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Solution Generating Algorithms in General Relativity

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As the candidate's supervisor, I have approved this dissertation for submission

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Dedicated to the living memory of my son

Rowan Krupanandan

Declaration

The research done in this dissertation is original work and has not been previously submitted to any other institution. Where reference to the work of other researchers was made, it has been duly acknowledged.

This thesis was completed under the supervision of Dr S. Hansraj.

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Abstract

We conduct a comprehensive investigative review of solution generating algorithms for the Einstein field equations governing the gravitational behaviour of an isolated neutral static spherical distribution of perfect fluid matter. Traditionally, the master field equation generated from the condition of pressure isotropy has been interpreted as a second order ordinary differential equation. However, since the pioneering work of Wyman (1949) it was observed that more success can be enjoyed by regarding the equation as a first order linear differential equation. There was a resurgence of the ideas of Wyman in 2000 and various researchers have been able to generate complete solutions to the field equations up to certain integrations. These have been accomplished by working in Schwarzschild (curvature) coordinates, isotropic coordinates, area coordinates and a coordinate system written in terms of the redshift parameter. We have utilised Durgapal–Banerjee (1983) coordinates and produced a new algorithm. The algorithm is used to generate new classes of perfect fluid solutions as well as to regain familiar particular solutions reported in the literature. We find that our solution is well behaved according to elementary physical requirements. The pressure vanishes for a certain radius and this establishes the boundary of the distribution. Additionally the pressure and energy density are both positive inside the radius. The energy conditions are shown to be satisfied and it is particularly pleasing to have the causality criterion satisfied to ensure that the speed of light is not exceeded by the speed of sound. We also report some new solutions using the algorithms proposed by Lake (2006).

Contents

Chapter 1

Introduction

Gravitation is a natural phenomenon by which physical bodies attract with a force proportional to their mass. While it is the most familiar of the the four fundamental interactions or forces of nature (electromagnetism, the nuclear strong and weak forces being the other three) it is the least understood. Consequently there have been ongoing attempts at devising theories that will explain the effects of gravity fully.

It should be noted that a variety of approaches have been followed in studying the effects of the gravitational field. The following are some theories that are in use today:

- **Einstein's general theory of relativity** is a theory where the effects of gravitation are ascribed to the curvature of spacetime. Einstein proposed that spacetime is curved by matter, and that free-falling objects are moving along locally straight paths in curved spacetime. Modern physics makes extensive use of Einstein's theory and it is widely held to be the most successful theory of gravitation till now.

- **Brans-Dicke theory of gravitation** (1961) is an alternative theoretical framework to explain gravitation. It is a well-known contrast of Einstein's more popular theory of general relativity. It is an example of a scalar-tensor theory, a gravitational theory in which the gravitational interaction is dependent on the tensor field of general relativity and is supplemented by a scalar field. Both Brans-Dicke theory and general relativity are examples of a class of relativistic classical field theories of gravitation, called metric theories.

- **F(R) gravity** is a type of rearranged gravity theory first proposed in 1959 by Buchdahl (1959) as a generalisation of Einstein's general relativity. Although it is an active field of research, there are known problems with the theory. It has the potential, in principle, to explain the accelerated expansion of the universe without adding unknown forms of dark energy or dark matter.

- **Einstein–Gauss-Bonnet (Lovelock) gravity** (1971) is a generalization of Einstein's theory of general relativity introduced in 1971. It is the most general metric theory of gravity yielding conserved second order equations of motion in arbitrary number of spacetime dimensions D. In this sense, Lovelock's theory is the natural generalization of Einstein's general relativity to higher dimensions. In dimension three and four (D = 3 and 4), Lovelock's theory coincides with Einstein's theory, but in higher dimensions the theories diverge.

Einstein's general relativity depicts the universe as a geometric system of three spatial and one time dimensions. The presence of mass, energy, and momentum (collectively quantified as mass-energy density or stress-energy) result in a bending of this space-time coordinate system. The geometric and dynamical quantities are related tonsorially via the Einstein field equations. These are, in the worst case scenario, a highly coupled system of ten partial differential equations. The governing field equations are in general nonlinear and this accounts for the extreme difficulty that exists in finding exact solutions to the system of equations. By exact solutions we mean solutions of the field equations that are obtained and no recourse to numerical methods are necessary. Why are exact solutions so important? The answer is that the solutions allow us to extract important information about the evolution of celestial phenomena in time. The literature contains a vast array of solutions of the Einstein field equations for a large variety of matter configurations (Kramer *et al* 2003). The difficulty though, with most solutions is that they fail to satisfy even very elementary requirements for physical plausibility. Some well-known solutions in general relativity applicable in astrophysics include:

- **The Schwarzschild exterior solution (1916a)** was amongst the first useful exact solutions of the Einstein field equations. This solution describes the gravitational field outside a spherical, uncharged, non-rotating massive object. It is also a good approximation to the gravitational field of a slowly rotating body like the Earth or Sun. According to Birkhoff's theorem (1923), the Schwarzschild exterior solution is the most general spherically symmetric, solution of the vacuum Einstein field equations.

- **The Schwarzschild interior solution (1916b)** describes the interior gravitational field of static spheres. The Schwarzschild interior and exterior solutions match smoothly across the boundary of the star. The interior solution for a neutral sphere is not unique and hence we are investigating this in our present work. It turns out that the system of field equations governing static spheres is a system of three equations in four unknowns. Therefore one of the unknowns has to be specified at the outset and the other three have to be obtained via integration. This accounts for the non-uniqueness of the interior metric. The Schwarzschild interior solution was obtained by assuming that the sphere is of constant energy density.

- **The Reissner–Nordstrom solution** (1918) is a static solution to the Einstein-Maxwell field equations, which corresponds to the exterior gravitational field of a charged, non-rotating, spherically symmetric body. The Reissner–Nordstrom model reduces to the Schwarzschild exterior when the charge vanishes. Numerous interior solutions have been found that match the exterior Reissner–Nordstrom solution across its pressure free hypersurface. The reason for a large number of solutions is that the presence of charge introduces an extra freedom of choice in the model. Now there are six equations in four unknowns so any two may be selected at the outset. Again, despite the rich variety of solutions only a few are worthwhile as physically reasonable models.

- **The Vaidya solution** (1951) represents the exterior gravitational field of a radiating neutral sphere. All previous solutions assumed the exterior of the star to be empty. Vaidya generalized this case to incorporate the radiation from the star, and the resulting solution was the famous Vaidya shining star metric. Interior solutions have been found, however, a major stumbling block was the Israel (1966)– Darmois (1927) junction conditions - these were only fully understood by Santos (1985) who obtained the conditions to be satisfied so that interior solutions could be matched to the exterior Vaidya spacetime.

- **The Kerr solution** (1963) describes the exterior gravitational field of a rotating, axially symmetric gravitating body. The Kerr solution reduces to the Schwarzschild exterior solution in the limit of vanishing angular momentum. This solution was an important milestone in relativity history as a large number of celestial bodies are rotating. Unfortunately, finding an interior solution that matches smoothly to the Kerr solution is still an open problem. It is widely regarded as one of the most important problems in classical general relativity. (Wiltshire *et al* (2009)).

Most attempts at solving the field equations for various matter configurations have been motivated by mathematical considerations. That is, *ad hoc* forms for some of the variables have been chosen which allow for a complete solution of the entire system of field equations. Often the result is an unphysical solution. Another approach is to impose some physical constraints at the outset by, for example, assuming a functional dependence of the pressure or the energy density. The caveat in this approach is that the resulting system becomes very difficult to solve. If the pressure is a function of the energy density this is referred to as an equation of state. If the relationship is linear it is called a barotropic equation of state and if the energy density contains a power then we call this a polytropic equation of state. Exact solutions in the latter case have been extremely rare. In some cases, such as for spherically symmetric fluids that are static, the entire system may be solved in general up to integrations. In other words, solutions to the system may be crafted in an algorithmic way.

We briefly trace the history of solution generating algorithms for static spherically symmetric perfect fluid distributions of matter. Observe, that the chronology here follows the actual reporting of the solutions on web platforms and does not necessarily follow the order in which the articles were eventually published.

- **1949** Wyman solves the field equations and proposes the earliest solution generating algorithm on record.
- **2000** Fodor publishes a method involving one generating function and a technique requiring no integrations. Only differentiation and algebraic operations are required.
- **2002** Rahman and Visser obtain an algorithm using isotropic coordinates. The algorithm utilises one differentiation and one integration. The caveat in this method is the appearance of a square root which is severely restrictive.

- **2003** Lake extends the methods first proposed by Wyman (1949) and obtains algorithms for curvature and isotropic coordinates. In curvature coordinates, the difficulty lies in the fact that two integrations are called for whereas in isotropic coordinates one integration is needed, however, the obstructive square root appears in the integral. The algorithm for isotropic coordinates closely follows the approach of Rahman and Visser (2002) and the transformations linking the two algorithms is provided.

- **2004** Martin and Visser produce another algorithm using so–called Schwarzschild coordinates

- **2005** Boonserm, Visser and Weinfurtner propose the simplest of the algorithms known.

- **2011** Hansraj uses Durgapal–Banerjee coordinates to obtain a new algorithm involving one integration.

Our purpose in this thesis is to investigate the efficiency of some solution generating algorithms in an effort to construct new solutions to the Einstein field equations. Our work is arranged as follows:

- In chapter 2 we give a broad overview of the aspects of Riemannian geometry relevant to our work. We examine the spherically symmetric spacetime in its standard form and then consider it using curvature and isotropic coordinates. We display the Einstein field equations in each of these three coordinate systems.

- In Chapter 3 we review some well known solution generating algorithms. We comment on their advantages and drawbacks. In all cases, we need to specify a certain source function and then invariably perform some integrations to determine the remaining geometric and physical quantities.

- In chapter 4 we endeavour to establish new solutions using the algorithms. We do have success and are able to report new classes of solutions for Durgapal–Banerjee, curvature and isotropic coordinates. However, there still remains a wide variety of source functions that could be used to unlock the entire system of field equations. Finally, we briefly comment on the physical behaviour of some solutions.

Chapter 2

Mathematical Preliminaries

2.1 Introduction

In this chapter we collect some of the main aspects of differential geometry and surface theory which are relevant to our study. We commence by establishing the line element or the metric tensor and then we compute the required Christoffel symbols, Riemann tensor, Ricci tensor, Ricci scalar and Einstein tensor. We also note the Weyl conformal tensor which is useful for checking whether solutions are conformally flat or not. We perform these calculations for the spherically symmetric line element in standard form and then for curvature and isotropic coordinates. These calculations then allow us to write the Einstein field equations in these three coordinate formulations. While the choice of coordinates is irrelevant to the eventual values of the geometric and physical quantities, the presentation of the field equations allow for a variety of approaches to obtain new exact solutions.

2.2 Differential Geometry

We take spacetime M to be a 4–dimensional differentiable manifold endowed with a symmetric, nonsingular metric field $\mathbf{g}$ of signature $(-+++)$. As the metric tensor field is indefinite the manifold is pseudo–Riemannian. Points in M are labeled by the real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$ where x^0 is timelike and x^1, x^2, x^3 are spacelike. The line element is given by

$$ds^2 = g_{ab} dx^a dx^b \tag{2.1}$$

which defines the invariant distance between neighbouring points of a curve in M. The fundamental theorem of Riemannian geometry guarantees the existence of a unique symmetric connection that preserves inner products under parallel transport. This is called the metric connection $\mathbf{\Gamma}$ or the Christoffel symbol of the second kind. The coefficients of the metric connection $\mathbf{\Gamma}$ are given by

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(g_{cd,b} + g_{db,c} - g_{bc,d} \right) \tag{2.2}$$

where commas denote partial differentiation.

The quantity

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec} \Gamma^e{}_{bd} - \Gamma^a{}_{ed} \Gamma^e{}_{bc} \tag{2.3}$$

is a $(1,3)$ tensor field and is called the Riemann tensor or the curvature tensor. Upon contraction of the Riemann tensor (2.3) we obtain

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^d{}_{ab,d} - \Gamma^d{}_{ad,b} + \Gamma^e{}_{ab} \Gamma^d{}_{ed} - \Gamma^e{}_{ad} \Gamma^d{}_{eb} \qquad (2.4) \end{aligned}$$

where R_{ab} is the Ricci tensor. The Riemann tensor indicates the amount of deviation from flatness and the vanishing of this tensor suggests a flat spacetime. On contracting the Ricci tensor (2.4) we obtain

$$\begin{aligned} R &= g^{ab}R_{ab} \\ &= R^a{}_a \end{aligned} \tag{2.5}$$

where R is the Ricci scalar. The Einstein tensor $\mathbf{G}$ is constructed in terms of the Ricci tensor (2.4) and the Ricci scalar (2.5) as follows:

$$G^{ab} = R^{ab} - \frac{1}{2}Rg^{ab} \tag{2.6}$$

The Einstein tensor has zero divergence:

$$G^{ab}{}_{;b} = 0 \tag{2.7}$$

a property referred to in the literature as the contracted Bianchi identity. This identity is useful when studying the conservation of matter which arises as a consequence of the field equations.

An arbitrary rank two tensor can be decomposed into its symmetric and anti–symmetric parts. Similarly the Riemann tensor (2.3) decomposes into the Weyl tensor (or conformal curvature tensor) and parts which involve the Ricci tensor and the curvature scalar. This decomposition is given by

$$\begin{aligned} R_{abcd} &= C_{abcd} - \frac{1}{6}R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \\ &+ \frac{1}{2}\left(g_{ac}R_{bd} - g_{bc}R_{ad} + g_{bd}R_{ac} - g_{ad}R_{bc}\right) \end{aligned} \tag{2.8}$$

where $\mathbf{C}$ is the Weyl tensor. The Weyl tensor is trace-free,

$$C^{ab}{}_{ad} = 0$$

and inherits all the symmetry properties of the curvature tensor (2.3). The vanishing of the Weyl tensor is an indication of conformal flatness. This means that the spacetime can be cast into a trivial scaling of the Minkowski spacetime.

The distribution of matter is specified by the energy–momentum tensor $\mathbf{T}$ which is given by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} \tag{2.9}$$

for neutral matter. In the above μ is the energy density, p is the isotropic pressure, q_a is the heat flow vector and π_{ab} represents the stress tensor. These quantities are measured relative to a fluid four–velocity $\mathbf{u}$ ($u^a u_a = -1$). The heat flow vector and stress tensor satisfy the conditions

$$q^a u_a = 0$$

$$\pi^{ab} u_b = 0$$

In the simpler case of a perfect fluid, which is the case for most cosmological models, the energy–momentum tensor (2.9) has the form

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} \tag{2.10}$$

The energy–momentum tensor (2.9) is coupled to the Einstein tensor (2.6) via the Einstein field equations

$$G_{ab} = T_{ab} \tag{2.11}$$

We utilise geometric units where the speed of light and the coupling constant are taken to be unity. The field equations (2.11) relate the gravitational field to the matter content. This is a system of coupled partial differential equations which are highly nonlinear and consequently difficult to integrate in general. Here we have provided only a brief outline of the results necessary for later work. For a comprehensive treatment of differential geometry applicable to general relativity the reader is referred to de Felice and Clarke (1990), Hawking and Ellis (1973) and Misner *et al* (1973).

2.3 Static Spherically Symmetric Spacetimes

The most general line element for static spherically symmetric spacetimes, in coordinates $(x^a) = (t, r, \theta, \phi)$, is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.12}$$

where the gravitational potentials ν and λ are functions only of the spacetime coordinate r. These coordinates are called standard or canonical coordinates. It is reasonable to assume that the interior and exterior gravitational fields of an isolated charged star are described by (2.12) in the absence of other matter. Whereas (2.12) is perhaps the metric form that has been most used, it must be noted that the spherically symmetric line element can be cast into a variety of forms. We will later resort to Durgapal–Banerjee (1983) coordinates, curvature (or Schwarzschild coordinates) as well as isotropic coordinates for our investigations.

For the metric (2.12) we may now evaluate the Ricci tensor (2.5), utilising the above connection coefficients, to yield the following non–zero components:

$$R_{tt} = e^{2(\nu-\lambda)}\left[\nu'' - \nu'\lambda' + \lambda'^2 + \frac{2\nu'}{r}\right] \tag{2.13}$$

$$R_{rr} = -\left[\nu'' + \nu'^2 - \frac{2\lambda'}{r} - \nu'\lambda'\right] \tag{2.14}$$

$$R_{\theta\theta} = 1 - \left[1 + r\nu' - r\lambda\right] e^{-2\lambda} \tag{2.15}$$

$$R_{\phi\phi} = \sin^2\theta R_{22} \tag{2.16}$$

Then the Ricci tensor components (2.13) and the definition (2.5) yield the following form for the Ricci scalar:

$$R = 2\left[\frac{1}{r^2} - \left(\nu'' - \nu'\lambda' + \lambda'^2 + \frac{1}{r^2} + \frac{2\nu'}{r} - \frac{2\lambda'}{r}\right) e^{-2\lambda}\right] \tag{2.17}$$

for the spherically symmetric spacetime (4.18). The Ricci tensor components (2.13) and the Ricci scalar (2.17) generate the corresponding non–vanishing components of the Einstein tensor (2.6). These are given by

$$G_{tt} = \frac{e^{2\nu}}{r^2}\left[r\left(1 - e^{-2\lambda}\right)\right]' \tag{2.18}$$

$$G_{rr} = -\frac{e^{2\nu}}{r^2}\left(1 - e^{-2\lambda}\right) + \frac{2\nu'}{r} \tag{2.19}$$

$$G_{\theta\theta} = \frac{r^2}{e^{2\lambda}}\left[\frac{\nu'}{r} - \frac{\lambda'}{r} + \nu'' - \nu'\lambda' + \lambda'^2\right] \tag{2.20}$$

$$G_{\phi\phi} = \sin^2\theta\, G_{22} \tag{2.21}$$

for the line element (2.12).

2.4 Curvature Coordinates

The standard metric of a spherically symmetric spacetime (2.12) may also be written in curvature coordinates and is given by

$$g_{ab} = \begin{pmatrix} -e^{2\Phi(r)} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2m(r)}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}. \tag{2.22}$$

where the metric potentials are given in terms of the functions $m(r)$ and $\Phi(r)$. This form is also referred to as Schwarzschild coordinates as one of the metric potentials is expressed in the form of the Schwarzschild exterior solution. It must be remembered that these alternative formulations of the metric tensor through coordinate redefinitions leave the Einstein field equations invariant - a property of tensor transformation laws. The purpose in investigating a variety of forms is that the appearance of the field equations in a different coordinate system could offer simplifications that could lead to the discovery of new exact solutions. We therefore compute the geometric quantities necessary for setting up the field equations. The Christoffel symbols (2.2) are given by

$$\Gamma^r{}_{tt} = -\frac{(r-2m)e^{2\Phi}\Phi'}{r} \qquad \Gamma^t{}_{rt} = \Phi$$

$$\Gamma^t{}_{tr} = -\Phi \qquad \Gamma^r{}_{rr} = \frac{m'r - m}{r(r-2m)}$$

$$\Gamma^\theta{}_{\theta r} = -\frac{1}{r} \qquad \Gamma^\phi{}_{\phi r} = -\frac{1}{r}$$

$$\Gamma^\theta{}_{r\theta} = \frac{1}{r} \qquad \Gamma^r{}_{\theta\theta} = r - 2m$$

$$\Gamma^\phi{}_{\phi\theta} = -\cot\theta \qquad \Gamma^\phi{}_{r\phi} = \frac{1}{r}$$

$$\Gamma^\phi{}_{\theta\phi} = \cot\theta \qquad \Gamma^r{}_{\phi\phi} = (r-2m)\sin^2\theta$$

$$\Gamma^\theta{}_{\phi\phi} = \sin\theta \; cos\theta$$

The Ricci tensor components are

$$R_{tt} = \frac{e^{2\Phi}\left((2r - rm' - 3m)\Phi' + (r(r-2m)((\Phi^2)' + \Phi'')\right)}{r^2} \tag{2.23}$$

$$R_{rr} = -\frac{r^2(r-2m)(\Phi'' - \Phi'^2) - (r\Phi + 2)(rm' - m)}{r^2(r-2m)} \tag{2.24}$$

$$R_{\theta\theta} = r(r-2m)\Phi' - m'r - m \tag{2.25}$$

$$R_{\phi\phi} = -\frac{1}{r}\sin^2\theta(\Phi' r^2 - 2\Phi' rm - m'r - m) \tag{2.26}$$

The Ricci Scalar is given by

$$R = -\frac{2}{r^2}\left((2r - rm' - 3m - 2m') + r(r-2m)\left((\Phi')^2 + \Phi''\right)\right) \tag{2.27}$$

The non-vanishing components of the Einstein tensor are given by

$$G_{tt} = \frac{2e^{2\Phi}m'}{r^2} \tag{2.28}$$

$$G_{rr} = \frac{2\left(r(r-2m)\Phi' - m\right)}{r^2(r-2m)} \tag{2.29}$$

$$G_{\theta\theta} = \frac{m}{r} - m' + (r - m + rm')\Phi' + r(r-2m)\left((\Phi')^2 + \Phi''\right) \tag{2.30}$$

$$G_{\phi\phi} = \sin^2\theta\, G_{\theta\theta} \tag{2.31}$$

The components of the Weyl conformal tensor are

$$C^t{}_{rrt} = 2C^\theta_{rr\theta} = 2C^\phi_{rr\phi} = -\frac{g(r)}{3r^2(r-2m)} \tag{2.32}$$

$$2C^t{}_{t\theta\theta} = 2C^r_{r\theta\theta} = C^\phi_{\phi\theta\theta} = \frac{g(r)}{3r} \tag{2.33}$$

$$-2C^t{}_{t\phi\phi} = -2C^r_{r\phi\phi} = C^\theta_{\theta\phi\phi} = \frac{\sin^2\theta g(r)}{3r} \tag{2.34}$$

$$C^r{}_{rtt} = -2C^\theta_{\theta tt} = -2C^\phi_{\phi tt} = -\frac{e^{2\Phi}g(r)}{3r^3} \tag{2.35}$$

where we have put

$$g(r) = rm' - 3m + r(3m - r - rm') + r^2(r-2m)\left((\Phi')^2 + \Phi''\right). \tag{2.36}$$

2.5 Isotropic Coordinates

The metric (2.12) may be transformed into a variety of equivalent forms by redefining the coordinates. Of course, the field equations also remain equivalent, however, utilising variations in the form of the line element result in ostensibly different (but equivalent) manifestations of the field equations. A popular alternative version of (2.12) involves so called isotropic coordinates. The metric tensor for static spherically symmetric spacetimes, using isotropic coordinates, is given by

$$g_{ab} = \begin{pmatrix} -e^{2\Phi(r)-B(r)} & 0 & 0 & 0 \\ 0 & e^{2B(r)} & 0 & 0 \\ 0 & 0 & r^2e^{2B(r)} & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta e^{2B(r)} \end{pmatrix}. \tag{2.37}$$

where B and Φ are yet to be determined functions of the spacetime coordinate r. This form has been utilised by Lake (2006) to obtain a solution generating algorithm.

We use this algorithm to find new exact solutions for static fluid spheres. For the line element (2.37) the non–zero coefficients of the metric connection (2.2) are given by

$$\Gamma^r{}_{tt} = \frac{e^{2\nu}\,(\Psi' - B')}{e^{4B}} \qquad \Gamma^t{}_{rt} = \Psi' - B'$$

$$\Gamma^t{}_{tr} = 1(\Psi' - B') \qquad \Gamma^r{}_{rr} = B'$$

$$\Gamma^\theta{}_{\theta r} = -\frac{rB' + 1}{r} \qquad \Gamma^\phi{}_{\phi r} = -\frac{rB' + 1}{r}$$

$$\Gamma^\theta{}_{r\theta} = \frac{rB' + 1}{r} \qquad \Gamma^r{}_{\theta\theta} = r(rB' + 1)$$

$$\Gamma^\phi{}_{\phi\theta} = -\tan\theta \qquad \Gamma^r{}_{\phi\phi} = r\sin^2\theta(rB' + 1)$$

$$\Gamma^\theta{}_{\phi\phi} = \sin\theta\cos\theta$$

where primes denote differentiation with respect to r.

The components of the Ricci tensor R_{ab} (2.4) have the form

$$R_{tt} = re^{-4B}(e^{2\Psi}(-rB'(\Psi) + (\Psi')^2 r + r(\Psi'' - rB'' + 2\Psi' - 2B')) \qquad (2.38)$$

$$R_{rr} = -\frac{1}{r}(r\Psi'' + rB'' + (\Psi')^2 r - 3rB'\Psi' + 2r(B')^2 + 2B') \qquad (2.39)$$

$$R_{\theta\theta} = -r^2\Psi' B' - \Psi' r - 2rB' - r^2 B'' \qquad (2.40)$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \tag{2.41}$$

The Ricci Scalar (2.5) is evaluated as

$$R = -\frac{1}{re^{2B}}\left(-2rB\Psi' + (\Psi')^2 r + r\Psi'' + rB'' + 2\Psi' + 2B' + r(B')^2\right) \tag{2.42}$$

The Einstein tensor (2.6) is given by

$$G_{tt} = -\frac{e^{2\Psi}\left(2rB'' + 4B' + r(B')^2\right)}{re^{4B}} \tag{2.43}$$

$$G_{rr} = -\frac{-2rB'\Psi' + r(B')^2 - 2\Psi}{r} \tag{2.44}$$

$$G_{\theta\theta} = -2r^2B'\Psi' + r\Psi' + r^2(\Psi')^2 + r^2\Psi'' + r^2(B')^2 \tag{2.45}$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta} \tag{2.46}$$

The Weyl conformal tensor (2.8) has the form

$$C^t_{trr} = 2C^\theta_{rr\theta} = 2C^\phi_{rr\phi} = -\frac{f(r)}{3r} \tag{2.47}$$

$$2C^t_{t\theta\theta} = 2C^r_{r\theta\theta} = C^\phi_{\theta\theta\phi} = \frac{rf(r)}{3} \tag{2.48}$$

$$2C^t_{t\phi\phi} = 2C^r_{r\phi\phi} = -C^\theta_{\theta\phi\phi} = \frac{r\sin^2\theta f(r)}{3} \tag{2.49}$$

$$-2C^r_{rtt} = C^\theta_{tt\theta} = C^\phi_{tt\phi} = \frac{e^{2(\Phi-2B)}f(r)}{6r} \tag{2.50}$$

where we have put

$$f(r) = r(\Psi')^2 - 4rB'\Psi' + r\Psi'' - 2rB'' + 4r(B')^2 + 2B' - \Psi'. \quad (2.51)$$

It should be noted that the Weyl tensor is non-zero in general. The vanishing of the Weyl tensor corresponds to conformally flat sapcetimes. We will be interested in non-conformally flat solutions in our study as all conformally flat solutions of the Einstein field equations have been found. They are either generalisations of the Schwarzschild interior solution, shown to be conformally flat by Buchdahl (1971) in the case of no expansion or Stephani and Krasinski (1983) stars if the solutions are expanding. We will utilise these geometric components to construct the Einstein field equations for static fluid spheres in chapter 4.

2.6 Einstein Field Equations

We are now in a position to generate the Einstein field equations for the spherically symmetric spacetime (2.12).
Using (2.12) it is easy to verify that (2.7) is identically satisfied. The Einstein equations (2.11) may be expressed as the system

$$\left[r(1 - e^{-2\lambda})\right]' = \rho r^2 \quad (2.52)$$

$$-\left(1 - e^{-2\lambda}\right) + 2\nu' r e^{-2\lambda} = p r^2 \quad (2.53)$$

$$r e^{-2\lambda}\left[\nu' - \lambda' + r\nu'' - r\nu'\lambda' + r(\nu')^2\right] = p r^2 \quad (2.54)$$

for the static spherically symmetric spacetime (2.12).

The conservation laws ${T^{ab}}_{;b} = 0$ reduce to the equation

$$p' + (\rho + p)\nu' = 0 \tag{2.55}$$

which can be used in the place of one of the field equations in the system (2.52) to (2.54). The exterior gravitational field for a static, spherically symmetric neutral distribution is governed by the Schwarzschild (1916a) solution. The Schwarzschild exterior line element has the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.56}$$

where M is associated with the mass of the sphere. Once a solution to the Einstein field equations have been discovered, it needs to be checked that the solutions match smoothly with the exterior Schwarzschild solution. That is, suitable constants must exist such that the metric potentials are continuous across the pressure-free boundary interface. This condition is called the Israel–Darmois junction condition. Note that the vanishing of the pressure on the boundary of a static sphere is not true for all configurations of matter. For example, radiating spheres have more complicated junction conditions and these were discovered by Santos (1985) and Herrera *et al* (1985).

2.6.1 Durgapal–Bannerjee Coordinates

We utilise the following transformation, which has been used by Durgapal and Bannerji (1983), Durgapal and Fuloria (1985) and Finch and Skea (1989), to generate new solutions in the case of neutral matter. A new coordinate x and two metric functions $y(x)$ and $Z(x)$ are defined as follows

$$x = Cr^2$$

$$\begin{aligned} Z(x) &= e^{-2\lambda(r)} \\ A^2y^2(x) &= e^{2\nu(r)} \end{aligned}$$

where A and C are constants. For this transformation the Einstein field equations (2.11) assume the form

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} \tag{2.57}$$

$$\frac{Z-1}{x} + \frac{4Z\dot{y}}{y} = \frac{p}{C} \tag{2.58}$$

$$4x^2Z\ddot{y} + 2x^2\dot{Z}\dot{y} + \left(\dot{Z}x - Z + 1\right)y = 0 \tag{2.59}$$

where dots represent differentiation with respect to x. We shall use this form of the field equations to generate new solutions in chapter 4.

The field equation (2.59) may be viewed as the master equation for this system. Once a form for $Z(x)$ is chosen, we may proceed with the possible integration of the second order linear differential equation in $y(x)$. A large number of exact solutions have been discovered in this manner. For example see Thirukannesh and Maharaj (2006), Finch and Skea (1985), Maharaj and Mkhwanazi (1996). The last named authors actually regained the Schwarzschild interior solution and demonstrated the equivalence of their solution with the Schwarzschild interior solution.

It is strange that no one was able to recognise (2.59) as a linear first order differential equation in $Z(x)$. That is, once a form for $y(x)$ is chosen it may be possible to integrate the linear first order differential equation to reveal the function $Z(x)$.

This has not been exploited before and we demonstrate in chapter 4 how the solution may be obtained explicitly. An algorithm is devised to find $Z(x)$ once $y(x)$ is chosen. Of course, the algorithm is only successful if the integration of (2.57) may be accomplished explicitly. Sadly only a small number of functions have been found that allow this. We report on these in chapter 4.

2.6.2 Curvature Coordinates

With the aid of the geometric quantities (2.29) to (2.31) and the energy-momentum tensor (2.9) we obtain the following Einstein field equations:

$$\frac{2m'}{r^2} = \mu \tag{2.60}$$

$$\frac{2(r(r-2m)\Phi' - m)}{r^3} = p \tag{2.61}$$

$$\frac{1}{r^2}\left(\frac{m}{r} - m' + (r - m - rm')\Phi' - r(r-2m)\left(\Phi'' + (\Phi')^2\right)\right) = p \tag{2.62}$$

for the line element in curvature coordinates. We have also selected a comoving fluid velocity vector of the form $u_a = e^{-\Phi(r)}\delta^a_0$.

The isotropy of pressure condition (2.61) = (2.62) yields the differential equation

$$r^2(r-2m)\left(\Phi'' + (\Phi')^2\right) + r(3m - r - rm')\Phi' + (3m - rm') = 0 \tag{2.63}$$

which is a Ricatti equation if considered as a first order equation in Φ'. If a form for m is specified then, in theory, it should be possible to integrate equation (2.63) to yield the function $\Phi(r)$ and consequently the remaining geometric and dynamical quantities governed by the Einstein field equations. However, this is not always possible as the Ricatti equation has been solved only for a few simple cases. For example see Reid(1972) and Reissner (1916).

On the other hand, if (2.63) is viewed as a first order differential equation in $m(r)$ then specifying $\Phi(r)$ *a priori* would, theoretically, allow for the integration of (2.63) as it is a linear first order differential equation. This is precisely the route chosen by Lake (2006) in constructing exact solutions to the Einstein field equations for static spherically symmetric neutral fluid spheres in an algorithmic way. Rearranging equation (2.63) we get

$$r(r\Phi' + 1)m' - (3r\Phi' - 2r^2\left(\Phi'' + (\Phi)^2\right) + 3)m + r^3\left(\Phi'' + (\Phi)^2\right) - r^2\Phi' = 0 \quad (2.64)$$

which is a first order equation in $m(r)$. Now selecting forms for $\Phi(r)$ should allow us to obtain $m(r)$ by integration. Of course, one cannot choose $\Phi(r)$ in an arbitrary manner as the integration of equation (2.64) must still be performed explicitly to yield the metric potentials and the dynamical quantities. Some suitable choices for $\Phi(r)$ were proposed by Lake (2006) and we have considered other possibilities in Chapter 4. We will return to these later.

2.6.3 Isotropic Coordinates

We now consider the Einstein field equations for static fluid spheres in isotropic coordinates. We select the fluid 4–velocity as $u_a = e^{\Psi(r)-B(r)}\ \delta^a_0$. Utilising the Einstein tensor (2.44) to (2.46) and the energy momentum tensor (2.9) we obtain the set of equations:

$$-e^{-2B}\left(2rB'' + 4B' + r(B')^2\right) = \mu \quad (2.65)$$

$$e^{-2B}\left(2B'\Psi' - (B')^2 + 2\frac{\Psi'}{r}\right) \quad = \quad p \tag{2.66}$$

$$e^{-2B}\left(\Psi'' + (\Psi')^2 - 2B'\Psi' + \frac{\Psi'}{r} + (B')^2\right) \quad = \quad p \tag{2.67}$$

Again the pressure isotropy condition (2.66) = (2.67) yields the equation

$$\Psi'' + (\Psi')^2 - \frac{\Psi}{r} - 4B'\Psi' + 2(B')^2 = 0 \tag{2.68}$$

If we introduce a transformation, for example $\Psi'(r) = Q(r)$ then the equation is rewritten as

$$Q' + Q^2 - \frac{1}{r}Q - 4B'Q + 2(B')^2 = 0$$

which is a Ricatti equation in $Q(r)$. Once a form for $B(r)$ is chosen, then the above equation may be solved to reveal $Q(r)$. In turn $Q(r)$ must be integrated to yield $B(r)$ which can then be used to obtain all the remaining geometric and dynamical quantities.

On the other hand if the equation (2.68) is rearranged as

$$2(B')^2 - 4\Psi'B' + \left(\Psi'' + (\Psi')^2 - \frac{1}{r}\Psi'\right) = 0 \tag{2.69}$$

then it may be interpreted as a standard algebraic quadratic equation in $B'(r)$. This is readily solved to give

$$B' = \Psi' \pm \frac{1}{\sqrt{2}}\sqrt{(\Psi')^2 - \Psi'' + \frac{1}{r}\Psi'}. \tag{2.70}$$

Finally integrating (2.70) gives

$$B(r) = \Psi \pm \frac{1}{\sqrt{2}}\int\sqrt{(\Psi^2) - \Psi'' + \frac{1}{r}\Psi'}dr + C \tag{2.71}$$

where C is an integration constant. Now if $\Psi(r)$ is prescribed then we proceed to carry out the integration of (2.71) in order to yield the functional form for $B(r)$. The advantage of this particular algorithm is that only a single integration needs to be performed as opposed to the the use of curvature coordinates which necessitates two integrations. The caveat in this algorithm, however, is the fact that we need to perform the integration of functions appearing under a square root. This is usually very difficult and as a start, it will be prudent to select forms for Ψ that result in the integrand of (2.71) being free of square roots. This form of $B(r)$ has been used by Lake (2006) to obtain an algorithm for generating new exact solutions of the Einstein field equations for static fluid spheres. We will examine this in greater detail in chapter 4.

2.7 Conditions for Physical Admissibility

We now consider briefly the conditions that have to be satisfied for solutions of the Einstein system to be physically admissable. The system (2.57) admits an infinite number of exact solutions as there are more variables than equations. Unfortunately many of the solutions reported in the literature correspond to unrealistic distributions of charged matter. It is desirable to isolate those solutions which are physically reasonable as these can then be used to model charged stars. Often the following constraints are imposed on solutions of the Einstein–Maxwell system in order to obtain models of stellar configurations that are physically plausible:

(a) Positivity and finiteness of pressure and energy density everywhere in the interior of the star including the origin and boundary:

$$0 \leq p < \infty \qquad 0 < \rho < \infty$$

(b) The pressure and energy density should be monotonic decreasing functions of the coordinate r. The pressure vanishes at the boundary $r = R$:

$$\frac{dp}{dr} \leq 0 \qquad \frac{d\rho}{dr} \leq 0 \qquad p(R) = 0$$

(c) Continuity of gravitational potentials across the boundary of the star. The interior line element should be matched smoothly to the exterior Schwarzschild line element at the boundary:

$$e^{2\nu(R)} = e^{-2\lambda(R)} = 1 - \frac{2M}{R}$$

(d) The principle of causality must be satisfied, i.e., the speed of sound should be everywhere less than the speed of light in the interior:

$$0 \leq \frac{dp}{d\rho} \leq 1$$

(e) The metric functions $e^{2\nu}$ and $e^{2\lambda}$should be positive and non–singular everywhere in the interior of the star.

(f) The following energy conditions should be satisfied:

- Weak energy condition: $\rho - p > 0$
- Strong energy condition: $\rho + p > 0$
- Dominant energy condition: $\rho + 3p > 0$

(g) Surface Redshift. For static fluid spheres with a monotonically decreasing and positive pressure profile, the surface redshift has been shown to be less than 2. (Buchdahl 1959, Ivanov 2002)

(h) Mass-Radius Ratio: The maximum mass to radius ratio for a static fluid sphere must satisfy the condition

$$\frac{\text{mass}}{\text{radius}} < \frac{8}{9}$$

to ensure the stability of the sphere (Buchdahl 1959).

Note that most solutions do not satisfy all the conditions (a) to (h) throughout the interior of the charged star. Additionally, some of the above conditions may be overly restrictive. For example, observational evidence suggests that in particular stars the energy density ρ is not a strictly monotonically decreasing function (Shapiro and Teukolsky 1983).

Many solutions presented in the literature are singular at the centre and are valid only for restricted regions of spacetime. Such solutions have to be treated as an envelope of the core and need to be matched to another solution valid for the core. For example the solutions by Herrera and Ponce de Leon (1985), Pant and Sah (1979), Tikekar (1984) and Whitman and Burch (1981) all suffer the drawback of a singularity at the stellar centre. The solution by Bannerjee and Santos (1981) becomes singular at a point in the interior of the distribution. Some solutions presented are regular at the centre but are not stable. For example the solution by Maartens and Maharaj (1990) violates the positivity of pressures condition; these solutions should not be rejected as negative pressures may have occurred in the early universe and thus such models may be acceptable in cosmology. Bonnor (1960, 1964, 1965), Bonnor and Wickramasuriya (1975) and Raychaudhuri (1975) showed that it is possible to generate realistic solutions with vanishing pressure. In such charged dust distributions the Coulombic repulsion is the force responsible for holding the matter in equilibrium in the absence of isotropic particle pressure. De and Raychaudhari

(1968) have verified that in order to guarantee the equilibrium of a static charged dust sphere the relation $\sigma = \pm\rho$ must be satisfied. Other configurations of spherically symmetric distributions include the presence of anisotropic pressures. Such cases were examined by Maharaj and Maartens (1989) and Ruderman (1972) in the case of neutral spheres, and by Herrera and Ponce de Leon (1985) and Maartens and Maharaj (1990) in the presence of charge.

Chapter 3

A Review of Solution Generating Algorithms

3.1 Introduction

In this chapter we review some of the important achievements in the area of solution generating algorithms for the static neutral spherically symmetric fluid from the earliest proposal of Wyman (1949). The idea of Wyman sadly lay dormant for over fifty years until Fodor (2000) resuscitated it and went on to produce an efficient algorithm for finding new exact solutions for the Einstein field equations. We do not present the developments chronologically in this section and instead elect to consider them in order of usefulness, with the most useful ones dealt with first. Arguably the algorithm of Lake (2003, 2006) is most promising although no method offers serious advantages without some side effects. One of Lake's algorithms has the drawback of involving two integrations while a second algorithm is at the mercy of square roots of functions. Nevertheless functional forms are postulated which do indeed result in new exact solutions as well as the regaining of familiar results. The contribution of Boonserm *et al* (2005) was to prove four theorems which show

how to construct new exact solutions of the Einstein field equations from existing solutions. Fodor (2000), working in area coordinates, observed that his choice of metric formulation resulted in the solving of an algebraic equation and consequently not requiring explicit integration. His approach was also beset by the appearance of square roots in the analysis. However, the major achievement of Fodor was that the energy density and pressure and consequently the adiabatic sound-speed index could be computed without a full integration of the potential functions. Finally we consider the Martin and Visser (2008) algorithm which is a variation of Lake's algorithm however again two integrations are necessary to unlock all the required functions.

3.2 Algorithm of Lake

We now examine two algorithms proposed by Lake (2006). Each of these has advantages and disadvantages.

First Lake writes the spherically symmetric metric as

$$ds_M^2 = \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2 - e^{2\Phi(r)} dt^2. \tag{3.1}$$

which is referred to as 'curvature coordinates' or because of its resemblance to the Schwarzschild exterior solution it is also called Schwarzschild coordinates. Note that the 2–sphere is represented by $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The field equations governing the gravitational behaviour of a neutral sphere, in these coordinates, have been given in (2.57) to (2.59). A single function $\Phi(r)$ must be nominated *a priori* and then the quantity $m(r)$ may be established or vice-versa. Finally the energy density and pressure are obtained via (3.5) and (3.6).

The function $m(r)$ is defined as follows

$$m(r) = \frac{\int b(r) e^{\int a(r) dr} dr + C}{e^{\int a(r)} dr} \tag{3.2}$$

where

$$a(r) \equiv \frac{2r^2(\Phi''(r) + \Phi'(r)^2) - 3r\Phi'(r) - 3}{r(r\Phi'(r) + 1)} \tag{3.3}$$

and

$$b(r) \equiv \frac{r(r(\Phi''(r) + \Phi'(r)^2) - \Phi'(r))}{r\Phi'(r) + 1} \tag{3.4}$$

with $' \equiv \frac{d}{dr}$ and C a constant. Observe that the quantity $m(r)$ has a clear interpretation as the gravitational mass of the perfect fluid sphere.

We are now in a position to obtain the dynamical quantities. The energy density is given by

$$\rho = \frac{m'(r)}{4\pi r^2} \tag{3.5}$$

while the pressure has the form

$$p = \frac{r\Phi'(r)[r - 2m(r)] - m(r)}{4\pi r^3}. \tag{3.6}$$

We have followed Lake's use of units for the geometric quantities. Note that it is expected that both these quantities should be positive for a realistic matter configuration.

It is also argued that the source function $\Phi(r)$ must be a monotone increasing function with a regular minimum at $r = 0$ based upon the behaviour of the distribution's central conditions. Additionally to ensure continuity of the metric potentials across the pressure-free boundary hypersurface ($p(r) = 0$) the necessary and sufficient condition is obtained by setting $m(r=R) \equiv M$, that is

$$\Phi'(r=R) = \frac{M}{R(R-2M)}.$$

This matches the interior solution to the unique exterior Schwarzschild solution (Birkhoff 1923).

Lake (2006) concedes that while the number of source functions $\Phi(r)$ for which (3.2) can be evaluated exactly is finite, it should be noted, however, that the generation of an exact solution does not necessarily mean that the equation $p(r=R)=0$ can be solved exactly.

By way of an example, the form

$$\Phi(r) = \frac{1}{2} N \ln \left(1 + \frac{r^2}{\alpha}\right) \tag{3.7}$$

is postulated for Φ. Here N is an integer ≥ 1 and α is a constant > 0 . The function (3.7) is monotone increasing with a regular minimum at $r=0$. With the source function (3.7), equation (3.2) can be evaluated exactly for any N. Interestingly this function (3.7) produces known solutions for $N = 1, ..., 5$. It is remarked by Lake that " these solutions with, $N =$ 1,...5, in fact constitute half of all the previously known physically interesting solutions in curvature coordinates. For $N \geq 5$ the solutions are acceptable on physical grounds and even exhibit a monotonically decreasing subluminal adiabatic sound speed."

Next we consider the spherically symmetric line element expressed in "isotropic coordinates". The metric has the form

$$ds_M^2 = e^{2B(r)}(dr^2 + r^2 d\Omega^2) - e^{2(\Psi(r) - B(r))} dt^s \tag{3.8}$$

where $\Psi(r)$ and $B(r)$ are two unknown functions to be determined. The advantage of 'curvature' coordinates was that the function $m(r)$ had a physical interpretation connected to the stellar mass. However, in this formulation (3.8) no particular physical meaning can be attached to the functions Ψ and B at the outset. Nevertheless, it will be noted that this structural version offers a more efficient approach to finding new exact solutions of the Einstein field equations.

If a sufficiently smooth function $\Psi(r)$ is selected at the outset then the function $B(r)$ may be calculated via

$$B(r) = \Psi(r) + \int c(r)dx + C \tag{3.9}$$

where

$$c(r) \equiv \frac{\epsilon}{\sqrt{2}}\sqrt{\Psi'(r)^2 - \Psi"(r) + \frac{\Psi'(r)}{r}} \tag{3.10}$$

with $\epsilon = \pm 1$, $' \equiv \frac{d}{dr}$ and C is a constant. The clear advantage of this algorithm is the fact that only one integration must be performed. However, the presence of the square root in the integrand is a negative feature. Nevertheless we may now compute the energy density in the form

$$\rho = \frac{-1}{8\pi e^{2B(\mathrm{r})}}\left(2B''(r) + \frac{4B'(r)}{r} + (B'(r))^2\right) \tag{3.11}$$

while the pressure is given by

$$p = \frac{-1}{8\pi e^{2B(\mathrm{r})}}\left(-B'(r)\Psi(r) + (B'(r))^2 - 2\frac{\Psi'(r)}{r}\right). \tag{3.12}$$

As usual it is desired that both energy density and pressure remain positive throughout the fluid's distribution. Additionally it is argued that the source function must be a monotonically increasing function for physical constraints to be satisfied. An added bonus of this prescription, in contrast with 'curvature' coordinates, is that the adiabatic sound speed index may be obtained explicitly without the need to find the function $B(r)$ - that is $\frac{dp}{d\rho}$ may be found without having to perform the integration (3.9). The details are contained in the work of Lake (2006) and are accordingly omitted.

By way of a demonstration Lake provides the following functional form for the source function Ψ:

$$\Psi(r) = \alpha \ln \frac{f(r)}{g(r)} \tag{3.13}$$

where $\alpha > 0$ is a constant and f and g are functions of r. An example, of suitable functions f and g that satisfy the criteria that Ψ is monotonic and increasing is $g(r) = (\delta + \epsilon r^2)^\zeta$ and $f(r) = \delta^\zeta + \gamma r^2$ with δ, ϵ, γ constants such that $\delta > 0$ and $\delta^(1 - \zeta)\gamma > \zeta\epsilon$. It is stated by Lake (2006) that this class of solutions includes a number of known solutions including the Schwarzschild interior solution and the Rahman-Visser (2002) general quadratic ansatz.

While the algorithms above do indeed represent all possible spherically symmetric perfect fluid spacetimes, it is acknowledged that there exists no systematic method of selecting the source functions in each case. The advantages and drawbacks of each proposal has been alluded to above.

3.3 Boonserm, Visser and Weinfurtner (BVW) Algorithm

Boonserm *et al* (2005) write the spherically symmetric line element in the form

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2 \tag{3.14}$$

where $\zeta(r)$ and $B(r)$ are two functions to be determined from the pressure isotropy condition. In addition $d\Omega^2$ is the unit 2-sphere $d\theta^2 + \sin^2 d\phi^2$. The associated master Einstein field equation reduces to

$$[r(r\zeta)']B' + [2r^2\zeta'' - 2(r\zeta)']B + 2\zeta = 0 \tag{3.15}$$

which is an ordinary differential equation in $B(r)$. Therefore once you have chosen a form for $\zeta(r)$, this equation must be solved for $B(r)$. Note that this same equation may be arranged differently as

$$2r^2B\zeta'' + (r^2B' - 2rB)\zeta' + (rB' - 2B + 2)\zeta = 0 \tag{3.16}$$

which is now a second order ordinary differential equation in $\zeta(r)$. The construction of new solutions may now proceed systematically by assuming the existence of a solution to any one of (3.15) or (3.16) above. In other words, new solutions may be constructed from old solutions in an algorithmic fashion. Suppose we have a spacetime metric

$$ds^2 = -\zeta_0(r)^2dt^2 + \frac{dr^2}{B_0(r)} + r^2d\Omega^2 \tag{3.17}$$

and assume it represents perfect fluid sphere. The idea of BVW (2005) is to "deform" this solution by applying four different transformation theorems on (ζ, B_0), such that the outcomes still presents a perfect fluid sphere. The outcome of this process will depend on one or more free parameters, and so automatically generates an entire family of perfect fluid spheres of which the original starting point is only one member. They also attempt to find a connection between all different transformation theorems. We omit the proofs here and the interested reader may refer to the work of BVW (2005) for the details. We adjudicate it prudent to state the main theorems which give an idea of how new solutions may be constructed algorithmically.

- **Theorem 1**

  Suppose $(\zeta_0(r), B_0(r))$ represents a perfect fluid sphere. Define

$$\Delta_{0(r)} = \left(\frac{\zeta_0(r)}{\zeta_0(r) + r\zeta_0'(r)}\right)^2 r^2 \exp\left\{2\int \frac{\zeta_0'(r)\zeta_0(r) - r\zeta_0'(r)}{\zeta_0(r)\zeta_0(r) + r\zeta_0'(r)} dr\right\} \tag{3.18}$$

  Then for all λ, the geometry defined by holding $\zeta_0(r)$ fixed and setting

$$ds^2 = -\zeta_0(r)^2 dt^2 + \frac{dr^2}{B_0(r) + \lambda\Delta_0(r)} + r^2 d\Omega^2 \tag{3.19}$$

  is also a perfect fluid spheres. That is, the mapping

$$T_1(\lambda) : \{\zeta_0, B_0\} \mapsto \{\zeta_0, B_0 + \lambda\Delta_0(\zeta_0)\} \tag{3.20}$$

  takes perfect fluid spheres into perfect fluid spheres.

- **Theorem 2**

  Let $\{\zeta_0, B_0\}$ describe a perfect fluid sphere. Define

$$Z_0(r) = \sigma + \epsilon \int \frac{r dr}{\zeta_0(r)^2 \sqrt{B_0(r)}} \tag{3.21}$$

  Then for all σ and ϵ, the geometry defined by holding $B_0(r)$ fixed and setting

$$ds^2 = -\zeta_0(r)^2 Z_0(r)^2 dt^2 + \frac{dr^2}{B_0(r)} + r^2 d\Omega^2 \tag{3.22}$$

  is also a perfect fluid sphere.

- **Theorem 3**

  If $\{\zeta_0, B_0\}$ denotes a perfect fluid sphere, then for all σ, ϵ, and λ, the three parameter geometry defined by

$$ds^2 = -\zeta_0(r)^2 \left\{\sigma + \epsilon \int \frac{r dr}{\zeta_0(r)^2 \sqrt{B_0(r)}}\right\}^2 dt^2 + \frac{dr^2}{B_0(r) + \lambda\Delta_0(\zeta_1, r)} + r^2 d\Omega^2 \tag{3.23}$$

is also a perfect fluid sphere, where Δ_0 is given by

$$\Delta_0(\zeta_1, r) = \left(\frac{\zeta_1(r)}{\zeta_1(r) + r\zeta_1'(r)}\right)^2 r^2 \exp\left\{2\int \frac{\zeta_1'(r)\zeta_1(r) - r\zeta_1'(r)}{\zeta_1(r)\zeta_1(r) + r\zeta_1'(r)} dr\right\}, \tag{3.24}$$

That is

$$T_3 = T_0 o T_1 : \{\zeta_0, B_0\} \mapsto \{\zeta_0, B_0 + \lambda\Delta_0(\zeta_0)\} \mapsto \{\zeta_0 Z_0(\zeta_0, B_0 + \lambda\Delta_0(\zeta_0)), B_0 + \lambda\Delta_0(\zeta_0)\} \tag{3.25}$$

takes perfect fluid spheres into perfect fluid spheres.

- **Theorem 4**

If $\{\zeta_0, B_0\}$ denotes a perfect fluid sphere, then for all σ, ϵ, and λ, three parameter geometry defined by

$$ds^2 = -\zeta_0(r)^2 \left\{\sigma + \epsilon \int \frac{rdr}{\zeta_0(r)^2\sqrt{B_0(r)}}\right\}^2 dt^2 + \frac{dr^2}{B_0(r) + \lambda\Delta_0(\zeta_1, r)} + r^2 d\Omega^2 \tag{3.26}$$

is also a perfect fluid sphere, where $\Delta_0(\zeta_1, r)$ is defined as

$$\Delta_0(\zeta_1, r) = \left(\frac{\zeta_1(r)}{\zeta_1(r) + r\zeta_1'(r)}\right)^2 r^2 \exp\left\{2\int \frac{\zeta_1'(r)\zeta_1(r) - r\zeta_1'(r)}{\zeta_1(r)\zeta_1(r) + r\zeta_1'(r)} dr\right\}, \tag{3.27}$$

depending on $\zeta_1 = \zeta_0 Z_0$, where as before

$$Z_0(r) = \sigma + \epsilon \int \frac{rdr}{\zeta_0(r)^2\sqrt{B_0(r)}} \tag{3.28}$$

That is

$$T_4 = T_1 o T_2 : \{\zeta_0, B_0\} \mapsto \{\zeta_0 Z_0(\zeta_0, B_0), B_0\} \mapsto \{\zeta_0 Z_0(\zeta_0, B_0), B_0 + \lambda\Delta_0(\zeta_0 Z_0(\zeta_0, B_0))\} \tag{3.29}$$

takes perfect fluid spheres into perfect fluid spheres.

Using these theorems, one can find all possible solutions to Einstein's field equations for a static perfect fluid, in principle. Naturally, known solutions may be recovered, however novel solutions may develop from existing metrics that are known to solve the field equations.

3.4 Fodor Algorithm in Area Coordinates

The interesting feature of Fodor's (2000) algorithm is that solutions of the field equations may be found without actual integration. The general stationary spherically symmetric configuration is written in area coordinates as

$$ds^2 = -e^{2\nu}dt^2 + \frac{1}{B}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{3.30}$$

where $\nu = \nu(r)$ and $B = B(r)$ are functions of the radical coordinate r. The energy density μ has the form

$$\mu = \frac{1}{8\pi r^2}(1 - B - rB') \tag{3.31}$$

the radial pressure p_r is given by

$$p_r = \frac{1}{8\pi r^2}(2rB\nu' + B - 1) \tag{3.32}$$

while the angular pressure has the form

$$p_\vartheta = \frac{1}{16\pi r}\left(2rB\nu'' + 2rB(\nu')^2 + r\nu' B' + 2B\nu' + B'\right) \tag{3.33}$$

where the prime denotes derivatives with respect to the radial coordinate r. Here Fodor has taken the more general form of the field equations where the fluid is anisotropic in general. However, if we assume pressure isotropy $p_\vartheta = p_r$ then we obtain the master field equation in the form

$$r(rv' + 1)B' + [2r^2v'' + 2r^2(v')^2 - 2rv' - 2]B + 2 = 0 \tag{3.34}$$

It is useful to introduce a new function $\beta(r)$ as

$$\beta = rv' + 1 \tag{3.35}$$

Then the field equation (3.34) takes the simple form

$$r\beta B' + 2rB\beta' + 2\beta^2 B - 8\beta B + 4B + 2 = 0 \tag{3.36}$$

and the pressure becomes

$$p = \frac{1}{8\pi r^2}\left(2\beta B - B - 1\right). \tag{3.37}$$

Introducing the transformations

$$\alpha = \beta^2 B \tag{3.38}$$

equation (3.36) becomes a second order algebraic equation in β ,

$$2(\alpha + 1)\beta^2 + (r\alpha' - 8\alpha)\beta + 4\alpha = 0 \tag{3.39}$$

Now if instead of α one introduces its square root, $z = \sqrt{\alpha}$ as a new function, and denote $\sqrt{B} = b$, then it is possible to get from (3.39) an equation in b, namely

$$2b^2 + (rz' - 4z)b + z^2 + 1 = 0 \tag{3.40}$$

which is, in fact, a second order algebraic equation in β. This equation was also reported by Burlankov (1993). Theoretically, once a form for z is chosen the function $b(r)$ may be calculated from (3.40).

While in theory this appears simple, in practice the problem may become intractable on account of the presence of square roots. More seriously Fodor (2000) admits that the most simple polynomial forms for z do not appear to work. Consider any function α for which

$$(8\alpha - r\alpha')^2 > 32\alpha(\alpha + 1) \tag{3.41}$$

then the quadratic equation (3.40) has two solutions for the function β which we denote by β_+ and β_-,

$$\beta_\pm = \frac{1}{4(\alpha + 1} \left[8\alpha - r\alpha' \pm \sqrt{(8\alpha - r\alpha')^2 - 32\alpha(\alpha+1)} \right] \tag{3.42}$$

Using (3.38) and (3.35) the metric functions B_+ and ν_-, can be calculated as

$$B_\pm = \frac{\alpha}{\beta_\pm^2} = \frac{(\alpha+1)^2}{4\alpha}\beta_\pm^2, \tag{3.43}$$

and

$$\nu_\pm = \int_0^r \frac{1}{r}\left(\beta_\pm - 1\right) dr + C_\pm \tag{3.44}$$

where C_+ and C_- are constants.The integral (3.44) generally cannot be resolved in terms of elementary functions, but as can be seen from (3.31) and (3.37), the physically important pressure and density can be calculated without performing integrals. If we denote the pressure and density belonging to β_- by p_+ and μ_+ ,and those belonging to β_- by p_- and μ_-, we obtain

$$p_\pm = \frac{1}{8\pi r^2}(2\beta_\pm B_\pm - 1) \tag{3.45}$$

and

$$\mu_\pm = \frac{1}{8\pi r^2}\left(1 - B_\pm - rB_\pm'\right). \tag{3.46}$$

So while this algorithm has the drawback of not admitting even simple polynomial type solutions for z, it has a distinct advantage from a physical analysis point of view. The dynamical quantities pressure and energy density may be computed even though the integration (3.44) may not be achievable. This allows one to also compute the adiabatic sound speed index $\frac{dp}{d\mu}$ explicitly and then to test if the model supports causality. That is we must check if $0 < \frac{dp}{d\mu} < 1$ is satisfied everywhere in the interior of the distribution. Most other solution generating techniques require that the metric potentials be explicitly known before the dynamical quantities are calculated.

3.5 Martin and Visser Algorithm

This approach constitutes a variation of Lake's algorithm. The spherically symmetric spacetime is chosen to have the form

$$ds^2 = -exp\left[-2\int_r^\infty g(r)d\tilde{r}\right]dt^2 + \frac{dr^2}{1-2m(r)/r} + r^2[d\theta^2 + sin^2\theta d\phi^2]. \tag{3.47}$$

where $g(r)$ is the so called "gravity profile". It is related to the gravitational redshift z by

$$1 + z = \exp\left[\int_r^\infty g(\tilde{r})d\tilde{r}\right] \tag{3.48}$$

and is related to the locally measured acceleration due to gravity by

$$a = \sqrt{1 - \frac{2m(r)}{r}}g(r) \tag{3.49}$$

Given $g(r)$ positive for a downward acceleration in the vacuum region beyond the surface of the star-like object, the Schwarzschild solution gives $g(r) = (M/r^2)/(1 - 2M/r)$ and $m(r) = M$. Martin and Visser find it more convenient to write the metric in the form:

$$ds^2 = -\exp\left[-2\int_r^\infty g(r)d\tilde{r}\right]dt^2 + \frac{dr^2}{1-2m(r)/r} + r^2[d\theta^2 + sin^2\theta d\phi^2]. \tag{3.50}$$

where $\mu(r) = \frac{4\pi}{3}\tilde{p}(r)$ is proportional to the average density inside radius r. In terms of these variables, the Einstein equations are

$$8\pi\rho = 2m'(r)/r^2 = 2[r\mu'(r) + 3\mu(r)] \tag{3.51}$$

$$8\pi p = 2\left\{\frac{g(r)[1 - 2\mu(r)r^2]}{r} - \mu(r)\right\} \tag{3.52}$$

$$8\pi p = -r[1+rg(r)]\frac{d\mu(r)}{dr} - 2\left\{[1+rg(r)]^2 + r^2\frac{dg(r)}{dr}\right\}\mu(r)$$

$$+\left[\frac{dg(r)}{dr} + \frac{g(r)}{r} + g(r)^2\right]. \qquad (3.53)$$

Equation (3.51) integrates to give

$$\mu(r) = \frac{1}{r^3}\int_0^r 4\pi\rho(\tilde{r})\tilde{r}^2 d\tilde{r}, \qquad (3.54)$$

which justifies the choice of notation $m(r) = \mu(r)r^3$. On the other hand the pressure isotropy condition (3.52) = (3.53) yields the differential equation

$$\frac{dg}{dr} = -g^2 + \frac{1+\mu' r^3}{r(1-2\mu r^2)}g + \frac{r\mu'}{1-2\mu r^2} \qquad (3.55)$$

which is a Riccati equation, for which there is no general solution. Rearranging (3.55) to extract $d\mu/dr$ we find

$$\frac{d\mu}{dr} = -\frac{2r(g^2+g')}{1+rg}\mu + \frac{(g/r)' + g^2/r}{1+rg}. \qquad (3.56)$$

which is now a simple first-order linear ordinary differential equation and hence explicitly solvable. The general solution is given by

$$\begin{aligned}\mu(r) &= \exp\left[-2\int\frac{r[g^2(r)+g'(r)]}{1+rg(r)}dr\right] \\ &\times \left\{C_1 + \int\frac{-g(r)+rg'(r)+rg(r)^2}{r^2[1+rg(r)]}\exp\left[2\int\frac{r[g^2(r)+g'(r)]}{1+rg(r)}dr\right]\right\}\end{aligned} \qquad (3.57)$$

Now once a form for the function $g(r)$ is chosen *ab initio* then the potential function $\mu(r)$ may then be established with the aid of (3.57). In turn the energy density and pressure may be calculated via (3.51) and (3.52) respectively. As can be seen, this process calls for two different integrations. So one would have to make an extremely fortuitous choice for $g(r)$ in order for all the integrals to be obtainable in elementary analytic forms.

Chapter 4

New Exact Solutions

4.1 Introduction

In this chapter we report new exact solutions of the Einstein field equations for static spherically symmetric distributions of matter. Based on the success of others in the past decade, we devise our own algorithm for finding new exact solution utilising a coordinate transformation first used by Durgapal and Banerjee (1983). The master field equation resulting from the pressure isotropy condition is rearranged in the form of a linear first order differential equation that is explicitly solvable. In order to obtain viable complete solutions the generating function must be chosen so as to facilitate the integration of the master field equation. We exhibit some new solutions using this method. We then investigate the algorithm of Lake (2006) utilising curvature and isotropic coordinates. In both cases we are able to obtain new exact solutions by postulating forms of the generating functions that have not previously yielded success. Finally we study some of our solutions for physical plausibility.

4.2 Solutions in Durgapal–Bannerjee Coordinates

As remarked in Chapter 2, the Einstein field equations (2.57) to (2.59) govern the gravitational behaviour of static spherical perfect fluids. The most common approach to obtaining a complete solution of the field equations has been to specify the gravitational potential $Z(x)$ and then to integrate (2.59) to obtain the function $y(x)$. For example, the choice $Z = 1 + x$ has been shown to lead to the Schwarzschild interior solution by Maharaj and Mkhwanazi (1996). Additionally Finch and Skea (1985) utilised the form $Z = (1 + x)^2$ and generated a new class of solutions which was shown to conform to the realistic behaviour of stars according to the theory of Walecka (1975). Thirukannesh and Maharaj (2006) studied the general form $Z = (1+x)^n$ and produced new classes of exact solutions, albeit some of which were expressed as series as the Frobenius method of solving differential equations was invoked. In the approaches followed so far in the literature, a form for $Z(x)$ is usually postulated and then $y(x)$ is calculated by solving the second order differential equation. An alternative approach is to consider (2.59) as a first order linear differential equation in $Z(x)$. It has the form

$$(2x^2\dot{y} + xy)\dot{Z} + (4x^2\ddot{y} - y)Z + y = 0 \tag{4.1}$$

which may be readily solved. The general solution to (4.1) is given by

$$Z(x) = ke^F + e^F \int e^{-F} \frac{y}{2x^2\dot{y} + xy} dx \tag{4.2}$$

where

$$F(x) = \int \frac{4x^2\ddot{y} - y}{2x^2\dot{y} + xy} dx$$

and k is an integration constant. This represents all possible solutions to the Einstein field equations for a static spherically symmetric configuration of matter. We must now select suitable forms for the function $y(x)$, compute $F(x)$ and then via (4.1) establish the gravitational potential $Z(x)$. This will then generate a complete model of the relativistic fluid sphere. Naturally, there is still no clear way to select suitable source functions $y(x)$ that will allow for the complete integration of the field equations and so the familiar *ad hoc* methods must still be used.

4.2.1 The Choice $y = a + bx$

As an example, we consider the simple choice

$$y(x) = a + bx \tag{4.3}$$

which is a linear function of x. This is equivalent to the choice $e^{2\nu} = (v + wr^2)^2$, v and w being constants, in the canonical coordinate system if we bear in mind that $x = Cr^2$. On examining the literature, we observe that our choice corresponds to the form of the Kuchowicz (1970)

$$e^{2\nu} = \left(Ar^{\alpha} + Br^{\beta}\right)^2 \tag{4.4}$$

if we put $\alpha = 1$ and $\beta = 2$. Kuchowicz (1970) was, however, only able to solve the field equations explicitly for specific choices of α and β. Therefore our solutions will only coincide with those of Kuchowicz for appropriate values of the constants. Another form postulated by Heintzman (1969) namely

$$e^{2\nu} = (a + br^2)^n$$

bears a resemblance to our work. Nevertheless exact solutions have only been reported for the cases $n = 3, \pm 1, -2$ none of which coincides with our class of solutions. With the form (4.3) we obtain

$$F(x) = \log \frac{(a + 3bx)^{\frac{2}{3}}}{x}$$

and substituting into (4.1) we obtain

$$Z(x) = \frac{(1+c_1)(a+3bx)^{\frac{2}{3}}}{x} \tag{4.5}$$

where c_1 is an integration constant. In order to facilitate a study of the field equations for physical plausibility, we elect to set the constants to the values $c_1 = 1$, $a = 1$ and $b = 3$. Now it is possible to find the various dynamical quantities :

The energy density ρ is obtained via (2.57) and is given by

$$\frac{\rho}{C} = \frac{2 + x(-2 + (1+3x)^{\frac{1}{3}})}{x^2(1+3x)^{\frac{1}{3}}} \tag{4.6}$$

The pressure is found with the help of (2.58) and has the form

$$\frac{p}{C} = \frac{2(1+x)(1+3x)^{\frac{2}{3}} + x(-1-x+8(1+3x)^{\frac{2}{3}})}{x(1+x)} \tag{4.7}$$

The rates of change of each of these dynamical quantities is given by

$$\frac{d\rho}{Cdx} = \frac{-4 + 4x^2(8 - 3(1+3x)^{\frac{1}{3}}) - x(12 + (1+3x)^{\frac{1}{3}})}{x^3(1+3x)^{\frac{4}{3}}} \tag{4.8}$$

$$\frac{dp}{Cdx} = \frac{-4 + x^3(-40 + (1+3x)^{\frac{1}{3}}) + x(-24 + (1+3x)^{\frac{1}{3}}) + 2x^2(-22 + (1+3x)^{\frac{1}{3}})}{x^3(1+x)^2(1+3x)^{\frac{1}{3}}} \tag{4.9}$$

This allows us to compute the adiabatic sound speed index

$$\frac{dp}{d\rho} = -\frac{(1+3x)(-4 + x^3(-40 + (1+3x)^{\frac{1}{3}}) + x(-24 + (1+3x)^{\frac{1}{3}}) + 2x^2(-22 + (1+3x)^{\frac{1}{3}}))}{(1+x)^2(4 + x(12 + (1+3x)^{\frac{1}{3}}) + x^2(-8 + 3(1+3x)^{\frac{1}{3}}))} \tag{4.10}$$

which we must check to see is constrained by $0 < \frac{dp}{d\rho} < 1$. That is the sound speed must be sub–luminal.

In addition we investigate the energy conditions. The following quantities are relevant:

$$\rho - p = \frac{1}{x + x^2}\left(2 + 2x - \frac{16 + 32x}{(1 + 3x)^{\frac{1}{3}}}\right) \tag{4.11}$$

$$\rho + p = \frac{4(1 + 4x + 7x^2)}{x^2(1 + x)(1 + 3x)^{\frac{1}{3}}} \tag{4.12}$$

$$\rho + 3p = \frac{8 - 2x^2(-44 + (1 + 3x)^{\frac{1}{3}}) - 2x(-24 + (1 + 3x)^{\frac{1}{3}})}{x^2(1 + x)(1 + 3x)^{\frac{1}{3}}} \tag{4.13}$$

and graphical plots will give us an indication whether the physical requirements are satisfied or not. Regrettably, the plots show a generic defect in that we are unable to obtain a vanishing pressure hypersurface. Additionally one of the energy conditions is always violated and the causality cannot be ensured. Accordingly there is little value in exhibiting all the plots.

We conclude this section by listing the complete solution to the Einstein field equations for static fluid spheres for our choice of source funcion. It is given by

$$y(x) = a + bx \tag{4.14}$$

$$Z(x) = \frac{(1 + C)(a + 3bx)^{\frac{2}{3}}}{x} \tag{4.15}$$

$$\rho = \frac{2 + x(-2 + (1 + 3x)^{\frac{1}{3}})}{x^2(1 + 3x)^{\frac{1}{3}}} \tag{4.16}$$

$$p = \frac{2(1 + x)(1 + 3x)^{\frac{2}{3}} + x(-1 - x + 8(1 + 3x)^{\frac{2}{3}})}{x(1 + x)} \tag{4.17}$$

The line element may be written as

$$ds^2 = -(a+bx)^2dt^2 + \left(\frac{(1+C)(a+3bx)^{\frac{2}{3}}}{x}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (4.18)$$

4.2.2 The Choice $y = (a+bx)^{-1}$

This simple choice also does not appear to fall into any general class treated previously. If we set $a = 1$ in our work the case corresponds to Heintzmann's (1969) case for $n = -1$ in his work. There is also the problem of an extra constant q in Heintzmann's formulation $e^{2\nu} = q(1+br^2)^n$. Therefore our solutions coincide only for specific values of appropriate constants and not in general. With the choice

$$y = \frac{1}{a+bx} \quad (4.19)$$

which is equivalent to $e^{2\nu} = 1/(v+wr^2)$, the form of Z is easily obtained as

$$Z(x) = \frac{1 + x + 4x^2 + \frac{20x^3}{3} - 16x^5 - \frac{64x^6}{3} - \frac{64x^7}{7}}{(1-2x)^2 x(1+2x)^4} \quad (4.20)$$

where we have set $a = 1$, $b = 2$, $C = 1$ and $k = 1$. Observe that the solution for general a and b is easily found, but we make these definitions of the constants to make the physical analysis more transparent. The energy density is now given by

$$\rho = \frac{-21 - 168x + 1092x^2 + 1120x^3 + 2912x^4 + 896x^5 - 6272x^6 - 5632x^7 + 5120x^8 + 6144x^9}{21x^2(-1+2x)^3(1+2x)^5} \quad (4.21)$$

while the pressure has the form

$$p = -\frac{-21 + 126x + 168x^2 + 448x^3 - 1344x^5 - 896x^6 + 1536x^7 + 1536x^8}{21(1-2x)^2x^2(1+2x)^5} \quad (4.22)$$

The sound speed index is calculated as

$$\frac{dp}{d\rho} = \Big(21 - 42x - 756x^2 + 3248x^3 - 2464x^4 + 1344x^5 - 4480x^6 - 8704x^7$$

$$+3072x^8 + 12288x^9 - 6144x^{11}\Big) \,/\, \Big(21 + 126x - 84x^2 - 4648x^3 + 18144x^4$$

$$+11200x^5 + 20608x^6 + 7424x^7 - 23552x^8 - 22528x^9 + 8192x^{10} + 12288x^{11}\Big) \quad (4.23)$$

Finally, we give the expressions for the analysis of the energy conditions. These have the form

$$\rho - p = \frac{8\left(-42 + 147x + 126x^2 + 476x^3 + 280x^4 - 1008x^5 - 1120x^6 + 832x^7 + 1152x^8\right)}{21x(-1+2x)^3(1+2x)^5} \quad (4.24)$$

$$\rho + p = \frac{2\left(-21 + 504x^2 + 616x^3 + 1008x^4 - 224x^5 - 2240x^6 - 1152x^7 + 1792x^8 + 1536x^9\right)}{21x^2(-1+2x)^3(1+2x)^5} \quad (4.25)$$

$$\rho + 3p = \frac{4\left(21 - 84x - 210x^2 - 364x^3 - 56x^4 + 784x^5 + 224x^6 - 1088x^7 - 128x^8 + 768x^9\right)}{21x^2(-1+2x)^3(1+2x)^5} \quad (4.26)$$

and these quantities should be positive for a realistic stellar configuration.

4.2.3 Qualitative Physical Analysis

In view of the complicated expressions for the geometric and dynamical quantities above, we have resorted to the use of mathematical software packages to plot the profiles of the main physical quantities. Figure 4.2 displays the pressure profile. It is pleasing to note that the pressure vanishes for the value $x = 0,13$ and this defines the boundary of the fluid sphere. Furthermore, it can be observed that the pressure is everywhere positive in the interior of the sphere and is monotonically decreasing towards the boundary of the sphere. A negative feature, though, is that the pressure is singular at the centre of the distribution.

Figure 4.1 exhibits the energy density and it is pleasing to see that the energy density is everywhere positive within the radius obtained from the pressure free hypersurface. Additionally it is also decreasing outwardly and has the similar drawback as the pressure of being infinite at the centre.

Perhaps the most noteworthy feature of this solution is Figure 4.3. This shows that the adiabatic sound speed criterion $0 < \frac{dp}{d\rho} < 1$ is satisfied everywhere inside the spherical configuration. This ensures that the speed of sound never exceeds the speed of light which is a fundamental postulate in Einstein's theory of gravity.

The graphs Figures 4.4 – 4.6 indicate that the energy conditions are satisfied everywhere inside the sphere. This is evidenced by all three expressions $\rho - p$, $\rho + p$ and $\rho + 3p$ are always positive within the radius of the sphere. It should be pointed out that it is indeed rare to find all these elementary requirements being satisfied in the same model. Most models reported in the literature contain one or other defect that renders them non-physical. In the work of Delgaty and Lake (1998) it is argued that of over 100 exact solutions for the static spherically symmetric fluid sphere only about 8 succeed as viable models of realistic phenomena. That is only this small subset satisfy the basic requirements for physical plausibility. In our case, if we are prepared to give up regularity at the centre, then all physical requirements are met.

Physically this means that our model may serve as a core–envelope model– in other words, our solution may be used to model a spherical shell of perfect fluid. This solution needs to be matched with the Schwarzschild exterior on the outer boundary and some other solution on the inside boundary. The matter surrounded by our shell may be another fluid described by another appropriate exact solution. For example it could be the Finch-Skea perfect fluid (Hansraj and Maharaj (2008)).

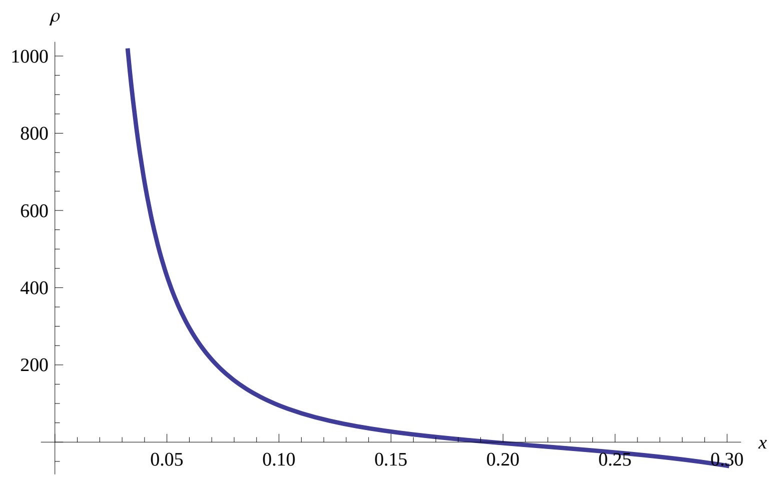

Figure 4.1: Graph of Energy density versus radial coordinate x

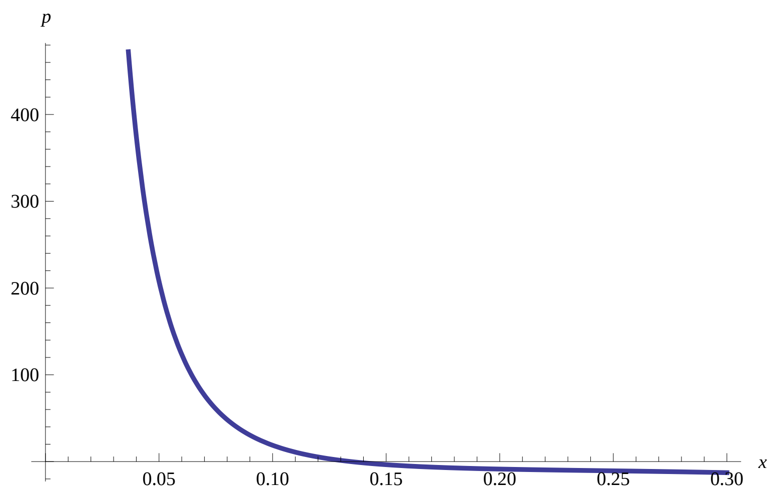

Figure 4.2: Graph of Pressure versus radial coordinate x

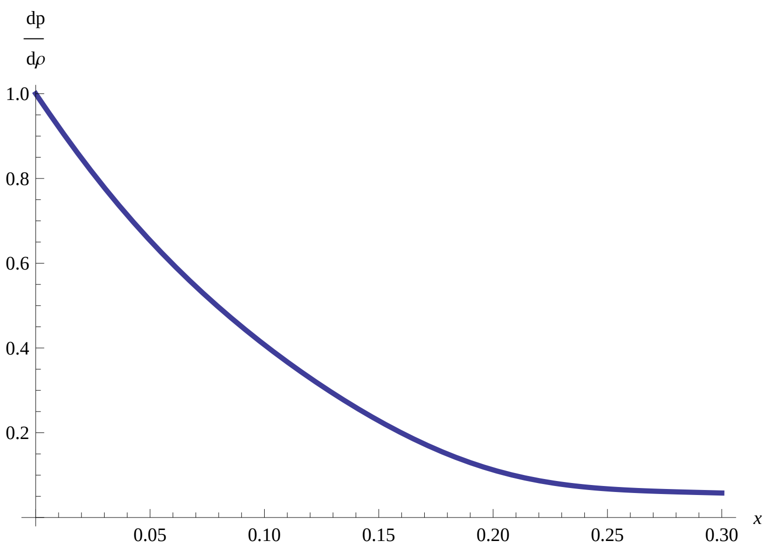

Figure 4.3: Graph of $\frac{dp}{d\rho}$ versus radial coordinate x

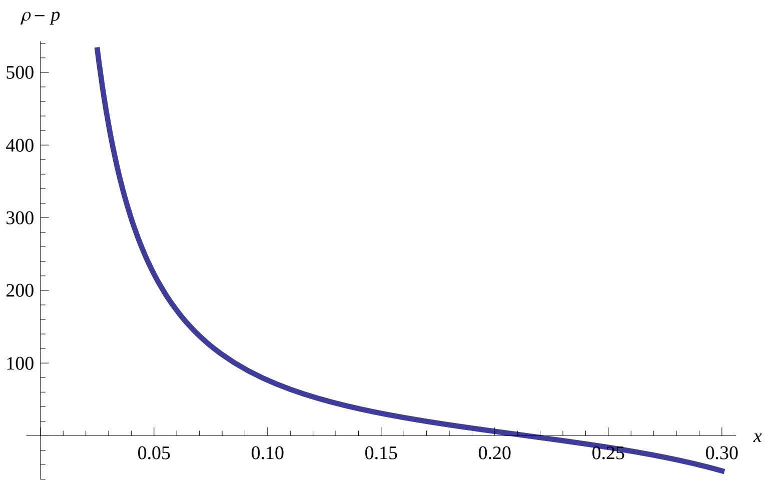

Figure 4.4: Graph of $\rho - p$ versus radial coordinate x

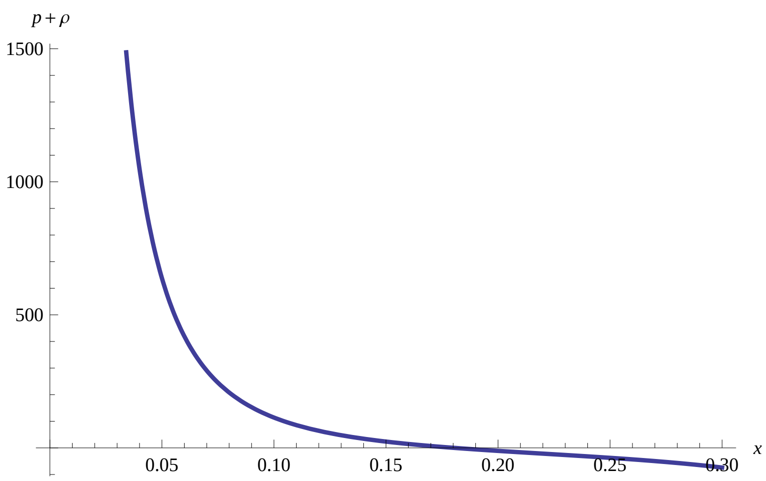

Figure 4.5: Graph of $\rho + p$ versus radial coordinate x

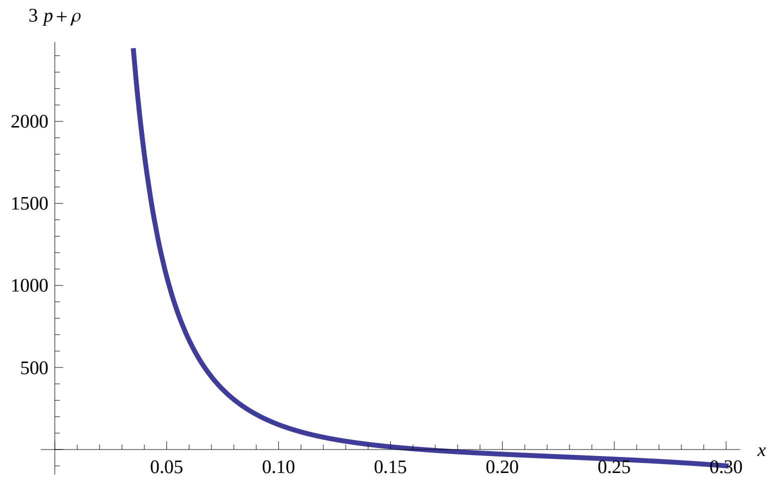

Figure 4.6: Graph of $\rho + 3p$ versus radial coordinate x

4.3 Solutions for Curvature Coordinates

We now investigate Lake's algorithm using curvature coordinates in order to find new exact solutions. We make the prescription

$$\Phi(r) = \log(2 + r^3) \tag{4.27}$$

which is inspired by Lake's own form $\log(1 + r^2)$. Observe that we are able to solve the more general form $\Phi(r) = \log(a + br^3)$ with the aid of Mathematica (Wolfram Research 2010) however we have set $a = 2$ and $b = 1$ to make the solution more lucid. The general forms are lengthy and unwieldy. It is also noteworthy that this general class of solutions does not appear to have been treated previously from an examination of the comprehensive record of Delgaty and Lake (1998) and the smaller collection in Stephani *et al* (2003). Then following the algorithm (3.2) we find that

$$a(r) = -\frac{3}{r + 2r^4} \qquad \text{and} \qquad B(r) = \frac{3r^2}{2 + 4r^3} \tag{4.28}$$

for this choice of Φ.

Now it is possible to find the various dynamical quantities. Using (3.5) the energy density ρ is given by

$$\rho = \frac{3(1+2r+r^4)}{4\pi(1+2r^3)^2} \tag{4.29}$$

while (3.6) allows us to calculate the pressure p as

$$p = -\frac{r(2+3r) + 2(-1+2r^2+r^3)\log(2+r^3)}{8\pi r(1+2r^3)} \tag{4.30}$$

This allows us to compute the adiabatic sound speed index after taking the derivatives of (4.30) and (4.29) and then dividing. We obtain

$$\frac{dp}{d\rho} = -\Big((1+2r^3)(3r^2(-2+2r+8r^2+3r^3+2r^4+4r^5-4r^7)\Big)$$

$$+2\Big((-2-4r^2-21r^3+14r^5-6r^6+8r^8+2r^9)\log 10(2+r^3)\Big) /$$

$$\Big(12r^2(2+r^3)(-1+6r^2+8r^3+2r^6)\Big) \tag{4.31}$$

for the causality index. In order to assess whether the model satisfies the energy conditions the following quantities are relevant:

$$\rho - p = \frac{r(8+15r+4r^3+12r^4) + (-2+4r^2-2r^3+8r^5+4r^6)\log(10(2+r^3))}{8\pi r(1+2r^3)^2} \tag{4.32}$$

$$\rho + p = \frac{r(4+9r-4r^3) + (2-4r^2+2r^3-8r^5-4r^6)\log(10(2+r^3))}{8\pi r(1+2r^3)^2} \tag{4.33}$$

$$\rho + 3p = -\frac{3(r^2(-1+4r^2+4r^3) + (-2+4r^2-2r^3+8r^5+4r^6)\log(10(2+r^3))}{8\pi r(1+2r^3)^2} \tag{4.34}$$

The complete solution of the Einstein field equations is finally expressible as

$$\Phi(r) = \log(2 + r^3) \tag{4.35}$$

$$m(r) = \frac{r^3(2 + 3r)}{2(1 + 2r^3)} \tag{4.36}$$

$$\rho = \frac{3(1 + 2r + r^4)}{4\pi(1 + 2r^3)^2} \tag{4.37}$$

$$p = -\frac{r(2 + 3r) + 2(-1 + 2r^2 + r^3)\log(2 + r^3)}{8\pi r(1 + 2r^3)} \tag{4.38}$$

and the associated line element is given by

$$ds_M^2 = \frac{(1 + 2r^3)dr^2}{(1 - 2r^2 - r^3)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) - (2 + r^3)^2 dt^2. \tag{4.39}$$

Now substituting $m(r)$ and $\Phi(r)$ into $g(r)$ below,

$$g(r) = rm' - 3m + r(3m - r - rm') + r^2(r - 2m)\left((\Phi')^2 + \Phi''\right).$$

we obtain:

$$g(r) = \frac{-2r^2 + 3r^4 - 11r^5 - 12r^6 + 4r^8}{2(1 + 2r^3)} + \frac{r^2(1 - 2r^2 - r^3)}{(1 + 2r^3)(2 + r^3)} \tag{4.40}$$

for(2.36) which is non-zero. This establishes that this model is not conformally flat.

4.4 Solutions for Isotropic Coordinates

We select the form $\Psi(r) = a + br^2$ for the purpose of running Lake's algorithm in isotropic coordinates. This form is not novel and is contained in the solutions of Nariai (1950) and Tolman (1939) – however, it gives a powerful application of the algorithm of Lake. Empirical testing with software packages suggest that we put $a = 1$ and $b = -1$ for physically reasonable models. Accordingly we commence with the form

$$\Psi(r) = 1 - r^2. \tag{4.41}$$

This allows us to obtain the function

$$c(r) = \sqrt{2}r \tag{4.42}$$

and consequently we are able to establish

$$B(r) = 2 - r^2\left(1 - \frac{1}{\sqrt{2}}\right) \tag{4.43}$$

The energy density ρ is given by:

$$\rho = \frac{e^{-4+(2-\sqrt{2})r^2}}{4\pi}\left(6 - 3\sqrt{2} + (2\sqrt{2} - 3)r^2\right) \tag{4.44}$$

with the help of (3.11). The pressure p has the form

$$p = \frac{e^{-4+(2-\sqrt{2})r^2}}{4\pi}\left(-2 + (\sqrt{2} - 1)r^2\right) \tag{4.45}$$

by using (3.12). Observe that while these forms for ρ and p are reasonably simple, it is not possible to write p explicitly in terms of ρ. The condition $p = p(\rho)$ is an equation of state and is highly desirable in constructing models of stars. Unfortunately the form for neither p nor ρ allow for them to be solved in the form $r = f(\rho)$ for some function f as they are not algebraic in r. Nevertheless we may compute the adiabatic sound speed index as

$$\frac{dp}{d\rho} = \frac{3\sqrt{2} - 5 + (3\sqrt{2} - 4)r^2}{-10\sqrt{2} - 15 + (7\sqrt{2} - 10)r^2} \tag{4.46}$$

which must lie between 0 and 1 for a causal fluid. The energy conditions require us to obtain the following quantities which are supposed to be positive for a realistic fluid sphere:

$$\rho - p = \frac{e^{-4+(2-\sqrt{2})r^2}}{4\pi}\left(8 - 3\sqrt{2} + (\sqrt{2} - 2)r^2\right) \tag{4.47}$$

$$\rho + p = \frac{e^{-4+(2-\sqrt{2})r^2}}{4\pi}\left(4 - 2\sqrt{2} + (3\sqrt{2} - 4)r^2\right) \tag{4.48}$$

$$\rho + 3p = \frac{e^{-4+(2-\sqrt{2})r^2}}{4\pi}\left(-3\sqrt{2} + (5\sqrt{2} - 6)r^2\right) \tag{4.49}$$

The complete solution to the Einstein field equations may now be given by

$$\Psi(r) = 1 - r^2 \tag{4.50}$$

$$B(r) = 2 - r^2\left(1 - \frac{1}{\sqrt{2}}\right) \tag{4.51}$$

$$\rho = \frac{e^{-4+(2-\sqrt{2})r^2}}{4\pi}\left(6 - 3\sqrt{2} + (2\sqrt{2} - 3)r^2\right) \tag{4.52}$$

$$p = \frac{e^{-4+(2-\sqrt{2})r^2}}{4\pi}\left(-2 + (\sqrt{2} - 1)r^2\right) \tag{4.53}$$

The line element for this solution has the form

$$ds_M^2 = e^{(4-r^2(2+\sqrt{2}))}(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) - e^{(-2-\sqrt{2}r^2)}dt^2 \qquad (4.54)$$

Now substituting $B(r)$ and $\Psi(r)$ into $f(r)$ below,

$$f(r) = r(\Psi')^2 - 4rB'\Psi' + r\Psi'' - 2rB'' + 4r(B')^2 + 2B' - \Psi' \qquad (4.55)$$

we have

$$f(r) = -4(-3 + 2\sqrt{2})r^3 \qquad (4.56)$$

for the Weyl tensor (2.51). That this function is non-zero suggests that the Weyl tensor does not vanish in general and so this model is not conformally flat.

Chapter 5

Conclusion

In this thesis, we have investigated solution generating algorithms for the Einstein field equations in the case of a static configuration of neutral perfect fluid. The problem of finding exact solutions commenced over a hundred years ago and in 1949 the first algorithm emerged with the work of Wyman (1949). Thereafter the topic lay dormant until the work of Fodor (2000) who revived interest in the method. Prior, to this exact solutions were sought in a purely *ad hoc* fashion with various researchers imposing constraints on the geometry of the fluid or by attempting to impose a functional dependence of the pressure on the energy density - this is called an equation of state. The field equations are under–determined in this instance as they form a system of three partial differential equations in four unknowns. This means that one of the unknowns must be chosen upfront and the remaining ones are to be found on integrating the system of field equations.

Solution generating algorithms offer a systematic approach to solving the system of field equations. The key observation has been that in most analyses, the master field equation arising out of the pressure isotropy condition, has been perceived as a second order ordinary differential equation. Strangely, researchers (aside from those mentioned above) failed to exploit the fact that the master field equation may also be interpreted as a first order differential equation and new possibilities for finding solutions could emerge. This is the line that we have pursued in this work.

After declaring the mathematical formalism in use (chapter 2) we have conducted a review in chapter 3 of the solution generating algorithms from Wyman (1949) to Fodor (2000), Rahman and Visser (2002), Lake (2003), Martin and Visser (2004) and then Boonserm *et al* (2005). It was noted that while these algorithms correctly gave all spherically symmetric perfect fluid models, in practice each of the algorithms suffered from some difficulty. For example some involved integrals of functions appearing under square roots. Hence the problem of finding new more efficient algorithms is still a challenge.

Finally in chapter 4 we presented a new algorithm making use of a coordinate transformation used by Durgapal and Banerjee (1983). One metric potential function must be selected upfront and then the remaining potential has to be found by a single integration. We have demonstrated a new solution by finding a suitable function that allowed the complete integration of the Einstein field equations. Additionally, we have investigated the algorithm of Lake and in each case of curvature coordinates and isotropic coordinates –we were able to construct new exact solutions.

This study reflects the view that while a large volume of literature exists on exact solutions of the Einstein field equations, the area is still vibrant as a research area. The reason for this is that exact solutions in themselves are not important. They are only useful if they can be used to model realistic phenomena. Therefore exact solutions must satisfy certain stringent conditions to succeed as astrophysical or cosmological models. Regrettably, only a very small subset of published solutions display the major requirements. Therefore this area remains a fertile research domain.

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