UNIVERSITY OF KWAZULU-NATAL

EXACT SOLUTIONS FOR RELATIVISTIC MODELS

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As the candidate's supervisor, I have approved this dissertation for submission.

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Abstract

In this thesis we study spherically symmetric spacetimes related to the Einstein field equations. We consider only neutral matter and apply the Einstein field equations with isotropic pressures. Our object is to model relativistic stellar systems. We express the Einstein field equations and the condition of pressure isotropy in terms of Schwarzschild coordinates and isotropic coordinates. For Schwarzschild coordinates we consider the transformations due to Buchdahl (1959), Durgapal and Bannerji (1983), Fodor (2000) and Tewari and Pant (2010). The condition of pressure isotropy is integrated and new exact solutions of the field equations are obtained utilizing the transformations of Buchdahl (1959) and Tewari and Pant (2010). These exact solutions are given in terms of elementary functions. For isotropic coordinates we can express the condition of pressure isotropy as a Riccati equation or a linear equation. An algorithm is developed that produces a new solution if a particular solution is known. The transformations reduce to a nonlinear Bernoulli equation in most instances. There are fundamentally three new classes of solutions to the condition of pressure isotropy.

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I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

Sifiso Allan Ngubelanga

December 2011

To

My late Grandmother,

for encouraging me to study further.

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Contents

Chapter 1

Introduction

Currently, the theory of general relativity provides the best description of the behaviour of the gravitational field. The predictions of general relativity have been shown to be consistent with observational data in relativistic astrophysics and cosmology. In general relativity the curvature of spacetime is described by the Riemann tensor. The matter content is described by the symmetric energy momentum tensor; in this thesis we consider only neutral perfect fluid matter. The Einstein field equations relate the matter content to the curvature. In the presence of an electromagnetic field these equations have to be supplemented with Maxwell's equations which incorporate charge and current. The Einstein tensor is crucial for the description of the gravitational field. The Einstein field equations satisfy the conservation laws through the Bianchi identity. Determining explicit solutions to the Einstein field equations is necessary for astrophysical and cosmological applications.

Our aim in this thesis is to consider relativistic stellar models in static spherically symmetric fields. We seek exact solutions to the Einstein field equations with isotropic pressures. Exact solutions to the Einstein field equations are essential because they throw light on the qualitative features of the gravitational fields. The Einstein field equations alleviate the investigation and the discussion of the physical properties of relativistic stars (Schutz 1985, Shapiro and Teukolsky 1983). A physical analysis is easier with an exact solution since in general it is difficult to study these features in the general Einstein field equations. It is important to note that an exact solution is only a first step in the modeling process. The criteria for physical acceptability must also be satisfied which is valid only for a small class of exact solutions as pointed out by Delgaty and Lake (1998).

For a relativistic stellar model additional physical constraints are required in the integration of the field equations. Exact solutions of the field equations are crucial for applications in cosmology and relativistic astrophysics. The following are some of the fundamental exact solutions in general relativity for astrophysics:

(a) The *Schwarzschild exterior* solution: This solution describes the gravitational field outside a spherically symmetric matter distribution. The Schwarzschild exterior solution is used in analysing the bending of light, perihelion advance, spectral shift and time delay for the classical tests of general relativity. It can be demonstrated that every spherically symmetric asymptotically flat exterior solution is static (which is given by the Schwarzschild exterior line element) even for the case where the solution is nonstatic. This general result is referred to as Birkoff's theorem.

(b) The *Schwarzschild interior* solution: This solution describes the interior of the gravitational field for a static spherical symmetric body. The Schwarzschild exterior and the Schwarzschild interior solutions match smoothly at the boundary of the star. The Schwarzschild interior solution can be used to model relativistic stars for which the fluctuations in the energy are small and is an effective approximation for small stars in which the pressures are not large.

(c) The *Reissner-Nordstrom* solution: This solution describes the gravitational field outside a static spherically charged body. The Reissner-Nordstrom solution reduces to the Schwarzschild exterior solution in the absence of charge. Astrophysical bodies are uncharged and consequently the influence of the electromagnetic field may be ignored in general. Nevertheless this solution is essential as a simple example of an exact solution of the Einstein-Maxwell system of equations and may be used as a first approximation in some physical situations.

(d) The *Kerr* solution: This solution is used to describe the exterior of a rotating body. This rotating solution has a complex form with interesting physical features. It is crucial to note that an interior solution that matches smoothly to the exterior Kerr line element has not yet been found (Stephani 2004). The Kerr solution reduces to the Schwarzschild solution in the appropriate limit.

Detailed information of the different known exact solutions is provided by Krasinski (1997) and Stephani *et al* (2003).

This dissertation is organised as follows:

• Chapter 1: Introduction.

• Chapter 2: In this chapter we briefly discuss the concepts of general relativity essential for this thesis. We briefly consider the spacetime geometry and the matter distribution that lead to the Einstein field equations. The formulation of the Einstein field equations is introduced in this chapter. We highlight the crucial physical concepts that are essential for the determination of a realistic relativistic stellar model.

• Chapter 3: In this chapter we derive the Einstein field equations for neutral perfect fluids in static spherically symmetric spacetimes. We use new variables and transformations to write the Einstein field equations in equivalent forms. In particular we consider the transformations of Buchdahl (1959), Durgapal and Bannerji (1983), Fodor (2000) and Tewari and Pant (2010). We rewrite the condition of pressure isotropy in terms of the new variables in order to generate new exact solutions in terms of elementary functions. Particular new solutions to the field equations are found.

• Chapter 4: In this chapter we consider the relativistic stellar model in isotropic coordinates. We generate the Einstein field equations and the condition of pressure isotropy for the shear-free spacetime in isotropic coordinates. We use new variables to transform these equations into equivalent forms that lead to the generation of new exact solutions. The condition of pressure isotropy can be written as a Riccati equation or a linear equation. We also generate an algorithm that enables us to find new exact solutions to the Einstein field equations if a particular solution is specified.

• Chapter 5: Conclusion.

Chapter 2

Differential geometry

2.1 Introduction

A variety of matter distributions arise in physical applications in relativistic astrophysics in different scenarios as pointed out by Will (1981). Spherically symmetric matter distributions are best described by Einstein's theory of general relativity for strong gravitational fields. In this chapter we briefly consider the background theory that provides us with the structure to generate a model of a relativistic star. We give a brief outline of the differential geometry and the matter distribution that lead to the Einstein field equations. For more detailed information on differential manifolds and tensor analysis the reader is referred to Bishop and Goldberg (1968), Misner *et al* (1973) and Wald (1984). The metric tensor field and the metric connection coefficients are introduced in §2.2. Then the Riemann tensor, the Ricci tensor, Ricci scalar and the Einstein tensor are defined. In §2.3, we consider matter fields by introducing the general energy momentum tensor and the special case for a perfect fluid. We also introduce the barotropic equation of state relating the pressure to the energy density. The Einstein field equations are generated by relating the Einstein tensor to the energy momentum tensor. In §2.4 we provide the conditions necessary for the physical acceptability of a relativistic stellar model.

2.2 Spacetime geometry

The local neighbourhood of a point in the spacetime manifold possesses the same structure as the open neighbourhood of a point in $\Re^n$. The global structure of the spacetime manifold in general is different from $\Re^n$. A pseudo-Riemannian manifold is a manifold with an indefinite metric tensor field. In general relativity, we assume that the spacetime $\mathbf{M}$ is a four-dimensional differentiable manifold endowed with a metric tensor field $\mathbf{g}$. The symmetric and nonsingular metric tensor field $\mathbf{g}$ has signature $(-+++)$. The metric tensor field $\mathbf{g}$ represents the gravitational potentials. Points in the manifold are labelled by the real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$, where $x^0 = ct$ (where c is the speed of the light in vacuum) is the timelike coordinate, and x^1, x^2, x^3 are spacelike coordinates. In this dissertation we use the convention that the speed of light $c = 1$.

The line element is given by

$$ds^2 = g_{ab} dx^a dx^b \tag{2.1}$$

which measures the infinitesimal interval between neighbouring points on a curve. In the line element (2.1), $\mathbf{g}$ represents the metric tensor field. We use the line element

(2.1) to generate the metric connection coefficients

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} (g_{cd,b} + g_{bd,c} - g_{bc,d}) \tag{2.2}$$

where the commas denote partial differentiation. We use the definition of the connection coefficients in equation (2.2) to generate the Riemann curvature tensor **R** which is given by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} \tag{2.3}$$

which is nonvanishing in general since the covariant derivative is not commutative. We contract the Riemann curvature (2.3) to get the Ricci tensor as follows

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{db}\Gamma^d{}_{ac} \end{aligned} \tag{2.4}$$

A second contraction yields the Ricci scalar R. This has the form

$$\begin{aligned} R &= R^a{}_a \\ &= g^{ab} R_{ab} \end{aligned} \tag{2.5}$$

We use the Ricci tensor (2.4) and the Ricci scalar (2.5) to form the Einstein tensor **G** which is given by

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} \tag{2.6}$$

Note that the divergence of the Einstein tensor is zero, *i.e.*

$$G^{ab}{}_{;b} = 0 \tag{2.7}$$

This is sometimes called the Bianchi identity and generates the conservation laws via the field equations.

2.3 Matter fields

The matter content is described by the energy momentum tensor $\mathbf{T}$. The energy momentum tensor is given by

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab} + q^a u^b + q^b u^a + \pi^{ab} \tag{2.8}$$

where ρ is the energy density, p is the isotropic pressure, $\mathbf{q}$ is the heat flux vector ($q_a u^a = 0$), π^{ab} is the anisotropic stress tensor ($\pi_{ab}u^a = 0 = \pi^a{}_a$) and $\mathbf{u}$ is a timelike four-velocity ($u^a u_a = -1$). The terms for the heat flux and the anisotropic stress vanish in perfect fluids ($q^a = 0, \pi^{ab} = 0$). Then the energy momentum tensor for a perfect fluid has the form

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab} \tag{2.9}$$

For many applications it is required that the matter distribution satisfies the barotropic equation of state

$$p = p(\rho) \tag{2.10}$$

A particular case is the equation of state

$$p = a\rho + b \tag{2.11}$$

where a and b are constants. The above form is often assumed in cosmology and is called the linear equation of state. The parameter a (with $b = 0$) has different values which describe familiar matter distributions: dust ($a = 0$), radiation ($a = \frac{1}{3}$) and stiff matter ($a = 1$). When $a \neq 0$ and $b \neq 0$ then the equation of state (2.11) includes matter distributions for quark, strange and exotic configurations (Komathiraj and Maharaj 2007, Mak and Harko 2004, Sharma and Maharaj 2007). Another case is the polytropic equation of state which has the form

$$p = k\rho^{1+\frac{1}{n}} \tag{2.12}$$

where k and n are constants. This equation of state is assumed in relativistic astrophysics (Shapiro and Teukolsky 1983).

The Einstein field equations follow by relating (2.6) to (2.8) so that

$$
\begin{aligned}
G^{ab} &= R^{ab} - \frac{1}{2}Rg^{ab} \\
&= T^{ab}
\end{aligned}
\tag{2.13}
$$

where the coupling constant is set to unity. The field equations (2.13) govern the interaction between the curvature of the spacetime and the matter distribution. From (2.13) and (2.7) we have the result

$$
T^{ab}{}_{;b} = 0 \tag{2.14}
$$

which is the conservation law for matter. In general the field equations (2.13) are a highly nonlinear system of differential equations which are difficult to integrate without making simplifying assumptions. For detailed information on general relativity and the formulation of the Einstein field equations the reader is referred to de Felice and Clark (1990), Narlikar (2002) and Stephani (2004). Exact solutions to the field equations which are applicable in many physically relevant relativistic models are listed in Krasinski (1997) and Stephani *et al* (2003).

2.4 Physical conditions

In this section we briefly consider the physical conditions relevant for a relativistic stellar model. Any stellar interior solution should match to the appropriate exterior spacetime. The spacetime surrounding the static spherically symmetric body is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where M is the mass of the stellar body measured by an observer at infinity which is given by $M = m(R)$, where R is the stellar radius. The above metric is famously known as the Schwarzschild (1916a,1916b) exterior line element. The gravitational field outside a static, charged spherically symmetric body has the form

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where Q is a constant related to the total charge of the sphere. This is called the Reissner-Nordstrom metric (Reissner 1916, Nordstrom 1918).

For isotropic matter the realistic stellar models are often assumed to satisfy:

(a) The energy density ρ and the pressure p should be positive and finite throughout the interior of the star.

(b) The radial pressure should disappear at the boundary $r = R$:

$$0 < \rho < \infty$$

$$0 < p < \infty$$

$$p(R) = 0$$

(c) The energy density ρ and the pressure p should be monotonically decreasing functions from the centre to the boundary:

$$\frac{d\rho}{dr} \leq 0$$

$$\frac{dp}{dr} \leq 0$$

(d) The gradients $\frac{d\rho}{dr}$ and $\frac{dp}{dr}$ should be regular in the interior.

(e) The speed of light should be subluminal throughout the interior of the star so that we have

$$0 \leq \frac{dp}{d\rho} \leq 0$$

The above condition is essential to maintain causality.

(f) The metric functions $e^{2\nu}$ and $e^{2\lambda}$ should be positive and nonsingular throughout the interior of the star.

(g) At the boundary the interior metric functions should match to the exterior Schwarzschild solution:

$$\begin{aligned} e^{2\nu} &= e^{-2\lambda} \\ &= 1 - \frac{2M}{r} \end{aligned}$$

(h) The solution should be stable with respect to radial perturbations.

It should be noted that not all exact solutions to the Einstein field equations satisfy the above conditions. However it is important to compare the physical features of individual stellar masses with the conditions listed above. A comprehensive list of perfect fluid models for static spherically symmetric fields is provided by Delgaty and Lake (1998) and Stephani *et al* (2003). Some other well known relativistic models are given by Finch and Skea (1989), Gupta and Kumar (2005), Maharaj and Leach (1996), Tikekar and Jotania (2005) and Yilmaz and Baysal (2005).

Chapter 3

Schwarzschild coordinates

3.1 Introduction

In this chapter we consider the Einstein field equations in terms of Schwarzschild coordinates which are comoving. The field equations are then expressed in several equivalent forms which may be easier to integrate. In §3.2 we analyse the spacetime geometry for static spherically symmetric gravitational fields by specifying the line element in a form that was first introduced by Schwarzschild. The components of the connection coefficients, the Ricci tensors, the Ricci scalar and the Einstein tensors are explicitly generated in this section. In §3.3 we compute the Einstein field equations by relating the components of the energy momentum tensor for the perfect fluid to the components of the Einstein tensor. The condition of pressure isotropy is also found in this section. It is possible to write the Einstein field equations in different forms by introducing new variables. In this section we consider particular transformations that are relevant to relativistic stellar models. These transformations were first introduced by Buchdahl (1959), Durgapal and Bannerji (1983), Fodor (2000) and Tewari and Pant (2010). The condition of pressure isotropy is also written in new variables using the relevant transformations. Particular exact solutions are found in §3.4 in terms of elementary functions for the condition of pressure isotropy. These functions are new solutions to the field equations.

3.2 Spacetime geometry

We consider a spacetime which is static and spherically symmetric and define local coordinates $(x^a) = (t, r, \theta, \phi)$. Then the line element in comoving coordinates can be written as

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.1}$$

where $\nu(r)$ and $\lambda(r)$ are arbitrary functions representing the gravitational potentials. This line element was first introduced by Schwarzschild (1916) and is often used to describe relativistic compact objects such as neutron stars and superdense stars in astrophysics (Komathiraj and Maharaj 2007, Sharma *et al* 2001, Thirukkanesh and Maharaj 2008).

The line element (3.1) is essential for the determination of the connection coefficients $\Gamma^a{}_{bc}$ which are defined in equation (2.2). The nonvanishing connection coefficients for the metric (3.1) are

$$
\begin{aligned}
\Gamma^0{}_{01} &= \nu' \\
\Gamma^1{}_{00} &= \nu' e^{2(\nu-\lambda)} \\
\Gamma^1{}_{11} &= \lambda' \\
\Gamma^1{}_{22} &= -re^{-2\lambda} \\
\Gamma^1{}_{33} &= -re^{-2\lambda}\sin^2\theta \\
\Gamma^2{}_{12} &= \frac{1}{r} \\
\Gamma^2{}_{33} &= -\sin\theta\cos\theta \\
\Gamma^3{}_{13} &= \frac{1}{r} \\
\Gamma^3{}_{23} &= \cot\theta
\end{aligned}
$$

In the above equations, primes denote differentiation with respect to the radial coordinate r.

On using the above connection coefficients we generate the nonvanishing Ricci tensor components for the line element (3.1). Substituting the above connection coefficients in the definition of the Ricci tensor equation (2.4) we obtain the following components

$$R_{00} = e^{2(\nu-\lambda)}\left(\nu'' + \nu'^2 - \nu'\lambda' + 2\frac{\nu'}{r}\right) \tag{3.2a}$$

$$R_{11} = -\left(\nu'' + \nu'^2 - \nu'\lambda' - 2\frac{\lambda'}{r}\right) \tag{3.2b}$$

$$R_{22} = 1 - \frac{1}{e^{2\lambda}}\left(1 + r(\nu' - \lambda')\right) \tag{3.2c}$$

$$R_{33} = \sin^2\theta\, R_{22} \tag{3.2d}$$

with $R_{ab} = 0$ for $a \neq b$. We then compute the Ricci scalar which is given in (2.5) from the nonvanishing Ricci tensor components in (3.2). The Ricci scalar has the form

$$R = 2\left[\frac{1}{r^2} - \frac{1}{e^{2\lambda}}\left(\nu'' + \nu'^2 - \nu'\lambda' + 2\frac{\nu'}{r} - 2\frac{\lambda'}{r} + \frac{1}{r^2}\right)\right] \tag{3.3}$$

for a static spherically symmetric metric.

We defined the Einstein tensor in equation (2.6). We now use equations (3.2) and (3.3) to compute the components of the Einstein tensor. The relevant components are given by

$$G_{00} = \frac{1}{r^2}e^{2\nu}\left[r\left(1-\frac{1}{e^{2\lambda}}\right)\right]' \tag{3.4a}$$

$$G_{11} = -\frac{1}{r^2}(e^{2\lambda}-1)+2\frac{\nu'}{r} \tag{3.4b}$$

$$G_{22} = \frac{r^2}{e^{2\lambda}}\left(\nu''+\nu'^2+\frac{\nu'}{r}-\nu'\lambda'-\frac{\lambda'}{r}\right) \tag{3.4c}$$

$$G_{33} = \sin^2\theta G_{22} \tag{3.4d}$$

with $G_{ab}=0$ for $a \neq b$.

3.3 Einstein field equations

Since the fluid four-velocity is comoving we have $u^a = e^{-\nu}\delta^a_0$ for the metric (3.1). The nonvanishing components of the energy momentum tensor are given by

$$T_{00} = \rho e^{2\nu} \tag{3.5a}$$

$$T_{11} = pe^{2\lambda} \tag{3.5b}$$

$$T_{22} = pr^2 \tag{3.5c}$$

$$T_{33} = \sin^2\theta T_{22} \tag{3.5d}$$

with $T_{ab} = 0$ for $a \neq b$. We use the Einstein tensor components from equation (3.4), in conjunction with the energy momentum tensor components from equation (3.5), to compute the Einstein field equations. These field equations take the form

$$\frac{1}{r^2}\left[r\left(1-\frac{1}{e^{2\lambda}}\right)\right]' = \rho \tag{3.6a}$$

$$-\frac{1}{r^2}\left(1-\frac{1}{e^{2\lambda}}\right)+2\frac{\nu'}{r}\frac{1}{e^{2\lambda}} = p \tag{3.6b}$$

$$\frac{1}{e^{2\lambda}}\left(\nu''+\nu'^2+\frac{\nu'}{r}-\nu'\lambda'-\frac{\lambda'}{r}\right) = p \tag{3.6c}$$

for the static spherically symmetric metric (3.1).

The conservation law for matter (2.14) can be written in the following form

$$\frac{dp}{dr} = -(\rho+p)\frac{d\nu}{dr} \tag{3.7}$$

Equation (3.7) also follows directly from the field equations (3.6). Therefore equation (3.7) can be used instead of one of the equations in (3.6). In the system of equations (3.6) we have four unknowns ρ, p, ν and λ, but we have only three independent equations. Consequently we need to choose one of ρ, p, ν or λ to integrate the field equations. Also on physical grounds we could impose a particular barotropic equation of state (2.10) following the approach of Hansraj and Maharaj (2006).

On equating equation (3.6b) to (3.6c) we obtain the following equation

$$\nu'' + \nu'^2 - \nu'\lambda' - \frac{(\nu' + \lambda')}{r} + \frac{(e^{2\lambda} - 1)}{r^2} = 0 \tag{3.8}$$

which is the condition of pressure isotropy. Note that (3.8) is the master equation determining the behaviour of the model for static spherically symmetric gravitational fields. The above equation (3.8) can be written in the more compact form as

$$\frac{d}{dr}\left(\frac{e^{-2\lambda} - 1}{r^2}\right) + \frac{d}{dr}\left(\frac{e^{-2\lambda}\nu'}{r}\right) + e^{-2\lambda-2\nu}\frac{d}{dr}\left(\frac{e^{2\nu}\nu'}{r}\right) = 0 \tag{3.9}$$

which was first derived by Tolman (1939). In particular applications the form (3.9) leads to solutions more easily.

It is possible to write equations (3.6) in several equivalent forms by introducing new variables. The condition of pressure isotropy is also transformed by the new variables. The resultant forms generate different solutions to the Einstein field equations corresponding to particular physical relativistic models. We present a number of different transformations that have been used by researchers over the years. This is not a complete list of known transformations; we have focused on those transformations that are well known and which have proved to be useful in generating physically reasonable models for a relativistic stellar system.

3.3.1 Durgapal and Bannerji

Here we use the transformation

$$x = Cr^2 \tag{3.10a}$$

$$Z(x) = e^{-2\lambda(r)} \tag{3.10b}$$

$$A^2y^2(x) = e^{2\nu(r)} \tag{3.10c}$$

where A and C are constants. Therefore x is a new coordinate and $y(x)$ and $Z(x)$ are new metric functions. We use the above transformation (3.10) to write equations (3.6) as

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} \tag{3.11a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} \tag{3.11b}$$

$$4xZ\frac{\ddot{y}}{y} + 2(x\dot{Z} + 2Z)\frac{\dot{y}}{y} + \dot{Z} = \frac{p}{C} \tag{3.11c}$$

where the dots denote differentiation with respect to x. The condition of pressure isotropy (3.8) becomes

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1)y = 0 \tag{3.12}$$

in terms of y and Z.

The transformation (3.10) was first introduced by Durgapal and Bannerji (1983). The above transformation has been used by Durgapal and Fuloria (1985), Finch and Skea (1989), Maharaj and Komathiraj (2007), Maharaj and Thirukkanesh (2006, 2009) and Thirukkanesh and Maharaj (2006) to generate new solutions. We note that equation (3.12) is first order and linear in variable Z. If we have an analytic form for y then (3.12) can be integrated to generate Z. Then the unknowns ρ and p can be obtained from equations (3.11a) and (3.11b) respectively. Alternatively, we can choose a form for the potential Z and equation (3.12) becomes a second order and a linear equation in the variable y. Both approaches lead to exact solutions of the Einstein field equations.

3.3.2 Buchdahl

In this case we utilise the transformation

$$x = r^2 \tag{3.13a}$$

$$Y = e^{\nu} \tag{3.13b}$$

$$1 - 2xw = e^{-2\lambda} \tag{3.13c}$$

where we have set

$$w = mr^{-3}$$

$$m = \frac{1}{2}\int^{r} \rho\tilde{r}^2 d\tilde{r}$$

In the above transformation (3.13), x is the new coordinate and w and Y are the new metric functions. We use the transformation (3.13) to transform the Einstein field equations (3.6) to the new form

$$\rho = 6w + 4xw_x \tag{3.14a}$$

$$p = -2w + (4 - 8xw)\frac{Y_x}{Y} \tag{3.14b}$$

$$p = 4x(1 - 2xw)\frac{Y_{xx}}{Y} + 4(1 - 3xw - x^2 w_x)\frac{Y_x}{Y} - 2(w + xw_x) \tag{3.14c}$$

where the subscript x represents differentiation with respect to the new coordinate x. The condition of pressure isotropy (3.8) becomes

$$(1 - 2xw)Y_{xx} - (w + xw_x)Y_x - \frac{1}{2}w_x Y = 0 \tag{3.15}$$

in terms of w and Y.

The transformation (3.13) was first introduced by Buchdahl (1959). This transformation has been helpful in studying models of stellar structures and black holes; some of these studies are contained in the recent works of de Avellar and Horvath (2010), Lemos and Zanchin (2010), Rahaman *et al* (2010) and Yazadjiev (2011). Equation (3.15) is first order and linear equation in the variable w if the quantity Y is a specified function. Equation (3.15) is a second order and a linear equation in the variable Y if w is a specified function. Given w we can integrate to find Y and alternatively given Y, we can integrate to find w. Particular solutions found using the transformation (3.13) have been used to model dark energy stars, anisotropic fluids and strange matter configurations.

3.3.3 Fodor

In this case we introduce a new function α by defining

$$e^{2\lambda} = \frac{(1+r\nu')^2}{\alpha} \tag{3.16}$$

The advantage of this transformation is that we can replace the metric function λ by the new function α. We utilise (3.16) to rewrite the field equations (3.6) as the equivalent system

$$\rho = \frac{1}{r^2}\left(r\left[\frac{(1+r\nu')^2-\alpha}{(1+r\nu')^2}\right]\right)' \tag{3.17a}$$

$$p = \frac{1}{r^2}\left[\frac{\alpha(1+2r\nu')-(1+r\nu')^2}{(1+r\nu')^2}\right] \tag{3.17b}$$

$$p = \left[\nu''+\nu'^2+\frac{\nu'}{r}-\left(\nu'+\frac{1}{r}\right)\left(\frac{\nu'+r\nu''}{1+r\nu'}-\frac{\alpha'}{2\alpha}\right)\right]\left[\frac{\alpha}{(1+r\nu')^2}\right] \tag{3.17c}$$

where primes denote differentiation with respect to the radial coordinate r. The condition of pressure isotropy equation (3.8) has the form

$$r(1 + r\nu')\alpha' + 2[(1 - r\nu')^2 - 2]\alpha + 2(1 + r\nu')^2 = 0 \tag{3.18}$$

in terms of ν and α.

The transformation (3.16) was first introduced by Fodor (2000). The transformation (3.16) has been useful in analyses involving relativistic fluid spheres and compact objects; a sample of the investigations are given by Boehmer (2008), Lake (2003), Rahaman *et al* (2010) and Rahman and Visser (2002). Equation (3.18) is a first order and linear equation in the variable α if the quantity ν' is known. If α is a known quantity then (3.18) becomes a quadratic equation in $r\nu'$. The form of the condition of pressure isotropy (3.18) helps in generating simple relativistic stellar models since the integration process has been simplified.

3.3.4 Tewari and Pant

Another useful transformation involves the new variables

$$U = e^{\nu} \tag{3.19a}$$

$$V = e^{-2\lambda} \tag{3.19b}$$

We apply the transformation (3.19) to express the Einstein field equations (3.6) in the equivalent form as

$$-\frac{1}{r^2}(rV' + V - 1) \quad = \quad \rho \tag{3.20a}$$

$$\frac{1}{r^2}\left[\left(2r\frac{U'}{U} + 1\right)V - 1\right] \quad = \quad p \tag{3.20b}$$

$$\frac{1}{2}\left(\frac{U'}{U} + \frac{1}{r}\right)V' + \left(\frac{U''}{U} + \frac{1}{r}\frac{U'}{U}\right)V \quad = \quad p \tag{3.20c}$$

where primes denote differentiation with respect to the radial coordinate r. The condition of pressure isotropy equation (3.8) becomes

$$\left(\frac{U + rU'}{r^2 U}\right)V' + 2\left(\frac{U''}{rU} - \frac{U'}{r^2 U} - \frac{1}{r^3}\right)V = -\frac{2}{r^3} \tag{3.21}$$

in terms of U and V.

We can express the condition of pressure isotropy (3.21) in a simpler form. We multiply equation (3.21) with the factor

$$\frac{r^2 U}{U + rU'}$$

to eliminate the coefficient of V'. Then we obtain the compact result

$$V' - 2\left[\frac{d}{dr}\left(\log\left(\frac{r^3}{rU' + U}\right)\right) - \frac{2rU}{r^2(rU' + U)}\right]V = -\frac{2U}{r(rU' + U)} \tag{3.22}$$

The advantage of equation (3.22) is that it is first order and a linear equation in the variable V. The transformation (3.19) was first applied by Tewari and Pant (2010). The form (3.22), which is equivalent to (3.8) and (3.9), is often helpful in generating solutions for given particular forms of U. The transformation (3.19) has proved to be very useful in producing models of static fluid balls in general relativity and highly compact spheres; some particular solutions for stellar interiors were found by Pant (1994), Pant and Sah (1982, 1985) and Pant and Tewari (1990).

3.4 Exact solutions

In this section we provide some new classes of exact solutions to the Einstein field equations using the transformations generated in §3.3.

3.4.1 New Buchdahl models

Firstly we let

$$\rho = A \tag{3.23}$$

which corresponds to a constant density. Then (3.14a) can be integrated to give

$$w = \frac{1}{6}A + Kx^{-\frac{3}{2}} \tag{3.24}$$

where K is a constant. We substitute equation (3.24) into equation (3.15) to obtain the following

$$\left(1-\frac{1}{3}Ax-2Kx^{-\frac{1}{2}}\right)Y_{xx}-\left(\frac{1}{6}A-\frac{1}{2}Kx^{-\frac{3}{2}}\right)Y_x+\left(\frac{3}{4}Kx^{-\frac{5}{2}}\right)Y=0 \tag{3.25}$$

To avoid singularities in the gravitational potential w we set $K=0$. Then equation (3.25) simplifies to

$$\left(1-\frac{1}{3}Ax\right)Y_{xx}-\left(\frac{1}{6}A\right)Y_x=0 \tag{3.26}$$

which is a simple homogeneous linear equation in Y. We integrate equation (3.26) to get

$$Y(x)=-\frac{\alpha_1}{A}\sqrt{6-2Ax}+\alpha_2 \tag{3.27}$$

where α_1 and α_2 are arbitrary constants. The result (3.27) represents the known constant density Schwarzschild interior solution (Schwarzschild 1916a, 1916b).

Secondly we let

$$\rho=A+Bx \tag{3.28}$$

which corresponds to a linear density. Then (3.14a) can be solved to give

$$w=\frac{1}{6}A+\frac{1}{10}Bx+Kx^{-\frac{3}{2}} \tag{3.29}$$

Then the condition of pressure isotropy (3.15) becomes

$$\left(1-\frac{1}{3}Ax-\frac{1}{5}Bx^2-2Kx^{-\frac{1}{2}}\right)Y_{xx}$$

$$-\left(\frac{1}{6}A+\frac{1}{5}Bx-\frac{1}{2}Kx^{-\frac{3}{2}}\right)Y_x-\left(\frac{1}{20}B-\frac{3}{4}Kx^{-\frac{5}{2}}\right)Y=0 \qquad (3.30)$$

For a nonsingular metric function w we must set $K=0$. This gives the reduced result

$$\left(1-\frac{1}{3}Ax-\frac{1}{5}Bx^2\right)Y_{xx}-\left(\frac{1}{6}A+\frac{1}{5}Bx\right)Y_x-\left(\frac{1}{20}B\right)Y=0 \qquad (3.31)$$

We can integrate equation (3.31) to obtain

$$Y(x)=\beta_1\cos\left[\frac{1}{2}\log\left[5A+6Bx+2\sqrt{3B(-15+5Ax+3Bx^2)}\right]\right]$$

$$+\beta_2\sin\left[\frac{1}{2}\log\left[5A+6Bx+2\sqrt{3B(-15+5Ax+3Bx^2)}\right]\right] \qquad (3.32)$$

where β_1 and β_2 are arbitrary constants. We believe that this result corresponds to a new solution of the Einstein field equations.

We should point out that solutions to the Einstein field equations with a linear equation of state have been found by Lobo (2006), Sharma and Maharaj (2007) and Thirukkanesh and Maharaj (2008), amongst others. However note that in those cases the pressures are anisotropic and field equations are easily satisfied. In our case the pressure is isotropic and an additional differential equation, the condition of pressure isotropy, must be integrated. We believe that our new result (3.32) is one of the few regular exact models with a linear equation of state with isotropic pressures. The solutions considered by Krasinski (1997) and Stephani *et al* (2003) have a singularity at the stellar centre with these properties.

Thirdly we let ρ take the form

$$\rho = A + Bx + Cx^2 \tag{3.33}$$

which is quadratic in the density. Then the field equation (3.14a) can be integrated to obtain

$$w = \frac{1}{6}A + \frac{1}{10}Bx + \frac{1}{14}Cx^2 + Kx^{-\frac{3}{2}} \tag{3.34}$$

The condition of pressure isotropy (3.15) becomes

$$\begin{aligned} &\left(1 - \frac{1}{3}Ax - \frac{1}{5}Bx^2 - \frac{1}{7}Cx^3 - 2Kx^{-\frac{1}{2}}\right) Y_{xx} \\ &- \left(\frac{1}{6}A + \frac{1}{5}Bx + \frac{3}{14}Cx^2 - \frac{1}{2}Kx^{-\frac{3}{2}}\right) Y_x \\ &- \left(\frac{1}{20}B + \frac{1}{14}Cx - \frac{3}{4}Kx^{-\frac{5}{2}}\right) Y = 0 \end{aligned} \tag{3.35}$$

To avoid singularities in the gravitational potential w we set $K = 0$. We obtain the reduced equation

$$\left(1-\frac{1}{3}Ax-\frac{1}{5}Bx^2-\frac{1}{7}Cx^3\right)Y_{xx}$$

$$-\left(\frac{1}{6}A+\frac{1}{5}Bx+\frac{3}{14}Cx^2\right)Y_x-\left(\frac{1}{20}B+\frac{1}{14}Cx\right)Y=0 \quad (3.36)$$

The above equation (3.36) is of second order and a linear equation in variable Y.

Equation (3.36) is difficult to solve in general as it does not fall into any of the standard types listed in handbooks. Since the point $x = 0$ is a regular point, solutions in terms of Taylor series exist which can be obtained by the method of Frobenius. This is a topic of research that can be pursued in future work.

3.4.2 New Tewari and Pant models

Equation (3.22) is of first order and a linear equation in variable V. It can be integrated in principle and we get the form

$$\begin{aligned} V &= e^{-2\lambda} \\ &= \frac{r^6}{(rU'+U)^2}\left[A-2\int\frac{(rU'+U)Ue^{\int\frac{4U}{r(rU'+U)}dr}}{r^7}dr\right]e^{-\int\frac{4U}{r(rU'+U)}dr} \end{aligned} \quad (3.37)$$

where A is an arbitrary constant. We need to choose U so that the right hand side of (3.37) can be integrated. In the next section we discuss two choices of U that lead to solutions.

Solution I

For this class of solution we take the choice

$$e^{\int \frac{4U}{r(rU'+U)}dr} = r^l(rU'+U)^n \tag{3.38}$$

where l and n are arbitrary constants. The above equation (3.38), can be transformed into a second order homogeneous differential equation in U. In particular we obtain

$$nr^2U'' + (l+2n)rU' + (l-4)U = 0 \tag{3.39}$$

which can be solved since it is of second order and a linear equation in variable U. The equation (3.39) is solved in terms of elementary functions, and the solution is of the form

$$U = C_1 r^{a+b-1} + C_2 r^{a-b-1} \tag{3.40}$$

where C_1 and C_2 are arbitrary constants and

$$a = \frac{n-l}{2n} \tag{3.41a}$$

$$b = \frac{1}{2n}\sqrt{(n-l)^2 + 16n} \tag{3.41b}$$

where $n \neq 0$. Then equation (3.37) is simplified and we get

$$
\begin{aligned}
V &= e^{-2\lambda} \\
&= \frac{r^{8+n-l-(a-b)(n+2)}[A-2I]}{[(a+b)C_1 r^{2b}+(a-b)C_2]^{n+2}} \qquad (3.42)
\end{aligned}
$$

where

$$
I \equiv \int r^{l-n-9+(a-b)(n+2)}[(a+b)C_1 r^{2b}+(a-b)C_2]^{n+1}[C_1 r^{2b}+C_2]dr \qquad (3.43)
$$

The integral I in equation (3.43) can be evaluated explicitly if the parameter n is specified. For example if we choose $n=1$ we get that

$$
\begin{aligned}
I &= \int r^{l-10+3(a-b)}[(a+b)C_1 r^{2b}+(a-b)C_2]^2[C_1 r^{2b}+C_2]dr \\
&= \int (a+b)^2 C_1^3 r^{l-10+3(a+b)}dr + \int (3a-b)(a+b)C_1^2 C_2 r^{l-10+(3a+b)}dr \\
&\quad + \int (3a+b)(a-b)C_1 C_2^2 r^{l-10+(3a-b)}dr + \int (a-b)^2 C_2^3 r^{l-10+3(a-b)}dr \qquad (3.44)
\end{aligned}
$$

If we integrate equation (3.44) then we get

$$
\begin{aligned}
I &= \frac{(a+b)^2 C_1^3 r^{l-9+3(a+b)}}{l-9+3(a+b)} + \frac{(3a-b)(a+b)C_1^2 C_2 r^{l-9+(3a+b)}}{l-9+(3a+b)} \\
&\quad + \frac{(3a+b)(a-b)C_1 C_2^2 r^{l-9+(3a-b)}}{l-9+(3a-b)} + \frac{(a-b)^2 C_2^3 r^{l-9+3(a-b)}}{l-9+3(a-b)} + C_3 \qquad (3.45)
\end{aligned}
$$

where C_1, C_2 and C_3 are constants.

Hence the potential V has the form

$$
\begin{aligned}
V &= e^{-2\lambda} \\
&= \frac{r^{8+n-l-(a-b)(n+2)}[A-2I]}{[(a+b)C_1 r^{2b}+(a-b)C_2]^{n+2}} \\
&= A\frac{r^{\kappa-3(a-b)}}{\Psi(r)} - \frac{2(a+b)^2C_1^3 r^{6b}}{[3(a+b)-\kappa]\Psi(r)} - \frac{2(3a-b)(a+b)C_1^2C_2r^{4b}}{[(3a+b)-\kappa]\Psi(r)} \\
&\quad -\frac{2(3a+b)(a-b)C_1C_2^2r^{2b}}{[(3a-b)-\kappa]\Psi(r)} - \frac{2(a-b)^2C_2^3}{[3(a-b)-\kappa]\Psi(r)} - \frac{2r^{\kappa-3(a-b)}}{\Psi(r)}C_3 \qquad (3.46)
\end{aligned}
$$

where we have set

$$
\begin{aligned}
\kappa &= 9-l \\
\Psi(r) &= [(a+b)C_1r^{2b}+(a-b)C_2]^3
\end{aligned}
$$

We believe that the expressions for U and V generate a new solution to the Einstein field equations.

Solution II

For the second class of solution we take the choice

$$U = ar + b \tag{3.47}$$

where a and b are arbitrary constants. Then on substitution of U in the expression of equation (3.37), we have the following

$$e^{\int \frac{4U}{r(rU'+U)} dr} = e^{\int \frac{4(ar+b)}{r(2ar+b)} dr}$$

Note that the right hand side of the above integral can be evaluated using partial fractions. We have

$$\frac{4(ar+b)}{r(2ar+b)} = \frac{A}{r} + \frac{B}{2ar+b} \tag{3.48}$$

where A and B are constants. On evaluating the above fractions we get

$$\begin{aligned} A &= 4 \\ B &= -4a \end{aligned}$$

This gives

$$\begin{aligned} e^{\int \frac{4U}{r(rU'+U)} dr} &= e^{\int \frac{A}{r} dr + \int \frac{B}{2ar+b} dr} \\ &= r^A (2ar+b)^{\frac{B}{2a}} \end{aligned} \tag{3.49}$$

Hence we have

$$e^{\int \frac{4U dr}{r(rU'+U)}} = r^4(2ar+b)^{-2} \tag{3.50}$$

We substitute the above equation (3.50) in equation (3.37) to get the following form for V:

$$\begin{aligned} V &= e^{-2\lambda} \\ &= r^2\left[A - 2\int \frac{ar+b}{r^3(2ar+b)}dr\right] \end{aligned} \tag{3.51}$$

The integral above can be evaluated, if we write

$$\frac{ar+b}{r^3(2ar+b)} = \frac{\alpha}{2ar+b} + \frac{\beta r^2 + \xi r + \eta}{r^3}$$

where α, β, ξ and η are arbitrary constants. On evaluating the above fractions we get

$$\begin{aligned} \alpha &= -4a\left(\frac{a}{b}\right)^2 \\ \beta &= 2\left(\frac{a}{b}\right)^2 \\ \xi &= -\frac{a}{b} \\ \eta &= 1 \end{aligned}$$

The quantity

$$\frac{ar+b}{r^3(2ar+b)}$$

can be simplified by the above constants α, β, ξ and η to get

$$\frac{ar+b}{r^3(2ar+b)} = -4\left(\frac{a}{b}\right)^2 \frac{a}{(2ar+b)} + 2\left(\frac{a}{b}\right)^2 \frac{1}{r} - \left(\frac{a}{b}\right)\frac{1}{r^2} + \frac{1}{r^3}$$

Consequently the potential V in (3.37) is given by

$$\begin{aligned} V &= e^{-2\lambda} \\ &= r^2\left[A + \left(\frac{2a}{b}\right)^2 \log\left(\frac{2ar+b}{r}\right) - \frac{2}{r}\left(\frac{2ar-b}{2br}\right)\right] \end{aligned} \quad (3.52)$$

We point out that the expressions for U and V in this section generate another new solution to the Einstein field equations.

An infinite family of exact solutions to the Einstein field equations maybe generated utilising the transformation of Tewari and Pant (2010). This procedure depends on evaluating the integral in equation (3.37) which may not be easy to complete in practice.

Chapter 4

Isotropic coordinates

4.1 Introduction

In this chapter we consider the relativistic stellar model in isotropic coordinates. Particular solutions in isotropic coordinates have been found which are useful in astrophysical applications (Stephani *et al* 2003). In §4.2, we consider the spacetime geometry of the shear-free spacetime in isotropic coordinates. We generate the components of connection coefficients, the Ricci tensor, the Ricci scalar and the Einstein tensor. We consider the energy momentum for the perfect fluid, and generate the Einstein field equations in §4.3. The components of the energy momentum tensor are related to the components of the Einstein tensor to generate the Einstein field equations. We deduce the condition of pressure isotropy from the Einstein field equations. The condition of pressure isotropy is a second order differential equation with variable coefficients. In §4.4 we analyse two sets of transformations that enable us to express the condition of pressure isotropy in equivalent form. The first transformation leads to Riccati equations. The second transformation produces a linear equation. In §4.5 we develop an algorithm that enables us to produce a new solution to the Einstein field equations if a particular solution is specified. Several new classes of solutions are found.

4.2 Spacetime geometry

In this section we consider the isotropic line element which has the following form

$$ds^2 = -A^2(r)dt^2 + B^2(r)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi)] \tag{4.1}$$

where $A(r)$ and $B(r)$ are arbitrary functions. The line element is used to describe relativistic compact objects such as neutron stars in astrophysics.

The line element (4.1) is important for the determination of the connection coefficients $\Gamma^a{}_{bc}$. We use equation (2.2) and the above isotropic line element (4.1) to determine the nonvanishing connection coefficients:

$$\begin{aligned}
\Gamma^0{}_{10} &= \frac{A'}{A} \\
\Gamma^1{}_{00} &= \frac{AA'}{B^2} \\
\Gamma^1{}_{11} &= \frac{B'}{B}
\end{aligned}$$

$$
\begin{aligned}
\Gamma^{1}{}_{22} &= -r^{2}\left(\frac{B'}{B}+\frac{1}{r}\right) \\
\Gamma^{1}{}_{33} &= -r^{2}\sin^{2}\theta\left(\frac{B'}{B}+\frac{1}{r}\right) \\
\Gamma^{2}{}_{12} &= \frac{B'}{B}+\frac{1}{r} \\
\Gamma^{2}{}_{33} &= -\sin\theta\cos\theta \\
\Gamma^{3}{}_{13} &= \frac{B'}{B}+\frac{1}{r} \\
\Gamma^{3}{}_{23} &= \cot\theta
\end{aligned}
$$

The primes denote differentiation with respect to the radial coordinate r.

By using the above connection coefficients we generate the Ricci tensor components for the line element (4.1). We substitute the above connection coefficients in equation (2.4) which is the general form for the Ricci tensor in order to obtain the following nonvanishing components

$$R_{00} = \frac{A}{B^2}\left[A'' + A'\left(\frac{B'}{B} + \frac{2}{r}\right)\right] \tag{4.2a}$$

$$R_{11} = -\left(\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B}\right) - 2\left[\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{1}{r}\right)\right] \tag{4.2b}$$

$$R_{22} = -r^2\left(\frac{A'}{A}\frac{B'}{B} + \frac{1}{r}\frac{A'}{A}\right) - r^2\left(\frac{B''}{B} + \frac{3}{r}\frac{B'}{B}\right) \tag{4.2c}$$

$$R_{33} = \sin^2\theta R_{22} \tag{4.2d}$$

with $R_{ab} = 0$ for $a \neq b$.

We use the Ricci tensor components (4.2) and equation (2.5), which is the definition of the Ricci scalar, to compute the value

$$R = -\frac{2}{B^2}\left[\frac{A''}{A} + \frac{A'}{A}\left(\frac{B'}{B} + \frac{2}{r}\right)\right] - \frac{2}{B^2}\left[2\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{4}{r}\right)\right] \tag{4.3}$$

in terms of the potentials A and B.

In equation (2.6) we defined the Einstein tensor. For isotropic coordinates we use the Ricci tensor components (4.2) and the Ricci scalar (4.3) to generate the nonvanishing components of the Einstein tensor. These are given by the following:

$$G_{00} = -\left(\frac{A}{B}\right)^2 \left[2\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{4}{r}\right)\right] \tag{4.4a}$$

$$G_{11} = 2\frac{A'}{A}\left(\frac{B'}{B} + \frac{1}{r}\right) + \frac{B'}{B}\left(\frac{B'}{B} + \frac{2}{r}\right) \tag{4.4b}$$

$$G_{22} = r^2\left(\frac{A''}{A} + \frac{1}{r}\frac{A'}{A}\right) + r^2\left[\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{1}{r}\right)\right] \tag{4.4c}$$

$$G_{33} = \sin^2\theta G_{22} \tag{4.4d}$$

with $G_{ab} = 0$ for $a \neq b$.

4.3 Einstein field equations

Since the fluid four velocity is comoving we have $u^a = \frac{1}{A}\delta^a_0$ for the metric (4.1). The nonvanishing energy momentum tensors are given by

$$T_{00} = \rho A^2 \tag{4.5a}$$

$$T_{11} = pB^2 \tag{4.5b}$$

$$T_{22} = pB^2r^2 \tag{4.5c}$$

$$T_{33} = \sin^2\theta T_{22} \tag{4.5d}$$

with $T_{ab} = 0$ for $a \neq b$.

We use the Einstein tensor components (4.4) in conjunction with the energy momentum tensor components (4.5) in isotropic coordinates to generate the Einstein field equations. We thus obtain the following field equations

$$\rho = -\frac{1}{B^2}\left[2\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{4}{r}\right)\right] \tag{4.6a}$$

$$p = 2\frac{A'}{A}\left(\frac{B'}{B^3} + \frac{1}{r}\frac{1}{B^2}\right) + \frac{B'}{B^3}\left(\frac{B'}{B} + \frac{2}{r}\right) \tag{4.6b}$$

$$p = \frac{1}{B^2}\left(\frac{A''}{A} + \frac{1}{r}\frac{A'}{A}\right) + \frac{1}{B^2}\left[\frac{B''}{B} - \frac{B'}{B}\left(\frac{B'}{B} - \frac{1}{r}\right)\right] \tag{4.6c}$$

for isotropic coordinates.

Equating equation (4.6b) to (4.6c) gives the following equation

$$\frac{A''}{A}\frac{1}{B^2} + \frac{B''}{B^3} - \frac{1}{r}\frac{A'}{A}\frac{1}{B^2} - 2\frac{B'^2}{B^4} - \frac{B'}{B^3}\left(2\frac{A'}{A} + \frac{1}{r}\right) = 0 \tag{4.7}$$

which is the condition of pressure isotropy. The above equation (4.7) is the master equation for the gravitating model in isotropic coordinates. The above equation (4.7) can also be written in a more compact form as

$$\frac{A''}{A} + \frac{B''}{B} = \left(\frac{A'}{A} + \frac{B'}{B}\right)\left(2\frac{B'}{B} + \frac{1}{r}\right) \tag{4.8}$$

in terms of the potentials A and B.

4.4 Condition of pressure isotropy

Different transformations may be used to transform (4.8) into another equivalent form. In this section we present two transformations that have proved to be helpful in relativistic stellar stars.

4.4.1 Riccati

Note that (4.8) can be treated as a first order Riccati equation. If we let

$$z = \frac{A'}{A}$$

then (4.8) becomes

$$z' - \left(\frac{1}{r} + 2\frac{B'}{B}\right) z + z^2 = \left(\frac{1}{r} + 2\frac{B'}{B}\right)\left(\frac{B'}{B}\right) - \frac{B''}{B} \tag{4.9}$$

which is a Riccati equation in z. If the gravitational potential B is specified then we can integrate (4.9) in principle and obtain z.

Equivalently we can let

$$y = \frac{B'}{B}$$

and (4.8) becomes

$$y' - \left(\frac{1}{r} + 2\frac{A'}{A}\right) y - y^2 = \frac{1}{r}\frac{A'}{A} - \frac{A''}{A} \tag{4.10}$$

which is a Riccati equation in y. If the metric function A is specified then we can integrate equation (4.10) in principle and find y.

4.4.2 Kustaanheimo and Qvist

We now consider another familiar transformation which has been used by Kustaanheimo and Qvist (1948). The relevant quantities are

$$x \equiv r^2 \tag{4.11a}$$

$$L \equiv B^{-1} \tag{4.11b}$$

$$G \equiv LA \tag{4.11c}$$

With the assistance of the transformation (4.11), the field equations (4.6) become

$$\rho = 4[2xLL_{xx} - 3(xL_x - L)L_x] \tag{4.12a}$$

$$p = 4L(L - 2xL_x)\frac{G_x}{G} - 4(2L - 3xL_x)L_x \tag{4.12b}$$

$$p = 4xL^2\frac{G_{xx}}{G} + 4L(L - 2xL_x)\frac{G_x}{G} - 4(2L - 3xL_x)L_x - 8xLL_{xx} \tag{4.12c}$$

where subscript x represents differentiation with respect to the coordinate x.

From (4.12b) and (4.12c) we obtain the condition of pressure isotropy in the form

$$LG_{xx} = 2GL_{xx} \tag{4.13}$$

which is very compact. This equation (4.13) is linear in both L and G, and solutions may be found by prescribing one of these two functions appropriately.

4.5 Exact solutions: New Kustaanheimo and Qvist models

In this section we provide some new classes of exact solutions to the Einstein field equations using the transformations generated in §4.4.2.

Suppose that $(\bar{L}, \bar{G})$ is a known solution to (4.13). Then we have that

$$\bar{L}\bar{G}_{xx} = 2\bar{G}\bar{L}_{xx} \tag{4.14}$$

holds.

4.5.1 Algorithm I

We take a new solution (L, G) of the form

$$L = \bar{L} \tag{4.15a}$$

$$G = \bar{G}e^{f(x)} \tag{4.15b}$$

where $f(x)$ is an arbitrary function. Substituting (4.15) in (4.13) gives

$$\bar{L}(\bar{G}_{xx} + 2\bar{G}_x f_x + \bar{G}f_{xx} + \bar{G}f_x^2) = 2\bar{G}\bar{L}_{xx}$$

Since (4.14) holds the above equation simplifies to

$$\bar{G}f_{xx} + 2\bar{G}_x f_x + \bar{G}f_x^2 = 0 \tag{4.16}$$

which is a second order nonlinear equation in f. It is convenient to let

$$f_x = H$$

Then (4.16) becomes

$$H_x + 2\left(\frac{\bar{G}_x}{\bar{G}}\right)H + H^2 = 0 \tag{4.17}$$

which is a first order Bernoulli equation in H.

We can write (4.17) in the form

$$\left(\frac{1}{H}\right)_x - 2\left(\frac{\bar{G}_x}{\bar{G}}\right)\left(\frac{1}{H}\right) = 1 \tag{4.18}$$

which is a linear equation in $\frac{1}{H}$. We can integrate (4.18) to get the solution

$$H = \left[\bar{G}^2 \left(\int \bar{G}^{-2} dx + c_1\right)\right]^{-1}$$

Then the function f follows from integration of $f_x = H$. We obtain the result

$$f(x) = \int \left[\bar{G}^2 \left(\int \bar{G}^{-2} dx + c_1\right)\right]^{-1} dx + c_2 \tag{4.19}$$

where c_1 and c_2 are constants.

Hence a new solution to (4.13) is given by

$$L = \bar{L} \tag{4.20a}$$

$$G = \bar{G} \exp\left(\int \left[\bar{G}^2 \left(\int \bar{G}^{-2} dx + c_1\right)\right]^{-1} dx + c_2\right) \tag{4.20b}$$

We have shown that if a solution $(\bar{L}, \bar{G})$ to the field equations is given then a new solution (L, G) is given by (4.20) provided the transformation (4.15) applies.

4.5.2 Algorithm II

Alternatively we can take a new solution (L, G) of the form

$$G = \bar{G} \tag{4.21a}$$

$$L = \bar{L}e^{g(x)} \tag{4.21b}$$

where $g(x)$ is an arbitrary function. Substituting (4.21) in the condition of pressure isotropy (4.13) we obtain

$$\bar{L}\bar{G}_{xx} = 2\bar{G}(\bar{L}_{xx} + 2\bar{L}_x g_x + \bar{L}g_{xx} + \bar{L}g_x^2)$$

Since (4.14) holds then the above equation simplifies to

$$\bar{L}g_{xx} + 2\bar{L}_x g_x + \bar{L}g_x^2 = 0 \tag{4.22}$$

We note that equation (4.22) is second order and nonlinear equation in the variable g. We let

$$g_x = H$$

Then equation (4.22) becomes

$$H_x + 2\left(\frac{\bar{L}_x}{\bar{L}}\right)H + H^2 = 0 \tag{4.23}$$

which is a first order Bernoulli equation in H.

We can rewrite (4.23) in an equivalent form as

$$\left(\frac{1}{H}\right)_x - 2\left(\frac{\bar{L}_x}{\bar{L}}\right)\left(\frac{1}{H}\right) = 1 \tag{4.24}$$

which is a linear equation in $\frac{1}{H}$. We integrate (4.24) to obtain the following

$$H = \left[\bar{L}^2\left(\int \bar{L}^{-2}dx + c_1\right)\right]^{-1} \tag{4.25}$$

Then since $g_x = H$, we integrate the function of g to obtain

$$g(x) = \int \left[\bar{L}^2\left(\int \bar{L}^{-2}dx + c_1\right)\right]^{-1} dx + c_2 \tag{4.26}$$

where c_1 and c_2 are constants.

Hence a new solution to (4.13) is given by

$$G = \bar{G} \tag{4.27a}$$

$$L = \bar{L}\exp\left(\int \left[\bar{L}^2\left(\int \bar{L}^{-2}dx + c_1\right)\right]^{-1} dx + c_2\right) \tag{4.27b}$$

Therefore we have determined that if a solution $(\bar{L}, \bar{G})$ to the field equations is known then a new solution (L, G) is provided by (4.27) provided the transformation (4.21) holds. This solution is different from that presented in §4.5.1.

4.5.3 Algorithm III

Thirdly we make a new solution (L, G) of the form

$$L = \bar{L}e^{g(x)} \tag{4.28a}$$

$$G = \bar{G}e^{f(x)} \tag{4.28b}$$

where $f(x)$ and $g(x)$ are arbitrary functions. When substituting (4.28) into (4.13) we obtain

$$(\bar{L}\bar{G}_{xx} - 2\bar{G}\bar{L}_{xx}) + 2(\bar{L}\bar{G}_x f_x - 2\bar{G}\bar{L}_x g_x) + \bar{L}\bar{G}(f_{xx} - 2g_{xx}) + \bar{L}\bar{G}(f_x^2 - 2g_x^2) = 0$$

which is in terms of two arbitrary functions $f(x)$ and $g(x)$ unlike algorithms I and II. We use equation (4.14) which holds for the above equation to obtain the reduced result

$$(f_{xx} - 2g_{xx}) + 2\left(\frac{\bar{G}_x}{\bar{G}}f_x - 2\frac{\bar{L}_x}{\bar{L}}g_x\right) + (f_x^2 - 2g_x^2) = 0 \tag{4.29}$$

Since equation (4.29) is in terms of both $f(x)$ and $g(x)$ it is difficult to integrate in general. We consider three cases for equation (4.29) for which we have been able to complete the integration.

Case I: $g(x) = f(x)$

Firstly we let

$$g = f$$

in equation (4.29) to obtain

$$f_{xx} - 2\left(\frac{\bar{G}_x}{\bar{G}} - 2\frac{\bar{L}_x}{\bar{L}}\right) f_x + f_x^2 = 0 \tag{4.30}$$

It is convenient to let

$$f_x = H$$

Then (4.30) simplifies to

$$H_x - 2\left(\frac{\bar{G}_x}{\bar{G}} - 2\frac{\bar{L}_x}{\bar{L}}\right) H + H^2 = 0 \tag{4.31}$$

which is a first order Bernoulli equation in H.

The above equation (4.31) can be written in the form

$$\left(\frac{1}{H}\right)_x + 2\left(\frac{\bar{G}_x}{\bar{G}} - 2\frac{\bar{L}_x}{\bar{L}}\right)\left(\frac{1}{H}\right) = 1 \tag{4.32}$$

We note that (4.32) is a linear equation in $\frac{1}{H}$ and we integrate to get the solution

$$H = \frac{\bar{G}^2}{\bar{L}^4}\left(\int \frac{\bar{G}^2}{\bar{L}^4} dx + c_1\right)^{-1}$$

Since $f_x = H$, we integrate to obtain the function of f as

$$f(x) = \int \left[\frac{\bar{G}^2}{\bar{L}^4}\left(\int \frac{\bar{G}^2}{\bar{L}^4} dx + c_1\right)^{-1}\right] dx + c_2 \tag{4.33}$$

where c_1 and c_2 are constants.

Then the new solution to (4.13) has the form

$$L = \bar{L}\exp\left(\int\left[\frac{\bar{G}^2}{\bar{L}^4}\left(\int\frac{\bar{G}^2}{\bar{L}^4}dx + c_1\right)^{-1}\right]dx + c_2\right) \tag{4.34a}$$

$$G = \bar{G}\exp\left(\int\left[\frac{\bar{G}^2}{\bar{L}^4}\left(\int\frac{\bar{G}^2}{\bar{L}^4}dx + c_1\right)^{-1}\right]dx + c_2\right) \tag{4.34b}$$

Therefore we have shown that if a solution $(\bar{L}, \bar{G})$ to the field equations is known then a new solution (L, G) is provided by (4.34) provided the transformation (4.28) holds. Note that this algorithm provides a new solution different from §4.5.1 (using Algorithm I) and §4.5.2 (using Algorithm II) since $g = f$ and we cannot regain the transformations (4.15) and (4.21) except trivially.

Case II: $g(x) = \alpha f(x)$

(Note: case II includes case I as a special case)

Secondly we let

$$g(x) = \alpha f(x)$$

where α is an arbitrary constant. On substituting the above equation in (4.29) we obtain

$$f_{xx} + \frac{2}{1-2\alpha}\left(\frac{\bar{G}_x}{\bar{G}} - 2\alpha\frac{\bar{L}_x}{\bar{L}}\right)f_x + \left(\frac{1-2\alpha^2}{1-2\alpha}\right)f_x^2 = 0 \tag{4.35}$$

We let

$$f_x = H$$

in (4.35) to obtain the following

$$H_x + \frac{2}{1-2\alpha}\left(\frac{\bar{G}_x}{\bar{G}} - 2\alpha\frac{\bar{L}_x}{\bar{L}}\right)H + \left(\frac{1-2\alpha^2}{1-2\alpha}\right)H^2 = 0 \tag{4.36}$$

which is a first order Bernoulli equation in H.

We can write (4.36) in the form

$$\left(\frac{1}{H}\right)_x - \frac{2}{1-2\alpha}\left(\frac{\bar{G}_x}{\bar{G}} - 2\alpha\frac{\bar{L}_x}{\bar{L}}\right)\left(\frac{1}{H}\right) = \frac{(1-2\alpha^2)}{(1-2\alpha)} \tag{4.37}$$

which is linear in $\frac{1}{H}$. We integrate (4.37) to obtain the solution

$$H = \left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\frac{2}{1-2\alpha}}\left[\left(\frac{1-2\alpha^2}{1-2\alpha}\right)\int\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\frac{2}{1-2\alpha}}dx + c_1\right]^{-1} \tag{4.38}$$

We then integrate $f_x = H$ to obtain

$$f(x) = \int\left(\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\frac{2}{1-2\alpha}}\left[\left(\frac{1-2\alpha^2}{1-2\alpha}\right)\int\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\frac{2}{1-2\alpha}}dx + c_1\right]^{-1}\right)dx + c_2 \tag{4.39}$$

where c_1 and c_2 are constants.

We now have the new solution to (4.13) as

$$L = \bar{L}\exp\alpha\left[\int\left(\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\frac{2}{1-2\alpha}}\left[\zeta\int\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\frac{2}{1-2\alpha}}dx + c_1\right]^{-1}\right)dx + c_2\right] \tag{4.40a}$$

$$G = \bar{G}\exp\left[\int\left(\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\frac{2}{1-2\alpha}}\left[\zeta\int\left(\frac{\bar{L}^{2\alpha}}{\bar{G}}\right)^{\frac{2}{1-2\alpha}}dx + c_1\right]^{-1}\right)dx + c_2\right] \tag{4.40b}$$

where we have set

$$\zeta = \left(\frac{1 - 2\alpha^2}{1 - 2\alpha} \right)$$

Therefore we have demonstrated that if a solution $(\bar{L}, \bar{G})$ to the field equations is known then a new solution (L, G) is provided by (4.40) provided that the transformation (4.28) holds. If we set $\alpha = 1$ then (4.40) becomes

$$L = \bar{L} \exp\left(\int \left[\frac{\bar{G}^2}{\bar{L}^4} \left(\int \frac{\bar{G}^2}{\bar{L}^4} dx + c_1 \right)^{-1} \right] dx + c_2 \right) \qquad (4.41a)$$

$$G = \bar{G} \exp\left(\int \left[\frac{\bar{G}^2}{\bar{L}^4} \left(\int \frac{\bar{G}^2}{\bar{L}^4} dx + c_1 \right)^{-1} \right] dx + c_2 \right) \qquad (4.41b)$$

We observe that equation (4.41) is the same as (4.34). Hence the new class of solutions found here are a generalisation of the result in Case I.

Case III: $f_x^2 = 2g_x^2$

Thirdly we let

$$f_x^2 = 2g_x^2$$

in equation (4.29). This eliminates the nonlinearity and we get

$$f = \pm\sqrt{2}g + k$$

where k is a constant. Equation (4.29) becomes

$$(1-\sqrt{2})f_{xx} = 2\left(\sqrt{2}\frac{\bar{L}_x}{\bar{L}} - \frac{\bar{G}_x}{\bar{G}}\right) f_x \tag{4.42}$$

This is a simple homogeneous linear equation. We integrate (4.42) to obtain

$$f(x) = c_1 \int \left(\frac{\bar{L}^{\sqrt{2}}}{\bar{G}}\right)^{\frac{2}{1-\sqrt{2}}} dx + c_2 \tag{4.43}$$

where c_1 and c_2 are constants.

Hence a new solution to (4.13) has the form

$$L = \bar{L}\exp\left[k \pm \frac{1}{\sqrt{2}}\left(c_1 \int \left(\frac{\bar{L}^{\sqrt{2}}}{\bar{G}}\right)^{\frac{2}{1-\sqrt{2}}} dx + c_2\right)\right] \tag{4.44a}$$

$$G = \bar{G}\exp\left[c_1 \int \left(\frac{\bar{L}^{\sqrt{2}}}{\bar{G}}\right)^{\frac{2}{1-\sqrt{2}}} dx + c_2\right] \tag{4.44b}$$

Therefore we have shown that if a solution $(\bar{L}, \bar{G})$ to the field equations is known then a new solution (L, G) is provided by (4.44) provided the transformation (4.28) holds with $f = \pm\sqrt{2}g + k$. Note that if we set $k = 0$ in (4.44) and $\alpha = \pm\frac{1}{\sqrt{2}}$ in (4.40) we have the same result. Hence Case II and Case III are the same for these parameter values; in general they have different functional forms.

Chapter 5

Conclusion

Our aim in this thesis was to examine the static spherically symmetric spacetimes and the Einstein field equations in relativistic astrophysics. Our main objective was to generate new exact solutions of the Einstein field equations with isotropic pressures. Since the Einstein field equations are highly nonlinear in general we used new variables in order to transform the field equations to equivalent forms. We transformed the condition of pressure isotropy by reducing it to less complicated second order differential equations with variable coefficients. We obtained some several new exact solutions in terms of elementary functions by choosing specific gravitational potentials in order to solve the master equation. We generated a number of algorithms that produce a new solution if a particular model is specified. The new exact solutions are useful in many applications for general relativity and realistic stellar models.

We now provide a brief outline of the dissertation by giving the main results achieved in our course of study:

• In chapter 2, we briefly introduced the concepts of differential geometry and the matter distribution that are essential for generating the Einstein field equations. We formulated the Einstein field equations for neutral perfect fluid matter distributions. We also briefly introduced the barotropic equation of state relating the pressure to the energy density. We outlined the physical conditions that are relevant for a realistic relativistic stellar model.

• In chapter 3, we generated the Einstein field equations in terms of Schwarzschild coordinates for neutral perfect fluid matter distributions in static spherically symmetric spacetimes. As a result of the high nonlinearity of the field equations we expressed them in equivalent forms using the transformations of Buchdahl (1959), Durgapal and Bannerji (1983), Fodor (2000) and Tewari and Pant (2010). We found particular new classes of exact solutions to the Einstein field equations in terms of elementary functions by integration of the condition of pressure isotropy. With the help of the Buchdahl (1959) transformation we obtained the exact solution

$$Y(x) = \beta_1 \cos\left[\frac{1}{2}\log\left[5A + 6Bx + 2\sqrt{3B(-15 + 5Ax + 3Bx^2)}\right]\right]$$

$$+\beta_2 \sin\left[\frac{1}{2}\log\left[5A + 6Bx + 2\sqrt{3B(-15 + 5Ax + 3Bx^2)}\right]\right]$$

which we believe is a new solution of Einstein's equations. This model is characterised by a linear barotropic equation of state and is regular at the origin. With the assistance of the Tewari and Pant (2010) transformation we obtained two exact solutions. The first is given by

$$
\begin{aligned}
V &= e^{-2\lambda} \\
&= A\frac{r^{\kappa-3(a-b)}}{\Psi(r)} - \frac{2(a+b)^2C_1^3r^{6b}}{[3(a+b)-\kappa]\Psi(r)} - \frac{2(3a-b)(a+b)C_1^2C_2r^{4b}}{[(3a+b)-\kappa]\Psi(r)} \\
&\quad - \frac{2(3a+b)(a-b)C_1C_2^2r^{2b}}{[(3a-b)-\kappa]\Psi(r)} - \frac{2(a-b)^2C_2^3}{[3(a-b)-\kappa]\Psi(r)} - \frac{2r^{\kappa-3(a-b)}}{\Psi(r)}C_3
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa &= 9-l \\
\Psi(r) &= [(a+b)C_1r^{2b} + (a-b)C_2]^3
\end{aligned}
$$

We also found the second exact solution

$$
\begin{aligned}
V &= e^{-2\lambda} \\
&= r^2\left[A + \left(\frac{2a}{b}\right)^2 \log\left(\frac{2ar+b}{r}\right) - \frac{2}{r}\left(\frac{2ar-b}{2br}\right)\right]
\end{aligned}
$$

The above two classes of exact models are new solutions to the Einstein field equations.

- In chapter 4, we considered the stellar model in isotropic coordinates. We generated the Einstein field equations by using the energy momentum for the perfect fluid. From the Einstein field equations we deduced the condition of pressure isotropy which is a second order differential equation with variable coefficients. The condition of pressure isotropy was also expressed into two equivalent forms by using new variables. The result of the first transformation leads to the Riccati equation. The second transformation was first introduced by Kustaanheimo and Qvist (1948). Using this transformation we generated an algorithm that enables us to produce three new classes of exact solutions to the Einstein field equations. With the help of Algorithm I we showed if $(\bar{L}, \bar{G})$ is a given solution to the condition of pressure isotropy then a new solution (L, G) can be found subject to the consistency condition

$$\bar{G} f_{xx} + 2\bar{G}_x f_x + \bar{G} f_x^2 = 0$$

We integrated this equation in general. Similarly with Algorithm II we demonstrated that a new solution can be found subject to integrating the consistency condition

$$\bar{L} g_{xx} + 2\bar{L}_x g_x + \bar{L} g_x^2 = 0$$

We solved this differential equation in general. With Algorithm III new solutions are possible subject to the integration of the consistency condition

$$(f_{xx} - 2g_{xx}) + 2\left(\frac{\bar{G}_x}{\bar{G}} f_x - 2\frac{\bar{L}_x}{\bar{L}} g_x\right) + (f_x^2 - 2g_x^2) = 0$$

It is not possible to integrate this equation in general. However we showed that for the special cases

$$
\begin{aligned}
g &= f
\end{aligned}
$$

$$
\begin{aligned}
g &= \alpha f
\end{aligned}
$$

$$
\begin{aligned}
f_x^2 &= g_x^2
\end{aligned}
$$

integration is possible and exact solutions are generated. It is remarkable that the procedure developed in §4.5 produces infinite families of new exact solutions provided a particular model $(\bar{L}, \bar{G})$ is known.

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