ℂharacter 𝕋ables of the 𝔾eneral 𝕃inear 𝔾roup and 𝕊ome of its 𝕊ubroups

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`A project submitted in the fulfillment of the requirements for the Masters Degree of Science in the School of Mathematical Sciences, University of KwaZulu-Natal, Pietermaritzburg.`

November 2008

Abstract

The aim of this dissertation is to describe the conjugacy classes and some of the ordinary irreducible characters of the finite general linear group $GL(n,q)$, together with character tables of some of its subgroups. We study the structure of $GL(n,q)$ and some of its important subgroups such as $SL(n,q)$, $UT(n,q)$, $SUT(n,q)$, $Z(GL(n,q))$, $Z(SL(n,q))$, $GL(n,q)'$, $SL(n,q)'$, the Weyl group W and parabolic subgroups P_λ. In addition, we also discuss the groups $PGL(n,q)$, $PSL(n,q)$ and the affine group $\mathrm{Aff}(n,q)$, which are related to $GL(n,q)$. The character tables of $GL(2,q)$, $SL(2,q)$, $SUT(2,q)$ and $UT(2,q)$ are constructed in this dissertation and examples in each case for $q=3$ and $q=4$ are supplied.
A complete description for the conjugacy classes of $GL(n,q)$ is given, where the theories of irreducible polynomials and partitions of $i \in \{1,2,\cdots,n\}$ form the atoms from where each conjugacy class of $GL(n,q)$ is constructed. We give a special attention to some elements of $GL(n,q)$, known as *regular semisimple*, where we count the number and orders of these elements. As an example we compute the conjugacy classes of $GL(3,q)$. Characters of $GL(n,q)$ appear in two series namely, *principal* and *discrete* series characters. The process of the *parabolic induction* is used to construct a large number of irreducible characters of $GL(n,q)$ from characters of $GL(m,q)$ for $m<n$. We study some particular characters such as *Steinberg characters* and *cuspidal characters* (characters of the discrete series). The latter ones are of particular interest since they form the atoms from where each character of $GL(n,q)$ is constructed. These characters are parameterized in terms of the Galois orbits of non-decomposable characters of $\mathbb{F}_{q^n}^*$. The values of the cuspidal characters on classes of $GL(n,q)$ will be computed. We describe and list the full character table of $GL(3,q)$. There exists a duality between the irreducible characters and conjugacy classes of $GL(n,q)$, that is to each irreducible character, one can associate a conjugacy class of $GL(n,q)$. Some aspects of this duality will be mentioned.

Preface

The work described in this dissertation was carried out under the supervision and direction of Professor Jamshid Moori, School of Mathematical Sciences, University of KwaZulu Natal, Pietermaritzburg, from February 2007 to November 2008.
The dissertation represent original work of the author and has not been otherwise been submitted in any form for any degree or diploma to any University. Where use has been made of the work of others it is duly acknowledged in the text.

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Signature (Student) Date

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Signature (Supervisor) Date

Dedication

TO MY PARENTS, MY FAMILY, MUSA COMTOUR, LAYLA SORKATTI, SHOSHO, EMAN NASR AND TO THE SOUL OF MY DEAR FRIEND JEAN CLAUDE (ABD ALKAREEM), I DEDICATE THIS WORK.

Acknowledgements

First of all, I thank ALLAH for his Grace and Mercy showered upon me. I heartily express my profound gratitude to my supervisor, Professor Jamshid Moori, for his invaluable learned guidance, advises, encouragement, understanding and continued support he has provided me throughout the duration of my studies which led to the compilation of this dissertation. I will be always indebted to him for introducing me to this fascinating area of Mathematics and creating my interest in Group Theory. I have learnt so much from him, not only in the academic orientation, but in various walks of life.

I am grateful for the facilities made available to me by the School of Mathematical Sciences of University of KwaZulu Natal, Pietermaritzburg. I am also grateful for the scholarship I have received from University of Stellenbosch through the African Institute for Mathematical Sciences (AIMS). My thanks also go to the National Research Foundation (NRF) for a grant holder bursary through Professor Moori and to the University of KwaZulu Natal for the graduate assistantship. My thanks extend to the administration of University of Khartoum, in particular to Dr Mohsin, dean of Faculty of Mathematical Sciences for releasing me to do my MSc and also for the full salary I have received from them during the period of my MSc. I would like to thank my officemates TT. Seretlo and Habyarimana Faustin for creating a pleasant working environment. Finally I sincerely thank my entire family represented by Basheer, Suaad, Eihab, Adeeb, Nada, Khalid, Tayseer, Rana, Amro, Mustafa.

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List of Notations

$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integer numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
V	vector space
dim	dimension of a vector space
$End(V)$	set of endomorphisms of a vector space V
∂f	degree of a polynomial f
$\mathcal{F}$	set of all irreducible polynomials of degree $\leq n$, except $f(t) = t$
$U_1(f)$	companion matrix of polynomial f
$U_m(f)$	matrix with $U_1(f)$ in the main diagonal and $I_{\partial f}$ in the super diagonal
$U_\lambda(f)$	direct sum of Jordan blocks correspond to the parts of λ
$I_m(q)$	number of irreducible polynomials of degree m over $\mathbb{F}_q$
det	determinant of a matrix
tr	trace of a matrix
$\mathbb{F}$	field
$\overline{\mathbb{F}}$	algebraic closure of $\mathbb{F}$
$\mathbb{F}^*$	multiplicative group of $\mathbb{F}$
$\mathbb{F}_q$	Galois field of q elements
$\mathbb{F}_{q^n} : \mathbb{F}_q$	field extension
$\Gamma,\ \Gamma(\mathbb{F}_{q^n} : \mathbb{F}_q)$	Galois group of field extension
$(\mathbb{F}_q^*)^2$	subgroup of $\mathbb{F}_q^*$ consisting of square elements
G	finite group
$e,\ 1_G$	identity element of G
$\|G\|$	order of G
$o(g)$	order of $g \in G$
$\cong$	isomorphism of groups
$H \leq G$	H is a subgroup of G
$[G : H]$	index of H in G

$N \trianglelefteq G$	N is a normal subgroup of G
$K \times Q,\ \bigotimes$	direct product of groups
$K{:}Q$	split extension of K by Q (semidirect product)
G/N	quotient group
$\mathcal{C}_1, \cdots, \mathcal{C}_k$	distinct conjugacy classes of a finite group G
$[g],\ C_g$	conjugacy class of g in G
$C_G(g)$	centralizer of $g \in G$
G_x	stabilizer of $x \in X$ when G acts on X
x^G	orbit of $x \in X$
$\|Fix(g)\|$	number of elements in a set X fixed by $g \in G$ under group action
$Aut(G)$	automorphism group of G
$Holo(G)$	holomorph of G
$[a,b]$	commutator of $a, b \in G$
$G^{'}$	derived or commutator subgroup of G
$Z(G)$	center of G
D_{2n}	dihedral group consisting of $2n$ elements
$Syl_p(G)$	set of Sylow $p-$subgroups of G
$\mathbb{Z}_n$	group $\{\overline{0}, \overline{1}, \cdots, \overline{n-1}\}$ under addition modulo n
S_n	symmetric group of n objects
A_n	alternating group of n objects
$GL(n, \mathbb{F})$	general linear group
$GL(n, q)$	finite general linear group
$SL(n, \mathbb{F})$	special linear group
$SD(n, q)$	subgroup of $GL(n, q)$ consisting of elements with square determinant
$UT(n, \mathbb{F})$	standard Borel subgroup of $GL(n, q)$ or group of invertible upper triangular matrices
$U_n(\mathbb{F})$	unitary group
$SUT(n, \mathbb{F})$	special upper triangular group ($SL(n, \mathbb{F}) \cap UT(n, \mathbb{F})$)
$SUUT(n, \mathbb{F})$	special upper triangular group with 1's along the main diagonal
$PGL(n, \mathbb{F})$	projective general linear group $GL(n, \mathbb{F})/Z(GL(n, \mathbb{F}))$
$PSL(n, \mathbb{F})$	projective special linear group $SL(n, \mathbb{F})/Z(SL(n, \mathbb{F}))$
$\mathrm{Aff}(n, \mathbb{F})$	affine group
$PG(n, \mathbb{F})$	projective geometry
$G_n^{\epsilon}(q)$	$GL(n, q)$ if $\epsilon = 1$ or $U_n(q)$ if $\epsilon = -1$
W	Weyl group
$\mathcal{A}$	root subgroup of $GL(n, \mathbb{F})$
$\lambda \vdash n$	λ is a partition of n
$l(\lambda)$	length of λ

λ'	conjugate partition of λ
$n(\lambda)$	$\sum_{i=1}^{l(\lambda')} \frac{\lambda'_i(\lambda'_1 - 1)}{2}$
$\phi_m(q)$	$\prod_{i=1}^{m}(1-q^i)$ if $m \geq 1$ and 1 if $m = 0$
$\phi_\lambda(q)$	$\prod_{i=1}^{k} \phi_{m_{\lambda_i}}(q)$, where $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \cdots, \lambda_k^{m_k})$
$\mathcal{P}(n)$	set of all partitions of n
$\mathfrak{F}$, $\mathfrak{F}_\lambda$	flag of a vector space V
P_λ	parabolic subgroup of $GL(n, \mathbb{F})$
U_λ	unipotent radical of P_λ
L_λ	levi complement of P_λ
$c(n, q)$	number of conjugacy classes of $GL(n, q)$
$t(n)$	number of types of conjugacy classes of $GL(n, q)$
$\mathcal{T}^{(i)}$	type of conjugacy classes of $GL(n, q)$
$\#\mathcal{T}^{(i)}$	number of conjugacy classes of $GL(n, q)$ of type $\mathcal{T}^{(i)}$
c^λ	regular semisimple class of $GL(n, q)$ of type λ
$F(\lambda)$	number of regular semisimple classes of type λ
$Reg(G_n^\epsilon(q))$	number of regular semisimple classes of $G_n^\epsilon(q)$
$g^c_{c_1,c_2,\cdots,c_k}$	Hall polynomial
ch_c	characteristic function of a class c
χ	character of finite group
χ_ρ	character afforded by a representation ρ of G
$\mathbf{1}$	trivial character of G
deg	degree of a representation or a character
$Irr(G)$	set of the ordinary irreducible characters of G
$Ch(\mathbb{F}^*_{q^n})$	character group of $\mathbb{F}^*_{q^n}$
$\langle\ ,\ \rangle$	inner product of class functions or a group generated by two elements (depends on the context of the discussion)
$\otimes$	tensor product of representations
$\oplus$, $\bigoplus$	direct sum
$\bigodot_{i=1}^{k} \psi_i$	$\odot-$product of class functions (or characters) ψ_i of $GL(n, q)$
$\mathfrak{C}$	algebra of class functions of $GL(n, q)$
$\mathfrak{R}$	algebra of characters of $GL(n, q)$
$\mathfrak{S}$	algebra of symmetric functions
$\chi^{(i)}$	type of characters of $GL(n, q)$
$\#\chi^{(i)}$	number of characters of $GL(n, q)$ of type $\chi^{(i)}$

$\chi\uparrow_H^G$	character induced from a subgroup H to G
$\chi\downarrow_H^G$	character restricted from a group G to its subgroup H
$St^{(\lambda)}$	Steinberg character of $GL(n,q)$
$S^{(\lambda)}$	permutation character $\mathbf{1}\uparrow_{S_\lambda}^{S_n}$
$C^{(\lambda)}$	permutation character $\mathbf{1}\uparrow_{P_\lambda}^{GL(n,q)}$
S_λ	$S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$, where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$
$\ker \phi$	kernel of a homomorphism ϕ
$Im\ \phi$	image of a function ϕ
V_c	module corresponds to a conjugacy class c of $GL(n,q)$
$V_{\langle f_i \rangle}$	characteristic submodule
$\mathbb{F}[x]$	ring of polynomials over $\mathbb{F}$
$\mathbb{Z}[t_1, t_2, \cdots, t_k]$	ring of symmetric polynomials over $\mathbb{Z}$ in indeterminates $t_1, t_2, \cdots, t_k$
$Ann(v)$	annihilator of v in a ring.
$N_{n,d}$	norm map
$ND(\mathbb{F}_q^*)$	set of non-decomposable characters of $\mathbb{F}_q^*$

1

Introduction

The *general linear group* $GL(V)$ is the automorphism group of a vector space V. The term *linear* comes because of the linear transformations and the term *general* comes because it is the largest group with the property of invertibility. When $V = V(n, \mathbb{F})$, the $n-$dimensional space over a field $\mathbb{F}$, we identify $GL(V)$ with the group $GL(n, \mathbb{F})$ consisting of all invertible $n \times n$ matrices. Moreover, if $\mathbb{F} = \mathbb{F}_q$, the *Galois field* of q elements, we write $GL(n, q)$ in place of $GL(n, \mathbb{F}_q)$. If $n = 1$, then $GL(1, \mathbb{F}) \cong \mathbb{F}^*$, which is abelian. The smallest general linear group is $GL(1, 2) \cong \mathbb{F}_2^* = \{1\}$. If $n > 1$, then $GL(n, \mathbb{F})$ is not abelian and the smallest non-abelian general linear group is $GL(2, 2) \cong S_3$. Also $GL(n, \mathbb{F})$ is not a simple group in general as it has many normal subgroups such as $SL(n, q)$, the special linear group. In 1907, H. Jordan [35] and I. Schur [67] separately calculated the ordinary character table of $GL(2, q)$. It was not until 1950 that the character table of $GL(3, q)$ was known, when Steinberg [72] determined the character tables of $GL(3, q)$ and $GL(4, q)$. Many attempts to calculate the ordinary character tables of $GL(n, q)$ for arbitrary n were made. For example partial results found by Steinberg, namely the *Steinberg characters* of $GL(n, q)$. In 1955, J. A. Green in a celebrated paper [27] was able to give a complete description for the character tables of $GL(n, q)$ for any positive integer n. To construct the characters of $GL(n, q)$, Green [27] combined the Frobenius method of induced characters from certain subgroups, together with Brauer's theorem of modular representations. The use of subgroups is similar to the Frobenius treatment of the character table of the Symmetric group S_n. In fact the work of Green [27] on $GL(n, q)$ inspired other authors, like Deligne - Lusztig [16] in their search for the characters of *reductive groups*. This was to generalize some of the aspects defined by Green [27] such as *Green polynomials* and *degeneracy rule.*

Below is a detailed description for the work carried on this dissertation:

In Chapter 2 we review the fundamental tools required for the theories of representations and characters, which will be used in the other chapters. This includes basic definitions and elementary results of representations and characters (Sections 2.1 and 2.2). Also we study some results of constructing new characters from characters we already know. In Section 2.3 we show that the product of two characters of a group G is again a character of G. In Section 2.4 we show that if G has a normal subgroup N then irreducible characters of the quotient G/N extend (lift) irreducibly to G. In Section 2.5 we study the dual operations known as *induction* and *restriction* of characters. We conclude Chapter 2 by studying an important type of characters of a group G known as the permutation character, which is associated with the group action. For instance if we have a subgroup $H \leq G$, then there exists a permutation character of G. Conversely if we have a permutation character of G, then under some certain conditions, we show the existence of a subgroup $H \leq G$.

Chapter 3 concerns with the structure of $GL(n, \mathbb{F})$ and some of its important subgroups such as $SL(n, \mathbb{F})$, $UT(n, \mathbb{F})$, $SUT(n, \mathbb{F})$, $Z(GL(n, \mathbb{F}))$, $Z(SL(n, \mathbb{F}))$, $GL(n, \mathbb{F})'$, $SL(n, \mathbb{F})'$, Weyl group W and parabolic subgroups P_λ. In addition we also discuss the groups $PGL(n, \mathbb{F})$, $PSL(n, \mathbb{F})$ and the affine group $\mathrm{Aff}(n, \mathbb{F})$, which are related to $GL(n, \mathbb{F})$. In most of these groups we focus on the case $\mathbb{F} = \mathbb{F}_q$. In the last section of this chapter we discuss the concept of the BN pair structure and we show that $GL(n, \mathbb{F})$, $SL(n, \mathbb{F})$ and $PSL(n, \mathbb{F})$ have BN structures.

In Chapter 4 we determine the character table of $GL(2, q)$, where in Section 4.2, we discuss the conjugacy classes of $GL(2, q)$ and see that there are $q^2 - 1$ classes fall into four families (Theorem 4.2.1). Also the orders of elements of $GL(2, q)$ will be given (Proposition 4.2.2). In Section 4.3 the irreducible characters of $GL(2, q)$ will be listed. These characters fall also in four families. The character table of $GL(2, q)$ will be used to construct character tables of $SL(2, q)$, $SUT(2, q)$ and $UT(2, q)$ in Sections 4.4, 4.5 and 4.6 respectively. In Section 4.4 the treatment of obtaining the character table of $SL(2, q)$ will depends on the parity of q. When q is even, $SL(2, q)$ has $q + 1$ irreducible characters, which are obtained from restriction of some characters of $GL(2, q)$. When q is odd, $SL(2, q)$ has $q + 4$ irreducible characters. Of these, q are obtained directly from the restriction of some of the characters of $GL(2, q)$. To find the other 4 characters of $SL(2, q)$, a subgroup of $GL(2, q)$ containing $SL(2, q)$ will enter to complete the picture. This subgroup, which is denoted by $SD(2, q)$ has index 2 in $GL(2, q)$. We list all the conjugacy classes and some of the irreducible characters of $SD(2, q)$. In Section 4.5 we prove that $SUT(2, q)$, q odd, has $q+3$ irreducible characters, while if q is even, then $SUT(2, q)$ has q irreducible characters. In the latter case, the character table of $SUT(2, q)$ will be constructed in two different methods. First we use the fact that $SUT(2, q)$ is one of the *Frobenius groups*, whose representations are known. The other approach is through the technique of the *coset analysis* together with *Clifford-Fischer* theory (see Moori [52] and Whitely [76]). In Section 4.6 we show that $UT(2, q)$ has $q^2 - q$ irreducible characters and we list the values of these characters on classes of $UT(2, q)$. An extensive number of examples of character tables of $GL(2, q)$, $SL(2, q)$, $SUT(2, q)$ and $UT(2, q)$ for $q = 3$ and $q = 4$ will be given in Section 4.7.

Chapter 5 contains the main results of this dissertation. In this chapter, we consider $GL(n, q)$ in general for any n. Section 5.1 is devoted to the study partitions of a positive integer n and some functions defined in terms of partitions, which will be used throughout the sequel of chapter 5. In Section 5.2 the conjugacy classes of $GL(n,q)$ will be determined completely, where we give a source for the representatives of the classes (Jordan Canonical Form, Theorem 5.2.1). We also calculate the size of any conjugacy class of $GL(n,q)$ (Equation 5.10). The classes of $GL(n,q)$ fall within several types and all classes of the same type have same size. There are some elements of $GL(n,q)$, called *regular semisimple*, which are of particular interest. We count the number and orders of these elements (Theorems 5.2.13 and 5.2.17). The types of regular semisimple classes of $GL(n,q)$ are in $1-1$ correspondence with partitions of n. Also we count the number of *primary classes* of $GL(n,q)$ (Proposition 5.2.14). As an application, we construct the conjugacy classes of $GL(3,q)$, count the number and orders of regular semisimple elements of $GL(3,q)$. We show that the ratio between the number of regular semisimple classes of $GL(3,q)$ of partition type $(n) \vdash n$ and those classes of $GL(3,q)$, which are not regular semisimple of type $(n) \vdash n$ is given by

$$\frac{\text{Number of regular semisimple classes of } GL(3,q) \text{ of type } (n) \vdash n}{\text{Number of non-regular semisimple classes of } GL(3,q) \text{ of type } (n) \vdash n} = \frac{\frac{1}{3}(q^3-q)}{\frac{2}{3}(q^3-q)} = \frac{1}{2}.$$

In Section 5.3 we discuss the process of *parabolic induction*, which produces a large number of characters of $GL(n,q)$ from characters of $GL(m,q)$ for $m < n$. The parametrization of such characters is, in some sense, related to the character theory of the Symmetric group S_n, where some characters of S_n are obtained by induction from characters of Young subgroups. The remaining characters of $GL(n,q)$, which cannot be obtained by parabolic induction, are called *cuspidal* characters or characters of the *discrete series*. Section 5.4 is devoted to the cuspidal characters of $GL(n,q)$, which have nice parametrization in terms of the Galois orbits of non-decomposable characters of $\mathbb{F}_{q^n}^*$ (Subsection 5.4.1). We also calculate the values of these characters on classes of $GL(n,q)$ (Theorem 5.4.4 and Equation (5.19)) and finally we show the importance of the cuspidal characters for all characters of $GL(n,q)$ (Theorem 5.4.6). In Section 5.5 we study the so-called *Steinberg characters* of $GL(n,q)$. For any partition of n, Steinberg found an irreducible character of $GL(n,q)$. He used simple properties of the underlying geometry of a vector space V. We list the values of Steinberg characters of $GL(2,q)$, $GL(3,q)$ and $GL(4,q)$. In Section 5.6 we go briefly over Green construction of characters, which is based on modular characters of $GL(n,q)$ (Theorem 5.6.2). We also prove that the number of linear characters of $GL(n,q)$, $(n,q) \neq (2,2)$ is $q-1$ (Theorem 5.6.3). The last section of this chapter is an application to the character table of $GL(3,q)$. The maximal parabolic subgroup $MP(3,q)$ of $GL(3,q)$ will produce a considerable number of irreducible characters of $GL(3,q)$. In fact this number is $\frac{2}{3}|Irr(GL(3,q))| = \frac{2}{3}(q^3-q)$, which is equal to the number of principal series characters of $GL(3,q)$. Therefore we have

$$\frac{\text{Number of cuspidal characters of } GL(3,q)}{\text{Number of principal series characters of } GL(3,q)} = \frac{\frac{1}{3}(q^3-q)}{\frac{2}{3}(q^3-q)} = \frac{1}{2}.$$

Green [27] established a duality between the irreducible characters and conjugacy classes of $GL(n,q)$, that is to each irreducible character of $GL(n,q)$, one can associate a conjugacy class of $GL(n,q)$; a property that not many groups have. We conclude Chapter 5 by mentioning some aspects of this duality (Table 5.14).

Finally a list of character tables, conjugacy classes and other relevant material are supplied in the Appendix.

We would also like to mention that 77 relevant references are listed under the Bibliography.

2

$\mathbb{E}$lementary $\mathbb{T}$heories of $\mathbb{R}$epresentations and $\mathbb{C}$haracters

In this dissertation, G means a finite group unless otherwise stated.

The theories of representations and characters of finite groups were developed by the end of the 19th century. Frobenius, Burnside, Schur and Brauer have contributed largely to these theories.

"The year 1897 was marked by two important mathematical events: the publication of the first paper on representations of finite groups by Ferdinand Georg Frobenius (1849-1917) and the appearance of the first treatise in English on the theory of finite groups by William Burnside (1852-1927). Burnside soon developed his own approach to representations of finite groups. In the next few years, working independently, Frobenius and Burnside explored the new subject and its applications to finite group theory. They were soon joined in this enterprise by Issai Schur (1875-1941) and some years later, by Richard Brauer (1901-1977). These mathematicians' pioneering research is the subject of this book. $\cdots$" Curtis [10].

The material that will be covered in this chapter is to illustrate the basics and fundamentals of representations and characters of finite groups. As general references, this can be found in Curtis and Reiner [9], Isaacs [38], James [39], Moori [54] and Sagan [66].

2.1. Preliminaries

There are two kinds of representations, namely *permutation* and *matrix* representations. An example of a permutation representation is given by the known Theorem of **Cayley**, which asserts that any group G (not necessarily finite) can be embedded into the Symmetric group S_G. The matrix representation of a finite group is of particular interest.

Definition 2.1.1. *Any homomorphism $\rho : G \longrightarrow GL(n, \mathbb{F})$, where $GL(n, \mathbb{F})$ is the group consisting of all $n \times n$ non-singular matrices is called a* ***matrix representation*** *or simply a* ***representation*** *of G. If $\mathbb{F} = \mathbb{C}$, then ρ is called an* ***ordinary*** *representation. The integer n is called the degree of ρ. Two representations ρ and σ are said to be* ***equivalent*** *if there exists $P \in GL(n, \mathbb{F})$ such that $\sigma(g) = P\rho(g)P^{-1}$, $\forall g \in G$.*

From now on, we restrict ourselves to ordinary representations only, unless an explicit exception is made.

Definition 2.1.2. *If $\rho : G \longrightarrow GL(n, \mathbb{C})$ is a representation. Then ρ affords a complex valued function $\chi_\rho : G \longrightarrow \mathbb{C}$ defined by $\chi_\rho(g) = trace(\rho(g)), \forall g \in G$. The function χ_ρ is called a* ***character*** *afforded by the representation ρ of G or simply a character of G. The integer n is called the* ***degree*** *of χ_ρ. If $n = 1$, then χ_ρ is said to be* ***linear***.

A function $\phi : G \longrightarrow \mathbb{C}$ which is invariant over every conjugacy class of G, that is $\phi(ghg^{-1}) = \phi(h)$, $\forall g, h \in G$, is called a *class function* of G.

Proposition 2.1.1. *Any character of G is a class function.*

PROOF. Immediate since similar matrices have same trace. ■

Now over the set of class functions of a group G we define addition and multiplication of two class functions ψ_1 and ψ_2 by

$$\begin{aligned} (\psi_1 + \psi_2)(g) &= \psi_1(g) + \psi_2(g), \quad \forall g \in G, \\ \psi_1\psi_2(g) &= \psi_1(g)\psi_2(g), \quad \forall g \in G. \end{aligned}$$

It is clear that $\psi_1 + \psi_2$ and $\psi_1\psi_2$ are class functions of G. Also if $\lambda \in \mathbb{C}$, then $\lambda\psi$ is a class function of G whenever ψ is. Therefore the set of all class functions of a group G forms an *algebra*, which we denote by $\mathfrak{C}(G)$. The set of all characters of G forms a *subalgebra* of $\mathfrak{C}(G)$. However, it may not be clear that the product of two characters is again a character. This fact will be shown in Section 2.3. Now we prove that the sum of two characters is again a character.

Proposition 2.1.2. *If χ_ψ and χ_ϕ are two characters of G, then so is $\chi_\psi + \chi_\phi$.*

PROOF. Let ψ and ϕ be representations of G affording the characters χ_ψ and χ_ϕ respectively. Define the function ξ on G by $\xi(g) = \begin{pmatrix} \psi(g) & 0 \\ 0 & \phi(g) \end{pmatrix} = \psi(g) \oplus \phi(g)$. It is obvious that ξ is a homomorphism (representation) of G with $\chi_\xi = \chi_\psi + \chi_\phi$. ■

The above proposition motivates the following definition.

Definition 2.1.3. *A representation ρ of G is said to be* ***irreducible*** *if it is not a direct sum of other representations of G. Also a character χ of G is said to be* ***irreducible*** *if it is not a sum of other characters of G.*

Example 2.1.1. For any G, consider the function $\rho : G \longrightarrow GL(1, \mathbb{C})$ given by $\rho(g) = 1, \ \forall g \in G$. It is clear that ρ is a representation of G and $\chi_\rho(g) = 1, \ \forall g \in G$. Obviously ρ is irreducible. This character is called the *trivial character* and sometimes we may denote it by **1**.

The Theorem of Maschke and Schur's Lemma (see Theorem 5.1.6 and Corollary 5.1.9 of Moori [54]) are two pillars on which the edifice of representation theory rests. Maschke Theorem ensures that under certain conditions, any representation splits up into irreducible pieces. Schur's Lemma leads to the orthogonality of representations and hence characters. We mention the statement of Maschke Theorem only.

Theorem 2.1.3 (**Maschke Theorem**)**.** *Let $\rho : G \longrightarrow GL(n, \mathbb{F})$ be a representation of G. If the characteristic of $\mathbb{F}$ is zero or does not divide $|G|$, then $\rho = \bigoplus_{i=1}^{r} \rho_i$, where ρ_i are irreducible representations of G.*

Over $\mathfrak{C}(G)$ one can define an inner product $\langle , \rangle : \mathfrak{C}(G) \times \mathfrak{C}(G) \longrightarrow \mathbb{C}$ by

$$\langle \psi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g) \overline{\phi(g)},$$

where $\overline{z}$ stands for the complex conjugate of z.

Among the important properties of characters of a group we can mention:

Proposition 2.1.4.

1. *Let χ_ρ be a character afforded by an irreducible representation ρ of G. Then $\langle \chi_\rho, \chi_\rho \rangle = 1$.*
2. *If χ_ρ and $\chi_{\rho'}$ are the irreducible characters of two non equivalent representations of G, then $\left\langle \chi_\rho, \chi_{\rho'} \right\rangle = 0$.*
3. *If $\rho \cong \bigoplus_{i=1}^{k} d_i \rho_i$, then $\chi_\rho = \sum_{i=1}^{k} d_i \chi_{\rho_i}$.*
4. *If $\rho \cong \bigoplus_{i=1}^{k} d_i \rho_i$, then $d_i = \langle \chi_\rho, \chi_{\rho_i} \rangle$.*
5. *ρ is irreducible if and only if $\langle \chi_\rho, \chi_\rho \rangle = 1$.*

PROOF. See Baker [5], James [40], Joshi [41] or Moori [54]. ■

We shall use the notation $Irr(G)$ to denote the set of all ordinary irreducible characters of G.

Corollary 2.1.5. *The set $Irr(G)$ forms an orthonormal basis for $\mathfrak{C}(G)$ over $\mathbb{C}$.*

PROOF. Omitted. See James [40]. ∎

Note 2.1.1. Observe that Corollary 2.1.5 asserts that if ψ is a class function of G, then $\psi = \sum_{i=1}^{k} \lambda_i \chi_i$, where $\lambda_i \in \mathbb{C}$ and $Irr(G) = \{\chi_1, \chi_2, \cdots, \chi_k\}$. If $\lambda_i \in \mathbb{Z}$, $\forall i$, then ψ is called a *generalized character*. Moreover, if $\lambda_i \in \mathbb{N} \cup \{0\}$, then ψ is a character of G.

The following theorem counts the number of irreducible characters of G.

Theorem 2.1.6. *The number of irreducible characters of G is equal to the number of conjugacy classes of G.*

PROOF. See Feit [19], James [40] or Moori [54]. ∎

2.2. Character Tables and Orthogonality Relations

Definition 2.2.1 (Character Table). *The **character table** of a group G is a square matrix, its columns correspond to the conjugacy classes, while its rows correspond to the irreducible characters.*

The character table of G is very powerful tool to prove results about representations of G and G itself. For example, the character table of G enables us to

- decide the simplicity of G,
- determine all the normal subgroups and hence can help to decide solvability of the group (in particular we are able to find the center and commutator subgroup of G),
- determine the sizes of conjugacy classes of G,
- determine the degrees of all representations of G.

Corollary 2.2.1. *The character table of G is an invertible matrix.*

PROOF. Direct result from the fact that the irreducible characters, and hence the rows of the character table are linearly independent. ∎

Proposition 2.2.2. *The following properties hold.*

1. $\chi(1_G) | |G|, \ \forall \chi \in Irr(G)$.
2. $\displaystyle\sum_{i=1}^{|Irr(G)|} (\chi_i(1_G))^2 = |G|$.
3. *If* $\chi \in Irr(G)$, *then* $\overline{\chi} \in Irr(G)$, *where* $\overline{\chi}(g) = \overline{\chi(g)}, \ \forall g \in G$.
4. $\chi(g^{-1}) = \overline{\chi(g)}, \ \forall g \in G$. *In particular if* $g^{-1} \in [g]$, *then* $\chi(g) \in \mathbb{R}, \ \forall \chi$.

PROOF. See James [40] or Moori [54]. ■

In addition to the properties mentioned in Proposition 2.2.2, the character table satisfies certain orthogonality relations mentioned in the next Theorem.

Theorem 2.2.3. *Let* $Irr(G) = \{\chi_1, \chi_2, \cdots, \chi_k\}$ *and* $\{g_1, g_2, \cdots, g_k\}$ *be a collection of representatives for the conjugacy classes of* G. *For each* $1 \le i \le k$ *let* $C_G(g_i)$ *be the centralizer of* g_i. *Then we have the following relations:*

1. *The row orthogonality relation:*
   For each $1 \le i, j \le k$,
$$\sum_{s=1}^{k} \frac{\chi_i(g_s)\overline{\chi_j(g_s)}}{|C_G(g_s)|} = \langle \chi_i, \chi_j \rangle = \delta_{ij}.$$
2. *The column orthogonality relation:*
   For each $1 \le i, j \le k$,
$$\sum_{s=1}^{k} \frac{\chi_s(g_i)\overline{\chi_s(g_j)}}{|C_G(g_i)|} = \delta_{ij}.$$

PROOF.

1. Using Proposition 2.1.4(2) we have
$$\delta_{ij} = \langle \chi_i, \chi_j \rangle = \frac{1}{|G|}\sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \frac{1}{|G|}\sum_{s=1}^{k} \frac{|G|}{|C_G(g_s)|}\chi_i(g_s)\overline{\chi_j(g_s)} = \sum_{s=1}^{k} \frac{\chi_i(g_s)\overline{\chi_j(g_s)}}{|C_G(g_s)|}.$$
2. For fixed $1 \le t \le k$, define $\psi_t : G \to \mathbb{C}$ by $\psi_t(g) = \begin{cases} 1 & \text{if } g \in [g_t], \\ 0 & \text{otherwise.} \end{cases}$
   It is clear that ψ_t is a class function on G. Since $Irr(G)$ form an orthonormal basis for $\mathfrak{C}(G)$, then $\exists \ \lambda_s's \in \mathbb{C}$ such that $\psi_t = \sum_{s=1}^{k} \lambda_s \chi_s$. Now for $1 \le j \le k$ we have
$$\lambda_j = \langle \psi_t, \chi_j \rangle = \frac{1}{|G|}\sum_{g \in G} \psi_t(g)\overline{\chi_j(g)} = \sum_{s=1}^{k} \frac{\psi_t(g_s)\overline{\chi_j(g_s)}}{|C_G(g_s)|} = \frac{\overline{\chi_j(g_t)}}{|C_G(g_t)|}.$$

Hence $\psi_t = \sum_{j=1}^{k} \frac{\overline{\chi_j(g_t)}}{|C_G(g_t)|}\chi_j$. Thus we have the required formula :

$$\delta_{ts} = \psi_t(g_s) = \sum_{j=1}^{k} \frac{\chi_j(g_s)\overline{\chi_j(g_t)}}{|C_G(g_s)|}.$$

This completes the proof. ■

We conclude this section by giving the character table of the cyclic group $\mathbb{F}_q^*$.

Theorem 2.2.4. *The group* $\mathbb{F}_q^* = \langle\theta\rangle$ *has* $q-1$ *irreducible characters* χ_k, $0 \leq k \leq q-2$ *given at* θ^j, *by* $\chi_k(\theta^j) = e^{\frac{2\pi jk}{q-1}i}$.

PROOF. If $\rho(\theta) = (c)_{1\times 1} = c \in \mathbb{C}$ is a $1-$dimensional matrix representation, then the values of the representation ρ over all elements of $\mathbb{F}_q^*$ are determined by $\rho(\theta^j) = c^j$. By the definition of representation, we have

$$c^{q-1} = \rho(\theta^{q-1}) = \rho(1_{\mathbb{F}_q^*}) = 1.$$

It follows that c must be a $(q-1)th$ root of unity. Therefore each root of unity gives an irreducible representation and the result follows since $\chi_\rho = \rho$. ■

2.3. Tensor Product of Characters

In this section we follow precisely the description of Moori [54]. Given two matrices $P = (p_{ij})_{m\times m}$ and $Q = (q_{ij})_{n\times n}$, we define the *tensor product* of P and Q to be the $mn \times mn$ matrix $P \otimes Q$

$$P \otimes Q = (p_{ij})Q = \begin{pmatrix} p_{11}Q & p_{12}Q & \cdots & p_{1m}Q \\ p_{21}Q & p_{22}Q & \cdots & p_{2m}Q \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}Q & p_{m2}Q & \cdots & p_{mm}Q \end{pmatrix}.$$

Then

$$trace(P \otimes Q) = p_{11}trace(Q) + p_{22}trace(Q) + \cdots + p_{mm}trace(Q) = trace(P)trace(Q).$$

Definition 2.3.1. *Let* U *and* T *be two representations of* G. *We define the tensor product of* $T \otimes U$ *by*

$$(T \otimes U)(g) = T(g) \otimes U(g),\ \forall g \in G.$$

Theorem 2.3.1. *Let* T *and* U *be representations of* G. *Then*

(i) *$T \otimes U$ is a representation of G,*

(ii) $\chi_{T\otimes U} = \chi_T \chi_U$.

PROOF.

(i) $\forall g, h \in G$, we have

$$\begin{aligned}(T\otimes U)(gh) &= T(gh)\otimes U(gh)\\ &= (T(g)T(h))\otimes(U(g)U(h))\\ &= (T(g)\otimes U(g))(T(h)\otimes U(h))\\ &= (T\otimes U)(g)(T\otimes U)(h).\end{aligned}$$

(ii)

$$\begin{aligned}\chi_{T\otimes U}(g) &= trace((T\otimes U)(g))\\ &= trace((T(g)\otimes U(g)))\\ &= trace(T(g))trace(U(g))\\ &= \chi_T(g)\chi_U(g).\end{aligned}$$

Hence $\chi_{T\otimes U} = \chi_T \chi_U$.

This proves the Theorem. ■

Note 2.3.1. Observe that $T\otimes U \neq U \otimes T$ in general, but $\chi_{T\otimes U} = \chi_T\chi_U = \chi_U\chi_T = \chi_{U\otimes T}$. Thus the tensor product of characters is commutative.

Now we show that knowing the character tables of two groups K and H, then the tensor products can be used to obtain the character table of $K \times H$.

Theorem 2.3.2. *Let H_1 and H_2 be two groups with conjugacy classes $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_r$ and $\mathcal{C}_1^{'}, \mathcal{C}_2^{'}, \cdots, \mathcal{C}_s^{'}$ respectively. Suppose that $Irr(H_1) = \{\chi_1, \chi_2, \cdots, \chi_r\}$ and $Irr(H_2) = \{\chi_1^{'}, \chi_2^{'}, \cdots, \chi_s^{'}\}$. The conjugacy classes of $H_1 \times H_2$ are $\mathcal{C}_i \times \mathcal{C}_j^{'}$ and $Irr(H_1 \times H_2) = \{\chi_i \times \chi_j^{'} |\ \chi_i \in Irr(H_1),\ \chi_j^{'} \in Irr(H_2)\}$ for $1 \leq i \leq r$ and $1 \leq j \leq s$.*

PROOF. For all $x, h_1 \in H_1$ and $y, h_2 \in H_2$, we have

$$(x, y)^{-1}(h_1, h_2)(x, y) = (x^{-1}h_1x, y^{-1}h_2y).$$

Therefore two elements (h_1, h_2) and $(h_1^{'}, h_2^{'})$ of $H_1 \times H_2$ are conjugate if and only if $h_1 \sim_{H_1} h_1^{'}$ and $h_2 \sim_{H_2} h_2^{'}$, where $\sim_H$ denotes the conjugation of two elements in a group H. Thus

$$\mathcal{C}_i \times \mathcal{C}_j^{'}, \quad 1 \leq i \leq r,\ 1 \leq j \leq s,$$

are the conjugacy classes of $H_1 \times H_2$. In particular, there are exactly rs conjugacy classes of $H_1 \times H_2$. On the other hand for all i, j, k, l,

$$\begin{aligned}
\left\langle \chi_i \times \chi_j^{'}, \chi_k \times \chi_l^{'} \right\rangle &= \frac{1}{|H_1 \times H_2|} \sum_{h \in H_1,\ \widehat{h} \in H_2} \chi_i(h)\chi_j^{'}(\widehat{h})\overline{\chi_k(h)\chi_l^{'}(\widehat{h})} \\
&= \left(\frac{1}{|H_1|} \sum_{h \in H_1} \chi_i(h)\overline{\chi_k(h)} \right) \left(\frac{1}{|H_2|} \sum_{\widehat{h} \in H_2} \chi_j^{'}(\widehat{h})\overline{\chi_l^{'}(\widehat{h})} \right) \\
&= \langle \chi_i, \chi_k \rangle \left\langle \chi_j^{'}, \chi_l^{'} \right\rangle = \delta_{ik}\delta_{jl}.
\end{aligned}$$

Thus the rs characters $\chi_i \times \chi_j^{'}$ are distinct and irreducible. This completes the proof. ■

Note 2.3.2. Observe that if $\chi, \psi \in Irr(G)$, then in general $\chi\psi \notin Irr(G)$. In the special case when $\deg(\psi) = 1$, we have the following proposition.

Proposition 2.3.3. *Let ψ be a linear character of G and $\chi \in Irr(G)$. Then $\chi\psi \in Irr(G)$.*

PROOF. Suppose that ψ is a linear character of G. Then we know that $\psi(g)$ is a root of unity for any $g \in G$. In particular, we have $1 = |\psi(g)| = \psi(g)\overline{\psi(g)}$ for every $g \in G$. Now assume that χ is an irreducible character of G. It follows that

$$\begin{aligned}
\langle \psi\chi, \psi\chi \rangle &= \frac{1}{|G|} \sum_{g \in G} \psi\chi(g)\overline{\psi\chi(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi}(g)\psi(g)\overline{\psi}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi}(g) = \langle \chi, \chi \rangle = 1.
\end{aligned}$$

Hence $\psi\chi$ is an irreducible character of G. ■

Proposition 2.3.4. *The number of linear characters of a group G is given by $|G|/|G^{'}|$, where $G^{'}$ is the derived subgroup of G.*

PROOF. See Theorem 17.11 of James [40]. ■

2.4. Lifting of Characters

In this section, we present a method for constructing characters of G when it has a proper normal subgroup N. We may look at the quotient group G/N, which is of a smaller order than $|G|$. Therefore it becomes reasonable to assume that the irreducible characters of G/N are known. From this assumption we may construct characters of G in a process known as *lifting of characters*. Thus the normal subgroups help to find characters of G and conversely the character table of G enables us to determine all the normal subgroups of G.

Proposition 2.4.1. *Let $N \lhd G$ and $\widetilde{\chi}$ be a character of G/N. The function $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \widetilde{\chi}(gN), \forall g \in G$ is a character of G with $\deg(\chi) = \deg(\widetilde{\chi})$. Moreover; if $\widetilde{\chi} \in Irr(G/N)$, then $\chi \in Irr(G)$.*

PROOF. Assume that $\widetilde{\rho} : G/N \to GL(n, \mathbb{C})$ is a representation which affords the character $\widetilde{\chi}$. Define the function $\rho : G \to GL(n, \mathbb{C})$ by $\rho(g) = \widetilde{\rho}(gN),\ \forall g \in G$. Then ρ defines a representation on G since

$$\rho(gh) = \widetilde{\rho}(ghN) = \widetilde{\rho}(gNhN) = \widetilde{\rho}(gN)\widetilde{\rho}(hN) = \rho(g)\rho(h),\ \forall g, h \in G.$$

Hence the character χ, which is afforded by ρ, satisfies

$$\chi(g) = tr(\rho(g)) = tr(\widetilde{\rho}(gN)) = \widetilde{\chi}(gN)\ \forall g \in G.$$

and so χ is a character of G. For the degree of χ, we have

$$\deg(\chi) = \chi(1_G) = \widetilde{\chi}(1_G N) = \widetilde{\chi}(N) = \deg(\widetilde{\chi}).$$

Now let S be a transversal of N in G. Then

$$\begin{aligned}
1 = \langle \widetilde{\chi}, \widetilde{\chi} \rangle &= \frac{1}{|G/N|} \sum_{gN \in G/N} \widetilde{\chi}(gN)\widetilde{\chi}(gN)^{-1} \\
&= \frac{1}{|G|} \sum_{gN \in G/N} |N|\widetilde{\chi}(gN)\widetilde{\chi}(gN)^{-1} \\
&= \frac{1}{|G|} \sum_{g \in S} |N|\widetilde{\chi}(gN)\widetilde{\chi}(g^{-1}N) \\
&= \frac{1}{|G|} \sum_{g \in S} |N|\chi(g)\chi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi(g^{-1}) \\
&= \langle \chi, \chi \rangle .
\end{aligned}$$

This completes the proof. ■

Definition 2.4.1. *The character χ defined in the above Proposition is called the* ***lift*** *of $\widetilde{\chi}$ to G.*

One of the advantages given by the character table of G is that it supplies us with all normal subgroups of G. This is the assertion of the next theorem.

Theorem 2.4.2. *Let $N \lhd G$. Then there exist irreducible characters $\chi_1, \chi_2, \cdots, \chi_s$ of G such that $N = \cap_{i=1}^{s} ker\chi_i$.*

PROOF. Firstly, we have the following observation. If $\chi_1, \chi_2, \cdots, \chi_k$ are the irreducible characters of G, then $\bigcap_{i=1}^{k} \ker \chi_i = \{1_G\}$. Now suppose that G/N has s distinct irreducible characters $\widetilde{\chi}_1, \widetilde{\chi}_2, \cdots, \widetilde{\chi}_s$. So $\bigcap_{i=1}^{s} \ker \widetilde{\chi}_i = \{N\}$. For $1 \leq i \leq s$, suppose that χ_i are the lifts to G of $\widetilde{\chi}_i$. Thus if $g \in \ker \chi_i$, then

$$\widetilde{\chi}_i(N) = \chi_i(1_G) = \chi_i(g) = \widetilde{\chi}_i(gN),$$

and hence $gN \in \ker \widetilde{\chi}_i$. Therefore if $g \in \bigcap_{i=1}^{|Irr(G)|} \ker \chi_i$, then $gN \in \bigcap_{i=1}^{s} \ker \widetilde{\chi}_i = \{N\}$, and so $g \in N$. Hence $N = \bigcap_{i=1}^{s} \ker \chi_i$. ■

The converse of the above theorem is also true, *i.e.* every normal subgroup of G arises in this way.

Corollary 2.4.3. *G is simple if and only if for every $\chi_r \in Irr(G)$, where $\chi_r \neq \chi_1$, and for all $1_G \neq g \in G$, we have $\chi_r(g) \neq \chi_r(1_G)$.*

PROOF. See Alperin [3] or Moori [54]. ■

Hence the character table can be used to decide whether G is simple group or not.

2.5. Restriction and Induction of Characters

Given a group G and a subgroup $H \leq G$. Knowing characters of G, one can get some characters of H and vice versa. These two dual operations are known as *restriction* and *induction* of characters.

2.5.1 Restriction of Characters

Let $H \leq G$ and let $\rho : G \longrightarrow GL(n, \mathbb{C})$ be a representation of G. The *restriction* of ρ to H, denoted by $\rho\downarrow_H^G$ is defined by

$$\rho\downarrow_H^G(h) = \rho(h), \ \forall h \in H.$$

If χ_ρ is the character afforded by ρ, then it is not difficult to see that $\chi_\rho\downarrow_H^G$ is a character of H. Also if $\chi_\rho \in Irr(G)$, then it is not necessarily that $\chi_\rho\downarrow_H^G \in Irr(H)$.

Theorem 2.5.1. *Let $H \leq G$. Let $\chi \in Irr(G)$ and let $Irr(H) = \{\psi_1, \psi_2, \cdots, \psi_r\}$. Then $\chi\downarrow_H^G = \sum_{i=1}^{r} d_i\psi_i$, where $d_i \in \mathbb{N} \cup \{0\}$ and $\sum_{i=1}^{r} d_i^2 \leq [G : H]$. The equality holds in the previous $\sum$ if and only if $\chi(g) = 0, \ \forall g \in G \setminus H$.*

PROOF. We have

$$\sum_{i=1}^{r} d_i^2 = \langle \chi\downarrow_H^G, \chi\downarrow_H^G \rangle = \frac{1}{|H|} \sum_{h \in H} \chi(h)\overline{\chi(h)}.$$

Since $\chi \in Irr(G)$, we have

$$\begin{aligned}\langle \chi, \chi \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} \\ &= \frac{1}{|G|} \sum_{g \in H} \chi(g)\overline{\chi(g)} + \frac{1}{|G|} \sum_{g \in G \setminus H} \chi(g)\overline{\chi(g)} \\ &= \frac{|H|}{|G|} \sum_{i=1}^{r} d_i^2 + K,\end{aligned}$$

where $K = \frac{1}{|G|} \sum_{g \in G \setminus H} \chi(g)\overline{\chi(g)}$. Since $K = \frac{1}{|G|} \sum_{g \in G \setminus H} |\chi(g)|^2$, we have that $K \geq 0$. Thus

$$\frac{|H|}{|G|} \sum_{i=1}^{r} d_i^2 = 1 - K \leq 1,$$

so

$$\sum_{i=1}^{r} d_i^2 \leq \frac{|G|}{|H|} = [G : H].$$

Also

$$K = 0 \Longleftrightarrow |\chi(g)|^2 = 0 \Longleftrightarrow \chi(g) = 0, \ \forall g \in G \setminus H.$$

This completes the proof. ∎

Theorem 2.5.1 asserts that the number of irreducible constituents of $\chi\downarrow_H^G$ is bounded above by $[G : H]$. Therefore if $[G : H]$ is fairly small, the character tables of H and G are closely related. For example if $[G : H] = 2$ and $\chi \in Irr(G)$, then either $\chi\downarrow_H^G \in Irr(H)$ or $\chi\downarrow_H^G = \psi_1 + \psi_2$ where $\psi_1, \psi_2 \in Irr(H)$.

2.5.2 Induction of Characters

Let $H \leq G$ such that the set $\{x_1, x_2, \cdots, x_r\}$ is a transversal for H in G. Let ϕ be a representation of H of degree n. Then we define ϕ^* on G as follows:

$$\phi^*(g) = \begin{pmatrix} \phi(x_1 g x_1^{-1}) & \phi(x_1 g x_2^{-1}) & \cdots & \phi(x_1 g x_r^{-1}) \\ \phi(x_2 g x_1^{-1}) & \phi(x_2 g x_2^{-1}) & \cdots & \phi(x_2 g x_r^{-1}) \\ \vdots & \cdots & \ddots & \vdots \\ \phi(x_r g x_1^{-1}) & \phi(x_r g x_2^{-1}) & \cdots & \phi(x_r g x_r^{-1}) \end{pmatrix}$$

where $\phi(x_i g x_j^{-1})$ is $n \times n$ block satisfying the property that

$$\phi(x_i g x_j^{-1}) = 0_{n\times n} \quad \forall x_i g x_j^{-1} \notin H.$$

It is possible to show that ϕ^* is a representation of G of degree n.

Definition 2.5.1. *With the above, the representation ϕ^* is called the representation of G **induced** from the representation ϕ of H and is denoted by $\phi^* = \phi\uparrow_H^G$.*

Definition 2.5.2. *Let ϕ be a class function of H. Then $\phi\uparrow_H^G$, the **induced** class function on G, is defined by*

$$\phi\uparrow_H^G(g) = \frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1})$$

where ϕ^0 is defined on G by

$$\phi^0(h) = \begin{cases} \phi(h) & \text{if } h \in H, \\ 0 & \text{if } h \notin H. \end{cases}$$

Note that $\deg(\phi\uparrow_H^G) = [G : H]\deg(\phi)$.

Theorem 2.5.2. *If ϕ is a character of H where $H \leq G$, then $\phi\uparrow_H^G$ is a character of G.*

PROOF. See Moori [54] or Whitley [76]. ■

Theorem 2.5.3 (Frobenius Reciprocity Theorem). *Let G be a group, $H \leq G$ and suppose that ϕ is a character of H and θ a character of G. Then*

$$\langle \phi, \theta\downarrow_H^G\rangle_H = \langle \phi\uparrow_H^G, \theta\rangle_G .$$

PROOF. We obtain that

$$\langle \phi\uparrow_H^G, \theta\rangle_G = \frac{1}{|G|}\sum_{g\in G}\phi\uparrow_H^G(g)\overline{\theta(g)} = \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{x\in G}\phi^0(xgx^{-1})\overline{\theta(g)}$$

Putting $y = xgx^{-1}$, then for fixed x, as g runs through G, so does y, and $\theta(y) = \theta(g)$, since θ is a class function on G. Hence

$$\begin{aligned}\langle \phi\uparrow_H^G, \theta\rangle_G &= \frac{1}{|G|}\frac{1}{|H|}\sum_{x\in G}\sum_{y\in G}\phi^0(y)\overline{\theta(y)} = \frac{1}{|G|}\frac{1}{|H|}\sum_{y\in G}\sum_{x\in G}\phi^0(y)\overline{\theta(y)} \\ &= \frac{1}{|H|}\sum_{y\in H}\phi(y)\overline{\theta(y)} = \langle \phi, \theta\downarrow_H^G\rangle_H .\end{aligned}$$

Hence the result. ■

Corollary 2.5.4. *Let $Irr(G) = \{\chi_1, \chi_2, \cdots, \chi_r\}$ and $Irr(H) = \{\psi_1, \psi_2, \cdots, \psi_s\}$ where $H \leq G$. Assume that $\chi_i\downarrow_H^G = \sum_{j=1}^{s} a_{ij}\psi_j$ and $\psi_j\uparrow_H^G = \sum_{i=1}^{r} b_{ij}\chi_i$. Then $a_{ij} = b_{ij}$ for all i, j.*

PROOF. Using the Frobenius Reciprocity Theorem we get $a_{ij} = \langle \chi_i\downarrow_H^G, \psi_j \rangle = \langle \chi_i, \psi_j\uparrow_H^G \rangle = b_{ij}$. ■

Next we compute the values of induced character $\phi\uparrow_H^G$ on classes of G.

Proposition 2.5.5. *Let ϕ be a character of H and let $\phi\uparrow_H^G$ be the induced character from H to G. Let $g \in G$ and suppose that $[g]$ breaks into m classes in H with representatives $x_1, x_2, \cdots, x_m$. If $H \cap [g] = \varnothing$, the empty set, then $\phi\uparrow_H^G(g) = 0$, while if $H \cap [g] \neq \varnothing$, then*

$$\phi\uparrow_H^G(g) = |C_G(g)| \sum_{i=1}^{m} \frac{\phi(x_i)}{|C_H(x_i)|}.$$

PROOF. We have

$$\phi\uparrow_H^G(g) = \frac{1}{|H|} \sum_{x \in G} \phi^0(xgx^{-1}).$$

If $H \cap [g] = \varnothing$, then $xgx^{-1} \notin H$ for all $x \in G$ and thus $\phi^0(xgx^{-1}) = 0$ for all $x \in G$ and $\phi\uparrow_H^G(g) = 0$. Now if $H \cap [g] \neq \varnothing$, then let $h \in H \cap [g]$. As x runs over G, we have $xgx^{-1} = h$ for exactly $|C_G(g)|$ times, so $\phi\uparrow_H^G(g) = \frac{|C_G(g)|}{|H|} \sum_{y \in [g]} \phi^0(y)$. Now $\phi^0(y) = 0$ if $y \notin H$, and $[g] \cap H$ contains $[H : C_H(x_i)]$ conjugates of each x_i. Therefore $\phi\uparrow_H^G(g) = |C_G(g)| \sum_{i=1}^{m} \frac{\phi(x_i)}{|C_H(x_i)|}$. ■

We conclude this section by remarking that the operations of restriction and induction of characters do not necessarily preserve irreducibility of characters.

2.6. Permutation Character

Let G acts on a finite set $\Omega = \{\omega_1, \omega_2, \cdots, \omega_k\}$ and for each $g \in G$ define the $k \times k$ matrix $\pi_g = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } \omega_i^g = \omega_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then π_g is a permutation matrix of the action of g and $P : G \longrightarrow GL(k, \mathbb{C})$ given by $P(g) = \pi_g$ is a representation of G.

The character χ_P afforded by this representation is called a *permutation character*, and $\chi_P(g) = |\{\omega \in \Omega | \ \omega^g = \omega\}|$, that is, $\chi_P(g)$ is the number of points of Ω left fixed by $g \in G$. Therefore $\chi_P(g) \in \mathbb{N} \cup \{0\}$, $\forall g \in G$.

Note 2.6.1. Observe that

$$\deg(\chi_P) = \chi_P(1_G) = |\{\omega \in \Omega | \ \omega^{1_G} = \omega\}| = |\Omega| = k,$$

since by definition of group action we have $\omega^{1_G} = \omega, \ \forall \omega \in \Omega$.

Recall that an action of G on a set X is called transitive if $\forall x, y \in X, \ \exists g \in G$ such that $x^g = y$. Now let $H \leq G$ and $S = \{a_1, a_2, \cdots, a_r\}$ be a left transversal for H in G. Then G acts on the set of left cosets of H in G by $(a_iH)^g = ga_iH$. It is clear that this action is transitive since for any $a_i, a_j \in S$, we have $(a_iH)^{a_ja_i^{-1}} = a_jH$. The resulting permutation character of this action is of degree $[G : H] = |S| = r$. In fact this permutation character is $\mathbf{1}\uparrow_H^G$. To see this we have

$$(a_iH)^g = a_iH \Longleftrightarrow ga_iH = a_iH \Longleftrightarrow a_i^{-1}ga_iH = H \Longleftrightarrow a_i^{-1}ga_i \in H.$$

Thus

$$\chi_P(g) = \sum_{i=1}^{r} \phi^0(a_i^{-1}ga_i),$$

where

$$\phi^0(y) = \begin{cases} 1 & \text{if } \ y \in H, \\ 0 & \text{if } y \notin H. \end{cases}$$

Hence $\chi_P = \mathbf{1}\uparrow_H^G$. This shows that for any subgroup H, there exists a permutation character of G. Conversely, if G acts transitively on any set X, then the associated permutation character represents $\mathbf{1}\uparrow_H^G$ for some subgroup H of G. This is the assertion of the following theorem.

Theorem 2.6.1. *Let G acts transitively on a set Ω and let $\omega \in \Omega$. Then $\mathbf{1}\uparrow_{G_\omega}^G$ is the permutation character of the action.*

PROOF. Since G acts transitively on Ω, we have $\omega^G = \Omega$. It follows by the *Orbit-Stabilizer Theorem* (see Moori [54] for example) that there is a $1-1$ correspondence between Ω and the set of left cosets of G_ω in G, given by $\omega^t \longmapsto tG_\omega$ for $t \in G$. Now for $g \in G$ we have

$$(\omega^t)^g = \omega^t \Longleftrightarrow \omega^{t^{-1}gt} = \omega \Longleftrightarrow t^{-1}gt \in G_\omega \Longleftrightarrow tG_\omega = gtG_\omega \Longleftrightarrow tG_\omega = (tG_\omega)^g,$$

where G acts on the set of left cosets of G_ω in G as given above. Therefore the permutation character of the action of G on Ω is the same as the permutation character of the action of G on the left cosets of G_ω in G, which is $\mathbf{1}\uparrow_{G_\omega}^G$. ■

Corollary 2.6.2. *Let G acts on Ω with a permutation character χ. Suppose Ω decomposes into exactly k orbits under the action of G. Then $\langle \chi, \mathbf{1} \rangle = k$, where $\mathbf{1}$ is the trivial character of G.*

PROOF. Write $\Omega = \bigcup_{i=1}^{k} \Delta_i$ where the Δ_i are orbits. Let χ_i be the permutation character of G on Δ_i so that $\chi = \sum_{i=1}^{k} \chi_i$. For $\omega \in \Delta_i$, we have $\chi_i = \mathbf{1}\uparrow_{G_\omega}^{G}$ by Theorem 2.6.1. Thus

$$\langle \chi_i, \mathbf{1} \rangle_G = \langle \mathbf{1}\uparrow_{G_\omega}^{G}, \mathbf{1} \rangle_G = \langle \mathbf{1}, \mathbf{1}\downarrow_{G_\omega}^{G} \rangle_{G_\omega} = 1$$

by Frobenius reciprocity. Therefore $\langle \chi, \mathbf{1} \rangle = \sum_{i=1}^{k} \langle \chi_i, \mathbf{1} \rangle = k$, completing the proof. ■

Lemma 2.6.3. *If G acts transitively on Ω, then all subgroups G_ω, $\omega \in \Omega$ of G are conjugate in G.*

PROOF. Since G acts transitively on Ω, there is some $h \in G$ such that $\omega^h = \kappa$ for any $\omega, \kappa \in \Omega$. Now

$$g \in G_\omega \iff \omega^g = \omega \iff \kappa^{gh^{-1}} = \kappa^{h^{-1}} \iff \kappa^{hgh^{-1}} = \kappa \iff hgh^{-1} \in G_\kappa \iff g \in (G\kappa)^h.$$

Thus $G_\omega = (G_\kappa)^h$, which shows that $G_\omega = hG_\kappa h^{-1}$. That is G_ω and G_κ are conjugate in G. ■

Because $\mathbf{1}\uparrow_H^G$ is a transitive permutation character, it must satisfy certain necessary conditions mentioned in the following theorem.

Theorem 2.6.4. *Let $H \leq G$ and $\chi = \mathbf{1}\uparrow_H^G$. Then*

(i) $\deg(\chi) | |G|$.

(ii) $\langle \chi, \psi \rangle \leq \deg(\psi), \ \forall \psi \in Irr(G)$.

(iii) $\langle \chi, \mathbf{1} \rangle = 1$.

(iv) $\chi(g) \in \mathbb{N} \cup \{0\}, \ \forall g \in G$.

(v) $\chi(g) \leq \chi(g^m), \ \forall g \in G, \ \forall m \in \mathbb{N} \cup \{0\}$.

(vi) $o(g) \nmid \frac{|G|}{\chi(1_G)} \Longrightarrow \chi(g) = 0$.

(vii) $\chi(g)\frac{|[g]|}{\chi(1_G)} \in \mathbb{Z}, \ \forall g \in G$.

PROOF. Let Ω be the set of the left cosets of H in G. Thus χ is the permutation character of G on Ω.

(i) Since $\deg(\chi) = [G : H]$, we have$\deg(\chi) | |G|$.

(ii) Using Frobenius reciprocity we get $\langle \chi, \psi \rangle_G = \langle \mathbf{1}\uparrow_H^G, \psi \rangle_G = \langle \mathbf{1}\downarrow_H^G, \psi\downarrow_H^G \rangle_H \leq \deg(\psi)$.

(iii) Since χ is a transitive permutation character, it follows by Corollary 2.6.2 that $\langle \chi, \mathbf{1} \rangle = 1$.

(iv) This follows because $\chi(g)$ is the number of points left fixed by g and hence is non-negative.

(v) Let $g \in G_\omega$, that is $\omega^g = \omega$. It is clear that $\omega^{g^m} = \omega$. Thus any point of Ω left fixed by g is also fixed by g^m. Therefore the number of points fixed by g does not exceed the number of points fixed by g^m.

(vi) We know that $\frac{|G|}{\chi(1_G)} = |H|$ so if $o(g) \nmid |H|$, then $[g] \cap H = \varnothing$, the empty set. Hence $\mathbf{1}\uparrow_H^G(g) = 0$.

(vii) Let $\mathcal{B} = \{(\omega, x) |\ \omega \in \Omega,\ x \in [g],\ \omega^x = \omega\}$. Since χ is constant on $[g]$, we have

$$|[g]|\chi(g) = |\mathcal{B}| = \sum_{\omega \in \Omega} |[g] \cap G_\omega|$$

By Lemma 2.6.3 all subgroups G_ω are conjugate in G. Thus $|[g] \cap G_\omega| = m$ is independent of ω, and $\chi(g)|[g]| = m|\Omega| = m\chi(1_G)$.

This completes the proof. ■

Corollary 2.6.5. *Let $H \leq G$ with $\chi = \mathbf{1}\uparrow_H^G$. Let $g \in G$ and assume that $[g]$ splits in H into m classes with representatives $h_1, h_2, \cdots, h_m$. Then*

$$\chi\uparrow_H^G(g) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_H(h_i)|}.$$

PROOF. Immediate by Proposition 2.5.5. ■

3

$\mathbb{S}$tructure $\mathbb{O}$f $\mathbb{T}$he $\mathbb{G}$eneral $\mathbb{L}$inear $\mathbb{G}$roup

In this chapter, we go briefly over the basic and elementary properties and the structure of the general linear group $GL(n, \mathbb{F})$ and some of its subgroups. Also some of the groups associated with $GL(n, \mathbb{F})$ will be studied. In most of the work, we follow the notation in Alperin [3], Cameron [12] and Rotman [65].

3.1. Subgroups and Associated Groups

In this section, we study the general features of the general linear group $GL(n, \mathbb{F})$ and some of its subgroups. We focus mainly in the case where $\mathbb{F}$ is finite; that is $\mathbb{F} = \mathbb{F}_q$, the Galois Field of q elements.

3.1.1 The General and Special Linear Groups

Definition 3.1.1. *Let V be a vector space over the field $\mathbb{F}$, the* ***General Linear Group*** *of V, written $GL(V)$ or $Aut(V)$, is the group of all automorphisms of V, i.e. the set of all bijective linear transformations $V \longrightarrow V$, together with composition of functions as group operation.*

If $V(n, \mathbb{F})$ denotes the $n-$dimensional vector space over a field $\mathbb{F}$, then $GL(V)$ is identified with group $GL(n, \mathbb{F})$ consisting of the $n \times n$ nonsingular matrices defined over the field $\mathbb{F}$. Moreover; if $\mathbb{F} = \mathbb{F}_q$, then we write $GL(n, q)$ in place of $GL(n, \mathbb{F}_q)$. The following proposition counts the elements of the group $GL(n, q)$.

Proposition 3.1.1. *The number of the elements of $GL(n, q)$ is* $\prod_{k=0}^{n-1}(q^n - q^k)$.

PROOF. This holds by counting the $n \times n$ matrices whose rows are linearly independent. The i^{th} row can be any vector not in the linear span of the first $i - 1$ rows and thus has $q^n - q^{i-1}$ possibilities. Hence, there are $(q^n - 1)(q^n - q)\cdots(q^n - q^{n-1}) = \prod_{k=0}^{n-1}(q^n - q^k)$ invertible $n \times n$ matrices. ∎

For any positive integers n and m with $n < m$ and fixed field $\mathbb{F}$, the group $GL(n, \mathbb{F})$ is embedded into $GL(m, \mathbb{F})$ by sending $A \in GL(n, \mathbb{F})$ to the $m \times m$ matrix having A in the left upper corner, I_{m-n} in the right lower corner and zeros elsewhere.

Definition 3.1.2. *An invertible linear transformation $A : V \longrightarrow V$ with determinant 1 is called a **unimodular**. For the finite n-dimensional vector space $V(n, \mathbb{F})$, the **Special Linear Group**, written $SL(n, \mathbb{F})$, is the subgroup of $GL(n, \mathbb{F})$ consisting of all unimodular transformations.*

We omit showing that $SL(n, \mathbb{F})$ satisfies the subgroup axioms. Moreover; we can see that $SL(n, \mathbb{F})$ is the kernel of the homomorphism $\det : GL(n, \mathbb{F}) \longrightarrow \mathbb{F}^*$ and hence $SL(n, \mathbb{F}) \trianglelefteq GL(n, \mathbb{F})$. Thus the group $GL(n, \mathbb{F})$ is not simple group in general.

Proposition 3.1.2. $|SL(n, q)| = \prod_{k=1}^{n-1}(q^{n+1} - q^k) = q^{\frac{n(n-1)}{2}}(q^n - 1)(q^{n-1} - 1)\cdots(q^2 - 1)$.

PROOF. By the first isomorphism theorem of groups, $GL(n, \mathbb{F})/\ker(\det) \cong Im(\det)$. Now det is surjective. Therefore, $Im(\det) = \mathbb{F}^*$ and $\ker(\det) = SL(n, \mathbb{F})$. Thus $GL(n, \mathbb{F})/SL(n, \mathbb{F}) \cong \mathbb{F}^*$ and hence, when $\mathbb{F}$ is finite with q elements, $|SL(n, q)| = |GL(n, q)|/q - 1$ and the result follows by Proposition 3.1.1. ■

Now, let K be an isomorphic copy of the group $\mathbb{F}^* \cong GL(1, \mathbb{F})$ in the group $GL(n, \mathbb{F})$, where the embedding is defined as in the comment after Proposition 3.1.1. That is

$$K = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & I_{n-1} \end{pmatrix} \mid \alpha \in \mathbb{F}^* \right\}. \tag{3.1}$$

This embedding makes K not normal subgroup in $GL(n, \mathbb{F})$ in general. The next theorem relates the elements of $GL(n, \mathbb{F})$ and $SL(n, \mathbb{F})$.

Theorem 3.1.3. *The group $GL(n, \mathbb{F}) = SL(n, \mathbb{F}){:}K$.*

PROOF. Assume that $g \in GL(n, \mathbb{F})$ and $\det(g) = \delta \in \mathbb{F}^*$. The element $k_{\delta^{-1}} = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & I_{n-1} \end{pmatrix}$ is in K. Let $h = gk_{\delta^{-1}}$. Then $\det(h) = \det(gk_{\delta^{-1}}) = \det(g)\det(k_{\delta^{-1}}) = \delta\delta^{-1} = 1$, which shows that $h \in SL(n, \mathbb{F})$ and therefore $g = h(k_{\delta^{-1}})^{-1} = hk_{\delta}$. Hence we have that $GL(n, \mathbb{F}) = SL(n, \mathbb{F})K$. On the other hand, since $SL(n, \mathbb{F}) \cap K = \{I_n\}$ and normality of $SL(n, \mathbb{F})$ in $GL(n, \mathbb{F})$ was established above, we have $GL(n, \mathbb{F}) = SL(n, \mathbb{F}) : K$. ■

Lemma 3.1.4. *Half of the elements of $\mathbb{F}_q^*$, q is odd, are squares while if q is even, then all the elements of $\mathbb{F}_q^*$ are squares.*

PROOF. See Hill [32]. ■

Corollary 3.1.5. *If q is even, then $GL(2,q) = SL(2,q) \times H,\ H \cong \mathbb{F}_q^*$.*

PROOF. Let $H = \{\alpha I_2 |\ \alpha \in \mathbb{F}_q^*\}$. Then H is normal in $GL(2,q)$ since $g\alpha I_2 g^{-1} = \alpha I_2,\ \forall g \in GL(2,q)$. By Lemma 3.1.4, every element of $\mathbb{F}_q^*$ is a square because q is even. Thus we may assume that $\det(g) = \delta^2$ for $g \in GL(n,q)$ and $\delta^2 \in \mathbb{F}_q^*$. Then similar steps used in Theorem 3.1.3 to show that $GL(n,\mathbb{F}) = SL(n,\mathbb{F})K$ can be applied here also. Therefore, $GL(2,q) = SL(2,q)H$. Now $\alpha I_2 \in SL(2,q) \iff det(\alpha I_2) = \alpha^2 = 1 \iff \alpha = 1$. Note that $-1 = 1$ because q has characteristic 2. Therefore $SL(2,q) \cap H = \{I_2\}$ and the result follows. ■

3.1.2 Upper Triangular, $p-$Sylow and Parabolic Subgroups

Definition 3.1.3. *The set $UT(n,\mathbb{F})$ consisting of all $n \times n$ invertible upper triangular matrices over the field $\mathbb{F}$ forms a subgroup of $GL(n,\mathbb{F})$, which we call the* ***Upper Triangular Subgroup****.*

The group $UT(n,q)$ has order $|UT(n,q)| = q^{\frac{n(n-1)}{2}}(q-1)^n$, since elements in the main diagonal are taken from $\mathbb{F}_q^*$ and elements above to the main diagonal can be any element of $\mathbb{F}_q$.

An important subgroup of the group $UT(n,\mathbb{F})$ is $UT(n,\mathbb{F}) \cap SL(n,\mathbb{F})$, which we denote by $SUT(n,\mathbb{F})$ and we call the ***Special Upper Triangular Group***. The group $SUT(n,q)$ has order $q^{\frac{n(n-1)}{2}}(q-1)^{n-1}$, since elements above the main diagonal can be chosen arbitrarily from $\mathbb{F}_q$, while all elements of the main diagonal are taken from $\mathbb{F}_q^*$ in arbitrary way, except the element in the $(n,n)th$ position, which must be $\left(\prod_{i=1}^{n-1} a_{ii}\right)^{-1}$ to make $\det(g) = 1$. Hence $SUT(n,q)$ is of index $(q-1)$ in $UT(n,q)$.

In what follows, we give our attention to the Sylow $p-$subgroups of the general linear group $GL(n,q)$, where p is the characteristic of the field of q elements.

Definition 3.1.4. *The subset of $SUT(n,\mathbb{F})$, where each element have 1's in the main diagonal, forms a subgroup of $SUT(n,\mathbb{F})$, called* ***Special Upper Unitriangular Group*** *and is denoted by $SUUT(n,\mathbb{F})$.*

Remark 3.1.1. The group $SUUT(n,q)$ have just been defined is easily seen to belong to $Syl_p(GL(n,q))$, the set of Sylow $p-$subgroups of $GL(n,q)$, since the order of $SUUT(n,q)$ is $q^{\frac{n(n-1)}{2}}$, which is the highest power of q in the order of $GL(n,q)$. Hence, any Sylow $p-$subgroup of $GL(n,q)$ is conjugate to $SUUT(n,q)$ and by Sylow's Theorem, the number of Sylow $p-$subgroups divides the number $[GL(n,q) : SUUT(n,q)]$. Moreover, the group $SUUT(n,q)$ represents a Sylow $p-$subgroup of the groups $SL(n,q)$, $UT(n,q)$ and $SUT(n,q)$. We see later that it is also a Sylow $p-$subgroup of the parabolic subgroup P_λ.

Definition 3.1.5. *An element u of $GL(n,\mathbb{F})$ is called* ***unipotent*** *if its characteristic polynomial is $(t-1)^n$. A subgroup H of $GL(n,\mathbb{F})$ is called a* ***unipotent subgroup*** *if all its elements are unipotent.*

Remark 3.1.2. The subgroup $SUUT(n,\mathbb{F})$ is a unipotent subgroup of $GL(n,\mathbb{F})$ since every element of this subgroup has all eigenvalues equal to 1. It is proved by Kolchin (see Alperin [3]) that any unipotent subgroup of $GL(n,\mathbb{F})$ is conjugate with the subgroup $SUUT(n,\mathbb{F})$.

The subgroup $SUUT(n,\mathbb{F})$ will be used to give a factorization of the group $UT(n,\mathbb{F})$, namely we will see that (see Corollary 3.1.9)

$$UT(n,\mathbb{F}) = SUUT(n,\mathbb{F}) : \bigotimes_{n \text{ copies}} \mathbb{F}^*.$$

Note 3.1.1. Note that for $n > 1$, the group $\bigotimes_{n \text{ copies}} \mathbb{F}^*$ is not normal in $UT(n,\mathbb{F})$, except when $\mathbb{F} = \mathbb{F}_2$. Hence $UT(n,\mathbb{F})$ is not the direct product of $SUUT(n,\mathbb{F})$ and $\bigotimes_{n \text{ copies}} \mathbb{F}^*$, in general. To see that $\bigotimes_{n \text{ copies}} \mathbb{F}^*$ is not normal, take $UT(n,\mathbb{F}) \ni g = \begin{pmatrix} 1 & \cdots & 0 & 1 \\ 0 & 1 & \vdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$ and $\bigotimes_{n \text{ copies}} \mathbb{F}^* \ni h = \begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$, for some $b \neq 1$. Then we have

$$\begin{pmatrix} 1 & \cdots & 0 & 1 \\ 0 & 1 & \vdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 & -1 \\ 0 & 1 & \vdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & 0 & \cdots & 1-b \\ 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \notin \bigotimes_{n \text{ copies}} \mathbb{F}^*.$$

Observe that as long as the field $\mathbb{F}$ contains more than two elements, then we have such b, which makes $ghg^{-1} \notin \bigotimes_{n \text{ copies}} \mathbb{F}^*$. In the case, when $q = 2$, the subgroup $SUUT(n,2) = UT(n,2)$, while the subgroup $\bigotimes_{n \text{ copies}} \mathbb{F}_2^*$ reduces to the neutral group. In this case, Theorem 3.1.9 is satisfied trivially.

We conclude this subsection by discussing the parabolic subgroups of $GL(n,\mathbb{F})$. We start by defining the *flags* of a vector space.

Definition 3.1.6. *A* ***flag*** $\mathfrak{F}$ *is an increasing sequence of subspaces of an* $n-$*dimensional vector space* $V_n = V(n,\mathbb{F})$, *which satisfies the proper containment; that is to say*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_r = V(n,\mathbb{F}).$$

Hence

$$0 < \dim V_1 < \dim V_2 < \cdots < \dim V_r = n. \tag{3.2}$$

If $\dim V_i = i, \ \forall i$, then the flag $\mathfrak{F}$ is called a *complete flag* or *full flag.*

Let F be the set of all flags of an $n-$dimensional vector space $V_n = V(n, \mathbb{F})$. We define an equivalence relation $\sim$ on F by

$$(V_0 \subset V_1 \subset \cdots \subset V_r) \sim (W_0 \subset W_1 \subset \cdots \subset W_s)$$
$$\text{if and only if } r = s \text{ and } \dim V_i = \dim W_i, \forall i.$$

From equation (3.2), we have

$$(\dim V_1 - 0) + (\dim V_2 - \dim V_1) + \cdots + (\dim V_r - \dim V_{r-1}) = \dim V_r - 0 = \dim V_r = n.$$

Therefore each equivalence class of $\sim$ defines a partition $\sigma \vdash n$, whose parts are $(\dim V_i - \dim V_{i-1})$, $1 \leq i \leq r$. Conversely, to any partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$, written in ascending order, one can associate (up to equivalence of flags) a flag $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_r = V(n, \mathbb{F})$ such that the subspaces $V_i, \ i \geq 1$, of V_n contains of the vectors whose first $\lambda_1 + \lambda_2 + \cdots + \lambda_i$ components are nonzero. We summarize this in the following proposition.

Proposition 3.1.6. *There is a $1-1$ correspondence between the set of equivalence classes defined by $\sim$ above and the set of partitions of n.*

PROOF. Established above. ■

In terms of the above proposition, we can write without ambiguity $\mathfrak{F}_\lambda$ to denote the flag corresponds to the partition λ. We may also call $\mathfrak{F}_\lambda$ by the $\lambda-$*flag.*

Note 3.1.2. The complete flag $\mathfrak{F}_\lambda$ is the flag corresponding to the partition $\lambda = 1^n$.

Let us denote the $\lambda-$flag $\mathfrak{F}_\lambda$ given in the above definition by $\mathfrak{F}_\lambda = (V_1, V_2, \cdots, V_k)$. The general linear group $G_n = GL(n, q)$ acts on a natural way on the set of all flags of the vector space $V(n, q)$ by $g(V_1, V_2, \cdots, V_r) = (gV_1, gV_2, \cdots, gV_r)$, where $g \in G_n$ can be viewed as an invertible linear transformation. This action by g preserves the proper containment and $\dim gV_i = \dim V_i, \ \forall i$.

The action of G_n on $\mathfrak{F}$ is intransitive and two flags $\mathfrak{F}_\lambda = (V_1, V_2, \cdots, V_k)$ and $\mathfrak{F}_\mu = (W_1, W_2, \cdots, W_s)$ belong to the same orbit if and only if $k = s$ and $\dim V_i = \dim W_i, \ \forall i$. The stabilizer of a flag $\mathfrak{F}_\lambda$ on the action of the group G_n on the set of flags, consists of the elements $g \in G_n$ such that $\mathfrak{F}_\lambda^g = \mathfrak{F}_\lambda$ or $g(V_1, V_2, \cdots, V_k) = (V_1, V_2, \cdots, V_k)$. This motivates the following definition.

Definition 3.1.7. *The stabilizer of a flag $\mathfrak{F}_\lambda$ which is associated with a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$ is called a* ***Standard Parabolic Subgroup*** *of G_n and we denote it by P_λ. More generally, any subgroup of G_n conjugates to P_λ is called a* ***Parabolic Subgroup****.*

It is proved (see Alperin [3], Bump [11], Green [27], Macdonald [50] or Springer [71]) that a parabolic subgroup P_λ of G_n consists of the elements of the form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & A_{kk} \end{pmatrix}, \tag{3.3}$$

where $A_{ii} \in GL(\lambda_i, \mathbb{F}),\ 1 \leq i \leq k$, and A_{ij} for $i < j$ is a block matrix of size $\lambda_i \times \lambda_j$.

Next we would like to count the number of $\lambda-$flags $\mathfrak{F}_\lambda$ of a vector space $V(n, q)$, where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$. For this we define $[r],\ r \in \mathbb{Z}$ by

$$[r] = \begin{cases} \frac{q^r-1}{q-1} = q^{r-1} + q^{r-2} + \cdots + 1 & \text{if } r \in \mathbb{Z}^+, \\ 0 & \text{if } r = 0, \\ -q^r[-r] & \text{if } r \in \mathbb{Z}^-. \end{cases} \tag{3.4}$$

Also let $\{r\} = [r]! = [1][2]\cdots[r]$. Moreover by $\begin{bmatrix} s \\ t \end{bmatrix}$ we mean

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{cases} \frac{[s]!}{[t]![s-t]!} & \text{if } s \geq t, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following Proposition.

Proposition 3.1.7. *Let $s, t \in \mathbb{N}$. Then $\begin{bmatrix} s \\ t \end{bmatrix}$ counts the number of $t-$dimensional subspaces W of an $s-$dimensional vector space V over $\mathbb{F}_q$.*

PROOF. See James [39]. ■

Definition 3.1.8. *The polynomial $\begin{bmatrix} s \\ t \end{bmatrix}$ is known as the* ***Gaussian Polynomial****.*

We recall that if $\mathfrak{F}_\lambda = (0 \subset V_1 \subset V_2 \subset \cdots \subset V_k)$, then $\dim V_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i,\ \forall 1 \leq i \leq k$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$. Now for each $i > 1$, the number of subspaces V_{i-1} of V_i is given by

$$\begin{bmatrix} \dim V_i \\ \dim V_{i-1} \end{bmatrix} = \begin{bmatrix} \lambda_1 + \lambda_2 + \cdots + \lambda_i \\ \lambda_1 + \lambda_2 + \cdots + \lambda_{i-1} \end{bmatrix} = \frac{\{\lambda_1 + \lambda_2 + \cdots + \lambda_i\}}{\{\lambda_1 + \lambda_2 + \cdots + \lambda_{i-1}\}}.$$

Therefore the number of the $\lambda-$flags is given by

$$\begin{aligned}\prod_{i=2}^{k}\begin{bmatrix}\dim V_i\\ \dim V_{i-1}\end{bmatrix} &= \frac{\{\lambda_1+\lambda_2\}}{\{\lambda_1\}\{\lambda_2\}}\cdot\frac{\{\lambda_1+\lambda_2+\lambda_3\}}{\{\lambda_3\}\{\lambda_1+\lambda_2\}}\cdot\frac{\{\lambda_1+\lambda_2+\lambda_3+\lambda_4\}}{\{\lambda_4\}\{\lambda_1+\lambda_2+\lambda_3\}}\\ &\cdots\frac{\{\lambda_1+\lambda_2+\cdots+\lambda_k\}}{\{\lambda_k\}\{\lambda_1+\lambda_2+\cdots+\lambda_{k-1}\}} = \frac{\{\lambda_1+\lambda_2+\cdots+\lambda_k\}}{\{\lambda_1\}\{\lambda_2\}\cdots\{\lambda_k\}}\\ &= \frac{\{n\}}{\{\lambda_1\}\{\lambda_2\}\cdots\{\lambda_k\}}.\end{aligned}$$

Thus $|\mathfrak{F}_\lambda{}^{GL(n,q)}| = \{n\}/\{\lambda_1\}\{\lambda_2\}\cdots\{\lambda_k\}$ and it follows by the *Orbit-Stabilizer* Theorem (see Theorem 1.2.2 of Moori [54]) that $|GL(n,q)_{\mathfrak{F}_\lambda}| = |P_\lambda| = |GL(n,q)|/|\mathfrak{F}_\lambda{}^{GL(n,q)}|$. Hence

$$[GL(n,q):P_\lambda] = \frac{\{n\}}{\{\lambda_1\}\{\lambda_2\}\cdots\{\lambda_k\}}. \tag{3.5}$$

From the definition of $[r]$, we can see that $q \nmid \{n\}/\{\lambda_1\}\{\lambda_2\}\cdots\{\lambda_k\}$. We deduce that if $P \in Syl_p(P_\lambda)$ (p is the characteristic of $\mathbb{F}_q$), then $|P| = q^{\frac{n(n-1)}{2}}$. Since $SUUT(n,q) \leq P_\lambda$ (by taking $A_{ii} \in SUUT(\lambda_i,q)$), it follows that $SUUT(n,q) \in Syl_p(P_\lambda)$ for any parabolic subgroup P_λ of $GL(n,q)$. It is possible to show that $|P_\lambda| = q^{\frac{n(n-1)}{2}}\prod_{m=1}^{k}\prod_{s=1}^{\lambda_m}(q^s-1)$, but this is not straightforward neither from (3.3) nor from (3.5) and we omit the verification.

Two important subgroups of any parabolic subgroup P_λ, namely the *unipotent radical* and the *standard levi complement* of P_λ, are of great importance. The unipotent radical, which we denote by U_λ, is defined to be the set of all invertible linear transformations which induce the identity on the successive quotient V_i/V_{i-1}, $\forall i$, where V_i are the components of the flag $\mathfrak{F}_\lambda$ on which the parabolic subgroup P_λ is defined. In terms of matrices, the unipotent radical U_λ consists of the matrices

$$\begin{pmatrix} I_{\lambda_1} & A_{12} & \cdots & A_{1k}\\ 0 & I_{\lambda_2} & \cdots & A_{2k}\\ \vdots & \cdots & \ddots & \vdots\\ 0 & 0 & 0 & I_{\lambda_k}\end{pmatrix}. \tag{3.6}$$

It follows that, if $\mathbb{F} = \mathbb{F}_q$, then the order of the unipotent radical U_λ is $q^{\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\lambda_i\lambda_j}$.

On the other hand the standard levi complement, denoted by L_λ consists of the matrices of the form

$$\begin{pmatrix} A_{11} & 0 & \cdots & 0\\ 0 & A_{22} & \cdots & 0\\ \vdots & \cdots & \ddots & \vdots\\ 0 & 0 & 0 & A_{kk}\end{pmatrix}, \tag{3.7}$$

where as before, $A_{ii} \in GL(\lambda_i, \mathbb{F}),\ 1 \leq i \leq k$.

Clearly $L_\lambda \cong \bigotimes_{i=1}^{k} GL(\lambda_i, \mathbb{F})$ and has order $\prod_{i=1}^{k} |GL(\lambda_i, \mathbb{F})|$ if $\mathbb{F}$ is the finite field of q elements. More generally, any non-normal subgroup of P_λ that conjugates to L_λ is called a *levi complement.*

Example 3.1.1. If $\mathfrak{F}_{1^n}$ is the complete flag, then the parabolic subgroup P_{1^n} is just the group of upper triangular matrices $UT(n, \mathbb{F})$, while $U_{1^n} = SUUT(n, \mathbb{F})$ and $L_{1^n} = \bigotimes_{n \text{ copies}} \mathbb{F}^* =$ the subgroup of the diagonal matrices (some people refer to L_{1^n} as the **torus**).

Example 3.1.2.

1. Let $n = 2$. Then the two parabolic subgroups corresponding to the partitions $\lambda = (2)$ and $\mu = 1^2$ are $P_{(2)} = GL(2, \mathbb{F})$ with unipotent radical $U_{(2)} = I_2$ and levi complement $L_{(2)} = GL(2, \mathbb{F})$, while the parabolic subgroup P_{1^2} is the group $UT(2, \mathbb{F})$ with $SUUT(2, \mathbb{F})$ as its unipotent radical and $GL(1, \mathbb{F}) \times GL(1, \mathbb{F}) \cong \mathbb{F}^* \times \mathbb{F}^*$ as its levi complement.

2. For $n = 3$, the three parabolic subgroups corresponding to the partitions $\lambda = (3);\ \mu = (1, 2)$ and $\nu = 1^3$ are

$$
\begin{aligned}
P_{(3)} &= GL(3, \mathbb{F}),\ U_{(3)} = I_3 \text{ and } L_{(3)} = GL(3, \mathbb{F});\\
P_{(1,2)} &= \left\{ \begin{pmatrix} \alpha & g & f \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \mid a, b, c, d, g, f \in \mathbb{F},\ \alpha \in \mathbb{F}^*,\ ad - bc \neq 0 \right\},\\
U_{(1,2)} &= \left\{ \begin{pmatrix} 1 & s & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid s, t \in \mathbb{F} \right\},\\
L_{(1,2)} &= \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{F},\ \alpha \in \mathbb{F}^*,\ ad - bc \neq 0 \right\} \cong GL(1, \mathbb{F}) \times GL(2, \mathbb{F});\\
P_{1^3} &= \left\{ \begin{pmatrix} \alpha_1 & a & b \\ 0 & \alpha_2 & c \\ 0 & 0 & \alpha_3 \end{pmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}^*,\ a, b, c, d \in \mathbb{F} \right\},\\
U_{1^3} &= \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F} \right\},\\
L_{1^3} &= \left\{ \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}^* \right\} \cong \mathbb{F}^* \times \mathbb{F}^* \times \mathbb{F}^*.
\end{aligned}
$$

Theorem 3.1.8. *With P_λ being a parabolic subgroup of G_n, then $P_\lambda = U_\lambda{:}L_\lambda$. Furthermore, $P_\lambda = N_{P_\lambda}(U_\lambda)$, the normalizer of U_λ in P_λ.*

PROOF. It is immediate to see from (3.6) and (3.7), that $U_\lambda \bigcap L_\lambda = \{I_n\}$. Normality of U_λ in P_λ follows from the fact that U_λ represents the kernel of the homomorphism $\psi : P_\lambda \to L_\lambda$, where ψ acts on P_λ by sending the main diagonal of an element A of P_λ to the diagonal matrix having the same diagonal of A. Let $A \in P_\lambda$, be an arbitrary element. Then $A\psi(A)^{-1} \in U_\lambda$. It follows that $A \in U_\lambda\psi(A) \subseteq U_\lambda L_\lambda$. Thus $P_\lambda \subseteq U_\lambda L_\lambda$, and the equality of P_λ and $U_\lambda L_\lambda$ is established. Since $U_\lambda \trianglelefteq P_\lambda$, then $P_\lambda = N_{P_\lambda}(U_\lambda)$. This completes the proof of the theorem. ■

Corollary 3.1.9. $UT(n,\mathbb{F}) = SUUT(n,\mathbb{F}) \colon \bigotimes_{n \text{ copies}} \mathbb{F}^*$.

PROOF. The proof is a special case of combining Example 3.1.1 and Theorem 3.1.8. ■

Since the levi complement $L_\lambda \cong \bigotimes_{i=1}^{k} GL(\lambda_i, \mathbb{F})$, then by Theorem 2.3.2, the irreducible characters of L_λ are

$$Irr(L_\lambda) = \left\{ \bigotimes_{i=1}^{k} \chi_i | \ \chi_i \in Irr(GL(\lambda_i, \mathbb{F})) \right\}, \tag{3.8}$$

where in the last equation, $\bigotimes$ is to be understood the tensor product of characters.

Theorem 3.1.8 asserts that the exact sequence

$$L_\lambda \longrightarrow P_\lambda \longrightarrow P_\lambda/U_\lambda$$

is an isomorphism, where the first map is inclusion and the second projection. This means that an irreducible character of L_λ extends irreducibly to P_λ, by using the method of lifting of characters described in Section 2.4. By equation (3.8), we get $\prod_{i=1}^{k} |Irr(GL(\lambda_i, q))|$ irreducible characters of P_λ. The preceding irreducible characters of P_λ comes from characters of L_λ are used as a base for Frobenius method of induction of characters to build up characters of the group $GL(n,q)$. The characters of the group $GL(n,q)$ appear into two series, namely *Principal* and *Discrete* series. The **Principal Series** characters are those which are obtained from characters of parabolic subgroups of $GL(n,q)$. Any character which is not in the principal series characters is said to belong to the **Discrete Series**. The discussion of obtaining characters of $GL(n,q)$ from those of P_λ, $\lambda \vdash n$ will be continued in Section 5.3. The discrete series characters will be discussed in Section 5.4.

3.1.3 Weyl Group of $GL(n,\mathbb{F})$

We recall that a *permutation matrix* is a matrix obtained from the identity matrix by switching some columns (rows). The set of all permutation matrices forms a subgroup W of $GL(n,\mathbb{F})$ called the *Weyl Group*.

Theorem 3.1.10. *The Weyl group W is isomorphic to the symmetric group S_n.*

PROOF. Let $\mathcal{B} = \{e_1, e_2, \cdots, e_n\}$ be the standard basis of $V(n, \mathbb{F})$. The Weyl group W act on $\mathcal{B}$ on a natural way; that is if $w \in W$, then $we_i = e_k$, $1 \leq i, k \leq n$. Let $X = \{1, 2, \cdots, n\}$. For each $w \in W$, the function $\varphi_w : X \longrightarrow X$ given by $\varphi_w(i) = k$, for $1 \leq i, k \leq n$ is such that $we_i = e_k$, is well defined and a bijective. Hence $\varphi_w \in S_n$. Now if we define $\varphi : W \longrightarrow S_n$ by $\varphi(w) = \varphi_w$, then it is not difficult to see that φ is a bijective homomorphism and hence it is an isomorphism. The result follows. ■

Remark 3.1.3. The above theorem asserts that the Weyl group of $GL(n, q)$ is independent of the choice of the field $\mathbb{F}$. It is characterized by the dimension n only.

In the next context, we introduce a special kind of matrices of $GL(n, \mathbb{F})$ which are of great importance in order to describe the elements of $GL(n, \mathbb{F})$ and consequently $SL(n, \mathbb{F})$.

Definition 3.1.9. *A **transvection** is a linear transformation T on $V(n, \mathbb{F})$ with eigenvalues equal to 1 and satisfying rank$(T - I_n) = 1$, where I_n is the identity transformation on $V(n, \mathbb{F})$.*

In matrix language, a transvection $A_{ij}(\alpha)$ where $i \neq j$ and $\alpha \in \mathbb{F}$, is a matrix different from the identity matrix only that it has α in the $(i, j)th$ position. It turns out that all transvections are elements of $SL(n, \mathbb{F})$.

One can easily verify the following properties of transvections.

Lemma 3.1.11. *For $\alpha, \beta \in \mathbb{F}, i \neq j$,*

1. $A_{ij}(0) = I_n$.
2. $\det(A_{ij}(\alpha)) = 1$.
3. *If $\alpha \neq 0$, then $A_{ij}(\alpha) \in UT(n, \mathbb{F}) \Longleftrightarrow i < j$.*
4. $A_{ij}(\alpha)A_{ij}(\beta) = A_{ij}(\alpha + \beta)$.
5. $(A_{ij}(\alpha))^{-1} = A_{ij}(-\alpha)$.
6. *For $i \neq j \neq k \neq i$, the commutator $[A_{ij}(\alpha), A_{jk}(\beta)] = A_{ik}(\alpha\beta)$.*

PROOF. Direct results from the definition. ■

As a quick result of this lemma, we have

Corollary 3.1.12. *For fixed i and j, the set $\mathcal{A}_{ij} = \{A_{ij}(\alpha) \mid \alpha \in \mathbb{F}\}$ forms a subgroup of $SL(n, \mathbb{F})$.*

PROOF. It follows directly by parts (2), (3) and (4) of Lemma 3.1.11. ■

The subgroups defined this way are refer as the *root subgroups of* $GL(n, \mathbb{F})$.

Now, we come to a known theorem concerning the structure of the group $G_n = GL(n, \mathbb{F})$.

Theorem 3.1.13 (Bruhat Decomposition Theorem). $GL(n, \mathbb{F}) = UT(n, \mathbb{F}) \cdot W \cdot UT(n, \mathbb{F})$.

PROOF. In Singh [70], it is shown that any matrix $A \in GL(n, \mathbb{F})$ splits into a product $A = L_1 w d L_2$, where $L_1, L_2 \in SUUT(n, \mathbb{F})$, $d \in \bigotimes_{n \text{ copies}} \mathbb{F}^*$ and $w \in W$. It follows that any element of $GL(n, \mathbb{F})$ is a product of an upper triangular matrix, a permutation matrix, and another upper triangular matrix. One can refer also to Alperin [3] for the details. ■

Remark 3.1.4. Bruhat Decomposition Theorem asserts that $GL(n, \mathbb{F})$ is a union (disjoint) of the double cosets $UT(n, \mathbb{F})wUT(n, \mathbb{F})$ as w ranges over all elements of W. Thus $GL(n, \mathbb{F})$ is a union of $n!$ disjoint double cosets $UT(n, \mathbb{F})w\ UT(n, \mathbb{F})$.

The next theorem gives a smaller generating set for $GL(n, \mathbb{F})$ than that given by Bruhat Decomposition Theorem, but we first mention a lemma without proof, which will be helpful in the proof of the theorem.

Lemma 3.1.14. *For each* $b \in UT(n, \mathbb{F})$, *there exists a product* T *of transvections such that* Tb *is a diagonal matrix having the same main diagonal entries as* b.

PROOF. See Alperin [3]. ■

Theorem 3.1.15. *The group* $GL(n, \mathbb{F})$ *is generated by the set of all invertible diagonal matrices and all transvections.*

PROOF. By Bruhat Decomposition Theorem, we have $GL(n, \mathbb{F}) = UT(n, \mathbb{F}) \cdot W \cdot UT(n, \mathbb{F})$. Thus if we could write all the elements of $UT(n, \mathbb{F})$ and W in terms of diagonal matrices and transvection, then we done. Using Lemma 3.1.14, we can see that $UT(n, \mathbb{F})$ has this property. By Theorem 3.1.10, every permutation matrices can be written in terms of permutations of S_n, which are generated by the set of transpositions. The action of a transposition on the standard basis $\mathcal{B} = \{e_1, e_2, \cdots, e_n\}$ is that it sends $e_i \longmapsto e_j \longmapsto e_i$ for some $i \neq j$ and fixes the rest of $\mathcal{B}$. Now the action of the matrix $A_{ji}(1)A_{ij}(-1)A_{ji}(1)$ on $\mathcal{B}$ is that it sends $e_i \longmapsto e_j \longmapsto -e_i$ for $i \neq j$ and fixes the other elements of $\mathcal{B}$. Multiplying this latter matrix by the diagonal matrix $diag(1, \cdots, 1, -1, 1, \cdots, 1)$, where -1 is in the (i, i) position, the resulting matrix sends $e_i \longmapsto e_j \longmapsto e_i$ for $i \neq j$ and fixes the other elements of $\mathcal{B}$, which shows that W can be written in terms of diagonal matrices and transvections. The result follows. ■

Theorem 3.1.16. *The group* $SL(n, \mathbb{F})$ *is generated by the root subgroups* $\mathcal{A}_{ij}$.

PROOF. We give the idea of the proof, which rests on the following three main points. Full details of the proof can be found in Alperin [3].

- Every element of the group $SL(n, \mathbb{F})$ can be transformed into an element of the group $UT(n, \mathbb{F})$ by multiplying by some suitably transvections.
- Every element of the group $UT(n, \mathbb{F})$ can be transformed into an element of the group $SUUT(n, \mathbb{F})$ by multiplying by some suitably transvections.
- Every element of the group $SUUT(n, \mathbb{F})$ can be transformed into the identity element I_n by multiplying by some suitably transvections.

Thus any element of $SL(n, \mathbb{F})$ is a product of transvections, which completes the proof. ■

Theorem 3.1.17. *All transvections are conjugate in $GL(n, q)$ and if $n \geq 3$, then all transvections are conjugate in $SL(n, q)$.*

PROOF. See Alperin [3] or Rotman [65]. ■

3.1.4 Center and Derived Subgroups of $GL(n, \mathbb{F})$ and $SL(n, \mathbb{F})$

Two normal subgroups of any group G, namely the center of the group $Z(G)$ and the commutator or derived subgroup $G^{'}$, are of particular interest. In what follows, we mention some important facts about these two normal subgroups for the case when G is $GL(n, \mathbb{F})$ or $SL(n, \mathbb{F})$.

Theorem 3.1.18. *The center $Z(GL(n, \mathbb{F}))$ consists of all invertible scalar matrices and hence isomorphic to the group $\mathbb{F}^*$, while the center of $Z(SL(n, \mathbb{F}))$ is $SL(n, \mathbb{F}) \cap Z(GL(n, \mathbb{F}))$.*

PROOF. Two different proofs are given in Alperin [3] and Rotman [65]. ■

Now, we attack the commutator subgroups of $GL(n, q)$ and $SL(n, q)$.

Theorem 3.1.19. *The commutator subgroup $GL(n, q)^{'}$ is $SL(n, q)$, except in the case $n = q = 2$.*

PROOF. Suppose that $n \neq 2$ or $q \neq 2$. Then by Dieudonné [17], $GL(n, q)/GL(n, q)^{'} \cong GL(1, q)/GL(1, q)^{'}$ which is $\mathbb{F}_q^*$. This shows that $[GL(n, q) : GL(n, q)^{'}] = q - 1$. Now, $GL(n, q)^{'} \leq SL(n, q)$ (This follows from the fact that if $aba^{-1}b^{-1}$ is a commutator of $GL(n, q)$, then $\det(aba^{-1}b^{-1}) = 1$, which implies that $GL(n, q)^{'} \subseteq SL(n, q)$ and hence $GL(n, q)^{'} \leq SL(n, q)$). Since $GL(n, q)^{'}$ and $SL(n, q)$ have the same orders, this forces $GL(n, q)^{'}$ to be $SL(n, q)$.
If $n = q = 2$, then $GL(2, 2) \cong SL(2, 2) \cong S_3$, but it is easy to see that $S_3^{'} = A_3 \ncong SL(2, 2)$, which completes the proof. ■

To deal with the commutator subgroup of $SL(n, q)$, we need the following lemma.

Lemma 3.1.20. *If $n \geq 2$, then every transvection $A_{ij}(\alpha)$ is a commutator of elements of $SL(n,q)$, except when $n = 2$ and ($q = 2$ or $q = 3$).*

PROOF. We start with the exceptional cases. Let $n = 2$. Possible transvections are $A_{12}(\alpha)$ and $A_{21}(\alpha)$ for $\alpha \in \mathbb{F}_q^*$. We consider the case $A_{12}(\alpha)$ and the other one follows similarly. Assume that $a = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$, $\beta \in \mathbb{F}_q^*$ and $b = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}$, $\theta \in \mathbb{F}_q$. The commutator of a and b is $[a,b] = aba^{-1}b^{-1} = \begin{pmatrix} 1 & (\beta^2-1)\theta \\ 0 & 1 \end{pmatrix}$. Therefore expressing the transvection $A_{12}(\alpha)$ as a commutator of two elements a and b of $SL(n,q)$ is conditionally connected with the existence of $\beta \in \mathbb{F}_q^*$, $\theta \in \mathbb{F}_q$ such that $\alpha = (\beta^2-1)\theta$. This is satisfied if $\theta \neq 0$ and $\beta^2 \neq 1$. If $|\mathbb{F}_q| > 3$, then existence of such β is guaranteed and we can take $\theta = \alpha(\beta^2-1)^{-1}$.
On the other hand if $n > 2$, then $A_{ij}(\alpha) = [A_{ik}(\alpha), A_{kj}(1)]$ for distinct i, j and k, by part (6) of Lemma 3.1.11. ■

Theorem 3.1.21. *The commutator subgroup $SL(n,q)'$ is $SL(n,q)$ itself, except in the cases $n = 2$ and ($q = 2$ or 3).*

PROOF. If $n \neq 2$, then Theorem 3.1.16 asserts that $SL(n,q)$ is generated by the set of all transvections in $GL(n,q)$. Lemma 3.1.20 states that every transvestion is a commutator of elements of $SL(n,q)$. Combining these two results, we deduce that $SL(n,q) \subseteq SL(n,q)'$. Since $SL(n,q)' \leq SL(n,q)$, we have $SL(n,q)' = SL(n,q)$.
If $n = 2$ and ($q = 2$ or 3), then $SL(2,2)$ and $SL(2,3)$ are isomorphic to S_3 and S_4 respectively. Again S_3' and S_4' are A_3 and A_4 respectively, which furnishes the case. ■

Corollary 3.1.22. *In the case $n \neq 2$ or $q \notin \{2,3\}$, the group $SL(n,q)$ is perfect.*

3.1.5 Groups Related To $GL(n,\mathbb{F})$

The Projective General and Special Linear Groups

It is known from elementary group theory that the center of a group G is a normal subgroup. So, the quotient is defined. This motivates the following definition.

Definition 3.1.10. *The groups $GL(n,\mathbb{F})/Z(GL(n,\mathbb{F}))$ and $SL(n,\mathbb{F})/Z(SL(n,\mathbb{F}))$ are known as the **Projective General Linear Group** and **Projective Special Linear Group**. These groups are denoted by $PGL(n,\mathbb{F})$ and $PSL(n,\mathbb{F})$ respectively.*

The group $PGL(n,q)$ has order equal to that of $SL(n,q)$ given in Proposition 3.1.2, while the order of the group $PSL(n,q)$ is given by $|PSL(n,q)| = |SL(n,q)|/\gcd(n,q-1)$, where $\gcd(n,q-1)$ is the greatest common divisor of n and $q-1$. In particular, if $n=2$, then

$$|PSL(2,q)| = \begin{cases} \frac{q^3-q}{2} & \text{if } q \text{ is odd,} \\ q^3-q & \text{if } q \text{ is even.} \end{cases}$$

Note 3.1.3. If q is even, then $PSL(2,q) \cong SL(2,q)$, since $Z(SL(2,q)) = \{I_2\}$.

It was proved (see Rotman [65]) by Jordan-Moore that $PSL(2,q)$ is simple for $q \geq 4$. In 1896, L. E. Dickson showed (see Cameron [12] or Rotman [65]) that $PSL(n,q)$ for any $n \geq 2$ is simple except when $n=2$ and ($q=2$ or $q=3$). There are many trends to characterize finite simple groups by their character tables. This problem has been solved completely for the infinite family of Alternating groups A_n, $n \geq 5$ by T. Oyama [59]. Lambert ([43], Theorem 5.1) proved that the infinite family of groups $PSL(2,q)$ can be characterized by their character tables; that is if G is a group with the same character table of $PSL(2,q)$, then $G \cong PSL(2,q)$. He solved the same problem for $PSL(3,q)$ in [44]. In [45], he proved that if G is a group with the same character table as $PSL(n,q)$, q even, then $G \cong PSL(n,q)$.

Example 3.1.3. Here, we have some of the isomorphisms between $PSL(n,q)$ and some other familiar groups.

1. $PSL(2,4) \cong SL(2,4) \cong A_5 \cong PSL(2,5)$.
2. $PSL(4,2) \cong A_8$, while $PSL(3,4)$ and A_8 are non-isomorphic simple groups of the same orders. This result due to Scottenfels in 1900, (see Rotman [65]).
3. $PSL(2,7) \cong GL(3,2)$ and $PSL(2,9) \cong A_6$.

The character tables of all the above groups are given in the appendix of Isaac [38].

The Affine group Aff(n,q)

An *affine transformation* from a finite dimensional vector space $V(n,\mathbb{F}) = V$ to itself is a map $\phi_{A,b}$ consisting of a linear transformation followed by a translation; that is $\phi_{A,b}(u) = Au + b$, where $A \in GL(n,\mathbb{F})$ and $b \in V$.

The set of all affine transformations of a vector space V form a group under the composition of functions. We call this group the *Affine Group* and we denote it by Aff$(n,\mathbb{F})$. Formally the affine group reads

$$\text{Aff}(n,\mathbb{F}) = \{\phi_{A,b} \mid \ A \in GL(n,\mathbb{F}),\ b \in V\}. \tag{3.9}$$

One can obtain all invertible linear transformations of V; that is $GL(n,\mathbb{F})$, by setting b to be the zero vector, $b=0$, in the preceding equation, then $\phi_{A,0}(u) = Au + 0 = Au$. A result which one can say that $GL(n,\mathbb{F}) \subseteq \text{Aff}(n,\mathbb{F})$. On the other hand, one can also obtain the set of all translations τ_b

of V, by setting A to be the identity transformation $A = I_n$, in the same equation. Then we get $\phi_{I_n,b}(u) = I_n u + b = u + b$. We deduce that the set of translations $\tau_b : V \longrightarrow V,\ \tau_b(u) = u + b$ form an abelian subgroup of $\mathrm{Aff}(n, \mathbb{F})$.

Proposition 3.1.23. *The abelian group T consisting of all translations τ_b of a vector space V is isomorphic to the additive group V.*

PROOF. The function $\theta : V \longrightarrow S_V$ defined by $\theta(b) = \tau_b,\ b \in V$ is a monomorphism. Its image $Im(\theta)$ is easily seen to be all T. Thus by the first isomorphism theorem, $V/\ker(\theta) \cong Im(\theta) = T$. But θ is one to one function. Therefore $\ker(\theta) = \{0_V\}$. Hence $V \cong T$ as claimed. ■

The affine group $\mathrm{Aff}(n, \mathbb{F})$ can be embedded as a subgroup of the general linear group of degree $n + 1$. This is the statement of the following theorem.

Theorem 3.1.24. $\mathrm{Aff}(n, \mathbb{F}) \leq GL(n + 1, \mathbb{F})$.

PROOF. Suppose that $\phi_{A,b}$ and $\phi'_{A',b'}$ are two elements of $\mathrm{Aff}(n, \mathbb{F})$. Then

$$\phi_{A,b}\phi'_{A',b'}(u) = \phi_{A,b}(A'u + b') = AA'u + Ab' + b = A''u + b'',$$

where $A'' = AA'$ and $b'' = Ab' + b$. Now define the function $\varphi : \mathrm{Aff}(n, \mathbb{F}) \longrightarrow GL(n + 1, \mathbb{F})$, by

$$\varphi(\phi_{A,b}) = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}.$$

Then φ is a group homomorphism since

$$\varphi(\phi_{A,b}\phi'_{A',b'}) = \begin{pmatrix} A'' & b'' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A' & b' \\ 0 & 1 \end{pmatrix} = \varphi(\phi_{A,b})\varphi(\phi'_{A',b'}).$$

It can also be shown that φ is injective. Therefore, φ is a monomorphism with kernel $\ker(\varphi) = \{I_n\}$. Hence

$$\mathrm{Aff}(n, \mathbb{F}) \cong Im(\varphi) = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in GL(n, \mathbb{F}),\ b \in V(n, \mathbb{F}) \right\} \leq GL(n + 1, \mathbb{F}),$$

which completes the proof of the Theorem. ■

The next theorem, which is stated without proof, is of great importance for the purpose of the computation of the character tables of $\mathrm{Aff}(n, \mathbb{F})$, by using the Clifford-Fischer Method.

Theorem 3.1.25. *The group $\mathrm{Aff}(n, \mathbb{F})$ is a split extension of $V(n, \mathbb{F})$ by $GL(n, \mathbb{F})$.*

PROOF. See Neumann [58] or Rodrigues [63]. ■

In the finite case when $\mathbb{F} = \mathbb{F}_q$, then from the above theorem we have $\text{Aff}(n, q) = q^n{:}GL(n, q)$. In her M.Sc dissertation, Whitley [76], calculated the character table of $\text{Aff}(3, 2) = 2^3{:}GL(3, 2)$. Iranmanesh [36], had calculated the full character tables of the groups $\text{Aff}(2, q)$, $\text{Aff}(3, q)$ and $\text{Aff}(4, q)$. The same author in [37] determined the character table of $\text{Aff}(n, q)$ for arbitrary positive integer n.

3.2. The BN Pair Structure of The General Linear Group

The notion of BN pair structure comes from the theory of Lie algebra (see Curtis and Reiner [9]).

Definition 3.2.1. *A $BN-$ **pair** (or **Tits System**) is an ordered quadruple (G, B, N, S) where:*

1. *G is a group generated by subgroups B and N.*
2. *$T := B \cap N \trianglelefteq N$.*
3. *S is a subset of $W = N/T$ consisting of involutions (elements of order 2), such that $\langle S \rangle = W$.*
4. *If $\sigma,\ \mu \in N$ and $\mu T \in S$, then $\mu B \sigma \subseteq B\sigma B \cup B\mu\sigma B$.*
5. *If $\mu T \in S$, then $\mu B \mu \neq B$.*

If (G, B, N, S) is a $BN - pair$, the subgroups B and T, and the group $W = N/T$ are known as the *Borel subgroup*, *Cartan subgroup* and *Weyl group* of G respectively. The number $|S|$ is called the *rank* of the system.
Now, the group $G = GL(n, \mathbb{F})$, $n \geq 2$, has a $BN-$pair structure. For B, we take the group of upper triangular matrices $UT(n, q)$. For N, we consider the group of monomial matrices, those are the matrices having exactly one nonzero element in each row and column. The Cartan group $T = B \cap N$ [1] consists of the diagonal matrices and it is normal in N. We identify $W = N/T$ with the group of permutation matrices. Finally, we may take S to be the subset of W consisting of those permutation matrices that obtained from the identity matrix by switching two adjacent columns; that is S consists of all the transpositions of S_n. Satisfying the conditions of the $BN-$pair structure for the group $GL(n, q)$ with the above groups B, N, T and the set S, are exhausted by Bruhat Decomposition Theorem given in Theorem 3.1.13.

Hence $(GL(n, \mathbb{F}), UT(n, \mathbb{F}), Monomials(n, \mathbb{F}), Transpositions(S_n))$ is a Tits system with rank $\binom{n}{2} = \frac{n(n-1)}{2}$.

Likewise the group $SL(n, \mathbb{F})$ has also a $BN-$pair structure. For this, let B, N, T and W be the groups, which together with the set S define the $BN-$pair structure of $GL(n, \mathbb{F})$. Take $B_0 =$

[1]The group T is known also as the minimal torus.

$B \cap SL(n, \mathbb{F}) \cong SUT(n, \mathbb{F})$, $N_0 = N \cap SL(n, \mathbb{F})$, $T_0 = T \cap SL(n, \mathbb{F})$ and $W_0 = N_0/T_0$. By Alperin [3], $W_0 \cong S_n$. Thus we may take S to be the set of transpositions of S_n. Now, $(SL(n, \mathbb{F}), SUT(n, \mathbb{F}), N_0, S)$ is a Tits system with rank $\binom{n}{2} = \frac{n(n-1)}{2}$.

Finally, the group $PSL(n, \mathbb{F})$ has a $BN-$pair structure. Refer to Alperin [3] for the details.

4

$GL(2, q)$ and $\mathbb{S}$ome of its $\mathbb{S}$ubgroups

4.1. Introduction

In this chapter, we will construct character tables of $GL(2, q)$ and some of its subgroups. This will include character tables of the following groups

1. $GL(2, q)$, the general linear group,
2. $SL(2, q)$, the special linear group,
3. $SUT(2, q)$, the special upper triangular group and
4. $UT(2, q)$, the standard Borel group (group of non-singular upper triangular matrices).

These groups and some other subgroups, have a lattice diagram shown in Figure 4.1. In each of the above four groups, two specific examples when $q = 3$ and $q = 4$ will be illustrated as the determination of some of the character tables of some of these groups will depend on the parity of q. The character table of the group $GL(2, q)$ will be used as a base to construct the character tables of the above mentioned groups. Also, the irreducible characters of $GL(2, q)$ will be used to construct the character tables of the groups $GL(m, q)$, for $m \geq 3$. In particular, in this dissertation we will use the irreducible characters of $GL(2, q)$ to produce a large number of irreducible characters of the group $GL(3, q)$ as we shall see in Section 5.7.

Systematic use of the dual operations, namely induction and restriction of characters from some subgroups to the main groups and conversely, will be made. All irreducible characters of the group $GL(2, q)$ will be obtained from induced characters of two subgroups; namely $\mathbb{F}_q^* \times \mathbb{F}_q^*$ and $\mathbb{F}_{q^2}^*$ with some suitable embedding into $GL(2, q)$. Following to that, the character table of the group $SL(2, q)$, q even, is obtained directly from that of $GL(2, q)$ because of Corollary 3.1.5. When q is odd, $SL(2, q)$ has $q + 4$ irreducible characters. Of these, q characters will be obtained from restriction of irreducible characters of $GL(2, q)$, while for the remaining four characters, the group

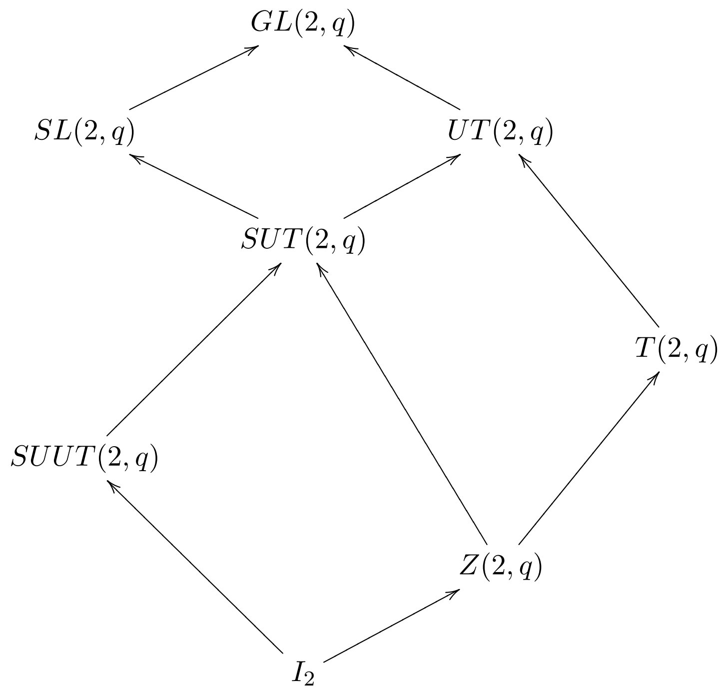

Figure 4.1: The lattice diagram of some subgroups of $GL(2,q)$.

$SD(2,q)$, consisting of elements with square determinants, will complete the picture. The group $SUT(2,q)$ is an split extension group and therefore the methods of the coset analysis and Clifford-Fischer theory can be applied to obtain its character table. We apply these theories to the case $SUT(2,q)$, q even. However, when q is odd, these theories are still applicable but are not done in this dissertation. Also when q is even, the group $SUT(2,q)$ will be one of the Frobenius groups, whose representations are known. Using this fact, the character table of $SUT(2,q)$, where q is even, will be constructed.

In the following, we give our attention to the character table of the group $GL(2,q)$, which was done firstly by Jordan [35] and Schur [67] separately in 1907. This has been studied extensively by many authors, for example one can find the description of these tables in Aburto [1], Adams [2], Alperin [3], Drobotenko [18], James, [40], Prasad [60], Reyes [61] or Steinberg [72].

Also, in describing the conjugacy classes and irreducible characters of the group $GL(2,q)$, we follow mainly James [40] and Steinberg [72].

4.2. Conjugacy Classes of $GL(2,q)$

In this section, we give representatives for the conjugacy classes of $GL(2,q)$, which is a group of order $(q^2-1)(q^2-q) = q(q-1)^2(q+1)$ by Proposition 3.1.1, together with the sizes of centralizers, classes and the orders of the class representatives.

Theorem 4.2.1. *The group $GL(2,q)$ has q^2-1 conjugacy classes described in* Table 4.1.

Class	$\mathcal{T}_k^{(1)}$	$\mathcal{T}_k^{(2)}$	$\mathcal{T}_{k,l}^{(3)}$	$\mathcal{T}_k^{(4)}$
Rep g	$\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -r^{q+1} & r+r^q\end{pmatrix}$
No. of CC	$q-1$	$q-1$	$(q-1)(q-2)/2$	$q(q-1)/2$
$\lvert C_g\rvert$	1	q^2-1	$q(q+1)$	$q(q-1)$
$\lvert C_{GL(2,q)}(g)\rvert$	$(q^2-1)(q^2-q)$	$q(q-1)$	$(q-1)^2$	q^2-1

Table 4.1: The conjugacy classes of $GL(2,q)$

where, in Table 4.1*,*

- *by* CC *we mean conjugacy classes of a prescribed type of classes,*
- $\alpha,\beta\in\mathbb{F}_q^*,\ \alpha\neq\beta$,
- $r\in\mathbb{F}_{q^2}\setminus\mathbb{F}_q$ *and r^q is excluded whenever r is included,*
- *in $\mathcal{T}_k^{(1)},\mathcal{T}_k^{(2)}$ and $\mathcal{T}_{k,l}^{(3)}$, k,l denote the integers for which $\alpha=\varepsilon^k$, $\beta=\varepsilon^l$ and ε being a generator of $\mathbb{F}_q^*$,*
- *in $\mathcal{T}_k^{(4)}$, k denotes the integer for which $r=\theta^k$, where θ is a generator of $\mathbb{F}_{q^2}^*$.*

PROOF. We claim that

- no two different classes of the same types can be conjugate,
- no two classes of different types can be conjugate, and
- the representatives given in the table have the stated sizes of classes and centralizers.

First, it is clear that the classes of the first type consists of the central elements of $GL(2,q)$. Therefore, each element form a self class and clearly there are $q-1$ such classes corresponding to each $\alpha\in\mathbb{F}_q^*$. We consider the other three types of classes through the following set of equations. With $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL(2,q)$ and $\alpha,\alpha',\beta,\beta'\in\mathbb{F}_q^*$, we have

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}=\begin{pmatrix}a\alpha & a+b\alpha\\ c\alpha & c+d\alpha\end{pmatrix}, \tag{4.1}$$

$$\begin{pmatrix}\alpha' & 1\\ 0 & \alpha'\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a\alpha'+c & d+b\alpha'\\ c\alpha' & d\alpha'\end{pmatrix}, \tag{4.2}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} a\alpha & b\beta \\ c\alpha & d\beta \end{pmatrix}, \tag{4.3}$$

$$\begin{pmatrix} \alpha' & 0 \\ 0 & \beta' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\alpha' & b\alpha' \\ c\beta' & d\beta' \end{pmatrix}, \tag{4.4}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -r^{q+1} & r+r^q \end{pmatrix} = \begin{pmatrix} -br^{q+1} & a+b(r+r^q) \\ -dr^{q+1} & c+d(r+r^q) \end{pmatrix}, \tag{4.5}$$

$$\begin{pmatrix} 0 & 1 \\ -r^{q+1} & r+r^q \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ -ar^{q+1}+c(r+r^q) & -br^{q+1}+d(r+r^q) \end{pmatrix}. \tag{4.6}$$

Now, assume that $\alpha \neq \alpha'$. Then we have the following implications.

$$\begin{aligned} \text{Equation (4.1)} = \text{Equation (4.2)} &\iff a\alpha = a\alpha' + c,\ a + b\alpha = d + b\alpha'\ c\alpha = c\alpha',\ \&\ c + d\alpha = d\alpha' \\ &\iff a(\alpha - \alpha') - c = 0,\ a - d + b(\alpha - \alpha') = 0,\ c(\alpha - \alpha') = 0, \\ &\qquad c + d(\alpha - \alpha') = 0 \end{aligned}$$

Since $\alpha \neq \alpha'$, we must have $c = 0$ and consequently, $a = b = d = 0$; that is $g = 0_{2\times 2}$, which contradicts that $g \in GL(2,q)$. Thus interchanging α with another element of $\mathbb{F}_q^*$ in the typical element of type $\mathcal{T}_k^{(2)}$, gives another class which is not conjugate to that one obtained by α. Hence there are $q-1$ conjugacy classes of the second type for $GL(2,q)$.

On the other hand, if $\alpha = \alpha'$, then

$$\begin{aligned} \text{Equation (4.1)} = \text{Equation (4.2)} &\iff a\alpha = a\alpha + c,\ a + b\alpha = d + b\alpha\ c\alpha = c\alpha,\ c + d\alpha = d\alpha \\ &\iff c = 0,\ a = d. \end{aligned}$$

Thus the centralizer of an element of the second type of classes of $GL(2,q)$ consists of the elements of the form $t = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ and therefore the invertibility of t implies that a must be in $\mathbb{F}_q^*$ while b can be any element of $\mathbb{F}_q$. So, there are $q(q-1)$ elements in the centralizer of an element of the second type and consequently $q^2 - 1$ conjugates as mentioned in the table.

Similarly, for classes of the third type of $GL(2,q)$, suppose firstly that $\{\alpha, \beta\} \neq \{\alpha', \beta'\}$. Then we have the following possibilities

- $\alpha = \alpha',\ \beta \neq \beta'$,
- $\alpha = \beta',\ \beta \neq \alpha'$,
- $\alpha \neq \alpha'$ and $\{\beta \neq \beta'$ or $\beta = \beta'\}$,
- $\alpha \neq \alpha'$, and $\{\beta = \alpha'$ or $\beta \neq \alpha'\}$.

Now

$$\begin{aligned}\text{Equation (4.3)} = \text{Equation (4.4)} &\iff a\alpha = a\alpha^{'},\ b\beta = b\alpha^{'}\ c\alpha = c\beta^{'},\ d\beta = d\beta^{'}\\ &\iff a(\alpha-\alpha^{'}) = 0,\ b(\beta-\alpha^{'}) = 0,\ c(\alpha-\beta^{'}) = 0,\\ &\qquad d(\beta-\beta^{'}) = 0.\end{aligned}$$

We consider two cases out of the above six possibilities of $\{\alpha,\beta\} \neq \{\alpha^{'},\beta^{'}\}$. Suppose that $\alpha = \alpha^{'}$ and $\beta \neq \beta^{'}$. Then $d = 0$ and we have $c(\alpha-\beta^{'}) = 0$. replacing α with $\alpha^{'}$ in the last equality, we get $c(\alpha^{'}-\beta^{'}) = 0$. Since $\alpha^{'} \neq \beta^{'}$, we must have $c = 0$. Now $c = d = 0$. This contradicts the invertibility of g. Let us consider the case where take $\alpha \neq \alpha^{'}$ and $\beta \neq \beta^{'}$. Here we have further subcases corresponding to $\alpha = \beta^{'}$ or $\alpha \neq \beta^{'}$. In either cases, equality of equations (4.3) and (4.4) implies that $a = 0$ and $d = 0$ with either $b = 0$ or $c = 0$ respectively. So, we have contradictions in these cases too. The other remaining cases are very similar and we omit the verifications. Therefore, whenever, $\{\alpha,\beta\} \neq \{\alpha^{'},\beta^{'}\}$, the elements $\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$ and $\begin{pmatrix}\alpha^{'} & 0\\ 0 & \beta^{'}\end{pmatrix}$ are not conjugate in $GL(2,q)$.

On the other hand, if $(\alpha,\beta) = (\alpha^{'},\beta^{'})$, then

$$\begin{aligned}\text{Equation (4.3)} = \text{Equation (4.4)} &\iff a\alpha = a\alpha,\ b\beta = b\alpha,\ c\alpha = c\beta,\ d\beta = d\beta\\ &\iff a\alpha = a\alpha,\ b(\beta-\alpha) = 0,\ c(\alpha-\beta) = 0\ d\beta = d\beta\\ &\iff b = c = 0\end{aligned}$$

Thus the centralizer of an element of the classes of third type for $GL(2,q)$ consists of the elements of the form $t = \begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}$ and therefore the invertibility of t forces a and d to be in $\mathbb{F}_q^*$. So, there are $(q-1)^2$ elements in the centralizer and consequently, $q(q+1)$ conjugates of an element in each of the classes of the third type. Now any class of the third type determined by a pair $(\alpha,\beta),\ \alpha \neq \beta$ is conjugate to the class determined by the pair (β,α) under the conjugation by the involution $\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$. In other words, each unordered pair $\{\alpha,\beta\}_{\alpha\neq\beta}$ gives one conjugacy class of this type. Since there are $q-1$ choices for α and $q-2$ choices for β, there are $\frac{(q-1)(q-2)}{2}$ conjugacy classes of the third type.

Since the size of a conjugacy class of an element of the second type is $(q-1)(q+1)$, which is different from $q(q+1)$, the size of a conjugacy class of a typical element of the third type, then we deduce that elements of the second type can not be conjugate to elements of the third type.

For the last case, where we consider an element of the fourth type $A_r = \begin{pmatrix}0 & 1\\ -r^{q+1} & r+r^q\end{pmatrix}$ and $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. The characteristic polynomial of A_r is $\lambda^2 - (r+r^q)\lambda + r^{q+1}$, which decomposes into $(\lambda-r)(\lambda-r^q)$. Therefore A_r has eigenvalues $\lambda = r$ and $\lambda = r^q$. Thus the Jordan form of A_r is

$\begin{pmatrix} r & 0 \\ 0 & r^q \end{pmatrix}$. We deduce that two elements A_s and A_r are conjugate in $GL(2,q)$ if and only if A_s and A_r have the same Jordan form; that is if and only if $\{s, s^q\} = \{r, r^q\}$. Clearly A_r is not conjugate to any of the preceding classes we discussed since eigenvalues of an element of the first three types are in $\mathbb{F}_q^*$, while the eigenvalues of an element of the fourth type are in $\mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$.
The next step is to count the size of the centralizer (isotropy group) of A_r. Equating equations (4.5) and (4.6), we obtain that $c = -br^{q+1}$ and $d = a + b(r + r^q)$. Thus the isotropy group of A_r consists of the elements $t = \begin{pmatrix} a & b \\ -br^{q+1} & a + b(r + r^q) \end{pmatrix}$, $(a, b) \neq (0, 0)$. Since a and b can be any elements of $\mathbb{F}_q$, but not both zero, we have $|C_{GL(2,q)}(A_r)| = q^2 - 1$ and consequently $|C_{A_r}| = q(q-1)$. We observe that for elements of the fourth type, we have $A_r = A_{r^q}$. Since r has $q^2 - q$ choices and $A_r = A_{r^q}$, this restricts the number of classes of this type to $\frac{q^2-q}{2}$.

As a final step, we count the number of elements in all classes that we have found so far:

$$\begin{aligned} & (q-1) + (q-1)(q^2-1) + \frac{(q-1)(q-2)}{2} q(q+1) + \frac{q^2-q}{2} q(q-1) \\ = \quad & (q-1)\left(q^2 + \frac{q(q+1)(q-2)}{2} + \frac{q^3-q^2}{2}\right) \\ = \quad & (q-1)\left(\frac{2q^2 + q^3 - q^2 - 2q + q^3 - q^2}{2}\right) \\ = \quad & (q-1)(q^3-q) = q(q-1)^2(q+1) = |GL(2,q)|. \end{aligned}$$

And the number of conjugacy classes is

$$\begin{aligned} & (q-1) + (q-1) + \frac{(q-1)(q-2)}{2} + \frac{q^2-q}{2} \\ = \quad & (q-1)\left(\frac{4 + (q-2) + q}{2}\right) = (q-1)(q+1) = q^2 - 1. \end{aligned}$$

This shows that the classes listed in Table 4.1, are the full conjugacy classes of $GL(2,q)$. ■

In Proposition 4.2.2, we calculate the orders of the elements in each of the conjugacy classes.

Proposition 4.2.2. *The elements of the group $GL(2,q)$ have the following orders,*

$$o(g) = \begin{cases} \frac{q-1}{\gcd(k,\ q-1)} & \text{if } g \text{ is of type } \mathcal{T}^{(1)}, \\ \frac{p(q-1)}{\gcd(k,\ q-1)} & \text{if } g \text{ is of type } \mathcal{T}^{(2)}, \\ lcm\left(\frac{(q-1)}{\gcd(k,\ q-1)}, \frac{(q-1)}{\gcd(l,\ q-1)}\right) & \text{if } g \text{ is of type } \mathcal{T}^{(3)}, \\ lcm\left(\frac{(q-1)}{\gcd(k,\ q-1)}, \frac{(q^2-1)}{\gcd(k,\ q^2-1)}, \frac{(q^2-1)}{\gcd(kq,\ q^2-1)}\right) & \text{if } g \text{ is of type } \mathcal{T}^{(4)}. \end{cases}$$

PROOF. We divide the proof into four parts. In each part we prove the stated order for a typical element of one of the four types of classes of $GL(2,q)$.

(i) Let us consider an element g of the first type $\mathcal{T}_k^{(1)}$, where $g = \alpha I_2 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ and $\alpha = \varepsilon^k$ for some k. Assume that g has order t. Then

$$g^t = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}^t = \begin{pmatrix} \alpha^t & 0 \\ 0 & \alpha^t \end{pmatrix} = \alpha^t I_2 = I_2 \Longleftrightarrow \alpha^t = 1 \Longleftrightarrow o(\alpha)|t. \tag{4.7}$$

Let $t^{'}$ be the order of α. Then

$$g^{t^{'}} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}^{t^{'}} = \begin{pmatrix} \alpha^{t^{'}} & 0 \\ 0 & \alpha^{t^{'}} \end{pmatrix} = I_2. \tag{4.8}$$

Therefore, $o(g)|t^{'}$; that is $t|t^{'}$. From this and equation (4.7), we deduce that $t = t^{'}$; that is αI_2 has same order as of $\alpha = \varepsilon^k$ for some $1 \leq k \leq q-1$. From elementary group theory, we have $o(\varepsilon^k) = (q-1)/\gcd(k, q-1)$. This gives the required order of a typical element of the first type.

(ii) Suppose that $g = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ has order t. Then

$$g^t = \begin{pmatrix} \alpha^t & t \\ 0 & \alpha^t \end{pmatrix} = I_2 \Longleftrightarrow \alpha^t = 1 \text{ and } p|t \Longleftrightarrow o(\alpha)|t \text{ and } p|t. \tag{4.9}$$

Since $\gcd(o(\alpha), p) = 1$, by (4.9), we have $p.o(\alpha)|t$. It is easy to see that

$$g^{p.o(\alpha)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$

so that $t|p.o(\alpha)$. Hence $t = p.o(\alpha) = p(q-1)/\gcd(k, q-1)$.

(iii) Now let $g = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and let t be the order of g. Then

$$\begin{aligned} g^t = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}^t = \begin{pmatrix} \alpha^t & 0 \\ 0 & \beta^t \end{pmatrix} = I_2 \Longleftrightarrow \alpha^t = 1,\ \beta^t = 1 \Longleftrightarrow o(\alpha)|t,\quad o(\beta)|t \\ \Longleftrightarrow lcm(o(\alpha), o(\beta))|t. \end{aligned} \tag{4.10}$$

Let $a = o(\alpha),\ b = o(\beta)$ and $d = \gcd(a, b)$. Suppose also that $s = lcm(o(\alpha), o(\beta)) = lcm(a, b)$. Then $s = \frac{ab}{d} = \frac{a^{'}db^{'}d}{d} = a^{'}b^{'}d = a^{'}b = b^{'}a$ and $\gcd(a^{'}, b^{'}) = 1$. Now

$$g^s = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}^s = \begin{pmatrix} \alpha^s & 0 \\ 0 & \beta^s \end{pmatrix} = \begin{pmatrix} \alpha^{b^{'}a} & 0 \\ 0 & \beta^{a^{'}b} \end{pmatrix} = I_2. \tag{4.11}$$

This implies that order of g which is t divides s. We have seen from equation (4.10) that $s|t$. Therefore

$$\begin{aligned} t = s &= lcm(o(\alpha), o(\beta)) = lcm(o(\varepsilon^k), o(\varepsilon^l)) = \frac{o(\varepsilon^k)o(\varepsilon^l)}{\gcd(o(\varepsilon^k), o(\varepsilon^l))} \\ &= \frac{(q-1)^2}{\gcd(k, q-1)\gcd(l, q-1)\gcd\left(\frac{q-1}{\gcd(k,q-1)}, \frac{q-1}{\gcd(l,q-1)}\right)}. \end{aligned}$$

(iv) Finally let $g = \begin{pmatrix} 0 & 1 \\ -r^{q+1} & r+r^q \end{pmatrix}$. The eigenvalues of g are $\lambda = r$ and $\lambda = r^q$. Since $g \sim \begin{pmatrix} r & 0 \\ 0 & r^q \end{pmatrix}$, we have $o(g) = o\left(\begin{pmatrix} r & 0 \\ 0 & r^q \end{pmatrix}\right)$. So $o(g) = lcm(o(r), o(r^q))$.

If θ is a generator of the group $\mathbb{F}^*_{q^2}$ and $r = \theta^k$ for some k, such that $q+1 \nmid k$, then

$$o(g) = \frac{(q^2-1)^2}{\gcd(k, q^2-1)\gcd(kq, q^2-1)\gcd\left(\frac{q^2-1}{\gcd(k,q^2-1)}, \frac{q^2-1}{\gcd(kq,q^2-1)}\right)}.$$

Hence all elements of the group $GL(2,q)$ have the stated orders, which completes the proof of the Proposition. ■

4.3. Irreducible Characters of $GL(2,q)$

We have seen that there are q^2-1 conjugacy classes of $GL(2,q)$ and hence there must be the same number of irreducible characters. In Table 4.2, we list the values of these irreducible characters on the conjugacy classes of $GL(2,q)$. For this, we identify the group $\mathbb{F}^*_q$ with the subgroup $U < \mathbb{C}$ consisting of the $(q-1)th$ roots of unity by $\widehat{\varepsilon} = \kappa$, where ε and κ are generators of $\mathbb{F}^*_q$ and U respectively.

Theorem 4.3.1. *The group $G = GL(2,q)$ has q^2-1 distinct conjugacy classes fall within four types $\mathcal{T}^{(1)}$, $\mathcal{T}^{(2)}$, $\mathcal{T}^{(3)}$ and $\mathcal{T}^{(4)}$. The q^2-1 irreducible characters fall also in four distinct types $\chi^{(1)}$, $\chi^{(2)}$, $\chi^{(3)}$ and $\chi^{(4)}$ described in* Table 4.2.

Table 4.2: Character table of $GL(2,q)$

	$\mathcal{T}^{(1)}_s$	$\mathcal{T}^{(2)}_s$	$\mathcal{T}^{(3)}_{s,t}$	$\mathcal{T}^{(4)}_k$
Rep g	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -r^{q+1} & r+r^q \end{pmatrix}$
No. of CC	$q-1$	$q-1$	$(q-1)(q-2)/2$	$q(q-1)/2$
$\lvert C_g \rvert$	1	q^2-1	$q(q+1)$	$q(q-1)$
$\lvert C_G(g) \rvert$	$(q^2-1)(q^2-q)$	$q(q-1)$	$(q-1)^2$	q^2-1
$\lambda_k = \chi^{(1)}_k$	$\widehat{\alpha}^{2k}$	$\widehat{\alpha}^{2k}$	$\widehat{\alpha}^k \widehat{\beta}^k$	$\widehat{r}^{k(q+1)}$
$\psi_k = \chi^{(2)}_k$	$q\widehat{\alpha}^{2k}$	0	$\widehat{\alpha}^k \widehat{\beta}^k$	$-\widehat{r}^{k(q+1)}$
$\psi_{k,l} = \chi^{(3)}_{k,l}$	$(q+1)\widehat{\alpha}^{k+l}$	$\widehat{\alpha}^{k+l}$	$\widehat{\alpha}^k \widehat{\beta}^l + \widehat{\alpha}^l \widehat{\beta}^k$	0
$\pi_k = \chi^{(4)}_k$	$(q-1)\widehat{\alpha}^k$	$-\widehat{\alpha}^k$	0	$-(\widehat{r}^k + \widehat{r}^{kq})$

where, in Table 4.2*,*

- $\alpha, \beta \in \mathbb{F}_q^*,\ \alpha \neq \beta$,
- s, t *denote the integers for which* $\alpha = \varepsilon^s$ *and* $\beta = \varepsilon^t$, ε *is a generator of* $\mathbb{F}_q^*$,
- $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ *and* r^q *is excluded whenever* r *is included,*
- *in* $\chi_k^{(1)},\ \chi_k^{(2)},\ k = 0, 1, \cdots, q-2$,
- *in* $\chi_{k,l}^{(3)},\ 0 \leq k < l \leq q-2$,
- *in* $\chi_k^{(4)},\ q+1 \nmid k,\ k = 1, 2, \cdots, q^2-1$ *and* kq *is excluded whenever* k *is included,*
- $\widehat{\ }$ *is the homomorphism* : $\mathbb{F}_{q^d}^* = \langle \varepsilon_d \rangle \longrightarrow \mathbb{C}^*$ *given by* $\widehat{\ }(\varepsilon_d^j) = \widehat{\varepsilon_d^j} = e^{\frac{2\pi j}{q^d-1}i}$, *for* $d = 1, 2$ *and* $0 \leq j \leq q^d - 2$.

PROOF. The function $\det : GL(2,q) \longrightarrow \mathbb{F}_q^*$ defines a group homomorphism. For $k = 0, 1, \cdots, q-2$, we set $\lambda_k : GL(2,q) \longrightarrow U$ to be $\lambda_k(g) := \chi_k(\det(g))$, where χ_k is an irreducible character of $\mathbb{F}_q^*$. It is clear that the composition $\chi_k \circ \det : GL(2,q) \longrightarrow U$ is a group homomorphism. Thus it is an ordinary representation of degree 1 and consequently is an irreducible character of $GL(2,q)$. We recall by Theorem 3.1.19 that the derived subgroup $GL(n,q)'$ is $SL(n,q)$ except when $n = q = 2$. By Proposition 2.3.4, the number of linear characters of $GL(2,q)$ is $|GL(2,q)|/|GL(2,q)'| = |GL(2,q)|/|SL(2,q)| = q-1$. Thus, apart from the case $GL(2,2)$, the $q-1$ linear characters given by λ_k are all the linear characters of $GL(2,q)$. In the case $GL(2,2) \cong S_3$, we have the extra linear character corresponding to the sign of the permutations of S_3.
The next table shows the values of the linear characters λ_k on the conjugacy classes of $GL(2,q)$.

Table 4.3: Values of the linear characters on elements of $GL(2,q)$

Class	$\mathcal{T}_s^{(1)}$	$\mathcal{T}_s^{(2)}$	$\mathcal{T}_{s,t}^{(3)}$	$\mathcal{T}_r^{(4)}$
λ_k	$\chi_k^2(\alpha)$	$\chi_k^2(\alpha)$	$\chi_k(\alpha)\chi_k(\beta)$	$\chi_k(r^{(q+1)})$

where $0 \leq k \leq q-2$.

In fact, the $q-1$ linear characters given by the powers of the determinants comprise all the linear characters of the group $GL(n,q)$, for any $n \in \mathbb{N}$ and any prime power q, excepting the case $GL(2,2)$. This will be proved in Theorem 5.6.3.

Let T be the torus in $GL(2,q)$, which by Section 3.2 consists of the 2×2 diagonal matrices; that is

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{F}_q^* \right\}.$$

We can see clearly that $T \cong \mathbb{F}_q^* \times \mathbb{F}_q^*$ and hence

$$Irr(T) = \{\chi_k\chi_l |\ \chi_k,\ \chi_l \in Irr(\mathbb{F}_q^*)\}.$$

We recall by Definition 3.1.4 that the group $SUUT(2,q)$ consists of the elements of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in \mathbb{F}_q$. By Theorem 3.1.9, this group is normal in $UT(2,q)$, where a typical element of $UT(2,q)$ will have the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $b \in \mathbb{F}_q,\ a,d \in \mathbb{F}_q^*$. The quotient $UT(2,q)/SUUT(2,q)$ is isomorphic to $T \cong \mathbb{F}_q^* \times \mathbb{F}_q^*$. We will use the method of lifting of characters from the quotient of a group by a normal subgroup, to get characters of the main group as described in Section 2.4. Hence $Irr(T) \subseteq Irr(UT(2,q))$, where

$$\chi_k\chi_l\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \chi_k(a)\chi_l(d). \tag{4.12}$$

The following table shows the values of these characters on classes of $UT(2,q)$, which they can be deduced easily from Table 4.1 [of course we need to check whether a class in $GL(2,q)$ will remain as it is or will break into classes in $UT(2,q)$]. The full character table of $UT(2,q)$ will be discussed in Section 4.6.

Table 4.4: Conjugacy classes and some irreducible characters of $UT(2,q)$

Type	$\mathcal{T}_k^{(1)}$	$\mathcal{T}_k^{(2)}$	$\mathcal{T}_{k,l}^{(3)}$
Rep g	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$
No. of CC	$q-1$	$q-1$	$(q-1)(q-2)$
$\lvert C_g\rvert$	1	$q-1$	q
$\lvert C_{UT(2,q)}(g)\rvert$	$q(q-1)^2$	$q(q-1)$	$(q-1)^2$
$\chi_k\chi_l$	$\chi_k(\alpha)\chi_l(\alpha)$	$\chi_k(\alpha)\chi_l(\alpha)$	$\chi_k(\alpha)\chi_l(\beta)$

Now, we use these linear characters of $UT(2,q)$ as a base for Frobenius method of induction of characters to get characters of $GL(2,q)$. Thus let $\chi_{k,l} = \chi_k\chi_l\uparrow_{UT(2,q)}^{GL(2,q)}$. We consider the following two cases

Case I: Suppose that $\chi_k \neq \chi_l$. Then we have

(i) If g is in a class of the first type; that is $g = \alpha I_2$ for some $\alpha \in \mathbb{F}_q^*$, then

$$\chi_{k,l}(g) = \frac{|C_{GL(2,q)}(g)|}{|C_{UT(2,q)}(g)|}\chi_k(\alpha)\chi_l(\alpha) = (q+1)\chi_k(\alpha)\chi_l(\alpha).$$

(ii) If g is in a class of the second type; that is $g = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ for some $\alpha \in \mathbb{F}_q^*$, then

$$\chi_{k,l}(g) = \frac{|C_{GL(2,q)}(g)|}{|C_{UT(2,q)}(g)|}\chi_k(\alpha)\chi_l(\alpha) = \frac{q(q-1)}{q(q-1)}\chi_k(\alpha)\chi_l(\alpha) = \chi_k(\alpha)\chi_l(\alpha).$$

(iii) If g is in a class of the third type; that is $g = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ for some $\alpha, \beta \in \mathbb{F}_q^*,\ \alpha \neq \beta$. Then we can check that the conjugacy class represented by $g' = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}$ which is conjugate to g in $GL(2,q)$ is no longer conjugate to g in $UT(2,q)$. In this case we have

$$\begin{aligned} \chi_{k,l}(g) &= |C_{GL(2,q)}(g)| \left(\frac{\chi_k(\alpha)\chi_l(\beta)}{|C_{UT(2,q)}(g)|} + \frac{\chi_k(\beta)\chi_l(\alpha)}{|C_{UT(2,q)}(g')|} \right) \\ &= (q-1)^2 \left(\frac{\chi_k(\alpha)\chi_l(\beta)}{(q-1)^2} + \frac{\chi_k(\beta)\chi_l(\alpha)}{(q-1)^2} \right) = \chi_k(\alpha)\chi_l(\beta) + \chi_k(\beta)\chi_l(\alpha). \end{aligned}$$

(iv) If g is in a class of the fourth type; that is $g = \begin{pmatrix} 0 & 1 \\ -r^{q+1} & r + r^q \end{pmatrix}$ for some $r \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$, then $\chi_{k,l}(g) = 0$, since there is no intersection between a class of this type and the group $UT(2,q)$. Note that an element of the fourth type has eigenvalues r and r^q which are in $\mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$, while the eigenvalues of any element of $UT(2,q)$ are in $\mathbb{F}_q^*$.

Let us now check the irreducibility of the above characters.

$$\begin{aligned} \langle \chi_{k,l}, \chi_{k,l} \rangle &= \frac{1}{|GL(2,q)|} \sum_{g \in GL(2,q)} \chi_{k,l}(g)\overline{\chi_{k,l}}(g) \\ &= \frac{(q-1)(q+1)^2}{q(q-1)^2(q+1)}\chi_k(\alpha)\chi_l(\alpha)\overline{\chi_k}(\alpha)\overline{\chi_l}(\alpha) + \frac{(q-1)(q^2-1)}{q(q-1)^2(q+1)}\chi_k(\alpha)\chi_l(\alpha)\overline{\chi_k}(\alpha)\overline{\chi_l}(\alpha) \\ &+ \frac{(q-1)(q-2)q(q+1)}{2q(q-1)^2(q+1)}(\chi_k(\alpha)\chi_l(\beta) + \chi_k(\beta)\chi_l(\alpha))(\overline{\chi_k}(\alpha)\overline{\chi_l}(\beta) + \overline{\chi_k}(\beta)\overline{\chi_l}(\alpha)) \\ &= \frac{(q+1)}{q(q-1)} + \frac{1}{q} \\ &+ \frac{1}{2(q-1)^2} \sum_{\alpha \neq \beta}(\chi_k(\alpha)\chi_l(\beta) + \chi_k(\beta)\chi_l(\alpha))(\overline{\chi_k}(\alpha)\overline{\chi_l}(\beta) + \overline{\chi_k}(\beta)\overline{\chi_l}(\alpha)). \end{aligned} \tag{4.13}$$

In the last term of (4.13), we have divided by 2 because interchanging α with β in the main diagonal of an element in a class of type $\mathcal{T}^{(3)}$, does give the same conjugacy class.

To evaluate the last sum of the right hand side of equation (4.13), we will use the abelian group $T \cong \mathbb{F}_q^* \times \mathbb{F}_q^*$ of order $(q-1)^2$ and hence of index $q(q+1)$ in $GL(2,q)$. Let $t_{\alpha,\beta} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ be an element of T. With $\chi_k\chi_l$ be an irreducible character of T, defined as before and for fixed k and l, the function $\xi : T \longrightarrow GL(2,\mathbb{C})$ given by

$$\xi(t_{\alpha,\beta}) = \begin{pmatrix} \chi_k(\alpha)\chi_l(\beta) & 0 \\ 0 & \chi_k(\beta)\chi_l(\alpha) \end{pmatrix}$$

is a representation of T. Then $\xi = \xi_1 \oplus \xi_2$, where $\xi_1, \xi_2 : T \longrightarrow GL(1,\mathbb{C})$ given by

$$\xi_1(t_{\alpha,\beta}) = \chi_k(\alpha)\chi_l(\beta), \qquad \xi_2(t_{\alpha,\beta}) = \chi_k(\beta)\chi_l(\alpha).$$

It is clear that $\xi_1, \xi_2 \in Irr(T)$. Also

$$\chi_\xi(t_{\alpha,\beta}) = \chi_k(\alpha)\chi_l(\beta) + \chi_k(\beta)\chi_l(\alpha).$$

Thus $\langle \chi_\xi, \chi_\xi \rangle = 2$ and hence

$$\begin{aligned} 2 = \langle \chi_\xi, \chi_\xi \rangle &= \frac{1}{|T|}\sum_{g\in T} \chi_\xi(g)\overline{\chi_\xi}(g) = \frac{1}{(q-1)^2}\sum_{\alpha,\beta\in\mathbb{F}_q^*} \chi_\xi(t_{\alpha,\beta})\overline{\chi_\xi}(t_{\alpha,\beta}) \\ &= \frac{1}{(q-1)^2}\sum_{\alpha=\beta} \chi_\xi(t_{\alpha,\alpha})\overline{\chi_\xi}(t_{\alpha,\alpha}) + \frac{1}{(q-1)^2}\sum_{\alpha\neq\beta} \chi_\xi(t_{\alpha,\beta})\overline{\chi_\xi}(t_{\alpha,\beta}) \\ &= \frac{1}{(q-1)^2}\left(\sum_{\alpha\in\mathbb{F}_q^*} (\chi_k(\alpha)\chi_l(\alpha) + \chi_k(\alpha)\chi_l(\alpha))(\overline{\chi_k}(\alpha)\overline{\chi_l}(\alpha) + \overline{\chi_k}(\alpha)\overline{\chi_l}(\alpha))\right) \\ &+ \frac{1}{(q-1)^2}\left(\sum_{\alpha\neq\beta} (\chi_k(\alpha)\chi_l(\beta) + \chi_k(\beta)\chi_l(\alpha))(\overline{\chi_k}(\alpha)\overline{\chi_l}(\beta) + \overline{\chi_k}(\beta)\overline{\chi_l}(\alpha))\right) \\ &= \frac{4(q-1)}{(q-1)^2} + \frac{1}{(q-1)^2}\left(\sum_{\alpha\neq\beta} (\chi_k(\alpha)\chi_l(\beta) + \chi_k(\beta)\chi_l(\alpha))(\overline{\chi_k}(\alpha)\overline{\chi_l}(\beta) + \overline{\chi_k}(\beta)\overline{\chi_l}(\alpha))\right). \end{aligned}$$

Therefore

$$\sum_{\alpha\neq\beta} (\chi_k(\alpha)\chi_l(\beta) + \chi_k(\beta)\chi_l(\alpha))(\overline{\chi_k}(\alpha)\overline{\chi_l}(\beta) + \overline{\chi_k}(\beta)\overline{\chi_l}(\alpha)) = 2(q-1)^2 - 4(q-1) = 2(q-1)(q-3).$$

Turning back to equation (4.13), we get

$$\begin{aligned} \langle \chi_{k,l}, \chi_{k,l} \rangle &= \frac{(q+1)}{q(q-1)} + \frac{1}{q} + \frac{1}{2(q-1)^2}2(q-1)(q-3) = \frac{(q+1)}{q(q-1)} + \frac{1}{q} + \frac{(q-3)}{(q-1)} \\ &= \frac{(q+1)+(q-1)+q(q-3)}{q(q-1)} = \frac{2q+q^2-3q}{q^2-q} = \frac{q^2-q}{q^2-q} = 1. \end{aligned}$$

This shows that $\chi_{k,l}$ for $0 \le k < l \le q-2$ are all irreducible.

It is easily seen that $\chi_{k,l} = \chi_k\chi_l = \chi_l\chi_k = \chi_{l,k}$. Since k and l are distinct and they range between 0 and $q-2$, there are $\frac{(q-1)(q-2)}{2}$ irreducible characters $\chi_{k,l}$.

To see that two characters $\chi_{k,l}$ and $\chi_{k',l'}$ for $0 \leq k < l \leq q-2$ and $0 \leq k' < l' \leq q-2$ with $(k,l) \neq (k',l')$ are distinct, refer to Corollary 28.11 of James [40].

This completes the proof for **Case I**.

Case II: Suppose that $\chi_l = \chi_k$; that is $l = k$. By $g \in C \in \mathcal{T}^{(i)}$ we mean an element $g \in GL(2,q)$ in the conjugacy class C of type $\mathcal{T}^{(i)}$. Then by computations similar to **Case I**, we have

- If $g \in C \in \mathcal{T}^{(1)}$, then $\chi_{k,k}(g) = (q+1)\chi_k^2(\alpha)$.
- If $g \in C \in \mathcal{T}^{(2)}$, then $\chi_{k,k}(g) = \chi_k^2(\alpha)$.
- If $g \in C \in \mathcal{T}^{(3)}$, then $\chi_{k,k}(g) = 2\chi_k(\alpha)\chi_k(\beta)$.
- If $g \in C \in \mathcal{T}^{(4)}$, then $\chi_{k,k}(g) = 0$.

Now

$$\begin{aligned}
\langle \chi_{k,k}, \chi_{k,k} \rangle &= \frac{1}{|GL(2,q)|}\sum_{g\in GL(2,q)} \chi_{k,k}(g)\overline{\chi_{k,k}}(g) = \frac{(q-1)(q+1)^2}{q(q-1)^2(q+1)}\chi_k^2(\alpha)\overline{\chi_k^2}(\alpha) \\
&+ \frac{(q-1)(q^2-1)}{q(q-1)^2(q+1)}\chi_k^2(\alpha)\overline{\chi_k^2}(\alpha) + \frac{(q-1)(q-2)q(q+1)}{2q(q-1)^2(q+1)}(2\chi_k(\alpha)\chi_k(\beta))(\overline{2\chi_k}(\alpha)\overline{\chi_k}(\beta)) \\
&= \frac{(q+1)}{q(q-1)} + \frac{1}{q} + \frac{2(q-2)}{(q-1)} = \frac{(q+1)+(q-1)+2q^2-4q}{q(q-1)} = \frac{2q^2-2q}{q^2-q)} = 2.
\end{aligned}$$

Thus $\chi_{k,k}$ is not an irreducible character of $GL(2,q)$. We next set $St := \chi_{0,0} - \mathbf{1}$. Therefore

- If $g \in C \in \mathcal{T}^{(1)}$, then $St(g) = (q+1) - 1 = q$.
- If $g \in C \in \mathcal{T}^{(2)}$, then $St(g) = 1 - 1 = 0$.
- If $g \in C \in \mathcal{T}^{(3)}$, then $St(g) = 2 - 1 = 1$.
- If $g \in C \in \mathcal{T}^{(4)}$, then $St(g) = 0 - 1 = -1$.

Since

$$\begin{aligned}\langle St, St\rangle &= \frac{1}{|GL(2,q)|}\sum_{g\in GL(2,q)} St(g)\overline{St}(g) = \frac{(q-1)q^2}{q(q-1)^2(q+1)}\\ &+ \frac{(q-1)(q-2)q(q+1)}{2q(q-1)^2(q+1)} + \frac{(q^2-q)(q^2-q)}{2q(q-1)^2(q+1)}\\ &= \frac{q}{(q-1)(q+1)} + \frac{(q-2)}{2(q-1)} + \frac{q}{2(q+1)}\\ &= \frac{2q^2+q(q+1)(q-2)+q^2(q-1)}{2q(q+1)(q-1)} = \frac{2q^2+q^3-q^2-2q+q^3-q^2}{2q^3-2q}\\ &= \frac{2q^3-2q}{2q^3-2q} = 1,\end{aligned}$$

we deduce that St is an irreducible character of $GL(2,q)$. The character St defined above is called an *Steinberg character* of $GL(2,q)$. In fact for the group $GL(n,q)$, Steinberg [72] defined $|\mathcal{P}(n)|$ irreducible characters corresponding to the partitions of n. These characters defined by Steinberg come from the action of the group $GL(n,q)$ on some geometric entities. The character St corresponds to the partition $(1,1)\vdash 2$. The other Steinberg character of $GL(2,q)$ corresponding to $2\vdash 2$ is the trivial character of $GL(2,q)$. Steinberg characters will be studied in more details in Section 5.5.

We know from Proposition 2.3.3 that a product of a linear character by an irreducible character is an irreducible character. Thus by tensoring the $q-1$ linear characters λ_k with the Steinberg character St, we get $q-1$ irreducible characters of degree q. In the following we give the values of these $q-1$ irreducible characters on the classes $GL(2,q)$

- If $g\in C\in\mathcal{T}^{(1)}$, then $\lambda_k St(g) = q\chi_k^2(\alpha)$.
- If $g\in C\in\mathcal{T}^{(2)}$, then $\lambda_k St(g) = 0$.
- If $g\in C\in\mathcal{T}^{(3)}$, then $\lambda_k St(g) = \chi_k(\alpha)\chi_k(\beta)$.
- If $g\in C\in\mathcal{T}^{(4)}$, then $\lambda_k St(g) = -\chi_k(r^{q+1})$.

The $q-1$ irreducible characters $\lambda_k St$ are all distinct, because for the primitive $(q-1)th$ root of unity ε, we have $\lambda_k St\left(\begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix}\right) = \chi_k(\varepsilon)$, which gives distinct values for $0\le k\le q-2$.

Note that $\lambda_k St(g) = \chi_{k,k} - \lambda_k$, $\forall\ 0\le k\le q-2$. We may denote $\lambda_k St$ by ψ_k.

So far we have found

- $q-1$ linear characters.
- $q-1$ irreducible characters of degree q.

- $\frac{(q-1)(q-2)}{2}$ irreducible characters of degree $q+1$.

Thus so far we have $(q-1)+(q-1)+\frac{(q-1)(q-2)}{2}=\frac{(q-1)(q+2)}{2}$ irreducible characters. Since there are q^2-1 irreducible characters of $GL(2,q)$, we need to find $\frac{q^2-q}{2}$ additional irreducible characters. Moreover, if we add up the squares of the degrees of the characters we have found so far, we get

$$\begin{aligned}(q-1)+(q-1)q^2+\frac{(q-1)(q-2)}{2}(q+1)^2 &= \frac{2q-2+2q^3-2q^2+q^4-q^3-3q^2+q+2}{2}\\ &= \frac{q^4+q^3-5q^2+3q}{2}.\end{aligned}$$

Now

$$|GL(2,q)|-\frac{q^4+q^3-5q^2+3q}{2}=q(q-1)^2(q+1)-\frac{q^4+q^3-5q^2+3q}{2}=\frac{q^2-q}{2}(q-1)^2.$$

It will be shown that each of the remaining $\frac{q^2-q}{2}$ characters will have the degree $q-1$.

We aim to find the remaining $\frac{q^2-q}{2}$ irreducible characters of $GL(2,q)$ by using the characters of the group $\mathbb{F}^*_{q^2}$. The group $\mathbb{F}^*_{q^2}=\langle\sigma\rangle$ is embedded into $GL(2,q)$ by $\sigma\mapsto k_\sigma$, where $k_\sigma=\begin{pmatrix}\sigma & 0\\ 0 & \sigma^q\end{pmatrix}$. Now the group $K=\langle k_\sigma\rangle=\left\{\begin{pmatrix}\sigma^s & 0\\ 0 & \sigma^{qs}\end{pmatrix}\mid 1\le s\le q^2-1\right\}$ is an isomorphic copy of $\mathbb{F}^*_{q^2}$ in $GL(2,q)$ of index q^2-q. If s is a multiple of $q+1$, that is $s=(q+1)j$ for some $1\le j\le q-1$, then $\sigma^{qs}=\sigma^{q(q+1)j}=\sigma^{q^2j}\sigma^{qj}=\sigma^j\sigma^{qj}=\sigma^{(q+1)j}=\sigma^s$. Thus when $s=(q+1)j$, the elements ${k_\sigma}^s=k_{\sigma^s}=\begin{pmatrix}\sigma^{(q+1)j} & 0\\ 0 & \sigma^{(q+1)j}\end{pmatrix}$ are the scalar matrices in $GL(2,q)$. On the other hand, if s is not a multiple of $q+1$, then $\sigma^{qs}\neq\sigma^s$. Note that there are $(q^2-1)-(q-1)=q^2-q$ such integers s and consequently, q^2-q non-scalar k_σ in K.

Therefore, K meets $GL(2,q)$ only on classes of type $\mathcal{T}^{(1)}$ and $\mathcal{T}^{(4)}$.

Since K is cyclic, its irreducible characters are known (Theorem 2.2.4). If $\theta_k\in Irr(K)$, then we let $\phi_k=\theta_k\uparrow_K^{GL(2,q)}$ and $\chi_k=\theta_k\downarrow^K_{\mathbb{F}^*_q}$. Now using Proposition 2.5.5 we obtain

- If $g\in C\in\mathcal{T}^{(1)}$, then $\phi_k(g)=\frac{|C_{GL(2,q)}(g)|}{|C_K(g)|}\theta_k(\alpha)=(q^2-q)\chi_k(\alpha)$.
- If $g\in C\in\mathcal{T}^{(2)}$, then $\phi_k(g)=0$.
- If $g\in C\in\mathcal{T}^{(3)}$, then $\phi_k(g)=0$.
- If $g\in C\in\mathcal{T}^{(4)}$, then $\phi_k(g)=|C_{GL(2,q)}(g)|\left(\frac{\theta_k(r)}{|C_K(r)|}+\frac{\theta_k(r^q)}{|C_K(r^q)|}\right)=\theta_k(r)+\theta_k(r^q)$.

In the next step, we find the remaining irreducible characters of $GL(2,q)$. For $1 \le k \le q^2-1$ such that $q+1 \nmid k$, let $\pi_k := \chi_{0,-k}\psi_k - \phi_k - \chi_{0,k}$ and note that π_k is a combination of known characters of $GL(2,q)$. In Table 4.5 we list the values of π_k on classes of $GL(2,q)$.

Table 4.5: Values of π_k on classes of $GL(2,q)$

	$\mathcal{T}^{(1)}$	$\mathcal{T}^{(2)}$	$\mathcal{T}^{(3)}$	$\mathcal{T}^{(4)}$
$\chi_{0,-k}$	$(q+1)\chi_{-k}(\alpha)$	$\chi_{-k}(\alpha)$	$\chi_{-k}(\alpha)+\chi_{-k}(\beta)$	0
ψ_k	$q\chi_k^2(\alpha)$	0	$\chi_k(\alpha)\chi_k(\beta)$	$-\chi_k(r^{q+1})$
$\chi_{0,-k}\psi_k$	$(q^2+q)\chi_{-k}(\alpha)$	0	$\chi_k(\alpha)+\chi_k(\beta)$	0
$\chi_{0,k}$	$(q+1)\chi_k(\alpha)$	$\chi_k(\alpha)$	$\chi_k(\alpha)+\chi_k(\beta)$	0
ϕ_k	$q^2-q\chi_k(\alpha)$	0	0	$\theta_k(r)+\theta_k(r^q)$
π_k	$(q-1)\chi_k(\alpha)$	$-\chi_k(\alpha)$	0	$-(\theta_k(r)+\theta_k(r^q))$

There are $(q^2-1)-(q-1) = q^2-q$ characters π_k of $GL(2,q)$. We have

$$\begin{aligned}
\langle \pi_k, \pi_k \rangle &= \frac{1}{|GL(2,q)|}\sum_{g\in GL(2,q)} \pi_k(g)\overline{\pi_k}(g) = \frac{(q-1)^2(q-1)}{q(q-1)^2(q+1)} \\
&+ \frac{(q-1)(q-1)(q+1)}{q(q-1)^2(q+1)} + \frac{q(q-1)}{2q(q-1)^2(q+1)} \sum_{r\in\mathbb{F}_{q^2}^*\setminus\mathbb{F}_q^*} (\theta_k(r)+\theta_k(r^q))(\overline{\theta_k}(r)+\overline{\theta_k}(r^q)) \\
&= \frac{(q-1)}{q(q+1)} + \frac{1}{q} + \frac{1}{2(q-1)(q+1)} \sum_{r\in\mathbb{F}_{q^2}^*\setminus\mathbb{F}_q^*} (\theta_k(r)+\theta_k(r^q))(\overline{\theta_k}(r)+\overline{\theta_k}(r^q)) \qquad (4.14)
\end{aligned}$$

and note that replacing r with r^q in an element of a class of type $\mathcal{T}^{(4)}$ produces the same conjugacy class. To evaluate the sum in the last term of the right hand side of equation (4.14), we use the following two groups. Let K be the subgroup of $GL(2,q)$ defined earlier, which is isomorphic to $\mathbb{F}_{q^2}^*$ and suppose that $r \in \mathbb{F}_{q^2}^*$. Also, let

$$H = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^q \end{pmatrix} \mid r \in \mathbb{F}_q^* \right\} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \mid r \in \mathbb{F}_q^* \right\}.$$

Then K and H are both abelian groups of orders q^2-1 and $q-1$ respectively with $H < K$. Now for any $r \in \mathbb{F}_{q^2}^*$ and for fixed k, the function $\gamma^{(k)} : K \longrightarrow GL(2,\mathbb{C})$ given by

$$\gamma^{(k)}\left(\begin{pmatrix} r & 0 \\ 0 & r^q \end{pmatrix}\right) = \begin{pmatrix} \theta_k(r) & 0 \\ 0 & \theta_k(r^q) \end{pmatrix}$$

is a representation of K. Then $\gamma^{(k)} = \gamma_1^{(k)} \oplus \gamma_2^{(k)}$, where $\gamma_1^{(k)}, \gamma_2^{(k)} : K \longrightarrow GL(1,\mathbb{C})$ given by

$$\begin{aligned}\gamma_1^{(k)}\left(\begin{pmatrix} r & 0\\ 0 & r^q\end{pmatrix}\right) &= \theta_k(r),\\ \gamma_2^{(k)}\left(\begin{pmatrix} r & 0\\ 0 & r^q\end{pmatrix}\right) &= \theta_k(r^q).\end{aligned}$$

We have $\gamma_1^{(k)}, \gamma_2^{(k)} \in Irr(K)$. Also

$$\chi_{\gamma^{(k)}}\left(\begin{pmatrix} r & 0\\ 0 & r^q\end{pmatrix}\right) = \theta_k(r) + \theta_k(r^q).$$

We deduce that $\chi_{\gamma^{(k)}}$ is a sum of two non-equivalent irreducible characters. Also note that $\chi_{\gamma^{(k)}}\downarrow_H^K = 2\theta_k\downarrow_H^K$.

Now

$$\begin{aligned}2 &= \left\langle \chi_{\gamma^{(k)}}, \chi_{\gamma^{(k)}}\right\rangle = \frac{1}{|K|}\sum_{g\in K}\chi_{\gamma^{(k)}}(g)\overline{\chi_{\gamma^{(k)}}}(g)\\ &= \frac{1}{q^2-1}\sum_{r\in\mathbb{F}_{q^2}^*}(\theta_k(r)+\theta_k(r^q))(\overline{\theta_k}(r)+\overline{\theta_k}(r^q)).\end{aligned}$$

Thus

$$\sum_{r\in\mathbb{F}_{q^2}^*}(\theta_k(r)+\theta_k(r^q))(\overline{\theta_k}(r)+\overline{\theta_k}(r^q)) = 2(q^2-1).$$

Similarly

$$\begin{aligned}4 &= \langle 2\theta_k\downarrow_H^K, 2\theta_k\downarrow_H^K\rangle_H = \left\langle \chi_{\gamma^{(k)}}\downarrow_H^K, \chi_{\gamma^{(k)}}\downarrow_H^K\right\rangle_H = \frac{1}{|H|}\sum_{g\in H}\chi_{\gamma^{(k)}}\downarrow_H^K(g)\overline{\chi_{\gamma^{(k)}}\downarrow_H^K}(g)\\ &= \frac{1}{q-1}\sum_{r\in\mathbb{F}_q^*}(\theta_k(r)+\theta_k(r^q))(\overline{\theta_k}(r)+\overline{\theta_k}(r^q)).\end{aligned}$$

Thus

$$\sum_{r\in\mathbb{F}_q^*}(\theta_k(r)+\theta_k(r^q))(\overline{\theta_k}(r)+\overline{\theta_k}(r^q)) = 4(q-1).$$

Hence

$$\sum_{r\in\mathbb{F}_{q^2}^*\setminus\mathbb{F}_q^*}(\theta_k(r)+\theta_k(r^q))(\overline{\theta_k}(r)+\overline{\theta_k}(r^q)) = 2(q^2-1)-4(q-1) = 2(q-1)^2.$$

Now returning back to equation (4.14), we get

$$\begin{aligned}\langle \pi_k, \pi_k\rangle &= \frac{(q-1)}{q(q+1)} + \frac{1}{q} + \frac{2(q-1)^2}{2(q-1)(q+1)} = \frac{(q-1)^2+(q-1)(q+1)+q(q-1)^2}{q(q-1)(q+1)}\\ &= \frac{(q-1)[q-1+q+1+q^2-q]}{q(q-1)(q+1)} = \frac{q(q+1)}{q(q+1)} = 1.\end{aligned}$$

Thus $\pi_k \in Irr(GL(2, q))$. From Table 4.5, we can see that $\pi_{kq} = \pi_k$. This restricts the number of π_k's to $\frac{q^2-q}{2}$. We claim that these characters are all distinct. Assume that $q + 1 \nmid k, l$ and $k \not\equiv l,\ lq\ mod(q^2 - 1)$. We show that $\pi_k \neq \pi_l$. Recall that the character $\chi_{\gamma^{(k)}}$ of K and H is given on element $g = \begin{pmatrix} r & 0 \\ 0 & r^q \end{pmatrix}$ by $\chi_{\gamma^{(k)}}(g) = \theta_k(r) + \theta_k(r^q)$. Now

- if $g \in H$, that is $r \in \mathbb{F}_q^*$ (thus $r^q = r$), then $\chi_{\gamma^{(k)}}(g) = 2\theta_k(r)$
- if $g \in K \setminus H$, that is $r \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$ (thus $r^q \neq r$), then $\chi_{\gamma^{(k)}}(g) = \theta_k(r) + \theta_k(r^q)$.

Since $k \not\equiv l,\ lq\ mod(q^2-1)$, we have $\chi_{\gamma^{(k)}}$ and $\chi_{\gamma^{(l)}}$ are distinct. Thus either $\theta_k(r) \neq \theta_l(r)$ for some $r \in \mathbb{F}_q^*$ or $\theta_k(r) + \theta_k(r^q) \neq \theta_l(r) + \theta_l(r^q)$ for some $r \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$. Thus $\pi_k \neq \pi_l$.

Finally, to be consistent with the notation given in the character table of $GL(2, q)$, shown in Table 4.2, we use $\widehat{\alpha}^k,\ \alpha \in \mathbb{F}_q^*$ to denote $\chi_k(\alpha)$. The same applies for the elements $r \in \mathbb{F}_{q^2}^*$, where $\widehat{r}^k$ means $\theta_k(r)$ for $\theta_k \in Irr(\mathbb{F}_{q^2}^*)$. Also, the irreducible characters $\lambda_k,\ \lambda_k St,\ \chi_{k,l}$ and π_k will be renamed to $\chi_k^{(1)},\ \chi_k^{(2)},\ \chi_{k,l}^{(3)}$ and $\chi_k^{(4)}$ respectively. Hence, Table 4.2 is the character table of $GL(2, q)$. This completes the proof of the Theorem. ■

Summary and Discussion

It is well known from elementary theory of ordinary representations that the number of irreducible characters is the same as the number of the conjugacy classes of the finite group G. In general, there is no way of associating a conjugacy class to each irreducible character. However, we do have a very natural correspondence between the conjugacy classes and the irreducible characters of the group $GL(2, q)$. The groups $\mathbb{F}_q^*$, $\mathbb{F}_{q^2}^*$; and their character groups $Ch(\mathbb{F}_q^*)$ and $Ch(\mathbb{F}_{q^2}^*)$; are used respectively to parameterize the conjugacy classes and the irreducible characters of $GL(2, q)$ as follows. To give a representative of a class in the first two types of classes, we use only one element $\alpha \in \mathbb{F}_q^*$ and the same for the first two types of characters, we use only one character $\chi \in Ch(\mathbb{F}_q^*)$. Note that the union of conjugacy classes of the first type will give the center of the group $GL(2, q)$, while the union of characters of the first type form a group isomorphic to the center of the group $GL(2, q)$. To represent a class of type $\mathcal{T}^{(3)}$, we have used two distinct elements $\alpha, \beta \in \mathbb{F}_q^*$ where the conjugacy class is unaltered if we interchange α with β in the class. We have used two distinct characters $\chi_k, \chi_l \in Ch(\mathbb{F}_q^*)$ to parameterize a character of the third type and we have seen that the product of $\chi_k, \chi_l \in Ch(\mathbb{F}_q^*)$ is commutative. Finally, to obtain a class of type $\mathcal{T}^{(4)}$, we made use of the elements $r \in \mathbb{F}_{q^2}^*$ which are not in $\mathbb{F}_q^*$ and whenever we choose such r, we exclude r^q because r and r^q give the same conjugacy class. Also to produce characters of the fourth type, we used characters $\theta \in Ch(\mathbb{F}_{q^2}^*)$ which do not decompose into characters in $Ch(\mathbb{F}_q^*)$ and whenever we choose such k to index a character of the fourth type, we exclude kq from the indexing set because k and

kq give the same character. So we can say that there is a complete duality between the conjugacy classes and the irreducible characters of the group $GL(2,q)$. Let ε and θ be generators of the groups $\mathbb{F}_q^*$ and $\mathbb{F}_{q^2}^*$ respectively. Table 4.6 shows the association of a class to an irreducible character of $GL(2,q)$.

Table 4.6: Duality between irreducible characters and conjugacy classes of $GL(2,q)$

Irreducible Character	Corresponding Conjugacy Class
$\chi_k^{(1)}$	$\mathcal{T}_k^{(1)} = \begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^k \end{pmatrix}$
$\chi_k^{(2)}$	$\mathcal{T}_k^{(2)} = \begin{pmatrix} \varepsilon^k & 1 \\ 0 & \varepsilon^k \end{pmatrix}$
$\chi_{k,l}^{(3)}$	$\mathcal{T}_{k,l}^{(3)} = \begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^l \end{pmatrix}$
$\chi_k^{(4)}$	$\mathcal{T}_k^{(4)} = \begin{pmatrix} 0 & 1 \\ -\theta^{(k+1)q} & \theta^k + \theta^{kq} \end{pmatrix}$

As a final remark, the above duality between the conjugacy classes and irreducible characters of the group $GL(2,q)$ will be satisfied in general for all groups $GL(n,q)$. A similar table of duality for the group $GL(3,q)$ will be given in Table 5.13.

4.4. Character Table of $SL(2,q)$

We recall by Definition 3.1.2 that the group $SL(2,q)$ consists of the 2×2 matrices with determinant 1. This group is normal in $GL(2,q)$ and the quotient $GL(2,q)/SL(2,q) \cong \mathbb{F}_q^*$. Also by Proposition 3.1.2 we have $|SL(2,q)| = q(q-1)(q+1) = (q^3-q)$.

The character table of the group $SL(2,q)$ has been studied extensively by many authors, for example Adams [2], Berckovich [6], Collins [13], Fulton [22], Gehles [24], Hageman [31], Humphreys [33], Prasad [60] and Springer [71] and others (some of the authors studied the character table of $SL(2,q)$ for only one case of q, even or odd. As we shall see later that the character table depends on the parity of q).

For determining the character table of the group $SL(2,q)$, we first need to find its conjugacy classes. These conjugacy classes come from those of the group $GL(2,q)$ that have determinant 1. We need to check whether a class of $GL(2,q)$ with determinant 1 splits in $SL(2,q)$ or remain as it is. Note that two non-conjugate elements in $GL(2,q)$ can not be conjugate in $SL(2,q)$. The four types of classes of $GL(2,q)$ have determinants given in Table 4.7.

Table 4.7: Determinants of the elements of $GL(2,q)$

g	$\mathcal{T}^{(1)}$	$\mathcal{T}^{(2)}$	$\mathcal{T}^{(3)}$	$\mathcal{T}^{(4)}$
$\det(g)$	α^2	α^2	$\alpha\beta$	r^{q+1}

Elements in classes of type $\mathcal{T}^{(1)}$, $\mathcal{T}^{(2)}$, $\mathcal{T}^{(3)}$ and $\mathcal{T}^{(4)}$ are in $SL(2,q)$ if and only if $\alpha^2 = 1$, $\alpha^2 = 1$, $\alpha\beta = 1$ and $r^{(q+1)} = 1$ respectively. If $\alpha^2 = \varepsilon^{2k} = 1$, then $k \in \{0, \frac{q-1}{2}\}$. Thus $\alpha = \pm 1$. Consequently, if q is an odd prime power, then $1 \neq -1$, while if $q = 2^s$, $s \geq 1$, then $1 = -1$. Thus let us consider each case separately.

4.4.1 Character Table of $SL(2,q)$, $q = p^m$, p an odd prime, $m \geq 1$

Conjugacy Classes of $SL(2,q)$

In this case, $Z(SL(2,q)) = \{I_2, -I_2\}$. Thus the first family of classes of $GL(2,q)$ gives us two classes in $SL(2,q)$ namely I_2 and $-I_2$. We will use the notation $\mathcal{T}_0^{(1)}$ instead of I_2. Clearly both these classes have size 1 each.

Now $\mathcal{T}_k^{(2)} \subseteq SL(2,q)$ iff $k \in \{0, \frac{q-1}{2}\}$. Hence we need only to consider $\mathcal{T}_0^{(2)}$ and $\mathcal{T}_{\frac{q-1}{2}}^{(2)}$ of $GL(2,q)$. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{T}_0^{(2)}$ and $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \in \mathcal{T}_{\frac{q-1}{2}}^{(2)}$, we denote the $\mathcal{T}_{\frac{q-1}{2}}^{(2)}$ by $-\mathcal{T}_0^{(2)}$. In the following we claim that each of $\mathcal{T}_0^{(2)}$ or $-\mathcal{T}_0^{(2)}$ splits into 2 conjugacy classes of $SL(2,q)$ respectively. Hence we obtain 4 classes of type $\mathcal{T}^{(2)}$ in $SL(2,q)$, namely $\mathcal{T}_{01}^{(2)}$, $\mathcal{T}_{0\varepsilon}^{(2)}$, $-\mathcal{T}_{01}^{(2)}$, $-\mathcal{T}_{0\varepsilon}^{(2)}$, with representatives

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -\varepsilon \\ 0 & -1 \end{pmatrix} \quad \text{respectively.}$$

We calculate the size of the conjugacy class $\mathcal{T}_{01}^{(2)}$. We need to find $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,q)$ such that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}. \tag{4.15}$$

So we must have $c = 0$ and $a = d$. Thus $g = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. But we know that $ad - bc = a^2 = 1$. Therefore $a = \pm 1$ and b can be any element of $\mathbb{F}_q$. This gives in total $2q$ elements in the centralizer and consequently $|\mathcal{T}_{01}^{(2)}| = \frac{q^2-1}{2}$.

Similarly $\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \in \mathcal{T}_{0\varepsilon}^{(2)}$, which is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $GL(2,q)$ but not in $SL(2,q)$, has a centralizer size $2q$. Hence $|\mathcal{T}_{0\varepsilon}^{(2)}| = \frac{q^2-1}{2}$ in $SL(2,q)$. By similar argument the class $-\mathcal{T}_0^{(2)}$ of $GL(2,q)$ splits into two classes, namely $-\mathcal{T}_{01}^{(2)}$ and $-\mathcal{T}_{0\varepsilon}^{(2)}$, each of size $\frac{q^2-1}{2}$.

Any element $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in \mathcal{T}_{k,l}^{(3)}$, $\alpha = \varepsilon^k \neq \varepsilon^l = \beta$ in $GL(2,q)$ has determinant $\alpha\beta$. Therefore

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in SL(2,q) \Longleftrightarrow \alpha\beta = 1 \Longleftrightarrow \beta = \alpha^{-1} \Longleftrightarrow l = -k.$$

Since $\alpha = 1$ and $\alpha = -1$ are the only self inverse elements in $\mathbb{F}_q^*$, the third family of classes of $GL(2,q)$ gives $\frac{q-3}{2}$ classes $\mathcal{T}_{k,-k}^{(3)}$, $k \in \{0, \frac{q-1}{2}\}$ with elements of type $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. Exclusion of $\alpha = 1$ and $\alpha = -1$ imply that there are $q-3$ choices for α and observing that swapping α with α^{-1} in $\mathcal{T}_{k,-k}^{(3)}$ produces the same conjugacy class. The next two equations are used to compute the size of the centralizer of an element of this type. Let $x = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in SL(2,q)$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,q)$ such that

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\alpha & b\alpha \\ c\alpha^{-1} & d\alpha^{-1} \end{pmatrix}, \tag{4.16}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} a\alpha & b\alpha^{-1} \\ c\alpha & d\alpha^{-1} \end{pmatrix}. \tag{4.17}$$

Equating equations (4.16) and (4.17), we obtain $c\alpha = c\alpha^{-1}$ and $b\alpha = b\alpha^{-1}$, which has unique solution $c = b = 0$. (Note that $\alpha \neq 1$ or -1). Since $ad - bc = ad = 1$, we have $d = a^{-1}$. Therefore $g \in C_{SL(2,q)}(x)$ if and only if $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $a \in \mathbb{F}_q^*$. So $|C_{SL(2,q)}(x)| = q-1$ and it follows that $|\mathcal{T}_{k,-k}^{(3)}| = q(q+1)$.

The element $A_r = \begin{pmatrix} 0 & 1 \\ -r^{q+1} & r + r^q \end{pmatrix} \in \mathcal{T}_k^{(4)}$ of $GL(2,q)$ has determinant r^{q+1}. Therefore this element is in $SL(2,q)$ if and only if $r^{q+1} = 1$. Let θ be a generator of $\mathbb{F}_{q^2}^*$; that is $o(\theta) = q^2 - 1$. Then θ^{q+1} is a generator of $\mathbb{F}_q^*$. So that $\mathbb{F}_q^* = \langle \theta^{q+1} \rangle \cong \mathbb{Z}_{q-1}$. Now the elements $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ which satisfy $r^{q+1} = 1$ are the elements of the form $r = \theta^{(q-1)j}$ for $j = 1, 2, \cdots, \frac{q-1}{2}$. Note that $r.r^q = \theta^{(q-1)j}\theta^{q(q-1)j} = \theta^{(q^2-1)j} = 1$. Also note that if $j = \frac{q-1}{2} + 1 = \frac{q+1}{2}$, then $r = \theta^{(q-1)\frac{q+1}{2}} = \theta^{\frac{(q^2-1)}{2}} = -1$, which is an element of $\mathbb{F}_q^*$, but our choice of r is in $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$. We conclude that there are at least $\frac{q-1}{2}$ conjugacy classes in $SL(2,q)$, which come from $\mathcal{T}_k^{(4)}$ classes of $GL(2,q)$.

Let us calculate the size of the centralizer of an element $A_r = \begin{pmatrix} 0 & 1 \\ -1 & r+r^q \end{pmatrix}$ of $SL(2,q)$ of type $\mathcal{T}_k^{(4)}$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,q)$ we have

$$\begin{pmatrix} 0 & 1 \\ -1 & r+r^q \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ -a+c(r+r^q) & -b+d(r+r^q) \end{pmatrix}, \tag{4.18}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & r+r^q \end{pmatrix} = \begin{pmatrix} -b & a+b(r+r^q) \\ -d & c+d(r+r^q) \end{pmatrix}. \tag{4.19}$$

Thus

$$g \in C_{SL(2,q)}(A_r) \Longleftrightarrow c=-b,\ a+b(r+r^q)=d,\ -d=-a+c(r+r^q),\ c+d(r+r^q)=-b+d(r+r^q).$$

Hence $g = \begin{pmatrix} a & b \\ -b & a+b(r+r^q) \end{pmatrix}$ and we have $a^2+ab(r+r^q)+b^2-1=0$. If $a=0$, then $b=\pm 1$. Hence $g = \begin{pmatrix} 0 & 1 \\ -1 & (r+r^q) \end{pmatrix}$ or $g = \begin{pmatrix} 0 & -1 \\ 1 & -(r+r^q) \end{pmatrix}$. On the other hand, if $a \neq 0$, then b depends on a and the fixed element $r+r^q$ and hence we have $q-1$ possibilities for g. Thus we have $(q-1)+2=q+1$ candidates for g in total. Hence $|C_{SL(2,q)}(A_r)| = q+1$ and $|[A_r]| = q(q-1)$.

We summarize the foregoing elements of $SL(2,q)$ in the following.

Type	Rep g	No of Conj. Classes	$\lvert C_{SL(2,q)}(g)\rvert$	$\lvert[g]\rvert$
$\mathcal{T}_0^{(1)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	q^3-q	1
$-\mathcal{T}_0^{(1)}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	1	q^3-q	1
$\mathcal{T}_{01}^{(2)}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	1	$2q$	$\frac{q^2-1}{2}$
$-\mathcal{T}_{01}^{(2)}$	$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$	1	$2q$	$\frac{q^2-1}{2}$
$\mathcal{T}_{0\varepsilon}^{(2)}$	$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$	1	$2q$	$\frac{q^2-1}{2}$
$-\mathcal{T}_{0\varepsilon}^{(2)}$	$\begin{pmatrix} -1 & -\varepsilon \\ 0 & -1 \end{pmatrix}$	1	$2q$	$\frac{q^2-1}{2}$
$\mathcal{T}_{k,-k}^{(3)}$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$	$\frac{q-3}{2}$	$q-1$	$q(q+1)$
$-\mathcal{T}_k^{(4)}$	$\begin{pmatrix} 0 & 1 \\ -1 & -(r+r^q) \end{pmatrix}$	$\frac{q-1}{2}$	$q+1$	$q(q-1)$

By counting the number of elements in the conjugacy classes we have found so far, we get

$$\begin{aligned} 1+1 &+ \frac{q^2-1}{2}+\frac{q^2-1}{2}+\frac{q^2-1}{2}+\frac{q^2-1}{2}+\frac{(q-3)q(q+1)}{2}+\frac{(q-1)q(q-1)}{2} \\ &= \frac{4+4q^2-4+q^3-2q^2-3q+q^3-2q^2+q}{2} \\ &= \frac{2q^3-2q}{2}=q^3-q=|SL(2,q)|. \end{aligned}$$

This shows that we have found all the conjugacy classes of $SL(2,q)$. The number of classes of this group is

$$1+1+1+1+1+1+\frac{q-3}{2}+\frac{q-1}{2}=\frac{12+2q-4}{2}=q+4.$$

Proposition 4.4.1. *Let $t_{01}^{(2)}$, $-t_{01}^{(2)}$, $t_{0\varepsilon}^{(2)}$, $-t_{0\varepsilon}^{(2)}$, $t_{k,-k}^{(3)}$, $t_k^{(4)}$ be elements of types $\mathcal{T}_{01}^{(2)}$, $-\mathcal{T}_{01}^{(2)}$, $\mathcal{T}_{0\varepsilon}^{(2)}$, $-\mathcal{T}_{0\varepsilon}^{(2)}$, $\mathcal{T}_{k,-k}^{(3)}$, $\mathcal{T}_k^{(4)}$ respectively. With p being the characteristic of $\mathbb{F}_q$, then*

(i) $o(\mathcal{T}_0^{(1)})=1$,

(ii) $o(-\mathcal{T}_0^{(1)})=2$,

(iii) $o(t_{01}^{(2)})=p$,

(iv) $o(-t_{01}^{(2)})=2p$,

(v) $o(t_{0\varepsilon}^{(2)})=p$,

(vi) $o(-t_{0\varepsilon}^{(2)})=2p$,

(vii) $o(t_{k,-k}^{(3)})=(q-1)/\gcd(k,q-1)$,

(viii) $o(t_k^{(4)})=q+1$.

PROOF. All follows by Proposition 4.2.2 as follows

1. (i) and (ii) are trivial.
2. For $t_{01}^{(2)}$, we have $\varepsilon^k=1 \Longrightarrow k=q-1$. Therefore $o(t_{01}^{(2)})=p(q-1)/\gcd(q-1,q-1)=p$.
3. For $-t_{01}^{(2)}$, we have $\varepsilon^k=-1 \Longrightarrow k=\frac{q-1}{2}$. Therefore $o(t_{01}^{(2)})=p(q-1)/\gcd(\frac{q-1}{2},q-1)=p(q-1)/\frac{q-1}{2}=2p$.
4. (v) and (vi) are similar to (2) and (3).
5. Since $l=-k$, we have $o(t_{k,-k}^{(3)})=lcm\left(\frac{q-1}{\gcd(k,q-1)},\frac{q-1}{\gcd(|-k|,q-1)}\right)=\frac{q-1}{\gcd(k,q-1)}$.
6. For fixed $t_k^{(4)}\in\mathcal{T}_k^{(4)}$ we have $k=(q-1)j$ for some $j\in\{1,2,\cdots,\frac{q-1}{2}\}$. Now

$$\begin{aligned} o(t_k^{(4)}) &= lcm\left(\frac{(q-1)}{\gcd((q-1)j,(q-1))},\frac{(q^2-1)}{\gcd((q-1)j,(q^2-1))},\frac{(q^2-1)}{\gcd(q(q-1)j,(q^2-1))}\right) \\ &= lcm(1,q+1,q+1)=q+1. \end{aligned}$$

Hence the result. ■

Irreducible Characters of $SL(2,q)$

We aim to find all the irreducible characters of $SL(2,q)$. Note that $|Irr(SL(2,q))| = q+4$. In Table 4.8, we list the restriction of the characters of $GL(2,q)$ to $SL(2,q)$.

Table 4.8: The conjugacy classes of $SL(2,q)$, q is odd; and the restriction of the characters of $GL(2,q)$.

Class	$\mathcal{T}_0^{(1)}$	$-\mathcal{T}_0^{(1)}$	$\mathcal{T}_{01}^{(2)}$	$-\mathcal{T}_{01}^{(2)}$
Rep g	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&-1\\0&-1\end{pmatrix}$
No. of CC	1	1	1	1
$\lvert C_{SL(2,q)}(g)\rvert$	q^3-q	q^3-q	$2q$	$2q$
$\lvert C_g\rvert$	1	1	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$
λ_k	1	1	1	1
ψ_k	q	q	0	0
$\psi_{k,l}$	$q+1$	$(-1)^{k+l}(q+1)$	1	$(-1)^{k+l}$
π_k	$q-1$	$(-1)^k(q-1)$	-1	$(-1)^{k+1}$

Table 4.8 (continued)

Class	$\mathcal{T}_{0\varepsilon}^{(2)}$	$-\mathcal{T}_{0\varepsilon}^{(2)}$	$\mathcal{T}_{k,-k}^{(3)}$	$\mathcal{T}_k^{(4)}$
Rep g	$\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&-\varepsilon\\0&-1\end{pmatrix}$	$\begin{pmatrix}\alpha&0\\0&\alpha^{-1}\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&r+r^q\end{pmatrix}$
No. of CC	1	1	$\frac{q-3}{2}$	$\frac{q-1}{2}$
$\lvert C_{SL(2,q)}(g)\rvert$	$2q$	$2q$	$q-1$	$q+1$
$\lvert C_g\rvert$	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$	$q(q+1)$	$q(q-1)$
λ_k	1	1	1	1
ψ_k	0	0	1	-1
$\psi_{k,l}$	1	$(-1)^{k+l}$	$\widehat{\alpha}^{(k-l)}+\widehat{\alpha}^{-(k-l)}$	0
π_k	-1	$(-1)^{k+1}$	0	$-(\widehat{r}^k+\widehat{r}^{kq})$

The next duty will be to check the irreducibility of the characters of $GL(2,q)$ restricted to $SL(2,q)$.

Firstly since all linear characters of $GL(2,q)$ correspond to the powers of the determinants and since each element of $SL(2,q)$ is of determinant 1, then $\lambda_k\downarrow^{GL(2,q)}_{SL(2,q)} = \lambda$, the trivial character of $SL(2,q)$, $k = 0,1,\cdots,q-2$.

It is clear from Table 4.8 that the values of ψ_k do not depend on k, $0 \leq k \leq q-2$. So we may rename it to ψ. Now

$$\begin{aligned}\langle \psi, \psi \rangle &= \frac{1}{q^3-q}\left(2q^2 + q\frac{q-3}{2}(q+1) + q\frac{(q-1)^2}{2}\right)\\ &= \frac{1}{q^3-q}\left(\frac{4q^2+q^3-2q^2-3q+q^3-2q^2+q}{2}\right)\\ &= \frac{1}{q^3-q}\left(\frac{2q^3-2q}{2}\right) = \frac{q^3-q}{q^3-q} = 1.\end{aligned}$$

Thus $\psi \in Irr(SL(2,q))$. We observe that ψ is the Steinberg character of $GL(2,q)$.

For the third family of characters $\psi_{k,l}$, firstly we fix $l=1$ and for each $k \in \{0, 1, \cdots, \frac{q-3}{2}\}$ we have

$$\langle \psi_{k,1}, \psi_{k,1} \rangle = \frac{1}{q^3-q}\left((q+1)^2 + (q+1)^2 + 4\frac{q^2-1}{2} + \sum_{1}^{\frac{q-3}{2}} q(q+1)|\widehat{\alpha}^{(k-1)} + \widehat{\alpha}^{-(k-1)}|^2\right)$$

To evaluate the term $\sum_{1}^{\frac{q-3}{2}} |\widehat{\alpha}^{(k-1)} + \widehat{\alpha}^{-(k-1)}|^2$ we use

$$\begin{aligned}\sum_{1}^{\frac{q-3}{2}} |\widehat{\alpha}^{(k-1)} + \widehat{\alpha}^{-(k-1)}|^2 &= \sum_{1}^{\frac{q-3}{2}} \widehat{\alpha}^{2(k-1)} + \sum_{1}^{\frac{q-3}{2}} 2 + \sum_{1}^{\frac{q-3}{2}} \widehat{\alpha}^{-2(k-1)}\\ &= (q-3) + \sum_{1}^{\frac{q-3}{2}} \widehat{\alpha}^{2(k-1)} + \sum_{1}^{\frac{q-3}{2}} \widehat{\alpha}^{-2(k-1)}.\end{aligned}$$

We recall that $\sum_{j=1}^{q-1} \widehat{\varepsilon}^j = 0$, where ε is a generator of $\mathbb{F}_q^*$ and $\widehat{\varepsilon}$ is a generator of the group consisting of the $(q-1)th$ roots of unity in $\mathbb{C}$. Also for any $s \in \mathbb{Z}$ we have $\sum_{j=1}^{q-1} \widehat{\varepsilon}^{sj} = 0$. Thus $\sum_{j=1}^{\frac{q-1}{2}} \widehat{\varepsilon}^{2j(k-1)} = 0$ and consequently $\sum_{j=1}^{\frac{q-3}{2}} \widehat{\varepsilon}^{2j(k-1)} = -\widehat{\varepsilon}^{2(k-1)\frac{q-1}{2}} = -(\widehat{\varepsilon}^{q-1})^{k-1} = -1$. Now if $\widehat{\alpha} = \widehat{\varepsilon}^j$ for some j, then we have $\sum_{1}^{\frac{q-3}{2}} \widehat{\alpha}^{2(k-1)} = \sum_{j=1}^{\frac{q-3}{2}} \widehat{\varepsilon}^{2j(k-1)}$. Hence $\sum_{1}^{\frac{q-3}{2}} \widehat{\alpha}^{2(k-1)} = -1$. Similarly $\sum_{1}^{\frac{q-3}{2}} \widehat{\alpha}^{-2(k-1)} = -1$. Therefore $\sum_{1}^{\frac{q-3}{2}} |\widehat{\alpha}^{(k-1)} + \widehat{\alpha}^{-(k-1)}|^2 = (q-3) - 1 - 1 = q-5$.

It follows that

$$\begin{aligned}
\langle \psi_{k,1}, \psi_{k,1} \rangle &= \frac{1}{q^3-q}\left(4q^2+4q+q(q+1)(q-5)\right)\\
&= \frac{1}{q^3-q}\left(4q^2+4q+q^3-4q^2-5q\right)\\
&= \frac{1}{q^3-q}\left(q^3-q\right)=1.
\end{aligned}$$

Hence we obtain $\frac{q-3}{2}$ irreducible characters of $SL(2,q)$ this way.

Setting $k=l=0$ in the characters $\psi_{k,l}$ of $GL(2,q)$ we get

$$\begin{aligned}
\langle \psi_{0,0}, \psi_{0,0} \rangle &= \frac{1}{q^3-q}\left((q+1)^2+(q+1)^2+4\frac{q^2-1}{2}+4\frac{q-3}{2}q(q+1)\right)\\
&= \frac{1}{q^3-q}(2q^2+4q+2+2q^2-2+2q^3-4q^2-6q)\\
&= \frac{1}{q^3-q}(2q^3-2q)=2.
\end{aligned}$$

Hence $\psi_{0,0} \notin Irr(SL(2,q))$. It is clear that $\psi_{0,0}=\lambda+\psi=\mathbf{1}+\psi$.

From another side, if we let $k=\frac{q-1}{2}$ and $l=0$ we get $\left\langle \psi_{\frac{q-1}{2},0}, \psi_{\frac{q-1}{2},0}\right\rangle = 2$ as follows

$$\begin{aligned}
\left\langle \psi_{\frac{q-1}{2},0}, \psi_{\frac{q-1}{2},0}\right\rangle &= \frac{1}{q^3-q}\left((q+1)^2+(q+1)^2+4\frac{q^2-1}{2}+\sum_{1}^{\frac{q-3}{2}} q(q+1)|\widehat{\alpha}^{\frac{q-1}{2}}+\widehat{\alpha}^{-\frac{q-1}{2}}|^2\right)\\
&= \frac{1}{q^3-q}\left(2q^2+4q+2+2q^2-2+q(q+1)\left(\sum_{1}^{\frac{q-3}{2}}\widehat{\alpha}^{2\frac{q-1}{2}}+\sum_{1}^{\frac{q-3}{2}}2+\sum_{1}^{\frac{q-3}{2}}\widehat{\alpha}^{-2\frac{q-1}{2}}\right)\right)\\
&= \frac{1}{q^3-q}\left(4q^2+4q+q(q+1)\left(\frac{q-3}{2}+2\frac{q-3}{2}+\frac{q-3}{2}\right)\right)\\
&= \frac{1}{q^3-q}\left(4q^2+4q+2q(q+1)(q-3)\right)=\frac{1}{q^3-q}\left(4q^2+4q+2q^3-4q^2-6q\right)\\
&= \frac{1}{q^3-q}\left(2q^3-2q\right)=2.
\end{aligned}$$

Thus $\psi_{\frac{q-1}{2},0} \notin Irr(SL(2,q))$. Here we have $\psi_{\frac{q-1}{2},0}=\widetilde{\xi_1}+\widetilde{\xi_2}$, where $\widetilde{\xi_1},\widetilde{\xi_2} \in Irr(SL(2,q))$ such that $\deg(\widetilde{\xi_1})=\deg(\widetilde{\xi_2})=\frac{q+1}{2}$ (to be shown later). The values of $\widetilde{\xi_i}$, $i=1,2$ on classes of $SL(2,q)$ are not easy to compute. We determine these values later. Now $\widetilde{\xi_1}+\widetilde{\xi_2}$ have values given by Table 4.9.

Table 4.9: Values of $\widetilde{\xi}_1+\widetilde{\xi}_2$ on classes of $SL(2,q)$

Class	$\mathcal{T}_0^{(1)}$	$-\mathcal{T}_0^{(1)}$	$\mathcal{T}_{01}^{(2)}$	$-\mathcal{T}_{01}^{(2)}$	$\mathcal{T}_{0\varepsilon}^{(2)}$	$-\mathcal{T}_{0\varepsilon}^{(2)}$	$\mathcal{T}_{k,-k}^{(3)}$	$\mathcal{T}_k^{(4)}$
$\widetilde{\xi}_1+\widetilde{\xi}_2$	$q+1$	$(-1)^{\frac{q-1}{2}}(q+1)$	1	$(-1)^{\frac{q-1}{2}}$	1	$(-1)^{\frac{q-1}{2}}$	$2(-1)^k$	0

Next we turn to the fourth family of characters of $GL(2,q)$. By similar computations used in the last step, we can show that π_k for $k=1,2,\cdots,\frac{q-1}{2}$ restrict irreducibly to be characters of $SL(2,q)$. But if $k=\frac{q+1}{2}$, we get $\left\langle \pi_{\frac{q+1}{2}},\pi_{\frac{q+1}{2}}\right\rangle=2$ and thus $\pi_{\frac{q+1}{2}}=\widetilde{\vartheta}_1+\widetilde{\vartheta}_2$, where $\widetilde{\vartheta}_1,\ \widetilde{\vartheta}_2\in Irr(SL(2,q))$ and $\deg(\widetilde{\vartheta}_1)=\deg(\widetilde{\vartheta}_2)=\frac{q-1}{2}$ (to be shown later). The values of $\widetilde{\vartheta}_1+\widetilde{\vartheta}_2$ on classes of $SL(2,q)$ are given in Table 4.10

Table 4.10: Values of $\widetilde{\vartheta}_1+\widetilde{\vartheta}_2$ on classes of $SL(2,q)$

Class	$\mathcal{T}_0^{(1)}$	$-\mathcal{T}_0^{(1)}$	$\mathcal{T}_{01}^{(2)}$	$-\mathcal{T}_{01}^{(2)}$	$\mathcal{T}_{0\varepsilon}^{(2)}$	$-\mathcal{T}_{0\varepsilon}^{(2)}$	$\mathcal{T}_{k,-k}^{(3)}$	$\mathcal{T}_k^{(4)}$
$\widetilde{\vartheta}_1+\widetilde{\vartheta}_2$	$q-1$	$(-1)^{\frac{q+1}{2}}(q-1)$	-1	$(-1)^{\frac{q-1}{2}}$	-1	$(-1)^{\frac{q-1}{2}}$	0	$2(-1)^{m+1}$

where $1\le m\le\frac{q-1}{2}$ is the integer for which $\mathbb{F}_{q^2}^*\setminus\mathbb{F}_q^*\ni r=\theta^{(q-1)m}$ and $\mathbb{F}_{q^2}^*=\langle\theta\rangle$.

Now we count the number of irreducible characters of $SL(2,q)$ we have obtained so far. This is built on the assumption that we have determined the values of $\widetilde{\xi}_1,\ \widetilde{\xi}_2,\ \widetilde{\vartheta}_1$ and $\widetilde{\vartheta}_2$ on classes of $SL(2,q)$ and we have proved their irreducibility. The characters we have found till now are

$$\lambda;\quad \psi;\quad \psi_{k,1},\ 1\le k\le\frac{q-3}{2};\quad \pi_k,\ 1\le k\le\frac{q-1}{2};\quad \widetilde{\xi}_1,\ \widetilde{\xi}_2;\quad \widetilde{\vartheta}_1,\ \widetilde{\vartheta}_2.$$

Thus

$$1+1+\frac{q-3}{2}+\frac{q-1}{2}+1+1+1+1=\frac{2q+8}{2}=q+4.$$

This is equal to the number of conjugacy classes of $SL(2,q)$, which tells that we have found all the irreducible characters of $SL(2,q)$. To complete the character table, we need to find the values of $\widetilde{\xi}_1,\ \widetilde{\xi}_2,\ \widetilde{\vartheta}_1$ and $\widetilde{\vartheta}_2$ on the classes of $SL(2,q)$. To do this, we invoke another subgroup of $GL(2,q)$, which contains $SL(2,q)$.

Let $(\mathbb{F}_q^*)^2$ denotes the subset of $\mathbb{F}_q^*$ consisting of the square elements of $\mathbb{F}_q^*$. By Lemma 3.1.4 we have

$$|(\mathbb{F}_q^*)^2|=\begin{cases} q-1 & \text{if } q \text{ is even,}\\ \frac{q-1}{2} & \text{if } q \text{ is odd.}\end{cases}$$

Lemma 4.4.2. *We have*

(i) *If $a,b\in(\mathbb{F}_q^*)^2$, then so is ab.*

(ii) *If* $a, b \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$, *then* $ab \in (\mathbb{F}_q^*)^2$.

(iii) *If* $a \in (\mathbb{F}_q^*)^2$ *and* $b \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$, *then* $ab \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$.

PROOF. Easy and omitted. One can refer to Wan [75]. ■

Remark 4.4.1. Since $\mathbb{F}_q^*$ is abelian, $(\mathbb{F}_q^*)^2 \trianglelefteq \mathbb{F}_q^*$.

Proposition 4.4.3. *Let* $SD(2,q) = \{g \in GL(2,q) | \det(g) \in (\mathbb{F}_q^*)^2\}$. *Then* $SD(2,q) \leq GL(2,q)$.

PROOF. It is clear that $\det(I_2) = 1 \in (\mathbb{F}_q^*)^2$. Thus $I_2 \in SD(2,q)$. Let $A, B \in SD(2,q)$ with $\det(A)$, $\det(B) \in (\mathbb{F}_q^*)^2$. Therefore $\det(A) = \gamma_1^2$, and $\det(B) = \gamma_2^2$, for some $\gamma_1^2, \gamma_2^2 \in (\mathbb{F}_q^*)^2$. Now $\det(AB^{-1}) = \det(A)\det(B^{-1}) = \gamma_1^2(\gamma_2^{-1})^2 = (\gamma_1\gamma_2^{-1})^2 = \gamma^2 \in (\mathbb{F}_q^*)^2$. The result follows. ■

Proposition 4.4.4.

$$|SD(2,q)| = \begin{cases} q(q-1)^2(q+1) & \text{if } q \text{ is even,} \\ \frac{1}{2}q(q-1)^2(q+1) & \text{if } q \text{ is odd.} \end{cases}$$

PROOF. If q is even, then by Lemma 3.1.4, we have $\det(g) \in (\mathbb{F}_q^*)^2$, $\forall g \in GL(2,q)$. Thus $GL(2,q) \subseteq SD(2,q)$. Hence $SD(2,q) = GL(2,q)$ and the result follows. On the other hand, if q is odd, then the function $\omega : GL(2,q) \longrightarrow \{1,-1\}$ defined by

$$\omega(g) = \begin{cases} 1 & \text{if } \det(g) \in (\mathbb{F}_q^*)^2, \\ -1 & \text{otherwise,} \end{cases}$$

is a group homomorphism as follows. Let $A, B \in GL(2,q)$.

- If $\det(A), \det(B) \in (\mathbb{F}_q^*)^2$, then by Lemma 4.4.2(i), we have $\det(AB) = \det(A)\det(B) \in (\mathbb{F}_q^*)^2$. Therefore $\omega(AB) = 1 = 1 \times 1 = \omega(A)\omega(B)$.
- If $\det(A), \det(B) \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$, then by Lemma 4.4.2(ii), we have $\det(AB) = \det(A)\det(B) \in (\mathbb{F}_q^*)^2$. Therefore $\omega(AB) = 1 = -1 \times -1 = \omega(A)\omega(B)$.
- If $\det(A) \in (\mathbb{F}_q^*)^2$ and $\det(B) \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$, then by Lemma 4.4.2(iii), we have $\det(AB) = \det(A)\det(B) \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$. Therefore $\omega(AB) = -1 = 1 \times -1 = \omega(A)\omega(B)$.

Furthermore, $Im(\omega) = \{1,-1\}$, since existence of $g, h \in GL(2,q)$ such that $\det(g) \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$ and $\det(h) \in (\mathbb{F}_q^*)^2$ is guaranteed. To see this, let $\sigma \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$ (such σ exists by Lemma 3.1.4). Now it is cleat that $\omega\left(\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}\right) = -1$. Take $h = I_2$. Then $\omega(h) = \omega(I_2) = 1 \in (\mathbb{F}_q^*)^2$. It follows that

$Im(\omega) = \{1, -1\}$. By the definition of ω we can see that $\ker(\omega) = SD(2,q)$. Hence by the 1^{st} Isomorphism Theorem $GL(2,q)/SD(2,q) \cong Im(\omega) = \{1,-1\}$. Thus $|SD(2,q)| = |GL(2,q)|/|\{1,-1\}| = \frac{1}{2}q(q-1)^2(q+1)$ as required. ■

Note that the proof of Proposition 4.4.4 asserts that $SD(2,q) = \ker(\omega)$. Therefore we have the following corollary.

Corollary 4.4.5. *$SD(2,q) \trianglelefteq GL(2,q)$.*

In the following, we focus on $SD(2,q)$ when q is odd.

We define an equivalence relation $\sim$ on $SD(2,q)$ by

$$A \sim B \text{ in } SD(2,q) \Longleftrightarrow \det(A) = \det(B).$$

Obviously $\sim$ is an equivalence relation. Suppose that $\mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_k$ are the equivalence classes defined by $\sim$. Let $(\mathbb{F}_q^*)^2 = \{\delta_1^2, \delta_2^2, \cdots, \delta_{\frac{q-1}{2}}^2\}$ with $\delta_1^2 = 1$. The following proposition counts the number of the equivalence classes of $\sim$.

Proposition 4.4.6. *There is $1-1$ correspondence between the equivalence classes defined by $\sim$ above and elements of $(\mathbb{F}_q^*)^2$.*

PROOF. For each $\delta_i^2 \in (\mathbb{F}_q^*)^2$ there corresponds an equivalence class represented by $\delta_i I_2$. Conversely, since all elements of $[\delta_i I_2]$ have the same determinant δ_i^2, then the equivalence classes defined of $\sim$ are in $1-1$ correspondence with elements of $(\mathbb{F}_q^*)^2$. ■

From the above proposition, it follows that we can denote the equivalence classes of $\sim$ by $\mathcal{M}_{\delta_1^2}, \mathcal{M}_{\delta_2^2}, \cdots, \mathcal{M}_{\delta_{\frac{q-1}{2}}^2}$, where all elements of $\mathcal{M}_{\delta_i^2}$ have determinant δ_i^2. Note that $SL(2,q) = \mathcal{M}_{\delta_1^2} = \mathcal{M}_1$.

The group $SD(2,q)$ is of particular interest since $[GL(2,q) : SD(2,q)] = 2$ and we know all irreducible characters of such subgroups (see page 219 of James [40]). We do not attempt to find $Irr(SD(2,q))$. We need only four of its characters to finish the character table of $SL(2,q)$. However, we need all the conjugacy classes of $SD(2,q)$ and to see how the classes of $SL(2,q)$ fuse into them.

Any class of type $\mathcal{T}^{(1)}$ in $GL(2,q)$ will form a class in $SD(2,q)$ since these are central classes. Conjugacy classes of type $\mathcal{T}^{(2)}$ are all in $SD(2,q)$ but one can easily check that for a fixed α, the elements $g_1 = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ and $g_2 = \begin{pmatrix} \alpha & \varepsilon \\ 0 & \alpha \end{pmatrix}$ of $SD(2,q)$, which are conjugate in $GL(2,q)$, are no longer conjugate in $SD(2,q)$. In $SD(2,q)$, we have $|[g_1]| = |[g_2]| = \frac{q^2-1}{2}$. To see that $g_1 \not\sim g_2$ in

$SD(2,q)$, let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SD(2,q)$. Hence $ad - bc = y^2 \in (\mathbb{F}_q^*)^2$. Now

$$\begin{aligned}
\begin{pmatrix} \alpha & \varepsilon \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a\alpha + c\varepsilon & b\alpha + d\varepsilon \\ c\alpha & d\alpha \end{pmatrix}, \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} &= \begin{pmatrix} a\alpha & a + b\alpha \\ c\alpha & c + d\alpha \end{pmatrix}.
\end{aligned}$$

If $gg_1g^{-1} = g_2$, then $c = 0$ and $a = d\varepsilon$. Therefore $ad - bc = d^2\varepsilon = y^2$. This yields that $\varepsilon = (yd^{-1})^2 \in (\mathbb{F}_q^*)^2$. Therefore $o(\varepsilon) | \frac{q-1}{2}$, which contradicts the fact that $o(\varepsilon) = q - 1$.

Next we consider classes of $GL(2,q)$ of type $\mathcal{T}^{(3)}$, where a typical element $t_{\alpha,\beta}$, $\alpha \neq \beta$ will be $t_{\alpha,\beta} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Now for a fixed α, we have

$$t_{\alpha,\beta} \in SD(2,q) \iff \beta = \alpha^{-1}x^2, \text{ for some } x \in \mathbb{F}_q^* \setminus \{\alpha, -\alpha\}.$$

Note that in this case we have $l = -k + 2m$, where $m \in \mathbb{Z} \setminus \{k, -k\}$ such that $x = \varepsilon^m$. It is easy to check that $[t_{\alpha,\beta}] = [t_{\beta,\alpha}]$ in $SD(2,q)$. Thus we have $\frac{(q-1)(q-3)}{4}$ class of this type in $SD(2,q)$, where it can be shown that each class has size $q(q+1)$.

It is possible to prove that each $\mathcal{M}_{\delta_i^2}$, $1 \leq i \leq \frac{q-1}{2}$ contains $\frac{q-1}{2}$ classes of type $\mathcal{T}^{(4)}$. Also, each class $\mathcal{T}_k^{(4)}$ in $GL(2,q)$ is a non-split class in $SD(2,q)$. Therefore we have $\frac{(q-1)^2}{4}$ such classes in $SD(2,q)$ and $|\mathcal{T}_k^{(4)}| = q(q-1)$.

Adding up the number of elements in $SD(2,q)$ we have found so far, we get

$$\begin{aligned}
(q-1) \quad &+ \quad (q-1)\frac{q^2-1}{2} + (q-1)\frac{q^2-1}{2} + \frac{(q-1)(q-3)}{4}q(q+1) + \frac{(q-1)^2}{4}q(q-1) \\
&= \quad \frac{q(q-1)^2(q+1)}{2} = |SD(2,q)|.
\end{aligned}$$

This tells that we have found all the classes of $SD(2,q)$, which we list in Table 4.11.

The number of classes of $SD(2,q)$ is

$$(q-1) + (q-1) + (q-1) + \frac{(q-1)(q-3)}{4} + \frac{(q-1)^2}{4} = \frac{(q-1)}{2}(q+4).$$

Table 4.11: Conjugacy classes of $SD(2,q)$

Class	$\mathcal{T}_s^{(1)}$	$\mathcal{T}_{s1}^{(2)}$	$\mathcal{T}_{s\varepsilon}^{(2)}$	$\mathcal{T}_{s,-s+2m}^{(3)}$	$\mathcal{T}_s^{(4)}$
Rep g	$\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & \varepsilon\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}x^2\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -r^{q+1} & r+r^{q+1}\end{pmatrix}$
No. of CC	$q-1$	$q-1$	$q-1$	$\frac{(q-1)(q-3)}{4}$	$\frac{(q-1)^2}{4}$
$\lvert C_g\rvert$	1	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$	$q(q+1)$	$q(q-1)$
$\lvert C_{SD(2,q)}(g)\rvert$	$\frac{(q^2-1)(q^2-q)}{2}$	$\frac{q(q-1)}{2}$	$\frac{q(q-1)}{2}$	$\frac{(q-1)^2}{2}$	$\frac{q^2-1}{2}$

where, in Table 4.11,

- $\alpha, x \in \mathbb{F}_q^*,\ x \neq \pm\alpha,$
- in $\mathcal{T}_{s,-s+2m}^{(3)}$, s, m are the positive integers for which $\alpha = \varepsilon^s,\ x = \varepsilon^m,$
- in $\mathcal{T}_s^{(4)}$, if $\mathbb{F}_{q^2}^* = \langle\theta\rangle$, then s is the integer for which $r = \theta^s$ and $-\theta^{s(q+1)} \in (\mathbb{F}_q^*)^2,$
- in $\mathcal{T}_s^{(1)}, \mathcal{T}_{s1}^{(2)}, \mathcal{T}_{s\varepsilon}^{(2)}$, s has the same explanations as in Table 4.1.

Remark 4.4.2. Recall that $SL(2,q)$ has $q+4$ distinct conjugacy classes. Now we have seen that $SD(2,q) = \bigcup_{i=1}^{\frac{q-1}{2}} \mathcal{M}_{\delta_i^2}$ has $\frac{(q-1)}{2}(q+4)$ conjugacy classes. We note that the central classes of $GL(2,q)$ are distributed equally into the sets $\mathcal{M}_{\delta_i^2}$; that is each $\mathcal{M}_{\delta_i^2}$ contains 2 central classes namely, $diag(\delta_i,\delta_i)$ and $diag(-\delta_i,-\delta_i)$. More generally, by defining an equivalence relation $\approx$ on each $\mathcal{M}_{\delta_i^2}$ by

$$m_1 \approx m_2 \text{ in } \mathcal{M}_{\delta_i^2} \iff \exists\, x \in \mathcal{M}_{\delta_i^2} \text{ such that } m_2 = x m_1 x^{-1},$$

then we can see that for fixed $1 \le i \le \frac{q-1}{2}$, we have

- $\mathcal{M}_{\delta_i^2}$ contains two equivalence classes $[x_{1i}]$ and $[-x_{1i}]$, where $x_{1i} = \begin{pmatrix}\delta_i & 0\\ 0 & \delta_i\end{pmatrix}$, and $|[x_{1i}]| = |[-x_{1i}]| = 1,$
- $\mathcal{M}_{\delta_i^2}$ contains two equivalence classes $[x_{2i}]$ and $[-x_{2i}]$, where $x_{2i} = \begin{pmatrix}\delta_i & 1\\ 0 & \delta_i\end{pmatrix}$, and $|[x_{2i}]| = |[-x_{2i}]| = \frac{q^2-1}{2},$
- $\mathcal{M}_{\delta_i^2}$ contains two equivalence classes $[x_{3i}]$ and $[-x_{3i}]$, where $x_{3i} = \begin{pmatrix}\delta_i & \varepsilon\\ 0 & \delta_i\end{pmatrix}$, and $|[x_{3i}]| = |[-x_{3i}]| = \frac{q^2-1}{2},$

- $\mathcal{M}_{\delta_i^2}$ contains $\frac{q-3}{2}$ equivalence classes $[x_{4i\alpha}]$, where $x_{4i\alpha} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1}\delta_i^2 \end{pmatrix}$ and $\alpha \in \mathbb{F}_q^* \setminus \{\delta_i, -\delta_i\}$. Also $|[x_{4i\alpha}]| = q(q+1)$.

- $\mathcal{M}_{\delta_i^2}$ contains $\frac{q-1}{2}$ equivalence classes $[x_{5ir}]$, where $x_{5ir} = \begin{pmatrix} 0 & 1 \\ -r^{q+1} & r+r^q \end{pmatrix}$ and $r \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$ such that $r^{q+1} = \delta_i^2$. Also $|[x_{5ir}]| = q(q-1)$

Note 4.4.1. For $\mathcal{M}_1 = SL(2,q)$, the equivalence classes defined by $\approx$ are the conjugacy classes of $SL(2,q)$.

The set $Irr(SD(2,q))$ can be derived from $Irr(GL(2,q))$ since $[GL(2,q) : SD(2,q)] = 2$ (see page 219 of James [40] to see how to extract $Irr(N)$, where $[G : N] = 2$ from $Irr(G)$). We do not require the full set $Irr(SD(2,q))$, but we will use four irreducible characters of $SD(2,q)$ to produce $\widetilde{\xi}_1,\ \widetilde{\xi}_2,\ \widetilde{\vartheta}_1,\ \widetilde{\vartheta}_2 \in Irr(SL(2,q))$. In Table 4.12 we list few of the irreducible characters of $SD(2,q)$.

Table 4.12: Some of the irreducible characters of $SD(2,q)$

Class	$\mathcal{T}_s^{(1)}$	$\mathcal{T}_{s1}^{(2)}$	$\mathcal{T}_{s\varepsilon}^{(2)}$	$\mathcal{T}_{s,-s+2m}^{(3)}$	$\mathcal{T}_s^{(4)}$
$\chi_k^{(1)}$	$\widehat{\alpha}^{2k}$	$\widehat{\alpha}^{2k}$	$\widehat{\alpha}^{2k}$	$\widehat{x}^{2k}$	$\widehat{r}^{k(q+1)}$
$\chi_k^{(2)}$	$q\widehat{\alpha}^{2k}$	0	0	$\widehat{x}^{2k}$	$-\widehat{r}^{k(q+1)}$
ϑ_1	$\frac{(q-1)}{2}\widehat{\alpha}^{1+\epsilon}$	$(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})\widehat{\alpha}^{1+\epsilon}$	$(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})\widehat{\alpha}^{1+\epsilon}$	0	$(-1)^{s+1}$
ϑ_2	$\frac{(q-1)}{2}\widehat{\alpha}^{1+\epsilon}$	$(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})\widehat{\alpha}^{1+\epsilon}$	$(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})\widehat{\alpha}^{1+\epsilon}$	0	$(-1)^{s+1}$
ξ_1	$\frac{(q+1)}{2}\widehat{\alpha}^{\epsilon}$	$(\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})\widehat{\alpha}^{\epsilon}$	$(\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})\widehat{\alpha}^{\epsilon}$	$(-1)^s$	0
ξ_2	$\frac{(q+1)}{2}\widehat{\alpha}^{\epsilon}$	$(\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})\widehat{\alpha}^{\epsilon}$	$(\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})\widehat{\alpha}^{\epsilon}$	$(-1)^s$	0

where, in Table 4.12,

- in $\chi_k^{(1)},\ \chi_k^{(2)},\ k = 0, 1, \cdots, q-2$,
- $\epsilon = (-1)^{\frac{q-1}{2}} = \begin{cases} 1 & \text{if } q \equiv 1(\text{mod}4), \\ -1 & \text{if } q \equiv 3(\text{mod}4). \end{cases}$

Next we determine how classes of $SL(2,q)$ fuse into classes of $SD(2,q)$. In Table 4.13 we illustrate this fusion.

From Table 4.12 we consider $\vartheta_1, \vartheta_2, \xi_1,\ \xi_2 \in Irr(SD(2,q))$. Let us use for simplicity of notations $\widehat{\vartheta}_1,\ \widehat{\vartheta}_2,\ \widehat{\xi}_1$ and $\widehat{\xi}_2$ to denote

$$\widehat{\vartheta}_1 = \vartheta_1\downarrow_{SL(2,q)}^{SD(2,q)}, \quad \widehat{\vartheta}_2 = \vartheta_2\downarrow_{SL(2,q)}^{SD(2,q)}, \quad \widehat{\xi}_1 = \xi_1\downarrow_{SL(2,q)}^{SD(2,q)} \quad \text{and} \quad \widehat{\xi}_2 = \xi_2\downarrow_{SL(2,q)}^{SD(2,q)}.$$

Table 4.13: Fusion of classes of $SL(2,q)$ into classes of $SD(2,q)$

Type of classes of $GL(2,q)$	Class of $SL(2,q)$	$\longrightarrow$	Class of $SD(2,q)$
$\mathcal{T}^{(1)}$	$\mathcal{T}_0^{(1)}$		$\mathcal{T}_0^{(1)}$
	$-\mathcal{T}_0^{(1)}$		$-\mathcal{T}_0^{(1)}$
$\mathcal{T}^{(2)}$	$\mathcal{T}_{01}^{(2)}$		$\mathcal{T}_{01}^{(2)}$
	$-\mathcal{T}_{01}^{(2)}$		$-\mathcal{T}_{01}^{(2)}$
$\mathcal{T}^{(2)}$	$\mathcal{T}_{0\varepsilon}^{(2)}$		$\mathcal{T}_{0\varepsilon}^{(2)}$
	$-\mathcal{T}_{0\varepsilon}^{(2)}$		$-\mathcal{T}_{0\varepsilon}^{(2)}$
$\mathcal{T}^{(3)}$	$\mathcal{T}_{k,-k}^{(3)}$		$\mathcal{T}_{k,-k}^{(3)}$
$\mathcal{T}^{(4)}$	$\mathcal{T}_k^{(4)}$		$\mathcal{T}_k^{(4)}$

The values of $\widehat{\vartheta}_1$, $\widehat{\vartheta}_2$, $\widehat{\xi}_1$ and $\widehat{\xi}_2$ on classes of $SL(2,q)$ are shown in Table 4.14. It is clear from Table 4.14 that $\widehat{\vartheta}_1$, $\widehat{\vartheta}_2$, $\widehat{\xi}_1$ and $\widehat{\xi}_2$ are $\mathbb{C}-$valued characters if $q \equiv 3(\text{mod}4)$ and $\mathbb{R}-$valued characters if $q \equiv 1(\text{mod}4)$.

Table 4.14: Values of $\widehat{\vartheta}_1$, $\widehat{\vartheta}_2$, $\widehat{\xi}_1$ and $\widehat{\xi}_2$ on classes of $SL(2,q)$

Class	$\mathcal{T}_0^{(1)}$	$-\mathcal{T}_0^{(1)}$	$\mathcal{T}_{01}^{(2)}$	$-\mathcal{T}_{01}^{(2)}$
$\widehat{\xi}_1$	$\frac{(q+1)}{2}$	$\epsilon\frac{(q+1)}{2}$	$(\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$\epsilon(\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$
$\widehat{\xi}_2$	$\frac{(q+1)}{2}$	$\epsilon\frac{(q+1)}{2}$	$(\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$\epsilon(\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$
$\widehat{\vartheta}_1$	$\frac{(q-1)}{2}$	$-\epsilon\frac{(q-1)}{2}$	$(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$-\epsilon(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$
$\widehat{\vartheta}_2$	$\frac{(q-1)}{2}$	$-\epsilon\frac{(q-1)}{2}$	$(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$-\epsilon(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$

Class	$\mathcal{T}_{0\varepsilon}^{(2)}$	$-\mathcal{T}_{0\varepsilon}^{(2)}$	$\mathcal{T}_{k,-k}^{(3)}$	$\mathcal{T}_k^{(4)}$
$\widehat{\xi}_1$	$(\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$\epsilon(\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$(-1)^k$	0
$\widehat{\xi}_1$	$(\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$\epsilon(\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$(-1)^k$	0
$\widehat{\vartheta}_1$	$(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$-\epsilon(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	0	$(-1)^{m+1}$
$\widehat{\vartheta}_2$	$(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$-\epsilon(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	0	$(-1)^{m+1}$

Firstly let $q \equiv 1(\text{mod}4)$, that is $\epsilon = 1$. In this case we know that $\widehat{\xi_1}(g) \in \mathbb{R},\ \forall g \in SL(2,q)$. We have

$$\begin{aligned}\left\langle \widehat{\xi_1}, \widehat{\xi_1} \right\rangle &= \frac{1}{q^3-q}\left(2\frac{(q+1)^2}{4} + 2\frac{q^2-1}{2}(\frac{1+q+2\sqrt{q}}{4}) + 2\frac{q^2-1}{2}(\frac{1+q-2\sqrt{q}}{4}) + \frac{q(q-1)(q-3)}{2}\right) \\ &= \frac{1}{q^3-q}\left(\frac{(q+1)^2}{2} + \frac{q^2-1}{2}\frac{(q+1)}{2} + \frac{(q+1)}{2}q(q-3)\right) \\ &= \frac{(q+1)}{2(q^3-q)}\left((q+1) + (q^2-1) + (q^2-3q)\right) = \frac{(q+1)}{2(q^3-q)}\left(2q^2-2q\right) = 1.\end{aligned}$$

Thus $\widehat{\xi_1} \in Irr(SL(2,q))$. On the other hand, if $q \equiv 3(\text{mod}4)$ then we know that $\exists g \in SL(2,q)$ such that $\widehat{\xi_1}(g) \in \mathbb{C} \setminus \mathbb{R}$ and we have

$$\begin{aligned}\left\langle \widehat{\xi_1}, \widehat{\xi_1} \right\rangle &= \frac{1}{q^3-q}\left(2\frac{(q+1)^2}{4} + 2\frac{q^2-1}{2}(\frac{1+q}{4}) + 2\frac{q^2-1}{2}(\frac{1+q}{4}) + \frac{q(q-1)(q-3)}{2}\right) \\ &= \frac{1}{q^3-q}\left(\frac{(q+1)^2}{2} + \frac{q^2-1}{2}\frac{(q+1)}{2} + \frac{(q+1)}{2}q(q-3)\right) \\ &= \frac{(q+1)}{2(q^3-q)}\left((q+1) + (q^2-1) + (q^2-3q)\right) = \frac{(q+1)}{2(q^3-q)}\left(2q^2-2q\right) = 1,\end{aligned}$$

which shows that $\widehat{\xi_1} \in Irr(SL(2,q))$.

Similar arguments can be used to show that $\widehat{\xi_2},\ \widehat{\vartheta_1},\ \widehat{\vartheta_2} \in Irr(SL(2,q))$.

Finally we are in the position to give the required values of $\widetilde{\xi_1},\ \widetilde{\xi_2},\ \widetilde{\vartheta_1}$ and $\widetilde{\vartheta_2}$ on the classes of $SL(2,q)$. First note that

$$\begin{aligned}\widehat{\xi_1}(g) + \widehat{\xi_2}(g) &= (\widetilde{\xi_1} + \widetilde{\xi_2})(g),\ \forall g \in SL(2,q), \\ \widehat{\vartheta_1}(g) + \widehat{\vartheta_2}(g) &= (\widetilde{\vartheta_1} + \widetilde{\vartheta_2})(g),\ \forall g \in SL(2,q),\end{aligned}$$

where $(\widetilde{\xi_1} + \widetilde{\xi_2})(g)$ and $(\widetilde{\vartheta_1} + \widetilde{\vartheta_2})(g),\ \forall g \in SL(2,q)$ are given in Tables 4.9, 4.10 respectively.

Therefore we may take

$$\widetilde{\xi_1} = \widehat{\xi_1} = \xi_1\downarrow^{SD(2,q)}_{SL(2,q)},\quad \widetilde{\xi_2} = \widehat{\xi_2} = \xi_2\downarrow^{SD(2,q)}_{SL(2,q)},\quad \widetilde{\vartheta_1} = \widehat{\vartheta_1} = \vartheta_1\downarrow^{SD(2,q)}_{SL(2,q)} \quad\text{and}\quad \widetilde{\vartheta_2} = \widehat{\vartheta_2} = \vartheta_2\downarrow^{SD(2,q)}_{SL(2,q)}.$$

This completes the character table of $SL(2,q)$, q odd. We list the character table of $SL(2,q)$, q odd, in Table 4.15.

Table 4.15: The character table of $SL(2,q)$, q is odd

Class	$\mathcal{T}_0^{(1)}$	$-\mathcal{T}_0^{(1)}$	$\mathcal{T}_{01}^{(2)}$	$-\mathcal{T}_{01}^{(2)}$
Rep g	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&-1\\0&-1\end{pmatrix}$
No. of CC	1	1	1	1
$\lvert C_{SL(2,q)}(g)\rvert$	q^3-q	q^3-q	$2q$	$2q$
$\lvert C_g\rvert$	1	1	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$
λ	1	1	1	1
ψ	q	q	0	0
$\psi_{k,1}$	$q+1$	$(-1)^{k+1}(q+1)$	1	1
π_k	$q-1$	$(-1)^k(q-1)$	-1	$(-1)^{k+1}$
$\widetilde{\xi}_1$	$\frac{(q+1)}{2}$	$\epsilon\frac{(q+1)}{2}$	$(\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$\epsilon(\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$
$\widetilde{\xi}_2$	$\frac{(q+1)}{2}$	$\epsilon\frac{(q+1)}{2}$	$(\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$\epsilon(\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$
$\widetilde{\vartheta}_1$	$\frac{(q-1)}{2}$	$-\epsilon\frac{(q-1)}{2}$	$(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$-\epsilon(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$
$\widetilde{\vartheta}_2$	$\frac{(q-1)}{2}$	$-\epsilon\frac{(q-1)}{2}$	$(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$-\epsilon(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$

Class	$\mathcal{T}_{0\varepsilon}^{(2)}$	$-\mathcal{T}_{0\varepsilon}^{(2)}$	$\mathcal{T}_{k,-k}^{(3)}$	$\mathcal{T}_k^{(4)}$
Rep g	$\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&-\varepsilon\\0&-1\end{pmatrix}$	$\begin{pmatrix}\alpha&0\\0&\alpha^{-1}\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&r+r^q\end{pmatrix}$
No. of CC	1	1	$\frac{q-3}{2}$	$\frac{q-1}{2}$
$\lvert C_{SL(2,q)}(g)\rvert$	$2q$	$2q$	$q-1$	$q+1$
$\lvert C_g\rvert$	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$	$q(q+1)$	$q(q-1)$
λ	1	1	1	1
ψ	0	0	1	-1
$\psi_{k,1}$	1	$(-1)^{k+1}$	$\widehat{\alpha}^{(k-1)}+\widehat{\alpha}^{-(k-1)}$	0
π_k	-1	$(-1)^{k+1}$	0	$-(\widehat{r}^k+\widehat{r}^{kq})$
$\widetilde{\xi}_1$	$(\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$\epsilon(\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$(-1)^k$	0
$\widetilde{\xi}_2$	$(\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$\epsilon(\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$(-1)^k$	0
$\widetilde{\vartheta}_1$	$(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$-\epsilon(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	0	$(-1)^{m+1}$
$\widetilde{\vartheta}_2$	$(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$-\epsilon(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	0	$(-1)^{m+1}$

where, in Table 4.15,

- $\epsilon = (-1)^{\frac{q-1}{2}} = \begin{cases} 1 & \text{if } q \equiv 1(\text{mod}4), \\ -1 & \text{if } q \equiv 3(\text{mod}4). \end{cases}$
- m is the positive integer for which $r = \theta^m$, θ being a generator of $\mathbb{F}_{q^2}^*$.

4.4.2 Character Table of $SL(2,q)$, q even

In the case $SL(2,q)$, q odd, we have seen that the number of conjugacy classes is $q+4$. If the summand 4 is interpreted as $|Z(SL(2,q))|^2$, then the same interpretation will be valid for the case when q is even, that is the number of conjugacy classes of $SL(2,2^t), t \geq 1$ is $q+1$. This can be shown through the next Proposition.

Proposition 4.4.7. *The group $SL(2,q)$, $q = 2^t$, $t \geq 1$ has $q+1$ distinct conjugacy classes described in* Table 4.16.

PROOF. The following statements are the results of direct computations from the conjugacy classes of $GL(2,q)$.

(i) The identity class I_2 is the only class of type $\mathcal{T}^{(1)}$ which is in $SL(2,2^t)$.

(ii) The class represented by $\mathcal{T}_0^{(2)}$ is the only class of type $\mathcal{T}^{(2)}$ which is in $SL(2,2^t)$. In equation (4.15), if we replace $SL(2,q)$, q odd by $SL(2,2^t)$, then we get $g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Thus $|C_{SL(2,2^t)}(x)| = q$, where $x \in \mathcal{T}_0^{(2)}$. Hence $|\mathcal{T}_0^{(2)}| = q^2 - 1$, which shows that the class $\mathcal{T}_0^{(2)}$ of $GL(2,q)$ is a non-split class in $SL(2,2^t)$.

(iii) If $t_{k,l} = \begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^l \end{pmatrix}$ is an element in class $\mathcal{T}_{k,l}^{(3)}$ of $GL(2,q)$, then $t_{k,l} \in SL(2,2^t) \Longleftrightarrow l = -k$. Excluding $k = 0$ and note that $t_{k,-k} \sim t_{-k,k}$ in $SL(2,2^t)$, we get $\frac{q-2}{2}$ such classes. It is straightforward to show that $|\mathcal{T}_{k,-k}^{(3)}| = q(q+1)$.

(iv) If $A_r = \begin{pmatrix} 0 & 1 \\ r^{q+1} & r + r^q \end{pmatrix} \in GL(2,q)$ is of type $\mathcal{T}^{(4)}$, then for $j = 1, 2, \cdots, \frac{q}{2}$, we have $A_r \in SL(2,2^t)$ since $\det(A_r) = r.r^q = \theta^{(q-1)j}.\theta^{q(q-1)j} = \theta^{(q^2-1)j} = 1$.

Now counting the elements we have found so far, we get

$$1 \cdot 1 + 1 \cdot (q^2 - 1) + \frac{q-2}{2} \cdot q(q+1) + \frac{q}{2} \cdot q(q-1) = q^3 - q = |SL(2,q)|.$$

Thus Table 4.16 lists the full conjugacy classes of $SL(2,q)$. ■

Table 4.16: The conjugacy classes of $SL(2,q)$, q is even

Class	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_{k,-k}^{(3)}$	$\mathcal{T}_k^{(4)}$
Rep g	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$	$\begin{pmatrix}\alpha & 0\\0 & \alpha^{-1}\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & r+r^q\end{pmatrix}$
No. of CC	1	1	$\frac{q-2}{2}$	$\frac{q}{2}$
$\lvert C_{SL(2,2^t)}(g)\rvert$	q^3-q	q	$q-1$	$q+1$
$\lvert C_g\rvert$	1	q^2-1	$q(q+1)$	$q(q-1)$

where $\alpha \in \mathbb{F}_q^*$, $\alpha = \varepsilon^k$, $k \neq 0$, and if $\mathbb{F}_{q^2}^* = \langle\theta\rangle$, then $r = \theta^{(q-1)j}$ for $j = 1, 2, \cdots, \frac{q}{2}$.

Irreducible Characters of $SL(2,2^t)$

Proposition 4.4.8. *The characters table of $SL(2,2^t)$ is given in* Table 4.17.

PROOF. It is easy to check that $\langle\psi,\psi\rangle = 1$. Similar to the case $SL(2,q)$, q odd in Subsection 4.4.1, where we proved that $\psi_{k,1} \in Irr(SL(2,q))$, we can prove that $\psi_{k,0} \in Irr(SL(2,2^t))$. Same arguments can also be used to show that $\langle\pi_k,\pi_k\rangle = 1$ for the appropriate k. ■

Table 4.17: The Character table of $SL(2,q)$, q is even

Class	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_{k,-k}^{(3)}$	$\mathcal{T}_k^{(4)}$
Rep g	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$	$\begin{pmatrix}\alpha & 0\\0 & \alpha^{-1}\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & r+r^q\end{pmatrix}$
No. of CC	1	1	$\frac{q-2}{2}$	$\frac{q}{2}$
$\lvert C_{SUT(2,2^t)}(g)\rvert$	q^3-q	q	$q-1$	$q+1$
$\lvert C_g\rvert$	1	q^2-1	$q(q+1)$	$q(q-1)$
λ	1	1	1	1
ψ	q	0	1	-1
$\psi_{k,0}$	$q+1$	1	$\widehat{\alpha}^k + \widehat{\alpha}^{-k}$	0
π_k	$q-1$	-1	0	$-(\widehat{r}^k + \widehat{r}^{kq})$

where, in Table 4.17,

- $\alpha \in \mathbb{F}_q^*,\ \alpha \neq 1$,
- with θ being a generator of $\mathbb{F}_{q^2}^*$, then $r = \theta^{(q-1)j}$ for $j = 1, 2, \cdots, \frac{q}{2}$,
- in $\psi_{k,0},\ k = 1, 2, \cdots, \frac{q-2}{2}$,
- in $\pi_k,\ k = 1, 2, \cdots, \frac{q}{2}$.

Remark 4.4.3. Note that $SL(2,q)$ (q even or odd and $q = p^s$) possesses only one irreducible character χ such that $p|\deg(\chi)$. It is shown in unpublished paper by Berkovich and Kazarin (see Berckovich [6]) that if a group possesses only one non-linear irreducible character of p'−degree, then it is solvable.

4.5. Character Table of $SUT(2,q)$

We recall by Subsection 3.1.2 that $SUT(2,q)$ has the form

$$SUT(2,q) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^*,\, b \in \mathbb{F}_q \right\}.$$

Therefore $|SUT(2,q)| = q(q-1)$ and hence $[SL(2,q) : SUT(2,q)] = q+1$. This group has the structure of group extension (nontrivial). To see this, we recall Schur-Zassenhaus Lemma.

Lemma 4.5.1 (**Schur-Zassenhaus Lemma**). *Let G be a finite group and $N \trianglelefteq G$ such that* $\gcd(|N|, |G/N|) = 1$. *Then $G = N{:}(G/N)$.*

PROOF. See Robinson [62] or Rotman [65]. ∎

Let

$$K = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_q \right\}, \quad H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^* \right\}. \tag{4.20}$$

Then $K \cong \mathbb{F}_q$ and $H \cong \mathbb{F}_q^*$. Note that since $|K| = q$ and $|H| = q-1$, we have $\gcd(|K|, |H|) = 1$.

Proposition 4.5.2. *$SUT(2,q) \cong K{:}H$.*

PROOF. Let $A = \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \in SUT(2,q)$ and $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in K$. Then

$$ABA^{-1} = \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & -c \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & a^2 b \\ 0 & 1 \end{pmatrix} \in K.$$

Thus $K \trianglelefteq SUT(2,q)$. The result now follows from Schur-Zassenhaus Lemma. ∎

Next we determine the character table of $SUT(2,q)$ in both cases of q odd or even.

4.5.1 Character Table of $SUT(2,q)$, q odd

Conjugacy Classes of $SUT(2,q)$

From Table 4.15, we can see that the classes $\mathcal{T}_0^{(1)}$, $-\mathcal{T}_0^{(1)}$, $\mathcal{T}_{01}^{(2)}$, $-\mathcal{T}_{01}^{(2)}$, $\mathcal{T}_{0\varepsilon}^{(2)}$, $-\mathcal{T}_{0\varepsilon}^{(2)}$ and $\mathcal{T}_{k,-k}^{(3)}$ are in $SUT(2,q)$. We recall that any class $\mathcal{T}_{k,-k}^{(3)}$ in $SL(2,q)$ does not change if we replace $\alpha \in \mathbb{F}_q^*$, $\alpha \notin \{1,-1\}$, by α^{-1} in a typical element of the class. This is not the case in $SUT(2,q)$, where each $\alpha \in \mathbb{F}_q^*$, $\alpha \notin \{1,-1\}$ gives a new conjugacy class since

$$\begin{aligned}
\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} &= \begin{pmatrix} a\alpha & b\alpha^{-1} \\ 0 & a^{-1}\alpha^{-1} \end{pmatrix},\\
\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \begin{pmatrix} a\alpha^{-1} & b\alpha^{-1} \\ 0 & a^{-1}\alpha \end{pmatrix}.
\end{aligned}$$

Thus if $\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \sim \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ in $SUT(2,q)$, we would have

$$a\alpha = a\alpha^{-1} \iff a(\alpha-\alpha^{-1}) = 0 \iff a = 0 \text{ or } \alpha = \alpha^{-1} \iff a = 0 \text{ or } \alpha \in \{1,-1\},$$

which contradicts the facts that $a \neq 0$ and $\alpha \notin \{1,-1\}$. Hence there are at least $6+(q-3) = q+3$ conjugacy classes of $SUT(2,q)$.

Now $|\mathcal{T}_0^{(1)}| = |-\mathcal{T}_0^{(1)}| = 1$. Suppose that $g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SUT(2,q)$ and let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{T}_{01}^{(2)}$. Then

$$\begin{aligned}
\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a & a+b \\ 0 & a^{-1} \end{pmatrix},\\
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \begin{pmatrix} a & b+a^{-1} \\ 0 & a^{-1} \end{pmatrix}.
\end{aligned}$$

Thus

$$g \in C_{SUT(2,q)}(A) \iff a+b = b+a^{-1} \iff a = a^{-1} \iff a \in \{1,-1\}.$$

Therefore $|C_{SUT(2,q)}(A)| = 2q$ and hence $|[A]| = \frac{q-1}{2}$. Similar computations show that

$$|-\mathcal{T}_{01}^{(2)}| = |\mathcal{T}_{0\varepsilon}^{(2)}| = |-\mathcal{T}_{0\varepsilon}^{(2)}| = \frac{q-1}{2}.$$

If $t = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ is any element in the class $\mathcal{T}_{k,-k}^{(3)}$, then

$$\begin{aligned}
\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} &= \begin{pmatrix} a\alpha & b\alpha^{-1} \\ 0 & a^{-1}\alpha^{-1} \end{pmatrix},\\
\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \begin{pmatrix} a\alpha & b\alpha \\ 0 & a^{-1}\alpha^{-1} \end{pmatrix}.
\end{aligned}$$

Thus

$$g \in C_{SUT(2,q)}(t) \iff b\alpha^{-1} = b\alpha \iff b(\alpha - \alpha^{-1}) = 0 \iff b = 0 \text{ or } \alpha = \alpha^{-1} \iff b = 0 \text{ or } \alpha \in \{1,-1\}.$$

Since $\alpha \notin \{1,-1\}$, we must have $b = 0$. Therefore $|C_{SUT(2,q)}(t)| = q-1$ and consequently, $|\mathcal{T}^{(3)}_{k,-k}| = q$.

By calculating the number of elements we have obtained so far, we get

$$2 + 4 \times \frac{q-1}{2} + q(q-3) = 2 + 2q - 2 + q^2 - 3q = q^2 - q = q(q-1) = |SUT(2,q)|.$$

This tells that there can not be further classes of $SUT(2,q)$. We list these classes in Table 4.18.

Table 4.18: The conjugacy classes of $SUT(2,q)$, q is odd

Class	$\mathcal{T}^{(1)}_0$	$-\mathcal{T}^{(1)}_0$	$\mathcal{T}^{(2)}_{01}$	$-\mathcal{T}^{(2)}_{01}$
Rep g	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$
No. of CC	1	1	1	1
$\lvert C_{SUT(2,q)}(g)\rvert$	q^2-q	q^2-q	$2q$	$2q$
$\lvert C_g\rvert$	1	1	$\frac{q-1}{2}$	$\frac{q-1}{2}$

Table 4.18 (continued)

Class	$\mathcal{T}^{(2)}_{0\varepsilon}$	$-\mathcal{T}^{(2)}_{0\varepsilon}$	$\mathcal{T}^{(3)}_{k,-k}$
Rep g	$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -\varepsilon \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$
No. of CC	1	1	$q-3$
$\lvert C_{SUT(2,q)}(g)\rvert$	$2q$	$2q$	$q-1$
$\lvert C_g\rvert$	$\frac{q-1}{2}$	$\frac{q-1}{2}$	q

where ε is a generator of the group $\mathbb{F}_q^*$ and $\alpha \in \mathbb{F}_q^* \setminus \{1,-1\}$.

Irreducible Characters of $SUT(2,q)$

Let K be the group defined in (4.20). By Proposition 4.5.2 we have $SUT(2,q)/K \cong H$. Hence by Section 2.4 we have $Irr(H) \subset Irr(SUT(2,q))$. Therefore we get $q-1$ linear characters of $SUT(2,q)$ and in fact these are all the linear characters, because of Proposition 4.5.3.

Proposition 4.5.3. $SUT(2,q)' = K$.

PROOF. Let $A = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SUT(2,q)$ and $C = \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \in SUT(2,q)$. The commutator of A and C is

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} c^{-1} & -d \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & cd(a^2-1) - ab(c^2-1) \\ 0 & 1 \end{pmatrix} \in K. \quad (4.21)$$

Thus $SUT(2,q)' \subseteq K$.

Conversely we aim to show that any $w \in K$ is a commutator. Let $w = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in K$ and assume that $a \in \mathbb{F}_q^*$, $a \neq 1$ (such a exists since $q \geq 3$). Now

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x(a^2-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & -x(a^2-1) \\ 0 & 1 \end{pmatrix} \in SUT(2,q)'.$$

Thus $K \subseteq SUT(2,q)'$. Therefore $SUT(2,q)' = K$ establishes the result. ∎

The $q-1$ linear characters χ_k, $1 \leq k \leq q-1$ are given on representatives of classes of $SUT(2,q)$ by

- $\chi_k\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 1, \qquad \chi_k\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) = (-1)^k,$
- $\chi_k\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = 1, \qquad \chi_k\left(\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right) = (-1)^k,$
- $\chi_k\left(\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}\right) = 1, \qquad \chi_k\left(\begin{pmatrix} -1 & -\varepsilon \\ 0 & -1 \end{pmatrix}\right) = (-1)^k,$
- $\chi_k\left(\begin{pmatrix} \varepsilon^s & 0 \\ 0 & \varepsilon^{-s} \end{pmatrix}\right) = e^{\frac{2\pi k s}{q-1} i}, \quad s = 2, 3, \cdots, q-2, \; 0 \leq k \leq q-2.$

We recall that $|Irr(SUT(2,q))| = q+3$. Hence 4 further characters left to be found.

Proposition 4.5.4. *Let* $\widetilde{\vartheta}_1$ *and* $\widetilde{\vartheta}_2$ *be the irreducible characters of* $SL(2,q)$ *given in* Table 4.15. *Then* $\widetilde{\vartheta}_1\downarrow^{SL(2,q)}_{SUT(2,q)}$, $\widetilde{\vartheta}_2\downarrow^{SL(2,q)}_{SUT(2,q)} \in Irr(SUT(2,q))$.

PROOF. For simplicity of notations, let us denote $\widetilde{\vartheta}_1\downarrow^{SL(2,q)}_{SUT(2,q)}$ and $\widetilde{\vartheta}_2\downarrow^{SL(2,q)}_{SUT(2,q)}$ by $\widehat{\vartheta}_1$ and $\widehat{\vartheta}_2$ respectively. We have seen that if $q \equiv 1(mod4)$, then $\widetilde{\vartheta}_1(g)$, $\widetilde{\vartheta}_2(g) \in \mathbb{R}$, $\forall g \in SL(2,q)$, while if $q \equiv 3(mod4)$, then $\exists g \in SL(2,q)$ such that $\widetilde{\vartheta}_1(g)$, $\widetilde{\vartheta}_2(g) \in \mathbb{C} \setminus \mathbb{R}$. Now suppose that $q \equiv 3(mod4)$. We have

$$\begin{aligned}
\left\langle \widehat{\vartheta}_1, \widehat{\vartheta}_1 \right\rangle &= \frac{1}{|SUT(2,q)|} \sum_{g \in SUT(2,q)} \widehat{\vartheta}_1(g)\overline{\widehat{\vartheta}_1(g)} = \frac{1}{q(q-1)} \left(\frac{(q-1)^2}{4} + \frac{(q-1)^2}{4} \right. \\
&+ \frac{q-1}{2}\left(-\frac{1}{2}+\frac{i\sqrt{q}}{2}\right)\left(-\frac{1}{2}-\frac{i\sqrt{q}}{2}\right) + \frac{q-1}{2}\left(-\frac{1}{2}+\frac{i\sqrt{q}}{2}\right)\left(-\frac{1}{2}-\frac{i\sqrt{q}}{2}\right) + \\
&+ \left. \frac{q-1}{2}\left(-\frac{1}{2}-\frac{i\sqrt{q}}{2}\right)\left(-\frac{1}{2}+\frac{i\sqrt{q}}{2}\right) + \frac{q-1}{2}\left(-\frac{1}{2}-\frac{i\sqrt{q}}{2}\right)\left(-\frac{1}{2}+\frac{i\sqrt{q}}{2}\right)\right) \\
&= \frac{1}{q(q-1)}\left(\frac{(q-1)^2}{2} + 4 \times \frac{q-1}{2}\frac{q+1}{4}\right) = \frac{1}{q(q-1)}\left(\frac{(q-1)^2}{2} + \frac{q-1}{2}q+1\right) \\
&= \frac{1}{q(q-1)}\left(\frac{(q-1)}{2}(q-1+q+1)\right) = \frac{1}{q(q-1)}q(q-1) = 1.
\end{aligned}$$

Hence $\widehat{\vartheta}_1 \in Irr(SUT(2,q))$. Similarly $\widehat{\vartheta}_1 \in Irr(SUT(2,q))$ when $q \equiv 1(mod4)$. This applies as well for the character $\widehat{\vartheta}_2$. ■

By tensoring the $q-1$ linear characters χ_k by $\widehat{\vartheta}_1$ and $\widehat{\vartheta}_2$, we can see that if k is even for $0 \leq k \leq q-2$, then $\chi_k\widehat{\vartheta}_1 = \widehat{\vartheta}_1$ and $\chi_k\widehat{\vartheta}_2 = \widehat{\vartheta}_2$, while if k is odd, then $\chi_k\widehat{\vartheta}_1 = \chi_1\widehat{\vartheta}_1$ and $\chi_k\widehat{\vartheta}_2 = \chi_1\widehat{\vartheta}_2$.

This gives the required 4 irreducible characters of $SUT(2,q)$. In Table 4.19 we list the complete character table of $SUT(2,q)$.

Now we turn to the other case when $q = 2^t$ for some positive integer t.

4.5.2 Character Table of $SUT(2, 2^t)$

We construct the character table of $SUT(2,2^t)$ in two different ways. In the first approach we show that $SUT(2,2^t)$ is one of the Frobenius groups, which have known representations. In the second approach we use Clifford-Fischer theory together with the technique of coset analysis.

I: Character Table of $SUT(2,q)$, where $SUT(2,q)$ Viewed as a Frobenius Group

We review the basic properties and the characters of Frobenius groups.

Definition 4.5.1. *A group* G *is called a* ***Frobenius group*** *if it has a proper subgroup* H *such that* $H \cap H^r = \{1_G\}$, $\forall r \in G \setminus H$. *The subgroup* H *will be referred as the* ***Frobenius complement****.*

Table 4.19: The character table of $SUT(2,q)$, q is odd.

Class	$\mathcal{T}_0^{(1)}$	$-\mathcal{T}_0^{(1)}$	$\mathcal{T}_{01}^{(2)}$	$-\mathcal{T}_{01}^{(2)}$
Rep g	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&-1\\0&-1\end{pmatrix}$
No. of CC	1	1	1	1
$\lvert C_{SUT(2,q)}(g)\rvert$	q^2-q	q^2-q	$2q$	$2q$
$\lvert C_g\rvert$	1	1	$\frac{q-1}{2}$	$\frac{q-1}{2}$
χ_k	1	$(-1)^k$	1	$(-1)^k$
$\widehat{\vartheta}_1$	$\frac{q-1}{2}$	$-\epsilon\frac{q-1}{2}$	$(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$-\epsilon(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$
$\widehat{\vartheta}_2$	$\frac{q-1}{2}$	$-\epsilon\frac{q-1}{2}$	$(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$-\epsilon(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$
$\chi_1\widehat{\vartheta}_1$	$\frac{q-1}{2}$	$\epsilon\frac{q-1}{2}$	$(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$\epsilon(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$
$\chi_1\widehat{\vartheta}_2$	$\frac{q-1}{2}$	$\epsilon\frac{q-1}{2}$	$(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$\epsilon(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$

Class	$\mathcal{T}_{0\varepsilon}^{(2)}$	$-\mathcal{T}_{0\varepsilon}^{(2)}$	$\mathcal{T}_{s,-s}^{(3)}$
Rep g	$\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&-\varepsilon\\0&-1\end{pmatrix}$	$\begin{pmatrix}\varepsilon^s&0\\0&\varepsilon^{-s}\end{pmatrix}$
No. of CC	1	1	$q-3$
$\lvert C_{SUT(2,q)}(g)\rvert$	$2q$	$2q$	$q-1$
$\lvert C_g\rvert$	$\frac{q-1}{2}$	$\frac{q-1}{2}$	q
χ_k	1	$(-1)^k$	$e^{\frac{2\pi iks}{q-1}}$
$\widehat{\vartheta}_1$	$(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$-\epsilon(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	0
$\widehat{\vartheta}_2$	$(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$-\epsilon(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	0
$\chi_1\widehat{\vartheta}_1$	$(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	$\epsilon(-\frac{1}{2}-\frac{\sqrt{\epsilon q}}{2})$	0
$\chi_1\widehat{\vartheta}_2$	$(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	$\epsilon(-\frac{1}{2}+\frac{\sqrt{\epsilon q}}{2})$	0

where, in Table 4.19,

- ε is a generator of the group $\mathbb{F}_q^*$,
- $s = 2, 3\cdots, q-2$,
- in $\chi_k, k = 0, 2, \cdots, q-2$,
- $\epsilon = (-1)^{\frac{q-1}{2}} = \begin{cases} 1 & \text{if } q \equiv 1(mod4), \\ -1 & \text{if } q \equiv 3(mod4). \end{cases}$

The *Frobenius kernel* K of G with respect to H is defined by

$$K = \left(G - \bigcup_{r \in G} H^r \right) \cup \{1_G\}.$$

We leave showing that $K \trianglelefteq G$ but one can refer to Flavell [20] or Grove [29] for the proof.

Example 4.5.1.

1. The smallest Frobenius group is S_3. It has A_3 as a Frobenius kernel K and $\mathbb{Z}_2$ as its complement H.
2. The Dihedral group D_{2n}, $n > 1$ is odd, is a Frobenius group. The subgroup generated by the element of order n acts as a kernel while $\mathbb{Z}_2$ is the complement.

The following Proposition gives a structure of finite Frobenius groups.

Proposition 4.5.5. *Any finite Frobenius group is a split extension of its kernel K by its complement H.*

PROOF. See page 243 of Robinson [62]. ∎

Frobenius groups have many other nice properties. One can refer to either Collins [13], Grove [29] or Robinson [62]. The irreducible characters of a Frobenius group G can be constructed from those of H and K. They appear in two types

- By Proposition 4.5.5, any irreducible representation ϕ of H gives an irreducible representation of G by using the quotient map from G to H. This gives the irreducible representations of G with K in their kernel.
- If ψ is any non-trivial irreducible representation of K, then the corresponding induced representation of G is also irreducible. This gives the irreducible representations of G with K not in their kernel.

To see that any irreducible representation of a Frobenius group has one of the above forms, refer to Grove [29].

In the following we show that $SUT(2, 2^t)$ is a Frobenius group.

Theorem 4.5.6. *$SUT(2, q)$ is a Frobenius group for even q.*

PROOF. Let H and K be the subgroups of $SUT(2, q)$ defined in (4.20). We aim to show that H and K are the Frobenius complement and kernel of $SUT(2, q)$ respectively. From the definition of Frobenius group, let $r \in G \setminus H$. Then a typical element r will have the form

$$r = \begin{pmatrix} c & b \\ 0 & c^{-1} \end{pmatrix}, \ b \neq 0.$$

Thus

$$H^r = rHr^{-1} = \left\{\begin{pmatrix} c & b \\ 0 & c^{-1}\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix} c^{-1} & -b \\ 0 & c\end{pmatrix}\right\}$$
$$= \left\{\begin{pmatrix} a & -abc + a^{-1}bc \\ 0 & a^{-1}\end{pmatrix}\right\} = \left\{\begin{pmatrix} a & abc + a^{-1}bc \\ 0 & a^{-1}\end{pmatrix}\right\}.$$

Therefore

$$H^r \text{ intersects } H \iff abc + a^{-1}bc = 0 \iff bc(a^{-1} + a) = 0 \iff a = -a^{-1} = a^{-1} \iff a = 1.$$

Thus $H^r \cap H = \{1_{SUT(2,q)}\} = I_2$ and hence H is a Frobenius complement of $SUT(2,q)$.
To show that K is a Frobenius kernel of $SUT(2,q)$, we use Theorems 9.2.1 and 9.8.2 of Grove [29], where the first theorem asserts that a group G is Frobenius if and only if it has nontrivial proper normal subgroup K such that if $1_G \neq x \in K$ then $C_G(x) \leq K$. The second theorem establishes the uniqueness of Frobenius kernel of a Frobenius group. We have proved in Proposition 4.5.2 that the subgroup K, defined in (4.20), is a normal subgroup of $SUT(2,q)$. Therefore to show that K is a Frobenius kernel of $SUT(2,q)$, it suffices to prove that $C_{SUT(2,q)}(x) \leq K$, $\forall\, x \in K \setminus \{1_{SUT(2,q)}\}$.
Suppose that $x = \begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}$ for some $b \in \mathbb{F}_q^*$. Let $g = \begin{pmatrix} a & c \\ 0 & a^{-1}\end{pmatrix} \in SUT(2,q)$. Then

$$\begin{pmatrix} a & c \\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} = \begin{pmatrix} a & ab + c \\ 0 & a^{-1}\end{pmatrix}, \tag{4.22}$$

$$\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & c \\ 0 & a^{-1}\end{pmatrix} = \begin{pmatrix} a & a^{-1}b + c \\ 0 & a^{-1}\end{pmatrix}. \tag{4.23}$$

Now

$$g \in C_{SUT(2,q)}(x) \iff (4.22) = (4.23) \iff ab = a^{-1}b \iff a = a^{-1} \iff a = 1.$$

Thus

$$C_{SUT(2,q)}(x) = \left\{\begin{pmatrix} 1 & c \\ 0 & 1\end{pmatrix} \mid c \in \mathbb{F}_q\right\} = K.$$

Hence K is a Frobenius kernel of $SUT(2,q)$, which completes the proof of the Theorem. ■

We now show that $SUT(2,q)$, for odd q, is not a Frobenius group.

Lemma 4.5.7. *If G is a Frobenius group, then $Z(G) = \{1_G\}$.*

PROOF. See Robinson [62]. ■

Corollary 4.5.8. *The group $SUT(2,q)$ for odd q is not a Frobenius group.*

PROOF. The contrapositive of Lemma 4.5.7 asserts that if the $|Z(G)| > 1$, then G is not a Frobenius group. It is clear that $I_2,\ -I_2 \in Z(SUT(2,q))$ and the result follows. ■

The following theorem deals with the conjugacy classes of $SUT(2,2^t)$.

Theorem 4.5.9. *The conjugacy classes of $SUT(2,2^t)$ are given in* Table 4.20.

PROOF. It is straightforward to see that for any $s \in \{1,2,\cdots,q-2\}$ we get a conjugacy class $\mathcal{T}^{(3)}_{s,-s}$ such that $|\mathcal{T}^{(3)}_{s,-s}| = q$. Also we can verify that the conjugacy class of $SL(2,2^t)$ represented by $\mathcal{T}^{(2)}_{0}$ where $|\mathcal{T}^{(2)}_{0}| = q^2-1$ is a split class in $SUT(2,2^t)$. If we let $\mathcal{T}^{(2)}_{01} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be a a class of $SUT(2,2^t)$, then $|\mathcal{T}^{(2)}_{01}| = q-1$ and $\mathcal{T}^{(2)}_{01}$ fuse to $\mathcal{T}^{(2)}_{0}$ in $SL(2,2^t)$. Adding elements of classes $\mathcal{T}^{(1)}_{0}$, $\mathcal{T}^{(2)}_{01}$, $\mathcal{T}^{(3)}_{1,-1}$, $\mathcal{T}^{(3)}_{2,-2}, \cdots, \mathcal{T}^{(3)}_{q-2,-(q-2)}$ we get

$$1 + (q-1) + q(q-2) = q + q(q-2) = q(q-1) = |SUT(2,q)|.$$

Hence the classes given in Table 4.20 are all the conjugacy classes of $SUT(2,q)$. ■

Table 4.20: The conjugacy classes of $SUT(2,q)$

	$\mathcal{T}^{(1)}_{0}$	$\mathcal{T}^{(2)}_{0}$	$\mathcal{T}^{(3)}_{s,-s}$
Rep g	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \varepsilon^s & 0 \\ 0 & \varepsilon^{-s} \end{pmatrix}$
No. of CC	1	1	$q-2$
$\lvert C_{SUT(2,2^t)}(g)\rvert$	q^2-q	q	$q-1$
$\lvert C_g\rvert$	1	$q-1$	q

where ε is a generator of $\mathbb{F}_q^*$ and $s = 1,2,\cdots,q-2$.

Irreducible Characters of $SUT(2,2^t)$

As there are q conjugacy classes of $SUT(2,q)$, we seek q irreducible characters.

We recall that by Proposition 4.5.3 the derived subgroup $SUT(2,q)'$ for odd q is K. We can prove similarly that $SUT(2,2^t)' = K$. In terms of the description of the irreducible characters of Frobenius groups, there are $q-1$ linear characters of $SUT(2,q)$ coming from those characters of H through the quotient map and therefore are all linear characters of $SUT(2, q)$. Let $t_{01}^{(2)} \in \mathcal{T}_{01}^{(2)}$ and $t_{s,-s}^{(3)} \in \mathcal{T}_{s,-s}^{(3)}$. The values of linear characters χ_k, $k = 1, \cdots, q-1$ on classes of $SUT(2, q)$ are given by

$$\chi_k(\mathcal{T}_0^{(1)}) = \chi_k(t_{01}^{(2)}) = 1 \quad \text{and} \quad \chi_k(t_{s,-s}^{(3)}) = e^{\frac{2\pi sk}{q-1}i}, \; 1 \leq s \leq q-2. \tag{4.24}$$

Since $|SUT(2, q)| = q(q-1)$, it turns out that the last character χ_q is of degree $q-1$. At this stage we can use elementary properties of the character tables (like the orthogonality relations) to produce the values of χ_q on classes of $SUT(2, q)$. We do not go this way since the purpose here is to use the Frobeniusity of $SUT(2, q)$. Therefore we use the nontrivial characters of $K \cong \mathbb{F}_q \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{t \text{ times}}$ to obtain χ_q. The character table of K is given by $\bigotimes_{t \text{ times}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Let $\psi \in Irr(K) \setminus \{\mathbf{1}\}$. Then

- $\psi\uparrow_K^{SUT(2,q)}(\mathcal{T}_0^{(1)}) = \frac{|C_{SUT(2,q)}(\mathcal{T}_0^{(1)})|}{|C_K(\mathcal{T}_0^{(1)})|}\psi(\mathcal{T}_0^{(1)}) = \frac{q(q-1)}{q} \cdot 1 = q-1,$
- $\psi\uparrow_K^{SUT(2,q)}(t_{01}^{(2)}) = \frac{|C_{SUT(2,q)}(t_{01}^{(2)})|}{|C_K(t_{01}^{(2)})|}\psi(t_{01}^{(2)}) = \frac{q}{q} \cdot -1 = -1,$
- $\psi\uparrow_K^{SUT(2,q)}(t_{s,-s}^{(3)}) = \frac{|C_{SUT(2,q)}(t_{s,-s}^{(3)})|}{|C_K(t_{s,-s}^{(3)})|}\psi(t_{s,-s}^{(3)}) = 0, \; s = 1, 2, \cdots, q-2.$

Now let $\chi_q = \psi\uparrow_K^{SUT(2,q)}$. Then χ_q reads the following values

$$\chi_q(\mathcal{T}_0^{(1)}) = q-1, \qquad \chi_q(t_{01}^{(2)}) = -1 \quad \text{and} \quad \chi_q(t_{s,-s}^{(3)}) = 0. \tag{4.25}$$

The complete character table of $SUT(2, 2^t)$ is shown in Table 4.21.

Table 4.21: The character table of $SUT(2, q)$

	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_{01}^{(2)}$	$\mathcal{T}_{s,-s}^{(3)}$
Rep g	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \varepsilon^s & 0 \\ 0 & \varepsilon^{-s} \end{pmatrix}$
No. of CC	1	1	$q-2$
$\lvert C_{SUT(2,2^t)}(g)\rvert$	q^2-q	q	$q-1$
$\lvert C_g\rvert$	1	$q-1$	q
χ_k	1	1	$e^{\frac{2\pi isk}{q-1}}$
χ_q	$q-1$	-1	0

where ε is a generator of $\mathbb{F}_q^*$, $s = 1, 2, \cdots, q-2$ and $k = 1, 2, \cdots, q-1$.

II: Character Table of $SUT(2, q)$ From Clifford-Fischer Theory

We recall that a *holomorph* of a group G, $Holo(G)$, is the group extension of G by its automorphism group $Aut(G)$; that is $Holo(G) = G{:}Aut(G)$. In her Masters dissertation, [76], Whitley determined the character tables of the holomorph of $\mathbb{Z}_p$, which is $\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$. In her work, she used the theory of Clifford-Fischer matrices together with the method of the *coset analysis*, a method for computing the conjugacy classes of group extension which was first described and used by Moori [52]. The two theories of the coset analysis and Clifford-Fischer matrices are out of the scope of this dissertation. However, we will use these theories to obtain the conjugacy classes and character table of $SUT(2, q)$. For a complete description of these theories we refer to either Moori [52], [53], Mpono [55], Rodrigues [63], or Whitley [76].

Proposition 4.5.10. $Holo(\mathbb{F}_q) \cong \mathbb{F}_q{:}\mathbb{F}_q^*$.

PROOF. We prove that $Aut(\mathbb{F}_q) = \mathbb{F}_q^*$. This is immediate since $Aut(V(n, q)) \cong GL(n, q)$. In particular, $Aut(V(1, q)) = Aut(\mathbb{F}_q) \cong GL(1, q) \cong \mathbb{F}_q^*$. ■

We start by describing the conjugacy classes of $SUT(2, q) \cong \mathbb{F}_q{:}\mathbb{F}_q^* \cong K{:}H$ using the coset analysis method (Moori [52]). To be consistent with the notation of Whitley [76], let us denote by $\overline{G}$, N and G, the groups $SUT(2, q)$, $\mathbb{F}_q$ and $\mathbb{F}_q^*$ respectively. Thus $\overline{G} = N{:}G$. Note that this extension is split and $\overline{G} = \bigcup_{g \in G} Ng$. Hence $\overline{G}$ is a union of $q - 1$ distinct cosets. Using the fact that N is elementary abelian group, then it was shown (Moori [52]) that

$$|C_{\overline{G}}(x)| = \frac{k|C_G(g)|}{f}, \tag{4.26}$$

where k, f and x are defined below.

(i) $k = |C_N(g)|$, the size of the stabilizer of an arbitrary element $g \in G$ in the action of G on N. Since N is abelian group, k represents the number of the orbits $Q_1, Q_2, \cdots, Q_k$ of the action of N on Ng.

(ii) f represents the number of the orbits Q_j, $j \geq 1$ that fuse to form an orbit Ω, when we act $C_G(g)$ on $Ng = \bigcup_{j=1}^{k} Q_j$.

(iii) x is an arbitrary element of Ω, which will be a representative of a class of $\overline{G}$.

Now, let us consider the cosets Ng for $g \in G$.

1. $g = 1_G$: The identity element 1_G fixes all elements of N, so $k = q$. Then under the action of $C_G(g) = G$, we have two orbits with $f = 1$ and $f = q - 1$. Hence the first coset $N1_G = N$

gives two conjugacy classes of $\overline{G}$. The first class is the identity class of $\overline{G}$ corresponding to $f=1$. For $f=q-1$, we have a class $\mathcal{T}_{01}^{(2)}$ of $\overline{G}$ containing the element $t_{01}=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ of order p, the characteristic of $\mathbb{F}_q$. We have $|C_{\overline{G}}(t_0)|=\frac{q(q-1)}{q-1}=q$.

2. $g\neq 1_G$: There are $q-2$ non-identity elements $g\in G$ and consequently $q-2$ distinct cosets Ng. A typical element g can be regarded as the element $\mathcal{T}_{s,-s}^{(3)}$ defined in the proof of Theorem 4.5.9, where $1\leq s\leq q-2$. This g fixes only the zero element of N as follows. Let $g=\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}$, $a\neq 1$ and $n=\begin{pmatrix}1&b\\0&1\end{pmatrix}$, $b\in\mathbb{F}_q$. Then

$$gng^{-1}=\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}\begin{pmatrix}1&b\\0&1\end{pmatrix}\begin{pmatrix}a^{-1}&0\\0&a\end{pmatrix}=\begin{pmatrix}1&a^2b\\0&1\end{pmatrix}=\begin{pmatrix}1&b\\0&1\end{pmatrix}\Longleftrightarrow b=0.$$

Thus $k=1$ and consequently $f=1$. This means that each $Ng,\ g\neq 1_G$, produce only one class in $\overline{G}$. Hence $|C_{\overline{G}}(x)|=\frac{1\times(q-1)}{1}=q-1$, so that $C_{\overline{G}}(x)=G$.

The conjugacy classes of $\overline{G}$ are given by the following table.

Table 4.22: The conjugacy classes of $\overline{G}=SUT(2,q)$

Class of G	1_G		$\mathcal{T}_{s,-s}^{(3)}$, $1\leq s\leq q-2$
Class of $\overline{G}$	$1_{\overline{G}}$	$\mathcal{T}_{01}^{(2)}$	$\mathcal{T}_{s,-s}^{(3)}$, $1\leq s\leq q-2$
$\lvert C_{\overline{G}}(x)\rvert$	q^2-q	q	$q-1$
$\lvert[x]\rvert$	1	$q-1$	q

Now, we determine the Fischer matrices. Since G acts transitively on the non-zero elements of N, G has two orbits on N and hence two orbits on $Irr(N)$ by a theorem of Brauer (for example see Lemma 4.2.1 of Whitley [76]). These orbits must have lengths 1 and $q-1$. The inertia groups are $\overline{H}_1=\overline{G}$ and $\overline{H}_2=N$. Let H_1 and H_2 be $\overline{H}_1/N$ and $\overline{H}_2/N$ respectively. Then $H_1=G$ and $H_2=\{1_G\}$.

We have used Theorem 4.2.5 of Whitley [76] to calculate the Fischer matrices and we have:

Corresponding to the identity 1_G of G, the Fischer matrix is

$$M(1_G)=\begin{pmatrix}1&1\\q-1&-1\end{pmatrix}.$$

The characters values in the $\overline{G}-$block at $\overline{G}-$classes 1_G and $\mathcal{T}_{01}^{(2)}$ are:

$$\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 1\\\vdots & \vdots\\1 & 1\end{pmatrix},$$

and the character values in the $\overline{H}_2-$block at 1_G and $\mathcal{T}_{01}^{(2)}$ are:

$$\begin{pmatrix}1\end{pmatrix}\begin{pmatrix}q-1 & -1\end{pmatrix} = \begin{pmatrix}q-1 & -1\end{pmatrix}.$$

For $g \neq 1_G$, since 1_G does not fuse to any non-identity class of G, then the Fischer matrix is

$$M(g) = \begin{pmatrix}1\end{pmatrix}.$$

The $\overline{G}-$block consists of the character table of G, while the $\overline{H}_2-$block consists of zeros.

The complete character table of $\overline{G}$ is shown in Table 4.23.

Table 4.23: The character table of $\overline{G} = SUT(2,q)$

Class	$1_{\overline{G}}$	$\mathcal{T}_{01}^{(2)}$	C_1	C_2	$\cdots$	C_{q-2}
$\lvert C_{\overline{G}}(x)\rvert$	q^2-q	q	$q-1$	$q-1$	$\cdots$	$q-1$
$\lvert C_x^{\overline{G}}\rvert$	1	$q-1$	q	q	$\cdots$	q
χ_1	1	1				
χ_2	1	1			X	
$\vdots$	$\vdots$	$\vdots$				
χ_{q-1}	1	1				
χ_q	$q-1$	-1	0	0	$\cdots$	0

where, in Table 4.23,

- $C_1, C_2, \cdots, C_{q-2}$ are the non-identity classes of $G = \mathbb{F}_q^*$,
- X denotes the values of the character table on the nonidentity classes of the group G.

If we look well at the character table of $\overline{G} = SUT(2,q)$, we can see clearly that this table coincides with the one we obtained using the technique for Frobenius groups. Of course, this is natural, since the character table of an arbitrary finite group G is unique. Also if we look at Table 5.14 of Whitley [76], we see that our table for the holomorph of $\mathbb{F}_q^*$ is similar to the holomorph of $\mathbb{Z}_p$ given in Whitley, only p is replaced by q. Thus our table is a generalization of Whitley's table.

We conclude this section of the character table of the group $SUT(2,q)$ by making a connection to the work has been done by Muktibodh [57]. He defined a new notion in group thoery by saying that a group G is called a *Con-Cos* group if it has a proper normal subgroup N and $\forall x \in G\setminus\{1_G\}$, the coset Nx forms a conjugacy class of G. Moreover, a group G is called a *2-Con-Cos* group if it is a Con-Cos group and N splits into exactly two conjugacy classes in G. Furthermore, he classified all the 2-Con-Cos groups.

4.6. Character Table of $UT(2,q)$

By computations similar to ones performed in the previous sections, we can easily get the conjugacy classes of $UT(2,q)$. These are listed, together with some irreducible characters of $UT(2,q)$ in Table 4.4 appeared in the proof of Theorem 4.3.1. In Table 4.24, we list the complete character table of $UT(2,q)$.

Theorem 4.6.1. *The character table of $UT(2,q)$ is shown in* Table 4.24*.*

PROOF. In equation (4.12), we produced $(q-1)(q-2)$ irreducible characters $\chi_k\chi_l,\ k\neq l$ of $UT(2,q)$, for which the values are listed in Table 4.4. Let us denote $\chi_k\chi_l$ by $\chi^{(2)}_{k,l}$. On the other hand if $l=k$, then we get $q-1$ irreducible characters of $UT(2,q)$ which we call $\chi^{(1)}_k$. Note that these characters are the powers of the determinants of elements of $UT(2,q)$. From Table 4.2, we have seen that $\chi^{(4)}_{q-1}\in Irr(GL(2,q))$. We try $\chi^{(4)}_{q-1}\downarrow^{GL(2,q)}_{UT(2,q)}$, which we denote by χ. Its values on classes of $UT(2,q)$ are given by

$$\chi(t_1)=q-1, \qquad \chi(t_2)=-1 \quad \textit{and} \quad \chi(t_{3_k})=0,$$

where t_1, t_2, t_3 are elements of the classes $\mathcal{T}^{(1)}_s$, $\mathcal{T}^{(2)}_s$, $\mathcal{T}^{(3)}_{s,t}$ respectively.

Now

$$\begin{aligned}\langle\chi,\chi\rangle=\frac{1}{|UT(2,q)|}\sum_{g\in UT(2,q)}\chi(g)\overline{\chi}(g) &= \frac{1}{q(q-1)^2}\left((q-1)(q-1)^2+(q-1)(q-1)\right)\\ &= \frac{1}{q(q-1)^2}(q-1)^2\left((q-1)+1\right)=1.\end{aligned}$$

Thus $\chi\in Irr(UT(2,q))$. By tensoring χ with the $q-1$ linear characters $\chi^{(1)}_k$, we get $q-1$ irreducible characters $\chi\chi^{(1)}_k$ of $UT(2,q)$ of degrees $q-1$. We rename $\chi\chi^{(1)}_k$ to $\chi^{(3)}_k$. Hence there are q^2-q irreducible characters, which is the same number of conjugacy classes of $UT(2,q)$. This finishes the character table of $UT(2,q)$. ■

Table 4.24: The character table of $UT(2,q)$

Class	$\mathcal{T}_s^{(1)}$	$\mathcal{T}_s^{(2)}$	$\mathcal{T}_{s,t}^{(3)}$
Rep g	$\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$
No. of CC	$q-1$	$q-1$	$(q-1)(q-2)$
$\lvert C_g\rvert$	1	$q-1$	q
$\lvert C_{UT(2,q)}(g)\rvert$	$q(q-1)^2$	$q(q-1)$	$(q-1)^2$
$\chi_k^{(1)}$	$\widehat{\alpha}^{2k}$	$\widehat{\alpha}^{2k}$	$\widehat{\alpha}^k\widehat{\beta}^k$
$\chi_{k,l}^{(2)}$	$\widehat{\alpha}^{k+l}$	$\widehat{\alpha}^{k+l}$	$\widehat{\alpha}^k\widehat{\beta}^l$
$\chi_k^{(3)}$	$(q-1)\widehat{\alpha}^k$	$-\widehat{\alpha}^k$	0

where the notations are as in Table 4.2 except for the characters $\chi_k^{(3)}$, where we have $k = 0, 1, \cdots, q-2$.

4.7. Examples

In this section, $\mathcal{T}_j^{(i)} : \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ means that $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is a representative of the conjugacy class $\mathcal{T}_j^{(i)}$.

4.7.1 $GL(2,3)$

$$\begin{aligned}
\mathbb{F}_{q^2} &= \mathbb{F}_9 = \{0, 1, \theta, \theta^2, \cdots, \theta^7\},\\
\mathbb{F}_{q^2}^* &= \mathbb{F}_9^* = \{1, \theta, \theta^2, \cdots, \theta^7\} = \langle\theta\rangle,\ \theta^8 = 1,\\
\mathbb{F}_q &= \mathbb{F}_3 = \{0, 1, \theta^4\},\\
\mathbb{F}_q^* &= \mathbb{F}_3^* = \{1, \theta^4\} = \langle\theta^4\rangle \cong \mathbb{Z}_2.
\end{aligned}$$

The group $GL(2,3)$ has order $q(q-1)^2(q+1) = 48$ and $q^2-1 = 8$ conjugacy classes. By Theorem 4.3.1, these classes lie in four types as follows:

classes of type $\mathcal{T}^{(1)}$ are $\quad \mathcal{T}_0^{(1)} : \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \mathcal{T}_1^{(1)} : \begin{pmatrix}\theta^4 & 0\\ 0 & \theta^4\end{pmatrix},$

classes of type $\mathcal{T}^{(2)}$ are $\quad \mathcal{T}_0^{(2)} : \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad \mathcal{T}_1^{(2)} : \begin{pmatrix}\theta^4 & 1\\ 0 & \theta^4\end{pmatrix},$

classes of type $\mathcal{T}^{(3)}$ are $\quad \mathcal{T}_{0,1}^{(3)} : \begin{pmatrix}1 & 0\\ 0 & \theta^4\end{pmatrix}.$

For the last type of classes, we have:

$$\mathbb{F}_9 \setminus \mathbb{F}_3 = \{\theta, \theta^2, \theta^3, \theta^5, \theta^6, \theta^7\}$$

These 6 elements of $\mathbb{F}_9 \setminus \mathbb{F}_3$ are partitioned into three sets each contains θ^j and θ^{3j}. Thus

$$\mathbb{F}_9 \setminus \mathbb{F}_3 = \{\theta, \theta^3\} \cup \{\theta^2, \theta^6\} \cup \{\theta^5, \theta^7\}.$$

We take θ, θ^2 and θ^5 to form the three remaining conjugacy classes of the family $\mathcal{T}^{(4)}$. Hence classes of type $\mathcal{T}^{(4)}$ are $\mathcal{T}_1^{(4)} : \begin{pmatrix} 0 & 1 \\ -\theta^4 & \theta+\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\quad \mathcal{T}_2^{(4)} : \begin{pmatrix} 0 & 1 \\ -1 & \theta^2+\theta^6 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$$\mathcal{T}_5^{(4)} : \begin{pmatrix} 0 & 1 \\ -\theta^4 & \theta^5+\theta^7 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \theta^4 \end{pmatrix}.$$

Table 4.25: Basic information of the conjugacy classes of $GL(2,3)$

Class	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_1^{(1)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_1^{(2)}$	$\mathcal{T}_{0,1}^{(3)}$	$\mathcal{T}_1^{(4)}$	$\mathcal{T}_2^{(4)}$	$\mathcal{T}_5^{(4)}$
$o(g)$	1	2	3	6	2	8	4	8
$\lvert C_g\rvert$	1	1	8	8	12	6	6	6
$\lvert C_{GL(2,3)}(g)\rvert$	48	48	6	6	4	8	8	8

Table 4.26: The power maps of $GL(2,3)$

$p\lvert o(g)$	2	3	$p\lvert o(g)$	2	3	$p\lvert o(g)$	2	3	$p\lvert o(g)$	2	3
$\mathcal{T}_0^{(1)}$	-	-	$\mathcal{T}_0^{(2)}$	-	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_{0,1}^{(3)}$	$\mathcal{T}_0^{(1)}$	-	$\mathcal{T}_2^{(4)}$	$\mathcal{T}_1^{(1)}$	-
$\mathcal{T}_1^{(1)}$	$\mathcal{T}_0^{(1)}$	-	$\mathcal{T}_1^{(2)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_1^{(1)}$	$\mathcal{T}_1^{(4)}$	$\mathcal{T}_2^{(4)}$	-	$\mathcal{T}_5^{(4)}$	$\mathcal{T}_2^{(4)}$	-

Since there are 8 conjugacy classes of $GL(2,3)$, there are 8 irreducible characters. These characters fall into four types $\chi_k^{(1)}$, $\chi_k^{(2)}$, $\chi_{k,l}^{(3)}$ and $\chi_k^{(4)}$ described as follows:

$\chi_k^{(1)}$: There are $q-1=2$ linear characters $\chi_0^{(1)}$ and $\chi_1^{(1)}$.
$\chi_k^{(2)}$: There are $q-1=2$ irreducible characters $\chi_0^{(2)}$ and $\chi_1^{(2)}$ each of degree $q=3$.
$\chi_{k,l}^{(3)}$: There are $\frac{(q-1)(q-2)}{2}=1$ irreducible character $\chi_{0,1}^{(3)}$ of degree 4.
$\chi_k^{(4)}$: There are $\frac{q^2-q}{2}=3$ irreducible characters $\chi_1^{(4)}$, $\chi_2^{(4)}$ and $\chi_5^{(4)}$ each of degree $q-1=2$.

We explain how to calculate the values of the irreducible characters on the conjugacy classes of $GL(2,3)$. We do this for $\chi_1^{(2)}$ and $\chi_5^{(4)}$ and the other characters follow similarly. Let the hat function $\widehat{}$ be the isomorphism from $\mathbb{F}_3^*$ to the group of second roots of unity in the complex numbers, that is $\widehat{\theta^4} = e^{\frac{2\pi}{2}i} = e^{\pi i} = -1$. Let $t_0^{(2)}$, $t_1^{(2)}$, $t_{0,1}^{(3)}$, $t_1^{(4)}$, $t_2^{(4)}$, $t_5^{(4)}$ be elements in the classes $\mathcal{T}_0^{(2)}$, $\mathcal{T}_1^{(2)}$, $\mathcal{T}_{0,1}^{(3)}$, $\mathcal{T}_1^{(4)}$, $\mathcal{T}_2^{(4)}$, $\mathcal{T}_5^{(4)}$ respectively. From Table 4.2 we obtain

$$\begin{array}{llll}
\chi_1^{(2)}(\mathcal{T}_0^{(1)}) = & q\widehat{\theta^4}^2 = 3\widehat{1}^2 = 3, & \chi_1^{(2)}(\mathcal{T}_1^{(1)}) = & q\widehat{\theta^4}^2 = 3(-1)^2 = 3, \\
\chi_1^{(2)}(t_0^{(2)}) = & 0, & \chi_1^{(2)}(t_1^{(2)}) = & 0, \\
\chi_1^{(2)}(t_{0,1}^{(3)}) = & \widehat{1}\widehat{\theta^4} = -1, & \chi_1^{(2)}(t_1^{(4)}) = & -\widehat{r}^{k(q+1)} = -\widehat{\theta^4} = 1, \\
\chi_1^{(2)}(t_2^{(4)}) = & -\widehat{r}^{k(q+1)} = -\widehat{\theta^4}^2 = -1, & \chi_1^{(2)}(t_5^{(4)}) = & -\widehat{r}^{k(q+1)} = -\widehat{\theta^4}^5 = 1.
\end{array}$$

For the character $\chi_5^{(4)}$, let the hat function $\widehat{}$ be the isomorphism from $\mathbb{F}_9^*$ to the group of $8th$ roots of unity in the complex numbers, that is $\widehat{\theta} = e^{\frac{2\pi}{8}i} = e^{\frac{\pi}{4}i} = \frac{1+i}{\sqrt{2}}$. Then

$$\begin{array}{ll}
\chi_5^{(4)}(\mathcal{T}_0^{(1)}) = & (q-1)\widehat{\theta^4}^k = 2\widehat{1}^5 = 2(1)^5 = 2, \\
\chi_5^{(4)}(\mathcal{T}_1^{(1)}) = & (q-1)\widehat{\theta^4}^5 = 2(-1)^5 = -2, \\
\chi_5^{(4)}(t_0^{(2)}) = & -\widehat{1}^5 = -(1)^5 = -1, \\
\chi_5^{(4)}(t_1^{(2)}) = & -\widehat{\theta^4}^5 = -(-1)^5 = 1, \\
\chi_5^{(4)}(t_{0,1}^{(3)}) = & 0, \\
\chi_5^{(4)}(t_1^{(4)}) = & -(\widehat{r}^k + \widehat{r}^{kq}) = -(\widehat{\theta}^5 + \widehat{\theta}^{15}) = -(e^{\frac{5\pi i}{4}} + e^{\frac{7\pi i}{4}}) = \frac{1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}} = i\sqrt{2}, \\
\chi_5^{(4)}(t_2^{(4)}) = & -(\widehat{r}^k + \widehat{r}^{kq}) = -(\widehat{\theta^2}^5 + \widehat{\theta^2}^{15}) = -(e^{\frac{10\pi i}{4}} + e^{\frac{30\pi i}{4}}) = -(-1+1) = 0, \\
\chi_5^{(4)}(t_5^{(4)}) = & -(\widehat{r}^k + \widehat{r}^{kq}) = -(\widehat{\theta^5}^5 + \widehat{\theta^5}^{15}) = -(e^{\frac{25\pi i}{4}} + e^{\frac{75\pi i}{4}}) = -(\frac{1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}}) = -i\sqrt{2}.
\end{array}$$

The complete character table of the group $GL(2,3)$ is listed in Table 4.27. This table can be obtained from GAP [23] through the command

```
gap> Display(CharacaterTable(GL(2,3)));
```

A table of correspondence between our table obtained manually through the theory and the one obtained by using GAP is also given.

Table 4.27: The character table of $GL(2,3)$

Class	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_1^{(1)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_1^{(2)}$	$\mathcal{T}_{0,1}^{(3)}$	$\mathcal{T}_1^{(4)}$	$\mathcal{T}_2^{(4)}$	$\mathcal{T}_5^{(4)}$
$o(g)$	1	2	3	6	2	8	4	8
$\lvert C_g\rvert$	1	1	8	8	12	6	6	6
$\lvert C_{GL(2,3)}(g)\rvert$	48	48	6	6	4	8	8	8
$\chi_0^{(1)}$	1	1	1	1	1	1	1	1
$\chi_1^{(1)}$	1	1	1	1	-1	-1	1	-1
$\chi_0^{(2)}$	3	3	0	0	1	-1	-1	-1
$\chi_1^{(2)}$	3	3	0	0	-1	1	-1	1
$\chi_{0,1}^{(3)}$	4	-4	1	-1	0	0	0	0
$\chi_1^{(4)}$	2	-2	-1	1	0	$-i\sqrt{2}$	0	$i\sqrt{2}$
$\chi_2^{(4)}$	2	2	-1	-1	0	0	2	0
$\chi_5^{(4)}$	2	-2	-1	1	0	$i\sqrt{2}$	0	$-i\sqrt{2}$

Table 4.28: Correspondence of conjugacy classes of $GL(2,3)$ in our notation and GAP notation

Our Notation	GAP Notation	Our Notation	GAP Notation	Our Notation	GAP Notation	Our Notation	GAP Notation
$\mathcal{T}_0^{(1)}$	$1a$	$\mathcal{T}_0^{(2)}$	$3a$	$\mathcal{T}_{0,1}^{(3)}$	$2b$	$\mathcal{T}_2^{(4)}$	$4a$
$\mathcal{T}_1^{(1)}$	$2a$	$\mathcal{T}_1^{(2)}$	$6a$	$\mathcal{T}_1^{(4)}$	$8a$	$\mathcal{T}_5^{(4)}$	$8b$

Table 4.29: Correspondence of irreducible characters of $GL(2,3)$ in our notation and GAP notation

Our Notation	GAP Notation	Our Notation	GAP Notation	Our Notation	GAP Notation	Our Notation	GAP Notation
$\chi_0^{(1)}$	χ_1	$\chi_0^{(2)}$	χ_6	$\chi_{0,1}^{(3)}$	χ_8	$\chi_2^{(4)}$	χ_3
$\chi_1^{(1)}$	χ_2	$\chi_1^{(2)}$	χ_7	$\chi_1^{(4)}$	χ_4	$\chi_5^{(4)}$	χ_5

4.7.2 $GL(2,4)$

Let $\mathbb{F}_4^* \cong \mathbb{Z}_3$ and $\mathbb{F}_{16}^* \cong \mathbb{Z}_{15}$ be generated by α and θ respectively. Then a set of representatives of the conjugacy classes of $GL(2,4)$ can be given as follows.

$$\mathcal{T}_0^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T}_1^{(1)} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad \mathcal{T}_2^{(1)} = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix},$$

$$\mathcal{T}_0^{(2)} : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T}_1^{(2)} : \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \quad \mathcal{T}_2^{(2)} : \begin{pmatrix} \alpha^2 & 1 \\ 0 & \alpha^2 \end{pmatrix},$$

$$\mathcal{T}_0^{(3)} : \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad \mathcal{T}_1^{(3)} : \begin{pmatrix} 1 & 0 \\ 0 & \alpha^2 \end{pmatrix}, \quad \mathcal{T}_2^{(3)} : \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{pmatrix},$$

$$\begin{aligned}
\mathcal{T}_1^{(4)} &: \begin{pmatrix} 0 & 1 \\ \theta^5 & \theta+\theta^4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha & 1 \end{pmatrix}, &\quad \mathcal{T}_2^{(4)} &: \begin{pmatrix} 0 & 1 \\ \theta^{10} & \theta^2+\theta^8 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 1 \end{pmatrix}, \\
\mathcal{T}_3^{(4)} &: \begin{pmatrix} 0 & 1 \\ 1 & \theta^3+\theta^{12} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \alpha^2 \end{pmatrix}, &\quad \mathcal{T}_6^{(4)} &: \begin{pmatrix} 0 & 1 \\ 1 & \theta^6+\theta^9 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix}, \\
\mathcal{T}_7^{(4)} &: \begin{pmatrix} 0 & 1 \\ \theta^5 & \theta^7+\theta^{13} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha & \alpha \end{pmatrix}, &\quad \mathcal{T}_{11}^{(4)} &: \begin{pmatrix} 0 & 1 \\ \theta^{10} & \theta^{11}+\theta^{14} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & \alpha^2 \end{pmatrix}.
\end{aligned}$$

Table 4.30 gives the basic information about these representatives.

Table 4.30: Basic information of the conjugacy classes of $GL(2,4)$

Class	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_1^{(1)}$	$\mathcal{T}_2^{(1)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_1^{(2)}$	$\mathcal{T}_2^{(2)}$	$\mathcal{T}_0^{(3)}$	$\mathcal{T}_1^{(3)}$	$\mathcal{T}_2^{(3)}$
$o(g)$	1	3	3	2	6	6	3	3	3
$\lvert C_g\rvert$	1	1	1	15	15	15	20	20	20
$\lvert C_{GL(2,4)}(g)\rvert$	180	180	180	12	12	12	9	9	9

Table 4.30 (continued)

Class	$\mathcal{T}_1^{(4)}$	$\mathcal{T}_2^{(4)}$	$\mathcal{T}_3^{(4)}$	$\mathcal{T}_6^{(4)}$	$\mathcal{T}_7^{(4)}$	$\mathcal{T}_{11}^{(4)}$
$o(g)$	15	15	5	5	15	15
$\lvert C_g\rvert$	12	12	12	12	12	12
$\lvert C_{GL(2,4)}(g)\rvert$	15	15	15	15	15	15

The power maps of theses conjugacy classes are given in Table 4.31.

Table 4.31: The power maps of $GL(2,4)$

$p\|o(g)$	2	3	5	$p\|o(g)$	2	3	5	$p\|o(g)$	2	3	5
$\mathcal{T}_0^{(1)}$	-	-	-	$\mathcal{T}_2^{(2)}$	$\mathcal{T}_1^{(1)}$	$\mathcal{T}_0^{(2)}$	-	$\mathcal{T}_2^{(4)}$	-	$\mathcal{T}_6^{(4)}$	$\mathcal{T}_1^{(2)}$
$\mathcal{T}_1^{(1)}$	-	$\mathcal{T}_0^{(1)}$	-	$\mathcal{T}_0^{(3)}$	-	$\mathcal{T}_0^{(1)}$	-	$\mathcal{T}_3^{(4)}$	-	-	$\mathcal{T}_0^{(1)}$
$\mathcal{T}_2^{(1)}$	-	$\mathcal{T}_0^{(1)}$	-	$\mathcal{T}_1^{(3)}$	-	$\mathcal{T}_0^{(1)}$	-	$\mathcal{T}_6^{(4)}$	-	-	$\mathcal{T}_0^{(1)}$
$\mathcal{T}_0^{(2)}$	$\mathcal{T}_0^{(1)}$	-	-	$\mathcal{T}_2^{(3)}$	-	$\mathcal{T}_0^{(1)}$	-	$\mathcal{T}_7^{(4)}$	-	$\mathcal{T}_6^{(4)}$	$\mathcal{T}_2^{(1)}$
$\mathcal{T}_1^{(2)}$	$\mathcal{T}_2^{(1)}$	$\mathcal{T}_0^{(2)}$	-	$\mathcal{T}_1^{(4)}$	-	$\mathcal{T}_3^{(4)}$	$\mathcal{T}_2^{(1)}$	$\mathcal{T}_{11}^{(4)}$	-	$\mathcal{T}_3^{(4)}$	$\mathcal{T}_1^{(2)}$

Let $A = e^{\frac{2i\pi}{3}} = -\frac{1}{2}+\frac{\sqrt{3}i}{2}$, $E = -(e^{\frac{2i\pi}{5}}+e^{\frac{8i\pi}{5}})$, $E^* = 2-e^2 = -(e^{\frac{4i\pi}{5}}+e^{\frac{6i\pi}{5}})$, $F = -(e^{\frac{4i\pi}{15}}+e^{\frac{16i\pi}{15}})$ and $G = -(e^{\frac{22i\pi}{15}}+e^{\frac{28i\pi}{15}})$. The character table of $GL(2,4)$ is shown in Table 4.32.

Table 4.32: The character table of $GL(2,4)$

	Class	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_1^{(1)}$	$\mathcal{T}_2^{(1)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_1^{(2)}$	$\mathcal{T}_2^{(2)}$	$\mathcal{T}_0^{(3)}$	$\mathcal{T}_1^{(3)}$	$\mathcal{T}_2^{(3)}$
	$o(g)$	1	3	3	2	6	6	3	3	3
	$\|C_g\|$	1	1	1	15	15	15	20	20	20
	$\|C_{GL(2,4)}(g)\|$	180	180	180	12	12	12	9	9	9
χ_1	$\chi_0^{(1)}$	1	1	1	1	1	1	1	1	1
χ_2	$\chi_1^{(1)}$	1	A	$\overline{A}$	1	A	$\overline{A}$	$\overline{A}$	A	1
χ_3	$\chi_2^{(1)}$	1	$\overline{A}$	A	1	$\overline{A}$	A	A	$\overline{A}$	1
χ_4	$\chi_0^{(2)}$	4	4	4	0	0	0	1	1	1
χ_5	$\chi_1^{(2)}$	4	$4A$	$4\overline{A}$	0	0	0	$\overline{A}$	A	1
χ_6	$\chi_2^{(2)}$	4	$4\overline{A}$	$4A$	0	0	0	A	$\overline{A}$	1
χ_7	$\chi_{0,1}^{(3)}$	5	$5\overline{A}$	$5A$	1	$\overline{A}$	A	$-A$	$-\overline{A}$	-1
χ_8	$\chi_{0,2}^{(3)}$	5	$5A$	$5\overline{A}$	1	A	$\overline{A}$	$-\overline{A}$	$-A$	-1
χ_9	$\chi_{1,2}^{(3)}$	5	5	5	1	1	1	-1	-1	-1
χ_{10}	$\chi_1^{(4)}$	3	$3\overline{A}$	$3A$	-1	$-\overline{A}$	$-A$	0	0	0
χ_{11}	$\chi_2^{(4)}$	3	$3A$	$3\overline{A}$	-1	$-A$	$-\overline{A}$	0	0	0
χ_{12}	$\chi_3^{(4)}$	3	3	3	-1	-1	-1	0	0	0
χ_{13}	$\chi_6^{(4)}$	3	3	3	-1	-1	-1	0	0	0
χ_{14}	$\chi_7^{(4)}$	3	$3\overline{A}$	$3A$	-1	$-\overline{A}$	$-A$	0	0	0
χ_{15}	$\chi_{11}^{(4)}$	3	$3A$	$3\overline{A}$	-1	$-A$	$-\overline{A}$	0	0	0

Table 4.32 (continued)

	Class	$\mathcal{T}_1^{(4)}$	$\mathcal{T}_2^{(4)}$	$\mathcal{T}_3^{(4)}$	$\mathcal{T}_6^{(4)}$	$\mathcal{T}_7^{(4)}$	$\mathcal{T}_{11}^{(4)}$
	$o(g)$	15	15	5	5	15	15
	$\|C_g\|$	12	12	12	12	12	12
	$\|C_{GL(2,4)}(g)\|$	15	15	15	15	15	15
χ_1	$\chi_0^{(1)}$	1	1	1	1	1	1
χ_2	$\chi_1^{(1)}$	A	$\overline{A}$	1	1	A	$\overline{A}$
χ_3	$\chi_2^{(1)}$	$\overline{A}$	A	1	1	$\overline{A}$	A
χ_4	$\chi_0^{(2)}$	-1	-1	-1	-1	-1	-1
χ_5	$\chi_1^{(2)}$	$-A$	$-\overline{A}$	-1	-1	$-A$	$-\overline{A}$
χ_6	$\chi_2^{(2)}$	$-\overline{A}$	$-A$	-1	-1	$-\overline{A}$	$-A$
χ_7	$\chi_{0,1}^{(3)}$	0	0	0	0	0	0
χ_8	$\chi_{0,2}^{(3)}$	0	0	0	0	0	0
χ_9	$\chi_{1,2}^{(3)}$	0	0	0	0	0	0
χ_{10}	$\chi_1^{(4)}$	$\overline{G}$	F	E	E^*	$\overline{F}$	G
χ_{11}	$\chi_2^{(4)}$	F	$\overline{G}$	E^*	E	G	$\overline{F}$
χ_{12}	$\chi_3^{(4)}$	E	E^*	E^*	E	E^*	E
χ_{13}	$\chi_6^{(4)}$	E^*	E	E	E^*	E	E^*
χ_{14}	$\chi_7^{(4)}$	$\overline{F}$	G	E^*	E	$\overline{G}$	F
χ_{15}	$\chi_{11}^{(4)}$	G	$\overline{F}$	E	E^*	F	$\overline{G}$

Table 4.33: Correspondence of conjugacy classes of $GL(2,4)$ in our notation and GAP notation

Our Notation	GAP Notation	Our Notation	GAP Notation	Our Notation	GAP Notation
$\mathcal{T}_0^{(1)}$	$1a$	$\mathcal{T}_2^{(2)}$	$6b$	$\mathcal{T}_2^{(4)}$	$15c$
$\mathcal{T}_1^{(1)}$	$3a$	$\mathcal{T}_0^{(3)}$	$3c$	$\mathcal{T}_3^{(4)}$	$5b$
$\mathcal{T}_2^{(1)}$	$3b$	$\mathcal{T}_1^{(3)}$	$3d$	$\mathcal{T}_6^{(4)}$	$5a$
$\mathcal{T}_0^{(2)}$	$2a$	$\mathcal{T}_2^{(3)}$	$3e$	$\mathcal{T}_7^{(4)}$	$15b$
$\mathcal{T}_1^{(2)}$	$6a$	$\mathcal{T}_1^{(4)}$	$15a$	$\mathcal{T}_{11}^{(4)}$	$15d$

Table 4.34: Correspondence of irreducible characters of $GL(2,4)$ in our notation and GAP notation

Our Notation	GAP Notation	Our Notation	GAP Notation	Our Notation	GAP Notation
χ_1	χ_1	χ_6	χ_{10}	χ_{11}	χ_6
χ_2	χ_2	χ_7	χ_{15}	χ_{12}	χ_5
χ_3	χ_3	χ_8	χ_{14}	χ_{13}	χ_9
χ_4	χ_{13}	χ_9	χ_{11}	χ_{14}	χ_7
χ_5	χ_{12}	χ_{10}	χ_4	χ_{15}	χ_8

4.7.3 $SL(2,3)$

The group $SL(2,3)$ has order $q(q-1)(q+1) = 24$ and according to Table 4.15 it has $q+4 = 7$ distinct conjugacy classes described as follows

$$\mathcal{T}_0^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad -\mathcal{T}_0^{(1)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{T}_{01}^{(2)} : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad -\mathcal{T}_{01}^{(2)} : \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix},$$

$$\mathcal{T}_{0\varepsilon}^{(2)} : \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}, \quad -\mathcal{T}_{0\varepsilon}^{(2)} : \begin{pmatrix} -1 & -\varepsilon \\ 0 & -1 \end{pmatrix}, \quad \mathcal{T}_1^{(4)} : \begin{pmatrix} 0 & 1 \\ -1 & \theta^2+\theta^6 \end{pmatrix}.$$

The orders and size of conjugacy classes and centralizers of the above representatives are given in Table 4.35.

Table 4.35: The conjugacy classes of $SL(2,3)$

Class	$\mathcal{T}_0^{(1)}$	$-\mathcal{T}_0^{(1)}$	$\mathcal{T}_0^{(2)}$	$-\mathcal{T}_0^{(2)}$	$\mathcal{T}_1^{(2)}$	$-\mathcal{T}_1^{(2)}$	$\mathcal{T}_1^{(4)}$
$o(g)$	1	2	3	6	3	6	4
$\lvert C_{SL(2,3)}(g)\rvert$	24	24	6	6	6	6	4
$\lvert C_g\rvert$	1	1	4	4	4	4	6

Now in terms of Table 4.15, the complete character table of $SL(2,3)$ is given in Table 4.36.

Table 4.36: The character table of $SL(2,3)$

Class	$\mathcal{T}_0^{(1)}$	$-\mathcal{T}_0^{(1)}$	$\mathcal{T}_{01}^{(2)}$	$-\mathcal{T}_{01}^{(2)}$	$\mathcal{T}_{0\varepsilon}^{(2)}$	$-\mathcal{T}_{0\varepsilon}^{(2)}$	$\mathcal{T}_1^{(4)}$
$o(g)$	1	2	3	6	3	6	4
$\lvert C_{SL(2,3)}(g)\rvert$	24	24	6	6	6	6	4
$\lvert C_g\rvert$	1	1	4	4	4	4	6
λ	1	1	1	1	1	1	1
ψ	3	3	0	0	0	0	-1
π_1	2	-2	-1	1	-1	1	0
$\widetilde{\xi}_1$	2	-2	$\frac{1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$	$\frac{1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	0
$\widetilde{\xi}_2$	2	-2	$\frac{1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	$\frac{1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$	0
$\widetilde{\vartheta}_1$	1	1	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$	1
$\widetilde{\vartheta}_2$	1	1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	1

4.7.4 $SL(2,4)$

This group has order 60 and has $q+1=5$ conjugacy classes described as follows:

$$\mathcal{T}_0^{(1)} = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \qquad \mathcal{T}_0^{(2)} : \begin{pmatrix}1&1\\0&1\end{pmatrix}, \qquad \mathcal{T}_{1,-1}^{(3)} : \begin{pmatrix}\theta^5&0\\0&\theta^{-5}\end{pmatrix},$$
$$\mathcal{T}_3^{(4)} : \begin{pmatrix}0&1\\1&\theta^3+\theta^{12}\end{pmatrix}, \qquad \mathcal{T}_6^{(4)} : \begin{pmatrix}0&1\\1&\theta^6+\theta^9\end{pmatrix},$$

where $\mathbb{F}_{16}^* = \langle\theta\rangle$.

Now in terms of Table 4.17, the character table of $SL(2,4)$ is given by Table 4.37.

From the the character table of $SL(2,q)$, we can see that this group is simple $(\chi_i(t_k^{(j)}) \neq \chi_i(\mathcal{T}_0^{(1)}))$, for all $2 \leq i \leq 5,\ 2 \leq j \leq 4,\ k \in \{0,3,6\}$, where $t_k^{(j)}$ denotes an element in the class $\mathcal{T}_k^{(j)})$. Since any simple group of order 60 is isomorphic to A_5 (see for example Rotman [65]), then $SL(2,4) \cong A_5$ (we have mentioned this in Example 3.1.3).

Table 4.37: The character table of $SL(2,4)$

Class	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_0^{(3)}$	$\mathcal{T}_3^{(4)}$	$\mathcal{T}_6^{(4)}$
$\lvert C_g\rvert$	1	15	20	12	12
$\lvert C_G(g)\rvert$	60	4	3	5	5
$o(g)$	1	2	3	5	5
χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	5	1	-1	0	0
χ_4	3	-1	0	E	$\overline{E}$
χ_5	3	-1	0	$\overline{E}$	E

where $E = -(e^{\frac{2i\pi}{5}} + e^{\frac{8i\pi}{5}})$.

4.7.5 $SUT(2,3)$

The group $SUT(2,3)$ has order $q(q-1) = 6$ and according to Table 4.19, has $q+3 = 6$ distinct conjugacy classes described as follows

$$\mathcal{T}_0^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad -\mathcal{T}_0^{(1)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{T}_0^{(2)} : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad -\mathcal{T}_0^{(2)} : \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix},$$
$$\mathcal{T}_1^{(2)} : \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}, \quad -\mathcal{T}_1^{(2)} : \begin{pmatrix} -1 & -\varepsilon \\ 0 & -1 \end{pmatrix}$$

where ε is a generator of $\mathbb{F}_3^* \cong \mathbb{Z}_2$. Since $|SUT(2,3)| = 6$, it follows that $SUT(2,3) \cong S_3$ or $SUT(2,3) \cong \mathbb{Z}_6$. We have 6 conjugacy classes, which is $|SUT(2,3)|$. Thus $SUT(2,3) \cong \mathbb{Z}_6$. The character table of $\mathbb{Z}_6$ is given by Theorem 2.2.4. However the idea here is to use the character table of $SUT(2,q)$, which is given by Table 4.19. We show the character table of $SUT(2,3)$ in Table 4.38.

4.7.6 $SUT(2,4)$

Let α be a generator of $\mathbb{F}_4^* \cong \mathbb{Z}_3$. Then the classes of $SUT(2,4)$ are

$$\mathcal{T}_0^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T}_0^{(2)} : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T}_1^{(3)} : \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \mathcal{T}_2^{(3)} : \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}.$$

Table 4.38: The character table of $SUT(2,3)$

Rep g	$\mathcal{T}_0^{(1)}$	$-\mathcal{T}_0^{(1)}$	$\mathcal{T}_{01}^{(2)}$	$-\mathcal{T}_{01}^{(2)}$	$\mathcal{T}_{0\varepsilon}^{(2)}$	$-\mathcal{T}_{0\varepsilon}^{(2)}$
$\lvert C_{SUT(2,3)}(g)\rvert$	6	6	6	6	6	6
$\lvert C_g\rvert$	1	1	1	1	1	1
$\lvert o(g)\rvert$	1	2	3	6	3	6
χ_0	1	1	1	1	1	1
χ_1	1	-1	1	-1	1	-1
ϑ_1	1	1	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$
ϑ_2	1	1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$
$\chi_1\vartheta_1$	1	-1	$\frac{-1+i\sqrt{3}}{2}$	$\frac{1-i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$	$\frac{1+i\sqrt{3}}{2}$
$\chi_1\vartheta_2$	1	-1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{1+i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$	$\frac{1-i\sqrt{3}}{2}$

Table 4.39: The character table of $SUT(2,4)$

	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_1^{(3)}$	$\mathcal{T}_2^{(3)}$
$\lvert C_g\rvert$	1	3	4	4
$\lvert C_{SUT(2,4)}(g)\rvert$	12	4	3	3
$o(g)$	1	2	3	3
χ_1	1	1	1	1
χ_2	1	1	A	$\overline{A}$
χ_3	1	1	$\overline{A}$	A
χ_4	3	-1	0	0

where $A = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Corollary 4.7.1. *$SUT(2,4) \cong A_4$.*

PROOF. By Theorem 2.2.13 of Moori [54] we know that there are three non-abelian groups (up to isomorphism) of order 12. Two of these groups have subgroups of order 6 while the other group A_4 does not have this property. It can be easily checked that $SUT(2,4)$ is a non-abelian group. If $\exists H \leq SUT(2,4)$ such that $|H| = 6$, then $H \trianglelefteq SUT(2,q)$. This implies that H is a union of conjugacy classes of $SUT(2,4)$. If we look at the attached information to the character table of $SUT(2,4)$, concerning the conjugacy classes of $SUT(2,4)$, we can see clearly that there is no combination of classes including the identity class that give a subgroup of order 6. Therefore, we deduce that

$SUT(2, 4) \cong A_4$. ■

4.7.7 $UT(2, 3)$

With $\mathbb{F}_3^* = \mathbb{Z}_2 = \langle \alpha \rangle$, then the conjugacy classes of $UT(2, 3)$ are

$$\mathcal{T}_0^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T}_1^{(1)} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

$$\mathcal{T}_0^{(2)} : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T}_1^{(2)} : \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix},$$

$$\mathcal{T}_1^{(3)} : \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad \mathcal{T}_2^{(3)} : \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

Since all elements of $UT(2, 3)$ are of prime orders, we do not need to give a table for the power maps. The complete character table of $UT(2, 3)$ is as follows.

Table 4.40: The character table of $UT(2, 3)$

Class	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_1^{(1)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_1^{(2)}$	$\mathcal{T}_1^{(3)}$	$\mathcal{T}_2^{(3)}$
$o(g)$	1	2	3	6	2	2
$\lvert C_g \rvert$	1	1	2	2	3	3
$\lvert C_{UT(2,3)}(g) \rvert$	12	12	6	6	4	4
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	−1	−1
χ_3	1	−1	1	−1	−1	1
χ_4	1	−1	1	−1	1	−1
χ_5	2	2	−1	−1	0	0
χ_6	2	−2	−1	1	0	0

Remark 4.7.1. It is unfortunate that the library of GAP [23] does not contain the group $UT(n, q)$ like the cases $GL(n, q)$, $SL(n, q)$, $PSL(n, q), \cdots$, etc, which are known there. We have written a small subroutine to construct the character table of $UT(2, 3)$, which is attached to the Appendix.

We can see clearly that the character table of $UT(2, 3)$ coincides with the character table of the Dihedral group D_{12}. We have the following Corollary.

Corollary 4.7.2. $UT(2,3) \cong D_{12}$.

PROOF. Let $a = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$. Then $a^6 = b^2 = I_2 = 1_{UT(2,3)}$. Also $a^{-1} = a^5 = \begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix}$. Now we have

$$ab = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix} = ba^5.$$

Since $|\langle a\rangle| = 6 = \frac{1}{2}|UT(2,3)|$, then $\langle a\rangle \trianglelefteq UT(2,3)$ and therefore $\langle a\rangle\langle b\rangle \leq UT(2,3)$. Now it is easy to check that $\langle a\rangle \cap \langle b\rangle = \{I_2\}$. Therefore $|\langle a\rangle\langle b\rangle| = |\langle a\rangle||\langle b\rangle| = 12$. Hence $UT(2,3) = \langle a\rangle\langle b\rangle$. We deduce that

$$UT(2,3) = \left\langle a, b \mid a^6 = b^2 = I_2,\ ab = ba^5 = ba^{-1}\right\rangle.$$

Hence $UT(2,3) \cong D_{12}$. ∎

4.7.8 $UT(2,4)$

With $\mathbb{F}_4^* = \mathbb{Z}_3 = \langle \alpha \rangle$, then the conjugacy classes of $UT(2,4)$ are

$$\mathcal{T}_0^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T}_1^{(1)} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad \mathcal{T}_2^{(1)} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$

$$\mathcal{T}_0^{(2)} : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T}_1^{(2)} : \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \quad \mathcal{T}_2^{(2)} : \begin{pmatrix} \alpha^{-1} & 1 \\ 0 & \alpha^{-1} \end{pmatrix},$$

$$\mathcal{T}_1^{(3)} : \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad \mathcal{T}_2^{(3)} : \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T}_3^{(3)} : \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$

$$\mathcal{T}_4^{(3)} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T}_5^{(3)} : \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \mathcal{T}_6^{(3)} : \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}.$$

Suppose that $A = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. The character table of the group $UT(2,4)$ is given by Table 4.41.

Table 4.41: The character table of $UT(2,4)$

	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_1^{(1)}$	$\mathcal{T}_2^{(1)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_1^{(2)}$	$\mathcal{T}_2^{(2)}$	$\mathcal{T}_1^{(3)}$	$\mathcal{T}_2^{(3)}$	$\mathcal{T}_3^{(3)}$	$\mathcal{T}_4^{(3)}$	$\mathcal{T}_5^{(3)}$	$\mathcal{T}_6^{(3)}$
$\lvert C_g \rvert$	1	1	1	3	3	3	4	4	4	4	4	4
$\lvert C_{UT(2,4)}(g) \rvert$	36	36	36	12	12	12	9	9	9	9	9	9
$o(g)$	1	3	3	2	6	6	3	3	3	3	3	3
$\chi_0^{(1)}$	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_1^{(1)}$	1	$\overline{A}$	A	1	$\overline{A}$	A	A	A	$\overline{A}$	$\overline{A}$	1	1
$\chi_2^{(1)}$	1	A	$\overline{A}$	1	A	$\overline{A}$	$\overline{A}$	$\overline{A}$	A	A	1	1
$\chi_{0,1}^{(2)}$	1	A	$\overline{A}$	1	A	$\overline{A}$	A	1	$\overline{A}$	1	$\overline{A}$	A
$\chi_{1,0}^{(2)}$	1	A	$\overline{A}$	1	A	$\overline{A}$	1	A	1	$\overline{A}$	A	$\overline{A}$
$\chi_{0,2}^{(2)}$	1	$\overline{A}$	A	1	$\overline{A}$	A	$\overline{A}$	1	A	1	A	$\overline{A}$
$\chi_{2,0}^{(2)}$	1	$\overline{A}$	A	1	$\overline{A}$	A	1	$\overline{A}$	1	A	$\overline{A}$	A
$\chi_{1,2}^{(2)}$	1	1	1	1	1	1	$\overline{A}$	A	A	$\overline{A}$	$\overline{A}$	A
$\chi_{2,1}^{(2)}$	1	1	1	1	1	1	A	A	$\overline{A}$	A	A	$\overline{A}$
$\chi_0^{(3)}$	3	3	3	-1	-1	-1	0	0	0	0	0	0
$\chi_1^{(3)}$	3	$3A$	$3\overline{A}$	-1	$-A$	$-\overline{A}$	0	0	0	0	0	0
$\chi_2^{(3)}$	3	$3\overline{A}$	$3A$	-1	$-\overline{A}$	$-A$	0	0	0	0	0	0

5

$\mathbb{T}$he $\mathbb{C}$haracter $\mathbb{T}$able of $GL(n,q)$

In this chapter we study the conjugacy classes of $GL(n,q)$ in general. This is achieved by giving a source for the representatives of the classes (Theorem 5.2.1) and then looking at Green's formula (equation (5.9)) for calculating the size of the centralizers of the representatives. The theories of irreducible polynomials and partitions of $i \in \{1, 2, \cdots, n\}$ form the atoms from which each conjugacy classes is build up. We give a special attention to some elements of $GL(n,q)$, known as *regular semisimple* and we calculate the number and the orders of these elements. Also we count the number of the *primary classes* of $GL(n,q)$. As an example we compute the conjugacy classes of $GL(3,q)$.

We have seen in the last chapter that a large number of irreducible characters of $GL(2,q)$ (characters of types $\chi^{(1)}$, $\chi^{(2)}$ and $\chi^{(3)}$) were obtained by considering those characters of $P_{(1,1)} = UT(2,q)$, which are obtained through lifting the characters of the quotient $UT(2,q)/SUUT(2,q) \cong T = \mathbb{F}_q^* \times \mathbb{F}_q^* \cong GL(1,q) \times GL(1,q) \cong L_{(1,1)}$ the levi complement of the parabolic subgroup $P_{(1,1)}$. The idea in this chapter is to use the irreducible characters of levi complements $L_\lambda = \bigotimes_{i=1}^{k} GL(\lambda_i, q)$ of parabolic subgroups P_λ, where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$, to construct characters of $GL(n,q)$. Any character of $GL(n,q)$ obtained in this way is referred to as, a *principal series* character and the process of obtaining the characters is referred to as, a *parabolic induction.* This process produces a large number of irreducible characters of $GL(n,q)$ from characters of $GL(m,q)$ for $m < n$. Any character of $GL(n,q)$ which can not obtained parabolically is referred to as a *discrete series* or a *cuspidal* character. The most important fact is that cuspidal characters of $GL(n,q)$ exist. In other words parabolic induction does not produce all $Irr(GL(n,q))$ for any n. Furthermore, the cuspidal characters are of great importance for characters of $GL(n,q)$ (Theorem 5.4.6) since each character of $GL(n,q)$ is build up from cuspidal characters. However Green, who constructed all the irreducible characters of $GL(n,q)$ in his great paper [27], did not start with the cuspidal characters. Instead he took as his building blocks some generalized characters that are *lifts* of modular characters. We mention Green construction of characters in Theorem 5.6.2. There are some certain characters of $GL(n,q)$ that have been found by Steinberg [72] and they bear his name. These characters are discussed in Section 5.5. Although we do not attempt to describe fully the set $Irr(GL(n, q))$, but we are able to find all irreducible characters of $GL(3, q)$. This has been done in Section 5.7. Green [27] showed that there exists a complete duality between the conjugacy classes and irreducible characters of $GL(n, q)$. That is to any conjugacy class one can associate an irreducible character, a property that not many groups have. Some aspects of this duality will be shown at the end of this chapter in Table 5.14.

5.1. Partitions

In this short section we go briefly over partitions of a positive integer n and some functions defined in terms of partitions, which will be used through the sequel of this chapter. A whole chapter in MacDonald [50] is devoted to study the theory of partitions. One can also refer to Goldschmidt [25].

Definition 5.1.1. *A **partition** $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ of a positive integer n is a decreasing (consequently increasing) sequence of nonnegative integers λ_i, whose sum is n, i.e. $\sum_{i=1}^{m} \lambda_i = n$.*

We will be using

- $\lambda \vdash n$ to denote that λ is a partition of n,
- $\{1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} \cdots n^{\alpha_n}\}$ to denote also partition of n, where i^{α_i} means that the positive integers i appears α_i times,
- $|\lambda|$ means the positive integer for which λ is a partition for and finally
- $\mathcal{P}(n)$ is the set of all partitions of n.

Definition 5.1.2. *Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m) \vdash n$. The integers λ_i are called the **parts** of λ, while m is called the **length** of λ, which sometimes written $l(\lambda)$.*

Any partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ written in descending order have a geometrical diagram known as a *Ferrers diagram*. This diagram is defined to be the set of points $(i, j) \in \mathbb{Z}^2$, such that $1 \leq j \leq \lambda_i$. To sketch the diagram, we let i (indexes a row) increases as going from top to bottom, while j (indexes a column) increases as going from left to right. For example the Ferrers diagram of $(4, 2, 2, 1) \vdash 9$ is shown in Figure 5.1.

The *conjugate* partition λ' of λ is the partition obtained by transposing the diagram of λ. For example, $(4, 2, 2, 1)' = (4, 3, 1, 1)$. The Ferrers diagram of $(4, 2, 2, 1)$ and $(4, 3, 1, 1)$ are shown in Figure 5.1.

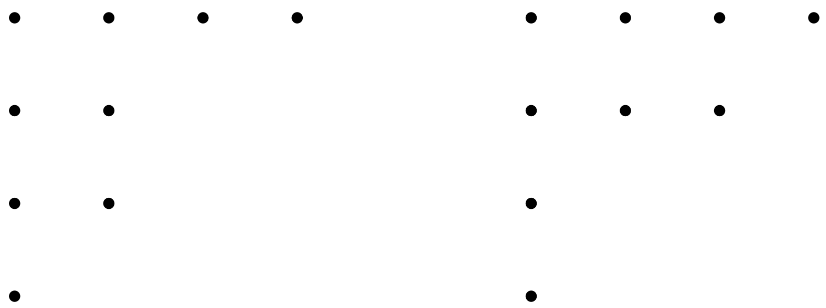

Figure 5.1: Ferrers diagrams of $(4, 2, 2, 1) \vdash 9$ (left) and $(4, 2, 2, 1)'$ (right)

To any partition λ, we attach an integer $n(\lambda)$, which depends essentially on λ'. This integer is given by

$$n(\lambda) = \sum_{i=1}^{l(\lambda')} \binom{\lambda'_i}{2}, \tag{5.1}$$

where λ'_i, $i \geq 1$ are the parts of λ'. For example consider $(1, 1, \cdots, 1) \vdash n$. Then $(1, 1, \cdots, 1)' = (n)$. Therefore, $n((1, 1, \cdots, 1)) = \binom{n}{2} = \frac{n(n-1)}{2}$. From another side if $\lambda = (n) \vdash n$, then $(n)' = (1, 1, \cdots, 1)$. It follows that $n((n)) = 0$ since $\binom{1}{2} = 0$. The integer $n(\lambda)$ is of great importance for Green's formula for $|C_{GL(n,q)}(g)|$ for $g \in G$. We have calculated the values of $n(\lambda)$, $\lambda \vdash n$ for $n = 1, 2, 3, 4, 5$ which are listed in Table 6.1.

For any $m \in \mathbb{N} \cup \{0\}$ we define ϕ_m by

$$\phi_m(t) = \begin{cases} \prod_{i=1}^{m}(1 - t^i) & \text{if } m \geq 1, \\ 1 & \text{if } m = 0. \end{cases} \tag{5.2}$$

If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$ with $\lambda_i \geq \lambda_{i+1}$, $\forall i$, we let m_{λ_i} be the multiplicity of the part λ_i in λ. We define $\phi_\lambda(q)$ by

$$\phi_\lambda(q) = \prod_{i=i}^{k} \phi_{m_{\lambda_i}}(q). \tag{5.3}$$

We conclude this section by defining an ordering on $\mathcal{P}(n)$. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_l) \vdash n$ be distinct partitions both written in descending order. We say that $\lambda < \mu$ if $k < l$ and if $k = l$, then $\lambda < \mu$ if the first non-vanishing difference $\lambda_i - \mu_i$ is positive. For example elements of $\mathcal{P}(5)$ can be ordered as follows

$$(5) < (4, 1) < (3, 2) < (3, 1, 1) < (2, 2, 1) < (2, 1, 1, 1) < (1, 1, 1, 1, 1).$$

5.2. Conjugacy Classes of $GL(n, q)$

This section is divided into four subsections, where in the first one we give representatives of the conjugacy classes of $GL(n, q)$. In the next subsection we calculate the size of the centralizers and hence the size of conjugacy classes. The third subsection is devoted to some results concerning the *regular semisimple* elements and *primary classes* of $GL(n, q)$, while in the last subsection we give an example for conjugacy classes of $GL(3, q)$ and we interpret how we got the representatives of classes of $GL(2, q)$, which were studied in the previous chapter.

5.2.1 Representatives of Conjugacy Classes of $GL(n, q)$

Construction of the conjugacy classes of $GL(n, q)$ depends essentially on the theories of irreducible polynomials and partitions.

Definition 5.2.1 (Companion matrix)**.** *Let* $f(t) = \sum_{i=0}^{d} a_i t^i \in \mathbb{F}_q[t],\ a_d = 1$. *The* $d \times d$ ***companion matrix*** $U(f) = U_1(f)$ *of* $f(t)$ *is defined to be*

$$U_1(f) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{d-1} \end{pmatrix}.$$

For any $m \in \mathbb{N}$, the *Jordan block* $U_m(f)$ is the $md \times md$ matrix

$$U_m(f) = \begin{pmatrix} U_1(f) & I_d & \underline{0} & \cdots & \underline{0} \\ \underline{0} & U_1(f) & I_d & \cdots & \underline{0} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & I_d \\ \underline{0} & \underline{0} & \underline{0} & \cdots & U_1(f) \end{pmatrix}.$$

Moreover, if $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$, then $U_\lambda(f)$ is defined to be

$$U_\lambda(f) = \begin{pmatrix} U_{\lambda_1}(f) & \underline{0} & \cdots & \underline{0} \\ \underline{0} & U_{\lambda_2}(f) & \cdots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \cdots & U_{\lambda_k}(f) \end{pmatrix} = \bigoplus_{i=1}^{k} U_{\lambda_i}(f).$$

The direct sum of Jordan blocks is called the *Jordan Canonical Form.*

Next we set $\mathcal{F}$ to be

$$\mathcal{F} = \{f \in \mathbb{F}_q[t] | \ \partial f \leq n, \ f \text{ is irreducible over } \mathbb{F}_q, \ f(t) \neq t\}. \tag{5.4}$$

The next theorem produces representatives of conjugacy classes of $GL(n,q)$.

Theorem 5.2.1 (The Jordan Canonical Form). *Let $A \in GL(n,q)$ with characteristic polynomial $f_A = f_1^{z_1} f_2^{z_2} \cdots f_k^{z_k}$, where $f_i \in \mathcal{F}$, $1 \leq i \leq k$ and z_i is the multiplicity of f_i in this decomposition. Then A is conjugate to a matrix of the form $\bigoplus_{i=1}^{k} U_{\nu_i}(f_i)$, where $\nu_i \vdash z_i$.*

PROOF. See Rotman [65]. ■

From the above theorem we deduce that a conjugacy class of $GL(n,q)$ is determined by a sequence $\{f_i\}_{i=1}^k$ such that $f_i \in \mathcal{F}$ and $\partial f_i = d_i$, $\forall i$, together with a sequence of partitions $\{\nu_i\}_{i=1}^k$, where $\nu_i \vdash z_i$, $\forall i$ and $\{z_i\}_{i=1}^k$ is a sequence of positive integers such that

$$\sum_{i=1}^{k} z_i d_i = n.$$

Therefore any conjugacy class c of $GL(n,q)$ is defined by the data $(\{f_i\}, \{d_i\}, \{z_i\}, \{\nu_i\})$. The integer k is called the *length* of the data.

Two data $(\{f_i\}, \{d_i\}, \{z_i\}, \{\nu_i\})$ and $(\{g_i\}, \{e_i\}, \{w_i\}, \{\mu_i\})$ with lengths k and k' respectively parameterize the same conjugacy class if $k = k'$ and there exists $\sigma \in S_k$ such that

$$w_i = z_{\sigma(i)}, \ e_i = d_{\sigma(i)}, \ \mu_i = \nu_{\sigma(i)} \quad \text{and} \quad g_i = f_{\sigma(i)}, \quad \forall i.$$

Hence A is conjugate uniquely (up to ordering of Jordan blocks) to a Jordan Canonical Form.

On the other hand, two conjugacy classes of $GL(n,q)$ parameterized by the above data are said to be of the same *type* if $k = k'$ and there exists $\sigma \in S_k$ such that

$$w_i = z_{\sigma(i)}, \ e_i = d_{\sigma(i)} \quad \text{and} \quad \mu_i = \nu_{\sigma(i)} \quad (g_i \text{ and } f_i \text{ are allowed to differ}). \tag{5.5}$$

Therefor the conjugacy classes of $GL(n,q)$ are distributed into types.

Remark 5.2.1. The work inspired by Green [27] showed that the values of an irreducible character of $GL(n,q)$ on classes of the same type can be expressed by a single functional formula.

Example 5.2.1. All central classes of $GL(n,q)$ are of the same type.

Green [27] showed that the number of types $t(n)$ of conjugacy classes of $GL(n,q)$ is independent of q. The next Theorem gives the number of conjugacy classes $c(n,q)$ of $GL(n,q)$ and $t(n)$.

Theorem 5.2.2. *1. The integer $t(n)$ is given by the coefficient of x^n in the expansion of the series $\prod_{i=1}^{\infty} \mathcal{P}(x^i)^{|\mathcal{P}(n)|}$, where*

$$\begin{aligned} \mathcal{P}(x) = \sum_{n=0}^{\infty} |\mathcal{P}(n)| x^n &= \prod_{i=1}^{\infty} \frac{1}{1-x^i} \\ &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots . \end{aligned}$$

2. The integer $c(n,q)$ is given by the generating function

$$\sum_{n=0}^{\infty} c(n,q)x^n = \prod_{m=1}^{\infty} \mathcal{P}(x^m)^{I_m(q)},$$

where $I_m(q) = \frac{1}{m}\sum_{s|m} \mu(s) q^{\frac{m}{s}}$ and $\mu(s)$ is the Möbuis function.

PROOF. See Green [27]. ■

Note 5.2.1. The function $I_m(q)$ represents the number of irreducible polynomials of degree m over $\mathbb{F}_q$ and this result is due to Gauss.

If $q \leq n$, we do not have a class of type

$$(\{t-\alpha_1, t-\alpha_2, \cdots, t-\alpha_n\}, \underbrace{\{1,1,\cdots,1\}}_{n \text{ times}}, \underbrace{\{1,1,\cdots,1\}}_{n \text{ times}}, \underbrace{\{1,1,\cdots,1\}}_{n \text{ times}}),$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{F}_q^*$ are all distinct since $|\mathbb{F}_q^*| = q-1 < n$. Note that a typical class of the above type is represented by

$$\begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix}.$$

Therefore it is not necessarily that all types of classes appear

In Table 5.1 we give $t(n)$ and $c(n,q)$ of $GL(n,q)$ for $n = 1, 2, \cdots, 7$.

Table 5.1: Number of types of classes and number of classes of $GL(n,q)$

n	$t(n)$	$c(n,q)$
1	1	$q-1$
2	4	q^2-1
3	8	q^3-q
4	22	q^4-q
5	42	$q^5-(q^2+q-1)$
6	103	q^6-q^2
7	199	$q^7-(q^3+q^2-1)$

Definition 5.2.2. *Let c be a conjugacy class given by $(\{f_i\},\{d_i\},\{z_i\},\{\nu_i\})$ with length k, then*

1. *c is called **primary class** if and only if $k=1$.*
2. *c is called **regular class** if and only if $l(\nu_i)\leq 1,\ \forall\ 1\leq i\leq k$.*
3. *c is called **semisimple class** if and only if $l(\nu_i^{'})\leq 1,\ \forall\ 1\leq i\leq k$.*
4. *c is called **regular semisimple class** if it is both regular and semisimple. Alternatively a class is regular semisimple if and only if $\nu_i=1,\ \forall\ 1\leq i\leq k$.*

Note 5.2.2.

1. The definition of c being a primary class implies that for $g\in c$, the characteristic polynomial of g is $f(t)=(t^d+a_{d-1}t^{d-1}+\cdots+1)^s$ for some s and hence $\partial f=d$ and $d|n$. In particular if $f(t)=(t-1)^n$, then we call c a *unipotent*. Note that we have defined in Definition 3.1.5 the unipotency of an element $A\in GL(n,q)$ and of a subgroup $H\leq GL(n,q)$.
2. The definition of c being a regular semisimple class of $GL(n,q)$ implies that any element $g\in c$ has n distinct eigenvalues.

We classify all classes of $GL(2,q)$, $GL(3,q)$ and $GL(4,q)$ according to Definition 5.2.2. These classes have been given in Tables 4.1, 5.3 and 6.10 respectively.

5.2.2 Sizes of Conjugacy Classes of $GL(n,q)$

Definition 5.2.3. *The pair (V,Ω), where V is an abelian group and $\Omega\subseteq End(V)$, is called a* ***module.***

Table 5.2: Conjugacy classes of $GL(2,q)$, $GL(3,q)$ and $GL(4,q)$

n	Primary Classes	Unipotent Classes	Regular Classes	Semisimple Classes	Regular Semisimple Classes
2	$\mathcal{T}^{(1)}$, $\mathcal{T}^{(2)}$, $\mathcal{T}^{(4)}$	$\mathcal{T}^{(1)}$, $\mathcal{T}^{(2)}$, $\alpha=1$	$\mathcal{T}^{(2)}$, $\mathcal{T}^{(3)}$, $\mathcal{T}^{(4)}$	$\mathcal{T}^{(3)}$, $\mathcal{T}^{(4)}$	$\mathcal{T}^{(3)}$, $\mathcal{T}^{(4)}$
3	$\mathcal{T}^{(1)}$, $\mathcal{T}^{(2)}$, $\mathcal{T}^{(3)}$, $\mathcal{T}^{(8)}$	$\mathcal{T}^{(1)}$, $\mathcal{T}^{(2)}$, $\mathcal{T}^{(3)}$, $\alpha=1$	$\mathcal{T}^{(3)}$, $\mathcal{T}^{(4)}$, $\mathcal{T}^{(6)}$, $\mathcal{T}^{(7)}$, $\mathcal{T}^{(8)}$	$\mathcal{T}^{(5)}$, $\mathcal{T}^{(6)}$ $\mathcal{T}^{(7)}$, $\mathcal{T}^{(8)}$	$\mathcal{T}^{(6)}$, $\mathcal{T}^{(7)}$ $\mathcal{T}^{(8)}$
4	$\mathcal{T}^{(1)}$, $\mathcal{T}^{(2)}$ $\mathcal{T}^{(3)}$, $\mathcal{T}^{(4)}$ $\mathcal{T}^{(5)}$, $\mathcal{T}^{(19)}$ $\mathcal{T}^{(20)}$, $\mathcal{T}^{(22)}$	$\mathcal{T}^{(1)}$, $\mathcal{T}^{(2)}$ $\mathcal{T}^{(3)}$, $\mathcal{T}^{(4)}$ $\mathcal{T}^{(5)}$, $\alpha=1$	$\mathcal{T}^{(5)}$, $\mathcal{T}^{(8)}$ $\mathcal{T}^{(11)}$, $\mathcal{T}^{(13)}$ $\mathcal{T}^{(15)}$, $\mathcal{T}^{(16)}$ $\mathcal{T}^{(17)}$, $\mathcal{T}^{(18)}$ $\mathcal{T}^{(20)}$, $\mathcal{T}^{(21)}$ $\mathcal{T}^{(22)}$	$\mathcal{T}^{(1)}$, $\mathcal{T}^{(6)}$ $\mathcal{T}^{(9)}$, $\mathcal{T}^{(12)}$ $\mathcal{T}^{(14)}$, $\mathcal{T}^{(16)}$ $\mathcal{T}^{(17)}$, $\mathcal{T}^{(18)}$ $\mathcal{T}^{(19)}$, $\mathcal{T}^{(21)}$ $\mathcal{T}^{(22)}$	$\mathcal{T}^{(16)}$, $\mathcal{T}^{(17)}$ $\mathcal{T}^{(18)}$, $\mathcal{T}^{(21)}$ $\mathcal{T}^{(22)}$

Two modules (V,Ω) and (V',Ω') are said to be *isomorphic* or *equivalent* if and only if $V\cong V'$ and Ω and Ω' generate the same ring of endomorphisms of V.

For any $n\times n$ matrix A over $\mathbb{F}_q$ (not necessarily invertible), we define the module V_A of A to be

$$V_A=(V(n,q),R),$$

where $R=\langle A,\mathbb{F}_q\rangle$ is the ring generated by A, together with scalars from $\mathbb{F}_q$. That is

$$R=\left\{\sum_{i=0}^{k}a_iA^i|\ a_i\in\mathbb{F}_q\right\}\cong\mathbb{F}_q[t]$$

and $\mathbb{F}_q[t]$ operates on $V(n,q)$ by $t.v=Av,\ \forall v\in V(n,q)$. Note that t^j is the composition of t taken j times. Thus any $A\in\mathbb{M}_{n\times n}(\mathbb{F}_q)$ defines an $\mathbb{F}_q[t]-$module, which we denote by V_A.

Definition 5.2.4. *A function $f:V_A\longrightarrow V_B$ is said to be an $\mathbb{F}_q[t]-$**isomorphism** if it is homomorphism and bijection. The modules V_A and V_B are called $\mathbb{F}_q[t]-$**isomorphic**.*

Lemma 5.2.3. *Let V and W be vector spaces over $\mathbb{F}_q$ and let $T:V\longrightarrow V$ and $S:W\longrightarrow W$ be linear transformations that determine $\mathbb{F}_q[t]-$modules V_T and V_S respectively. A function $f:V_T\longrightarrow V_S$ is an $\mathbb{F}_q[t]-$homorphism if and only if*

1. *f is linear transformation of the vector spaces V and W,*
2. *$f(Tv)=S(f(v)),\ \forall v\in V$.*

PROOF. See Rotman [65]. ∎

Proposition 5.2.4. *Two matrices A and B are similar if and only if the corresponding $\mathbb{F}_q[t]-$modules V_A and V_B are isomorphic.*

PROOF. Let $T, S : V \longrightarrow V$ be linear transformations affording A and B respectively and also let V_T and V_S be the corresponding $\mathbb{F}_q[t]-$modules defined by T and S respectively. Suppose that A and B are similar matrices. Hence there exists $P \in GL(n, q)$ such that $B = PAP^{-1}$. If $f : V \longrightarrow V$ is the linear transformation corresponds to P, then we claim that f is an $\mathbb{F}_q[t]-$isomorphism between V_T and V_S. From Lemma 5.2.3, it suffices to show that $f(Tv) = S(f(v))$, $\forall v \in V$, i.e., $fT = Sf$. In terms of matrices, this represents $PA = BP$, which we have. Thus $V_A \cong V_B$.
Conversely, suppose that $f : V \longrightarrow V$ is an $\mathbb{F}_q[t]-$isomorphism between V_T and V_S. By Lemma 5.2.3, we have $Sf = fT$. Since f is an isomorphism, it follows that $S = fTf^{-1}$. If P is the matrix corresponding to the linear transformation f, then $B = PAP^{-1}$; that is A and B are similar matrices. This completes the proof. ∎

If $A, B \in GL(n, q)$ are in a conjugacy class c, then by Proposition 5.2.4 we have $V_A \cong V_B$. It follows that we can write V_c in place of V_A without any ambiguity. Next we review some notions from the elementary Ring Theory to learn more about the structure of V_c.

We recall that a *principal ideal domain* R is an integral domain such that all its ideals are principal ideals. That is if I is an ideal of R, then $I = \langle a \rangle = aR$ for $a \in R$. For any $v \in V$, where V is an $R-$module, the *annihilator* $Ann(v)$ is defined to be the set

$$Ann(v) = \{r \in R |\ rv = 0_V\}.$$

It is not difficult to see that $Ann(v) \underbrace{\trianglelefteq}_{\text{ideal}} R$. If R is a principal ideal domain, then

$$Ann(v) = \{ar|\ r \in R\} = aR, \text{ for some } a \in R.$$

Moreover, if p is an irreducible element of R (has no divisors in R except p and 1_R), then an $R-$module V is called a *$p-$primary* if for all $v \in V$,

$$Ann(v) = \langle p^\alpha \rangle = p^\alpha R, \text{ for some } \alpha \in \mathbb{N}.$$

Theorem 5.2.5. *Let R be a principal ideal domain and V be a finitely generated $R-$module. Then*

$$V = \bigoplus_{i=1}^{s} V_i,$$

where each V_i is cyclic submodule and isomorphic to either R or $R/p^m R$, for some irreducible element p of R. Moreover, the decomposition is unique up to the order of the factors.

PROOF. See Rotman [65]. ∎

We would like to apply the above discussion to the case $R = \mathbb{F}_q[t]$, which it can be shown that it is a principal ideal domain. Thus for the annihilator of v of the $\mathbb{F}_q[t]-$modules V_c we take the fixed element to be a monic polynomial of smallest degree in the ideal, that is if $v \in V_c$, then $Ann(v) = \langle f \rangle$, where f is a monic polynomial such that $\partial f \leq \partial g,\ \forall g \in \mathbb{F}_q[t]$. Let $f_i \in \mathcal{F}$. By $V_{\langle f_i \rangle}$, we mean the f_i-primary submodule of V_c; that is the submodule consisting of all $v \in V_c$ annihilated by some power of f_i. The submodules $V_{\langle f_1 \rangle}, V_{\langle f_2 \rangle}, \cdots, V_{\langle f_k \rangle}$ of V_c are referred as the *characteristic submodules* since $f_1, f_2, \cdots, f_k$ are the irreducible factors which appear in the characteristic polynomial of an element in the conjugacy class c. Thus giving $V_{\langle f_i \rangle}$ the name characteristic submodule becomes more appropriate.

If $c = (\{f_i\}, \{d_i\}, \{z_i\}, \{\nu_i\})$, then by Theorem 5.2.5 we have $V_c = \bigoplus_{i=1}^{k} V_{\langle f_i \rangle}$, where each $V_{\langle f_i \rangle}$ is of the form

$$V_{\langle f_i \rangle} = \bigoplus_{j=1}^{l(\nu_i)} \mathbb{F}_q[t]/ \langle f_i \rangle^{\nu_{i_j}}$$

and $\nu_i = \{\nu_{i_1}, \nu_{i_2}, \cdots, \nu_{i_{l(\nu_i)}}\}$ is the partition associated with f_i in c. Therefore

$$V_c = \bigoplus_{i=1}^{k} V_{\langle f_i \rangle} = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{l(\nu_i)} \mathbb{F}_q[t]/ \langle f_i \rangle^{\nu_{i_j}} . \tag{5.6}$$

It has been shown in Lemma 2.1 of Green [27] that if $Aut(V_c)$ is the automorphism group of V_c, then

$$Aut(V_c) = \bigotimes_{i=1}^{k} Aut(V_{\langle f_i \rangle}). \tag{5.7}$$

Now by equation (2.6) of MacDonald [50] we have

$$|Aut(V_{\langle f_i \rangle})| = q^{d_i(|\nu_i|+2n(\nu_i))} \phi_{\nu_i}(q^{-d_i}).$$

Consequently

$$|Aut(V_c)| = |\bigotimes_{i=1}^{k} Aut(V_{\langle f_i \rangle})| = \prod_{i=1}^{k} |Aut(V_{\langle f_i \rangle})| = \prod_{i=1}^{k} q^{d_i(|\nu_i|+2n(\nu_i))} \phi_{\nu_i}(q^{-d_i}). \tag{5.8}$$

The following Theorem is of great importance and is the main theorem of this subsection. It characterizes $C_{GL(n,q)}(A)$.

Theorem 5.2.6. *Let $A \in GL(n,q)$ lies in a conjugacy class c. Then $C_{GL(n,q)}(A) = Aut(V_c)$.*

PROOF. Suppose that $\sigma \in Aut(V_c)$. Then $\sigma : V_c \longrightarrow V_c$ and

$$\sigma(ru+sv) = r\sigma(u) + s\sigma(v),\ \forall r,s \in R = \langle A, \mathbb{F}_q\rangle,\ u,v \in V(n,q).$$

We know that $Aut(V(n,q)) = GL(n,q)$ and $\sigma(Au) = A\sigma(u)$ (since A is regarded as a scalar from the ring R). Thus

$$\begin{aligned}\sigma(Au) = A\sigma(u) &\Longrightarrow (\sigma A)u = (A\sigma)u,\ \forall u \in V(n,q)\\ &\Longrightarrow A\sigma = \sigma A\\ &\Longrightarrow \sigma \in C_{GL(n,q)}(A)\\ &\Longrightarrow Aut(V_c) \subseteq C_{GL(n,q)}(A).\end{aligned}$$

Conversely, if $\sigma \in C_{GL(n,q)}(A)$, then

$$\begin{aligned}A\sigma = \sigma A &\Longrightarrow (A\sigma)u = (\sigma A)u,\ \forall u \in V(n,q),\\ &\Longrightarrow \sigma(Au) = A\sigma(u)\\ &\Longrightarrow \sigma \in Aut(V_c)\\ &\Longrightarrow C_{GL(n,q)}(A) \subseteq Aut(V_c).\end{aligned}$$

Hence $C_{GL(n,q)}(A) = Aut(V_c)$. ■

Now in terms of equation (5.8) and Theorem 5.2.6 if $A \in GL(n,q)$ lies in $c = (\{f_i\},\{d_i\},\{z_i\},\{\nu_i\})$, then we deduce that

$$|C_{GL(n,q)}(A)| = \prod_{i=1}^{k} q^{d_i(|\nu_i|+2n(\nu_i))}\phi_{\nu_i}(q^{-d_i}). \tag{5.9}$$

It follows that

$$|C_A| = (\prod_{s=0}^{n-1}(q^n - q^s))/\prod_{i=1}^{k} q^{d_i(|\nu_i|+2n(\nu_i))}\phi_{\nu_i}(q^{-d_i}). \tag{5.10}$$

Sometimes we may write a_{ν_i} to denote $q^{d_i(|\nu_i|+2n(\nu_i))}\phi_{\nu_i}(q^{-d_i})$. That is $|C_{GL(n,q)}(A)| = \prod_{i=1}^{k} a_{\nu_i}$.

Corollary 5.2.7. *Two conjugacy classes of the same type have same size.*

PROOF. Suppose that $c_1 = (\{f_i\},\{d_i\},\{z_i\},\{\nu_i\})$ and $c_2 = (\{f_i^{'}\},\{d_i^{'}\},\{z_i^{'}\},\{\nu_i^{'}\})$ are two classes of the same type with length k (see Remark 5.2.2). It follows by (5.5) that there exists $\sigma \in S_k$ such that $z_i^{'} = z_{\sigma(i)}$, $d_i^{'} = d_{\sigma(i)}$ and $\nu_i^{'} = \nu_{\sigma(i)}$, $\forall 1 \le i \le k$. If $A_1 \in c_1$ and $A_2 \in c_2$, then by (5.9) we have

$$\begin{aligned}|C_{GL(n,q)}(A_2)| &= \prod_{i=1}^{k} q^{d_i^{'}(|\nu_i^{'}|+2n(\nu_i^{'}))}\phi_{\nu_i^{'}}(q^{-d_i^{'}}) = \prod_{i=1}^{k} q^{d_{\sigma(i)}(|\nu_{\sigma(i)}|+2n(\nu_{\sigma(i)}))}\phi_{\nu_{\sigma(i)}}(q^{-d_{\sigma(i)}})\\ &= \prod_{i=1}^{k} q^{d_i(|\nu_i|+2n(\nu_i))}\phi_{\nu_i}(q^{-d_i}) = |C_{GL(n,q)}(A_1)|.\end{aligned}$$

The result follows by (5.10). ■

Remark 5.2.2. Note that k is the length of the data parameterizing c. The term *length of conjugacy class* has another meaning.

5.2.3 Regular Semisimple Elements and Primary Classes of $GL(n, q)$

In this subsection we emphasize some of our results on counting the number and orders of the regular semisimple elements of $GL(n, q)$. Also we count the number of the primary classes of $GL(n, q)$. Fleischmann et al. [21] studied the number of regular semisimple classes of $G_n^\epsilon(q)$ for $n \geq 2$, where

$$G_n^\epsilon(q) = \begin{cases} GL(n, q) & \text{if } \epsilon = 1, \\ U_n(q) & \text{if } \epsilon = -1 \end{cases}$$

and $U_n(q)$ is the Unitary Group. The method given there uses the Theory of Lie Algebra and some topological notions as connectedness of groups. If $Reg(G_n^\epsilon(q))$ denotes the number of regular semisimple classes of $G_n^\epsilon(q)$, then Theorem 1.1 of [21] reads

$$Reg(G_n^\epsilon(q)) = (q - \epsilon)\frac{q^{n+1} - q^n + (-1)^{n+1}\epsilon^{[n/2]}(q - \epsilon^n)}{q^2 - \epsilon}. \tag{5.11}$$

Here we introduce a method to calculate the number of regular semisimple elements of $GL(n, q)$ using simple notions as all what we need is the elementary theory of partitions of a positive integer n. However formula (5.11) is faster in computations.

Number of Regular Semisimple Elements of $GL(n, q)$

Counting the number of regular semisimple elements of $GL(n, q)$ is achieved by

- counting the number of regular semisimple types,
- counting the number of classes contained in each of the regular semisimple type,
- counting the number of elements contained in each of the regular semisimple class.

Proposition 5.2.8. *There is a $1-1$ correspondence between the types of classes of regular semisimple elements of $GL(n, q)$ and partitions of n.*

PROOF. By the Jordan Canonical Form (Theorem 5.2.1) any class $c = (\{f_i\}, \{d_i\}, \{z_i\}, \{\nu_i\})$ of $GL(n, q)$ must satisfies $\sum_{i=1}^{k} z_i d_i = n$. We know that any regular semisimple class c_1 has the form

$c_1 = (\{f_i\}, \{d_i\}, \{1\}_{k \text{ times}}, \{1\}_{k \text{ times}})$. Therefore we have $\sum_{i=1}^{k} d_i = n$, that is $(d_1, d_2, \cdots, d_k) \vdash n$. If $c_2 = (\{f_i^{'}\}, \{d_i^{'}\}, \{1\}_{k \text{ times}}, \{1\}_{k \text{ times}})$ is any other regular semisimple class of same type of c_1, then $d_i^{'} = d_{\sigma(i)}$, for some $\sigma \in S_k$. Thus c_1 and c_2 determine the same partition. Hence any type of regular semisimple classes determines a partition of n. Conversely, any partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$ defines a type of regular semisimple classes, where a typical class c will have the form $c = (\{f_i\}, \{\lambda_i\}, \{1\}_{k \text{ times}}, \{1\}_{k \text{ times}})$, $1 \leq i \leq k$. Note that for any $\lambda_i \in \mathbb{N}$, there exists an irreducible polynomial of degree λ_i over $\mathbb{F}_q$. Hence types of regular semisimple classes are in one to one correspondence with the partitions of n as claimed. ∎

It turns out that we may denote any type of regular semisimple classes of $GL(n,q)$ by $\mathcal{T}^\lambda$ and a typical class by c^λ without any ambiguity.

To count the number of regular semisimple conjugacy classes contained in each type $\mathcal{T}^\lambda$, $\lambda \vdash n$, we put into our consideration the repetition of the parts of λ. Therefore if we let r_i be the multiplicity of the integer i, then we can write λ in the form $\lambda = 1^{r_1} 2^{r_2} \cdots n^{r_n}$, where $r_i \in \mathbb{N} \cup \{0\}$. We have the following lemma.

Lemma 5.2.9. *Let $f(t) = \sum_{i=0}^{m} a_i t^i \in \mathbb{F}_q[t]$, $a_m = 1$. If α is a root of f, then $\alpha^q, \alpha^{q^2}, \cdots, \alpha^{q^{m-1}}$ are the other roots.*

PROOF. The Galois group $\Gamma = \Gamma(\mathbb{F}_{q^m} : \mathbb{F}_q)$ is a cyclic group of order m and is generated by the Frobenius automorphism $\sigma_q : a \longmapsto a^q$, $\forall a \in \mathbb{F}_{q^m}$. We can see clearly that $\sigma_q^j = \sigma_{q^j}$. Now given that α is a root of f, then $\sum_{i=0}^{m} a_i \alpha^i = 0$. Acting by elements of Γ on both sides of the preceding equality we get, for all $j = 0, 1, \cdots, m-1$,

$$\begin{aligned} \sigma_{q^j}\left(\sum_{i=0}^{m} a_i \alpha^i\right) = \sigma_{q^j}(0) \quad &\Longleftrightarrow \quad \sum_{i=0}^{m} \sigma_{q^j}(a_i \alpha^i) = 0 \Longleftrightarrow \sum_{i=0}^{m} a_i \sigma_{q^j}(\alpha^i) = 0 \\ &\Longleftrightarrow \quad \sum_{i=0}^{m} a_i \alpha^{q^j i} = 0 \Longleftrightarrow \sum_{i=0}^{m} a_i (\alpha^{q^j})^i = 0. \end{aligned}$$

The last equality tells that α^{q^j} is a root of f whenever α is. ∎

Proposition 5.2.10. *The number of regular semisimple classes of type λ, which we denote by $F(\lambda)$, is given by*

$$F(\lambda) = \frac{\prod_{i=1}^{n} \prod_{s=0}^{r_i - 1} (I_i(q) - s)}{\prod_{i=1}^{n} r_i!},$$

where

- $I_i(q) = \frac{1}{i}\sum_{d|i} \mu(d) q^{\frac{i}{d}}$ *is the number of irreducible polynomials of degree i over $\mathbb{F}_q$,*
- *if $r_i - 1 < 0$, then the term $\prod_{s=0}^{r_i-1}(I_i(q) - s)$ is neglected.*

PROOF. Let $f(t)$ be an irreducible polynomial of degree i over $\mathbb{F}_q$. It is known by Gauss Lemma that the number of such polynomials is given by $I_i(q) = \frac{1}{i}\sum_{d|i} \mu(d) q^{\frac{i}{d}}$. Now let α_1 be a root of $f(t)$. It follows by Lemma 5.2.9 that the other roots of $f(t)$ are $\alpha_1^q, \alpha_1^{q^2}, \cdots, \alpha_1^{q^{i-1}}$. Thus if we choose α_1 as an eigenvalue of a representative of a regular semisimple class, then α_1 together with the former set of powers of α_1 form a complete set of eigenvalues of the Jordan block of size i. Since each $1 \leq i \leq n$ appears r_i times in the partition λ, it follows that we can choose α_1 in $I_i(q)$ ways, α_2 in $I_i(q) - 1$ ways, and so forth till the α_{r_i}, which we can choose in $I_i(q) - (r_i - 1)$ ways. We recall that a conjugacy class is unaltered by the arrangement of the eigenvalues in the Jordan block. Thus we divide by the number of all possible arrangements, which is $r_i!$. Repeating this for all $1 \leq i \leq n$, we get the required number mentioned in the statement of the Proposition. ∎

For any positive integer n, two partitions namely, $\underbrace{(1,1,\cdots,1)}_{n \text{ times}} \vdash n$ and $(n) \vdash n$ are of particular interest.

Corollary 5.2.11. *With $q > n$, then corresponding to the partitions $\lambda = \underbrace{(1,1,\cdots,1)}_{n \text{ times}} \vdash n$ and $\sigma = (n) \vdash n$, we have $F(\lambda) = \frac{(q-1)(q-2)\cdots(q-n)}{n!}$ and $F(\sigma) = \frac{1}{n}\sum_{d|n} \mu(d) q^{\frac{n}{d}}$.*

PROOF. Immediate from Proposition 5.2.10. ∎

Remark 5.2.3. Note that by Propositions 5.2.8 and 5.2.10 the number of regular semisimple classes of $GL(n,q)$ is given by $\sum_{\lambda \vdash n} F(\lambda)$. For fixed $n = 1,2,3,4,5$ one can calculate $\sum_{\lambda \vdash n} F(\lambda)$ from Table 6.12 and compare with $Reg(G_n^1(q))$. We can see that $\sum_{\lambda \vdash n} F(\lambda) = Reg(G_n^1(q))$.

Finally we count the number of regular semisimple elements contained in class c^λ.

Proposition 5.2.12. *Let c^λ be a regular semisimple class, where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$. Then*

$$|c^\lambda| = \frac{\prod_{s=0}^{n-1}(q^n - q^s)}{\prod_{i=1}^{k}(q^{\lambda_i} - 1)}. \tag{5.12}$$

PROOF. Let $g \in c^{\lambda} = (\{f_i\}, \{\lambda_i\}, \{1\}_{k \text{ times}}, \{1\}_{k \text{ times}})$. Since $\nu_i = 1$, $\forall 1 \leq i \leq k$, we obtain by substituting in equation (5.9) that

$$|C_{GL(n,q)}(g)| = \prod_{i=1}^{k} q^{\lambda_i}\phi_1(q^{-\lambda_i}) = \prod_{i=1}^{k} q^{\lambda_i}\left(\frac{q^{\lambda_i}-1}{q^{\lambda_i}}\right) = \prod_{i=1}^{k}(q^{\lambda_i}-1).$$

The result follows by (5.10). ■

Now we give the main theorem of this subsection which counts the number of regular semisimple elements of $GL(n,q)$.

Theorem 5.2.13. *With $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \equiv 1^{r_1}2^{r_2}\cdots n^{r_n}$ for $r_i \in \mathbb{N}\cup\{0\}$, the number of regular semisimple elements of $GL(n,q)$ is given by*

$$\sum_{\lambda \vdash n} \frac{\prod_{s=0}^{n-1}(q^n - q^s)\prod_{i=1}^{n}\prod_{s=0}^{r_i-1}(I_i(q)-s)}{\prod_{i=1}^{k}(q^{\lambda_i}-1)\prod_{i=1}^{n} r_i!}. \tag{5.13}$$

PROOF. Direct result from Propositions 5.2.8 and 5.2.10, together with equation (5.12). ■

Example 5.2.2. Let us consider the regular semisimple classes of type $\mathcal{T}^{(2,2)}$ of $GL(4,q)$. Each class $c^{(2,2)}$ will have the form $c^{(2,2)} = (\{f_1, f_2\}, \{2,2\}, \{1,1\}, \{1,1\})$ where $f_1(t) = t^2 + a_1 t + a_0$ and $f_2(t) = t^2 + b_1 t + b_0$ are two distinct irreducible polynomials over $\mathbb{F}_q$. To count the number of classes of this type, we follow Proposition 5.2.10. Thus we may write the partition $(2,2)$ in the form 2^2, that is $r_1 = r_3 = r_4 = 0$ and $r_2 = 2$. Therefore

$$\begin{aligned} F(2^2) &= \frac{\prod_{i=1}^{4}\prod_{s=0}^{r_i-1}(I_i(q)-s)}{\prod_{i=1}^{n} r_i!} = \frac{\prod_{s=0}^{1}(I_2(q)-s)}{0!\ 2!\ 0!\ 0!} \\ &= \frac{I_2(q)(I_2(q)-1)}{2} = \frac{1}{2}\frac{q^2-q}{2}\frac{q^2-q-2}{2} = \frac{q(q^2-1)(q-2)}{8}. \end{aligned}$$

Now applying equation (5.12) to any class of this type, we get

$$\begin{aligned} |c^{(2,2)}| &= \frac{\prod_{s=0}^{3}(q^4-q^s)}{\prod_{i=1}^{2}(q^{\lambda_i}-1)} = \frac{(q^4-1)(q^4-q)(q^4-q^2)(q^4-q^3)}{(q^2-1)(q^2-1)} \\ &= q^6(q-1)(q^2+1)(q^3-1). \end{aligned}$$

Hence we obtain

$$q^6(q-1)(q^2+1)(q^3-1)\frac{q(q^2-1)(q-2)}{8} = \frac{q^7(q^4-1)(q^3-1)(q-1)(q-2)}{8}$$

regular semisimple elements of type $(2,2)$. Repeating the above work for the other four partitions of 4, we get for $GL(4,q)$, a total number of regular semisimple elements given by

$$q^{16} - 2q^{15} + q^{13} + q^{12} - 2q^{10} - q^9 - q^8 + 2q^7 + q^6.$$

For example the group $GL(4,5)$, which is of order 116064000000 has 9299587000 regular semisimple elements.

In the Appendix we list the number of types, conjugacy classes, elements in each conjugacy class of regular semisimple elements of $GL(n,q)$ for $n = 1,2,3,4,5$.

Recall that a class $c = (\{f_i\}, \{d_i\}, \{z_i\}, \{\nu_i\})$ of $GL(n,q)$ with length k is called primary if and only if $k = 1$. The next Proposition counts the number of primary classes of $GL(n,q)$.

Proposition 5.2.14. *The number of primary classes of $GL(n,q)$ is given by*

$$\sum_{d|n} |\mathcal{P}(\frac{n}{d})| \cdot I_d(q). \tag{5.14}$$

PROOF. By definition a conjugacy class c of $GL(n,q)$ is primary if and only if $c = (f, d, \frac{n}{d}, \nu)$ for some $f \in \mathcal{F}$ with degree d, $d|n$ and $\nu \vdash \frac{n}{d}$. For fixed d and any $\nu \vdash \frac{n}{d}$ we have $I_d(q)$ irreducible polynomials f of degree d that defines a primary class. It follows that there are $|\mathcal{P}(\frac{n}{d})| \cdot I_d(q)$ conjugacy classes defined by the fixed integer d and partitions of $\frac{n}{d}$. The result follows by letting d runs over all divisors of n. ■

Corollary 5.2.15. *There are exactly $I_n(q) = \sum_{d|n} \mu(d) q^{\frac{n}{d}}$ primary regular semisimple classes of $GL(n,q)$.*

PROOF. A class c of $GL(n,q)$ is primary and regular semisimple if and only if $c = (f, n, 1, 1)$ for some $f \in \mathcal{F}$ with degree n. It follows by Proposition 5.2.8 that c is a regular semisimple class of type $\lambda = (n) \vdash n$. Hence by Corollary 5.2.11 we have $F((n)) = I_n(q) = \sum_{d|n} \mu(d) q^{\frac{n}{d}}$. In particular if $n = p'$ is a prime integer (whether $p' = p$, the characteristic of $\mathbb{F}_q$ or not), then there are $I_{p'}(q) = \frac{q^{p'} - q}{p'}$ primary regular semisimple classes of $GL(p', q)$. ■

Corollary 5.2.16. *The group $GL(p', q)$ has exactly $(q-1)|\mathcal{P}(p')| + \frac{q^{p'} - q}{p'}$ primary conjugacy classes.*

PROOF. By Proposition 5.2.14 the number of primary conjugacy classes of $GL(p',q)$ is given by $\sum_{d|p'} |\mathcal{P}(\frac{p'}{d})| \cdot I_d(q)$. We have $d \in \{1, p'\}$. If $d=1$, then there are $|\mathcal{P}(p')|$ types of primary classes each consists of $q-1$ conjugacy classes. On the other hand if $d=p'$, then $c=(f,p',1,1)$ for some $f \in \mathcal{F}$ with prime degree p'. Therefore by Corollary 5.2.15 we have $F((p')) = \frac{q^{p'}-q}{p'}$. Hence the result. ■

Example 5.2.3. See Table 5.2.

Orders of regular semisimple elements of $GL(n,q)$

Suppose that g is a regular semisimple element of $GL(n,q)$ in a conjugacy class c^λ, where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$. Assume that $\mathbb{F}^*_{q^{\lambda_i}} = \langle \varepsilon_i \rangle$, $\forall 1 \le i \le k$ and for each group $\mathbb{F}^*_{q^{\lambda_i}}$, we fix one generator ε_i, that is if $\lambda_j = \lambda_i$, for some j and i, then we identify ε_j with ε_i. It follows that the eigenvalues of g are

$$\varepsilon_1^{j_1}, \varepsilon_1^{j_1 q}, \cdots, \varepsilon_1^{j_1 q^{\lambda_1 - 1}};\ \varepsilon_2^{j_2}, \varepsilon_2^{j_2 q}, \cdots, \varepsilon_2^{j_2 q^{\lambda_2 - 1}};\ \cdots; \varepsilon_k^{j_k}, \varepsilon_k^{j_k q}, \cdots, \varepsilon_k^{j_k q^{\lambda_k - 1}} \tag{5.15}$$

for some integers $j_1, j_2, \cdots, j_k$.

Theorem 5.2.17. *With the above, the order of g is given by*

$$\begin{aligned} o(g) = lcm \Bigg(& \frac{q^{\lambda_1}-1}{\gcd(j_1, q^{\lambda_1}-1)}, \frac{q^{\lambda_1}-1}{\gcd(j_1 q, q^{\lambda_1}-1)}, \cdots, \frac{q^{\lambda_1}-1}{\gcd(j_1 q^{\lambda_1 - 1}, q^{\lambda_1}-1)}, \\ & \frac{q^{\lambda_2}-1}{\gcd(j_2, q^{\lambda_2}-1)}, \frac{q^{\lambda_2}-1}{\gcd(j_2 q, q^{\lambda_2}-1)}, \cdots, \frac{q^{\lambda_2}-1}{\gcd(j_2 q^{\lambda_2 - 1}, q^{\lambda_2}-1)}, \\ & \vdots \\ & \frac{q^{\lambda_k}-1}{\gcd(j_k, q^{\lambda_k}-1)}, \frac{q^{\lambda_k}-1}{\gcd(j_k q, q^{\lambda_k}-1)}, \cdots, \frac{q^{\lambda_k}-1}{\gcd(j_k q^{\lambda_k - 1}, q^{\lambda_k}-1)} \Bigg). \end{aligned}$$

PROOF. For $1 \le l \le k,\ 0 \le r \le \lambda_l - 1$, let $\underline{h}$ denotes the row of eigenvalues of g given by (5.15). Also let $\underline{t}$ denotes the row $o(\varepsilon_l^{j_l q^r})$ for $1 \le l \le k,\ 0 \le r \le \lambda_l - 1$.

We know that $g \sim D = diag(\underline{h})$. Assume that $o(g) = t$. Then

$$\begin{aligned} g^t \sim D^t &= (diag(\underline{h}))^t = diag(\underline{h}^t) = I_n \Longleftrightarrow \varepsilon_l^{j_l q^r t} = 1,\ \forall 1 \le l \le k,\ \forall 0 \le r \le \lambda_l - 1 \\ &\Longleftrightarrow o(\varepsilon_l^{j_l q^r}) | t,\ \forall 1 \le l \le k,\ \forall 0 \le r \le \lambda_l - 1 \\ &\Longleftrightarrow lcm(\underline{t}) | t. \end{aligned} \tag{5.16}$$

Let $o(\varepsilon_l^{j_l q^r}) = t_{lr}$ for each $1 \le l \le k,\ 0 \le r \le \lambda_l - 1$ and let $d = \gcd(\underline{t})$. Now for each $1 \le m \le k,\ 0 \le j \le \lambda_m - 1$ we have

$$lcm(\underline{t}) = \frac{\prod_{r,l} t_{lr}}{d} = t_{mj} \left(\prod_{(l,r) \ne (m,j)} \frac{t_{lr}}{d} \right).$$

Now

$$g^{lcm(\underline{t})} \sim D^{lcm(\underline{t})} = diag\Big((\varepsilon_1^{j_1})^{lcm(\underline{t})}, (\varepsilon_1^{j_1 q})^{lcm(\underline{t})}, \cdots, (\varepsilon_1^{j_1 q^{\lambda_1 - 1}})^{lcm(\underline{t})}$$
$$(\varepsilon_2^{j_2})^{lcm(\underline{t})}, (\varepsilon_2^{j_2 q})^{lcm(\underline{t})}, \cdots, (\varepsilon_2^{j_2 q^{\lambda_2 - 1}})^{lcm(\underline{t})}$$
$$\vdots$$
$$(\varepsilon_k^{j_k})^{lcm(\underline{t})}, (\varepsilon_k^{j_k q})^{lcm(\underline{t})}, \cdots, (\varepsilon_k^{j_k q^{\lambda_k - 1}})^{lcm(\underline{t})}\Big) = I_n,$$

where $lcm(\underline{t})$ in each diagonal entry $(\varepsilon_m^{j_m q^{r\lambda_m - 1}})^{lcm(\underline{t})}$ is replaced by $t_{mj}\left(\prod_{(l,r)\neq(m,j)} \frac{t_{lr}}{d}\right)$. This implies that $t|lcm(\underline{t})$, since $o(g) = t$. From equation (5.16) we have $lcm(\underline{t})|t$. Hence $t = o(g) = lcm(\underline{t})$. Now the result follows from elementary group theory, where we know that $o(\varepsilon_l^{j_l q^r}) = (q^{\lambda_l} - 1)/(\gcd(j_l q^r, q^{\lambda_l} - 1))$. ■

As a corollary of Theorem 5.2.17 we show the existence of an element of $GL(n,q)$ of order $q^n - 1$ (this has been mentioned in Darafasheh [15] without proof).

Corollary 5.2.18. *The group $GL(n,q)$, $n > 1$ has at least $q^{\frac{n(n-1)}{2}} \prod_{s=0}^{n-2}(q^{n-1} - q^s)$ elements of order $q^n - 1$ and at least twice of the previous number if q is even.*

PROOF. Let $g, h \in GL(n,q)$ such that $\{\varepsilon_n, \varepsilon_n^q, \cdots, \varepsilon_n^{q^{n-1}}\}$ and $\{\varepsilon_n^2, \varepsilon_n^{2q}, \cdots, \varepsilon_n^{2q^{n-1}}\}$ are the eigenvalues of g and h respectively. It follows from Theorem 5.2.17 that

$$\begin{aligned} o(g) &= lcm\left(\frac{q^n - 1}{\gcd(1, q^n - 1)}, \frac{q^n - 1}{\gcd(q, q^n - 1)}, \cdots, \frac{q^n - 1}{\gcd(q^{n-1}, q^n - 1)}\right) \\ &= lcm\,(q^n - 1, q^n - 1, \cdots, q^n - 1) = q^n - 1. \end{aligned}$$

If q is even, then $q^n - 1$ is odd and hence $\gcd(2, q^n - 1) = 1$, which yields that $\gcd(2q^m, q^n - 1) = 1,\ \forall\ 0 \le m \le n-1$. Hence $o(h) = q^n - 1$ by similar argument used for $o(g)$. The result follows since all conjugate elements have the same order. ■

5.2.4 Examples: Conjugacy Classes of $GL(3,q)$, $GL(4,q)$, and $GL(2,q)$ (Revisited)

Conjugacy Classes of $GL(3,q)$

We illustrate how to obtain the conjugacy classes of $GL(n,q)$ for $n = 3$. Any $A \in GL(3,q)$ has characteristic polynomial that is a monic polynomial of degree 3 of the form $f(t) = t^3 + a_2 t^2 + a_1 t + a_0 \in \mathbb{F}_q[t]$ and it splits over $\mathbb{F}_q$ into one of the following five forms:

1. $f(t) = (t - \alpha)^3$ for some $\alpha \in \mathbb{F}_q^*$.

2. $f(t) = (t-\alpha)^2(t-\beta)$ for some $\alpha, \beta \in \mathbb{F}_q^*$, $\alpha \neq \beta$.

3. $f(t) = (t-\alpha)(t-\beta)(t-\gamma)$ for some $\alpha, \beta, \gamma \in \mathbb{F}_q^*$, α, β and γ are distinct.

4. $f(t) = (t^2 + b_1 t + b_0)(t-\alpha)$ and $t^2 + b_1 t + b_0$ is irreducible over $\mathbb{F}_q$.

5. $f(t) = t^3 + a_2 t^2 + a_1 t + a_0$ remains irreducible over $\mathbb{F}_q$.

We had excluded the cases $f(t) = t \cdot (t^2 + c_1 t + c_0)$, $t^2 \cdot (t + c_0)$ and t^3 since these cases yield that A has some zero eigenvalue, which contradicts the invertibility of A. We consider each of the preceding five cases separately.

1. Suppose that $f(t) = (t-\alpha)^3$ for some $\alpha \in \mathbb{F}_q^*$. In this case $f_1(t) = (t-\alpha)$ and $f_i(t) = 1$, $\forall i > 1$ with $k_1 = 3$ and $k_i = 0$, $\forall i > 1$. We have $\nu_1 \in \mathcal{P}(3) = \{(1,1,1), (2,1), (3)\}$.

   (i) If $\nu_1 = 1^3 = (1,1,1) \vdash 3$, then

   $$A \sim diag(U_{(1)}(t-\alpha), U_{(1)}(t-\alpha), U_{(1)}(t-\alpha)) = diag(U_1(t-\alpha), U_1(t-\alpha), U_1(t-\alpha)).$$

   From the definition of the matrix $U_1(f)$, then $U_1(t-\alpha) = [\alpha]_{1\times 1} = \alpha$. Thus $A \sim \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$.

   Each $\alpha \in \mathbb{F}_q^*$ gives a new conjugacy class and it is clear that these classes are the central classes of $GL(3,q)$. This yields that there are $q-1$ distinct conjugacy class each of size 1. We denote this type of classes by $\mathcal{T}^{(1)}$.

   (ii) If $\nu = (2,1) \vdash 3$, then

   $$A \sim diag(U_2(f_1(t)), U_1(f_1(t))) = diag(U_2(t-\alpha), U_1(t-\alpha)).$$

   Since $U_1(t-\alpha) = \alpha$, it follows that $U_2(t-\alpha) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$. Thus $A \sim \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$. Each $\alpha \in \mathbb{F}_q^*$ gives a new conjugacy class. Thus there are $q-1$ conjugacy classes of this type, which we denote by $\mathcal{T}^{(2)}$. Next we calculate $|C_{GL(n,q)}(A)|$, where A is of type $\mathcal{T}^{(2)}$. In terms of (5.3) we have $m_2 = m_1 = 1$. The function $\phi_{(2,1)}$ is thus

   $$\phi_{(2,1)}\left(\frac{1}{q}\right) = \prod_{i=1}^{2} \phi_{m_i}\left(\frac{1}{q}\right) = \phi_{m_1}\left(\frac{1}{q}\right)\phi_{m_2}\left(\frac{1}{q}\right) = \phi_{m_1}^2\left(\frac{1}{q}\right) = \left(1 - \frac{1}{q}\right)^2 = \left(\frac{q-1}{q}\right)^2.$$

   From Table 6.1 (see Appendix) we have $n((2,1)) = 1$. Using this together with (5.9) we get

   $$|C_{GL(3,q)}(A)| = q^{|(2,1)|+2n((2,1))}\phi_{(2,1)}\left(\frac{1}{q}\right) = q^{3+2}\frac{(q-1)^2}{q^2} = q^3(q-1)^2.$$

It follows by (5.10) that

$$|C_A| = (q-1)(q+1)(q^2+q+1).$$

(iii) In the final subcase when $\nu = (3) \vdash 3$, we have

$$A \sim diag(U_3(f_1(t))) = diag(U_3(t-\alpha)).$$

Since $U_1(t-\alpha) = \alpha$, it follows that $U_3(t-\alpha) = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix} \sim A$. For any $\alpha \in \mathbb{F}_q^*$ there corresponds a conjugacy class. Therefore there are $q-1$ distinct conjugacy classes of this type, which we denote by $\mathcal{T}^{(3)}$.

By (5.3) we have $m_1 = m_2 = 0$ and $m_3 = 1$. The function $\phi_{(3)}$ is given by

$$\phi_{(3)}\left(\frac{1}{q}\right) = \prod_{i=1}^{3} \phi_{m_i}\left(\frac{1}{q}\right) = \phi_{m_1}\left(\frac{1}{q}\right)\phi_{m_2}\left(\frac{1}{q}\right)\phi_{m_3}\left(\frac{1}{q}\right) = \left(\phi_0\left(\frac{1}{q}\right)\right)^2 \phi_1\left(\frac{1}{q}\right) = \frac{q-1}{q}.$$

From Table 6.1 we have $n((3)) = 0$. Thus

$$|C_{GL(3,q)}(A)| = q^{|(3)|+2n((3))}\phi_{(3)}\left(\frac{1}{q}\right) = q^3\frac{(q-1)}{q} = q^2(q-1).$$

It follows by (5.10) that

$$|C_A| = q(q-1)^2(q+1)(q^2+q+1).$$

2. Suppose that $f(t) = (t-\alpha)^2(t-\beta)$, $\alpha, \beta \in \mathbb{F}_q^*$, $\alpha \neq \beta$. In this case, $f_1(t) = (t-\alpha)^2$, $k_1 = 1$, $f_2(t) = (t-\beta)$, $k_2 = 1$, $f_i(t) = 1$, $k_i = 0$, $\forall i \geq 3$. We have $\nu_1 \in \mathcal{P}(2) = \{(1,1),(2)\}$ and $\nu_2 \in \mathcal{P}(1) = \{(1)\}$. Therefore we have two subcases:

(i) If $\nu_1 = (1,1)$ and $\nu_2 = (1)$, then

$$A \sim diag(U_{\nu_1}(f_1(t)), U_{\nu_2}(f_2(t))) = diag(U_{(1,1)}(t-\alpha), U_{(1)}(t-\beta)).$$

Since $U_1(t-\alpha) = \alpha$, it follows that $U_{(1,1)}(t-\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$. Hence $A \sim \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$. This type of classes will be denoted by $\mathcal{T}^{(4)}$. Notice that

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \sim \begin{pmatrix} \beta & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \not\sim \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

Thus any ordered pair (α, β), $\alpha, \beta \in \mathbb{F}_q^*$, $\alpha \neq \beta$ presents a new conjugacy class of this type and hence there are $(q-1)(q-2)$ distinct conjugacy classes of this type.

We compute $|C_{GL(3,q)}(A)|$, A being an element in a class of type $\mathcal{T}^{(4)}$. For the partition $\nu_1 = (1,1)$ we have $m_1 = 2$ and $m_2 = 0$, while for $\nu_2 = (1)$ we have $m_1 = 0$ and $m_2 = 1$. The function $\phi_{(1,1)}$ is given by

$$\phi_{\nu_1}\left(\frac{1}{q}\right) = \phi_{(1,1)}\left(\frac{1}{q}\right) = \left(1-\frac{1}{q}\right)\left(1-\frac{1}{q^2}\right) = \frac{(q-1)(q^2-1)}{q^3}.$$

By Table 6.1 we have $n((1,1)) = 1$. Thus

$$a_{\nu_1} = q^{|\nu_1|+2n((\nu_1))}\phi_{(1,1)}\left(\frac{1}{q}\right) = \frac{q^4(q-1)(q^2-1)}{q^3} = q(q-1)^2(q+1).$$

Also

$$a_{\nu_2} = q^{|\nu_2|+2n((\nu_2))}\phi_1\left(\frac{1}{q}\right) = \frac{q(q-1)}{q} = q-1.$$

Therefore

$$|C_{GL(3,q)}(A)| = a_{\nu_1}a_{\nu_2} = q(q-1)^2(q+1)(q-1) = q(q-1)^3(q+1).$$

It follows that

$$|C_A| = q^2(q^2+q+1).$$

(ii) On the other hand, if $\nu_1 = (2)$ and $\nu_2 = (1)$, then

$$A \sim diag(U_{\nu_1}(f_1(t)), U_{\nu_2}(f_2(t))) = diag(U_{(2)}(t-\alpha), U_{(1)}(t-\beta)) = diag(U_2(t-\alpha), U_1(t-\beta)).$$

Since $U_1(t-\alpha) = \alpha$, it follows that $U_2(t-\alpha) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$. Hence $A \sim \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$. Type of classes of this format will be denoted by $\mathcal{T}^{(5)}$. As in previous subcase the number of distinct conjugacy classes of this type is $(q-1)(q-2)$.

Now

$$a_{\nu_1} = a_{(2)} = q^{|2|+2n((2))}\phi_{(2)}\left(\frac{1}{q}\right) = q^{2+2\cdot 0}\frac{q-1}{q} = q(q-1).$$

In the previous subcase 2(i) we have seen that

$$a_{\nu_2} = q^{|\nu_2|+2n((\nu_2))}\phi_1\left(\frac{1}{q}\right) = q^{1+2\cdot 0}\phi_1\left(\frac{1}{q}\right) = \frac{q(q-1)}{q} = (q-1).$$

By (5.9) it follows that

$$|C_{GL(3,q)}(A)| = a_{\nu_1}a_{\nu_2} = (q-1)q(q-1) = q(q-1)^2.$$

This shows that

$$|C_A| = \frac{|GL(3,q)|}{|C_{GL(3,q)}(A)|} = \frac{q^3(q-1)^3(q+1)(q^2+2+1)}{q(q-1)^2} = q^2(q-1)(q+1)(q^2+2+1).$$

3. Suppose that $f(t) = (t-\alpha)(t-\beta)(t-\gamma)$, where α, β and γ are distinct. We let $f_1(t) = (t-\alpha)$, $f_2(t) = (t-\beta)$, $f_3(t) = (t-\gamma)$ and $f_i(t) = 1$, $\forall i > 3$. Thus $k_1 = 1 = k_2$, $k_3 = 1$ and $k_i = 0$, $\forall i > 3$. Hence

$$\begin{aligned} A &\sim diag(U_{\nu_1}(f_1(t)), U_{\nu_2}(f_2(t)), U_{\nu_3}(f_3(t))) = diag(U_{(1)}(t-\alpha), U_{(1)}(t-\beta), U_{(1)}(t-\gamma)) \\ &= diag(U_1(t-\alpha), U_1(t-\beta), U_1(t-\gamma)) = diag(\alpha,\beta,\gamma) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \end{aligned}$$

Type of classes of this format will be denoted by $\mathcal{T}^{(6)}$.

Each unordered triple $\{\alpha,\beta,\gamma\}$ with $\alpha,\beta,\gamma \in \mathbb{F}_q^*$ and $\alpha \neq \beta \neq \gamma \neq \alpha$ introduces a new conjugacy class. Therefore there are $\frac{(q-1)(q-2)(q-3)}{6}$ distinct conjugacy classes of this type. The size of the centralizer of an element A of type $\mathcal{T}^{(6)}$ is given by $a_{\nu_1}a_{\nu_2}a_{\nu_3} = (a_{\nu_1})^3 = (a_{(1)})^3$, where $a_{(1)} = q-1$. Thus

$$|C_{GL(3,q)}(A)| = (q-1)^3.$$

Hence

$$|C_A| = q^3(q+1)(q^2+q+1).$$

4. Suppose that $f(t) = (t^2+b_1t+b_0)(t-\alpha)$, where $t^2+b_1t+b_0 \in \mathcal{F}$ and $\alpha \in \mathbb{F}_q^*$. We let $f_1(t) = t^2+b_1t+b_0$ and $f_2(t) = (t-\alpha)$. Then $k_1 = 1$ and $k_2 = 1$, which implies that $\nu_1 = \nu_2 = (1)$. Since $f_1(t) = t^2+b_1t+b_0 \in \mathcal{F}$, by Lemma 5.2.9, it splits completely over $\mathbb{F}_{q^2}$. Suppose that r is a root of $f_1(t)$. Then the other root is r^q. These two roots of $f_1(t)$ satisfy the relations $r + r^q = -b_1$ and $r^{1+q} = b_0$. Therefore $U_1(t^2+b_1t+b_0) = \begin{pmatrix} 0 & 1 \\ -b_0 & -b_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -r^{1+q} & r+r^q \end{pmatrix}$. Also we have $U_1(t-\alpha) = \alpha$. Hence every A corresponds to an $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and $\alpha \in \mathbb{F}_q^*$, will be of the form

$$\begin{aligned} A &\sim diag(U_{\nu_1}(f_1), U_{\nu_2}(f_2)) = diag(U_{(1)}(f_1), U_{(1)}(f_2)) \\ &= diag(U_1(t^2+b_1t+b_0), U_1(t-\alpha)) = \begin{pmatrix} 0 & 1 & 0 \\ -r^{1+q} & r+r^q & 0 \\ 0 & 0 & \alpha \end{pmatrix}. \end{aligned}$$

Type of classes of this format will be denoted by $\mathcal{T}^{(7)}$.

Using equation (5.9) we get

$$\begin{aligned}|C_{GL(3,q)}(A)| &= \prod_{f\in\mathcal{F}} a_{\nu(f)}(q^{deg(f)}) = a_{\nu_1(f_1)}(q^2).a_{\nu_2(f_2)}(q)\\ &= a_{(1)}(q^2).a_{(1)}(q) = \left(q^{2|(1)|+4n((1))}\phi_1\left(\frac{1}{q^2}\right)\right)\left(q^{|(1)|+2n((1))}\phi_1\left(\frac{1}{q}\right)\right)\\ &= \left(q^2\frac{(q^2-1)}{q^2}\right)\left(q\frac{(q-1)}{q}\right) = (q-1)^2(q+1).\end{aligned}$$

It follows by (5.10) that

$$|C_A| = \frac{q^3(q-1)^3(q+1)(q^2+2+1)}{(q-1)^2(q+1)} = q^3(q-1)(q^2+q+1).$$

Since $\alpha \in \mathbb{F}_q^*$ (there are $q-1$ possibilities for α) and $r \in \mathbb{F}_{q^2}\setminus\mathbb{F}_q$ (there are q^2-q choices for r), it follows that there are $q(q-1)^2$ classes of this type. But for fixed $\alpha \in \mathbb{F}_q^*$, replacing r^q with r gives the same conjugacy class. This restricts the number of conjugacy classes of this type to $\frac{q(q-1)^2}{2}$.

5. Suppose that $f(t) = t^3 + a_2t^2 + a_1t + a_0$ remains irreducible over $\mathbb{F}_q$. In this case, $f_1 = f$ and $f_i = 1,\ k_1 = 1$ and $k_i = 0\ \forall i \geq 2$. It follows that $\nu_1 = (1)$.
Since $f_1(t) \in \mathcal{F}$, it splits completely over $\mathbb{F}_{q^3}$. If s is a root of $f_1(t)$, then s^q and s^{q^2} are the other roots of $f_1(t)$ by Lemma 5.2.9. These roots satisfy the relations $s\cdot s^q\cdot s^{q^2} = s^{1+q+q^2} = -a_0,\ s\cdot s^q + s\cdot s^{q^2} + s^q\cdot s^{q^2} = s^{1+q} + s^{1+q^2} + s^{q+q^2} = a_1$ and $s + s^q + s^{q^2} = a_2$. Thus an element A corresponds to $s \in \mathbb{F}_{q^3}\setminus\mathbb{F}_q$ will have the form

$$\begin{aligned}A &\sim diag(U_{\nu_1}(f_1)) = diag(U_{(1)}(t^3+a_2t^2+a_1t+a_0))\\ &= \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ a_0 & a_1 & a_2\end{pmatrix}\\ &= \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -s^{1+q+q^2} & s^{1+q}+s^{1+q^2}+s^{q+q^2} & s+s^q+s^{q^2}\end{pmatrix}.\end{aligned}$$

Type of classes of this format will be denoted by $\mathcal{T}^{(8)}$.

By (5.9) we obtain

$$\begin{aligned}|C_{GL(3,q)}(A)| &= \prod_{f\in\mathcal{F}} a_{\nu(f)}(q^{deg(f)}) = a_{\nu_1(f_1)}(q^3) = \left(q^{3(|(1)|+2n((1)))}\phi_1\left(\frac{1}{q^3}\right)\right)\\ &= q^3\frac{(q^3-1)}{q^3} = q^3-1 = (q-1)(q^2+q+1).\end{aligned}$$

Hence

$$|C_A| = q^3(q-1)^2(q+1).$$

Since $s \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$, there are $q^3 - q$ choices for s. The conjugacy classes introduced by choosing s^q and s^{q^2} are the same class of that one introduced by choosing s. This gives a total of $\frac{q^3-q}{3}$ classes of this type.

Note 5.2.3. It can be easily checked that classes obtained in each type are not conjugate to classes in other types as well as classes of the same type are also not conjugate.

We had processed all the possible cases. As a final step, we check the total number of elements in conjugacy classes we have produced so far. Let $A_i \in C_{A_i}$, where C_{A_i} is a conjugacy class of type $\mathcal{T}^{(i)}$. By $\#\mathcal{T}^{(i)}$ we mean the number of conjugacy classes of type $\mathcal{T}^{(i)}$. Then we have

$$\begin{aligned}
\sum_{i=1}^{8} |C_{A_i}| \cdot \#\mathcal{T}^{(i)} &= 1 \cdot (q-1) + (q-1)(q+1)(q^2+q+1) \cdot (q-1) \\
&+ q(q-1)^2(q+1)(q^2+q+1) \cdot (q-1) + q^2(q^2+q+1) \cdot (q-1)(q-2) \\
&+ q^2(q-1)(q+1)(q^2+q+1) \cdot (q-1)(q-2) \\
&+ q^3(q+1)(q^2+q+1) \cdot \frac{(q-1)(q-2)(q-3)}{6} + q^3(q-1)(q^2+q+1) \cdot \frac{q(q-1)^2}{2} \\
&+ q^3(q-1)^2(q+1) \cdot \frac{q^3-q}{3} \\
&= q^7 - q^6 + q^8 - 2q^7 - q^5 + 2q^4 + \frac{q^9 - 4q^8 + q^7 + 5q^6 + 4q^5 - q^4 - 6q^3}{6} \\
&+ \frac{q^9 - 2q^8 + q^7 - q^6 + 2q^5 - q^4}{2} + \frac{q^9 - q^8 - 2q^7 + 2q^6 + q^5 - q^4}{3} \\
&= q^9 - q^8 - q^7 + q^5 + q^4 - q^3 = q^3(q-1)^3(q+1)(q^2+q+1) = |GL(3,q)|.
\end{aligned}$$

Also

$$\begin{aligned}
\sum_{i=1}^{8} \#\mathcal{T}^{(i)} &= (q-1) + (q-1) + (q-1) + (q-1)(q-2) \\
&+ (q-1)(q-2) + \frac{(q-1)(q-2)(q-3)}{6} + \frac{q(q-1)^2}{2} + \frac{q^3-q}{3} \\
&= 3q - 3 + 2q^2 - 6q + 4 + \frac{q^3 - 6q^2 + 11q - 6}{6} + \frac{q^3 - 2q^2 + q}{2} + \frac{q^3 - q}{3} \\
&= q^3 - q = c(3,q) = \text{The number of conjugacy classes of } GL(3,q).
\end{aligned}$$

Hence the conjugacy classes we have found are all the classes of $GL(3,q)$. In Table 5.3 we list these classes with the size of centralizers, conjugacy classes and the number of classes contained in each type.

Table 5.3: The conjugacy classes of $GL(3,q)$

	$\mathcal{T}^{(1)}$	$\mathcal{T}^{(2)}$	$\mathcal{T}^{(3)}$
Rep g	$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$
No. of CC	$q-1$	$q-1$	$(q-1)$
$\lvert C_{GL(3,q)}(g)\rvert$	$q^3(q-1)^3(q+1)(q^2+q+1)$	$q^3(q-1)^2$	$q^2(q-1)$
$\lvert C_g\rvert$	1	$(q^2-1)(q^2+q+1)$	$q(q^2-1)(q^3-1)$

Table 5.3 (continued)

	$\mathcal{T}^{(4)}$	$\mathcal{T}^{(5)}$	$\mathcal{T}^{(6)}$
Rep g	$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$
No. of CC	$(q-1)(q-2)$	$(q-1)(q-2)$	$\frac{(q-1)(q-2)(q-3)}{6}$
$\lvert C_{GL(3,q)}(g)\rvert$	$q(q-1)^3(q+1)$	$q(q-1)^2$	$(q-1)^3$
$\lvert C_g\rvert$	$q^2(q^2+q+1)$	$q^2(q^2-1)(q^2+q+1)$	$q^3(q+1)(q^2+q+1)$

Table 5.3 (continued)

	$\mathcal{T}^{(7)}$	$\mathcal{T}^{(8)}$
Rep g	$\begin{pmatrix} 0 & 1 & 0 \\ -r^{q+1} & r+r^{1+q} & 0 \\ 0 & & \alpha \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s^{1+q+q^2} & -(s^{1+q}+s^{1+q^2}+s^{q+q^2}) & s+s^q+s^{q^2} \end{pmatrix}$
No. of CC	$\frac{q}{2}(q-1)^2$	$\frac{1}{3}(q^3-q)$
$\lvert C_{GL(3,q)}(g)\rvert$	$(q-1)^2(q+1)$	$(q-1)(q^2+q+1)$
$\lvert C_g\rvert$	$q^3(q-1)(q^2+q+1)$	$q^3(q-1)^2(q+1)$

Note 5.2.4. The conjugacy classes of types $\mathcal{T}^{(6)}$, $\mathcal{T}^{(7)}$ and $\mathcal{T}^{(8)}$ comprise all the regular semisimple classes of $GL(3,q)$, while classes of types $\mathcal{T}^{(1)}$, $\mathcal{T}^{(2)}$, $\mathcal{T}^{(3)}$ and $\mathcal{T}^{(8)}$ are the primary classes of $GL(3,q)$. Note that the number of primary classes of $GL(3,q)$ that given by Corollary 5.2.16 is $(q-1)|\mathcal{P}(3)| + I_3(q) = 3(q-1) + \frac{q^3-q}{3} = \frac{(q-1)(q^2+q+9)}{3}$. Also the only type of regular semisimple classes, which are also primary, is $\mathcal{T}^{(8)}$, which consists of $\frac{q^3-q}{3}$ classes.

In terms of (5.11) the number of regular semisimple classes of $GL(3,q)$ is given by

$$(q-1)\frac{q^4-q^3+(q-1)}{q^2-1} = \frac{q^3(q-1)+(q-1)}{q+1} = \frac{(q^3+1)(q-1)}{q+1} = (q-1)(q^2-q+1).$$

Alternatively we can use our formula given in Proposition 5.2.10 to produce the number of regular semisimple classes as follows:

- if $\lambda = (1^3) \vdash 3$, then $F(1^3) = \frac{(q-1)(q-2)(q-3)}{6} = \#\mathcal{T}^{(6)}$,
- if $\lambda = (1^1 2^1) \vdash 3$, then $F(1^1 2^1) = I_1(q) I_2(q) = \frac{q(q-1)^2}{2} = \#\mathcal{T}^{(7)}$,
- if $\lambda = (3) \vdash 3$, then $F((3)) = \frac{(q^3-q)}{3} = \#\mathcal{T}^{(8)}$.

Therefore the number of regular semisimple classes is

$$\frac{(q-1)(q-2)(q-3)}{6} + \frac{q(q-1)^2}{2} + \frac{(q^3-q)}{3} = q^3 - 2q^2 + 2q - 1 = (q-1)(q^2-q+1).$$

As a direct application of Theorem 5.2.17 we calculate the orders of the regular semisimple elements of $GL(3,q)$. Let $\varepsilon_1, \varepsilon_2$ and ε_3 be generators of $\mathbb{F}_q^*$, $\mathbb{F}_{q^2}^*$ and $\mathbb{F}_{q^3}^*$ respectively. Suppose that g, h and t are representative elements for classes of types $\mathcal{T}^{(6)}$, $\mathcal{T}^{(7)}$ and $\mathcal{T}^{(8)}$ respectively. Then

$$g = \begin{pmatrix} \varepsilon_1^{j_1} & 0 & 0 \\ 0 & \varepsilon_1^{j_2} & 0 \\ 0 & 0 & \varepsilon_1^{j_3} \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 1 & 0 \\ -\varepsilon_2^{j(q+1)} & \varepsilon_2^{j} + \varepsilon_2^{j(1+q)} & 0 \\ 0 & & \varepsilon_1^{i} \end{pmatrix}$$

and

$$t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3^{j_3(1+q+q^2)} & -(\varepsilon_3^{j_3(1+q)} + \varepsilon_3^{j_3(1+q^2)} + \varepsilon_3^{j_3(q+q^2)}) & \varepsilon_3^{j_3} + \varepsilon_3^{j_3 q} + \varepsilon_3^{j_3 q^2} \end{pmatrix}$$

for some distinct integers i, j, j_1, j_2 and j_3. By Theorem 5.2.17 we get

$$\begin{aligned} o(g) &= lcm\left(\frac{(q-1)}{\gcd(j_1,\ q-1)}, \frac{(q-1)}{\gcd(j_2,\ q-1)}, \frac{(q-1)}{\gcd(j_3,\ q-1)}\right), \\ o(h) &= lcm\left(\frac{(q-1)}{\gcd(i,\ q-1)}, \frac{(q^2-1)}{\gcd(j,\ q^2-1)}, \frac{(q^2-1)}{\gcd(jq,\ q^2-1)}\right) \end{aligned}$$

and

$$o(t) = lcm\left(\frac{(q^3-1)}{\gcd(j_3,\ q^3-1)}, \frac{(q^3-1)}{\gcd(j_3 q,\ q^3-1)}, \frac{(q^3-1)}{\gcd(j_3 q^2,\ q^3-1)}\right).$$

In similar fashion to the proof of Proposition 4.2.2 we can calculate the orders of other elements of $GL(3,q)$. We have done this and we listed these orders in Table 6.2, in the Appendix.

Conjugacy Classes of $GL(4, q)$

Any $A \in GL(4, q)$ has characteristic polynomial $f(t)$ decomposes into one of the following forms:

1. $f(t) = (t - \alpha)^4, \ \alpha \in \mathbb{F}_q^*$,
2. $f(t) = (t - \alpha)^3(t - \beta), \ \alpha, \beta \in \mathbb{F}_q^*, \ \alpha \neq \beta$,
3. $f(t) = (t - \alpha)^2(t - \beta)^2, \ \alpha, \beta \in \mathbb{F}_q^*, \ \alpha \neq \beta$,
4. $f(t) = (t - \alpha)^2(t - \beta)(t - \gamma), \ \alpha, \beta, \gamma \in \mathbb{F}_q^*, \ \alpha, \beta, \gamma$ are distinct,
5. $f(t) = (t - \alpha)^2(t^2 + at + b), \ \alpha \in \mathbb{F}_q^*$ and $t^2 + at + b \in \mathcal{F}$,
6. $f(t) = (t - \alpha)(t - \beta)(t - \gamma)(t - \xi), \ \alpha, \beta, \gamma, \xi \in \mathbb{F}_q^*, \ \alpha, \beta, \gamma, \xi$ are distinct,
7. $f(t) = (t - \alpha)(t - \beta)(t^2 + at + b), \ \alpha, \beta \in \mathbb{F}_q^*, \ \beta \neq \alpha$ and $t^2 + at + b \in \mathcal{F}$,
8. $f(t) = (t^2 + at + b)(t^2 + ct + d), \ t^2 + at + b, \ t^2 + ct + d \in \mathcal{F}$,
9. $f(t) = (t^2 + at + b)^2, \ t^2 + at + b \in \mathcal{F}$,
10. $f(t) = (t - \alpha)(t^3 + at^2 + bt + c), \ \alpha \in \mathbb{F}_q^*$ and $t^3 + at^2 + bt + c \in \mathcal{F}$,
11. $f(t) = (t^4 + at^3 + bt^2 + ct + d), \ t^4 + at^3 + bt^2 + ct + d \in \mathcal{F}$,

Now one can build the Jordan Canonical Form of any $A \in GL(4, q)$ by using similar fashion used in the case $GL(3, q)$. One can also use equations (5.9) and (5.10) to calculate the size of the centralizers and conjugacy classes. Tables including the representatives of classes and size of these classes have been given in Tables 6.10 and 6.11 respectively (see Appendix).

Conjugacy Classes of $GL(2, q)$ (Revisited)

We conclude this subsection by revisiting the group $GL(2, q)$, which its conjugacy classes were given in Theorem 4.2.1. There we proved that the classes lie in four types $\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \mathcal{T}^{(3)}$ and $\mathcal{T}^{(4)}$, but we did not give a source for these classes. Now we are in good position to interpret how we got the representatives. The characteristic polynomial of any $A \in GL(2, q)$ splits into one of the following forms

1. $(t - \alpha)^2, \ \alpha \in \mathbb{F}_q^*$,
2. $(t - \alpha)(t - \beta), \ \alpha, \beta \in \mathbb{F}_q^*, \ \alpha \neq \beta$ or
3. $t^2 + at + b$, remains irreducible over $\mathbb{F}_q$.

When the characteristic polynomial is $(t-\alpha)^2$, $\alpha \in \mathbb{F}_q^*$, then corresponding to the partitions $\nu_1 = (1,1) \vdash 2$ and $\nu_2 = (2) \vdash 2$, we get representatives in respective way given by

$$A \sim diag(U_{(1)}(t-\alpha), U_{(1)}(t-\alpha)) = diag(U_1(t-\alpha), U_1(t-\alpha)) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

and

$$A \sim diag(U_{(2)}(t-\alpha) = diag(U_2(t-\alpha)) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}.$$

This gives the first two types of classes $\mathcal{T}^{(1)}$ and $\mathcal{T}^{(2)}$, where it is clear that $\mathcal{T}^{(1)}$ consists of the central elements of $GL(2,q)$. Thus the size of each conjugacy class of type $\mathcal{T}^{(1)}$ is 1, while if A is an element in a class of type $\mathcal{T}^{(2)}$, then

$$|C_{GL(2,q)}(A)| = q^{|(2)|+2\cdot n((2))}\phi_{(2)}\left(\frac{1}{q}\right) = q^2\left(\frac{q-1}{q}\right) = q(q-1)$$

which implies that $|C_A| = q^2-1$, exactly as in Theorem 4.2.1.

Corresponding to the factorization $(t-\alpha)(t-\beta)$, $\alpha, \beta \in \mathbb{F}_q^*$, $\alpha \neq \beta$, we get elements of type $\mathcal{T}^{(3)}$, while if the characteristic polynomial $t^2+at+b \in \mathcal{F}$, then we get elements of type $\mathcal{T}^{(4)}$. Similarly we can calculate the size of centralizers and hence conjugacy classes of elements of types $\mathcal{T}^{(3)}$ and $\mathcal{T}^{(4)}$ using (5.9) and (5.10) respectively. Thus using Green's formula for the size of the centralizer of element of $GL(2,q)$ coincides with the size computed from the definition directly as done in Section 4.2.

5.3. Induction From Parabolic Subgroups

We recall by Theorem 3.1.8 that a parabolic subgroup P_λ is a split extension of its unipotent radical U_λ by its levi complement L_λ, where the last one is isomorphic to $\bigotimes_{i=1}^{k} GL(\lambda_i, q)$.

Let ψ_i be a class function of $GL(\lambda_i, q)$, $1 \leq i \leq k$. The function ψ defined on L_λ by

$$\psi(A) = \bigotimes_{i=1}^{k} \psi_i(A_{ii}) = \psi_1(A_{11})\psi_2(A_{22})\cdots\psi_k(A_{kk}) \tag{5.17}$$

is a class function of L_λ. This class function can be inflated by the lifting method described in Section 2.4 to be a class function of P_λ by setting $\psi(ul) = \psi(l)$ for all $u \in U_\lambda$ and $l \in L_\lambda$. Furthermore, we define the $\odot-$product $\psi_1 \odot \psi_2 \odot \cdots \odot \psi_k = \bigodot_{i=1}^{k} \psi_i$ to be the class function of $GL(n,q)$, which is obtained by inducing ψ from P_λ to $GL(n,q)$. Formally $\bigodot_{i=1}^{k} \psi_i = \bigotimes_{i=1}^{k} \psi_i \uparrow_{P_\lambda}^{GL(n,q)}$.

Corollary 5.3.1. *If all ψ_i are characters of $GL(\lambda_i, q)$, $\forall 1 \leq i \leq k$ respectively, then $\bigodot_{i=1}^{k} \psi_i$ is a character of $GL(n, q)$.*

PROOF. Immediate. ■

Definition 5.3.1. *The process of obtaining characters of $GL(n, q)$ from those of P_λ, which in turns are obtained by lifting the characters of L_λ, is referred to as a* ***parabolic induction****.*

Note 5.3.1. Observe that parabolic induction produces characters of $GL(n, q)$ from characters of $GL(m, q)$ for $m < n$.

Let $\{B_1, B_2, \cdots, B_s\}$ be a left transversal of P_λ in $GL(n, q)$. Using the induction theorem of characters, which asserts that for $x \in GL(n, q) \setminus P_\lambda$, then $\phi\uparrow_{P_\lambda}^{GL(n,q)}(x) = 0$ for a character ϕ of P_λ. Then for all $A \in GL(n, q)$ we get $(\bigodot_{i=1}^{k} \psi_i)(A) = \sum_i \psi(B_i^{-1} A B_i)$, where the sum is made over all cosets $B_i P$ for which $B_i^{-1} A B_i \in P_\lambda$. Now

$$B^{-1}AB \in P_\lambda \Longleftrightarrow \mathfrak{F}_\lambda^{B^{-1}AB} = \mathfrak{F}_\lambda \Longleftrightarrow B^{-1}AB\mathfrak{F}_\lambda = \mathfrak{F}_\lambda \Longleftrightarrow AB\mathfrak{F}_\lambda = B\mathfrak{F}_\lambda \Longleftrightarrow B\mathfrak{F}_\lambda^A = B\mathfrak{F}_\lambda.$$

The last equality means that $B\mathfrak{F}_\lambda$ is a flag stabilized by the submodule V_A. For all $1 \leq i \leq k$, if we let $W_i = BV_i$, where V_i are the components of the flag $\mathfrak{F}_\lambda$ and if

$$B_i A B_i^{-1} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix}, \qquad (A_{ii} \in GL(\lambda_i, q)),$$

then the factor module W_i / W_{i-1} is isomorphic to $V_{A_{ii}}$ for all $1 \leq i \leq k$ (see Green [27] or MacDonald [50]).

Theorem 5.3.2. *Let ψ_i be a class function on $GL(\lambda_i, q)$, $1 \leq i \leq k$ and c be any conjugacy class of $GL(n, q)$. Then*

$$(\bigodot_{i=1}^{k} \psi_i)(c) = \sum_{\{c_1, c_2, \cdots, c_k\}} g^c_{c_1, c_2, \cdots, c_k} \prod_{i=1}^{k} \psi_i(c_i)$$

summed over all sequences $\{c_1, c_2, \cdots, c_k\}$, where c_i is a conjugacy class of $GL(\lambda_i, q)$ and $g^c_{c_1, c_2, \cdots, c_k}$ is the number of sequences

$$0 = W_0 \subset W_1 \subset \cdots \subset W_k = V(n, q)$$

of submodules of $V(n, q)$ such that the successive quotient $W_i / W_{i-1} \cong V_{c_i}$, $1 \leq i \leq k$. In particular,

$$(\bigodot_{i=1}^{k} \psi_i)(I_n) = \frac{\phi_n(q)}{\prod_{i=1}^{k} \phi_{\lambda_i}(q)} \psi_1(I_n)\psi_2(I_n) \cdots \psi_k(I_n). \tag{5.18}$$

Moreover, if c is a regular semisimple class of $GL(n, q)$, then it is sufficient to consider only the regular semisimple classes c_i of $GL(\lambda_i, q)$ to evaluate $(\bigodot_{i=1}^{k} \psi_i)(c)$.

PROOF. See Green [27] or MacDonald [50]. ■

Remark 5.3.1. The number $g^{c}_{c_1, c_2, \cdots, c_k}$ is known as the *Hall polynomial*. This polynomial, which will not be used in this dissertation again, is of great importance for the polynomial defined by Green, which they bear his name. One can refer to Green [27], Klein [42], MacDonald [50], Springer [71] or Zelevinsky [77] for more information concerning both of Green and Hall polynomials.

If ch_c is the characteristic function of a class c, which is defined over $x \in GL(n, q)$ by

$$ch_c(x) = \begin{cases} 1 & \text{if } x \in c, \\ 0 & \text{if } x \notin c, \end{cases}$$

then

$$\bigodot_{i=1}^{k} (ch_{c_i}) = \sum_{\{c_1, c_2, \cdots, c_k\}} g^{c}_{c_1, c_2, \cdots, c_k} ch_c.$$

Note 5.3.2. Note that the characteristic function ch_c is a class function and hence the $\odot-$product is well defined.

Proposition 5.3.3. *The $\odot-$product $\bigodot_{i=1}^{k} \psi_i$ is multilinear, associative and commutative.*

PROOF. See page 411 of Green [27]. ■

Let $\mathfrak{C}_n$ denotes the algebra of class functions of $GL(n, q)$ and let $\mathfrak{C} = \bigoplus_{n=0}^{\infty} \mathfrak{C}_n$, where $GL(0, q)$ is to be understood as the neutral group. Note that $\mathfrak{C}_0 = \mathbb{C}$. The $\odot-$product discussed above defines a multiplication on $\mathfrak{C}$. We can check that $\mathfrak{C}$ is a commutative and associative algebra over $\mathbb{C}$. The $\odot-$product generates characters of $GL(n, q)$ from characters of $GL(s, q)$ for $s < n$. In fact $\mathfrak{R} < \mathfrak{C}$, where $\mathfrak{R} = \bigoplus_{n=0}^{\infty} \mathfrak{R}_n$ and $\mathfrak{R}_n$ is the subalgebra consists of characters of $GL(n, q)$. Systematic use of $\mathfrak{C}$ has been made by Green [27] in showing that some functions (given by Definition 7.3 in his paper) of $GL(n, q)$ are integral linear combination of characters of $GL(n, q)$ and hence are characters themselves. For this purpose he defined a homomorphism from $\mathfrak{C}$ into $\mathfrak{S} = \bigoplus_{n=1}^{\infty} \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the algebra of symmetric polynomials in n variables.

5.4. Cuspidal Characters

We recall by Subsection 3.1.2 that a character of $GL(n, q)$ is said to be a *discrete series* or *cuspidal* character of $GL(n, q)$ if it is not a principal series character. In this section we discuss the cuspidal characters of $GL(n, q)$. These character have nice parametrization in terms of the non-decomposable characters of $\mathbb{F}_{q^n}^*$ (Subsection 5.4.1). We also discuss the values of these characters on classes of $GL(n, q)$ (Theorem 5.4.4 and Equation (5.19)) and finally we show the importance of the cuspidal characters for other characters of $GL(n, q)$ (Theorem 5.4.6).

5.4.1 Parametrization of the Cuspidal Characters

Let $\mathbb{F}_{q^n} : \mathbb{F}_q$ be a field extension. We recall that the Galois group $\Gamma = \Gamma(\mathbb{F}_{q^n} : \mathbb{F}_q)$ is a cyclic group of size n generated by the Frobenius automorphism $\sigma_q : a \longmapsto a^q, \ \forall a \in \mathbb{F}_{q^n}$. Note that $(\sigma_q)^j = \sigma_{q^j}, \forall 0 \leq j \leq n-1$. Also Γ acts on the maximal torus $\mathbb{F}_{q^n}^*$ by $a^{\sigma_q^j} = a^{q^j}$, for $\forall 0 \leq j \leq n-1$ and all $a \in \mathbb{F}_{q^n}^*$. On the other hand, if θ is a character of $\mathbb{F}_{q^n}^*$ and $\sigma_{q^j} \in \Gamma$, then we define $\theta^{\sigma_{q^j}}$ by $\theta^{\sigma_{q^j}} = \theta^{q^j}$, where $\theta^{q^j}(a) = \theta(a^{q^j}) = \theta(a^{\sigma_{q^j}}), \ a \in \mathbb{F}_{q^n}^*$.

Definition 5.4.1. *Let θ be a character of $\mathbb{F}_{q^n}^*$. For $0 \leq i \leq n-1$, the* ***conjugate character*** *$\overline{\theta}_i$ of θ is defined to be $\overline{\theta}_i = \theta^{\sigma_{q^i}} = \theta^{q^i}$.*

Note 5.4.1. It is clear that Γ has dual action on $\mathbb{F}_{q^n}^*$ and on its character group also.

Over $\mathbb{F}_{q^n}$ we know that $(a^d - 1)|(a^n - 1) \Longleftrightarrow d|n$. For any d dividing n we define the *norm map* $N_{n,d} : \mathbb{F}_{q^n}^* \longrightarrow \mathbb{F}_{q^d}^*$ by

$$N_{n,d}(a) = a^{\frac{q^n-1}{q^d-1}} = \prod_{i=0}^{\frac{n}{d}-1} a^{\sigma_q^{di}} = \prod_{i=0}^{\frac{n}{d}-1} a^{q^{di}}.$$

It is not difficult to check that $N_{n,d}$ is a group homomorphism.

Definition 5.4.2. *A character θ of $\mathbb{F}_{q^n}^*$ is said to be* ***non-decomposable*** *if there is not any d dividing n such that $\theta = N_{n,d} \circ \chi$ for any character χ of $\mathbb{F}_{q^d}^*$.*

This means that θ does not factor through the norm map $N_{n,d} : \mathbb{F}_{q^n}^* \longrightarrow \mathbb{F}_{q^d}^*$ for any d dividing n.

Remark 5.4.1. Some authors refer to a non-decomposable character by *regular* or *primitive* character.

The next Proposition detects the non-decomposable characters of $\mathbb{F}_{q^n}^*$.

Proposition 5.4.1. *A character θ of $\mathbb{F}_{q^n}^*$ is non-decomposable if and only if all of its conjugates are distinct.*

PROOF. See Prasad [60] or Reyes [61]. ■

Let $ND(\mathbb{F}_{q^n}^*)$ denotes the set of all non-decomposable characters of $\mathbb{F}_{q^n}^*$. Since Γ acts on the character group of $\mathbb{F}_{q^n}^*$, it follows that Γ acts on $ND(\mathbb{F}_{q^n}^*)$ and if $\theta', \theta \in ND(\mathbb{F}_{q^n}^*)$, then we have

$$\theta' \in \theta^\Gamma \text{ if and only if } \theta' = \theta^{q^i} \text{ for some } 0 \leq i \leq n-1.$$

Note 5.4.2. Note that each Galois orbit $\theta^\Gamma = \{\theta, \theta^q, \theta^{q^2}, \cdots, \theta^{q^{n-1}}\}$ and hence $|\theta^\Gamma| = n$.

Proposition 5.4.2. *The number of Galois orbits θ^Γ is given by $\frac{1}{n}\sum_{d|n} \mu(d) q^{\frac{n}{d}}$.*

PROOF. See Lemma 7.7 of Green [27]. ■

Remark 5.4.2. Green [27] constructed an entity called a *simplex* as follows. Let $\overline{\mathbb{F}}_{q^n}^* = \mathbb{F}_{q^{n!}}^* = \langle \varepsilon \rangle$ and $\varepsilon_s = \varepsilon^{\frac{q^{n!}-1}{q^s-1}}$, for $1 \leq s \leq n$. Then $\mathbb{F}_{q^s}^* = \langle \varepsilon_s \rangle$, $\forall s$. Thus every element of $\mathbb{F}_{q^s}^*$ has the form ε_s^k, where k is uniquely determined $mod(q^s-1)$. The element ε_s^k has s distinct conjugates $\varepsilon_s^k, \varepsilon_s^{kq}, \cdots, \varepsilon_s^{kq^{s-1}}$ if and only if the set $\mathfrak{S} = \{k, kq, \cdots, kq^{s-1}\}$ forms a complete set of residues $mod(q^s-1)$. The set $\mathfrak{S}$ is called an *s-simplex* $\mathfrak{S}$ or a *simplex* $\mathfrak{S}$ of degree s. Each of the integers kq^i, $0 \leq i \leq s-1$ is called a *root* of the simplex $\mathfrak{S}$ of degree s. Green [27] established a canonical bijection between the set $\mathcal{S}$ of all distinct simplexes $\mathfrak{S}$ of degree $\leq n$ and the set $\mathcal{F}$ of all distinct irreducible polynomials over $\mathbb{F}_q$ of degree $\leq n$.

We can see that $\omega : \theta^\Gamma \longrightarrow \mathfrak{S}$ given by $\omega(\theta^{q^j}) = kq^j$, $\forall 0 \leq j \leq n-1$ is a bijective function.

The cuspidal characters of $GL(n,q)$ have nice parametrization in terms of elements of $ND(\mathbb{F}_{q^n}^*)$. To each ${\theta_k}^\Gamma$, where $\theta_k \in ND(\mathbb{F}_{q^n}^*)$, we associate a cuspidal character χ_{θ_k} of $GL(n,q)$ as follows: Consider the integers $1 \leq k \leq q^n-1$ such that $\sum_{i=0}^{\frac{n}{d}-1} q^{di} \nmid k$, $\forall d$, d dividing n. This to grant that we are considering only the non-decomposable characters of $\mathbb{F}_{q^n}^*$. For such k, we exclude $kq, kq^2, \cdots, kq^{n-1} mod(q^n-1)$. In the computations, we use χ_k in place of χ_{θ_k} for the appropriate k.

We summarize the foregoing discussion in the following Theorem, which is the main theorem of this subsection.

Theorem 5.4.3. *The number of cuspidal characters of $GL(n,q)$ is same as the number of regular semisimple classes of $GL(n,q)$ of type $\lambda = (n) \vdash n$, which is equal to the number of irreducible polynomials of degree n over $\mathbb{F}_q$. This number is given by $\frac{1}{n}\sum_{d|n} \mu(d) q^{\frac{n}{d}}$.*

5.4.2 Values of the Cuspidal Characters on Classes of $GL(n, q)$

The values of the cuspidal characters of $GL(n, q)$ on all the conjugacy classes of $GL(n, q)$ are easy to compute. We follow the description of Green [28]. Suppose that $f \in \mathbb{F}_q[t]$ with a root α. By Lemma 5.2.9 we know that $\alpha^q, \alpha^{q^2} \cdots \alpha^{q^{\partial f-1}}$ are the other roots of f. Let θ be an arbitrary character of $\mathbb{F}_{q^n}^*$. We define $\theta(f)$ by $\theta(f) = \sum_{i=0}^{\partial f-1} \theta(\alpha^{q^i})$. Now if $d|n$, we identify θ with $\theta\downarrow_{\mathbb{F}_{q^d}^*}^{\mathbb{F}_{q^n}^*}$. Next we define the class function χ_θ on $g \in GL(n, q)$ by

$$\chi_\theta(g) = \begin{cases} \theta(f)\phi_{l(\sigma)-1}(q^{\partial f}) & \text{if } [g] \text{ is a primary class with } \sigma \vdash \frac{n}{\partial f}, \\ 0 & \text{if } [g] \text{ is not a primary class,} \end{cases} \tag{5.19}$$

where $\phi_{l(\sigma)-1}$ is the function defined in (5.2).

Theorem 5.4.4. *The class function χ_θ defined in (5.19) is a generalized character of $GL(n, q)$ for any character θ of $\mathbb{F}_{q^n}^*$ and if $\theta_k \in ND(\mathbb{F}_{q^n}^*)$, then $(-1)^{n-1}\chi_{\theta_k} \in Irr(GL(n, q))$.*

PROOF. See Fulton [22], Green [27] or Green [28]. ∎

Example 5.4.1. Consider the central elements $g = \alpha I_n = (t - \alpha, 1, n, 1^n)$, $\alpha \in \mathbb{F}_q^*$ of $GL(n, q)$, which are self-classes. Since the characteristic polynomial of g is $(t - \alpha)^n$, it follows by Definition 5.2.2 that $[g]$ is primary. If θ_k is any character of $\mathbb{F}_{q^n}^*$ and $\mathbb{F}_{q^n}^* = \langle\theta\rangle$, then

$$\theta_k(f) = \theta_k(t - \alpha) = \sum_{i=0}^{1-1} \theta_k(\alpha^{q^i}) = \theta_k(\alpha) = \theta(\alpha^k).$$

Also

$$\phi_{l(\sigma)-1}(q^{\partial f}) = \phi_{l(1^n)-1}(q) = \prod_{i=1}^{n-1}(1 - q^i) = (1 - q)(1 - q^2)\cdots(1 - q^{n-1}).$$

Now if $\theta_k \in ND(\mathbb{F}_{q^n}^*)$, then Theorem 5.4.4 asserts that $(-1)^{n-1}\chi_{\theta_k} \in Irr(GL(n, q))$ and at $g = \alpha I_n$, we have

$$\begin{aligned} (-1)^{n-1}\chi_{\theta_k}(\alpha I_n) &= (-1)^{n-1}(1 - q)(1 - q^2)\cdots(1 - q^{n-1})\theta(\alpha^k) \\ &= (q - 1)(q^2 - 1)\cdots(q^{n-1} - 1)\theta(\alpha^k). \end{aligned}$$

In Example 5.4.1 if $\alpha = 1$, that is the identity matrix of $GL(n, q)$, then we have the following Corollary.

Corollary 5.4.5. *The degree of a cuspidal character of $GL(n, q)$ is $(q - 1)(q^2 - 1)\cdots(q^{n-1} - 1)$.*

The next theorem is of great importance for characters of $GL(n, q)$. It shows that the cuspidal characters are the atoms from which any character of $GL(n, q)$ is build up.

Theorem 5.4.6. *Every character of $GL(n,q)$ is either cuspidal or a constituent of an $\bigodot$ −product of cuspidal characters.*

PROOF. The proof is inductively on n. By definition, all characters of $GL(1,q)$ are cuspidal. Assume the result is true $\forall 1 \leq m \leq n-1$. Let $\chi \in Irr(GL(n,q))$. If χ is cuspidal, then there is nothing to prove. Let χ be a principal series character. Then χ is a constituent of $\bigotimes_{i=1}^{k} \chi_i \uparrow_{P_\lambda}^{GL(n,q)}$ for some P_λ such that $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t) \in \mathcal{P}(n)$, $t \neq 1$ and $\chi_i \in Irr(GL(\lambda_i, q))$. By hypothesis each χ_i is either cuspidal or a constituent of $\bigodot$ −product of some cuspidal characters. Hence χ is a constituent of an $\bigodot$ −product of cuspidal characters. ■

Remark 5.4.3. By the above theorem (or by definition), we know that any $\chi_k \in Irr(GL(1,q)) = Irr(\mathbb{F}_q^*)$ is a cuspidal character. Recall that characters of $GL(2,q)$ fall into four types, where characters of type $\chi^{(4)}$ are the cuspidal characters of $GL(2,q)$. We also recall that $\chi_{k,l}^{(3)} = \chi_k \chi_l \uparrow_{UT(2,q)}^{GL(2,q)} = \chi_k \odot \chi_l$, while the characters $\chi_k^{(1)}$ and $\chi_k^{(2)}$ appeared as constituents of $\chi_{k,k}$ where $\chi_k \chi_k \uparrow_{UT(2,q)}^{GL(2,q)} = \chi_k \odot \chi_k$. This show that any $\chi_{k,l}^{(3)}$ is an $\bigodot$-product of cuspidal characters, while any character $\chi_k^{(1)}$ or $\chi_k^{(2)}$ appears as a constituent of an $\bigodot$-product of cuspidal characters. This confirms Theorem 5.4.6 for $GL(2,q)$.

We illustrate the indexing and the values of the cuspidal characters of $GL(2,q)$, $GL(3,q)$ and $GL(4,q)$ in the following examples.

Example 5.4.2. Consider $GL(2,q)$ and let θ_k be a character of $\mathbb{F}_{q^2}^*$. We determine the non-decomposable characters of $\mathbb{F}_{q^2}^*$. The norm map $N_{2,1} : \mathbb{F}_{q^2}^* \longrightarrow \mathbb{F}_q^*$ is given by $N_{2,1}(r) = r.r^q = r^{q+1}$, $r \in \mathbb{F}_{q^2}^*$. Now θ_k has two conjugate characters, namely θ_k itself and $\overline{\theta}_k = \theta_k^q$, where θ_k^q is given by $\theta_k^q(r) = \theta_k(r^q)$. If $q+1|k$, we can see that $\theta_k^q = \theta_k$. Therefore $\theta_k \in ND(\mathbb{F}_{q^2}^*)$ if and only if $k \in K = \{1, 2, \cdots, q^2-1\} - \{q+1, 2(q+1), \cdots, (q-1)(q+1)\}$. Thus $|K| = q^2 - q$. It easily to see that for any $k \in K$, we have $\theta_{kq} \in K$ and $\theta_{kq} = \theta_k$. Therefore in indexing the cuspidal characters of $GL(2,q)$, whenever we choose $k \in K$, we take off kq from K. Thus we get a set of $\frac{q^2-q}{2}$ such k to index the cuspidal characters χ_k of $GL(2,q)$.

Next we calculate the values of the cuspidal characters χ_{θ_k} on classes of $GL(2,q)$. Let g_1, g_2, g_3, g_4 be elements in classes of types $\mathcal{T}^{(1)}$, $\mathcal{T}^{(2)}$, $\mathcal{T}^{(3)}$ $\mathcal{T}^{(4)}$ respectively. By Example 5.4.1 we have $\chi_{\theta_k}(g_1) = (-1)^{2-1}(1-q)\theta_k(\alpha) = (q-1)\theta_k(\alpha) = (q-1)\widehat{\alpha}^k$. The characteristic polynomial of g_2 is $f(t) = (t-\alpha)$ for some $\alpha \in \mathbb{F}_q^*$ and the associated partition to $[g_2]$ is $\lambda = (2) \vdash 2$. Thus by (5.19), we have $\theta_k(f) = \theta_k(t-\alpha) = \theta_k(\alpha) = \widehat{\alpha}^k$. Also $\phi_{l(\lambda)-1}(q^{\partial f}) = \phi_{1-1}(q) = 1$. Now Theorem 5.4.4 asserts that $\chi_{\theta_k}(g_2) = (-1)^{2-1}\theta_k(\alpha) = -\widehat{\alpha}^k$. The characteristic polynomial of g_3 splits into two distinct linear factors. Immediately, $\chi_{\theta_k}(g_3) = 0$. The last case where g_4 has characteristic polynomial $f(t) = t^2 + at + b \in \mathbb{F}_q[t]$, which is irreducible, that is f has eigenvalues r and r^q where

$r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Here we have $\theta_k(f) = \sum_{i=0}^{\partial f-1} \theta_k(r^{q^i}) = \theta_k(r) + \theta_k(r^q)$. Also $\phi_{l(\lambda)-1}(q^{\partial f}) = \phi_{1-1}(q^2) = 1$. By Theorem 5.4.4 we have

$$\chi_{\theta_k}(g_4) = (-1)^{2-1}(\theta_k(r) + \theta_k(r^q)) = -(\theta_k(r) + \theta_k(r^q)) = -(\widehat{r}^k + \widehat{r}^{kq}).$$

This completes the cuspidal characters of $GL(2, q)$.

Sometimes we may write χ_k in place of χ_{θ_k}.

Example 5.4.3. The case $n = 3$ is very similar to the case $n = 2$ since 2 and 3 are both prime numbers. Consider $k \in \{1, 2, \cdots, q^3 - 1\}$ such that $q^2 + q + 1 \nmid k$. If we choose such k, we exclude kq and kq^2 from the set $\{1, 2, \cdots, q^3 - 1\}$. Note that if $q^2 + q + 1 \nmid k$, it does not also divide kq or kq^2. We get $\frac{q^3-q}{3}$ cuspidal characters of $GL(3, q)$, which their values are given by

$$\chi_k(g) = \begin{cases} (q-1)^2(q+1)\widehat{\alpha}^k & \text{if } g \text{ is of type } \mathcal{T}^{(1)}, \\ -(q-1)\widehat{\alpha}^k & \text{if } g \text{ is of type } \mathcal{T}^{(2)}, \\ \widehat{\alpha}^k & \text{if } g \text{ is of type } \mathcal{T}^{(3)}, \\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(4)}, \\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(5)}, \\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(6)}, \\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(7)}, \\ \widehat{s}^k + \widehat{s}^{kq} + \widehat{s}^{kq^2} & \text{if } g \text{ is of type } \mathcal{T}^{(8)}. \end{cases} \tag{5.20}$$

Example 5.4.4. We calculate the cuspidal characters of $GL(4, q)$. Firstly we determine the non-decomposable characters of $\mathbb{F}^*_{q^4}$. Assume that $k \in \{1, 2, \cdots, q^4 - 1\}$ and let θ_k be a character of $\mathbb{F}^*_{q^4}$. We consider the norm maps $N_{4,1} : \mathbb{F}^*_{q^4} \longrightarrow \mathbb{F}^*_q$ and $N_{4,2} : \mathbb{F}^*_{q^4} \longrightarrow \mathbb{F}^*_{q^2}$, which are given by $N_{4,1}(r) = r^{\frac{q^4-1}{q-1}} = r^{q^3+q^2+q+1}$ and $N_{4,2}(r) = r^{\frac{q^4-1}{q^2-1}} = r^{q^2+1}$, for all $r \in \mathbb{F}^*_{q^4}$. Now $\theta_k \in ND(\mathbb{F}^*_{q^4})$ if and only if $q^3 + q^2 + q + 1 \nmid k$ and $q^2 + 1 \nmid k$. Note that $q^3 + q^2 + q + 1 = (q^2 + 1) + q(q^2 + 1) = (q + 1)(q^2 + 1)$. This is reduced to say that $\theta_k \in ND(\mathbb{F}^*_{q^4})$ if and only if $q^2 + 1 \nmid k$. Equivalently $\theta_k \in ND(\mathbb{F}^*_{q^4}) \iff k \in \{1, 2, \cdots, q^4 - 1\} \setminus \{q^2 + 1, 2(q^2 + 1), \cdots, (q^2 - 1)(q^2 + 1)\}$. This gives $(q^4 - 1) - (q^2 - 1) = q^4 - q^2$ non-decomposable characters of $\mathbb{F}^*_{q^4}$. Now each orbit of the action of $\Gamma = \Gamma(\mathbb{F}_{q^4} : \mathbb{F}_q)$ on $ND(\mathbb{F}^*_{q^4})$ consists of four conjugate characters namely, ${\theta_k}^\Gamma = \{\theta_k, \theta_k^q, \theta_k^{q^2}, \theta_k^{q^3}\}$. To parameterize a cuspidal character of $GL(4, q)$, we choose from each ${\theta_k}^\Gamma$ a representative character θ_k since $\chi_{\theta_k} = \chi_{\theta_k^q} = \chi_{\theta_k^{q^2}} = \chi_{\theta_k^{q^3}}$. Therefore we have $\frac{1}{4}(q^4 - q^2)$ cuspidal characters χ_{θ_k} of $GL(4, q)$. Note that $I_4(q) = \frac{1}{4}(q^4 - q^2)$.

To evaluate the cuspidal characters on classes of $GL(4,q)$ we use Theorem 5.4.4 and similar steps used in calculating the values of the cuspidal characters of $GL(2,q)$. For example consider $[g]$ of type $\mathcal{T}^{(18)}$, which is given by the data $((t^2+at+b),2,2,(1,1))$. Let r and r^q be the roots of $f(t)=t^2+at+b$. Then $\theta_k(f)=\theta_k(r)+\theta_k(r^q)$ and $\phi_{l(\lambda)-1}(q^{\partial f})=\phi_{l((1,1))-1}(q^2)=(1-q^2)$. Therefore we have

$$\chi_k=\chi_{\theta_k}=(-1)^{4-1}(1-q^2)(\theta_k(r)+\theta_k(r^q))=(q^2-1)(\widehat{r}^k+\widehat{r}^{kq}).$$

Similarly we can calculate the values of the cuspidal characters on all other primary classes of $GL(4,q)$. Let $\Delta=\{6,7,8,9,10,11,12,13,14,15,16,17,18,21\}$. In Table 5.4 we have skipped giving relevant information for classes of types $\mathcal{T}^{(i)}$ for $i\in\Delta$, since all classes of these types are not primary and by Theorem 5.4.4 the values of the cuspidal characters on these classes are zero. The values of the cuspidal characters on classes of $GL(4,q)$ are given in Table 5.4.

Remark 5.4.4. Note that the values of the cuspidal characters of $GL(2,q)$ given by Example 5.4.2, where we used the non-decomposable characters of $\mathbb{F}_{q^2}^*$, the same as the values of $\chi_k^{(4)}=\pi_k$ given in Table 4.2, where π_k is written as a combination in terms of some characters of $GL(2,q)$.

Table 5.4: Values of the cuspidal characters of $GL(4,q)$

Type of classes	Roots of the characteristic polynomial of g	Associated partition λ with the class	$\theta_k(f)$	$\phi_{l(\lambda)-1}(q^{\partial f})$	$(-1)^{n-1}\chi_{\theta_k}$
$\mathcal{T}^{(1)}$	$\alpha,\ \alpha \in \mathbb{F}_q^*$	1^4	$\theta_k(\alpha)$	$(q-1)(q^2-1)(q^3-1)$	$(q-1)(q^2-1)(q^3-1)\theta_k(\alpha)$
$\mathcal{T}^{(2)}$	$\alpha,\ \alpha \in \mathbb{F}_q^*$	$(2,1,1)$	$\theta_k(\alpha)$	$(q-1)(q^2-1)$	$-(q-1)(q^2-1)\theta_k(\alpha)$
$\mathcal{T}^{(3)}$	$\alpha,\ \alpha \in \mathbb{F}_q^*$	$(2,2)$	$\theta_k(\alpha)$	$-(q-1)$	$(q-1)\theta_k(\alpha)$
$\mathcal{T}^{(4)}$	$\alpha,\ \alpha \in \mathbb{F}_q^*$	$(3,1)$	$\theta_k(\alpha)$	$-(q-1)$	$(q-1)\theta_k(\alpha)$
$\mathcal{T}^{(5)}$	$\alpha,\ \alpha \in \mathbb{F}_q^*$	(4)	$\theta_k(\alpha)$	-1	$\theta_k(\alpha)$
$\mathcal{T}^{(19)}$	$r, r^q,\ r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$	$(1,1)$	$\theta_k(r)+\theta_k(r^q)$	$-(q^2-1)$	$(q^2-1)(\theta_k(r)+\theta_k(r^q))$
$\mathcal{T}^{(20)}$	$r, r^q,\ r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$	(2)	$\theta_k(r)+\theta_k(r^q)$	1	$\theta_k(r)+\theta_k(r^q)$
$\mathcal{T}^{(22)}$	s, s^q, s^{q^2}, $s^{q^3},\ s \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$	(2)	$\theta_k(s)+\theta_k(s^q)$ $\theta_k(s^{q^2})+\theta_k(s^{q^3})$	1	$\theta_k(s)+\theta_k(s^q)$ $+\theta_k(s^{q^2})+\theta_k(s^{q^3})$
$\mathcal{T}^{(i)},\ i \in \Delta$					0

5.5. Steinberg Characters

Let V be an $(n+1)-$dimensional vector space over a field $\mathbb{F}$ and let V^* denotes the set of nonzero vectors $\mathbf{x} = (x_0, x_1, \cdots, x_n)$ of V, that is $V^* = V - \{\mathbf{0}\}$. We define an equivalence relation $\sim$ on V^* by

$$\mathbf{x} \sim \mathbf{y} \text{ in } V^* \text{ if and only if } \mathbf{x} = \lambda \mathbf{y}, \text{ for some } \lambda \in \mathbb{F}^*.$$

Obviously this is an equivalence relation. In the rest of this section let us focus on the case where $\mathbb{F} = \mathbb{F}_q$. It is clear that $|V^*| = q^{n+1} - 1$. If we denote the equivalence class containing $\mathbf{x}$ by $[\mathbf{x}]$, then by definition $|[\mathbf{x}]| = q - 1$. The set of all equivalence classes of V^* will be denoted by $\mathcal{D}$ and will be called *projective* $n-$*space*. Immediately we can see that

$$|\mathcal{D}| = \frac{q^{n+1} - 1}{q - 1} = q^n + q^{n-1} + \cdots + 1.$$

Each class $[\mathbf{x}]$ in the projective $n-$space will be called a *projective point* or simply a *point* of $\mathcal{D}$.

Note 5.5.1. Observe that the points of $\mathcal{D}$ are in fact the orbits of the action of $\mathbb{F}^*$ on V^* given by $\mathbf{x}^\lambda = \lambda \mathbf{x}$ for $\lambda \in \mathbb{F}^*$ and $\mathbf{x} \in V^*$.

Definition 5.5.1. *A* line $L(\alpha)$, *where* $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in V^*$ *is defined to be the set of all points* $[\mathbf{x}]$ *of* $\mathcal{D}$ *such that*

$$\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n = 0. \tag{5.21}$$

Note that if $\mathbf{x}$ satisfies (5.21), then so does $\lambda \mathbf{x}$, $\forall \lambda \in \mathbb{F}_q^*$. Thus $L(\alpha) = L(\lambda\alpha)$, $\forall \lambda \in \mathbb{F}_q^*$. For arbitrary $x_1, x_2, \cdots, x_n \in \mathbb{F}_q$ and for fixed $\alpha \in V^*$ where assumed that $\alpha_0 \neq 0$, then (5.21) has exactly $q^n - 1$ solutions $(x_0, x_1, x_2, \cdots, x_n) \in V^*$. Note that x_0 is governed by (5.21). Since each point $[\mathbf{x}]$ contains $q - 1$ vectors of V^*, there are exactly $\frac{q^n - 1}{q-1} = q^{n-1} + q^{n-2} + \cdots + 1$ points $[\mathbf{x}]$ satisfying (5.21). In other words, there are $q^{n-1} + q^{n-2} + \cdots + 1$ points on each line.
Finally it can be shown that any two distinct points $[\mathbf{x}]$ and $[\mathbf{y}]$ are contained in exactly $\frac{q^{n-1}-1}{q-1} = q^{n-2} + q^{n-3} + \cdots + 1$ lines.

More generally if W is a subspace of V, then we define

$$[W] = \{[x] | \ x \in W^*\} \subseteq \mathcal{D}.$$

If W is an $(m+1)-$dimensional subspace of $V(n+1, q)$, then W is called a *projective* $m-$*subspace*, and we say that W has *projective dimension* m. In particular if $m = 0, 1, 2$ or $n - 1$, that is W is a $1-$dimensional, $2-$dimensional, $3-$dimensional or $n-$dimensional subspace of V respectively, then W is called *point, line, plane* or *hyperplane* respectively. In general the $i-$dimensional subspace is called an $i-$*flat*.

Definition 5.5.2. *Let $V(n+1,q)$ be a vector space. The lattice of subspaces of $V(n+1,q)$ of dimension at least 1 is called a **Projective Geometry** and is denoted by $PG(n,q)$; i.e., the structure which contains points, lines, planes, hyperplanes, etc.*

Remark 5.5.1. The triple $\left(\frac{q^{n+1}-1}{q-1}, \frac{q^n-1}{q-1}, \frac{q^{n-1}-1}{q-1}\right)$ forms a design.

Example 5.5.1. Let $n=1$. The projective geometry $PG(1,q)$ consists of the vectors $(x_0,x_1) \in \mathbb{F}_q^2 - (0,0)$. These q^2-1 vectors divided into $\frac{q^2-1}{q-1} = q+1$ classes (points), which are all contained in one line.
Now the group $GL(2,q)$ acts on $PG(1,q)$ by the mean that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,q)$ and $\lambda \in \mathbb{F}_q^*$, then

$$g[\mathbf{x}] = g(\lambda x_0, \lambda x_1) = (\lambda(ax_0+bx_1), \lambda(cx_0+dx_1)) = \lambda(ax_0+bx_1, cx_0+dx_1).$$

This action is transitive and hence the resulting permutation character Δ contains the trivial character **1** once. For any $g \in GL(2,q)$, let $Fix(g)$ be the set

$$Fix(g) = \{[\mathbf{x}] \in \mathcal{D} |\ g[\mathbf{x}] = [\mathbf{x}]\}.$$

We consider the four types of elements of $GL(2,q)$ in this action.

1. Let $g = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ be any element of type $\mathcal{T}^{(1)}$. Then

$$g[\mathbf{x}] = g(\lambda x_0, \lambda x_1) = (\alpha\lambda x_0, \alpha\lambda x_1)) = \gamma(x_0,x_1) = [\mathbf{x}],$$

   i.e., any point of $PG(1,q)$ is fixed by the central elements of $GL(2,q)$, that is $|Fix(g)| = q+1$. Therefore $\deg \Delta = q+1$.

2. Let $g = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ be any element of type $\mathcal{T}^{(2)}$. Then

$$g[\mathbf{x}] = g(\lambda x_0, \lambda x_1) = (\alpha\lambda x_0 + \lambda x_1, \alpha\lambda x_1) \in [\mathbf{x}] = (\gamma x_0, \gamma x_1) \iff x_1 = 0,$$

   i.e, all the $q-1$ vectors $(x_0, 0)$ of the unique point $\mathbf{x}$ are fixed by elements of type $\mathcal{T}^{(2)}$. Therefore $|Fix(g)| = 1$.

3. Let $g = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ be any element of type $\mathcal{T}^{(3)}$. Then

$$g[\mathbf{x}] = g(\lambda x_0, \lambda x_1) = (\alpha\lambda x_0, \beta\lambda x_1) = \gamma(x_0,x_1) \in [\mathbf{x}] \iff \text{ either } x_0 = 0 \text{ or } x_1 = 0.$$

   The points $\mathbf{x}$ and $\mathbf{y}$ represented by the vectors $(1,0)$ and $(0,1)$ respectively are linearly independent in $\mathbb{F}_q^2$, since there exists no $\gamma \in \mathbb{F}_q^*$ such that $\mathbf{x} = \gamma\mathbf{y}$. We deduce that $|Fix(g)| = 2$.

4. Let $g = \begin{pmatrix} 0 & 1 \\ -r^{q+1} & r + r^q \end{pmatrix}$ be any element of type $\mathcal{T}^{(4)}$. Then

$$g[\mathbf{x}] = g(\lambda x_0, \lambda x_1) = (\lambda x_1, -\lambda r^{q+1} x_0 + \lambda(r + r^q)x_1) \neq \gamma(x_0, x_1) \in [\mathbf{x}], \text{ for some } \gamma \in \mathbb{F}_q^*.$$

Thus there is no point of $PG(1, q)$ fixed by an element of type $\mathcal{T}^{(4)}$, i.e., $|Fix(g)| = 0$.

For convenience we list the values of Δ in Table 5.5.

Table 5.5: Values of the permutation character Δ on classes of $GL(2, q)$

	$\mathcal{T}_k^{(1)}$	$\mathcal{T}_k^{(2)}$	$\mathcal{T}_{k,l}^{(3)}$	$\mathcal{T}_k^{(4)}$
No. of CC	$q-1$	$q-1$	$(q-1)(q-2)/2$	$q(q-1)/2$
$\lvert C_g\rvert$	1	q^2-1	$q(q+1)$	$q(q-1)$
Δ	$q+1$	1	2	0

Now

$$\begin{aligned}
\langle \Delta, \Delta \rangle &= \frac{1}{|G|} \sum_{g \in G} \Delta(g)\overline{\Delta(g)} \\
&= \frac{1}{q(q-1)^2(q+1)} \left((q-1)(q+1)^2 + (q-1)(q^2-1) + 4\frac{(q-1)(q-2)}{2} q(q+1) \right) \\
&= \frac{1}{q(q-1)^2(q+1)} \left(2q(q-1)^2(q+1)\right) = 2.
\end{aligned}$$

Therefore $\Delta = \mathbf{1} + ST$, where $ST \in Irr(GL(2, q))$. Hence $ST = \Delta - \mathbf{1}$.

Note 5.5.2. Note that the values of the character ST we have obtained recently is same as the values of the Steinberg character St have been found in page 50.

In [72] Steinberg found $|\mathcal{P}(n)|$ irreducible characters of $GL(n, q)$ corresponding to the partitions $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \vdash n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. He used the underlying geometry of a vector space $V(n, q)$. These characters are closely related to the irreducible characters of the Symmetric group S_n.

We recall from subsection 3.1.2 that $GL(n, q)$ acts on the set F consisting of all flags of $V(n, q)$ in a natural way. This action is intransitive and the resulting orbits are in fact the equivalence classes defined by $\sim$ in page 25. It follows by Proposition 3.1.6 that these orbits are in $1-1$ correspondence with $\mathcal{P}(n)$. Let $\mathfrak{F}_\lambda$ be a representative for the orbit $[\mathfrak{F}_\lambda]$. By definition $GL(n, q)_{\mathfrak{F}_\lambda} = P_\lambda$. Due to the action of $GL(n, q)$ on F, we get a permutation character $C^{(\lambda)}$, which is in fact $\mathbf{1}\uparrow_{P_\lambda}^{GL(n,q)}$. Using equation (3.5), the degree $\deg C^{(\lambda)}$ is given by

$$\deg C^{(\lambda)} = \frac{\{n\}}{\{\lambda_1\}\{\lambda_2\}\cdots\{\lambda_n\}}. \tag{5.22}$$

The character $C^{(\lambda)}$ of $GL(n,q)$ is not irreducible in general.

Note 5.5.3. Note that for the partition $\lambda = (n) \vdash n$, the permutation character $C^{(\lambda)}$ is the trivial character since $P_{(n)} = GL(n,q)$.

To see how $C^{(\lambda)}$ breaks into a sum of irreducible characters, we use of the analogy between $GL(n,q)$ and S_n. With λ being the previous partition, we partition the set $\{1,2,\cdots,n\}$ into subsets consisting of $\lambda_1, \lambda_2, \cdots, \lambda_n$ integers. Let $S_\lambda = \bigotimes_{i=1}^{n} S_{\lambda_i}$, where a factor S_0 is ignored. Then $S_\lambda \leq S_n$ and clearly has index in S_n given by

$$[S_n : S_\lambda] = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_n!}.$$

Next we let $S^{(\lambda)} = \mathbf{1}\uparrow_{S_\lambda}^{S_n}$. Thus $\deg S^{(\lambda)} = n!/\lambda_1!\lambda_2!\cdots\lambda_n!$. Now Corollary 1 of Steinberg [72] reads the following.

Theorem 5.5.1. *The permutation characters $C^{(\lambda)}$ and $S^{(\lambda)}$ split into irreducible characters in exactly the same manner. That is if $C^{(\lambda)} = \sum_{i=1}^{m} d_i\chi_i$, where $\chi_i \in Irr(GL(n,q))$, then $S^{(\lambda)} = \sum_{i=1}^{m} d_i\widetilde{\chi}_i$, where $\widetilde{\chi}_i \in Irr(S_n)$.*

To find the Steinberg characters we follow the following steps:

1. If $\mathcal{P}(n) = \{\lambda_1, \lambda_2, \cdots, \lambda_{|\mathcal{P}(n)|}\}$, then order the partitions λ_i in ascending order as defined in Section 5.1 and renumber them in such away that if $i < j$, then $\lambda_i < \lambda_j$. That is $\lambda_1 = (n) < \lambda_2 = (n-1,1) < \cdots < \lambda_{|\mathcal{P}(n)|} = (1,1,\cdots,1)$.

2. Determine the values of $C^{(\lambda)} = \mathbf{1}\uparrow_{P_\lambda}^{GL(n,q)}$, $\forall\lambda \in \mathcal{P}(n)$ on classes of $GL(n,q)$. We know that $C^{(n)} = \mathbf{1}$ the trivial character, since $P_{(n)} = GL(n,q)$. Consider $C^{(\lambda_1)}, C^{(\lambda_2)}, \cdots, C^{(\lambda_{|\mathcal{P}(n)|})}$ in this order.

3. Consider $S^{(\lambda_1)}, S^{(\lambda_2)}, \cdots, S^{(\lambda_{|\mathcal{P}(n)|})}$ in this order, where we know $S^{(n)} = \mathbf{1}$ the trivial character, since $S_{(n)} = S_n$.

4. Start by decomposing each $\mathbf{1}\uparrow_{S_\lambda}^{S_n}$ into its irreducible constituents in the order given above. For example $S^{(n)} = \mathbf{1}$, $S^{(n-1,1)} = \mathbf{1} + \chi$ and $\chi \in Irr(S_n)$. Then find $S^{(n-2,2)}$, $S^{(n-2,1,1)}$ and so forth till $S^{(1,1,\cdots,1)}$.

5. Theorem 5.5.1 asserts that $C^{(\lambda)} = \mathbf{1}\uparrow_{P_\lambda}^{GL(n,q)}$ and $S^{(\lambda)} = \mathbf{1}\uparrow_{S_\lambda}^{S_n}$ decompose exactly in the same manner. Therefore we start by decomposing each $C^{(\lambda)}$ in the order $C^{(\lambda_1)}$, $C^{(\lambda_2)}$, $\cdots$, $C^{(\lambda_{|\mathcal{P}(n)|})}$. For example $C^{(n)} = \mathbf{1}$, $C^{(n-1,1)} = \mathbf{1} + \chi$ and $\chi \in Irr(GL(n, q))$. This χ is called a *Steinberg character*. Then find $C^{(n-2,2)}$, $C^{(n-2,1,1)}$ and so forth till $C^{(1,1,\cdots,1)}$.

6. From each $C^{(\lambda)}$ we get a new irreducible character of $GL(n, q)$, which we denote by $St^{(\lambda)}$.

7. The irreducible character $St^{(\lambda)}$ is called a *Steinberg character*.

In the Examples 5.5.2 and 5.5.3 we determine the decomposition and the values of $C^{(\lambda)}$, $\lambda \vdash 3$ respectively.

Example 5.5.2. Consider $n = 3$ and let $\nu = (0, 0, 3)$, $\mu = (0, 1, 2)$ and $\lambda = (1, 1, 1)$. By (5.22) we have $\deg C^{(0,0,3)} = 1$, $\deg C^{(0,1,2)} = q^2+q+1$ and $\deg C^{(1,1,1)} = (q+1)(q^2+q+1) = q^3+2(q^2+q)+1$. We know that $C^{(0,0,3)} = \mathbf{1}$. Now let $S_\mu = S_{(0,1,2)} = S_1 \times S_2 \cong S_2$. Thus

$$\mathbf{1}\uparrow_{S_2}^{S_3}(1_{S_3}) = 3, \quad \mathbf{1}\uparrow_{S_2}^{S_3}((1\ 2)) = 1 \quad \text{and } \mathbf{1}\uparrow_{S_2}^{S_3}((1\ 2\ 3)) = 0,$$

which shows that $\mathbf{1}\uparrow_{S_2}^{S_3} = \mathbf{1} + \chi$, where χ is the irreducible character of S_3 of degree 2. Hence by Theorem 5.5.1, we have $C^\mu = C^{(0,1,2)} = \mathbf{1} + St^{(1,2)}$, where $St^{(1,2)} \in Irr(GL(3, q))$ and $\deg St^{(1,2)} = q^2 + q$. In the last case where $\lambda = (1, 1, 1)$, let $S_\lambda = S_1 \times S_1 \times S_1 \cong S_1$. Then $\mathbf{1} \uparrow_{S_1}^{S_3}$ is the regular character which have values given by

$$\mathbf{1}\uparrow_{S_1}^{S_3}(g) = \begin{cases} 6 & \text{if } g = 1_{S_3}, \\ 0 & \text{otherwise.} \end{cases}$$

Now by Theorem 5.2.8 of Moori [54] we know that the regular character χ_π of any finite group G with $Irr(G) = \{\chi_1, \chi_2, \cdots, \chi_k\}$ is equal to $\sum_{i=1}^{k} \chi_i(1_G)\chi_i$. Thus

$$\mathbf{1}\uparrow_{S_1}^{S_3} = \mathbf{1} + 2\chi + \text{ the sign character,}$$

and $\chi \in Irr(S_3)$ with $\deg \chi = 2$. Therefore by Theorem 5.5.1 we deduce that $C^{(1,1,1)} = \mathbf{1}+2St^{(1,2)}+St^{(1,1,1)}$. Hence

$$St^{(1,1,1)} = C^{(1,1,1)} - 2St^{(1,2)} - \mathbf{1}.$$

It follows that $\deg St^{(1,1,1)} = q^3 + 2(q^2 + q) + 1 - (2(q^2 + q) + 1) = q^3$.

Remark 5.5.2. Darafasheh [14] showed how to extract the 11 Steinberg characters of $GL(6, 2)$ from the permutation characters $\mathbf{1}\uparrow_{P_\lambda}^{GL(6,2)}$, $\lambda \vdash 6$.

Some Properties of Steinberg Characters

Let $St^{(\lambda)}$, $\lambda \in \mathcal{P}(n)$ denotes the Steinberg character of $GL(n,q)$ corresponding to λ.

1. The trivial character of $GL(n,q)$ is $St^{(n)}$.
2. $St^{(\lambda)}(g) \in \mathbb{Z}, \ \forall g \in GL(n,q), \ \forall \lambda \in \mathcal{P}(n)$.
3. Trivially characters are invariant over conjugate elements. Steinberg characters are invariant over conjugacy classes of the same type.

   The next two points give special attention to the Steinberg character $St^{(1^n)}$.

4. Corresponding to $\lambda = (1^n)$, we have $\deg St^{(1^n)} = q^{\frac{n(n-1)}{2}}$. We recall by Remark 3.1.1 that if $P \in Syl_p(GL(n,q))$, where p is the characteristic of $\mathbb{F}_q$, then $|P| = q^{\frac{n(n-1)}{2}} = \deg St^{(1^n)}$. This character is of particular interest for those who are working in modular representations of $GL(n,q)$. We do not propose any studies for this character in the direction of modular representations. However as an ordinary character of $GL(n,q)$, the values $St^{(1^n)}$ are easy to compute in principal. Excepting the identity element I_n of $GL(n,q)$, the value of $St^{(1^n)}$ at an element x of $GL(n,q)$ is given by

$$St^{(1^n)}(x) = \begin{cases} 0 & \text{if } x \text{ is a } p-singular, \\ \pm|P| & \text{if } x \text{ is a } \ p-regulare, \end{cases} \tag{5.23}$$

   where $P \in Syl_p(C_{GL(n,q)}(x))$. The sign of $St^{(1^n)}$ at regular elements is slightly tricky to be determined. For further details see Humphreys [34].

5. **Characterizing $St^{(1^n)}$:** The following theorem gives sufficient condition in order to characterize $St^{(1^n)}$.

   Theorem 5.5.2. *$St^{(1^n)}$ is the unique irreducible constituent of $\mathbf{1}\uparrow_{UT(n,q)}^{GL(n,q)}$, which fails to occur in any other $\mathbf{1}\uparrow_{P_\lambda}^{G}$ when $P_\lambda \neq UT(n,q)$.*

   PROOF. See Humphreys [34]. ■

Proposition 5.5.3. *The group $GL(n,q)$ has an irreducible character χ such that $p|\deg(\chi)$, where p is the characteristic of $\mathbb{F}_q$.*

PROOF. Let $\lambda = (n-1,1) \vdash n$. Then $S_{(n-1,1)} = S_{n-1} \times S_1 \cong S_{n-1}$. Let $S^{(n-1,1)} = \mathbf{1}\uparrow_{S_{n-1}}^{S_n}$. We know by Proposition 13.24 of James [40] that $\mathbf{1}\uparrow_{S_{n-1}}^{S_n} - \mathbf{1} \in Irr(S_n)$ and $\deg(\mathbf{1}\uparrow_{S_{n-1}}^{S_n} - \mathbf{1}) = n-1$. Now by Theorem 5.5.1, the permutation character $C^{(n-1,1)}$ splits exactly into irreducible constituents in the same way of $S^{(n-1,1)} = \mathbf{1}\uparrow_{S_{n-1}}^{S_n}$. Therefore $C^{(n-1,1)} = \mathbf{1} + St^{(n-1,1)}$ and $St^{(n-1,1)} \in Irr(GL(n,q))$ is a Steinberg character. Thus $St^{(n-1,1)} = C^{(n-1,1)} - \mathbf{1}$. In terms of equations (3.5) and (3.4) we have

$$\deg(C^{(n-1,1)}) = \frac{\{n\}}{\{n-1\}\{1\}} = \frac{[n][n-1]!}{[n-1]![1]!} = [n] = \frac{q^n - 1}{q-1} = q^{n-1} + q^{n-2} + \cdots + 1.$$

It follows that

$$\deg(St^{(n-1,1)}) = (q^{n-1} + q^{n-2} + \cdots + q + 1) - 1 = q(q^{n-2} + q^{n-3} + \cdots + q + 1).$$

Hence the result. ■

Remark 5.5.3. In fact we can go further and prove that $p|\deg(St^{(\lambda)}),\ \forall \lambda \in \mathcal{P}(n) \setminus \{(n)\}$. Note that $\deg(St^{(1^n)}) = q^{\frac{n(n-1)}{2}}$.

Example 5.5.3. In this example we would like to compute the Steinberg characters of $GL(3,q)$. Let $V = V(3,q)$. Then $|V^*| = q^3 - 1$ and by the equivalence relation $\sim$ defined on these vectors, we get $|\mathcal{D}| = q^2 + q + 1$ points. Each point $[\mathbf{x}]$ consists of the vectors of the form $\mathbf{x} = \begin{pmatrix} \lambda x_0 \\ \lambda x_1 \\ \lambda x_2 \end{pmatrix}$ for all $\lambda \in \mathbb{F}_q^*$. These points are stabilized by $P_{(2,1)}$ in the action of $GL(3,q)$ on $PG(2,q)$. Let $g = \begin{pmatrix} a & b & c \\ d & e & f \\ j & h & i \end{pmatrix} \in GL(3,q)$ and $[\mathbf{x}]$ be a point in $PG(2,q)$, then

$$g[\mathbf{x}] = \begin{pmatrix} a & b & c \\ d & e & f \\ j & h & i \end{pmatrix} \begin{pmatrix} \lambda x_0 \\ \lambda x_1 \\ \lambda x_2 \end{pmatrix} = \begin{pmatrix} a\lambda x_0 + b\lambda x_1 + c\lambda x_2 \\ d\lambda x_0 + e\lambda x_1 + f\lambda x_2 \\ j\lambda x_0 + h\lambda x_1 + i\lambda x_2 \end{pmatrix}.$$

As in Example 5.5.1, for $g \in GL(3,q)$, let $Fix(g)$ be the set

$$Fix(g) = \{[\mathbf{x}] \in \mathcal{D}|\ g[\mathbf{x}] = [\mathbf{x}]\}.$$

We know that $C^{(2,1)}(g) = \mathbf{1}\uparrow_{P_{(2,1)}}^{GL(3,q)}(g) = |Fix(g)|$. Also by Example 5.5.2, we know that $C^{(2,1)} = \mathbf{1} + St^{(2,1)}$ and consequently, $St^{(2,1)}(g) = |Fix(g)| - 1,\ \forall g \in GL(3,q)$.

We consider the 8 types of conjugacy classes of $GL(3,q)$, given in Table 5.3.

1. If $g \in GL(3,q)$ is of type $\mathcal{T}^{(1)}$, then

$$g[\mathbf{x}] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} \lambda x_0 \\ \lambda x_1 \\ \lambda x_2 \end{pmatrix} = \begin{pmatrix} \alpha\lambda x_0 \\ \alpha\lambda x_1 \\ \alpha\lambda x_2 \end{pmatrix} \in [\mathbf{x}].$$

   Therefore each point of $PG(2,q)$ is fixed by the central elements of $GL(3,q)$. Hence $|Fix(g)| = q^2 + q + 1$ and consequently $St^{(2,1)}(g) = q^2 + q$.

2. If $g \in GL(3,q)$ is of type $\mathcal{T}^{(2)}$, then

$$g[\mathbf{x}] = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} \lambda x_0 \\ \lambda x_1 \\ \lambda x_2 \end{pmatrix} = \begin{pmatrix} \alpha\lambda x_0 + \lambda x_1 \\ \alpha\lambda x_1 \\ \alpha\lambda x_2 \end{pmatrix} \in [\mathbf{x}] \iff x_1 = 0.$$

Therefore, non-zero vectors of the form $\begin{pmatrix} \lambda x_0 \\ 0 \\ \lambda x_2 \end{pmatrix}$ are fixed by elements of type $\mathcal{T}^{(2)}$. Clearly, there are q^2-1 such vectors. According to the equivalence relation defined on these non-zero vectors, we have $\frac{q^2-1}{q-1} = q+1 = |Fix(g)|$. Hence $St^{(2,1)}(g) = (q+1)-1 = q$.

3. If $g \in GL(3,q)$ is of type $\mathcal{T}^{(3)}$, then

$$g[\mathbf{x}] = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} \lambda x_0 \\ \lambda x_1 \\ \lambda x_2 \end{pmatrix} = \begin{pmatrix} \alpha\lambda x_0 + \lambda x_1 \\ \alpha\lambda x_1 + \lambda x_2 \\ \alpha\lambda x_2 \end{pmatrix} \in [\mathbf{x}] \Longleftrightarrow x_1 = x_2 = 0.$$

Therefore non-zero vectors of the form $\begin{pmatrix} \lambda x_0 \\ 0 \\ 0 \end{pmatrix}$ are fixed by elements of type $\mathcal{T}^{(3)}$. Clearly, there are $q-1$ such vectors and all lie in one point. Thus $|Fix(g)| = 1$ and it follows that $St^{(2,1)}(g) = 1-1 = 0$.

4. If $g \in GL(3,q)$ is of type $\mathcal{T}^{(4)}$, then

$$g[\mathbf{x}] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} \lambda x_0 \\ \lambda x_1 \\ \lambda x_2 \end{pmatrix} = \begin{pmatrix} \alpha\lambda x_0 \\ \alpha\lambda x_1 \\ \beta\lambda x_2 \end{pmatrix} \in [\mathbf{x}] \Longleftrightarrow \ (x_0 = x_1 = 0) \text{ or } x_2 = 0.$$

If $x_0 = x_1 = 0$, then the non-zero vectors will have the form $\begin{pmatrix} 0 \\ 0 \\ \lambda x_2 \end{pmatrix}$ and it follows that there is one point in this case. On the other hand, if $x_2 = 0$, then the non-zero vectors will have the form $\begin{pmatrix} \lambda x_0 \\ \lambda x_1 \\ 0 \end{pmatrix}$. It is immediate to see that there are q^2-1 non-zero vectors and they lie in $q+1$ points by the equivalence relation defined on V^*. Therefore $|Fix(g)| = (q+1)+1 = q+2$ and consequently $St^{(2,1)}(g) = (q+2)-1 = q+1$.

5. If $g \in GL(3,q)$ is of type $\mathcal{T}^{(5)}$, then

$$g[\mathbf{x}] = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} \lambda x_0 \\ \lambda x_1 \\ \lambda x_2 \end{pmatrix} = \begin{pmatrix} \alpha\lambda x_0 + \alpha\lambda x_1 \\ \alpha\lambda x_1 \\ \beta\lambda x_2 \end{pmatrix} \in [\mathbf{x}] \Longleftrightarrow x_1 = x_2 = 0 \text{ or } x_1 = x_0 = 0.$$

A typical non-zero vector in the case $x_1 = x_2 = 0$ will have the form $\begin{pmatrix} \lambda x_0 \\ 0 \\ 0 \end{pmatrix}$ and for the case $x_1 = x_0 = 0$, the vector will have the form $\begin{pmatrix} 0 \\ 0 \\ \lambda x_2 \end{pmatrix}$. Since these two vectors are linearly independent in V, they are in different points. We deduce that $|Fix(g)| = 2$ and consequently $St^{(2,1)}(g) = 2 - 1 = 1$.

6. If $g \in GL(3, q)$ is of type $\mathcal{T}^{(6)}$, then

$$g[\mathbf{x}] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \lambda x_0 \\ \lambda x_1 \\ \lambda x_2 \end{pmatrix} = \begin{pmatrix} \alpha\lambda x_0 \\ \beta\lambda x_1 \\ \gamma\lambda x_2 \end{pmatrix} \in [\mathbf{x}] \iff \quad (x_0 = x_1 = 0)$$
$$\text{or} \quad (x_0 = x_2 = 0) \text{ or } (x_1 = x_2 = 0).$$

Typical non-zero vectors will have one of the form $\begin{pmatrix} \lambda x_0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \lambda x_1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 0 \\ \lambda x_2 \end{pmatrix}$ in respective way to the cases $(x_1 = x_2 = 0)$, $(x_0 = x_2 = 0)$ or $(x_0 = x_1 = 0)$. Since these three vectors are linearly independent in V, they are in different points, we deduce that $|Fix(g)| = 3$ and hence $St^{(2,1)}(g) = 3 - 1 = 2$.

7. If $g \in GL(3, q)$ is of type $\mathcal{T}^{(7)}$, then

$$g[\mathbf{x}] = \begin{pmatrix} 0 & 1 & 0 \\ -r^{q+1} & r + r^q & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} \lambda x_0 \\ \lambda x_1 \\ \lambda x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ -r^{q+1}\lambda x_0 + (r + r^q)\lambda x_1 \\ \alpha\lambda x_2 \end{pmatrix} \in [\mathbf{x}] \iff x_0 = x_1 = 0.$$

A typical non-zero vector will have the form $\begin{pmatrix} 0 \\ 0 \\ \lambda x_2 \end{pmatrix}$. All these $q - 1$ non-zero vectors are in one point. Therefore $St^{(2,1)}(g) = 1 - 1 = 0$.

8. If $g \in GL(3, q)$ is of type $\mathcal{T}^{(8)}$, then

$$\begin{aligned} g[\mathbf{x}] &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s^{q^2+q+1} & -(s^{q+1} + s^{q^2+1} + s^{q^2+q}) & s + s^q + s^{q^2} \end{pmatrix} \begin{pmatrix} \lambda x_0 \\ \lambda x_1 \\ \lambda x_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda s^{q^2+q+1} x_0 - \lambda(s^{q+1} + s^{q^2+1} + s^{q^2+q}) x_1 + \lambda(s + s^q + s^{q^2}) x_2 \end{pmatrix} \notin [\mathbf{x}]. \end{aligned}$$

There is no point fixed by an element of this type of classes of $GL(3, q)$. Hence $St^{(2,1)}(g) = 0 - 1 = -1$.

This finishes the values of the Steinberg character $St^{(2,1)}$ of $GL(3, q)$.

To determine the values of $C^{(1,1,1)}$, we use the parabolic subgroup $P_{(1,1,1)} = UT(3, q) = GL(3, q)_{\mathfrak{F}_{1^n}}$. This subgroup has the form

$$UT(3,q) = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \mid a, d, f \in \mathbb{F}_q^*,\ b, c, e \in \mathbb{F}_q \right\}.$$

Therefore $|UT(3, q)| = q^3(q - 1)^3$ and hence

$$[GL(3, q) : UT(3, q)] = (q + 1)(q^2 + q + 1) = q^3 + 2(q^2 + q) + 1.$$

This subgroup is a minimal parabolic subgroup of $GL(3, q)$. From the conjugacy classes of $GL(3, q)$ obtained in Subsection 5.2.4, we can determine those of $UT(3, q)$. We do not list all classes here, but for example the elements

$$diag(\alpha, \beta, \gamma),\ diag(\alpha, \gamma, \beta),\ diag(\beta, \gamma, \alpha),\ diag(\beta, \alpha, \gamma),\ diag(\gamma, \alpha, \beta),\ diag(\gamma, \beta, \alpha) \tag{5.24}$$

which are conjugate in $GL(3, q)$ are no longer conjugate in $UT(3, q)$. We can show that

$$|C_{UT(3,q)}(A)| = |C_{GL(3,q)}(diag(\alpha, \beta, \gamma))| = (q - 1)^3,$$

where A denotes any matrix of (5.24).

Let $C^{(1,1,1)} = \mathbf{1}\uparrow^{GL(3,q)}_{UT(3,q)}$.

1. If g is of type $\mathcal{T}^{(1)}$, then $C^{(1,1,1)}(g) = [GL(3, q) : UT(3, q)] = q^3 + 2(q^2 + q) + 1$.
2. If g is of type $\mathcal{T}^{(2)}$, then $C^{(1,1,1)}(g) = 2q + 1$.
3. If g is of type $\mathcal{T}^{(3)}$, then $C^{(1,1,1)}(g) = 1$.
4. If g is of type $\mathcal{T}^{(4)}$, then $C^{(1,1,1)}(g) = 3(q + 1)$.
5. If g is of type $\mathcal{T}^{(5)}$, then $C^{(1,1,1)}(g) = 3$.
6. If g is of type $\mathcal{T}^{(6)}$, then $C^{(1,1,1)}(g) = 6$.
7. If g is of type $\mathcal{T}^{(7)}$, then $C^{(1,1,1)}(g) = 0$.
8. If g is of type $\mathcal{T}^{(8)}$, then $C^{(1,1,1)}(g) = 0$.

By Example 5.5.2, we know that $C^{(1,1,1)} = \chi + 2St^{(2,1)} + St^{(3)}$, where $\chi \in Irr(GL(3, q))$ and $\deg(\chi) = q^3$. In fact, χ is the third Steinberg character $St^{(1,1,1)}$. We have $St^{(1,1,1)} = C^{(1,1,1)} - 2St^{(2,1)} - \mathbf{1}$. Therefore

1. If g is of type $\mathcal{T}^{(1)}$, then $St^{(1,1,1)}(g) = q^3$.
2. If g is of type $\mathcal{T}^{(2)}$, then $St^{(1,1,1)}(g) = 0$.
3. If g is of type $\mathcal{T}^{(3)}$, then $St^{(1,1,1)}(g) = 0$.
4. If g is of type $\mathcal{T}^{(4)}$, then $St^{(1,1,1)}(g) = q$.
5. If g is of type $\mathcal{T}^{(5)}$, then $St^{(1,1,1)}(g) = 0$.
6. If g is of type $\mathcal{T}^{(6)}$, then $St^{(1,1,1)}(g) = 1$.
7. If g is of type $\mathcal{T}^{(7)}$, then $St^{(1,1,1)}(g) = -1$.
8. If g is of type $\mathcal{T}^{(8)}$, then $St^{(1,1,1)}(g) = 1$.

Note 5.5.4. We can see that $St^{(3)}$ and $St^{(2,1)}$ appeared as constituents in both $\mathbf{1}\uparrow_{P_{(2,1)}}^{GL(3,q)}$ and $\mathbf{1}\uparrow_{P_{(1,1,1)}}^{GL(3,q)}$, while $St^{(1,1,1)}$ appeared as a constituent for only $\mathbf{1}\uparrow_{P_{(1,1,1)}}^{GL(3,q)}$, which confirms Theorem 5.5.2.

Alternatively one can determine, up to sign, the values of $St^{(1,1,1)}$ using (5.23) since we have the orders of elements of $GL(3, q)$ in Table 6.2. From this table, elements of types $\mathcal{T}^{(2)}$, $\mathcal{T}^{(3)}$ and $\mathcal{T}^{(5)}$ are $p-$singular, while elements in other types are $p-$regular. Let $g_1, g_2, \cdots, g_8$ be elements of types $\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \cdots, \mathcal{T}^{(8)}$ with centralizers $C_1, C_2, \cdots, C_8$ in $GL(3, q)$ respectively. Also for $1 \leq i \leq 8$, let $P_i \in Syl_p(C_i)$. Then

$$St^{(1,1,1)}(g_2) = St^{(1,1,1)}(g_3) = St^{(1,1,1)}(g_5) = 0.$$

From Table 5.3 we can see that

$$|P_1| = q^3, \ |P_4| = q \text{ and } |P_6| = |P_7| = |P_8| = 1.$$

Hence

$$\begin{aligned} St^{(1,1,1)}(g_1) &= \pm q^3, \text{ in particular } St^{(1,1,1)}(I_n) = q^3; \quad St^{(1,1,1)}(g_4) = \pm q; \quad St^{(1,1,1)}(g_6) = \pm 1; \\ St^{(1,1,1)}(g_7) &= \pm 1; \qquad St^{(1,1,1)}(g_8) = \pm 1. \end{aligned}$$

This completes the Steinberg characters of $GL(3, q)$, which are listed in Table 5.7

To obtain the five Steinberg characters of $GL(4, q)$ one can use similar method to the one used in calculating the Steinberg characters of $GL(3, q)$ via the geometric entities or to determine directly the values of

$$\mathbf{1}\uparrow_{P_{(4)}}^{GL(4,q)}, \qquad \mathbf{1}\uparrow_{P_{(3,1)}}^{GL(4,q)}, \qquad \mathbf{1}\uparrow_{P_{(2,2)}}^{GL(4,q)}, \qquad \mathbf{1}\uparrow_{P_{(2,1,1)}}^{GL(4,q)}, \qquad \mathbf{1}\uparrow_{P_{(1,1,1,1)}}^{GL(4,q)}$$

on classes of $GL(4,q)$ (see Table 6.10) and then using

$$\mathbf{1}\uparrow_{S_{(4)}}^{S_4}, \quad \mathbf{1}\uparrow_{S_{(3,1)}}^{S_4}, \quad \mathbf{1}\uparrow_{S_{(2,2)}}^{S_4}, \quad \mathbf{1}\uparrow_{S_{(2,1,1)}}^{S_4}, \quad \mathbf{1}\uparrow_{S_{(1,1,1,1)}}^{S_4},$$

which we know their decompositions in terms of $Irr(S_4)$, to decide how $\mathbf{1}\uparrow_{P_{(\lambda)}}^{GL(4,q)}$, $\lambda \vdash 4$ will decompose. Each $\mathbf{1}\uparrow_{P_{(\lambda)}}^{GL(4,q)}$ affords a new irreducible character of $GL(4,q)$, which is $St^{(\lambda)}$.

Table 5.6: The values of Steinberg characters for $GL(2,q)$

Type	$\mathcal{T}^{(1)}$	$\mathcal{T}^{(2)}$	$\mathcal{T}^{(3)}$	$\mathcal{T}^{(4)}$
$St^{(2)}$	1	1	1	1
$St^{(1,1)}$	q	0	1	-1

Table 5.7: The values of Steinberg characters for $GL(3,q)$

Type	$\mathcal{T}^{(1)}$	$\mathcal{T}^{(2)}$	$\mathcal{T}^{(3)}$	$\mathcal{T}^{(4)}$	$\mathcal{T}^{(5)}$	$\mathcal{T}^{(6)}$	$\mathcal{T}^{(7)}$	$\mathcal{T}^{(8)}$
$St^{(3)}$	1	1	1	1	1	1	1	1
$St^{(2,1)}$	q^2+q	q	0	$q+1$	1	2	0	-1
$St^{(1,1,1)}$	q^3	0	0	q	0	1	-1	1

Table 5.8: The values of Steinberg characters for $GL(4,q)$

Type	$\mathcal{T}^{(1)}$	$\mathcal{T}^{(2)}$	$\mathcal{T}^{(3)}$	$\mathcal{T}^{(4)}$	$\mathcal{T}^{(5)}$	$\mathcal{T}^{(6)}$	$\mathcal{T}^{(7)}$	$\mathcal{T}^{(8)}$
$St^{(4)}$	1	1	1	1	1	1	1	1
$St^{(3,1)}$	$q(q^2+q+1)$	q^2+q	q	q	0	q^2+q+1	$q+1$	1
$St^{(2,2)}$	$q^2(q^2+1)$	q^2	q^2	0	0	q^2+q	q	0
$St^{(2,1,1)}$	$q^3(q^2+q+1)$	q^3	0	0	0	$q(q^2+q+1)$	q	0
$St^{(1,1,1,1)}$	q^6	0	0	0	0	q^3	0	0

Table 5.8 (continued)

Type	$\mathcal{T}^{(9)}$	$\mathcal{T}^{(10)}$	$\mathcal{T}^{(11)}$	$\mathcal{T}^{(12)}$	$\mathcal{T}^{(13)}$	$\mathcal{T}^{(14)}$	$\mathcal{T}^{(15)}$	$\mathcal{T}^{(16)}$
$St^{(4)}$	1	1	1	1	1	1	1	1
$St^{(3,1)}$	$2q+1$	$q+1$	1	$q+2$	2	q	0	3
$St^{(2,2)}$	q^2+1	1	1	$q+1$	1	$-q+1$	1	2
$St^{(2,1,1)}$	q^2+2q	q	0	$2q+1$	1	-1	-1	3
$St^{(1,1,1,1)}$	q^2	0	0	q	0	$-q$	0	1

Table 5.8 (continued)

Type	$\mathcal{T}^{(17)}$	$\mathcal{T}^{(18)}$	$\mathcal{T}^{(19)}$	$\mathcal{T}^{(20)}$	$\mathcal{T}^{(21)}$	$\mathcal{T}^{(22)}$
$St^{(4)}$	1	1	1	1	1	1
$St^{(3,1)}$	1	-1	-1	-1	0	-1
$St^{(2,2)}$	0	2	q^2+1	1	-1	0
$St^{(2,1,1)}$	-1	-1	$-q^2$	0	0	1
$St^{(1,1,1,1)}$	-1	1	q^2	0	1	-1

5.6. Construction of the Characters

Definition 5.6.1. *A group G is called $p-$**elementary** if $G = P \times \langle x \rangle$, where P is a $p-$group and $p \nmid o(x)$ for some $x \in G$.*

Definition 5.6.2. *For any finite group G, the subring of the space of complex valued class functions on G generated by the irreducible complex characters of G is called the **character ring of** G and denoted by $Ch(G)$.*

Theorem 5.6.1 **(Brauer's Characterization of Characters).** *A class function ϕ of a finite group G is a character of G if and only if $\phi\downarrow_H^G$ is a character for all elementary subgroups H of G.*

PROOF. See Brauer [8], Goldschmdit [25], Isaacs [38] or Serre [68]. ■

Thus by Brauer's Characterization of Characters Theorem, the elementary subgroups of G detect the character ring of G.

Definition 5.6.3. *Let $t_1, t_2, \cdots, t_m$ be a set of indeterminate variables. A function f on $t_1, t_2, \cdots, t_m$ is said to be **symmetric** if it is invariant under the action of permutations of S_n on the variables $t_1, t_2, \cdots, t_m$. That is for any $\sigma \in S_m$, we have $f(t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(m)}) = f(t_1, t_2, \ldots, t_m)$.*

The set of all symmetric polynomials in m indeterminate variables forms a subring

$$\Lambda_m = \mathbb{F}[t_1, t_2, \cdots, t_n]^{S_m}$$

of the polynomial ring $\mathbb{F}[t_1, t_2, \cdots, t_m]$.

Theorem 5.6.2 (Green's Theorem [27]). *Suppose that $\theta : \overline{\mathbb{F}}^*_{q^m} \longrightarrow \mathbb{C}^*$ is a character of the multiplicative group of the algebraic closure of $\mathbb{F}_{q^m}$. Let $\phi : G \longrightarrow GL(m,q)$ be a modular representation of a finite group G. For each $x \in G$, let $\lambda_i(x)$, $1 \leq i \leq m$, be the eigenvalues of $\phi(x)$. Suppose that $f \in \Lambda_m$ is a symmetric polynomial, then the function*

$$\chi_\phi : x \longmapsto f(\theta(\lambda_1(x)), \theta(\lambda_2(x)), \cdots, \theta(\lambda_m(x)))$$

is in $Ch(G)$.

PROOF. We recall that the ring of symmetric polynomials form an algebra, with the elementary symmetric polynomials as a basis (see Goldschmidt [25], MacDonald [50] or Sagan [66] for these polynomials). It follows that it is enough to prove the theorem for $f = e_r$, $1 \leq r \leq m$, an elementary symmetric polynomial. Replacing ϕ by its exterior powers, it is then enough to prove the theorem for $f = e_1 = \sum_{i=1}^{m} t_i$.

Suppose that G is an arbitrary finite group. Let H be any elementary subgroup of G. Then we know that $H = K \times P$, where P is a $p-$group and K is a cyclic group such that $p \nmid |K|$ (p is any prime not necessary to be the characteristic of $\mathbb{F}_q$). By page 414 of Green [27], we know that $\chi_\phi\downarrow_K \in Ch(K)$. The result will follow if we could show that $\chi_\phi(xy) = \chi_\phi(y)$, for $x \in K$, $y \in P$. Since $o(\phi(y)) = p^r$ for some r, it follows that the eigenvalues of $\phi(y)$ are $p-$powers roots of unity in $\overline{\mathbb{F}}^*_{q^m}$, hence are all equal to 1. Also $\phi(x)$ commutes with $\phi(y)$. Using the Jordan canonical form, we may find a basis for the representation space of ϕ such that $\phi(x)$ and $\phi(y)$ can be transformed simultaneously to triangular matrices, that is

$$\phi(x) = \begin{pmatrix} \lambda_1(x) & * & \cdots & * \\ 0 & \lambda_2(x) & \cdots & * \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_m(x) \end{pmatrix}, \qquad \phi(y) = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Since $\phi(xy) = \phi(x)\phi(y)$, $\phi(xy)$ has eigenvalues $\lambda_1(x), \lambda_2(x), \cdots, \lambda_m(x)$ and it follows that

$$\begin{aligned} \chi_\phi(xy) = \chi_\phi(x) &= \sum_{i=1}^{m} \lambda_i(x) \\ &= \sum_{i=1}^{m} \theta(\lambda_i(x)), \qquad \text{by page 155 of MacDonald [50]} \\ &= e_1(\theta(\lambda_1(x)), \theta(\lambda_2(x)), \cdots, \theta(\lambda_m(x))) = f(\theta(\lambda_1(x)), \theta(\lambda_2(x)), \cdots, \theta(\lambda_m(x))). \end{aligned}$$

Now the function $xy \longmapsto x$ is a homomorphic mapping of $H = K \times P$ onto K and therefore $\chi_\phi \in Ch(H)$. Since H is an arbitrary elementary subgroup of G, it follows by Brauer's Characterization of Characters Theorem that $\chi_\phi \in Ch(G)$. ■

The construction of characters given by Theorem 5.6.2 does not produce irreducible characters of $GL(n, q)$ in general. However, these generalized characters have two advantages:

- the values of these characters are easily described,
- these characters can be extended to generalized characters of $GL(n, q^k)$ for any $k \in \mathbb{N}$. The ordinary characters do not have this property. See Bump [11].

We apply Green's Theorem to the case $G = GL(n, q)$. Let $\phi : GL(n, q) \longrightarrow GL(n, q)$ be given by $\phi(A) = A$ and let f be the r^{th} elementary symmetric function $f = e_r$, $1 \leq r \leq n$. Also we shall assume that $\theta : \mathbb{F}_{q^n}^* \longrightarrow U$ is an isomorphism of $\mathbb{F}_{q^n}^*$ to the group U consisting of the $(q^n - 1)th$ roots of unity in $\mathbb{C}$. With $\lambda_1, \lambda_2, \cdots, \lambda_n$ being the eigenvalues of $A \in GL(n, q)$ (counted with multiplicity) and for any $k \in \mathbb{N} \cup \{0\}$, then

$$e_r^k(A) = \sum_{0}^{\binom{n}{r}} \prod_{i=1}^{r} \theta^k(\lambda_i), \quad r \leq n \tag{5.25}$$

is a character of $GL(n, q)$ by Green's Theorem. In particular if $r = n$, then

$$e_n^k(A) = \prod_{i=1}^{n} \theta^k(\lambda_i) = \theta^k(\prod_{i=1}^{n}(\lambda_i)) = \theta^k(\det(A)) = (\theta(\det(A)))^k. \tag{5.26}$$

The second and the last equalities of equation (5.26) come from the fact that θ is a homomorphism.

In turns, equation (5.26) gives $q-1$ linear characters of $GL(n, q)$ corresponding to $k = 0, 1, \cdots, q-2$. Excepting the case $n = q = 2$, the next theorem shows that these $q - 1$ linear characters are all the linear characters of $GL(n, q)$.

Theorem 5.6.3. *If $n \neq 2$ or $q \neq 2$, then the characters given by equation* (5.26) *are all the linear characters of $GL(n, q)$.*

PROOF. Suppose that $n \neq 2$ or $q \neq 2$. By Proposition 2.3.4 the number of linear characters of a group G is equal to $[G : G']$, where G' is the derived subgroup. From Theorem 3.1.19, we know that $GL(n, q)' = SL(n, q)$, except in the case $n = 2$ and $q \in \{2, 3\}$. It follows that the number of the linear characters of $GL(n, q)$ is $q-1$. Since equation (5.26) supplies us with $q-1$ distinct linear characters, therefore we know that there can be no further linear character.
In the case $n = q = 2$, where $GL(2, 2) \cong S_3$, we have the extra linear character corresponds to the sing of the permutations of S_3. ■

One can get from equation (5.26) the following commutative diagram. That is $e_n^k = \theta|_{\mathbb{F}_q^*} \circ \det$.

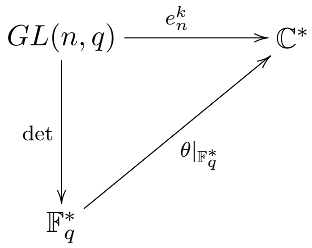

Figure 5.2: The commutative diagram of linear characters of $GL(n, q)$.

5.7. Application: Character Table of $GL(3, q)$

We have seen in Subsection 5.2.4 that the conjugacy classes of $GL(3, q)$ are distributed into 8 types. In this section we show that the irreducible characters of $GL(3, q)$ are also fall within 8 types.

5.7.1 Principal Series Characters of $GL(3, q)$

Linear Characters of $GL(3, q)$

According to Theorem 5.6.3 there are $q - 1$ linear characters of $GL(3, q)$. These characters are given by the powers of the determinants of elements of $GL(3, q)$ and we denote each character by $\chi_k^{(1)}$. We list the values of $\chi_k^{(1)}$ in Table 5.9.

Table 5.9: Linear characters of $GL(3, q)$

	$\mathcal{T}^{(1)}$	$\mathcal{T}^{(2)}$	$\mathcal{T}^{(3)}$	$\mathcal{T}^{(4)}$	$\mathcal{T}^{(5)}$	$\mathcal{T}^{(6)}$	$\mathcal{T}^{(7)}$	$\mathcal{T}^{(8)}$
$\chi_k^{(1)}$	$\widehat{\alpha}^{3k}$	$\widehat{\alpha}^{3k}$	$\widehat{\alpha}^{3k}$	$\widehat{\alpha}^{2k}\widehat{\beta}^k$	$\widehat{\alpha}^{2k}\widehat{\beta}^k$	$\widehat{\alpha}^k\widehat{\beta}^k\widehat{\gamma}^k$	$\widehat{r}^{k(q+1)}\widehat{\alpha}^k$	$\widehat{s}^{k(q^2+q+1)}$

where $k = 0, 1, \cdots, q - 2$.

Characters of $GL(3, q)$ obtained through Steinberg characters

In Table 5.7 we have listed the values of $St^{(3)} = \mathbf{1}$, $St^{(2,1)}$ and $St^{(1,1,1)}$ on classes of $GL(3, q)$. Forming the tensor product of $St^{(2,1)}$ and $St^{(1,1,1)}$ with $\chi_k^{(1)}$ we get two new types of irreducible characters of $GL(3, q)$, namely $\chi_k^{(2)} = St^{(2,1)}\chi_k^{(1)}$ and $\chi_k^{(3)} = St^{(1,1,1)}\chi_k^{(1)}$. We list the values of $\chi_k^{(2)}$ and $\chi_k^{(3)}$ on classes of $GL(3, q)$ in Table 5.10.

Table 5.10: Steinberg characters tensored by linear characters of $GL(3, q)$

	$\mathcal{T}^{(1)}$	$\mathcal{T}^{(2)}$	$\mathcal{T}^{(3)}$	$\mathcal{T}^{(4)}$	$\mathcal{T}^{(5)}$	$\mathcal{T}^{(6)}$	$\mathcal{T}^{(7)}$	$\mathcal{T}^{(8)}$
$\chi_k^{(2)}$	$(q^2+q)\widehat{\alpha}^{3k}$	$q\widehat{\alpha}^{3k}$	0	$(q+1)\widehat{\alpha}^{2k}\widehat{\beta}^k$	$\widehat{\alpha}^{2k}\widehat{\beta}^k$	$2\widehat{\alpha}^k\widehat{\beta}^k\widehat{\gamma}^k$	0	$-\widehat{s}^{k(q^2+q+1)}$
$\chi_k^{(3)}$	$q^3\widehat{\alpha}^{3k}$	0	0	$q\widehat{\alpha}^{2k}\widehat{\beta}^k$	0	$\widehat{\alpha}^k\widehat{\beta}^k\widehat{\gamma}^k$	$\widehat{r}^{k(q+1)}\widehat{\alpha}^k$	$\widehat{s}^{k(q^2+q+1)}$

where $k = 0, 1, \cdots, q-2$.

Characters of $GL(3, q)$ obtained from parabolic subgroups

Here we use the parabolic subgroup $P_{(2,1)}$ defined in Example 3.1.2 to construct some characters of $GL(3, q)$. In fact it will be shown later that $\chi_k^{(2)}$ and $\chi_k^{(3)}$ are also characters of $GL(3, q)$ come from characters of $P_{(2,1)}$. Recall that $P_{(2,1)}$ has the form

$$P_{(2,1)} = \left\{ \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & \alpha \end{pmatrix} \mid \alpha \in \mathbb{F}_q^*,\ a, b, c, d, e, f \in \mathbb{F}_q,\ ad - bc \neq 0 \right\}.$$

Therefore $|P_{(2,1)}| = q^3(q-1)^3(q+1)$ and hence $[GL(3, q) : P_{(2,1)}] = q^2+q+1$. This group, which we change its notation now from $P_{(2,1)}$ to $MP(3, q)$, is a maximal subgroup of $GL(3, q)$ by a Theorem of Aschbacher [4], since it is associated to a partition with 2 parts. From the conjugacy classes of $GL(3, q)$ obtained in Subsection 5.2.4, we can determine those of $MP(3, q)$. We do not list all classes of $MP(3, q)$ here, but for example the two elements

$$g = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \quad \text{and} \quad g' = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \tag{5.27}$$

of type $\mathcal{T}^{(4)}$, which are conjugate in $GL(3, q)$ are no longer conjugate in $MP(3, q)$ as follows. Let

$$h = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & t \end{pmatrix} \in MP(3, q).$$

Then

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & t \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} = \begin{pmatrix} a\alpha & b\alpha & e\beta \\ c\alpha & d\alpha & f\beta \\ 0 & 0 & t\beta \end{pmatrix}, \tag{5.28}$$

$$\begin{pmatrix} \beta & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & t \end{pmatrix} = \begin{pmatrix} a\beta & b\beta & e\beta \\ c\alpha & d\alpha & f\alpha \\ 0 & 0 & t\alpha \end{pmatrix}. \tag{5.29}$$

Now

$$g \sim g^{'} \text{ in } MP(3,q) \Longleftrightarrow (5.28) = (5.29) \Longleftrightarrow h = \begin{pmatrix} 0 & 0 & e \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which contradicts the facts that $ad - bc \neq 0$ and $t \neq 0$. Therefore g and $g^{'}$ are not conjugate in $MP(3,q)$.

To calculate $|C_{MP(3,q)}(g)|$ and $|C_{MP(3,q)}(g^{'})|$ we add the following two equations.

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & t \end{pmatrix} = \begin{pmatrix} a\alpha & b\alpha & e\alpha \\ c\alpha & d\alpha & f\alpha \\ 0 & 0 & t\beta \end{pmatrix}, \tag{5.30}$$

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & t \end{pmatrix}\begin{pmatrix} \beta & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} = \begin{pmatrix} a\beta & b\alpha & e\alpha \\ c\beta & d\alpha & f\alpha \\ 0 & 0 & t\alpha \end{pmatrix}. \tag{5.31}$$

If $(5.28) = (5.30)$ and $(5.29) = (5.31)$, then we obtain

$$|C_{MP(3,q)}(g)| = q(q-1)^3(q+1) = |C_{GL(3,q)}(g)| \quad \text{and} \quad |C_{MP(3,q)}(g^{'})| = q(q-1)^3.$$

Through similar steps we can determine those splitting and non-splitting classes of $GL(3,q)$ in $MP(3,q)$ and calculate the size of each class of $MP(3,q)$.

From the maximality of $MP(3,q)$ we expect to obtain large number of characters of $GL(3,q)$. In fact this group gives rise to all characters of the principal series of $GL(3,q)$. So in the following we determine some of the irreducible characters of $MP(3,q)$.

By Theorem 3.1.8 we have that $MP(3,q) = U_{(2,1)}{:}L_{(2,1)}$ and hence $MP(3,q)/U_{(2,1)} \cong L_{(2,1)}$, where $U_{(2,1)}$ and $L_{(2,1)}$ are the unipotent radical and levi complement of $MP(3,q)$ respectively. By Example 3.1.2, the former two subgroups have the forms

$$U_{(2,1)} = \left\{ \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \mid e, f \in \mathbb{F}_q \right\},$$

$$L_{(2,1)} = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \alpha \end{pmatrix} \mid \alpha \in \mathbb{F}_q^*,\ a, b, c, d, \in \mathbb{F}_q,\ ad - bc \neq 0 \right\}.$$

We know that $L_{(2,1)} \cong GL(2,q) \times GL(1,q)$. If $\chi \in Irr(GL(2,q))$ and $\psi \in Irr(GL(1,q))$, then

$$Irr(L_{(2,1)}) = \{\chi\psi |\ \chi \in Irr(GL(2,q)), \psi \in Irr(GL(1,q))\}.$$

Since irreducible characters of $GL(2,q)$ fall in four types as shown in Table 4.2, the irreducible characters of $L_{(2,1)}$ are also distributed into four types as follows

- $\lambda_l\chi_k$, where λ_l and χ_k are any linear characters of $GL(2,q)$ and $GL(1,q)$ respectively.
- $\psi_l\chi_k$, where ψ_l is any character of $GL(2,q)$ of type $\chi^{(2)}$ and χ_k is any character of $GL(1,q)$.
- $\psi_{l,m}\chi_k$, where $\psi_{l,m}$ is any character of $GL(2,q)$ of type $\chi^{(3)}$ and χ_k is any character of $GL(1,q)$.
- $\pi_l\chi_k$, where π_l is any cuspidal character of $GL(2,q)$ and χ_k is any character of $GL(1,q)$.

Since $MP(3,q)/U_{(2,1)} \cong L_{(2,1)}$, the above characters extend (lift) irreducibly to $MP(3,q)$, because they are characters of the quotient, where if $\chi \in Irr(GL(2,q))$ and $\chi_k \in Irr(GL(1,q))$, then

$$(\chi\chi_k)\left(\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & t \end{pmatrix}\right) = \chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\chi_k(t).$$

Next we consider $\chi\chi_k\uparrow_{MP(3,q)}^{GL(3,q)}$. Note that the operation of obtaining characters of $GL(3,q)$ this way is the parabolic induction described previously. Therefore we use the notation $\chi \odot \chi_k = \chi\chi_k\uparrow_{MP(3,q)}^{GL(3,q)}$, $\chi \in Irr(GL(2,q))$, $\chi_k \in Irr(GL(1,q))$. We give an example on how to calculate $\chi \odot \chi_k$ on classes of type $\mathcal{T}^{(4)}$. Let g and g' be the elements of $GL(3,q)$ defined in (5.27). To follow the next computations, we need to keep closely the character table of $GL(2,q)$ given in Table 4.2.

- Suppose that $\lambda_l \in Irr(GL(2,q))$ is any linear character and χ_k is any character of $GL(1,q)$, where $k \neq l$. Then

$$\begin{aligned}(\lambda_l \odot \chi_k)(g) &= \lambda_l\chi_k\uparrow_{MP(3,q)}^{GL(3,q)}(g) \\ &= \frac{|C_{GL(3,q)}(g)|}{|C_{MP(3,q)}(g)|}\lambda_l\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\right)\chi_k(\beta) + \frac{|C_{GL(3,q)}(g)|}{|C_{MP(3,q)}(g')|}\lambda_l\left(\begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}\right)\chi_k(\alpha) \\ &= \frac{q(q-1)^3(q+1)}{q(q-1)^3(q+1)}\widehat{\alpha}^{2l}\widehat{\beta}^k + \frac{q(q-1)^3(q+1)}{q(q-1)^3}\widehat{\beta}^l\widehat{\alpha}^l\widehat{\alpha}^k = \widehat{\alpha}^{2l}\widehat{\beta}^k + (q+1)\widehat{\alpha}^{(l+k)}\widehat{\beta}^l.\end{aligned}$$

- Suppose that $\psi_l \in Irr(GL(2,q))$ is of type $\chi^{(2)}$ and χ_k is any character of $GL(1,q)$, where $k \neq l$. Then

$$\begin{aligned}(\psi_l \odot \chi_k)(g) &= \psi_l\chi_k\uparrow_{MP(3,q)}^{GL(3,q)}(g) \\ &= \frac{|C_{GL(3,q)}(g)|}{|C_{MP(3,q)}(g)|}\psi_l\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\right)\chi_k(\beta) + \frac{|C_{GL(3,q)}(g)|}{|C_{MP(3,q)}(g')|}\psi_l\left(\begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}\right)\chi_k(\alpha) \\ &= \frac{q(q-1)^3(q+1)}{q(q-1)^3(q+1)}q\widehat{\alpha}^{2l}\widehat{\beta}^k + \frac{q(q-1)^3(q+1)}{q(q-1)^3}\widehat{\beta}^l\widehat{\alpha}^l\widehat{\alpha}^k = q\widehat{\alpha}^{2l}\widehat{\beta}^k + (q+1)\widehat{\alpha}^{(l+k)}\widehat{\beta}^l.\end{aligned}$$

- Suppose that $\psi_{l,m} \in Irr(GL(2,q))$ is of type $\chi^{(3)}$ and χ_k is any character of $GL(1,q)$, where $k \neq l \neq m \neq k$. Then

$$\begin{aligned}
(\psi_{l,m} \odot \chi_k)(g) &= \psi_{l,m}\chi_k\uparrow_{MP(3,q)}^{GL(3,q)}(g) \\
&= \frac{|C_{GL(3,q)}(g)|}{|C_{MP(3,q)}(g)|}\psi_{l,m}\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\right)\chi_k(\beta) + \frac{|C_{GL(3,q)}(g)|}{|C_{MP(3,q)}(g')|}\psi_{l,m}\left(\begin{pmatrix}\beta & 0\\ 0 & \alpha\end{pmatrix}\right)\chi_k(\alpha) \\
&= \frac{q(q-1)^3(q+1)}{q(q-1)^3(q+1)}(q+1)\widehat{\alpha}^{(l+m)}\widehat{\beta}^k + \frac{q(q-1)^3(q+1)}{q(q-1)^3}\left(\widehat{\beta}^l\widehat{\alpha}^m + \widehat{\beta}^m\widehat{\alpha}^l\right)\widehat{\alpha}^k \\
&= (q+1)\widehat{\alpha}^{(l+m)}\widehat{\beta}^k + (q+1)\left(\widehat{\beta}^l\widehat{\alpha}^m\widehat{\alpha}^k + \widehat{\beta}^m\widehat{\alpha}^l\widehat{\alpha}^k\right) \\
&= (q+1)\left(\widehat{\alpha}^{(l+m)}\widehat{\beta}^k + \widehat{\beta}^l\widehat{\alpha}^{(m+k)} + \widehat{\beta}^m\widehat{\alpha}^{(l+k)}\right).
\end{aligned}$$

- Suppose that $\pi_l \in Irr(GL(2,q))$ is any cuspidal character and χ_k is any character of $GL(1,q)$. Then

$$\begin{aligned}
(\pi_l \odot \chi_k)(g) &= \pi_l\chi_k\uparrow_{MP(3,q)}^{GL(3,q)}(g) \\
&= \frac{|C_{GL(3,q)}(g)|}{|C_{MP(3,q)}(g)|}\pi_l\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\right)\chi_k(\beta) + \frac{|C_{GL(3,q)}(g)|}{|C_{MP(3,q)}(g')|}\pi_l\left(\begin{pmatrix}\beta & 0\\ 0 & \alpha\end{pmatrix}\right)\chi_k(\alpha) \\
&= \frac{q(q-1)^3(q+1)}{q(q-1)^3(q+1)}(q-1)\widehat{\alpha}^l\widehat{\beta}^k + \frac{q(q-1)^3(q+1)}{q(q-1)^3}.0.\widehat{\alpha}^k = (q-1)\widehat{\alpha}^l\widehat{\beta}^k.
\end{aligned}$$

Due to the complexity of computations, we give the values of $\chi \odot \chi_k$ on conjugacy classes of $GL(3,q)$ without details of computations.

1. Let λ_l be a linear character of $GL(2,q)$ and χ_k be any character of $GL(1,q)$. Two cases appear.

   (a) Consider firstly the case where $l \neq k$. Then

$$(\lambda_l \odot \chi_k)(g) = \begin{cases}
(q^2+q+1)\widehat{\alpha}^{2l+k} & \text{if } g \text{ is of type } \mathcal{T}^{(1)},\\
(q+1)\widehat{\alpha}^{2l+k} & \text{if } g \text{ is of type } \mathcal{T}^{(2)},\\
\widehat{\alpha}^{2l+k} & \text{if } g \text{ is of type } \mathcal{T}^{(3)},\\
(q+1)\widehat{\alpha}^{l+k}\widehat{\beta}^l + \widehat{\alpha}^{2l}\widehat{\beta}^k & \text{if } g \text{ is of type } \mathcal{T}^{(4)},\\
\widehat{\alpha}^{l+k}\widehat{\beta}^l + \widehat{\alpha}^{2l}\widehat{\beta}^k & \text{if } g \text{ is of type } \mathcal{T}^{(5)},\\
\widehat{\alpha}^k\widehat{\beta}^l\widehat{\gamma}^l + \widehat{\alpha}^l\widehat{\beta}^k\widehat{\gamma}^l + \widehat{\alpha}^l\widehat{\beta}^l\widehat{\gamma}^k & \text{if } g \text{ is of type } \mathcal{T}^{(6)},\\
\widehat{\alpha}^k\widehat{r}^{l(q+1)} & \text{if } g \text{ is of type } \mathcal{T}^{(7)},\\
0 & \text{if } g \text{ is of type } \mathcal{T}^{(8)}.
\end{cases}$$

   We can check that each ordered pair (k,l), $k \neq l$ admits $\lambda_l \odot \chi_k \in Irr(GL(3,q))$. We get $(q-1)(q-2)$ irreducible characters $\lambda_l \odot \chi_k$ and we let $\chi_{k,l}^{(4)} = \lambda_l \odot \chi_k$.

(b) Consider the other case where $l = k$. Here we get

$$(\lambda_k \odot \chi_k)(g) = \begin{cases} (q^2+q+1)\widehat{\alpha}^{3k} & \text{if } g \text{ is of type } \mathcal{T}^{(1)}, \\ (q+1)\widehat{\alpha}^{3k} & \text{if } g \text{ is of type } \mathcal{T}^{(2)}, \\ \widehat{\alpha}^{3k} & \text{if } g \text{ is of type } \mathcal{T}^{(3)}, \\ (q+2)\widehat{\alpha}^{2k}\widehat{\beta}^{k} & \text{if } g \text{ is of type } \mathcal{T}^{(4)}, \\ 2\widehat{\alpha}^{2k}\widehat{\beta}^{k} & \text{if } g \text{ is of type } \mathcal{T}^{(5)}, \\ 3\widehat{\alpha}^{k}\widehat{\beta}^{k}\widehat{\gamma}^{k} & \text{if } g \text{ is of type } \mathcal{T}^{(6)}, \\ \widehat{\alpha}^{k}\widehat{r}^{k(q+1)} & \text{if } g \text{ is of type } \mathcal{T}^{(7)}, \\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(8)}. \end{cases}$$

We can see that $\lambda_k \odot \chi_k = C^{(2,1)}\chi_k^{(1)}$, where $C^{(2,1)}$ is the permutation character of $GL(3,q)$ discussed in Section 5.5. Therefore $\lambda_k \odot \chi_k \notin Irr(GL(3,q))$. Observe that $\lambda_k \odot \chi_k = C^{(2,1)}\chi_k^{(1)} = \chi_k^{(2)} + \chi_k^{(1)}$. This shows that $\chi_k^{(1)}$ and $\chi_k^{(2)}$ are principal series characters of $GL(3,q)$. In particular note that $St^{(2,1)} = \chi_0^{(2)} = \lambda_0 \odot \chi_0 - \mathbf{1}$.

2. Let ψ_l be any irreducible character of $GL(2,q)$ of degree q and $\chi_k \in GL(1,q)$. There are two cases appear according to $k = l$ or not. We consider only the case $k \neq l$. Here we have

$$(\psi_l \odot \chi_k)(g) = \begin{cases} q(q^2+q+1)\widehat{\alpha}^{2l+k} & \text{if } g \text{ is of type } \mathcal{T}^{(1)}, \\ q\widehat{\alpha}^{2l+k} & \text{if } g \text{ is of type } \mathcal{T}^{(2)}, \\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(3)}, \\ (q+1)\widehat{\alpha}^{(l+k)}\widehat{\beta}^{l} + q\widehat{\alpha}^{2l}\widehat{\beta}^{k} & \text{if } g \text{ is of type } \mathcal{T}^{(4)}, \\ \widehat{\alpha}^{(l+k)}\widehat{\beta}^{l} + \widehat{\alpha}^{2l}\widehat{\beta}^{k} & \text{if } g \text{ is of type } \mathcal{T}^{(5)}, \\ \widehat{\alpha}^{k}\widehat{\beta}^{l}\widehat{\gamma}^{l} + \widehat{\alpha}^{l}\widehat{\beta}^{k}\widehat{\gamma}^{l} + \widehat{\alpha}^{l}\widehat{\beta}^{l}\widehat{\gamma}^{k} & \text{if } g \text{ is of type } \mathcal{T}^{(6)}, \\ -\widehat{\alpha}^{k}\widehat{r}^{l(q+1)} & \text{if } g \text{ is of type } \mathcal{T}^{(7)}, \\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(8)}. \end{cases}$$

Each ordered pair (k,l), $k \neq l$ makes $\psi_l \odot \chi_k \in Irr(GL(3,q))$. We let $\chi_{k,l}^{(5)} = \psi_l \odot \chi_k$. Clearly we have $(q-1)(q-2)$ irreducible characters $\chi_{k,l}^{(5)}$.

3. Let $\psi_{l,m}$ be any irreducible character of $GL(2,q)$ of degree $q+1$ and $\chi_k \in Irr(GL(1,q))$. There are many cases appear according to $(l = m = k)$, $(l = m \neq k)$, $(l \neq m = k)$, $(l = k \neq m)$ or $(l \neq m \neq k \neq l)$. We consider only two cases out of these possibilities.

(a) Assume that $l \neq m \neq k \neq l$. We have

$$(\psi_{l,m}\odot\chi_k)(g)=\begin{cases}(q+1)(q^2+q+1)\widehat{\alpha}^{(l+m+k)} & \text{if } g \text{ is of type } \mathcal{T}^{(1)},\\ (2q+1)\widehat{\alpha}^{(l+m+k)} & \text{if } g \text{ is of type } \mathcal{T}^{(2)},\\ \widehat{\alpha}^{(l+m+k)} & \text{if } g \text{ is of type } \mathcal{T}^{(3)},\\ (q+1)\left(\widehat{\alpha}^{(l+m)}\widehat{\beta}^k+\widehat{\alpha}^{(l+k)}\widehat{\beta}^m+\widehat{\alpha}^{(m+k)}\widehat{\beta}^l\right) & \text{if } g \text{ is of type } \mathcal{T}^{(4)},\\ \left(\widehat{\alpha}^{(l+m)}\widehat{\beta}^k+\widehat{\alpha}^{(l+k)}\widehat{\beta}^m+\widehat{\alpha}^{(m+k)}\widehat{\beta}^l\right) & \text{if } g \text{ is of type } \mathcal{T}^{(5)},\\ \widehat{\alpha}^k\widehat{\beta}^l\widehat{\gamma}^m+\widehat{\alpha}^k\widehat{\beta}^m\widehat{\gamma}^l+\widehat{\alpha}^m\widehat{\beta}^l\widehat{\gamma}^k & \text{if } g \text{ is of type } \mathcal{T}^{(6)},\\ +\widehat{\alpha}^m\widehat{\beta}^k\widehat{\gamma}^l+\widehat{\alpha}^l\widehat{\beta}^k\widehat{\gamma}^m+\widehat{\alpha}^l\widehat{\beta}^m\widehat{\gamma}^k & \\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(7)},\\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(8)}.\end{cases}$$

For each $k\neq l\neq m\neq k$, we can check that $\psi_{l,m}\odot\chi_k$ is irreducible, but we do not do this here as this requires invoking some other subgroups of $GL(3,q)$ to evaluate some of the terms appear in $\langle\psi_{l,m}\odot\psi_k,\psi_{l,m}\odot\chi_k\rangle$. For example to evaluate

$$\frac{(q-1)(q-2)(q-3)}{6}q^3(q+1)(q^2+q+1)[\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}\left(\widehat{\alpha}^k\widehat{\beta}^l\widehat{\gamma}^m+\widehat{\alpha}^k\widehat{\beta}^m\widehat{\gamma}^l+\widehat{\alpha}^m\widehat{\beta}^l\widehat{\gamma}^k\right)\times \overline{\left(\widehat{\alpha}^k\widehat{\beta}^l\widehat{\gamma}^m+\widehat{\alpha}^k\widehat{\beta}^m\widehat{\gamma}^l+\widehat{\alpha}^m\widehat{\beta}^l\widehat{\gamma}^k\right)}],$$

two of subgroups of $GL(3,q)$ will be needed are the torus

$$T=\left\{\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}\mid a,b,c\in\mathbb{F}_q^*\right\}$$

and its subgroup

$$\left\{\begin{pmatrix}a&0&0\\0&a&0\\0&0&b\end{pmatrix}\mid a,b\in\mathbb{F}_q^*\right\}.$$

Then one can trace similar computations to ones done in page 49. Each unordered triple $\{k,l,m\}$, $k\neq l\neq m\neq k$ gives an irreducible character $\psi_{l,m}\odot\chi_k$, which we denote by $\chi^{(6)}_{k,l,m}$. Clearly there are $\frac{(q-1)(q-2)(q-3)}{6}$ such irreducible characters.

(b) Consider the case where $l=k=m$. Here we get

$$(\psi_{k,k}\odot\chi_k)(g)=\begin{cases}(q+1)(q^2+q+1)\widehat{\alpha}^{3k} & \text{if } g \text{ is of type } \mathcal{T}^{(1)},\\ (2q+1)\widehat{\alpha}^{3k} & \text{if } g \text{ is of type } \mathcal{T}^{(2)},\\ \widehat{\alpha}^{3k} & \text{if } g \text{ is of type } \mathcal{T}^{(3)},\\ 3(q+1)\widehat{\alpha}^{2k}\widehat{\beta}^k & \text{if } g \text{ is of type } \mathcal{T}^{(4)},\\ 3\widehat{\alpha}^{2k}\widehat{\beta}^k & \text{if } g \text{ is of type } \mathcal{T}^{(5)},\\ 6\widehat{\alpha}^k\widehat{\beta}^k\widehat{\gamma}^k & \text{if } g \text{ is of type } \mathcal{T}^{(6)},\\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(7)},\\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(8)}.\end{cases}$$

It is readily verified that $\psi_{k,k} \odot \chi_k = C^{(1,1,1)}\chi_k^{(1)}$, where $C^{(1,1,1)}$ is the permutation character discussed in Section 5.5. Therefore $\psi_{k,k} \odot \chi_k \notin Irr(GL(3, q))$. We can see that $\psi_{k,k} \odot \chi_k = C^{(1,1,1)}\chi_k^{(1)} = \chi_k^{(3)} + 2\chi_k^{(2)} + \chi_k^{(1)}$ that is $\chi_k^{(3)} = St^{(1,1,1)}\chi_k^{(1)} = \psi_{k,k} \odot \chi_k - 2\chi_k^{(2)} - \chi_k^{(1)}$. Thus $\chi_k^{(3)}$ is a principal series character of $GL(3, q)$. In particular note that $St^{(1,1,1)} = \psi_{0,0} \odot \chi_0 - 2\chi_0^{(2)} - \mathbf{1}$.

4. Let π_l be a cuspidal character of $GL(2, q)$ and $\chi_k \in Irr(GL(1, q))$. In this case we have

$$(\pi_l \odot \chi_k)(g) = \begin{cases} (q^3-1)\widehat{\alpha}^{(l+k)} & \text{if } g \text{ is of type } \mathcal{T}^{(1)}, \\ -\widehat{\alpha}^{(l+k)} & \text{if } g \text{ is of type } \mathcal{T}^{(2)}, \\ -\widehat{\alpha}^{(l+k)} & \text{if } g \text{ is of type } \mathcal{T}^{(3)}, \\ (q-1)\widehat{\alpha}^l\widehat{\beta}^k & \text{if } g \text{ is of type } \mathcal{T}^{(4)}, \\ -\widehat{\alpha}^l\widehat{\beta}^k & \text{if } g \text{ is of type } \mathcal{T}^{(5)}, \\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(6)}, \\ -\widehat{\alpha}^k(\widehat{r}^l + \widehat{r}^{lq}) & \text{if } g \text{ is of type } \mathcal{T}^{(7)}, \\ 0 & \text{if } g \text{ is of type } \mathcal{T}^{(8)}. \end{cases}$$

The characters $\pi_l \odot \chi_k$ provided by the pair $\{k, l\}$ where $k = 0, 1, \cdots, q-2$ and $q+1 \nmid l$, $l = 1, 2, \cdots, q^2-1$ and $lq(mod\ q^2-1)$ is excluded whenever l is included, are irreducible. It follows that there are $\frac{q(q-1)^2}{2}$ such characters, which we denote by $\chi_{k,l}^{(7)}$.

5.7.2 Discrete Series Characters of $GL(3, q)$

In Example 5.4.3 we have found the values of the cuspidal characters of $GL(3, q)$. Let us denote each cuspidal character of $GL(3, q)$ by $\chi_k^{(8)}$. We tabulate the values of $\chi_k^{(8)}$ on classes of $GL(3, q)$ in Table 5.11.

Table 5.11: The cuspidal characters of $GL(3, q)$

	$\mathcal{T}^{(1)}$	$\mathcal{T}^{(2)}$	$\mathcal{T}^{(3)}$	$\mathcal{T}^{(4)}$	$\mathcal{T}^{(5)}$	$\mathcal{T}^{(6)}$	$\mathcal{T}^{(7)}$	$\mathcal{T}^{(8)}$
$\chi_k^{(8)}$	$(q-1)^2(q+1)\widehat{\alpha}^k$	$-(q-1)\widehat{\alpha}^k$	$\widehat{\alpha}^k$	0	0	0	0	$\widehat{s}^k + \widehat{s}^{kq} + \widehat{s}^{kq^2}$

where $q^2 + q + 1 \nmid k$, $k = 1, 2, \cdots, q^3 - 1$ and kq, kq^2 are excluded whenever k is being chosen.

Alternatively one can get the values of $\chi_k^{(8)}$ as follows: Let $\langle f_\sigma \rangle$, where $f_\sigma = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma^q & 0 \\ 0 & 0 & \sigma^{q^2} \end{pmatrix}$ and $\mathbb{F}_{q^3}^* = \langle \sigma \rangle$, be an isomorphic copy of $\mathbb{F}_{q^3}^*$ in $GL(3, q)$. Suppose that $\theta_k \in Irr(\langle f_\sigma \rangle)$ and $\phi_k =$

$\theta_k\uparrow_{\langle f_\sigma\rangle}^{GL(3,q)}$. Then provided that $q^2+q+1 \nmid k$, one can easily check that

$$\widetilde{\chi}_k^{(8)} = \phi_k - \left(\chi_0^{(3)} - \chi_0^{(2)} + \chi_0^{(1)}\right)\chi_{0,k}^{(7)} \tag{5.32}$$

have the same values as $\chi_k^{(8)}$ over all $g \in GL(3,q)$. That is $\chi_k^{(8)} = \widetilde{\chi}_k^{(8)}$.

If $\#\chi^{(i)}$ means the number of irreducible characters of type $\chi^{(i)}$ and also $\#\mathcal{T}^{(i)}$ means the number of conjugacy classes of type $\mathcal{T}^{(i)}$, then we have

$$\#\chi^{(i)} = \#\mathcal{T}^{(i)}, \qquad \forall\, 1 \le i \le 8.$$

For example

$$\#\chi^{(5)} = (q-1)(q-2) = \#\mathcal{T}^{(5)}.$$

Therefore

$$\sum_{i=1}^{8} \#\chi^{(i)} = q^3 - q = \sum_{i=1}^{8} \#\mathcal{T}^{(i)}.$$

Since all characters we have found are distinct, it follows that we are done with the character table of $GL(3,q)$, which we list in Table 5.12.

Table 5.12: The character table of $GL(3, q)$

Type	$\mathcal{T}^{(1)}$	$\mathcal{T}^{(2)}$	$\mathcal{T}^{(3)}$
Rep g	$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$
No. of CC	$q-1$	$q-1$	$(q-1)$
$\|C_{GL(3,q)}(g)\|$	$q^3(q-1)^3(q+1)(q^2+q+1)$	$q^3(q-1)^2$	$q^2(q-1)$
$\|C_g\|$	1	$(q^2-1)(q^2+q+1)$	$q(q^2-1)(q^3-1)$
$\chi_k^{(1)}$	$\widehat{\alpha}^{3k}$	$\widehat{\alpha}^{3k}$	$\widehat{\alpha}^{3k}$
$\chi_k^{(2)}$	$(q^2+q)\widehat{\alpha}^{3k}$	$q\widehat{\alpha}^{3k}$	0
$\chi_k^{(3)}$	$q^3\widehat{\alpha}^{3k}$	0	0
$\chi_{k,l}^{(4)}$	$(q^2+q+1)\widehat{\alpha}^{k+2l}$	$(q+1)\widehat{\alpha}^{k+2l}$	$\widehat{\alpha}^{k+2l}$
$\chi_{k,l}^{(5)}$	$q(q^2+q+1)\widehat{\alpha}^{k+2l}$	$q\widehat{\alpha}^{k+2l}$	0
$\chi_{k,l,m}^{(6)}$	$(q+1)(q^2+q+1)\widehat{\alpha}^{k+l+m}$	$(2q+1)\widehat{\alpha}^{k+l+m}$	$\widehat{\alpha}^{k+l+m}$
$\chi_{k,l}^{(7)}$	$(q^3-1)\widehat{\alpha}^{(k+l)}$	$-\widehat{\alpha}^{(k+l)}$	$-\widehat{\alpha}^{(k+l)}$
$\chi_k^{(8)}$	$(q-1)^2(q+1)\widehat{\alpha}^k$	$-(q-1)\widehat{\alpha}^k$	$\widehat{\alpha}^k$

Table 5.12 (continued)

Type	$\mathcal{T}^{(4)}$	$\mathcal{T}^{(5)}$	$\mathcal{T}^{(6)}$
Rep g	$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$
No. of CC	$(q-1)(q-2)$	$(q-1)(q-2)$	$\frac{(q-1)(q-2)(q-3)}{6}$
$\lvert C_{GL(3,q)}(g)\rvert$	$q(q-1)^3(q+1)$	$q(q-1)^2$	$(q-1)^3$
$\lvert C_g\rvert$	$q^2(q^2+q+1)$	$q^2(q^2-1)(q^2+q+1)$	$q^3(q+1)(q^2+q+1)$
$\chi_k^{(1)}$	$\widehat{\alpha}^{2k}\widehat{\beta}^k$	$\widehat{\alpha}^{2k}\widehat{\beta}^k$	$\widehat{\alpha}^k\widehat{\beta}^k\widehat{\gamma}^k$
$\chi_k^{(2)}$	$(q+1)\widehat{\alpha}^{2k}\widehat{\beta}^k$	$\widehat{\alpha}^{2k}\widehat{\beta}^k$	$2\widehat{\alpha}^k\widehat{\beta}^k\widehat{\gamma}^k$
$\chi_k^{(3)}$	$q\widehat{\alpha}^{2k}\widehat{\beta}^k$	0	$\widehat{\alpha}^k\widehat{\beta}^k\widehat{\gamma}^k$
$\chi_{k,l}^{(4)}$	$(q+1)\widehat{\alpha}^{(k+l)}\widehat{\beta}^l+\widehat{\alpha}^{2l}\widehat{\beta}^k$	$\widehat{\alpha}^{(k+l)}\widehat{\beta}^l+\widehat{\alpha}^{2l}\widehat{\beta}^k$	$\widehat{\alpha}^k\widehat{\beta}^l\widehat{\gamma}^l+\widehat{\alpha}^l\widehat{\beta}^k\widehat{\gamma}^l$ $+\widehat{\alpha}^l\widehat{\beta}^l\widehat{\gamma}^k$
$\chi_{k,l}^{(5)}$	$(q+1)\widehat{\alpha}^{(k+l)}\widehat{\beta}^l+q\widehat{\alpha}^{2l}\widehat{\beta}^k$	$\widehat{\alpha}^{(k+l)}\widehat{\beta}^l$	$\widehat{\alpha}^k\widehat{\beta}^l\widehat{\gamma}^l+\widehat{\alpha}^l\widehat{\beta}^k\widehat{\gamma}^l$ $+\widehat{\alpha}^l\widehat{\beta}^l\widehat{\gamma}^k$
$\chi_{k,l,m}^{(6)}$	$(q+1)\left(\widehat{\alpha}^{(k+l)}\widehat{\beta}^m+\widehat{\alpha}^{(k+m)}\widehat{\beta}^l\right.$ $\left.+\widehat{\alpha}^{(l+m)}\widehat{\beta}^k\right)$	$\widehat{\alpha}^{(k+l)}\widehat{\beta}^m+\widehat{\alpha}^{(k+m)}\widehat{\beta}^l$ $+\widehat{\alpha}^{(l+m)}\widehat{\beta}^k$	$\widehat{\alpha}^k\widehat{\beta}^l\widehat{\gamma}^m+\widehat{\alpha}^k\widehat{\beta}^m\widehat{\gamma}^l$ $+\widehat{\alpha}^m\widehat{\beta}^l\widehat{\gamma}^k+\widehat{\alpha}^m\widehat{\beta}^k\widehat{\gamma}^l$ $+\widehat{\alpha}^l\widehat{\beta}^k\widehat{\gamma}^m+\widehat{\alpha}^l\widehat{\beta}^m\widehat{\gamma}^k$
$\chi_{k,l}^{(7)}$	$(q-1)\widehat{\alpha}^l\widehat{\beta}^k$	$-\widehat{\alpha}^l\widehat{\beta}^k$	0
$\chi_k^{(8)}$	0	0	0

Table 5.12 (continued)

Type	$\mathcal{T}^{(7)}$	$\mathcal{T}^{(8)}$
Rep g	$\begin{pmatrix} 0 & 1 & 0 \\ -r^{q+1} & r+r^{1+q} & 0 \\ 0 & 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s^{1+q+q^2} & -(s^{1+q}+s^{1+q^2}+s^{q+q^2}) & s+s^q+s^{q^2} \end{pmatrix}$
No. of CC	$\frac{q}{2}(q-1)^2$	$\frac{1}{3}(q^3-q)$
$\lvert C_{GL(3,q)}(g)\rvert$	$(q-1)^2(q+1)$	$(q-1)(q^2+q+1)$
$\lvert C_g\rvert$	$q^3(q-1)(q^2+q+1)$	$q^3(q-1)^2(q+1)$
$\chi_k^{(1)}$	$\widehat{\alpha}^k\widehat{r}^{k(q+1)}$	$\widehat{s}^{k(q^2+q+1)}$
$\chi_k^{(2)}$	0	$-\widehat{s}^{k(q^2+q+1)}$
$\chi_k^{(3)}$	$-\widehat{\alpha}^k\widehat{r}^{k(q+1)}$	$\widehat{s}^{k(q^2+q+1)}$
$\chi_{k,l}^{(4)}$	$\widehat{\alpha}^k\widehat{r}^{l(q+1)}$	0
$\chi_{k,l}^{(5)}$	$-\widehat{\alpha}^k\widehat{r}^{l(q+1)}$	0
$\chi_{k,l,m}^{(6)}$	0	0
$\chi_{k,l}^{(7)}$	$-\widehat{\alpha}^k(\widehat{r}^l+\widehat{r}^{ql})$	0
$\chi_k^{(8)}$	0	$\widehat{s}^k+\widehat{s}^{kq}+\widehat{s}^{kq^2}$

where, in Table 5.12,

- $\alpha, \beta, \gamma \in \mathbb{F}_q^*,\ \alpha \neq \beta \neq \gamma \neq \alpha$,
- $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q,\ r^q$ is excluded whenever r is included,
- $s \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q,\ s^q$ and s^{q^2} are excluded whenever s is included,
- in $\chi_k^{(1)},\ \chi_k^{(2)}$ and $\chi_k^{(3)},\ k = 0, 1, \cdots, q-2$,
- in $\chi_{k,l}^{(4)}$ and $\chi_{k,l}^{(5)},\ k, l = 0, 1, \cdots, q-2,\ k \neq l$,
- in $\chi_{k,l,m}^{(6)},\ 0 \leq k < l < m \leq q-2$,
- in $\chi_{k,l}^{(7)},\ k = 0, 1, \cdots, q-2,\ l = 1, 2, \cdots, q^2-1,\ q+1 \nmid l$ and $lq (mod\ q^2-1)$ is excluded whenever l is included,
- in $\chi_k^{(8)},\ k = 1, 2, \cdots, q^3-1,\ q^2+q+1 \nmid k$ and $kq, kq^2 (mod\ q^3-1)$ are excluded whenever k is included, and finally,
- $\widehat{\ }$ is the character : $\mathbb{F}_{q^d}^* = \langle \varepsilon_d \rangle \longrightarrow \mathbb{C}^*$ given by $\widehat{\ }(\varepsilon_d^j) = e^{\frac{2\pi j}{q^d-1}i}$, for $d = 1, 2, 3$ and $0 \leq j \leq q^d-2$.

Remark 5.7.1. We have seen in Remark 5.4.3 that $\lambda_k,\ \psi_k,\ \psi_{k,l},\ \pi_k \in Irr(GL(2,q))$ appeared as either cuspidal characters of $GL(2,q)$ or constituents of an $\bigodot$ −product of cuspidal characters of $GL(1,q)$. Now for $GL(3,q)$ we have

1. $\lambda_k \odot \chi_k = \chi_k^{(2)} + \chi_k^{(1)}$. Thus every $\chi_k^{(1)}$ and $\chi_k^{(2)}$ appears as a constituent of $\lambda_k \odot \chi_k$, where λ_k itself is a constituent of an $\bigodot$ −product of cuspidal characters. Thus $\chi_k^{(1)}$ and $\chi_k^{(2)}$ are constituents of an $\bigodot$ −product of cuspidal characters of $GL(2,q)$ and $GL(1,q)$.

2. $\psi_{k,k} \odot \chi_k = \chi_k^{(3)} + 2\chi_k^{(2)} + \mathbf{1}$. Thus every $\chi_k^{(3)}$ appears as a constituent of $\psi_{k,k} \odot \chi_k$, where $\psi_{k,k}$ is an $\bigodot$ −product of cuspidal characters of $GL(1,q)$. Thus every $\chi_k^{(3)}$ is a constituent of an $\bigodot$ −product of cuspidal characters of $GL(2,q)$ and $GL(1,q)$.

3. $\lambda_l \odot \chi_k = \chi_{k,l}^{(4)}$. Thus every $\chi_{k,l}^{(4)}$ is an $\bigodot$ −product of cuspidal characters of $GL(2,q)$ and $GL(1,q)$.

4. $\psi_l \odot \chi_k = \chi_{k,l}^{(5)}$. Thus every $\chi_{k,l}^{(5)}$ is an $\bigodot$ −product of cuspidal characters of $GL(2,q)$ and $GL(1,q)$.

5. $\psi_{l,m} \odot \chi_k = \chi_{k,l,m}^{(6)}$. Thus every $\chi_{k,l,m}^{(6)}$ is an $\bigodot$ −product of distinct cuspidal characters of $GL(1,q)$.

6. $\pi_l \odot \chi_k = \chi_{k,l}^{(7)}$. Thus every $\chi_{k,l}^{(7)}$ is an $\bigodot$ −product of cuspidal characters π_l of $GL(2,q)$ and the cuspidal characters χ_k of $GL(1,q)$.

7. $\chi_k^{(8)}$ are the cuspidal characters of $GL(3,q)$.

Hence every character of $GL(3,q)$ is either a cuspidal or an $\bigodot$ −product of cuspidal characters or a constituent of an $\bigodot$ −product of cuspidal characters of $GL(2,q)$ and $GL(1,q)$.

We have indicated before that there is a duality between the irreducible characters and the conjugacy classes of $GL(3,q)$. In Table 5.13 we attach to every irreducible character a conjugacy class of $GL(3,q)$.

In Chapter 4 we gave examples for the character tables of $GL(2,q)$ for $q = 3,4$. Here we give the character table of $GL(3,3)$ and the conjugacy classes of $GL(3,4)$. We list these tables in the Appendix.

We conclude this chapter by mentioning that there exists a complete duality between the irreducible characters and the conjugacy classes of $GL(n,q)$. That is to each irreducible character, one can associate a conjugacy class. Some of the aspects of this duality is shown in Table 5.14. There exists an exact numeric in each case. For example the number of primary classes is the same as the number of *monatomic irreducible* characters (see Definition 5.7.1).

Table 5.13: Duality between irreducible characters and conjugacy classes of $GL(3, q)$

Irreducible Character	Associated Conjugacy Class
$\chi_k^{(1)}$	$\mathcal{T}_k^{(1)} = \begin{pmatrix} \varepsilon_1^k & 0 & 0 \\ 0 & \varepsilon_1^k & 0 \\ 0 & 0 & \varepsilon_1^k \end{pmatrix}$
$\chi_k^{(2)}$	$\mathcal{T}_k^{(2)} = \begin{pmatrix} \varepsilon_1^k & 0 & 0 \\ 0 & \varepsilon_1^k & 0 \\ 0 & 0 & \varepsilon_1^k \end{pmatrix}$
$\chi_k^{(3)}$	$\mathcal{T}_k^{(3)} = \begin{pmatrix} \varepsilon_1^k & 0 & 0 \\ 0 & \varepsilon_1^k & 0 \\ 0 & 0 & \varepsilon_1^k \end{pmatrix}$
$\chi_{k,l}^{(4)}$	$\mathcal{T}_{k,l}^{(4)} = \begin{pmatrix} \varepsilon_1^k & 0 & 0 \\ 0 & \varepsilon_1^k & 0 \\ 0 & 0 & \varepsilon_1^l \end{pmatrix}$
$\chi_{k,l}^{(5)}$	$\mathcal{T}_{k,l}^{(5)} = \begin{pmatrix} \varepsilon_1^k & 1 & 0 \\ 0 & \varepsilon_1^k & 0 \\ 0 & 0 & \varepsilon_1^l \end{pmatrix}$
$\chi_{k,l,m}^{(6)}$	$\mathcal{T}_{k,l,m}^{(6)} = \begin{pmatrix} \varepsilon_1^k & 0 & 0 \\ 0 & \varepsilon_1^l & 0 \\ 0 & 0 & \varepsilon_1^m \end{pmatrix}$
$\chi_{k,l}^{(7)}$	$\mathcal{T}_{k,l}^{(7)} = \begin{pmatrix} 0 & 1 & 0 \\ -\varepsilon_2^{l(1+q)} & \varepsilon_2^l + \varepsilon_2^{lq} & 0 \\ 0 & 0 & \varepsilon_1^k \end{pmatrix}$
$\chi_k^{(8)}$	$\mathcal{T}_k^{(8)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3^{k(1+q+q^2)} & -(\varepsilon_3^{k(1+q)} + \varepsilon_3^{k(1+q^2)} + \varepsilon_3^{k(q+q^2)}) & \varepsilon_3^k + \varepsilon_3^{kq} + \varepsilon_3^{kq^2} \end{pmatrix}$

where $\varepsilon_1, \varepsilon_2$ and ε_3 are generators of $\mathbb{F}_q^*, \mathbb{F}_{q^2}^*$ and $\mathbb{F}_{q^3}^*$ respectively.

Definition 5.7.1. *An irreducible character* χ *of* $GL(n, q)$ *is called a* ***monatomic*** *if for some* d *divides* n*, then* χ *appears as a constituent of* $\bigodot_{\frac{n}{d} \text{ times}} \pi = \underbrace{\pi \odot \pi \odot \cdots \odot \pi}_{\frac{n}{d} \text{ times}}$ *such that* π *is a cuspidal character of* $GL(d, q)$.

Remark 5.7.2. Green [27] refers to a monatomic character by a *primary* character.

Table 5.14: Some aspects of the duality between irreducible characters and conjugacy classes of $GL(n, q)$

Irreducible character	Associated conjugacy class	No. of conjugacy classes
linear characters	central classes	$q - 1$
unipotent characters †	unipotent classes	$\lvert\mathcal{P}(n)\rvert$
monatomic characters	primary classes	$\sum_{d\mid n} \lvert\mathcal{P}(\frac{n}{d})\rvert \cdot I_d(q)$
cuspidal characters	regular semisimple classes of type $(n) \vdash n$	$\frac{1}{n}\sum_{d\mid n} \mu(d) q^{\frac{n}{d}}$
induced from distinct cuspidal characters of $GL(1, q)$	regular semisimple classes of type $(1, 1, \cdots, 1) \vdash n$	$\prod_{i=1}^{n} (q - i)/n!$

† The definition of a *unipotent* character can be found in Bump [11] or Zelevinsky [77].

6

Appendix

Table 6.1: Values of $n(\lambda)$, $\lambda \vdash n$, $n = 1, 2, 3, 4, 5$

n	$\lambda \vdash n$	$n(\lambda)$
1	1	0
2	$(1,1)$	1
	2	0
3	$(1,1,1)$	3
	$(2,1)$	1
	3	0
4	$(1,1,1,1)$	6
	$(2,1,1)$	3
	$(2,2)$	2
	$(3,1)$	1
	4	0
5	$(1,1,1,1,1)$	10
	$(2,1,1,1)$	6
	$(2,2,1)$	4
	$(3,1,1)$	3
	$(3,2)$	2
	$(4,1)$	1
	5	0

Table 6.2: The orders of elements of $GL(3,q)$

Type		Order
$\mathcal{T}^{(1)}$	$\alpha = \varepsilon_1^{j}$	$\frac{q-1}{\gcd(j,\ q-1)}$
$\mathcal{T}^{(2)}$	$\alpha = \varepsilon_1^{j}$	$\frac{p(q-1)}{\gcd(j,\ q-1)}$
$\mathcal{T}^{(3)}$	$\alpha = \varepsilon_1^{j}$	$\frac{p(q-1)^2}{\gcd(j,\ q-1)}$
$\mathcal{T}^{(4)}$	$\alpha = \varepsilon_1^{j_1},\ \beta = \varepsilon_1^{j_2}$	$lcm\left(\frac{(q-1)}{\gcd(j_1,\ q-1)}\ ,\ \frac{(q-1)}{\gcd(j_2,\ q-1)}\right)$
$\mathcal{T}^{(5)}$	$\alpha = \varepsilon_1^{j_1},\ \beta = \varepsilon_1^{j_2}$	$p\ lcm\left(\frac{(q-1)}{\gcd(j_1,\ q-1)}\ ,\ \frac{(q-1)}{\gcd(j_2,\ q-1)}\right)$
$\mathcal{T}^{(6)}$	$\alpha = \varepsilon_1^{j_1},\ \beta = \varepsilon_1^{j_2},\ \gamma = \varepsilon_1^{j_3}$	$lcm\left(\frac{(q-1)}{\gcd(j_1,\ q-1)}\ ,\ \frac{(q-1)}{\gcd(j_2,\ q-1)},\ \frac{(q-1)}{\gcd(j_3,\ q-1)}\right)$
$\mathcal{T}^{(7)}$	$\alpha = \varepsilon_1^{j_1},\ r = \varepsilon_2^{j_2}$	$lcm\left(\frac{(q-1)}{\gcd(j_1,\ q-1)}\ ,\ \frac{(q^2-1)}{\gcd(j_2,\ q^2-1)},\ \frac{(q^2-1)}{\gcd(j_2 q,\ q^2-1)}\right)$
$\mathcal{T}^{(8)}$	$s = \varepsilon_3^{j}$	$lcm\left(\frac{(q^3-1)}{\gcd(j,\ q^3-1)}\ ,\ \frac{(q^3-1)}{\gcd(jq,\ q^3-1)},\ \frac{(q^3-1)}{\gcd(jq^2,\ q^3-1)}\right)$

where $\varepsilon_1,\ \varepsilon_2$ and ε_3 are generators of $\mathbb{F}_q^*$, $\mathbb{F}_{q^2}^*$ and $\mathbb{F}_{q^3}^*$ respectively.

Conjugacy Classes of $GL(3,3)$ and $GL(3,4)$

The Group $GL(3,3)$

Let $\mathbb{F}_3^* = \langle\alpha\rangle$, $\mathbb{F}_9^* = \langle\theta\rangle$ and $\mathbb{F}_{27}^* = \langle\zeta\rangle$. Representatives of the conjugacy classes of $GL(3,3)$ are given by:

$$\mathcal{T}_0^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{T}_1^{(1)} = \begin{pmatrix} \zeta^{13} & 0 & 0 \\ 0 & \zeta^{13} & 0 \\ 0 & 0 & \zeta^{13} \end{pmatrix},$$

$$\mathcal{T}_0^{(2)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{T}_1^{(2)} = \begin{pmatrix} \zeta^{13} & 1 & 0 \\ 0 & \zeta^{13} & 0 \\ 0 & 0 & \zeta^{13} \end{pmatrix},$$

$$\mathcal{T}_0^{(3)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{T}_1^{(3)} = \begin{pmatrix} \zeta^{13} & 1 & 0 \\ 0 & \zeta^{13} & 1 \\ 0 & 0 & \zeta^{13} \end{pmatrix},$$

$$\mathcal{T}_{0,1}^{(4)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^{13} & 0 \\ 0 & 0 & \zeta^{13} \end{pmatrix}, \qquad \mathcal{T}_{1,0}^{(4)} = \begin{pmatrix} \zeta^{13} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{T}_{0,1}^{(5)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^{13} & 1 \\ 0 & 0 & \zeta^{13} \end{pmatrix}, \qquad \mathcal{T}_{1,0}^{(5)} = \begin{pmatrix} \zeta^{13} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{T}_{0,1}^{(7)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\theta^4 & \theta+\theta^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\mathcal{T}_{0,2}^{(7)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & \theta^2+\theta^6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \zeta^{13} & 0 \end{pmatrix},$$

$$\mathcal{T}_{0,5}^{(7)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\theta^4 & \theta^5+\theta^7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \zeta^{13} \end{pmatrix},$$

$$\mathcal{T}_{1,1}^{(7)} = \begin{pmatrix} \zeta^{13} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\theta^4 & \theta+\theta^3 \end{pmatrix} = \begin{pmatrix} \zeta^{13} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\mathcal{T}_{1,2}^{(7)} = \begin{pmatrix} \zeta^{13} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & \theta^2+\theta^6 \end{pmatrix} = \begin{pmatrix} \zeta^{13} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \zeta^{13} & 0 \end{pmatrix},$$

$$\mathcal{T}_{1,5}^{(7)} = \begin{pmatrix} \zeta^{13} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\theta^4 & \theta^5+\theta^7 \end{pmatrix} = \begin{pmatrix} \zeta^{13} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \zeta^{13} \end{pmatrix},$$

$$\mathcal{T}_{1}^{(8)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta^{13} & -(\zeta^4+\zeta^{10}+\zeta^{12}) & \zeta+\zeta^3+\zeta^9 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta^{13} & 1 & 0 \end{pmatrix},$$

$$\mathcal{T}_{2}^{(8)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -(\zeta^8+\zeta^{20}+\zeta^{24}) & \zeta^2+\zeta^6+\zeta^{18} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & \zeta^{13} & \zeta^{13} \end{pmatrix},$$

$$\mathcal{T}_{4}^{(8)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -(\zeta^{16}+\zeta^{14}+\zeta^{22}) & \zeta^4+\zeta^{12}+\zeta^{10} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & \zeta^{13} \end{pmatrix},$$

$$\mathcal{T}_{5}^{(8)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta^{13} & -(\zeta^{20}+\zeta^{24}+\zeta^8) & \zeta^5+\zeta^{15}+\zeta^{19} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta^{13} & \zeta^{13} & 1 \end{pmatrix},$$

$$\mathcal{T}_{7}^{(8)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta^{13} & -(\zeta^2+\zeta^{18}+\zeta^6) & \zeta^7+\zeta^{21}+\zeta^{11} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta^{13} & 1 & \zeta^{13} \end{pmatrix},$$

$$\mathcal{T}_8^{(8)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -(\zeta^6+\zeta^2+\zeta^{18}) & \zeta^8+\zeta^{24}+\zeta^{20} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$\mathcal{T}_{14}^{(8)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -(\zeta^4+\zeta^{10}+\zeta^{12}) & \zeta^{14}+\zeta^{16}+\zeta^{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$\mathcal{T}_{17}^{(8)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta^{13} & -(\zeta^{14}+\zeta^{16}+\zeta^{22}) & \zeta^{17}+\zeta^{25}+\zeta^{23} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \zeta^{13} & 0 & 1 \end{pmatrix}.$$

The basic information about the classes of $GL(3,3)$ are given in Table 6.3.

Table 6.3: Basic information for conjugacy classes of $GL(3,3)$

g	$o(g)$	$\lvert C_{GL(3,3)}(g)\rvert$	$\lvert[g]\rvert$	g	$o(g)$	$\lvert C_{GL(3,3)}(g)\rvert$	$\lvert[g]\rvert$
$\mathcal{T}_0^{(1)}$	1	11232	1	$\mathcal{T}_{0,5}^{(7)}$	8	16	702
$\mathcal{T}_1^{(1)}$	2	11232	1	$\mathcal{T}_{1,1}^{(7)}$	8	16	702
$\mathcal{T}_0^{(2)}$	3	108	104	$\mathcal{T}_{1,2}^{(7)}$	8	16	702
$\mathcal{T}_1^{(2)}$	6	108	104	$\mathcal{T}_{1,5}^{(7)}$	8	16	702
$\mathcal{T}_0^{(3)}$	3	18	624	$\mathcal{T}_1^{(8)}$	26	26	432
$\mathcal{T}_1^{(3)}$	6	18	624	$\mathcal{T}_2^{(8)}$	13	26	432
$\mathcal{T}_{0,1}^{(4)}$	2	96	117	$\mathcal{T}_4^{(8)}$	13	26	432
$\mathcal{T}_{1,0}^{(4)}$	2	96	117	$\mathcal{T}_5^{(8)}$	26	26	432
$\mathcal{T}_{0,1}^{(5)}$	6	12	936	$\mathcal{T}_7^{(8)}$	26	26	432
$\mathcal{T}_{1,0}^{(5)}$	6	12	936	$\mathcal{T}_8^{(8)}$	13	26	432
$\mathcal{T}_{0,1}^{(7)}$	8	16	702	$\mathcal{T}_{14}^{(8)}$	13	26	432
$\mathcal{T}_{0,2}^{(7)}$	4	16	702	$\mathcal{T}_{17}^{(8)}$	26	26	432

Table 6.4: The power maps of $GL(3,3)$

$p \mid o(g)$	2	3	13
$\mathcal{T}_0^{(1)}$	-	-	-
$\mathcal{T}_1^{(1)}$	$\mathcal{T}_0^{(1)}$	-	-
$\mathcal{T}_0^{(2)}$	-	$\mathcal{T}_0^{(1)}$	-
$\mathcal{T}_1^{(2)}$	$\mathcal{T}_0^{(3)}$	$\mathcal{T}_1^{(1)}$	-
$\mathcal{T}_0^{(3)}$	-	$\mathcal{T}_0^{(1)}$	-
$\mathcal{T}_1^{(3)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_1^{(1)}$	-
$\mathcal{T}_{0,1}^{(4)}$	$\mathcal{T}_0^{(1)}$	-	-
$\mathcal{T}_{1,0}^{(4)}$	$\mathcal{T}_0^{(1)}$	-	-
$\mathcal{T}_{0,1}^{(5)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_{0,1}^{(4)}$	-
$\mathcal{T}_{1,0}^{(5)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_{1,0}^{(4)}$	-
$\mathcal{T}_{0,1}^{(7)}$	$\mathcal{T}_{0,2}^{(7)}$	-	-
$\mathcal{T}_{0,2}^{(7)}$	$\mathcal{T}_{0,1}^{(4)}$	-	-
$\mathcal{T}_{0,5}^{(7)}$	$\mathcal{T}_{0,2}^{(7)}$	-	-
$\mathcal{T}_{1,1}^{(7)}$	$\mathcal{T}_{0,2}^{(7)}$	-	-
$\mathcal{T}_{1,2}^{(7)}$	$\mathcal{T}_{0,1}^{(4)}$	-	-
$\mathcal{T}_{1,5}^{(7)}$	$\mathcal{T}_{0,2}^{(7)}$	-	-
$\mathcal{T}_1^{(8)}$	$\mathcal{T}_2^{(8)}$	-	$\mathcal{T}_1^{(1)}$
$\mathcal{T}_2^{(8)}$	-	-	$\mathcal{T}_0^{(1)}$
$\mathcal{T}_4^{(8)}$	-	-	$\mathcal{T}_0^{(1)}$
$\mathcal{T}_5^{(8)}$	$\mathcal{T}_4^{(8)}$	-	$\mathcal{T}_1^{(1)}$
$\mathcal{T}_7^{(8)}$	$\mathcal{T}_{14}^{(8)}$	-	$\mathcal{T}_1^{(1)}$
$\mathcal{T}_8^{(8)}$	-	-	$\mathcal{T}_0^{(1)}$
$\mathcal{T}_{14}^{(8)}$	-	-	$\mathcal{T}_0^{(1)}$
$\mathcal{T}_{17}^{(8)}$	$\mathcal{T}_8^{(8)}$	-	$\mathcal{T}_1^{(1)}$

Table 6.5: The character table of $GL(3,3)$

Class	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_1^{(1)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_1^{(2)}$	$\mathcal{T}_0^{(3)}$	$\mathcal{T}_1^{(3)}$	$\mathcal{T}_{0,1}^{(4)}$	$\mathcal{T}_{1,0}^{(4)}$	$\mathcal{T}_{0,1}^{(5)}$	$\mathcal{T}_{1,0}^{(5)}$
$\|C_g\|$	1	1	104	104	624	624	117	117	936	936
$\|C_{GL(3,3)}(g)\|$	11232	11232	108	108	18	18	96	96	12	12
$o(g)$	1	2	3	6	3	6	2	2	6	6
$\chi_0^{(1)}$	1	1	1	1	1	1	1	1	1	1
$\chi_1^{(1)}$	1	−1	1	−1	1	−1	−1	1	−1	1
$\chi_0^{(2)}$	12	12	3	3	0	0	4	4	1	1
$\chi_1^{(2)}$	12	−12	3	−3	0	0	−4	4	−1	1
$\chi_0^{(3)}$	27	27	0	0	0	0	3	3	0	0
$\chi_1^{(3)}$	27	−27	0	0	0	0	−3	3	0	0
$\chi_{0,1}^{(4)}$	13	13	4	4	1	1	−3	−3	0	0
$\chi_{1,0}^{(4)}$	13	−13	4	−4	1	−1	3	−3	0	0
$\chi_{0,1}^{(5)}$	39	39	3	3	0	0	−1	−1	−1	−1
$\chi_{1,0}^{(5)}$	39	−39	3	−3	0	0	1	−1	1	−1
$\chi_{0,1}^{(7)}$	26	−26	−1	1	−1	1	2	−2	−1	1
$\chi_{0,2}^{(7)}$	26	26	−1	−1	−1	−1	2	2	−1	−1
$\chi_{0,5}^{(7)}$	26	−26	−1	1	−1	1	2	−2	−1	1
$\chi_{1,1}^{(7)}$	26	26	−1	−1	−1	−1	−2	−2	1	1
$\chi_{1,2}^{(7)}$	26	−26	−1	1	−1	1	−2	2	1	−1
$\chi_{1,5}^{(7)}$	26	26	−1	−1	−1	−1	−2	−2	1	1
$\chi_1^{(8)}$	16	−16	−2	2	1	−1	0	0	0	0
$\chi_2^{(8)}$	16	16	−2	−2	1	1	0	0	0	0
$\chi_4^{(8)}$	16	16	−2	−2	1	1	0	0	0	0
$\chi_5^{(8)}$	16	−16	−2	2	1	−1	0	0	0	0
$\chi_7^{(8)}$	16	−16	−2	2	1	−1	0	0	0	0
$\chi_8^{(8)}$	16	16	−2	−2	1	1	0	0	0	0
$\chi_{14}^{(8)}$	16	16	−2	−2	1	1	0	0	0	0
$\chi_{17}^{(8)}$	16	−16	−2	2	1	−1	0	0	0	0

Table 6.5 (continued)

Class	$\mathcal{T}_{0,1}^{(7)}$	$\mathcal{T}_{0,2}^{(7)}$	$\mathcal{T}_{0,5}^{(7)}$	$\mathcal{T}_{1,1}^{(7)}$	$\mathcal{T}_{1,2}^{(7)}$	$\mathcal{T}_{1,5}^{(7)}$
$\|C_g\|$	702	702	702	702	702	702
$\|C_{GL(3,3)}(g)\|$	16	16	16	16	16	16
$o(g)$	8	4	8	8	4	8
$\chi_0^{(1)}$	1	1	1	1	1	1
$\chi_1^{(1)}$	1	-1	1	-1	1	-1
$\chi_0^{(2)}$	0	0	0	0	0	0
$\chi_1^{(2)}$	0	0	0	0	0	0
$\chi_0^{(3)}$	-1	-1	-1	-1	-1	-1
$\chi_1^{(3)}$	-1	1	-1	1	-1	1
$\chi_{0,1}^{(4)}$	-1	1	-1	-1	1	-1
$\chi_{1,0}^{(4)}$	1	1	1	-1	-1	-1
$\chi_{0,1}^{(5)}$	1	-1	1	1	-1	1
$\chi_{1,0}^{(5)}$	-1	-1	-1	1	1	1
$\chi_{0,1}^{(7)}$	$-i\sqrt{2}$	0	$i\sqrt{2}$	$-i\sqrt{2}$	0	$i\sqrt{2}$
$\chi_{0,2}^{(7)}$	0	2	0	0	2	0
$\chi_{0,5}^{(7)}$	$i\sqrt{2}$	0	$-i\sqrt{2}$	$i\sqrt{2}$	0	$-i\sqrt{2}$
$\chi_{1,1}^{(7)}$	$-i\sqrt{2}$	0	$i\sqrt{2}$	$i\sqrt{2}$	0	$-i\sqrt{2}$
$\chi_{1,2}^{(7)}$	0	2	0	0	-2	0
$\chi_{1,5}^{(7)}$	$i\sqrt{2}$	0	$-i\sqrt{2}$	$-i\sqrt{2}$	0	$i\sqrt{2}$
$\chi_1^{(8)}$	0	0	0	0	0	0
$\chi_2^{(8)}$	0	0	0	0	0	0
$\chi_4^{(8)}$	0	0	0	0	0	0
$\chi_5^{(8)}$	0	0	0	0	0	0
$\chi_7^{(8)}$	0	0	0	0	0	0
$\chi_8^{(8)}$	0	0	0	0	0	0
$\chi_{14}^{(8)}$	0	0	0	0	0	0
$\chi_{17}^{(8)}$	0	0	0	0	0	0

Table 6.5 (continued)

Class	$\mathcal{T}_1^{(8)}$	$\mathcal{T}_2^{(8)}$	$\mathcal{T}_4^{(8)}$	$\mathcal{T}_5^{(8)}$	$\mathcal{T}_7^{(8)}$	$\mathcal{T}_8^{(8)}$	$\mathcal{T}_{14}^{(8)}$	$\mathcal{T}_{17}^{(8)}$
$\lvert C_g\rvert$	432	432	432	432	432	432	432	432
$\lvert C_{GL(3,3)}(g)\rvert$	26	26	26	26	26	26	26	26
$o(g)$	26	13	13	26	26	13	13	26
$\chi_0^{(1)}$	1	1	1	1	1	1	1	1
$\chi_1^{(1)}$	-1	1	1	-1	-1	1	1	-1
$\chi_0^{(2)}$	-1	-1	-1	-1	-1	-1	-1	-1
$\chi_1^{(2)}$	1	-1	-1	1	1	-1	-1	1
$\chi_0^{(3)}$	1	1	1	1	1	1	1	1
$\chi_1^{(3)}$	-1	1	1	-1	-1	1	1	-1
$\chi_{0,1}^{(4)}$	0	0	0	0	0	0	0	0
$\chi_{1,0}^{(4)}$	0	0	0	0	0	0	0	0
$\chi_{0,1}^{(5)}$	0	0	0	0	0	0	0	0
$\chi_{1,0}^{(5)}$	0	0	0	0	0	0	0	0
$\chi_{0,1}^{(7)}$	0	0	0	0	0	0	0	0
$\chi_{0,2}^{(7)}$	0	0	0	0	0	0	0	0
$\chi_{0,5}^{(7)}$	0	0	0	0	0	0	0	0
$\chi_{1,1}^{(7)}$	0	0	0	0	0	0	0	0
$\chi_{1,2}^{(7)}$	0	0	0	0	0	0	0	0
$\chi_{1,5}^{(7)}$	0	0	0	0	0	0	0	0
$\chi_1^{(8)}$	$-B$	A	$\overline{B}$	$-A$	$-\overline{A}$	$\overline{A}$	B	$-\overline{B}$
$\chi_2^{(8)}$	A	$\overline{B}$	$\overline{A}$	$\overline{B}$	B	B	A	$\overline{A}$
$\chi_4^{(8)}$	$\overline{B}$	$\overline{A}$	B	$\overline{A}$	A	A	$\overline{B}$	B
$\chi_5^{(8)}$	$-A$	$\overline{B}$	$\overline{A}$	$-\overline{B}$	$-B$	B	A	$-\overline{A}$
$\chi_7^{(8)}$	$-\overline{A}$	B	A	$-B$	$-\overline{B}$	$\overline{B}$	$\overline{A}$	$-A$
$\chi_8^{(8)}$	$\overline{A}$	B	A	B	$\overline{B}$	$\overline{B}$	$\overline{A}$	A
$\chi_{14}^{(8)}$	B	A	$\overline{B}$	A	$\overline{A}$	$\overline{A}$	B	$\overline{B}$
$\chi_{17}^{(8)}$	$-\overline{B}$	$\overline{A}$	$\overline{B}$	$-\overline{A}$	$-A$	A	$\overline{B}$	$-B$

where $A = e^{\frac{2\pi}{13}i} + e^{\frac{5\pi}{13}i} + e^{\frac{6\pi}{13}i}$ and $B = e^{\frac{\pi}{13}i} + e^{\frac{3\pi}{13}i} + e^{\frac{9\pi}{13}i}$.

Table 6.6: Correspondence of conjugacy classes of $GL(3,3)$ in our notation and GAP notation

Our Notation	GAP Notation	Our Notation	GAP Notation	Our Notation	GAP Notation	Our Notation	GAP Notation
$\mathcal{T}_0^{(1)}$	$1a$	$\mathcal{T}_{0,1}^{(4)}$	$2c$	$\mathcal{T}_{0,5}^{(7)}$	$8c$	$\mathcal{T}_4^{(8)}$	$13a$
$\mathcal{T}_1^{(1)}$	$2a$	$\mathcal{T}_{1,0}^{(4)}$	$2b$	$\mathcal{T}_{1,1}^{(7)}$	$8b$	$\mathcal{T}_5^{(8)}$	$26b$
$\mathcal{T}_0^{(2)}$	$3a$	$\mathcal{T}_{0,1}^{(5)}$	$6d$	$\mathcal{T}_{1,2}^{(7)}$	$4a$	$\mathcal{T}_7^{(8)}$	$26d$
$\mathcal{T}_1^{(2)}$	$6b$	$\mathcal{T}_{1,0}^{(5)}$	$6c$	$\mathcal{T}_{1,5}^{(7)}$	$8a$	$\mathcal{T}_8^{(8)}$	$13d$
$\mathcal{T}_0^{(3)}$	$3b$	$\mathcal{T}_{0,1}^{(7)}$	$8d$	$\mathcal{T}_1^{(8)}$	$26c$	$\mathcal{T}_{14}^{(8)}$	$13c$
$\mathcal{T}_1^{(3)}$	$6a$	$\mathcal{T}_{0,2}^{(7)}$	$4b$	$\mathcal{T}_2^{(8)}$	$13b$	$\mathcal{T}_{17}^{(8)}$	$26a$

Table 6.7: Correspondence of irreducible characters of $GL(3,3)$ in our notation and GAP notation

Our Notation	GAP Notation	Our Notation	GAP Notation	Our Notation	GAP Notation	Our Notation	GAP Notation
$\chi_0^{(1)}$	$\chi 1$	$\chi_{0,1}^{(4)}$	$\chi 5$	$\chi_{0,5}^{(7)}$	$\chi 19$	$\chi_4^{(8)}$	$\chi 9$
$\chi_1^{(1)}$	$\chi 2$	$\chi_{1,0}^{(4)}$	$\chi 6$	$\chi_{1,1}^{(7)}$	$\chi 17$	$\chi_5^{(8)}$	$\chi 12$
$\chi_0^{(2)}$	$\chi 3$	$\chi_{0,1}^{(5)}$	$\chi 23$	$\chi_{1,2}^{(7)}$	$\chi 16$	$\chi_7^{(8)}$	$\chi 11$
$\chi_1^{(2)}$	$\chi 4$	$\chi_{1,0}^{(5)}$	$\chi 24$	$\chi_{1,5}^{(7)}$	$\chi 18$	$\chi_8^{(8)}$	$\chi 7$
$\chi_0^{(3)}$	$\chi 21$	$\chi_{0,1}^{(7)}$	$\chi 20$	$\chi_1^{(8)}$	$\chi 14$	$\chi_{14}^{(8)}$	$\chi 10$
$\chi_1^{(3)}$	$\chi 22$	$\chi_{0,2}^{(7)}$	$\chi 15$	$\chi_2^{(8)}$	$\chi 8$	$\chi_{17}^{(8)}$	$\chi 13$

The Group $GL(3,4)$

Suppose that $\mathbb{F}_{64}^{*} \cong \mathbb{Z}_{63} = \langle \zeta \rangle$. The conjugacy classes of $GL(3,4)$ are given in Table 6.8.

Table 6.8: The conjugacy classes of $GL(3,4)$

Class	Rep g	$o(g)$	$\lvert[g]\rvert$	$\lvert C_G(g)\rvert$	Class	Rep g	$o(g)$	$\lvert[g]\rvert$	$\lvert C_G(g)\rvert$
$\mathcal{T}_0^{(1)}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	1	1	181440	$\mathcal{T}_1^{(1)}$	$\begin{pmatrix} \zeta^{21} & 0 & 0 \\ 0 & \zeta^{21} & 0 \\ 0 & 0 & \zeta^{21} \end{pmatrix}$	3	1	181440
$\mathcal{T}_2^{(1)}$	$\begin{pmatrix} \zeta^{42} & 0 & 0 \\ 0 & \zeta^{42} & 0 \\ 0 & 0 & \zeta^{42} \end{pmatrix}$	3	1	181440	$\mathcal{T}_0^{(2)}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	2	315	576
$\mathcal{T}_1^{(2)}$	$\begin{pmatrix} \zeta^{21} & 1 & 0 \\ 0 & \zeta^{21} & 0 \\ 0 & 0 & \zeta^{21} \end{pmatrix}$	6	315	576	$\mathcal{T}_2^{(2)}$	$\begin{pmatrix} \zeta^{42} & 1 & 0 \\ 0 & \zeta^{42} & 0 \\ 0 & 0 & \zeta^{42} \end{pmatrix}$	6	315	576
$\mathcal{T}_0^{(3)}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	4	3780	48	$\mathcal{T}_1^{(3)}$	$\begin{pmatrix} \zeta^{21} & 1 & 0 \\ 0 & \zeta^{21} & 1 \\ 0 & 0 & \zeta^{21} \end{pmatrix}$	12	3780	48
$\mathcal{T}_2^{(3)}$	$\begin{pmatrix} \zeta^{42} & 1 & 0 \\ 0 & \zeta^{42} & 1 \\ 0 & 0 & \zeta^{42} \end{pmatrix}$	12	3780	48	$\mathcal{T}_{0,1}^{(4)}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^{21} \end{pmatrix}$	3	336	540
$\mathcal{T}_{1,0}^{(4)}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^{42} \end{pmatrix}$	3	336	540	$\mathcal{T}_{0,2}^{(4)}$	$\begin{pmatrix} \zeta^{21} & 0 & 0 \\ 0 & \zeta^{21} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	3	336	540
$\mathcal{T}_{2,0}^{(4)}$	$\begin{pmatrix} \zeta^{21} & 0 & 0 \\ 0 & \zeta^{21} & 0 \\ 0 & 0 & \zeta^{42} \end{pmatrix}$	3	336	540	$\mathcal{T}_{1,2}^{(4)}$	$\begin{pmatrix} \zeta^{42} & 0 & 0 \\ 0 & \zeta^{42} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	3	336	540
$\mathcal{T}_{2,1}^{(4)}$	$\begin{pmatrix} \zeta^{42} & 0 & 0 \\ 0 & \zeta^{42} & 0 \\ 0 & 0 & \zeta^{21} \end{pmatrix}$	3	336	540	$\mathcal{T}_{0,1}^{(5)}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^{21} \end{pmatrix}$	6	5040	36
$\mathcal{T}_{1,0}^{(5)}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^{42} \end{pmatrix}$	6	5040	36	$\mathcal{T}_{0,2}^{(5)}$	$\begin{pmatrix} \zeta^{21} & 1 & 0 \\ 0 & \zeta^{21} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	6	5040	36
$\mathcal{T}_{2,0}^{(5)}$	$\begin{pmatrix} \zeta^{21} & 1 & 0 \\ 0 & \zeta^{21} & 0 \\ 0 & 0 & \zeta^{42} \end{pmatrix}$	6	5040	36	$\mathcal{T}_{1,2}^{(5)}$	$\begin{pmatrix} \zeta^{42} & 1 & 0 \\ 0 & \zeta^{42} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	6	5040	36
$\mathcal{T}_{2,1}^{(5)}$	$\begin{pmatrix} \zeta^{42} & 1 & 0 \\ 0 & \zeta^{42} & 0 \\ 0 & 0 & \zeta^{42} \end{pmatrix}$	6	5040	36	$\mathcal{T}_{0,1,2}^{(6)}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^{21} & 0 \\ 0 & 0 & \zeta^{42} \end{pmatrix}$	3	6720	27
$\mathcal{T}_{0,1}^{(7)}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \zeta^{21} & 1 \end{pmatrix}$	15	4032	45	$\mathcal{T}_{0,2}^{(7)}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \zeta^{41} & 1 \end{pmatrix}$	15	4032	45
$\mathcal{T}_{0,3}^{(7)}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \zeta^{42} \end{pmatrix}$	5	4032	45	$\mathcal{T}_{0,6}^{(7)}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \zeta^{21} \end{pmatrix}$	5	4032	45

Table 6.8 (continued)

Class	Rep g	$o(g)$	$\lvert[g]\rvert$	$\lvert C_G(g)\rvert$	Class	Rep g	$o(g)$	$\lvert[g]\rvert$	$\lvert C_G(g)\rvert$
$\mathcal{T}^{(7)}_{0,7}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&\zeta^{21}&\zeta^{21}\end{pmatrix}$	15	4032	45	$\mathcal{T}^{(7)}_{0,11}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&\zeta^{42}&\zeta^{42}\end{pmatrix}$	15	4032	45
$\mathcal{T}^{(7)}_{1,1}$	$\begin{pmatrix}\zeta^{21}&0&0\\0&0&1\\0&\zeta^{21}&1\end{pmatrix}$	15	4032	45	$\mathcal{T}^{(7)}_{1,2}$	$\begin{pmatrix}\zeta^{21}&0&0\\0&0&1\\0&\zeta^{42}&1\end{pmatrix}$	15	4032	45
$\mathcal{T}^{(7)}_{1,3}$	$\begin{pmatrix}\zeta^{21}&0&0\\0&0&1\\0&1&\zeta^{42}\end{pmatrix}$	15	4032	45	$\mathcal{T}^{(7)}_{1,6}$	$\begin{pmatrix}\zeta^{21}&0&0\\0&0&1\\0&1&\zeta^{21}\end{pmatrix}$	15	4032	45
$\mathcal{T}^{(7)}_{1,7}$	$\begin{pmatrix}\zeta^{21}&0&0\\0&0&1\\0&\zeta^{21}&\zeta^{21}\end{pmatrix}$	15	4032	45	$\mathcal{T}^{(7)}_{1,11}$	$\begin{pmatrix}\zeta^{21}&0&0\\0&0&1\\0&\zeta^{42}&\zeta^{42}\end{pmatrix}$	5	4032	45
$\mathcal{T}^{(7)}_{2,1}$	$\begin{pmatrix}\zeta^{42}&0&0\\0&0&1\\0&\zeta^{21}&1\end{pmatrix}$	5	4032	45	$\mathcal{T}^{(7)}_{2,2}$	$\begin{pmatrix}\zeta^{42}&0&0\\0&0&1\\0&\zeta^{42}&1\end{pmatrix}$	15	4032	45
$\mathcal{T}^{(7)}_{2,3}$	$\begin{pmatrix}\zeta^{42}&0&0\\0&0&1\\0&1&\zeta^{42}\end{pmatrix}$	15	4032	45	$\mathcal{T}^{(7)}_{2,6}$	$\begin{pmatrix}\zeta^{42}&0&0\\0&0&1\\0&1&\zeta^{21}\end{pmatrix}$	15	4032	45
$\mathcal{T}^{(7)}_{2,7}$	$\begin{pmatrix}\zeta^{42}&0&0\\0&0&1\\0&\zeta^{21}&\zeta^{21}\end{pmatrix}$	15	4032	45	$\mathcal{T}^{(7)}_{2,11}$	$\begin{pmatrix}\zeta^{42}&0&0\\0&0&1\\0&\zeta^{42}&\zeta^{42}\end{pmatrix}$	15	4032	45
$\mathcal{T}^{(8)}_{1}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{21}&1&1\end{pmatrix}$	63	2880	63	$\mathcal{T}^{(8)}_{2}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{42}&1&1\end{pmatrix}$	63	2880	63
$\mathcal{T}^{(8)}_{3}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&\zeta^{21}\end{pmatrix}$	21	2880	63	$\mathcal{T}^{(8)}_{5}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{42}&\zeta^{21}&1\end{pmatrix}$	63	2880	63
$\mathcal{T}^{(8)}_{6}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&\zeta^{42}\end{pmatrix}$	21	2880	63	$\mathcal{T}^{(8)}_{7}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{21}&0&0\end{pmatrix}$	9	2880	63
$\mathcal{T}^{(8)}_{9}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&1&0\end{pmatrix}$	7	2880	63	$\mathcal{T}^{(8)}_{10}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{21}&\zeta^{42}&1\end{pmatrix}$	63	2880	63
$\mathcal{T}^{(8)}_{11}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{42}&\zeta^{21}&\zeta^{42}\end{pmatrix}$	63	2880	63	$\mathcal{T}^{(8)}_{13}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{21}&1&\zeta^{42}\end{pmatrix}$	63	2880	63
$\mathcal{T}^{(8)}_{14}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{42}&0&0\end{pmatrix}$	9	2880	63	$\mathcal{T}^{(8)}_{15}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&\zeta^{21}&0\end{pmatrix}$	21	2880	63
$\mathcal{T}^{(8)}_{22}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{21}&\zeta^{42}&\zeta^{21}\end{pmatrix}$	63	2880	63	$\mathcal{T}^{(8)}_{23}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{42}&\zeta^{42}&\zeta^{21}\end{pmatrix}$	63	2880	63
$\mathcal{T}^{(8)}_{26}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{42}&1&\zeta^{21}\end{pmatrix}$	63	2880	63	$\mathcal{T}^{(8)}_{27}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&1\end{pmatrix}$	7	2880	63
$\mathcal{T}^{(8)}_{30}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&\zeta^{42}&0\end{pmatrix}$	21	2880	63	$\mathcal{T}^{(8)}_{31}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{21}&\zeta^{21}&\zeta^{21}\end{pmatrix}$	63	2880	63
$\mathcal{T}^{(8)}_{43}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{21}&\zeta^{21}&\zeta^{42}\end{pmatrix}$	63	2880	63	$\mathcal{T}^{(8)}_{47}$	$\begin{pmatrix}0&1&0\\0&0&1\\\zeta^{42}&\zeta^{42}&\zeta^{42}\end{pmatrix}$	63	2880	63

Table 6.9: The power maps of $GL(3,4)$

$p \mid o(g)$	2	3	5	7	$p \mid o(g)$	2	3	5	7
$\mathcal{T}_{0}^{(1)}$	-	-	-	-	$\mathcal{T}_{1,3}^{(7)}$	-	$\mathcal{T}_{0,6}^{(7)}$	$\mathcal{T}_{1,0}^{(4)}$	-
$\mathcal{T}_{1}^{(1)}$	-	$\mathcal{T}_{0}^{(1)}$	-	-	$\mathcal{T}_{1,6}^{(7)}$	-	$\mathcal{T}_{0,3}^{(7)}$	$\mathcal{T}_{1,0}^{(4)}$	-
$\mathcal{T}_{2}^{(1)}$	-	$\mathcal{T}_{0}^{(1)}$	-	-	$\mathcal{T}_{1,7}^{(7)}$	-	$\mathcal{T}_{0,6}^{(7)}$	$\mathcal{T}_{2,0}^{(4)}$	-
$\mathcal{T}_{0}^{(2)}$	$\mathcal{T}_{0}^{(1)}$	-	-	-	$\mathcal{T}_{1,11}^{(7)}$	-	$\mathcal{T}_{0,3}^{(7)}$	$\mathcal{T}_{2}^{(1)}$	-
$\mathcal{T}_{1}^{(2)}$	$\mathcal{T}_{2}^{(1)}$	$\mathcal{T}_{0}^{(2)}$	-	-	$\mathcal{T}_{2,1}^{(7)}$	-	$\mathcal{T}_{0,3}^{(7)}$	$\mathcal{T}_{1}^{(1)}$	-
$\mathcal{T}_{2}^{(2)}$	$\mathcal{T}_{1}^{(1)}$	$\mathcal{T}_{0}^{(2)}$	-	-	$\mathcal{T}_{2,2}^{(7)}$	-	$\mathcal{T}_{0,6}^{(7)}$	$\mathcal{T}_{2,1}^{(4)}$	-
$\mathcal{T}_{0}^{(3)}$	$\mathcal{T}_{0}^{(2)}$	-	-	-	$\mathcal{T}_{2,3}^{(7)}$	-	$\mathcal{T}_{0,6}^{(7)}$	$\mathcal{T}_{0,1}^{(4)}$	-
$\mathcal{T}_{1}^{(3)}$	$\mathcal{T}_{2}^{(2)}$	$\mathcal{T}_{0}^{(3)}$	-	-	$\mathcal{T}_{2,6}^{(7)}$	-	$\mathcal{T}_{0,3}^{(7)}$	$\mathcal{T}_{0,1}^{(4)}$	-
$\mathcal{T}_{2}^{(3)}$	$\mathcal{T}_{1}^{(2)}$	$\mathcal{T}_{0}^{(3)}$	-	-	$\mathcal{T}_{2,7}^{(7)}$	-	$\mathcal{T}_{0,6}^{(7)}$	$\mathcal{T}_{1}^{(1)}$	-
$\mathcal{T}_{0,1}^{(4)}$	-	$\mathcal{T}_{0}^{(1)}$	-	-	$\mathcal{T}_{2,11}^{(7)}$	-	$\mathcal{T}_{0,3}^{(7)}$	$\mathcal{T}_{2,1}^{(4)}$	-
$\mathcal{T}_{1,0}^{(4)}$	-	$\mathcal{T}_{0}^{(1)}$	-	-	$\mathcal{T}_{1}^{(8)}$	-	$\mathcal{T}_{3}^{(8)}$	-	$\mathcal{T}_{7}^{(8)}$
$\mathcal{T}_{0,2}^{(4)}$	-	$\mathcal{T}_{0}^{(1)}$	-	-	$\mathcal{T}_{2}^{(8)}$	-	$\mathcal{T}_{6}^{(8)}$	-	$\mathcal{T}_{14}^{(8)}$
$\mathcal{T}_{2,0}^{(4)}$	-	$\mathcal{T}_{0}^{(1)}$	-	-	$\mathcal{T}_{3}^{(8)}$	-	$\mathcal{T}_{9}^{(8)}$	-	$\mathcal{T}_{1}^{(1)}$
$\mathcal{T}_{1,2}^{(4)}$	-	$\mathcal{T}_{0}^{(1)}$	-	-	$\mathcal{T}_{5}^{(8)}$	-	$\mathcal{T}_{15}^{(8)}$	-	$\mathcal{T}_{14}^{(8)}$
$\mathcal{T}_{2,1}^{(4)}$	-	$\mathcal{T}_{0}^{(1)}$	-	-	$\mathcal{T}_{6}^{(8)}$	-	$\mathcal{T}_{9}^{(8)}$	-	$\mathcal{T}_{2}^{(1)}$
$\mathcal{T}_{0,1}^{(5)}$	$\mathcal{T}_{1,0}^{(4)}$	$\mathcal{T}_{0}^{(2)}$	-	-	$\mathcal{T}_{7}^{(8)}$	-	$\mathcal{T}_{1}^{(1)}$	-	-
$\mathcal{T}_{1,0}^{(5)}$	$\mathcal{T}_{0,1}^{(4)}$	$\mathcal{T}_{0}^{(2)}$	-	-	$\mathcal{T}_{9}^{(8)}$	-	-	-	$\mathcal{T}_{0}^{(1)}$
$\mathcal{T}_{0,2}^{(5)}$	$\mathcal{T}_{1,2}^{(4)}$	$\mathcal{T}_{0}^{(2)}$	-	-	$\mathcal{T}_{10}^{(8)}$	-	$\mathcal{T}_{30}^{(8)}$	-	$\mathcal{T}_{7}^{(8)}$
$\mathcal{T}_{2,0}^{(5)}$	$\mathcal{T}_{2,1}^{(4)}$	$\mathcal{T}_{0}^{(2)}$	-	-	$\mathcal{T}_{11}^{(8)}$	-	$\mathcal{T}_{6}^{(8)}$	-	$\mathcal{T}_{14}^{(8)}$
$\mathcal{T}_{1,2}^{(5)}$	$\mathcal{T}_{0,2}^{(4)}$	$\mathcal{T}_{0}^{(2)}$	-	-	$\mathcal{T}_{13}^{(8)}$	-	$\mathcal{T}_{30}^{(8)}$	-	$\mathcal{T}_{7}^{(8)}$
$\mathcal{T}_{2,1}^{(5)}$	$\mathcal{T}_{2,0}^{(4)}$	$\mathcal{T}_{0}^{(2)}$	-	-	$\mathcal{T}_{14}^{(8)}$	-	$\mathcal{T}_{2}^{(1)}$	-	-
$\mathcal{T}_{0,1,2}^{(6)}$	-	$\mathcal{T}_{0}^{(1)}$	-	-	$\mathcal{T}_{15}^{(8)}$	-	$\mathcal{T}_{27}^{(8)}$	-	$\mathcal{T}_{2}^{(1)}$
$\mathcal{T}_{0,1}^{(7)}$	-	$\mathcal{T}_{0,3}^{(7)}$	$\mathcal{T}_{0,2}^{(4)}$	-	$\mathcal{T}_{22}^{(8)}$	-	$\mathcal{T}_{3}^{(8)}$	-	$\mathcal{T}_{7}^{(8)}$
$\mathcal{T}_{0,2}^{(7)}$	-	$\mathcal{T}_{0,6}^{(7)}$	$\mathcal{T}_{1,2}^{(4)}$	-	$\mathcal{T}_{23}^{(8)}$	-	$\mathcal{T}_{6}^{(8)}$	-	$\mathcal{T}_{14}^{(8)}$
$\mathcal{T}_{0,3}^{(7)}$	-	-	$\mathcal{T}_{0}^{(1)}$	-	$\mathcal{T}_{26}^{(8)}$	-	$\mathcal{T}_{15}^{(8)}$	-	$\mathcal{T}_{14}^{(8)}$
$\mathcal{T}_{0,6}^{(7)}$	-	-	$\mathcal{T}_{0}^{(1)}$	-	$\mathcal{T}_{27}^{(8)}$	-	-	-	$\mathcal{T}_{0}^{(1)}$
$\mathcal{T}_{0,7}^{(7)}$	-	$\mathcal{T}_{0,3}^{(7)}$	$\mathcal{T}_{0,2}^{(4)}$	-	$\mathcal{T}_{30}^{(8)}$	-	$\mathcal{T}_{27}^{(8)}$	-	$\mathcal{T}_{1}^{(1)}$
$\mathcal{T}_{0,11}^{(7)}$	-	$\mathcal{T}_{0,3}^{(7)}$	$\mathcal{T}_{1,2}^{(4)}$	-	$\mathcal{T}_{31}^{(8)}$	-	$\mathcal{T}_{30}^{(8)}$	-	$\mathcal{T}_{7}^{(8)}$
$\mathcal{T}_{1,1}^{(7)}$	-	$\mathcal{T}_{0,3}^{(7)}$	$\mathcal{T}_{2,0}^{(4)}$	-	$\mathcal{T}_{43}^{(8)}$	-	$\mathcal{T}_{3}^{(8)}$	-	$\mathcal{T}_{7}^{(8)}$
$\mathcal{T}_{1,2}^{(7)}$	-	$\mathcal{T}_{0,6}^{(7)}$	$\mathcal{T}_{2}^{(1)}$	-	$\mathcal{T}_{47}^{(8)}$	-	$\mathcal{T}_{15}^{(8)}$	-	$\mathcal{T}_{14}^{(8)}$

Table 6.10: Conjugacy classes of $GL(4,q)$

Type	Representative	Conditions	No of Classes
$\mathcal{T}^{(1)}$	$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$\alpha \in \mathbb{F}_q^*$	$q-1$
$\mathcal{T}^{(2)}$	$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$\alpha \in \mathbb{F}_q^*$	$q-1$
$\mathcal{T}^{(3)}$	$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$\alpha \in \mathbb{F}_q^*$	$q-1$
$\mathcal{T}^{(4)}$	$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$\alpha \in \mathbb{F}_q^*$	$q-1$
$\mathcal{T}^{(5)}$	$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$\alpha \in \mathbb{F}_q^*$	$q-1$
$\mathcal{T}^{(6)}$	$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$\alpha, \beta \in \mathbb{F}_q^*,\ \alpha \neq \beta$	$(q-1)(q-2)$
$\mathcal{T}^{(7)}$	$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$\alpha, \beta \in \mathbb{F}_q^*,\ \alpha \neq \beta$	$(q-1)(q-2)$
$\mathcal{T}^{(8)}$	$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$\alpha, \beta \in \mathbb{F}_q^*,\ \alpha \neq \beta$	$(q-1)(q-2)$
$\mathcal{T}^{(9)}$	$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$\alpha, \beta \in \mathbb{F}_q^*,\ \alpha \neq \beta$	$\frac{(q-1)(q-2)}{2}$
$\mathcal{T}^{(10)}$	$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$\alpha, \beta \in \mathbb{F}_q^*,\ \alpha \neq \beta$	$(q-1)(q-2)$
$\mathcal{T}^{(11)}$	$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$\alpha, \beta \in \mathbb{F}_q^*,\ \alpha \neq \beta$	$\frac{(q-1)(q-2)}{2}$

Table 6.10 (continued)

Type	Representative	Conditions	No of Classes
$\mathcal{T}^{(12)}$	$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$	$\alpha, \beta, \gamma \in \mathbb{F}_q^*,\ \alpha \neq \beta \neq \gamma \neq \alpha$	$\frac{(q-1)(q-2)(q-3)}{2}$
$\mathcal{T}^{(13)}$	$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$	$\alpha, \beta \in \mathbb{F}_q^*,\ \alpha \neq \beta \neq \gamma \neq \alpha$	$\frac{(q-1)(q-2)(q-3)}{2}$
$\mathcal{T}^{(14)}$	$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -r^{q+1} & r+r^q \end{pmatrix}$	$\alpha \in \mathbb{F}_q^*,\ r \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$	$\frac{q(q-1)^2}{2}$
$\mathcal{T}^{(15)}$	$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -r^{q+1} & r+r^q \end{pmatrix}$	$\alpha \in \mathbb{F}_q^*,\ r \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$	$\frac{q(q-1)^2}{2}$
$\mathcal{T}^{(16)}$	$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \xi \end{pmatrix}$	$\alpha, \beta, \gamma, \xi \in \mathbb{F}_q^*,\ \alpha, \beta, \gamma, \xi$ are distinct	$\frac{(q-1)(q-2)(q-3)(q-4)}{24}$
$\mathcal{T}^{(17)}$	$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -r^{q+1} & r+r^q \end{pmatrix}$	$\alpha, \beta \in \mathbb{F}_q^*,\ \alpha \neq \beta,\ r \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$	$\frac{q(q-1)^2(q-2)}{4}$
$\mathcal{T}^{(18)}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -r^{q+1} & r+r^q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -s^{q+1} & s+s^q \end{pmatrix}$	$r, s \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*,\ r \neq s$	$\frac{q(q^2-1)(q-2)}{8}$
$\mathcal{T}^{(19)}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -r^{q+1} & r+r^q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -r^{q+1} & r+r^q \end{pmatrix}$	$r \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$	$\frac{(q^2-q)}{2}$
$\mathcal{T}^{(20)}$	$\begin{pmatrix} 0 & 1 & 1 & 0 \\ -r^{q+1} & r+r^q & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -r^{q+1} & r+r^q \end{pmatrix}$	$r \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$	$\frac{(q^2-q)}{2}$
$\mathcal{T}^{(21)}$	$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \theta & 0 & 0 \\ 0 & 0 & \theta^q & 0 \\ 0 & 0 & 0 & \theta^{q^2} \end{pmatrix}$	$\alpha, \in \mathbb{F}_q^*, \theta \in \mathbb{F}_{q^3}^* \setminus \mathbb{F}_q^*,\ \theta^q, \theta^{q^2}$ are excluded whenever θ is included	$\frac{q(q-1)^2(q+1)}{3}$
$\mathcal{T}^{(22)}$	$\begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \kappa^q & 0 & 0 \\ 0 & 0 & \kappa^{q^2} & 0 \\ 0 & 0 & 0 & \kappa^{q^3} \end{pmatrix}$	$\kappa \in \mathbb{F}_{q^4}^* \setminus \mathbb{F}_{q^2}^*,\ \kappa^q, \kappa^{q^2}, \kappa^{q^3}$ are excluded whenever κ is included	$\frac{q^4-q^2}{4}$

Table 6.11: Sizes of classes and centralizers of $GL(4, q)$

g	$\lvert C_{GL(4,q)}(g)\rvert$	$\lvert C_g\rvert$
$\mathcal{T}^{(1)}$	$q^6(q-1)(q^2-1)(q^3-1)(q^4-1)$	1
$\mathcal{T}^{(2)}$	$q^6(q-1)^2(q^2-1)$	$(q^2+q+1)(q^4-1)$
$\mathcal{T}^{(3)}$	$q^5(q-1)(q^2-1)$	$q(q^3-1)(q^4-1)$
$\mathcal{T}^{(4)}$	$q^4(q-1)^2$	$q^2((q+1)(q^3-1)(q^4-1)$
$\mathcal{T}^{(5)}$	$q^3(q-1)$	$q^3(q^2-1)(q^3-1)(q^4-1)$
$\mathcal{T}^{(6)}$	$q^3(q-1)^2(q^2-1)(q^3-1)$	$q^3(q^3+q^2+q+1)$
$\mathcal{T}^{(7)}$	$q^3(q-1)^3$	$q^3(q+1)(q^2+q+1)(q^4-1)$
$\mathcal{T}^{(8)}$	$q^2(q-1)^2$	$q^4(q+1)(q^3-1)(q^4-1)$
$\mathcal{T}^{(9)}$	$q^2(q-1)^2(q^2-1)^2$	$q^4(q^2+1)(q^2+q+1)$
$\mathcal{T}^{(10)}$	$q^2(q-1)^2(q^2-1)$	$q^4(q+1)(q^2+q+1)(q-1)(q^2+1)$
$\mathcal{T}^{(11)}$	$q^2(q-1)^2$	$q^4(q+1)(q^3-1)(q^4-1)$
$\mathcal{T}^{(12)}$	$q^2(q-1)^4$	$q^4(q+1)(q^2+q+1)(q^3+q^2+q+1)$
$\mathcal{T}^{(13)}$	$q(q-1)^3$	$q^5(q+1)(q^2+q+1)(q^4-1)$
$\mathcal{T}^{(14)}$	$q(q-1)(q^2-1)^2$	$q^5(q^2+q+1)(q^2+1)$
$\mathcal{T}^{(15)}$	$q(q-1)(q^2-1)$	$q^5(q^3-1)(q^4-1)$
$\mathcal{T}^{(16)}$	$(q-1)^4$	$q^6(q+1)(q^2+q+1)(q^3+q^2+q+1)$
$\mathcal{T}^{(17)}$	$(q-1)^2(q^2-1)$	$q^6(q^2+q+1)(q^4-1)$
$\mathcal{T}^{(18)}$	$(q^2-1)^2$	$q^6(q-1)(q^3-1)(q^2+1)$
$\mathcal{T}^{(19)}$	$q^2(q^2-1)(q^4-1)$	$q^4(q-1)(q^3-1)$
$\mathcal{T}^{(20)}$	$q^2(q^2-1)$	$q^4(q-1)(q^3-1)(q^4-1)$
$\mathcal{T}^{(21)}$	$(q-1)(q^3-1)$	$q^6(q^2-1)(q^4-1)$
$\mathcal{T}^{(22)}$	(q^4-1)	$q^6(q-1)(q^2-1)(q^3-1)$

A programm to calculate the character table of $UT(2,3)$ using GAP [23]

```
gap> G:= GL(2,3);
GL(2,3)
gap> K:=Elements(G);;
gap> H:= [];
[  ]
gap> for  i in [1..Size(K)]do
> if K[i][2][1]= 0*Z(3) then
> Add(H, K[i]);
> fi;
> od;
gap> Size(H);
12
gap> for i in [1..Size(H)]do
> for j in [1..Size(H)]do
> if Order(Group(H[i], H[j])) = Size(H) then
> break;
> fi;
> od;
> od;
gap> M:= Group(H[i], H[j]);
Group([ [ [ Z(3), Z(3) ], [ 0*Z(3),
Z(3) ] ], [ [ Z(3)^0, 0*Z(3) ], [ 0*Z(3), Z(3) ] ] ])
gap>Order(M); 12
gap> T:= CharacterTable(M);
CharacterTable(Group([ [ [ Z(3), Z(3) ], [ 0*Z(3), Z(3) ] ], [ [ Z(3)^0, 0*Z(3)
], [ 0*Z(3), Z(3) ] ] ]) )
gap> Display(T); CT1
            2  2  2  1  2  2  1
            3  1  .  1  .  1  1
              1a 2a 3a 2b 2c 6a
        X.1     1  1  1  1  1  1
        X.2     1 -1  1 -1  1  1
        X.3     1  1  1 -1 -1 -1
        X.4     1 -1  1  1 -1 -1
        X.5     2  . -1  . -2  1
        X.6     2  . -1  .  2 -1
```

Table 6.12: Number of regular semisimple elements of $GL(n,q),\ n = 1, 2, \cdots, 5,\ q > n$

n	$\lambda \vdash n$	$F(\lambda)$	$\lvert c^\lambda \rvert$	No. of regular semisimple elements
1	$(1) \equiv 1^1$	$q-1$	1	$q-1$
2	$(1,1) \equiv 1^2$	$(q-1)(q-2)/2$	$q(q+1)$	$q^4 - 2q^3 + q$
	$(2) \equiv 2^1$	$q(q-1)/2$	$q(q-1)$	
3	$(1,1,1) \equiv 1^3$	$(q-1)(q-2)(q-3)/6$	$q^3(q+1)(q^2+q+1)$	$q^9 - 2q^8 + q^6 + 2q^5 - q^4 - q^3$
	$(2,1) \equiv 1^1 2^1$	$q(q-1)^2/2$	$q^3(q-1)(q^2+q+1)$	
	$(3) \equiv 3^1$	$(q^3-q)/3$	$q^3(q-1)^2(q+1)$	
4	$(1,1,1,1) \equiv 1^4$	$(q-1)(q-2)(q-3)(q-4)/24$	$q^6(q+1)^2(q^2+1)(q^2+q+1)$	$q^{16} - 2q^{15} + q^{13} + q^{12} - 2q^{10}$ $-q^9 - q^8 + 2q^7 + q^6$
	$(2,1,1) \equiv 1^2 2^1$	$q(q-1)^2(q-2)/4$	$q^6(q^2+q+1)(q^4-1)$	
	$(2,2) \equiv 2^2$	$q(q^2-1)(q-2)/8$	$q^6(q-1)^2(q^2+1)(q^2+q+1)$	
	$(3,1) \equiv 1^1 3^1$	$q(q-1)^2(q+1)/3$	$q^6(q^2-1)(q^4-1)$	
	$(4) \equiv 4^1$	$(q^4-q^2)/4$	$q^6(q-1)(q^2-1)(q^3-1)$	
5	$(1,1,1,1,1) \equiv 1^5$	$((q-1)(q-2)(q-3)$ $(q-4)(q-5)/120)$	$(q^{10}(q+1)^2(q^2+1)$ $(q^2+q+1)(q^4+q^3+q^2+q+1))$	$q^{25} - q^{24} + q^{22} + q^{19} + 2q^{16} + q^{15}$ $+q^{14} - q^{12} - 3q^{11} - q^{10}$
	$(2,1,1,1) \equiv 1^3 2^1$	$q(q-1)^2(q-2)(q-3)/12$	$q^{10}(q^4-1)(q^2+q+1)$ $(q^4+q^3+q^2+q+1)$	
	$(2,2,1) \equiv 1^1 2^2$	$q(q-1)^2(q-2)(q+1)/8$	$q^{10}(q^2+1)(q^3-1)(q^5-1)$	
	$(3,1,1) \equiv 1^2 3^1$	$q(q-1)^2(q+1)(q-2)/6$	$(q^{10}(q^2-1)^2(q^2+1)$ $(q^4+q^3+q^2+q+1)$	
	$(3,2) \equiv 2^1 3^1$	$q^2(q-1)^2(q+1)/6$	$(q^{10}(q-1)^3(q+1)(q^2+1)$ $(q^4+q^3+q^2+q+1)$	
	$(4,1) \equiv 1^1 4^1$	$(q^4-q^2)(q-1)/4$	$q^{10}(q^2-1)(q^3-1)(q^5-1)$	
	$(5) \equiv 5^1$	$(q^5-q)/5$	$q^{10}(q^2-1)^2(q^2+1)(q-1)(q^3-1)$	

Bibliography

[1] L. Aburto, R. Johnson and J. Pantoja, *The complex linear representations of $GL(2,k)$, k a finite field* , Proyecciones, **25** (2006), 307 - 329.

[2] J. Adams, *Character Tables for $GL(2)$, $SL(2)$, $PGL(2)$ and $PSL(2)$ over a finite field*, Lecture Notes, University of Maryland, 2002.

[3] J. L. Alperin and R. B. Bell, *Groups and Representations*, Springer-Verlag, New York, Graduate Texts in Mathematics, **162**, 1995.

[4] M. Aschbacher, *Finite Groups Theory*, Cambridge University press, Cambridge, Cambridge Studies in Advanced Mathematics, **10**, 1986.

[5] A. Baker, *Representations of Finite Groups*, University of Glasgow, 2005. Available at: http://www.maths.gla.ac.uk/ ajb/dvi-ps/groupreps.pdf

[6] G. Ya. Berkovich and E. M. Zhmud, *Characters of Finite Groups, Part 2* , American Mathematical Society, Translation of Mathematical Monographs, **181**, 1991.

[7] N. Biggs, *Finite Groups of Automorphisms*, London Mathematical Society, Cambridge University Press, Lecture Note Series. **6**, 1971.

[8] R. Brauer, *A Characterization of the Characters of Groups of Finite Order*, Annals of Mathematics, **57**, no. 2, (1953).

[9] C. W. Curtis and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders*, John Wiley and Sons, New York, A Wiley-Interscience Series of Texts, Monographs and Tracts, **2**, 1987.

[10] C. W. Curtis, *Pioneers of Representation Theory: Frobenius, Burnside, Schur, and Brauer*, American Mathematical Society, London Mathematical Society, 1999.

[11] D. Bump, *Lie Groups*, Springer-Verlag New York, Graduate Texts in Mathematics, **225**, 2004.

[12] P. J. Cameron, *Notes on Classical Groups*, Lecture Notes given at Queen Mary and Westfield College, London, January to March 2000.

[13] M. J. Collins, *Represenatation and Characters of Finite Groups*, Cambridge University Press, Cambridge Studies in Advanced Mathematics, **22**, 1990.

[14] M. R. Darafasheh, *Computing the irreducible characters of the group $GL_6(2)$*, American Mathematical Society, **46** (1986), 301 - 319.

[15] M. R. Darafasheh, *Orders of elements in the groups related to the general linear group*, Finite Fields and Their Applications, **11** (2005), 738 - 747.

[16] P. Deligne and G. Lusztig, *Representations of Reductive Groups over Finite Fields*, Ann. of Math, **103** (1976), 103 - 161.

[17] J. Dieudonné, *Les déterminants sur un corps non commutatif*, Bull. Soc. Math. France, **71** (1943), 27 - 45.

[18] E. S. Drobotenko, *Irreducible representations of the groups $GL(2, q)$ and $GL(3, q)$ over an arbitrary field of characteristic zero*, Springer New York, **21** (1969), 687 - 693.

[19] W. Feit, *Characters of Finite Groups*, W. A. Benjamin, New York, 1967.

[20] P. Flavell, *A note on Frobenius groups*, Journal of Algebra, **228** (2000), 367 - 376.

[21] P. Fleischmann, I. Janiszczak and R. Knörr, *The number of regular semisimple classes of special linear and unitary groups*, Linear Algebra and Its Applications, **274** (1998), 17-26.

[22] W. Fulton and J. Harris, *Representation Theory, A first Course*, Springer-Verlag New Yourk, Graduate Texts in Mathematics, **129**, 1991.

[23] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.10; 2007.* (http://www.gap-system.org)

[24] K. E. Gehles, *Ordinary Characters of Finite Speical Linear Groups*, MSc Dissertation, School of Mathematics and Statistics, University of St Andrews, 2002.

[25] D. M. Goldschmidt, *Group Characters, Symmetric Functions, and the Hecke Algebras* , American Mathematical Society, University Lecture Series, **4** 1993.

[26] D. Gorenstein, *Finite Groups*, Harper & Row New York, Evanston and London, Harper's Series in Modern Mathematics, 1968.

[27] J. A. Green, *The characters of the finite general linear groups*, American Mathematical Society, **80** (1956), 402 - 447.

[28] J. A. Green, *Discrete series characters for $GL(n, q)$*, Algebras and Representation Theory, **2** (1999) 61 - 82.

[29] L. C. Grove, *Groups and Characters*, John Wiley & Sons, New York, 1997.

[30] F. J. Grunewald, B. Hartly, T. O. Hawkes, I. G. Macdonald, D. S. Passman, D. J. S. Robinson, D. Segal, R. Strebel and J. G. Thompson, *Group Theory: essays for Philip Hall*, London Mathematical Society, 1984.

[31] L. A. Hageman, *Green's formula for $SL(2, k)$*, Communication in Algebra, **24** (1996), 3489 - 3499.

[32] R. Hill, *First Course in Coding Theory*, Oxford, England, Oxford University Press, 1986.

[33] J. E. Humphreys, *Representations of $SL(2, p)$*, Amer. Math. Monthly, **82** (1975), 21 - 39. Available on "http://www.jstor.org/view/00029890/di991598/99p0873y/0".

[34] J. E. Humphreys, *The Steinberg representation*, Bultin of the American Mathematical Society, **16** (1987), 247 - 263.

[35] H. E. Jordan, *Group-characters of various types of linear groups*, Amer. J. Math. **29** (1907), 387405.

[36] A. Iranmanesh, *Characters of the affine groups in low dimensions*, Southeast Asian Bultin of Mathematics, **25** (1997), 121 - 128.

[37] A. Iranmanesh, *Fischer matrices of the affine groups*, Southeast Asian Bultin of Mathematics, **21** (2001), 27 - 62.

[38] I. M. Isaacs, *Character Theory of Finite Groups*, Academic press, New York, San Fransisco, London, 1976.

[39] G. D. James, *Representations of the General Linear Groups*, London Mathematical Society, Cambridge University Press, Lecture Note Series. **94**, 1984.

[40] G. James and M. Liebeck, *Representations and Characters of Groups*, 2^{nd} edition, Cambridge Mathematical Textbooks, Cambridge University Press, 2001.

[41] A. W. Joshi, *Elements of Group Theory for Physicists*, 3^{rd} edition, Wiley Eastern Limited, New Delhi, India, 1982.

[42] T. Klein, *The Hall polynomial*, Journal of Algebra, **12** (1969), 61 - 78.

[43] P. J Lambert, *Characterizing groups by their character tables I*, Quart J. Math Oxford (2), **23** (1972), 427 - 433.

[44] P. J Lambert, *Characterizing groups by their character tables II*, Quart J. Math Oxford (2), **24** (1973), 223 - 240.

[45] P. J Lambert, *Projective special linear groups in characteristic 2*, Comm. Algebra (9), **4** (1976), 873 - 879.

[46] W. Ledermann, *Introduction to Group Characters*, 2^{nd} edition, Cambridge University Press, Cambridge, 1987.

[47] G. I. Lehrer, *The characters of the finite special linear groups* , Journal of Algebra, **26** (1972), 564 - 583.

[48] R. Lidle and H. Niederreiter, *Introduction to Finite Fields and Their Applications*, Cambridge University Press, Cambridge, 1986.

[49] D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, Oxford, 1940.

[50] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 1^{st} edition, Clarendon Press, Oxford, Oxford Mathematical Monographs, 1979.

[51] K. Mcgerty, *The General Linear Group*, Department of Mathematics, University of Chicago, Lecture Notes, 2006.

[52] J. Moori, *On the Groups G^+ and $\overline{G}$ of the form $2^{10}{:}M_{22}$ and $2^{10}{:}\overline{M}_{22}$*, PhD Thesis, University of Birmingham, 1975.

[53] J. Moori, *On certain groups associated with the smallest Fischer group*, J. London Math. Soc., **2** (1981), 61 - 67.

[54] J. Moori, *Finite Groups and Representation Theory*, Lecture Notes, University of KawZulu-Natal, Pietermaritzburg, 2008.

[55] Z. E. Mpono, *Fischer Clifford Theory and Characters Tables of Group Extensions*, PhD Thesis, University of Natal, Pietermaritzburg, 1998.

[56] A. O. Morris, *The characters of the group $GL(n,q)$*, Math . Zeitschrift, **81** (1963), 112 - 123.

[57] A. S. Muktibodh, *Characterization of Frobenius groups of special type*, Math. J. Okayama Univ., **48** (2006), 73 - 76.

[58] P. M. Neumann, G. A. Stoy and E. C. Thompson, *Groups and Geometry*, Oxford University Press, Oxford, 1994.

[59] T. Oyama, *On the groups with the same table of characters as the alternating groups*, Osaka, J. Math., **1** (1964), 91 - 101.

[60] A. Prasad, *Representations of $GL_2(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_q)$, and some remarks about $GL_n(\mathbb{F}_q)$*, Lecture Notes, Advanced Instructional School on Representation Theory and Related Topics held at the Bhaskaracharya Pratishthana and the University of Pune, July 2007.

[61] E. N. Reyes, *The representations of $GL(2, K)$ where K is a finite field*, Proceedings of the Louisiana-Mississippi Section of the Mathematical Association of America, 2001. Available on "http://www.mc.edu/campus/users/travis/maa/proceedings/spring2001/reyes.pdf".

[62] J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag New York, Heidelberg, Berlin, Graduate Texts in Mathematics, **80**, 1982.

[63] B. G. Rodrigues, *On The Theory and Examples of Group Extensions* , MSc Thesis, University of Natal, Pietermaritzburg, 1999.

[64] J. S. Rose, *A Course on Group Theory*, Cambridge University Press, Cambridge, 1978.

[65] J. J. Rotman, *An Introduction to the Theory of Groups*, 4^{th} edition, Springer-Verlag New York, Graduate Texts in Mathematics, **148**, 1995.

[66] B. E. Sagan, *The Symmetric Group Representataions, Combintorial Algorithms, and Symmetric Functions*, *2nd* edition, Springer-Verlag New York, Graduate Texts in Mathematics, **203**, 2001.

[67] I. Schur, *Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen* , J. Reine Angew. Math., **132** (1907), 85 - 137.

[68] J. P. Serre, *Linear Representataions of Finite Groups*, Springer-Verlag New York, Graduate Texts in Mathematics, **42**, 1977.

[69] A. J. Silberger and E. W. Zink, *The characters of the generalized Steinberg representations of finite general linear groups on the regular elliptic set*, Transactions of the American Mathematical Society, **352**, (2000), 3339 - 3356.

[70] A. Singh, *Structure of GL_n and SL_n*, Lecture Notes, Department of Mathematics, The Institute of Mathematical Sciences C.I.T. Campus, Taramani, INDIA, 2007. Available on "http://anupamk18.googlepages.com/lecture1.pdf".

[71] T. A. Springer, *Characters of Special Groups*, Springer, Lecture Notes, **131** (1970), 121 - 156.

[72] R. Steinberg, *The representations of $GL(3, q)$, $GL(4, q)$, $PGL(3, q)$ and $PGL(4, q)$*, American J. Math., **3** (1951), 225 - 235.

[73] R. Steinberg, *A geometric approach to the representations of the full linear group over a Galois field*, Trans. Amer. Math. Soc., **71** (1951), 274 - 282.

[74] R. C. Thompson, *Commutators in the special and general linear groups*, American Mathematical Society, **7** (1960), 565 - 582.

[75] Z. X. Wan, *Lectures on Finite Fields and Galois Rings*, World Scientific Pub Co Inc, 2003.

[76] N. S. Whitely, *Fischer Matrices and Character Tables of Group Extensions* , MSc Thesis, University of Natal, Pietermaritzburg, 1993.

[77] A. V. Zelevinsky, *Representations of Finite Classical Groups*, Springer-Verlag Berlin Heidelberg, Lecture Notes in Mathematics, **869**, 1981.