Real Options: Duopoly dynamics with more than one source of randomness

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by

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Abstract

The valuation of real options has been of interest for some time. Recently, the model has been revised to include more than one source of randomness, e.g. Paxson and Pinto (2005). In this dissertation, we present a model with more than one diffusion process to analyze strategic interaction in a duopolistic framework. We consider a complete market where the profit per unit and the number of units sold are assumed to evolve according to distinct, but possibly correlated, geometric Brownian motions, and aim to extend Paxson and Pinto's research to a wider context by adjusting the model to include the effect of the covariance between the stochastic factors. In particular, we present results in both the pre-emptive and non pre-emptive equilibrium case pertaining to the follower's and leader's value function. We also investigate the consequences for the model in relation to traditional net present value theory, and include an analysis of the comparative static relationships that exist between the parameters. We then conclude with a chapter that extends our two-variable model to three sources of randomness - first by allowing the investment cost to be modelled as a random once-off payment, and then by considering it to be a stochastically variable ongoing cost.

Keywords

Real options, complete markets, more than one stochastic process, competitive games, duopoly

Preface

The work described in this dissertation was carried out in the School of Statistics and Actuarial Science at the University of KwaZulu-Natal, under the supervision of Prof. J.G. O'Hara.

Theses studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

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Natalie MacKenzie
December 2009

As the candidate's supervisor I have approved this dissertation for submission.

Signed: ________________ Name:________________ Date:________________

Declaration

I, Natalie MacKenzie, declare that

1. The research reported in this dissertation, except where otherwise indicated, is my original research.

2. The dissertation has not been submitted for any degree or examination at any other university.

3. This dissertation does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

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Lastly, I would like to thank my family and friends for their encouragement and support.

Chapter 1

An Introduction to Real Options

Investment opportunities may also be described as real options - the valuation of which has been of interest for some time. A real option represents the right, but not the obligation, to undertake some business decision - typically the option to purchase an asset at some future time of choice. This asset may be tangible, or it may merely relate to obtaining a position within a new market. However, in contrast to financial options, real options are not tradable. In particular, a firm cannot sell its individual right to enter a market to another party. Only the firm itself can make this decision.

Real options are thus defined in the literal sense of "choice", and their valuation relies heavily upon economic theory. The aim of this first chapter is to therefore provide an overview of this theory, in order to lay the foundation for further development later in our dissertation. We shall also introduce the various techniques that have been used for the valuation of investment opportunities - both past and present - in order to highlight the contribution of real-option research to the advances that have been made in investment analysis.

In Chapter 2, we present a real-option model with two sources of randomness to analyze strategic interaction in a pre-emptive duopolistic framework, and then examine the results for cases of co-operative equilibria in Chapter 3. We investigate the consequences for each of these models in relation to traditional net present value theory, and include an analysis of the comparative static relationships that exist between the parameters in Chapter 4. We then conclude with a chapter that extends our two-variable model for the follower to three sources of randomness.

1.1 An Overview of Economic Theory

1.1.1 Perfect Competition

In economic theory, perfect competition arises when the individual market participants (i.e. buyers and sellers) are unable to influence the price of the good or service concerned. Individual firms have no market power and are therefore price-takers, given that the market-clearing price is determined by the interaction of demand and supply. This means that they have to accept the price as given, and can only decide what quantities to supply or demand at that price.

In (7), Fourie *et al* state that perfect competition exists only when each of the following conditions are met:

- There must be a large number of buyers and sellers of the product - so large that no individual participant can affect the market price.
- Each seller must act independently so that there is no collusion between them.
- All the goods sold in the market must be homogenous (or identical) so that there is no reason for buyers to prefer the product of one seller over another.
- There must be complete freedom of entry in to and exit out of the market.
- All buyers and sellers must have perfect knowledge of market conditions so that if one seller raises its price above the ruling market price, it is assumed that all buyers will know that the other sellers are charging less and will not purchase any goods from the firm in question.
- There must be no government intervention.
- All factors of production (e.g. labour) must be perfectly mobile and free to move from one market to another.

These requirements are clearly very restrictive, so it is hardly surprising that approximations to these characteristics are only found in a small percentage of markets, such as agriculture. Nevertheless, it is important for us to provide an overview of perfect competition, as it serves as a standard or norm against which we can now compare the functioning of all other markets[1].

1.1.2 Imperfect Competition

Imperfect competition refers to a situation in which at least one of the conditions for perfect competition is not satisfied. Within such a market, an individual participant can be affected by the actions of other individual players and, depending on the form of imperfect competition that prevails, each firm has a

[1] We must emphasize that perfect competition is not necessarily the most desirable form of competition - it is simply the highest or most complete degree of competition.

varying degree of control over the price of their product (but never absolute). Imperfectly competitive firms are therefore price-makers.

In this dissertation, we will be examining real options within the context of imperfectly competitive markets. In particular, intermediate market structures will be of interest in later chapters. We shall therefore proceed with a detailed introduction to the various imperfect market types, based once again on the literature of Fourie *et al* in (7).

Monopoly

The word "monopoly" is derived from the Greek words *monos*, meaning "single", and *polein*, meaning "sell". In its pure form, monopoly is a market structure in which there is only one seller of a particular good or service that has no close substitutes, with the further defining feature that entry into the market is completely blocked to other aspiring competitors. Monopoly is therefore at the opposite extreme to perfect competition.

Given that such an industry only consists of a single firm, market demand is actually just the demand for the good or service of the monopolist. The monopolist is thus able to fix the price at which it offers its product, but thereafter will only be able to sell an additional quantity of output if it lowers this price. Hence, it is indeed possible for a monopolist to make a loss, as the quantity sold by the firm remains constrained by the market demand for the product that it vendors, despite the lack of competition.

However, pure monopoly is a relatively rare occurrence. Although there may often be only one seller of a particular good or service in the market, that product is likely to have substitutes. Most "monopolies" are therefore actually near-monopolies. Furthermore, a monopoly may also be virtual if the geographic location of a particular vendor is simply isolated from the competition. Hence, we must emphasize that a single firm can only be classified as a monopolist if entry into the market is blocked.

Most markets exhibit elements of competition and monopoly. For this reason, it has been necessary for us to explore the theories of both of these extreme market forms, so that we can now develop a better understanding of how the majority of intermediate markets operate.

Oligopoly

Oligopoly refers to another situation of imperfect competition, and is the most prevalent market form in any modern economy. The word oligopoly is derived from the Greek words *oligoi*, meaning "few", and *polein* which, as we have already explained, means "sell". Hence, an oligopoly comprises a small number of large firms[2], each with varying ability to influence the market price of their product. If all of the firms sell identical products, then the oligopoly is said to be pure or homogeneous, as opposed to a differentiated oligopoly in which firms produce goods that are only similar[3].

The firms within an oligopoly may act independently, or they may explicitly agree to co-ordinate their activities in an attempt to limit competition within the market. This may include agreements to restrict output, set prices or share the market. Consequently, an oligopolistic industry is usually marked by substantial barriers to entry[4]. Such collusive behaviour is unique to an oligopoly, but is only successful if the colluding group - or cartel - can enforce agreements.

Despite cartel agreements, intense competition remains a defining feature of oligopoly, but it is usually non-price competition. Unlike the extreme market structures of perfect competition and monopoly, there is a high degree of interdependence between or among the activities of imperfectly competitive firms. The need to anticipate the subsequent effects of rival actions complicates the profit maximization decision for individual firms, and it is this strategic interaction that we will later be addressing within the context of a duopolistic framework.

From henceforth, we shall therefore only be considering duopolistic markets in which each firm is concerned with earning the largest possible profit or payoff at the expense of its rival. It is thus important for us to next spend some time examining specific models of oligopoly (or duopoly) that have been developed through the implementation of game theory to analyze the decisions of firms that are engaging in strategic conflict.

1.1.3 The Nash Equilibrium

When either perfect competition or monopoly prevails, the market is said to be in equilibrium if no firm has any desire to change its output level, given what everyone else is doing. Based on the account given in (27), the Nobel prize-winning economist and mathematician John Nash (1951) similarly defined an equilibrium for an oligopolistic market in the following formal manner: *"A set of strategies is a Nash equilibrium if, holding the strategies of all players (firms) constant, no player (firm) can obtain a higher payoff (profit) by choosing a different strategy."*

In such a Nash equilibrium, there is no desire for any firm to alter its strategy, because each firm is simultaneously applying its best response. This means that each firm is either setting the price or quantity that maximizes its profit,

[2] A duopoly is a special form of oligopoly in which there are only two firms operating within the market.

[3] Economists also consider products to be heterogeneous if consumers merely perceive that they differ.

[4] These barriers may also take the form of government licenses or patents.

given its beliefs about rival strategies. Now there are many different theories of the pricing and output decisions of firms under oligopoly, each based on varying assumptions about the reactions of rivals to the choices of the firm under consideration. We shall provide an outline of only the most plausible models, pertaining to a duopoly in a single-period game[5].

Within this framework, the opposing firms engage in a non co-operative game of imperfect information. The failure to collude in a single-period game arises from a lack of trust in the absence of a binding agreement. If the firms engage in the game only once, there is a substantial profit incentive for each to cheat on any cartel agreement. However, if the same game is played repeatedly, the strategy that either firm devises for a particular period will depend on the actions of its rival in previous periods. Each firm can therefore influence its rival's behaviour in a multi-period game by threatening to punish them. However, it will still be difficult to maintain the cartel if the game has a known stopping point.

Collusion is therefore more likely to feature in game that will continue forever, or that will end at an indefinite time. Although situations do occur in which the roles of the players or firms in a single-period game are exogenously pre-assigned, we shall nevertheless focus our attention on models of non co-operative oligopoly (or duopoly).

The Cournot Model of Non Co-operative Oligopoly

Antoine Augustin Cournot introduced the first formal model of oligopoly in 1838. To simplify our discussion of this model, we shall impose the following assumptions:

- The market comprises a duopoly - no other firm can enter.
- Both firms sell homogeneous products.
- The game unfolds in a single period - the product or service cannot be stored and then sold later.

The Cournot (Nash) equilibrium is defined in (27) as "*a set of quantities sold by firms such that, holding the quantities of all other firms constant, no firm can obtain a higher profit by choosing a different quantity.*" This equilibrium is the only plausible Nash outcome when both firms set their output levels simultaneously and independently, and allow the market to determine the price thereafter. The quantity produced by each firm directly affects the profit of its rival, because the market price depends on total output. Thus, when selecting its strategy, each firm must anticipate the behaviour of its competitor.

If each firm correctly predicts the output choice of the other, both will be maximizing their profit, and neither will have any incentive to alter their level of

[5]For more detail on the complexities of game-theoretic models, refer to Fudenberg and Tirole in (8).

production. However, if either firm incorrectly anticipates its rival's behaviour and is not producing its best-response quantity, it will change its output to increase its profit. This may subsequently induce its rival to alter production. These adjustments will continue until the firms are producing the Cournot levels of output, and equilibrium is established. Any other set of quantities is not a Cournot equilibrium.

The Stackelberg Model of Non Co-operative Oligopoly

In contrast to the Cournot model which is based upon simultaneous action, a sequential decision-making game arises naturally if one firm enters the market before another. The first entrant, called the leader, is able to set its output before its rival, the follower. In order to determine the equilibrium levels of production in such a single-period game, Heinrich von Stackelberg modified the Cournot model in 1934 for this highly probable economic scenario.

Given that the follower will have upfront knowledge of the leader's chosen quantity before entering the market itself, the leader realizes that the follower will implement its Cournot best-response strategy to the leader's chosen level of production. The leader is therefore able to predict the follower's reaction prior to their entry, and hence manipulates this knowledge to benefit at the follower's expense.

The Stackelberg (Nash) equilibrium thus deems that the follower will always adopt a strategy that maximizes its profit, so that it does not have any desire to alter its behaviour. Similarly, given that the leader is aware of how the follower will respond to each possible output level that it chooses, the leader will always ensure that it cannot earn a higher profit than selling at its current level of production.

The quantity sold and profit earned by the Stackelberg follower are both less than the corresponding levels in the Cournot equilibrium. The opposite is true for the Stackelberg leader, as this firm enjoys the benefits of a monopoly, prior to the entry of the follower. Total Stackelberg output is greater than overall Cournot production, resulting in a lower market price. The total profit earned in the Stackelberg model is also less than the corresponding Cournot level, as the follower is far worse off than if it enters simultaneously with the leader.

The Bertrand Price-setting Model

In 1883, Joseph Bertrand argued that oligopolistic firms set prices rather than quantities, and thereafter allow consumer demand to determine production. Returning to the context of simultaneous decision-making, the resulting Nash equilibrium is called a Bertrand equilibrium and is defined in (27) as "*a set of prices such that no firm can obtain a higher profit by choosing a different price, providing that all other firms continue to charge these prices.*"

The Bertrand equilibrium price and quantity differ from those in the Cournot model, and are highly dependent on whether the oligopoly is pure or differentiated. To illustrate this, we shall first consider a price-setting duopoly in which the two firms produce homogeneous goods and are therefore subject to identical costs. If either one of these firms is undercut by its rival, the rival firm will capture the entire market because it will be selling identical products at a lower price. In turn, the firm's best response will be to undercut its opposition.

Theoretically, the competing firms will continue to undercut one another until the price drops to the level of marginal cost. If the price is further undercut, the respective firm will monopolize the market, but it will make a loss on each unit of output sold. The firms will thus choose to split the market instead, so that equilibrium will be established with both firms charging only the marginal cost, and earning zero profit.

Thus, firms that produce homogeneous goods can be expected to set quantities rather than prices, rendering the Cournot model more plausible for our study of pure duopoly when we consider simultaneous market entry later in the dissertation. However, markets with differentiated goods are extremely common, as is price-setting by firms. For completeness, we will therefore briefly explain the usefulness of the Bertrand model within this context.

If the price charged by one of the firms in a differentiated duopoly falls slightly relative to that of the other, not all consumers will choose to purchase the cheaper alternative. If there is at least a perceived difference in the goods, consumer preference for the more expensive brand will sustain a certain level of demand[6]. Thus, neither firm will have to match a price cut by its rival exactly, making it possible for larger profits to be earned. The resulting Nash equilibrium dictates that each firm will set its best response, given the price that its rival is charging. Neither firm will desire to alter its price because neither firm can increase its profit by so doing.

Now that we are armed with a general understanding of the economic theory of investment and market behaviour, we shall next introduce the various techniques that are used for the valuation of investment opportunities.

1.2 The Valuation of Investment Opportunities

In (2), Dixit and Pindyck define an investment as "*the act of incurring an immediate cost in the expectation of future rewards.*" In reality, such an act is either partially or completely irreversible. The initial cost is sunk and cannot be entirely recovered in the event of withdrawal from the investment. Uncertainty also prevails over the value of future rewards. It is therefore necessary to assess the probabilities of all alternative outcomes before committing to a decision to invest. In some instances, it may be possible to postpone the decision until

[6]The less substitutable are the products, the greater will be the sustained demand.

further information is revealed about future economic conditions, but there will never be complete certainty.

These features of investment interact to determine the optimal choices of investors and, in particular, of those contemplating entry into a new market. However, the traditional (or Marshallian) method for the analysis of investment decisions is based on the idea of net present value. All future cash flows are assumed to be known so that there is zero uncertainty. In brief, the model dictates that a firm will invest immediately if the discounted value of future earnings exceeds the present value of future costs. This orthodox approach clearly ignores the true characteristics of investment, which may explain the observed deviation of firm behaviour from the theory.

However, before we examine alternative valuation methods, it will be useful for us to obtain a better understanding of the ways in which irreversibility, uncertainty and timing affect the decisions of investors.

1.2.1 The Characteristics of Investment

Firms obtain investment opportunities from a variety of sources, including patents, the ownership of land or other natural resources, managerial expertise, technological knowledge, reputation, or their position within an existing market. The factors favouring investment are usually highly dependent on market conditions, and there is thus no guarantee that they will last indefinitely. This phenomenon is referred to as economic hysteresis, and is explained in (2) as "*the failure of investment decisions to reverse themselves when the underlying causes are fully reversed.*"

Typically, investment expenditures are sunk costs when they pertain to a particular firm or industry. If we consider the steel industry, the specificity of the plant and equipment renders it useless in any other market. Thus, if the entire steel industry experiences an economic slump, the resale value of any particular plant is low, and the irreversibility is large.

Irreversibility thus creates an opportunity cost to investing, if the future value of the project is uncertain. This opportunity cost is highly sensitive to changing economic conditions which affect the perceived risk of future cash flows and hence, have a large impact on investment spending. To incorporate such an opportunity cost into the valuation of an investment project, the ability to invest in the future is required as an alternative to investing immediately. This creates an option to wait.

However, it is not always possible for firms to delay investment, particularly when strategic consideration is involved. In the Stackelberg equilibrium, there is a significant cost to delaying entry into the market as the leader gains a competitive advantage over the follower. This cost not only takes the form of foregone cash flows, but also the risk of being pre-empted by the rival firm. The cost of delaying entry must therefore be weighed against the benefit of waiting. The less leeway there is with the timing of an investment, and the greater the cost of delaying, the less will irreversibility affect the investment decision.

Much emphasis has been placed on the important roles that time, future uncertainty and irreversibility play in the analysis of investment opportunities. If possible, any investor should include the option of postponement in their array of choices. The mathematical techniques that are implemented to model investment decisions must therefore be capable of incorporating all of these considerations.

Dynamic programming is a very general optimization tool that is particularly useful for the treatment of the uncertainty that is inherent in dynamic decisions and, up until recent times, has been used as the main method of valuation for investment opportunities. The technique essentially involves a systematic comparison of the present values that result from immediate investment, to those that result from delay - the details of which we shall now present in the following section.

1.2.2 Dynamic Programming

Although firms make, implement and revise investment decisions continuously through time, we shall initially develop the theory of dynamic programming in a discrete framework, before extending it to a continuous setting. Thus, we shall first model state-variable uncertainty using a discrete-time Markov process, or random walk. As defined by Hull in (14), "*a Markov process is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variable and the way that the present has emerged from the past are irrelevant.*"

Of eventual interest will be the application of the theory to situations where the state-variable uncertainty takes the form of a Wiener process, or more general diffusion process. Such continuous-time stochastic processes are also Markov in nature, and can therefore be derived from limiting random walks. To express these processes more formally using the definitions in (14), a variable z follows a Wiener process [or Brownian motion] if it has the following two properties:

1. The change Δz during a small period of time Δt is

   $$\Delta z = \epsilon\sqrt{\Delta t}$$

   where ϵ has a standardized $N(0,1)$ distribution. Thus,

   $$\xi(\Delta z) = 0$$

   and

   $$Var(\Delta z) = \Delta t,$$

   where ξ denotes an expectation operator.

2. The values of Δz for any two different short intervals of time, Δt, are independent.

As the small changes become closer to zero, then we refer to the Wiener process as dz, meaning that it has the properties for Δz given above in the limit as

$$\Delta t \to 0.$$

We can now define a generalized diffusion process for a variable Y in terms of dz:

$$dY = \mu(Y)dt + \sigma(Y)dz$$

where the parameters μ and σ are functions of the value of the underlying variable Y.

A Two-period Example

In this section, we shall follow the two-period example of Dixit and Pindyck in (2) to first develop the theory of dynamic programming in a discrete framework. We shall let K denote the sunk cost of an investment that promises to produce one unit of output per period in perpetuity, we shall let $\alpha > 0$ be the interest rate, and we shall suppose that the current (period 0) price at which each unit of output is sold is P_0. In the next period, this price will change to either

$$(1 + a)P_0$$

with probability q, or to

$$(1 - b)P_0$$

with probability $(1 - q)$. The price will remain constant thereafter.

If we consider the situation in which a firm contemplating the above investment is forced to decide on a now-or-never basis, the expected present value of the revenues that the firm will earn if it chooses to invest is given by

$$\begin{aligned}
EPV_0 &= P_0 + [q(1+a)P_0 + (1-q)(1-b)P_0]\left[\frac{1}{1+\alpha} + \frac{1}{(1+\alpha)^2} + ...\right] \\
&= P_0 + P_0[1 + q(a+b) - b]\left[\frac{\frac{1}{1+\alpha}}{1 - \frac{1}{1+\alpha}}\right] \\
&= P_0 \frac{[1 + \alpha + q(a+b) - b]}{\alpha}.
\end{aligned}$$

Now, the firm will only proceed with the investment if

$$EPV_0 > K.$$

If

$$EPV_0 = K,$$

then the firm will display indifference towards the decision, as it will receive zero whether it chooses to invest or not. The termination value at the current time, Ω_0, denotes the net payoff from investing when is not possible to delay the decision for any length of time. Thus,

$$\Omega_0 = \max[EPV_0 - K, 0].$$

If the investment opportunity remains available in the future, the firm's decision becomes complicated by the option of postponement. The price will not change beyond the next period, so there is no point in delaying any profitable projects further than this.

Rather than investing immediately, we shall suppose that the firm chooses to wait. This allows for the opportunity to be re-assessed once the period 1 price, P_1, has been observed. Discounting back to period 1, the present value of the stream of revenues that the firm will receive if it invests at this point is given by

$$PV_1 = P_1 + \frac{P_1}{(1+\alpha)} + \frac{P_1}{(1+\alpha)^2} + ... = P_1 \frac{(1+\alpha)}{\alpha}.$$

Again, for each of the possible prices

$$P_1 \quad = \quad \begin{cases} (1+a)P_0 & \text{with probability } q \\ (1-b)P_0 & \text{with probability } 1-q \end{cases},$$

the firm will invest only if

$$PV_1 > K.$$

The net payoff that is realized as a consequence of such future optimal decisions is therefore defined by

$$F_1 = \max[PV_1 - K, 0],$$

where F_1 is referred to as the continuation value.

Now the price P_1, and hence the values PV_1 and $\mathcal{F}_1$, are all random variables from the perspective of period 0. Thus, the expected continuation value with respect to current and available information must be of the form

$$\xi_0(\mathcal{F}_1) = q \max\left[(1+a)P_0\frac{(1+\alpha)}{\alpha} - K, 0\right] + (1-q)\max\left[(1-b)P_0\frac{(1+\alpha)}{\alpha} - K, 0\right].$$

If we return to the decision at period 0, the firm is faced with two choices. Immediate investment will earn the net payoff

$$EPV_0 - K,$$

whilst postponement will yield the continuation value

$$\xi_0(\mathcal{F}_1)$$

derived above. This expectation pertains to period 1 however, and must therefore be discounted to the current time. The firm's rational choice is obviously the action that yields the larger value. Hence, we may reduce the optimal deployment of the entire investment opportunity to the assessment of the following net present value:

$$\mathcal{F}_0 = \max\left\{EPV_0 - K, \frac{1}{1+\alpha}\xi_0(\mathcal{F}_1)\right\}.$$

To effect this calculation, it is necessary for us to work in reverse. At the last relevant decision point, period 1, we must identify the best choice by comparing the net payoff from investing to that if the firm never invests at all. If it is not profitable for the firm to invest in period 1, then the firm should never invest, as the conditions are to remain constant from period 1 onwards.

Having determined the continuation value $\mathcal{F}_1$, it is then possible for us to calculate its discounted expectation at the next prior decision point, period 0. A comparison of this value to the proceeds that would be earned from immediate investment yields the optimal choice for the firm at the current time. The firm will then know whether to capitalize on the opportunity, or whether to delay the decision for one period.

There are many factors that affect the value of the option to postpone

$$\mathcal{F}_0 - \Omega_0.$$

By delaying investment, the firm foregoes the period 0 revenue, P_0, which increases the desirability of immediate action. However, postponing the decision also postpones the payment of the investment cost K. This favours waiting since the interest rate is positive. More importantly, waiting will allow the firm a separate optimization for each possible outcome of the price P_1, whereas immediate action can be based only on the expected price.

This contingent ability adds value to the freedom of delay. The magnitude of the difference between the values F_0 and Ω_0 will vary with fluctuating conditions, and depends largely on the nature of the investment decision. When the value of the full opportunity, F_0, equals the termination value, Ω_0, immediate investment becomes optimal.

In an environment where conditions continue to change, postponement beyond period 1 may be favourable. When there are many periods involved, the optimization demands repeated application of the same procedure. This captures the essence of dynamic programming: an entire sequence of decisions is reduced to an immediate choice, whilst the effects of the remaining decisions are summarized in the continuation value.

Generalizing the Theory

Now that we have covered the basic building blocks of dynamic programming, we must next extend the theory to more generalized investment decisions. We shall assume the existence of a firm that possesses the opportunity to expand its operations, with the status of this hypothetical firm being described by the state variable Y. The evolution of this variable assumes a Markov process and, at any time t, the current value Y_t is known, whilst future values $Y_{t+1}, Y_{t+2}, \ldots$ are random variables.

The choices available to the firm in each period shall be represented by the control variable(s) c. These decisions may relate to any aspect of the firm's operation, including the quantity of labour hired, raw materials purchased, or even the scale of investment. If the firm is simply contemplating whether to invest or wait, we shall assign c as a scalar binary variable with the value 0 indicating delay, and the value 1 signalling immediate action.

The chosen value of the control at time t must depend only on the information summarized in the current state Y_t. The state and the control thus determine the firm's immediate profit flow

$$\pi_t(Y_t, c_t),$$

and thereby influence the probability distribution of future states. We shall let

$$\Psi_t(Y_{t+1}|Y_t, c_t)$$

denote the cumulative probability distribution function of the state in the next period, conditional upon the current information, and the discount factor between any two periods shall be

$$\frac{1}{1+\omega},$$

where ω is the rate of return such that the firm aims to choose the sequence of controls c_t that will maximize the expected net present value of its revenues.

In order for us to apply the dynamic programming technique in a fashion similar to the two-period example, we must also let $\digamma_t(Y_t)$ represent the expected net present value of all cash flows resulting from optimal decision-making. Once the firm chooses the control variables c_t, it will earn an immediate profit flow $\pi_t(Y_t, c_t)$. At the next period $(t+1)$, the status of the firm will be Y_{t+1}, and optimal decisions thereafter will yield $\digamma_{t+1}(Y_{t+1})$. This value is random from the perspective of period t, so the firm must consider its discounted expectation. The continuation value is calculated as

$$\xi_t[\digamma_{t+1}(Y_{t+1})] = \int \digamma_{t+1}(Y_{t+1}) d\Psi_t(Y_{t+1}|Y_t, c_t),$$

where the range of integration is that over which Y_{t+1} is distributed. The firm will thus choose c_t to maximize the sum of its immediate payoff and the discounted continuation value. This will just result in the value $\digamma_t(Y_t)$, thus giving rise to the following Bellman equation, or fundamental equation of optimality:

$$\digamma_t(Y_t) = \max_{c_t} \left\{ \pi_t(Y_t, c_t) + \frac{1}{1+\omega} \xi_t[\digamma_{t+1}(Y_{t+1})] \right\}. \qquad (1.2.2.1)$$

In (2), Bellman's Principle of Optimality states this decomposition formally: "*An optimal policy has the property that, whatever the initial action, the remaining choices constitute an optimal policy with respect to the subproblem starting at the state that results from the initial actions.*"

If the decision process has a fixed finite time horizon T, the firm will receive a final termination payoff $\Omega_T(Y_T)$, and we may then implement the same backward calculation to solve the maximization problem. This procedure will thus begin with the solution of

$$\digamma_{T-1}(Y_{T-1}) = \max_{c_{T-1}} \left\{ \pi_{T-1}(Y_{T-1}, c_{T-1}) + \frac{1}{1+\omega} \xi_{T-1}[\Omega_T(Y_T)] \right\}.$$

Of greater interest to us, however, is the valuation of an investment opportunity that remains available in perpetuity.

The Case of Infinite Horizon

An infinitely repeating decision process actually simplifies the Bellman equation (1.2.2.1) by removing its dependence on time. The current state Y_t is obviously still of significance, but the calendar date t by itself becomes irrelevant.

Provided π, Ψ and ω are independent of time, the value function is thus common to all periods, except that it is evaluated at different starting states Y_t. The Bellman equation for the recursive dynamic programming problem therefore simplifies to

$$F(Y) = \max_c \left\{ \pi(Y, c) + \frac{1}{1+\omega} \xi[F(Y^+)|Y, c] \right\},$$

where Y and Y^+ represent any two possible consecutive states. In the absence of a final decision point, however, there is no known termination payoff from which we can work backward. Without knowing the function F, it is impossible for us to determine the optimal control c by solving the maximization problem.

Nevertheless, the recursive Bellman equation fortunately has a very special structure that facilitates the proof of the existence and uniqueness of a solution function $F(Y)$, under conditions that are typical of the economic applications that will follow later in our dissertation. A brief explanation of this proof is given in (2), but the full technical argument is provided by Lucas *et al* in (20).

Continuous Time

We are now in a position to extend our dynamic programming analysis of investment decisions to a continuous framework. Suppose that each time period is of length Δt. Of interest is the limit where Δt tends to zero, and time is continuous. The function

$$\pi(Y, c, t)$$

shall now represent the *rate* of profit flow, so that the actual profit earned over the time period Δt is

$$\pi(Y, c, t)\Delta t.$$

Similarly, we shall once again let ω be the rate of return per unit time, so that the discount factor for an interval of length Δt is given by

$$\frac{1}{1+\omega\Delta t}.$$

The Bellman equation (1.2.2.1) therefore becomes

$$F(Y, t) = \max_c \left\{ \pi(Y, c, t)\Delta t + \frac{1}{1+\omega\Delta t} \xi[F(Y,^+ t + \Delta t)|Y, c] \right\},$$

and if we multiply by $(1 + \omega\Delta t)$ and rearrange terms, we get

$$\begin{aligned} \omega\Delta t F(Y, t) &= \max_c \left\{ \pi(Y, c, t)\Delta t(1 + \omega\Delta t) + \xi[F(Y,^+ t + \Delta t) - F(Y, t)|Y, c] \right\} \\ &= \max_c \left\{ \pi(Y, c, t)\Delta t(1 + \omega\Delta t) + \xi[\Delta F] \right\}. \end{aligned}$$

Dividing through by Δt and letting

$$\Delta t \to 0$$

finally gives

$$\omega F(Y,t) = \max_c \left\{ \pi(Y,c,t) + \frac{1}{dt}\xi[dF(Y,t)] \right\}, \tag{1.2.2.2}$$

where $\frac{1}{dt}\xi[dF(Y,t)]$ is the limit of $\frac{1}{\Delta t}\xi[\Delta F]$.

This form of the Bellman equation conveys the entitlement to the profit flow as an explicit asset, with a value $F(Y,t)$. The left-hand side may be interpreted as the normal return per unit time that a decision-maker would require for holding the asset, while the right-hand side represents the total expected return per unit time. The first term is the immediate dividend from the asset, and the second term is the asset's expected rate of capital gain (loss if negative). The equality thus constitutes a no-arbitrage or equilibrium condition which expresses the investor's willingness to hold the asset.

The above analysis is local to the short time interval $(t, t+dt)$, and the resulting equation is maintained for any t. The derivation may be completed by choosing a finite time horizon and imposing a terminal payoff, or by specifying a perpetual decision process and implementing a recursive structure.

Under either circumstance, it is difficult for us to prove the existence and uniqueness of a solution in continuous time. Only two classes of continuous stochastic processes, namely the Itô and Poisson processes, permit a solution of the function $F(Y,t)$. However, we shall omit the rigourous mathematics of Fleming and Rishel in (6), and of Krylov in (17).

Itô Processes

The Itô process is of great significance for many economic applications, as it yields a simple form for equation (1.2.2.2). If we allow the state variable Y to assume such a stochastic process, then the drift and diffusion parameters will depend on the control variable c and time t, as well as the state variable itself. An Itô process for Y can therefore be written algebraically as

$$dY = \mu(Y,c,t)dt + \sigma(Y,c,t)dz,$$

where the uncertainty is modelled as the increment of a standard Wiener process dz. Now if we suppose that Y is the observed state at time t, with

$$Y^{+} = Y + dY$$

denoting the random position at the end of the small interval dt, then Itô's Lemma[7] shows that the value function F of Y and t also follows an Itô process of the form

$$\begin{aligned} dF(Y,t) &= F_t(Y,t)dt + F_Y(Y,t)dY + \frac{1}{2}F_{YY}(Y,t)(dY)^2 \\ &= \left\{ F_t(Y,t) + \mu(Y,c,t)F_Y(Y,t) + \frac{1}{2}\sigma^2(Y,c,t)F_{YY}(Y,t) \right\} dt \\ &\quad +\sigma(Y,c,t)F_Y(Y,t)dz. \end{aligned}$$

We hence get that

$$\xi[dF(Y,t)] = \left\{ F_t(Y,t) + \mu(Y,c,t)F_Y(Y,t) + \frac{1}{2}\sigma^2(Y,c,t)F_{YY}(Y,t) \right\} dt,$$

since $\xi(dz) = 0$. The "return equilibrium" condition (1.2.2.2) thus becomes

$$\omega F(Y,t) = \max_c \left\{ \pi(Y,c,t) + F_t(Y,t) + \mu(Y,c,t)F_Y(Y,t) + \frac{1}{2}\sigma^2(Y,c,t)F_{YY}(Y,t) \right\}.$$

Furthermore, if the horizon is infinite and the functions π, μ and σ do not depend explicitly on time, then neither does the value function. Thus

$$F_t = 0,$$

and the return equation reduces to an ordinary differential equation, with Y as its only independent variable:

$$\omega F(Y) = \max_c \left\{ \pi(Y,c) + \mu(Y,c)F_Y(Y) + \frac{1}{2}\sigma^2(Y,c)F_{YY}(Y) \right\}.$$

We will develop appropriate solution methods when the need to formulate such equations arises later in our application of this theory. Of current importance is a special control that is particularly relevant for the analysis of investment opportunities - namely, the optimal stopping of an Itô process.

[7] Itô's Lemma is an important result in the study of the behaviour of functions of stochastic variables that was discovered by the mathematician K. Itô in 1951. A completely rigorous proof of Itô's Lemma is given in (15).

Optimal Stopping

This class of control limits dynamic programming to a binary decision problem. At every instant, one alternative corresponds to stopping the process to receive the termination payoff $\Omega(Y, t)$, whilst the other entails continuation of the current situation to earn a profit flow $\pi(Y, t)$. Both the profit flow and the termination payoff can depend on the state variable Y and on time t, where Y follows an Itô process of the form

$$dY = \mu(Y, t)dt + \sigma(Y, t)dz. \tag{1.2.2.3}$$

Our consideration of an infinite horizon will obviously induce independence from time in later chapters, but we shall first develop the theory within a general framework.

A natural example of an optimal stopping problem pertains to a firm contemplating shutdown in bad economic conditions. The decision to cease operation will yield a termination payoff equal to the scrap value of equipment, minus any severance payments that the firm is required to make. If the firm chooses to continue operation, a profit flow (positive or negative) will be generated. The Bellman equation for such an optimal stopping problem is thus given by

$$\mathcal{F}(Y, t) = \max\left\{\Omega(Y, t), \pi(Y, t) + \frac{1}{1 + \omega dt}\xi[\mathcal{F}(Y + dY, t + dt)|Y]\right\}. \tag{1.2.2.4}$$

An investment opportunity to enter a new market places the optimal control problem within a differing context that has the utmost relevance to this dissertation. Continuation refers to the decision to wait - so that the profit flow is zero - whilst stopping corresponds to investing. The termination payoff is just the expected present value of future profits, net of the investment cost.

Returning to a non-specific framework, the intervals of Y values that render termination optimal could alternate with ranges in which the right-hand side of equation (1.2.2.4) is maximized through continuation. However, it is possible to impose certain conditions on the profit flow and termination payoff functions to ensure the existence of a single cutoff $Y^*(t)$. Dixit and Pindyck offer a formal explanation in (2), but most economic applications are naturally structured with a well-defined division, and we will simply assume in later chapters that the necessary conditions are satisfied.

Thus, for each t, there will be a unique critical value $Y^*(t)$ rendering continuation optimal if Y_t lies on one side of $Y^*(t)$, and termination optimal if it lies on the other side. The critical values $Y^*(t)$ for various t will therefore form a curve that divides the (Y, t) space into two regions. Depending on the nature of the optimal stopping problem, continuation will be optimal above (below) the curve, and termination will be optimal below (above) it. The equation of the curve

$$Y = Y^*(t)$$

is referred to as an unknown free boundary, and will need to be determined as part of the solution to the dynamic programming problem.

In the continuation region, the second term on the right-hand side of equation (1.2.2.4) is the larger of the two. Thus,

$$F(Y,t) = \pi(Y,t)dt + \frac{1}{1+\omega dt}\xi\left[F(Y+dY,t+dt)|Y\right].$$

If we expand this by Itô's Lemma and simplify as before, the value function will be seen to satisfy the following partial differential equation:

$$\frac{1}{2}\sigma^2(Y,t)F_{YY}(Y,t) + \mu(Y,t)F_Y(Y,t) + F_t(Y,t) - \omega F(Y,t) + \pi(Y,t) = 0, \quad \text{(1.2.2.5)}$$

which holds for

$$Y < Y^*(t)$$

when specifically analyzing an investment opportunity to enter a new market. An initial condition for this equation is given by

$$F(0,t) = 0, \quad \text{(1.2.2.6)}$$

which results from the properties of a diffusion process[8]. Thus, if the firm ever ceases operation, the value function $F(Y,t)$ will be worthless. However, to solve equation (1.2.2.5), boundary conditions must also be identified along the dividing curve. In the stopping region,

$$F(Y,t) = \Omega(Y,t)$$

so, by continuity, a value-matching condition can be imposed $\forall t$:

$$F(Y^*(t),t) = \Omega(Y^*(t),t). \quad \text{(1.2.2.7)}$$

As the free boundary

$$Y = Y^*(t)$$

also requires solution, the region in (Y,t) space over which the partial differential equation is valid is itself endogenous. This form of dynamic programming

[8]Recalling that the state variable Y is assumed to follow the Itô process (1.2.2.3), if Y ever goes to zero, it will remain at zero forever.

is thus referred to as a free-boundary problem, and even though the partial differential equation (1.2.2.5) is only of second order, a third condition is necessary for the joint determination of the boundary $Y^*(t)$ with the function $F(Y,t)$. A high-order contact or smooth-pasting condition completes the specification, insisting that, $\forall t$, the functions $F(Y,t)$ and $\Omega(Y,t)$ should meet tangentially at the boundary $Y^*(t)$:

$$F_Y(Y^*(t),t) = \Omega_Y(Y^*(t),t). \tag{1.2.2.8}$$

This completes our introduction to the theory of dynamic programming, and we shall now turn our attention to more modern methods of investment analysis.

1.2.3 Contingent Claims Analysis

Although real options are not derivative instruments, the description of investment opportunities as such has led to the development of another more analytic method for their valuation. In contrast to dynamic programming, the option-pricing approach to investment analysis exploits the analogy of an investment opportunity to that of a perpetual financial call option on a dividend-paying stock.

The irreversible investment expenditure exercises or "kills" the option to invest, and the firm surrenders the possibility of waiting for new information that might affect the desirability or timing of its decision. The resulting payout stream from the investment corresponds to the continuous dividend yield earned on a stock, but the firm is unable to disinvest should market conditions sway adversely. The lost option value is therefore an opportunity cost that must be included as part of the cost of investment.

Immediate exercise is thus deemed optimal only when the real option is sufficiently deep in-the-money, so that the expected present value of the proceeds is at least as large as the full cost - namely the direct investment expenditure, plus the opportunity cost of investing. At this threshold, the cost of waiting (the sacrifice of immediate profit) outweighs the benefit of delay (the ability to observe new information and avoid any losses the firm would suffer should earnings fall).

Now the dynamic programming approach to the optimal stopping problem interprets $F(Y,t)$ as the market value of an asset that entitles the owner to the firm's future profit flows $\pi(Y,t)$. For an investor holding the asset over a short interval of time, the "return equilibrium" condition (1.2.2.2) expresses that the immediate profit flow and the expected capital gain should together provide a total rate of return ω. This arbitrary rate is simply specified exogenously at the outset of the development of the theory, without any indication of where it should come from or whether it should even remain constant over time.

In practice, a rate of return represents the opportunity cost of capital. To prevent the exploitation of any discrepancy by arbitrageurs, ω should hence match the return that could have been earned on other investment opportunities with comparable risk characteristics. In this section, we shall therefore use the option-pricing approach - or contingent claims technique - to re-examine the optimal stopping problem in order to provide a better treatment of risk and the subsequent return that it yields.

Spanning Assets: The Risk-adjusted Rate of Return

In a fashion analogous to option pricing, we shall construct an appropriate portfolio[9] to replicate the risk and return characteristics of the asset or firm, $F(Y, t)$, in the continuation region. In order to eliminate any possibility of arbitrage, the price of the asset must then equal the market value of our portfolio.

Let us suppose that the profit flow at time t depends on the firm's output price Y_t, where the underlying uncertainty is described by an Itô process. In particular, it is convenient for us to assume that Y_t follows a geometric Brownian motion, as we shall be considering proportional rates of return. Thus,

$$dY_t = \mu' Y_t dt + \sigma Y_t dz \tag{1.2.3.1}$$

where μ' is the real-world growth rate parameter, σ the volatility, and dz the increment of a standard Wiener process.

If the firm produces a commodity such as oil or copper, then its output will literally be traded as an asset in financial markets, and the variable price Y_t may then be continuously and directly observed. Now although most firms do not produce tradable goods, it will still nevertheless suffice to trade some other asset whose stochastic fluctuations are perfectly correlated with the stochastic process for Y_t.

The replicating asset is referred to as the spanning, since its own risk characteristics track or span the uncertainty in Y_t. The spanning could be a simple asset such as a stock or futures contract, or else a dynamic portfolio of simple assets whose contents are continuously adjusted so that the value of the portfolio is perfectly correlated with the process for Y_t.

To illustrate the use of a spanning, we shall let S_t denote the traded price of an asset or dynamic portfolio that is perfectly correlated with Y_t. If we assume that this replicating asset pays no dividends so that its entire return is earned from capital gains, then S_t will evolve according to

$$dS_t = \theta S_t dt + \sigma S_t dz,$$

[9]The selection of suitable portfolio constituents requires the existence of a diverse economy of traded assets.

where the drift rate θ represents the expected rate of return that investors would require as compensation for bearing the asset's inherent systematic risk.

This risk-adjusted return is the equilibrium rate that is established by the capital market, which incorporates an appropriate risk premium that is determined by the covariance of the return on asset S_t, with that on the entire market portfolio[10]. To be more specific, the fundamental condition of equilibrium from the Capital Asset Pricing Model (see either (2) or (5)) requires

$$\theta = r + \phi\sigma\rho_{SM}, \tag{1.2.3.2}$$

where r is the riskless rate of return[11], ϕ is another exogenous parameter denoting the aggregate market price of risk, and ρ_{SM} is the coefficient of correlation between returns on the particular asset S_t and that on the whole market portfolio M.

Now since S_t is perfectly correlated with Y_t, we must have

$$\rho_{SM} = \rho_{YM}.$$

Both assets also exhibit a constant volatility of return σ, and their stochastic fluctuations are modelled by the same Wiener process dz. Hence, to prevent any arbitrage opportunities from arising, asset Y_t must also provide a total expected rate of return θ.

Whereas the dynamic programming approach to investment analysis stipulates an exogenous rate of return ω, the contingent claims method employs a fundamental equation to derive the corresponding quantity. This deems a more accurate treatment of uncertainty. The total rate of return θ is endogenous to the analysis, and depends directly on the risk profile of the asset under consideration.

The Dividend Rate δ

Part of the risk-adjusted return θ that asset Y_t must provide will arise from the expected price appreciation μ'. Now for reasons that soon will be made clear, it is necessary for us to assume that

$$\theta > \mu'.$$

[10]As the market portfolio comprises of every existing traded asset, it therefore offers the maximum available diversification.

[11]The return on government bonds is often considered to be risk-free, but in reality there is always some uncertainty due to inflation.

Thus, in order to compensate investors sufficiently, asset Y_t must also pay a dividend at rate

$$\delta = \theta - \mu' > 0.$$

This dividend flow plays an important role in the model, either bearing direct monetary benefits or else taking the form of an implicit "convenience yield" from storage[12]. To highlight the significance of δ, it will be useful for us to draw on the analogy of an investment decision to that of a financial call option.

If the dividend rate is positive, the foregone dividend stream creates an opportunity cost to keeping the option alive, rather than exercising. Since δ is a proportional rate, the higher is the stock price Y_t, the greater is the flow of dividends. At a sufficiently high price level, the opportunity cost will eventually become great enough to render exercise more profitable than delay.

Even for a fixed price Y_t, the opportunity cost of waiting becomes large as

$$\delta \to \infty,$$

because the value of the option approaches zero and the holder must decide whether to invest immediately or never at all. Thus, only in such extreme circumstances does the standard net present value (or Marshallian) rule apply to the investment choice.

Conversely, there would be no opportunity cost to keeping the option alive if δ were equal to zero. Further delay would always be the optimal policy and firms would never invest. It is therefore essential for the expected price appreciation μ' to be less than the risk-adjusted return θ, so as to ensure a dividend rate

$$\delta > 0.$$

Portfolio Replication

We shall now proceed with our replicating argument[13]. If we consider a portfolio that consists of the firm and n units of a short position in the asset Y, then the cost of this portfolio is given by

$$F(Y, t) - nY.$$

[12]The holder of the investment might be a firm that plans to use the asset as an input and finds it convenient to store its own inventory rather than rely on the spot market; then the dividend is the implicit "convenience yield".

[13]The subscript t shall be dropped from this point on, and we shall simply take it as a given that the state variable Y pertains to time t.

If this portfolio is held over a short interval of time dt, the capital gain may be explained by

$$\begin{aligned} dF(Y,t) - ndY &= \left\{F_t(Y,t) + \mu' Y F_Y(Y,t) + \frac{1}{2}\sigma^2 Y^2 F_{YY}(Y,t)\right\} dt \\ &\quad + \sigma Y F_Y(Y,t) dz - n\left(\mu' Y dt + \sigma Y dz\right) \\ &= \left\{F_t(Y,t) + \mu' Y F_Y(Y,t) + \frac{1}{2}\sigma^2 Y^2 F_{YY}(Y,t) - n\mu' Y\right\} dt \\ &\quad + \left\{\sigma Y F_Y(Y,t) - n\sigma Y\right\} dz, \end{aligned}$$

where we have used Itô's Lemma for the expansion. During this same time interval dt, the firm also earns a profit of

$$\pi(Y,t)dt.$$

Furthermore, the holder of the short position must pay the holder of the corresponding long position a dividend to the amount of

$$(\theta - \mu')nY dt = \delta n Y dt,$$

where θ is determined by the equilibrium condition (1.2.3.2), and μ' is the observed drift in the stochastic process (1.2.3.1). This ensures that the asset yields the required rate of return

$$\mu' + \delta = \theta$$

that could otherwise have been earned on another source of investment with comparable risk characteristics. The total change in the portfolio during the time interval dt is thus given by

$$\left\{F_t(Y,t) + \mu' Y F_Y(Y,t) + \frac{1}{2}\sigma^2 Y^2 F_{YY}(Y,t) - n\mu' Y - n\delta Y + \pi(Y,t)\right\} dt$$

$$+ \left\{\sigma Y F_Y(Y,t) - n\sigma Y\right\} dz.$$

To eliminate the random increment dz, it is obvious that

$$n = F_Y(Y,t)$$

must be chosen. With fluctuations in the asset price Y, the composition of the portfolio may thus alter dynamically from one short interval of time to the next.

However, to simplify matters, the number of units shall be held fixed over each short interval of length dt to yield

$$\left\{F_t(Y,t) + \frac{1}{2}\sigma^2 Y^2 F_{YY}(Y,t) - \delta Y F_Y(Y,t) + \pi(Y,t)\right\} dt,$$

where the trend term $\mu' Y F_Y(Y,t)$ has also been cancelled.

Since the total return on the portfolio has been rendered non-stochastic, it must now equal the riskless return on its initial cost to avoid the possibility of arbitrage. Hence, we next set

$$\left\{F_t(Y,t) + \frac{1}{2}\sigma^2 Y^2 F_{YY}(Y,t) - \delta Y F_Y(Y,t) + \pi(Y,t)\right\} dt = r\left\{F(Y,t) - Y F_Y(Y,t)\right\} dt.$$

If we divide by dt and rearrange terms, we then see that $F(Y,t)$ satisfies the following partial differential equation (1.2.3.3):

$$\frac{1}{2}\sigma^2 Y^2 F_{YY}(Y,t) + (r-\delta) Y F_Y(Y,t) + F_t(Y,t) - rF(Y,t) + \pi(Y,t) = 0.$$

This partial differential equation is almost identical to the corresponding equation (1.2.2.5) that was derived by dynamic programming methods. After substituting

$$\mu(Y,t) = \mu' Y$$

and

$$\sigma(Y,t) = \sigma Y,$$

the only remaining difference between the equations is that the riskless interest rate r has replaced the exogenously specified rate of return ω, and the coefficient of $F_Y(Y,t)$ now contains

$$(r-\delta)$$

instead of μ'. Hence, the solution of partial differential equation (1.2.3.3) will hinge on the same initial and boundary conditions that were specified in the dynamic programming approach.

These striking similarities between the alternately derived partial differential equations that $F(Y,t)$ satisfies are not coincidental, and warrant some discussion. In conclusion to this chapter, we shall therefore undertake a further exploration of the close parallels between the Bellman equation of dynamic programming, and that of contingent claims analysis, in the hope that it will lead us to a useful technique for obtaining and assessing solutions to these similar partial differential equations.

1.2.4 Equivalent Risk-neutral Evaluation

In a familiar context, we shall suppose that the profit flow

$$\pi(Y, t)$$

of a hypothetical firm depends once again on a state variable Y. This state variable shall assume the geometric Brownian motion of equation (1.2.3.1), and termination shall be forced at a finite time T to yield a payoff

$$\Omega(Y_T, T).$$

Now given that the current state Y pertains to time t, we shall first derive the value of the firm's title to the stated stream of profits using dynamic programming. The stipulation of an exogenous rate of return ω immediately gives rise to the expected present value

$$F(Y, t) = \xi_t \left\{ \int_t^T \pi(Y_\tau, \tau) e^{-\omega(\tau - t)} d\tau + \Omega(Y_T, T) e^{-\omega(T-t)} \right\}, \tag{1.2.4.1}$$

where ξ_t denotes the expectation conditional on information as of time t. After a short interval of time dt, the state variable will evolve to

$$(Y + dY),$$

and the value of the asset will change to

$$F(Y + dY, t + dt).$$

As our derivation involves continuous time, the discount factor[14]

$$e^{-\omega dt}$$

must be used. Furthermore, dY is a random increment from the current perspective, so an expectation must be taken[15]. Thus, we get

$$F(Y, t) = \pi(Y, t)dt + e^{-\omega dt} \xi_t [F(Y + dY, t + dt)]. \tag{1.2.4.2}$$

In this instance, the Bellman equation (1.2.4.2) is trivial. No action is taken during the interval dt, eliminating the need for any maximization. Thus, if we

[14] Recall the Taylor series expansion $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ Ignoring terms of order $(dt)^2$ and higher, the discount factor $e^{-\omega dt}$ may thus be approximated by $(1 - \omega dt)$.

[15] Although the profit flow $\pi(Y, t)$ could also randomly evolve over a short interval in continuous time, the difference made by this consideration is of negligible magnitude $(dt)^2$ and shall be ignored.

expand by Itô's Lemma and omit terms that tend to zero faster than dt, the right-hand side of the equation yields

$$\pi(Y,t)dt + (1-\omega dt)\left\{F(Y,t) + F_t(Y,t)dt + F_Y(Y,t)\mu' Y dt + \frac{1}{2}F_{YY}(Y,t)\sigma^2 Y^2 dt\right\}$$

$$= F(Y,t) + \left\{\frac{1}{2}\sigma^2 Y^2 F_{YY}(Y,t) + \mu' Y F_Y(Y,t) + F_t(Y,t) - \omega F(Y,t) + \pi(Y,t)\right\} dt.$$

Substituting back into (1.2.4.2), final simplification produces the following partial differential equation:

$$\frac{1}{2}\sigma^2 Y^2 F_{YY}(Y,t) + \mu' Y F_Y(Y,t) + F_t(Y,t) - \omega F(Y,t) + \pi(Y,t) = 0.$$

This is exactly analogous to the dynamic programming result (1.2.2.5) for the present case of geometric Brownian motion. By construction, expression (1.2.4.1) also satisfies the value-matching condition[16]

$$F(Y,T) = \Omega(Y,T)$$

for all Y. Hence, expression (1.2.4.1) is the solution to the partial differential equation. Of course, much work is still required to evaluate the expectation, and this often necessitates numerical calculation. However, expression (1.2.4.1) has significant conceptual use, even when $\pi(Y,t)$ and $\Omega(Y,t)$ have inconvenient functional forms that do not permit explicit evaluation.

If we now revert to the contingent claims approach, the value function has already been shown to satisfy partial differential equation (1.2.3.3), which we shall restate for ease of reference:

$$\frac{1}{2}\sigma^2 Y^2 F_{YY}(Y,t) + (r-\delta) Y F_Y(Y,t) + F_t(Y,t) - rF(Y,t) + \pi(Y,t) = 0.$$

Although the same value-matching condition serves as the final boundary, the solution $F(Y,t)$ is not known in advance here. Nevertheless, it can be extracted immediately by noticing the formal analogy between this partial differential equation and the one obtained using dynamic programming methods.

As we have already highlighted, the riskless market rate r has been substituted for the exogenously specified rate of return ω, and the growth rate μ' from the geometric Brownian motion of Y has been replaced by

$$(r-\delta).$$

[16] The smooth pasting boundary is derived from the value-matching condition and is therefore satisfied too.

We may therefore evaluate the future payoff by discounting it at the riskless rate r, under the pretense that Y follows a process with a different trend

$$\mu = r - \delta.$$

The solution is thus identified as

$$F(Y,t) = \xi_t \left[\int_t^T \pi(Y_\tau, \tau) e^{-r(\tau - t)} d\tau + \Omega(Y_T, T) e^{-r(T-t)} \right],$$

where Y is now artificially assumed to follow a new geometric Brownian motion

$$dY = \mu Y dt + \sigma Y dz.$$

The above procedure provides an illustration of "equivalent risk-neutral valuation", and serves to explain the close relationship between the two approaches. The methods will lead to identical results under the assumption of risk-neutrality, but an estimate of the equilibrium return θ is still needed to determine the dividend rate δ. Thus, only if capital markets are complete can the investment decision be viewed as an option-pricing problem and solved using the techniques of contingent claims analysis.

As dynamic programming makes no such demand, it provides a method of valuation when markets are incomplete and risk can neither be traded nor replicated. However, the existence of arbitrage opportunities becomes highly probable. Without spanning, there is no theory for determining the "correct value" for the required rate of return ω. The perceived level of risk becomes subjective and the objective function may simply reflect the decision-maker's private preferences.

Despite the similarities, the alternative approaches are thus based on different assumptions about financial markets, and both have off-setting advantages and disadvantages. In specific situations, one may be more convenient in practice than the other, but together they can handle quite a large variety of applications - in particular, the study of real options has given rise to numerous examples.

In (32), Shibata investigates the impact of volatility on the value of a real option, assuming that the underlying revenue process follows a non-linear stochastic differential equation. When geometric Brownian motion (typically used in almost all standard real-option models) is replaced by a general diffusion process with concave drift, Shibata demonstrates numerically that convexity of the drift is not necessary for the proposition.

Grenadier (11) "*provides a tractable approach for deriving equilibrium investment strategies in a continuous-time Cournot-Nash framework*", but only considers one source of uncertainty. The market demand is assumed to suffer exogenous shocks which are modelled according to a geometric Brownian motion. Hence, the inverse demand function (or unitary output price) fluctuates in response to the underlying randomness.

Kong and Kwok (16) provide a complete characterization of pre-emptive, dominant and simultaneous equlibria by imposing externalities on two rivalling asymmetric firms. They demonstrate richer strategic interactions and produce a more complex set of entry decisions by combining asymmetry in the stochastic revenue flows with asymmetry in the non-random sunk costs of investment.

In (18), Lambrecht and Perraudin introduce incomplete information to a preemptive equilibrium in which investment in an indivisible technology yields an aggregate, stochastically evolving flow of income.

These are but a few of the authors who have considered a single stochastic variable in their research of real options. Shackleton *et al* (31) have been among the first to incorporate a second stochastic factor into real-option modelling, but their game-theoretic approach is based on a market that can only accommodate one active firm. The idle firm possesses the option to claim the entire market.

The two-variable analysis undertaken by Paxson and Pinto (26) differs from the work of Shackleton *et al* in that the existence of a duopoly is permitted. They consider two firms that are contemplating the option to enter a new market, using both pre-emptive and non pre-emptive strategies.

In the chapters that follow, we shall present a similar model with more than one diffusion process to analyze strategic interaction in a duopolistic framework. The profit per unit and the number of units sold shall be assumed to evolve according to distinct, but possibly correlated, geometric Brownian motions, so that the profit flow is a product of two disaggregated stochastic factors. However, we shall extend Paxson and Pinto's research to a wider context by adjusting the model to include the effect of the covariance between the stochastic factors.

For simplicity, we shall assume that symmetric firms rival under a state of complete information (see (9)). Dissimilar games will be examined in which the roles of the players are first endogenous to the models, and then later preassigned. The first mover shall acquire a competitive advantage in market share over the follower, and all investment will be characterized by irreversibility, uncertainty and the option to delay.

Chapter 2

Pre-emptive Equilibrium

In this chapter, we aim to develop a real-option model that improves on the accuracy of Paxson and Pinto's results for the timing of pre-emptive market entry into a duopoly. Pre-emption describes a non co-operative game in which the roles of the leader and the follower are endogenous to the strategies that are implemented. As the first entrant acquires a lasting competitive advantage over its rival, each firm strives for the position of leader. This creates a pre-emption effect which induces sequential entry, and resembles a Stackelberg-Nash equilibrium.

The two firms are thus defined to be *ex-ante* symmetrical but asymmetrical *post-ante*. The value functions of both payers are equal at the start of the game, and they are faced with complete information, identical costs and hence, identical unitary profits. Both must incur the same fixed investment cost K, and the marginal cost of production is assumed constant. Once the game concludes, there is imbalance in the payoff functions of the firms. The leader's competitive advantage yields a market share and subsequent profit total that is greater than that of the follower.

2.1 The Model

We assume the existence of a new market in which the investing firms will compete in the manufacture and sale of homogeneous goods. We let P_t represent the profit per unit, and we let Q_t denote the quantity sold in the market *by the follower*. Although these variables are usually combined and treated as an aggregate return variable in traditional real options literature, we follow Paxson and Pinto (26) and choose to consider them separately[1].

We believe that this decision is sensible as the economic factors that influence the cost of manufacturing each unit, and hence the profit per unit, can differ from those that determine the actual number of units that are eventually sold. As an example, consumers have no choice but to bear the price inflation of es-

[1]Examples of other authors that have considered more than one stochastic factor in a single setting include Lee and Paxson (19), Clark *et al* (1), Williams (34) and Quigg (28).

sential goods and services. When demand is relatively inelastic, a change in price (and hence in revenue, assuming all other factors remain constant) will result in a proportionately smaller change in the quantity sold.

The magnitude of the correlation between volume and unit price will obviously vary according to market structure and (dis)economies of scale, but we will apply a fairly conservative range of correlation coefficients in our practical examples and graphical illustrations. Intuitively, we would expect a negative correlation to prevail between these respective variables, with increases in price resulting in a decrease in the number of units sold in the market, and *vice versa*. However, Paxson and Pinto (26) hypothesize that "*conditions of both excess demand and economies of scale may lead to positive correlation*", so both scenarios shall be considered.

Thus, in reality, it is evident that the paths followed by the profit per unit and the quantity sold may not necessarily relate. Significantly different drifts and volatilities may be observed, and the realized values of the variables may even evolve according to different stochastic processes. Indeed, we immediately notice that the profit per unit is a continuous variable, whilst the number of units is technically discrete. However, this distinction is less important for high volume products where any unit increase in the quantity sold is proportionately very small.

The model developed by Paxson and Pinto in (26) is best suited for application by such a typefied industry where business focuses on high volumes of production. The telecommunications market is cited as a prime example of a service sector in which the profit per unit and the number of units sold are both heavily dependent on the volatility of the market. As our own model has been derived from their research, we may thus treat the number of units as a continuous variable, and are therefore able to assume that P_t and Q_t follow different, but possibly correlated, geometric Brownian motions.

Both variables are constrained to a domain from zero to infinity. Although the lower limit unrealistically implies that a firm cannot earn a negative profit, it is necessary to impose this restriction due to the assumed stochastic process. Under the real-world probability measure,

$$dP_t = \mu'_P P_t dt + \sigma_P P_t dz_P \tag{2.1.1}$$

and

$$dQ_t = \mu'_Q Q_t dt + \sigma_Q Q_t dz_Q, \tag{2.1.2}$$

where μ'_P and μ'_Q represent the expected multiplicative trends of P_t and Q_t respectively, whilst σ_P and σ_Q denote the volatilities. The magnitude of the correlation between the Wiener process increments dz_P and dz_Q is described by the coefficient ρ_{PQ}, and all parameters are assumed constant.

Now according to Fudenberg and Tirole (9), if one player applies a Markovian strategy, its rival should employ as best reaction a Markovian strategy as well. Hence, a Markovian equilibrium will result, even when history dependent strategies are permitted and other equilibria exist. As the state variables P_t and Q_t follow Markov processes, the payoff-relevant strategies of both players (firms) are also assumed to be Markovian. Their actions will thus yield a Nash equilibrium in every proper sub game.

2.2 The Follower's Value Function

As is usual in dynamic contexts, the stopping time game is solved backwards. We therefore begin our analysis of the pre-emptive equilibrium by deriving the follower's option-value function using a replicating argument similar to the one described in Section 1.2.3.

For simplicity, the investing firms are assumed to be risk-neutral, so that all cash flows may be discounted at the constant risk-free rate $r > 0$. We let θ_P and θ_Q denote the equilibrium rates of return that the Capital Asset Pricing Model would otherwise require from P_t and Q_t respectively in the real world.

Under the risk-neutral probability measure, the state variables evolve according to the stochastic processes

$$dP_t = \mu_P P_t dt + \sigma_P P_t dz_P \tag{2.2.1}$$

and

$$dQ_t = \mu_Q Q_t dt + \sigma_Q Q_t dz_Q, \tag{2.2.2}$$

where

$$\mu_P = r - \delta_P = r - (\theta_P - \mu'_P)$$

and

$$\mu_Q = r - \delta_Q = r - (\theta_Q - \mu'_Q)$$

define the risk-neutral growth rates of P_t and Q_t respectively, whilst the quantities δ_P and δ_Q describe the dividend rates for these same respective assets[2]. By rearranging the defining equalities for μ_P and μ_Q, we arrive at the following relationships:

$$\delta_P = r - \mu_P = \theta_P - \mu'_P \tag{2.2.3}$$

and

$$\delta_Q = r - \mu_Q = \theta_Q - \mu'_Q. \tag{2.2.4}$$

These equilibrium conditions must continue to hold in the event of any underlying parameter change.

[2]The time subscripts of P_t and Q_t will be suppressed from now on.

Let us focus our attention on equation (2.2.3). When some parameter of the model varies, we must question what happens to δ_P. Various possibilities can be explored. Following Dixit and Pindyck (2), we will always suppose that the riskless interest rate r is fixed by the larger considerations of the entire capital market, and thus remains independent of what happens to any one asset, firm, or even industry. Hence, we will limit our discussion to the relationship

$$\delta_P = \theta_P - \mu'_P.$$

Now suppose σ_P increases. The Capital Asset Pricing Model dictates that this higher volatility will raise the risk-adjusted discount rate θ_P. In order to preserve equilibrium in the market for asset P, either μ'_P or δ_P must subsequently adjust.

We shall consider two extreme outcomes. First, μ'_P might be a fundamental fact about P, so that δ_P must respond to the change in θ_P (for example, the dividend rate might depend on the quantity of the asset held). Alternatively, δ_P might be a basic behavioural parameter, so that the price process of P is subject to change as μ'_P shifts to restore equilibrium.

A third possibility is that both μ'_P and δ_P take up part of the adjustment. However, in the analysis of our model we will adopt the approach of Dixit and Pindyck who regard δ_P as a basic parameter that is independent of σ_P, and likewise for δ_Q. Bearing this in mind, the derivation of the follower's option-value function shall proceed under the risk-neutral structure, with the parameters μ_P and μ_Q not needing to effect change at any stage due to the assumed constancy of the risk-free rate r.

2.2.1 A Partial Differential Equation

We assume the existence of a complete market such that $\Pi(P, Q)$ denotes a replicating portfolio for the value function of an idle follower. The portfolio is constructed so that it is long on an underlying option and short on n_P and n_Q units of P and Q respectively.

Although P and Q do not describe assets, they may be treated as such under our assumption of a complete market. Within a sufficiently diverse economy, it is possible to replicate the risk characteristics of P and Q with a pair of traded assets[3] whose stochastic fluctuations are perfectly correlated with the stochastic processes (2.1.1) and (2.1.2).

[3]These may be simple assets such as stocks or futures contracts, or else dynamic portfolios of assets whose contents are continuously adjusted to maintain perfect correlation with the stochastic processes for P and Q.

The value of the replicating portfolio is thus given by

$$\Pi(P,Q) = f_0(P,Q) - n_P P - n_Q Q,$$

where $f_0(P,Q)$ denotes the worth of a perpetual option to enter second in a given market[4]. We assume further that

$$r > \mu_P, \mu_Q,$$

which is equivalent to assuming that

$$\theta_P > \mu'_P$$

and

$$\theta_Q > \mu'_Q.$$

Thus, the respective drift rates μ_P and μ_Q will only provide part of the risk-free return r that is required from each of the assets P and Q. In order to ensure that the corresponding holders of the long positions in these assets are fully compensated, assets ***P*** and ***Q*** must therefore each pay a dividend at rate

$$\delta_P = r - \mu_P > 0$$

and

$$\delta_Q = r - \mu_Q > 0$$

respectively[5]. In contrast, the portfolio itself only offers a return in the form of capital gains. Any change in this portfolio over a time interval of length dt may thus be explained by

$$d\Pi(P,Q) = df_0(P,Q) - n_P dP - \delta_P n_P P dt - n_Q dQ - \delta_Q n_Q Q dt, \qquad (2.2.1.1)$$

where $\delta_P n_P P dt$ and $\delta_Q n_Q Q dt$ reflect the required dividend payments. We next expand $df_0(P,Q)$ using Itô's Lemma:

$$\begin{aligned} df_0(P,Q) &= \frac{\partial f_0(P,Q)}{\partial P} dP + \frac{\partial f_0(P,Q)}{\partial Q} dQ + \frac{1}{2}\frac{\partial^2 f_0(P,Q)}{\partial P^2}(dP)^2 \\ &\quad + \frac{1}{2}\frac{\partial^2 f_0(P,Q)}{\partial Q^2}(dQ)^2 + \frac{\partial^2 f_0(P,Q)}{\partial P \partial Q}(dP)(dQ) \end{aligned}$$

[4]The zero subscript indicates that the value of the option pertains to the continuation region, i.e. prior to the follower's entry.

[5]We recall from Section 1.2.3 that it is essential for the dividend flows to be positive in order to prevent the model from becoming trivial.

$$
\begin{aligned}
= \quad & \frac{\partial f_0(P,Q)}{\partial P}(\mu_P P dt + \sigma_P P dz_P) + \frac{\partial f_0(P,Q)}{\partial Q}(\mu_Q Q dt + \sigma_Q Q dz_Q) \\
& + \frac{1}{2}\frac{\partial^2 f_0(P,Q)}{\partial P^2}\sigma_P^2 P^2 dt + \frac{1}{2}\frac{\partial^2 f_0(P,Q)}{\partial Q^2}\sigma_Q^2 Q^2 dt \\
& + \frac{\partial^2 f_0(P,Q)}{\partial P \partial Q}\rho_{PQ}\sigma_P\sigma_Q PQ dt
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \left\{\mu_P P \frac{\partial f_0(P,Q)}{\partial P} + \mu_Q Q \frac{\partial f_0(P,Q)}{\partial Q} + \frac{1}{2}\frac{\partial^2 f_0(P,Q)}{\partial P^2}\sigma_P^2 P^2 \right. \\
& \left. + \frac{1}{2}\frac{\partial^2 f_0(P,Q)}{\partial Q^2}\sigma_Q^2 Q^2 + \rho_{PQ}\sigma_P\sigma_Q PQ \frac{\partial^2 f_0(P,Q)}{\partial P \partial Q}\right\} dt \\
& + \sigma_P P \frac{\partial f_0(P,Q)}{\partial P} dz_P + \sigma_Q Q \frac{\partial f_0(P,Q)}{\partial Q} dz_Q.
\end{aligned}
$$

By substituting the preceding expression into equation (2.2.1.1) and recollecting like terms, we then obtain

$$d\Pi(P,Q)$$

$$
\begin{aligned}
= \quad & \left\{\mu_P P \frac{\partial f_0(P,Q)}{\partial P} + \mu_Q Q \frac{\partial f_0(P,Q)}{\partial Q} + \frac{1}{2}\frac{\partial^2 f_0(P,Q)}{\partial P^2}\sigma_P^2 P^2 \right. \\
& \left. + \frac{1}{2}\frac{\partial^2 f_0(P,Q)}{\partial Q^2}\sigma_Q^2 Q^2 + \rho_{PQ}\sigma_P\sigma_Q PQ \frac{\partial^2 f_0(P,Q)}{\partial P \partial Q}\right\} dt \\
& + \sigma_P P \frac{\partial f_0(P,Q)}{\partial P} dz_P - n_P(\mu_P P dt + \sigma_P P dz_P) - \delta_P n_P P dt \\
& + \sigma_Q Q \frac{\partial f_0(P,Q)}{\partial Q} dz_Q - n_Q(\mu_Q Q dt + \sigma_Q Q dz_Q) - \delta_Q n_Q Q dt
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \left\{\mu_P P \frac{\partial f_0(P,Q)}{\partial P} - n_P\mu_P P - \delta_P n_P P + \mu_Q Q \frac{\partial f_0(P,Q)}{\partial Q} - n_Q \mu_Q Q \right. \\
& - \delta_Q n_Q Q + \frac{1}{2}\frac{\partial^2 f_0(P,Q)}{\partial P^2}\sigma_P^2 P^2 + \frac{1}{2}\frac{\partial^2 f_0(P,Q)}{\partial Q^2}\sigma_Q^2 Q^2 \\
& \left. + \rho_{PQ}\sigma_P\sigma_Q PQ \frac{\partial^2 f_0(P,Q)}{\partial P \partial Q}\right\} dt + \left\{\sigma_P P \frac{\partial f_0(P,Q)}{\partial P} - n_P\sigma_P P\right\} dz_P \\
& + \left\{\sigma_Q Q \frac{\partial f_0(P,Q)}{\partial Q} - n_Q\sigma_Q Q\right\} dz_Q.
\end{aligned}
$$

Now if we construct a delta-hedged portfolio by choosing

$$n_P = \frac{\partial f_0(P,Q)}{\partial P}$$

and

$$n_Q = \frac{\partial f_0(P,Q)}{\partial Q},$$

the expression for $d\Pi(P,Q)$ thus simplifies to

$$\left\{\frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 f_0(P,Q)}{\partial P^2} - \delta_P P \frac{\partial f_0(P,Q)}{\partial P} + \frac{1}{2}\sigma_Q^2 Q^2 \frac{\partial^2 f_0(P,Q)}{\partial Q^2} - \delta_Q Q \frac{\partial f_0(P,Q)}{\partial Q} \right.$$
$$\left. + \rho_{PQ}\sigma_P\sigma_Q PQ \frac{\partial^2 f_0(P,Q)}{\partial P \partial Q}\right\} dt.$$

As the random components have been perfectly hedged, the return on the portfolio should therefore equal the risk-free rate r. Hence, we have

$$\begin{aligned} d\Pi(P,Q) &= r\Pi(P,Q)dt \\ &= r\left\{f_0(P,Q) - P\frac{\partial f_0(P,Q)}{\partial P} - Q\frac{\partial f_0(P,Q)}{\partial Q}\right\} dt. \end{aligned}$$

By substituting our simplified expression for $d\Pi(P,Q)$ into the return equation and grouping like terms, we eventually arrive at the following partial differential equation after some slight rearranging:

$$\begin{aligned} \frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 f_0(P,Q)}{\partial P^2} + \mu_P P \frac{\partial f_0(P,Q)}{\partial P} + \frac{1}{2}\sigma_Q^2 Q^2 \frac{\partial^2 f_0(P,Q)}{\partial Q^2} &+ \mu_Q Q \frac{\partial f_0(P,Q)}{\partial Q} \\ + \rho_{PQ}\sigma_P\sigma_Q PQ \frac{\partial^2 f_0(P,Q)}{\partial P \partial Q} - r f_0(P,Q) &= 0. \end{aligned} \qquad (2.2.1.2)$$

This partial differential equation explains the movements in the idle follower's value function, and his subsequent opportunity to invest. In Section 2.2.3, we will resort to methods of similarity to obtain a closed-form solution to equation (2.2.1.2).

2.2.2 The Profit Flow, X

Although we have assigned separate geometric Brownian motion processes to the disaggregated state variables P and Q, it is at this point that we shall turn our attention to the follower's total profit (or profit flow)

$$X = PQ.$$

In order to examine the stochastic process according to which the profit flow X evolves, we note that

$$\begin{aligned} dX = d(PQ) &= \frac{\partial X}{\partial P}dP + \frac{\partial X}{\partial Q}dQ + \frac{\partial^2 X}{\partial P \partial Q}(dP)(dQ) \\ &= Q(\mu_P P dt + \sigma_P P dz_P) + P(\mu_Q Q dt + \sigma_Q Q dz_Q) + \rho_{PQ}\sigma_P\sigma_Q PQdt \\ &= PQ(\mu_P + \mu_Q + \rho_{PQ}\sigma_P\sigma_Q)dt + PQ(\sigma_P dz_P + \sigma_Q dz_Q) \\ &= \mu_X X dt + \sigma_X X dz_X, \end{aligned}$$

where dz_X is the increment of a standard Wiener process.

Thus, under the imposed risk-neutral structure, X is also seen to follow a geometric Brownian motion, but with expected trend

$$\mu_X = \mu_P + \mu_Q + \rho_{PQ}\sigma_P\sigma_Q,$$

and volatility σ_X such that

$$\sigma_X^2 = \sigma_P^2 + \sigma_Q^2 + 2\rho_{PQ}\sigma_P\sigma_Q.$$

This diffusion for X differs from that presented by Paxson and Pinto in (26), as they do not incorporate the covariance term $\rho_{PQ}\sigma_P\sigma_Q$ into the risk-adjusted drift. In this dissertation, we shall make comparison to their results, and demonstrate how the contribution of the covariance term improves the effectiveness of the model.

Having identified the random process for X, it seems reasonable for us to conjecture that the quantity

$$r - \mu_X = r - (\mu_P + \mu_Q + \rho_{PQ}\sigma_P\sigma_Q)$$

should therefore denote the corresponding dividend rate, δ_X, for the profit flow in the risk-neutral framework. In order to confirm the validity of this statement, it is necessary for us to investigate the equilibrium rate of return, θ_X, that the Capital Asset Pricing Model specifies for the profit flow under the real-world probability measure.

Following the methodology of McDonald and Siegel in (22), we now show that the equilibrium expected rate of return on the profit flow X is directly dependent on the equilibrium expected rates of return on assets with the same risk as P and Q in a complete market. Risk aversion by the follower is here introduced by supposing that the option to invest is owned by a well-diversified investor who needs only to be compensated for the systematic component of the risk that is associated with the aggregate profit flow, and with the option to invest.

In standard asset-pricing models, the risk premium earned on an asset is proportional to the volatility of the asset. Hence, from the Capital Asset Pricing Model, the risk premium earned on each of the disaggregated variables is given by

$$\theta_P - r = \phi\rho_{PM}\sigma_P$$

and

$$\theta_Q - r = \phi\rho_{QM}\sigma_Q,$$

where ϕ is the market price of risk, and ρ_{PM} and ρ_{QM} are the co-efficients of correlation between the rate of return on the market portfolio and that on assets P and Q respectively.

The actual rate of return on the aggregate profit flow X is given by the Itô derivative

$$\frac{dX}{X} = \mu'_X dt + \sigma_X dz_X = (\mu'_P + \mu'_Q + \rho_{PQ}\sigma_P\sigma_Q)dt + \sigma_P dz_P + \sigma_Q dz_Q,$$

where the unanticipated component of the return on X is a summation of the stochastic components in the rates of change of P and Q. It thus follows that the risk premium earned on X must be a direct summation of the individual premiums earned on P and Q, i.e.

$$\theta_X - r = (\theta_P - r) + (\theta_Q - r).$$

Hence, the total equilibrium expected rate of return on the profit flow is given by

$$\theta_X = \theta_P + \theta_Q - r.$$

Having defined θ_X, it is now possible for us to determine the exact dividend rate for the profit flow under the real-world measure, i.e.

$$\begin{aligned}\delta_X = \theta_X - \mu'_X &= (\theta_P + \theta_Q - r) - (\mu'_P + \mu'_Q + \rho_{PQ}\sigma_P\sigma_Q) \\ &= (\theta_P - \mu'_P) + (\theta_Q - \mu'_Q) - r - \rho_{PQ}\sigma_P\sigma_Q. \qquad (2.2.2.1)\end{aligned}$$

From conditions (2.2.3) and (2.2.4), we know that

$$\theta_P - \mu'_P = r - \mu_P$$

and

$$\theta_Q - \mu'_Q = r - \mu_Q.$$

Thus,

$$\begin{aligned} \delta_X &= (r - \mu_P) + (r - \mu_Q) - r - \rho_{PQ}\sigma_P\sigma_Q \\ &= r - (\mu_P + \mu_Q + \rho_{PQ}\sigma_P\sigma_Q) \\ &= r - \mu_X, \end{aligned}$$

thereby giving rise to the following equilibrium condition:

$$\delta_X = r - \mu_X = \theta_X - \mu'_X. \tag{2.2.2.2}$$

Unlike the disaggregated dividend rates δ_P and δ_Q, the dividend rate δ_X cannot be viewed as a basic parameter that remains independent of the volatilities σ_P and σ_Q.

Consider definition (2.2.2.1). It is clear that the dividend rate for the profit flow may also be given in the form

$$\delta_X = \delta_P + \delta_Q - r - \rho_{PQ}\sigma_P\sigma_Q.$$

Recalling the assumed constancy of the parameters δ_P, δ_Q and r, the dependence of the dividend rate δ_X on either of the volatilities σ_P and σ_Q arises due to the non-zero correlation ρ_{PQ} between the state variables P and Q. In the event of an underlying change in volatility, δ_X will thus be seen to adjust accordingly in order to restore equilibrium. The magnitude and direction of the response will be determined by the degree of correlation, and by the extent and direction of the shift in either σ_P or σ_Q.

However, the dividend rate will not always assume the entire adjustment necessary for the preservation of equilibrium. To illustrate this point, we restate the real-world drift for ease of reference:

$$\mu'_X = \mu'_P + \mu'_Q + \rho_{PQ}\sigma_P\sigma_Q.$$

Now suppose $\rho_{PQ} > 0$. An increase in σ_P will thus result in an increase in the covariance term $\rho_{PQ}\sigma_P\sigma_Q$, whilst at the same time raising the value of μ'_P in order to preserve equilibrium for asset P. Hence, the growth rate for X will be seen to increase in conjunction with the lowering of the dividend rate δ_X.

If $\rho_{PQ} < 0$, a higher value of σ_P will raise the dividend rate δ_X, but the increased magnitude of the negative covariance term $\rho_{PQ}\sigma_P\sigma_Q$ will dampen the effect of the higher value of μ'_P. However, the negative shift in covariance is unlikely to counter the positive response in μ'_P completely. Hence, the growth rate μ'_X may still be seen to take up part of the adjustment necessary for condition (2.2.2.2) to hold.

Thus, it is clear that the adjustments that result from an underlying change in volatility depend entirely on the values of the parameters in the model. We shall now return to the risk-neutral solution of the follower's option-value function.

2.2.3 A Closed-Form Solution

As we have already mentioned in Section 2.2.1, it is necessary for us to apply methods of similarity to partial differential equation (2.2.1.2) in order to obtain a closed-form solution for the idle follower's value function. Through the appropriate substitution of the profit flow

$$X = PQ$$

in Appendix A.1, we are able to reduce equation (2.2.1.2) to the following second order Cauchy-Euler ordinary differential equation:

$$\frac{1}{2}\sigma_X^2 X^2 \frac{d^2 F_0(X)}{dX^2} + \mu_X X \frac{dF_0(X)}{dX} - rF_0(X) = 0, \qquad (2.2.3.1)$$

where F_0 is now the function to be determined.

Following the solution technique prescribed in (29), we next apply the transformation

$$X = e^{t_1}$$

in Appendix A.2. Substitution back into equation (2.2.3.1) then obtains a linear ordinary differential equation with constant coefficients:

$$\frac{1}{2}\sigma_X^2 \frac{d^2 G_0(t_1)}{dt_1^2} + (\mu_X - \frac{1}{2}\sigma_X^2)\frac{dG_0(t_1)}{dt_1} - rG_0(t_1) = 0. \qquad (2.2.3.2)$$

The auxiliary equation is given by the fundamental quadratic

$$\frac{1}{2}\sigma_X^2 \beta^2 + (\mu_X - \frac{1}{2}\sigma_X^2)\beta - r = 0,$$

the solution of which yields two roots:

$$\begin{aligned}\beta_{1,2} &= \frac{-(\mu_X - \frac{1}{2}\sigma_X^2) \pm \sqrt{(\mu_X - \frac{1}{2}\sigma_X^2)^2 - 4(\frac{1}{2}\sigma_X^2)(-r)}}{2(\frac{1}{2}\sigma_X^2)} \\ &= \frac{-(\mu_X - \frac{1}{2}\sigma_X^2) \pm \sqrt{2r\sigma_X^2 + (\mu_X - \frac{1}{2}\sigma_X^2)^2}}{\sigma_X^2}.\end{aligned}$$

Upon closer inspection of the characteristic quadratic function

$$\Upsilon(\beta) = \frac{1}{2}\sigma_X^2\beta^2 + (\mu_X - \frac{1}{2}\sigma_X^2)\beta - r,$$

we note that the coefficient of β^2 is positive, with

$$\Upsilon(0) = -r < 0$$

and

$$\Upsilon(1) = \mu_X - r = -(r - \mu_X) = -\delta_X < 0.$$

From these observations[6] we easily deduce that $\Upsilon(\beta)$ crosses the horizontal axis to the right of $\beta = 1$, and to the left of $\beta = 0$ i.e. $\beta_1 > 1$ and $\beta_2 < 0$.

We illustrate this conclusion using parameter values that are similar to those chosen by Paxson and Pinto: $r = 0.05$, $\mu_P = 0.01$, $\mu_Q = 0.01$, $\sigma_P = 0.1$, $\sigma_Q = 0.2$ and $\rho_{PQ} = -0.2$, which in turn yield $\sigma_X = 0.20494$, $\mu_X = 0.016$, $\delta_X = 0.034$, $\beta_1 = 1.66667$ and $\beta_2 = -1.42857$.

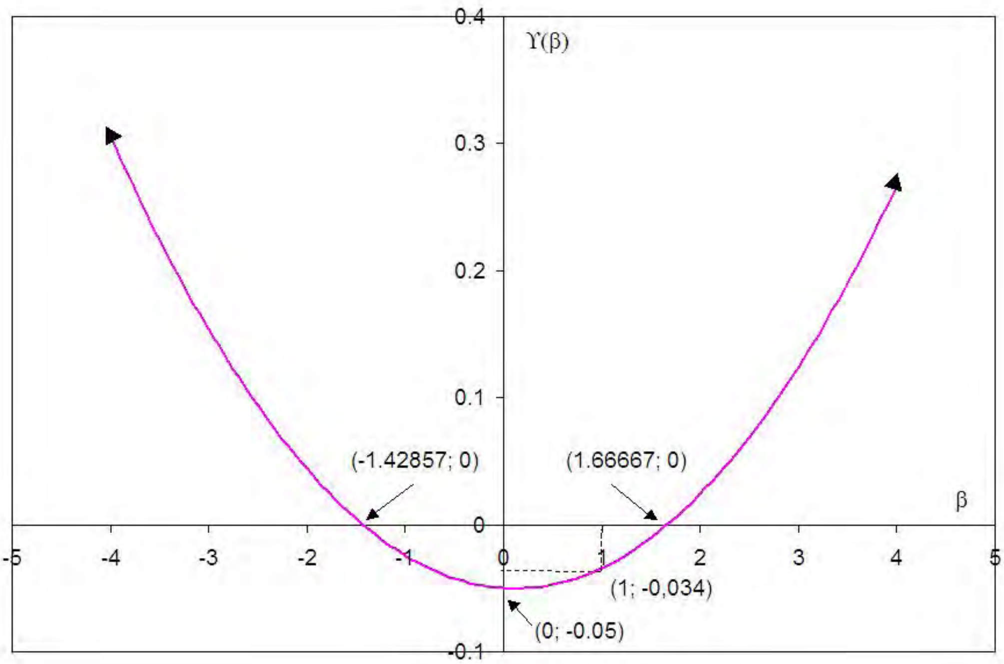


Figure 2.1: Characteristic quadratic function

[6]The variables ρ_{PQ}, σ_P and σ_Q will always be constrained to ensure that $\delta_X > 0$ (or equivalently that $r > \mu_X$) for any fixed set of parameters r, μ_P and μ_Q, where $r > \mu_P$ and $r > \mu_Q$.

As we have shown the roots of the auxiliary equation to be real and distinct, we become immediately aware that equation (2.2.3.2) must therefore admit a general solution of the form

$$G_0(t_1) = Ae^{\beta_1 t_1} + Be^{\beta_2 t_1},$$

or equivalently

$$F_0(X) = AX^{\beta_1} + BX^{\beta_2}. \tag{2.2.3.3}$$

This general solution for the function F_0 is subject to the usual boundary conditions that are typically present in any optimal control problem (refer to Section 1.2.2). The first boundary is the value-matching condition which provides the recommended function value at which the idle follower should invest and become active in the market.

Intuitively speaking, market entry will only become optimal for the follower when the value of the follower's option to invest is equal to the expected present discounted value (EPV) of the future profits that exercise would yield (minus the investment cost), so that there is no longer any incentive for further delay. Our value-matching condition may thus be stated as

$$\begin{aligned}
F_0(X_F) &= EPV_{\text{(future profits)}} - K \\
&= \xi\left(\int_0^\infty X_t e^{-rt} dt\right) - K \\
&= \int_0^\infty \xi(X_t) e^{-rt} dt - K \\
&= \int_0^\infty X_F e^{(\mu_X)t} e^{-rt} dt - K \\
&= \int_0^\infty X_F e^{-(r-\mu_X)t} dt - K \\
&= \int_0^\infty X_F e^{-(\delta_X)t} dt - K \\
&= \frac{X_F}{\delta_X} - K, \qquad (2.2.3.4)
\end{aligned}$$

where X_F is the corresponding value of the profit flow X that yields this precise function value, and triggers optimal investment.

The value-matching condition therefore marks the follower's change of state from an idle to an active market participant. To ensure a smooth transition from the continuation region to the stopping region, the slopes of the idle and active value functions must also be equal at the point of investment. This gives rise to our second typical boundary condition.

The smooth-pasting boundary equates the derivatives of the two functions at the unique "trigger value" X_F, and is thus given by

$$\left.\frac{dF_0(X)}{dX}\right|_{X=X_F} = \frac{1}{\delta_X}. \tag{2.2.3.5}$$

As the trigger itself requires solution, the region in X-space over which the Cauchy-Euler equation (2.2.3.1) remains valid is subsequently endogenous to the model. This form of optimal control is thus referred to as a free-boundary problem, and even though the differential equation is only of second order, a third condition is necessary for our joint determination of the free boundary X_F with the function $F_0(X)$.

Recalling that X evolves according to a geometric Brownian motion, a convenient property of all such diffusion processes lies in the existence of an absorbing barrier at zero. Thus, if the profit flow X ever falls to zero, it will remain there forever and the follower's option to invest will be rendered worthless. It therefore follows that our third boundary should be provided by the initial condition

$$F_0(0) = 0. \tag{2.2.3.6}$$

Now it is evident that the general solution (2.2.3.3) has to be finite, and as X decreases, the value function of the follower must also necessarily decrease. Hence, in order for condition (2.2.3.6) to hold, we next take $B = 0$ to eliminate the negative exponent β_2. This leaves

$$F_0(X) = AX^{\beta_1}, \tag{2.2.3.7}$$

so that we are finally in a position to apply the value-matching and smooth-pasting conditions. Our simultaneous solution of the coefficient A and of the trigger X_F is given in Appendix B.1, yielding

$$A = X_F^{-\beta_1}(\lambda - 1)K$$

and

$$X_F = \lambda\delta_X K,$$

where

$$\lambda = \frac{\beta_1}{\beta_1 - 1}.$$

By substituting our expression for A into the general equation (2.2.3.7), the follower's option-value function

$$F_0(X) = f_0(P, Q)$$

is obtained[7].

[7] We recall from Appendix A.1 that this transformation is permissible due to the simple relationship between the variables X, P and Q.

This in turn gives rise to a complete piecewise solution, where the active component is described by the net present value of cash flows:

$$f(P,Q) = F(X) \quad = \quad \begin{cases} (\lambda - 1)K\left(\dfrac{X}{X_F}\right)^{\beta_1} & X < X_F \\ \dfrac{X}{\delta_X} - K & X \geq X_F \end{cases} \tag{2.2.3.8}$$

where

$$X_F = \lambda \delta_X K. \tag{2.2.3.9}$$

It is interesting to note that the idle follower's value function is analogous to that of a monopolist American option to enter second in the hypothetical new market. As we have only assumed the existence of two firms that possess the opportunity to invest, our model therefore allows the follower to enter optimally at it's value-maximizing point, after having been pre-empted by its rival. Thus, in the absence of any further competition, the follower essentially holds a monopoly on the right to enter second, and should only invest when the optimal trigger X_F is reached. We summarize these conclusions in the subsequent proposition.

Proposition 1: *The optimal strategy for the follower, conditional on the leader's previous entry, is to invest as soon as the profits reach X_F. The optimal time for the follower to invest may thus be stated as*

$$T_F = \inf\{t \geq 0 : X = \lambda \delta_X K\}.$$

2.3 The Option Value Multiple, λ

In this section, we shall refer back to the traditional practice of capital budgeting, or net present value (NPV) analysis, in comparison to our use of real-option valuation. In the combined words of Dixit and Pindyck in (3), and Mason and Trigeorgis in (21) respectively, the aim of our discussion is "*to examine the shortcomings of the conventional [approach] to decision making about investment*", and to highlight the strides that have been taken to "*bridge the gap between financial theory and strategic planning*" through the research of real options.

In essence, NPV analysis deems investment to be economically viable as long as the expected present value of future profits is at least as large as the required sunk investment cost. Based on this theory, the follower in our hypothetical scenario would enter the market as soon as

$$\frac{X}{\delta_X} \geq K,$$

or

$$X \geq \delta_X K. \qquad (2.3.1)$$

Now if we compare this NPV trigger for the follower to the corresponding profit trigger X_F in our real-option model, we see that our optimal investment rule is actually the product of the traditional trigger given by (2.3.1) above, and the factor λ. The quantity

$$\lambda = \frac{\beta_1}{\beta_1 - 1}$$

is referred to as the option value multiple, and is graphed below as a function of the auxiliary root $\beta_1 > 1$:

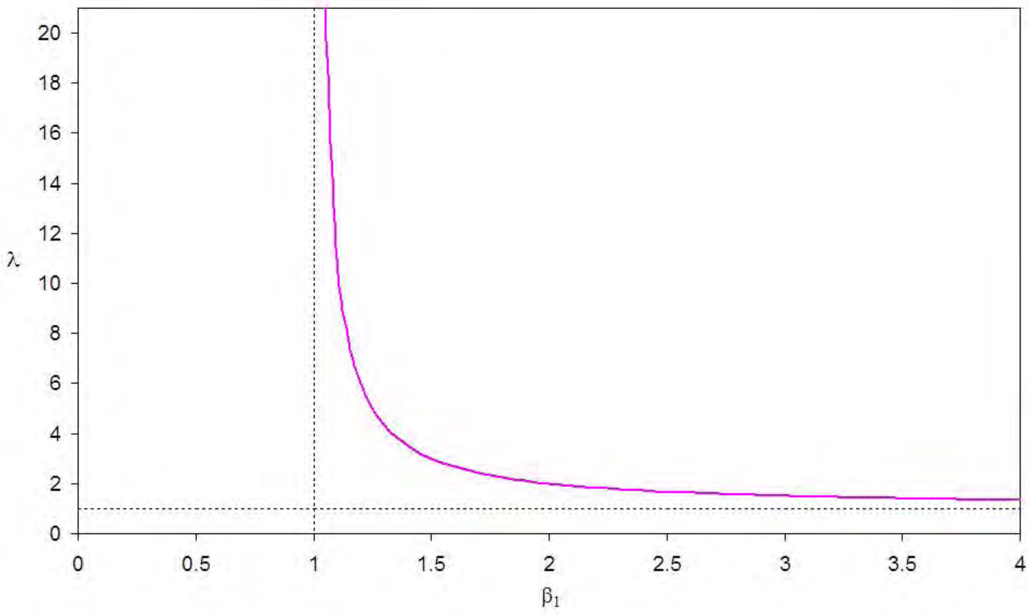


Figure 2.2: Option value multiple

By examining Figure 2.2, we see that the option value multiple decreases exponentially with increases in β_1, which therefore means that

$$\frac{\partial \lambda}{\partial \beta_1} < 0.$$

Of utmost interest, however, is the asymptotic tendency of the option value multiple as

$$\beta_1 \to \infty.$$

A key implication of the real-option approach to investment valuation stems from the following limiting result:

$$\lim_{\beta_1 \to \infty} \lambda = 1,$$

so that $\forall \beta_1 (> 1)$, we must have

$$\lambda > 1.$$

Hence, regardless of the parameters that are imposed, our optimal trigger X_F will always exceed the required level of profit that the corresponding NPV rule specifies for the follower's entry. Real-option valuation thus delays investment beyond the prerequisite of a positive NPV.

Further to the referenced work of Dixit and Pindyck, and Mason and Trigeorgis, there is extensive literature that discusses the faulty assumptions upon which the NPV method is based. The full compilation of papers in (30) provides a comprehensive overview of real options and the various inadequacies of traditional investment valuation that they address. The practical example below highlights just the basic weakness of NPV analysis, using the same default parameter values together with $K = 5$, $P = 0.01$ and $Q \in [0; 100]$.

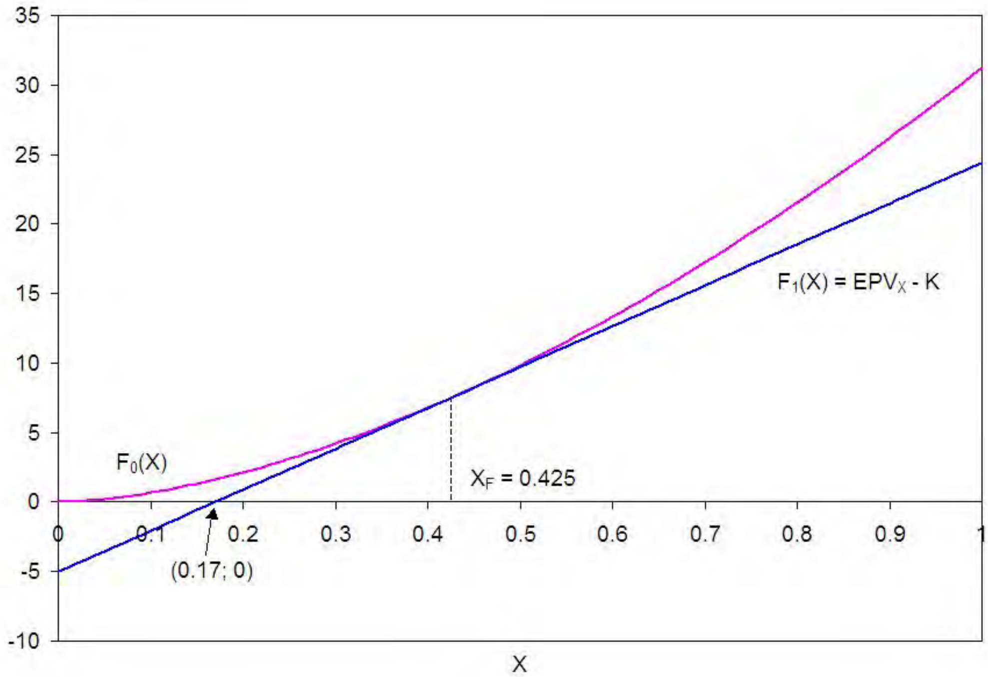


Figure 2.3: The optimal trigger as a point of tangency

Figure 2.3 demonstrates a graphical method of solution for the optimal trigger X_F, as an alternative to applying the closed-form expression (2.2.3.9). The tangential point of contact between the follower's continuation strategy or option value $F_0(X)$, and the follower's value from immediate investment $F_1(X)$, provides an illustrative interpretation of the value-matching and smooth-pasting boundaries.

For our chosen set of parameters, the option value multiple

$$\lambda = 2.5$$

thus determines that the profit flow X must be at least *two and half times* as large as the corresponding NPV trigger before the follower should invest. The simple NPV rule is therefore grossly in error. Based on the closed-form solution (2.2.3.8), the value of the follower's investment opportunity is given by

$$f(P,Q) = F(X) \quad = \quad \begin{cases} 31.21857X^{-1.67} & \text{for} \quad X < 0.425 \\ \\ 29.41176X - 5 & \text{for} \quad X \geq 0.425 \end{cases} .$$

Figure 2.3 also portrays the fundamental flaw of NPV analysis. In line with Dixit and Pindyck's explanation in (2), the simple NPV rule must be modified to incorporate the opportunity cost of immediate investment, as opposed to delay. We may quantify that opportunity cost exactly. When $X < X_F$,

$$F_0(X) > F_1(X) = \frac{X}{\delta_X} - K$$

and therefore

$$\frac{X}{\delta_X} < K + F_0(X).$$

This inequality exposes the inefficiency of the traditional approach. The option-valuation technique deems market entry to be sub-optimal as long as the expected present value of the investment opportunity is less than its *full* cost, *viz*. the direct cost K plus the opportunity cost $F_0(X)$. This revelation accounts for the presence of the option value multiple in equation (2.2.3.9).

Upon entering the market, the follower surrenders the ability to revise its operating strategy should future uncertainty resolve unfavourably. The value of this foregone managerial flexibility is embedded in the option premium $F_0(X)$. In (21), Mason and Trigeorgis introduce an expanded NPV criterion which combines this additional source of value with the static NPV of directly measurable cash flows:

Expanded NPV = Static NPV + Option Premium.

By implementing this expanded definition of NPV, the traditional rule for investment valuation should theoretically produce a trigger level of profit that coincides with the real-option trigger X_F. Hence, the revised criterion succeeds in rectifying the fundamental inadequacy of NPV analysis, but many other complications arise in the application of the traditional method (see (30)).

2.4 The Leader's Value Function

In this section we shall attempt to derive a closed-form solution for the leader's value function which, together with the follower's value function, will then define the Stackelberg-Nash equilibrium that arises in our pre-emptive game.

Until the follower's optimal profit trigger X_F is hit, the first entrant will remain alone in the market and will earn monopolistic revenues. In addition to these profits, we assume that the leader will also acquire a lasting competitive advantage in market share over the follower, thereby creating a significant incentive for either player to invest first. In fear of being pre-empted by its rival, the leader will subsequently enter the market sooner than would otherwise be optimal in a monopolistic framework. The leader's option to maximize its value function and delay investment thus diminishes in our competitive setting.

Given the strategic interaction that unfolds between the leader and the follower in a pre-emptive environment, we shall therefore use indifference rather than optimization techniques to determine the leader's profit trigger for market entry. Now at the moment in time when the profit flow *does* eventually hit the leader's trigger, both firms will want to invest in order to obtain the competitive advantage. To induce a game of sequential exercise and avoid simultaneous entry (which would be sub-optimal at this point according to Weeds in (33)), we assume further that the leader will be arbitrarily determined by the toss of a coin[8].

As the roles of the two identical firms are endogenous to the model, a random procedure in which the rivals have equal chance of success has to be implemented in defining these roles. The winner will assume the position of the leader and will have no further action to take after investing. Monopoly profits will be enjoyed until the follower enters, at which point the leader will be forced to share the market. In order to maximize its value, however, the losing firm will only invest when the profit trigger X_F is reached. Hence, only if the initial value of the state variable X is greater than trigger X_F, will a non co-operative simultaneous equilibrium arise in which one player enters and the other immediately follows.

For the sake of clarity, we shall begin our derivation by first reiterating that the state variable Q refers to the number of units sold in the market by the

[8]This fair game is also used in Grenadier (10) and Weeds (33).

follower. Hence, if we consider $\overline{m}$ and m to be multipliers of Q in the magnitude

$$1 < m < \overline{m},$$

then $\overline{m}Q$ may be said to denote the number of units sold by the leader whilst alone in the market, with mQ thereafter describing the quantity sold by the leader once the follower invests. The subsequent expression

$$Q(m-1)$$

therefore characterizes the first mover's advantage in market share over the follower. The value function of the pre-emptive leader, prior to the second mover's entry, may thus be explained by the following equation[9]:

$$L_0^P(X) = \xi\left(\int_0^{T_F} \overline{m}Xe^{-rt}dt\right) + \xi\left(e^{-rT_F}\right)\frac{mX_F}{\delta_X} - K. \tag{2.4.1}$$

The first term in equation (2.4.1) models the total profit that the leader can expect to earn when alone in the market, whilst the second term denotes the expected value of the leader's instantaneous profit at the moment that the follower enters. Now since T_F depends on the realization of X, we shall let

$$h(X) = \xi\left(e^{-rT_F}\right),$$

where X has already been shown to follow a geometric Brownian motion. After a short interval of time dt, the state variable will evolve to

$$(X + dX),$$

and the function will assume the value

$$h(X + dX).$$

As our model is based within a continuous framework, we must apply the discount factor[10]

$$e^{-rdt} \approx 1 - rdt$$

[9]In a game of pre-emption, the zero subscript pertains strictly to the *follower's* continuation region.

[10]By ignoring terms of order $(dt)^2$ and higher, the Taylor series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

may be approximated by

$$e^x = 1 + x.$$

in order to calculate the present value of the function. Furthermore, as dX is a random increment from the current perspective, an expectation of the evolution in the function value must also be taken. We therefore arrive at the following expression:

$$\begin{aligned} h(X) &= e^{-rdt}\xi[h(X+dX)] \\ &= e^{-rdt}\xi[h(X)+dh(X)] \\ &= e^{-rdt}\{h(X)+\xi[dh(X)]\} \end{aligned} \tag{2.4.2}$$

where the term $\xi[dh(X)]$ requires some expansion. By applying Itô's Lemma and recalling that $\xi(dz_X)=0$, we obtain

$$\begin{aligned} \xi[dh(X)] &= \xi\left[\left\{\mu_X X\frac{dh(X)}{dX}+\frac{1}{2}\sigma_X^2 X^2\frac{d^2h(X)}{dX^2}\right\}dt+\sigma_X X\frac{dh(X)}{dX}dz_X\right] \\ &= \left\{\mu_X X\frac{dh(X)}{dX}+\frac{1}{2}\sigma_X^2 X^2\frac{d^2h(X)}{dX^2}\right\}dt, \end{aligned}$$

so that expression (2.4.2) then becomes

$$\begin{aligned} h(X) &= e^{-rdt}\left\{h(X)+\left\{\mu_X X\frac{dh(X)}{dX}+\frac{1}{2}\sigma_X^2 X^2\frac{d^2h(X)}{dX^2}\right\}dt\right\} \\ &= (1-rdt)\left\{h(X)+\left\{\mu_X X\frac{dh(X)}{dX}+\frac{1}{2}\sigma_X^2 X^2\frac{d^2h(X)}{dX^2}\right\}dt\right\}. \end{aligned}$$

If we choose to discard terms of negligible magnitude $(dt)^2$ after expansion, simplification then yields another second order Cauchy-Euler ordinary differential equation that is analogous to equation (2.2.3.1):

$$\frac{1}{2}\sigma_X^2 X^2\frac{d^2h(X)}{dX^2}+\mu_X X\frac{dh(X)}{dX}-rh(X)=0.$$

Thus, we may immediately conclude that the general solution to this differential equation is once again of the form

$$h(X)=CX^{\beta_1}+DX^{\beta_2}, \tag{2.4.3}$$

where β_1 and β_2 remain as previously defined. Now, in order to establish the boundary conditions that are necessary for the determination of the coefficients C and D, we next consider the effects of an evolving profit flow on our function $h(X)$.

As the profit flow approaches the follower's trigger X_F, the remaining time until the follower invests, T_F, will tend to zero. Hence, we must have

$$\lim_{X \to X_F} h(X) = \lim_{T_F \to 0} \xi(e^{-rT_F}) = 1. \tag{2.4.4}$$

In contrast, a second boundary may be defined as the profit flow X tends to zero. If X is absorbed at the zero barrier, the follower's trigger X_F will never be reached. The optimal time for the follower to invest will thus be delayed *ad infinitum* so that

$$\lim_{X \to 0} h(X) = \lim_{T_F \to \infty} \xi(e^{-rT_F}) = 0, \tag{2.4.5}$$

which then implies that the second coefficient D in equation (2.4.3) must again be equal to zero:

$$h(X) = CX^{\beta_1}. \tag{2.4.6}$$

By applying the first boundary condition (2.4.4) to our general solution (2.4.6), it follows that

$$\begin{aligned} \lim_{X \to X_F} h(X) = h(X_F) &= 1 \\ CX_F^{\beta_1} &= 1 \\ C &= X_F^{-\beta_1}, \end{aligned}$$

and an explicit solution for $h(X)$ is finally obtained:

$$h(X) = \left(\frac{X}{X_F} \right)^{\beta_1}. \tag{2.4.7}$$

Returning our attention to the leader's value equation (2.4.1), we shall next let

$$g(X) = \xi \left(\int_0^{T_F} \overline{m} X e^{-rt} dt \right).$$

Over a short interval of time dt prior to the follower's entry, the leader will earn monopolistic profits amounting to

$$\overline{m} X dt,$$

and our function will assume the random value

$$g(X + dX).$$

Now in theory, the profit flow $X\overline{m}$ could also stochastically evolve over this short interval of continuous time, but the difference made by this consideration is of negligible magnitude $(dt)^2$ and shall be ignored. Thus,

$$\begin{aligned} g(X) &= \overline{m}Xdt + e^{-rdt}\xi[g(X+dX)] \\ &= \overline{m}Xdt + e^{-rdt}\xi[g(X)+dg(X)] \\ &= \overline{m}Xdt + e^{-rdt}\{g(X)+\xi[dg(X)]\}. \end{aligned} \tag{2.4.8}$$

If we once again expand the expected value by Itô's Lemma and substitute it back into equation (2.4.8), simplification will then show that $g(X)$ satisfies the following ordinary differential equation:

$$\frac{1}{2}\sigma_X^2 X^2 \frac{d^2g(X)}{dX^2} + \mu_X X \frac{dg(X)}{dX} - rg(X) + \overline{m}X = 0.$$

The presence of the term $\overline{m}X$ renders the equation non-homogeneous, and the general solution is therefore given by

$$g(X) = EX^{\beta_1} + IX^{\beta_2} + \frac{\overline{m}X}{\delta_X}, \tag{2.4.9}$$

where the final term represents the expected present value of the profit that the leader earns whilst alone in the market, and serves as a particular solution.

Analogous boundary conditions may then be imposed. The first arises intuitively from the fact that when the follower enters, the market will be shared and the leader's excess profits will dry up. Thus,

$$g(X_F) = 0. \tag{2.4.10}$$

In contrast, the leader will permanently retain its monopoly on the market if the state variable X is absorbed at zero, but will never earn any profit. The second boundary condition is therefore given by

$$g(0) = 0, \tag{2.4.11}$$

and subsequently deems that the coefficient I in our general solution (2.4.9) must assume a value of zero:

$$g(X) = EX^{\beta_1} + \frac{\overline{m}X}{\delta_X}. \tag{2.4.12}$$

By combining equation (2.4.12) with condition (2.4.10), we get that

$$EX_F^{\beta_1} + \frac{\overline{m}X_F}{\delta_X} = 0$$

$$EX_F^{\beta_1} = -\frac{\overline{m}X_F}{\delta_X}$$

$$E = -\left(\frac{1}{X_F}\right)^{\beta_1}\frac{\overline{m}X_F}{\delta_X},$$

which in turn yields the following precise solution for $g(X)$:

$$g(X) = -\frac{\overline{m}X_F}{\delta_X}\left(\frac{X}{X_F}\right)^{\beta_1} + \frac{\overline{m}X}{\delta_X}, \tag{2.4.13}$$

where the first term captures the negative effect that the follower's entry will have on the leader's value function.

Upon substituting solutions (2.4.7) and (2.4.13) back into equation (2.4.1), we eventually obtain an explicit expression for the pre-emptive leader's value function prior to the follower's entry. The entire piecewise value function is given by:

$$L^P(X) \quad = \quad \begin{cases} \left(\frac{X}{X_F}\right)^{\beta_1}\lambda K(m-\overline{m}) + \frac{\overline{m}X}{\delta_X} - K & X < X_F \\ \\ \frac{mX}{\delta_X} - K & X \geq X_F \end{cases}, \tag{2.4.14}$$

where the first expression applies over the *follower's* continuation region, and the second in the *follower's* stopping region. This closed-form solution thus completes the characterization of the Stackelberg-Nash equilibrium that arises in our pre-emptive game, but we have yet to define the profit trigger for the leader's entry into the market. In Figure 2.4, a graphical representation of our classical pre-emptive equilibrium[11] leads us to the answer using our default parameter values, together with $m = 1.5$ and $\overline{m} = 2.5$.

The value function of the leader is almost always larger than that of the follower, except when total profits are very low. The follower remains idle over this domain, and if the leader were already in the market, it would be better off being a follower. Hence, it is sub-optimal for either firm to invest at such levels of profit. It is therefore intuitive that the leader's pre-emptive trigger X_L^P should be defined by the profit level at which the two functions meet[12]. At this investment point, the expected payoffs of the two firms must be equal to ensure that the proposed outcome is an equilibrium in which neither firm has any incentive to deviate.

[11] The same shape would obtain if the number of units were held fixed and the unit profits were allowed to vary.

[12] It is at this point that a coin should be tossed to determine the leader.

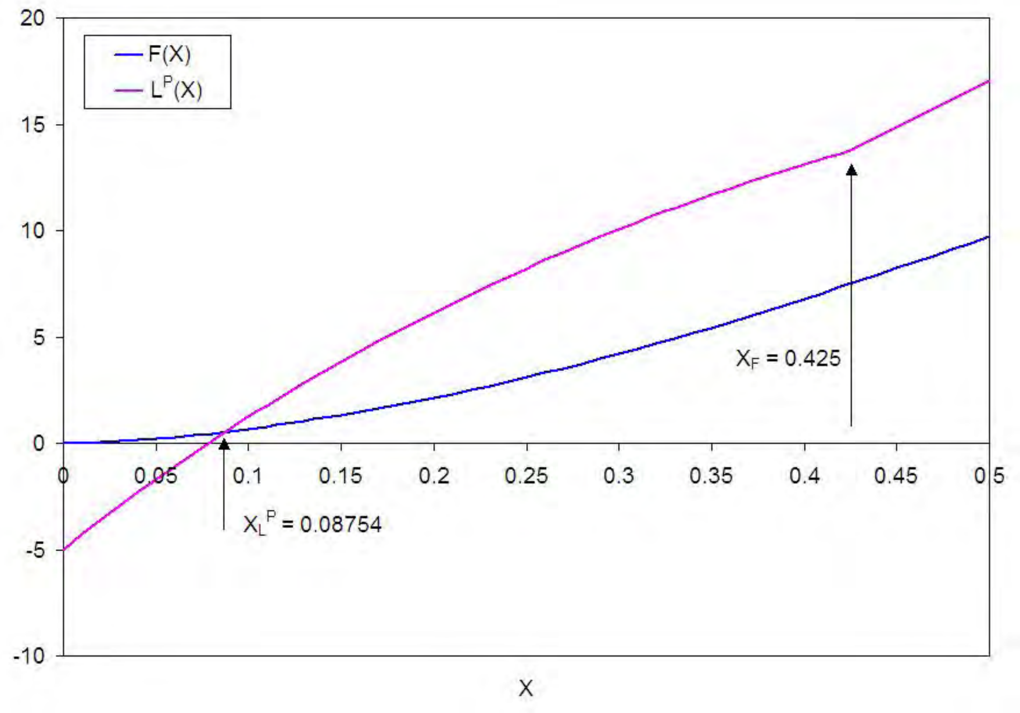


Figure 2.4: Pre-emptive equilibrium

The equality

$$L_0^P(X_L^P) = F_0(X_L^P)$$

thus arises from the fact that the threat of pre-emption equalizes rents in a duopoly, and has therefore been described as an illustration of the rent equalization principle of Fudenberg and Tirole in (8), with Kong and Kwok having adapted the principle in (16) to define rent equalization as a "*state equilibrium*" in which the benefit of being the leader is equal to that of being the follower.

Although the firms are indifferent to their roles at the profit level X_L^P, it is preferable to be the leader from here on after. Hence, this profit level defines the trigger for the leader's pre-emptive entry, but it is impossible to derive a closed-form solution for X_L^P because the resulting equation is highly non-linear. The trigger can nevertheless be obtained numerically[13], and it is possible to prove the existence of a unique root strictly below X_F. The detailed proof given in Appendix C justifies the following proposition.

***Proposition* 2**: *The optimal strategy for the pre-emptive leader is to invest as soon as the profits reach X_L^P. The optimal time for the leader to invest may thus be stated as*

$$T_L^P = \inf\left\{t > 0 : X \in [X_L^P, X_F)\right\}.$$

[13]For our chosen parameter values, *Solver* provides the numerical solution $X_L^P = 0.08754$, whilst $X_F = 0.425$.

2.5 Sensitivity Analysis of the Pre-emptive Equilibrium

Although Figure 2.4 only offers one particular illustration of our pre-emptive equilibrium, it is effective nonetheless in highlighting the predictable shape of the follower's value function. An examination of the first derivative of solution (2.2.3.8) readily confirms that, regardless of the parameters that are imposed upon our model, the follower's value function will always exhibit a positive slope and increase linearly with total profit once the follower becomes active in the market:

$$\frac{dF(X)}{dX} = \begin{cases} \dfrac{1}{\delta_X}\left(\dfrac{X}{X_F}\right)^{\beta_1 - 1} > 0 & X < X_F \\ \dfrac{1}{\delta_X} > 0 & X \geq X_F \end{cases}.$$

The value function of the pre-emptive leader is more complicated than that of the follower. It is concave until the moment that the follower invests, at which point its slope becomes discontinuous. To explain this discontinuity, it is necessary for us to examine the sensitivity of the leader's value function with respect to the state variable:

$$\frac{dL^P(X)}{dX} = \begin{cases} \left(\dfrac{X}{X_F}\right)^{\beta_1 - 1} \dfrac{\beta_1(m - \overline{m})}{\delta_X} + \dfrac{\overline{m}}{\delta_X} \neq 0 & X < X_F \\ \dfrac{m}{\delta_X} > 0 & X \geq X_F \end{cases}.$$

Prior to the follower's entry, the slope of the leader's value function is dependent on the magnitude of the ratio

$$\left(\frac{X}{X_F}\right)^{\beta_1 - 1}.$$

As the profit flow X approaches the trigger X_F, the value of this ratio increases so that the negative effect

$$(m - \overline{m})$$

thus becomes significant and offsets the leader's rising monopoly profits to such an extent that a negative gradient results. Once the follower becomes an active market participant, the value functions of either player are only distinguishable due to the leader's permanent competitive advantage. If the follower was to capture an equal share of the market, the value functions would be identical from X_F onwards.

Although the invalidation of the standard NPV rule is perhaps the most well-known result that has arisen from mainstream real-option research, the overview given in Section 2.3 ignores the strategic interactions between the rivaling firms. In the absence of any further competition, the follower essentially holds a monopoly on the option to enter the market *second*. The follower's investment strategy is thus formulated in isolation, without regard to the potential impact of any other firm's choice of action.

With reference to the example used in Section 2.3, the real-option approach posits that such a monopolist should not enter the market until the expected present value of future profits is at least two and half times the initial cost of investment. However, this result is highly dependent on the lack of competition. In (11), Grenadier demonstrates that, "*even for industries with only a few competitors, the [fear of pre-emption] drastically erodes the value of the option to wait and leads to investment at very near the zero net present value threshold.*" Our game-theoretic analysis supports this conclusion.

In the pre-emptive equilibrium displayed in Figure 2.4, the leader enters the market at the sub-optimal profit level

$$X_L^P = 0.08754.$$

As we shall later explain in Section 3.3, a leader that ignores the possibility of pre-emption will choose to invest as if it were a monopolist. Now the expected present discounted value of the leader's monopoly profits is given by the expression

$$\begin{aligned}
\xi\left(\int_0^\infty \overline{m}X_t e^{-rt}dt\right) &= \int_0^\infty \overline{m}\xi(X_t)e^{-rt}dt \\
&= \int_0^\infty \overline{m}Xe^{(\mu_X)t}e^{-rt}dt \\
&= \int_0^\infty \overline{m}Xe^{-(r-\mu_X)t}dt \\
&= \int_0^\infty \overline{m}Xe^{-(\delta_X)t}dt \\
&= \frac{\overline{m}X}{\delta_X},
\end{aligned}$$

where $X = X_0$ denotes the flow of profit earned by the leader upon immediate exercise. Hence, traditional NPV theory would encourage the first mover to invest as soon as

$$\frac{\overline{m}X}{\delta_X} > K,$$

or

$$X > \frac{\delta_X K}{\overline{m}} = 0.068. \tag{2.5.1}$$

By comparing this value to the corresponding pre-emptive trigger X_L^P, we see that the equalization of rents requires a level of profit that is only 1.28736 times larger than the static NPV threshold. This is in stark contrast to the option value multiple of 2.5 that we obtained in the case of the follower.

Hence, we may conclude that the option to wait becomes less valuable as the degree of rivalry intensifies, thereby providing further evidence to support Grenadier's theory, and validating our use of indifference methods, rather than optimization, to derive the pre-emptive leader's value function.

Chapter 3

Co-operative Equilibria

In this chapter, we aim to model the timing of market entry for both firms when there is no race to invest first. In order for such a non pre-emptive environment to exist, the roles of the leader and the follower must be assigned exogenously so that each player is secure in the knowledge of its rival's future actions, and may thus optimize its pre-assigned position in the market. Hence, the leader's adoption point is no longer defined by indifference, but rather by the profit level that maximizes its value function. The leader holds the right to enter first, and is aware that the follower will invest when its own value-maximizing trigger is reached.

3.1 Sequential Investment

We shall first consider an equilibrium in which the leader and the follower agree to enter the market at different points in time. Although the players are no longer competing in a game of pre-emption, the resulting pattern of sequential investment that we are going to model once again resembles that of a Stackelberg-Nash equilibrium.

Within this context, the follower's value function is not affected by the co-operative game. The follower will remain idle until it becomes worthwhile to exercise its American call option to enter second, thereby maximizing its value function. By implementing the same replicating argument that we applied in the pre-emptive game, the follower's value function is thus given by equation (2.2.3.8) and, conditional on the leader's prior entry, the optimal time for the follower to invest is once again defined by the profit trigger (2.2.3.9).

Since the leader does not have to act under the fear of pre-emption, this firm can also delay investment to optimize its market entry. In a co-operative framework, the leader's value function is therefore characterized by three distinct components:

1. Before the leader enters, the firm holds an American option to invest first in the new market.

2. Once active, the leader earns monopoly profits until the follower's trigger is hit.

3. After the follower enters, the leader shares the market but maintains a permanent first-mover advantage.

The second and third components are analogous to the leader's piecewise value function in the pre-emptive game. The first component describes the leader's continuation region and requires some discussion.

As the leader has the option to wait in a non pre-emptive environment, we may immediately deduce that its value function prior to entry[1], $L_0(X)$, must be explained by an ordinary differential equation of the exact form as equation (2.2.3.1). Hence, the following general solution results:

$$L_0(X) = JX^{\beta_1} + NX^{\beta_2}$$

where the roots remain as previously defined. Since β_2 is negative and the value function $L_0(X)$ must necessarily decrease as the profit level X decreases, previous exposition deems that N equals zero:

$$L_0(X) = JX^{\beta_1}. \tag{3.1.1}$$

Now, the leader will be alone in the market when it enters. Thus, investment will only be optimal when the net present value of the *monopoly* profits that the leader expects to receive[2] is at least as large as the value of its option to enter. Hence, the first component of equation (2.4.14) must be used to impose the value-matching condition:

$$L_0(X_L) = \left(\frac{X_L}{X_F}\right)^{\beta_1} \lambda K(m - \overline{m}) + \frac{\overline{m}X_L}{\delta_X} - K \tag{3.1.2}$$

where X_L denotes the leader's non pre-emptive trigger. The smooth-pasting boundary follows naturally:

$$\left.\frac{dL_0(X)}{dX}\right|_{X=X_L} = \left(\frac{X_L}{X_F}\right)^{\beta_1 - 1} \frac{\lambda K \beta_1}{X_F}(m - \overline{m}) + \frac{\overline{m}}{\delta_X} \tag{3.1.3}$$

thus enabling the simultaneous solution of the coefficient J and of the trigger X_L. The calculation in Appendix B.2 produces

$$J = X_F^{-\beta_1} \lambda K(m - \overline{m}) + X_L^{-\beta_1}(\lambda - 1)K$$

[1] In a co-operative game, the zero subscript pertains to the respective firm's continuation region.

[2] This net present value incorporates the adverse effect of the follower's impending entry.

and

$$X_L = \frac{\lambda \delta_X K}{\overline{m}}.$$

Combining this expression for J with equation (3.1.1) characterizes the leader's option to enter the market. The non pre-emptive leader's entire piecewise value function is therefore given by

$$L(X) = \begin{cases} \left(\frac{X}{X_F}\right)^{\beta_1} \lambda K(m - \overline{m}) + \left(\frac{X}{X_L}\right)^{\beta_1} (\lambda - 1)K & X < X_L \\ \left(\frac{X}{X_F}\right)^{\beta_1} \lambda K(m - \overline{m}) + \frac{\overline{m}X}{\delta_X} - K & X \in [X_L; X_F) \\ \frac{mX}{\delta_X} - K & X \geq X_F \end{cases}, \quad (3.1.4)$$

where

$$X_L = \frac{\lambda \delta_X K}{\overline{m}} \quad (3.1.5)$$

and X_F is once again given by equation (2.2.3.9).

The repeated term in the first and second components of equation (3.1.4) models the adverse effect of the follower's option to invest. The leader's value function is thus diminished by the follower's impending entry, even before the leader is active in the market. The first component of equation (3.1.4) displays the value of the leader's American call option to enter first, the second models the leader's expected monopoly profits, and the third describes the value of the market when shared with the follower.

Based on the forecast developments in the game, the actions of the pre-assigned leader may thus be defined in anticipation of the follower's response.

***Proposition* 3**: *The optimal strategy for the non pre-emptive leader is to invest as soon as the profits reach X_L. The optimal time for the leader to invest may thus be stated as*

$$T_L = \inf\left\{t > 0 : X = \frac{\lambda \delta_X K}{\overline{m}}\right\}.$$

3.2 Simultaneous Investment

We shall next consider an alternative form of non pre-emptive equilibrium in which the leader and the follower collude on investment timing. If the firms agree to enter simultaneously, a Cournot-Nash equilibrium will arise in which we assume that both entities will capture an equal share of the market. We shall let

$$1 < m_s < m$$

such that $m_s Q$ represents the instantaneous number of units that each colluding investor will sell. This framework thus describes a situation in which it is beneficial for the follower to collude, whilst sequential investment remains preferable for the leader.

If the firms conspire, they will choose to delay investment until simultaneous entry becomes optimal[3]. As mutual investment will render their value functions indistinguishable, it is easiest to treat colluding firms as a single investor. The combined entity thus possesses a monopoly on the option to enter the new market. Hence, it makes sense for us to employ the same methodology that we used to derive the follower's value function in the pre-emptive game, and the leader's value function in the co-operative sequential equilibrium, in order to model the value function of each idle firm, $S_0(X)$, under collusive investment. Without any preamble, we are therefore able to state that the general solution is given by

$$\begin{aligned} S_0(X) &= WX^{\beta_1} + UX^{\beta_2} \\ &= WX^{\beta_1}, \end{aligned} \tag{3.2.1}$$

with value-matching condition

$$\begin{aligned} S_0(X_S) &= EPV_{\text{(future profits)}} - K \\ &= \xi \left(\int_0^\infty m_s X_t e^{-rt} dt \right) - K \\ &= \int_0^\infty m_s \xi(X_t) e^{-rt} dt - K, \\ &= \int_0^\infty m_s X_S e^{(\mu_X)t} e^{-rt} dt - K \\ &= \int_0^\infty m_s X_S e^{-(r-\mu_X)t} dt - K \\ &= \int_0^\infty m_s X_S e^{-(\delta_X)t} dt - K \\ &= \frac{m_s X_S}{\delta_X} - K, \end{aligned} \tag{3.2.2}$$

such that X_S denotes the level of profit at which it becomes optimal for the idle

[3]A non pre-emptive environment thus enables both firms to maximize their value functions, whether sequential or collusive strategies are implemented.

firms to enter the market together. The subsequent smooth-pasting boundary

$$\left.\frac{dS_0(X)}{dX}\right|_{X=X_S} = \frac{m_s}{\delta_X} \tag{3.2.3}$$

completes the requirements for the simultaneous determination of the coefficient W and of the trigger X_S in Appendix B.3:

$$W = X_S^{-\beta_1}(\lambda - 1)K$$

and

$$X_S = \frac{\lambda \delta_X K}{m_s}.$$

By combining the above expression for W with the general solution (3.2.1), we are able to determine the option-value function $S_0(X)$ of each colluding firm, and thus arrive at the following piecewise equation:

$$S(X) \quad = \quad \begin{cases} (\lambda - 1)K\left(\dfrac{X}{X_S}\right)^{\beta_1} & X < X_S \\ \dfrac{m_s X}{\delta_X} - K & X \geq X_S \end{cases} \tag{3.2.4}$$

where

$$X_S = \frac{\lambda \delta_X K}{m_s}. \tag{3.2.5}$$

Equation (3.2.4) closely resembles the value function of the follower in either a pre-emptive or co-operative sequential investment game. The only structural difference is the presence of the factor

$$m_s > 1,$$

which could potentially be attributed to the advantages that arise from networking. This factor thus provides an input into the measure for the follower's benefit from collusive investment:

$$Q(m_s - 1).$$

As the value functions of both players are maximized under collusive investment, we must question why any two firms would ever choose to participate in a game of pre-emption, when an implicit agreement to invest jointly at X_S produces a more favourable equilibrium for both parties. The answer lies within the realm of game theory. Fudenberg and Tirole explain in (9) how the difficulty of enforcing a cartel creates a form of "prisoner's dilemma". Both firms would benefit by colluding to enter after X_S, but either would have an incentive to break the agreement by investing over the domain

$$[X_L; X_S).$$

Of course, simultaneous equilibrium would arise naturally if the initial value of the state variable was already greater than the follower's profit trigger X_F. Thus, collusive entry has only been considered as a special case of the non pre-emptive game, where joint investment is imposed despite the initial value of the state variable. The next proposition therefore defines the best choice of action for conspiring firms, based on our conclusions.

Proposition 4: *The optimal strategy for colluding investors is to enter as soon as the profits reach X_S. The optimal time for simultaneous investment in a non pre-emptive environment may thus be stated as*

$$T_S = \inf\left\{t > 0 : X = \frac{\lambda \delta_X K}{m_s}\right\}.$$

3.3 Sensitivity Analysis of the Co-operative Equilibria

3.3.1 Sequential Equilibrium

Although the value function of the active leader is the same irrespective of whether the game unfolds in a pre-emptive or co-operative environment, the idle leader's value function is dependent on the nature of the strategic play. As we have already discussed, the fear of pre-emption drastically erodes the value of the leader's option to wait. However, the non pre-emptive leader possesses the ability to delay investment until its value-maximizing point is reached. By referring back to the first component of equation (3.1.4), we see that the derivative of the non pre-emptive leader's option to invest is given by

$$\frac{dL_0(X)}{dX} = \left(\frac{X}{X_F}\right)^{\beta_1 - 1} \frac{\beta_1(m - \overline{m})}{\delta_X} + \left(\frac{X}{X_L}\right)^{\beta_1 - 1} \frac{\overline{m}}{\delta_X},$$

where we recall that

$$X_F = \lambda \delta_X K$$

and

$$X_L = \frac{\lambda \delta_X K}{\overline{m}}.$$

Now since

$$X_F = \overline{m} X_L,$$

it follows that this derivative may be re-expressed in the form

$$\begin{aligned}\frac{dL_0(X)}{dX} &= \left(\frac{X}{\overline{m}X_L}\right)^{\beta_1-1}\frac{\beta_1(m-\overline{m})}{\delta_X}+\left(\frac{X}{X_L}\right)^{\beta_1-1}\frac{\overline{m}}{\delta_X}\\ &= \frac{1}{\delta_X}\left(\frac{X}{\overline{m}X_L}\right)^{\beta_1-1}\left[(m-\overline{m})\beta_1+\overline{m}^{\beta_1}\right].\end{aligned}$$

Although the first term

$$(m-\overline{m})\beta_1$$

is negative, the *magnitude* of the second term will always be greater, i.e.

$$|(m-\overline{m})\beta_1| < \overline{m}^{\beta_1}.$$

Hence, we must have

$$(m-\overline{m})\beta_1+\overline{m}^{\beta_1} > 0,$$

which allows us to conclude that the pre-assigned leader's option to invest will always display as an increasing function of total profit. As the second and third components of value function (3.1.4) have already been examined in Section 2.5, the corresponding derivative functions do not warrant any further discussion.

Given that the leader is able to enter optimally in a non pre-emptive environment, the presence of the option value multiple in trigger (3.1.5) is not coincidental. Substitution of our default parameter values yields the maximizing profit level

$$X_L = 0.17$$

which, as we should now expect, is two and half times larger than the corresponding NPV trigger

$$\frac{\delta_X K}{\overline{m}} = 0.068$$

that we derived for the leader in Section 2.5. Hence, we repeat that neither the follower nor the leader's option value of waiting suffers any erosion within a co-operative framework. However, irrespective of the level of competition, there are several other factors upon which first-mover entry is contingent.

Figure 3.1 illustrates the dependence of the leader's investment trigger on the monopolized market volume $\overline{m}Q$, using the same parameter values that were applied to generate Figure 2.3, but with $m = 1.5$, $Q = 30$ and $\overline{m} \in [2; 24]$.

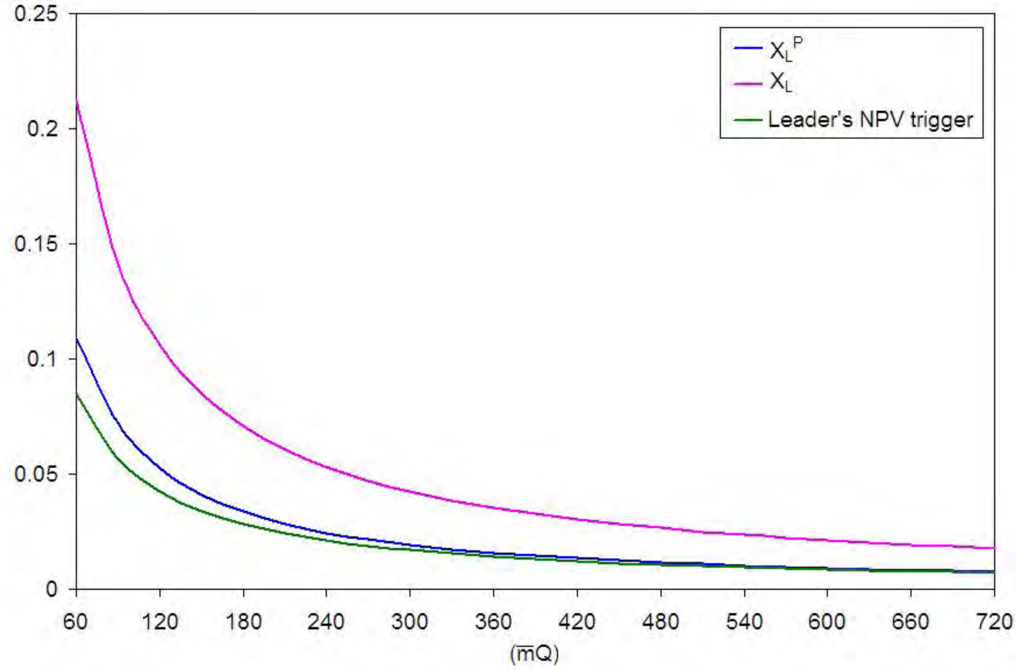


Figure 3.1: Trigger dependence on monopoly volume

Regardless of the degree of rivalry that prevails between the players, or the methodology that we use to prescribe the first-mover's entry, we see that the value of the leader's trigger decreases exponentially with increases in the number of units sold whilst alone in the market. Hence, the higher the monopoly profits, the sooner the leader will enter[4].

Although this relationship is fairly intuitive, we must refer back to value functions (3.1.4) and (2.4.14) in order to understand why the optimal and pre-emptive triggers become less distinct as the quantity $\overline{m}Q$ increases. As the value functions of the active leaders are identical whether in a pre-emptive or co-operative environment, the monopolistic components of both will increase by the same amount as the factor $\overline{m}$ is raised. We therefore turn to the idle component of the non pre-emptive leader's value function in search of an answer.

As the factor $\overline{m}$ is also implicitly present within the profit trigger X_L, we deduce that the value of the pre-assigned leader's option to invest will also rise with increases in $\overline{m}$. Now although we have shown that an increase in option value delays exercise, we notice that the increase in the monopolistic component of

[4]The follower's trigger X_F remains independent of the number of units that the leader sells whilst alone in the market.

the leader's value function will be even greater than the increase in the "opportunity cost" of entering the market. Hence, the non pre-emptive leader will actually become less eager to delay investment when the factor $\overline{m}$ is raised, which thus results in a proportionately larger decrease in the adoption point X_L, than in the pre-emptive trigger X_L^P.

Despite the inadequacies of classic NPV theory, Figure 3.1 also demonstrates the rapid convergence of the pre-emptive trigger X_L^P towards the corresponding NPV trigger for the leader. Thus, for high values of $\overline{m}$, the traditional investment rule provides a fairly accurate approximation for the first mover's entry in a non co-operative framework.

Another quantity of interest is the first-mover advantage

$$Q(m-1).$$

In order to examine the influence of this advantage on the timing of market entry, we once again apply the parameter values of Figure 2.3, but now with $\overline{m} = 2.5$, $Q = 30$ and $m \in [1; 1.6]$.

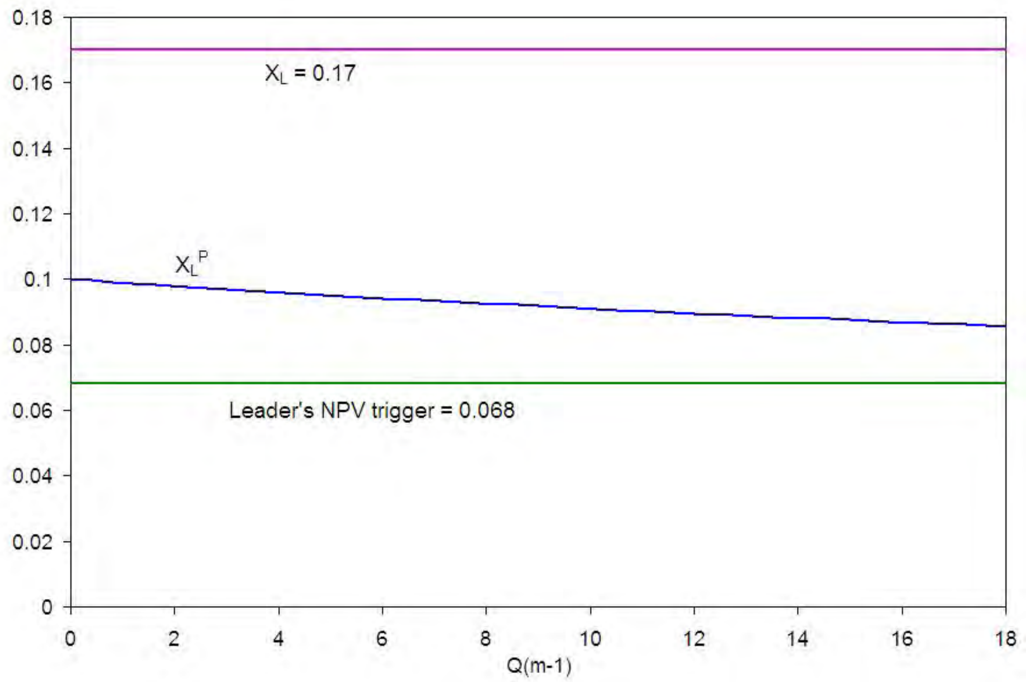


Figure 3.2: Trigger dependence on first-mover advantage

Although we would anticipate the leader's trigger to display similar dependence on this measure, the relationships depicted in Figure 3.2 are counter-intuitive. The non pre-emptive leader's indifference to the magnitude of its competitive advantage is an unexpected feature. If we re-examine value function (3.1.4), we see that the first derivative of both the idle and the monopolistic component with respect to m is given by

$$\left(\frac{X}{X_F}\right)^{\beta_1} \lambda K > 0.$$

Hence, as the first-mover advantage increases, the value of the pre-assigned leader's option to invest increases in exactly the same proportion as the active component prior to the follower's entry. The opposing effects are thus in balance, rendering the first mover indifferent to the magnitude of the competitive advantage that it will hold over the follower in the future. Thus, the non pre-emptive leader's investment strategy corresponds to that of a permanent monopolist, but is devised in full awareness that the "state of bigness" will only prevail for a stochastically finite duration[5].

Given that the optimal profit trigger (3.1.5) is but a multiple of the corresponding NPV threshold, the above discussion serves as a joint explanation for the traditional investment rule's lack of contingence on the first-mover advantage. Although not displayed, we may also surmise the independence of the follower's entry with respect to this measure. The market share Q that the active follower captures is not influenced by the number of units mQ that the leader sells. In actual fact, the converse is true.

Thus, it is only the pre-emptive trigger X_L^P that demonstrates any sensitivity to changes in the magnitude of the leader's competitive advantage. With reference to value function (2.4.14), we see that increases in the market volume multiple m reduce the detrimental effect of the follower's potential entry latter in the game, by rendering the term

$$(m - \overline{m})$$

less negative. As the pre-emptive leader's strategy is devised in anticipation of its rival's best response, a higher first-mover advantage will therefore induce earlier investment because the follower's impending entry will be less damaging to the leader's market volume.

3.3.2 Simultaneous Equilibrium

As our value function for colluding investors is simply a multiple m_s of the follower's value function, the simultaneous equilibrium will therefore respond in a similar fashion to changes in parameter values, and will increase with total profit:

$$\frac{dS(X)}{dX} = \begin{cases} \dfrac{1}{\delta_X}\left(\dfrac{X}{X_S}\right)^{\beta_1 - 1} > 0 & X < X_S \\ \dfrac{m_s}{\delta_X} > 0 & X \geq X_S \end{cases}.$$

[5]This synonymous catch phrase was coined by Fourie *et al* in (7).

Substitution of our default parameter values into trigger (3.2.5), with

$$m_s = 1.1,$$

yields the value-maximizing profit level

$$X_S = 0.38636.$$

For completeness, traditional NPV analysis deems collusive investment to be economically viable if

$$\begin{aligned}
\xi\left(\int_0^{\infty} m_s X_t e^{-rt} dt\right) &> K \\
\int_0^{\infty} m_s \xi(X_t) e^{-rt} dt &> K \\
\int_0^{\infty} m_s X e^{(\mu_X)t} e^{-rt} dt &> K \\
\int_0^{\infty} m_s X e^{-(r-\mu_X)t} dt &> K \\
\int_0^{\infty} m_s X e^{-(\delta_X)t} dt &> K \\
\frac{m_s X}{\delta_X} &> K
\end{aligned}$$

or

$$X > \frac{\delta_X K}{m_s} = 0.154545455, \tag{3.3.2.1}$$

where the optimal trigger X_S is two and half times larger, and $X = X_0$ once again denotes the profit from immediate investment.

In summary, the differing investment triggers may thus be ranked. Since

$$1 < m_s < m < \overline{m},$$

it follows that

$$X_F > X_S > X_L,$$

but within a non co-operative framework, the leader enters sooner than is optimal due to the fear of pre-emption. Hence, we must have

$$X_F > X_S > X_L > X_L^P.$$

Figure 3.3 offers an illustrative comparison of the varying equilibria that may arise in a duopoly, depending on the strategic interactions that unfold between the players.

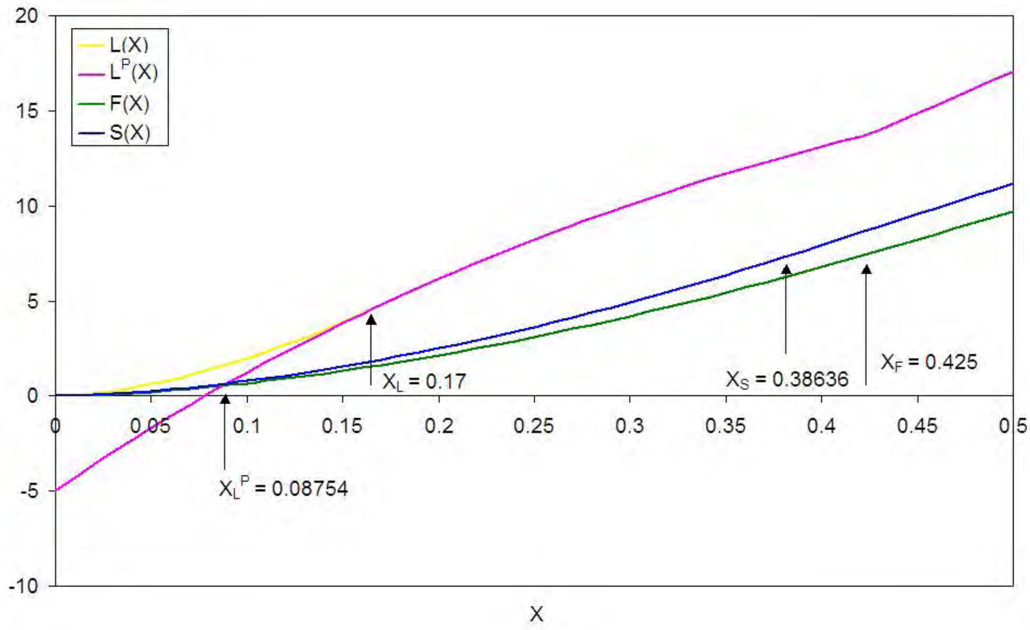


Figure 3.3: Comparison of equilibria with $Q \in [0; 50]$

Chapter 4

Comparative Statics

In order for our model to serve as an effective tool for investment valuation, we must be able to predict the ways in which our profit triggers will vary within such a dynamic economic environment. Comparative statics analysis explores the dependence of our optimal investment rules on the values of the underlying parameters, and allows for direct comparison to qualitative results from standard option pricing models.

To acquire an understanding of the investment rules that our model produces, we shall examine the response of the root β_1, and of the dividend rate δ_X, to changes in the individual parameters ρ_{PQ}, σ_P and σ_Q[1]. This is an extension of Paxson and Pinto's research, as their approach deems the dividend rate δ_X to be independent of each of these parameters.

We shall make frequent reference to the fundamental quadratic

$$\Upsilon(\beta) = \frac{1}{2}\sigma_X^2\beta^2 + (\mu_X - \frac{1}{2}\sigma_X^2)\beta - r,$$

where we recall that β_1 represents the positive solution to the auxiliary equation

$$\Upsilon(\beta) = 0.$$

Given that the non pre-emptive leader's trigger X_L, and the simultaneous trigger X_S, are both multiples of the investment rule X_F, the comparative static relationships that the follower's trigger yields shall therefore extend to these investment rules too. Hence, the follower's trigger shall be used to demonstrate the trends in the variation of the above-mentioned set.

[1]Recall that the risk-free rate r and the risk-neutral growth rates μ_P and μ_Q are all assumed to remain constant.

4.1 The Dependence of the Investment Rule on the Correlation Coefficient ρ_{PQ}

Of utmost interest is the partial differential equation

$$\frac{\partial}{\partial \rho_{PQ}} \Upsilon(\beta)\bigg|_{\beta=\beta_1} = 0,$$

i.e.

$$\frac{\partial \Upsilon}{\partial \sigma_X}\bigg|_{\beta=\beta_1} \frac{\partial \sigma_X}{\partial \rho_{PQ}} + \frac{\partial \Upsilon}{\partial \beta}\bigg|_{\beta=\beta_1} \frac{\partial \beta_1}{\partial \rho_{PQ}} + \frac{\partial \Upsilon}{\partial \rho_{PQ}}\bigg|_{\beta=\beta_1} = 0. \tag{4.1.1}$$

Now

$$\frac{\partial \Upsilon}{\partial \sigma_X}\bigg|_{\beta=\beta_1} = \sigma_X \beta_1^2 - \sigma_X \beta_1 = \sigma_X \beta_1 (\beta_1 - 1) > 0,$$

since $\beta_1 > 1$ and $\sigma_X > 0$ by definition[2]. Similarly, given that $\sigma_P, \sigma_Q > 0$, we also have

$$\frac{\partial \sigma_X}{\partial \rho_{PQ}} = \frac{\sigma_P \sigma_Q}{(\sigma_P^2 + \sigma_Q^2 + 2\rho_{PQ}\sigma_P\sigma_Q)^{\frac{1}{2}}} = \frac{\sigma_P \sigma_Q}{\sigma_X} > 0$$

and

$$\frac{\partial \Upsilon}{\partial \rho_{PQ}}\bigg|_{\beta=\beta_1} = \sigma_P \sigma_Q \beta_1 > 0.$$

Finally, we may also infer from Figure 2.1 in Section 2.2.3 that

$$\frac{\partial \Upsilon}{\partial \beta}\bigg|_{\beta=\beta_1} > 0$$

and thus, it follows that we must have

$$\frac{\partial \beta_1}{\partial \rho_{PQ}} < 0 \tag{4.1.2}$$

in order for partial differential equation (4.1.1) to hold. Hence, the higher the (positive) correlation between the state variables P and Q, the lower the root β_1, and therefore the larger the option value multiple[3]:

$$\frac{\partial \lambda}{\partial \rho_{PQ}} > 0.$$

[2]Volatility is a measure of standard deviation and hence, is non-negative by definition.

[3]Refer to Figure 2.2 for a depiction of the exponentially decreasing relationship between the root β_1 and the option value multiple.

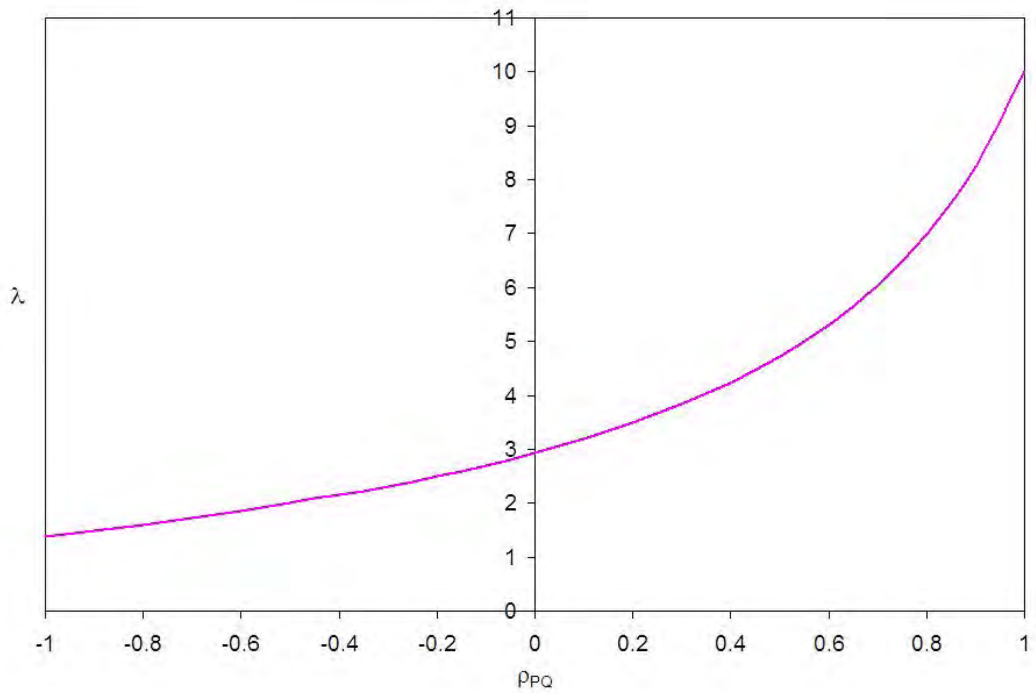


Figure 4.1: Option value multiple as a function of correlation

Although the option value multiple is an increasing function of the correlation coefficient ρ_{PQ}, this relationship does not extend at once to our optimal investment rule for the follower, as it does in Paxson and Pinto's model. The relationship

$$\frac{\partial \delta_X}{\partial \rho_{PQ}} = -\sigma_P \sigma_Q < 0 \tag{4.1.3}$$

plays a significant role in that it counters the variation in the option value multiple, and hence dampens the overall response in our investment rule to changes in correlation. The resultant effect on our profit trigger X_F is therefore dependent on whether the rate of change in the option value multiple λ is lesser than, greater than, or equal to the rate of change in the dividend rate δ_X. For example,

$$\frac{\partial X_F}{\partial \rho_{PQ}} \geq 0$$

if, and only if,

$$\left(\frac{\partial \lambda}{\partial \rho_{PQ}} \delta_X + \frac{\partial \delta_X}{\partial \rho_{PQ}} \lambda \right) K \geq 0$$

$$\frac{\partial \lambda}{\partial \rho_{PQ}} \delta_X + \frac{\partial \delta_X}{\partial \rho_{PQ}} \lambda \geq 0$$

$$\frac{\partial\lambda}{\partial\rho_{PQ}}\delta_X \geq -\frac{\partial\delta_X}{\partial\rho_{PQ}}\lambda$$
$$\frac{\frac{\partial\lambda}{\partial\rho_{PQ}}}{\lambda} \geq -\frac{\frac{\partial\delta_X}{\partial\rho_{PQ}}}{\delta_X}.$$

A further insight that may be drawn from relationship (4.1.3) relates to the follower's NPV trigger (2.3.1). Classical NPV theory evidently proposes that investment becomes economically viable even earlier at higher levels of (positive) correlation. The trend observed in our optimal trigger X_F contradicts this result, and hence exposes the inefficiency of the traditional methodology under circumstances of extreme positive correlation. This discrepancy is further exaggerated by the elasticity of Paxson and Pinto's investment rule for the follower under these same conditions.

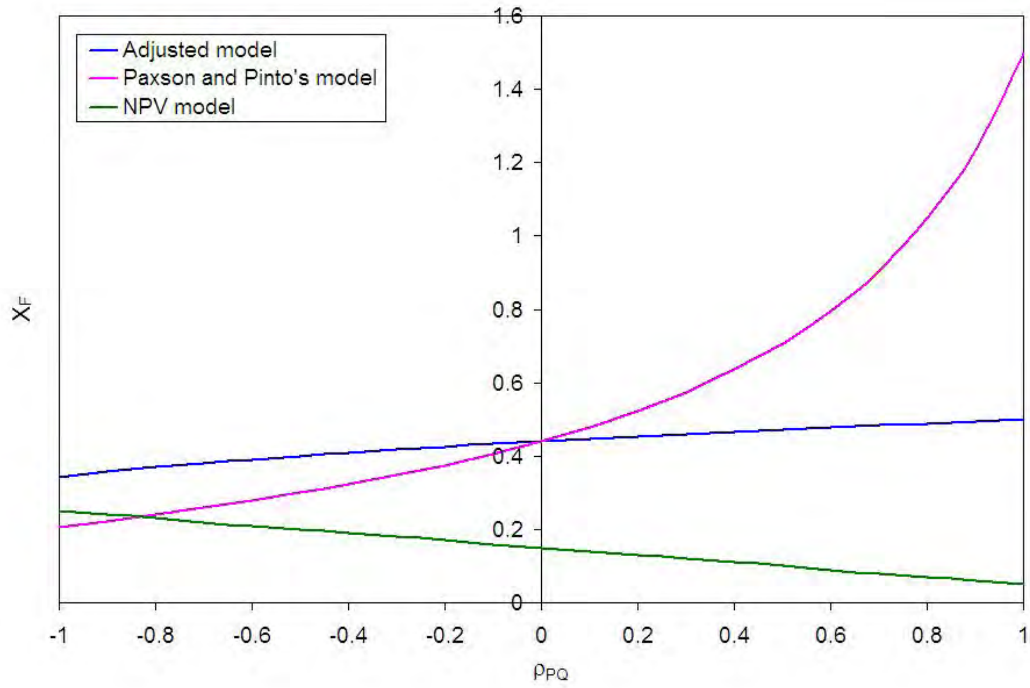


Figure 4.2: Trigger dependence on correlation

These comparisons are summarized in Figure 4.2. It is interesting to note that, for the chosen set of parameters, Paxson and Pinto's model deems investment to be optimal even earlier than the NPV model suggests under conditions of extreme negative correlation. This finding demonstrates the importance of incorporating the covariance term

$$\rho_{PQ}\sigma_P\sigma_Q$$

into our analysis, in order to improve on the accuracy of the investment rules that Paxson and Pinto's version of the model produces.

4.2 The Dependence of the Investment Rule on the Volatilities σ_P, σ_Q

In similar fashion, we shall focus our attention on the partial differential equation

$$\frac{\partial}{\partial \sigma_P} \Upsilon(\beta)\bigg|_{\beta=\beta_1} = 0,$$

i.e.

$$\frac{\partial \Upsilon}{\partial \sigma_X}\bigg|_{\beta=\beta_1} \frac{\partial \sigma_X}{\partial \sigma_P} + \frac{\partial \Upsilon}{\partial \beta}\bigg|_{\beta=\beta_1} \frac{\partial \beta_1}{\partial \sigma_P} + \frac{\partial \Upsilon}{\partial \sigma_P}\bigg|_{\beta=\beta_1} = 0, \tag{4.2.1}$$

with all findings also holding true for the model's dependence on the volatility σ_Q. We have already shown that

$$\frac{\partial \Upsilon}{\partial \sigma_X}\bigg|_{\beta=\beta_1} > 0$$

and

$$\frac{\partial \Upsilon}{\partial \beta}\bigg|_{\beta=\beta_1} > 0.$$

Now

$$\frac{\partial \sigma_X}{\partial \sigma_P} = \frac{\sigma_P + \rho_{PQ}\sigma_Q}{(\sigma_P^2 + \sigma_Q^2 + 2\rho_{PQ}\sigma_P\sigma_Q)^{\frac{1}{2}}} = \frac{\sigma_P + \rho_{PQ}\sigma_Q}{\sigma_X},$$

and

$$\frac{\partial \Upsilon}{\partial \sigma_P}\bigg|_{\beta=\beta_1} = \rho_{PQ}\sigma_Q\beta_1.$$

However, in order for us to proceed, we will need to consider these components of the fundamental quadratic over two separate intervals of correlation between the state variables P and Q.

4.2.1 Non-negative Correlation

If $\rho_{PQ} \geq 0$, then

$$\frac{\partial \sigma_X}{\partial \sigma_P} > 0$$

and

$$\frac{\partial \Upsilon}{\partial \sigma_P}\bigg|_{\beta=\beta_1} \geq 0,$$

so that we must have

$$\frac{\partial \beta_1}{\partial \sigma_P} < 0 \tag{4.2.1.1}$$

in order for partial differential equation (4.2.1) to hold. Hence, when the stochastic variables P and Q are positively correlated (or indeed, entirely uncorrelated), an increase in volatility will lower the value of the root β_1, and subsequently raise the option value multiple:

$$\frac{\partial \lambda}{\partial \sigma_P} > 0.$$

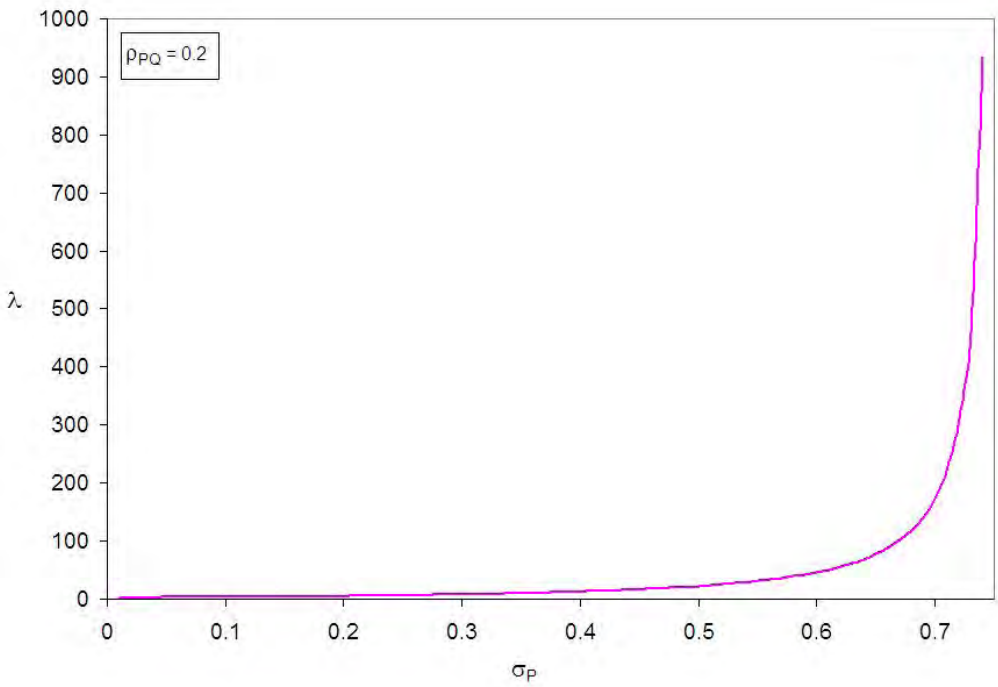


Figure 4.3: Option value multiple as a function of volatility, under positive correlation

However, the inverse relationship

$$\frac{\partial \delta_X}{\partial \sigma_P} = -\rho_{PQ}\sigma_Q \leq 0 \tag{4.2.1.2}$$

is once again likely to counter this variation (unless $\rho_{PQ} = 0$), so that the profit trigger in each of the three models demonstrates a similar trend in response to that depicted in Figure 4.2. Thus, the relationship between our optimal profit trigger X_F and the volatility σ_P (or σ_Q) is yet again dependent on whether the rate of change in the option value multiple λ is lesser than, greater than, or equal to the rate of change in the dividend rate δ_X, i.e.

$$\frac{\partial X_F}{\partial \sigma_P} \geq 0$$

if, and only if,

$$\frac{\frac{\partial \lambda}{\partial \sigma_P}}{\lambda} \geq -\frac{\frac{\partial \delta_X}{\partial \sigma_P}}{\delta_X}.$$

In Figure 4.4, an increase in the volatility σ_P is seen to delay the follower's market entry in both real-option models. However, Paxson and Pinto's investment rule is seen to increase exponentially at high levels of positive volatility, thus highlighting the role that our inclusion of the covariance term

$$\rho_{PQ}\sigma_P\sigma_Q$$

plays in dampening the variation in the option value multiple. The fallibility of the NPV Model is also evident, in that, contrary to existing literary propositions, the value of the investment opportunity actually decreases with any increase in volatility, thereby inducing even earlier and almost immediate market entry for the follower under positive correlation.

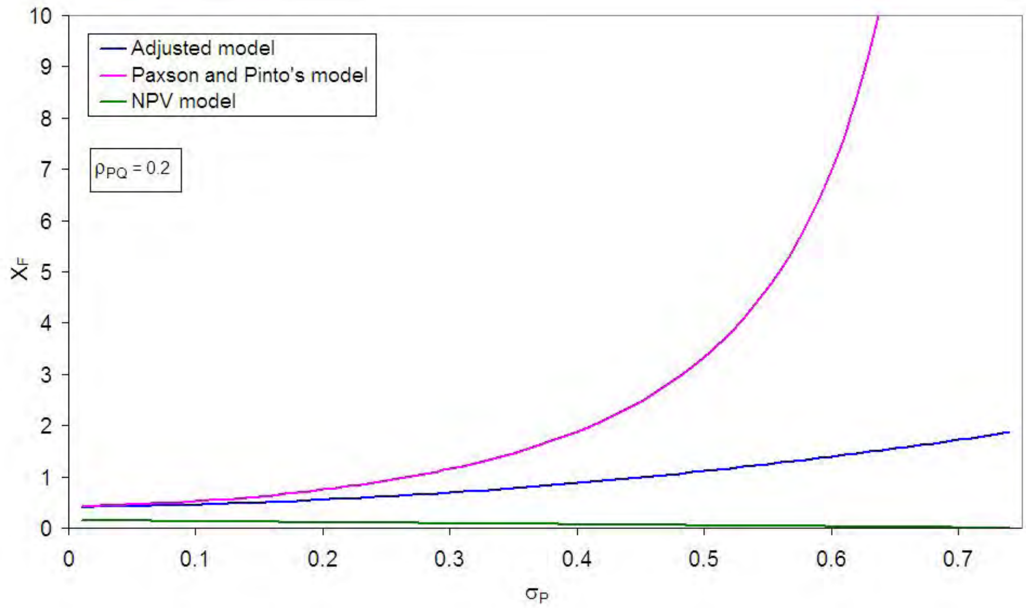


Figure 4.4: Trigger dependence on volatility, under positive correlation

4.2.2 Negative Correlation

If $\rho_{PQ} < 0$, the comparative static relationship between the root β_1 and the volatility σ_P becomes less easy to deduce, as

$$\left.\frac{\partial \Upsilon}{\partial \sigma_P}\right|_{\beta=\beta_1} < 0,$$

whilst the partial derivative

$$\frac{\partial \sigma_X}{\partial \sigma_P}$$

may assume either a positive *or* a negative value. To be more specific, an increase in the underlying volatility σ_P will actually bring about a reduction in the composite volatility σ_X, if the correlation that prevails between the state variables P and Q satisfies the following variable constraint:

$$\rho_{PQ} < -\frac{\sigma_P}{\sigma_Q}.$$

However, if we reconsider partial differential equation (4.2.1), it is obvious that

$$\frac{\partial \beta_1}{\partial \sigma_P} > 0$$

only when

$$\left.\frac{\partial \Upsilon}{\partial \sigma_X}\right|_{\beta=\beta_1} \frac{\partial \sigma_X}{\partial \sigma_P} + \left.\frac{\partial \Upsilon}{\partial \sigma_P}\right|_{\beta=\beta_1} < 0, \tag{4.2.2.1}$$

given that

$$\left.\frac{\partial \Upsilon}{\partial \beta}\right|_{\beta=\beta_1} > 0$$

irrespective of the value of ρ_{PQ}. Inequality (4.2.2.1) thus yields the condition necessary for an increasing relationship between the root β_1 and the volatility σ_P. Substituting for the partial derivatives, we obtain:

$$\sigma_X \beta_1 (\beta_1 - 1) \frac{\sigma_P + \rho_{PQ}\sigma_Q}{\sigma_X} + \rho_{PQ}\sigma_Q\beta_1 < 0$$

$$\beta_1(\beta_1 - 1)(\sigma_P + \rho_{PQ}\sigma_Q) + \rho_{PQ}\sigma_Q\beta_1 < 0$$

$$\rho_{PQ}\sigma_Q\beta_1 < -\beta_1(\beta_1 - 1)(\sigma_P + \rho_{PQ}\sigma_Q)$$

$$\rho_{PQ}\sigma_Q < -(\beta_1 - 1)(\sigma_P + \rho_{PQ}\sigma_Q)$$

$$\frac{\rho_{PQ}\sigma_Q}{\beta_1 - 1} < -\sigma_P - \rho_{PQ}\sigma_Q$$

$$\frac{\rho_{PQ}\sigma_Q + \rho_{PQ}\sigma_Q(\beta_1 - 1)}{\beta_1 - 1} < -\sigma_P$$

$$\frac{\rho_{PQ}\sigma_Q\beta_1}{\beta_1 - 1} < -\sigma_P$$

$$\rho_{PQ}\sigma_Q\lambda < -\sigma_P,$$

i.e.

$$\rho_{PQ} < -\frac{\sigma_P}{\sigma_Q\lambda} < 0. \quad (4.2.2.2)$$

Hence, only for those intervals of volatility over which the correlation coefficient satisfies the above variable constraint, will an increase in the volatility σ_P raise the value of the root β_1, and subsequently lower the option value multiple:

$$\frac{\partial\lambda}{\partial\sigma_P} < 0.$$

For

$$-\frac{\sigma_P}{\sigma_Q\lambda} \leq \rho_{PQ} < 0,$$

the option value multiple will be seen to increase, as in the case of non-negative correlation above, or else remain unchanged should the equality hold:

$$\frac{\partial\lambda}{\partial\sigma_P} \geq 0.$$

However, the rate of change in the option value multiple over this region of increase[4] is far less extreme under negative correlation, as becomes obvious if we compare Figures 4.3 and 4.5.

Despite the complexity of these findings, we have yet to consider the dependence of the dividend rate δ_X on the volatility σ_P. When the state variables P and Q are negatively correlated, the relationship

$$\frac{\partial\delta_X}{\partial\sigma_P} = -\rho_{PQ}\sigma_Q > 0 \quad (4.2.2.3)$$

will only counter the movement in the option value multiple if constraint (4.2.2.2) is satisfied, in which case the resultant effect on our profit trigger X_F will yet

[4]In Figure 4.5, $\sigma_P = 0.1$ clearly represents the divide between the respective intervals of decrease and increase for the option value multiple, in this instance of negative correlation.

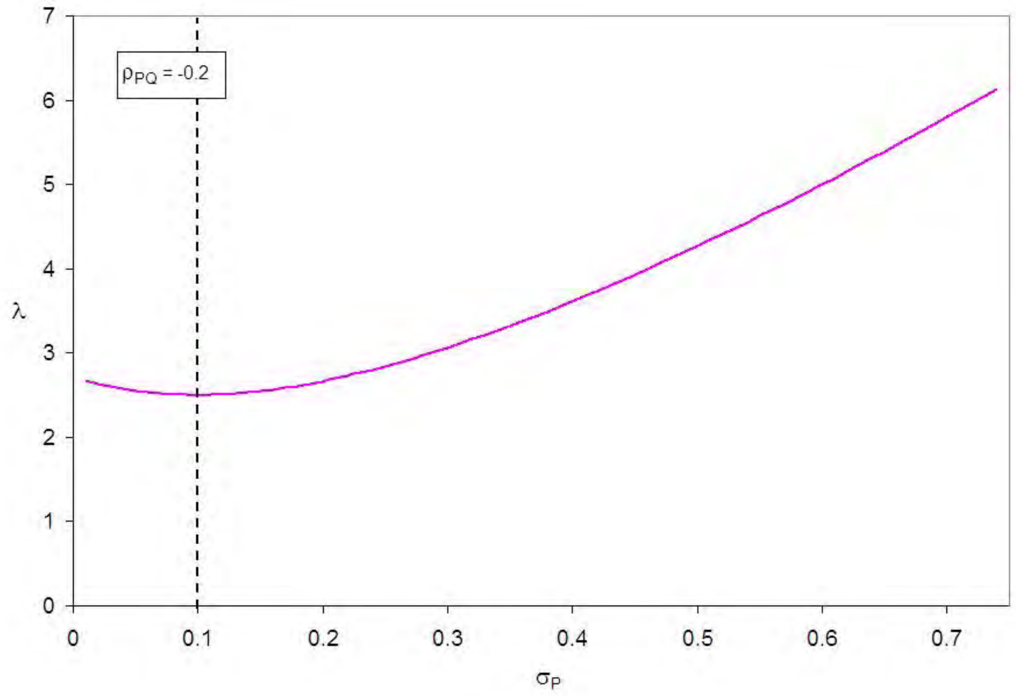


Figure 4.5: Option value multiple as a function of volatility, under negative correlation

again be dependent on the relative magnitudes of the rate of change in the option value multiple λ, and of that in the dividend rate δ_X, i.e.

$$\frac{\partial X_F}{\partial \sigma_P} \geq 0$$

if, and only if,

$$\frac{\dfrac{\partial \delta_X}{\partial \sigma_P}}{\delta_X} \geq -\frac{\dfrac{\partial \lambda}{\partial \sigma_P}}{\lambda}.$$

When

$$-\frac{\sigma_P}{\sigma_Q \lambda} \leq \rho_{PQ} < 0,$$

the change in the dividend rate δ_X will otherwise be seen to exacerbate the overall response in our investment rule X_F, thereby giving rise to the following conclusive result:

$$\frac{\partial X_F}{\partial \sigma_P} \geq 0.$$

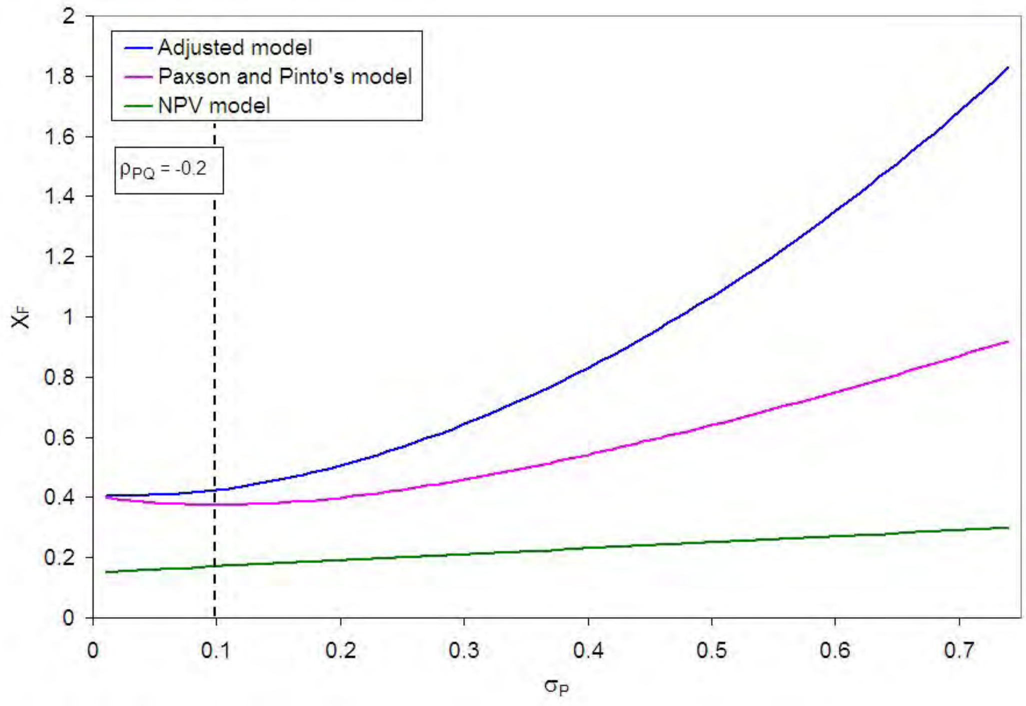


Figure 4.6: Trigger dependence on volatility, under negative correlation

In the particular example that Figure 4.6 illustrates, all three models display a tendency towards delayed market entry for the follower under conditions of increasing volatility and negative correlation. However, our adjusted real-option model yields the most conservative investment rule at every level of volatility, due to the added cautionary effect of the covariance term. Whilst the NPV trigger is also seen to increase with volatility, the linearity of the relationship, as opposed to the concavity of the responses in the other two investment rules, yet again leads us to question the usefulness of NPV theory when such discrepancies are evident in the results that it produces.

Chapter 5

An Extension of the Model to Three Sources of Randomness

Up until this point, we have regarded the investment cost K as a fixed once-off payment that remains independent of the stochastically variable number of units sold in the market either by the follower, Q, or by the leader, $\overline{m}Q$. Given that we have assumed the profit per unit P, and the number of units Q, to be the only variables to display any volatility, our game-theoretic model is thus far best suited to service industries where business is dependent on high volumes of production. In (25), Paxson and Pinto provide an empirical application of their version of the model, by determining the optimal time for a Portuguese mobile phone company Optimus to enter the market as a follower.

However, several authors (including (28) and (34)) have also taken strides to develop real options models that are more appropriate for industries where the cost of investment has a high probability of changing with time. In particular, Research and Development firms represent a natural selection for the application of such models, as the high levels of uncertainty that surround the timing of any form of progressive development subsequently result in high levels of uncertainty concerning returns and investment costs.

In light of these characteristics, the real options models that have been derived to accommodate such industries do not treat the number of manufactured units as a stochastic variable, but instead hinge on the assumption that the quantity produced is deterministic. The profit per unit, and hence the profit flow, is considered to be a single stochastic factor, whilst the second source of randomness is assigned to the cost of investment. However, in many of these models, the capital investment is still regarded as an upfront once-off payment, so that the evolution of this variable only remains relevant up until the point of market entry.

For example, in (28), Quigg considers the option to develop a piece of land. The value of the real estate opportunity incorporates the option to delay development, and is treated as a function of both the developed building, as well as the initial cost of investment. More recently in (24), Paxson and Pinto have modified this model to consider strategic implications, but various opportunities remain for further extension.

Consider the Fast Moving Consumer Goods sector. Companies that operate within this realm are heavily dependent on their volumes of production, with forecasts having to be adjusted at frequent intervals in order to meet the ever-changing levels of demand. Naturally, the profit per unit produced is also seen to vary in line with the cost of raw materials, as well as the breadth and depth of trade spend. However, if players really wish to grow and establish themselves in such a market, high levels of innovation are required in order to satisfy evolving consumer needs, and thereby obtain some form of competitive advantage. Research and Development therefore also forms an integral part of the Fast Moving Consumer Goods sector, and requires continual investment.

The telecommunications market presents another example of an industry where high but fluctuating volumes of production are coupled with uncertain investment costs. Although Paxson and Pinto have based their empirical application in (25) on such a typefied service industry due to the fact that it is often characterized by license race entry, their choice of model should not be seen as prescriptive given that the telecommunications market is also subject to rapid and major technological change.

Hence, although each of the above-mentioned types of model only caters for the possibility of two stochastic variables, many industries are actually faced with three typical sources of randomness - namely, the volatility associated with output from production, unitary profit, as well as ongoing capital investment. In this chapter we shall attempt to address this reality by extending our model for the follower[1] to include a stochastically variable investment cost K_t. To allow for direct comparison to our two-variable model, we shall first treat this random investment as a once-off payment, and then later consider the case where it assumes ongoing relevance.

5.1 A Model with Three Stochastic Variables

Continuing with the hypothetical game described in Section 2.1, we assume the existence of a new market in which the investing firms will compete in the manufacture and sale of homogeneous goods. The profit per unit P_t remains disaggregated from the quantity sold by the follower Q_t. However, we now introduce a third stochastic variable K_t in order to denote the uncertain cost of market entry - the payment of which we initially consider to be once-off.

Under the real-world probability measure, P_t, Q_t and K_t are each assumed to

[1]Given that the follower's investment decision remains the same whether in a pre-emptive or co-operative environment, the value function that we derive will be applicable within either framework.

evolve according to different but possibly correlated geometric Brownian motions:

$$dP_t = \mu'_P P_t dt + \sigma_P P_t dz_P, \tag{5.1.1}$$

$$dQ_t = \mu'_Q Q_t dt + \sigma_Q Q_t dz_Q \tag{5.1.2}$$

and

$$dK_t = \mu'_K K_t dt + \sigma_K K_t dz_K \tag{5.1.3}$$

where μ'_P, μ'_Q and μ'_K represent the expected multiplicative trends of P_t, Q_t and K_t respectively, whilst σ_P, σ_Q and σ_K denote the disaggregated volatilities. All parameters are assumed to remain constant, unless otherwise specified.

The magnitude of the correlation between each of the Wiener process increments dz_P, dz_Q and dz_K is described by the respective coefficients ρ_{PQ}, ρ_{PK} and ρ_{QK}, giving rise to four possible scenarios or "combinations of correlation":

Table 5.1: Correlation scenarios for the extended model

Scenario (i)	$0 < \rho_{PQ} \leq 1$	$0 < \rho_{PK} \leq 1$	$0 < \rho_{QK} \leq 1$
Scenario (ii)	$-1 \leq \rho_{PQ} < 0$	$0 < \rho_{PK} \leq 1$	$-1 \leq \rho_{QK} < 0$
Scenario (iii)	$-1 \leq \rho_{PQ} < 0$	$-1 \leq \rho_{PK} < 0$	$0 < \rho_{QK} \leq 1$
Scenario (iv)	$0 < \rho_{PQ} \leq 1$	$-1 \leq \rho_{PK} < 0$	$-1 \leq \rho_{QK} < 0$

For simplicity, the scenarios presented in Table 5.1 exclude the possibility of zero correlation. In the probable event that one of the correlation coefficients assumes a zero value, then at least one (if not both) of the remaining coefficients will also display a lack of correlation between the respective state variables. Aside from this exclusion, the above combinations form an exhaustive set, as it is intuitive that all three coefficients will not be able to assume a negative value at the same time.

As we are considering a market that is characterized by high volume production despite changing costs of investment, any unit increase in the quantity sold by the follower is proportionately very small, so that the number of units produced, Q_t, may once again be treated as a continuous variable, together with the profit per unit, P_t, and the cost of investment, K_t. However, we recall that all three variables need to be constrained to a domain from zero to infinity, due to the nature of the diffusion process assumed for each.

5.2 Extending the Follower's Value Function

In order to simplify the derivation of our three-variable model, we shall proceed under the assumption that the investing firms are risk-neutral. All cash flows may thus be discounted at the constant risk-free rate $r > 0$, where θ_P, θ_Q and now also θ_K denote the equilibrium rates of return that the Capital Asset Pricing Model would otherwise require from P_t, Q_t and K_t respectively.

Under the risk-neutral probability measure, the state variable K_t evolves according to the stochastic process

$$dK_t = \mu_K K_t dt + \sigma_K K_t dz_K, \tag{5.2.1}$$

where

$$\mu_K = r - \delta_K = r - (\theta_K - \mu'_K)$$

defines the risk-adjusted growth rate for K_t, and yields the equilibrium condition

$$\delta_K = r - \mu_K = \theta_K - \mu'_K. \tag{5.2.2}$$

The risk-adjusted diffusions for P_t and Q_t remain as defined[2] by equations (2.2.1) and (2.2.2), with respective equilibrium conditions (2.2.3) and (2.2.4).

In the event of an underlying change in the volatility σ_K, the Capital Asset Pricing Model will dictate a similar change in the risk-adjusted discount rate θ_K. However, as the dividend rate δ_K is regarded as a basic behavioural parameter that remains fixed, the real-world stochastic process of K will thus be subject to change as μ'_K shifts accordingly to restore equilibrium. On the contrary, the risk-neutral growth rate μ_K will not be seen to adjust at any stage, due to the fixing of the risk-free rate r by the larger considerations of the entire capital market.

Similar dynamics for P and Q have already been discussed in Section 2.2. From henceforth, our three-variable model shall only be regarded within the context of a risk-neutral framework.

5.2.1 Deriving the Partial Differential Equation

Under the assumption of a complete market, we shall let $\Pi(P, Q, K)$ denote the replicating portfolio of an idle follower that is long on an option and short on n^*_P, n^*_Q and n^*_K units of "assets" P, Q and K respectively. The value of the replicating portfolio is thus given by

$$\Pi(P, Q, K) = f_0(P, Q, K) - n^*_P P - n^*_Q Q - n^*_K K,$$

[2]The time subscripts of P_t, Q_t and K_t will once again be suppressed from this point on.

where $f_0(P,Q,K)$ now denotes the worth of the perpetual option to enter second in the given market. In order to avoid triviality, we impose the restrictions

$$r > \mu_P,$$

$$r > \mu_Q$$

and

$$r > \mu_K.$$

As we recall, these conditions ensure that further delay of the follower's market entry will eventually become sub-optimal, with each of the assets P, Q and K paying a dividend at the respective rate

$$\delta_P = r - \mu_P > 0,$$

$$\delta_Q = r - \mu_Q > 0$$

and

$$\delta_K = r - \mu_K > 0$$

to fully compensate the corresponding holders in the long positions. As the portfolio itself only offers a return in the form of capital gains, any change in the portfolio over a time interval of length dt may thus be explained by

$$\begin{aligned} d\Pi(P,Q,K) &= df_0(P,Q,K) - n_P^* dP - \delta_P n_P^* P dt - n_Q^* dQ \\ &\quad -\delta_Q n_Q^* Q dt - n_K^* dK - \delta_K n_K^* K dt, \end{aligned} \tag{5.2.1.1}$$

where $\delta_P n_P^* P dt$, $\delta_Q n_Q^* Q dt$ and $\delta_K n_K^* K dt$ reflect the required dividend payments.

Following the methodology applied in Section 2.2.1, we next expand $df_0(P,Q,K)$ using Itô's Lemma to obtain

$$df_0(P,Q,K)$$

$$\begin{aligned} &= \frac{\partial f_0(P,Q,K)}{\partial P} dP + \frac{\partial f_0(P,Q,K)}{\partial Q} dQ + \frac{\partial f_0(P,Q,K)}{\partial K} dK \\ &\quad + \frac{1}{2}\frac{\partial^2 f_0(P,Q,K)}{\partial P^2}(dP)^2 + \frac{1}{2}\frac{\partial^2 f_0(P,Q,K)}{\partial Q^2}(dQ)^2 \\ &\quad + \frac{1}{2}\frac{\partial^2 f_0(P,Q,K)}{\partial K^2}(dK)^2 + \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial Q}(dP)(dQ) \\ &\quad + \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial K}(dP)(dK) + \frac{\partial^2 f_0(P,Q,K)}{\partial Q \partial K}(dQ)(dK) \end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{\partial f_0(P,Q,K)}{\partial P}(\mu_{_P} P dt + \sigma_{_P} P dz_{_P}) + \frac{\partial f_0(P,Q,K)}{\partial Q}(\mu_{_Q} Q dt + \sigma_{_Q} Q dz_{_Q}) \\
& + \frac{\partial f_0(P,Q,K)}{\partial K}(\mu_{_K} K dt + \sigma_{_K} K dz_{_K}) + \frac{1}{2}\frac{\partial^2 f_0(P,Q,K)}{\partial P^2}\sigma_{_P}^2 P^2 dt \\
& + \frac{1}{2}\frac{\partial^2 f_0(P,Q,K)}{\partial Q^2}\sigma_{_Q}^2 Q^2 dt + \frac{1}{2}\frac{\partial^2 f_0(P,Q,K)}{\partial K^2}\sigma_{_K}^2 K^2 dt \\
& + \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial Q}\rho_{_{PQ}}\sigma_{_P}\sigma_{_Q} PQ dt + \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial K}\rho_{_{PK}}\sigma_{_P}\sigma_{_K} PK dt \\
& + \frac{\partial^2 f_0(P,Q,K)}{\partial Q \partial K}\rho_{_{QK}}\sigma_{_Q}\sigma_{_K} QK dt
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \left\{ \mu_{_P} P \frac{\partial f_0(P,Q,K)}{\partial P} + \mu_{_Q} Q \frac{\partial f_0(P,Q,K)}{\partial Q} + \mu_{_K} K \frac{\partial f_0(P,Q,K)}{\partial K} \right. \\
& + \frac{1}{2}\frac{\partial^2 f_0(P,Q,K)}{\partial P^2}\sigma_{_P}^2 P^2 + \frac{1}{2}\frac{\partial^2 f_0(P,Q,K)}{\partial Q^2}\sigma_{_Q}^2 Q^2 \\
& + \frac{1}{2}\frac{\partial^2 f_0(P,Q,K)}{\partial K^2}\sigma_{_K}^2 K^2 + \rho_{_{PQ}}\sigma_{_P}\sigma_{_Q} PQ \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial Q} \\
& \left. + \rho_{_{PK}}\sigma_{_P}\sigma_{_K} PK \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial K} + \rho_{_{QK}}\sigma_{_Q}\sigma_{_K} QK \frac{\partial^2 f_0(P,Q,K)}{\partial Q \partial K} \right\} dt \\
& + \sigma_{_P} P \frac{\partial f_0(P,Q,K)}{\partial P} dz_{_P} + \sigma_{_Q} Q \frac{\partial f_0(P,Q,K)}{\partial Q} dz_{_Q} \\
& + \sigma_{_K} K \frac{\partial f_0(P,Q,K)}{\partial K} dz_{_K}.
\end{aligned}
$$

Substituting into equation (5.2.1.1) and recollecting like terms yields

$$d\Pi(P,Q,K)$$

$$
\begin{aligned}
= \quad & \left\{ \mu_{_P} P \frac{\partial f_0(P,Q,K)}{\partial P} + \mu_{_Q} Q \frac{\partial f_0(P,Q,K)}{\partial Q} + \mu_{_K} K \frac{\partial f_0(P,Q,K)}{\partial K} \right. \\
& + \frac{1}{2}\frac{\partial^2 f_0(P,Q,K)}{\partial P^2}\sigma_{_P}^2 P^2 + \frac{1}{2}\frac{\partial^2 f_0(P,Q,K)}{\partial Q^2}\sigma_{_Q}^2 Q^2 \\
& + \frac{1}{2}\frac{\partial^2 f_0(P,Q,K)}{\partial K^2}\sigma_{_K}^2 K^2 + \rho_{_{PQ}}\sigma_{_P}\sigma_{_Q} PQ \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial Q} \\
& \left. + \rho_{_{PK}}\sigma_{_P}\sigma_{_K} PK \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial K} + \rho_{_{QK}}\sigma_{_Q}\sigma_{_K} QK \frac{\partial^2 f_0(P,Q,K)}{\partial Q \partial K} \right\} dt \\
& + \sigma_{_P} P \frac{\partial f_0(P,Q,K)}{\partial P} dz_{_P} - n_{_P}^*(\mu_{_P} P dt + \sigma_{_P} P dz_{_P}) - \delta_{_P} n_{_P}^* P dt \\
& + \sigma_{_Q} Q \frac{\partial f_0(P,Q,K)}{\partial Q} dz_{_Q} - n_{_Q}^*(\mu_{_Q} Q dt + \sigma_{_Q} Q dz_{_Q}) - \delta_{_Q} n_{_Q}^* Q dt \\
& + \sigma_{_K} K \frac{\partial f_0(P,Q,K)}{\partial K} dz_{_K} - n_{_K}^*(\mu_{_K} K dt + \sigma_{_K} K dz_{_K}) - \delta_{_K} n_{_K}^* K dt
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \left\{ \mu_P P \frac{\partial f_0(P,Q,K)}{\partial P} - n_P^* \mu_P P - \delta_P n_P^* P + \mu_Q Q \frac{\partial f_0(P,Q,K)}{\partial Q} \right. \\
& - n_Q^* \mu_Q Q - \delta_Q n_Q^* Q + \mu_K K \frac{\partial f_0(P,Q,K)}{\partial K} - n_K^* \mu_K K - \delta_K n_K^* K \\
& + \frac{1}{2} \frac{\partial^2 f_0(P,Q,K)}{\partial P^2} \sigma_P^2 P^2 + \frac{1}{2} \frac{\partial^2 f_0(P,Q,K)}{\partial Q^2} \sigma_Q^2 Q^2 \\
& + \frac{1}{2} \frac{\partial^2 f_0(P,Q,K)}{\partial K^2} \sigma_K^2 K^2 + \rho_{PQ} \sigma_P \sigma_Q PQ \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial Q} \\
& \left. + \rho_{PK} \sigma_P \sigma_K PK \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial K} + \rho_{QK} \sigma_Q \sigma_K QK \frac{\partial^2 f_0(P,Q,K)}{\partial Q \partial K} \right\} dt \\
& + \left\{ \sigma_P P \frac{\partial f_0(P,Q,K)}{\partial P} - n_P^* \sigma_P P \right\} dz_P \\
& + \left\{ \sigma_Q Q \frac{\partial f_0(P,Q,K)}{\partial Q} - n_Q^* \sigma_Q Q \right\} dz_Q \\
& + \left\{ \sigma_K K \frac{\partial f_0(P,Q,K)}{\partial K} - n_K^* \sigma_K K \right\} dz_K
\end{aligned}
$$

If we once again construct a delta-hedged portfolio by choosing

$$n_P^* = \frac{\partial f_0(P,Q,K)}{\partial P},$$

$$n_Q^* = \frac{\partial f_0(P,Q,K)}{\partial Q}$$

and

$$n_K^* = \frac{\partial f_0(P,Q,K)}{\partial K},$$

the expression for $d\Pi(P,Q,K)$ thus simplifies to

$$
\begin{aligned}
& \left\{ \frac{1}{2} \sigma_P^2 P^2 \frac{\partial^2 f_0(P,Q,K)}{\partial P^2} - \delta_P P \frac{\partial f_0(P,Q,K)}{\partial P} + \frac{1}{2} \sigma_Q^2 Q^2 \frac{\partial^2 f_0(P,Q,K)}{\partial Q^2} \right. \\
& \quad - \delta_Q Q \frac{\partial f_0(P,Q,K)}{\partial Q} + \frac{1}{2} \sigma_K^2 K^2 \frac{\partial^2 f_0(P,Q,K)}{\partial Q^2} - \delta_K K \frac{\partial f_0(P,Q,K)}{\partial K} \\
& \qquad + \rho_{PQ} \sigma_P \sigma_Q PQ \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial Q} + \rho_{PK} \sigma_P \sigma_K PK \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial K} \\
& \qquad\qquad \left. + \rho_{QK} \sigma_Q \sigma_K QK \frac{\partial^2 f_0(P,Q,K)}{\partial Q \partial K} \right\} dt.
\end{aligned}
$$

However, we know that the return on the portfolio is now actually just equal to the risk-free rate r. Hence,

$$\begin{aligned} d\Pi(P,Q,K) &= r\Pi(P,Q,K)dt \\ &= r\left\{ f_0(P,Q,K) - P\frac{\partial f_0(P,Q,K)}{\partial P} - Q\frac{\partial f_0(P,Q,K)}{\partial Q} \right. \\ & \left. -K\frac{\partial f_0(P,Q,K)}{\partial K} \right\} dt. \end{aligned}$$

Equating and rearranging each of these expressions for $d\Pi(P,Q,K)$ yields partial differential equation (5.2.1.2):

$$\begin{aligned} &\frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 f_0(P,Q,K)}{\partial P^2} + \mu_P P \frac{\partial f_0(P,Q,K)}{\partial P} + \frac{1}{2}\sigma_Q^2 Q^2 \frac{\partial^2 f_0(P,Q,K)}{\partial Q^2} + \mu_Q Q \frac{\partial f_0(P,Q,K)}{\partial Q} \\ &+ \frac{1}{2}\sigma_K^2 K^2 \frac{\partial^2 f_0(P,Q,K)}{\partial K^2} + \mu_K K \frac{\partial f_0(P,Q,K)}{\partial K} + \rho_{PQ}\sigma_P\sigma_Q PQ \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial Q} \\ &+ \rho_{PK}\sigma_P\sigma_K PK \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial K} + \rho_{QK}\sigma_Q\sigma_K QK \frac{\partial^2 f_0(P,Q,K)}{\partial Q \partial K} - r f_0(P,Q,K) = 0. \end{aligned}$$

Equation (5.2.1.2) presents a more complex explanation of the movements in the idle follower's value function. In order to obtain a closed-form solution for the value of the perpetual option $f_0(P,Q,K)$, we must once again resort to methods of similarity.

5.2.2 The Return on Investment, R

Now that we are allowing for three random variables to affect the firm's investment decision, we have to find the whole space of values of (P,Q,K) where investment will occur, the whole space where it will not occur, and the critical boundary or threshold surface (as opposed to two-dimensional curve) separating each space. Needless to say, this is mathematically more difficult than in the two-variable model, and generally requires numerical methods of some complexity.

However, in (2), Dixit and Pindyck specify that certain examples with special features - in particular, some form of homogeneity - can be solved by again reducing the problem to one state variable. Following their technique, we will now illustrate this theory to arrive at a closed-form solution for partial differential equation (5.2.1.2). As in Section 2.2.2, we let

$$X = PQ$$

denote the total profit (or profit flow) for the follower, where X has already been shown to evolve according to a geometric Brownian motion with expected trend

$$\mu_X = \mu_P + \mu_Q + \rho_{PQ}\sigma_P\sigma_Q$$

under the risk-neutral measure, and volatility σ_X such that

$$\sigma_X^2 = \sigma_P^2 + \sigma_Q^2 + 2\rho_{PQ}\sigma_P\sigma_Q.$$

However, as the value of the follower's option to invest now depends on the uncertain capital amount K, as well as the profit flow X, the firm's decision will essentially be based on its ability to leverage this investment. Intuitively, we would expect the follower to delay market entry when X is low or K is high (relatively speaking), and conversely, to exercise the real option when X becomes sufficiently high for given K, or K becomes sufficiently low for given X. The optimal threshold should therefore depend only on the ratio

$$R = \frac{X}{K}.$$

This quantity describes the follower's return on investment and serves as an indicator of operating performance, bearing close resemblance to the measures of profitability that are discussed in (5) by Firer *et al.* In order to identify the stochastic process that this financial ratio follows, we apply Itô's Lemma:

$$\begin{aligned}
dR &= d\left(\frac{X}{K}\right) \\
&= \frac{\partial R}{\partial X}dX + \frac{\partial R}{\partial K}dK + \frac{1}{2}\frac{\partial^2 R}{\partial X^2}(dX)^2 + \frac{1}{2}\frac{\partial^2 R}{\partial K^2}(dK)^2 + \frac{\partial^2 R}{\partial X \partial K}(dX)(dK) \\
&= \frac{1}{K}(\mu_X X dt + \sigma_X X dz_X) - \frac{X}{K^2}(\mu_K K dt + \sigma_K K dz_K) + \frac{X}{K^3}\sigma_K^2 K^2 dt \\
&\quad - \frac{1}{K^2}\rho_{XK}\sigma_X\sigma_K X K dt \\
&= \frac{X}{K}(\mu_X - \mu_K + \sigma_K^2 - \rho_{XK}\sigma_X\sigma_K)dt + \frac{X}{K}(\sigma_X dz_X - \sigma_K dz_K) \\
&= R(\mu_X - \mu_K + \sigma_K^2 - \rho_{XK}\sigma_X\sigma_K)dt + R\sigma_R dz_R
\end{aligned}$$

where dz_R is the increment of a standard Wiener process, or alternatively

$$\begin{aligned}
dR &= d\left(\frac{PQ}{K}\right) \\
&= \frac{\partial R}{\partial P}dP + \frac{\partial R}{\partial Q}dQ + \frac{\partial R}{\partial K}dK + \frac{1}{2}\frac{\partial^2 R}{\partial P^2}(dP)^2 + \frac{1}{2}\frac{\partial^2 R}{\partial Q^2}(dQ)^2 \\
&\quad + \frac{1}{2}\frac{\partial^2 R}{\partial K^2}(dK)^2 + \frac{\partial^2 R}{\partial P \partial Q}(dP)(dQ) + \frac{\partial^2 R}{\partial P \partial K}(dP)(dK) \\
&\quad + \frac{\partial^2 R}{\partial Q \partial K}(dQ)(dK)
\end{aligned}$$

$$\begin{aligned}
&= \frac{Q}{K}(\mu_P P dt + \sigma_P P dz_P) + \frac{P}{K}(\mu_Q Q dt + \sigma_Q Q dz_Q) \\
&\quad - \frac{PQ}{K^2}(\mu_K K dt + \sigma_K K dz_K) + \frac{PQ}{K^3}\sigma_K^2 K^2 dt + \frac{1}{K}\rho_{PQ}\sigma_P\sigma_Q PQ dt \\
&\quad - \frac{Q}{K^2}\rho_{PK}\sigma_P\sigma_K PK dt - \frac{P}{K^2}\rho_{QK}\sigma_Q\sigma_K QK dt \\
&= \frac{PQ}{K}(\mu_P + \mu_Q - \mu_K + \sigma_K^2 + \rho_{PQ}\sigma_P\sigma_Q - \rho_{PK}\sigma_P\sigma_K - \rho_{QK}\sigma_Q\sigma_K)dt \\
&\quad + \frac{PQ}{K}(\sigma_P dz_P + \sigma_Q dz_Q - \sigma_K dz_K) \\
&= \frac{X}{K}(\mu_X - \mu_K + \sigma_K^2 - \rho_{PK}\sigma_P\sigma_K - \rho_{QK}\sigma_Q\sigma_K)dt \\
&\quad + \frac{X}{K}(\sigma_X dz_X - \sigma_K dz_K) \\
&= R(\mu_X - \mu_K + \sigma_K^2 - \rho_{PK}\sigma_P\sigma_K - \rho_{QK}\sigma_Q\sigma_K)dt + R\sigma_R dz_R.
\end{aligned}$$

Comparing the parametric trend expressions in these equivalent diffusions, we conclude that

$$\rho_{XK}\sigma_X\sigma_K = \rho_{PK}\sigma_P\sigma_K + \rho_{QK}\sigma_Q\sigma_K$$

or[3]

$$\rho_{XK} = \frac{\rho_{PK}\sigma_P + \rho_{QK}\sigma_Q}{\sigma_X},$$

so that the follower's return on investment may thus be described by a geometric Brownian motion with risk-neutral drift

$$\mu_R = \mu_X - \mu_K + \sigma_K^2 - \rho_{XK}\sigma_X\sigma_K,$$

and volatility σ_R such that

$$\sigma_R^2 = \sigma_X^2 + \sigma_K^2 - 2\rho_{XK}\sigma_X\sigma_K.$$

As in the two-variable model, we now conjecture that the quantity

$$r - \mu_R = r - (\mu_X - \mu_K + \sigma_K^2 - \rho_{XK}\sigma_X\sigma_K)$$

should therefore be assigned as the risk-neutral representation of the dividend rate δ_R that the return on investment yields. However, in order to provide some validation of this claim, we must once again refer back to the work of McDonald and Siegel (see (22)).

[3]Given that $\sigma_X > 0$, the sign of ρ_{XK} is solely dependent on the signs of ρ_{PK} and ρ_{QK}, together with the magnitudes of the volatilities σ_P and σ_Q relative to each other.

Risk-aversion by the follower is re-introduced with the purpose of investigating the real-world representation

$$\theta_R - \mu'_R = \theta_R - (\mu'_X - \mu'_K + \sigma_K^2 - \rho_{XK}\sigma_X\sigma_K)$$

of the above-mentioned dividend rate. In order to determine the equilibrium rate of return θ_R that the Capital Asset Pricing Model expects from R in the real world, we first need to establish the dependence of θ_R on the individual equilibrium rates that are required from assets with the same risk as the profit flow X and the investment cost K.

Proceeding under the constant assumption of a complete market, we recall that the risk premium earned on an asset is proportional to the volatility of the asset, so that

$$\theta_X - r = \phi\rho_{XM}\sigma_X$$

and

$$\theta_K - r = \phi\rho_{KM}\sigma_K,$$

where ϕ continues to denote the market price of risk, and ρ_{XM}[4] and ρ_{KM} are the co-efficients of correlation between the rate of return on the market portfolio and that on assets X and K respectively.

Now, the actual rate of return on R is given by the Itô derivative

$$\begin{aligned}\frac{dR}{R} &= \mu'_R dt + \sigma_R dz_R \\ &= (\mu'_X - \mu'_K + \sigma_K^2 - \rho_{XK}\sigma_X\sigma_K)dt + \sigma_X dz_X - \sigma_K dz_K,\end{aligned}$$

where the random component of the return is described by the difference between the stochastic components in the rates of change of X and K. Thus, it follows that the risk premium earned on the return on investment must also be given by the difference between the individual premiums earned on X and K, i.e.

$$\theta_R - r = (\theta_X - r) - (\theta_K - r),$$

[4]As the risk premium earned on the profit flow X has already been shown to be a direct summation of the individual premiums earned on assets P and Q, we may in fact deduce that

$$\rho_{XM} = \frac{\rho_{PM}\sigma_P + \rho_{QM}\sigma_Q}{\sigma_X},$$

since

$$\phi\rho_{XM}\sigma_X = \phi\rho_{PM}\sigma_P + \phi\rho_{QM}\sigma_Q.$$

so that the total equilibrium expected rate of return on R is given by

$$\theta_R = \theta_X - \theta_K + r.$$

Hence, it is now possible for us to determine the exact dividend rate that the return on investment yields under the real-world measure:

$$\begin{aligned} \delta_R &= (\theta_X - \theta_K + r) - (\mu'_X - \mu'_K + \sigma_K^2 - \rho_{XK}\sigma_X\sigma_K) \\ &= (\theta_X - \mu'_X) - (\theta_K - \mu'_K) + r - \sigma_K^2 + \rho_{XK}\sigma_X\sigma_K. \end{aligned} \tag{5.2.2.1}$$

Further to this, we know from conditions (2.2.2.2) and (5.2.2) that

$$\theta_X - \mu'_X = r - \mu_X$$

and

$$\theta_K - \mu'_K = r - \mu_K.$$

Thus, the dividend rate for R may be re-expressed as

$$\begin{aligned} \delta_R &= (r - \mu_X) - (r - \mu_K) + r - \sigma_K^2 + \rho_{XK}\sigma_X\sigma_K \\ &= r - (\mu_X - \mu_K + \sigma_K^2 - \rho_{XK}\sigma_X\sigma_K) \\ &= r - \mu_R, \end{aligned}$$

thereby validating our conjecture and giving rise to the following equilibrium condition:

$$\delta_R = r - \mu_R = \theta_R - \mu'_R. \tag{5.2.2.2}$$

For completeness, we note that the dividend rate δ_R cannot be regarded as a basic behavioural parameter that remains independent of the volatilities σ_X and σ_K. Consider definition (5.2.2.1). It follows that the dividend rate earned on the return on investment may also be given in the form

$$\delta_R = \delta_X - \delta_K + r - \sigma_K^2 + \rho_{XK}\sigma_X\sigma_K.$$

Recalling the assumed constancy of the parameter δ_K and the risk-free rate r, the dependence of the dividend rate δ_R on either of the volatilities σ_X and σ_K arises due to the non-zero correlation $\rho_{XK}\sigma_X\sigma_K$ between the state variables X and K, as well as the variability[5] of the dividend rate δ_X with respect to changes in the volatility σ_X. In the event of an underlying change in volatility, δ_R will

[5]Refer to Section 2.2.2 for the discussion of the non-parametric characterization of δ_X.

thus be seen to adjust accordingly. However, it is obvious that the real-world drift

$$\mu'_R = \mu'_X - \mu'_K + \sigma_K^2 - \rho_{XK}\sigma_X\sigma_K$$

will also take up part of the adjustment necessary for the preservation of equilibrium.

5.2.3 A Closed-Form Solution for the Follower's Three-Variable Value Function

Returning our attention to the risk-neutral solution of equation (5.2.1.2), appropriate substitution of the return on investment ratio R in Appendix D.1 gives rise to the following second order Cauchy-Euler ordinary differential equation:

$$\frac{1}{2}\sigma_R^2 R^2 \frac{d^2 F_0^*(R)}{dR^2} + (\mu_X - \mu_K) R \frac{dF_0^*(R)}{dR} - \delta_K F_0^*(R) = 0, \qquad (5.2.3.1)$$

where F_0^* is now the function to be determined.

Although this simplified explanation of the movements in the idle follower's value function bears close resemblance in structure to the corresponding Cauchy-Euler equation (2.2.3.1) in the case when the investment cost is fixed, it must be pointed out that the coefficient of the first-order derivative does not reduce to the risk-neutral drift of the new state variable R, with the risk-free rate r also having been replaced by the dividend rate δ_K.

Nevertheless, we are able to proceed using the same solution techniques as before, applying next the transformation

$$R = e^{t_3}$$

to obtain a linear ordinary differential equation with constant coefficients (refer to Appendix D.2):

$$\frac{1}{2}\sigma_R^2 \frac{d^2 G_0^*(t_3)}{dt_3^2} + (\mu_X - \mu_K - \frac{1}{2}\sigma_R^2) \frac{dG_0^*(t_3)}{dt_3} - \delta_K G_0^*(t_3) = 0. \qquad (5.2.3.2)$$

Thus, the auxiliary equation is now given by the fundamental quadratic

$$\Gamma(\beta) = \frac{1}{2}\sigma_R^2 \beta^2 + (\mu_X - \mu_K - \frac{1}{2}\sigma_R^2)\beta - \delta_K = 0,$$

where

$$\beta_{3,4} = \frac{-(\mu_X - \mu_K - \frac{1}{2}\sigma_R^2) \pm \sqrt{(\mu_X - \mu_K - \frac{1}{2}\sigma_R^2)^2 - 4(\frac{1}{2}\sigma_R^2)(-\delta_K)}}{2(\frac{1}{2}\sigma_R^2)}$$

$$= \frac{-(\mu_X - \mu_K - \frac{1}{2}\sigma_R^2) \pm \sqrt{2\delta_K \sigma_R^2 + (\mu_X - \mu_K - \frac{1}{2}\sigma_R^2)^2}}{\sigma_R^2}.$$

describe the solutions to this characteristic equation such that β_3 denotes the larger of the two roots.

Similar to the function $\Upsilon(\beta)$ in the two-variable model, we observe that the coefficient of β^2 in $\Gamma(\beta)$ is positive, with

$$\Gamma(0) = -\delta_K < 0$$

and

$$\Gamma(1) = \mu_X - \mu_K - \delta_K = \mu_X - \mu_K - (r - \mu_K) = -(r - \mu_X) = -\delta_X < 0.$$

Hence, we likewise conclude that the characteristic quadratic function $\Gamma(\beta)$ crosses the horizontal axis to the right of $\beta = 1$, and to the left of $\beta = 0$ i.e. $\beta_3 > 1$ and $\beta_4 < 0$. However, despite the similarities in their defining features, a plot of $\Upsilon(\beta)$ and $\Gamma(\beta)$ on the same Cartesian plane illustrates the subtle influence of the stochastic investment cost on the characteristic quadratic function.

Figures 5.1(a) and 5.1(b) demonstrate the respective cases of positive and negative correlation between the state variables P and Q. The characteristic functions that are based on correlation scenarios (i) and (iv), and (ii) and (iii), have therefore been grouped for comparison to the relevant quadratic function $\Upsilon(\beta)$ in the two-variable model.

Although the influence of a third stochastic variable does not appear significant (regardless of the correlation that it exhibits with each of the other two variables), an inspection of the precise values of the associated roots provides meaningful insight. In Figure 5.1(a), we find that

$$\beta_1 = 1.40210 < \beta_3^{(iv)} = 1.42596 < \beta_3^{(i)} = 1.54888$$

and

$$\beta_2 = -1.22968 < \beta_4^{(i)} = -0.69174 < \beta_4^{(iv)} = -0.52596,$$

where $\beta_3^{(i)}$ and $\beta_4^{(i)}$ denote the positive and negative solutions respectively to the fundamental quadratic

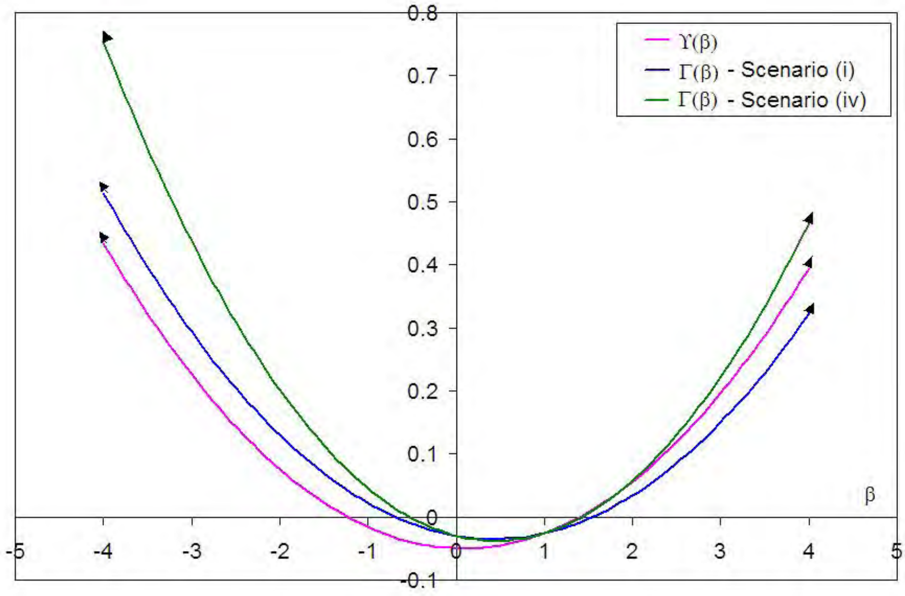
$$\Gamma(\beta) = 0$$

under correlation scenario (i). Similarly, in Figure 5.1(b) we find that

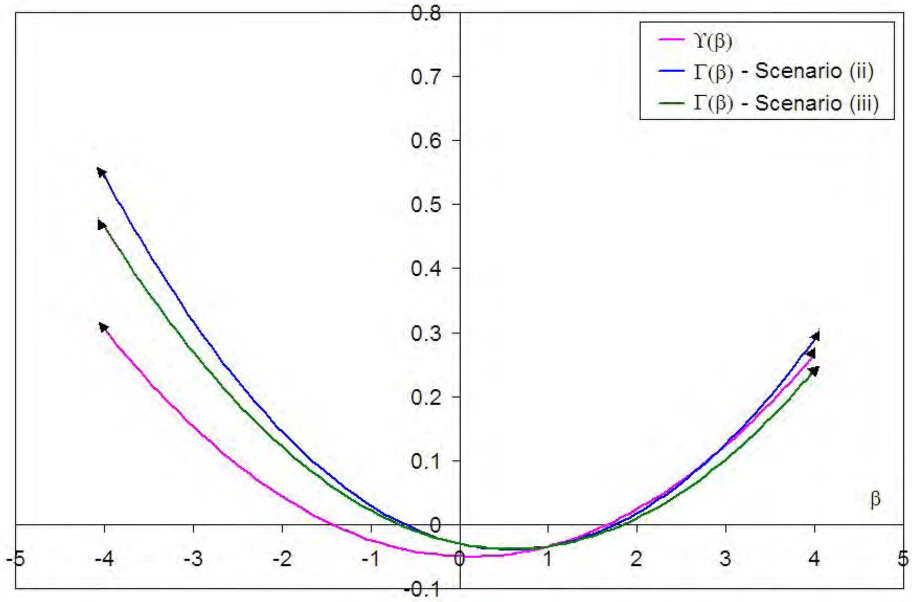
$$\beta_1 = 1.66667 < \beta_3^{(ii)} = 1.75378 < \beta_3^{(iii)} = 1.8444,$$

and

$$\beta_2 = -1.42857 < \beta_4^{(iii)} = -0.67773 < \beta_4^{(ii)} = -0.61092.$$



(a) $\rho_{PQ} = 0.2$



(b) $\rho_{PQ} = -0.2$

Figure 5.1: A comparison of characteristic quadratic functions

The following values have been assigned: $\mu_P = 0.01$, $\mu_Q = 0.01$, $\mu_K = 0.02$, $\sigma_P = 0.1$, $\sigma_Q = 0.2$, $\sigma_K = 0.1$, $r = 0.05$, $\rho_{PQ} = \pm 0.2$, $\rho_{PK} = \pm 0.2$ and $\rho_{QK} = \pm 0.2$.

Thus, for the chosen set of parameters, we may conclude that

$$\beta_1 < \beta_3$$

and

$$\beta_2 < \beta_4,$$

irrespective[6] of the correlation that prevails between each of the state variables P, Q and K.

The implication of this particular result will become clear as we progress with the development of our three-variable model. However, we may already surmise that, for the chosen parameter values, the additional influence of the stochastic investment cost induces earlier entry by the follower than in the case when the investment cost is fixed.

This conjecture stems from our existing knowledge of the relationship between the option value multiple, and the positive root of the auxiliary equation (refer to Figure 2.2). Given that our three-variable model produces a larger root in this instance, we subsequently anticipate the corresponding option value multiple to be lower. However, before drawing any final conclusions, we first need to determine an explicit expression for the follower's three-variable value function.

As we have shown the roots of the fundamental quadratic

$$\Gamma(\beta) = 0$$

to be real and distinct, we may therefore assume that equation (5.2.3.2) admits a general solution of the form

$$G_0^*(t_3) = A_1 e^{\beta_3 t_3} + B_1 e^{\beta_4 t_3},$$

or equivalently

$$F_0^*(R) = A_1 R^{\beta_3} + B_1 R^{\beta_4}. \tag{5.2.3.3}$$

[6]Although the case of zero correlation is not shown in Figures 5.1(a) and 5.1(b), we easily verify that these results will continue to hold if the stochastic investment cost K is entirely uncorrelated with each of the state variables P and Q:

$$\beta_1 = 1.40210 < \beta_3 = 1.47896$$

and

$$\beta_2 = -1.22968 < \beta_4 = -0.59660$$

when $\rho_{PQ} = 0.2$ and $\rho_{XK} = 0$, and similarly

$$\beta_1 = 1.66667 < \beta_3 = 1.79622$$

and

$$\beta_2 = -1.42857 < \beta_4 = -0.64237$$

when $\rho_{PQ} = -0.2$ and $\rho_{XK} = 0$.

This solution has exactly the same structure as the familiar equation (2.2.3.3) for the case when only the profit flow X is uncertain. Moreover, its boundary conditions are also similar. The value-matching condition is given by

$$F_0^*(R_F) = \frac{1}{K_F} F_0(X_F^*, K_F),$$

where R_F denotes the unique value of the composite state variable R that triggers optimal market entry for the follower, whilst X_F^* refers to the value of the profit flow, and K_F to the cost of investment, at this same point[7]. The factor

$$F_0(X_F^*, K_F)$$

represents the net present value of cash flows and is almost identical to the value-matching condition (2.2.3.4) in our two-variable model, but with the fixed investment cost K replaced by the random payment K_F at the point of market entry. Naturally, the unique trigger value X_F must also be replaced by the appropriate profit flow X_F^* to ensure that investment yields the optimal return R_F. Hence, the value-matching condition becomes

$$F_0^*(R_F) = \frac{1}{K_F}\left(\frac{X_F^*}{\delta_X} - K_F\right) = \frac{R_F}{\delta_X} - 1, \tag{5.2.3.4}$$

with corresponding smooth-pasting condition[8]

$$\left.\frac{dF_0^*(R)}{dR}\right|_{R=R_F} = \frac{1}{\delta_X}. \tag{5.2.3.5}$$

Although we are now dealing with a free-boundary problem that involves the added complexity of a third source of randomness, it is fortunate that the diffusion process for our new state variable R once again permits the initial condition

$$F_0^*(0) = 0, \tag{5.2.3.6}$$

so that we may immediately take $B_1 = 0$ to eliminate the negative exponent β_4:

$$F_0^*(R) = A_1 R^{\beta_3}. \tag{5.2.3.7}$$

[7] Note that there are an infinite number of solutions for both X_F^* and K_F in this three-variable model. The solution set describes a linear relationship, the slope of which is determined by the unique value of the optimal return on investment R_F:

$$X_F^* = R_F K_F.$$

[8] It is interesting to note that the smooth-pasting boundary for the follower's optimal entry remains unchanged regardless of whether the investment cost is stochastic or fixed.

Applying conditions (5.2.3.4) and (5.2.3.5) to equation (5.2.3.7) in Appendix E.1, we then arrive at the following simultaneous solution for the coefficient A_1 and of the optimal trigger R_F:

$$A_1 = R_F^{-\beta_3}(\gamma - 1)$$

and

$$R_F = \gamma\delta_X,$$

where

$$\gamma = \frac{\beta_3}{\beta_3 - 1}.$$

Combining this expression for A_1 with the general solution (5.2.3.7), the function of interest $F_0^*(R)$ is determined, yielding the following complete piecewise solution:

$$F^*(R) = \begin{cases} (\gamma - 1)\left(\dfrac{R}{R_F}\right)^{\beta_3} & R < R_F \\ \dfrac{R}{\delta_X} - 1 & R \geq R_F; K = K_F \end{cases}. \quad (5.2.3.8)$$

Hence, the follower's three-variable value function is therefore given by

$$f(P,Q,K) = KF^*(R) = \begin{cases} (\gamma - 1)K\left(\dfrac{R}{R_F}\right)^{\beta_3} & R < R_F \\ \dfrac{X}{\delta_X} - K & R \geq R_F; K = K_F \end{cases}. \quad (5.2.3.9)$$

where

$$R_F = \gamma\delta_X. \quad (5.2.3.10)$$

The free boundary R_F may be directly compared to the optimal point of market entry for the follower in the case when the investment cost is fixed. From trigger (2.2.3.9), we know that the optimal return on investment in the two-variable model is given by

$$\frac{X_F}{K} = \lambda\delta_X.$$

Hence, we may conclude that uncertainty in the once-off investment payment will only induce earlier entry by the follower if

$$\gamma < \lambda,$$

or equivalently if

$$\beta_3 > \beta_1.$$

This result confirms our preconceptions in the earlier example, but in each particular case, the timing of market entry relative to the trigger in the fixed-cost scenario will be dependent on the values of the parameter inputs. The following generic proposition therefore arises.

***Proposition* 5**: *Conditional on the leader's previous entry, the optimal strategy for a follower that is subject to a once-off random capital outlay is to invest as soon as the return on investment ratio reaches R_F. The optimal time for such a follower to enter the market may thus be stated as*

$$T_F^* = \inf\{t \geq 0 : R = \gamma\delta_X\}.$$

5.3 A Model with Three Stochastic Variables where the Cost of Investment Assumes Ongoing Relevance

We next consider the uncertain investment cost K to be an ongoing capital outlay from the point of market entry. Despite the fact that this third source of randomness now exhibits lasting influence on the model, the evolution of this stochastic variable in the risk-neutral world remains as described by the diffusion process (5.2.1), with all other assumptions and parameter characterizations also holding fixed.

In order to derive a partial differential equation that explains the movements in the idle follower's extended three-variable value function, we adopt the same replicating argument to construct a portfolio

$$\Pi^{cont}(P, Q, K)$$

that is long on an option $f_0^{cont}(P, Q, K)$, and short on n_P^{cont}, n_Q^{cont} and n_K^{cont} units of "assets" P, Q and K respectively.

By applying Itô's Lemma and employing the hedging strategy

$$n_P^{cont} = \frac{\partial f_0^{cont}(P, Q, K)}{\partial P},$$

$$n_Q^{cont} = \frac{\partial f_0^{cont}(P, Q, K)}{\partial Q}$$

and

$$n_K^{cont} = \frac{\partial f_0^{cont}(P,Q,K)}{\partial K},$$

we obtain a partial differential equation that is identical to equation (5.2.1.2), but with the idle follower's value function instead given by $f_0^{cont}(P,Q,K)$. Hence, by persisting with the same solution techniques as before[9], we are once again able to reduce the problem to a second order Cauchy-Euler ordinary differential equation, and subsequently arrive at the following general solution:

$$F_0^{**}(R) = C_1 R^{\beta_3} + D_1 R^{\beta_4}, \tag{5.3.1}$$

where F_0^{**} is now the function of interest.

Thus far, the extension of our three-variable model has not seen any structural change, and even the same parameters are present. However, the value-matching boundary proves to be the crucial point of difference. As the follower is expected to outlay a continuous stream of uncertain amount upon becoming active in the market, the net present value of cash flows at the stopping point is therefore given by

$$\begin{aligned}
F_0^{cont}(X_F^{**}, K_F^{cont}) &= EPV_{\text{(future profits)}} - EPV_{\text{(future investments)}} \\
&= \xi\left(\int_0^\infty X_t e^{-rt} dt\right) - \xi\left(\int_0^\infty K_t e^{-rt} dt\right) \\
&= \int_0^\infty \xi(X_t) e^{-rt} dt - \int_0^\infty \xi(K_t) e^{-rt} dt \\
&= \int_0^\infty X_F^{**} e^{(\mu_X)t} e^{-rt} dt - \int_0^\infty K_F^{cont} e^{(\mu_K)t} e^{-rt} dt \\
&= \int_0^\infty X_F^{**} e^{-(r-\mu_X)t} dt - \int_0^\infty K_F^{cont} e^{-(r-\mu_K)t} dt \\
&= \int_0^\infty X_F^{**} e^{-(\delta_X)t} dt - \int_0^\infty K_F^{cont} e^{-(\delta_K)t} dt \\
&= \frac{X_F^{**}}{\delta_X} - \frac{K_F^{cont}}{\delta_K},
\end{aligned}$$

where X_F^{**} refers to the initial value of the profit flow, and K_F^{cont} to the initial

[9]We let

$$R = \frac{X}{K} = \frac{PQ}{K},$$

so that

$$f_0^{cont}(P,Q,K) = F_0^{cont}(X,K) = F_0^{cont}(RK,K) = KF_0^{**}(R).$$

cost of market entry, when investment first commences. The value-matching condition hence becomes

$$F_0^{**}(R_F^{cont}) = \frac{1}{K_F^{cont}}\left(\frac{X_F^{**}}{\delta_X} - \frac{K_F^{cont}}{\delta_K}\right) = \frac{R_F^{cont}}{\delta_X} - \frac{1}{\delta_K}, \tag{5.3.2}$$

where R_F^{cont} denotes the optimal return on investment that triggers market entry for the follower when random capital outlay is deemed to continue[10].

Whilst the non-unique quantities X_F^{**} and K_F^{cont} may very well assume the same values as their respective counterparts X_F^{*} and K_F for a particular set of parameters, we will soon show that

$$R_F^{cont} > R_F,$$

so that the following relationship may be stated with certainty: If

$$X_F^{**} = X_F^{*}$$

then

$$K_F^{cont} < K_F,$$

but if

$$K_F^{cont} = K_F$$

then

$$X_F^{**} > X_F^{*}.$$

Despite the slight change in the value-matching boundary (5.3.2), the corresponding smooth-pasting condition

$$\left.\frac{dF_0^{**}(R)}{dR}\right|_{R=R_F^{cont}} = \frac{1}{\delta_X} \tag{5.3.3}$$

assumes the usual form, as does the initial condition

$$F_0^{**}(0) = 0. \tag{5.3.4}$$

[10]The solution set for the critical values of the profit flow and of the investment cost is once again described by a linear relationship, the slope of which is now determined by the unique value of the optimal return on investment R_F^{cont}:

$$X_F^{**} = R_F^{cont} K_F^{cont}.$$

Thus, we are immediately able to conclude that $D_1 = 0$, so that the value-matching and smooth-pasting boundaries may then be applied to the equation

$$F_0^{**}(R) = C_1 R^{\beta_3} \tag{5.3.5}$$

for the simultaneous determination of the coefficient C_1 and of the optimal ratio R_F^{cont}. The calculation is given in Appendix E.2, yielding

$$C_1 = \frac{\gamma - 1}{\delta_K}(R_F^{cont})^{-\beta_3}$$

and

$$R_F^{cont} = \gamma \frac{\delta_X}{\delta_K},$$

where γ remains as previously defined. Combining this expression for C_1 with the general solution (5.3.5), the function of interest $F_0^{**}(R)$ is obtained, giving rise to the piecewise equation below:

$$F^{**}(R) \quad = \quad \begin{cases} \dfrac{\gamma - 1}{\delta_K}\left(\dfrac{R}{R_F^{cont}}\right)^{\beta_3} & R < R_F^{cont} \\ \dfrac{R}{\delta_X} - \dfrac{1}{\delta_K} & R \geq R_F^{cont} \end{cases}. \tag{5.3.6}$$

Hence, the follower's extended three-variable value function is given by

$$f^{cont}(P, Q, K) = KF^{**}(R) \quad = \quad \begin{cases} \dfrac{(\gamma - 1)K}{\delta_K}\left(\dfrac{R}{R_F^{cont}}\right)^{\beta_3} & R < R_F^{cont} \\ \dfrac{X}{\delta_X} - \dfrac{K}{\delta_K} & R \geq R_F^{cont} \end{cases} \tag{5.3.7}$$

where

$$R_F^{cont} = \gamma \frac{\delta_X}{\delta_K}. \tag{5.3.8}$$

Now from trigger equation (5.2.3.10), we recall that

$$R_F = \gamma \delta_X,$$

so that the free boundary R_F^{cont} may alternatively be expressed in the form

$$R_F^{cont} = \frac{R_F}{\delta_K}.$$

As the dividend rate δ_K denotes a percentage return (i.e. $\delta_K < 1$), we therefore deduce that

$$R_F^{cont} > R_F$$

for any given set of parameters. This result seems intuitive as we would expect the follower to require a higher return on investment from the outset if the cost of playing in the market is an ongoing uncertain outlay, rather than a once-off stochastic payment. Hence, market entry occurs later when there are three *permanent* sources of randomness to consider[11]. The following proposition therefore arises.

***Proposition* 6**: *Conditional on the leader's previous entry, the optimal strategy for a follower that is expected to outlay a continuous stream of uncertain amount upon becoming active in the market is to invest as soon as the return on investment ratio reaches R_F^{cont}. The optimal time for such a follower to enter the market may thus be stated as*

$$T_F^{**} = \inf\left\{ t \geq 0 : R = \gamma \frac{\delta_X}{\delta_K} \right\},$$

where

$$T_F^{**} > T_F^{*}.$$

[11]Note that we are once again unable to derive a rule for the timing of market entry relative to the two-variable model.

Chapter 6

Conclusion

In this dissertation, we have studied the timing of sequential market entry for each player in both a pre-emptive and non pre-emptive environment, as well as the optimal timing for simultaneous investment into a duopoly. Under all three circumstances, we have extended the work of Paxson and Pinto by adjusting their two-variable model to include the effect of the covariance term $\rho_{PQ}\sigma_P\sigma_Q$ on the value functions of the firms, and on the subsequent timing of their investment.

Now regardless of whether the covariance term is incorporated into the two-variable model or not, we have found that the *relative* timing of market entry for the participants remains unchanged, i.e.

$$X_F > X_S > X_L > X_L^P.$$

In a sequential game, the leader will always enter earlier than if they had chosen to invest simultaneously with their rivalling counterpart, whilst the opposite is true of the follower. Furthermore, if there is a race to invest first, the leader will enter even sooner at the sub-optimal profit flow X_L^P.

We have also found that the higher are the monopoly profits that the leader will earn whilst alone in the market, the more immediate will the first-mover's investment be. This result holds true irrespective of whether the leader's role is pre-assigned, or endogenous to the model. On the contrary - although the same can be said of the pre-emptive leader's trigger dependence on the magnitude of the first-mover advantage that they will acquire - the non pre-emptive leader displays indifference towards this measure.

Now unlike the relative timing of investment, the *absolute* value of each optimal profit trigger differs between Paxson and Pinto's model and our adjusted model, with findings remaining consistent despite the nature of the game, or the position of the player. In particular, our inclusion of the covariance term has improved on the accuracy of Paxson and Pinto's results by deeming market entry to be optimal slightly later (earlier) than their model suggests, under conditions of negative (positive) correlation. Only when the state variables are entirely uncorrelated, will the models produce identical triggers.

In Chapter 4, we have also seen that the presence of the covariance term in our adjusted model dampens the response in each of our investment rules to changes in correlation and volatility. To be more specific, Paxson and Pinto's proposed timings for optimal market entry become far less accurate under conditions of extreme positive correlation - where investment is significantly delayed - and high levels of positive volatility - where investment is far earlier (later) than our model suggests under negative (positive) correlation.

In Chapter 5, we have extended our adjusted model to include a third source of randomness. Whilst we have not been able to derive a rule for the timing of the follower's entry relative to the fixed-cost scenario, we may conclude with certainty that investment will occur later when there are three *permanent* sources of randomness to consider, as opposed to the investment cost being a once-off stochastic payment. For simplicity, we have only extended the follower's value function to three variables, but the extension of the pre-emptive and co-operative leader value functions could serve as possible areas for further investigation.

Although we have made promising progress in this dissertation with enhancing existing results, it is important to bear in mind that our adjusted model has been developed under the assumption of a complete market. As such, our conclusions are only likely to be relevant under these circumstances and hence, the development of similar models for the incomplete case could pose as another potential area for future research. In fact, Ewald and Yang (4), Henderson and Hobson (see (12) and (13)) and Miao and Wang (23) are among a few of the authors to have already made strides in this area by adopting a utility-based approach.

Appendix A

Methods of Similarity

A.1 Derivation of Equation (2.2.3.1)

For ease of reference, we shall restate the partial differential equation (2.2.1.2):

$$\frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 f_0(P,Q)}{\partial P^2} + \mu_P P \frac{\partial f_0(P,Q)}{\partial P} + \frac{1}{2}\sigma_Q^2 Q^2 \frac{\partial^2 f_0(P,Q)}{\partial Q^2} + \mu_Q Q \frac{\partial f_0(P,Q)}{\partial Q}$$
$$+\rho_{PQ}\sigma_P\sigma_Q PQ \frac{\partial^2 f_0(P,Q)}{\partial P \partial Q} - r f_0(P,Q) = 0.$$

Now if we let

$$X = PQ$$

denote the total profit (or profit flow) for the follower, we may then write

$$f_0(P,Q) = F_0(X),$$

where F_0 is now the function to be determined. This transformation is permissible due to the simple relationship between the variables X, P and Q, and hence it follows that

$$\begin{aligned} \frac{\partial f_0(P,Q)}{\partial Q} &= \frac{\partial F_0(X)}{\partial X}\frac{\partial X}{\partial Q} \\ &= \frac{\partial F_0(X)}{\partial X} P, \end{aligned}$$

$$\begin{aligned} \frac{\partial f_0(P,Q)}{\partial P} &= \frac{\partial F_0(X)}{\partial X}\frac{\partial X}{\partial P} \\ &= \frac{\partial F_0(X)}{\partial X} Q, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 f_0(P,Q)}{\partial Q^2} &= \frac{\partial}{\partial X}\left[\frac{\partial F_0(X)}{\partial X}P\right]\frac{\partial X}{\partial Q} \\
&= \frac{\partial^2 F_0(X)}{\partial X^2}P^2,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f_0(P,Q)}{\partial P^2} &= \frac{\partial}{\partial X}\left[\frac{\partial F_0(X)}{\partial X}Q\right]\frac{\partial X}{\partial P} \\
&= \frac{\partial^2 F_0(X)}{\partial X^2}Q^2
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 f_0(P,Q)}{\partial P\partial Q} &= \frac{\partial}{\partial P}\left[\frac{\partial F_0(X)}{\partial X}P\right] \\
&= \frac{\partial F_0(X)}{\partial X} + \frac{\partial}{\partial X}\left[\frac{\partial F_0(X)}{\partial X}\right]\frac{\partial X}{\partial P}P \\
&= \frac{\partial F_0(X)}{\partial X} + \frac{\partial^2 F_0(X)}{\partial X^2}X.
\end{aligned}
$$

After making the appropriate substitutions, equation (2.2.1.2) simplifies to the following second order Cauchy-Euler ordinary differential equation:

$$\frac{1}{2}\sigma_X^2 X^2\frac{d^2F_0(X)}{dX^2} + \mu_X X\frac{dF_0(X)}{dX} - rF_0(X) = 0,$$

where

$$\sigma_X^2 = \sigma_P^2 + \sigma_Q^2 + 2\rho_{PQ}\sigma_P\sigma_Q$$

and

$$\mu_X = \mu_P + \mu_Q + \rho_{PQ}\sigma_P\sigma_Q.$$

A.2 Derivation of Equation (2.2.3.2)

Let us reconsider equation (2.2.3.1) and apply the transformation

$$X = e^{t_1}$$

so that

$$F_0(X) = G_0(t_1).$$

Since $X > 0$, the inverse transformation may then be defined as

$$t_1 = \ln X.$$

Thus,

$$\frac{dF_0(X)}{dX} = \frac{dG_0(t_1)}{dt_1}\frac{dt_1}{dX} = \frac{1}{X}\frac{dG_0(t_1)}{dt_1}$$

which gives

$$X\frac{dF_0(X)}{dX} = \frac{dG_0(t_1)}{dt_1},$$

and similarly

$$\begin{aligned}
\frac{d^2F_0(X)}{dX^2} &= \frac{d}{dX}\left[\frac{1}{X}\frac{dG_0(t_1)}{dt_1}\right] \\
&= \frac{1}{X}\frac{d}{dX}\left[\frac{dG_0(t_1)}{dt_1}\right] - \frac{1}{X^2}\frac{dG_0(t_1)}{dt_1} \\
&= \frac{1}{X}\frac{d^2G_0(t_1)}{dt_1^2}\frac{dt_1}{dX} - \frac{1}{X^2}\frac{dG_0(t_1)}{dt_1} \\
&= \frac{1}{X^2}\left[\frac{d^2G_0(t_1)}{dt_1^2} - \frac{dG_0(t_1)}{dt_1}\right]
\end{aligned}$$

so that

$$X^2\frac{d^2F_0(X)}{dX^2} = \frac{d^2G_0(t_1)}{dt_1^2} - \frac{dG_0(t_1)}{dt_1}.$$

Final substitution back into equation (2.2.3.1) obtains the following linear ordinary differential equation with constant coefficients:

$$\frac{1}{2}\sigma_X^2\frac{d^2G_0(t_1)}{dt_1^2} + (\mu_X - \frac{1}{2}\sigma_X^2)\frac{dG_0(t_1)}{dt_1} - rG_0(t_1) = 0.$$

Appendix B

Solution of the Option-value Functions, and of the Corresponding Optimal Triggers

B.1 Solution of the Coefficient A, and of the Trigger X_F

For the general solution (2.2.3.7), boundary conditions (2.2.3.4) and (2.2.3.5) give rise to the following set of simultaneous equations:

$$AX_F^{\beta_1} = \frac{X_F}{\delta_X} - K \tag{B.1.1}$$

and

$$A\beta_1 X_F^{\beta_1 - 1} = \frac{1}{\delta_X}. \tag{B.1.2}$$

Dividing (B.1.1) by (B.1.2) yields

$$\frac{X_F}{\beta_1} = X_F - K\delta_X$$

$$X_F = X_F\beta_1 - K\beta_1\delta_X$$

$$X_F(1 - \beta_1) = -K\beta_1\delta_X,$$

so that

$$X_F = \frac{\beta_1}{\beta_1 - 1}\delta_X K = \lambda\delta_X K$$

where

$$\lambda = \frac{\beta_1}{\beta_1 - 1}.$$

Now if we reconsider (B.1.1) and substitute for X_F, we get

$$\begin{aligned} A &= X_F^{-\beta_1}\left(\frac{X_F}{\delta_X} - K\right) \\ &= X_F^{-\beta_1}(\lambda K - K) \\ &= X_F^{-\beta_1}(\lambda - 1)K. \end{aligned}$$

B.2 Solution of the Coefficient J, and of the Trigger X_L

To determine a particular solution of the general form (3.1.1), we must consider boundary conditions (3.1.2) and (3.1.3). Combining equation (3.1.1) with the value matching condition yields

$$JX_L^{\beta_1} = \left(\frac{X_L}{X_F}\right)^{\beta_1} \lambda K(m - \overline{m}) + \frac{X_L\overline{m}}{\delta_X} - K, \tag{B.2.1}$$

and similarly, the smooth-pasting boundary may be written as

$$J\beta_1 X_L^{\beta_1 - 1} = \left(\frac{X_L}{X_F}\right)^{\beta_1 - 1} \frac{\lambda K\beta_1}{X_F}(m - \overline{m}) + \frac{\overline{m}}{\delta_X}. \tag{B.2.2}$$

Now from (B.2.1), we get

$$J = X_F^{-\beta_1}\lambda K(m - \overline{m}) + \frac{X_L^{1-\beta_1}\overline{m}}{\delta_X} - KX_L^{-\beta_1}. \tag{B.2.3}$$

Substituting (B.2.3) into (B.2.2) and cancelling like terms produces

$$\frac{K\beta_1}{X_L} = \frac{\overline{m}(\beta_1 - 1)}{\delta_X}$$

or

$$\frac{X_L}{K\beta_1} = \frac{\delta_X}{\overline{m}(\beta_1 - 1)},$$

so that

$$X_L = \frac{\beta_1}{\beta_1 - 1}\frac{\delta_X K}{\overline{m}} = \frac{\lambda\delta_X K}{\overline{m}}.$$

Final substitution of X_L back into (B.2.3) yields

$$\begin{aligned} J &= X_F^{-\beta_1}\lambda K(m - \overline{m}) + X_L^{-\beta_1}\left(\frac{X_L\overline{m}}{\delta_X} - K\right) \\ &= X_F^{-\beta_1}\lambda K(m - \overline{m}) + X_L^{-\beta_1}(\lambda K - K) \\ &= X_F^{-\beta_1}\lambda K(m - \overline{m}) + X_L^{-\beta_1}(\lambda - 1)K. \end{aligned}$$

B.3 Solution of the Coefficient W, and of the Trigger X_S

For the general solution (3.2.1), boundary conditions (3.2.2) and (3.2.3) give rise to the following set of simultaneous equations:

$$WX_S^{\beta_1} = \frac{m_s X_S}{\delta_X} - K \tag{B.3.1}$$

and

$$W\beta_1 X_S^{\beta_1 - 1} = \frac{m_s}{\delta_X}. \tag{B.3.2}$$

Dividing (B.3.1) by (B.3.2) yields

$$\begin{aligned}
\frac{X_S}{\beta_1} &= X_S - \frac{K\delta_X}{m_s} \\
X_S &= X_S\beta_1 - \frac{K\beta_1\delta_X}{m_s} \\
X_S(1-\beta_1) &= -\frac{K\beta_1\delta_X}{m_s},
\end{aligned}$$

so that

$$X_S = \frac{\beta_1}{\beta_1 - 1}\frac{\delta_X K}{m_s} = \frac{\lambda\delta_X K}{m_s}.$$

Now if we reconsider (B.3.1) and substitute for X_S, we get

$$\begin{aligned}
W &= X_S^{-\beta_1}\left(\frac{m_s X_S}{\delta_X} - K\right) \\
&= X_S^{-\beta_1}(\lambda K - K) \\
&= X_S^{-\beta_1}(\lambda - 1)K.
\end{aligned}$$

Appendix C

Proof of the Uniqueness of the Pre-emptive Trigger X_L^P

To prove the existence of a unique pre-emptive trigger strictly below X_F, we shall define

$$\begin{aligned} V(X) &= L_0^P(X) - F_0(X) \\ &= \left(\frac{X}{X_F}\right)^{\beta_1} \lambda K(m - \overline{m}) + \frac{X\overline{m}}{\delta_X} - K - (\lambda - 1)K\left(\frac{X}{X_F}\right)^{\beta_1}, \end{aligned}$$

where the trigger X_L^P implicitly solves the non-linear equation

$$V(X) = 0.$$

Now

$$V(0) = -K < 0,$$

but

$$\begin{aligned} V(X_F) &= \lambda K(m - \overline{m}) + \frac{X_F\overline{m}}{\delta_X} - K - (\lambda - 1)K \\ &= \frac{X_F}{\delta_X}(m - \overline{m}) + \frac{X_F\overline{m}}{\delta_X} - \frac{X_F}{\delta_X} \\ &= \frac{X_F(m-1)}{\delta_X} \\ &= \lambda K(m - 1) > 0. \end{aligned}$$

Thus, there must exist at least one root in the domain

$$X \in (0, X_F).$$

By examining the second derivative of the function $V(X)$, we see that

$$\begin{aligned}\frac{d^2V}{dX^2} &= \beta_1(\beta_1-1)\frac{X^{\beta_1-2}}{X_F^{\beta_1}}\lambda K(m-\overline{m}) - (\lambda-1)K\beta_1(\beta_1-1)\frac{X^{\beta_1-2}}{X_F^{\beta_1}} \\ &= \frac{K\beta_1^2(m-\overline{m})\left(\frac{X}{X_F}\right)^{\beta_1-2} - K\beta_1\left(\frac{X}{X_F}\right)^{\beta_1-2}}{X_F^2} < 0,\end{aligned}$$

thereby revealing the strict monotonicity of the function over the respective interval. Hence, the root X_L^P must be unique, with

$$V(X) < 0$$

for

$$X \in (0, X_L^P),$$

and

$$V(X) > 0$$

for

$$X \in (X_L^P, X_F).$$

Appendix D

Extended Methods of Similarity

D.1 Derivation of Equation (5.2.3.1)

For ease of reference, we shall restate the partial differential equation (5.2.1.2):

$$\frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 f_0(P,Q,K)}{\partial P^2} + \mu_P P \frac{\partial f_0(P,Q,K)}{\partial P} + \frac{1}{2}\sigma_Q^2 Q^2 \frac{\partial^2 f_0(P,Q,K)}{\partial Q^2} + \mu_Q Q \frac{\partial f_0(P,Q,K)}{\partial Q}$$
$$+ \frac{1}{2}\sigma_K^2 K^2 \frac{\partial^2 f_0(P,Q,K)}{\partial K^2} + \mu_K K \frac{\partial f_0(P,Q,K)}{\partial K} + \rho_{PQ}\sigma_P\sigma_Q PQ \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial Q}$$
$$+ \rho_{PK}\sigma_P\sigma_K PK \frac{\partial^2 f_0(P,Q,K)}{\partial P \partial K} + \rho_{QK}\sigma_Q\sigma_K QK \frac{\partial^2 f_0(P,Q,K)}{\partial Q \partial K} - r f_0(P,Q,K) = 0,$$

and we shall once again let

$$X = PQ$$

denote the total profit (or profit flow) for the follower. Now if we were to double the current values of both X and K, this would merely double the profit flow as well as the cost of investing, so that the return on investment

$$R = \frac{X}{K}$$

would remain unchanged. Hence, with reference to the literature of Dixit and Pindyck (2), the follower's option-value function should therefore be homogeneous of degree 1 in (X, K), enabling us to write

$$\begin{aligned} f_0(P,Q,K) &= F_0(X,K) \\ &= F_0(RK,K) \\ &= KF_0^*(R), \end{aligned}$$

so that F_0^* is now the function to be determined. We therefore have

$$\begin{aligned}
\frac{\partial f_0(P,Q,K)}{\partial P} &= K\frac{\partial F_0^*(R)}{\partial R}\frac{\partial R}{\partial P} \\
&= K\frac{\partial F_0^*(R)}{\partial R}\frac{Q}{K} \\
&= Q\frac{\partial F_0^*(R)}{\partial R},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f_0(P,Q,K)}{\partial Q} &= K\frac{\partial F_0^*(R)}{\partial R}\frac{\partial R}{\partial Q} \\
&= K\frac{\partial F_0^*(R)}{\partial R}\frac{P}{K} \\
&= P\frac{\partial F_0^*(R)}{\partial R},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f_0(P,Q,K)}{\partial K} &= K\frac{\partial F_0^*(R)}{\partial R}\frac{\partial R}{\partial K} + F_0^*(R) \\
&= F_0^*(R) - K\frac{\partial F_0^*(R)}{\partial R}\frac{PQ}{K^2} \\
&= F_0^*(R) - R\frac{\partial F_0^*(R)}{\partial R},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f_0(P,Q,K)}{\partial P^2} &= \frac{\partial}{\partial P}\left[Q\frac{\partial F_0^*(R)}{\partial R}\right] \\
&= \frac{\partial}{\partial R}\left[Q\frac{\partial F_0^*(R)}{\partial R}\right]\frac{\partial R}{\partial P} \\
&= \frac{Q^2}{K}\frac{\partial^2 F_0^*(R)}{\partial R^2},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f_0(P,Q,K)}{\partial Q^2} &= \frac{\partial}{\partial Q}\left[P\frac{\partial F_0^*(R)}{\partial R}\right] \\
&= \frac{\partial}{\partial R}\left[P\frac{\partial F_0^*(R)}{\partial R}\right]\frac{\partial R}{\partial Q} \\
&= \frac{P^2}{K}\frac{\partial^2 F_0^*(R)}{\partial R^2},
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 f_0(P,Q,K)}{\partial K^2} &= \frac{\partial}{\partial K}\left[F_0^*(R) - R\frac{\partial F_0^*(R)}{\partial R}\right] \\
&= \frac{\partial F_0^*(R)}{\partial R}\frac{\partial R}{\partial K} - \left[R\frac{\partial^2 F_0^*(R)}{\partial R^2}\frac{\partial R}{\partial K} - \frac{PQ}{K^2}\frac{\partial F_0^*(R)}{\partial R}\right] \\
&= -\frac{PQ}{K^2}\frac{\partial F_0^*(R)}{\partial R} - R\frac{\partial^2 F_0^*(R)}{\partial R^2}\left(-\frac{PQ}{K^2}\right) + \frac{PQ}{K^2}\frac{\partial F_0^*(R)}{\partial R} \\
&= \frac{R^2}{K}\frac{\partial^2 F_0^*(R)}{\partial R^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f_0(P,Q,K)}{\partial P \partial Q} &= \frac{\partial}{\partial Q}\left[Q\frac{\partial F_0^*(R)}{\partial R}\right] \\
&= \frac{\partial F_0^*(R)}{\partial R} + Q\frac{\partial^2 F_0^*(R)}{\partial R^2}\frac{\partial R}{\partial Q} \\
&= \frac{\partial F_0^*(R)}{\partial R} + R\frac{\partial^2 F_0^*(R)}{\partial R^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f_0(P,Q,K)}{\partial P \partial K} &= \frac{\partial}{\partial K}\left[Q\frac{\partial F_0^*(R)}{\partial R}\right] \\
&= Q\frac{\partial^2 F_0^*(R)}{\partial R^2}\frac{\partial R}{\partial K} \\
&= Q\frac{\partial^2 F_0^*(R)}{\partial R^2}\left(-\frac{PQ}{K^2}\right) \\
&= -\frac{PQ^2}{K^2}\frac{\partial^2 F_0^*(R)}{\partial R^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 f_0(P,Q,K)}{\partial Q \partial K} &= \frac{\partial}{\partial K}\left[P\frac{\partial F_0^*(R)}{\partial R}\right] \\
&= P\frac{\partial^2 F_0^*(R)}{\partial R^2}\frac{\partial R}{\partial K} \\
&= P\frac{\partial^2 F_0^*(R)}{\partial R^2}\left(-\frac{PQ}{K^2}\right) \\
&= -\frac{P^2 Q}{K^2}\frac{\partial^2 F_0^*(R)}{\partial R^2}.
\end{aligned}
$$

After the appropriate substitution and grouping of terms, equation (5.2.1.2) eventually simplifies to the following second order Cauchy-Euler ordinary differential equation:

$$
\frac{1}{2}\sigma_R^2 R^2 \frac{d^2 F_0^*(R)}{dR^2} + (\mu_X - \mu_K) R \frac{dF_0^*(R)}{dR} - \delta_K F_0^*(R) = 0,
$$

where

$$\begin{aligned}\sigma_R^2 &= \sigma_X^2 + \sigma_K^2 - 2\rho_{XK}\sigma_X\sigma_K \\ &= \sigma_P^2 + \sigma_Q^2 + \sigma_K^2 + 2\rho_{PQ}\sigma_P\sigma_Q - 2\rho_{PK}\sigma_P\sigma_K - 2\rho_{QK}\sigma_Q\sigma_K\end{aligned}$$

and

$$\mu_X = \mu_P + \mu_Q + \rho_{PQ}\sigma_P\sigma_Q.$$

D.2 Derivation of Equation (5.2.3.2)

Let us reconsider equation (5.2.3.1). Recalling that the composite stochastic variable R is defined by the ratio

$$\frac{X}{K},$$

we shall next apply the transformations

$$X = e^{t_1}$$

and

$$K = e^{t_2},$$

so that

$$R = e^{t_1 - t_2} = e^{t_3},$$

where $t_3 = t_1 - t_2$. Hence,

$$F_0^*(R) = G_0^*(t_3).$$

Now since $X > 0$ and $K > 0$, the inverse transformations may then be defined as

$$t_1 = \ln X$$

and

$$t_2 = \ln K,$$

so that

$$t_3 = \ln X - \ln K = \ln\left(\frac{X}{K}\right) = \ln R.$$

Thus,

$$\frac{dF_0^*(R)}{dR} = \frac{dG_0^*(t_3)}{dt_3}\frac{dt_3}{dR} = \frac{1}{R}\frac{dG_0^*(t_3)}{dt_3}$$

or

$$R\frac{dF_0^*(R)}{dR} = \frac{dG_0^*(t_3)}{dt_3},$$

and similarly,

$$\begin{aligned}
\frac{d^2F_0^*(R)}{dR^2} &= \frac{d}{dR}\left[\frac{1}{R}\frac{dG_0^*(t_3)}{dt_3}\right] \\
&= \frac{1}{R}\frac{d}{dR}\left[\frac{dG_0^*(t_3)}{dt_3}\right] - \frac{1}{R^2}\frac{dG_0^*(t_3)}{dt_3} \\
&= \frac{1}{R}\frac{d^2G_0^*(t_3)}{dt_3^2}\frac{dt_3}{dR} - \frac{1}{R^2}\frac{dG_0^*(t_3)}{dt_3} \\
&= \frac{1}{R^2}\left[\frac{d^2G_0^*(t_3)}{dt_3^2} - \frac{dG_0^*(t_3)}{dt_3}\right]
\end{aligned}$$

so that

$$R^2\frac{d^2F_0^*(R)}{dR^2} = \frac{d^2G_0^*(t_3)}{dt_3^2} - \frac{dG_0^*(t_3)}{dt_3}.$$

Final substitution back into equation (5.2.3.1) obtains the following linear ordinary differential equation with constant coefficients:

$$\frac{1}{2}\sigma_R^2\frac{d^2G_0^*(t_3)}{dt_3^2} + (\mu_X - \mu_K - \frac{1}{2}\sigma_R^2)\frac{dG_0^*(t_3)}{dt_3} - \delta_K G_0^*(t_3) = 0.$$

Appendix E

Solution of the Follower's Extended Option-value Function, and of the Corresponding Optimal Trigger

E.1 Solution of the Coefficient A_1, and of the Trigger R_F

For the general solution (5.2.3.7), boundary conditions (5.2.3.4) and (5.2.3.5) give rise to the following set of simultaneous equations:

$$A_1 R_F^{\beta_3} = \frac{R_F}{\delta_X} - 1 \tag{E.1.1}$$

and

$$A_1 \beta_3 R_F^{\beta_3 - 1} = \frac{1}{\delta_X}. \tag{E.1.2}$$

Dividing (E.1.1) by (E.1.2) yields

$$\begin{aligned} \frac{R_F}{\beta_3} &= R_F - \delta_X \\ R_F &= R_F \beta_3 - \beta_3 \delta_X \\ R_F(1 - \beta_3) &= -\beta_3 \delta_X, \end{aligned}$$

so that

$$R_F = \frac{\beta_3}{\beta_3 - 1}\delta_X = \gamma \delta_X$$

where

$$\gamma = \frac{\beta_3}{\beta_3 - 1}.$$

Now if we reconsider (E.1.1) and substitute for R_F, we get

$$A_1 = R_F^{-\beta_3}\left(\frac{R_F}{\delta_X} - 1\right) = R_F^{-\beta_3}(\gamma - 1).$$

E.2 Solution of the Coefficient C_1, and of the Trigger R_F^{cont}

For the general solution (5.3.5), boundary conditions (5.3.2) and (5.3.3) give rise to the following set of simultaneous equations:

$$C_1(R_F^{cont})^{\beta_3} = \frac{R_F^{cont}}{\delta_X} - \frac{1}{\delta_K} \tag{E.2.1}$$

and

$$C_1\beta_3(R_F^{cont})^{\beta_3-1} = \frac{1}{\delta_X}. \tag{E.2.2}$$

Dividing (E.2.1) by (E.2.2) yields

$$\begin{aligned} \frac{R_F^{cont}}{\beta_3} &= R_F^{cont} - \frac{\delta_X}{\delta_K} \\ R_F^{cont} &= R_F^{cont}\beta_3 - \frac{\delta_X}{\delta_K}\beta_3 \\ R_F^{cont}(1-\beta_3) &= -\frac{\delta_X}{\delta_K}\beta_3, \end{aligned}$$

so that

$$R_F^{cont} = \frac{\beta_3}{\beta_3 - 1}\frac{\delta_X}{\delta_K} = \gamma\frac{\delta_X}{\delta_K}.$$

Now if we reconsider (E.2.1) and substitute for R_F^{cont}, we get

$$C_1 = (R_F^{cont})^{-\beta_3}\left(\frac{R_F^{cont}}{\delta_X} - \frac{1}{\delta_K}\right) = \frac{\gamma - 1}{\delta_K}(R_F^{cont})^{-\beta_3}.$$

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