$, where $i \leq j$. Then (V, f) is a non-degenerate symplectic space of dimension $2n$. Let T be an isometry of (V, f) and*

$$G(n) = SP(2n,q) = \{T \mid f(T(x), T(y)) = f(x,y) \ \forall x, y \in V\} \quad .$$

Then $G(n)$ acts transitively on V^*. Let $\alpha \in V^*$ and $A(n)$ be the stabilizer of α in $G(n)$. Then we obtain that

$$A(n) = \{T \in G(n) \mid T(\alpha) = \alpha\} \quad .$$

Then $A(n) \leq G(n)$ and $A(n)$ is called the affine subgroup of $G(n)$.

Remark 4.4.2 In any finite dimensional non-degenerate symplectic space (V, f) we can find a suitable basis such that f can be given as in Definition 4.4.1 above.

Since $A(n) \leq G(n)$, then we obtain that $[G(n){:}A(n)] = q^{2n} - 1$ and $A(n)$ is the subgroup of $G(n)$ that fixes a nonzero vector $\alpha \in V^*$.

Let G be a group. Then the intersection of all maximal subgroups of G is itself a subgroup of G. We denote this intersection by $\Phi(G)$ and write

$$\Phi(G) = \bigcap_{M \overset{\max}{\leq} G} M \quad .$$

Then $\Phi(G)$ is called the *Frattini subgroup* of G. However we have that $\Phi(G)$ is a normal subgroup of G. Now suppose that $G = P$ is a p-group. Then $P' \leq \Phi(P)$. We say that P is a *special p-group* if we have that $Z(P) = P' = \Phi(P)$ is elementary abelian.

Lemma 4.4.3 *[53] Let q be a power of an odd prime p. Then $A(n)$ is a split extension of a special p-group $P(n)$ of order q^{2n-1} by a subgroup H of $G(n)$ such that $H \cong G(n-1) \cong SP(2n-2, q)$.*

Proof. We have that the symplectic form f can be given by $f(e_i, e_j) = \delta(i, 2n+1-j)$, where $i \leq j$ and $\{e_1, e_2, \dots, e_{2n}\}$ is a suitable basis for V. Since $G(n)$ acts transitively on V^*, then we let $A(n)$ to be the stabilizer of e_1 in $G(n)$. Thus we have $A(n) = \{T \in G(n) \mid T(e_1) = e_1\}$. Let $P(n)$ be the subgroup of $A(n)$ consisting of elements $T \in G(n)$, such that

$$T(e_1) = e_1$$

$$T(e_i) = \alpha_i e_1 + e_i, \quad 2 \leq i \leq 2n-1$$

$$T(e_{2n}) = \sum_{i=1}^{2n} \beta_i e_i$$

where β_1 is arbitrary and

$$\alpha_j = \begin{cases} -\beta_{2n+1-j} & 2 \leq j \leq n \\ \beta_{2n+1-j} & n < j \leq 2n-1 \end{cases} .$$

Let H be the subgroup of $A(n)$ which fixes e_{2n}. Then H fixes both e_1 and e_{2n} and acts on $W = \langle e_2, e_3, \dots, e_{2n-1} \rangle$ as $G(n-1)$ and we obtain that $H \cong G(n-1) \cong SP(2n-2, q)$. It can be shown that H is a complement of $P(n)$ in $A(n)$. Hence we obtain that

$$A(n) = P(n){:}H = P(n){:}SP(2n-2, q) .$$

□

Remark 4.4.4 It is not difficult to see that if $p = 2$, then $P(n)$ is an elementary abelian 2-group.

Theorem 4.4.5 *[53] Let q be a power of an odd prime p. Then for any $i \in \mathbb{Z}$ satisfying $1 \leq i \leq n-1$, $A(n)$ has non-faithful irreducible characters of degree $(q^{2n-2} - 1) \cdots (q^{2n-2i} - 1)$. The kernel of these characters is the centre $Z(P(n))$ of $P(n)$.*

Proof. The existence of these characters follows by induction similar to the proof of Theorem 1 in [53]. $G(n-1)$ acts transitively on the non-identity linear characters of $P(n)$ and thus we can take the subgroup fixing such a character to be $A(n-1)$. $Z(P(n))$ sits in the kernel of any of the characters. However $P(n)/Z(P(n))$ is the unique minimal normal subgroup of $A(n)/Z(P(n))$ and $P(n)$ cannot be contained in the kernel of any character. Hence these kernels cannot be larger than $Z(P(n))$ and therefore they must be equal to $Z(P(n))$. Hence the result. □

For q a power of an odd prime p, then $P(n)$ has $q-1$ irreducible characters of degree q^{n-1} and these are all invariant under the action of $G(n-1)$.

Theorem 4.4.6 *Let q be a power of 2. Then $A(n)$ has non-faithful irreducible characters of degree $(q^{2n-2}-1)\cdots(q^{2n-2i}-1)$ for any $i \in \mathbb{Z}$ satisfying $1 \leq i \leq n-1$.*

Proof. The proof is similar to Theorem 4.4.5 for the odd characteristic case although the subgroup $P(n)$ is now elementary abelian. □

Let q be a power of 2. Then there are two different quadratic forms, denoted by f^+ and f^- which can be defined on V. The two groups leaving these forms invariant are denoted by $O^+(2n,q)$ and $O^-(2n,q)$ respectively and they are subgroups of $GL(2n,q)$ which sit maximally in $SP(2n,q)$. The groups $O^+(2n,q)$ and $O^-(2n,q)$ are orthogonal groups.

Since $A(n) = P(n){:}G(n-1)$, where $G(n-1) \cong SP(2n-2,q)$, then the two orthogonal groups which sit inside $G(n-1)$ are $O^+(2n-2,q)$ and $O^-(2n-2,q)$. Hence we can obtain two characters of $A(n)$ of degree $[G(n-1):O^+(2n-2,q)]$ and $[G(n-1):O^-(2n-2,q)]$. These characters are irreducible with degrees $\frac{1}{2}q^{n-1}(q^{n-1}+1)$ and $\frac{1}{2}q^{n-1}(q^{n-1}-1)$ respectively. (See Theorem 4 of [53]). We can also obtain further characters of $A(n)$ by using the characters of $O^+(2n-2,q)$ and $O^-(2n-2,q)$. For example these orthogonal groups have each a character of degree $q^{(n-1)(n-2)}$ which is known as the *Steinberg character.* Using the Steinberg character of these groups, we can obtain characters of $A(n)$ of degrees $\frac{1}{2}q^{(n-1)^2}(q^{n-1}\pm 1)$.

Remark 4.4.7 Let $q = 2^k$ for some $k \in \mathbb{N}$. Then $P(n)$ is an elementary abelian 2-group. The group $A(n)$ has $2q$ orbits $\Delta_1, \ldots, \Delta_{2q}$ on $P(n)$ such that

$$|\Delta_1| = |\Delta_2| = \cdots = |\Delta_q| = 1$$

$$|\Delta_{q+1}| = |\Delta_{q+2}| = \cdots = |\Delta_{2q}| = q^{2n-2} - 1 \quad .$$

Furthermore the action of $A(n)$ on $Irr(P(n))$ produces $2q$ orbits $\Gamma_1, \ldots, \Gamma_{2q}$ such that

$$|\Gamma_1| = 1 \quad \text{and} \quad |\Gamma_2| = q^{2n-2} - 1$$

$$|\Gamma_3| = |\Gamma_4| = \cdots = |\Gamma_{q+1}| = \frac{1}{2}q^{n-1}(q^{n-1}+1)$$

$$|\Gamma_{q+2}| = |\Gamma_{q+3}| = \cdots = |\Gamma_{2q}| = \frac{1}{2}q^{n-1}(q^{n-1}-1)$$

with corresponding stabilizers as:

$$G(n-1);\ A(n-1);\ O^{+}(2n-2,q), q-1 \ \ copies;\ O^{-}(2n-2,q), q-1 \ \ copies \quad .$$

The corresponding indices of these stabilizers in $G(n-1)$ are:

$$1;\ q^{2n-2}-1;\ \frac{1}{2}q^{n-1}(q^{n-1}+1), q-1 \ \ copies;\ \frac{1}{2}q^{n-1}(q^{n-1}-1), q-1 \ \ copies \quad .$$

Chapter 5

The Fischer-Clifford Matrices

Character tables of finite groups can be constructed using various techniques. However B. Fischer studied a technique which can be used to construct character tables of group extensions. This technique derives its fundamentals from the Clifford Theory. This technique which is known as the technique of the *Fischer-Clifford* matrices, provides very powerful information for constructing character tables. In this thesis we apply this technique mainly to split extensions. Given a group extension $\overline{G} = N{\cdot}G$ such that every irreducible character of N can be extended to its inertia group then for each class representative $g \in G$, we are able to construct a matrix $M(g)$ called the Fischer-Clifford matrix. By using these matrices together with the fusion maps and character tables of some subgroups of G which are inertia factors of the inertia groups in $\overline{G}$, we are able to construct the complete character table of $\overline{G}$. In this chapter, we shall discuss the theory behind the technique of the Fischer-Clifford matrices. We shall however begin by discussing the Clifford Theory and then go on to discuss the theory of the Fischer-Clifford matrices. Then the character table of $\overline{G}$ can be constructed using these matrices and the character tables of factor groups of the inertia groups. This technique has also been discussed and used in [30], [31], [41], [42], [43], [75], [76], [98], [106] and [116]. In the subsequent chapters, we will use this technique and other group theoretic and character theoretic information that have been discussed in the previous chapters to construct the character tables of the groups which have been studied in this thesis. For the Fischer-Clifford matrices, we shall follow the work of Whitley [116] very closely.

5.1 The Clifford Theory

Definition 5.1.1 *Let G be a group, $H \leq G$ and θ be a character of H. Then for $g \in G$, we define $\theta^g : gHg^{-1} \longrightarrow \mathbb{C}$ by $\theta^g(t) = \theta(gtg^{-1})$ for all $t \in gHg^{-1}$. Then θ^g is said to be a G-conjugate of θ. If H is a normal subgroup of G and $\theta^g = \theta$ for all $g \in G$, then θ is said to be G-invariant.*

If $H \leq G$ and $g \in G$, then θ^g is a character of gHg^{-1}. However if H is normal in G, θ^g becomes a character of H.

Remark 5.1.2 Let G be a group, H a normal subgroup of G and θ a character of H. Then for $g \in G$, it is not difficult to see that $\theta^g \in Irr(H)$ if and only if $\theta \in Irr(H)$.

Theorem 5.1.3 *[60](***Clifford's Theorem***) Let G be a group, H a normal subgroup of G and $\chi \in Irr(G)$. Let θ be an irreducible constituent of χ_H and that $\theta_1, \theta_2, \ldots, \theta_n$ are the distinct conjugates of θ in G such that $\theta_1 = \theta$. Then*

$$\chi_H = e\sum_{i=1}^{n}\theta_i, \quad where \quad e = \langle \chi_H, \theta\rangle \quad .$$

Proof. We have that for $h \in H$

$$\theta^G(h) = \frac{1}{|H|}\sum_{x\in G}\theta^\circ(xhx^{-1}) = \frac{1}{|H|}\sum_{x\in G}\theta^x(h) \quad .$$

Thus we obtain that

$$(\theta^G)_H = \frac{1}{|H|}\sum_{x\in G}\theta^x \quad .$$

Let $\phi \in Irr(H)$ such that $\phi \notin \{\theta_i \mid 1 \leq i \leq n\}$. Then we obtain that

$$\langle\sum_{x\in G}\theta^x, \phi\rangle = 0$$

and hence $\langle(\theta^G)_H, \phi\rangle = 0$. However by the Frobenius reciprocity, we obtain that $\langle\chi_H, \theta\rangle = \langle\chi, \theta^G\rangle$. Hence χ is an irreducible constituent of θ^G. Since $\langle(\theta^G)_H, \phi\rangle = 0$, then $\langle\chi_H, \phi\rangle = 0$. Thus ϕ is not an irreducible constituent of χ_H. Hence all the irreducible constituents of χ_H are among the θ_i and thus we obtain that

$$\chi_H = \sum_{i=1}^{n}\langle\chi_H, \theta_i\rangle\theta_i = \sum_{i=1}^{n}\langle\chi_H, \theta\rangle\theta_i = \langle\chi_H, \theta\rangle\sum_{i=1}^{n}\theta_i = e\sum_{i=1}^{n}\theta_i \quad ,$$

where $e = \langle \chi_H, \theta \rangle$. Hence result $\square$

Clifford's Theorem asserts that for H a normal subgroup of G, $\chi \in Irr(G)$ and $\theta \in Irr(H)$ an irreducible constituent of χ_H, then every G-conjugate of θ will also be an irreducible constituent of χ_H.

Theorem 5.1.4 *[70] Let G be a group, $K, H \leq G$ such that $K \leq H \leq G$ and χ be a character of K. Then for all $g \in G$ we have*

(i) $(\chi^H)^g = (\chi^g)^{g^{-1}Hg}$

(ii) $(\chi^g)^G = \chi^G$.

Proof. (i) Let T be a transversal for K in H. Then gTg^{-1} is a transversal for gKg^{-1} in gHg^{-1}. Define χ° as follows

$$\chi^\circ(x) = \begin{cases} \chi(x) & x \in K \\ 0 & x \notin K \end{cases} .$$

Let $\lambda = \chi^{g^{-1}}$, then define λ° similarly as follows

$$\lambda^\circ(x) = \begin{cases} \lambda(x) & x \in gKg^{-1} \\ 0 & x \notin gKg^{-1} \end{cases} .$$

We obtain that $x \in gKg^{-1}$ if and only if $g^{-1}xg \in K$ and thus we obtain that $\lambda^\circ(x) = (\chi^\circ)^{g^{-1}}(x) = \chi^\circ(g^{-1}xg)$ for all $x, g \in G$. Thus for any $x \in gHg^{-1}$, we obtain that

$$\begin{aligned} \lambda^{gHg^{-1}}(x) &= \sum_{t \in T} \lambda^\circ((gtg^{-1})x(gtg^{-1})^{-1}) = \sum_{t \in T} \chi^\circ(g^{-1}(gtg^{-1})x(gtg^{-1})^{-1}g) \\ &= \sum_{t \in T} \chi^\circ(t(g^{-1}xg)t^{-1}) = \chi^H(g^{-1}xg) = (\chi^H)^{g^{-1}}(x) \end{aligned}$$

Hence we obtain that $(\chi^H)^{g^{-1}}(x) = \lambda^{gHg^{-1}}(x) = (\chi^{g^{-1}})^{gHg^{-1}}(x)$, for all $x, g \in G$ and therefore we have that $(\chi^H)^g = (\chi^g)^{g^{-1}Hg}$. Hence (i) is established.

(ii) We know that $\chi^G = (\chi^H)^G$. Thus

$$\chi^G(x) = \frac{1}{|K|} \sum_{t \in G} \chi^\circ(txt^{-1}) \quad ,$$

where $txt^{-1} \in K$. Also we obtain that

$$(\chi^G)^g(x) = \chi^G(gxg^{-1}) = \frac{1}{|K|}\sum_{y\in G}\chi^\circ(y(gxg^{-1})y^{-1}) = \frac{1}{|K|}\sum_{y\in G}\chi^\circ(ygx(yg)^{-1}) \quad .$$

Taking $t = yg$, then we obtain that

$$(\chi^G)^g(x) = \frac{1}{|K|}\sum_{t\in G}\chi^\circ(txt^{-1}) = \chi^G(x) \quad .$$

Thus we obtain that $(\chi^G)^g = \chi^G$. However by (i) above, we have that $(\chi^G)^g = (\chi^g)^{g^{-1}Gg} = (\chi^g)^G$. Hence we obtain that $(\chi^g)^G = \chi^G$ and (ii) is established. □

Let ϕ be a representation of G and α an automorphism of G. Then ϕ^α is a representation of G given by

$$\phi^\alpha(x) = \phi(x^\alpha) \quad \text{and} \quad \phi^\alpha(xy) = \phi^\alpha(x)\phi^\alpha(y)$$

for $x, y \in G$. If the representation ϕ affords a character χ of G, then the representation ϕ^α affords a character χ^α of G which is given by $\chi^\alpha(x) = \chi(x^\alpha)$ for $x \in G$. Then the representation ϕ^α and the character χ^α are called the *algebraic conjugates* of ϕ and χ respectively induced by the automorphism α. Let $X = (\chi_i(x_j))$ be the character table of G, where $\chi_i \in Irr(G)$, $1 \leq i \leq n$ and x_j, $1 \leq j \leq n$ are representatives of the conjugacy classes of elements of G. Then the automorphism α of G induces a permutation on the conjugacy classes of G and thus induces a permutation on the columns of X. For each $\chi_i \in Irr(G)$, we deduce that $\chi_i^\alpha \in Irr(G)$. Hence α induces a permutation on the irreducible characters χ_i of G and thus induces a permutation on the rows of X. Moreover since $\chi_i^\alpha(x_j) = \chi_i(x_j^\alpha)$, then the matrices obtained from X by these two operations are identical. Hence we obtain the following theorem known as Brauer's Theorem.

Theorem 5.1.5 *[52](***Brauer's Theorem***) Let G be a group and K be a group of automorphisms of G. Then the number of orbits of K as a group of permutations on the irreducible characters of G is the same as the number of orbits of K as a group of permutations on the conjugacy classes of G.*

Proof. Let X be the character table of G. Then as a matrix, X is square and nonsingular. Let α be an automorhism of G such that $\alpha \in K$. Then α induces a permutation on the conjugacy classes of G and thus induces a permutation on the columns of X. Hence K acts on the conjugacy classes of G. Since $\alpha \in K$, then to each character χ of G, we obtain a character χ^{α} of G such that $\chi^{\alpha} \in Irr(G)$ whenever $\chi \in Irr(G)$. For $y \in G$, we obtain that $\chi^{\alpha}(y) = \chi(y^{\alpha})$. Thus α induces a permutation on the rows of X. Hence K acts on the irreducible characters of G. Let X^{α} denote the image of X under α. Then we obtain that

$$P(\alpha)X = X^{\alpha} = XQ(\alpha) \quad ,$$

where $P(\alpha), Q(\alpha)$ are appropriate permutation matrices which are uniquely determined by $\alpha \in K$. Suppose that $\alpha, \beta \in K$. Then we obtain that $X^{\alpha\beta} = (X^{\alpha})^{\beta}$. Also we have that

$$P(\alpha\beta)X = X^{\alpha\beta} = (X^{\alpha})^{\beta} = (P(\alpha)X)^{\beta} = P(\beta)P(\alpha)X$$

and hence $P(\alpha\beta) = P(\beta)P(\alpha)$. We also have that $X^{\alpha\beta} = XQ(\alpha\beta)$ and $(X^{\alpha})^{\beta} = (XQ(\alpha))^{\beta} = XQ(\alpha)Q(\beta)$. Since $X^{\alpha\beta} = (X^{\alpha})^{\beta}$, we obtain that $XQ(\alpha\beta) = XQ(\alpha)Q(\beta)$. The nonsingularity of X implies that $Q(\alpha\beta) = Q(\alpha)Q(\beta)$. Define mappings π_1 and π_2 on K by $\pi_1(\alpha) = (P(\alpha))^t$ and $\pi_2(\alpha) = Q(\alpha)$, where t denotes the transpose operation on matrices. Then π_1 and π_2 are permutation representations of K. Let θ_1 and θ_2 be the permutation characters afforded by π_1 and π_2 respectively. Since $X^{-1}P(\alpha)X = Q(\alpha)$, $P(\alpha)$ and $Q(\alpha)$ are similar and thus have the same trace. Since $trace(P(\alpha))^t = trace(P(\alpha))$, we have that $trace(P(\alpha))^t = trace(Q(\alpha))$. Hence $\theta_1 = \theta_2$ and π_1 and π_2 are equivalent. Let d_1, d_2 be the number of orbits of K on the irreducible characters and on the conjugacy classes of G respectively. Thus we observe that d_1 is the number of orbits of $\pi_1(K)$ in its action as a group of permutations. Also d_2 is the number of orbits of $\pi_2(K)$ in its action as a group of permutations. Since θ_1 is the permutation character of K acting on the irreducible characters of G, we obtain that $\langle \theta_1, I_K \rangle = d_1$. Also for θ_2, we obtain that $\langle \theta_2, I_K \rangle = d_2$. However $\theta_1 = \theta_2$ and thus $\langle \theta_1, I_K \rangle = \langle \theta_2, I_K \rangle$ and hence $d_1 = d_2$. Hence the result. $\square$

Definition 5.1.6 *Let G be a group and $H \leq G$. Then for a character χ of H, we define*

$$I_G(\chi) = \{g \in N_G(H) \mid \chi^g = \chi\}$$

and we call $I_G(\chi)$ the inertia group of χ in G. If H .is normal in G, then

$$I_G(\chi) = \{g \in G \mid \chi^g = \chi\} \quad .$$

We observe that $N_G(H)$ acts on the characters of H by $g : \chi \longmapsto \chi^g$ for all $g \in N_G(H)$. Then the inertia group of χ is the stabilizer of χ in $N_G(H)$. Hence $I_G(\chi) \leq N_G(H) \leq G$ and it is not difficult to see that H is a normal subgroup of $I_G(\chi)$.

Theorem 5.1.7 *[60] Let G be a group, H a normal subgroup of G, $\theta \in Irr(H)$ and $T = I_G(\theta)$. Let*

$$A = \{\psi \in Irr(T) \mid \langle \psi_H, \theta \rangle \neq 0\}$$

$$B = \{\chi \in Irr(G) \mid \langle \chi_H, \theta \rangle \neq 0\}$$

Then

(a) If $\psi \in A$, then $\psi^G \in Irr(G)$.

(b) If $\psi^G = \chi$ and $\psi \in A$, then $\langle \psi_H, \theta \rangle = \langle \chi_H, \theta \rangle$.

(c) If $\psi^G = \chi$ and $\psi \in A$, then ψ is the unique irreducible constituent of χ_T which sits in A.

(d) The map $\psi \longmapsto \psi^G$ is a bijection of A to B.

Proof. (a) Let $\psi \in A$ and χ be an irreducible constituent of ψ^G. Then ψ is an irreducible constituent of χ_T. Since θ is an irreducible constituent of ψ_H, θ is an irreducible constituent of χ_H and thus $\chi \in B$. Now suppose that $\theta_1, \theta_2, \ldots, \theta_n$ are the distinct conjugactes of θ in G, where $\theta_1 = \theta$. Then we obtain that $[G : T] = n$ and by Clifford's theorem, we obtain that $\chi_H = e \sum_{i=1}^{n} \theta_i$ for some $e \in \mathbb{N}$, where $e = \langle \chi_H, \theta \rangle$. Since θ is invariant in T, θ is self-conjugate in T. Hence by Clifford's theorem (applied to T, H and ψ) we get that $\psi_H = k\theta$ for some $k \in \mathbb{N}$ where $k = \langle \psi_H, \theta \rangle$. Since ψ is an irreducible constituent of χ_T, then we obtain that $k \leq e$. Hence we have

$$en\theta(1_H) = \chi(1_G) \leq \psi^G(1_G) = n\psi(1_T) = kn\theta(1_H) \leq en\theta(1_H)$$

and thus equality holds throughout. In particular, from this equality we obtain that $\psi^G(1_G) = \chi(1_G)$ and hence we obtain that $\psi^G = \chi$. Therefore $\psi^G \in Irr(G)$.

(b) We have that $\langle \chi_H, \theta \rangle = e$ and $\langle \psi_H, \theta \rangle = k$ and from the equality in part(a), we obtain that $k = e$ and thus $\langle \chi_H, \theta \rangle = \langle \psi_H, \theta \rangle$.

(c) Let $\phi \in A, \phi \neq \psi$ and ϕ is an irreducible constituent of χ_T. Then we obtain that

$$\langle \chi_H, \theta \rangle \geq \langle (\phi + \psi)_H, \theta \rangle = \langle \phi_H, \theta \rangle + \langle \psi_H, \theta \rangle > \langle \psi_H, \theta \rangle$$

which is a contradiction by part(b). Hence the result.

(d) The map $\psi \longmapsto \psi^G$ is well-defined by part(a). Also we obtain that for any $\psi \in A$, $\psi^G \in B$ by part(b). By the uniqueness assertion given by part(c), the map $\psi \longmapsto \psi^G$ is one-to-one. Then suffices to show that the map is onto B. Let $\chi \in B$. Then θ is an irreducible constituent of χ_H and hence there exists an irreducible constituent ψ of χ_T such that $\langle \psi_H, \theta \rangle \neq 0$. Thus $\psi \in A$ and we have that χ is an irreducible constituent of ψ^G. Hence we obtain that $\chi = \psi^G$ since $\psi^G \in Irr(G)$ by part(a). $\square$

Remark 5.1.8 By Theorem 5.1.7 we deduce that induction to G maps the irreducible characters of T that contain θ in their restriction to H faithfully onto the irreducible characters of G that contain θ in their restriction to H.

Definition 5.1.9 *Let G be a group, H a normal subgroup of G, $\theta \in Irr(H)$ and $T = I_G(\theta)$. Since H is normal in T, then the factor group T/H is called the inertia factor of T.*

Let $\overline{G} = N{:}G$. Then for all $\theta \in Irr(N)$, define

$$\overline{H} = \{x \in \overline{G} \mid \theta^x = \theta\} = I_{\overline{G}}(\theta)$$

$$H = \{y \in G \mid \theta^y = \theta\} = I_G(\theta) \quad .$$

Then it can be shown that $\overline{H} = N{:}H$.

Remark 5.1.10 The inertia factor $\overline{H}/N \cong H$ can be regarded as the inertia group of θ in the factor group $\overline{G}/N \cong G$.

Definition 5.1.11 *Let G be a group, H a subgroup of G, $\theta \in Irr(H)$ and $\chi \in Irr(G)$ such that $\chi_H = \theta$. Then θ is said to be extendible to an irreducible character of G.*

If θ is extendible to an irreducible character of G, we will simply say that θ is extendible to G. There are various conditions which have to be satisfied in order that θ can be extended to G. For our purposes, the cornerstone of those conditions is given in Mackey's Theorem which will be proved later. Readers can also consult [47], [48], [69] and many other relevant sources for further reading and information on extendibility of characters.

Definition 5.1.12 *Let G be a group and F be a field. Then the map $\rho : G \longrightarrow GL(n, F)$ such that*

(i) $\rho(1_G) = I$, where I is the identity $n \times n$ matrix.

(ii) for all $x, y \in G$, there exists a map $\alpha : G \times G \longrightarrow F^*$ such that

$$\rho(x)\rho(y) = \alpha(x,y)\rho(xy) \quad \text{where} \quad \alpha(x,y) \in F^* \quad .$$

Then ρ is called a projective representation of G over F of degree n. The map α is called the factor set associated with ρ.

From the above definition, we observe that

$$\alpha(x,y) = \rho(x)\rho(y)(\rho(xy))^{-1} \quad .$$

Thus for the factor set α associated with ρ, if $\alpha(x,y) = 1_F$ for all $x, y \in G$, then we obtain that

$$\rho(xy) = \rho(x)\rho(y)$$

and hence ρ becomes an ordinary representation of G. Sometimes a pair (ρ, α) is used to indicate a projective representation ρ and its associated factor set α.

Theorem 5.1.13 *[70] Let N a normal subgroup of G, $F = \mathbb{C}$, $\chi \in Irr(N)$, where χ is G- invariant and let Γ be a matrix representation of N which affords χ. Then* *(i) there exists a projective representation ρ of G such that $\Gamma(n) = \rho(n)$ and $(\rho(g))^{o(g)} = I$, for all $n \in N, g \in G$ where I is the identity matrix.*

(ii) If $G = N{\cdot}H$ for some $H \leq G$ and if ρ_H is an ordinary representation of H, then χ can be extended to G.

Proof. (i) Let $g \in G$. Since χ is G-invariant, then the representations Γ and Γ^g of N are equivalent. Hence there is an invertible matrix $\theta(g)$ such that $(\theta(g))^{-1}\Gamma(n)\theta(g) = \Gamma^g(n)$, where $g \in G, n \in N$. We may assume that $\theta(n) = \Gamma(n)$ for all $n \in N$. We have that $\theta : G \longrightarrow GL(k, F)$, where $k = deg(\Gamma)$, and that $\theta_N = \Gamma$. Now let $g_1, g_2 \in G$, then we obtain that

$$\begin{aligned}(\theta(g_1g_2))^{-1}\Gamma(n)\theta(g_1g_2) &= \Gamma^{g_1g_2}(n) = (\Gamma^{g_1})^{g_2}(n) = (\theta(g_2))^{-1}\Gamma^{g_1}(n)\theta(g_2)\\ &= (\theta(g_2))^{-1}(\theta(g_1))^{-1}\Gamma(n)\theta(g_1)\theta(g_2).\end{aligned}$$

So that

$$\theta(g_1)\theta(g_2)(\theta(g_1g_2))^{-1}\Gamma(n) = \Gamma(n)\theta(g_1)\theta(g_2)(\theta(g_1g_2))^{-1} \quad .$$

Thus for all $n \in N$, $\theta(g_1)\theta(g_2)(\theta(g_1g_2))^{-1}$ commutes with $\Gamma(n)$ and thus by the Corollary 3.1.3, we can define a function $\alpha : G \times G \longrightarrow F^*$ such that $\theta(g_1)\theta(g_2) = \alpha(g_1, g_2)\theta(g_1g_2)$. Since Γ is a representation of N, then we obtain that $\theta(1_N) = \Gamma(1_N) = I$. Hence θ is a projective representation of G with associated factor set α. Let $o(g) = m$ and if $g \in N$, then we obtain that $(\theta(g))^m = I$. However if $g \in G - N$, then since $\theta(g^m) = \theta(1_G) = I$, then there exists $\lambda(g) \in F^*$ such that $(\theta(g))^m = \lambda(g)I$. Now let $\mu(g) \in F^*$ such that $(\mu(g))^m = (\lambda(g))^{-1}$ and let $\mu(n) = 1_F$ for all $n \in N$. Then the projective representation ρ of G given by $\rho(g) = \mu(g)\theta(g)$ is such that $\rho(n) = \mu(n)\theta(n) = \theta(n) = \Gamma(n)$ for all $n \in N$. Also we have that

$$(\rho(g))^m = (\mu(g)\theta(g))^m = (\mu(g))^m(\theta(g))^m = (\lambda(g))^{-1}\lambda(g)I = I \quad .$$

Hence property (i) is established.

(ii) Let T be a transversal for $N \cap H$ in H containing 1_H. Then every $g \in G$ has a unique expression of the form $g = tn$, where $t \in T, n \in N$. Now let $g_1 \in G, g_1 \neq g$ be given by $g_1 = t_1n_1$, where $t_1 \in T, n_1 \in N$. Since $t, t_1 \in T$, then $t, t_1 \in H$ and hence $tt_1 \in H$. Now let $tt_1 = t_2n_2$, where $t_2 \in T$ and $n_2 \in N \cap H$. Define ψ on G by $\psi(g) = \rho(t)\rho(n)$. Since $n_2t_1^{-1}nt_1n_1 \in N$, we obtain that

$$\psi(gg_1) = \psi(tnt_1n_1) = \psi(tt_1t_1^{-1}nt_1n_1) = \psi(t_2n_2t_1^{-1}nt_1n_1) = \rho(t_2)\rho(n_2t_1^{-1}nt_1n_1) \quad .$$

Also we have

$$\begin{aligned}\psi(g)\psi(g_1) &= \rho(t)\rho(n)\rho(t_1)\rho(n_1) = \rho(t)\rho(t_1)(\rho(t_1))^{-1}\rho(n)\rho(t_1)\rho(n_1)\\ &= \rho(t)\rho(t_1)[(\rho(t_1))^{-1}\rho(n)\rho(t_1)]\rho(n_1).\end{aligned}$$

However from the proof of part(i) above we have that $(\rho(g))^{-1}\Gamma(n)\rho(g) = \Gamma^g(n)$ and $\rho(n) = \Gamma(n)$ for all $n \in N, g \in G$. Since $t_1^{-1}nt_1 \in N$, then we obtain that

$$\rho(t_1^{-1}nt_1) = \Gamma(t_1^{-1}nt_1) = \Gamma^{t_1}(n) = (\rho(t_1))^{-1}\Gamma(n)\rho(t_1) = (\rho(t_1))^{-1}\rho(n)\rho(t_1) \quad .$$

Since by the assumption ρ is an ordinary representation on H we have $\rho(tt_1) = \rho(t)\rho(t_1)$ since $tt_1 \in H$. We deduce that

$$\begin{aligned}\psi(g)\psi(g_1) &= \rho(t)\rho(n)\rho(t_1)\rho(n_1)\\ &= \rho(t)\rho(t_1)(\rho(t_1))^{-1}\rho(n)\rho(t_1)\rho(n_1)\\ &= \rho(t)\rho(t_1)[(\rho(t_1))^{-1}\rho(n)\rho(t_1)]\rho(n_1)\\ &= \rho(t)\rho(t_1)\rho(t_1^{-1}nt_1)\rho(n_1) = \rho(tt_1)\rho(t_1^{-1}nt_1)\rho(n_1)\\ &= \rho(t_2n_2)\rho(t_1^{-1}nt_1)\rho(n_1) = \rho(t_2)\rho(n_2t_1^{-1}nt_1n_1).\end{aligned}$$

Hence we obtain that $\psi(gg_1) = \psi(g)\psi(g_1)$. Therefore ψ is an ordinary representation of G. However $\forall\ n \in N$, we obtain that $\psi(n) = \rho(n) = \Gamma(n)$ and thus the character afforded by the representation ψ of G, extends χ to G. Hence the result. □

Theorem 5.1.14 *[70] Let $\overline{G} = N{\cdot}G$ where N is a normal subgroup of $\overline{G}$, and $G \leq \overline{G}$ such that $N \cap G \subseteq N'$. If θ is an irreducible $\overline{G}$-invariant character of N such that $(deg(\theta), |G|) = 1$, then θ can be extended to $\overline{G}$.*

Proof. For a detailed proof which uses the previous theorem, see Corollary 7.1.2 of [70] □

Theorem 5.1.15 *([27],[116])(***Mackey's Theorem***) Let N be a normal subgroup of $\overline{G}$ and θ be a $\overline{G}$-invariant irreducible character of N. If N is abelian and $\overline{G}$ splits over N, then θ can be extended to $\overline{G}$.*

Proof. Let $\overline{G} = N{:}G$. Since $\overline{G}$ is a semidirect product of N by G, then any $x \in \overline{G}$ can be expressed uniquely as $x = ng$, where $n \in N, g \in G$. Define χ on $\overline{G}$ by $\chi(ng) = \theta(n)$. Since N is abelian, θ has degree 1 and thus is linear. The invariance of θ in $\overline{G}$ implies that $\theta(n) = \theta(xnx^{-1})$ for all $x \in \overline{G}$. Now let $x_1 = n_1g_1, x_2 = n_2g_2$ be elements of $\overline{G}$. Then we obtain that

$$\begin{aligned}\chi(x_1x_2) &= \chi(n_1g_1n_2g_2) = \chi(n_1n_2^{g_1}g_1g_2) = \theta(n_1n_2^{g_1}) \\ &= \theta(n_1)\theta(n_2^{g_1}) = \theta(n_1)\theta(n_2) = \chi(x_1)\chi(x_2).\end{aligned}$$

Therefore χ is a linear character of $\overline{G}$ such that $\chi_N = \theta$. □

Remark 5.1.16 We give a different proof of Mackey's theorem by applying Theorem 5.1.14. Let $\overline{G} = N{:}G$. Since N is abelian, then $N' = \{1\}$ and $deg(\theta) = 1$. Also since extension is split, we have $N \cap G = \{1\}$. Thus we obtain that $N \cap G \subseteq N'$ and $(deg(\theta), |G|) = 1$. Thus the conditions of Theorem 5.1.14 are satisfied and hence θ can be extended to $\overline{G}$.

Another extension result is given in the following theorem proved by Gagola in [47].

Theorem 5.1.17 *Let N be a normal subgroup of a finite group $\overline{G}$ and θ be an irreducible character of N which is invariant in $\overline{G}$, then θ is extendible to a character of $\overline{G}$ if $([\overline{G} : N], \frac{|N|}{deg(\theta)}) = 1$.*

Proof. See [47]. □

Theorem 5.1.18 *Suppose $\overline{G}$ is a splitting extension of a normal subgroup N, then any linear character $\theta \in Irr(N)$ can be extended to its inertia group $I_{\overline{G}}(\theta)$.*

Proof. Let $\overline{G} = N{:}G$ and $\theta \in Irr(N)$ be linear. Let $\overline{H} = I_{\overline{G}}(\theta)$, then we obtain that $\overline{H} = N{:}H$, where $H = I_G(\theta)$. Since $\overline{H}$ is a split extension, we obtain that $N \cap H = \{1\} \leq N'$. Also we have that $(deg(\psi), |H|) = (1, |H|) = 1$ and clearly θ is $\overline{H}$-invariant. Thus the conditions of Theorem 5.1.14 are satisfied and hence θ can be extended to $\overline{H}$. $\square$

Theorem 5.1.18 is proved in a different way as Lemma 2.2 in [102]. Also Mackey's theorem is reinforced by Theorem 5.1.18 since for N abelian, all its irreducible characters are linear and hence are extendible to their inertia groups.

Theorem 5.1.19 *([48],[60],[116])(***Gallagher's Theorem***) Let N a normal subgroup of $\overline{G}$, $\theta \in Irr(N)$ and $\overline{H} = I_{\overline{G}}(\theta)$. If θ can be extended to $\psi \in Irr(\overline{H})$ then as β ranges over all the irreducible characters of $\overline{H}$ which contain N in their kernels, $\beta\psi$ ranges over all the irreducible characters of $\overline{H}$ which contain θ in their restriction to N.*

Proof. Since $\overline{H} = I_{\overline{G}}(\theta)$, then θ is self-conjugate in $\overline{H}$ and thus by Clifford's theorem we obtain that $(\theta^{\overline{H}})_N = f\theta$ for some positive integer f. Comparing degrees we have $(\theta^{\overline{H}})_N = [\overline{H} : N]\theta$ and so $\langle \theta^{\overline{H}}, \theta^{\overline{H}} \rangle = \langle \theta, (\theta^{\overline{H}})_N \rangle = [\overline{H} : N]$. Now we claim that $\theta^{\overline{H}} = \sum_\beta \beta(1_{\overline{G}})\beta\psi$, where β ranges over all the irreducible characters of $\overline{H}$ that contain N in their kernels. Both $\theta^{\overline{H}}$ and $\sum_\beta \beta(1_{\overline{G}})\beta\psi$ are zero off N since for $g \notin N, xgx^{-1} \notin N$ for all $x \in \overline{G}$ and thus $\theta^{\overline{H}}(g) = 0$. Also for $g \notin N$, by the orthogonality of the columns of the character table of $\overline{H}/N$ we have that $\sum_\beta \beta(1_{\overline{G}})(\beta\psi)(g) = [\sum_\beta \beta(1_{\overline{G}})\beta(g)]\psi(g) = 0$. We also have that $(\theta^{\overline{H}})_N = [\overline{H} : N]\theta = (\sum_\beta \beta(1_{\overline{G}})\beta\psi)_N$ since for $g \in N$, $\sum_\beta \beta(1_{\overline{G}})\beta(g)\psi(g) = \sum_\beta(\beta(1_{\overline{G}}))^2\psi(g) = [\overline{H} : N]\psi(g) = [\overline{H} : N]\theta(g)$. Hence we obtain that $\theta^{\overline{H}} = \sum_\beta \beta(1_{\overline{G}})\beta\psi$. So we have

$$[\overline{H} : N] = \langle \theta^{\overline{H}}, \theta^{\overline{H}} \rangle = \langle \sum_\beta \beta(1_{\overline{G}})\beta\psi, \sum_\tau \tau(1_{\overline{G}})\tau\psi \rangle = \sum_{\beta,\tau} \beta(1_{\overline{G}})\tau(1_{\overline{G}})\langle \beta\psi, \tau\psi \rangle \quad .$$

The diagonal terms contribute at least $\sum(\beta(1_{\overline{G}}))^2 = [\overline{H} : N]$, so the $\beta\psi$ are irreducible and distinct, and are all the irreducible constituents of $\theta^{\overline{H}}$ and so are all the irreducible characters of $\overline{H}$ that contain θ in their restriction to N, since for $\phi \in Irr(\overline{H})$ such that $\langle \phi_N, \theta \rangle \neq 0$, we obtain that $\langle \phi_N, \theta \rangle = \langle \phi, \theta^{\overline{H}} \rangle$ which implies that ϕ is an irreducible constituent of $\theta^{\overline{H}}$ and hence is of the form $\beta\psi$. $\square$

5.2 The Fischer-Clifford matrices

Let $\overline{G} = N{\cdot}G$ such that every irreducible character of N is extendible to its inertia group. We have that $\overline{G}$ permutes $Irr(N)$ by $x : \theta \longmapsto \theta^x$, where $x \in \overline{G}$ and $\theta \in Irr(N)$. Now let $\theta_1, \theta_2, \dots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $Irr(N)$, $\overline{H}_i = I_{\overline{G}}(\theta_i), 1 \leq i \leq t, \psi_i \in Irr(\overline{H}_i)$ be an extension of θ_i to $\overline{H}_i$ and $\beta \in Irr(\overline{H}_i)$ such that $N \subseteq ker(\beta)$. Then by Gallagher's theorem, Theorem 5.1.7 and Remark 5.1.8 all irreducible characters of $\overline{G}$ will be of the form $(\beta\psi_i)^{\overline{G}}, 1 \leq i \leq t$. So

$$Irr(\overline{G}) = \bigcup_{i=1}^{t}\{(\beta\psi_i)^{\overline{G}} \mid \beta \in Irr(\overline{H}_i),\ N \subseteq ker(\beta)\} \quad .$$

Hence the irreducible characters of $\overline{G}$ will be divided into blocks, where each block corresponds to an inertia group $\overline{H}_i$.

5.2.1 Definition and Preliminaries

Let $\overline{G} = N{\cdot}G$ with the property that every irreducible character of N can be extended to its inertia group. Let $\overline{g} \in \overline{G}$ be a lifting of $g \in G$ under the natural homomorphism $\overline{G} \longrightarrow G$ and $[g]$ be a conjugacy class of elements of G with representative g. Let $X(g) = \{x_1, x_2, \dots, x_{c(g)}\}$ be a set of representatives of the conjugacy classes of $\overline{G}$ from the coset $N\overline{g}$ whose images under the natural homomorphism $\overline{G} \longrightarrow G$ are in $[g]$ and we take $x_1 = \overline{g}$. Let $\{\theta_1, \theta_2, \dots, \theta_t\}$ be a set of representatives of the orbits of $\overline{G}$ on $Irr(N)$ such that for $1 \leq i \leq t$, we have $\overline{H}_i = I_{\overline{G}}(\theta_i)$ with $H_i = \overline{H}_i/N \leq G$ and that $\psi_i \in Irr(\overline{H}_i)$ is an extension of θ_i to $\overline{H}_i$. Then without loss of generality suppose that $\theta_1 = I_N$ is the identity character of N. Then $\overline{H}_1 = \overline{G}$ and $H_1 = G$. Now choose $y_1, y_2, \dots, y_r$ to be the representatives of the conjugacy classes of elements of H_i which fuse into $[g]$ in G. Since $y_k \in H_i$ for $1 \leq k \leq r$, then we define $y_{\ell_k} \in \overline{H}_i$ such that y_{ℓ_k} ranges over all the representatives of the conjugacy classes of elements of $\overline{H}_i$ which map to y_k under the homomorphism $\overline{H}_i \longrightarrow H_i$ whose kernel is N. Let $\beta \in Irr(\overline{H}_i)$ such that $N \subseteq ker(\beta)$. Then β is a lifting of $\hat{\beta} \in Irr(H_i)$ such that $\beta(y_{\ell_k}) = \hat{\beta}(y_k)$ for any lifting $y_{\ell_k} \in \overline{H}_i$ of $y_k \in H_i$. Then we obtain that

$$(\psi_i\beta)^{\overline{G}}(x_j) \quad = \quad \sum_{1 \leq k \leq r} {\sum_{\ell}}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|}\psi_i\beta(y_{\ell_k})$$

$$= \sum_{1\leq k\leq r} \sum_{\ell}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|}\psi_i(y_{\ell_k})\beta(y_{\ell_k})$$

$$= \sum_{1\leq k\leq r} (\sum_{\ell}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|}\psi_i(y_{\ell_k}))\hat{\beta}(y_k)$$

where $\sum_\ell{}'$ is the summation over all ℓ for which $y_{\ell_k} \sim x_j$ in $\overline{G}$. Now we define a matrix $M_i(g)$ by $M_i(g) = (a_{uv})$, where $1 \leq u \leq r$ and $1 \leq v \leq c(g)$, and

$$a_{uv} = \sum_{\ell}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|}\psi_i(y_{\ell_k}) \quad .$$

Then we obtain that

$$(\psi_i\beta)^{\overline{G}}(x_j) = \sum_{1\leq k\leq r} a_{uv}\hat{\beta}(y_k) \quad .$$

By doing this for all $1 \leq i \leq t$ such that H_i contains an element in $[g]$ we obtain the matrix $M(g)$ given by

$$M(g) = \begin{bmatrix} M_1(g) \\ M_2(g) \\ \vdots \\ M_t(g) \end{bmatrix} ,$$

where $M_i(g)$ is the submatrix corresponding to the inertia group $\overline{H}_i$ and its inertia factor H_i. If $H_i \cap [g] = \emptyset$, then $M_i(g)$ will not exist and $M(g)$ does not contain $M_i(g)$. The size of the matrix $M(g)$ is $p \times c(g)$ where p is the number of conjugacy classes of elements of the inertia factors H_i's for $1 \leq i \leq t$ which fuse into $[g]$ in G and $c(g)$ is the number of conjugacy classes of elements of $\overline{G}$ which correspond to the coset $N\overline{g}$. Then $M(g)$ is the *Fischer-Clifford matrix* of $\overline{G}$ corresponding to the coset $N\overline{g}$. We will see later that $M(g)$ is a $c(g) \times c(g)$ nonsingular matrix. Let

$$R(g) = \{(i, y_k) \mid 1 \leq i \leq t \ , \ H_i \cap [g] \neq \emptyset \ , \ 1 \leq k \leq r\}$$

and we note that y_k runs over representatives of the conjugacy classes of elements of H_i which fuse into $[g]$ in G. Following the notation used in [43] and [116] we denote $M(g)$ by writing $M(g) = (a_j^{(i,y_k)})$, where

$$a_j^{(i,y_k)} = \sum_{\ell}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|}\psi_i(y_{\ell_k}) \quad ,$$

with columns indexed by $X(g)$ and rows indexed by $R(g)$. Then the partial character table of $\overline{G}$ on the classes $\{x_1, x_2, \ldots, x_{c(g)}\}$ is given by

$$\begin{bmatrix} C_1(g)M_1(g) \\ C_2(g)M_2(g) \\ \vdots \\ C_t(g)M_t(g) \end{bmatrix}$$

where the Fischer-Clifford matrix $M(g)$ is divided into blocks with each block corresponding to an inertia group $\overline{H}_i$ and $C_i(g)$ is the partial character table of H_i consisting of the columns corresponding to the classes that fuse into $[g]$ in G. We can also observe that the number of irreducible characters of $\overline{G}$ is the sum of the numbers of irreducible characters of the inertia factors H_i's.

5.2.2 Properties of Fischer-Clifford matrices

We shall discuss the properties which are useful in the computation of the Fischer-Clifford matrices. These properties have been discussed in [41], [75], [76], [106], [98], [116].

Let K be a group and $A \leq Aut(K)$. Then by Brauer's theorem A acts on the conjugacy classes of elements of K and on the irreducible characters of K resulting in the same number of orbits.

Lemma 5.2.1 *Suppose we have the following matrix describing the above actions:*

$$\begin{array}{c} \\ s_1 \\ s_2 \\ \vdots \\ s_i \\ \vdots \\ s_t \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1=l_1 & l_2 & \cdots & l_j & \cdots & l_t \end{array} \\
\begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2t} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{it} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{tj} & \cdots & a_{tt} \end{pmatrix}
\end{array}$$

where $a_{1j} = 1$ for $j \in \{1, 2, \ldots, t\}$, l_j's are lengths of orbits of A on the conjugacy classes of K, s_i's are lengths of orbits of A on $Irr(K)$ and a_{ij} is the sum of s_i irreducible characters of K on the element x_j, where x_j is an element of the orbit of

length l_j. Then the following relation holds for $i, i' \in \{1, 2, \ldots, t\}$:
$\sum_{j=1}^{t} a_{ij}\overline{a_{i'j}}l_j = |K|s_i\delta_{ii'}$.

Proof. This result has been proved as Lemma 2.2.2 in [106] and as Lemma 4.2.2 in [116]. □

Let $x_j \in X(g)$ and define $m_j = [C_{\overline{g}} : C_{\overline{G}}(x_j)]$. The Fischer-Clifford matrix $M(g)$ is partitioned row-wise into blocks, where each block corresponds to an inertia group. The columns of $M(g)$ are indexed by $X(g)$ and for each $x_j \in X(g)$, at the top of the columns of $M(g)$, we write $|C_{\overline{G}}(x_j)|$ and at the bottom we write m_j. The rows of $M(g)$ are indexed by $R(g)$ and on the left of each row we write $|C_{H_i}(y_k)|$, where y_k fuses into $[g]$ in G. Then in general we can write $M(g)$ with corresponding weights for rows and columns as follows, where blocks corresponding to the inertia groups are separated by horizontal lines.

$$
\begin{array}{c}
\\
|C_G(g)| \\
\\
|C_{H_2}(y_1)| \\
|C_{H_2}(y_2)| \\
\vdots \\
\\
|C_{H_i}(y_1)| \\
|C_{H_i}(y_2)| \\
\vdots \\
\\
|C_{H_t}(y_1)| \\
|C_{H_t}(y_2)| \\
\vdots \\
\\
\\
\end{array}
\begin{array}{c}
\begin{array}{cccc}
|C_{\overline{G}}(x_1)| & |C_{\overline{G}}(x_2)| & \cdots & |C_{\overline{G}}(x_{c(g)})|
\end{array} \\
\left(\begin{array}{cccc}
a_1^{(1,g)} & a_2^{(1,g)} & \ldots & a_{c(g)}^{(1,g)} \\
\hline
a_1^{(2,y_1)} & a_2^{(2,y_1)} & \ldots & a_{c(g)}^{(2,y_1)} \\
a_1^{(2,y_2)} & a_2^{(2,y_2)} & \ldots & a_{c(g)}^{(2,y_2)} \\
\vdots & \vdots & \vdots & \vdots \\
\hline
a_1^{(i,y_1)} & a_2^{(i,y_1)} & \ldots & a_{c(g)}^{(i,y_1)} \\
a_1^{(i,y_2)} & a_2^{(i,y_2)} & \ldots & a_{c(g)}^{(i,y_2)} \\
\vdots & \vdots & \vdots & \vdots \\
\hline
a_1^{(t,y_1)} & a_2^{(t,y_1)} & \ldots & a_{c(g)}^{(t,y_1)} \\
a_1^{(t,y_2)} & a_2^{(t,y_2)} & \ldots & a_{c(g)}^{(t,y_2)} \\
\vdots & \vdots & \vdots & \vdots \\
\hline
\end{array}\right) \\
\begin{array}{cccc}
m_1 & m_2 & \ldots & m_{c(g)}
\end{array}
\end{array}
$$

From the theory of coset analysis for computing the conjugacy classes of elements of $\overline{G} = N{\cdot}G$ where N is abelian, we observe that

$$m_j = [C_{\overline{g}} : C_{\overline{G}}(x_j)] = \frac{f.|N|}{k} \quad .$$

Remark 5.2.2 It can be shown that the Fischer-Clifford matrix $M(g)$ satisfies complex conjugation.

The following result gives the orthogonality relation for $M(g)$. Its proof was obtained from Whitley [116], Proposition 4.2.3.

Proposition 5.2.3 *[116](*Column orthogonality*) Let $\overline{G} = N{\cdot}G$, then*

$$\sum_{(i,y_k)\in R(g)} |C_{H_i}(y_k)| a_j^{(i,y_k)} \overline{a_{j'}^{(i,y_k)}} = \delta_{jj'} |C_{\overline{G}}(x_j)| \quad .$$

Proof. The partial character table of $\overline{G}$ at classes $x_1, \ldots, x_{c(g)}$ is given by

$$\begin{bmatrix} C_1(g)M_1(g) \\ C_2(g)M_2(g) \\ \vdots \\ C_t(g)M_t(g) \end{bmatrix}$$

By column orthogonality of the character table of $\overline{G}$, we have

$$\begin{aligned}
|C_{\overline{G}}(x_j)|\delta_{jj'} &= \sum_{i=1}^{t} \sum_{\beta_i \in Irr(H_i)} \Big(\sum_{y_k:(i,y_k)\in R(g)} a_j^{(i,y_k)} \beta_i(y_k) \Big) \overline{\Big(\sum_{y'_k:(i,y'_k)\in R(g)} a_{j'}^{(i,y'_k)} \beta_i(y'_k) \Big)} \\
&= \sum_{i=1}^{t} \sum_{\beta_i \in Irr(H_i)} \Big(\sum_{y_k} a_j^{(i,y_k)} \overline{a_{j'}^{(i,y'_k)}} \beta_i(y_k) \overline{\beta_i(y_k)} \; + \\
&\quad \sum_{y_k} \sum_{y'_k \neq y_k} a_j^{(i,y_k)} \overline{a_{j'}^{(i,y'_k)}} \beta_i(y_k) \overline{\beta_i(y'_k)} \Big) \\
&= \sum_{i=1}^{t} \Big(\sum_{y_k} a_j^{(i,y_k)} \overline{a_{j'}^{(i,y_k)}} \sum_{\beta_i \in Irr(H_i)} \beta_i(y_k) \overline{\beta_i(y_k)} \; + \\
&\quad \sum_{y_k} \sum_{y'_k \neq y_k} a_j^{(i,y_k)} \overline{a_{j'}^{(i,y'_k)}} \sum_{\beta_i \in Irr(H_i)} \beta_i(y_k) \overline{\beta_i(y'_k)} \Big)
\end{aligned}$$

$$= \sum_{i=1}^{t}(\sum_{y_k} a_j^{(i,y_k)}\overline{a_{j'}^{(i,y_k)}}|C_{H_i}(y_k)| \ + \ 0)$$
$$= \sum_{(i,y_k)\in R(g)} a_j^{(i,y_k)}\overline{a_{j'}^{(i,y_k)}}|C_{H_i}(y_k)|.$$

□

Theorem 5.2.4 $a_j^{(1,g)} = 1$ *for all* $j = \{1, 2, \ldots, c(g)\}$

Proof. For $y_{\ell_k} \sim x_j$ in $\overline{G}$, we have $|C_{\overline{G}}(x_j)| = |C_{\overline{H}_1}(y_{\ell_k})|$. Thus we obtain that

$$a_j^{(1,g)} = \sum_{\ell}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_1}(y_{\ell_k})|}\psi_1(y_{\ell_k}) = \sum_{\ell}{}' 1 = 1 \quad .$$

Hence the result. □

Proposition 5.2.5 *([75], [116]) The matrix $M(1_G)$ is the matrix with rows equal to the orbit sums of the action of $\overline{G}$ on $Irr(N)$ with duplicate columns discarded. For this matrix we have $a_j^{(i,1_G)} = [G : H_i]$, and an orthogonality relation for rows:*

$$\sum_{j=1}^{t} \frac{1}{|C_{\overline{G}}(x_j)|} a_j^{(i,1_G)} a_j^{(i',1_G)} = \frac{1}{|C_{H_i}(1_G)|}\delta_{ii'} = \frac{1}{|H_i|}\delta_{ii'} \quad .$$

Proof. The $(i, 1_G), j^{\text{th}}$ entry of $M(1_G)$ is given by

$$a_j^{(i,1_G)} = \sum_{\ell}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|}\psi_i(y_{\ell_k}) \quad ,$$

where we sum over representatives of conjugacy classes of $\overline{H}_i$ which fuse into $[x_j]$ in $\overline{G}$. Therefore $a_j^{(i,1_G)} = \psi_i^{\overline{G}}(x_j)$. By Theorem 5.1.7 we have $\psi_i^{\overline{G}} \in Irr(\overline{G})$ and we obtain that $\langle(\psi_i^{\overline{G}})_N, \theta_i\rangle = \langle(\psi_i)_N, \theta_i\rangle = 1$. Therefore by Clifford's theorem $(\psi_i^{\overline{G}})_N = \sum_\alpha \theta_\alpha$, where the summation is taken over all $\theta_\alpha \in Irr(N)$ such that θ_α is conjugate to θ_i. So for $x_j \in N$ we obtain that $a_j^{(i,1_G)} = \sum_\alpha \theta_\alpha(x_j)$. The orthogonality relation follows by Lemma 5.2.1. □

As a consequence of Lemma 5.2.1, Proposition 5.2.3 and the results proved by Fischer in [43], the Fischer-Clifford matrix $M(g)$ satisfies the following properties:

(a) $|X(g)| = |R(g)|$

(b) $\sum_{j=1}^{c(g)} m_j a_j^{(i,y_k)} \overline{a_j^{(i',y'_k)}} = \delta_{(i,y_k),(i',y'_k)} \frac{|C_G(g)|}{|C_{H_i}(y_k)|} |N|$

(c) $\sum_{(i,y_k)\in R(g)} a_j^{(i,y_k)} \overline{a_{j'}^{(i,y_k)}} |C_{H_i}(y_k)| = \delta_{jj'} |C_{\overline{G}}(x_j)|$

(d) $M(g)$ is square and nonsingular.

If N is elementary abelian, then we obtain the following additional properties of $M(g)$.

(e) $a_1^{(i,y_k)} = \frac{|C_G(g)|}{|C_{H_i}(y_k)|}$

(f) $|a_1^{(i,y_k)}| \geq |a_j^{(i,y_k)}|$

Remark 5.2.6 Suppose that N is an elementary abelian p-group. Let $\overline{g} \in \overline{G}$. Then the map $\phi_{\overline{g}} : n \longmapsto n\overline{g}n^{-1}(\overline{g})^{-1}$ defines an endomorphism of N. It is not difficult to see that $Im(\phi_{\overline{g}})$ and $ker(\phi_{\overline{g}})$ are $C_{\overline{g}}$-submodules of N. Let $Im(\phi_{\overline{g}}) = M$. Then N acts on $N\overline{g}$ by conjugation and M acts on $N\overline{g}$ by left multiplication such that the resulting orbits of the two actions are the same. Hence the action of $C_{\overline{g}}$ on the orbits of N acting on $N\overline{g}$ is the same as the action of $C_{\overline{g}}$ on the module N/M. Thus the orbits of the action of M on $N\overline{g}$ can be identified with the elements of N/M. Let $\theta_i \in Irr(N), \psi_i \in Irr(\overline{H}_i)$ and ψ_i be an extension of θ_i to $\overline{H}_i$. Then ψ_i is constant on the orbits of N acting on $N\overline{g}$. So we may define a class function μ on N/M by $\mu(Mn_j\overline{g}) = \psi_i(n_j\overline{g})$, where $n_j \in N, n_j\overline{g} \in Q_j$ is a representative of the j-th orbit of N acting on $N\overline{g}$ and $n_1 = 1_N$. Then $\mu(M\overline{g}) = \psi_i(\overline{g})$. Let $\hat{\mu}$ be an extension of μ to the inertia group of μ in $C_{\overline{g}}$. Then induction of $\hat{\mu}$ to $\overline{G}$ evaluated on the elements of $N\overline{g}$ is equivalent to the induction of $\hat{\mu}$ to $C_{\overline{g}}/M$ evaluated on the elements of N/M. If $\overline{G}$ is a split extension, then it can be shown (see [75]) that the Fischer-Clifford matrix at a nonidentity coset of N in $\overline{G}$ is the matrix of orbit sums of $C_{\overline{g}}$ acting on the rows of the character table of N/M with duplicating columns discarded. However for $\overline{G}$ a non-split extension, it may happen that μ is not a character of N/M. Then $\xi\mu$ will be a character of N/M, where ξ is an appropriate p-th root of unity. Thus for $\overline{G}$ a non-split extension, the Fischer-Clifford matrix is the matrix of orbit sums of $C_{\overline{g}}$ acting on the rows of the character table of N/M with duplicate columns discarded and with each row multiplied by an appropriate p-th root of unity. It may happen that the p-th root of unity for each row is 1. (For more details see [75]).

Proposition 5.2.7 *If N is elementary abelian and $M = Im(\phi_{\overline{g}})$, then $[N : M] = k$ where k is the number of elements of N fixed by a class representative g of G.*

Proof. We have that the orbits $Q_1, Q_2, \ldots, Q_k$ of N acting on $N\overline{g}$ are the same as the orbits $D_1, D_2, \ldots, D_k$ of M acting on $N\overline{g}$ by left multiplication. Also the the orbits $D_1, D_2, \ldots, D_k$ can be identified with the elements of N/M. Then it immediately follows that $|N/M| = [N : M] = k$. $\square$

Remark 5.2.8 If N is an elementary abelian p-group, then from the theory of coset analysis for the group $\overline{G} = N{\cdot}G$, we obtain that $k = p^m$ for $0 \leq m \leq n$, where $|N| = p^n$ and k is the number of elements of N fixed by a class representative g of G. Suppose for some class representative $g \in G$ that we obtain orbits $Q_1, Q_2, \ldots, Q_k$ of N acting on $N\overline{g}$. Then for $h \in C_G(g)$ and $\overline{h}$ being a lifting of h in $\overline{G}$, suppose that on acting $\{\overline{h} \mid h \in C_G(g)\}$ on the orbits $Q_1, Q_2, \ldots, Q_k$, we obtain $f_1 = f_2 = \cdots = f_k = 1$ and that the entries of the first column of $M(g)$ are 1. Then in this case, the Fischer-Clifford matrix $M(g)$ coincides with the character table of the abelian group N/M of order $k = p^m$, where $M = Im(\phi_{\overline{g}})$ as defined in Remark 5.2.6.

Let $\overline{G} = N{:}G$ be a split extension and N be an elementary abelian 2-group. Then for $g \in G$, a lifting of g is g itself. Then C_g acts on N/M where $M = Im(\phi_g)$. By Remark 5.2.6 the Fischer-Clifford matrix $M(g)$ is given by

$$M(g) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2j} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ d_{i1} & d_{i2} & d_{i3} & \cdots & d_{ij} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ d_{t1} & d_{t2} & d_{t3} & \cdots & d_{tj} & \cdots & \end{pmatrix}.$$

where d_{ij}'s are the orbit sums of C_g acting on the rows of the character table of N/M.

Proposition 5.2.9 $d_{i1} \in \mathbb{N}$ *for all* $i \in \{2, 3, \ldots, t\}$.

Proof. By Remark 5.2.6, we obtain that

$$d_{i1} = \sum_{\chi \in \Delta_i} \chi(1_{N/M})$$

where Δ_i's are the orbits of C_g acting on $Irr(N/M)$. Since $\chi(1_{N/M}) = deg(\chi)$, we have $d_{i1} \in \mathbb{N}\ \forall\, i \in \{2, 3, \ldots, t\}$. $\square$

For $j \geq 2$, we obtain that

$$d_{ij} = \sum_{\chi \in \Delta_i} \chi(\overline{x}_j)$$

where $\overline{x}_j \in N/M$ is a representative of the j-th orbit under the action of C_g on the elements of N/M. Since $\chi(\overline{x}_j) \in \{-1, 1\}$, we have $d_{ij} \in \mathbb{Z}$.

Proposition 5.2.10 $d_{ij} \equiv d_{i1}(mod\ \ 2)$ *for all* $j \geq 2$.

Proof. Since N is an elementary abelian 2-group, then N/M is also an elementary abelian 2-group. We obtain that

$$\begin{aligned} d_{ij} &= \sum_{\chi \in \Delta_i} \chi(\overline{x}_j) = \sum_{r=1}^{d_{i1}} \pm 1 \\ &= \underbrace{1 + 1 + \cdots + 1}_{m_{ij}-times} + \underbrace{-1 - 1 - \cdots - 1}_{n_{ij}-times} \\ &= m_{ij} - n_{ij}. \end{aligned}$$

However we have that $0 \leq m_{ij}, n_{ij} \leq d_{i1}$ and that $m_{ij} + n_{ij} = d_{i1}$. Thus we obtain that

$$d_{ij} = m_{ij} - n_{ij} = d_{i1} - 2n_{ij} \quad .$$

Hence we deduce that

$$d_{ij} \equiv d_{i1}(mod\ \ 2) \quad .$$

$\square$

Since $d_{ij} \in \mathbb{Z}$, we deduce that the Fischer-Clifford matrix $M(g)$ will have integer entries d_{ij} such that $d_{i1} \geq |d_{ij}|$ and $d_{ij} \equiv d_{i1}(mod\ \ 2)$. If $d_{i1} = n$ for some $n \in \mathbb{N}$, then for $j \geq 2$ we have $d_{ij} \in \{\pm 1, \pm 3, \ldots, \pm n\}$ if n is odd and $d_{ij} \in \{0, \pm 2, \pm 4, \ldots, \pm n\}$ if n is even. It is easy to see that for a fixed n there are $n + 1$ possible values for each d_{ij} with $j \geq 2$. We also notice that $\sum_i d_{i1} = |N/M| = k$.

Proposition 5.2.11 *For any j-th column of $M(g)$ for which $j \geq 2$, we obtain that* $\sum_i d_{ij} = 0$.

Proof. For any j-th column of $M(g)$, where $j \geq 2$, we have that

$$\sum_i d_{ij} = \sum_i \Big(\sum_{\chi \in \Delta_i} \chi(\overline{x}_j) \Big) = \sum_{\chi \in Irr(N/M)} \chi(\overline{x}_j) = 0$$

by the orthogonality of the columns of the character table of N/M. $\square$

Chapter 6

A maximal subgroup of Fi_{22}

In this chapter we study the group $2^6{:}SP(6,2)$ which is a maximal subgroup of the smallest Fischer simple group Fi_{22} of index 694980. Let $\overline{G} = 2^6{:}SP(6,2)$ be the split extension of $N = 2^6$ by $G = SP(6,2)$, where N is the vector space of dimension 6 over $GF(2)$ on which G acts naturally. Although the character table of $2^6{:}SP(6,2)$ is known, it was however constructed using a different method and its Fischer-Clifford matrices had not been determined. We therefore use the technique of the Fischer-Clifford matrices to reconstruct its character table. This character table will be divided row-wise into blocks where each block corresponds to an inertia group $\overline{H}_i = N{:}H_i$, where the H_i's are the inertia factors. The character table of $\overline{G}$ can be constructed by finding the Fischer-Clifford matrix $M(g)$ for each class representative g of G and using the character tables of the inertia factors. We use the properties of the Fischer-Clifford matrices which have been discussed in Section 5.2.2 of Chapter 5 to compute their entries. In some cases we need to use the following additional information to compute these entries:

(i) For χ a character of any group H and $h \in H$, we have $|\chi(h)| \leq \chi(1_H)$, where 1_H is the identity element of H.

(ii) For χ a character of any group H and h a p-singular element of H, where p is a prime, then we have $\chi(h) \equiv \chi(h^p) mod(p)$.

(iii) For any irreducible character χ of a group H and for $h_i \in C_i$ then $d_i = \frac{b_i\chi(h_i)}{\chi(1_H)}$ is an algebraic integer, where C_i is the i-th conjugacy class of H and $b_i = |C_i| = [H : C_H(h_i)]$. Obviously if $d_i \in \mathbb{Q}$, then $d_i \in \mathbf{Z}$.

We also study a group of the form $2^5{:}S_6$ which is maximal and affine in $SP(6,2)$ of index 63. We construct the character table of this affine subgroup using the technique of the Fischer-Clifford matrices. This character table is necessary since it will be used to construct the character table of $\overline{G}$. In the process we also construct the character table of $3^2{:}D_4$ which is maximal in S_6 of index 10. This character table is used in the construction of the character table of $2^5{:}S_6$. The Fischer-Clifford matrices and the character table of $2^6{:}SP(6,2)$ are given in Section 6.4. In Sections 6.5 and 6.6 we deal with the fusion of $2^6{:}SP(6,2)$ into Fi_{22} and the permutation character of Fi_{22} on $2^6{:}SP(6,2)$ respectively.

6.1 The conjugacy classes of $\overline{G} = 2^6{:}SP(6,2)$

In this section we use the method of coset analysis discussed in Chapter 2, Section 2.3, to determine the conjugacy classes of $\overline{G}$. Let $N = 2^6$ and $G = SP(6,2)$ and let us view N as the vector space of dimension 6 over $GF(2)$ on which $SP(6,2)$ acts naturally. Then G has 30 conjugacy classes and thus for each $[g]$ in G with representative $g \in G$, we analyse the coset Ng to obtain the classes of $\overline{G}$ which correspond to the class $[g]$ of G. However G is generated by two 6×6 matrices over $GF(2)$, namely

$$\alpha = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

where $o(\alpha) = 2$ and $o(\beta) = 6$. We also give the class representatives $g \in G$ in terms of 6×6 matrices over $GF(2)$ in the following table, where M is the matrix which represents that particular class.

$[g]_G$	M	$\lvert[g]_G\rvert$	$[g]_G$	M	$\lvert[g]_G\rvert$
$1A$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	1	$2A$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 1&0&1&1&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 1&0&0&1&0&1 \end{pmatrix}$	63
$2B$	$\begin{pmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 1&0&1&0&0&0\\ 1&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&1&1&1&1 \end{pmatrix}$	315	$2C$	$\begin{pmatrix} 1&1&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&1&1&0&1\\ 0&1&0&0&1&1\\ 0&0&0&0&0&1 \end{pmatrix}$	945
$2D$	$\begin{pmatrix} 0&1&0&1&1&1\\ 1&1&0&0&0&1\\ 1&0&1&0&0&1\\ 0&1&1&1&0&0\\ 1&1&1&0&1&1\\ 1&1&0&1&1&0 \end{pmatrix}$	3780	$3A$	$\begin{pmatrix} 0&1&1&0&1&1\\ 0&1&1&0&1&0\\ 0&0&1&0&0&0\\ 0&0&1&1&1&0\\ 1&1&1&0&0&1\\ 1&1&1&0&1&0 \end{pmatrix}$	672
$3B$	$\begin{pmatrix} 0&0&1&0&1&1\\ 0&1&1&1&0&1\\ 1&1&0&0&1&0\\ 0&1&0&1&1&1\\ 1&0&1&1&0&0\\ 1&1&0&1&0&1 \end{pmatrix}$	2240	$3C$	$\begin{pmatrix} 0&1&1&1&1&1\\ 0&0&1&1&0&1\\ 1&1&1&0&1&0\\ 1&1&0&1&1&0\\ 1&0&0&0&1&1\\ 0&1&0&0&0&1 \end{pmatrix}$	13440
$4A$	$\begin{pmatrix} 1&0&0&0&0&0\\ 1&1&1&0&1&0\\ 0&1&1&1&0&0\\ 0&0&1&1&1&0\\ 1&1&0&1&1&0\\ 1&0&1&1&0&1 \end{pmatrix}$	3780	$4B$	$\begin{pmatrix} 1&0&1&0&0&1\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&1&1&0&1\\ 0&0&0&0&0&1 \end{pmatrix}$	7560
$4C$	$\begin{pmatrix} 0&1&0&0&1&1\\ 1&0&0&0&1&1\\ 0&1&1&1&1&1\\ 0&1&1&0&1&1\\ 1&1&0&0&0&1\\ 1&0&0&1&0&1 \end{pmatrix}$	7560	$4D$	$\begin{pmatrix} 1&0&0&0&0&0\\ 1&0&0&1&0&0\\ 1&1&0&0&1&0\\ 0&1&0&0&0&0\\ 1&0&1&1&0&0\\ 0&1&1&0&0&1 \end{pmatrix}$	11340
$4E$	$\begin{pmatrix} 0&1&1&0&1&1\\ 1&0&1&0&1&1\\ 0&1&0&1&0&0\\ 1&0&0&0&1&1\\ 1&1&1&1&1&0\\ 0&1&0&0&0&1 \end{pmatrix}$	45360	$5A$	$\begin{pmatrix} 0&1&0&0&1&0\\ 0&1&1&1&1&0\\ 1&1&0&0&1&1\\ 1&1&1&1&1&0\\ 1&1&0&1&1&1\\ 1&1&0&1&0&0 \end{pmatrix}$	48384

$[g]_G$	M	$\lvert[g]_G\rvert$	$[g]_G$	M	$\lvert[g]_G\rvert$
$6A$	$\begin{pmatrix} 0&1&0&0&0&1\\ 1&1&1&1&1&1\\ 0&1&0&1&0&0\\ 1&0&0&0&1&0\\ 0&1&0&1&0&1\\ 0&0&1&0&1&0 \end{pmatrix}$	10080	$6B$	$\begin{pmatrix} 1&1&1&0&0&0\\ 1&1&0&0&1&1\\ 0&0&1&0&0&0\\ 1&0&1&1&1&0\\ 0&1&0&0&1&1\\ 1&0&0&0&1&0 \end{pmatrix}$	10080
$6C$	$\begin{pmatrix} 0&0&1&1&1&0\\ 0&1&1&0&1&0\\ 1&0&1&1&1&0\\ 1&1&1&0&1&1\\ 1&1&1&0&0&0\\ 1&1&1&1&0&1 \end{pmatrix}$	20160	$6D$	$\begin{pmatrix} 0&0&0&1&1&1\\ 1&0&1&1&1&0\\ 1&1&0&1&1&0\\ 1&1&0&0&1&1\\ 1&0&1&1&0&1\\ 1&1&1&1&1&1 \end{pmatrix}$	30240
$6E$	$\begin{pmatrix} 1&1&0&0&1&0\\ 1&1&1&1&0&1\\ 1&0&0&1&1&1\\ 1&1&1&1&1&0\\ 1&0&1&1&0&0\\ 1&1&1&0&1&1 \end{pmatrix}$	40320	$6F$	$\begin{pmatrix} 1&1&0&0&0&0\\ 1&0&1&0&1&1\\ 1&0&0&0&1&1\\ 0&1&0&1&0&0\\ 1&0&1&1&1&1\\ 1&0&0&0&0&1 \end{pmatrix}$	40320
$6G$	$\begin{pmatrix} 1&0&1&0&1&1\\ 1&1&0&0&1&0\\ 1&0&0&1&0&0\\ 0&1&0&0&0&1\\ 0&1&0&1&0&0\\ 0&0&0&1&0&0 \end{pmatrix}$	120960	$7A$	$\begin{pmatrix} 0&1&0&0&1&0\\ 1&0&0&1&0&1\\ 0&1&1&0&1&1\\ 1&0&1&0&0&0\\ 1&0&1&1&0&1\\ 1&1&1&0&0&0 \end{pmatrix}$	207360
$8A$	$\begin{pmatrix} 1&0&0&0&0&0\\ 1&1&0&1&1&0\\ 1&0&0&1&0&0\\ 1&1&1&1&0&0\\ 0&1&0&0&0&0\\ 1&0&1&0&0&1 \end{pmatrix}$	90720	$8B$	$\begin{pmatrix} 0&0&1&1&1&0\\ 0&0&1&1&0&1\\ 1&1&0&1&1&1\\ 1&1&0&1&0&1\\ 1&0&0&0&1&0\\ 1&0&0&1&1&0 \end{pmatrix}$	90720
$9A$	$\begin{pmatrix} 0&1&1&1&0&0\\ 1&1&0&1&1&1\\ 0&1&1&0&1&1\\ 0&0&1&0&1&0\\ 0&1&1&1&0&1\\ 1&1&1&1&1&0 \end{pmatrix}$	161280	$10A$	$\begin{pmatrix} 1&0&0&1&1&0\\ 0&0&1&0&1&1\\ 1&1&0&0&1&1\\ 0&0&0&0&1&0\\ 0&0&1&1&1&1\\ 1&0&0&1&0&1 \end{pmatrix}$	145152
$12A$	$\begin{pmatrix} 1&1&0&0&0&0\\ 1&1&0&1&1&1\\ 1&0&1&1&1&1\\ 1&0&1&1&0&0\\ 1&1&1&0&1&1\\ 1&1&1&1&1&0 \end{pmatrix}$	60480	$12B$	$\begin{pmatrix} 1&1&0&1&0&0\\ 0&1&1&1&0&1\\ 1&0&0&1&0&0\\ 0&0&0&1&1&1\\ 0&0&0&1&0&0\\ 1&1&0&0&1&0 \end{pmatrix}$	60480

$[g]_G$	M	$\lvert[g]_G\rvert$	$[g]_G$	M	$\lvert[g]_G\rvert$
$12C$	$\begin{pmatrix} 0&1&0&1&0&0\\ 1&1&0&1&0&0\\ 1&0&0&1&0&0\\ 1&1&1&0&1&0\\ 1&1&1&1&1&1\\ 1&1&1&0&0&1 \end{pmatrix}$	120960	$15A$	$\begin{pmatrix} 1&0&0&1&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&1&0\\ 0&1&0&1&1&0\\ 0&1&1&0&0&1\\ 0&0&1&1&1&0 \end{pmatrix}$	96768

When G acts on N and invariably on the classes of N, then by Corollary 4.3.7 G fixes the zero vector and acts transitively on the remaining 63 nonzero vectors of N. Thus we obtain two orbits of lengths 1 and 63 with two corresponding stabilizers $S_1$and S_2 of indices 1 and 63 respectively in G. Obviously $S_1 = G$ and S_2 must sit inside one of the maximal subgroups of G. However any maximal subgroup of G which contains S_2 must have its order divisible by $|S_2|$ and its index in G must divide 63. From the ATLAS we obtain that up to isomorphism and conjugacy there is only one maximal subgroup of G which would contain S_2 and that subgroup is isomorphic to $2^5{:}S_6$. However we have that $|S_2| = |2^5{:}S_6|$ and thus $S_2 \cong 2^5{:}S_6$. Let X be the set of all non-zero vectors of N. Then G acts on X transitively with the stabilizer $G_x = S_2$, for $x \in X$. The action of G on X is the same as the action of G on the cosets of S_2 and this action gives rise to a permutation representation which affords a permutation character $\chi(G|S_2)$ of degree 63. For each $g \in G$, the number of fixed points of $g \in G$ in N is equal to $k = |C_N(g)|$. Since the zero vector of N is fixed by every $g \in G$, we have $k = 1 + \chi(G|S_2)(g)$ and hence we obtain that

$$k = 1 + (1a + 27a + 35b)(g) \quad ,$$

where $\chi(G|S_2) = 1a + 27a + 35b$ is written in terms of the irreducible characters of $SP(6,2)$. However since $C_N(g) \leq N$, we must have $k = 2^n$, where $n \in \{0,1,2,3,4,5,6\}$. Hence we obtain the values of the k's for the various classes of G and these are given below.

$[g]_G$	$1A$	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$4E$	$5A$	$6A$
k	64	32	16	16	8	16	1	4	4	8	8	4	4	4	4
$[g]_G$	$6B$	$6C$	$6D$	$6E$	$6F$	$6G$	$7A$	$8A$	$8B$	$9A$	$10A$	$12A$	$12B$	$12C$	$15A$
k	8	1	4	2	4	2	1	2	2	1	2	2	2	1	1

Having obtained the values of the k's for the various classes of G, then we need to calculate the f_j's corresponding to these various k's. For this purpose we use Programme A given in Chapter 2, Section 2.3.

```
V : vector space(6, GF(2));
S : symplectic(6, GF(2));
c : classes(S);
O : matrix orbit(S, vec(1,1,1,1,1,1), false);
for i = 1 to 30 do;
print c[i], '$N';
e = null;
w = vec(0) of V;
while O - e ne [ ] do;
d = null;
for each x in O do;
y = [x + w + (x * c[i])];
d = d join y;
end;
print d, '$N';
print '******';
e = d join e;
if O - e ne [ ] then;
w = setrep(O - e);
end;
end;

r = null;
u = vec(0) of V;
while O - r ne [ ] do;
m = null;
for each g in centralizer(S, c[i]) do;
l = [u * g];
m = m join l;
end;
print 'A block for the vectors under the action of centralizer :';
```

print m;
$r = m\ join\ r$;
if $O - r$ *ne* [ ] *then*;
$u = setrep(O - r)$;
end;
end;
print '**********************************';
end;

From the programme output we calculate f_j the number of orbits Q_i's for $1 \leq i \leq k$, which have come together under the action of $C_G(g)$ to form one orbit Δ_j. Having obtained the f_j's, we therefore deduce that the group $\overline{G} = 2^6{:}SP(6,2)$ has altogether 67 conjugacy classes of elements. These values are listed in Table 6.1. In this table we also list the d_j's where $d_j g$ is a representative of the Δ_j. Now for each class representative $g \in G$, we calculate the lengths of the corresponding classes $[x]_{\overline{G}}$ of $\overline{G}$ by using the theory of the conjugacy classes of the group extensions which has been discussed in Chapter 2 (Section 2.3). For each $[x]_{\overline{G}}$, the order of $C_{\overline{G}}(x)$ is also given. The conjugacy classes $[x]_{\overline{G}}$ of $\overline{G}$ are listed in column 6 of Table 6.1.

For example if $g = 2A$, then $k = 32$, $f_1 = 1$, $f_2 = 15$ and $f_3 = 16$. Hence we produce three corresponding classes $[x_1]_{\overline{G}}$, $[x_2]_{\overline{G}}$ and $[x_3]_{\overline{G}}$. For $[x_1]_{\overline{G}}$, we have

$$|C_{\overline{G}}(x_1)| = \frac{k|C_G(g)|}{f_1} = \frac{32 \times 23040}{1} = 737280$$

and

$$|[x_1]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x_1)|} = 126 \ .$$

For $[x_2]_{\overline{G}}$, we have

$$|C_{\overline{G}}(x_2)| = \frac{k|C_G(g)|}{f_2} = \frac{32 \times 23040}{15} = 49152$$

and

$$|[x_2]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x_2)|} = 1890 \ .$$

Similarly for $[x_3]_{\overline{G}}$, we have

$$|C_{\overline{G}}(x_3)| = \frac{k|C_G(g)|}{f_3} = \frac{32 \times 23040}{16} = 46080$$

and

$$|[x_3]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x_3)|} = 2016 \ .$$

For a class representative $dg \in \overline{G}$ where $d \in 2^6$, $g \in SP(6,2)$ and $o(g) = m$, by Theorem 2.3.10 and Remark 2.3.11 we have

$$o(dg) = \begin{cases} m & \text{if } w = 1_N \\ 2m & \text{otherwise} \end{cases} .$$

To calculate the orders of the class representatives $dg \in \overline{G}$, we use Programme B given in Chapter 2 to compute w for each $d \in N$ and each class representative $g \in SP(6,2)$. For example for $g = 2A$ and $[x_1]_{\overline{G}}$ we have

$V : vector\ space(6, GF(2));$
$S : symplectic(6, GF(2));$
$c : classes(S);$
$g = c[2];$
$d = vec(0,0,0,0,0,0);$
$w = d + d * g + d * (g^2) + d * (g^3) + \ldots + d * (g^{m-1});$
$print\ w;$

Observe that $g = 2A = c[2]$ is an involution of $SP(6,2)$ and thus $m = 2$. Then we obtain that $w = (0,0,0,0,0,0) = 1_N$ and hence $o(dg) = 2$ and we obtain the class $2B$ of $\overline{G}$. For $[x_2]_{\overline{G}}$ we have

$V : vector\ space(6, GF(2));$
$S : symplectic(6, GF(2));$
$c : classes(S);$
$g = c[2];$
$d = vec(1,1,1,1,1,1);$
$w = d + d * g + d * (g^2) + d * (g^3) + \ldots + d * (g^{m-1});$
$print\ w;$

Since $g = c[2]$ and $m = 2$, we obtain that $w = (0,0,0,0,0,0) = 1_N$ and hence $o(dg) = 2$ and we obtain the class $2C$ of $\overline{G}$. For $[x_3]_{\overline{G}}$ we have

$V : vector\ space(6, GF(2));$
$S : symplectic(6, GF(2));$

$c : classes(S);$
$g = c[2];$
$d = vec(1,1,1,1,1,0);$
$w = d + d * g + d * (g^2) + d * (g^3) + \ldots + d * (g^{m-1});$
$print\ w;$

Since $g = c[2]$ and $m = 2$, we obtain that $w = (1,0,0,1,0,0) \neq 1_N$ and hence $o(dg) = 2 \times 2 = 4$ and we obtain the class $4A$ of $\overline{G}$. Table 6.1 below gives detailed information about the conjugacy classes of $\overline{G}$.

Table 6.1: The conjugacy classes of elements of $2^6{:}SP(6,2)$

$[g]_G$	k	f_j	d_j	w	$[x]_{\overline{G}}$	$\lvert[x]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(x)\rvert$
$1A$	64	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$1A$	1	92897280
		$f_2 = 63$	$(1,1,1,1,1,1)$	$(1,1,1,1,1,1)$	$2A$	63	1474560
$2A$	32	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2B$	126	737280
		$f_2 = 15$	$(1,1,1,1,1,1)$	$(0,0,0,0,0,0)$	$2C$	1890	49152
		$f_3 = 16$	$(1,1,1,1,1,0)$	$(1,0,0,1,0,0)$	$4A$	2016	46080
$2B$	16	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2D$	1260	73728
		$f_2 = 12$	$(0,0,0,0,0,1)$	$(0,1,1,1,1,0)$	$4B$	15120	6144
		$f_3 = 3$	$(1,1,1,0,0,0)$	$(0,0,0,0,0,0)$	$2E$	3780	24576
$2C$	16	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2F$	3780	24576
		$f_2 = 3$	$(1,1,0,1,1,0)$	$(0,0,0,0,0,0)$	$2G$	11340	8192
		$f_3 = 4$	$(0,1,0,1,1,0)$	$(0,1,1,0,0,0)$	$4C$	15120	6144
		$f_4 = 8$	$(0,1,0,0,1,0)$	$(0,1,0,0,0,1)$	$4D$	30240	3072
$2D$	8	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2H$	30240	3072
		$f_2 = 3$	$(1,0,0,1,1,1)$	$(1,0,0,0,0,1)$	$4E$	90720	1024
		$f_3 = 3$	$(0,0,1,0,0,1)$	$(0,1,0,1,1,0)$	$4F$	90720	1024
		$f_4 = 1$	$(0,1,0,1,0,0)$	$(1,1,1,0,0,1)$	$4G$	30240	3072
$3A$	16	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3A$	2688	34560
		$f_2 = 15$	$(0,0,0,0,0,1)$	$(0,0,1,0,1,1)$	$6A$	40320	2304
$3B$	1	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3B$	143360	648
$3C$	4	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3C$	215040	432
		$f_2 = 3$	$(1,0,0,1,0,0)$	$(1,1,1,0,0,1)$	$6B$	645120	144
$4A$	4	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4H$	60480	1536
		$f_2 = 3$	$(0,1,0,0,1,1)$	$(0,0,0,0,0,0)$	$4I$	181440	512
$4B$	8	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4J$	60480	1536
		$f_2 = 3$	$(1,1,0,1.1,0)$	$(0,0,0,0,0,0)$	$4K$	181440	512
		$f_3 = 4$	$(1,1,1,1,0,1)$	$(0,1,1,0,0,0)$	$8A$	241920	384

Table 6.1: The conjugacy classes of elements of $2^6{:}SP(6,2)$(continued)

$[g]_G$	k	f_j	d_j	w	$[x]_{\overline{G}}$	$\lvert[x]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(x)\rvert$
$4C$	8	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4L$	60480	1536
		$f_2 = 3$	$(0,1,0,0,0,0)$	$(0,0,0,0,0,0)$	$4M$	181440	512
		$f_3 = 4$	$(1,1,0,1,1,0)$	$(1,1,1,0,1,1)$	$8B$	241920	384
$4D$	4	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4N$	181440	512
		$f_2 = 1$	$(1,0,0,1,1,1)$	$(0,0,0,0,0,0)$	$4O$	181440	512
		$f_3 = 2$	$(0,1,1,0,1,1)$	$(0,0,0,0,0,0)$	$4P$	362880	256
$4E$	4	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4Q$	725760	128
		$f_2 = 1$	$(1,0,1,0,0,0)$	$(0,0,0,0,0,0)$	$4R$	725760	128
		$f_3 = 1$	$(0,0,1,0,0,1)$	$(1,1,0,0,0,0)$	$8C$	725760	128
		$f_4 = 1$	$(1,0,1,0,1,0)$	$(1,1,0,0,0,0)$	$8D$	725760	128
$5A$	4	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$5A$	774144	120
		$f_2 = 3$	$(0,0,1,1,0,0)$	$(0,0,0,0,1,1)$	$10A$	2322432	40
$6A$	4	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6C$	161280	576
		$f_2 = 3$	$(0,0,0,1,1,1)$	$(1,0,1,1,0,1)$	$12A$	483840	192
$6B$	8	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6D$	80640	1152
		$f_2 = 3$	$(1,1,0,1,1,0)$	$(0,0,0,0,0,0)$	$6E$	241920	384
		$f_3 = 4$	$(1,0,0,1,0,0)$	$(0,1,0,0,0,1)$	$12B$	322560	288
$6C$	1	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6F$	1290240	72
$6D$	4	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6G$	483840	192
		$f_2 = 1$	$(0,1,1,1,1,0)$	$(0,1,1,0,0,0)$	$12C$	483840	192
		$f_3 = 2$	$(1,1,1,0,1,0)$	$(0,0,1,0,0,1)$	$12D$	967680	96
$6E$	2	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6H$	1290240	72
		$f_2 = 1$	$(1,1,0,1,1,0)$	$(0,0,0,1,0,1)$	$12E$	1290240	72
$6F$	4	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6I$	645120	144
		$f_2 = 3$	$(1,1,1,0,0,0)$	$(0,0,0,0,0,0)$	$6J$	1935360	48
$6G$	2	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6K$	3870720	24
		$f_2 = 1$	$(1,1,1,1,1,0)$	$(1,1,1,0,0,1)$	$12F$	3870720	24
$7A$	1	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$7A$	13271040	7
$8A$	2	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$8E$	2903040	32
		$f_2 = 1$	$(0,1,1,0,1,1)$	$(0,0,0,0,0,0)$	$8F$	2903040	32
$8B$	2	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$8G$	2903040	32
		$f_2 = 1$	$(1,1,0,0,1,0)$	$(0,0,0,0,0,0)$	$8H$	2903040	32
$9A$	1	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$9A$	10321920	9
$10A$	2	$f_1 = 1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$10B$	4644864	20
		$f_2 = 1$	$(1,1,1,1,0,0)$	$(1,0,0,1,0,0)$	$20A$	4644864	20

Table 6.1: The conjugacy classes of elements of $2^6{:}SP(6,2)$(continued)

$[g]_G$	k	f_j	d_j	w	$[x]_{\overline{G}}$	$\lvert[x]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(x)\rvert$
$12A$	2	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$12G$	1935360	48
		$f_2=1$	$(0,1,0,1,0,1)$	$(0,1,1,0,0,0)$	$24A$	1935360	48
$12B$	2	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$12H$	1935360	48
		$f_2=1$	$(1,1,1,1,0,0,)$	$(1,1,1,0,1,1)$	$24B$	1935360	48
$12C$	1	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$12I$	7741440	12
$15A$	1	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$15A$	6193152	15

6.2 The Inertia Groups of $2^6{:}SP(6,2)$

Since G has two orbits on N of lengths 1 and 63 respectively, then by Brauer's theorem (Theorem 5.1.5) G acts on $Irr(N)$ with the same number of orbits. Hence the lengths of these orbits will also be 1 and 63 with corresponding point stabilizers H_1 and H_2 as subgroups of G such that $[G : H_1] = 1$ and $[G : H_2] = 63$. Thus we obtain that $H_1 = SP(6,2)$ and $H_2 = 2^5{:}S_6$. Since H_2 is a split extension, we construct its character table using the technique of the Fischer-Clifford matrices.

6.2.1 The character table of $H_2 = 2^5{:}S_6$

The group S_6 acts naturally on a module of dimension 6 by permuting the basis elements which generate the module. Let V be the 6-dimensional natural module of S_6 over $GF(2)$, where $V = \langle e_1,\ e_2,\ e_3\ , e_4,\ e_5,\ e_6\rangle$, and $e_i^2 = 1$ for $i \in \{1,2,3,4,5,6\}$ where we regard V as a multiplicative elementary abelian 2-group of order 2^6.

Theorem 6.2.1 *Let V be the natural module of S_6 over $GF(2)$. Then there exist S_6-invariant submodules M_1 and M_2 of V such that $V \supset M_2 \supset M_1 \supset 0$ and that*

$$dim(M_2) = 5 \quad and \quad dim(M_1) = 1 \quad .$$

Proof. Let $V = \langle e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6\rangle$ with $e_i^2 = 1$ for $i \in \{1,2,3,4,5,6\}$. Then S_6 acts naturally on V and this natural action results in the following orbits:

1. $O_0 = \{1_V\}$ and $|O_0| = 1$.

2. $O_1 = \{e_i | \ 1 \leq i \leq 6\}$, $|O_1| = 6$.

3. $O_2 = \{e_ie_j | \ 1 \leq i, j \leq 6, i \neq j\}$, $|O_2| = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 15$.

4. $O_3 = \{e_ie_je_k | \ 1 \leq i, j, k \leq 6, distinct \ \ i, j, k\}$, $|O_3| = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 20$.

5. $O_4 = \{e_ie_je_ke_\ell | \ 1 \leq i, j, k, \ell \leq 6, distinct \ \ i, j, k, \ell\}$, $|O_4| = \begin{pmatrix} 6 \\ 4 \end{pmatrix} = 15$.

6. $O_5 = \{e_ie_je_ke_\ell e_s | \ 1 \leq i, j, k, \ell, s \leq 6, distinct \ \ i, j, k, \ell, s\}$, $|O_5| = \begin{pmatrix} 6 \\ 5 \end{pmatrix} = 6$.

7. $O_6 = \{e_1e_2e_3e_4e_5e_6\}$, $|O_6| = \begin{pmatrix} 6 \\ 6 \end{pmatrix} = 1$.

Thus S_6 forms 7 orbits on V. Set $M_1 = \langle e_1e_2e_3e_4e_5e_6 \rangle$. Then M_1 is an S_6-invariant submodule of V with $dim(M_1) = 1$. Now set $M_2 = O_0 \cup O_2 \cup O_4 \cup O_6$. Then M_2 is an S_6-invariant submodule of V and since $|M_2| = 32$, we have $dim(M_2) = 5$. Since $M_1 = O_0 \cup O_6$, we obtain that $V \supset M_2 \supset M_1 \supset 0$. □

Remark 6.2.2 M_2 is not irreducible, however M_2/M_1 is an S_6-invariant irreducible module of dimension 4. Let $M_3 = V/M_1$. Then M_3 is an S_6-invariant module of dimension 5. Thus we obtain two groups of the form $2^5{:}S_6$ which are $M_2{:}S_6$ and $M_3{:}S_6$, where M_2 and M_3 are regarded as elementary abelian groups of order 2^5.

Theorem 6.2.3 *The group $M_2{:}S_6$ is such that under the action of S_6 on M_2, there are four orbits of lengths 1, 1, 15, 15.*

Proof. In the proof of Theorem 6.2.1, we set $M_2 = O_0 \cup O_2 \cup O_4 \cup O_6$. So the orbits of S_6 acting on M_2 are O_0, O_2, O_4, O_6 with

$$|O_0| = 1, \quad |O_2| = 15, \quad |O_4| = 15, \quad |O_6| = 1$$

Thus we obtain four orbits of lengths 1, 15, 15, 1. Hence the result. □

Remark 6.2.4 We observe that $M_2 = \langle e_1e_2, e_1e_3, e_1e_4, e_1e_5, e_1e_6 \rangle$. Call these vectors $\gamma_1 = e_1e_2, \gamma_2 = e_1e_3, \gamma_3 = e_1e_4, \gamma_4 = e_1e_5$ and $\gamma_5 = e_1e_6$. However we have that $S_6 = \langle \alpha,\ \beta \rangle$, where $\alpha = (1\ \ 2)$ and $\beta = (1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6)$. Then we obtain that

$$\begin{aligned} \alpha\ :\ \gamma_1 &\longrightarrow \gamma_1 \\ \gamma_2 &\longrightarrow \gamma_1 + \gamma_2 \\ \gamma_3 &\longrightarrow \gamma_1 + \gamma_3 \\ \gamma_4 &\longrightarrow \gamma_1 + \gamma_4 \\ \gamma_5 &\longrightarrow \gamma_1 + \gamma_5 \end{aligned}$$

and hence α can be represented by the following matrix

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Similarly for β we have that

$$\begin{aligned} \beta\ :\ \gamma_1 &\longrightarrow \gamma_1 + \gamma_2 \\ \gamma_2 &\longrightarrow \gamma_1 + \gamma_3 \\ \gamma_3 &\longrightarrow \gamma_1 + \gamma_4 \\ \gamma_4 &\longrightarrow \gamma_1 + \gamma_5 \\ \gamma_5 &\longrightarrow \gamma_1 \end{aligned}$$

and we obtain β in matrix form as follows:

$$\beta = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The group $H_2 = 2^5{:}S_6$ is a maximal subgroup of $SP(6,2)$ which is isomorphic to $C_{SP(6,2)}(x)$, where x is an element of the $2A$-conjugacy class of $SP(6,2)$. By direct calculation within the group $SP(6,2)$ using CAYLEY and by the above results relating to the group $M_2{:}S_6$, it is not difficult to see that H_2 and $M_2{:}S_6$ can be identified.

We give the conjugacy class representatives of S_6 in terms of 5×5 matrices over $GF(2)$ in the following table, where M is the matrix which represents that particular conjugacy class.

$[g]_{S_6}$	M	$\lvert[g]_{S_6}\rvert$	$[g]_{S_6}$	M	$\lvert[g]_{S_6}\rvert$
$1A$	$\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$	1	$2A$	$\begin{pmatrix} 0&0&1&1&0\\ 0&0&1&0&1\\ 0&0&1&0&0\\ 1&0&1&0&0\\ 0&1&1&0&0 \end{pmatrix}$	15
$2B$	$\begin{pmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 1&0&1&0&0\\ 1&0&0&1&0\\ 1&0&0&0&1 \end{pmatrix}$	15	$2C$	$\begin{pmatrix} 1&0&0&0&1\\ 0&0&1&0&1\\ 0&1&0&0&1\\ 0&0&0&1&1\\ 0&0&0&0&1 \end{pmatrix}$	45
$3A$	$\begin{pmatrix} 0&1&1&0&0\\ 0&1&0&1&0\\ 0&1&0&0&1\\ 0&1&0&0&0\\ 1&1&0&0&0 \end{pmatrix}$	40	$3B$	$\begin{pmatrix} 0&0&0&1&0\\ 0&1&0&0&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1 \end{pmatrix}$	40
$4A$	$\begin{pmatrix} 0&1&0&1&0\\ 0&1&0&0&1\\ 0&1&0&0&0\\ 1&1&0&0&0\\ 0&1&1&0&0 \end{pmatrix}$	90	$4B$	$\begin{pmatrix} 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 1&0&0&0&0 \end{pmatrix}$	90
$5A$	$\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&0&0&0&0 \end{pmatrix}$	144	$6A$	$\begin{pmatrix} 1&1&0&0&0\\ 1&0&1&0&0\\ 1&0&0&1&0\\ 1&0&0&0&1\\ 1&0&0&0&0 \end{pmatrix}$	120
$6B$	$\begin{pmatrix} 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix}$	120			

Theorem 6.2.5 *Under the action of S_6 on $Irr(M_2)$, we obtain four orbits of lengths 1, 6, 10, 15.*

Proof. We know from Theorem 6.2.3 that S_6 has four orbits on the conjugacy classes of M_2. Then by Brauer's theorem (Theorem 5.1.5), we obtain the same number of orbits of S_6 on $Irr(M_2)$. Suppose that V is the 6-dimensional natural module of $SP(6,2)$ over $GF(2)$ and let χ be an irreducible Brauer character of $SP(6,2)$ over $GF(2)$ such that $deg(\chi)=6$. Then χ_{M_2} can be expressed as a sum of six irreducible characters of M_2. Moreover χ_{M_2} is invariant under the action of S_6 on $Irr(M_2)$. Thus we obtain an orbit of length 6. Hence we have two orbits of lengths 1 and 6. Then using the indices of the maximal subgroups of S_6 listed in the ATLAS, the only possibility for the two remaining orbit lengths are 10 and 15. Hence the result. □

Remark 6.2.6 Since we obtain four orbits from the action of S_6 on $Irr(M_2)$, then we obtain four inertia groups $\overline{I}_i = M_2{:}I_i$ in $M_2{:}S_6$, where $i \in \{1,2,3,4\}$ of indices 1, 6, 10, 15 respectively such that

$$I_1 = S_6, I_2 = S_5, I_3 = 3^2{:}D_4, I_4 = S_4 \times 2 \quad ,$$

where D_4 is the dihedral group of order 8.

We had that when S_6 acts on the classes of M_2, this action gives rise to four orbits of lengths 1, 1, 15, 15 with the corresponding stabilizers $S_6, S_6, S_4 \times 2, S_4 \times 2$ respectively. Now let $\chi(S_6|2^5)$ be the permutation character of S_6 acting on 2^5. Then we obtain that

$$\chi(S_6|2^5) = 1 + 1 + I_{S_4\times 2}^{S_6} + I_{S_4\times 2}^{S_6} \quad ,$$

where $I_{S_4\times 2}^{S_6}$ is the identity character of $S_4 \times 2$ induced to S_6. However both $I_{S_4\times 2}^{S_6}$ are the permutation characters of S_6 of degree 15 which we denote by χ_{ρ_i}, where $i \in \{1,2\}$. Then from the ATLAS, we obtain that

$$\chi_{\rho_i} \in \{1a + 5a + 9a, 1a + 5b + 9a\} \quad .$$

Then we obtain that

$$\chi(S_6|2^5) = \begin{cases} 1a + 1a + \chi_{\rho_1} + \chi_{\rho_2} & \text{if } \chi_{\rho_1} \neq \chi_{\rho_2} \\ 1a + 1a + 2\chi_{\rho_i} & \text{where } i \in \{1,2\} \text{ and } \chi_{\rho_1} = \chi_{\rho_2} \end{cases} .$$

Now using the character table of S_6 we obtain

$[g]_{S_6}$	$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$4A$	$4B$	$5A$	$6A$	$6B$
$1a+5a+9a$	15	3	7	3	0	3	1	1	0	0	1
$1a+5b+9a$	15	7	3	3	3	0	1	1	0	1	0

However if $\chi(S_6|2^5) = 1a + 1a + \chi_{\rho_1} + \chi_{\rho_2}$, then $\chi(S_6|2^5)(2A) = 12 \neq 2^n$ for any $n \in \mathbb{N} \cup \{0\}$. This contradicts the fact that $\chi(S_6|2^5)(g) = 2^n$ for all $g \in S_6$ and some $n \in \{0,1,2,3,4,5\}$. Thus we must have

$$\chi(S_6|2^5) = 1a + 1a + 2\chi_{\rho_i} \quad, i \in \{1,2\} \quad \text{with} \quad \chi_{\rho_1} = \chi_{\rho_2} \quad .$$

Hence we obtain that

$$\chi(S_6|2^5) = 4 \times 1a + 2 \times 5a + 2 \times 9a \quad \text{or} \quad \chi(S_6|2^5) = 4 \times 1a + 2 \times 5b + 2 \times 9a \quad .$$

Therefore we obtain the following possible values of $\chi(S_6|2^5)$ on the classes of S_6.

$[g]_{S_6}$	$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$4A$	$4B$	$5A$	$6A$	$6B$
$\chi(S_6\|2^5)$	32	8	16	8	2	8	4	4	2	2	4
$\chi(S_6\|2^5)$	32	16	8	8	8	2	4	4	2	4	2

Thus the values of $\chi(S_6|2^5)$ give us the values of the k's which we need for computing the conjugacy classes of $H_2 = 2^5{:}S_6$ for the various classes of S_6 (see Chapter 2, Section 2.3). In Remark 6.2.4 we constructed the group S_6 as a matrix group over $GF(2)$ generated by 5×5 matrices α and β. Using the action of S_6 on $M_2 = \langle \gamma_1, \gamma_2, \ldots, \gamma_5 \rangle$, and the method developed in Chapter 2, Section 2.3, we are able to compute the exact values of the k's which are listed in the following table.

$[g]_{S_6}$	$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$4A$	$4B$	$5A$	$6A$	$6B$
k	32	8	16	8	2	8	4	4	2	2	4

and we deduce that $\chi(S_6|2^5) = 4 \times 1a + 2 \times 5a + 2 \times 9a$. We again use Programme A from Chapter 2, Section 2.3, to obtain the f_j's and hence the conjugacy classes of elements of $2^5{:}S_6$. See Appendix, Programme A for $2^5{:}S_6$.

We then obtain the values for the f_j's, the corresponding vectors d_j's and w's. Table 6.2 provides detailed information for the conjugacy classes $[x]_{H_2}$ of elements of $H_2 = 2^5{:}S_6$.

Table 6.2: The conjugacy classes of elements of $2^5{:}S_6$

$[g]_{S_6}$	k	f_j	d_j	w	$[x]_{H_2}$	$\|[x]_{H_2}\|$	$\|C_{H_2}(x)\|$
$1A$	32	$f_1 = 1$	$(0,0,0,0,0)$	$(0,0,0,0,0)$	$1A$	1	23040
		$f_2 = 1$	$(1,1,1,1,1)$	$(1,1,1,1,1)$	$2A$	1	23040
		$f_3 = 15$	$(1,1,1,1,0)$	$(1,1,1,1,0)$	$2B$	15	1536
		$f_4 = 15$	$(0,0,0,0,1)$	$(0,0,0,0,1)$	$2C$	15	1536
$2A$	8	$f_1 = 1$	$(0,0,0,0,0)$	$(0,0,0,0,0)$	$2D$	60	384
		$f_2 = 1$	$(1,1,1,1,1)$	$(0,0,0,0,0)$	$2E$	60	384
		$f_3 = 6$	$(1,1,1,1,0)$	$(0,1,1,0,1)$	$4A$	360	64
$2B$	16	$f_1 = 1$	$(0,0,0,0,0)$	$(0,0,0,0,0)$	$2F$	30	768
		$f_2 = 1$	$(1,1,1,1,1)$	$(0,0,0,0,0)$	$2G$	30	768
		$f_3 = 4$	$(0,0,1,1,1)$	$(1,0,0,0,0)$	$4B$	120	192
		$f_4 = 4$	$(1,0,0,0,1)$	$(1,0,0,0,0)$	$4C$	120	192
		$f_5 = 6$	$(0,0,0,1,1)$	$(0,0,0,0,0)$	$2H$	180	128
$2C$	8	$f_1 = 1$	$(0,0,0,0,0)$	$(0,0,0,0,0)$	$2I$	180	128
		$f_2 = 1$	$(1,1,1,1,1)$	$(0,0,0,0,0)$	$2J$	180	128
		$f_3 = 1$	$(0,1,0,0,0)$	$(0,1,1,0,1)$	$4D$	180	128
		$f_4 = 1$	$(1,1,0,1,1)$	$(0,1,1,0,1)$	$4E$	180	128
		$f_5 = 4$	$(0,1,1,1,1)$	$(0,0,0,0,1)$	$4F$	720	32
$3A$	2	$f_1 = 1$	$(0,0,0,0,0)$	$(0,0,0,0,0)$	$3A$	640	36
		$f_2 = 1$	$(1,0,1,1,1)$	$(1,1,1,1,1)$	$6A$	640	36
$3B$	8	$f_1 = 1$	$(0,0,0,0,0)$	$(0,0,0,0,0)$	$3B$	160	144
		$f_2 = 1$	$(1,1,1,1,1)$	$(1,1,1,1,1)$	$6B$	160	144
		$f_3 = 3$	$(1,1,1,0,0)$	$(0,1,0,0,0)$	$6C$	480	48
		$f_4 = 3$	$(1,1,1,1,0)$	$(1,1,1,1,0)$	$6D$	480	48
$4A$	4	$f_1 = 1$	$(0,0,0,0,0)$	$(0,0,0,0,0)$	$4G$	720	32
		$f_2 = 1$	$(1,1,1,1,1)$	$(0,0,0,0,0)$	$4H$	720	32
		$f_3 = 2$	$(0,0,0,1,1)$	$(0,1,1,0,1)$	$8A$	1440	16
$4B$	4	$f_1 = 1$	$(0,0,0,0,0)$	$(0,0,0,0,0)$	$4I$	720	32
		$f_2 = 1$	$(1,1,1,1,1)$	$(0,0,0,0,0)$	$4J$	720	32
		$f_3 = 2$	$(0,1,1,1,1)$	$(1,1,0,1,1)$	$8B$	1440	16
$5A$	2	$f_1 = 1$	$(0,0,0,0,0)$	$(0,0,0,0,0)$	$5A$	2304	10
		$f_2 = 1$	$(1,1,1,1,1)$	$(1,1,1,1,1)$	$10A$	2304	10
$6A$	2	$f_1 = 1$	$(0,0,0,0,0)$	$(0,0,0,0,0)$	$6E$	1920	12
		$f_2 = 1$	$(1,1,1,1,1)$	$(0,0,0,0,0)$	$6F$	1920	12
$6B$	4	$f_1 = 1$	$(0,0,0,0,0)$	$(0,0,0,0,0)$	$6G$	960	24
		$f_2 = 1$	$(0,0,0,1,0)$	$(0,0,0,0,0)$	$6H$	960	24
		$f_3 = 1$	$(1,0,1,1,1)$	$(0,1,0,0,1)$	$12A$	960	24
		$f_4 = 1$	$(0,0,1,1,1)$	$(0,1,0,0,1)$	$12B$	960	24

Table 6.2 shows that $H_2 = 2^5{:}S_6$ has altogether 37 conjugacy classes of elements.

6.2.2 The inertia groups of $2^5{:}S_6$

We proved that when S_6 acts on $Irr(2^5)$, then we obtain four orbits of lengths $1, 6, 10, 15$. Thus we obtain four inertia groups $\overline{I}_i = 2^5{:}I_i$ for $2^5{:}S_6$ where $i \in \{1,2,3,4\}$ of indices 1, 6, 10, 15 respectively in $2^5{:}S_6$ such that $I_1 = S_6, I_2 = S_5, I_3 = 3^2{:}D_4$ and $I_4 = S_4 \times 2$. We observe that I_3 is a split extension and thus we compute its character table using the Fischer-Clifford matrices.

We construct the group D_4 as a group of 2×2 matrices over $GF(3)$, that is as a subgroup of $GL(2,3)$ so that it acts on $V = 3^2$. Then D_4 is generated by two 2×2 matrices over $GF(3)$ as follows

$$a = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

where $o(a) = 4$ and $o(b) = 2$ such that $bab = a^{-1}$. We observe that D_4 has five conjugacy classes of elements. We give the conjugacy class representatives of D_4 in terms of 2×2 matrices over $GF(3)$ in the following table, where M is the matrix which represents that particular conjugacy class.

$[d]_{D_4}$	M	$\lvert[d]_{D_4}\rvert$	$[d]_{D_4}$	M	$\lvert[d]_{D_4}\rvert$
$1A$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	$2A$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	1
$2B$	$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	2	$2C$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	2
$4A$	$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$	2			

Lemma 6.2.7 *The action of D_4 on 3^2 gives rise to three orbits of lengths 1, 4, 4.*

Proof. We observe that $3^2{:}D_4 = (3^2{:}4){:}2$ where $3^2{:}4$ is a maximal subgroup of A_6 of index 10. Thus when 4 acts on 3^2, then it fixes the identity in 3^2. If again 4 fixes a non-identity element say $\alpha \in 3^2$, then α commutes with all the elements in $3^2{:}4$ and in particular α will commute with the element β, where $\langle \beta \rangle = 4$. Then we obtain an element $\alpha\beta \in A_6$ with $o(\alpha\beta) = 12$ which is a contradiction. Thus the possibilities for the orbit lengths under the action of 4 on 3^2 are $\{1,2,2,4\}$, $\{1,2,2,2,2\}$ or $\{1,4,4\}$. Suppose that we have the possibility $\{1,2,2,4\}$, then β has the cycle type 12^24 on 3^2. So assume that $(x\ y)$, where $x, y \in 3^2$, is one of the 2-cycles in the cycle type of β, then β^2 will fix both x and y. Then we obtain an element $x\beta^2 \in A_6$ with $o(x\beta^2) = 6$ which is a contradiction. Similarly $\{1,2,2,2,2\}$ is not possible. Hence we must have the possibility $\{1,4,4\}$. Now we consider $3^2{:}D_4$ and the action of D_4 on 3^2. Since $4 \subset D_4$ and under the action of 4 on 3^2 we have the orbit lengths $\{1,4,4\}$, when D_4 acts on 3^2 we get two possibilities of $\{1,4,4\}$ or $\{1,8\}$ for orbit lengths. Let $P = 3^2$. Then $P \in Syl_3(A_6)$ and $P \in Syl_3(S_6)$. Hence P contains representatives from all the classes of S_6 having elements of order 3. So we can assume that there exist x and y in P such that $x \in 3A$ and $y \in 3B$ where $3A$ and $3B$ are conjugacy classes of S_6. We deduce that x and y are not conjugate in S_6. Since $D_4 \leq S_6$ and $x, y \in 3^2$, the elements x and y lie in two distinct orbits under the action of D_4 on 3^2. Thus we must have the orbit lengths $\{1,4,4\}$. $\square$

Lemma 6.2.8 *The action of D_4 on $Irr(3^2)$ gives rise to three orbits of lengths 1, 4, 4.*

Proof. Since D_4 acting on the classes of 3^2 produces three orbits, D_4 acting on $Irr(3^2)$ will also produce three orbits of lengths $1, t, z$ where $t, z \in \mathbb{N}$ such that $1+t+z = 9$. However from the subgroup-indices in D_4, we obtain that $t, z \notin \{2,6,8\}$. Thus the only possibility is $t = z = 4$. Hence the result. $\square$

We had that D_4 acting on the classes of 3^2 produces three orbits of lengths 1, 4, 4. Then the point stabilizers corresponding to these orbits are D_4, $\mathbf{Z}_2$ and $\mathbf{Z}_2$ respectively. Now let $\chi(D_4|3^2)$ be the permutation character of D_4 acting on 3^2. Then we obtain that

$$\chi(D_4|3^2) = 1 + I_{\mathbf{Z}_2}^{D_4} + I_{\mathbf{Z}_2}^{D_4} ,$$

where $I_{\mathbf{Z}_2}^{D_4}$ is the identity character of $\mathbf{Z}_2$ induced to D_4. Thus for any class $[d]$ of D_4, we must have that $k = \chi(D_4|3^2)(d) = 3^m$, where $m \in \{0,1,2\}$. However both $I_{\mathbf{Z}_2}^{D_4}$ are the permutation characters of D_4 of degree 4. It is not difficult to see that we have three permutation characters of D_4 of degree 4 denoted by $\pi_i, i \in \{1,2,3\}$. Then we obtain the following table for these candidates:

$[d]_{D_4}$	$1A$	$2A$	$2B$	$2C$	$4A$
π_1	4	0	0	2	0
π_2	4	0	2	0	0
π_3	4	4	0	0	0

Since $\chi(D_4|3^2) = 1 + I_{\mathbf{Z}_2}^{D_4} + I_{\mathbf{Z}_2}^{D_4}$, we have $\chi(D_4|3^2) = 2\pi_i + 1, i \in \{1,2,3\}$ or $\chi(D_4|3^2) = \pi_i + \pi_j + 1, i \neq j, i, j \in \{1,2,3\}$. However $\chi(D_4|3^2) = \pi_i + \pi_3 + 1, i \in \{1,2\}$ and $\chi(D_4|3^2) = 2\pi_i + 1, i \in \{1,2,3\}$ produce values for k's for some classes of D_4 which are not of the form 3^m, $m \in \{0,1,2\}$. Thus the only working possibility is $\chi(D_4|3^2) = \pi_1 + \pi_2 + 1$ and we get the following table for the corresponding values of these k's.

$[d]_{D_4}$	$1A$	$2A$	$2B$	$2C$	$4A$
k	9	1	3	3	1

Using Programme A from Chapter 2, Section 2.3, we are able to obtain the f_j's and hence the conjugacy classes of elements of $3^2{:}D_4$. See Appendix, Programme A for $3^2{:}D_4$.

Having obtained the f_j's, we then use Programme B from Chapter 2 (Section 2.3) to determine the orders of the conjugacy class representatives. Table 6.3 below provides details of the conjugacy classes $[x]_{I_3}$ of elements of $I_3 = 3^2{:}D_4$.

Table 6.3: The conjugacy classes of elements of $3^2{:}D_4$

$[d]_{D_4}$	k	f_j	d_j	w	$[x]_{I_3}$	$\lvert[x]_{I_3}\rvert$	$\lvert C_{I_3}(x)\rvert$
$1A$	9	$f_1 = 1$	$(0,0)$	$(0,0)$	$1A$	1	72
		$f_2 = 4$	$(1,1)$	$(1,1)$	$3A$	4	18
		$f_3 = 4$	$(1,0)$	$(1,0)$	$3B$	4	18
$2A$	1	$f_1 = 1$	$(0,0)$	$(0,0)$	$2A$	9	8
$2B$	3	$f_1 = 1$	$(0,0)$	$(0,0)$	$2B$	6	12
		$f_2 = 2$	$(1,1)$	$(2,0)$	$6A$	12	6
$2C$	3	$f_1 = 1$	$(0,0)$	$(0,0)$	$2C$	6	12
		$f_2 = 2$	$(0,1)$	$(1,1)$	$6B$	12	6
$4A$	1	$f_1 = 1$	$(0,0)$	$(0,0)$	$4A$	18	4

Thus we observe that $I_3 = 3^2{:}D_4$ has altogether 9 conjugacy classes.

In order to compute the charater table of $3^2{:}D_4$, we need to obtain its inertia groups. We proved that when D_4 acts on $Irr(3^2)$ we obtain three orbits of lengths 1, 4, 4 and thus three corresponding inertia groups $\overline{T}_i = 3^2{:}T_i$, where $i \in \{1,2,3\}$ of indices 1, 4, 4 respectively in $3^2{:}D_4$. Thus we have $T_1 = D_4, T_2 = \mathbf{Z}_2, T_3 = \mathbf{Z}_2$. By looking at the conjugacy classes of $3^2{:}D_4$ listed above we obtain that no element of $2A$ fixes an element of order 3 in 3^2. But each elements of $2B$ and $2C$ fixes some elements of order 3 in 3^2 respectively, which give rise to the elements of order 6 in $6A$ and $6B$ classes of $3^2{:}D_4$. By considering the character table of 3^2 it is not difficult to see that

(a) there is no $\chi \in Irr(3^2)$ and no $\alpha \in 2A$ such that $\chi^\alpha = \chi$.

(b) for $x \in 2B$ and $y \in 2C$, there exist $\chi, \psi \in Irr(3^2)$ such that $\chi \neq \psi$ and $\chi^x = \chi$ and $\psi^y = \psi$.

Hence without loss of generality we can assume that $T_2 = \langle x \rangle$ and $T_3 = \langle y \rangle$ for some $x \in 2B$ and $y \in 2C$. Since T_2 and T_3 are subgroups of D_4, we deduce that x and y fuse to $2B$ and $2C$ classes of D_4 respectively. Thus we have obtained the complete fusions of T_2 and T_3 into D_4. Having obtained these fusions, we are now able to compute the Fischer-Clifford matrices of the group $3^2{:}D_4$. We will use the relations of Proposition 5.2.3, Theorem 5.2.4 and the properties (a) through (f) of the Fischer-Clifford matrices which are given in Chapter 5 (Section 5.2.2). Note that all the relations hold since 3^2 is an elementary abelian group. Consider the conjugacy class $2B$ of D_4. Then we obtain that $M(2B)$ has the following form with corresponding *weights* attached to the rows and columns

$$M(2B) = \begin{array}{c c} & \begin{array}{cc} 12 & 6 \end{array} \\ \begin{array}{c} 4 \\ 2 \end{array} & \begin{pmatrix} a & c \\ b & d \end{pmatrix} \\ & \begin{array}{cc} 3 & 6 \end{array} \end{array}$$

However by Theorem 5.2.4 we have $a = c = 1$ and by property (e) of the properties given in Chapter 5 (Section 5.2.2) of the Fischer-Clifford matrices, we obtain $b = 2$.

By the orthogonalities of the columns and rows respectively we must have $4+4d=0$ and $6+6d=0$. Hence $d=-1$ and we obtain $M(2B)$ to be given by

$$M(2B)=\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}.$$

Now we consider $M(1A)$. Then

$$M(1A)=\begin{array}{c}\\ 8\\ 2\\ 2\\ \\ \end{array}\begin{array}{c}\begin{array}{ccc}72 & 18 & 18\end{array}\\ \begin{pmatrix}1 & 1 & 1\\ 4 & a & c\\ 4 & b & d\end{pmatrix}\\ \begin{array}{ccc}1 & 4 & 4\end{array}\end{array}$$

such that $8+2a^2+2b^2=18$ and $8+8a+8b=0$. Hence we obtain that $a^2+b^2=5$ and $a+b=-1$. We deduce that $\{a=1,b=-2\}$ or $\{a=-2,b=1\}$. Similarly c and d must satisfy the relations $c^2+d^2=5$ and $c+d=-1$ and hence $\{c=1,d=-2\}$ or $\{c=-2,d=1\}$. Using the weights $m_1=1, m_2=4$ and $m_3=4$ for the orthogonality of the first and second rows we obtain $4+4a+4c=0$ and hence $a+c=-1$. Similarly we obtain $b+d=-1$. Thus $\{a=1,b=-2,c=-2,d=1\}$ or $\{a=-2,b=1,c=1,d=-2\}$. Hence we have the following two possibilities for $M(1A)$:

$$\begin{pmatrix}1 & 1 & 1\\ 4 & 1 & -2\\ 4 & -2 & 1\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}1 & 1 & 1\\ 4 & -2 & 1\\ 4 & 1 & -2\end{pmatrix}.$$

Since in $3^2{:}D_4$ we have $(6A)^2=3A$, for $\chi\in Irr(3^2{:}D_4)$ we must have $\chi(3A)\equiv\chi(6A)(mod\ 2)$. Checking the validity of this congruent relation for the portions of the character table of $3^2{:}D_4$ corresponding to $M(2B)$ and to the two candidates of $M(1A)$ we deduce that $M(1A)=\begin{pmatrix}1 & 1 & 1\\ 4 & 1 & -2\\ 4 & -2 & 1\end{pmatrix}$ is the only candidate.

We obtain all the Fischer-Clifford matrices for $3^2{:}D_4$ which are listed in Table 6.4 below.

Table 6.4: The Fischer-Clifford matrices of $3^2{:}D_4$

$M(d)$	$M(d)$	$M(d)$
$M(1A) = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 1 & -2 \\ 4 & -2 & 1 \end{pmatrix}$	$M(2A) = \begin{pmatrix} 1 \end{pmatrix}$	$M(2B) = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$
$M(2C) = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$	$M(4A) = \begin{pmatrix} 1 \end{pmatrix}$	

The character tables of $T_1 = D_4, T_2$ and T_3 are as follows:

The character table of T_1

$[x]_{T_1}$	$1A$	$2A$	$2B$	$2C$	$4A$
$\lvert[x]_{T_1}\rvert$	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	1	-1
χ_5	2	-2	0	0	0

The character table of T_2

$[x]_{T_2}$	$1A$	$2B$
$\lvert[x]_{T_2}\rvert$	1	1
χ_1	1	1
χ_2	1	-1

The character table of T_3

$[x]_{T_3}$	$1A$	$2C$
$\lvert[x]_{T_3}\rvert$	1	1
χ_1	1	1
χ_2	1	-1

We use the Fischer-Clifford matrices given in Table 6.4 and the character tables of $T_1 = D_4$, T_2 and T_3 together with the fusions of T_2 and T_3 into D_4 to obtain the character table of $3^2{:}D_4$. For example using $M(1A)$ and the portions of the character tables of the inertia factors which correspond to the classes that fuse into $1A$ in D_4, we compute the portion of the character table of $3^2{:}D_4$ which corresponds to the identity coset as follows:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -2 \\ 4 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 1 \\ 4 & -2 & 1 \end{bmatrix} .$$

Similarly we use $M(2B)$ to compute the portion of the character table of $3^2{:}D_4$ which corresponds to the coset $2B$:

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \\ -1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} .$$

The complete character table of $3^2{:}D_4$ is displayed in Table 6.5.

Table 6.5: The character table of $I_3 = 3^2{:}D_4$

$[d]_{D_4}$	$1A$			$2A$	$2B$		$2C$		$4A$
$[x]_{I_3}$	$1A$	$3A$	$3B$	$2A$	$2B$	$6A$	$2C$	$6B$	$4A$
$\|C_{I_3}(x)\|$	72	18	18	8	12	6	12	6	4
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1	1
χ_3	1	1	1	1	1	1	-1	-1	-1
χ_4	1	1	1	1	-1	-1	1	1	-1
χ_5	2	2	2	-2	0	0	0	0	0
χ_6	4	1	-2	0	2	-1	0	0	0
χ_7	4	1	-2	0	-2	1	0	0	0
χ_8	4	-2	1	0	0	0	2	-1	0
χ_9	4	-2	1	0	0	0	-2	1	0

6.2.3 The fusions of I_2, I_3 and I_4 into S_6

As we mentioned before there are four inertia groups $\overline{I}_1$, $\overline{I}_2$, $\overline{I}_3$ and $\overline{I}_4$ for the group $2^5{:}S_6$ such that $I_1 = S_6$, $I_2 = S_5$, $I_3 = 3^2{:}D_4$ and $I_4 = S_4 \times 2$. We first compute the power maps of the elements of $3^2{:}D_4$ which are given in Table 6.6.

Table 6.6: The power maps of the elements of $I_3 = 3^2{:}D_4$

$[d]_{D_4}$	$[x]_{I_3}$	2	3
$1A$	$1A$		
	$3A$		$1A$
	$3B$		$1A$
$2A$	$2A$	$1A$	
$2B$	$2B$	$1A$	
	$6A$	$3A$	$2B$
$2C$	$2C$	$1A$	
	$6B$	$3B$	$2C$
$4A$	$4A$	$2A$	

The power maps of the elements of I_2, I_4 and S_6 are easily obtainable. Using the character tables of I_2, I_3 and I_4 together with the power maps of their elements, the cycle structures, the permutation characters of S_6 of degrees 6, 10 and 15, and Corollary 3.5.4 we obtain the fusions of I_2, I_3 and I_4 into S_6 which are listed in Tables 6.7, 6.8 and 6.9 below. The entries of the tables are obtained by computing $|C_{S_6}(y)|/|C_{I_i}(x)|$ where y is a representative of a conjugacy class of S_6 and x a representative of a conjugacy class of I_i, where $i \in \{2,3,4\}$ and $o(x) = o(y)$. The entries of the boxes in the tables give the actual fusions. For example in the fusion of $3^2{:}D_4$ into S_6 we have $1A \longrightarrow 1A, 2A \longrightarrow 2C, 2B \longrightarrow 2A$ and so on. Similarly in the fusion of $S_4 \times 2$ into S_6, we have $1A \longrightarrow 1A, 2A \longrightarrow 2B, 2B \longrightarrow 2C, 2C \longrightarrow 2B$ and so on.

Table 6.7: The fusion of S_5 into S_6

Cycle of S_6		1^6	$1^4\,2$	2^3	$1^2\,2^2$	$1^3\,3$	3^2	$1^2\,4$	$2\,4$	$1\,5$	$1\,2\,3$	6
Class of S_6		$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$4A$	$4B$	$5A$	$6A$	$6B$
Class of S_5	Cycle of S_5											
$1A$	1^6	$\boxed{6}$										
$2A$	2^3		4	$\boxed{4}$								
$2B$	$1^2\,2^2$		6	6	$\boxed{2}$							
$3A$	3^2					3	$\boxed{3}$					
$4A$	$1^2\,4$							$\boxed{2}$	2			
$5A$	$1\,5$									$\boxed{1}$		
$6A$	6										1	$\boxed{1}$
$\chi(S_6\|S_5)$		6	0	4	2	0	3	2	0	1	0	1

Table 6.8: The fusion of $3^2{:}D_4$ into S_6

Cycle of S_6		1^6	$1^4\,2$	2^3	$1^2\,2^2$	$1^3\,3$	3^2	$1^2\,4$	$2\,4$	$1\,5$	$1\,2\,3$	6
Class of S_6		$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$4A$	$4B$	$5A$	$6A$	$6B$
Class of $3^2{:}D_4$	Cycle of $3^2{:}D_4$											
$1A$	1^6	$\boxed{10}$										
$2A$	$1^2\,2^2$		6	6	$\boxed{2}$							
$2B$	$1^4\,2$		$\boxed{4}$	4								
$2C$	2^3		4	$\boxed{4}$								
$3A$	$1^3\,3$					$\boxed{1}$	1					
$3B$	3^2					1	$\boxed{1}$					
$4A$	$2\,4$							2	$\boxed{2}$			
$6A$	$1\,2\,3$										$\boxed{1}$	1
$6B$	6										1	$\boxed{1}$
$\chi(S_6\|3^2{:}D_4)$		10	4	4	2	1	1	0	2	0	1	1

Table 6.9: The fusion of $S_4 \times 2$ into S_6

Cycle of S_6		1^6	$1^4\,2$	2^3	$1^2\,2^2$	$1^3\,3$	3^2	$1^2\,4$	$2\,4$	$1\,5$	$1\,2\,3$	6
Class of S_6		$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$4A$	$4B$	$5A$	$6A$	$6B$
Class of $S_4 \times 2$	Cycle of $S_4 \times 2$											
$1A$	1^6	$\boxed{15}$										
$2A$	2^3		6	$\boxed{6}$	2							
$2B$	$1^2\,2^2$		3	3	$\boxed{1}$							
$2C$	2^3		1	$\boxed{1}$								
$2D$	$1^2\,2^2$		6	6	$\boxed{2}$							
$2E$	$1^4\,2$		$\boxed{3}$	3	1							
$3A$	3^3					3	$\boxed{3}$					
$4A$	$1^2\,4$							$\boxed{1}$	1			
$4B$	$2\,4$							1	$\boxed{1}$			
$6A$	6										1	$\boxed{1}$
$\chi(S_6\|S_4 \times 2)$		15	3	7	3	0	3	1	1	0	0	1

6.2.4 The Fischer-Clifford Matrices of $2^5{:}S_6$

We use the fusions discussed in Section 6.2.3 together with the relations of Proposition 5.2.3, Theorem 5.2.4 and the properties (a) through (f) of the Fischer-Clifford matrices which are given in Chapter 5 (Section 5.2.2) to construct the Fischer-Clifford matrices of $2^5{:}S_6$. For each class representative $h \in S_6$, we construct a Fischer-Clifford matrix $M(h)$ and these are displayed in the following table.

Table 6.10: The Fischer-Clifford matrices of $2^5{:}S_6$

$M(h)$	$M(h)$	$M(h)$
$M(1A) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 6 & -6 & -2 & 2 \\ 10 & -10 & 2 & -2 \\ 15 & 15 & -1 & -1 \end{pmatrix}$	$M(2A) = \begin{pmatrix} 1 & 1 & 1 \\ 4 & -4 & 0 \\ 3 & 3 & -1 \end{pmatrix}$	$M(2B) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -4 & 2 & -2 & 0 \\ 4 & -4 & -2 & 2 & 0 \\ 6 & 6 & 0 & 0 & -2 \\ 1 & 1 & -1 & -1 & 1 \end{pmatrix}$
$M(2C) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & -2 & -2 & 2 & 0 \\ 2 & -2 & 2 & -2 & 0 \\ 1 & 1 & 1 & 1 & -1 \\ 2 & 2 & -2 & -2 & 0 \end{pmatrix}$	$M(3A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(3B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -3 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & 3 & -1 & -1 \end{pmatrix}$
$M(4A) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$	$M(4B) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$	$M(5A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$M(6A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(6B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$	

We used the above Fischer-Clifford matrices and the character tables of I_1, I_2, I_3 and I_4 together with the fusions of I_2, I_3 and I_4 into S_6 to obtain the character table of $H_2 = 2^5{:}S_6$. The set of irreducible characters of $2^5{:}S_6$ will be partitioned into four blocks B_1, B_2, B_3 and B_4 corresponding to the inertia factors I_1, I_2, I_3 and I_4 respectively. In fact $B_1 = \{\chi_i \mid 1 \leq i \leq 11\}, B_2 = \{\chi_i \mid 12 \leq i \leq 18\}, B_3 = \{\chi_i \mid 19 \leq i \leq 27\}, B_4 = \{\chi_i \mid 28 \leq i \leq 37\}$, where $Irr(2^5{:}S_6) = \bigcup_{i=1}^{4} B_i$. The complete character table of $2^5{:}S_6$ is given in Table 6.11. Please note that the centralizers of elements of $2^5{:}S_6$ are not listed here but are listed in the last column of Table 6.2.

Table 6.11: The character table of $2^5{:}S_6$

	$1A$				$2A$			$2B$					$2C$					$3A$	
	$1A$	$2A$	$2B$	$2C$	$2D$	$2E$	$4A$	$2F$	$2G$	$4B$	$4C$	$2H$	$2I$	$2J$	$4D$	$4E$	$4F$	$3A$	$6A$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	5	5	5	5	-3	-3	-3	1	1	1	1	1	1	1	1	1	1	2	2
χ_3	9	9	9	9	-3	-3	-3	-3	-3	-3	-3	-3	1	1	1	1	1	0	0
χ_4	5	5	5	5	-1	-1	-1	3	3	3	3	3	1	1	1	1	1	-1	-1
χ_5	10	10	10	10	-2	-2	-2	2	2	2	2	2	-2	-2	-2	-2	-2	1	1
χ_6	16	16	16	16	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	-2
χ_7	5	5	5	5	1	1	1	-3	-3	-3	-3	-3	1	1	1	1	1	-1	-1
χ_8	10	10	10	10	2	2	2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	1	1
χ_9	9	9	9	9	3	3	3	3	3	3	3	3	1	1	1	1	1	0	0
χ_{10}	5	5	5	5	3	3	3	-1	-1	-1	-1	-1	1	1	1	1	1	2	2
χ_{11}	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1
χ_{12}	6	-6	-2	2	0	0	0	4	-4	2	-2	0	2	-2	-2	2	0	0	0
χ_{13}	24	-24	-8	8	0	0	0	-8	8	-4	4	0	0	0	0	0	0	0	0
χ_{14}	30	-30	-10	10	0	0	0	-4	4	-2	2	0	2	-2	-2	2	0	0	0
χ_{15}	36	-36	-12	12	0	0	0	0	0	0	0	0	-4	4	4	-4	0	0	0
χ_{16}	30	-30	-10	10	0	0	0	4	-4	2	-2	0	2	-2	-2	2	0	0	0
χ_{17}	24	-24	-8	8	0	0	0	8	-8	4	-4	0	0	0	0	0	0	0	0
χ_{18}	6	-6	-2	2	0	0	0	-4	4	-2	2	0	2	-2	-2	2	0	0	0
χ_{19}	10	-10	2	-2	4	-4	0	4	-4	-2	2	0	2	-2	2	-2	0	1	-1
χ_{20}	10	-10	2	-2	-4	4	0	-4	4	2	-2	0	2	-2	2	-2	0	1	-1
χ_{21}	10	-10	2	-2	4	-4	0	-4	4	2	-2	0	2	-2	2	-2	0	1	-1
χ_{22}	10	-10	2	-2	-4	4	0	4	-4	-2	2	0	2	-2	2	-2	0	1	-1
χ_{23}	20	-20	4	-4	0	0	0	0	0	0	0	0	-4	4	-4	4	0	2	-2
χ_{24}	40	-40	8	-8	8	-8	0	0	0	0	0	0	0	0	0	0	0	1	-1
χ_{25}	40	-40	8	-8	-8	8	0	0	0	0	0	0	0	0	0	0	0	1	-1
χ_{26}	40	-40	8	-8	0	0	0	8	-8	-4	4	0	0	0	0	0	0	-2	2
χ_{27}	40	-40	8	-8	0	0	0	-8	8	4	-4	0	0	0	0	0	0	-2	2
χ_{28}	15	15	-1	-1	3	3	-1	7	7	-1	-1	-1	3	3	-1	-1	-1	0	0
χ_{29}	45	45	-3	-3	3	3	-1	-9	-9	3	3	-1	1	1	-3	-3	1	0	0
χ_{30}	30	30	-2	-2	-6	-6	2	-2	-2	2	2	-2	2	2	2	2	-2	0	0
χ_{31}	45	45	-3	-3	3	3	-1	3	3	3	3	-5	-3	-3	1	1	1	0	0
χ_{32}	15	15	-1	-1	-3	-3	1	5	5	1	1	-3	-1	-1	3	3	-1	0	0
χ_{33}	15	15	-1	-1	3	3	-1	-5	-5	-1	-1	3	-1	-1	3	3	-1	0	0
χ_{34}	45	45	-3	-3	-3	-3	1	-3	-3	-3	-3	5	-3	-3	1	1	1	0	0
χ_{35}	30	30	-2	-2	6	6	-2	2	2	-2	-2	2	2	2	2	2	-2	0	0
χ_{36}	45	45	-3	-3	-3	-3	1	9	9	-3	-3	1	1	1	-3	-3	1	0	0
χ_{37}	15	15	-1	-1	-3	-3	1	-7	-7	1	1	1	3	3	-1	-1	-1	0	0

Table 6.11: The character table of 2^5:S_6(continued)

	$3B$				$4A$			$4B$			$5A$		$6A$		$6B$			
	$3B$	$6B$	$6C$	$6D$	$4G$	$4H$	$8A$	$4I$	$4J$	$8B$	$5A$	$10A$	$6E$	$6F$	$6G$	$6H$	$12A$	$12B$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	1	1	1	1
χ_3	0	0	0	0	1	1	1	1	1	1	-1	-1	0	0	0	0	0	0
χ_4	2	2	2	2	1	1	1	-1	-1	-1	0	0	-1	-1	0	0	0	0
χ_5	1	1	1	1	0	0	0	0	0	0	0	0	1	1	-1	-1	-1	-1
χ_6	-2	-2	-2	-2	0	0	0	0	0	0	1	1	0	0	0	0	0	0
χ_7	2	2	2	2	-1	-1	-1	-1	-1	-1	0	0	1	1	0	0	0	0
χ_8	1	1	1	1	0	0	0	0	0	0	0	0	-1	-1	1	1	1	1
χ_9	0	0	0	0	-1	-1	-1	1	1	1	-1	-1	0	0	0	0	0	0
χ_{10}	-1	-1	-1	-1	1	1	1	-1	-1	-1	0	0	0	0	-1	-1	-1	-1
χ_{11}	1	1	1	1	-1	-1	-1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
χ_{12}	3	-3	-1	1	2	-2	0	0	0	0	1	-1	0	0	1	-1	-1	1
χ_{13}	3	-3	-1	1	0	0	0	0	0	0	-1	1	0	0	1	-1	-1	1
χ_{14}	-3	3	1	-1	2	-2	0	0	0	0	0	0	0	0	-1	1	1	-1
χ_{15}	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
χ_{16}	-3	3	1	-1	-2	2	0	0	0	0	0	0	0	0	1	-1	-1	1
χ_{17}	3	-3	-1	1	0	0	0	0	0	0	-1	1	0	0	-1	1	1	-1
χ_{18}	3	-3	-1	1	-2	2	0	0	0	0	1	-1	0	0	-1	1	1	-1
χ_{19}	1	-1	1	-1	0	0	0	2	-2	0	0	0	1	-1	1	-1	1	-1
χ_{20}	1	-1	1	-1	0	0	0	2	-2	0	0	0	-1	1	-1	1	-1	1
χ_{21}	1	-1	1	-1	0	0	0	-2	2	0	0	0	1	-1	-1	1	-1	1
χ_{22}	1	-1	1	-1	0	0	0	-2	2	0	0	0	-1	1	1	-1	1	-1
χ_{23}	2	-2	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{24}	-2	2	-2	2	0	0	0	0	0	0	0	0	-1	1	0	0	0	0
χ_{25}	-2	2	-2	2	0	0	0	0	0	0	0	0	1	-1	0	0	0	0
χ_{26}	1	-1	1	-1	0	0	0	0	0	0	0	0	0	0	-1	1	-1	1
χ_{27}	1	-1	1	-1	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1
χ_{28}	3	3	-1	-1	1	1	-1	1	1	-1	0	0	0	0	1	1	-1	-1
χ_{29}	0	0	0	0	1	1	-1	-1	-1	1	0	0	0	0	0	0	0	0
χ_{30}	-3	-3	1	1	0	0	0	0	0	0	0	0	0	0	1	1	-1	-1
χ_{31}	0	0	0	0	-1	-1	1	1	1	-1	0	0	0	0	0	0	0	0
χ_{32}	3	3	-1	-1	1	1	-1	-1	-1	1	0	0	0	0	-1	-1	1	1
χ_{33}	3	3	-1	-1	-1	-1	1	-1	-1	1	0	0	0	0	1	1	-1	-1
χ_{34}	0	0	0	0	1	1	-1	1	1	-1	0	0	0	0	0	0	0	0
χ_{35}	-3	-3	1	1	0	0	0	0	0	0	0	0	0	0	-1	-1	1	1
χ_{36}	0	0	0	0	-1	-1	1	-1	-1	1	0	0	0	0	0	0	0	0
χ_{37}	3	3	-1	-1	-1	-1	1	1	1	-1	0	0	0	0	-1	-1	1	1

6.3 The fusion of 2^5:S_6 into $SP(6,2)$

The conjugacy classes of $H_2 = 2^5$:S_6 are listed in Table 6.2 (Section 6.2.1). We used these classes and computed the power maps of the elements of 2^5:S_6 which are given in Table 6.12 below.

Table 6.12: The power maps of the elements of $H_2 = 2^5{:}S_6$

$[y]_{S_6}$	$[x]_{H_2}$	2	3	5	$[y]_{S_6}$	$[x]_{H_2}$	2	3	5
$1A$	$1A$				$2A$	$2D$	$1A$		
	$2A$	$1A$				$2E$	$1A$		
	$2B$	$1A$				$4A$	$2B$		
	$2C$	$1A$							
$2B$	$2F$	$1A$			$2C$	$2I$	$1A$		
	$2G$	$1A$				$2J$	$1A$		
	$2H$	$1A$				$4D$	$2B$		
	$4B$	$2C$				$4E$	$2B$		
	$4C$	$2C$				$4F$	$2C$		
$3A$	$3A$		$1A$		$3B$	$3B$		$1A$	
	$6A$	$3A$	$2A$			$6B$	$3B$	$2A$	
						$6C$	$3B$	$2C$	
						$6D$	$3B$	$2B$	
$4A$	$4G$	$2I$			$4B$	$4I$	$2I$		
	$4H$	$2I$				$4J$	$2I$		
	$8A$	$4E$				$8B$	$4D$		
$5A$	$5A$			$1A$	$6A$	$6E$	$3A$	$2D$	
	$10A$	$5A$		$2A$		$6F$	$3A$	$2E$	
$6B$	$6G$	$3B$	$2G$						
	$6H$	$3B$	$2H$						
	$12A$	$6C$	$4B$						
	$12B$	$6C$	$4C$						

The power maps of the elements of $SP(6,2)$ are given in the ATLAS. Using the information provided by the conjugacy classes of the elements of $2^5{:}S_6$ and $SP(6,2)$, the power maps and the permutation character of $SP(6,2)$ of degree 63, we are able to obtain partial fusion of $2^5{:}S_6$ into $SP(6,2)$. For example the classes $2A, 2B, 2C$ of $2^5{:}S_6$ fuse respectively to $2A, 2B, 2C$ in $SP(6,2)$. To complete the fusion map, we restrict irreducible characters of $SP(6,2)$ of small degrees to $2^5{:}S_6$. To determine the restrictions of irreducible characters of $SP(6,2)$ to $2^5{:}S_6$, we use the following technique of set intersections for characters which has been discussed and used in [80] and [81].

Let ρ be the character of S_6 afforded by the regular representation of S_6. Then we obtain that $\rho = \sum_{i=1}^{11} e_i\phi_i$, where $\phi_i \in Irr(S_6)$ and $e_i = deg(\phi_i)$. Then ρ can be regarded as a character of $2^5{:}S_6$ which contains 2^5 in its kernel such that

$$\rho(g) = \begin{cases} |S_6| & \text{if} \quad g \in 2^5 \\ 0 & \text{otherwise} \end{cases} .$$

If ψ is a character of $SP(6,2)$, then we obtain that

$$\begin{aligned}
\langle \rho, \psi \rangle_{2^5:S_6} &= \frac{1}{|2^5{:}S_6|} \{\rho(1A)\psi(1A) + \rho(2A)\psi(2A) + 15\rho(2B)\psi(2B) + 15\rho(2C)\psi(2C)\} \\
&= \frac{1}{|2^5{:}S_6|} \{|S_6|\psi(1A) + |S_6|\psi(2A) + 15|S_6|\psi(2B) + 15|S_6|\psi(2C)\} \\
&= \frac{1}{32} \{\psi(1A) + \psi(2A) + 15\psi(2B) + 15\psi(2C)\} \\
&= \langle \psi_{2^5}, \tau_1 \rangle
\end{aligned}$$

where τ_1 is the identity character of 2^5 and ψ_{2^5} is the restriction of ψ to 2^5. Also for ψ we obtain that

$$\psi_{2^5} = a_1\theta_1 + a_2\theta_2 + a_3\theta_3 + a_4\theta_4 \;,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{N} \cup \{0\}$ and $\theta_i, i \in \{1,2,3,4\}$ are the sums of the irreducible characters of 2^5 which are in one orbit under the action of S_6 on $Irr(2^5)$. Let $\tau_j \in Irr(2^5)$, where $j \in \{1, 2, \ldots, 32\}$. Then we obtain that (using Theorem 6.2.5)

$$\theta_1 = \tau_1 \;,\; deg(\theta_1) = 1$$

$$\theta_2 = \sum_{j=2}^{7} \tau_j \;,\; deg(\theta_2) = 6$$

$$\theta_3 = \sum_{j=8}^{17} \tau_j \;,\; deg(\theta_3) = 10$$

$$\theta_4 = \sum_{j=18}^{32} \tau_j \;,\; deg(\theta_4) = 15 \;.$$

Hence

$$\psi_{2^5} = a_1\tau_1 + a_2 \sum_{j=2}^{7} \tau_j + a_3 \sum_{j=8}^{17} \tau_j + a_4 \sum_{j=18}^{32} \tau_j$$

and

$$\langle \psi_{2^5}, \psi_{2^5} \rangle = a_1^2 + 6a_2^2 + 10a_3^2 + 15a_4^2 \;.$$

Notice that $a_1 = \langle \psi_{2^5}, \tau_1 \rangle = \langle \rho, \psi \rangle_{2^5:S_6}$. We also have that

$$\langle \psi_{2^5}, \psi_{2^5} \rangle = \frac{1}{32} \{\psi(1A)\psi(1A) + \psi(2A)\psi(2A) + 15\psi(2B)\psi(2B) + 15\psi(2C)\psi(2C)\} \;.$$

We now apply the above results to $\psi_1 = 7a$ and $\psi_2 = 15a$, irreducible characters of $SP(6,2)$ of degrees 7 and 15 respectively. For ψ_1 we obtain that

$$a_1 = \langle \rho, \psi_1 \rangle_{2^5:S_6} = \frac{1}{32}[7 + (-5) + 15(-1) + 15(3)] = \frac{1}{32}[32] = 1 \quad .$$

Since $deg(\psi_1) = 7$, we must have that

$$a_1 + 6a_2 + 10a_3 + 15a_4 = 7$$

and since $a_1 = 1$, then we must have that $a_2 = 1, a_3 = a_4 = 0$. Now based on the partial fusion of $2^5{:}S_6$ in $SP(6,2)$ which has already been determined, we obtain that

$$(\psi_1)_{2^5:S_6} = \chi_{11} + \chi_{12} \quad .$$

Similarly for ψ_2 we obtain that

$$a_1 = \langle \rho, \psi_2 \rangle_{2^5:S_6} = \frac{1}{32}[15 + (-5) + 15(7) + 15(3)] = \frac{1}{32}[160] = 5 \quad .$$

Since $deg(\psi_2) = 15$, we must have that

$$a_1 + 6a_2 + 10a_3 + 15a_4 = 15 \quad .$$

Since $a_1 = 5$, then we have $a_2 = a_4 = 0$ and $a_3 = 1$. Hence we get $(\psi_2)_{2^5:S_6} = \chi_{10} + \chi_{19}$.

Using the partial fusion which has already been determined, the values of ψ_1 and ψ_2 on the classes of $SP(6,2)$ and the values of $(\psi_1)_{2^5:S_6}$ and $(\psi_2)_{2^5:S_6}$ on the classes of $2^5{:}S_6$, we are able to complete the fusion of $H_2 = 2^5{:}S_6$ into $G = SP(6,2)$. This fusion is given in Table 6.13.

Table 6.13: The fusion of $2^5{:}S_6$ into $SP(6,2)$

$[g]_G$	$1A$	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$4E$	$5A$
$[h]_{H_2}$														
$1A$	$\boxed{63}$													
$2A$		$\boxed{1}$												
$2B$		15	$\boxed{3}$	1										
$2C$		15	3	$\boxed{1}$										
$2D$		60	$\boxed{12}$	4	1									
$2E$		60	12	4	$\boxed{1}$									
$2F$		30	6	$\boxed{2}$										
$2G$		$\boxed{30}$	6	2										
$2H$		180	36	12	$\boxed{3}$									
$2I$		180	36	$\boxed{12}$	3									
$2J$		180	36	12	$\boxed{3}$									
$3A$						60	18	$\boxed{3}$						
$3B$						$\boxed{15}$								
$4A$									6	3	3	$\boxed{2}$		
$4B$									2	$\boxed{1}$	1			
$4C$									2	1	$\boxed{1}$			
$4D$									3			$\boxed{1}$		
$4E$									$\boxed{3}$			1		
$4F$									12	6	6	4	$\boxed{1}$	
$4G$									12	6	6	4	$\boxed{1}$	
$4H$									12	6	$\boxed{6}$	4	1	
$4I$									12	6	6	4	$\boxed{1}$	
$4J$									12	$\boxed{6}$	6	4	1	
$5A$														$\boxed{3}$
$\chi(SP(6,2)\|2^5{:}S_6)$	63	31	15	15	7	15	0	3	3	7	7	3	3	3

Table 6.13: The fusion of $2^5{:}S_6$ into $SP(6,2)$(continued)

$[g]_G$	$6A$	$6B$	$6C$	$6D$	$6E$	$6F$	$6G$	$7A$	$8A$	$8B$	$9A$	$10A$	$12A$	$12B$	$12C$	$15A$
$[h]_{H_2}$																
$6A$	4	4	2		$\boxed{1}$	1										
$6B$	1	$\boxed{1}$														
$6C$	3	3		$\boxed{1}$												
$6D$	$\boxed{3}$	3		1												
$6E$	12	12	6	4	3	$\boxed{3}$	1									
$6F$	12	12	6	4	3	3	$\boxed{1}$									
$6G$	6	6	3	$\boxed{2}$												
$6H$	6	$\boxed{6}$	3	2												
$8A$									$\boxed{1}$	1						
$8B$									1	$\boxed{1}$						
$10A$												$\boxed{1}$				
$12A$													$\boxed{1}$	1		
$12B$													1	$\boxed{1}$		
$\chi(SP(6,2)\|2^5{:}S_6)$	3	7	0	3	1	3	1	0	1	1	0	1	1	1	0	0

6.4 The Fischer-Clifford matrices of $\overline{G}$

We use the fusion discussed in Section 6.3 together with the relations of Proposition 5.2.3, Theorem 5.2.4 and the properties (a) through (f) of the Fischer-Clifford matrices which are given in Chapter 5 (Section 5.2.2) to construct the Fischer-Clifford matrices of $2^6{:}SP(6,2)$. For each conjugacy class $[g]$ of G with representative $g \in G$, we construct the corresponding Fischer-Clifford matrix $M(g)$ and these matrices are given in Table 6.14 below.

Table 6.14: The Fischer-Clifford matrices of $\overline{G}$

$M(g)$	$M(g)$	$M(g)$
$M(1A) = \begin{pmatrix} 1 & 1 \\ 63 & -1 \end{pmatrix}$	$M(2A) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 30 & -2 & 0 \end{pmatrix}$	$M(2B) = \begin{pmatrix} 1 & 1 & 1 \\ 3 & -1 & 3 \\ 12 & 0 & -4 \end{pmatrix}$
$M(2C) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 2 & 2 & -2 & 0 \\ 12 & -4 & 0 & 0 \end{pmatrix}$	$M(2D) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & -1 & -1 & 3 \\ 3 & 1 & -1 & -3 \end{pmatrix}$	$M(3A) = \begin{pmatrix} 1 & 1 \\ 15 & -1 \end{pmatrix}$
$M(3B) = \begin{pmatrix} 1 \end{pmatrix}$	$M(3C) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	$M(4A) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$
$M(4B) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 6 & -2 & 0 \end{pmatrix}$	$M4C) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 6 & 0 & -2 \end{pmatrix}$	$M(4D) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$
$M(4E) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$	$M(5A) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	$M(6A) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$
$M(6B) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 6 & -2 & 0 \end{pmatrix}$	$M(6C) = \begin{pmatrix} 1 \end{pmatrix}$	$M(6D) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 6 & -2 & 0 \end{pmatrix}$
$M(6E) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(6F) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	$M(6G) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$M(7A) = \begin{pmatrix} 1 \end{pmatrix}$	$M(8A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(8B) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$M(9A) = \begin{pmatrix} 1 \end{pmatrix}$	$M(10A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(12A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$M(12B) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(12C) = \begin{pmatrix} 1 \end{pmatrix}$	$M(15A) = \begin{pmatrix} 1 \end{pmatrix}$

We used the above Fischer-Clifford matrices and the character tables of $SP(6,2)$ and $2^5{:}S_6$ together with the fusion of $2^5{:}S_6$ into $SP(6,2)$ to obtain the character table of $\overline{G} = 2^6{:}SP(6,2)$. The set of irreducible characters of $\overline{G} = 2^6{:}SP(6,2)$ will be partitioned into two blocks B_1 and B_2 corresponding to the inertia factors H_1 and

H_2 respectively. In fact $B_1 = \{\chi_i \mid 1 \leq i \leq 30\}, B_2 = \{\chi_i \mid 31 \leq i \leq 67\}$, where $Irr(2^6{:}SP(6,2)) = \bigcup_{i=1}^{2} B_i$. The complete character table of $\overline{G}$ is given in Table 6.15. Please note that the centralizers of elements of $\overline{G}$ are listed in the last column of Table 6.1.

Table 6.15: The character table of $2^6{:}SP(6,2)$

	$1A$		$2A$			$2B$			$2C$				$2D$			
	$1A$	$2A$	$2B$	$2C$	$4A$	$2D$	$4B$	$2E$	$2F$	$2G$	$4C$	$4D$	$2H$	$4E$	$4F$	$4G$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	7	7	-5	-5	-5	-1	-1	-1	3	3	3	3	-1	-1	-1	-1
χ_3	15	15	-5	-5	-5	7	7	7	3	3	3	3	-1	-1	-1	-1
χ_4	21	21	9	9	9	-3	-3	-3	1	1	1	1	-3	-3	-3	-3
χ_5	21	21	-11	-11	-11	5	5	5	5	5	5	5	-3	-3	-3	-3
χ_6	27	27	15	15	15	3	3	3	7	7	7	7	3	3	3	3
χ_7	35	35	-5	-5	-5	3	3	3	-5	-5	-5	-5	3	3	3	3
χ_8	35	35	15	15	15	11	11	11	7	7	7	7	3	3	3	3
χ_9	56	56	-24	-24	-24	-8	-8	-8	8	8	8	8	0	0	0	0
χ_{10}	70	70	-10	-10	-10	-10	-10	-10	6	6	6	6	-2	-2	-2	-2
χ_{11}	84	84	4	4	4	20	20	20	4	4	4	4	4	4	4	4
χ_{12}	105	105	-35	-35	-35	1	1	1	5	5	5	5	1	1	1	1
χ_{13}	105	105	25	25	25	-7	-7	-7	9	9	9	9	1	1	1	1
χ_{14}	105	105	5	5	5	17	17	17	-3	-3	-3	-3	-7	-7	-7	-7
χ_{15}	120	120	40	40	40	-8	-8	-8	8	8	8	8	0	0	0	0
χ_{16}	168	168	40	40	40	8	8	8	8	8	8	8	8	8	8	8
χ_{17}	189	189	21	21	21	-3	-3	-3	-11	-11	-11	-11	-3	-3	-3	-3
χ_{18}	189	189	-51	-51	-51	-3	-3	-3	13	13	13	13	-3	-3	-3	-3
χ_{19}	189	189	-39	-39	-39	21	21	21	1	1	1	1	-3	-3	-3	-3
χ_{20}	210	210	50	50	50	2	2	2	2	2	2	2	-6	-6	-6	-6
χ_{21}	210	210	10	10	10	-14	-14	-14	10	10	10	10	2	2	2	2
χ_{22}	216	216	-24	-24	-24	24	24	24	8	8	8	8	0	0	0	0
χ_{23}	280	280	-40	-40	-40	-8	-8	-8	-8	-8	-8	-8	8	8	8	8
χ_{24}	280	280	40	40	40	24	24	24	8	8	8	8	0	0	0	0
χ_{25}	315	315	-45	-45	-45	-21	-21	-21	3	3	3	3	3	3	3	3
χ_{26}	336	336	-16	-16	-16	16	16	16	-16	-16	-16	-16	0	0	0	0
χ_{27}	378	378	-30	-30	-30	-6	-6	-6	2	2	2	2	-6	-6	-6	-6
χ_{28}	405	405	45	45	45	-27	-27	-27	-3	-3	-3	-3	-3	-3	-3	-3
χ_{29}	420	420	20	20	20	4	4	4	-12	-12	-12	-12	4	4	4	4
χ_{30}	512	512	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 6.15: The character table of $2^6{:}SP(6,2)$(continued)

	$3A$		$3B$	$3C$		$4A$		$4B$			$4C$			$4D$		
	$3A$	$6A$	$3B$	$3C$	$6B$	$4H$	$4I$	$4J$	$4K$	$8A$	$4L$	$8B$	$4M$	$4N$	$4O$	$4P$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	4	4	-2	1	1	3	3	1	1	1	-3	-3	-3	-1	-1	-1
χ_3	0	0	-3	3	3	-1	-1	-3	-3	-3	1	1	1	3	3	3
χ_4	6	6	3	0	0	5	5	-1	-1	-1	3	3	3	1	1	1
χ_5	6	6	3	0	0	1	1	-3	-3	-3	-3	-3	-3	1	1	1
χ_6	9	9	0	0	0	3	3	1	1	1	5	5	5	-1	-1	-1
χ_7	5	5	-1	2	2	7	7	-1	-1	-1	-1	-1	-1	-1	-1	-1
χ_8	5	5	-1	2	2	-1	-1	5	5	5	1	1	1	3	3	3
χ_9	11	11	2	2	2	0	0	4	4	4	-4	-4	-4	0	0	0
χ_{10}	-5	-5	7	1	1	2	2	2	2	2	2	2	2	2	2	2
χ_{11}	-6	-6	3	3	3	4	4	0	0	0	0	0	0	4	4	4
χ_{12}	15	15	-3	-3	-3	5	5	-1	-1	-1	-5	-5	-5	1	1	1
χ_{13}	0	0	6	3	3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
χ_{14}	0	0	6	3	3	-3	-3	3	3	3	-1	-1	-1	1	1	1
χ_{15}	15	15	-6	0	0	0	0	-4	-4	-4	4	4	4	0	0	0
χ_{16}	6	6	6	-3	-3	0	0	0	0	0	0	0	0	0	0	0
χ_{17}	9	9	0	0	0	9	9	1	1	1	1	1	1	1	1	1
χ_{18}	9	9	0	0	0	-3	-3	1	1	1	1	1	1	-3	-3	-3
χ_{19}	9	9	0	0	0	-3	-3	-5	-5	-5	-1	-1	-1	1	1	1
χ_{20}	15	15	3	0	0	-2	-2	2	2	2	2	2	2	-2	-2	-2
χ_{21}	-15	-15	-6	3	3	6	6	-2	-2	-2	-2	-2	-2	-2	-2	-2
χ_{22}	-9	-9	0	0	0	0	0	-4	-4	-4	4	4	4	0	0	0
χ_{23}	10	10	10	1	1	0	0	0	0	0	0	0	0	0	0	0
χ_{24}	-5	-5	-8	-2	-2	0	0	4	4	4	-4	-4	-4	0	0	0
χ_{25}	0	0	-9	0	0	-5	-5	3	3	3	3	3	3	3	3	3
χ_{26}	6	6	-6	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{27}	-9	-9	0	0	0	6	6	2	2	2	2	2	2	-2	-2	-2
χ_{28}	0	0	0	0	0	-3	-3	-3	-3	-3	-3	-3	-3	5	5	5
χ_{29}	0	0	-3	3	3	-4	-4	0	0	0	0	0	0	-4	-4	-4
χ_{30}	-16	-16	8	-4	-4	0	0	0	0	0	0	0	0	0	0	0

Table 6.15: The character table of $2^6{:}SP(6,2)$(continued)

	$4E$				$5A$		$6A$		$6B$			$6C$	$6D$			$6E$	
	$4Q$	$4R$	$8C$	$8D$	$5A$	$10A$	$6C$	$12A$	$6D$	$6E$	$12B$	$6F$	$6G$	$12C$	$12D$	$6H$	$12E$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	2	2	2	2	-2	-2	-2	2	0	0	0	1	1
χ_3	1	1	1	1	0	0	-2	-2	-2	-2	-2	1	0	0	0	1	1
χ_4	-1	-1	-1	-1	1	1	0	0	0	0	0	3	-2	-2	-2	0	0
χ_5	1	1	1	1	1	1	2	2	-2	-2	-2	-1	2	2	2	-2	-2
χ_6	1	1	1	1	2	2	3	3	3	3	3	0	1	1	1	0	0
χ_7	-1	-1	-1	-1	0	0	-3	-3	1	1	1	3	1	1	1	-2	-2
χ_8	1	1	1	1	0	0	-1	-1	3	3	3	-1	1	1	1	0	0
χ_9	0	0	0	0	1	1	1	1	-3	-3	-3	-2	-1	-1	-1	0	0
χ_{10}	-2	-2	-2	-2	0	0	-1	-1	-1	-1	-1	-1	3	3	3	-1	-1
χ_{11}	0	0	0	0	-1	-1	2	2	-2	-2	-2	-1	-2	-2	-2	1	1
χ_{12}	-1	-1	-1	-1	0	0	1	1	1	1	1	1	-1	-1	-1	1	1
χ_{13}	1	1	1	1	0	0	-4	-4	4	4	4	2	0	0	0	1	1
χ_{14}	-1	-1	-1	-1	0	0	2	2	2	2	2	2	0	0	0	-1	-1
χ_{15}	0	0	0	0	0	0	1	1	1	1	1	-2	-1	-1	-1	-2	-2
χ_{16}	0	0	0	0	-2	-2	2	2	-2	-2	-2	2	2	2	2	1	1
χ_{17}	1	1	1	1	-1	-1	-3	-3	-3	-3	-3	0	1	1	1	0	0
χ_{18}	1	1	1	1	-1	-1	-3	-3	-3	-3	-3	0	1	1	1	0	0
χ_{19}	-1	-1	-1	-1	-1	-1	3	3	3	3	3	0	1	1	1	0	0
χ_{20}	-2	-2	-2	-2	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	2	2
χ_{21}	-2	-2	-2	-2	0	0	1	1	1	1	1	-2	1	1	1	1	1
χ_{22}	0	0	0	0	1	1	-3	-3	-3	-3	-3	0	-1	-1	-1	0	0
χ_{23}	0	0	0	0	0	0	-2	-2	2	2	2	-2	-2	-2	-2	-1	-1
χ_{24}	0	0	0	0	0	0	-3	-3	1	1	1	0	-1	-1	-1	-2	-2
χ_{25}	-1	-1	-1	-1	0	0	0	0	0	0	0	3	0	0	0	0	0
χ_{26}	0	0	0	0	1	1	-2	-2	2	2	2	-2	2	2	2	2	2
χ_{27}	2	2	2	2	-2	-2	3	3	3	3	3	0	-1	-1	-1	0	0
χ_{28}	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{29}	0	0	0	0	0	0	4	4	-4	-4	-4	1	0	0	0	-1	-1
χ_{30}	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0

Table 6.15: The character table of $2^6{:}SP(6,2)$(continued)

	$6F$		$6G$		$7A$	$8A$		$8B$		$9A$	$10A$		$12A$		$12B$		$12C$	$15A$
	$6I$	$6J$	$6K$	$12F$	$7A$	$8E$	$8F$	$8G$	$8H$	$9A$	$10B$	$20A$	$12G$	$24A$	$12H$	$24B$	$12I$	$15A$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	-1	-1	-1	-1	0	-1	-1	1	1	1	0	0	-2	-2	0	0	0	-1
χ_3	1	1	-1	-1	1	1	1	-1	-1	0	0	0	0	0	-2	-2	-1	0
χ_4	0	0	0	0	0	-1	-1	1	1	0	-1	-1	2	2	0	0	-1	1
χ_5	2	2	0	0	0	-1	-1	-1	-1	0	-1	-1	0	0	0	0	1	1
χ_6	0	0	0	0	-1	1	1	-1	-1	0	0	0	1	1	-1	-1	0	-1
χ_7	0	0	0	0	0	1	1	1	1	-1	0	0	-1	-1	-1	-1	1	0
χ_8	2	2	0	0	0	-1	-1	1	1	-1	0	0	-1	-1	1	1	-1	0
χ_9	-2	-2	0	0	0	0	0	0	0	-1	1	1	1	1	-1	-1	0	1
χ_{10}	-1	-1	1	1	0	0	0	0	0	1	0	0	-1	-1	-1	-1	-1	0
χ_{11}	-1	-1	1	1	0	0	0	0	0	0	-1	-1	0	0	0	0	1	-1
χ_{12}	1	1	1	1	0	1	1	-1	-1	0	0	0	-1	-1	1	1	-1	0
χ_{13}	-1	-1	1	1	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0
χ_{14}	-1	-1	-1	-1	0	1	1	-1	-1	0	0	0	0	0	2	2	0	0
χ_{15}	-2	-2	0	0	1	0	0	0	0	0	0	0	-1	-1	1	1	0	0
χ_{16}	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
χ_{17}	0	0	0	0	0	-1	-1	-1	-1	0	1	1	1	1	1	1	0	-1
χ_{18}	0	0	0	0	0	1	1	1	1	0	-1	-1	1	1	1	1	0	-1
χ_{19}	0	0	0	0	0	-1	-1	1	1	0	1	1	1	1	-1	-1	0	-1
χ_{20}	2	2	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	1	0
χ_{21}	1	1	-1	-1	0	0	0	0	0	0	0	0	1	1	1	1	0	0
χ_{22}	0	0	0	0	-1	0	0	0	0	0	1	1	-1	-1	1	1	0	1
χ_{23}	1	1	-1	-1	0	0	0	0	0	1	0	0	0	0	0	0	0	0
χ_{24}	0	0	0	0	0	0	0	0	0	1	0	0	1	1	-1	-1	0	0
χ_{25}	0	0	0	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	1	0
χ_{26}	-2	-2	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	1
χ_{27}	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	1
χ_{28}	0	0	0	0	-1	1	1	1	1	0	0	0	0	0	0	0	0	0
χ_{29}	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	-1	0
χ_{30}	0	0	0	0	1	0	0	0	0	-1	0	0	0	0	0	0	0	-1

Table 6.15: The character table of 2^6:$SP(6,2)$(continued)

	$1A$		$2A$			$2B$			$2C$				$2D$			
	$1A$	$2A$	$2B$	$2C$	$4A$	$2D$	$4B$	$2E$	$2F$	$2G$	$4C$	$4D$	$2H$	$4E$	$4F$	$4G$
χ_{31}	63	-1	31	-1	-1	15	-1	-1	15	-1	-1	-1	7	-1	-1	-1
χ_{32}	315	-5	35	3	-5	-21	-5	27	19	3	3	-5	3	3	-5	3
χ_{33}	567	-9	-81	15	-9	-9	-9	39	15	-1	15	-9	-9	7	-1	-9
χ_{34}	315	-5	95	-1	-5	3	-5	19	23	7	-1	-5	11	-1	-5	7
χ_{35}	630	-10	70	6	-10	6	-10	38	-10	22	6	-10	-2	-2	-2	14
χ_{36}	1008	-16	16	16	-16	48	-16	48	16	16	16	-16	0	0	0	0
χ_{37}	315	-5	-85	11	-5	27	-5	11	11	-5	11	-5	-5	3	3	-13
χ_{38}	630	-10	-50	14	-10	54	-10	22	-18	14	14	-10	-10	-2	6	-2
χ_{39}	567	-9	99	3	-9	63	-9	15	27	11	3	-9	15	-5	-1	3
χ_{40}	315	-5	-25	7	-5	51	-5	3	15	-1	7	-5	3	-1	3	-9
χ_{41}	63	-1	-29	3	-1	-9	-1	7	11	-5	3	-1	-1	3	-1	-5
χ_{42}	378	-6	-126	2	6	-6	2	-6	34	2	-6	-2	-6	-2	2	6
χ_{43}	1512	-24	216	-40	24	-24	8	-24	-8	-8	24	-8	0	0	0	0
χ_{44}	1890	-30	90	-38	30	-30	10	-30	26	-6	18	-10	-6	-2	2	6
χ_{45}	2268	-36	-36	-36	36	-36	12	-36	-36	28	12	-12	12	4	-4	-12
χ_{46}	1890	-30	-150	-22	30	-30	10	-30	42	10	2	-10	-6	-2	2	6
χ_{47}	1512	-24	-264	-8	24	-24	8	-24	24	24	-8	-8	0	0	0	0
χ_{48}	378	-6	114	-14	6	-6	2	-6	18	-14	10	-2	-6	-2	2	6
χ_{49}	630	-10	-130	-2	10	54	-2	-10	30	-2	-10	2	-10	2	-2	10
χ_{50}	630	-10	110	-18	10	-42	-2	22	14	-18	6	2	-2	-6	6	2
χ_{51}	630	-10	110	-18	10	54	-2	-10	14	-18	6	2	-10	2	-2	10
χ_{52}	630	-10	-130	-2	10	-42	-2	22	30	-2	-10	2	-2	-6	6	2
χ_{53}	1260	-20	-20	-20	20	12	-4	12	-52	12	-4	4	12	4	-4	-12
χ_{54}	2520	-40	-40	-40	40	120	-8	-8	-8	-8	-8	8	-8	8	-8	8
χ_{55}	2520	-40	-40	-40	40	-72	-8	56	-8	-8	-8	8	8	-8	8	-8
χ_{56}	2520	-40	-280	-24	40	24	-8	24	8	8	-24	8	0	0	0	0
χ_{57}	2520	-40	200	-56	40	24	-8	24	-24	-24	8	8	0	0	0	0
χ_{58}	945	-15	225	1	-15	33	1	-15	49	1	-15	1	9	1	1	-15
χ_{59}	2835	-45	-225	63	-45	27	3	-21	-9	-25	15	3	3	-1	3	-9
χ_{60}	1890	-30	-30	34	-30	-78	2	18	18	-14	2	2	-6	10	-6	-6
χ_{61}	2835	-45	135	39	-45	27	3	-21	-33	15	-9	3	-21	-1	11	-9
χ_{62}	945	-15	165	5	-15	-39	1	9	-3	13	-11	1	-15	5	1	-3
χ_{63}	945	-15	-135	25	-15	33	1	-15	-23	-7	9	1	9	-7	1	9
χ_{64}	2835	-45	-45	51	-45	-45	3	3	-45	3	3	3	3	-5	-5	27
χ_{65}	1890	-30	90	26	-30	66	2	-30	26	-6	-6	2	18	-6	2	-6
χ_{66}	2835	-45	315	27	-45	-45	3	3	27	11	-21	3	3	3	-5	3
χ_{67}	945	-15	-195	29	-15	-39	1	9	21	-27	13	1	9	5	-7	-3

Table 6.15: The character table of $2^6{:}SP(6,2)$(continued)

	$3A$		$3B$	$3C$		$4A$		$4B$			$4C$			$4D$		
	$3A$	$6A$	$3B$	$3C$	$6B$	$4H$	$4I$	$4J$	$4K$	$8A$	$4L$	$8B$	$4M$	$4N$	$4O$	$4P$
χ_{31}	15	-1	0	3	-1	3	-1	7	-1	-1	7	-1	-1	3	-1	-1
χ_{32}	-15	1	0	6	-2	3	-1	-5	3	-1	-5	-1	3	-5	7	-1
χ_{33}	0	0	0	0	0	3	-1	3	-5	3	3	3	-5	-5	7	-1
χ_{34}	30	-2	0	-3	1	3	-1	-3	5	-3	9	-3	1	-1	3	-1
χ_{35}	15	-1	0	3	-1	-6	2	2	2	-2	2	-2	2	-6	2	2
χ_{36}	-30	2	0	-6	2	0	0	0	0	0	0	0	0	0	0	0
χ_{37}	30	-2	0	-3	1	3	-1	-9	-1	3	-9	3	-1	3	-1	-1
χ_{38}	15	-1	0	3	-1	-6	2	-2	-2	2	-2	2	-2	2	-6	2
χ_{39}	0	0	0	0	0	3	-1	9	1	-3	-3	-3	5	7	-5	-1
χ_{40}	-15	1	0	6	-2	3	-1	-7	1	1	5	1	-3	7	-5	-1
χ_{41}	15	-1	0	3	-1	3	-1	5	-3	1	-7	1	1	-1	3	-1
χ_{42}	45	-3	0	0	0	6	-2	2	2	-2	-14	2	2	-2	-2	2
χ_{43}	45	-3	0	0	0	0	0	-4	-4	4	4	-4	4	0	0	0
χ_{44}	-45	3	0	0	0	6	-2	-2	-2	2	-10	-2	6	-2	-2	2
χ_{45}	0	0	0	0	0	-12	4	0	0	0	0	0	0	4	4	-4
χ_{46}	-45	3	0	0	0	6	-2	2	2	-2	10	2	-6	-2	-2	2
χ_{47}	45	-3	0	0	0	0	0	4	4	-4	-4	4	-4	0	0	0
χ_{48}	45	-3	0	0	0	6	-2	-2	-2	2	14	-2	-2	-2	-2	2
χ_{49}	15	-1	0	3	-1	-6	2	-14	2	2	2	-2	2	2	2	-2
χ_{50}	15	-1	0	3	-1	-6	2	-10	6	-2	-2	2	-2	2	2	-2
χ_{51}	15	-1	0	3	-1	-6	2	14	-2	-2	-2	2	-2	2	2	-2
χ_{52}	15	-1	0	3	-1	-6	2	10	-6	2	2	-2	2	2	2	-2
χ_{53}	30	-2	0	6	-2	12	-4	0	0	0	0	0	0	-4	-4	4
χ_{54}	-30	2	0	3	-1	0	0	0	0	0	0	0	0	0	0	0
χ_{55}	-30	2	0	3	-1	0	0	0	0	0	0	0	0	0	0	0
χ_{56}	15	-1	0	-6	2	0	0	-4	-4	4	4	-4	4	0	0	0
χ_{57}	15	-1	0	-6	2	0	0	4	4	-4	-4	4	-4	0	0	0
χ_{58}	45	-3	0	0	0	-3	1	5	-3	1	5	1	-3	-3	1	1
χ_{59}	0	0	0	0	0	-9	3	-3	5	-3	9	-3	1	-5	-1	3
χ_{60}	-45	3	0	0	0	6	-2	2	2	-2	2	-2	2	6	-2	-2
χ_{61}	0	0	0	0	0	3	-1	9	1	-3	-3	-3	5	-1	3	-1
χ_{62}	45	-3	0	0	0	9	-3	-5	3	-1	7	-1	-1	5	1	-3
χ_{63}	45	-3	0	0	0	9	-3	-7	1	1	-7	1	1	1	5	-3
χ_{64}	0	0	0	0	0	3	-1	3	-5	3	3	3	-5	3	-1	-1
χ_{65}	-45	3	0	0	0	6	-2	-2	-2	2	-2	2	-2	-2	6	-2
χ_{66}	0	0	0	0	0	-9	3	-9	-1	3	-9	3	-1	-1	-5	3
χ_{67}	45	-3	0	0	0	-3	1	7	-1	-1	-5	-1	3	1	-3	1

Table 6.15: The character table of $2^6{:}SP(6,2)$(continued)

	$4E$				$5A$		$6A$		$6B$			$6C$	$6D$			$6E$	
	$4Q$	$4R$	$8C$	$8D$	$5A$	$10A$	$6C$	$12A$	$6D$	$6E$	$12B$	$6F$	$6G$	$12C$	$12D$	$6H$	$12E$
χ_{31}	3	-1	-1	-1	3	-1	3	-1	7	-1	-1	0	3	-1	-1	1	-1
χ_{32}	-1	3	-1	-1	0	0	-3	1	5	-3	1	0	1	-3	1	2	-2
χ_{33}	3	-1	-1	-1	-3	1	0	0	0	0	0	0	0	0	0	0	0
χ_{34}	1	1	-3	1	0	0	6	-2	2	2	-2	0	2	2	-2	-1	1
χ_{35}	-2	-2	2	2	0	0	3	-1	-5	3	-1	0	-1	3	-1	1	-1
χ_{36}	0	0	0	0	3	-1	-6	2	-2	-2	2	0	-2	-2	2	-2	2
χ_{37}	-1	3	-1	-1	0	0	6	-2	2	2	-2	0	2	2	-2	-1	1
χ_{38}	-2	-2	2	2	0	0	3	-1	7	-1	-1	0	3	-1	-1	1	-1
χ_{39}	1	1	1	-3	-3	1	0	0	0	0	0	0	0	0	0	0	0
χ_{40}	1	1	-3	1	0	0	-3	1	-7	1	1	0	-3	1	1	2	-2
χ_{41}	1	1	1	-3	3	-1	3	-1	-5	3	-1	0	-1	3	-1	1	-1
χ_{42}	2	-2	-2	2	3	-1	3	-1	-9	-1	3	0	1	-3	1	0	0
χ_{43}	0	0	0	0	-3	1	3	-1	-9	-1	3	0	1	-3	1	0	0
χ_{44}	2	-2	-2	2	0	0	-3	1	9	1	-3	0	-1	3	-1	0	0
χ_{45}	0	0	0	0	3	-1	0	0	0	0	0	0	0	0	0	0	0
χ_{46}	-2	2	2	-2	0	0	-3	1	-3	5	-3	0	3	-1	-1	0	0
χ_{47}	0	0	0	0	-3	1	3	-1	3	-5	3	0	-3	1	1	0	0
χ_{48}	-2	2	2	-2	3	-1	3	-1	3	-5	3	0	-3	1	1	0	0
χ_{49}	2	-2	2	-2	0	0	-3	1	-7	1	1	0	3	-1	-1	-1	1
χ_{50}	2	-2	2	-2	0	0	-3	1	5	-3	1	0	-1	3	-1	-1	1
χ_{51}	-2	2	-2	2	0	0	-3	1	5	-3	1	0	-1	3	-1	-1	1
χ_{52}	-2	2	-2	2	0	0	-3	1	-7	1	1	0	3	-1	-1	-1	1
χ_{53}	0	0	0	0	0	0	-6	2	-2	-2	2	0	2	2	-2	-2	2
χ_{54}	0	0	0	0	0	0	6	-2	2	2	-2	0	-2	-2	2	-1	1
χ_{55}	0	0	0	0	0	0	6	-2	2	2	-2	0	-2	-2	2	-1	1
χ_{56}	0	0	0	0	0	0	-3	1	5	-3	1	0	-1	3	-1	2	-2
χ_{57}	0	0	0	0	0	0	-3	1	-7	1	1	0	3	-1	-1	2	-2
χ_{58}	1	-3	1	1	0	0	-3	1	9	1	-3	0	1	-3	1	0	0
χ_{59}	1	1	-3	1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{60}	-2	-2	2	2	0	0	3	-1	3	-5	3	0	3	-1	-1	0	0
χ_{61}	1	1	1	-3	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{62}	-1	-1	-1	3	0	0	-3	1	-3	5	-3	0	-3	1	1	0	0
χ_{63}	-3	1	1	1	0	0	-3	1	9	1	-3	0	1	-3	1	0	0
χ_{64}	3	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{65}	-2	-2	2	2	0	0	3	-1	-9	-1	3	0	-1	3	-1	0	0
χ_{66}	-1	3	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{67}	-1	-1	3	-1	0	0	-3	1	-3	5	-3	0	-3	1	1	0	0

Table 6.15: The character table of 2^6:$SP(6,2)$(continued)

	$6F$		$6G$		$7A$	$8A$		$8B$		$9A$	$10A$		$12A$		$12B$		$12C$	$15A$
	$6I$	$6J$	$6K$	$12F$	$7A$	$8E$	$8F$	$8G$	$8H$	$9A$	$10B$	$20A$	$12G$	$24A$	$12H$	$24B$	$12I$	$15A$
χ_{31}	3	-1	1	-1	0	1	-1	1	-1	0	1	-1	1	-1	1	-1	0	0
χ_{32}	0	0	0	0	0	-1	1	-1	1	0	0	0	1	-1	1	-1	0	0
χ_{33}	0	0	0	0	0	1	-1	1	-1	0	-1	1	0	0	0	0	0	0
χ_{34}	-3	1	-1	1	0	1	-1	-1	1	0	0	0	0	0	0	0	0	0
χ_{35}	3	-1	1	-1	0	0	0	0	0	0	0	0	-1	1	-1	1	0	0
χ_{36}	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
χ_{37}	3	-1	1	-1	0	-1	1	-1	1	0	0	0	0	0	0	0	0	0
χ_{38}	-3	1	-1	1	0	0	0	0	0	0	0	0	1	-1	1	-1	0	0
χ_{39}	0	0	0	0	0	-1	1	1	-1	0	-1	1	0	0	0	0	0	0
χ_{40}	0	0	0	0	0	1	-1	-1	1	0	0	0	-1	1	-1	1	0	0
χ_{41}	-3	1	-1	1	0	-1	1	1	-1	0	1	-1	-1	1	-1	1	0	0
χ_{42}	0	0	0	0	0	0	0	0	0	0	-1	1	-1	1	1	-1	0	0
χ_{43}	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	1	-1	0	0
χ_{44}	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	0	0
χ_{45}	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0
χ_{46}	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	1	-1	0	0
χ_{47}	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	-1	1	0	0
χ_{48}	0	0	0	0	0	0	0	0	0	0	-1	1	1	-1	-1	1	0	0
χ_{49}	3	-1	-1	1	0	0	0	0	0	0	0	0	1	-1	-1	1	0	0
χ_{50}	-3	1	1	-1	0	0	0	0	0	0	0	0	-1	1	1	-1	0	0
χ_{51}	3	-1	-1	1	0	0	0	0	0	0	0	0	-1	1	1	-1	0	0
χ_{52}	-3	1	1	-1	0	0	0	0	0	0	0	0	1	-1	-1	1	0	0
χ_{53}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{54}	-3	1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{55}	3	-1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{56}	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	1	-1	0	0
χ_{57}	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	0	0
χ_{58}	0	0	0	0	0	-1	1	-1	1	0	0	0	-1	1	-1	1	0	0
χ_{59}	0	0	0	0	0	-1	1	1	-1	0	0	0	0	0	0	0	0	0
χ_{60}	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	-1	1	0	0
χ_{61}	0	0	0	0	0	1	-1	-1	1	0	0	0	0	0	0	0	0	0
χ_{62}	0	0	0	0	0	-1	1	1	-1	0	0	0	1	-1	1	-1	0	0
χ_{63}	0	0	0	0	0	1	-1	1	-1	0	0	0	-1	1	-1	1	0	0
χ_{64}	0	0	0	0	0	-1	1	-1	1	0	0	0	0	0	0	0	0	0
χ_{65}	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	0	0
χ_{66}	0	0	0	0	0	1	-1	1	-1	0	0	0	0	0	0	0	0	0
χ_{67}	0	0	0	0	0	1	-1	-1	1	0	0	0	1	-1	1	-1	0	0

6.5 The fusion of 2^6:$SP(6,2)$ into Fi_{22}

We used the results in Section 6.1 to compute the power maps of the elements of 2^6:$SP(6,2)$ which are listed in Table 6.16 below.

Table 6.16: The power maps of the elements of $2^6{:}SP(6,2)$

$[g]_{SP(6,2)}$	$[x]_{2^6:SP(6,2)}$	2	3	5	7	$[g]_{SP(6,2)}$	$[x]_{2^6:SP(6,2)}$	2	3	5	7
1A	1A					2A	2B	1A			
	2A	1A					2C	1A			
							4A	2A			
2B	2D	1A				2C	2F	1A			
	2E	1A					2G	1A			
	4B	2A					4C	2A			
							4D	2A			
2D	2H	1A				3A	3A		1A		
	4E	2A					6A	3A	2A		
	4F	2A									
	4G	2A									
3B	3B		1A			3C	3C		1A		
							6B	3C	2A		
4A	4H	2D				4B	4J	2F			
	4I	2E					4K	2F			
							8A	4C			
4C	4L	2F				4D	4N	2D			
	4M	2F					4O	2E			
	8B	4C					4P	2E			
4E	4Q	2F				5A	5A			1A	
	4R	2G					10A	5A		2A	
	8C	4C									
	8D	4C									
6A	6C	3A	2D			6B	6D	3A	2B		
	12A	6A	4B				6E	3A	2C		
							12B	6A	4A		
6C	6F	3B	2D			6D	6G	3A	2F		
							12C	6A	4C		
							12D	6A	4D		
6E	6H	3C	2B			6F	6I	3C	2D		
	12E	6B	4A				6J	3C	2E		
6G	6K	3C	2H			7A	7A				1A
	12F	6B	4G								
8A	8E	4H				8B	8G	4N			
	8F	4I					8H	4O			
9A	9A		3B			10A	10B	5A		2B	
							20A	10A		4A	
12A	12G	6G	4J			12B	12H	6G	4L		
	24A	12C	8A				24B	12C	8B		
12C	12I	6F	4H			15A	15A		5A	3A	

The power maps of the elements of Fi_{22} are given in the ATLAS. We make use of the power maps and conjugacy classes of elements for both groups to obtain the partial fusion of $2^6{:}SP(6,2)$ into Fi_{22}. To complete the fusion map we use the restrictions of the irreducible characters of Fi_{22} of small degrees to $2^6{:}SP(6,2)$. To determine these restrictions, we again use the technique of set intersections for characters. Thus we restrict two irreducible characters $78a$ and $429a$ of degrees 78 and 429 respectively of Fi_{22} to $2^6{:}SP(6,2)$.

Let ρ be the character of $SP(6,2)$ afforded by the regular representation of $SP(6,2)$. Then we obtain that $\rho = \sum_{i=1}^{30} e_i\phi_i$, where $\phi_i \in Irr(SP(6,2))$ and $e_i = deg(\phi_i)$. Then ρ can be regarded as a character of $2^6{:}SP(6,2)$ which contains 2^6 in its kernel such that

$$\rho(g) = \begin{cases} |SP(6,2)| & \text{if} \quad g \in 2^6 \\ 0 & \text{otherwise} \end{cases} .$$

If ψ is a character of Fi_{22}, then we obtain that

$$\begin{aligned} \langle \rho, \psi \rangle_{2^6:SP(6,2)} &= \frac{1}{|2^6{:}SP(6,2)|}\{\rho(1A)\psi(1A) + 63\rho(2A)\psi(2A)\} \\ &= \frac{1}{|2^6{:}SP(6,2)|}\{|SP(6,2)|\psi(1A) + 63|SP(6,2)|\psi(2A)\} \\ &= \frac{1}{64}\{\psi(1A) + 63\psi(2A)\} \\ &= \langle \psi_{2^6}, \tau_1 \rangle \end{aligned} .$$

where τ_1 is the identity character of 2^6 and ψ_{2^6} is the restriction of ψ to 2^6. Also for ψ we obtain that

$$\psi_{2^6} = a_1\theta_1 + a_2\theta_2 \quad ,$$

where $a_1, a_2 \in \mathbb{N} \cup \{0\}$ and $\theta_i, i \in \{1, 2\}$ are the sums of the irreducible characters of 2^6 which are in one orbit under the action of $SP(6,2)$ on $Irr(2^6)$. Let $\tau_j \in Irr(2^6)$, where $j \in \{1, 2, \ldots, 64\}$. Then we obtain that

$$\theta_1 = \tau_1 \ , \ deg(\theta_1) = 1$$

$$\theta_2 = \sum_{j=2}^{64} \tau_j \ , \ deg(\theta_2) = 63$$

and thus we have

$$\psi_{2^6} = a_1\tau_1 + a_2\sum_{j=2}^{64}\tau_j$$

and hence

$$\langle \psi_{2^6}, \psi_{2^6} \rangle = a_1^2 + 63a_2^2 \quad ,$$

where $a_1 = \langle \psi_{2^6}, \tau_1 \rangle = \langle \rho, \psi \rangle_{2^6:SP(6,2)}$. We also have that

$$\langle \psi_{2^6}, \psi_{2^6} \rangle = \frac{1}{64}\{\psi(1A)\psi(1A) + 63\psi(2A)\psi(2A)\} \quad .$$

Let $78a$ and $429a$ be the irreducible characters of Fi_{22} of degrees 78 and 429 respectively. First let $\psi_1 = 78a$. Then we obtain that

$$\langle \rho, \psi_1 \rangle_{2^6:SP(6,2)} = \frac{1}{64}[78 + 63\psi_1(2A)] \quad .$$

However Fi_{22} has three classes of involutions namely $2A, 2B, 2C$. The $2A$ class of $2^6{:}SP(6,2)$ must fuse into one of these classes of involutions of Fi_{22} such that the condition $\langle \rho, \psi_1 \rangle \in \mathbb{N} \cup \{0\}$ is satisfied. But the values of $78a$ on the classes $2A, 2C$ of Fi_{22} violate this condition and only the value on $2B$ fulfills the condition. Hence we obtain that $2A$ of $2^6{:}SP(6,2)$ fuses into $2B$ of Fi_{22} and that

$$a_1 = \langle \rho, \psi_1 \rangle_{2^6:SP(6,2)} = \frac{1}{64}[78 + 63 \times 14] = \frac{1}{64}[960] = 15 \quad .$$

Since $deg(\psi_1) = 78$, we must have that $a_1 + 63a_2 = 78$ and since $a_1 = 15$, we must have that $a_2 = 1$. Hence based on the partial fusion of $2^6{:}SP(6,2)$ into Fi_{22} which has already been determined, we obtain that $(\psi_1)_{2^6:SP(6,2)} = \chi_3 + \chi_{41}$.

Now let $\psi_2 = 429a$. Then we obtain that

$$a_1 = \langle \rho, \psi_2 \rangle_{2^6:SP(6,2)} = \frac{1}{64}[429 + 63 \times 45] = 51 \quad .$$

Since $deg(\psi_2) = 429$, we must have that $a_1 + 63a_2 = 429$ and since $a_1 = 51$, we must have $a_2 = 6$. Hence we obtain

$$(\psi_2)_{2^6:SP(6,2)} = \chi_1 + \chi_3 + \chi_8 + \chi_{31} + \chi_{32} \quad .$$

Using the partial fusion already determined and the values of ψ_1 and ψ_2 on the classes of Fi_{22} and the values of $(\psi_1)_{2^6:SP(6,2)}$ and $(\psi_2)_{2^6:SP(6,2)}$ on the classes of $2^6{:}SP(6,2)$, we are able to complete the fusion map of $2^6{:}SP(6,2)$ into Fi_{22} and this is given in Table 6.17.

Table 6.17: The fusion of $2^6{:}SP(6,2)$ into Fi_{22}

$[g]_{SP(6,2)}$	$[x]_{2^6:SP(6,2}$	$\longrightarrow$	$[h]_{Fi_{22}}$	$[g]_{SP(6,2)}$	$[x]_{2^6:SP(6,2}$	$\longrightarrow$	$[h]_{Fi_{22}}$
1A	1A		1A	2A	2B		2A
	2A		2B		2C		2C
					4A		4B
2B	2D		2C	2C	2F		2B
	2E		2B		2G		2C
	4B		4A		4C		4A
					4D		4E
2D	2H		2C	3A	3A		3A
	4E		4E		6A		6D
	4F		4C				
	4G		4B				
3B	3B		3D	3C	3C		3C
					6B		6I
4A	4H		4D	4B	4J		4E
	4I		4C		4K		4B
					8A		8B
4C	4L		4B	4D	4N		4D
	4M		4E		4O		4A
	8B		8A		4P		4E
4E	4Q		4E	5A	5A		5A
	4R		4D		10A		10B
	8C		8A				
	8D		8B				
6A	6C		6F	6B	6D		6A
	12A		12C		6E		6F
					12B		12D
6C	6F		6K	6D	6G		6D
					12C		12B
					12D		12I
6E	6H		6E	6F	6I		6H
	12E		12J		6J		6I
6G	6K		6J	7A	7A		7A
	12F		12J				
8A	8E		8D	8B	8G		8D
	8F		8C		8H		8B
9A	9A		9C	10A	10B		10A
					20A		20A
12A	12G		12I	12B	12H		12D
	24A		24A		24B		24B
12C	12I		12K	15A	15A		15A

6.6 The permutation character of Fi_{22} on $2^6{:}SP(6,2)$

We have that $2^6{:}SP(6,2)$ is a maximal subgroup of Fi_{22} of index 694980 in Fi_{22}. Thus when Fi_{22} acts on the cosets of $2^6{:}SP(6,2)$, then this action gives rise to a permutation representation which affords a permutation character of degree 694980 and let $\chi(Fi_{22}|2^6{:}SP(6,2))$ be this permutation character. To determine $\chi(Fi_{22}|2^6{:}SP(6,2))$, we use the fusion of $2^6{:}SP(6,2)$ into Fi_{22} and the restrictions of $\chi_i \in Irr(Fi_{22})$ to $2^6{:}SP(6,2)$, where $deg(\chi_i) \leq 694980$. However from the ATLAS, we need only restrict $\chi_i \in Irr(Fi_{22})$, where $i \in \{1,2,3,\ldots,45\}$ to $2^6{:}SP(6,2)$. Let ψ_1 be the identity character of $2^6{:}SP(6,2)$. Having restricted $\chi_i \in Irr(Fi_{22}), i \in \{1,2,\ldots,45\}$ to $2^6{:}SP(6,2)$, then we compute the inner product of each χ_i with ψ_1. We thus obtain the following table for this information.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}
$\langle\chi_i,\psi_1\rangle$	1	0	1	0	1	0	1	0	1	1	0	0	1	0	0

	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}	χ_{27}	χ_{28}	χ_{29}	χ_{30}
$\langle\chi_i,\psi_1\rangle$	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0

	χ_{31}	χ_{32}	χ_{33}	χ_{34}	χ_{35}	χ_{36}	χ_{37}	χ_{38}	χ_{39}	χ_{40}	χ_{41}	χ_{42}	χ_{43}	χ_{44}	χ_{45}
$\langle\chi_i,\psi_1\rangle$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Using the above table and the Frobenius-Reciprocity (Theorem 3.4.3), we obtain that the permutation character $\chi(Fi_{22}|2^6{:}SP(6,2))$ is given by

$$\begin{aligned}\chi(Fi_{22}|2^6{:}SP(6,2)) &= 1a + 429a + 1430a + 3080a + 13650a + 30030a + \\ &\quad 45045a + 75075a + 205920a + 320320a \quad .\end{aligned}$$

The work of Ivanov et. al. in [65] and of Ivanov and Saxl in [66] shows that the group Fi_{22} acting on the cosets of $2^6{:}SP(6,2)$ has rank 10 with subdegrees 1, 135, 1260, 2304, 8640, 10080, 45360, 143360 and 241920(twice).

Chapter 7

A maximal subgroup of $2^6{:}SP(6,2)$

The sporadic simple group Fi_{22} is generated by a conjugacy class D of 3510 involutions which are called 3-transpositions such that the product of any noncommuting pair is an element of order 3. The full automorphism group of Fi_{22} is denoted by $\overline{F}i_{22}$ and it is given by $\overline{F}i_{22} = Fi_{22}{:}\langle e \rangle$, where e is an involutory outer automorphism of Fi_{22}. In $\overline{F}i_{22}$ there are three classes of involutory outer automorphisms of Fi_{22} which are denoted by e, f and θ and represented in the ATLAS by $2D$, $2F$ and $2E$ respectively. In this chapter, we study the group $C_{Fi_{22}}(\theta) \cong 2^6{:}O^-(6,2)$ which is a maximal subgroup of $2^6{:}SP(6,2)$ of index 28. We determine its Fischer-Clifford matrices and hence construct its character table. We use the properties of the Fischer-Clifford matrices which have been discussed in Chapter 5 (Section 5.2.2) and in some cases we also use the additional information discussed in the introduction of Chapter 6, to compute the entries of the Fischer-Clifford matrices. Motivation for this problem came from Moori's papers [83] and [85]. Moori in [83] obtained the generators for the groups $C_{Fi_{22}}(e)$, $C_{Fi_{22}}(f)$ and $C_{Fi_{22}}(\theta)$, where

$$C_{Fi_{22}}(e) \cong O^+(8,2){:}S_3,\ C_{Fi_{22}}(f) \cong SP(6,2) \times 2 \quad \text{and} \quad C_{Fi_{22}}(\theta) \cong 2^6{:}O^-(6,2) \quad .$$

From [83] we obtain that the above groups are D-subgroups of Fi_{22} generated by $C_D(e)$, $C_D(f)$ and $C_D(\theta)$ respectively. The complete fusion of $2^6{:}O^-(6,2)$ into $2^6{:}SP(6,2)$ will be fully determined.

7.1 The conjugacy classes of $2^6{:}O^-(6,2)$

In this section we use the method of coset analysis discussed in Chapter 2, Section 2.3, to determine the conjugacy classes of elements of $2^6{:}O^-(6,2)$. The group $O^-(6,2)$ is a maximal subgroup of $SP(6,2)$ of index 28. From the conjugacy classes of elements of $SP(6,2)$, obtained using CAYLEY, we generated $O^-(6,2)$ by two elements α and β of $SP(6,2)$ which are given by:

$$\alpha = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

such that $o(\alpha) = 4$ and $o(\beta) = 9$. We also give the class representatives $g \in O^-(6,2)$ in terms of 6×6 matrices over $GF(2)$ in the following table, where M is the matrix which represents that particular class.

$[g]_G$	M	$\lvert[g]_G\rvert$	$[g]_G$	M	$\lvert[g]_G\rvert$
$1A$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	1	$2A$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	36
$2B$	$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$	45	$2C$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$	270
$2D$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	540	$3A$	$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	80

$[g]_G$	M	$\|[g]_G\|$	$[g]_G$	M	$\|[g]_G\|$
$3B$	$\begin{pmatrix} 1&1&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&1&1&0&1\\ 1&1&1&0&1&0\\ 1&0&1&0&0&0 \end{pmatrix}$	240	$3C$	$\begin{pmatrix} 0&0&1&1&1&0\\ 1&1&0&1&1&0\\ 0&1&1&0&1&1\\ 0&1&1&0&0&0\\ 1&1&0&1&1&1\\ 0&0&1&0&0&1 \end{pmatrix}$	480
$4A$	$\begin{pmatrix} 1&1&0&1&0&1\\ 0&1&0&1&1&1\\ 1&0&1&1&0&0\\ 0&0&1&0&1&0\\ 0&1&1&1&1&0\\ 1&1&1&1&1&0 \end{pmatrix}$	540	$4B$	$\begin{pmatrix} 1&0&0&0&0&0\\ 1&1&0&1&1&0\\ 1&0&1&1&1&0\\ 1&1&0&0&1&0\\ 1&0&1&1&0&0\\ 1&0&0&1&1&1 \end{pmatrix}$	540
$4C$	$\begin{pmatrix} 1&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&1&1&0&1\\ 1&1&0&0&1&0\\ 1&1&1&0&0&1 \end{pmatrix}$	1620	$4D$	$\begin{pmatrix} 0&1&1&0&1&0\\ 0&1&0&1&0&0\\ 1&1&1&0&1&0\\ 0&0&1&1&1&1\\ 1&1&1&1&0&0\\ 1&1&1&1&1&1 \end{pmatrix}$	3240
$5A$	$\begin{pmatrix} 0&0&1&0&0&1\\ 1&0&1&1&1&0\\ 0&1&0&0&1&1\\ 1&1&0&1&1&0\\ 0&1&1&0&1&1\\ 1&1&0&0&0&1 \end{pmatrix}$	5184	$6A$	$\begin{pmatrix} 1&0&1&0&1&0\\ 1&0&1&1&1&1\\ 1&0&1&1&0&1\\ 1&1&1&1&0&0\\ 1&1&0&1&1&0\\ 0&0&1&0&0&1 \end{pmatrix}$	720
$6B$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&1&0&0\\ 1&1&1&0&1&0\\ 0&1&0&0&0&0\\ 1&1&1&1&0&0\\ 1&0&0&1&0&1 \end{pmatrix}$	1440	$6C$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 1&0&1&1&0&0\\ 0&0&0&1&0&0\\ 1&1&0&0&1&0\\ 0&0&0&1&1&1 \end{pmatrix}$	1440
$6D$	$\begin{pmatrix} 1&1&0&1&0&0\\ 0&0&0&1&0&0\\ 0&1&0&0&1&1\\ 1&1&0&1&1&1\\ 0&0&1&1&0&1\\ 0&1&0&0&1&0 \end{pmatrix}$	1440	$6E$	$\begin{pmatrix} 1&0&0&0&1&1\\ 1&1&0&0&1&0\\ 0&1&1&1&0&1\\ 0&0&0&1&0&0\\ 0&0&0&1&1&0\\ 1&0&0&1&0&0 \end{pmatrix}$	1440
$6F$	$\begin{pmatrix} 0&0&1&0&0&1\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 1&0&1&1&0&1\\ 1&0&0&0&0&1 \end{pmatrix}$	2160	$6G$	$\begin{pmatrix} 0&0&0&1&0&1\\ 1&0&1&1&0&1\\ 1&1&1&1&0&1\\ 0&1&0&1&1&1\\ 1&1&1&0&1&1\\ 0&1&1&0&0&1 \end{pmatrix}$	4320

$[g]_G$	M	$\lvert [g]_G \rvert$	$[g]_G$	M	$\lvert [g]_G \rvert$
$8A$	$\begin{pmatrix} 0&0&0&1&0&1 \\ 0&0&0&1&0&0 \\ 0&0&0&1&1&0 \\ 0&1&0&1&1&1 \\ 1&1&1&0&1&1 \\ 1&0&0&0&1&0 \end{pmatrix}$	6480	$9A$	$\begin{pmatrix} 1&0&0&0&1&0 \\ 0&0&0&1&1&0 \\ 0&0&0&0&1&0 \\ 1&1&1&1&0&1 \\ 1&0&1&1&1&0 \\ 0&0&0&1&0&1 \end{pmatrix}$	5760
$10A$	$\begin{pmatrix} 0&1&0&0&1&0 \\ 0&0&1&0&0&0 \\ 1&1&0&0&1&1 \\ 0&1&0&0&1&1 \\ 0&1&1&1&1&0 \\ 1&1&1&0&0&0 \end{pmatrix}$	5184	$12A$	$\begin{pmatrix} 1&1&0&0&0&1 \\ 1&0&1&1&1&0 \\ 1&1&0&1&1&0 \\ 1&1&1&1&0&1 \\ 1&1&0&0&1&0 \\ 0&1&0&1&1&0 \end{pmatrix}$	4320
$12B$	$\begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&1&1&1&1 \\ 0&1&1&0&1&1 \\ 0&0&1&1&0&1 \\ 0&1&0&0&1&1 \\ 1&0&1&0&1&0 \end{pmatrix}$	4320			

We obtain that $O^-(6,2)$ has 25 conjugacy classes and that when $O^-(6,2)$ acts on 2^6 it gives rise to three orbits of lengths 1, 27 and 36 and hence three point stabilizers $O^-(6,2)$, $2^4{:}S_5$ and $S_6 \times 2$ of indices 1, 27 and 36 respectively in $O^-(6,2)$. Let $\chi(O^-(6,2)|2^4{:}S_5)$ and $\chi(O^-(6,2)|S_6 \times 2)$ be the permutation characters of $O^-(6,2)$ on $2^4{:}S_5$ and $S_6 \times 2$ respectively. Then from the ATLAS, we obtain that

$$\chi(O^-(6,2)|2^4{:}S_5) = 1a + 6a + 20a \quad \text{and} \quad \chi(O^-(6,2)|S_6 \times 2) = 1a + 15b + 20a \quad .$$

Now let $\chi(O^-(6,2)|2^6)$ be the permutation character of $O^-(6,2)$ on 2^6. Then we obtain that

$$\chi(O^-(6,2)|2^6) = 1 + I_{2^4:S_5}^{O^-(6,2)} + I_{S_6\times 2}^{O^-(6,2)} \quad ,$$

where $I_{2^4:S_5}^{O^-(6,2)}$ and $I_{S_6\times 2}^{O^-(6,2)}$ are the identity characters of $2^4{:}S_5$ and $S_6 \times 2$ respectively induced to $O^-(6,2)$ and we observe that

$$I_{2^4:S_5}^{O^-(6,2)} = \chi(O^-(6,2)|2^4{:}S_5) \quad \text{and} \quad I_{S_6\times 2}^{O^-(6,2)} = \chi(O^-(6,2)|S_6 \times 2) \quad .$$

Hence $\chi(O^-(6,2)|2^6) = 1 + \chi(O^-(6,2)|2^4{:}S_5) + \chi(O^-(6,2)|S_6 \times 2)$. Thus the values of $\chi(O^-(6,2)|2^6)$ on the various classes of $O^-(6,2)$ give us the number k of fixed points of each $g \in O^-(6,2)$ in 2^6. The following table provides us with a complete list of the k's, which we need for calculating the conjugacy classes of $2^6{:}O^-(6,2)$.

$[g]_{O^-(6,2)}$	$1A$	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$5A$
$\chi(O^-(6,2)\|2^4{:}S_5)$	27	15	3	7	3	0	9	0	3	1	5	1	2
$\chi(O^-(6,2)\|S_6\times 2)$	36	16	12	8	4	0	6	3	0	6	2	2	1
k	64	32	16	16	8	1	16	4	4	8	8	4	4
$[g]_{O^-(6,2)}$	$6A$	$6B$	$6C$	$6D$	$6E$	$6F$	$6G$	$8A$	$9A$	$10A$	$12A$	$12B$	
$\chi(O^-(6,2)\|2^4{:}S_5)$	0	0	3	0	3	1	0	1	0	0	1	0	
$\chi(O^-(6,2)\|S_6\times 2)$	0	1	4	3	0	2	1	0	0	1	0	0	
k	1	2	8	4	4	4	2	2	1	2	2	1	

Having obtained the values of the k's for each class representative $g \in O^-(6,2)$, we then need to compute the f_j's corresponding to these various k's. For this purpose we use Progrmme A given in Chapter 2 (Section 2.3). See Appendix, Programme A for $2^6{:}O^-(6,2)$. From the programme output we calculate the number f_j of orbits Q_i's, $1 \leq i \leq k$ which have come together under the action of $C_{O^-(6,2)}(g)$ for each class representative $g \in O^-(6,2)$. Having obtained the f_j's, we deduce that $2^6{:}O^-(6,2)$ has altogether 65 conjugacy classes of elements. These values are listed in Table 7.1. In this table we also list the d_j's where d_jg is a representative of the Δ_j. For each class representative $g \in O^-(6,2)$, we calculate the lengths of the corresponding classes $[x]_{2^6:O^-(6,2)}$ of $2^6{:}O^-(6,2)$ by using the theory of the conjugacy classes of group extensions which has been discussed in Chapter 2 (Section 2.3). For each $[x]_{2^6:O^-(6,2)}$, the order of $C_{2^6:O^-(6,2)}(x)$ is also given in the last column of Table 7.1. Table 7.1 provides complete details of the conjugacy classes of elements of $2^6{:}O^-(6,2)$.

Table 7.1: The conjugacy classes of $2^6{:}O^-(6,2)$

$[g]_{O^-(6,2)}$	k	f_j	d_j	w	$[x]_{2^6:O^-(6,2)}$	$\|[x]_{2^6:O^-(6,2)}\|$	$\|C_{2^6:O^-(6,2)}(x)\|$
$1A$	64	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$1A$	1	3317760
		$f_2=27$	$(1,1,1,1,0,1)$	$(1,1,1,1,0,1)$	$2A$	27	122880
		$f_3=36$	$(1,1,1,1,1,1)$	$(1,1,1,1,1,1)$	$2B$	36	92160
$2A$	32	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2C$	72	46080
		$f_2=6$	$(1,1,1,1,0,1)$	$(1,0,0,0,0,0)$	$4A$	432	7680
		$f_3=10$	$(1,1,1,1,1,1)$	$(1,0,0,0,0,0)$	$4B$	720	4608
		$f_4=15$	$(1,0,1,0,1,0)$	$(0,0,0,0,0,0)$	$2D$	1080	3072
$2B$	16	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2E$	180	18432
		$f_2=3$	$(1,1,1,1,1,1)$	$(0,0,0,0,0,0)$	$2F$	540	6144
		$f_3=12$	$(1,0,1,1,1,1)$	$(1,0,1,1,0,1)$	$4C$	2160	1536

Table 7.1: The conjugacy classes of $2^6{:}O^-(6,2)$(continued)

$[g]_{O^-(6,2)}$	k	f_j	d_j	w	$[x]_{2^6:O^-(6,2)}$	$\lvert[x]_{2^6:O^-(6,2)}\rvert$	$\lvert C_{2^6:O^-(6,2)}(x)\rvert$
$2C$	16	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2G$	1080	3072
		$f_2=1$	$(1,1,0,1,1,0)$	$(0,1,1,0,0,0)$	$4D$	1080	3072
		$f_3=3$	$(1,1,1,1,1,1)$	$(0,1,1,0,0,0)$	$4E$	3240	1024
		$f_4=3$	$(0,0,1,0,0,1)$	$(0,0,0,0,0,0)$	$2H$	3240	1024
		$f_5=8$	$(1,0,1,0,1,0)$	$(1,0,0,1,1,0)$	$4F$	8640	384
$2D$	8	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2I$	4320	768
		$f_2=1$	$(1,1,1,1,1,1)$	$(1,1,0,0,0,1)$	$4G$	4320	768
		$f_3=3$	$(0,1,0,1,0,1)$	$(1,0,0,1,0,1)$	$4H$	12960	256
		$f_4=3$	$(1,1,1,1,1,0)$	$(0,1,0,1,0,0)$	$4I$	12960	256
$3A$	1	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3A$	5120	648
$3B$	16	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3B$	960	3456
		$f_2=6$	$(1,1,1,1,0,1)$	$(0,0,0,1,0,1)$	$6A$	5760	576
		$f_3=9$	$(1,1,1,1,1,1)$	$(1,0,1,1,1,0)$	$6B$	8640	384
$3C$	4	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3C$	7680	432
		$f_2=3$	$(1,1,1,1,1,1)$	$(1,0,1,0,1,1)$	$6C$	23040	144
$4A$	4	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4J$	8640	384
		$f_2=3$	$(1,1,1,1,0,1)$	$(0,0,0,0,0,0)$	$4K$	25920	128
$4B$	8	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4L$	4320	768
		$f_2=3$	$(1,1,1,1,1,1)$	$(0,0,0,0,0,0)$	$4M$	12960	256
		$f_3=4$	$(1,0,0,1,0,0)$	$(0,1,1,0,0,0)$	$8A$	17280	192
$4C$	8	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4N$	12960	256
		$f_2=1$	$(0,1,1,0,1,1)$	$(0,0,0,0,0,0)$	$4O$	12960	256
		$f_3=2$	$(1,0,1,0,1,0)$	$(0,0,0,0,0,0)$	$4P$	25920	128
		$f_4=4$	$(1,1,1,1,1,1)$	$(0,0,1,0,0,0)$	$8B$	51840	64
$4D$	4	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4Q$	51840	64
		$f_2=1$	$(1,1,1,1,1,1)$	$(1,1,0,0,1,0)$	$8C$	51840	64
		$f_3=1$	$(1,0,1,0,1,0)$	$(1,1,0,0,1,0)$	$8D$	51840	64
		$f_4=1$	$(1,0,0,1,0,0)$	$(0,0,0,0,0,0)$	$4R$	51840	64
$5A$	4	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$5A$	82944	40
		$f_2=1$	$(1,1,1,0,0,0)$	$(1,0,0,0,1,1)$	$10A$	82944	40
		$f_3=2$	$(1,1,1,1,1,1)$	$(0,0,1,1,1,1)$	$10B$	165888	20
$6A$	1	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6D$	46080	72
$6B$	2	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6E$	46080	72
		$f_2=1$	$(1,1,1,1,1,1)$	$(1,0,0,0,0,0)$	$12A$	46080	72
$6C$	8	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6F$	11520	288
		$f_2=1$	$(0,1,1,1,1,0)$	$(1,0,0,1,0,0)$	$12B$	11520	288
		$f_3=3$	$(1,1,1,1,1,1)$	$(0,0,0,0,0,0)$	$6G$	34560	96
		$f_4=3$	$(1,0,1,0,1,0)$	$(1,0,0,1,0,0)$	$12C$	34560	96

Table 7.1: The conjugacy classes of $2^6{:}O^-(6,2)$(continued)

$[g]_{O^-(6,2)}$	k	f_j	d_j	w	$[x]_{2^6:O^-(6,2)}$	$\lvert[x]_{2^6:O^-(6,2)}\rvert$	$\lvert C_{2^6:O^-(6,2)}(x)\rvert$
$6D$	4	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6H$	23040	144
		$f_2=3$	$(1,1,1,1,1,1)$	$(0,0,0,0,0,0)$	$6I$	69120	48
$6E$	4	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6J$	23040	144
		$f_2=3$	$(1,1,1,1,1,1)$	$(1,1,0,0,1,1)$	$12D$	69120	48
$6F$	4	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6K$	34560	96
		$f_2=1$	$(1,1,1,1,1,1)$	$(0,1,1,0,0,0)$	$12E$	34560	96
		$f_2=2$	$(1,0,1,0,1,0)$	$(1,0,0,1,1,0)$	$12F$	69120	48
$6G$	2	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6L$	138240	24
		$f_2=1$	$(1,1,1,1,1,1)$	$(1,1,0,0,0,1)$	$12G$	138240	24
$8A$	2	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$8E$	207360	16
		$f_2=1$	$(1,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$8F$	207360	16
$9A$	1	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$9A$	368640	9
$10A$	2	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$10C$	165888	20
		$f_2=1$	$(1,0,0,1,0,0)$	$(1,0,0,1,1,1)$	$20A$	165888	20
$12A$	2	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$12H$	138240	24
		$f_2=1$	$(1,0,0,1,0,0)$	$(0,1,1,0,0,0)$	$24A$	138240	24
$12B$	1	$f_1=1$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$12I$	276480	12

7.2 The inertia groups of $2^6{:}O^-(6,2)$

When $O^-(6,2)$ acts on the conjugacy classes of 2^6, it forms three orbits of lengths 1, 27 and 36. Hence $O^-(6,2)$ acting on $Irr(2^6)$ will form three orbits of lengths 1, t and z such that t + z = 63. Since $O^-(6,2) \cong U_4(2).2$, then from the ATLAS we obtain that t = 27 and z = 36. We deduce that there are three inertia groups $\overline{H}_i = 2^6{:}H_i$ of indices 1, 27, 36 in $2^6{:}O^-(6,2)$ respectively, where $i \in \{1,2,3\}$ and $H_i \leq O^-(6,2)$ are the inertia factors. Then we obtain that the inertia factors are given by $H_1 = O^-(6,2)$, $H_2 = 2^4{:}S_5$ and $H_3 = S_6 \times 2$, where $H_2 = \langle \alpha_1, \alpha_2 \rangle$ and $H_3 = \langle \beta_1, \beta_2 \rangle$, where α_1, α_2, β_1 and β_2 are given by

$$\alpha_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\beta_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

such that $o(\alpha_1) = 2$, $o(\alpha_2) = 12$, $o(\beta_1) = 2$ and $o(\beta_2) = 6$.

7.3 The fusions of $2^4{:}S_5$ and $S_6 \times 2$ into $O^-(6,2)$

The groups $2^4{:}S_5$ and $S_6 \times 2$ are maximal subgroups of $O^-(6,2)$ of indices 27 and 36 respectively. Thus using direct matrix conjugation in $O^-(6,2)$ and the permutation characters of $O^-(6,2)$ on $2^4{:}S_5$ and $S_6 \times 2$ of degrees 27 and 36 respectively, we obtain the fusions of the inertia factors $H_2 = 2^4{:}S_5$ and $H_3 = S_6 \times 2$ into $O^-(6,2)$. These are given in Tables 7.2 and 7.3 respectively. We follow the techniques already discussed and used in Chapter 6 for the fusions.

Table 7.2: The fusion of $2^4{:}S_5$ into $O^-(6,2)$

$[g]_{O^-(6,2)}$	$1A$	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$5A$
$[h]_{2^4:S_5}$													
$1A$	$\boxed{27}$												
$2A$			$\boxed{3}$										
$2B$			6	$\boxed{1}$									
$2C$		$\boxed{15}$	12	2	1								
$2D$		45	36	6	$\boxed{3}$								
$2E$		45	36	$\boxed{6}$	3								
$3A$						27	$\boxed{9}$						
$4A$									1	$\boxed{1}$			
$4B$									$\boxed{3}$	3	1		
$4C$									3	3	$\boxed{1}$		
$4D$									6	6	2	$\boxed{1}$	
$4E$									12	12	$\boxed{4}$	2	
$5A$													$\boxed{2}$
$\chi(O^-(6,2)\|2^4{:}S_5)$	27	15	3	7	3	0	9	0	3	1	5	1	2

Table 7.2: The fusion of $2^4{:}S_5$ into $O^-(6,2)$(continued)

$[g]_{O^-(6,2)}$ / $[h]_{2^4:S_5}$	$6A$	$6B$	$6C$	$6D$	$6E$	$6F$	$6G$	$8A$	$9A$	$10A$	$12A$	$12B$
$6A$	3					$\boxed{1}$						
$6B$	6	3	$\boxed{3}$	3	3	2	1					
$6C$	6	3	3	3	$\boxed{3}$	2	1					
$8A$								$\boxed{1}$				
$12A$											$\boxed{1}$	1
$\chi(O^-(6,2)\|2^4{:}S_5)$	0	0	3	0	3	1	0	1	0	0	1	0

Table 7.3: The fusion of $S_6 \times 2$ into $O^-(6,2)$

$[g]_{O^-(6,2)}$ / $[h]_{S_6\times 2}$	$1A$	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$5A$
$1A$	$\boxed{36}$												
$2A$		$\boxed{1}$											
$2B$		15	12	$\boxed{2}$	1								
$2C$		15	$\boxed{12}$	2	1								
$2D$		$\boxed{15}$	12	2	1								
$2E$		15	12	2	$\boxed{1}$								
$2F$		45	36	$\boxed{6}$	3								
$2G$		45	36	6	$\boxed{3}$								
$3A$						18	$\boxed{6}$	3					
$3B$						18	6	$\boxed{3}$					
$4A$									6	6	2	$\boxed{1}$	
$4B$									6	6	2	$\boxed{1}$	
$4C$									6	$\boxed{6}$	2	1	
$4D$									6	6	$\boxed{2}$	1	
$5A$													$\boxed{1}$
$\chi(O^-(6,2)\|S_6 \times 2)$	36	16	12	8	4	0	5	3	0	6	2	2	1

Table 7.3: The fusion of $S_6 \times 2$ into $O^-(6,2)$(continued)

$[g]_{O^-(6,2)}$ / $[h]_{S_6\times 2}$	$6A$	$6B$	$6C$	$6D$	$6E$	$6F$	$6G$	$8A$	$9A$	$10A$	$12A$	$12B$
$6A$	2	1	$\boxed{1}$	1	1							
$6B$	2	$\boxed{1}$	1	1	1							
$6C$	6	3	3	3	3	$\boxed{2}$	1					
$6D$	6	3	3	$\boxed{3}$	3	2	1					
$6E$	6	3	3	3	3	2	$\boxed{1}$					
$6F$	6	3	$\boxed{3}$	3	3	2	1					
$10A$										$\boxed{1}$		
$\chi(O^-(6,2)\|S_6 \times 2)$	0	1	4	3	0	2	1	0	0	1	0	0

7.4 The Fischer-Clifford matrices of $2^6{:}O^-(6,2)$

We use the fusions discussed in Section 7.3 together with the relations of Proposition 5.2.3, Theorem 5.2.4 and the properties (a) through (f) of the Fischer-Clifford matrices which are given in Chapter 5 (Section 5.2.2) to construct the Fischer-Clifford matrices of $2^6{:}O^-(6,2)$. For each conjugacy class $[g]$ of $O^-(6,2)$ with representative $g \in O^-(6,2)$, we construct the corresponding Fischer-Clifford matrix $M(g)$ and these are given in Table 7.4 below.

Table 7.4: The Fischer-Clifford matrices of $2^6{:}O^-(6,2)$

$M(g)$	$M(g)$	$M(g)$
$M(1A) = \begin{pmatrix} 1 & 1 & 1 \\ 27 & -5 & 3 \\ 36 & 4 & -4 \end{pmatrix}$	$M(2A) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 15 & -5 & 3 & -1 \\ 1 & -1 & -1 & 1 \\ 15 & 5 & -3 & -1 \end{pmatrix}$	$M(2B) = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & -1 \\ 12 & -4 & 0 \end{pmatrix}$
$M(2C) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 6 & -6 & 2 & -2 & 0 \\ 2 & -2 & -2 & 2 & 0 \\ 6 & 6 & -2 & -2 & 0 \end{pmatrix}$	$M(2D) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 3 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 3 & -3 & 1 & -1 \end{pmatrix}$	$M(3A) = \begin{pmatrix} 1 \end{pmatrix}$
$M(3B) = \begin{pmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ 6 & 2 & -2 \end{pmatrix}$	$M(3C) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	$M(4A) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$
$M(4B) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 6 & -2 & 0 \end{pmatrix}$	$M(4C) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 4 & -4 & 0 & 0 \\ 2 & 2 & -2 & 0 \end{pmatrix}$	$M(4D) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$
$M(5A) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$	$M(6A) = \begin{pmatrix} 1 \end{pmatrix}$	$M(6B) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$M(6C) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -3 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & 3 & -1 & -1 \end{pmatrix}$	$M(6D) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	$M(6E) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$
$M(6F) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}$	$M(6G) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(8A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$M(9A) = \begin{pmatrix} 1 \end{pmatrix}$	$M(10A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(12A) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$M(12B) = \begin{pmatrix} 1 \end{pmatrix}$		

We use the above Fischer-Clifford matrices and the character tables of $H_1 = O^-(6,2)$, H_2 and H_3, together with the fusions of H_2 and H_3 into $O^-(6,2)$ to obtain the character table of $2^6{:}O^-(6,2)$. The set of irreducible characters of $2^6{:}O^-(6,2)$ will be partitioned into three blocks B_1, B_2 and B_3 corresponding to the inertia factors

H_1, H_2 and H_3 respectively. In fact $B_1 = \{\chi_i \mid 1 \leq i \leq 25\}, B_2 = \{\chi_i \mid 26 \leq i \leq 43\}, B_3 = \{\chi_i \mid 44 \leq i \leq 65\}$, where $Irr(2^6{:}O^-(6,2)) = \bigcup_{i=1}^{3} B_i$. The complete character table of $2^6{:}O^-(6,2)$ is displayed in Table 7.5. Please note that the centralizers of the elements of $2^6{:}O^-(6,2)$ are listed in the last column of Table 7.1.

Table 7.5: The character table of $2^6{:}O^-(6,2)$

	$1A$			$2A$				$2B$			$2C$				
	$1A$	$2A$	$2B$	$2C$	$4A$	$4B$	$2D$	$2E$	$2F$	$4C$	$2G$	$4D$	$4E$	$2H$	$4F$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1
χ_3	6	6	6	4	4	4	4	-2	-2	-2	2	2	2	2	2
χ_4	6	6	6	-4	-4	-4	-4	-2	-2	-2	2	2	2	2	2
χ_5	10	10	10	0	0	0	0	-6	-6	-6	2	2	2	2	2
χ_6	15	15	15	-5	-5	-5	-5	7	7	7	3	3	3	3	3
χ_7	15	15	15	5	5	5	5	-1	-1	-1	-1	-1	-1	-1	-1
χ_8	15	15	15	-5	-5	-5	-5	-1	-1	-1	-1	-1	-1	-1	-1
χ_9	15	15	15	5	5	5	5	7	7	7	3	3	3	3	3
χ_{10}	20	20	20	-10	-10	-10	-10	4	4	4	4	4	4	4	4
χ_{11}	20	20	20	10	10	10	10	4	4	4	4	4	4	4	4
χ_{12}	20	20	20	0	0	0	0	4	4	4	-4	-4	-4	-4	-4
χ_{13}	24	24	24	-4	-4	-4	-4	8	8	8	0	0	0	0	0
χ_{14}	24	24	24	4	4	4	4	8	8	8	0	0	0	0	0
χ_{15}	30	30	30	-10	-10	-10	-10	-10	-10	-10	2	2	2	2	2
χ_{16}	30	30	30	10	10	10	10	-10	-10	-10	2	2	2	2	2
χ_{17}	60	60	60	0	0	0	0	12	12	12	4	4	4	4	4
χ_{18}	60	60	60	-10	-10	-10	-10	-4	-4	-4	4	4	4	4	4
χ_{19}	60	60	60	10	10	10	10	-4	-4	-4	4	4	4	4	4
χ_{20}	64	64	64	-16	-16	-16	-16	0	0	0	0	0	0	0	0
χ_{21}	64	64	64	16	16	16	16	0	0	0	0	0	0	0	0
χ_{22}	80	80	80	0	0	0	0	-16	-16	-16	0	0	0	0	0
χ_{23}	81	81	81	-9	-9	-9	-9	9	9	9	-3	-3	-3	-3	-3
χ_{24}	81	81	81	9	9	9	9	9	9	9	-3	-3	-3	-3	-3
χ_{25}	90	90	90	0	0	0	0	-6	-6	-6	-6	-6	-6	-6	-6

Table 7.5: The character table of $2^6{:}O^-(6,2)$(continued)

	$2D$				$3A$	$3B$			$3C$		$4A$		$4B$		
	$2I$	$4G$	$4H$	$4I$	$3A$	$3B$	$6A$	$6B$	$3C$	$6C$	$4J$	$4K$	$4L$	$4M$	$8A$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1
χ_3	0	0	0	0	-3	3	3	3	0	0	2	2	-2	-2	-2
χ_4	0	0	0	0	-3	3	3	3	0	0	2	2	2	2	2
χ_5	0	0	0	0	1	-2	-2	-2	4	4	2	2	0	0	0
χ_6	-1	-1	-1	-1	-3	0	0	0	3	3	-1	-1	-3	-3	-3
χ_7	-3	-3	-3	-3	6	3	3	3	0	0	3	3	1	1	1
χ_8	3	3	3	3	6	3	3	3	0	0	3	3	-1	-1	-1
χ_9	1	1	1	1	-3	0	0	0	3	3	-1	-1	3	3	3
χ_{10}	-2	-2	-2	-2	2	5	5	5	-1	-1	0	0	-2	-2	-2
χ_{11}	2	2	2	2	2	5	5	5	-1	-1	0	0	2	2	2
χ_{12}	0	0	0	0	-7	2	2	2	2	2	4	4	0	0	0
χ_{13}	-4	-4	-4	-4	6	0	0	0	3	3	0	0	0	0	0
χ_{14}	4	4	4	4	6	0	0	0	3	3	0	0	0	0	0
χ_{15}	2	2	2	2	3	3	3	3	3	3	-2	-2	4	4	4
χ_{16}	-2	-2	-2	-2	3	3	3	3	3	3	-2	-2	-4	-4	-4
χ_{17}	0	0	0	0	-3	-6	-6	-6	0	0	4	4	0	0	0
χ_{18}	-2	-2	-2	-2	6	-3	-3	-3	-3	-3	0	0	2	2	2
χ_{19}	2	2	2	2	6	-3	-3	-3	-3	-3	0	0	-2	-2	-2
χ_{20}	0	0	0	0	-8	4	4	4	-2	-2	0	0	0	0	0
χ_{21}	0	0	0	0	-8	4	4	4	-2	-2	0	0	0	0	0
χ_{22}	0	0	0	0	-10	-4	-4	-4	2	2	0	0	0	0	0
χ_{23}	3	3	3	3	0	0	0	0	0	0	-3	-3	-3	-3	-3
χ_{24}	-3	-3	-3	-3	0	0	0	0	0	0	-3	-3	3	3	3
χ_{25}	0	0	0	0	9	0	0	0	0	0	2	2	0	0	0

Table 7.5: The character table of $2^6{:}O^-(6,2)$(continued)

	$4C$				$4D$				$5A$			$6A$	$6B$		$6C$			
	$4N$	$4O$	$4P$	$8B$	$4Q$	$8C$	$8D$	$4R$	$5A$	$10A$	$10B$	$6D$	$6E$	$12A$	$6F$	$12B$	$6G$	$12C$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
χ_3	2	2	2	2	0	0	0	0	1	1	1	1	-2	-2	1	1	1	1
χ_4	-2	-2	-2	-2	0	0	0	0	1	1	1	1	2	2	-1	-1	-1	-1
χ_5	0	0	0	0	-2	-2	-2	-2	0	0	0	-3	0	0	0	0	0	0
χ_6	1	1	1	1	1	1	1	1	0	0	0	1	1	1	-2	-2	-2	-2
χ_7	1	1	1	1	-1	-1	-1	-1	0	0	0	2	2	2	-1	-1	-1	-1
χ_8	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	2	-2	-2	1	1	1	1
χ_9	-1	-1	-1	-1	1	1	1	1	0	0	0	1	-1	-1	2	2	2	2
χ_{10}	-2	-2	-2	-2	0	0	0	0	0	0	0	-2	-1	-1	-1	-1	-1	-1
χ_{11}	2	2	2	2	0	0	0	0	0	0	0	-2	1	1	1	1	1	1
χ_{12}	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
χ_{13}	0	0	0	0	0	0	0	0	-1	-1	-1	2	-1	-1	2	2	2	2
χ_{14}	0	0	0	0	0	0	0	0	-1	-1	-1	2	1	1	-2	-2	-2	-2
χ_{15}	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1
χ_{16}	0	0	0	0	0	0	0	0	0	0	0	-1	1	1	1	1	1	1
χ_{17}	0	0	0	0	0	0	0	0	0	0	0	-3	0	0	0	0	0	0
χ_{18}	2	2	2	2	0	0	0	0	0	0	0	2	-1	-1	-1	-1	-1	-1
χ_{19}	-2	-2	-2	-2	0	0	0	0	0	0	0	2	1	1	1	1	1	1
χ_{20}	0	0	0	0	0	0	0	0	-1	-1	-1	0	2	2	2	2	2	2
χ_{21}	0	0	0	0	0	0	0	0	-1	-1	-1	0	-2	-2	-2	-2	-2	-2
χ_{22}	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
χ_{23}	1	1	1	1	-1	-1	-1	-1	1	1	1	0	0	0	0	0	0	0
χ_{24}	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	0	0	0	0	0	0	0
χ_{25}	0	0	0	0	2	2	2	2	0	0	0	-3	0	0	0	0	0	0

Table 7.5: The character table of 2^6:$O^-(6,2)$(continued)

	$6D$		$6E$			$6F$		$6G$		$8A$		$9A$	$10A$		$12A$		$12B$
	$6H$	$6I$	$6J$	$12D$	$6K$	$12E$	$12F$	$6L$	$12G$	$8E$	$8F$	$9A$	$10C$	$20A$	$12H$	$24A$	$12I$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	-1	-1	-1	-1	1	-1	-1	-1	-1	1
χ_3	-2	-2	1	1	-1	-1	-1	0	0	0	0	0	-1	-1	1	1	-1
χ_4	-2	-2	1	1	-1	-1	-1	0	0	0	0	0	1	1	-1	-1	-1
χ_5	0	0	0	0	2	2	2	0	0	0	0	1	0	0	0	0	-1
χ_6	1	1	-2	-2	0	0	0	-1	-1	1	1	0	0	0	0	0	-1
χ_7	2	2	-1	-1	-1	-1	-1	0	0	-1	-1	0	0	0	1	1	0
χ_8	2	2	-1	-1	-1	-1	-1	0	0	1	1	0	0	0	-1	-1	0
χ_9	1	1	-2	-2	0	0	0	1	1	-1	-1	0	0	0	0	0	-1
χ_{10}	1	1	1	1	1	1	1	1	1	0	0	-1	0	0	1	1	0
χ_{11}	1	1	1	1	1	1	1	-1	-1	0	0	-1	0	0	-1	-1	0
χ_{12}	-2	-2	-2	-2	2	2	2	0	0	0	0	-1	0	0	0	0	1
χ_{13}	-1	-1	2	2	0	0	0	-1	-1	0	0	0	1	1	0	0	0
χ_{14}	-1	-1	2	2	0	0	0	1	1	0	0	0	-1	-1	0	0	0
χ_{15}	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	1	1	1
χ_{16}	-1	-1	-1	-1	-1	-1	-1	1	1	0	0	0	0	0	-1	-1	1
χ_{17}	0	0	0	0	-2	-2	-2	0	0	0	0	0	0	0	0	0	1
χ_{18}	-1	-1	-1	-1	1	1	1	1	1	0	0	0	0	0	-1	-1	0
χ_{19}	-1	-1	-1	-1	1	1	1	-1	-1	0	0	0	0	0	1	1	0
χ_{20}	0	0	0	0	0	0	0	0	0	0	0	1	-1	-1	0	0	0
χ_{21}	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0
χ_{22}	2	2	2	2	0	0	0	0	0	0	0	-1	0	0	0	0	0
χ_{23}	0	0	0	0	0	0	0	0	0	-1	-1	0	1	1	0	0	0
χ_{24}	0	0	0	0	0	0	0	0	0	1	1	0	-1	-1	0	0	0
χ_{25}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1

Table 7.5: The character table of $2^6{:}O^-(6,2)$(continued)

	$1A$			$2A$				$2B$			$2C$				
	$1A$	$2A$	$2B$	$2C$	$4A$	$4B$	$2D$	$2E$	$2F$	$4C$	$2G$	$4D$	$4E$	$2H$	$4F$
χ_{26}	27	-5	3	15	-5	3	-1	3	3	-1	7	-5	3	-1	-1
χ_{27}	27	-5	3	-15	5	-3	1	3	3	-1	7	-5	3	-1	-1
χ_{28}	108	-20	12	30	-10	6	-2	12	12	-4	4	4	4	4	-4
χ_{29}	108	-20	12	-30	10	-6	2	12	12	-4	4	4	4	4	-4
χ_{30}	135	-25	15	-15	5	-3	1	15	15	-5	11	-1	7	3	-5
χ_{31}	135	-25	15	15	-5	3	-1	15	15	-5	11	-1	7	3	-5
χ_{32}	135	-25	15	45	-15	9	-3	-9	-9	3	7	-5	3	-1	-1
χ_{33}	135	-25	15	-45	15	-9	3	-9	-9	3	7	-5	3	-1	-1
χ_{34}	162	-30	18	0	0	0	0	18	18	-6	-6	18	2	10	-6
χ_{35}	270	-50	30	60	-20	12	-4	6	6	-2	10	-14	2	-6	2
χ_{36}	270	-50	30	-30	10	-6	2	6	6	-2	-14	10	-6	2	2
χ_{37}	270	-50	30	0	0	0	0	-18	-18	6	14	-10	6	-2	-2
χ_{38}	270	-50	30	-60	20	-12	4	6	6	-2	10	-14	2	-6	2
χ_{39}	270	-50	30	30	-10	6	-2	6	6	-2	-14	10	-6	2	2
χ_{40}	405	-75	45	45	-15	9	-3	-27	-27	9	-3	9	1	5	-3
χ_{41}	405	-75	45	-45	15	-9	3	-27	-27	9	-3	9	1	5	-3
χ_{42}	540	-100	60	-30	10	-6	2	12	12	-4	-4	-4	-4	-4	4
χ_{43}	540	-100	60	30	-10	6	-2	12	12	-4	-4	-4	-4	-4	4
χ_{44}	36	4	-4	16	4	-4	0	12	-4	0	8	4	-4	0	0
χ_{45}	36	4	-4	14	6	-2	-2	-12	4	0	4	8	0	-4	0
χ_{46}	36	4	-4	-14	-6	2	2	-12	4	0	4	8	0	-4	0
χ_{47}	36	4	-4	-16	-4	4	0	12	-4	0	8	4	-4	0	0
χ_{48}	180	20	-20	-10	-10	-2	6	36	-12	0	4	8	0	-4	0
χ_{49}	180	20	-20	20	0	-8	4	-36	12	0	8	4	-4	0	0
χ_{50}	180	20	-20	-50	-10	14	-2	-12	4	0	12	0	-8	4	0
χ_{51}	180	20	-20	-40	-20	4	8	12	-4	0	0	12	4	-8	0
χ_{52}	180	20	-20	10	10	2	-6	36	-12	0	4	8	0	-4	0
χ_{53}	180	20	-20	50	10	-14	2	-12	4	0	12	0	-8	4	0
χ_{54}	180	20	-20	-20	0	8	-4	-36	12	0	8	4	-4	0	0
χ_{55}	180	20	-20	40	20	-4	-8	12	-4	0	0	12	4	-8	0
χ_{56}	324	36	-36	-36	-24	0	12	-36	12	0	0	12	4	-8	0
χ_{57}	324	36	-36	54	6	-18	6	36	-12	0	12	0	-8	4	0
χ_{58}	324	36	-36	36	24	0	-12	-36	12	0	0	12	4	-8	0
χ_{59}	324	36	-36	-54	-6	18	-6	36	-12	0	12	0	-8	4	0
χ_{60}	360	40	-40	40	0	-16	8	-24	8	0	-8	-16	0	8	0
χ_{61}	360	40	-40	-40	0	16	-8	-24	8	0	-8	-16	0	8	0
χ_{62}	360	40	-40	-20	-20	-4	12	24	-8	0	-16	-8	8	0	0
χ_{63}	360	40	-40	20	20	4	-12	24	-8	0	-16	-8	8	0	0
χ_{64}	576	64	-64	-16	16	16	-16	0	0	0	0	0	0	0	0
χ_{65}	576	64	-64	16	-16	-16	16	0	0	0	0	0	0	0	0

Table 7.5: The character table of $2^6{:}O^-(6,2)$(continued)

	$2D$				$3A$	$3B$			$3C$		$4A$		$4B$		
	$2I$	$4G$	$4H$	$4I$	$3A$	$3B$	$6A$	$6B$	$3C$	$6C$	$4J$	$4K$	$4L$	$4M$	$8A$
χ_{26}	3	3	-1	-1	0	9	-3	1	0	0	3	-1	1	1	-1
χ_{27}	-3	-3	1	1	0	9	-3	1	0	0	3	-1	-1	-1	1
χ_{28}	6	6	-2	-2	0	9	-3	1	0	0	0	0	2	2	-2
χ_{29}	-6	-6	2	2	0	9	-3	1	0	0	0	0	-2	-2	2
χ_{30}	-3	-3	1	1	0	-9	3	-1	0	0	3	-1	-1	-1	1
χ_{31}	3	3	-1	-1	0	-9	3	-1	0	0	3	-1	1	1	-1
χ_{32}	-3	-3	1	1	0	18	-6	2	0	0	3	-1	-3	-3	3
χ_{33}	3	3	-1	-1	0	18	-6	2	0	0	3	-1	3	3	-3
χ_{34}	0	0	0	0	0	0	0	0	0	0	-6	2	0	0	0
χ_{35}	0	0	0	0	0	9	-3	1	0	0	-6	2	2	2	-2
χ_{36}	6	6	-2	-2	0	9	-3	1	0	0	6	-2	-4	-4	4
χ_{37}	0	0	0	0	0	-18	6	-2	0	0	6	-2	0	0	0
χ_{38}	0	0	0	0	0	9	-3	1	0	0	-6	2	-2	-2	2
χ_{39}	-6	-6	2	2	0	9	-3	1	0	0	6	-2	4	4	-4
χ_{40}	-3	-3	1	1	0	0	0	0	0	0	-3	1	-3	-3	3
χ_{41}	3	3	-1	-1	0	0	0	0	0	0	-3	1	3	3	-3
χ_{42}	-6	-6	2	2	0	-9	3	-1	0	0	0	0	2	2	-2
χ_{43}	6	6	-2	-2	0	-9	3	-1	0	0	0	0	-2	-2	2
χ_{44}	4	-4	0	0	0	6	2	-2	3	-1	0	0	6	-2	0
χ_{45}	-2	2	-2	2	0	6	2	-2	3	-1	0	0	-6	2	0
χ_{46}	2	-2	2	-2	0	6	2	-2	3	-1	0	0	6	-2	0
χ_{47}	-4	4	0	0	0	6	2	-2	3	-1	0	0	-6	2	0
χ_{48}	6	-6	-2	2	0	-6	-2	2	6	-2	0	0	-6	2	0
χ_{49}	0	0	4	-4	0	-6	-2	2	6	-2	0	0	-6	2	0
χ_{50}	-2	2	-2	2	0	12	4	-4	-3	1	0	0	6	-2	0
χ_{51}	4	-4	0	0	0	12	4	-4	-3	1	0	0	-6	2	0
χ_{52}	-6	6	2	-2	0	-6	-2	2	6	-2	0	0	6	-2	0
χ_{53}	2	-2	2	-2	0	12	4	-4	-3	1	0	0	-6	2	0
χ_{54}	0	0	-4	4	0	-6	-2	2	6	-2	0	0	6	-2	0
χ_{55}	-4	4	0	0	0	12	4	-4	-3	1	0	0	6	-2	0
χ_{56}	0	0	4	-4	0	0	0	0	0	0	0	0	6	-2	0
χ_{57}	6	-6	-2	2	0	0	0	0	0	0	0	0	6	-2	0
χ_{58}	0	0	-4	4	0	0	0	0	0	0	0	0	-6	2	0
χ_{59}	-6	6	2	-2	0	0	0	0	0	0	0	0	-6	2	0
χ_{60}	-8	8	0	0	0	6	2	-2	3	-1	0	0	0	0	0
χ_{61}	8	-8	0	0	0	6	2	-2	3	-1	0	0	0	0	0
χ_{62}	-4	4	-4	4	0	6	2	-2	3	-1	0	0	0	0	0
χ_{63}	4	-4	4	-4	0	6	2	-2	3	-1	0	0	0	0	0
χ_{64}	0	0	0	0	0	-12	-4	4	-6	2	0	0	0	0	0
χ_{65}	0	0	0	0	0	-12	-4	4	-6	2	0	0	0	0	0

Table 7.5: The character table of $2^6{:}O^-(6,2)$(continued)

	$4C$				$4D$				$5A$			$6A$	$6B$		$6C$			
	$4N$	$4O$	$4P$	$8B$	$4Q$	$8C$	$8D$	$4R$	$5A$	$10A$	$10B$	$6D$	$6E$	$12A$	$6F$	$12B$	$6G$	$12C$
χ_{26}	5	-3	1	-1	1	-1	-1	1	2	-2	0	0	0	0	3	-3	-1	1
χ_{27}	-5	3	-1	1	1	-1	-1	1	2	-2	0	0	0	0	-3	3	1	-1
χ_{28}	2	2	2	-2	0	0	0	0	-2	2	0	0	0	0	-3	3	1	-1
χ_{29}	-2	-2	-2	2	0	0	0	0	-2	2	0	0	0	0	3	-3	-1	1
χ_{30}	3	-5	-1	1	1	-1	-1	1	0	0	0	0	0	0	-3	3	1	-1
χ_{31}	-3	5	1	-1	1	-1	-1	1	0	0	0	0	0	0	3	-3	-1	1
χ_{32}	5	-3	1	-1	-1	1	1	-1	0	0	0	0	0	0	0	0	0	0
χ_{33}	-5	3	-1	1	-1	1	1	-1	0	0	0	0	0	0	0	0	0	0
χ_{34}	0	0	0	0	-2	2	2	-2	2	-2	0	0	0	0	0	0	0	0
χ_{35}	-2	-2	-2	2	0	0	0	0	0	0	0	0	0	0	3	-3	-1	1
χ_{36}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	-3	-1	1
χ_{37}	0	0	0	0	-2	2	2	-2	0	0	0	0	0	0	0	0	0	0
χ_{38}	2	2	2	-2	0	0	0	0	0	0	0	0	0	0	-3	3	1	-1
χ_{39}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-3	3	1	-1
χ_{40}	-3	5	1	-1	1	-1	-1	1	0	0	0	0	0	0	0	0	0	0
χ_{41}	3	-5	-1	1	1	-1	-1	1	0	0	0	0	0	0	0	0	0	0
χ_{42}	2	2	2	-2	0	0	0	0	0	0	0	0	0	0	3	-3	-1	1
χ_{43}	-2	-2	-2	2	0	0	0	0	0	0	0	0	0	0	-3	3	1	-1
χ_{44}	2	2	-2	0	2	0	0	-2	1	1	-1	0	1	-1	4	2	0	-2
χ_{45}	2	2	-2	0	0	2	-2	0	1	1	-1	0	-1	1	2	4	-2	0
χ_{46}	-2	-2	2	0	0	2	-2	0	1	1	-1	0	1	-1	-2	-4	2	0
χ_{47}	-2	-2	2	0	2	0	0	-2	1	1	-1	0	-1	1	-4	-2	0	2
χ_{48}	2	2	-2	0	0	-2	2	0	0	0	0	0	2	-2	-4	-2	0	2
χ_{49}	-2	-2	2	0	-2	0	0	2	0	0	0	0	2	-2	2	4	-2	0
χ_{50}	-2	-2	2	0	0	-2	2	0	0	0	0	0	1	-1	-2	2	-2	2
χ_{51}	-2	-2	2	0	-2	0	0	2	0	0	0	0	-1	1	2	-2	2	-2
χ_{52}	-2	-2	2	0	0	-2	2	0	0	0	0	0	-2	2	4	2	0	-2
χ_{53}	2	2	-2	0	0	-2	2	0	0	0	0	0	-1	1	2	-2	2	-2
χ_{54}	2	2	-2	0	-2	0	0	2	0	0	0	0	-2	2	-2	-4	2	0
χ_{55}	2	2	-2	0	-2	0	0	2	0	0	0	0	1	-1	-2	2	-2	2
χ_{56}	2	2	-2	0	2	0	0	-2	-1	-1	1	0	0	0	0	0	0	0
χ_{57}	-2	-2	2	0	0	2	-2	0	-1	-1	1	0	0	0	0	0	0	0
χ_{58}	-2	-2	2	0	2	0	0	-2	-1	-1	1	0	0	0	0	0	0	0
χ_{59}	2	2	-2	0	0	2	-2	0	-1	-1	1	0	0	0	0	0	0	0
χ_{60}	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	-2	-4	2	0
χ_{61}	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	2	4	-2	0
χ_{62}	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	4	2	0	-2
χ_{63}	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	-4	-2	0	2
χ_{64}	0	0	0	0	0	0	0	0	1	1	-1	0	2	-2	2	-2	2	-2
χ_{65}	0	0	0	0	0	0	0	0	1	1	-1	0	-2	2	-2	2	-2	2

Table 7.5: The character table of $2^6{:}O^-(6,2)$(continued)

	$6D$		$6E$		$6F$			$6G$		$8A$		$9A$	$10A$		$12A$		$12B$
	$6H$	$6I$	$6J$	$12D$	$6K$	$12E$	$12F$	$6L$	$12G$	$8E$	$8F$	$9A$	$10C$	$20A$	$12H$	$24A$	$12I$
χ_{26}	0	0	3	-1	1	1	-1	0	0	1	-1	0	0	0	1	-1	0
χ_{27}	0	0	3	-1	1	1	-1	0	0	-1	1	0	0	0	-1	1	0
χ_{28}	0	0	3	-1	1	1	-1	0	0	0	0	0	0	0	-1	1	0
χ_{29}	0	0	3	-1	1	1	-1	0	0	0	0	0	0	0	1	-1	0
χ_{30}	0	0	-3	1	-1	-1	1	0	0	1	-1	0	0	0	-1	1	0
χ_{31}	0	0	-3	1	-1	-1	1	0	0	-1	1	0	0	0	1	-1	0
χ_{32}	0	0	0	0	-2	-2	2	0	0	-1	1	0	0	0	0	0	0
χ_{33}	0	0	0	0	-2	-2	2	0	0	1	-1	0	0	0	0	0	0
χ_{34}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{35}	0	0	-3	1	1	1	-1	0	0	0	0	0	0	0	-1	1	0
χ_{36}	0	0	-3	1	1	1	-1	0	0	0	0	0	0	0	-1	1	0
χ_{37}	0	0	0	0	2	2	-2	0	0	0	0	0	0	0	0	0	0
χ_{38}	0	0	-3	1	1	1	-1	0	0	0	0	0	0	0	1	-1	0
χ_{39}	0	0	-3	1	1	1	-1	0	0	0	0	0	0	0	1	-1	0
χ_{40}	0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
χ_{41}	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0
χ_{42}	0	0	3	-1	-1	-1	1	0	0	0	0	0	0	0	-1	1	0
χ_{43}	0	0	3	-1	-1	-1	1	0	0	0	0	0	0	0	1	-1	0
χ_{44}	3	-1	0	0	2	-2	0	1	-1	0	0	0	1	-1	0	0	0
χ_{45}	-3	1	0	0	-2	2	0	1	-1	0	0	0	-1	1	0	0	0
χ_{46}	-3	1	0	0	-2	2	0	-1	1	0	0	0	1	-1	0	0	0
χ_{47}	3	-1	0	0	2	-2	0	-1	1	0	0	0	-1	1	0	0	0
χ_{48}	0	0	0	0	-2	2	0	0	0	0	0	0	0	0	0	0	0
χ_{49}	0	0	0	0	2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{50}	-3	1	0	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
χ_{51}	3	-1	0	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
χ_{52}	0	0	0	0	-2	2	0	0	0	0	0	0	0	0	0	0	0
χ_{53}	-3	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0
χ_{54}	0	0	0	0	2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{55}	3	-1	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0
χ_{56}	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
χ_{57}	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
χ_{58}	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0
χ_{59}	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0
χ_{60}	3	-1	0	0	-2	2	0	1	-1	0	0	0	0	0	0	0	0
χ_{61}	3	-1	0	0	-2	2	0	-1	1	0	0	0	0	0	0	0	0
χ_{62}	-3	1	0	0	2	-2	0	-1	1	0	0	0	0	0	0	0	0
χ_{63}	-3	1	0	0	2	-2	0	1	-1	0	0	0	0	0	0	0	0
χ_{64}	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
χ_{65}	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0

7.5 The fusion of $2^6{:}O^-(6,2)$ into $2^6{:}SP(6,2)$

Using the list of maximal subgroups of Fi_{22} given in the ATLAS, we can easily show that $2^6{:}O^-(6,2) \leq 2^6{:}SP(6,2)$, where $2^6{:}SP(6,2)$ is a maximal subgroup of Fi_{22}. In fact $2^6{:}O^-(6,2)$ is a maximal subgroup of $2^6{:}SP(6,2)$. We used the results in Section 7.1 to compute the power maps of the elements of $2^6{:}O^-(6,2)$ which are listed in Table 7.6 below.

Table 7.6: The power maps of the elements of $2^6{:}O^-(6,2)$

$[g]_{O^-(6,2)}$	$[x]_{2^6:O^-(6,2)}$	2	3	5	$[g]_{O^-(6,2)}$	$[x]_{2^6:O^-(6,2)}$	2	3	5
$1A$	$1A$				$2A$	$2C$	$1A$		
	$2A$	$1A$				$4A$	$2B$		
	$2B$	$1A$				$4B$	$2B$		
						$2D$	$1A$		
$2B$	$2E$	$1A$			$2C$	$2G$	$1A$		
	$2F$	$1A$				$4D$	$2A$		
	$4C$	$2A$				$4E$	$2A$		
						$2H$	$1A$		
						$4F$	$2B$		
$2D$	$2I$	$1A$			$3A$	$3A$		$1A$	
	$4G$	$2B$							
	$4H$	$2B$							
	$4I$	$2A$							
$3B$	$3B$		$1A$		$3C$	$3C$		$1A$	
	$6A$	$3B$	$2B$			$6C$	$3C$	$2B$	
	$6B$	$3B$	$2A$						
$4A$	$4J$	$2E$			$4B$	$4L$	$2G$		
	$4K$	$2F$				$4M$	$2G$		
						$8A$	$4D$		
$4C$	$4N$	$2G$			$4D$	$4Q$	$2G$		
	$4O$	$2G$				$8C$	$4D$		
	$4P$	$2G$				$8D$	$4E$		
	$8B$	$4E$				$4R$	$2H$		
$5A$	$5A$			$1A$	$6A$	$6D$	$3A$	$2E$	
	$10A$	$5A$		$2B$					
	$10B$	$5A$		$2A$					
$6B$	$6E$	$3C$	$2C$		$6C$	$6F$	$3B$	$2C$	
	$12A$	$6C$	$4B$			$12B$	$6A$	$4B$	
						$6G$	$3B$	$2D$	
						$12C$	$6A$	$4A$	
$6D$	$6H$	$3C$	$2E$		$6E$	$6J$	$3B$	$2E$	
	$6I$	$3C$	$2F$			$12D$	$6B$	$4C$	
$6F$	$6K$	$3B$	$2G$		$6G$	$6L$	$3C$	$2I$	
	$12E$	$6B$	$4D$			$12G$	$6C$	$4G$	
	$12F$	$6A$	$4F$						

Table 7.6: The power maps of the elements of $2^6{:}O^-(6,2)$(continued)

$[g]_{O^-(6,2)}$	$[x]_{2^6:O^-(6,2)}$	2	3	5	$[g]_{O^-(6,2)}$	$[x]_{2^6:O^-(6,2)}$	2	3	5
$8A$	$8E$	$4J$			$9A$	$9A$		$3A$	
	$8F$	$4K$							
$10A$	$10C$	$5A$		$2C$	$12A$	$12H$	$6K$	$4L$	
	$20A$	$10A$		$4A$		$24A$	$12E$	$8A$	
$12B$	$12I$	$6D$	$4J$						

The power maps of elements of $2^6{:}SP(6,2)$ are given in Chapter 6 (Section 6.5). Since the group $O^-(6,2)$ is a subgroup of $SP(6,2)$, then its fusion into $SP(6,2)$ will help to determine the fusion of $2^6{:}O^-(6,2)$ into $2^6{:}SP(6,2)$. For the restrictions of the irreducible characters of $2^6{:}SP(6,2)$ to $2^6{:}O^-(6,2)$, we use the technique of set intersections for characters. Using the permutation character of $SP(6,2)$ on $O^-(6,2)$ of degree 28, we obtain the partial fusion of $O^-(6,2)$ into $SP(6,2)$. For the remaining classes 4A, 4B, 6B, 6C, 6D, 6E, 12A and 12B, we used direct matrix conjugation in $SP(6,2)$. The complete fusion of $O^-(6,2)$ into $SP(6,2)$ is given in Table 7.7.

Table 7.7: The fusion of $O^-(6,2)$ into $SP(6,2)$

$[g]_{SP(6,2)}$	$1A$	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$4E$	$5A$
$[h]_{O^-(6,2)}$														
$1A$	$\boxed{28}$													
$2A$		$\boxed{16}$												
$2B$		20	$\boxed{4}$	1										
$2C$		120	24	$\boxed{8}$	2									
$2D$		240	48	16	$\boxed{4}$									
$3A$							$\boxed{1}$							
$3B$						$\boxed{10}$	3							
$3C$						20	3	$\boxed{1}$						
$4A$									$\boxed{4}$	2	2			
$4B$									4	$\boxed{2}$	2			
$4C$									12	6	$\boxed{6}$	4	1	
$4D$									24	12	12	8	$\boxed{2}$	
$5A$														$\boxed{3}$
$\chi(SP(6,2)\|O^-(6,2))$	28	16	4	8	4	10	1	1	4	2	6	0	2	3

Table 7.7: The fusion of $O^-(6,2)$ into $SP(6,2)$(continued)

$[g]_{SP(6,2)}$	$6A$	$6B$	$6C$	$6D$	$6E$	$6F$	$6G$	$7A$	$8A$	$8B$	$9A$	$10A$	$12A$	$12B$	$12C$	$15A$
$[h]_{O^-(6,2)}$																
$6A$	2	2	$\boxed{1}$													
$6B$	4	4	2		$\boxed{1}$	1										
$6C$	4	$\boxed{4}$	2		1	1										
$6D$	4	4	2		1	$\boxed{1}$										
$6E$	$\boxed{4}$	4	2		1	1										
$6F$	6	6	3	$\boxed{2}$												
$6G$	12	12	6	4	3	3	$\boxed{1}$									
$8A$									$\boxed{2}$	2						
$9A$											$\boxed{1}$					
$10A$												$\boxed{1}$				
$12A$													$\boxed{2}$	2	1	
$12B$													2	2	$\boxed{1}$	
$\chi(SP(6,2) \vert O^-(6,2))$	4	4	1	2	1	1	1	0	2	0	1	1	2	0	1	0

Proposition 7.5.1 *Let G, H and N be groups such that $H \leq G$ and that class kA of H fuses into class kB of G. Let $a \in kA$ and $b \in kB$. Then the classes of $N{:}H$ corresponding to the coset Na will fuse into the classes of $N{:}G$ corresponding to the coset Nb.*

Proof. Since kA fuses into kB, a and b are conjugate in G. Thus there exists $g \in G$ such that $a^g = gag^{-1} = b$. Then we obtain that

$$(Na)^g = \{gnag^{-1} \mid n \in N\} = \{gng^{-1}(gag^{-1}) \mid n \in N\} = \{gng^{-1}b \mid n \in N\} = Nb \quad .$$

Hence the result. $\square$

Remark 7.5.2 When H and G act on N, then a and b will have the same number of fixed points in N. This is true since a and b are conjugate in G and thus will have the same number of fixed points in N.

We used the information provided by the conjugacy classes and power maps of $2^6{:}O^-(6,2)$ and $2^6{:}SP(6,2)$ to partially compute the fusion map. Also the above proposition and remark provide information which is useful in computing the fusion map. In order to complete the fusion map, we restricted the irreducible characters $7a$, $63a$, $63b$, $315a$ and $315d$ of $2^6{:}SP(6,2)$ to $2^6{:}O^-(6,2)$. To determine these restrictions, we use the technique of set intersections for characters.

Let ρ be the character afforded by the regular representation of $O^-(6,2)$. Then we obtain that $\rho = \sum_{i=1}^{25} e_i\phi_i$, where $\phi_i \in Irr(O^-(6,2))$ and $e_i = deg(\phi_i)$. Then ρ can be regarded as a character of $2^6{:}O^-(6,2)$ which contains 2^6 in its kernel such that

$$\rho(g) = \begin{cases} |O^-(6,2)| & \text{if } \ g \in 2^6 \\ 0 & \text{otherwise} \end{cases} .$$

If ψ is a character of $2^6{:}SP(6,2)$, then we obtain that

$$\begin{aligned} \langle \rho, \psi \rangle_{2^6:O^-(6,2)} &= \frac{1}{|2^6{:}O^-(6,2)|}\{\rho(1A)\psi(1A) + 27\rho(2A)\psi(2A) + 36\rho(2B)\psi(2B)\} \\ &= \frac{1}{|2^6{:}O^-(6,2)|}\left\{|O^-(6,2)|\{\psi(1A) + 27\psi(2A) + 36\psi(2B)\}\right\} \\ &= \frac{1}{64}\{\psi(1A) + 27\psi(2A) + 36\psi(2B)\} \\ &= \langle \psi_{2^6}, \tau_1 \rangle \end{aligned}$$

where τ_1 is the identity character of 2^6 and ψ_{2^6} is the restriction of ψ to 2^6. Also for ψ we obtain that

$$\psi_{2^6} = a_1\theta_1 + a_2\theta_2 + a_3\theta_3 \quad ,$$

where $a_1, a_2, a_3 \in \mathbb{N} \cup \{0\}$ and $\theta_i, i \in \{1,2,3\}$, are the sums of the irreducible characters of 2^6 which are in the same orbit under the action of $O^-(6,2)$ on $Irr(2^6)$. Let $\tau_j \in Irr(2^6)$, where $j \in \{1, 2, \ldots, 64\}$. Then we obtain that

$$\theta_1 = \tau_1 \ , \ deg(\theta_1) = 1$$

$$\theta_2 = \sum_{j=2}^{28} \tau_j \ , \ deg(\theta_2) = 27$$

$$\theta_2 = \sum_{j=29}^{64} \tau_j \ , \ deg(\theta_2) = 36$$

and thus we have

$$\psi_{2^6} = a_1\tau_1 + a_2\sum_{j=2}^{28}\tau_j + a_3\sum_{j=29}^{64}\tau_j$$

and hence

$$\langle \psi_{2^6}, \psi_{2^6} \rangle = a_1^2 + 27a_2^2 + 36a_3^2 \quad ,$$

where $a_1 = \langle \psi_{2^6}, \tau_1 \rangle = \langle \rho, \psi \rangle_{2^6:O^-(6,2)}$. We also have that

$$\langle \psi_{2^6}, \psi_{2^6} \rangle = \frac{1}{64}\{\psi(1A)\psi(1A) + 27\psi(2A)\psi(2A) + 36\psi(2B)\psi(2B)\} \quad .$$

Also we obtain that $a_1 + 27a_2 + 36a_3 = deg(\psi)$.

Now let $7a$, $63a$, $63b$, $315a$ and $315d$ be the irreducible characters of $2^6{:}SP(6,2)$ of degrees 7, 63, 63, 315 and 315 respectively. Hence based on the partial fusion of $2^6{:}O^-(6,2)$ into $2^6{:}SP(6,2)$ which has already been determined and the above argument, we obtain that

$$(7a)_{2^6:O^-(6,2)} = 1b + 6b \ , \quad (63a)_{2^6:O^-(6,2)} = 27a + 36b \quad ,$$

$$(63b)_{2^6:O^-(6,2)} = 27b + 36c \ , \quad (315a)_{2^6:O^-(6,2)} = 135b + 180b \quad ,$$

$$(315d)_{2^6:O^-(6,2)} = 135a + 180a \quad .$$

Using the partial fusion already determined and the values of $7a, 63a, 63b, 315a$ and $315d$ on the classes of $2^6{:}SP(6,2)$ and the values of $(7a)_{2^6:O^-(6,2)}$, $(63a)_{2^6:O^-(6,2)}$, $(63b)_{2^6:O^-(6,2)}$, $(315a)_{2^6:O^-(6,2)}$ and $(315d)_{2^6:O^-(6,2)}$ on the classes of $2^6{:}O^-(6,2)$, we are able to complete the fusion map of $2^6{:}O^-(6,2)$ into $2^6{:}SP(6,2)$ which is given in Table 7.8.

Table 7.8: The fusion of $2^6{:}O^-(6,2)$ into $2^6{:}SP(6,2)$

$[g]_{O^-(6,2)}$	$[x]_{2^6:O^-(6,2)}$	$\longrightarrow$	$[h]_{2^6:SP(6,2)}$	$[g]_{O^-(6,2)}$	$[x]_{2^6:O^-(6,2)}$	$\longrightarrow$	$[h]_{2^6:SP(6,2)}$
$1A$	$1A$		$1A$	$2A$	$2C$		$2B$
	$2A$		$2A$		$4A$		$4A$
	$2B$		$2A$		$4B$		$4A$
					$2D$		$2C$
$2B$	$2E$		$2D$	$2C$	$2G$		$2F$
	$2F$		$2E$		$4D$		$4C$
	$4C$		$4B$		$4E$		$4C$
					$2H$		$2G$
					$4F$		$4D$
$2D$	$2I$		$2H$	$3A$	$3A$		$3B$
	$4G$		$4G$				
	$4H$		$4E$				
	$4I$		$4F$				
$3B$	$3B$		$3A$	$3C$	$3C$		$3C$
	$6A$		$6A$		$6C$		$6B$
	$6B$		$6A$				

Table 7.8: The fusion of $2^6{:}O^-(6,2)$ into $2^6{:}SP(6,2)$(continued)

$[g]_{O^-(6,2)}$	$[x]_{2^6:O^-(6,2)}$	$\longrightarrow$ $[h]_{2^6:SP(6,2)}$	$[g]_{O^-(6,2)}$	$[x]_{2^6:O^-(6,2)}$	$\longrightarrow$ $[h]_{2^6:SP(6,2)}$
$4A$	$4J$	$4H$	$4B$	$4L$	$4J$
	$4K$	$4I$		$4M$	$4K$
				$8A$	$8A$
$4C$	$4N$	$4L$	$4D$	$4Q$	$4Q$
	$4O$	$4M$		$8C$	$8C$
	$4P$	$4M$		$8D$	$8D$
	$8B$	$8B$		$4R$	$4R$
$5A$	$5A$	$5A$	$6A$	$6D$	$6F$
	$10A$	$10A$			
	$10B$	$10A$			
$6B$	$6E$	$6H$	$6C$	$6F$	$6D$
	$12A$	$12E$		$12B$	$12B$
				$6G$	$6E$
				$12C$	$12B$
$6D$	$6H$	$6I$	$6E$	$6J$	$6C$
	$6I$	$6J$		$12D$	$12A$
$6F$	$6K$	$6G$	$6G$	$6L$	$6K$
	$12E$	$12C$		$12G$	$12F$
	$12F$	$12D$			
$8A$	$8E$	$8E$	$9A$	$9A$	$9A$
	$8F$	$8F$			
$10A$	$10C$	$10B$	$12A$	$12H$	$12G$
	$20A$	$20A$		$24A$	$24A$
$12B$	$12I$	$12I$			

Since the group $2^6{:}O^-(6,2)$ is a subgroup of Fi_{22}, it must sit inside at least one maximal subgroup of Fi_{22}. The possible maximal subgroups of Fi_{22} which may contain $2^6{:}O^-(6,2)$ are $2{\cdot}U(6,2)$, $O^+(8,2){:}S_3$, $2^6{:}SP(6,2)$ and $(2\times 2_+^{1+8}{:}U(4,2)){:}2$ with indices 5544, 315, 28 and 16 respectively. If these maximal subgroups of Fi_{22} contain $2^6{:}O^-(6,2)$, then they must have permutation characters of degrees corresponding to the respective indices. However by computations using GAP, we obtain that the groups $2{\cdot}U(6,2)$, $O^+(8,2){:}S_3$ and $(2\times 2_+^{1+8}{:}U(4,2)){:}2$ do not have permutation characters of degrees 5544, 315 and 16 respectively. Hence $2^6{:}SP(6,2)$ is the only maximal subgroup of Fi_{22} which contains $2^6{:}O^-(6,2)$.

Chapter 8

A maximal subgroup of $\overline{F}i_{22}$

The maximal subgroup $2^6{:}SP(6,2)$ of Fi_{22}, where 2^6 is a $2B$-pure group and that $N_{Fi_{22}}(2^6) = 2^6{:}SP(6,2)$, is a 2-local subgroup of Fi_{22}. We have $2^6{:}SP(6,2) \leq N_{Fi_{22}}(2^6{:}SP(6,2))$ and since Fi_{22} is simple, the maximality of $2^6{:}SP(6,2)$ in Fi_{22} implies that $N_{Fi_{22}}(2^6{:}SP(6,2)) = 2^6{:}SP(6,2)$. In $\overline{F}i_{22}$, we obtain that $2^6{:}SP(6,2) \leq N_{\overline{F}i_{22}}(2^6{:}SP(6,2))$, but $N_{\overline{F}i_{22}}(2^6{:}SP(6,2)) \neq \overline{F}i_{22}, Fi_{22}$. By Theorem C in [118] and the results of [71], we deduce that $N_{\overline{F}i_{22}}(2^6{:}SP(6,2)) = 2^7{:}SP(6,2)$ and hence $2^7{:}SP(6,2) = (2^6{:}SP(6,2)){:}\langle e\rangle$. In Chapter 6, we computed the conjugacy classes and the Fischer-Clifford matrices of the group $2^6{:}SP(6,2)$. In this chapter, we construct the conjugacy classes and the character table of the group $2^7{:}SP(6,2)$ which is a maximal subgroup of $\overline{F}i_{22}$ of index 694980. We shall use the technique of the Fischer-Clifford matrices to construct this character table. We use the properties of the Fischer-Clifford matrices which have been discussed in Chapter 5 (Section 5.2.2) and in some cases we also use the additional information discussed in the introduction of Chapter 6, to compute their entries. It can be easily shown that

$$\overline{F}i_{22} = Fi_{22}{:}\langle e\rangle = Fi_{22}{:}\langle f\rangle = Fi_{22}{:}\langle \theta\rangle \quad ,$$

where e, f and θ are the involutory outer automorphisms of Fi_{22} in $\overline{F}i_{22}$ which are represented in the ATLAS by $2D$, $2F$ and $2E$ respectively.

8.1 The actions of $SP(6,2)$ on 2^6 and 2^7

We have that $O^-(6,2) = U_4(2){:}2$ is a maximal subgroup of $SP(6,2)$ of index 28. Consider the conjugacy classes $2D$, $5A$ and $7A$ of $SP(6,2)$. Let $a, x \in SP(6,2)$ such that $a \in 2D, x \in 5A$ are given by

$$a = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \text{ and } x = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Then we observe that $H = \langle a, x\rangle \cong O^-(6,2)$. We find $b \in 7A$ such that $b * a * b^6 \notin H$. Let $c = b * a * b^6$. Then $c \in 2D$ and $c \notin H$. So $\langle H, c\rangle = SP(6,2)$. We also deduce that $o(ax) = 8, o(cx) = 9, o(ac) = 15$, $SP(6,2) = \langle H, c\rangle = \langle a, x, c\rangle = \langle x, c\rangle$. We obtain

$$c = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Now let $\overline{a}, \overline{x}, \overline{c}$ be the following 7×7 matrices over $GF(2)$

$$\overline{a} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \overline{x} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\overline{c} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Then we obtain that $\langle \overline{a}, \overline{x} \rangle \cong O^-(6,2)$ and $\langle \overline{a}, \overline{x}, \overline{c} \rangle = \langle \overline{x}, \overline{c} \rangle \cong SP(6,2)$. We thus give the class representatives $g \in SP(6,2)$ in terms of 7×7 matrices over $GF(2)$ in the following table, where M is the matrix that represents that particular class.

$[g]_{SP(6,2)}$	M	$\lvert[g]_{SP(6,2)}\rvert$	$[g]_{SP(6,2)}$	M	$\lvert[g]_{SP(6,2)}\rvert$
$1A$	$\begin{pmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{pmatrix}$	1	$2A$	$\begin{pmatrix} 1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0 \\ 0&1&0&0&1&0&1 \end{pmatrix}$	63
$2B$	$\begin{pmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 1&0&0&1&1&0&0 \\ 0&1&0&1&0&1&0 \\ 1&1&0&0&0&0&1 \end{pmatrix}$	315	$2C$	$\begin{pmatrix} 0&0&0&1&1&1&0 \\ 1&1&0&1&1&1&0 \\ 1&0&1&1&1&1&0 \\ 0&1&1&1&0&0&0 \\ 0&1&1&0&1&0&0 \\ 1&0&0&1&1&0&0 \\ 1&0&0&1&1&1&1 \end{pmatrix}$	945
$2D$	$\begin{pmatrix} 1&1&0&1&0&1&0 \\ 1&0&0&0&1&0&0 \\ 1&1&0&0&0&1&0 \\ 1&0&1&0&0&0&0 \\ 1&0&0&1&0&1&0 \\ 0&1&1&1&1&1&0 \\ 1&0&0&1&1&1&1 \end{pmatrix}$	3780	$3A$	$\begin{pmatrix} 0&0&1&0&0&1&0 \\ 1&1&1&1&1&1&0 \\ 1&0&0&1&1&1&0 \\ 1&0&1&1&0&1&0 \\ 0&0&0&0&1&0&0 \\ 1&0&1&0&0&0&0 \\ 1&0&1&1&1&1&1 \end{pmatrix}$	672
$3B$	$\begin{pmatrix} 1&1&1&1&1&1&0 \\ 0&1&1&0&1&1&0 \\ 1&1&1&0&0&1&0 \\ 1&0&1&0&1&1&0 \\ 0&0&0&1&0&1&0 \\ 1&0&1&1&0&0&0 \\ 0&0&1&1&1&0&1 \end{pmatrix}$	2240	$3C$	$\begin{pmatrix} 1&0&0&0&0&0&0 \\ 1&1&0&1&0&0&0 \\ 1&1&1&0&1&0&0 \\ 0&1&0&0&0&0&0 \\ 0&1&1&1&0&0&0 \\ 1&0&1&1&0&1&0 \\ 1&0&0&1&1&0&1 \end{pmatrix}$	13440
$4A$	$\begin{pmatrix} 1&1&1&0&1&1&0 \\ 1&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0 \\ 0&1&1&1&0&0&0 \\ 0&1&0&0&0&1&0 \\ 1&0&1&0&1&1&0 \\ 0&1&0&0&0&0&1 \end{pmatrix}$	3780	$4B$	$\begin{pmatrix} 0&1&1&1&0&0&0 \\ 1&0&1&1&0&0&0 \\ 0&0&1&0&1&1&0 \\ 1&0&0&0&0&0&0 \\ 1&0&0&1&1&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&1&1&1 \end{pmatrix}$	7560

$[g]_{SP(6,2)}$	M	$\lvert[g]_{SP(6,2)}\rvert$	$[g]_{SP(6,2)}$	M	$\lvert[g]_{SP(6,2)}\rvert$
$4C$	$\begin{pmatrix} 0&1&0&0&0&1&0\\ 1&1&0&0&1&1&0\\ 1&0&0&1&1&0&0\\ 1&0&1&0&1&0&0\\ 1&1&1&1&1&0&0\\ 0&1&1&1&1&0&0\\ 0&1&0&0&1&0&1 \end{pmatrix}$	7560	$4D$	$\begin{pmatrix} 0&0&1&1&1&1&0\\ 0&0&0&0&0&1&0\\ 0&1&1&0&0&0&0\\ 0&1&0&1&0&1&0\\ 1&0&1&0&0&1&0\\ 0&1&0&0&0&0&0\\ 1&0&1&1&1&0&1 \end{pmatrix}$	11340
$4E$	$\begin{pmatrix} 0&0&0&1&1&1&0\\ 1&1&0&0&1&1&0\\ 1&0&1&0&0&1&0\\ 0&1&1&0&0&1&0\\ 0&1&1&1&1&1&0\\ 1&1&1&0&1&1&0\\ 1&0&0&0&0&1&1 \end{pmatrix}$	45360	$5A$	$\begin{pmatrix} 1&1&1&0&0&0&0\\ 0&1&0&1&0&1&0\\ 1&1&0&1&1&0&0\\ 1&1&1&1&0&1&0\\ 1&1&1&0&1&1&0\\ 1&0&0&1&1&1&0\\ 0&1&1&1&1&1&1 \end{pmatrix}$	48384
$6A$	$\begin{pmatrix} 1&0&1&1&1&0&0\\ 0&1&1&1&1&1&0\\ 0&1&0&1&0&0&0\\ 0&1&1&0&0&0&0\\ 0&1&0&0&0&1&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	10080	$6B$	$\begin{pmatrix} 1&1&1&1&0&1&0\\ 0&0&0&0&1&1&0\\ 0&0&1&0&0&1&0\\ 0&0&1&1&1&1&0\\ 0&1&1&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&1&1&1&0&1 \end{pmatrix}$	10080
$6C$	$\begin{pmatrix} 1&0&1&1&1&0&0\\ 0&0&1&0&1&1&0\\ 1&1&0&0&0&1&0\\ 0&1&0&1&1&0&0\\ 1&1&0&1&1&0&0\\ 0&1&1&1&0&0&0\\ 1&0&0&0&0&0&1 \end{pmatrix}$	20160	$6D$	$\begin{pmatrix} 1&1&1&1&1&0&0\\ 1&1&0&1&1&1&0\\ 1&0&1&1&1&1&0\\ 1&0&0&1&0&1&0\\ 1&0&1&0&1&0&0\\ 0&1&0&1&1&0&0\\ 1&0&1&1&1&0&1 \end{pmatrix}$	30240
$6E$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&1&0&1&1&0&0\\ 1&1&0&1&0&0&0\\ 0&0&1&0&1&0&0\\ 0&1&0&1&1&1&0\\ 1&0&1&1&1&0&1 \end{pmatrix}$	40320	$6F$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0\\ 1&1&0&0&0&0&0\\ 0&0&1&1&1&0&0\\ 0&1&0&1&0&0&0\\ 1&1&1&0&1&1&0\\ 1&0&1&0&1&0&1 \end{pmatrix}$	40320
$6G$	$\begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 1&1&0&0&0&0&0\\ 1&1&0&1&0&1&0\\ 1&1&1&0&0&0&0\\ 0&0&0&0&1&1&0\\ 0&1&1&1&0&1&1 \end{pmatrix}$	120960	$7A$	$\begin{pmatrix} 1&1&0&1&1&0&0\\ 0&0&1&1&1&0&0\\ 1&1&1&0&1&1&0\\ 1&0&0&0&0&0&0\\ 0&1&0&0&1&0&0\\ 0&1&1&0&1&0&0\\ 1&0&1&1&1&1&1 \end{pmatrix}$	207360

$[g]_{SP(6,2)}$	M	$\lvert[g]_{SP(6,2)}\rvert$	$[g]_{SP(6,2)}$	M	$\lvert[g]_{SP(6,2)}\rvert$
$8A$	$\begin{pmatrix} 1&0&0&1&0&0&0\\ 1&1&1&1&1&1&0\\ 1&0&1&1&0&1&0\\ 1&0&1&0&1&1&0\\ 1&0&0&1&1&0&0\\ 1&1&1&0&1&0&0\\ 0&0&1&1&1&0&1 \end{pmatrix}$	90720	$8B$	$\begin{pmatrix} 0&1&0&0&0&1&0\\ 1&1&0&0&1&1&0\\ 1&0&1&0&1&0&0\\ 1&1&1&1&1&1&0\\ 1&0&0&1&1&1&0\\ 0&1&1&1&1&0&0\\ 0&1&0&0&1&0&1 \end{pmatrix}$	90720
$9A$	$\begin{pmatrix} 0&1&0&0&1&1&0\\ 0&1&0&1&1&0&0\\ 1&1&1&1&0&0&0\\ 0&1&0&0&1&0&0\\ 0&0&1&1&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&1&1&1 \end{pmatrix}$	161280	$10A$	$\begin{pmatrix} 0&0&1&0&0&1&0\\ 1&0&1&1&1&0&0\\ 1&0&0&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&1&1&0&0&0\\ 1&1&0&0&1&1&0\\ 0&0&1&0&0&1&1 \end{pmatrix}$	145152
$12A$	$\begin{pmatrix} 0&0&1&1&0&1&0\\ 1&1&0&1&0&0&0\\ 0&1&0&0&1&1&0\\ 0&0&1&1&1&0&0\\ 1&1&1&0&0&0&0\\ 1&0&0&0&1&1&0\\ 1&0&1&0&1&1&1 \end{pmatrix}$	60480	$12B$	$\begin{pmatrix} 1&0&1&0&1&0&0\\ 1&0&1&1&0&1&0\\ 1&1&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0\\ 1&0&0&1&0&0&0\\ 0&0&1&1&0&1&1 \end{pmatrix}$	60480
$12C$	$\begin{pmatrix} 0&0&1&1&0&1&0\\ 0&1&0&0&1&1&0\\ 1&1&1&0&0&1&0\\ 0&0&0&0&0&1&0\\ 0&0&1&1&1&0&0\\ 0&1&0&1&1&0&0\\ 0&1&1&1&0&1&1 \end{pmatrix}$	120960	$15A$	$\begin{pmatrix} 1&1&0&1&0&0&0\\ 0&0&1&0&0&1&0\\ 1&0&1&1&0&1&0\\ 1&0&1&1&1&0&0\\ 0&0&0&1&1&1&0\\ 1&1&0&1&1&0&0\\ 0&1&1&0&1&1&1 \end{pmatrix}$	96768

Suppose that $N = 2^6 = \langle e_1, e_2, e_3, e_4, e_5, e_6\rangle$ and $W = 2^7 = \langle 2^6, \theta\rangle$, where $e_1 = [1,0,0,0,0,0], e_2 = [0,1,0,0,0,0], \ldots, e_6 = [0,0,0,0,0,1]$, $\theta = [0,0,0,0,0,0,1]$. Then we observe that $\overline{a}$ and $\overline{x}$ fix θ whereas $\overline{c} : \theta \to e_1 + e_4 + e_5 + e_6 + \theta$. Hence $\langle \overline{a}, \overline{x}, \overline{c}\rangle \cong SP(6,2)$ acts on 2^7. Note that $C_{\overline{F}i_{22}}(\theta) = C_{Fi_{22}}(\theta) \times \langle\theta\rangle = [2^6{:}O^-(6,2)] \times \langle\theta\rangle$ by Moori [83]. Considering e_1, θ and $e_3 + \theta = [0,0,1,0,0,0,1]$, then from the computations using CAYLEY and GAP we obtain the point stablizers in $SP(6,2)$ which are given by

$$[SP(6,2]_{e_1} \cong 2^5{:}S_6\ , \quad [SP(6,2)]_\theta \cong O^-(6,2) \quad \text{and} \quad [SP(6,2)]_{e_3+\theta} \cong S_8 \quad .$$

Thus when $SP(6,2)$ acts on 2^7, we obtain four orbits of lengths 1, 28, 36 and 63 with corresponding point stabilizers $SP(6,2)$, $O^-(6,2)$, S_8 and $2^5{:}S_6$ respectively.

Hence $SP(6,2)$ has four orbits on $Irr(2^7)$. We also note that N can be identified with the 6-dimensional irreducible module of $SP(6,2)$ over $GF(2)$. Furthermore $W \supset N \supset 0$. Let χ_1 denote the identity character of 2^7. Since $SP(6,2)$ fixes χ_1, $\{\chi_1\}$ forms an orbit of length 1 for the action of $SP(6,2)$ on $Irr(2^7)$. Consider $\chi_1' \in Irr(2^7)$ given by $\chi_1'(e_i) = 1$ for $1 \leq i \leq 6$ and $\chi_1'(\theta) = -1$. Then since $\overline{x}$ and $\overline{c}$ fix χ_1', $\langle \overline{x}, \overline{c} \rangle = SP(6,2)$ will fix χ_1' forming a second orbit of length 1 given by $\{\chi_1'\}$. Since $2^7 \supset 2^6$ and $SP(6,2)$ acting on $Irr(2^6)$ produces an orbit Δ of length 63, we can regard Δ as an orbit of $SP(6,2)$ on $Irr(2^7)$. Then the remaining orbit which we denote by Δ' also has length 63.

Since $2^7 = 2^6 \times \langle \theta \rangle$, the orbits of $SP(6,2)$ on $Irr(2^7)$ are $\{\chi_1\}$, $\{\chi_1'\}$, Δ and Δ', where $\Delta' = \{\chi \mid \chi \in Irr(2^7),\ \chi_{2^6} \in \Delta \ \text{ and } \ \chi(\theta) = -1\}$ and where χ_{2^6} is the restriction of χ to 2^6. Since $|\Delta| = |\Delta'| = 63$, $SP(6,2)$ produces four orbits of lengths 1, 1, 63 and 63 on $Irr(2^7)$ with corresponding point stabilizers $H_1 = SP(6,2)$, $H_2 = SP(6,2)$, $H_3 = 2^5{:}S_6$ and $H_4 = 2^5{:}S_6$ respectively. Let $\chi \in \Delta$. Then $\chi.\chi_1' \in \Delta'$ and we can easily see that $I_{SP(6,2)}(\chi_1) = I_{SP(6,2)}(\chi_1') = SP(6,2)$, $I_{SP(6,2)}(\chi) \cong 2^5{:}S_6$ and $I_{SP(6,2)}(\chi{\cdot}\chi_1') \cong 2^5{:}S_6$. So we deduce that $H_1 = H_2 = SP(6,2)$.

Proposition 8.1.1 *Let $H_3 = I_{SP(6,2)}(\chi)$ and $H_4 = I_{SP(6,2)}(\chi{\cdot}\chi_1')$. Then $H_3 = H_4$.*

Proof. We need to show that $\forall\, g \in H_3$, we have

$$(\chi{\cdot}\chi_1')^g(x) = \chi{\cdot}\chi_1'(x) \ \ \forall\, x \in 2^7 \ .$$

For $g \in H_3$ we have

$$\begin{aligned}(\chi{\cdot}\chi_1')^g(x) &= (\chi{\cdot}\chi_1')(x^g) = \chi(x^g).\chi_1'(x^g) = \chi^g(x).(\chi_1')^g(x) \\ &= \chi(x).\chi_1'(x) = \chi{\cdot}\chi_1'(x)\end{aligned}$$

Hence $H_3 = H_4$. $\square$

8.2 The conjugacy classes of $2^7{:}SP(6,2)$

In this section we use the method of coset analysis discussed in Chapter 2, Section 2.3, to determine the conjugacy classes of elements of $2^7{:}SP(6,2)$. We observe that $W =$

$N \cup Ne_7$, where $e_7 = \theta$, $N = \langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle$ and $W = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle$. Thus when $SP(6,2)$ acts on W, we obtain four orbits Δ_1, Δ_2, Δ_3 and Δ_4 of lengths 1, 28, 36 and 63 respetively such that

$$\Delta_1 \cup \Delta_4 = N \quad \text{and} \quad \Delta_2 \cup \Delta_3 = Ne_7 \ .$$

For a class representative $g \in SP(6,2)$, the coset Wg is given by $Wg = Ng \cup Ne_7g$. We would like to study the action of W on the coset Wg. Firstly notice that for $n \in N$ and $w \in W$, we have

$$n(wg)n = nwgng^{-1}g = nwn^g g \qquad (*)$$

and

$$ne_7(wg)ne_7 = ne_7w(ne_7)^g g = ne_7wn'e_7g = nn'wg \qquad (**)$$

where $(ne_7)^g = n'e_7$ for some $n' \in N$.

Secondly since $w \in W$ and $W = N \cup Ne_7$, we have $w = n_1$ or $w = n_1e_7$ for some $n_1 \in N$. If $w = n_1$ then by (*) we have $n(wg)n = n(n_1g)n = nn_1n^g g = nn_1n_2g \in Ng$, where $n^g = n_2 \in N$ and by (**) we have $ne_7(wg)ne_7 = ne_7n_1(ne_7)^g g = ne_7n_1n'e_7g = nn'n_1g \in Ng$. If $w = n_1e_7$, then by (*) we have $n(wg)n = n(n_1e_7g)n = n(n_1e_7)n^g g = nn_1n^g e_7g = nn_1n_2e_7g \in Ne_7g$ and by (**) we have $ne_7(wg)ne_7 = ne_7(n_1e_7g)ne_7 = ne_7n_1e_7(ne_7)^g g = nn_1(ne_7)^g g = nn_1n'e_7g = nn'n_1e_7g \in Ne_7g$.

The above argument shows that when W acts on Wg, the elements of Ng are sent to elements of Ng and those elements of Ne_7g are sent to elements of Ne_7g. Now applying the theory of coset analysis for the conjugacy classes of elements, we deduce that Wg splits into k blocks such that $\frac{k}{2}$ of these blocks correspond to Ng and the other $\frac{k}{2}$ blocks correspond to Ne_7g. Now we act $C_G(g)$ on these blocks where $G = SP(6,2)$. Let $x \in C_G(g)$ and we obtain that

(a) $x(ng)x^{-1} = xnx^{-1}g \in Ng$

(b) $x(ne_7g)x^{-1} = xne_7x^{-1}g \in Ne_7g$

Thus when $C_G(g)$ acts on the blocks, it either fixes a block or sends a block of Ng to a block of Ng or sends a block of Ne_7g to a block of Ne_7g.

The number of conjugacy classes of $2^7{:}SP(6,2)$ is equal to

$$\sum_{i=1}^{4} |Irr(H_i)| = 30 + 30 + 37 + 37 = 134 \quad .$$

When $SP(6,2)$ acts on 2^7, we obtain four orbits of lengths 1, 28, 36 and 63 with corresponding point stabilizers $SP(6,2)$, $O^-(6,2)$, S_8 and $2^5{:}S_6$ respectively. Let $\chi(SP(6,2)|2^7)$ be the permutation character of $SP(6,2)$ acting on 2^7. Then we obtain that

$$\begin{aligned}\chi(SP(6,2)|2^7) &= 1 + I_{O^-(6,2)}^{SP(6,2)} + I_{S_8}^{SP(6,2)} + I_{2^5:S_6}^{SP(6,2)} \\ &= 1a + 1a + 27a + 1a + 35b + 1a + 27a + 35b \\ &= 4 \times 1a + 2 \times 27a + 2 \times 35b\end{aligned}$$

where $I_{O^-(6,2)}^{SP(6,2)}, I_{S_8}^{SP(6,2)}$ and $I_{2^5:S_6}^{SP(6,2)}$ are the identity characters of $O^-(6,2)$, S_8 and $2^5{:}S_6$ respectively induced to $SP(6,2)$. For each class representative $g \in SP(6,2)$, $\chi(SP(6,2)|2^7)$ will give us the number k of fixed points of each g in 2^7. The following table provides us with the complete list of the k's which we need in order to be able to calculate the conjugacy classes of elements of $2^7{:}SP(6,2)$.

$[g]_{SP(6,2)}$	1A	2A	2B	2C	2D	3A	3B	3C	4A	4B	4C	4D	4E	5A	6A
k	128	64	32	32	16	32	2	8	8	16	16	8	8	8	16
$[g]_{SP(6,2)}$	6B	6C	6D	6E	6F	6G	7A	8A	8B	9A	10A	12A	12B	12C	15A
k	8	2	8	8	4	4	2	4	4	2	4	4	4	2	2

Having obtained the values of the k's for various class representatives of $SP(6,2)$, we then use Programme A of Chapter 2, Section 2.3, to obtain the f_j's. See Appendix, Programme A for $2^7{:}SP(6,2)$.

From the programme output we calculate the number f_j of orbits Q_i's for $1 \leq i \leq k$, which have come together under the action of $C_{SP(6,2)}(g), g \in SP(6,2)$ to form one orbit Δ_j. These values are listed in Table 8.1. In this table we also list the d_j's where $d_j g$ is a representative of the Δ_j. For each class representative $g \in SP(6,2)$, we calculate the lengths of the corresponding classes $[x]_{2^7:SP(6,2)}$ of $2^7{:}SP(6,2)$ by using the theory of conjugacy classes of group extensions which has been discussed in Chapter 2 (Section 2.3). For each $[x]_{2^7:SP(6,2)}$, the order of $C_{2^7:SP(6,2)}(x)$ is given in the last column of Table 8.1. Table 8.1 below provides details and a complete enumeration of the conjugacy classes of elements of $2^7{:}SP(6,2)$.

Table 8.1: The conjugacy classes of elements of $2^7{:}SP(6,2)$

$[g]_{SP(6,2)}$	k	f_j	d_j	w	$[x]_{2^7:SP(6,2)}$	$\|[x]_{2^7:SP(6,2)}\|$	$\|C_{2^7:SP(6,2)}(x)\|$
$1A$	128	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$1A$	1	185794560
		$f_2=28$	$(1,0,1,0,1,0,1)$	$(1,0,1,0,1,0,1)$	$2A$	28	6635520
		$f_3=36$	$(1,1,1,1,1,1,1)$	$(1,1,1,1,1,1,1)$	$2B$	36	5160960
		$f_4=63$	$(1,0,0,0,0,0,0)$	$(1,0,0,0,0,0,0)$	$2C$	63	2949120
$2A$	64	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$2D$	126	1474560
		$f_2=6$	$(0,1,0,0,1,0,1)$	$(0,1,0,0,1,0,1)$	$4A$	756	245760
		$f_3=10$	$(1,1,1,1,1,1,1)$	$(0,1,0,0,1,0,0)$	$4B$	1260	147456
		$f_4=15$	$(1,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$2E$	1890	98304
		$f_5=16$	$(1,0,1,0,1,0,0)$	$(0,1,0,0,1,0,0)$	$4C$	2016	92160
		$f_6=16$	$(0,1,0,1,0,0,1)$	$(0,0,0,0,0,0,0)$	$2F$	2016	92160
$2B$	32	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$2G$	1260	147456
		$f_2=1$	$(0,1,0,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$2H$	1260	147456
		$f_3=3$	$(1,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$2I$	3780	49152
		$f_4=3$	$(1,1,1,0,0,0,1)$	$(0,0,0,0,0,0,0)$	$2J$	3780	49152
		$f_5=12$	$(1,0,1,1,0,1,0)$	$(1,0,0,1,0,0,0)$	$4D$	15120	12288
		$f_6=12$	$(1,1,1,1,1,1,1)$	$(1,1,0,0,0,0,0)$	$4D$	15120	12288
$2C$	32	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$2K$	3780	49152
		$f_2=1$	$(1,0,1,0,1,0,1)$	$(1,1,1,1,1,1,0)$	$4F$	3780	49152
		$f_3=3$	$(1,1,0,1,1,0,0)$	$(0,0,0,0,0,0,0)$	$2L$	11340	16384
		$f_4=3$	$(1,1,1,0,1,1,1)$	$(1,1,1,1,1,1,0)$	$4G$	11340	16384
		$f_5=4$	$(1,1,1,1,0,0,0)$	$(1,1,1,1,1,1,0)$	$4H$	15120	12288
		$f_6=4$	$(1,1,0,0,0,1,1)$	$(0,0,0,0,0,0,0)$	$2M$	15120	12288
		$f_7=8$	$(1,0,0,0,0,0,0)$	$(1,0,0,1,1,1,0)$	$4I$	30240	6144
		$f_8=8$	$(1,1,1,1,1,1,1)$	$(1,0,0,1,1,1,0)$	$4J$	30240	6144
$2D$	16	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$2N$	30240	6144
		$f_2=1$	$(1,1,1,1,1,1,0)$	$(0,0,1,0,1,1,0)$	$4K$	30240	6144
		$f_3=1$	$(1,1,0,1,1,0,1)$	$(0,0,1,0,1,1,0)$	$4L$	30240	6144
		$f_4=1$	$(1,1,1,0,1,1,1)$	$(0,0,0,0,0,0,0)$	$2O$	30240	6144
		$f_5=3$	$(1,0,0,0,0,0,0)$	$(0,1,0,1,0,1,0)$	$4M$	90720	2048
		$f_6=3$	$(0,1,1,1,1,1,0)$	$(0,1,1,1,1,0,0)$	$4N$	90720	2048
		$f_7=3$	$(1,1,1,1,1,1,1)$	$(1,0,1,1,0,0,0)$	$4O$	90720	2048
		$f_8=3$	$(0,1,1,1,1,1,1)$	$(1,1,1,0,0,1,0)$	$4P$	90720	2048
$3A$	32	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$3A$	2688	69120
		$f_2=6$	$(1,1,0,1,1,0,1)$	$(1,1,0,1,1,0,1)$	$6A$	16128	11520
		$f_3=10$	$(1,1,1,1,1,1,1)$	$(0,1,0,0,0,0,1)$	$6B$	26880	6912
		$f_4=15$	$(1,0,0,0,0,0,0)$	$(1,0,0,1,1,0,0)$	$6C$	40320	4608
$3B$	2	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$3B$	143360	1296
		$f_2=1$	$(1,1,1,1,1,1,1)$	$(0,1,0,1,1,1,1)$	$6D$	143360	1296
$3C$	8	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$3C$	215040	864
		$f_2=1$	$(1,0,1,1,0,1,1)$	$(1,0,0,1,1,1,1)$	$6E$	215040	864
		$f_3=3$	$(1,0,0,0,0,0,0)$	$(1,0,0,0,0,0,0)$	$6F$	645120	288
		$f_4=3$	$(1,1,1,1,1,1,1)$	$(0,0,0,1,1,1,1)$	$6G$	645120	288

Table 8.1: The conjugacy classes of elements of $2^7{:}SP(6,2)$(continued)

$[g]_{SP(6,2)}$	k	f_j	d_j	w	$[x]_{2^7:SP(6,2)}$	$\lvert[x]_{2^7:SP(6,2)}\rvert$	$\lvert C_{2^7:SP(6,2)}(x)\rvert$
$4A$	8	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$4Q$	60480	3072
		$f_2=1$	$(1,0,1,1,0,1,1)$	$(0,0,0,0,0,0,0)$	$4R$	60480	3072
		$f_3=3$	$(1,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$4S$	181440	1024
		$f_4=3$	$(1,1,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$4T$	181440	1024
$4B$	16	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$4U$	60480	3072
		$f_2=1$	$(1,0,1,0,1,0,1)$	$(0,0,0,0,0,0,0)$	$4V$	60480	3072
		$f_3=3$	$(1,0,1,1,0,1,0)$	$(0,0,0,0,0,0,0)$	$4W$	181440	1024
		$f_4=3$	$(1,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0)$	$4X$	181440	1024
		$f_5=4$	$(1,0,0,0,0,0,0)$	$(1,1,1,1,1,1,0)$	$8A$	241920	768
		$f_6=4$	$(1,1,1,1,1,1,1)$	$(1,1,1,1,1,1,0)$	$8B$	241920	768
$4C$	16	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$4Y$	60480	3072
		$f_2=1$	$(0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0)$	$4Z$	60480	3072
		$f_3=3$	$(1,1,0,1,1,0,0)$	$(0,0,0,0,0,0,0)$	$4AA$	181440	1024
		$f_4=3$	$(1,0,0,1,0,0,1)$	$(0,0,0,0,0,0,0)$	$4AB$	181440	1024
		$f_5=4$	$(1,0,0,0,0,0,0)$	$(0,1,1,1,1,1,0)$	$8C$	241920	768
		$f_6=4$	$(1,1,1,1,1,1,1)$	$(0,1,1,1,1,1,0)$	$8D$	241920	768
$4D$	8	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$4AC$	181440	1024
		$f_2=1$	$(1,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$4AD$	181440	1024
		$f_3=1$	$(1,1,1,1,0,1,1)$	$(0,0,0,0,0,0,0)$	$4AE$	181440	1024
		$f_4=1$	$(1,1,0,1,1,0,1)$	$(0,0,0,0,0,0,0)$	$4AF$	181440	1024
		$f_5=2$	$(1,1,1,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$4AG$	362880	512
		$f_6=2$	$(1,1,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$4AH$	362880	512
$4E$	8	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$4AI$	725760	256
		$f_2=1$	$(1,0,0,0,0,0,0)$	$(1,0,0,1,0,1,0)$	$8E$	725760	256
		$f_3=1$	$(1,1,0,1,1,0,0)$	$(1,0,0,1,0,1,0)$	$8F$	725760	256
		$f_4=1$	$(1,0,1,0,1,0,0)$	$(0,0,0,0,0,0,0)$	$4AJ$	725760	256
		$f_5=1$	$(1,1,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$4AK$	725760	256
		$f_6=1$	$(1,0,1,0,1,0,1)$	$(1,0,0,1,0,1,0)$	$8G$	725760	256
		$f_7=1$	$(1,1,0,1,1,0,1)$	$(0,0,0,0,0,0,0)$	$4AL$	725760	256
		$f_8=1$	$(0,0,0,0,0,0,1)$	$(1,0,0,1,0,1,0)$	$8H$	725760	256
$5A$	8	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$5A$	774144	240
		$f_2=1$	$(0,0,1,1,1,1,1)$	$(1,0,1,1,0,0,1)$	$10A$	774144	240
		$f_3=3$	$(1,0,0,0,0,0,0)$	$(0,0,0,1,1,0,0)$	$10B$	2322432	80
		$f_4=3$	$(1,1,1,1,1,1,1)$	$(0,0,0,0,1,1,1)$	$10C$	2322432	80
$6A$	16	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$6H$	80640	2304
		$f_2=1$	$(1,1,1,1,0,1,1)$	$(0,0,0,0,0,1,0)$	$12A$	80640	2304
		$f_3=3$	$(0,0,1,0,0,1,0)$	$(0,0,0,0,0,0,0)$	$6I$	241920	768
		$f_4=3$	$(1,1,1,1,1,1,1)$	$(0,0,0,0,0,1,0)$	$12B$	241920	768
		$f_5=4$	$(1,0,0,0,0,0,0)$	$(0,0,0,0,0,1,0)$	$12C$	322560	576
		$f_6=4$	$(0,0,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$6J$	322560	576

Table 8.1: The conjugacy classes of elements of $2^7{:}SP(6,2)$(continued)

$[g]_{SP(6,2)}$	k	f_j	d_j	w	$[x]_{2^7:SP(6,2)}$	$\lvert[x]_{2^7:SP(6,2)}\rvert$	$\lvert C_{2^7:SP(6,2)}(x)\rvert$
$6B$	8	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$6K$	161280	1152
		$f_2=1$	$(1,1,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$6L$	161280	1152
		$f_3=3$	$(1,0,0,0,0,0,0)$	$(0,1,0,1,1,0,0)$	$12D$	483840	384
		$f_4=3$	$(1,0,0,1,0,0,1)$	$(0,0,0,0,0,1,0)$	$12E$	483840	384
$6C$	2	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$6M$	1290240	144
		$f_2=1$	$(1,1,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$6N$	1290240	144
$6D$	8	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$6O$	483840	384
		$f_2=1$	$(1,0,1,1,0,1,0)$	$(1,1,1,1,1,1,0)$	$12F$	483840	384
		$f_3=1$	$(0,0,0,1,0,0,1)$	$(1,1,1,1,1,1,0)$	$12G$	483840	384
		$f_4=1$	$(0,1,0,1,1,0,1)$	$(0,0,0,0,0,0,0)$	$6P$	483840	384
		$f_5=2$	$(1,0,0,0,0,0,0)$	$(1,0,0,1,1,1,0)$	$12H$	967680	192
		$f_6=2$	$(1,1,1,1,1,1,1)$	$(1,0,0,1,1,1,0)$	$12I$	967680	192
$6E$	8	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$6Q$	645120	288
		$f_2=1$	$(1,0,0,1,0,1,1)$	$(0,0,0,0,0,0,0)$	$6R$	645120	288
		$f_3=3$	$(1,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$6S$	1935360	96
		$f_4=3$	$(1,1,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$6T$	1935360	96
$6F$	4	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$6U$	1290240	144
		$f_2=1$	$(0,1,0,1,0,1,0)$	$(1,0,0,0,0,0,0)$	$12J$	1290240	144
		$f_3=1$	$(1,1,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$6V$	1290240	144
		$f_4=1$	$(0,1,1,1,1,0,1)$	$(1,0,0,0,0,0,0)$	$12K$	1290240	144
$6G$	4	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$6W$	3870720	48
		$f_2=1$	$(1,0,0,0,0,0,0)$	$(0,0,1,0,1,1,0)$	$12L$	3870720	48
		$f_3=1$	$(1,1,1,1,1,1,1)$	$(0,0,1,0,1,1,0)$	$12M$	3870720	48
		$f_4=1$	$(1,0,0,1,0,0,1)$	$(0,0,0,0,0,0,0)$	$6X$	3870720	48
$7A$	2	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$7A$	13271040	14
		$f_2=1$	$(1,1,1,1,1,1,1)$	$(0,0,0,1,1,1,1)$	$14A$	13271040	14
$8A$	4	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$8I$	2903040	64
		$f_2=1$	$(1,1,1,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$8J$	2903040	64
		$f_3=1$	$(1,1,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$8K$	2903040	64
		$f_4=1$	$(1,1,0,0,1,1,1)$	$(0,0,0,0,0,0,0)$	$8L$	2903040	64
$8B$	4	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$8M$	2903040	64
		$f_2=1$	$(1,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$8N$	2903040	64
		$f_3=1$	$(1,1,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$8O$	2903040	64
		$f_4=1$	$(1,0,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$8P$	2903040	64
$9A$	2	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$9A$	10321920	18
		$f_2=1$	$(1,1,1,1,1,1,1)$	$(0,0,1,1,0,1,1)$	$18A$	10321920	18
$10A$	4	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$10D$	4644864	40
		$f_2=1$	$(0,1,1,0,0,1,0)$	$(0,1,0,0,1,0,0)$	$20A$	4644864	40
		$f_3=1$	$(1,1,1,1,1,1,1)$	$(0,1,0,0,1,0,0)$	$20B$	4644864	40
		$f_4=1$	$(0,0,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$10E$	4644864	40

Table 8.1: The conjugacy classes of elements of $2^7{:}SP(6,2)$(continued)

$[g]_{SP(6,2)}$	k	f_j	d_j	w	$[x]_{2^7:SP(6,2)}$	$\lvert[x]_{2^7:SP(6,2)}\rvert$	$\lvert C_{2^7:SP(6,2)}(x)\rvert$
$12A$	4	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$12N$	1935360	96
		$f_2=1$	$(1,0,0,0,0,0,0)$	$(1,1,1,1,1,1,0)$	$24A$	1935360	96
		$f_3=1$	$(1,1,1,1,1,1,1)$	$(1,1,1,1,1,1,0)$	$24B$	1935360	96
		$f_4=1$	$(1,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0)$	$12O$	1935360	96
$12B$	4	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$12P$	1935360	96
		$f_2=1$	$(1,0,0,0,0,0,0)$	$(0,1,1,1,1,1,0)$	$24C$	1935360	96
		$f_3=1$	$(1,1,1,1,1,1,1)$	$(0,1,1,1,1,1,0)$	$24D$	1935360	96
		$f_4=1$	$(0,1,0,1,0,0,1)$	$(0,0,0,0,0,0,0)$	$12Q$	1935360	96
$12C$	2	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$12R$	7741440	24
		$f_2=1$	$(1,1,1,1,1,1,1)$	$(0,0,0,0,0,0,0)$	$12S$	7741440	24
$15A$	2	$f_1=1$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0)$	$15A$	6193152	30
		$f_2=1$	$(1,1,1,1,1,1,1)$	$(1,0,1,1,0,0,1)$	$30A$	6193152	30

8.3 The inertia groups of $2^7{:}SP(6,2)$

From the results of Section 8.1 we obtain four inertia groups $\overline{H}_i = 2^7{:}H_i$ of indices 1, 1, 63, 63 in $2^7{:}SP(6,2)$ respectively, where $i \in \{1,2,3,4\}$. We also observed that $H_1 = H_2 \cong SP(6,2)$ and $H_3 = H_4 \cong 2^5{:}S_6$ of indices 1, 1, 63, 63 in $SP(6,2)$ respectively. We used the generators $\overline{a}$, $\overline{x}$, $\overline{c}$ of $SP(6,2)$ to compute the class representatives of the elements of $SP(6,2)$ in terms of 7×7 matrices over $GF(2)$. Hence we were able to produce $\alpha, \beta \in SP(6,2)$ such that $\langle\alpha,\beta\rangle \cong 2^5{:}S_6$, $\alpha \in 2B$, $\beta \in 12A$, where $2B$ and $12A$ are two conjugacy classes of elements of $SP(6,2)$. We have

$$\alpha = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 1&1&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 1&0&0&1&1&0&0\\ 0&1&0&1&0&1&0\\ 1&1&0&0&0&0&1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0&0&1&1&0&1&0\\ 1&1&0&1&0&0&0\\ 0&1&0&0&1&1&0\\ 0&0&1&1&1&0&0\\ 1&1&1&0&0&0&0\\ 1&0&0&0&1&1&0\\ 1&0&1&0&1&1&1 \end{pmatrix}.$$

8.4 The fusion of $2^5{:}S_6$ into $SP(6,2)$

The group $2^5{:}S_6$ is a maximal subgroup of $SP(6,2)$ of index 63. Using the permutation character of $SP(6,2)$ of degree 63, we are able to obtain the partial fusion of $2^5{:}S_6$

into $SP(6,2)$. We completed the fusion by using matrix conjugation in $SP(6,2)$. The complete fusion of $2^5{:}S_6$ into $SP(6,2)$ is given in Table 8.2. We follow the techniques already discussed and used in Chapter 6 for the fusion.

Table 8.2: The fusion of $2^5{:}S_6$ into $SP(6,2)$

$[g]_{SP(6,2)}$ / $[h]_{2^5:S_6}$	$1A$	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$4E$	$5A$
$1A$	$\boxed{63}$													
$2A$		$\boxed{1}$												
$2B$		15	3	$\boxed{1}$										
$2C$		15	$\boxed{3}$	1										
$2D$		$\boxed{30}$	6	2										
$2E$		30	6	$\boxed{2}$										
$2F$		60	12	4	$\boxed{1}$									
$2G$		60	$\boxed{12}$	4	1									
$2H$		180	36	$\boxed{12}$	3									
$2I$		180	36	12	$\boxed{3}$									
$2J$		180	36	12	$\boxed{3}$									
$3A$						$\boxed{15}$								
$3B$						60	18	$\boxed{3}$						
$4A$									2	$\boxed{1}$	1			
$4B$									2	1	$\boxed{1}$			
$4C$									$\boxed{3}$			1		
$4D$									3			$\boxed{1}$		
$4E$									6	3	3	$\boxed{2}$		
$4F$									12	$\boxed{6}$	6	4	1	
$4G$									12	6	6	4	$\boxed{1}$	
$4H$									12	6	6	4	$\boxed{1}$	
$4I$									12	6	$\boxed{6}$	4	1	
$4J$									12	6	6	4	$\boxed{1}$	
$5A$														$\boxed{3}$
$\chi(SP(6,2)\vert 2^5{:}S_6)$	63	31	15	15	7	15	0	3	3	7	7	3	3	3

Table 8.2: The fusion of $2^5{:}S_6$ into $SP(6,2)$(continued)

$[g]_{SP(6,2)}$ / $[h]_{2^5:S_6}$	$6A$	$6B$	$6C$	$6D$	$6E$	$6F$	$6G$	$7A$	$8A$	$8B$	$9A$	$10A$	$12A$	$12B$	$12C$	$15A$
$6A$	$\boxed{1}$	1														
$6B$	3	3		$\boxed{1}$												
$6C$	3	$\boxed{3}$		1												
$6D$	4	4	2		1	$\boxed{1}$										
$6E$	$\boxed{6}$	6	3	2												
$6F$	6	6	3	$\boxed{2}$												
$6G$	12	12	6	4	3	3	$\boxed{1}$									
$6H$	12	12	6	4	$\boxed{3}$	3	1									
$8A$									1	$\boxed{1}$						
$8B$									$\boxed{1}$	1						
$10A$												$\boxed{1}$				
$12A$													$\boxed{1}$	1		
$12B$													1	$\boxed{1}$		
$\chi(SP(6,2)\vert 2^5{:}S_6)$	7	3	0	3	3	1	1	0	1	1	0	1	1	1	0	0

8.5 The Fischer-Clifford matrices of $2^7{:}SP(6,2)$

We use the fusion discussed in Section 8.4 together with the relations of Proposition 5.2.3, Theorem 5.2.4 and the properties (a) through (f) of the Fischer-Clifford matrices which are given in Chapter 5 (Section 5.2.2) to construct the Fischer-Clifford matrices of $2^7{:}SP(6,2)$. For each conjugacy class [g] of $SP(6,2)$ with representative $g \in SP(6,2)$, we construct the corresponding Fischer-Clifford matrix $M(g)$. These matrices are given in Table 8.3.

Table 8.3: The Fischer-Clifford matrices of $2^7{:}SP(6,2)$

$M(g)$	$M(g)$
$M(1A) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 63 & -9 & 7 & -1 \\ 63 & 9 & -7 & -1 \end{pmatrix}$	$M(2A) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 30 & -10 & 6 & -2 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 30 & 10 & -6 & -2 & 0 & 0 \end{pmatrix}$
$M(2B) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 3 & 3 & 3 & 3 & -1 & -1 \\ 12 & -12 & -4 & 4 & 0 & 0 \\ 3 & -3 & 3 & -3 & -1 & 1 \\ 12 & 12 & -4 & -4 & 0 & 0 \end{pmatrix}$	$M(2C) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 2 & 2 & 2 & 2 & -2 & -2 & 0 & 0 \\ 12 & -12 & -4 & 4 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 2 & -2 & 2 & -2 & -2 & 2 & 0 & 0 \\ 12 & 12 & -4 & -4 & 0 & 0 & 0 & 0 \end{pmatrix}$
$M(2D) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 \\ 3 & -3 & 3 & -3 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 3 & 3 & -3 & -3 & -1 & -1 & 1 & 1 \\ 3 & -3 & -3 & 3 & 1 & -1 & 1 & -1 \end{pmatrix}$	$M(3A) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 15 & -5 & 3 & -1 \\ 15 & 5 & -3 & -1 \end{pmatrix}$
$M(3B) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$M(3C) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & -3 & -1 & 1 \\ 3 & 3 & -1 & -1 \end{pmatrix}$
$M(4A) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & 3 & -1 & -1 \\ 3 & -3 & -1 & 1 \end{pmatrix}$	$M(4B) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 6 & -6 & -2 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 6 & 6 & -2 & -2 & 0 & 0 \end{pmatrix}$
$M(4C) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 6 & -6 & -2 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 6 & 6 & -2 & -2 & 0 & 0 \end{pmatrix}$	$M(4D) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 2 & -2 & 2 & -2 & 0 & 0 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 2 & -2 & -2 & 2 & 0 & 0 \end{pmatrix}$

Table 8.3: The Fischer-Clifford matrices of $2^7{:}SP(6,2)$(continued)

$M(g)$	$M(g)$
$M(4E)=\begin{pmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&-1&-1&-1&-1\\ 1&-1&-1&1&1&-1&1&-1\\ 1&-1&1&-1&1&1&-1&-1\\ 1&1&-1&-1&1&-1&-1&1\\ 1&-1&-1&1&-1&1&-1&1\\ 1&-1&1&-1&-1&-1&1&1\\ 1&1&-1&-1&-1&1&1&-1 \end{pmatrix}$	$M(5A)=\begin{pmatrix} 1&1&1&1\\ 1&-1&1&-1\\ 3&-3&-1&1\\ 3&3&-1&-1 \end{pmatrix}$
$M(6A)=\begin{pmatrix} 1&1&1&1&1&1\\ 1&-1&1&-1&1&-1\\ 1&1&1&1&-1&-1\\ 6&-6&-2&2&0&0\\ 1&-1&1&-1&-1&1\\ 6&6&-2&-2&0&0 \end{pmatrix}$	$M(6B)=\begin{pmatrix} 1&1&1&1\\ 1&-1&1&-1\\ 3&3&-1&-1\\ 3&-3&-1&1 \end{pmatrix}$
$M(6C)=\begin{pmatrix} 1&1\\ 1&-1 \end{pmatrix}$	$M(6D)=\begin{pmatrix} 1&1&1&1&1&1\\ 1&1&-1&-1&1&-1\\ 1&1&1&1&-1&-1\\ 2&-2&2&-2&0&0\\ 1&1&-1&-1&-1&1\\ 2&-2&-2&2&0&0 \end{pmatrix}$
$M(6E)=\begin{pmatrix} 1&1&1&1\\ 1&-1&1&-1\\ 3&-3&-1&1\\ 3&3&-1&-1 \end{pmatrix}$	$M(6F)=\begin{pmatrix} 1&1&1&1\\ 1&1&-1&-1\\ 1&-1&-1&1\\ 1&-1&1&-1 \end{pmatrix}$
$M(6G)=\begin{pmatrix} 1&1&1&1\\ 1&1&-1&-1\\ 1&-1&1&-1\\ 1&-1&-1&1 \end{pmatrix}$	$M(7A)=\begin{pmatrix} 1&1\\ 1&-1 \end{pmatrix}$
$M(8A)=\begin{pmatrix} 1&1&1&1\\ 1&1&-1&-1\\ 1&-1&-1&1\\ 1&-1&1&-1 \end{pmatrix}$	$M(8B)=\begin{pmatrix} 1&1&1&1\\ 1&1&-1&-1\\ 1&-1&-1&1\\ 1&-1&1&-1 \end{pmatrix}$
$M(9A)=\begin{pmatrix} 1&1\\ 1&-1 \end{pmatrix}$	$M(10A)=\begin{pmatrix} 1&1&1&1\\ 1&1&-1&-1\\ 1&-1&1&-1\\ 1&-1&-1&1 \end{pmatrix}$
$M(12A)=\begin{pmatrix} 1&1&1&1\\ 1&1&-1&-1\\ 1&-1&-1&1\\ 1&-1&1&-1 \end{pmatrix}$	$M(12B)=\begin{pmatrix} 1&1&1&1\\ 1&1&-1&-1\\ 1&-1&-1&1\\ 1&-1&1&-1 \end{pmatrix}$
$M(12C)=\begin{pmatrix} 1&1\\ 1&-1 \end{pmatrix}$	$M(15A)=\begin{pmatrix} 1&1\\ 1&-1 \end{pmatrix}$

We use the above Fischer-Clifford matrices and the character tables of the inertia factors $H_1 = H_2 = SP(6,2)$ and $H_3 = H_4 = 2^5{:}S_6$, together with the fusion of $2^5{:}S_6$ into $SP(6,2)$ to obtain the character table of $2^7{:}SP(6,2)$. The set of irreducible characters of $2^7{:}SP(6,2)$ will be partitioned into four blocks B_1, B_2, B_3 and B_4 corresponding to the inertia factors H_1, H_2, H_3 and H_4 respectively. In fact $B_1 = \{\chi_i \mid 1 \le i \le 30\}, B_2 = \{\chi_i \mid 31 \le i \le 60\}, B_3 = \{\chi_i \mid 61 \le i \le 97\}, B_4 = \{\chi_i \mid 98 \le i \le 134\}$, where $Irr(2^7{:}SP(6,2)) = \bigcup_{i=1}^{4} B_i$. The complete character table of $2^7{:}SP(6,2)$ is given in Table 8.4. Please note that the centralizers of the elements of $2^7{:}SP(6,2)$ are listed in the last column of Table 8.1.

Table 8.4: The character table of $2^7{:}SP(6,2)$

	$1A$				$2A$						$2B$					
	$1A$	$2A$	$2B$	$2C$	$2D$	$4A$	$4B$	$2E$	$4C$	$2F$	$2G$	$2H$	$2I$	$2J$	$4D$	$4E$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	7	7	7	7	-5	-5	-5	-5	-5	-5	-1	-1	-1	-1	-1	-1
χ_3	15	15	15	15	-5	-5	-5	-5	-5	-5	7	7	7	7	7	7
χ_4	21	21	21	21	-11	-11	-11	-11	-11	-11	5	5	5	5	5	5
χ_5	21	21	21	21	9	9	9	9	9	9	-3	-3	-3	-3	-3	-3
χ_6	27	27	27	27	15	15	15	15	15	15	3	3	3	3	3	3
χ_7	35	35	35	35	-5	-5	-5	-5	-5	-5	3	3	3	3	3	3
χ_8	35	35	35	35	15	15	15	15	15	15	11	11	11	11	11	11
χ_9	56	56	56	56	-24	-24	-24	-24	-24	-24	-8	-8	-8	-8	-8	-8
χ_{10}	70	70	70	70	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10
χ_{11}	84	84	84	84	4	4	4	4	4	4	20	20	20	20	20	20
χ_{12}	105	105	105	105	-35	-35	-35	-35	-35	-35	1	1	1	1	1	1
χ_{13}	105	105	105	105	5	5	5	5	5	5	17	17	17	17	17	17
χ_{14}	105	105	105	105	25	25	25	25	25	25	-7	-7	-7	-7	-7	-7
χ_{15}	120	120	120	120	40	40	40	40	40	40	-8	-8	-8	-8	-8	-8
χ_{16}	168	168	168	168	40	40	40	40	40	40	8	8	8	8	8	8
χ_{17}	189	189	189	189	-51	-51	-51	-51	-51	-51	-3	-3	-3	-3	-3	-3
χ_{18}	189	189	189	189	21	21	21	21	21	21	-3	-3	-3	-3	-3	-3
χ_{19}	189	189	189	189	-39	-39	-39	-39	-39	-39	21	21	21	21	21	21
χ_{20}	210	210	210	210	10	10	10	10	10	10	-14	-14	-14	-14	-14	-14
χ_{21}	210	210	210	210	50	50	50	50	50	50	2	2	2	2	2	2
χ_{22}	216	216	216	216	-24	-24	-24	-24	-24	-24	24	24	24	24	24	24
χ_{23}	280	280	280	280	40	40	40	40	40	40	24	24	24	24	24	24
χ_{24}	280	280	280	280	-40	-40	-40	-40	-40	-40	-8	-8	-8	-8	-8	-8
χ_{25}	315	315	315	315	-45	-45	-45	-45	-45	-45	-21	-21	-21	-21	-21	-21
χ_{26}	336	336	336	336	-16	-16	-16	-16	-16	-16	16	16	16	16	16	16
χ_{27}	378	378	378	378	-30	-30	-30	-30	-30	-30	-6	-6	-6	-6	-6	-6
χ_{28}	405	405	405	405	45	45	45	45	45	45	-27	-27	-27	-27	-27	-27
χ_{29}	420	420	420	420	20	20	20	20	20	20	4	4	4	4	4	4
χ_{30}	512	512	512	512	0	0	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$2C$								$2D$							
	$2K$	$4F$	$2L$	$4G$	$4H$	$2M$	$4I$	$4J$	$2N$	$4K$	$4L$	$2O$	$4M$	$4N$	$4O$	$4P$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	3	3	3	3	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1
χ_3	3	3	3	3	3	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1
χ_4	5	5	5	5	5	5	5	5	-3	-3	-3	-3	-3	-3	-3	-3
χ_5	1	1	1	1	1	1	1	1	-3	-3	-3	-3	-3	-3	-3	-3
χ_6	7	7	7	7	7	7	7	7	3	3	3	3	3	3	3	3
χ_7	-5	-5	-5	-5	-5	-5	-5	-5	3	3	3	3	3	3	3	3
χ_8	7	7	7	7	7	7	7	7	3	3	3	3	3	3	3	3
χ_9	8	8	8	8	8	8	8	8	0	0	0	0	0	0	0	0
χ_{10}	6	6	6	6	6	6	6	6	-2	-2	-2	-2	-2	-2	-2	-2
χ_{11}	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
χ_{12}	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1
χ_{13}	-3	-3	-3	-3	-3	-3	-3	-3	-7	-7	-7	-7	-7	-7	-7	-7
χ_{14}	9	9	9	9	9	9	9	9	1	1	1	1	1	1	1	1
χ_{15}	8	8	8	8	8	8	8	8	0	0	0	0	0	0	0	0
χ_{16}	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
χ_{17}	13	13	13	13	13	13	13	13	-3	-3	-3	-3	-3	-3	-3	-3
χ_{18}	-11	-11	-11	-11	-11	-11	-11	-11	-3	-3	-3	-3	-3	-3	-3	-3
χ_{19}	1	1	1	1	1	1	1	1	-3	-3	-3	-3	-3	-3	-3	-3
χ_{20}	10	10	10	10	10	10	10	10	2	2	2	2	2	2	2	2
χ_{21}	2	2	2	2	2	2	2	2	-6	-6	-6	-6	-6	-6	-6	-6
χ_{22}	8	8	8	8	8	8	8	8	0	0	0	0	0	0	0	0
χ_{23}	8	8	8	8	8	8	8	8	0	0	0	0	0	0	0	0
χ_{24}	-8	-8	-8	-8	-8	-8	-8	-8	8	8	8	8	8	8	8	8
χ_{25}	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
χ_{26}	-16	-16	-16	-16	-16	-16	-16	-16	0	0	0	0	0	0	0	0
χ_{27}	2	2	2	2	2	2	2	2	-6	-6	-6	-6	-6	-6	-6	-6
χ_{28}	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
χ_{29}	-12	-12	-12	-12	-12	-12	-12	-12	4	4	4	4	4	4	4	4
χ_{30}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$3A$				$3B$		$3C$				$4A$			
	$3A$	$6A$	$6B$	$6C$	$3B$	$6D$	$3C$	$6E$	$6F$	$6G$	$4Q$	$4R$	$4S$	$4T$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	4	4	4	4	-2	-2	1	1	1	1	3	3	3	3
χ_3	0	0	0	0	-3	-3	3	3	3	3	-1	-1	-1	-1
χ_4	6	6	6	6	3	3	0	0	0	0	1	1	1	1
χ_5	6	6	6	6	3	3	0	0	0	0	5	5	5	5
χ_6	9	9	9	9	0	0	0	0	0	0	3	3	3	3
χ_7	5	5	5	5	-1	-1	2	2	2	2	7	7	7	7
χ_8	5	5	5	5	-1	-1	2	2	2	2	-1	-1	-1	-1
χ_9	11	11	11	11	2	2	2	2	2	2	0	0	0	0
χ_{10}	-5	-5	-5	-5	7	7	1	1	1	1	2	2	2	2
χ_{11}	-6	-6	-6	-6	3	3	3	3	3	3	4	4	4	4
χ_{12}	15	15	15	15	-3	-3	-3	-3	-3	-3	5	5	5	5
χ_{13}	0	0	0	0	6	6	3	3	3	3	-3	-3	-3	-3
χ_{14}	0	0	0	0	6	6	3	3	3	3	-3	-3	-3	-3
χ_{15}	15	15	15	15	-6	-6	0	0	0	0	0	0	0	0
χ_{16}	6	6	6	6	6	6	-3	-3	-3	-3	0	0	0	0
χ_{17}	9	9	9	9	0	0	0	0	0	0	-3	-3	-3	-3
χ_{18}	9	9	9	9	0	0	0	0	0	0	9	9	9	9
χ_{19}	9	9	9	9	0	0	0	0	0	0	-3	-3	-3	-3
χ_{20}	-15	-15	-15	-15	-6	-6	3	3	3	3	6	6	6	6
χ_{21}	15	15	15	15	3	3	0	0	0	0	-2	-2	-2	-2
χ_{22}	-9	-9	-9	-9	0	0	0	0	0	0	0	0	0	0
χ_{23}	-5	-5	-5	-5	-8	-8	-2	-2	-2	-2	0	0	0	0
χ_{24}	10	10	10	10	10	10	1	1	1	1	0	0	0	0
χ_{25}	0	0	0	0	-9	-9	0	0	0	0	-5	-5	-5	-5
χ_{26}	6	6	6	6	-6	-6	0	0	0	0	0	0	0	0
χ_{27}	-9	-9	-9	-9	0	0	0	0	0	0	6	6	6	6
χ_{28}	0	0	0	0	0	0	0	0	0	0	-3	-3	-3	-3
χ_{29}	0	0	0	0	-3	-3	3	3	3	3	-4	-4	-4	-4
χ_{30}	-16	-16	-16	-16	8	8	-4	-4	-4	-4	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$4B$						$4C$						$4D$					
	$4U$	$4V$	$4W$	$4X$	$8A$	$8B$	$4Y$	$4Z$	$4AA$	$4AB$	$8C$	$8D$	$4AC$	$4AD$	$4AE$	$4AF$	$4AG$	$4AH$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	-3	-3	-3	-3	-3	-3	-1	-1	-1	-1	-1	-1
χ_3	-3	-3	-3	-3	-3	-3	1	1	1	1	1	1	3	3	3	3	3	3
χ_4	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	1	1	1	1	1	1
χ_5	-1	-1	-1	-1	-1	-1	3	3	3	3	3	3	1	1	1	1	1	1
χ_6	1	1	1	1	1	1	5	5	5	5	5	5	-1	-1	-1	-1	-1	-1
χ_7	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
χ_8	5	5	5	5	5	5	1	1	1	1	1	1	3	3	3	3	3	3
χ_9	4	4	4	4	4	4	-4	-4	-4	-4	-4	-4	0	0	0	0	0	0
χ_{10}	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
χ_{11}	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4	4	4
χ_{12}	-1	-1	-1	-1	-1	-1	-5	-5	-5	-5	-5	-5	1	1	1	1	1	1
χ_{13}	3	3	3	3	3	3	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
χ_{14}	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
χ_{15}	-4	-4	-4	-4	-4	-4	4	4	4	4	4	4	0	0	0	0	0	0
χ_{16}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{17}	1	1	1	1	1	1	1	1	1	1	1	1	-3	-3	-3	-3	-3	-3
χ_{18}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_{19}	-5	-5	-5	-5	-5	-5	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
χ_{20}	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
χ_{21}	2	2	2	2	2	2	2	2	2	2	2	2	-2	-2	-2	-2	-2	-2
χ_{22}	-4	-4	-4	-4	-4	-4	4	4	4	4	4	4	0	0	0	0	0	0
χ_{23}	4	4	4	4	4	4	-4	-4	-4	-4	-4	-4	0	0	0	0	0	0
χ_{24}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{25}	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
χ_{26}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{27}	2	2	2	2	2	2	2	2	2	2	2	2	-2	-2	-2	-2	-2	-2
χ_{28}	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	5	5	5	5	5	5
χ_{29}	0	0	0	0	0	0	0	0	0	0	0	0	-4	-4	-4	-4	-4	-4
χ_{30}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$4E$								$5A$				$6A$					
	$4AI$	$8E$	$8F$	$4AJ$	$4AK$	$8G$	$4AL$	$8H$	$5A$	$10A$	$10B$	$10C$	$6H$	$12A$	$6I$	$12B$	$12C$	$6J$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	2	2	2	2	-2	-2	-2	-2	-2	-2
χ_3	1	1	1	1	1	1	1	1	0	0	0	0	-2	-2	-2	-2	-2	-2
χ_4	1	1	1	1	1	1	1	1	1	1	1	1	-2	-2	-2	-2	-2	-2
χ_5	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	0	0	0	0	0	0
χ_6	1	1	1	1	1	1	1	1	2	2	2	2	3	3	3	3	3	3
χ_7	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	1	1	1	1	1	1
χ_8	1	1	1	1	1	1	1	1	0	0	0	0	3	3	3	3	3	3
χ_9	0	0	0	0	0	0	0	0	1	1	1	1	-3	-3	-3	-3	-3	-3
χ_{10}	-2	-2	-2	-2	-2	-2	-2	-2	0	0	0	0	-1	-1	-1	-1	-1	-1
χ_{11}	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2
χ_{12}	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	1	1	1	1	1	1
χ_{13}	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	2	2	2	2	2	2
χ_{14}	1	1	1	1	1	1	1	1	0	0	0	0	4	4	4	4	4	4
χ_{15}	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
χ_{16}	0	0	0	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
χ_{17}	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-3	-3	-3	-3	-3	-3
χ_{18}	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-3	-3	-3	-3	-3	-3
χ_{19}	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	3	3	3	3	3	3
χ_{20}	-2	-2	-2	-2	-2	-2	-2	-2	0	0	0	0	1	1	1	1	1	1
χ_{21}	-2	-2	-2	-2	-2	-2	-2	-2	0	0	0	0	-1	-1	-1	-1	-1	-1
χ_{22}	0	0	0	0	0	0	0	0	1	1	1	1	-3	-3	-3	-3	-3	-3
χ_{23}	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
χ_{24}	0	0	0	0	0	0	0	0	0	0	0	0	2	2	2	2	2	2
χ_{25}	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0
χ_{26}	0	0	0	0	0	0	0	0	1	1	1	1	2	2	2	2	2	2
χ_{27}	2	2	2	2	2	2	2	2	-2	-2	-2	-2	3	3	3	3	3	3
χ_{28}	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
χ_{29}	0	0	0	0	0	0	0	0	0	0	0	0	-4	-4	-4	-4	-4	-4
χ_{30}	0	0	0	0	0	0	0	0	2	2	2	2	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$6B$				$6C$		$6D$						$6E$			
	$6K$	$6L$	$12D$	$12E$	$6M$	$6N$	$6O$	$12F$	$12G$	$6P$	$12H$	$12I$	$6Q$	$6R$	$6S$	$6T$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	2	2	2	2	2	2	0	0	0	0	0	0	-1	-1	-1	-1
χ_3	-2	-2	-2	-2	1	1	0	0	0	0	0	0	1	1	1	1
χ_4	2	2	2	2	-1	-1	2	2	2	2	2	2	2	2	2	2
χ_5	0	0	0	0	3	3	-2	-2	-2	-2	-2	-2	0	0	0	0
χ_6	3	3	3	3	0	0	1	1	1	1	1	1	0	0	0	0
χ_7	-3	-3	-3	-3	3	3	1	1	1	1	1	1	0	0	0	0
χ_8	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	2	2	2	2
χ_9	1	1	1	1	-2	-2	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2
χ_{10}	-1	-1	-1	-1	-1	-1	3	3	3	3	3	3	-1	-1	-1	-1
χ_{11}	2	2	2	2	-1	-1	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1
χ_{12}	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1
χ_{13}	2	2	2	2	2	2	0	0	0	0	0	0	-1	-1	-1	-1
χ_{14}	-4	-4	-4	-4	2	2	0	0	0	0	0	0	-1	-1	-1	-1
χ_{15}	1	1	1	1	-2	-2	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2
χ_{16}	2	2	2	2	2	2	2	2	2	2	2	2	-1	-1	-1	-1
χ_{17}	-3	-3	-3	-3	0	0	1	1	1	1	1	1	0	0	0	0
χ_{18}	-3	-3	-3	-3	0	0	1	1	1	1	1	1	0	0	0	0
χ_{19}	3	3	3	3	0	0	1	1	1	1	1	1	0	0	0	0
χ_{20}	1	1	1	1	-2	-2	1	1	1	1	1	1	1	1	1	1
χ_{21}	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	2	2	2	2
χ_{22}	-3	-3	-3	-3	0	0	-1	-1	-1	-1	-1	-1	0	0	0	0
χ_{23}	-3	-3	-3	-3	0	0	-1	-1	-1	-1	-1	-1	0	0	0	0
χ_{24}	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	1	1	1	1
χ_{25}	0	0	0	0	3	3	0	0	0	0	0	0	0	0	0	0
χ_{26}	-2	-2	-2	-2	-2	-2	2	2	2	2	2	2	-2	-2	-2	-2
χ_{27}	3	3	3	3	0	0	-1	-1	-1	-1	-1	-1	0	0	0	0
χ_{28}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{29}	4	4	4	4	1	1	0	0	0	0	0	0	1	1	1	1
χ_{30}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$6F$				$6G$				$7A$		$8A$				$8B$			
	$6U$	$12J$	$6V$	$12K$	$6W$	$12L$	$12M$	$6X$	$7A$	$14A$	$8I$	$8J$	$8K$	$8L$	$8M$	$8N$	$8O$	$8P$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1	0	0	1	1	1	1	-1	-1	-1	-1
χ_3	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	1	1	1	1
χ_4	-2	-2	-2	-2	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1
χ_5	0	0	0	0	0	0	0	0	0	0	1	1	1	1	-1	-1	-1	-1
χ_6	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	1	1	1	1
χ_7	-2	-2	-2	-2	0	0	0	0	0	0	1	1	1	1	1	1	1	1
χ_8	0	0	0	0	0	0	0	0	0	0	1	1	1	1	-1	-1	-1	-1
χ_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{10}	-1	-1	-1	-1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
χ_{11}	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
χ_{12}	1	1	1	1	1	1	1	1	0	0	-1	-1	-1	-1	1	1	1	1
χ_{13}	-1	-1	-1	-1	-1	-1	-1	-1	0	0	-1	-1	-1	-1	1	1	1	1
χ_{14}	1	1	1	1	1	1	1	1	0	0	-1	-1	-1	-1	-1	-1	-1	-1
χ_{15}	-2	-2	-2	-2	0	0	0	0	1	1	0	0	0	0	0	0	0	0
χ_{16}	1	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0
χ_{17}	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
χ_{18}	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1
χ_{19}	0	0	0	0	0	0	0	0	0	0	1	1	1	1	-1	-1	-1	-1
χ_{20}	1	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0
χ_{21}	2	2	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{22}	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0
χ_{23}	-2	-2	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{24}	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0
χ_{25}	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1
χ_{26}	2	2	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{27}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{28}	0	0	0	0	0	0	0	0	-1	-1	1	1	1	1	1	1	1	1
χ_{29}	-1	-1	-1	-1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
χ_{30}	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$9A$		$10A$				$12A$				$12B$				$12C$		$15A$	
	$9A$	$18A$	$10D$	$20A$	$20B$	$10E$	$12N$	$24A$	$24B$	$12O$	$12P$	$24C$	$24D$	$12Q$	$12R$	$12S$	$15A$	$30A$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	0	0	0	0	-2	-2	-2	-2	0	0	0	0	0	0	-1	-1
χ_3	0	0	0	0	0	0	0	0	0	0	-2	-2	-2	-2	-1	-1	0	0
χ_4	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	1	1	1	1
χ_5	0	0	-1	-1	-1	-1	2	2	2	2	0	0	0	0	-1	-1	1	1
χ_6	0	0	0	0	0	0	1	1	1	1	-1	-1	-1	-1	0	0	-1	-1
χ_7	-1	-1	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	1	1	0	0
χ_8	-1	-1	0	0	0	0	-1	-1	-1	-1	1	1	1	1	-1	-1	0	0
χ_9	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	0	0	1	1
χ_{10}	1	1	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0
χ_{11}	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	1	1	-1	-1
χ_{12}	0	0	0	0	0	0	-1	-1	-1	-1	1	1	1	1	-1	-1	0	0
χ_{13}	0	0	0	0	0	0	0	0	0	0	2	2	2	2	0	0	0	0
χ_{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{15}	0	0	0	0	0	0	-1	-1	-1	-1	1	1	1	1	0	0	0	0
χ_{16}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
χ_{17}	0	0	-1	-1	-1	-1	1	1	1	1	1	1	1	1	0	0	-1	-1
χ_{18}	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	-1	-1
χ_{19}	0	0	1	1	1	1	1	1	1	1	-1	-1	-1	-1	0	0	-1	-1
χ_{20}	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
χ_{21}	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	1	1	0	0
χ_{22}	0	0	1	1	1	1	-1	-1	-1	-1	1	1	1	1	0	0	1	1
χ_{23}	1	1	0	0	0	0	1	1	1	1	-1	-1	-1	-1	0	0	0	0
χ_{24}	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{25}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
χ_{26}	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	1	1
χ_{27}	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	0	0	1	1
χ_{28}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{29}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	0	0
χ_{30}	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$1A$				$2A$						$2B$					
	$1A$	$2A$	$2B$	$2C$	$2D$	$4A$	$4B$	$2E$	$4C$	$2F$	$2G$	$2H$	$2I$	$2J$	$4D$	$4E$
χ_{31}	1	-1	-1	1	1	-1	-1	1	1	-1	1	-1	1	-1	1	-1
χ_{32}	7	-7	-7	7	-5	5	5	-5	-5	5	-1	1	-1	1	-1	1
χ_{33}	15	-15	-15	15	-5	5	5	-5	-5	5	7	-7	7	-7	7	-7
χ_{34}	21	-21	-21	21	-11	11	11	-11	-11	11	5	-5	5	-5	5	-5
χ_{35}	21	-21	-21	21	9	-9	-9	9	9	-9	-3	3	-3	3	-3	3
χ_{36}	27	-27	-27	27	15	-15	-15	15	15	-15	3	-3	3	-3	3	-3
χ_{37}	35	-35	-35	35	-5	5	5	-5	-5	5	3	-3	3	-3	3	-3
χ_{38}	35	-35	-35	35	15	-15	-15	15	15	-15	11	-11	11	-11	11	-11
χ_{39}	56	-56	-56	56	-24	24	24	-24	-24	24	-8	8	-8	8	-8	8
χ_{40}	70	-70	-70	70	-10	10	10	-10	-10	10	-10	10	-10	10	-10	10
χ_{41}	84	-84	-84	84	4	-4	-4	4	4	-4	20	-20	20	-20	20	-20
χ_{42}	105	-105	-105	105	-35	35	35	-35	-35	35	1	-1	1	-1	1	-1
χ_{43}	105	-105	-105	105	5	-5	-5	5	5	-5	17	-17	17	-17	17	-17
χ_{44}	105	-105	-105	105	25	-25	-25	25	25	-25	-7	7	-7	7	-7	7
χ_{45}	120	-120	-120	120	40	-40	-40	40	40	-40	-8	8	-8	8	-8	8
χ_{46}	168	-168	-168	168	40	-40	-40	40	40	-40	8	-8	8	-8	8	-8
χ_{47}	189	-189	-189	189	-51	51	51	-51	-51	51	-3	3	-3	3	-3	3
χ_{48}	189	-189	-189	189	21	-21	-21	21	21	-21	-3	3	-3	3	-3	3
χ_{49}	189	-189	-189	189	-39	39	39	-39	-39	39	21	-21	21	-21	21	-21
χ_{50}	210	-210	-210	210	10	-10	-10	10	10	-10	-14	14	-14	14	-14	14
χ_{51}	210	-210	-210	210	50	-50	-50	50	50	-50	2	-2	2	-2	2	-2
χ_{52}	216	-216	-216	216	-24	24	24	-24	-24	24	24	-24	24	-24	24	-24
χ_{53}	280	-280	-280	280	40	-40	-40	40	40	-40	24	-24	24	-24	24	-24
χ_{54}	280	-280	-280	280	-40	40	40	-40	-40	40	-8	8	-8	8	-8	8
χ_{55}	315	-315	-315	315	-45	45	45	-45	-45	45	-21	21	-21	21	-21	21
χ_{56}	336	-336	-336	336	-16	16	16	-16	-16	16	16	-16	16	-16	16	-16
χ_{57}	378	-378	-378	378	-30	30	30	-30	-30	30	-6	6	-6	6	-6	6
χ_{58}	405	-405	-405	405	45	-45	-45	45	45	-45	-27	27	-27	27	-27	27
χ_{59}	420	-420	-420	420	20	-20	-20	20	20	-20	4	-4	4	-4	4	-4
χ_{60}	512	-512	-512	512	0	0	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$2C$								$2D$							
	$2K$	$4F$	$2L$	$4G$	$4H$	$2M$	$4I$	$4J$	$2N$	$4K$	$4L$	$2O$	$4M$	$4N$	$4O$	$4P$
χ_{31}	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	1	1	-1	-1
χ_{32}	3	-3	3	-3	3	-3	3	-3	-1	-1	1	1	-1	-1	1	1
χ_{33}	3	-3	3	-3	3	-3	3	-3	-1	-1	1	1	-1	-1	1	1
χ_{34}	5	-5	5	-5	5	-5	5	-5	-3	-3	3	3	-3	-3	3	3
χ_{35}	1	-1	1	-1	1	-1	1	-1	-3	-3	3	3	-3	-3	3	3
χ_{36}	7	-7	7	-7	7	-7	7	-7	3	3	-3	-3	3	3	-3	-3
χ_{37}	-5	5	-5	5	-5	5	-5	5	3	3	-3	-3	3	3	-3	-3
χ_{38}	7	-7	7	-7	7	-7	7	-7	3	3	-3	-3	3	3	-3	-3
χ_{39}	8	-8	8	-8	8	-8	8	-8	0	0	0	0	0	0	0	0
χ_{40}	6	-6	6	-6	6	-6	6	-6	-2	-2	2	2	-2	-2	2	2
χ_{41}	4	-4	4	-4	4	-4	4	-4	4	4	-4	-4	4	4	-4	-4
χ_{42}	5	-5	5	-5	5	-5	5	-5	1	1	-1	-1	1	1	-1	-1
χ_{43}	-3	3	-3	3	-3	3	-3	3	-7	-7	7	7	-7	-7	7	7
χ_{44}	9	-9	9	-9	9	-9	9	-9	1	1	-1	-1	1	1	-1	-1
χ_{45}	8	-8	8	-8	8	-8	8	-8	0	0	0	0	0	0	0	0
χ_{46}	8	-8	8	-8	8	-8	8	-8	8	8	-8	-8	8	8	-8	-8
χ_{47}	13	-13	13	-13	13	-13	13	-13	-3	-3	3	3	-3	-3	3	3
χ_{48}	-11	11	-11	11	-11	11	-11	11	-3	-3	3	3	-3	-3	3	3
χ_{49}	1	-1	1	-1	1	-1	1	-1	-3	-3	3	3	-3	-3	3	3
χ_{50}	10	-10	10	-10	10	-10	10	-10	2	2	-2	-2	2	2	-2	-2
χ_{51}	2	-2	2	-2	2	-2	2	-2	-6	-6	6	6	-6	-6	6	6
χ_{52}	8	-8	8	-8	8	-8	8	-8	0	0	0	0	0	0	0	0
χ_{53}	8	-8	8	-8	8	-8	8	-8	0	0	0	0	0	0	0	0
χ_{54}	-8	8	-8	8	-8	8	-8	8	8	8	-8	-8	8	8	-8	-8
χ_{55}	3	-3	3	-3	3	-3	3	-3	3	3	-3	-3	3	3	-3	-3
χ_{56}	-16	16	-16	16	-16	16	-16	16	0	0	0	0	0	0	0	0
χ_{57}	2	-2	2	-2	2	-2	2	-2	-6	-6	6	6	-6	-6	6	6
χ_{58}	-3	3	-3	3	-3	3	-3	3	-3	-3	3	3	-3	-3	3	3
χ_{59}	-12	12	-12	12	-12	12	-12	12	4	4	-4	-4	4	4	-4	-4
χ_{60}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	3A				3B		3C				4A			
	3A	6A	6B	6C	3B	6D	3C	6E	6F	6G	4Q	4R	4S	4T
χ_{31}	1	-1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_{32}	4	-4	-4	4	-2	2	1	-1	1	-1	3	-3	3	-3
χ_{33}	0	0	0	0	-3	3	3	-3	3	-3	-1	1	-1	1
χ_{34}	6	-6	-6	6	3	-3	0	0	0	0	1	-1	1	-1
χ_{35}	6	-6	-6	6	3	-3	0	0	0	0	5	-5	5	-5
χ_{36}	9	-9	-9	9	0	0	0	0	0	0	3	-3	3	-3
χ_{37}	5	-5	-5	5	-1	1	2	-2	2	-2	7	-7	7	-7
χ_{38}	5	-5	-5	5	-1	1	2	-2	2	-2	-1	1	-1	1
χ_{39}	11	-11	-11	11	2	-2	2	-2	2	-2	0	0	0	0
χ_{40}	-5	5	5	-5	7	-7	1	-1	1	-1	2	-2	2	-2
χ_{41}	-6	6	6	-6	3	-3	3	-3	3	-3	4	-4	4	-4
χ_{42}	15	-15	-15	15	-3	3	-3	3	-3	3	5	-5	5	-5
χ_{43}	0	0	0	0	6	-6	3	-3	3	-3	-3	3	-3	3
χ_{44}	0	0	0	0	6	-6	3	-3	3	-3	-3	3	-3	3
χ_{45}	15	-15	-15	15	-6	6	0	0	0	0	0	0	0	0
χ_{46}	6	-6	-6	6	6	-6	-3	3	-3	3	0	0	0	0
χ_{47}	9	-9	-9	9	0	0	0	0	0	0	-3	3	-3	3
χ_{48}	9	-9	-9	9	0	0	0	0	0	0	9	-9	9	-9
χ_{49}	9	-9	-9	9	0	0	0	0	0	0	-3	3	-3	3
χ_{50}	-15	15	15	-15	-6	6	3	-3	3	-3	6	-6	6	-6
χ_{51}	15	-15	-15	15	3	-3	0	0	0	0	-2	2	-2	2
χ_{52}	-9	9	9	-9	0	0	0	0	0	0	0	0	0	0
χ_{53}	-5	5	5	-5	-8	8	-2	2	-2	2	0	0	0	0
χ_{54}	10	-10	-10	10	10	-10	1	-1	1	-1	0	0	0	0
χ_{55}	0	0	0	0	-9	9	0	0	0	0	-5	5	-5	5
χ_{56}	6	-6	-6	6	-6	6	0	0	0	0	0	0	0	0
χ_{57}	-9	9	9	-9	0	0	0	0	0	0	6	-6	6	-6
χ_{58}	0	0	0	0	0	0	0	0	0	0	-3	3	-3	3
χ_{59}	0	0	0	0	-3	3	3	-3	3	-3	-4	4	-4	4
χ_{60}	-16	16	16	-16	8	-8	-4	4	-4	4	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$4B$						$4C$						$4D$					
	$4U$	$4V$	$4W$	$4X$	$8A$	$8B$	$4Y$	$4Z$	$4AA$	$4AB$	$8C$	$8D$	$4AC$	$4AD$	$4AE$	$4AF$	$4AG$	$4AH$
χ_{31}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	1	-1
χ_{32}	1	-1	1	-1	1	-1	-3	3	-3	3	-3	3	-1	-1	1	1	-1	1
χ_{33}	-3	3	-3	3	-3	3	1	-1	1	-1	1	-1	3	3	-3	-3	3	-3
χ_{34}	-3	3	-3	3	-3	3	-3	3	-3	3	-3	3	1	1	-1	-1	1	-1
χ_{35}	-1	1	-1	1	-1	1	3	-3	3	-3	3	-3	1	1	-1	-1	1	-1
χ_{36}	1	-1	1	-1	1	-1	5	-5	5	-5	5	-5	-1	-1	1	1	-1	1
χ_{37}	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	1	-1	1
χ_{38}	5	-5	5	-5	5	-5	1	-1	1	-1	1	-1	3	3	-3	-3	3	-3
χ_{39}	4	-4	4	-4	4	-4	-4	4	-4	4	-4	4	0	0	0	0	0	0
χ_{40}	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	2	-2	-2	2	-2
χ_{41}	0	0	0	0	0	0	0	0	0	0	0	0	4	4	-4	-4	4	-4
χ_{42}	-1	1	-1	1	-1	1	-5	5	-5	5	-5	5	1	1	-1	-1	1	-1
χ_{43}	3	-3	3	-3	3	-3	-1	1	-1	1	-1	1	1	1	-1	-1	1	-1
χ_{44}	-3	3	-3	3	-3	3	-3	3	-3	3	-3	3	-3	-3	3	3	-3	3
χ_{45}	-4	4	-4	4	-4	4	4	-4	4	-4	4	-4	0	0	0	0	0	0
χ_{46}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{47}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-3	-3	3	3	-3	3
χ_{48}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	1	-1
χ_{49}	-5	5	-5	5	-5	5	-1	1	-1	1	-1	1	1	1	-1	-1	1	-1
χ_{50}	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	-2	2	2	-2	2
χ_{51}	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	-2	-2	2	2	-2	2
χ_{52}	-4	4	-4	4	-4	4	4	-4	4	-4	4	-4	0	0	0	0	0	0
χ_{53}	4	-4	4	-4	4	-4	-4	4	-4	4	-4	4	0	0	0	0	0	0
χ_{54}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{55}	3	-3	3	-3	3	-3	3	-3	3	-3	3	-3	3	3	-3	-3	3	-3
χ_{56}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{57}	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	-2	-2	2	2	-2	2
χ_{58}	-3	3	-3	3	-3	3	-3	3	-3	3	-3	3	5	5	-5	-5	5	-5
χ_{59}	0	0	0	0	0	0	0	0	0	0	0	0	-4	-4	4	4	-4	4
χ_{60}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of 2^7:$SP(6,2)$(continued)

	$4E$								$5A$				$6A$					
	$4AI$	$8E$	$8F$	$4AJ$	$4AK$	$8G$	$4AL$	$8H$	$5A$	$10A$	$10B$	$10C$	$6H$	$12A$	$6I$	$12B$	$12C$	$6J$
χ_{31}	1	1	1	1	-1	-1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_{32}	1	1	1	1	-1	-1	-1	-1	2	-2	2	-2	-2	2	-2	2	-2	2
χ_{33}	1	1	1	1	-1	-1	-1	-1	0	0	0	0	-2	2	-2	2	-2	2
χ_{34}	1	1	1	1	-1	-1	-1	-1	1	-1	1	-1	-2	2	-2	2	-2	2
χ_{35}	-1	-1	-1	-1	1	1	1	1	1	-1	1	-1	0	0	0	0	0	0
χ_{36}	1	1	1	1	-1	-1	-1	-1	2	-2	2	-2	3	-3	3	-3	3	-3
χ_{37}	-1	-1	-1	-1	1	1	1	1	0	0	0	0	1	-1	1	-1	1	-1
χ_{38}	1	1	1	1	-1	-1	-1	-1	0	0	0	0	3	-3	3	-3	3	-3
χ_{39}	0	0	0	0	0	0	0	0	1	-1	1	-1	-3	3	-3	3	-3	3
χ_{40}	-2	-2	-2	-2	2	2	2	2	0	0	0	0	-1	1	-1	1	-1	1
χ_{41}	0	0	0	0	0	0	0	0	-1	1	-1	1	-2	2	-2	2	-2	2
χ_{42}	-1	-1	-1	-1	1	1	1	1	0	0	0	0	1	-1	1	-1	1	-1
χ_{43}	-1	-1	-1	-1	1	1	1	1	0	0	0	0	2	-2	2	-2	2	-2
χ_{44}	1	1	1	1	-1	-1	-1	-1	0	0	0	0	4	-4	4	-4	4	-4
χ_{45}	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	1	-1
χ_{46}	0	0	0	0	0	0	0	0	-2	2	-2	2	-2	2	-2	2	-2	2
χ_{47}	1	1	1	1	-1	-1	-1	-1	-1	1	-1	1	-3	3	-3	3	-3	3
χ_{48}	1	1	1	1	-1	-1	-1	-1	-1	1	-1	1	-3	3	-3	3	-3	3
χ_{49}	-1	-1	-1	-1	1	1	1	1	-1	1	-1	1	3	-3	3	-3	3	-3
χ_{50}	-2	-2	-2	-2	2	2	2	2	0	0	0	0	1	-1	1	-1	1	-1
χ_{51}	-2	-2	-2	-2	2	2	2	2	0	0	0	0	-1	1	-1	1	-1	1
χ_{52}	0	0	0	0	0	0	0	0	1	-1	1	-1	-3	3	-3	3	-3	3
χ_{53}	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	1	-1
χ_{54}	0	0	0	0	0	0	0	0	0	0	0	0	2	-2	2	-2	2	-2
χ_{55}	-1	-1	-1	-1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
χ_{56}	0	0	0	0	0	0	0	0	1	-1	1	-1	2	-2	2	-2	2	-2
χ_{57}	2	2	2	2	-2	-2	-2	-2	-2	2	-2	2	3	-3	3	-3	3	-3
χ_{58}	1	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0
χ_{59}	0	0	0	0	0	0	0	0	0	0	0	0	-4	4	-4	4	-4	4
χ_{60}	0	0	0	0	0	0	0	0	2	-2	2	-2	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$6B$				$6C$		$6D$						$6E$			
	$6K$	$6L$	$12D$	$12E$	$6M$	$6N$	$6O$	$12F$	$12G$	$6P$	$12H$	$12I$	$6Q$	$6R$	$6S$	$6T$
χ_{31}	1	-1	1	-1	1	-1	1	1	-1	-1	1	-1	1	-1	1	-1
χ_{32}	2	-2	2	-2	2	-2	0	0	0	0	0	0	-1	1	-1	1
χ_{33}	-2	2	-2	2	1	-1	0	0	0	0	0	0	1	-1	1	-1
χ_{34}	2	-2	2	-2	-1	1	2	2	-2	-2	2	-2	2	-2	2	-2
χ_{35}	0	0	0	0	3	-3	-2	-2	2	2	-2	2	0	0	0	0
χ_{36}	3	-3	3	-3	0	0	1	1	-1	-1	1	-1	0	0	0	0
χ_{37}	-3	3	-3	3	3	-3	1	1	-1	-1	1	-1	0	0	0	0
χ_{38}	-1	1	-1	1	-1	1	1	1	-1	-1	1	-1	2	-2	2	-2
χ_{39}	1	-1	1	-1	-2	2	-1	-1	1	1	-1	1	-2	2	-2	2
χ_{40}	-1	1	-1	1	-1	1	3	3	-3	-3	3	-3	-1	1	-1	1
χ_{41}	2	-2	2	-2	-1	1	-2	-2	2	2	-2	2	-1	1	-1	1
χ_{42}	1	-1	1	-1	1	-1	-1	-1	1	1	-1	1	1	-1	1	-1
χ_{43}	2	-2	2	-2	2	-2	0	0	0	0	0	0	-1	1	-1	1
χ_{44}	-4	4	-4	4	2	-2	0	0	0	0	0	0	-1	1	-1	1
χ_{45}	1	-1	1	-1	-2	2	-1	-1	1	1	-1	1	-2	2	-2	2
χ_{46}	2	-2	2	-2	2	-2	2	2	-2	-2	2	-2	-1	1	-1	1
χ_{47}	-3	3	-3	3	0	0	1	1	-1	-1	1	-1	0	0	0	0
χ_{48}	-3	3	-3	3	0	0	1	1	-1	-1	1	-1	0	0	0	0
χ_{49}	3	-3	3	-3	0	0	1	1	-1	-1	1	-1	0	0	0	0
χ_{50}	1	-1	1	-1	-2	2	1	1	-1	-1	1	-1	1	-1	1	-1
χ_{51}	-1	1	-1	1	-1	1	-1	-1	1	1	-1	1	2	-2	2	-2
χ_{52}	-3	3	-3	3	0	0	-1	-1	1	1	-1	1	0	0	0	0
χ_{53}	-3	3	-3	3	0	0	-1	-1	1	1	-1	1	0	0	0	0
χ_{54}	-2	2	-2	2	-2	2	-2	-2	2	2	-2	2	1	-1	1	-1
χ_{55}	0	0	0	0	3	-3	0	0	0	0	0	0	0	0	0	0
χ_{56}	-2	2	-2	2	-2	2	2	2	-2	-2	2	-2	-2	2	-2	2
χ_{57}	3	-3	3	-3	0	0	-1	-1	1	1	-1	1	0	0	0	0
χ_{58}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{59}	4	-4	4	-4	1	-1	0	0	0	0	0	0	1	-1	1	-1
χ_{60}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$6F$				$6G$				$7A$		$8A$				$8B$			
	$6U$	$12J$	$6V$	$12K$	$6W$	$12L$	$12M$	$6X$	$7A$	$14A$	$8I$	$8J$	$8K$	$8L$	$8M$	$8N$	$8O$	$8P$
χ_{31}	1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	-1
χ_{32}	1	1	-1	-1	-1	-1	1	1	0	0	1	1	-1	-1	-1	-1	1	1
χ_{33}	1	1	-1	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1	1	-1	-1
χ_{34}	-2	-2	2	2	0	0	0	0	0	0	-1	-1	1	1	-1	-1	1	1
χ_{35}	0	0	0	0	0	0	0	0	0	0	1	1	-1	-1	-1	-1	1	1
χ_{36}	0	0	0	0	0	0	0	0	-1	1	-1	-1	1	1	1	1	-1	-1
χ_{37}	-2	-2	2	2	0	0	0	0	0	0	1	1	-1	-1	1	1	-1	-1
χ_{38}	0	0	0	0	0	0	0	0	0	0	1	1	-1	-1	-1	-1	1	1
χ_{39}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{40}	-1	-1	1	1	1	1	-1	-1	0	0	0	0	0	0	0	0	0	0
χ_{41}	1	1	-1	-1	1	1	-1	-1	0	0	0	0	0	0	0	0	0	0
χ_{42}	1	1	-1	-1	1	1	-1	-1	0	0	-1	-1	1	1	1	1	-1	-1
χ_{43}	-1	-1	1	1	-1	-1	1	1	0	0	-1	-1	1	1	1	1	-1	-1
χ_{44}	1	1	-1	-1	1	1	-1	-1	0	0	-1	-1	1	1	-1	-1	1	1
χ_{45}	-2	-2	2	2	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
χ_{46}	1	1	-1	-1	-1	-1	1	1	0	0	0	0	0	0	0	0	0	0
χ_{47}	0	0	0	0	0	0	0	0	0	0	1	1	-1	-1	1	1	-1	-1
χ_{48}	0	0	0	0	0	0	0	0	0	0	-1	-1	1	1	-1	-1	1	1
χ_{49}	0	0	0	0	0	0	0	0	0	0	1	1	-1	-1	-1	-1	1	1
χ_{50}	1	1	-1	-1	-1	-1	1	1	0	0	0	0	0	0	0	0	0	0
χ_{51}	2	2	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{52}	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0
χ_{53}	-2	-2	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{54}	-1	-1	1	1	-1	-1	1	1	0	0	0	0	0	0	0	0	0	0
χ_{55}	0	0	0	0	0	0	0	0	0	0	-1	-1	1	1	-1	-1	1	1
χ_{56}	2	2	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{57}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{58}	0	0	0	0	0	0	0	0	-1	1	1	1	-1	-1	1	1	-1	-1
χ_{59}	-1	-1	1	1	1	1	-1	-1	0	0	0	0	0	0	0	0	0	0
χ_{60}	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$9A$		$10A$				$12A$				$12B$				$12C$		$15A$	
	$9A$	$18A$	$10D$	$20A$	$20B$	$10E$	$12N$	$24A$	$24B$	$12O$	$12P$	$24C$	$24D$	$12Q$	$12R$	$12S$	$15A$	$30A$
χ_{31}	1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	1	-1
χ_{32}	1	-1	0	0	0	0	-2	-2	2	2	0	0	0	0	0	0	-1	1
χ_{33}	0	0	0	0	0	0	0	0	0	0	-2	-2	2	2	-1	1	0	0
χ_{34}	0	0	-1	-1	1	1	0	0	0	0	0	0	0	0	1	-1	1	-1
χ_{35}	0	0	-1	-1	1	1	2	2	-2	-2	0	0	0	0	-1	1	1	-1
χ_{36}	0	0	0	0	0	0	1	1	-1	-1	-1	-1	1	1	0	0	-1	1
χ_{37}	-1	1	0	0	0	0	-1	-1	1	1	-1	-1	1	1	1	-1	0	0
χ_{38}	-1	1	0	0	0	0	-1	-1	1	1	1	1	-1	-1	-1	1	0	0
χ_{39}	-1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	0	0	1	-1
χ_{40}	1	-1	0	0	0	0	-1	-1	1	1	-1	-1	1	1	-1	1	0	0
χ_{41}	0	0	-1	-1	1	1	0	0	0	0	0	0	0	0	1	-1	-1	1
χ_{42}	0	0	0	0	0	0	-1	-1	1	1	1	1	-1	-1	-1	1	0	0
χ_{43}	0	0	0	0	0	0	0	0	0	0	2	2	-2	-2	0	0	0	0
χ_{44}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{45}	0	0	0	0	0	0	-1	-1	1	1	1	1	-1	-1	0	0	0	0
χ_{46}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-1
χ_{47}	0	0	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	0	0	-1	1
χ_{48}	0	0	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	0	0	-1	1
χ_{49}	0	0	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	0	0	-1	1
χ_{50}	0	0	0	0	0	0	1	1	-1	-1	1	1	-1	-1	0	0	0	0
χ_{51}	0	0	0	0	0	0	-1	-1	1	1	-1	-1	1	1	1	-1	0	0
χ_{52}	0	0	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	0	0	1	-1
χ_{53}	1	-1	0	0	0	0	1	1	-1	-1	-1	-1	1	1	0	0	0	0
χ_{54}	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{55}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0
χ_{56}	0	0	-1	-1	1	1	0	0	0	0	0	0	0	0	0	0	1	-1
χ_{57}	0	0	0	0	0	0	-1	-1	1	1	-1	-1	1	1	0	0	1	-1
χ_{58}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{59}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0
χ_{60}	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$1A$				$2A$						$2B$					
	$1A$	$2A$	$2B$	$2C$	$2D$	$4A$	$4B$	$2E$	$4C$	$2F$	$2G$	$2H$	$2I$	$2J$	$4D$	$4E$
χ_{61}	63	-9	7	-1	31	-9	7	-1	-1	-1	15	-9	-1	7	-1	-1
χ_{62}	63	-9	7	-1	-29	11	-5	3	-1	-1	-9	15	7	-1	-1	-1
χ_{63}	315	-45	35	-5	35	-5	11	3	-5	-5	-21	51	27	3	-5	-5
χ_{64}	315	-45	35	-5	-25	15	-1	7	-5	-5	51	-21	3	27	-5	-5
χ_{65}	315	-45	35	-5	95	-25	23	-1	-5	-5	3	27	19	11	-5	-5
χ_{66}	315	-45	35	-5	-85	35	-13	11	-5	-5	27	3	11	19	-5	-5
χ_{67}	378	-54	42	-6	114	-46	18	-14	6	6	-6	-6	-6	-6	2	2
χ_{68}	378	-54	42	-6	-126	34	-30	2	6	6	-6	-6	-6	-6	2	2
χ_{69}	567	-81	63	-9	-81	39	-9	15	-9	-9	-9	63	39	15	-9	-9
χ_{70}	567	-81	63	-9	99	-21	27	3	-9	-9	63	-9	15	39	-9	-9
χ_{71}	630	-90	70	-10	70	-10	22	6	-10	-10	6	54	38	22	-10	-10
χ_{72}	630	-90	70	-10	-130	30	-34	-2	10	10	54	-42	-10	22	-2	-2
χ_{73}	630	-90	70	-10	110	-50	14	-18	10	10	54	-42	-10	22	-2	-2
χ_{74}	630	-90	70	-10	110	-50	14	-18	10	10	-42	54	22	-10	-2	-2
χ_{75}	630	-90	70	-10	-50	30	-2	14	-10	-10	54	6	22	38	-10	-10
χ_{76}	630	-90	70	-10	-130	30	-34	-2	10	10	-42	54	22	-10	-2	-2
χ_{77}	945	-135	105	-15	225	-55	57	1	-15	-15	33	-39	-15	9	1	1
χ_{78}	945	-135	105	-15	-135	65	-15	25	-15	-15	33	-39	-15	9	1	1
χ_{79}	945	-135	105	-15	165	-35	45	5	-15	-15	-39	33	9	-15	1	1
χ_{80}	945	-135	105	-15	-195	85	-27	29	-15	-15	-39	33	9	-15	1	1
χ_{81}	1008	-144	112	-16	16	16	16	16	-16	-16	48	48	48	48	-16	-16
χ_{82}	1260	-180	140	-20	-20	-20	-20	-20	20	20	12	12	12	12	-4	-4
χ_{83}	1512	-216	168	-24	216	-104	24	-40	24	24	-24	-24	-24	-24	8	8
χ_{84}	1512	-216	168	-24	-264	56	-72	-8	24	24	-24	-24	-24	-24	8	8
χ_{85}	1890	-270	210	-30	90	10	42	26	-30	-30	66	-78	-30	18	2	2
χ_{86}	1890	-270	210	-30	-30	50	18	34	-30	-30	-78	66	18	-30	2	2
χ_{87}	1890	-270	210	-30	-150	10	-54	-22	30	30	-30	-30	-30	-30	10	10
χ_{88}	1890	-270	210	-30	90	-70	-6	-38	30	30	-30	-30	-30	-30	10	10
χ_{89}	2268	-324	252	-36	-36	-36	-36	-36	36	36	-36	-36	-36	-36	12	12
χ_{90}	2520	-360	280	-40	200	-120	8	-56	40	40	24	24	24	24	-8	-8
χ_{91}	2520	-360	280	-40	-40	-40	-40	-40	40	40	120	-72	-8	56	-8	-8
χ_{92}	2520	-360	280	-40	-280	40	-88	-24	40	40	24	24	24	24	-8	-8
χ_{93}	2520	-360	280	-40	-40	-40	-40	-40	40	40	-72	120	56	-8	-8	-8
χ_{94}	2835	-405	315	-45	-45	75	27	51	-45	-45	-45	27	3	-21	3	3
χ_{95}	2835	-405	315	-45	315	-45	99	27	-45	-45	-45	27	3	-21	3	3
χ_{96}	2835	-405	315	-45	-225	135	-9	63	-45	-45	27	-45	-21	3	3	3
χ_{97}	2835	-405	315	-45	135	15	63	39	-45	-45	27	-45	-21	3	3	3

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$2C$								$2D$							
	$2K$	$4F$	$2L$	$4G$	$4H$	$2M$	$4I$	$4J$	$2N$	$4K$	$4L$	$2O$	$4M$	$4N$	$4O$	$4P$
χ_{61}	15	-9	-1	7	-1	-1	-1	-1	7	-1	7	-1	-1	-1	-1	-1
χ_{62}	11	-13	-5	3	3	3	-1	-1	-1	-5	-1	-5	3	-1	-1	3
χ_{63}	19	-5	3	11	3	3	-5	-5	3	3	3	3	3	-5	-5	3
χ_{64}	15	-9	-1	7	7	7	-5	-5	3	-9	3	-9	-1	3	3	-1
χ_{65}	23	-1	7	15	-1	-1	-5	-5	11	7	11	7	-1	-5	-5	-1
χ_{66}	11	-13	-5	3	11	11	-5	-5	-5	-13	-5	-13	3	3	3	3
χ_{67}	18	-30	-14	2	10	10	-2	-2	-6	6	-6	6	-2	2	2	-2
χ_{68}	34	-14	2	18	-6	-6	-2	-2	-6	6	-6	6	-2	2	2	-2
χ_{69}	15	-9	-1	7	15	15	-9	-9	-9	-9	-9	-9	7	-1	-1	7
χ_{70}	27	3	11	19	3	3	-9	-9	15	3	15	3	-5	-1	-1	-5
χ_{71}	-10	38	22	6	6	6	-10	-10	-2	14	-2	14	-2	-2	-2	-2
χ_{72}	30	-18	-2	14	-10	-10	2	2	-10	10	-10	10	2	-2	-2	2
χ_{73}	14	-34	-18	-2	6	6	2	2	-10	10	-10	10	2	-2	-2	2
χ_{74}	14	-34	-18	-2	6	6	2	2	-2	2	-2	2	-6	6	6	-6
χ_{75}	-18	30	14	-2	14	14	-10	-10	-10	-2	-10	-2	-2	6	6	-2
χ_{76}	30	-18	-2	14	-10	-10	2	2	-2	2	-2	2	-6	6	6	-6
χ_{77}	49	-23	1	25	-15	-15	1	1	9	-15	9	-15	1	1	1	1
χ_{78}	-23	1	-7	-15	9	9	1	1	9	9	9	9	-7	1	1	-7
χ_{79}	-3	21	13	5	-11	-11	1	1	-15	-3	-15	-3	5	1	1	5
χ_{80}	21	-51	-27	-3	13	13	1	1	9	-3	9	-3	5	-7	-7	5
χ_{81}	16	16	16	16	16	16	-16	-16	0	0	0	0	0	0	0	0
χ_{82}	-52	44	12	-20	-4	-4	4	4	12	-12	12	-12	4	-4	-4	4
χ_{83}	-8	-8	-8	-8	24	24	-8	-8	0	0	0	0	0	0	0	0
χ_{84}	24	24	24	24	-8	-8	-8	-8	0	0	0	0	0	0	0	0
χ_{85}	26	-22	-6	10	-6	-6	2	2	18	-6	18	-6	-6	2	2	-6
χ_{86}	18	-30	-14	2	2	2	2	2	-6	-6	-6	-6	10	-6	-6	10
χ_{87}	42	-6	10	26	2	2	-10	-10	-6	6	-6	6	-2	2	2	-2
χ_{88}	26	-22	-6	10	18	18	-10	10	-6	6	-6	6	-2	2	2	-2
χ_{89}	-36	60	28	-4	12	12	-12	-12	12	-12	12	-12	4	-4	-4	4
χ_{90}	-24	-24	-24	-24	8	8	8	8	0	0	0	0	0	0	0	0
χ_{91}	-8	-8	-8	-8	-8	-8	8	8	-8	8	-8	8	8	-8	-8	8
χ_{92}	8	8	8	8	-24	-24	8	8	0	0	0	0	0	0	0	0
χ_{93}	-8	-8	-8	-8	-8	-8	8	8	8	-8	8	-8	-8	8	8	-8
χ_{94}	-45	27	3	-21	3	3	3	3	3	27	3	27	-5	-5	-5	-5
χ_{95}	27	3	11	19	-21	-21	3	3	3	3	3	3	3	-5	-5	3
χ_{96}	-9	-33	-25	-17	15	15	3	3	3	-9	3	-9	-1	3	3	-1
χ_{97}	-33	39	15	-9	-9	-9	3	3	-21	-9	-21	-9	-1	11	11	-1

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	3A				3B		3C				4A			
	3A	6A	6B	6C	3B	6D	3C	6E	6F	6G	4Q	4R	4S	4T
χ_{61}	15	-5	3	-1	0	0	3	-3	-1	1	3	3	-1	-1
χ_{62}	15	-5	3	-1	0	0	3	-3	-1	1	3	3	-1	-1
χ_{63}	-15	5	-3	1	0	0	6	-6	-2	2	3	3	-1	-1
χ_{64}	-15	5	-3	1	0	0	6	-6	-2	2	3	3	-1	-1
χ_{65}	30	-10	6	-2	0	0	-3	3	1	-1	3	3	-1	-1
χ_{66}	30	-10	6	-2	0	0	-3	3	1	-1	3	3	-1	-1
χ_{67}	45	-15	9	-3	0	0	0	0	0	0	6	6	-2	-2
χ_{68}	45	-15	9	-3	0	0	0	0	0	0	6	6	-2	-2
χ_{69}	0	0	0	0	0	0	0	0	0	0	3	3	-1	-1
χ_{70}	0	0	0	0	0	0	0	0	0	0	3	3	-1	-1
χ_{71}	15	-5	3	-1	0	0	3	-3	-1	1	-6	-6	2	2
χ_{72}	15	-5	3	-1	0	0	3	-3	-1	1	-6	-6	2	2
χ_{73}	15	-5	3	-1	0	0	3	-3	-1	1	-6	-6	2	2
χ_{74}	15	-5	3	-1	0	0	3	-3	-1	1	-6	-6	2	2
χ_{75}	15	-5	3	-1	0	0	3	-3	-1	1	-6	-6	2	2
χ_{76}	15	-5	3	-1	0	0	3	-3	-1	1	-6	-6	2	2
χ_{77}	45	-15	9	-3	0	0	0	0	0	0	-3	-3	1	1
χ_{78}	45	-15	9	-3	0	0	0	0	0	0	9	9	-3	-3
χ_{79}	45	-15	9	-3	0	0	0	0	0	0	9	9	-3	-3
χ_{80}	45	-15	9	-3	0	0	0	0	0	0	-3	-3	1	1
χ_{81}	-30	10	-6	2	0	0	-6	6	2	-2	0	0	0	0
χ_{82}	30	-10	6	-2	0	0	6	-6	-2	2	12	12	-4	-4
χ_{83}	45	-15	9	-3	0	0	0	0	0	0	0	0	0	0
χ_{84}	45	-15	9	-3	0	0	0	0	0	0	0	0	0	0
χ_{85}	-45	15	-9	3	0	0	0	0	0	0	6	6	-2	-2
χ_{86}	-45	15	-9	3	0	0	0	0	0	0	6	6	-2	-2
χ_{87}	-45	15	-9	3	0	0	0	0	0	0	6	6	-2	-2
χ_{88}	-45	15	-9	3	0	0	0	0	0	0	6	6	-2	-2
χ_{89}	0	0	0	0	0	0	0	0	0	0	-12	-12	4	4
χ_{90}	15	-5	3	-1	0	0	-6	6	2	-2	0	0	0	0
χ_{91}	-30	10	-6	2	0	0	3	-3	-1	1	0	0	0	0
χ_{92}	15	-5	3	-1	0	0	-6	6	2	-2	0	0	0	0
χ_{93}	-30	10	-6	2	0	0	3	-3	-1	1	0	0	0	0
χ_{94}	0	0	0	0	0	0	0	0	0	0	3	3	-1	-1
χ_{95}	0	0	0	0	0	0	0	0	0	0	-9	-9	3	3
χ_{96}	0	0	0	0	0	0	0	0	0	0	-9	-9	3	3
χ_{97}	0	0	0	0	0	0	0	0	0	0	3	3	-1	-1

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$4B$						$4C$						$4D$					
	$4U$	$4V$	$4W$	$4X$	$8A$	$8B$	$4Y$	$4Z$	$4AA$	$4AB$	$8C$	$8D$	$4AC$	$4AD$	$4AE$	$4AF$	$4AG$	$4AH$
χ_{61}	7	-5	-1	3	-1	-1	7	-5	-1	3	-1	-1	3	-1	3	-1	-1	-1
χ_{62}	5	-7	-3	1	1	1	-7	5	1	-3	1	1	-1	3	-1	3	-1	-1
χ_{63}	-5	7	3	-1	-1	-1	-5	7	3	-1	-1	-1	-5	7	-5	7	-1	-1
χ_{64}	-7	5	1	-3	1	1	5	-7	-3	1	1	1	7	-5	7	-5	-1	-1
χ_{65}	-3	9	5	1	-3	-3	9	-3	1	5	-3	-3	-1	3	-1	3	-1	-1
χ_{66}	-9	3	-1	-5	3	3	-9	3	-1	-5	3	3	3	-1	3	-1	-1	-1
χ_{67}	-2	-2	-2	-2	2	2	14	-10	-2	6	-2	-2	-2	-2	-2	-2	2	2
χ_{68}	2	2	2	2	-2	-2	-14	10	2	-6	2	2	-2	-2	-2	-2	2	2
χ_{69}	3	-9	-5	-1	3	3	3	-9	-5	-1	3	3	-5	7	-5	7	-1	-1
χ_{70}	9	-3	1	5	-3	-3	-3	9	5	1	-3	-3	7	-5	7	-5	-1	-1
χ_{71}	2	2	2	2	-2	-2	2	2	2	2	-2	-2	-6	2	-6	2	2	2
χ_{72}	-14	10	2	-6	2	2	2	2	2	2	-2	-2	2	2	2	2	-2	-2
χ_{73}	14	-10	-2	6	-2	-2	-2	-2	-2	-2	2	2	2	2	2	2	-2	-2
χ_{74}	-10	14	6	-2	-2	-2	-2	-2	-2	-2	2	2	2	2	2	2	-2	-2
χ_{75}	-2	-2	-2	-2	2	2	-2	-2	-2	-2	2	2	2	-6	2	-6	2	2
χ_{76}	10	-14	-6	2	2	2	2	2	2	2	-2	-2	2	2	2	2	-2	-2
χ_{77}	5	-7	-3	1	1	1	5	-7	-3	1	1	1	-3	1	-3	1	1	1
χ_{78}	-7	5	1	-3	1	1	-7	5	1	-3	1	1	1	5	1	5	-3	-3
χ_{79}	-5	7	3	-1	-1	-1	7	-5	-1	3	-1	-1	5	1	5	1	-3	-3
χ_{80}	7	-5	-1	3	-1	-1	-5	7	3	-1	-1	-1	1	-3	1	-3	1	1
χ_{81}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{82}	0	0	0	0	0	0	0	0	0	0	0	0	-4	-4	-4	-4	4	4
χ_{83}	-4	-4	-4	-4	4	4	4	4	4	4	-4	-4	0	0	0	0	0	0
χ_{84}	4	4	4	4	-4	-4	-4	-4	-4	-4	4	4	0	0	0	0	0	0
χ_{85}	-2	-2	-2	-2	2	2	-2	-2	-2	-2	2	2	-2	6	-2	6	-2	-2
χ_{86}	2	2	2	2	-2	-2	2	2	2	2	-2	-2	6	-2	6	-2	-2	-2
χ_{87}	2	2	2	2	-2	-2	10	-14	-6	2	2	2	-2	-2	-2	-2	2	2
χ_{88}	-2	-2	-2	-2	2	2	-10	14	6	-2	-2	-2	-2	-2	-2	-2	2	2
χ_{89}	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	4	-4	-4
χ_{90}	4	4	4	4	-4	-4	-4	-4	-4	-4	4	4	0	0	0	0	0	0
χ_{91}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{92}	-4	-4	-4	-4	4	4	4	4	4	4	-4	-4	0	0	0	0	0	0
χ_{93}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{94}	3	-9	-5	-1	3	3	3	-9	-5	-1	3	3	3	-1	3	-1	-1	-1
χ_{95}	-9	3	-1	-5	3	3	-9	3	-1	-5	3	3	-1	-5	-1	-5	3	3
χ_{96}	-3	9	5	1	-3	-3	9	-3	1	5	-3	-3	-5	-1	-5	-1	3	3
χ_{97}	9	-3	1	5	-3	-3	-3	9	5	1	-3	-3	-1	3	-1	3	-1	-1

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$4E$								$5A$				$6A$					
	$4AI$	$8E$	$8F$	$4AJ$	$4AK$	$8G$	$4AL$	$8H$	$5A$	$10A$	$10B$	$10C$	$6H$	$12A$	$6I$	$12B$	$12C$	$6J$
χ_{61}	3	-1	-1	-1	3	-1	-1	-1	3	-3	-1	1	7	-5	-1	3	-1	-1
χ_{62}	1	-3	1	1	1	1	1	-3	3	-3	-1	1	-5	7	3	-1	-1	-1
χ_{63}	-1	-1	-1	3	-1	-1	3	-1	0	0	0	0	5	-7	-3	1	1	1
χ_{64}	1	1	-3	1	1	-3	1	1	0	0	0	0	-7	5	1	-3	1	1
χ_{65}	1	1	-3	1	1	-3	1	1	0	0	0	0	2	2	2	2	-2	-2
χ_{66}	-1	-1	-1	3	-1	-1	3	-1	0	0	0	0	2	2	2	2	-2	-2
χ_{67}	-2	-2	2	2	-2	2	2	-2	3	-3	-1	1	3	-9	-5	-1	3	3
χ_{68}	2	2	-2	-2	2	-2	-2	2	3	-3	-1	1	-9	3	-1	-5	3	3
χ_{69}	3	-1	-1	-1	3	-1	-1	-1	-3	3	1	-1	0	0	0	0	0	0
χ_{70}	1	-3	1	1	1	1	1	-3	-3	3	1	-1	0	0	0	0	0	0
χ_{71}	-2	2	2	-2	-2	2	-2	2	0	0	0	0	-5	7	3	-1	-1	-1
χ_{72}	2	-2	2	-2	2	2	-2	-2	0	0	0	0	-7	5	1	-3	1	1
χ_{73}	-2	2	-2	2	-2	-2	2	2	0	0	0	0	5	-7	-3	1	1	1
χ_{74}	2	-2	2	-2	2	2	-2	-2	0	0	0	0	5	-7	-3	1	1	1
χ_{75}	-2	2	2	-2	-2	2	-2	2	0	0	0	0	7	-5	-1	3	-1	-1
χ_{76}	-2	2	-2	2	-2	-2	2	2	0	0	0	0	-7	5	1	-3	1	1
χ_{77}	1	1	1	-3	1	1	-3	1	0	0	0	0	9	-3	1	5	-3	-3
χ_{78}	-3	1	1	1	-3	1	1	1	0	0	0	0	9	-3	1	5	-3	-3
χ_{79}	-1	3	-1	-1	-1	-1	-1	3	0	0	0	0	-3	9	5	1	-3	-3
χ_{80}	-1	-1	3	-1	-1	3	-1	-1	0	0	0	0	-3	9	5	1	-3	-3
χ_{81}	0	0	0	0	0	0	0	0	3	-3	-1	1	-2	-2	-2	-2	2	2
χ_{82}	0	0	0	0	0	0	0	0	0	0	0	0	-2	-2	-2	-2	2	2
χ_{83}	0	0	0	0	0	0	0	0	-3	3	1	-1	-9	3	-1	-5	3	3
χ_{84}	0	0	0	0	0	0	0	0	-3	3	1	-1	3	-9	-5	-1	3	3
χ_{85}	-2	2	2	-2	-2	2	-2	2	0	0	0	0	-9	3	-1	-5	3	3
χ_{86}	-2	2	2	-2	-2	2	-2	2	0	0	0	0	3	-9	-5	-1	3	3
χ_{87}	-2	-2	2	2	-2	2	2	-2	0	0	0	0	-3	9	5	1	-3	-3
χ_{88}	2	2	-2	-2	2	-2	-2	2	0	0	0	0	9	-3	1	5	-3	-3
χ_{89}	0	0	0	0	0	0	0	0	3	-3	-1	1	0	0	0	0	0	0
χ_{90}	0	0	0	0	0	0	0	0	0	0	0	0	-7	5	1	-3	1	1
χ_{91}	0	0	0	0	0	0	0	0	0	0	0	0	2	2	2	2	-2	-2
χ_{92}	0	0	0	0	0	0	0	0	0	0	0	0	5	-7	-3	1	1	1
χ_{93}	0	0	0	0	0	0	0	0	0	0	0	0	2	2	2	2	-2	-2
χ_{94}	3	-1	-1	-1	3	-1	-1	-1	0	0	0	0	0	0	0	0	0	0
χ_{95}	-1	-1	-1	3	-1	-1	3	-1	0	0	0	0	0	0	0	0	0	0
χ_{96}	1	1	-3	1	1	-3	1	1	0	0	0	0	0	0	0	0	0	0
χ_{97}	1	-3	1	1	1	1	1	-3	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$6B$				$6C$		$6D$						$6E$			
	$6K$	$6L$	$12D$	$12E$	$6M$	$6N$	$6O$	$12F$	$12G$	$6P$	$12H$	$12I$	$6Q$	$6R$	$6S$	$6T$
χ_{61}	3	3	-1	-1	0	0	3	-1	3	-1	-1	-1	3	-3	-1	1
χ_{62}	3	3	-1	-1	0	0	-1	3	-1	3	-1	-1	-3	3	1	-1
χ_{63}	-3	-3	1	1	0	0	1	-3	1	-3	1	1	0	0	0	0
χ_{64}	-3	-3	1	1	0	0	-3	1	-3	1	1	1	0	0	0	0
χ_{65}	6	6	-2	-2	0	0	2	2	2	2	-2	-2	-3	3	1	-1
χ_{66}	6	6	-2	-2	0	0	2	2	2	2	-2	-2	3	-3	-1	1
χ_{67}	3	3	-1	-1	0	0	-3	1	-3	1	1	1	0	0	0	0
χ_{68}	3	3	-1	-1	0	0	1	-3	1	-3	1	1	0	0	0	0
χ_{69}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{70}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{71}	3	3	-1	-1	0	0	-1	3	-1	3	-1	-1	3	-3	-1	1
χ_{72}	-3	-3	1	1	0	0	3	-1	3	-1	-1	-1	3	-3	-1	1
χ_{73}	-3	-3	1	1	0	0	-1	3	-1	3	-1	-1	3	-3	-1	1
χ_{74}	-3	-3	1	1	0	0	-1	3	-1	3	-1	-1	-3	3	1	-1
χ_{75}	3	3	-1	-1	0	0	3	-1	3	-1	-1	-1	-3	3	1	-1
χ_{76}	-3	-3	1	1	0	0	3	-1	3	-1	-1	-1	-3	3	1	-1
χ_{77}	-3	-3	1	1	0	0	1	-3	1	-3	1	1	0	0	0	0
χ_{78}	-3	-3	1	1	0	0	1	-3	1	-3	1	1	0	0	0	0
χ_{79}	-3	-3	1	1	0	0	-3	1	-3	1	1	1	0	0	0	0
χ_{80}	-3	-3	1	1	0	0	-3	1	-3	1	1	1	0	0	0	0
χ_{81}	-6	-6	2	2	0	0	-2	-2	-2	-2	2	2	0	0	0	0
χ_{82}	-6	-6	2	2	0	0	2	2	2	2	-2	-2	0	0	0	0
χ_{83}	3	3	-1	-1	0	0	1	-3	1	-3	1	1	0	0	0	0
χ_{84}	3	3	-1	-1	0	0	-3	1	-3	1	1	1	0	0	0	0
χ_{85}	3	3	-1	-1	0	0	-1	3	-1	3	-1	-1	0	0	0	0
χ_{86}	3	3	-1	-1	0	0	3	-1	3	-1	-1	-1	0	0	0	0
χ_{87}	-3	-3	1	1	0	0	3	-1	3	-1	-1	-1	0	0	0	0
χ_{88}	-3	-3	1	1	0	0	-1	3	-1	3	-1	-1	0	0	0	0
χ_{89}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{90}	-3	-3	1	1	0	0	3	-1	3	-1	-1	-1	0	0	0	0
χ_{91}	6	6	-2	-2	0	0	-2	-2	-2	-2	2	2	-3	3	1	-1
χ_{92}	-3	-3	1	1	0	0	-1	3	-1	3	-1	-1	0	0	0	0
χ_{93}	6	6	-2	-2	0	0	-2	-2	-2	-2	2	2	3	-3	-1	1
χ_{94}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{95}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{96}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{97}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$6F$				$6G$				$7A$		$8A$				$8B$			
	$6U$	$12J$	$6V$	$12K$	$6W$	$12L$	$12M$	$6X$	$7A$	$14A$	$8I$	$8J$	$8K$	$8L$	$8M$	$8N$	$8O$	$8P$
χ_{61}	1	-1	-1	1	1	-1	1	-1	0	0	1	-1	-1	1	1	-1	-1	1
χ_{62}	1	-1	-1	1	-1	1	-1	1	0	0	1	-1	-1	1	-1	1	1	-1
χ_{63}	2	-2	-2	2	0	0	0	0	0	0	-1	1	1	-1	-1	1	1	-1
χ_{64}	2	-2	-2	2	0	0	0	0	0	0	-1	1	1	-1	1	-1	-1	1
χ_{65}	-1	1	1	-1	-1	1	-1	1	0	0	-1	1	1	-1	1	-1	-1	1
χ_{66}	-1	1	1	-1	1	-1	1	-1	0	0	-1	1	1	-1	-1	1	1	-1
χ_{67}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{68}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{69}	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	1	-1	-1	1
χ_{70}	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	-1	1	1	-1
χ_{71}	1	-1	-1	1	1	-1	1	-1	0	0	0	0	0	0	0	0	0	0
χ_{72}	-1	1	1	-1	-1	1	-1	1	0	0	0	0	0	0	0	0	0	0
χ_{73}	-1	1	1	-1	-1	1	-1	1	0	0	0	0	0	0	0	0	0	0
χ_{74}	-1	1	1	-1	1	-1	1	-1	0	0	0	0	0	0	0	0	0	0
χ_{75}	1	-1	-1	1	-1	1	-1	1	0	0	0	0	0	0	0	0	0	0
χ_{76}	-1	1	1	-1	1	-1	1	-1	0	0	0	0	0	0	0	0	0	0
χ_{77}	0	0	0	0	0	0	0	0	0	0	-1	1	1	-1	-1	1	1	-1
χ_{78}	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	1	-1	-1	1
χ_{79}	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	-1	1	1	-1
χ_{80}	0	0	0	0	0	0	0	0	0	0	-1	1	1	-1	1	-1	-1	1
χ_{81}	-2	2	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{82}	-2	2	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{83}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{84}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{85}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{86}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{87}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{88}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{89}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{90}	2	-2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{91}	-1	1	1	-1	1	-1	1	-1	0	0	0	0	0	0	0	0	0	0
χ_{92}	2	-2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{93}	-1	1	1	-1	-1	1	-1	1	0	0	0	0	0	0	0	0	0	0
χ_{94}	0	0	0	0	0	0	0	0	0	0	-1	1	1	-1	-1	1	1	-1
χ_{95}	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	1	-1	-1	1
χ_{96}	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	-1	1	1	-1
χ_{97}	0	0	0	0	0	0	0	0	0	0	-1	1	1	-1	1	-1	-1	1

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$9A$		$10A$				$12A$				$12B$				$12C$		$15A$	
	$9A$	$18A$	$10D$	$20A$	$20B$	$10E$	$12N$	$24A$	$24B$	$12O$	$12P$	$24C$	$24D$	$12Q$	$12R$	$12S$	$15A$	$30A$
χ_{61}	0	0	1	-1	1	-1	1	-1	-1	1	1	-1	-1	1	0	0	0	0
χ_{62}	0	0	1	-1	1	-1	-1	1	1	-1	-1	1	1	-1	0	0	0	0
χ_{63}	0	0	0	0	0	0	1	-1	-1	1	1	-1	-1	1	0	0	0	0
χ_{64}	0	0	0	0	0	0	-1	1	1	-1	-1	1	1	-1	0	0	0	0
χ_{65}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{66}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{67}	0	0	-1	1	-1	1	1	-1	-1	1	-1	1	1	-1	0	0	0	0
χ_{68}	0	0	-1	1	-1	1	-1	1	1	-1	1	-1	-1	1	0	0	0	0
χ_{69}	0	0	-1	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{70}	0	0	-1	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{71}	0	0	0	0	0	0	-1	1	1	-1	-1	1	1	-1	0	0	0	0
χ_{72}	0	0	0	0	0	0	1	-1	-1	1	-1	1	1	-1	0	0	0	0
χ_{73}	0	0	0	0	0	0	-1	1	1	-1	1	-1	-1	1	0	0	0	0
χ_{74}	0	0	0	0	0	0	-1	1	1	-1	1	-1	-1	1	0	0	0	0
χ_{75}	0	0	0	0	0	0	1	-1	-1	1	1	-1	-1	1	0	0	0	0
χ_{76}	0	0	0	0	0	0	1	-1	-1	1	-1	1	1	-1	0	0	0	0
χ_{77}	0	0	0	0	0	0	-1	1	1	-1	-1	1	1	-1	0	0	0	0
χ_{78}	0	0	0	0	0	0	-1	1	1	-1	-1	1	1	-1	0	0	0	0
χ_{79}	0	0	0	0	0	0	1	-1	-1	1	1	-1	-1	1	0	0	0	0
χ_{80}	0	0	0	0	0	0	1	-1	-1	1	1	-1	-1	1	0	0	0	0
χ_{81}	0	0	1	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{82}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{83}	0	0	1	-1	1	-1	-1	1	1	-1	1	-1	-1	1	0	0	0	0
χ_{84}	0	0	1	-1	1	-1	1	-1	-1	1	-1	1	1	-1	0	0	0	0
χ_{85}	0	0	0	0	0	0	1	-1	-1	1	1	-1	-1	1	0	0	0	0
χ_{86}	0	0	0	0	0	0	-1	1	1	-1	-1	1	1	-1	0	0	0	0
χ_{87}	0	0	0	0	0	0	-1	1	1	-1	1	-1	-1	1	0	0	0	0
χ_{88}	0	0	0	0	0	0	1	-1	-1	1	-1	1	1	-1	0	0	0	0
χ_{89}	0	0	-1	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{90}	0	0	0	0	0	0	1	-1	-1	1	-1	1	1	-1	0	0	0	0
χ_{91}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{92}	0	0	0	0	0	0	-1	1	1	-1	1	-1	-1	1	0	0	0	0
χ_{93}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{94}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{95}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{96}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{97}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$1A$				$2A$						$2B$					
	$1A$	$2A$	$2B$	$2C$	$2D$	$4A$	$4B$	$2E$	$4C$	$2F$	$2G$	$2H$	$2I$	$2J$	$4D$	$4E$
χ_{98}	63	9	-7	-1	31	9	-7	-1	-1	1	15	9	-1	-7	-1	1
χ_{99}	63	9	-7	-1	-29	-11	5	3	-1	1	-9	-15	7	1	-1	1
χ_{100}	315	45	-35	-5	35	5	-11	3	-5	5	-21	-51	27	-3	-5	5
χ_{101}	315	45	-35	-5	-25	-15	1	7	-5	5	51	21	3	-27	-5	5
χ_{102}	315	45	-35	-5	95	25	-23	-1	-5	5	3	-27	19	-11	-5	5
χ_{103}	315	45	-35	-5	-85	-35	13	11	-5	5	27	-3	11	-19	-5	5
χ_{104}	378	54	-42	-6	114	46	-18	-14	6	-6	-6	6	-6	6	2	-2
χ_{105}	378	54	-42	-6	-126	-34	30	2	6	-6	-6	6	-6	6	2	-2
χ_{106}	567	81	-63	-9	-81	-39	9	15	-9	9	-9	-63	39	-15	-9	9
χ_{107}	567	81	-63	-9	99	21	-27	3	-9	9	63	9	15	-39	-9	9
χ_{108}	630	90	-70	-10	70	10	-22	6	-10	10	6	-54	38	-22	-10	10
χ_{109}	630	90	-70	-10	-130	-30	34	-2	10	-10	54	42	-10	-22	-2	2
χ_{110}	630	90	-70	-10	110	50	-14	-18	10	-10	54	42	-10	-22	-2	2
χ_{111}	630	90	-70	-10	110	50	-14	-18	10	-10	-42	-54	22	10	-2	2
χ_{112}	630	90	-70	-10	-50	-30	2	14	-10	10	54	-6	22	-38	-10	10
χ_{113}	630	90	-70	-10	-130	-30	34	-2	10	-10	-42	-54	22	10	-2	2
χ_{114}	945	135	-105	-15	225	55	-57	1	-15	15	33	39	-15	-9	1	-1
χ_{115}	945	135	-105	-15	-135	-65	15	25	-15	15	33	39	-15	-9	1	-1
χ_{116}	945	135	-105	-15	165	35	-45	5	-15	15	-39	-33	9	15	1	-1
χ_{117}	945	135	-105	-15	-195	-85	27	29	-15	15	-39	-33	9	15	1	-1
χ_{118}	1008	144	-112	-16	16	-16	-16	16	-16	16	48	-48	48	-48	-16	16
χ_{119}	1260	180	-140	-20	-20	20	20	-20	20	-20	12	-12	12	-12	-4	4
χ_{120}	1512	216	-168	-24	216	104	-24	-40	24	-24	-24	24	-24	24	8	-8
χ_{121}	1512	216	-168	-24	-264	-56	72	-8	24	-24	-24	24	-24	24	8	-8
χ_{122}	1890	270	-210	-30	90	-10	-42	26	-30	30	66	78	-30	-18	2	-2
χ_{123}	1890	270	-210	-30	-30	-50	-18	34	-30	30	-78	-66	18	30	2	-2
χ_{124}	1890	270	-210	-30	-150	-10	54	-22	30	-30	-30	30	-30	30	10	-10
χ_{125}	1890	270	-210	-30	90	70	6	-38	30	-30	-30	30	-30	30	10	-10
χ_{126}	2268	324	-252	-36	-36	36	36	-36	36	-36	-36	36	-36	36	12	-12
χ_{127}	2520	360	-280	-40	200	120	-8	-56	40	-40	24	-24	24	-24	-8	8
χ_{128}	2520	360	-280	-40	-40	40	40	-40	40	-40	120	72	-8	-56	-8	8
χ_{129}	2520	360	-280	-40	-280	-40	88	-24	40	-40	24	-24	24	-24	-8	8
χ_{130}	2520	360	-280	-40	-40	40	40	-40	40	-40	-72	-120	56	8	-8	8
χ_{131}	2835	405	-315	-45	-45	-75	-27	51	-45	45	-45	-27	3	21	3	-3
χ_{132}	2835	405	-315	-45	315	45	-99	27	-45	45	-45	-27	3	21	3	-3
χ_{133}	2835	405	-315	-45	-225	-135	9	63	-45	45	27	45	-21	-3	3	-3
χ_{134}	2835	405	-315	-45	135	-15	-63	39	-45	45	27	45	-21	-3	3	-3

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$2C$								$2D$							
	$2K$	$4F$	$2L$	$4G$	$4H$	$2M$	$4I$	$4J$	$2N$	$4K$	$4L$	$2O$	$4M$	$4N$	$4O$	$4P$
χ_{98}	15	9	-1	-7	-1	1	-1	1	7	-1	-7	1	-1	-1	1	1
χ_{99}	11	13	-5	-3	3	-3	-1	1	-1	-5	1	5	3	-1	1	-3
χ_{100}	19	5	3	-11	3	-3	-5	5	3	3	-3	-3	3	-5	5	-3
χ_{101}	15	9	-1	-7	7	-7	-5	5	3	-9	-3	9	-1	3	-3	1
χ_{102}	23	1	7	-15	-1	1	-5	5	11	7	-11	-7	-1	-5	5	1
χ_{103}	11	13	-5	-3	11	-11	-5	5	-5	-13	5	13	3	3	-3	-3
χ_{104}	18	30	-14	-2	10	-10	-2	2	-6	6	6	-6	-2	2	-2	2
χ_{105}	34	14	2	-18	-6	6	-2	2	-6	6	6	-6	-2	2	-2	2
χ_{106}	15	9	-1	-7	15	-15	-9	9	-9	-9	9	9	7	-1	1	-7
χ_{107}	27	-3	11	-19	3	-3	-9	9	15	3	-15	-3	-5	-1	1	5
χ_{108}	-10	-38	22	-6	6	-6	-10	10	-2	14	2	-14	-2	-2	2	2
χ_{109}	30	18	-2	-14	-10	10	2	-2	-10	10	10	-10	2	-2	2	-2
χ_{110}	14	34	-18	2	6	-6	2	-2	-10	10	10	-10	2	-2	2	-2
χ_{111}	14	34	-18	2	6	-6	2	-2	-2	2	2	-2	-6	6	-6	6
χ_{112}	-18	-30	14	2	14	-14	-10	10	-10	-2	10	2	-2	6	-6	2
χ_{113}	30	18	-2	-14	-10	10	2	-2	-2	2	2	-2	-6	6	-6	6
χ_{114}	49	23	1	-25	-15	15	1	-1	9	-15	-9	15	1	1	-1	-1
χ_{115}	-23	-1	-7	15	9	-9	1	-1	9	9	-9	-9	-7	1	-1	7
χ_{116}	-3	-21	13	-5	-11	11	1	-1	-15	-3	15	3	5	1	-1	-5
χ_{117}	21	51	-27	3	13	-13	1	-1	9	-3	-9	3	5	-7	7	-5
χ_{118}	16	-16	16	-16	16	-16	-16	16	0	0	0	0	0	0	0	0
χ_{119}	-52	-44	12	20	-4	4	4	-4	12	-12	-12	12	4	-4	4	-4
χ_{120}	-8	8	-8	8	24	-24	-8	8	0	0	0	0	0	0	0	0
χ_{121}	24	-24	24	-24	-8	8	-8	8	0	0	0	0	0	0	0	0
χ_{122}	26	22	-6	-10	-6	6	2	-2	18	-6	-18	6	-6	2	-2	6
χ_{123}	18	30	-14	-2	2	-2	2	-2	-6	-6	6	6	10	-6	6	-10
χ_{124}	42	6	10	-26	2	-2	-10	10	-6	6	6	-6	-2	2	-2	2
χ_{125}	26	22	-6	-10	18	-18	-10	10	-6	6	6	-6	-2	2	-2	2
χ_{126}	-36	-60	28	4	12	-12	-12	12	12	-12	-12	12	4	-4	4	-4
χ_{127}	-24	24	-24	24	8	-8	8	-8	0	0	0	0	0	0	0	0
χ_{128}	-8	8	-8	8	-8	8	8	-8	-8	8	8	-8	8	-8	8	-8
χ_{129}	8	-8	8	-8	-24	24	8	-8	0	0	0	0	0	0	0	0
χ_{130}	-8	8	-8	8	-8	8	8	-8	8	-8	-8	8	-8	8	-8	8
χ_{131}	-45	-27	3	21	3	-3	3	-3	3	27	-3	-27	-5	-5	5	5
χ_{132}	27	-3	11	-19	-21	21	3	-3	3	3	-3	-3	3	-5	5	-3
χ_{133}	-9	33	-25	17	15	-15	3	-3	3	-9	-3	9	-1	3	-3	1
χ_{134}	-33	-39	15	9	-9	9	3	-3	-21	-9	21	9	-1	11	-11	1

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$3A$				$3B$		$3C$				$4A$			
	$3A$	$6A$	$6B$	$6C$	$3B$	$6D$	$3C$	$6E$	$6F$	$6G$	$4Q$	$4R$	$4S$	$4T$
χ_{98}	15	5	-3	-1	0	0	3	3	-1	-1	3	-3	-1	1
χ_{99}	15	5	-3	-1	0	0	3	3	-1	-1	3	-3	-1	1
χ_{100}	-15	-5	3	1	0	0	6	6	-2	-2	3	-3	-1	1
χ_{101}	-15	-5	3	1	0	0	6	6	-2	-2	3	-3	-1	1
χ_{102}	30	10	-6	-2	0	0	-3	-3	1	1	3	-3	-1	1
χ_{103}	30	10	-6	-2	0	0	-3	-3	1	1	3	-3	-1	1
χ_{104}	45	15	-9	-3	0	0	0	0	0	0	6	-6	-2	2
χ_{105}	45	15	-9	-3	0	0	0	0	0	0	6	-6	-2	2
χ_{106}	0	0	0	0	0	0	0	0	0	0	3	-3	-1	1
χ_{107}	0	0	0	0	0	0	0	0	0	0	3	-3	-1	1
χ_{108}	15	5	-3	-1	0	0	3	3	-1	-1	-6	6	2	-2
χ_{109}	15	5	-3	-1	0	0	3	3	-1	-1	-6	6	2	-2
χ_{110}	15	5	-3	-1	0	0	3	3	-1	-1	-6	6	2	-2
χ_{111}	15	5	-3	-1	0	0	3	3	-1	-1	-6	6	2	-2
χ_{112}	15	5	-3	-1	0	0	3	3	-1	-1	-6	6	2	-2
χ_{113}	15	5	-3	-1	0	0	3	3	-1	-1	-6	6	2	-2
χ_{114}	45	15	-9	-3	0	0	0	0	0	0	-3	3	1	-1
χ_{115}	45	15	-9	-3	0	0	0	0	0	0	9	-9	-3	3
χ_{116}	45	15	-9	-3	0	0	0	0	0	0	9	-9	-3	3
χ_{117}	45	15	-9	-3	0	0	0	0	0	0	-3	3	1	-1
χ_{118}	-30	-10	6	2	0	0	-6	-6	2	2	0	0	0	0
χ_{119}	30	10	-6	-2	0	0	6	6	-2	-2	12	-12	-4	4
χ_{120}	45	15	-9	-3	0	0	0	0	0	0	0	0	0	0
χ_{121}	45	15	-9	-3	0	0	0	0	0	0	0	0	0	0
χ_{122}	-45	-15	9	3	0	0	0	0	0	0	6	-6	-2	2
χ_{123}	-45	-15	9	3	0	0	0	0	0	0	6	-6	-2	2
χ_{124}	-45	-15	9	3	0	0	0	0	0	0	6	-6	-2	2
χ_{125}	-45	-15	9	3	0	0	0	0	0	0	6	-6	-2	2
χ_{126}	0	0	0	0	0	0	0	0	0	0	-12	12	4	-4
χ_{127}	15	5	-3	-1	0	0	-6	-6	2	2	0	0	0	0
χ_{128}	-30	-10	6	2	0	0	3	3	-1	-1	0	0	0	0
χ_{129}	15	5	-3	-1	0	0	-6	-6	2	2	0	0	0	0
χ_{130}	-30	-10	6	2	0	0	3	3	-1	-1	0	0	0	0
χ_{131}	0	0	0	0	0	0	0	0	0	0	3	-3	-1	1
χ_{132}	0	0	0	0	0	0	0	0	0	0	-9	9	3	-3
χ_{133}	0	0	0	0	0	0	0	0	0	0	-9	9	3	-3
χ_{134}	0	0	0	0	0	0	0	0	0	0	3	-3	-1	1

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$4B$						$4C$						$4D$					
	$4U$	$4V$	$4W$	$4X$	$8A$	$8B$	$4Y$	$4Z$	$4AA$	$4AB$	$8C$	$8D$	$4AC$	$4AD$	$4AE$	$4AF$	$4AG$	$4AH$
χ_{98}	7	5	-1	-3	-1	1	7	5	-1	-3	-1	1	3	-1	-3	1	-1	1
χ_{99}	5	7	-3	-1	1	-1	-7	-5	1	3	1	-1	-1	3	1	-3	-1	1
χ_{100}	-5	-7	3	1	-1	1	-5	-7	3	1	-1	1	-5	7	5	-7	-1	1
χ_{101}	-7	-5	1	3	1	-1	5	7	-3	-1	1	-1	7	-5	-7	5	-1	1
χ_{102}	-3	-9	5	-1	-3	3	9	3	1	-5	-3	3	-1	3	1	-3	-1	1
χ_{103}	-9	-3	-1	5	3	-3	-9	-3	-1	5	3	-3	3	-1	-3	1	-1	1
χ_{104}	-2	2	-2	2	2	-2	14	10	-2	-6	-2	2	-2	-2	2	2	2	-2
χ_{105}	2	-2	2	-2	-2	2	-14	-10	2	6	2	-2	-2	-2	2	2	2	-2
χ_{106}	3	9	-5	1	3	-3	3	9	-5	1	3	-3	-5	7	5	-7	-1	1
χ_{107}	9	3	1	-5	-3	3	-3	-9	5	-1	-3	3	7	-5	-7	5	-1	1
χ_{108}	2	-2	2	-2	-2	2	2	-2	2	-2	-2	2	-6	2	6	-2	2	-2
χ_{109}	-14	-10	2	6	2	-2	2	-2	2	-2	-2	2	2	2	-2	-2	-2	2
χ_{110}	14	10	-2	-6	-2	2	-2	2	-2	2	2	-2	2	2	-2	-2	-2	2
χ_{111}	-10	-14	6	2	-2	2	-2	2	-2	2	2	-2	2	2	-2	-2	-2	2
χ_{112}	-2	2	-2	2	2	-2	-2	2	-2	2	2	-2	2	-6	-2	6	2	-2
χ_{113}	10	14	-6	-2	2	-2	2	-2	2	-2	-2	2	2	2	-2	-2	-2	2
χ_{114}	5	7	-3	-1	1	-1	5	7	-3	-1	1	-1	-3	1	3	-1	1	-1
χ_{115}	-7	-5	1	3	1	-1	-7	-5	1	3	1	-1	1	5	-1	-5	-3	3
χ_{116}	-5	-7	3	1	-1	1	7	5	-1	-3	-1	1	5	1	-5	-1	-3	3
χ_{117}	7	5	-1	-3	-1	1	-5	-7	3	1	-1	1	1	-3	-1	3	1	-1
χ_{118}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{119}	0	0	0	0	0	0	0	0	0	0	0	0	-4	-4	4	4	4	-4
χ_{120}	-4	4	-4	4	4	-4	4	-4	4	-4	-4	4	0	0	0	0	0	0
χ_{121}	4	-4	4	-4	-4	4	-4	4	-4	4	4	-4	0	0	0	0	0	0
χ_{122}	-2	2	-2	2	2	-2	-2	2	-2	2	2	-2	-2	6	2	-6	-2	2
χ_{123}	2	-2	2	-2	-2	2	2	-2	2	-2	-2	2	6	-2	-6	2	-2	2
χ_{124}	2	-2	2	-2	-2	2	10	14	-6	-2	2	-2	-2	-2	2	2	2	-2
χ_{125}	-2	2	-2	2	2	-2	-10	-14	6	2	-2	2	-2	-2	2	2	2	-2
χ_{126}	0	0	0	0	0	0	0	0	0	0	0	0	4	4	-4	-4	-4	4
χ_{127}	4	-4	4	-4	-4	4	-4	4	-4	4	4	-4	0	0	0	0	0	0
χ_{128}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{129}	-4	4	-4	4	4	-4	4	-4	4	-4	-4	4	0	0	0	0	0	0
χ_{130}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{131}	3	9	-5	1	3	-3	3	9	-5	1	3	-3	3	-1	-3	1	-1	1
χ_{132}	-9	-3	-1	5	3	-3	-9	-3	-1	5	3	-3	-1	-5	1	5	3	-3
χ_{133}	-3	-9	5	-1	-3	3	9	3	1	-5	-3	3	-5	-1	5	1	3	-3
χ_{134}	9	3	1	-5	-3	3	-3	-9	5	-1	-3	3	-1	3	1	-3	-1	1

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$4E$								$5A$				$6A$					
	$4AI$	$8E$	$8F$	$4AJ$	$4AK$	$8G$	$4AL$	$8H$	$5A$	$10A$	$10B$	$10C$	$6H$	$12A$	$6I$	$12B$	$12C$	$6J$
χ_{98}	3	-1	-1	-1	-3	1	1	1	3	3	-1	-1	7	5	-1	-3	-1	1
χ_{99}	1	-3	1	1	-1	-1	-1	3	3	3	-1	-1	-5	-7	3	1	-1	1
χ_{100}	-1	-1	-1	3	1	1	-3	1	0	0	0	0	5	7	-3	-1	1	-1
χ_{101}	1	1	-3	1	-1	3	-1	-1	0	0	0	0	-7	-5	1	3	1	-1
χ_{102}	1	1	-3	1	-1	3	-1	-1	0	0	0	0	2	-2	2	-2	-2	2
χ_{103}	-1	-1	-1	3	1	1	-3	1	0	0	0	0	2	-2	2	-2	-2	2
χ_{104}	-2	-2	2	2	2	-2	-2	2	3	3	-1	-1	3	9	-5	1	3	-3
χ_{105}	2	2	-2	-2	-2	2	2	-2	3	3	-1	-1	-9	-3	-1	5	3	-3
χ_{106}	3	-1	-1	-1	-3	1	1	1	-3	-3	1	1	0	0	0	0	0	0
χ_{107}	1	-3	1	1	-1	-1	-1	3	-3	-3	1	1	0	0	0	0	0	0
χ_{108}	-2	2	2	-2	2	-2	2	-2	0	0	0	0	-5	-7	3	1	-1	1
χ_{109}	2	-2	2	-2	-2	-2	2	2	0	0	0	0	-7	-5	1	3	1	-1
χ_{110}	-2	2	-2	2	2	2	-2	-2	0	0	0	0	5	7	-3	-1	1	-1
χ_{111}	2	-2	2	-2	-2	-2	2	2	0	0	0	0	5	7	-3	-1	1	-1
χ_{112}	-2	2	2	-2	2	-2	2	-2	0	0	0	0	7	5	-1	-3	-1	1
χ_{113}	-2	2	-2	2	2	2	-2	-2	0	0	0	0	-7	-5	1	3	1	-1
χ_{114}	1	1	1	-3	-1	-1	3	-1	0	0	0	0	9	3	1	-5	-3	3
χ_{115}	-3	1	1	1	3	-1	-1	-1	0	0	0	0	9	3	1	-5	-3	3
χ_{116}	-1	3	-1	-1	1	1	1	-3	0	0	0	0	-3	-9	5	-1	-3	3
χ_{117}	-1	-1	3	-1	1	-3	1	1	0	0	0	0	-3	-9	5	-1	-3	3
χ_{118}	0	0	0	0	0	0	0	0	3	3	-1	-1	-2	2	-2	2	2	-2
χ_{119}	0	0	0	0	0	0	0	0	0	0	0	0	-2	2	-2	2	2	-2
χ_{120}	0	0	0	0	0	0	0	0	-3	-3	1	1	-9	-3	-1	5	3	-3
χ_{121}	0	0	0	0	0	0	0	0	-3	-3	1	1	3	9	-5	1	3	-3
χ_{122}	-2	2	2	-2	2	-2	2	-2	0	0	0	0	-9	-3	-1	5	3	-3
χ_{123}	-2	2	2	-2	2	-2	2	-2	0	0	0	0	3	9	-5	1	3	-3
χ_{124}	-2	-2	2	2	2	-2	-2	2	0	0	0	0	-3	-9	5	-1	-3	3
χ_{125}	2	2	-2	-2	-2	2	2	-2	0	0	0	0	9	3	1	-5	-3	3
χ_{126}	0	0	0	0	0	0	0	0	3	3	-1	-1	0	0	0	0	0	0
χ_{127}	0	0	0	0	0	0	0	0	0	0	0	0	-7	-5	1	3	1	-1
χ_{128}	0	0	0	0	0	0	0	0	0	0	0	0	2	-2	2	-2	-2	2
χ_{129}	0	0	0	0	0	0	0	0	0	0	0	0	5	7	-3	-1	1	-1
χ_{130}	0	0	0	0	0	0	0	0	0	0	0	0	2	-2	2	-2	-2	2
χ_{131}	3	-1	-1	-1	-3	1	1	1	0	0	0	0	0	0	0	0	0	0
χ_{132}	-1	-1	-1	3	1	1	-3	1	0	0	0	0	0	0	0	0	0	0
χ_{133}	1	1	-3	1	-1	3	-1	-1	0	0	0	0	0	0	0	0	0	0
χ_{134}	1	-3	1	1	-1	-1	-1	3	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$6B$				$6C$		$6D$						$6E$			
	$6K$	$6L$	$12D$	$12E$	$6M$	$6N$	$6O$	$12F$	$12G$	$6P$	$12H$	$12I$	$6Q$	$6R$	$6S$	$6T$
χ_{98}	3	-3	-1	1	0	0	3	-1	-3	1	-1	1	3	3	-1	-1
χ_{99}	3	-3	-1	1	0	0	-1	3	1	-3	-1	1	-3	-3	1	1
χ_{100}	-3	3	1	-1	0	0	1	-3	-1	3	1	-1	0	0	0	0
χ_{101}	-3	3	1	-1	0	0	-3	1	3	-1	1	-1	0	0	0	0
χ_{102}	6	-6	-2	2	0	0	2	2	-2	-2	-2	2	-3	-3	1	1
χ_{103}	6	-6	-2	2	0	0	2	2	-2	-2	-2	2	3	3	-1	-1
χ_{104}	3	-3	-1	1	0	0	-3	1	3	-1	1	-1	0	0	0	0
χ_{105}	3	-3	-1	1	0	0	1	-3	-1	3	1	-1	0	0	0	0
χ_{106}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{107}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{108}	3	-3	-1	1	0	0	-1	3	1	-3	-1	1	3	3	-1	-1
χ_{109}	-3	3	1	-1	0	0	3	-1	-3	1	-1	1	3	3	-1	-1
χ_{110}	-3	3	1	-1	0	0	-1	3	1	-3	-1	1	3	3	-1	-1
χ_{111}	-3	3	1	-1	0	0	-1	3	1	-3	-1	1	-3	-3	1	1
χ_{112}	3	-3	-1	1	0	0	3	-1	-3	1	-1	1	-3	-3	1	1
χ_{113}	-3	3	1	-1	0	0	3	-1	-3	1	-1	1	-3	-3	1	1
χ_{114}	-3	3	1	-1	0	0	1	-3	-1	3	1	-1	0	0	0	0
χ_{115}	-3	3	1	-1	0	0	1	-3	-1	3	1	-1	0	0	0	0
χ_{116}	-3	3	1	-1	0	0	-3	1	3	-1	1	-1	0	0	0	0
χ_{117}	-3	3	1	-1	0	0	-3	1	3	-1	1	-1	0	0	0	0
χ_{118}	-6	6	2	-2	0	0	-2	-2	2	2	2	-2	0	0	0	0
χ_{119}	-6	6	2	-2	0	0	2	2	-2	-2	-2	2	0	0	0	0
χ_{120}	3	-3	-1	1	0	0	1	-3	-1	3	1	-1	0	0	0	0
χ_{121}	3	-3	-1	1	0	0	-3	1	3	-1	1	-1	0	0	0	0
χ_{122}	3	-3	-1	1	0	0	-1	3	1	-3	-1	1	0	0	0	0
χ_{123}	3	-3	-1	1	0	0	3	-1	-3	1	-1	1	0	0	0	0
χ_{124}	-3	3	1	-1	0	0	3	-1	-3	1	-1	1	0	0	0	0
χ_{125}	-3	3	1	-1	0	0	-1	3	1	-3	-1	1	0	0	0	0
χ_{126}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{127}	-3	3	1	-1	0	0	3	-1	-3	1	-1	1	0	0	0	0
χ_{128}	6	-6	-2	2	0	0	-2	-2	2	2	2	-2	-3	-3	1	1
χ_{129}	-3	3	1	-1	0	0	-1	3	1	-3	-1	1	0	0	0	0
χ_{130}	6	-6	-2	2	0	0	-2	-2	2	2	2	-2	3	3	-1	-1
χ_{131}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{132}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{133}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{134}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$6F$				$6G$				$7A$		$8A$				$8B$			
	$6U$	$12J$	$6V$	$12K$	$6W$	$12L$	$12M$	$6X$	$7A$	$14A$	$8I$	$8J$	$8K$	$8L$	$8M$	$8N$	$8O$	$8P$
χ_{98}	1	-1	1	-1	1	-1	-1	1	0	0	1	-1	1	-1	1	-1	1	-1
χ_{99}	1	-1	1	-1	-1	1	1	-1	0	0	1	-1	1	-1	-1	1	-1	1
χ_{100}	2	-2	2	-2	0	0	0	0	0	0	-1	1	-1	1	-1	1	-1	1
χ_{101}	2	-2	2	-2	0	0	0	0	0	0	-1	1	-1	1	1	-1	1	-1
χ_{102}	-1	1	-1	1	-1	1	1	-1	0	0	-1	1	-1	1	1	-1	1	-1
χ_{103}	-1	1	-1	1	1	-1	-1	1	0	0	-1	1	-1	1	-1	1	-1	1
χ_{104}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{105}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{106}	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	1	-1	1	-1
χ_{107}	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	-1	1	-1	1
χ_{108}	1	-1	1	-1	1	-1	-1	1	0	0	0	0	0	0	0	0	0	0
χ_{109}	-1	1	-1	1	-1	1	1	-1	0	0	0	0	0	0	0	0	0	0
χ_{110}	-1	1	-1	1	-1	1	1	-1	0	0	0	0	0	0	0	0	0	0
χ_{111}	-1	1	-1	1	1	-1	-1	1	0	0	0	0	0	0	0	0	0	0
χ_{112}	1	-1	1	-1	-1	1	1	-1	0	0	0	0	0	0	0	0	0	0
χ_{113}	-1	1	-1	1	1	-1	-1	1	0	0	0	0	0	0	0	0	0	0
χ_{114}	0	0	0	0	0	0	0	0	0	0	-1	1	-1	1	-1	1	-1	1
χ_{115}	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	1	-1	1	-1
χ_{116}	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	-1	1	-1	1
χ_{117}	0	0	0	0	0	0	0	0	0	0	-1	1	-1	1	1	-1	1	-1
χ_{118}	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{119}	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{120}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{121}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{122}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{123}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{124}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{125}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{126}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{127}	2	-2	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{128}	-1	1	-1	1	1	-1	-1	1	0	0	0	0	0	0	0	0	0	0
χ_{129}	2	-2	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{130}	-1	1	-1	1	-1	1	1	-1	0	0	0	0	0	0	0	0	0	0
χ_{131}	0	0	0	0	0	0	0	0	0	0	-1	1	-1	1	-1	1	-1	1
χ_{132}	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	1	-1	1	-1
χ_{133}	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	-1	1	-1	1
χ_{134}	0	0	0	0	0	0	0	0	0	0	-1	1	-1	1	1	-1	1	-1

Table 8.4: The character table of $2^7{:}SP(6,2)$(continued)

	$9A$		$10A$				$12A$				$12B$				$12C$		$15A$	
	$9A$	$18A$	$10D$	$20A$	$20B$	$10E$	$12N$	$24A$	$24B$	$12O$	$12P$	$24C$	$24D$	$12Q$	$12R$	$12S$	$15A$	$30A$
χ_{98}	0	0	1	-1	-1	1	1	-1	1	-1	1	-1	1	-1	0	0	0	0
χ_{99}	0	0	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	0	0	0	0
χ_{100}	0	0	0	0	0	0	1	-1	1	-1	1	-1	1	-1	0	0	0	0
χ_{101}	0	0	0	0	0	0	-1	1	-1	1	-1	1	-1	1	0	0	0	0
χ_{102}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{103}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{104}	0	0	-1	1	1	-1	1	-1	1	-1	-1	1	-1	1	0	0	0	0
χ_{105}	0	0	-1	1	1	-1	-1	1	-1	1	1	-1	1	-1	0	0	0	0
χ_{106}	0	0	-1	1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{107}	0	0	-1	1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{108}	0	0	0	0	0	0	-1	1	-1	1	-1	1	-1	1	0	0	0	0
χ_{109}	0	0	0	0	0	0	1	-1	1	-1	-1	1	-1	1	0	0	0	0
χ_{110}	0	0	0	0	0	0	-1	1	-1	1	1	-1	1	-1	0	0	0	0
χ_{111}	0	0	0	0	0	0	-1	1	-1	1	1	-1	1	-1	0	0	0	0
χ_{112}	0	0	0	0	0	0	1	-1	1	-1	1	-1	1	-1	0	0	0	0
χ_{113}	0	0	0	0	0	0	1	-1	1	-1	-1	1	-1	1	0	0	0	0
χ_{114}	0	0	0	0	0	0	-1	1	-1	1	-1	1	-1	1	0	0	0	0
χ_{115}	0	0	0	0	0	0	-1	1	-1	1	-1	1	-1	1	0	0	0	0
χ_{116}	0	0	0	0	0	0	1	-1	1	-1	1	-1	1	-1	0	0	0	0
χ_{117}	0	0	0	0	0	0	1	-1	1	-1	1	-1	1	-1	0	0	0	0
χ_{118}	0	0	1	-1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{119}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{120}	0	0	1	-1	-1	1	-1	1	-1	1	1	-1	1	-1	0	0	0	0
χ_{121}	0	0	1	-1	-1	1	1	-1	1	-1	-1	1	-1	1	0	0	0	0
χ_{122}	0	0	0	0	0	0	1	-1	1	-1	1	-1	1	-1	0	0	0	0
χ_{123}	0	0	0	0	0	0	-1	1	-1	1	-1	1	-1	1	0	0	0	0
χ_{124}	0	0	0	0	0	0	-1	1	-1	1	1	-1	1	-1	0	0	0	0
χ_{125}	0	0	0	0	0	0	1	-1	1	-1	-1	1	-1	1	0	0	0	0
χ_{126}	0	0	-1	1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{127}	0	0	0	0	0	0	1	-1	1	-1	-1	1	-1	1	0	0	0	0
χ_{128}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{129}	0	0	0	0	0	0	-1	1	-1	1	1	-1	1	-1	0	0	0	0
χ_{130}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{131}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{132}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{133}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{134}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

8.6 The fusion of $2^7{:}SP(6,2)$ into $\overline{Fi}_{22}$

We use the results of Section 8.2 to compute the power maps of elements of $2^7{:}SP(6,2)$ which are listed in Table 8.5 below.

Table 8.5: The power maps of the elements of $2^7{:}SP(6,2)$

$[g]_{SP(6,2)}$	$[x]_{2^7:SP(6,2)}$	2	3	5	7
$1A$	$1A$				
	$2A$	$1A$			
	$2B$	$1A$			
	$2C$	$1A$			
$2A$	$2D$	$1A$			
	$4A$	$2C$			
	$4B$	$2C$			
	$2E$	$1A$			
	$4C$	$2C$			
	$2F$	$1A$			
$2B$	$2G$	$1A$			
	$2H$	$1A$			
	$2I$	$1A$			
	$2J$	$1A$			
	$4D$	$2C$			
	$4D$	$2C$			
$2C$	$2K$	$1A$			
	$4F$	$2C$			
	$2L$	$1A$			
	$4G$	$2C$			
	$4H$	$2C$			
	$2M$	$1A$			
	$4I$	$2C$			
	$4J$	$2C$			
$2D$	$2N$	$1A$			
	$4K$	$2C$			
	$4L$	$2C$			
	$2O$	$1A$			
	$4M$	$2C$			
	$4N$	$2C$			
	$4O$	$2C$			
	$4P$	$2C$			
$3A$	$3A$		$1A$		
	$6A$	$3A$	$2B$		
	$6B$	$3A$	$2A$		
	$6C$	$3A$	$2C$		
$3B$	$3B$		$1A$		
	$6D$	$3B$	$2A$		
$3C$	$3C$		$1A$		
	$6E$	$3C$	$2A$		
	$6F$	$3C$	$2C$		
	$6G$	$3C$	$2B$		
$4A$	$4Q$	$2G$			
	$4R$	$2G$			
	$4S$	$2I$			
	$4T$	$2I$			
$4B$	$4U$	$2K$			
	$4V$	$2K$			
	$4W$	$2K$			
	$4X$	$2K$			
	$8A$	$4H$			
	$8B$	$4H$			
$4C$	$4Y$	$2K$			
	$4Z$	$2K$			
	$4AA$	$2K$			
	$4AB$	$2K$			
	$8C$	$4H$			
	$8D$	$4H$			
$4D$	$4AC$	$2G$			
	$4AD$	$2I$			
	$4AE$	$2G$			
	$4AF$	$2I$			
	$4AG$	$2I$			
	$4AH$	$2I$			
$4E$	$4AI$	$2K$			
	$8E$	$4H$			
	$8F$	$4H$			
	$4AJ$	$2L$			
	$4AK$	$2L$			
	$8G$	$4H$			
	$4AL$	$2K$			
	$8H$	$4H$			
$5A$	$5A$			$1A$	
	$10A$	$5A$		$2B$	
	$10B$	$5A$		$2C$	
	$10C$	$5A$		$2A$	

Table 8.5: The power maps of the elements of $2^7{:}SP(6,2)$(continued)

$[g]_{SP(6,2)}$	$[x]_{2^7:SP(6,2)}$	2	3	5	7	$[g]_{SP(6,2)}$	$[x]_{2^7:SP(6,2)}$	2	3	5	7
$6A$	$6H$	$3A$	$2D$			$6B$	$6K$	$3A$	$2G$		
	$12A$	$6C$	$4B$				$6L$	$3A$	$2H$		
	$6I$	$3A$	$2E$				$12D$	$6C$	$4D$		
	$12B$	$6C$	$4A$				$12E$	$6C$	$4E$		
	$12C$	$6C$	$4C$								
	$6J$	$3A$	$2F$								
$6C$	$6M$	$3B$	$2G$			$6D$	$6O$	$3A$	$2K$		
	$6N$	$3B$	$2H$				$12F$	$6C$	$4H$		
							$12G$	$6C$	$4F$		
							$6P$	$3A$	$2M$		
							$12H$	$6C$	$4I$		
							$12I$	$6C$	$4J$		
$6E$	$6Q$	$3C$	$2G$			$6F$	$6U$	$3C$	$2D$		
	$6R$	$3C$	$2H$				$12J$	$6F$	$4C$		
	$6S$	$3C$	$2I$				$6V$	$3C$	$2F$		
	$6T$	$3C$	$2J$				$12K$	$6F$	$4B$		
$6G$	$6W$	$3C$	$2N$			$7A$	$7A$				$1A$
	$12L$	$6F$	$4K$				$14A$	$7A$			$2B$
	$12M$	$6F$	$4L$								
	$6X$	$3C$	$2O$								
$8A$	$8I$	$4AC$				$8B$	$8M$	$4Q$			
	$8J$	$4AD$					$8N$	$4S$			
	$8K$	$4AD$					$8O$	$4S$			
	$8L$	$4AC$					$8P$	$4Q$			
$9A$	$9A$		$3B$			$10A$	$10D$	$5A$		$2D$	
	$18A$	$9A$	$6D$				$20A$	$10B$		$4C$	
							$20B$	$10B$		$4A$	
							$10E$	$5A$		$2F$	
$12A$	$12N$	$6O$	$4U$			$12B$	$12P$	$6O$	$4Y$		
	$24A$	$12F$	$8A$				$24C$	$12F$	$8C$		
	$24B$	$12F$	$8B$				$24D$	$12F$	$8D$		
	$12O$	$6O$	$4V$				$12Q$	$6O$	$4Z$		
$12C$	$12R$	$6M$	$4Q$			$15A$	$15A$		$5A$	$3A$	
	$12S$	$6M$	$4R$				$30A$	$15A$	$10A$	$6A$	

The power maps of the elements of $\overline{F}i_{22}$ are given in the ATLAS. The conjugacy classes of elements of $\overline{F}i_{22}$ can be divided into two categories, those which are in Fi_{22} and those which are outside of Fi_{22}. Since $2^6{:}SP(6,2) \leq 2^7{:}SP(6,2)$, we first need to obtain the complete fusion of $2^6{:}SP(6,2)$ into $2^7{:}SP(6,2)$. This fusion enables us to identify those classes of $2^7{:}SP(6,2)$ which fuse into Fi_{22}. Hence we obtain the partial fusion of $2^7{:}SP(6,2)$ into $\overline{F}i_{22}$. The complete conjugacy classes of $2^6{:}SP(6,2)$ and the fusion into Fi_{22} were computed in Chapter 6. For $g \in SP(6,2)$ the classes of $2^6{:}SP(6,2)$ obtained from the coset Ng will fuse into the classes of $2^7{:}SP(6,2)$ obtained from the coset Wg. However since $Wg = Ng \cup Ne_7g$, the classes of $2^6{:}SP(6,2)$ obtained from the coset Ng will only fuse into the classes of $2^7{:}SP(6,2)$ corresponding to the Ng component of the coset Wg. The complete fusion of $2^6{:}SP(6,2)$ into $2^7{:}SP(6,2)$ is given in Table 8.6.

Table 8.6: The fusion of $2^6{:}SP(6,2)$ into $2^7{:}SP(6,2)$

$[g]_{SP(6,2)}$	$[x]_{2^6:SP(6,2)}$	$\longrightarrow$	$[y]_{2^7:SP(6,2)}$	$[g]_{SP(6,2)}$	$[x]_{2^6:SP(6,2)}$	$\longrightarrow$	$[y]_{2^7:SP(6,2)}$
$1A$	$1A$		$1A$	$2A$	$2B$		$2D$
	$2A$		$2C$		$2C$		$2E$
					$4A$		$4C$
$2B$	$2D$		$2G$	$2C$	$2F$		$2K$
	$4B$		$4D$		$2G$		$2L$
	$2E$		$2I$		$4C$		$4H$
					$4D$		$4I$
$2D$	$2H$		$2N$	$3A$	$3A$		$3A$
	$4E$		$4M$		$6A$		$6C$
	$4F$		$4N$				
	$4G$		$4K$				
$3B$	$3B$		$3B$	$3C$	$3C$		$3C$
					$6B$		$6F$
$4A$	$4H$		$4Q$	$4B$	$4J$		$4U$
	$4I$		$4S$		$4K$		$4W$
					$8A$		$8A$
$4C$	$4L$		$4Y$	$4D$	$4N$		$4AC$
	$8B$		$8C$		$4O$		$4AD$
	$4M$		$4AA$		$4P$		$4AG$
$4E$	$4Q$		$4AI$	$5A$	$5A$		$5A$
	$4R$		$4AJ$		$10A$		$10B$
	$8C$		$8F$				
	$8D$		$8E$				
$6A$	$6C$		$6K$	$6B$	$6D$		$6H$
	$12A$		$12D$		$6E$		$6I$
					$12B$		$12C$
$6C$	$6F$		$6M$	$6D$	$6G$		$6O$
					$12C$		$12F$
					$12D$		$12H$
$6E$	$6H$		$6U$	$6F$	$6I$		$6Q$
	$12E$		$12J$		$6J$		$6S$
$6G$	$6K$		$6W$	$7A$	$7A$		$7A$
	$12F$		$12L$				
$8A$	$8E$		$8M$	$8B$	$8G$		$8I$
	$8F$		$8N$		$8H$		$8J$
$9A$	$9A$		$9A$	$10A$	$10B$		$10D$
					$20A$		$20A$
$12A$	$12G$		$12N$	$12B$	$12H$		$12P$
	$24A$		$24A$		$24B$		$24C$
$12C$	$12I$		$12R$	$15A$	$15A$		$15A$

The conjugacy classes of elements of $2^7{:}SP(6,2)$ corresponding to the coset Wg for $g \in SP(6,2)$ are divided into two parts, the Ng and the Ne_7g parts respectively. The classes obtained from the Ng part will fuse into Fi_{22} and the others will fuse into $\overline{F}i_{22} - Fi_{22}$. As was mentioned above the fusion of the classes obtained from Ng into Fi_{22} is completely determined by the fusion of $2^6{:}SP(6,2)$ into $2^7{:}SP(6,2)$ and then into Fi_{22}. The fusion of the classes of $2^7{:}SP(6,2)$ obtained from Ne_7g into $\overline{F}i_{22}$ will be accomplished by using the information provided by the conjugacy classes and the power maps of $2^7{:}SP(6,2)$ and $\overline{F}i_{22}$ and also by using the restrictions of irreducible characters of $\overline{F}i_{22}$ of small degrees to $2^7{:}SP(6,2)$.

For every $\chi_i \in Irr(\overline{F}i_{22})$, there exists $\chi_i' \in Irr(\overline{F}i_{22})$ such that

$$\chi_i'(x) = \begin{cases} \chi_i(x) & x \in Fi_{22} \\ -\chi_i(x) & x \in \overline{F}i_{22} - Fi_{22} \end{cases}.$$

Using the partial fusion of $2^7{:}SP(6,2)$ into $\overline{F}i_{22}$ which has already been determined from the fusion of the classes of $2^7{:}SP(6,2)$ corresponding to Ng into the classes of Fi_{22}, we are able to restrict $78a, (78a)', 429a, (429a)' \in Irr(\overline{F}i_{22})$ to $2^7{:}SP(6,2)$. Using the theory of set intersections for characters, the fusion of the classes obtained from the Ne_7 part of the identity coset W into $\overline{F}i_{22}$, which is important for the restrictions of the irreducible characters of $\overline{F}i_{22}$ to $2^7{:}SP(6,2)$, was fully detremined.

Let ρ be the character afforded by the regular representation of $SP(6,2)$. Then we obtain that $\rho = \sum_{i=1}^{30} e_i\phi_i$, where $\phi_i \in Irr(SP(6,2))$ and $e_i = deg(\phi_i)$. Then ρ can be regarded as a character of $2^7{:}SP(6,2)$ which contains 2^7 in its kernel such that

$$\rho(g) = \begin{cases} |SP(6,2)| & \text{if} \quad g \in 2^7 \\ 0 & \text{otherwise} \end{cases}.$$

If ψ is a character of $\overline{F}i_{22}$, then we obtain that

$$\begin{aligned} \langle \rho, \psi \rangle_{2^7:SP(6,2)} &= \frac{1}{|2^7{:}SP(6,2)|}\{\rho(1A)\psi(1A) + 28\rho(2A)\psi(2A) + 36\rho(2B)\psi(2B) + \\ &\quad 63\rho(2C)\psi(2C)\} \\ &= \frac{1}{|2^7{:}SP(6,2)|}\{|SP(6,2)|\{\psi(1A) + 28\psi(2A) + 36\psi(2B) + \\ &\quad 63\psi(2C)\}\} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{128}\{\psi(1A)+28\psi(2A)+36\psi(2B)+63\psi(2C)\} \\ &= \langle \psi_{2^7}, \tau_1 \rangle \end{aligned}$$

where τ_1 is the identity character of 2^7 and ψ_{2^7} is the restriction of ψ to 2^7. Also for ψ we obtain that

$$\psi_{2^7} = a_1\theta_1 + a_2\theta_2 + a_3\theta_3 + a_4\theta_4 \quad ,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{N} \cup \{0\}$ and $\theta_i, i \in \{1,2,3,4\}$, are the sums of the irreducible characters of 2^7 which are in the same orbit under the action of $SP(6,2)$ on $Irr(2^7)$. Let $\tau_j \in Irr(2^7)$, where $j \in \{1,2,\ldots,128\}$. Then we obtain that

$$\theta_1 = \tau_1 \ , \quad deg(\theta_1) = 1$$

$$\theta_2 = \tau_2 \ , \quad deg(\theta_2) = 1$$

$$\theta_3 = \sum_{j=3}^{65} \tau_j \ , \quad deg(\theta_3) = 63$$

$$\theta_4 = \sum_{j=66}^{128} \tau_j \ , \quad deg(\theta_4) = 63 \ .$$

and thus we have

$$\psi_{2^7} = a_1\tau_1 + a_2\tau_2 + a_3\sum_{j=3}^{65}\tau_j + a_4\sum_{j=66}^{128}\tau_j$$

and hence

$$\langle \psi_{2^7}, \psi_{2^7} \rangle = a_1^2 + a_2^2 + 63a_3^2 + 63a_4^2 \quad ,$$

where $a_1 = \langle \rho, \psi \rangle_{2^7:SP(6,2)}$. Also we obtain that $a_1 + a_2 + 63a_3 + 63a_4 = deg(\psi)$.

Now let $\psi = 78a$ be the irreducible character of $\overline{F}i_{22}$ of degree 78. Then we obtain that

$$a_1 = \frac{1}{128}\{78 + 28(6) + 36(22) + 63(14)\} = 15$$

and $a_1+a_2+63a_3+63a_4 = 78$. Hence we obtain two possibilities ($a_2 = a_3 = 0, a_4 = 1$) or ($a_2 = a_4 = 0, a_3 = 1$). Hence without loss of generality we take $a_2 = a_4 = 0$ and $a_3 = 1$. We also know from Chapter 6 (Section 6.5) that $(78a)_{2^6:SP(6,2)} = \chi_3 + \chi_{41}$. Then based on the partial fusion of $2^7{:}SP(6,2)$ into $\overline{F}i_{22}$ which has already been determined, we obtain that $(78a)_{2^7:SP(6,2)} = \chi_3 + \chi_{62}$. Hence we have that

$$(78a)_{2^7:SP(6,2)} = \chi_3 + \chi_{62} \quad \text{and} \quad (78a)'_{2^7:SP(6,2)} = \chi_{33} + \chi_{99} \quad .$$

Similarly we can show that

$$(429a)_{2^7:SP(6,2)} = \chi_1 + \chi_3 + \chi_8 + \chi_{63} + \chi_{98}$$

and

$$(429a)'_{2^7:SP(6,2)} = \chi_{31} + \chi_{33} + \chi_{38} + \chi_{61} + \chi_{100} \quad .$$

Using the partial fusion already determined and the values of $78a, (78a)', 429a$ and $(429a)'$ on the classes of $\overline{F}i_{22}$ and the values of $(78a)_{2^7:SP(6,2)}$, $(78a)'_{2^7:SP(6,2)}$, $(429a)_{2^7:SP(6,2)}$ and $(429a)'_{2^7:SP(6,2)}$ on the classes of $2^7{:}SP(6,2)$, we are able to complete the fusion map of $2^7{:}SP(6,2)$ into $\overline{F}i_{22}$. This is given in Table 8.7 below.

Table 8.7: The fusion of $2^7{:}SP(6,2)$ into $\overline{F}i_{22}$

$[g]_{SP(6,2)}$	$[x]_{2^7:SP(6,2)}$	$\longrightarrow$	$[y]_{\overline{F}i_{22}}$	$[g]_{SP(6,2)}$	$[x]_{2^7:SP(6,2)}$	$\longrightarrow$	$[y]_{\overline{F}i_{22}}$
$1A$	$1A$		$1A$	$2A$	$2D$		$2A$
	$2A$		$2E$		$4A$		$4G$
	$2B$		$2D$		$4B$		$4F$
	$2C$		$2B$		$2E$		$2C$
					$4C$		$4B$
					$2F$		$2F$
$2B$	$2G$		$2C$	$2C$	$2K$		$2B$
	$2H$		$2D$		$4F$		$4F$
	$2I$		$2B$		$2L$		$2C$
	$2J$		$2E$		$4G$		$4G$
	$4D$		$4A$		$4H$		$4A$
	$4E$		$4G$		$2M$		$2E$
					$4I$		$4E$
					$4J$		$4H$
$2D$	$2N$		$2C$	$3A$	$3A$		$3A$
	$4K$		$4B$		$6A$		$6L$
	$4L$		$4I$		$6B$		$6Q$
	$2O$		$2F$		$6C$		$6D$
	$4M$		$4E$				
	$4N$		$4C$				
	$4O$		$4I$				
	$4P$		$4H$				
$3B$	$3B$		$3D$	$3C$	$3C$		$3C$
	$6D$		$6T$		$6E$		$6U$
					$6F$		$6I$
					$6G$		$6P$
$4A$	$4Q$		$4D$	$4B$	$4U$		$4E$
	$4R$		$4J$		$4V$		$4F$
	$4S$		$4C$		$4W$		$4B$
	$4T$		$4I$		$4X$		$4I$
					$8A$		$8B$
					$8B$		$8F$

Table 8.7: The fusion of 2^7:$SP(6,2)$ into $\overline{F}i_{22}$(continued)

$[g]_{SP(6,2)}$	$[x]_{2^7:SP(6,2)}$	$\longrightarrow$	$[y]_{\overline{F}i_{22}}$	$[g]_{SP(6,2)}$	$[x]_{2^7:SP(6,2)}$	$\longrightarrow$	$[y]_{\overline{F}i_{22}}$
$4C$	$4Y$		$4B$	$4D$	$4AC$		$4D$
	$4Z$		$4G$		$4AD$		$4A$
	$4AA$		$4E$		$4AE$		$4J$
	$4AB$		$4I$		$4AF$		$4G$
	$8C$		$8A$		$4AG$		$4E$
	$8D$		$8E$		$4AH$		$4H$
$4E$	$4AI$		$4E$	$5A$	$5A$		$5A$
	$8E$		$8B$		$10A$		$10C$
	$8F$		$8A$		$10B$		$10B$
	$4AJ$		$4D$		$10C$		$10E$
	$4AK$		$4J$				
	$8G$		$8E$				
	$4AL$		$4H$				
	$8H$		$8F$				
$6A$	$6H$		$6A$	$6B$	$6K$		$6F$
	$12A$		$12K$		$6L$		$6M$
	$6I$		$6F$		$12D$		$12C$
	$12B$		$12O$		$12E$		$12O$
	$12C$		$12D$				
	$6J$		$6O$				
$6C$	$6M$		$6K$	$6D$	$6O$		$6D$
	$6N$		$6S$		$12F$		$12B$
					$12G$		$12M$
					$6P$		$6Q$
					$12H$		$12H$
					$12I$		$12P$
$6E$	$6Q$		$6H$	$6F$	$6U$		$6E$
	$6R$		$6P$		$12J$		$12I$
	$6S$		$6I$		$6V$		$6V$
	$6T$		$6U$		$12K$		$12N$
$6G$	$6W$		$6J$	$7A$	$7A$		$7A$
	$12L$		$12I$		$14A$		$14B$
	$12M$		$12S$				
	$6X$		$6V$				
$8A$	$8I$		$8D$	$8B$	$8M$		$8D$
	$8J$		$8B$		$8N$		$8C$
	$8K$		$8F$		$8O$		$8G$
	$8L$		$8H$		$8P$		$8H$
$9A$	$9A$		$9C$	$10A$	$10D$		$10A$
	$18A$		$18G$		$20A$		$20A$
					$20B$		$20B$
					$10E$		$10D$

Table 8.7: The fusion of $2^7{:}SP(6,2)$ into $\overline{Fi}_{22}$(continued)

$[g]_{SP(6,2)}$	$[x]_{2^7:SP(6,2)}$	$\longrightarrow$ $[y]_{\overline{Fi}_{22}}$	$[g]_{SP(6,2)}$	$[x]_{2^7:SP(6,2)}$	$\longrightarrow$ $[y]_{\overline{Fi}_{22}}$
$12A$	$12N$	$12H$	$12B$	$12P$	$12D$
	$24A$	$24A$		$24C$	$24B$
	$24B$	$24D$		$24D$	$24C$
	$12O$	$12M$		$12Q$	$12O$
$12C$	$12R$	$12J$	$15A$	$15A$	$15A$
	$12S$	$12T$		$30A$	$30B$

8.7 The permutation character of $\overline{Fi}_{22}$ on $2^7{:}SP(6,2)$

The group $2^7{:}SP(6,2)$ is a maximal subgroup of $\overline{Fi}_{22}$ of index 694980. Thus when $\overline{Fi}_{22}$ acts on the cosets of $2^7{:}SP(6,2)$, then this action gives rise to a permutation representation which affords a permutation character of degree 694980 and we denote this permutation character by $\chi(\overline{Fi}_{22}|2^7{:}SP(6,2))$. We also know from Chapter 6 (Section 6.6) that

$$\begin{aligned}\chi(Fi_{22}|2^6{:}SP(6,2)) &= 1a + 429a + 1430a + 3080a + 13650a + 30030a + \\ &\quad 45045a + 75075a + 205920a + 320320a \ .\end{aligned}$$

The permutation character $\chi(\overline{Fi}_{22}|2^7{:}SP(6,2))$ is related to $\chi(Fi_{22}|2^6{:}SP(6,2))$ in that the irreducible characters involved in $\chi(\overline{Fi}_{22}|2^7{:}SP(6,2))$ are irreducible characters χ_i or χ_i' such that χ_i is involved in $\chi(Fi_{22}|2^6{:}SP(6,2))$. Using the values of the irreducible characters $1a, 429a, (429a)', 1430a, (1430a)', 3080a, (3080a)', 13650a, (13650a)', 30030a, (30030a)', 45045a, (45045a)', 75075a, (75075a)', 205920a, (205920a)', 320320a$ and $(320320a)'$ of $\overline{Fi}_{22}$ on the conjugacy classes of $2^7{:}SP(6,2)$ we deduce that

$$\begin{aligned}\chi(\overline{Fi}_{22}|2^7{:}SP(6,2)) &= 1a + 429a + 1430a + 3080a + 13650a + 30030a + \\ &\quad 45045a + 75075a + 205920a + 320320a \ .\end{aligned}$$

There is another group of the form $2^7{:}SP(6,2)$ which is an affine subgroup of $SP(8,2)$. This subgroup is maximal in $SP(8,2)$ of index 255 and is isomorphic to the centralizer of an element of the $2A$ conjugacy class of $SP(8,2)$. By the discussion following Theorem 4.4.6 and by Remark 4.4.7, for this affine subgroup of $SP(8,2)$ we would have four inertia groups of indices 1, 28, 36 and 63 in $2^7{:}SP(6,2)$ with inertia factors of indices 1, 28, 36 and 63 respectively in $SP(6,2)$. This group would have some irreducible characters of degrees 28 and 36. Therefore the group $2^7{:}SP(6,2)$ that has been studied in this chapter, is not the one that sits in $SP(8,2)$.

Appendix A

Programmes

A.1 Programme A for $2^5{:}S_6$

$V : vector\ space(5, GF(2));$
$S : matrix\ group(V);$
$S.generators : a = mat(1,0,0,0,0 : 1,1,0,0,0 : 1,0,1,0,0 : 1,0,0,1,0 : 1,0,0,0,1), b =$
$mat(1,1,0,0,0 : 1,0,1,0,0 : 1,0,0,1,0 : 1,0,0,0,1 : 1,0,0,0,0);$
$c : classes(S);$
$O1 : matrix\ orbit(S, vec(1,1,1,1,1), false);$
$O2 : matrix\ orbit(S, vec(1,1,1,1,0), false);$
$O3 : matrix\ orbit(S, vec(0,0,0,0,1), false);$
$O : O1\ join\ O2\ join\ O3;$
$for\ i = 1\ to\ 11\ do;$
$print\ c[i],\ '\$N';$
$e = null;$
$w = vec(0)\ of\ V;$
$while\ O - e\ ne\ [\,]\ do;$
$d = null;$
$for\ each\ x\ in\ O\ do;$
$y = [x + w + (x * c[i])];$
$d = d\ join\ y;$

```
end;
print d, '$N';
print '******';
e = d join e;
if O - e ne [] then;
w = setrep(O - e);
end;
end;
r = null;
u = vec(0) of V;
while O - r ne [] do;
m = null;
for each g in centralizer(S, c[i]) do;
l = [u * g];
m = m join l;
end;
print 'A block for the vectors under the action of centralizer :';
print m;
r = m join r;
if O - r ne [] then;
u = setrep(O - r);
end;
end;
print '*********************************';
end;
```

A.2 Programme A for $3^2{:}D_4$

```
V : vector space(2, GF(3));
S : matrix group(V);
S.generators : a = mat(0,1 : 2,0), b = mat(1,0 : 0,2);
c : classes(S);
```

```
O1 : matrix orbit(S, vec(1,1), false);
O2 : matrix orbit(S, vec(1,0), false);
O : O1 join O2;
for i = 1 to 5 do;
print c[i], '$N';
e = null;
w = vec(0) of V;
while O - e ne [ ] do;
d = null;
for each x in O do;
y = [x + w + (x * c[i])];
d = d join y;
end;
print d, '$N';
print '******';
e = d join e;
if O - e ne [ ] then;
w = setrep(O - e);
end;
end;
r = null;
u = vec(0) of V;
while O - r ne [ ] do;
m = null;
for each g in centralizer(S, c[i]) do;
l = [u * g];
m = m join l;
end;
print 'A block for the vectors under the action of centralizer :';
print m;
r = m join r;
if O - r ne [ ] then;
u = setrep(O - r);
```

```
end;
end;
print '********************************';
end;
```

A.3 Programme A for $2^6{:}O^-(6,2)$

```
V : vector space(6, GF(2));
S : symplectic(6, GF(2));
c : classes(S);
H : matrix group(V);
H.generators : c[10] = mat(1,0,1,0,0,1 : 0,0,1,0,0,0 : 0,1,0,0,0,0 : 0,0,0,0,1,0 :
0,0,1,1,0,1 : 0,0,0,0,0,1), c[25] = mat(0,1,1,1,0,0 : 1,1,0,1,1,1 : 0,1,1,0,1,1 :
0,0,1,0,1,0 : 0,1,1,1,0,1 : 1,1,1,1,1,0);
q : classes(H);
O1 : matrix orbit(H, vec(1,1,1,1,0,1), false);
O2 : matrix orbit(H, vec(1,1,1,1,1,1), false);
O : O1 join O2;
for i = 1 to 25 do;
print q[i], '$N';
e = null;
w = vec(0) of V;
while O - e ne [ ] do;
d = null;
for each x in O do;
y = [x + w + (x * q[i])];
d = d join y;
end;
print d, '$N';
print '******';
e = d join e;
if O - e ne [ ] then;
```

$w = setrep(O - e);$
$end;$
$end;$
$r = null;$
$u = vec(0)\ of\ V;$
$while\ O - r\ ne\ [\,]\ do;$
$m = null;$
$for\ each\ g\ in\ centralizer(H, q[i])\ do;$
$l = [u * g];$
$m = m\ join\ l;$
$end;$
$print\ 'A\ block\ for\ the\ vectors\ under\ the\ action\ of\ centralizer\ :';$
$print\ m;$
$r = m\ join\ r;$
$if\ O - r\ ne\ [\,]\ then;$
$u = setrep(O - r);$
$end;$
$end;$
$print\ '********************************';$
$end;$

A.4 Programme A for 2^7:$SP(6,2)$

$V : vector\ space(7, GF(2));$
$S : matrix\ group(V);$
$S.generators : \overline{a} = mat(1,1,0,0,1,0,0 : 1,1,0,0,0,1,0 : 0,0,0,1,0,0,0 : 0,0,1,0,0,0,0 : 1,0,0,0,1,1,0 : 0,1,0,0,1,1,0 : 0,0,0,0,0,0,1), \overline{x} = mat(0,1,0,1,1,1,0 : 0,1,1,1,0,1,0 : 0,1,1,1,0,0,0 : 1,0,1,0,0,0,0 : 1,1,1,0,1,0,0 : 0,1,1,0,1,0,0 : 0,0,0,0,0,0,1), \overline{c} = mat(0,0,1,1,0,0,0 : 1,1,1,1,0,0,0 : 1,1,0,0,1,1,0 : 0,1,0,0,1,1,0 : 0,1,1,1,1,0,0 : 1,0,0,1,1,0,0 : 1,0,0,1,1,1,1);$
$c : classes(S);$
$O1 : matrix\ orbit(S, vec(1,0,1,0,1,0,1), false);$

```
O2 : matrix orbit(S, vec(1,1,1,1,1,1,1), false);
O3 : matrix orbit(S, vec(1,0,0,0,0,0,0), false);
O : O1 join O2 join O3;
for i = 1 to 30 do;
print c[i], '$N';
e = null;
w = vec(0) of V;
while O - e ne [ ] do;
d = null;
for each x in O do;
y = [x + w + (x * c[i])];
d = d join y;
end;
print d, '$N';
print ' * * * * * *';
e = d join e;
if O - e ne [ ] then;
w = setrep(O - e);
end;
end;

r = null;
u = vec(0) of V;
while O - r ne [ ] do;
m = null;
for each g in centralizer(S, c[i]) do;
l = [u * g];
m = m join l;
end;
print 'A block for the vectors under the action of centralizer :';
print m;
r = m join r;
if O - r ne [ ] then;
u = setrep(O - r);
```

```
end;
end;
print '********************************';
end;
```

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