ASPECTS OF FUNCTIONAL VARIATIONS OF DOMINATION IN GRAPHS

by

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Abstract

Let $G = (V, E)$ be a graph. For any real valued function $f : V \to \mathbb{R}$ and $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. The weight of $f$ is defined as $f(V)$. A signed $k$-subdominating function (signed $kSF$) of $G$ is defined as a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for at least $k$ vertices of $G$, where $N[v]$ denotes the closed neighborhood of $v$. The signed $k$-subdomination number of a graph $G$, denoted by $\gamma_{ks}^{-11}(G)$, is equal to $\min\{f(V) : f$ is a signed $kSF$ of $G\}$. If instead of the range $\{-1, 1\}$, we require the range $\{-1, 0, 1\}$, then we obtain the concept of a minus $k$-subdominating function. Its associated parameter, called the minus $k$-subdomination number of $G$, is denoted by $\gamma_{ks}^{-101}(G)$.

A total signed dominating function (signed $TkSF$) of $G$ is defined as a function $f : V \to \{-1, 1\}$ such that $f(N(v)) \geq 1$ for at least $k$ vertices of $G$, where $N(v)$ denotes the open neighborhood of $v$. The total signed $k$-subdomination number of a graph $G$, denoted by $\gamma_{tks}^{-11}(G)$, is equal to $\min\{f(V) : f$ is a signed $TkSF$ of $G\}$. If instead of the range $\{-1, 1\}$, we require the range $\{-1, 0, 1\}$, then we obtain the concept of a total minus $k$-subdominating function. Its associated parameter, called the total minus $k$-subdomination number of $G$, is denoted by $\gamma_{tks}^{-101}(G)$.

In Chapter 2, we survey recent results on signed and minus $k$-subdomination in graphs.

In Chapter 3, we compute the signed and minus $k$-subdomination numbers for certain complete multipartite graphs and their complements, generalizing results due to Holm [30].

In Chapter 4, we give a lower bound on the total signed $k$-subdomination number in terms of the minimum degree, maximum degree and the order of the graph. A lower bound in terms of the degree sequence is also given. We then compute the
total signed $k$-subdomination number of a cycle, and present a characterization of
graphs $G$ with equal total signed $k$-subdomination and total signed $\ell$-subdomination
numbers. Finally, we establish a sharp upper bound on the total signed $k$-
subdomination number of a tree in terms of its order $n$ and $k$ where $1 \leq k < n$, and
characterize trees attaining these bounds for certain values of $k$. For this purpose,
we first establish the total signed $k$-subdomination number of simple structures,
including paths and spiders.

In Chapter 5, we show that the decision problem corresponding to the computation
of the total minus domination number of a graph is $\mathsf{NP}$-complete, even when
restricted to bipartite graphs or chordal graphs. For a fixed $k$, we show that
the decision problem corresponding to determining whether a graph has a total
minus domination function of weight at most $k$ may be $\mathsf{NP}$-complete, even when
restricted to bipartite or chordal graphs. Also in Chapter 5, linear time algorithms
for computing $\gamma^{-11}_{\text{tns}}(T)$ and $\gamma^{-101}_{\text{tns}}(T)$ for an arbitrary tree $T$ are presented, where
$n = n(T)$.

In Chapter 6, we present cubic time algorithms to compute $\gamma^{-11}_{\text{tk}s}(T)$ and $\gamma^{-101}_{\text{tk}s}(T)$
for a tree $T$. We show that the decision problem corresponding to the computation
of $\gamma^{-11}_{\text{tk}s}(G)$ is $\mathsf{NP}$-complete, and that the decision problem corresponding to the
computation of $\gamma^{-101}_{\text{tk}s}(T)$ is $\mathsf{NP}$-complete, even for bipartite graphs. In addition, we
present cubic time algorithms to compute $\gamma^{-11}_{k_s}(T)$ and $\gamma^{-101}_{k_s}(T)$ for a tree $T$, solving
problems appearing in [25].
For Jim and Michael
Preface

The work described in this thesis was carried out under the supervision and direction of Professor Michael A. Henning, School of Mathematics, Statistics and Information Technology, University of Natal, Pietermaritzburg and Professor Johannes H. Hattingh, Department of Mathematics and Statistics, Georgia State University, Atlanta, Georgia, USA, from May 1999 to September 2003.

The thesis represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

Signed:

Laura Marie Harris

Professor Michael A. Henning (Supervisor)

Professor Johannes H. Hattingh (Co-Supervisor)
Understanding and processing the raw text, we can see that this page contains the Acknowledgments section of a document. It lists individuals who have contributed to the work, including Johan Hattingh, Michael Henning, Wayne Goddard, Elna Ungerer, Robert Parris, and others. Each individual is thanked for their support, guidance, or assistance. The text is a heartfelt expression of gratitude. The document format is typical for academic or professional works where formal recognition of contributors is necessary.
Jim, my daily strength and support. Words cannot express my appreciation for everything he means to me.

Michael, the miracle in my life. He is an angel and the answer to my prayers. There is nothing I would rather be than his mother.

Mom and Dad. I honestly give thanks every day for them and the home they made for us. There is no way to repay the gifts they have given me. I pray that I will demonstrate my gratitude through the home I help build for Michael.

John, who has always represented more than a brother and a friend. Words cannot adequately describe my admiration and love for him. I am so proud to be his sister.

Priscilla, who has been a source of encouragement and friendship from the first time we met. I will always be grateful that she and her loving family are a part of my life.

Sophie and Will, my beautiful niece and nephew. They bring joy to the lives of all who know them and strengthen my hope in the future.

Ann and Bill, who raised two fine sons. I hope to do as well with Michael.

No word of thanks would be complete without remembering my grandparents. They were four very different people and each one had the ability to make me feel infinitely precious. I miss them more than I can say.
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Chapter 1

INTRODUCTION

In the first section of this chapter, we define the necessary concepts that will be used throughout the thesis. Then, in Section 1.2, we give a brief overview of the history of domination theory and define the necessary domination concepts that will be used.

1.1 Preliminary definitions

A graph $G$ is a finite nonempty set of objects called vertices (the singular is vertex) together with a (possibly empty) set of unordered pairs of distinct vertices of $G$ called edges. The vertex set of $G$ is denoted by $V(G)$ (or $V$ if no confusion is likely), while the edge set of $G$ is denoted by $E(G)$ (or $E$). The number of vertices in $V(G)$ is denoted by $n(G)$ which is also known as the order of the graph $G$, while the number of edges in $E(G)$ is denoted by $m(G)$. A graph $G$ is trivial if $n(G) = 1$ and non-trivial if $n(G) \geq 2$. For a graph $G$, if $n(G) = n$ and $m(G) = m$, then $G$ is called a $(n,m)$-graph. Unless otherwise specified, the symbols $n$ and $m$ (or $n(G)$ and $m(G)$) will be reserved exclusively for the order and number of edges, respectively,
of a graph $G$. By $G = (V, E)$ we will imply the graph $G$ with vertex set $V$ and edge set $E$.

The edge $e = uv$ is said to join the vertices $u$ and $v$. If $e = uv$ is an edge of $G$, then $u$ and $v$ are adjacent vertices, while $u$ and $e$ are incident as are $v$ and $e$. Furthermore, if $e_1$ and $e_2$ are distinct edges of $G$ incident with a common vertex, then $e_1$ and $e_2$ are adjacent edges.

The complement $\overline{G}$ of a graph $G$ is the graph with vertex set $V(G)$ and such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in $G$.

The degree of a vertex $v$ in a graph $G$ is the number of edges of $G$ incident with $v$ and is denoted by $\deg(v)$. If the graph $G$ with respect to which the degree is considered is not clear from the context, we shall write $\deg_G(v)$ to denote such a degree. The minimum degree of a vertex in $G$ is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. If there is a vertex $v \in V(G)$ such that $\deg(v) = 0$, then $v$ is called an isolated vertex, and if $\deg(v) = 1$, then $v$ is called an end-vertex of $G$. A vertex is called odd or even depending on whether its degree is odd or even.

The vertex adjacent to an end-vertex is called a remote vertex of $G$. A graph is regular of degree $r$ if for each vertex $v$ of $G$, $\deg(v) = r$; such graphs are also called $r$-regular. A graph is complete if every two of its vertices are adjacent. A complete $(n, m)$-graph is therefore a regular graph of degree $n - 1$ having $m = \frac{n(n-1)}{2}$ edges; we denote this graph by $K_n$. The complement $\overline{K_n}$ of $K_n$ has $n$ vertices and no edges and is referred to as the empty graph of order $n$.

A graph $H$ is a subgraph of a graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H$ is a subgraph of $G$, then we write $H \subseteq G$. If $U$ is a nonempty subset of the vertex set $V(G)$ of a graph $G$, then the subgraph $G[U]$ of $G$ induced by $U$ is the graph with vertex set $U$ and whose edge set consists of all those edges of $G$ incident with two elements of $U$. 

Let $u$ and $v$ be (not necessarily distinct) vertices of a graph $G$. A $u-v$ walk of $G$ is a finite, alternating sequence $u = v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n = v$ of vertices and edges, beginning with vertex $u$ and ending with vertex $v$, such that $e_i = v_{i-1}v_i$ for $i = 1, 2, \ldots, n$. The number $n$ (the number of occurrences of edges) is called the length of the walk. A trivial walk contains no edges. Often only the vertices of a walk are indicated since the edges present are then evident. A $u-v$ walk is closed or open depending on whether $u = v$ or $u \neq v$. A $u-v$ trail is a $u-v$ walk in which no edge is repeated, while a $u-v$ path is a $u-v$ walk in which no vertex is repeated. A nontrivial closed trail of a graph $G$ is referred to as a circuit of $G$, and a circuit $v_1, v_2, \ldots, v_n, v_1$ ($n \geq 3$) whose $n$ vertices are distinct is called a cycle. A graph of order $n$ that is a path (or a cycle) is denoted by $P_n$ (or $C_n$), respectively. Therefore, $P_n : v_1, v_2, \ldots, v_n$ indicates a path of length $n-1$ on the vertices $v_1, v_2, \ldots, v_n$, while $C_n$ indicates a cycle of length $n$ on the same vertices.

A graph $G$ is connected if for every pair of distinct vertices $u, v \in V(G)$ there exists a path between $u$ and $v$ in $G$, and disconnected if it is not connected. The relation 'is connected to' is an equivalence relation on the vertex set of every graph $G$. Each subgraph induced by the vertices in a resulting equivalence class is called a connected component or simply a component of $G$.

A tree is a connected graph which has no cycles. A directed tree is an asymmetric digraph whose underlying graph is a tree [7]. A directed tree $T$ is called a rooted tree if there exists a vertex $r$ of $T$, called the root, such that for every vertex $v$ of $T$, there is an $r-v$ path in $T$. If $T$ is a rooted tree, then it is customary to draw $T$ with root $r$ at the top, say level 0, the vertices adjacent to $r$ are placed one level below, at level 1, and any vertex adjacent to a vertex at level 1 is at level 2, etc. More formally, a vertex $x$ in a rooted tree with root $r$ is at level $i$ if and only if the $r-x$ path in $T$ has length $i$. Let $T$ be a rooted tree. If a vertex $v$ of $T$ is adjacent to $u$ and $u$ lies in the level below $v$, then $u$ is called a child of $v$, and $v$ is the parent of
An end-vertex of a tree, or a vertex with no children, is also called a leaf of the tree. The set of all of the leaves of a tree \( T \) is denoted by \( L(T) \). A vertex that is adjacent to a leaf is called a support vertex.

A spider is a tree with at most one vertex of degree greater than two. If a spider \( T \) is a path, we call one of the end-vertices of \( T \) the head of the spider; otherwise, we call the vertex of maximum degree the head of the spider. The paths emanating from the head of the spider we call the legs of the spider. An even spider is a spider with all legs of even length. In particular, a path of even length (and therefore odd order) is an even spider.

If \( U \subseteq V(G) \) then \( G - U \) indicates the subgraph of \( G \) induced by the vertices of \( V(G) - U \). If \( F \subseteq E(G) \) then \( G - F \) indicates the subgraph of \( G \) with vertex set equal to that of \( G \) and edge set consisting of all edges in \( E(G) - F \). Suppose \( G_1 \) and \( G_2 \) are two graphs with disjoint vertex sets. Then the union \( G = G_1 \cup G_2 \) has vertex set \( V(G) = V(G_1) \cup V(G_2) \) and edge set \( E(G) = E(G_1) \cup E(G_2) \). A graph \( G \) is isomorphic to a graph \( H \), denoted \( G \cong H \), if there exists a one-to-one mapping \( \phi \), called an isomorphism, from \( V(G) \) onto \( V(H) \) such that \( \phi \) preserves adjacency, that is \( uv \in E(G) \) if and only if \( \phi(u)\phi(v) \in E(H) \).

A graph \( G \) is \( r \)-partite, \( r \geq 1 \), if it is possible to partition \( V \) into \( r \) subsets \( V_1, V_2, \ldots, V_r \) (called partite sets) such that every element of \( E \) joins a vertex of \( V_i \) to a vertex of \( V_j \), \( i \neq j \). If \( G \) is a 1-partite graph of order \( n \), then \( G \cong \overline{K}_n \).

For \( r = 2 \), such graphs are called bipartite graphs, and where the specification of \( r \) is of no significance, an \( r \)-partite graph is also referred to as a multipartite graph.

A complete \( r \)-partite graph \( G \) is an \( r \)-partite graph with partite sets \( V_1, V_2, \ldots, V_r \) having the added property that if \( u \in V_i \) and \( v \in V_j \), \( i \neq j \), then \( uv \in E(G) \). If \( |V_i| = n_i \), then this graph is denoted by \( K(n_1, n_2, \ldots, n_r) \). (The order of the numbers \( n_1, n_2, \ldots, n_r \) is not important.) A complete bipartite graph with partite sets \( V_1 \) and \( V_2 \), where \( |V_1| = m \) and \( |V_2| = n \), is denoted by \( K(m, n) \) or \( K_{m,n} \). The
graph $K_{1,n-1}$ is called a star and its order is $n$; in this case, the vertex in the partite set containing only one vertex is referred to as the center of the star. The split graph, denoted by $S_{m,n}$, is defined as the complete multipartite graph with one partite set of cardinality $m$ and $n$ partite sets of cardinality one.

The following definitions and explanations concerning the theory of NP-completeness may be found in [12]. The theory of NP-completeness provides many straightforward techniques for proving that a given problem is “just as hard” as some other problems that are known to be NP-hard and have been confounding experts for years. The knowledge that a certain problem is NP-complete provides valuable information on approaching the problem. In short, the primary application of the theory of NP-completeness is to assist algorithm designers on deciding whether a particular problem is difficult or not. A problem is intractable if there is no polynomial time algorithm that can solve it. The principal technique used for demonstrating that two problems are related is that of transforming one to the other. Such a transformation provides the means for converting any algorithm that solves the second problem into a corresponding algorithm for solving the first problem. The class $\text{NP}$ of decision problems (problems whose solutions are either “yes” or “no”) is a class of problems that can be solved in polynomial time by a nondeterministic computer. The equivalence class consisting of the “hardest” problems in $\text{NP}$ is known as the class of $\text{NP}$-complete problems. However, problems outside of $\text{NP}$ may also be hard. A problem which is at least as hard as the NP-complete problems, is called NP-hard.

The honor of being the “first” NP-complete problem goes to a decision problem from Boolean logic known as the SATISFIABILITY problem (SAT, for short). Let $U = \{u_1, u_2, \ldots, u_m\}$ be a set of Boolean variables. A truth assignment for $U$ is a function $t : U \rightarrow \{T, F\}$. If $t(u) = T$ we say that $u$ is “true” under $t$, and if $t(u) = F$ we say that $u$ is “false”. If $u$ is a variable in $U$, then $u$ and $\bar{u}$ are literals
over \(u\). The literal \(\bar{u}\) is true if and only if the variable \(u\) is false.

A clause over \(U\) is a set of literals over \(U\), such as \(\{u_1, \bar{u}_3, u_8\}\). A collection \(C\) of clauses over \(U\) is satisfiable if and only if there exists some truth assignment for \(U\) that simultaneously satisfies all the clauses in \(C\).

For any further information on the subject of NP-completeness the reader is referred to [12].

1.2 Domination in graphs

The earliest ideas of dominating sets seem to date back to the origin of the game of chess in India over 400 years ago, in which one studies sets of chess pieces which cover or dominate various opposing pieces or various squares of the chessboard. There are many other examples of dominating sets.

One such example is the situation of a prison in which prisoners must be monitored by guards at all times. The assumption here is that the cells are set up in such a way that a guard in one cell could monitor the prisoners in the adjacent cells. The issues of budget and personnel dictate to have as few guards as possible. The question then becomes: what is the minimum number of guards necessary to monitor or "dominate" all the cells? The graph that represents this situation is one in which the vertices represent the cells and the edges indicate which cells are adjacent to each other. The question stated above can be answered by finding the domination number of the associated graph.

The concept of total domination can be illustrated by assuming that there is a suspicion of misconduct among the guards. Thus, not only does each prisoner need to be monitored by a guard, but each guard must be in the view of another guard. To find the minimum number of guards necessary in this situation, we need to find the total domination number of the associated graph.
Finally, the topic of domination was given formal mathematical definition with the publications of books by Berge [2] in 1958 and Ore [32] in 1962. Ore first referred to the domination number of a graph, while Berge used the term coefficient of extremal stability. Until 1977 relatively little work was done on this topic until Cockayne and Hedetniemi published a survey paper [5] of the results that had been obtained up to that time. Since that time, many papers on domination and variations of domination have been published. The vast literature on this subject has been surveyed and detailed in two books by Haynes et al. [24, 25].

One common definition of domination is given in terms of sets of vertices. Let $G$ be a graph and $D$ a set of vertices such that every vertex in $G$ is in $D$ or adjacent to at least one vertex in $D$. Then $D$ is called a dominating set of $G$, and the smallest cardinality of such a dominating set of $G$ is defined as the domination number of $G$, denoted by $\gamma(G)$.

A functional definition of domination is also possible and allows for several interesting variations. Let $G = (V,E)$ be a graph and let $v$ be a vertex in $V$. The open neighborhood of $v$ is defined as the set of vertices adjacent to $v$, i.e., $N(v) = \{u | uv \in E\}$. The closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. For a set $S$ of vertices, we define the open neighborhood, $N(S)$, of $S$ as $\bigcup_{v \in S} N(v)$, and the closed neighborhood, $N[S]$, of $S$ as $N(S) \cup S$. The sets $N_G(v)$ (or $N_G[v]$) will respectively denote the open (or closed) neighborhood of $v$ with respect to the graph $G$. For any real valued function $f : V \rightarrow \mathbb{R}$ and $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. The weight of $f$ is defined as $f(V)$. We say that $f : V \rightarrow \{0, 1\}$ is a dominating function of $G$ if $f(N[v]) \geq 1$ for all $v \in V$. Since the characteristic function of a dominating set is also a dominating function, $\gamma(G)$ can equivalently be defined as $\min\{f(V) | f$ is a dominating function of $G\}$.

Changing the range of $f$, defined above, to $\{-1, 1\}$ or $\{-1, 0, 1\}$ subject to the same restrictions on $f$ as above, lead to signed domination [11] and minus domination [10]
in graphs.

More specifically, a minus dominating function is defined in [10] as a function \( f : V \to \{-1, 0, 1\} \) such that \( f(N[v]) \geq 1 \) for every \( v \in V \). The minus domination number of a graph \( G \) is \( \gamma^-(G) = \min \{ f(V) \mid f \) is a minus dominating function of \( G \} \).

A signed dominating function is defined in [11] as a function \( f : V \to \{-1, 1\} \) such that \( f(N[v]) \geq 1 \) for every \( v \in V \). The signed domination number of a graph \( G \) is \( \gamma_s(G) = \min \{ f(V) \mid f \) is a signed dominating function of \( G \} \).

A majority dominating function of \( G \) is defined in [4] as a function \( f : V \to \{-1, 1\} \) such that \( f(N[v]) \geq 1 \) for at least half the vertices \( v \) in \( V \). The majority domination number of a graph \( G \) is \( \gamma_{\text{maj}}(G) = \min \{ f(V) \mid f \) is a majority dominating function of \( G \} \).

Let \( k \in \mathbb{Z}^+ \) such that \( 1 \leq k \leq |V| \). More generally, a signed \( k \)-subdominating function for \( G \), or signed \( k \)SF, is defined in [8] as a function \( f : V \to \{-1, 1\} \) such that \( f(N[v]) \geq 1 \) for at least \( k \) vertices of \( G \). The signed \( k \)-subdomination number of a graph \( G \), denoted by \( \gamma_{ks}^{-11}(G) \), is equal to \( \min \{ f(V) \mid f \) is a signed \( k \)SF of \( G \} \).

In the special cases where \( k = |V| \) and \( k = \lceil \frac{|V|}{2} \rceil \), \( \gamma_{ks}^{-11}(G) \) is respectively the signed domination number and the majority domination number.

A minus \( k \)-subdominating function for \( G \), or minus \( k \)SF, is defined in [3] as a function \( f : V \to \{-1, 0, 1\} \) such that \( f(N[v]) \geq 1 \) for at least \( k \) vertices of \( G \). The minus \( k \)-subdomination number of a graph \( G \), denoted by \( \gamma_{ks}^{-101}(G) \), is equal to \( \min \{ f(V) \mid f \) is a minus \( k \)SF of \( G \} \). In the special case where \( k = |V| \), \( \gamma_{ks}^{-101}(G) \) is the minus domination number.

If every vertex of a graph is adjacent to some vertex of a set \( S \), then \( S \) is called a total dominating set of \( G \). For \( \delta(G) \geq 1 \), the total domination number, denoted by \( \gamma_t(G) \), is defined as the minimum cardinality of a total dominating set of \( G \).
Alternatively, \( f: V \rightarrow \{0, 1\} \) is a total dominating function of \( G \) if \( f(N(v)) \geq 1 \) for all \( v \in V \). Since the characteristic function of a total dominating set is also a total dominating function, \( \gamma_t(G) \) can equivalently be defined as \( \min\{f(V) \mid f \text{ is a total dominating function of } G\} \).

An analogous theory for total \( k \)-subdominating functions arise when instead of using the "closed" neighborhood in the definition of a \( k \)-subdominating function, we use the "open" neighborhood. A total signed \( k \)-subdominating function for \( G \), or signed \( TkSF \), is defined as a function \( f: V \rightarrow \{-1, 1\} \) such that \( f(N(v)) \geq 1 \) for at least \( k \) vertices of \( G \). The total signed \( k \)-subdomination number of a graph \( G \), denoted by \( \gamma_{tk^s}^{-11}(G) \), is equal to \( \min\{f(V) \mid f \text{ is a signed } TkSF \text{ of } G\} \). In the special case where \( k = |V| \), \( \gamma_{tk^s}^{-11}(G) \) is the total signed domination number studied in [27, 37]. Similarly, a total minus \( k \)-subdominating function for \( G \), or minus \( TkSF \), is defined as a function \( f: V \rightarrow \{-1, 0, 1\} \) such that \( f(N(v)) \geq 1 \) for at least \( k \) vertices of \( G \). The total minus \( k \)-subdomination number of a graph \( G \), denoted by \( \gamma_{tk^s}^{-101}(G) \), is equal to \( \min\{f(V) \mid f \text{ is a minus } TkSF \text{ of } G\} \).

The motivation for studying the signed (minus, respectively) total \( k \)-subdomination number is rich and varied from a modeling perspective. For example, by assigning the values \(-1 \) or \(+1 \) (as well as \( 0 \), in the case for minus) to the vertices of a graph we can model networks of people or organizations in which global decisions must be made. We assume that each individual has one vote and that each individual has an initial opinion. We assign \(+1 \) to vertices (individuals) which have a positive opinion and \(-1 \) to vertices which have a negative opinion (as well as \( 0 \), in the case of minus, to vertices which have a neutral opinion). We also assume, however, that an individual's vote is affected by the opinions of neighboring individuals. In particular, each individual gives equal weight to the opinions of neighboring individuals (thus individuals of high degree have greater "influence"). A voter votes 'aye' if there are more vertices in its (open) neighborhood with positive opinion
than with negative opinion, otherwise the vote is 'nay'. We seek an assignment of opinions that guarantee at least $k$ vertices voting aye. We call such an assignment of opinions a $k$-positive assignment. Among all $k$-positive assignments of opinions, we are interested primarily in the minimum number of vertices (individuals) who have a positive opinion. The signed (minus, respectively) total $k$-subdomination number is the minimum possible sum of all opinions, $-1$ for a negative opinion and $+1$ for a positive opinion, in a $k$-positive assignment of opinions. The signed (minus, respectively) total $k$-subdomination number represents, therefore, the minimum number of individuals which can have positive opinions and in doing so force at least $k$ individuals to vote aye.

Suppose we are given real valued functions $f, g : V \to \mathbb{R}$. Then $g < f$ if and only if $g(v) \leq f(v)$ for every $v \in V$ and $g(w) < f(w)$ for at least one $w \in V$.

If $f$ is a $kSF$ (total $kSF$, respectively) of $G$, then the set of covered vertices of $f$ is denoted by $C_f$ and defined by $C_f = \{v \in V | f(N[v]) \geq 1\}$ ($C_f = \{v \in V | f(N(v)) \geq 1\}$, respectively). If $v \in C_f$, we say $v$ is covered by $f$; otherwise it is uncovered by $f$.

In Chapter 2, we survey recent results on signed and minus $k$-subdomination in graphs.

In Chapter 3, we compute the signed and minus $k$-subdomination numbers for certain complete multipartite graphs and their complements, generalizing results due to Holm [30].

In Chapter 4, we give a lower bound on the total signed $k$-subdomination number in terms of the minimum degree, maximum degree and the order of the graph. A lower bound in terms of the degree sequence is also given. We then compute the total signed $k$-subdomination number of a cycle, and present a characterization of graphs $G$ with equal total signed $k$-subdomination and total signed $\ell$-subdomination numbers. Finally, we establish a sharp upper bound on the total signed $k$-subdomination number of a tree in terms of its order $n$ and $k$ where $1 \leq k < n$, and
characterize trees attaining these bounds for certain values of $k$. For this purpose, we first establish the total signed $k$-subdomination number of simple structures, including paths and spiders.

In Chapter 5, we show that the decision problem corresponding to the computation of the total minus domination number of a graph is NP-complete, even when restricted to bipartite graphs or chordal graphs. For a fixed $k$, we show that the decision problem corresponding to determining whether a graph has a total minus domination function of weight at most $k$ may be NP-complete, even when restricted to bipartite or chordal graphs. Also in Chapter 5, linear time algorithms for computing $\gamma_{tns}^{-11}(T)$ and $\gamma_{tns}^{-101}(T)$ for an arbitrary tree $T$ are presented, where $n = n(T)$.

In Chapter 6, we present cubic time algorithms to compute $\gamma_{tk_s}^{-11}(T)$ and $\gamma_{tk_s}^{-101}(T)$ for a tree $T$. We show that the decision problem corresponding to the computation of $\gamma_{tk_s}^{-11}(G)$ is NP-complete, and that the decision problem corresponding to the computation of $\gamma_{tk_s}^{-101}(G)$ is NP-complete, even for bipartite graphs. In addition, we present cubic time algorithms to compute $\gamma_{ks}^{-11}(T)$ and $\gamma_{ks}^{-101}(T)$ for a tree $T$, solving problems appearing in [25].
Chapter 2

A LITERATURE SURVEY

In this chapter, we survey the literature on majority domination, signed \( k \)-subdomination, and minus \( k \)-subdomination in graphs.

2.1 Majority Domination in Classes of Graphs

The majority domination numbers of the following classes of graphs have been determined.

**Theorem 2.1** [4] For \( n \geq 1 \),

\[
\gamma_{maj}(K_n) = \begin{cases} 
1 & \text{if } n \text{ is odd} \\
2 & \text{if } n \text{ is even} 
\end{cases}
\]

**Theorem 2.2** [4] For \( n \geq 2 \),

\[
\gamma_{maj}(K_{1,n-1}) = \begin{cases} 
1 & \text{if } n \text{ is odd} \\
2 & \text{if } n \text{ is even} 
\end{cases}
\]
Theorem 2.3 [4] For $t \geq s \geq 2$,

$$\gamma_{\text{maj}}(K_{s,t}) = \begin{cases} 2 - t & \text{if } s \text{ is even} \\ 3 - t & \text{if } s \text{ is odd} \end{cases}$$

Theorem 2.4 [4] For $t \geq s \geq 1$,

$$\gamma_{\text{maj}}(K_s \cup K_t) = \begin{cases} 1 - s & \text{if } t \text{ is odd} \\ 2 - s & \text{if } t \text{ is even} \end{cases}$$

The majority domination number has also been determined for paths and cycles. This will be addressed in Section 2.2, where we look at the first generalization of majority domination in graphs.

Holm [30] determines the majority domination number of the following complete multipartite graphs.

Theorem 2.5 [30] For integers $m > n \geq 1$,

$$\gamma_{\text{maj}}(K_{\underbrace{1, \ldots, 1}_m, n}) = \begin{cases} 1 & \text{if } n + m \text{ is odd} \\ 2 & \text{if } n + m \text{ is even} \end{cases}$$

Theorem 2.6 [30] For integers $m \geq 2$ and $n \geq 3$,

$$\gamma_{\text{maj}}(K_{\underbrace{n, \ldots, n}_m}) = \begin{cases} 2 - n & \text{if } m \text{ and } n \text{ are even} \\ 3 - n & \text{if } m \text{ is even and } n \text{ is odd} \\ 4 - n & \text{if } m \text{ is odd} \end{cases}$$

Theorem 2.7 [30] $\gamma_{\text{maj}}(K_{\underbrace{2, \ldots, 2}_m}) = 0$.

The following result involves the disjoint union of a complete graph and a graph $G$. 
CHAPTER 2. A LITERATURE SURVEY

Theorem 2.8 [30] Suppose that \( n > m \geq 1 \). If \( G \) is a graph of order \( m \) and \( H = K_n \cup G \), then
\[
\gamma_{\text{maj}}(H) = \begin{cases} 
1 - m & \text{if } n \text{ is odd} \\
2 - m & \text{if } n \text{ is even}
\end{cases}
\]

Theorem 2.9 [30] For integers \( m > n \geq 1 \),
\[
\gamma_{\text{maj}}(K_m \cup K_n) = \begin{cases} 
1 - n & \text{if } m \text{ and } n \text{ are odd} \\
2 - n & \text{if } m \text{ is even} \\
3 - n & \text{if } n \text{ is even and } m \text{ is odd}
\end{cases}
\]

Holm also determines the majority domination number of the union of \( m \) complete graphs of order \( n \).

Theorem 2.10 [30] For integers \( m > 2 \) and \( n \geq 2 \),
\[
\gamma_{\text{maj}}(\bigcup_{i=1}^{m} K_n) = \begin{cases} 
\lceil m/2 \rceil - n\lceil m/2 \rceil & \text{if } n \text{ is odd} \\
2\lceil m/2 \rceil - n\lfloor m/2 \rfloor & \text{if } n \text{ is even}
\end{cases}
\]

2.2 Signed \( k \)-subdomination in Graphs

In this section, we survey some recent results concerning signed \( k \)-subdomination in graphs.

Let \( f \) be a signed \( kSF \) of \( G = (V,E) \). Let \( P_f = \{v \in V \mid f(v) = 1\} \) and \( B_f = \{v \in V \mid f(N[v]) \in \{1,2\}\} \). For \( A, B \subseteq V \), we say \( A \) dominates \( B \), denoted by \( A \supseteq B \), if for each \( b \in B \), \( N[b] \cap A \neq \emptyset \).

Theorem 2.11 [8] A signed \( kSF \) \( f \) is minimal if and only if for each \( k \)-subset \( K \) of \( C_f, K \cap B_f \supseteq P_f \).
Let $\gamma(n, k)$ be the minimum value of $\gamma_{ks}(T)$ taken over all trees $T$ of order $n$ ($n \geq k$) and $S(n, k)$ be the set of such trees $T$ with $\gamma_{ks}(T) = \gamma(n, k)$. Further, let $\sigma(T)$ be the degree sum of all vertices of $T$ with degree at least three and define $T(n, k) = \{T \in S(n, k) \mid \sigma(T) \text{ is minimum}\}$.

**Theorem 2.12** [8] For any $n$, $S(n, k) = \{P_n\}$.

**Theorem 2.13** [8] For $n \geq 2$ and $1 \leq k \leq n$,

$$\gamma_{ks}^{-1}(P_n) = 2[(2k + 4)/3] - n.$$

Note that if $k = \left\lfloor \frac{n}{2} \right\rfloor$, then we obtain the following result.

**Theorem 2.14** [8] For $n \geq 3$,

$$\gamma_{maj}(P_n) = \begin{cases} -2\left\lfloor \frac{n-4}{6} \right\rfloor & \text{for } n \text{ even} \\ 1 - 2\left\lfloor \frac{n-3}{6} \right\rfloor & \text{for } n \text{ odd} \end{cases}.$$

Using Theorems 2.12 and 2.13, Cockayne and Mynhardt established the following result.

**Theorem 2.15** [8] If $T$ is a tree of order $n \geq 2$ and $k$ is an integer such that $1 \leq k \leq n$, then

$$\gamma_{ks}^{-1}(T) \geq 2[(2k + 4)/3] - n$$

with equality for $T = P_n$.

This result sheds new light on the following result.

**Theorem 2.16** [11] Let $T$ be a tree of order $n \geq 2$. Then $\gamma_s(T) \geq \frac{n+4}{3}$ with equality if and only if $T$ is a path on $3j + 2$ vertices, for $j \geq 0$. 
Let \( n \geq 2 \) be an integer and let \( k \) be an integer such that \( 1 \leq k \leq n \). Trees \( T \) of order \( n \) for which \( \gamma_{ks}^{-11}(T) = 2\lfloor (2k + 4)/3 \rfloor - n \) were recently characterized by Hattingh and Ungerer. The statement of this result is rather intricate and the reader is therefore referred to [22] for the details.

The comet \( C_{s,t} \), where \( s \) and \( t \) are positive integers, denotes the tree obtained by identifying the center of the star \( K_{1,s} \) with an end-vertex of \( P_t \), the path of order \( t \). So \( C_{s,1} \cong K_{1,s} \) and \( C_{1,p-1} \cong P_p \). Beineke and Henning [1] computed \( \gamma_{ks}^{-11}(C_{s,t}) \) for \( k = s + t \) and for \( k = \lceil \frac{s+t}{2} \rceil + 1 \). Hattingh and Ungerer extended their result as follows.

**Theorem 2.17** [23] Let \( n, s \) and \( t \) be positive integers such that \( n = s + t \) and let \( G = C_{s,t} \). If \( s, t \geq 2 \), then

\[
\gamma_{ks}^{-11}(G) = \begin{cases} 
2\lfloor (2k + 4)/3 \rfloor - n & \text{if } k \leq t - 1 \\
2(k - \lceil \frac{t}{3} \rceil + 2) - n & \text{if } t \leq k \text{ and } (k \leq t + \lfloor \frac{t}{3} \rfloor - 2, \ t \equiv 0 \pmod{3} \text{ or } k \leq t + \lfloor \frac{t}{3} \rfloor + 2, \ t \equiv 1 \pmod{3} \text{ or } k \leq t + \lfloor \frac{t}{3} \rfloor - 1, \ t \equiv 2 \pmod{3}) \\
2(k - \lceil \frac{t}{3} \rceil + 1) - n & \text{otherwise.}
\end{cases}
\]

The value of \( \gamma_{ks}^{-11}(C_n) \) is calculated in [19].

**Theorem 2.18** [19] If \( n \geq 3 \) and \( 1 \leq k \leq n - 1 \), then

\[
\gamma_{ks}^{-11}(C_n) = \begin{cases} 
\frac{n-2}{3} & \text{if } k = n - 1 \text{ and } k \equiv 1 \pmod{3} \\
2\lfloor \frac{2k+4}{3} \rfloor - n & \text{otherwise.}
\end{cases}
\]

This result generalizes the following.

**Theorem 2.19** [4] If \( n \geq 3 \), then

\[
\gamma_{maj}(C_n) = \gamma_{maj}(P_n).
\]
Theorem 2.20 [19] If \( n \geq 3 \) and \( 1 \leq k \leq n \), then for every \( r \)-regular \((r \geq 2)\) graph \( G \) of order \( n \),

\[
\gamma_{ks}^{11}(G) \geq \begin{cases} 
  k \frac{r+3}{r+1} - n & \text{for odd } r \\
  k \frac{r+2}{r+1} - n & \text{for even } r,
\end{cases}
\]

and these bounds are best possible.

The following result combines results due to Dunbar, Henning, Hedetniemi and Slater \((r \text{ even})\) [11] and Henning and Slater \((r \text{ odd})\) [29].

Theorem 2.21 For every \( r \)-regular \((r \geq 2)\) graph \( G \) of order \( n \),

\[
\gamma_s(G) \geq \begin{cases} 
  \frac{2n}{r+1} & \text{for odd } r \\
  \frac{n}{r+1} & \text{for even } r.
\end{cases}
\]

Theorem 2.22 [36] For every cubic graph \( G \) of order \( n \), \( \gamma_{maj}(G) \geq -\frac{n}{4} \).

Since \( \gamma_{maj}(2K_4) = -2 = -\frac{8}{4} \), this bound is best possible.

Theorem 2.23 [26] For every \( r \)-regular \((r \geq 2)\) graph \( G = (V, E) \) of order \( n \),

\[
\gamma_{maj}(G) \geq \begin{cases} 
  \left( 1 - \frac{r}{2(r+1)} \right) n & \text{for odd } r \\
  \left( 1 - \frac{r}{2(r+1)} \right) n & \text{for even } r
\end{cases}
\]

and these bounds are best possible.

Note that, if \( k = n \) in the statement of Theorem 2.20, then we obtain the result of Theorem 2.21, and, if \( k = \lceil \frac{n}{2} \rceil \), then we obtain the result of Theorem 2.23.
Theorem 2.24 [4] For any connected graph $G$ of order $n$,

$$\gamma_{\text{maj}}(G) \leq \begin{cases} 
1 & \text{if } n \text{ is odd} \\
2 & \text{if } n \text{ is even}
\end{cases}.$$ 

Obviously, if $f$ is a majority dominating function of $G = (V, E)$, then $f$ is a signed $kSF$ for each $k \leq \lfloor |V|/2 \rfloor$. Hence we have the following corollary.

**Corollary 1** [8] For any connected graph $G$ of order $n$ and integer $k \leq \lceil \frac{n}{2} \rceil$,

$$\gamma^{-11}_{ks}(G) \leq \begin{cases} 
1 & \text{if } n \text{ is odd} \\
2 & \text{if } n \text{ is even}
\end{cases}.$$ 

That this bound is sharp can be seen by noting that $\gamma^{-11}_{ks}(K_{2t+1}) = 1$ and $\gamma^{-11}_{ks}(K_{2t}) = 2$ for each $k \leq t + 1$.

In [8], this bound is improved for trees and extended to an upper bound for $\gamma^{-11}_{ks}(T)$ for all $k \in \{1, \ldots, n\}$.

**Theorem 2.25** [8] For any tree $T$ of order $n$ and integer $k \in \{1, \ldots, n\}$, $\gamma^{-11}_{ks}(T) \leq 2(k + 1) - n$.

That this bound is exact for trees of order $n$ when $k \leq \frac{1}{2}n$ follows easily since $\gamma^{-11}_{ks}(K_{1,n-1}) = 2(k + 1) - n$ if $k \leq \frac{1}{2}n$. The following result, initially formulated as a conjecture by Cockayne and Mynhardt [8], was recently settled independently by Chang et al. and Kang et al.

**Theorem 2.26** [6, 31] For any tree $T$ of order $n$ and any $k$ with $\frac{1}{2}n < k \leq n$, $\gamma^{-11}_{ks}(T) \leq 2k - n$.

The following conjecture of Cockayne and Mynhardt is shown to be false in [28] for the special case when $k = \lceil \frac{n+1}{2} \rceil$. 
Conjecture 2.27 [8] For any connected graph $G$ of order $n$ and any $k$ with $\frac{1}{2}n < k \leq n$, $\gamma^{-11}_{ks}(G) \leq 2k - n$.

The remainder of Cockayne and Mynhardt's paper [8] is devoted to determining conditions on $k$ such that $\gamma^{-11}_{ks}(T) \leq 2k - n$ for certain classes of trees of order $n$.

Theorem 2.28 [31] For any connected graph $G$ of order $n$ and any $k$ with $\frac{n}{2} < k \leq n$, $\gamma^{-11}_{ks}(G) \leq 2\lceil \frac{k}{n-k+1} \rceil (n - k + 1) - n$.

In [6], Chang et al. give a lower bound for the signed $k$-subdomination number of a graph in terms of its order and degree sequence.

Theorem 2.29 [6] If $G$ is a graph of order $n$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$, then $\gamma^{-11}_{ks}(G) \geq -n + \frac{2}{n+1} \sum_{j=1}^{k} \lceil \frac{d_j+2}{2} \rceil$.

Lastly, consider the decision problem

**PARTIAL SIGNED DOMINATING FUNCTION (PSDF)**

**INSTANCE:** A graph $G$, positive rational number $r \leq 1$ (in its simplest form) and an integer $f$.

**QUESTION:** Is there a function $f : V(G) \to \{-1, 1\}$ of weight $\ell$ or less for $G$ such that $|C_f| \geq r|V(G)|$?

Hattingh, Henning and Ungerer [19] showed that PSDF is NP-complete by describing a polynomial transformation from the following problem [12]:

**PLANAR 4-REGULAR DOMINATING SET**

**INSTANCE:** A planar 4-regular graph $G = (V, E)$ and a positive integer $k \leq \frac{|V|}{2}$.

**QUESTION:** Is there a dominating set of cardinality $k$ or less for $G$?

If $r = 1$, then PSDF is the NP-complete problem SIGNED DOMINATION [18]. Hence, we also assume that $0 < r < 1$. 

Theorem 2.30 [19] The decision problem \( \text{PSDF} \) is NP-complete.

This result generalizes the corresponding result of [4].

### 2.3 Minus \( k \)-subdomination in Graphs

In this section, we survey some recent results concerning minus \( k \)-subdomination in graphs.

Let \( f \) be a minus \( kSF \) for the graph \( G = (V, E) \). We use three sets for such an \( f \):

\[
B_f = \{ v \in V | f[v] = 1 \},
\]

\[
P_f = \{ v \in V | f(v) \geq 0 \}
\]

and \( C_f = \{ v \in V | f[v] \geq 1 \} \).

As before, a vertex \( v \in C_f \) is covered by \( f \); all other vertices are uncovered by \( f \). Note that \( B_f \subseteq C_f \).

Theorem 2.31 [3] A minus \( kSF \) \( f \) is minimal if and only if for each \( k \)-subset \( K \) of \( C_f \) we have \( B_f \cap K \supseteq P_f \).

Theorem 2.32 [3] If \( n \geq 2 \) and \( 1 \leq k \leq n - 1 \), then \( \gamma_{ks}^{101}(P_n) = \left[ \frac{k}{3} \right] + k - n + 1 \).

The following result is proved in [10].

Theorem 2.33 [10] For the path \( P_n \), \( \gamma_{ks}^{101}(P_n) = \left[ \frac{n}{3} \right] \).

Hattingh and Ungerer [21] established the following result.
CHAPTER 2. A LITERATURE SURVEY

Theorem 2.34 [21] If \( T \) is a tree of order \( n \geq 2 \) and \( k \) is an integer such that \( 1 \leq k \leq n - 1 \), then

\[
\gamma_{ks}^{-101}(T) \geq k - n + 2.
\]

Moreover, this bound is best possible.

However, trees which achieve the lower bound were not characterized in [21]. The following result solves this problem.

Theorem 2.35 [20] Let \( n \geq 2 \) and let \( 1 \leq k \leq n - 1 \) be an integer. Then, for a tree \( T \) of order \( n \), \( \gamma_{ks}^{-101}(T) = k - n + 2 \) if and only if one of the following holds.

1. \( T \) has a vertex \( v \) adjacent to at least \( k \) end-vertices.
2. \( T \) has a vertex \( v \) with \( \deg(v) = k \) and at least \( k - 1 \) neighbors of \( v \) are end-vertices.
3. \( T \) has two adjacent vertices \( u \) and \( v \) with \( \deg(u) + \deg(v) = k + 2 \) such that \( u \) and \( v \) together are adjacent to at least \( k - 2 \) end-vertices.
4. \( T \) has a vertex \( w \) of degree three and two of the neighbors of \( w \) together are adjacent to exactly \( k - 3 \) other vertices, all of which are end-vertices.

This result supplements the following result of [10].

Theorem 2.36 [10] If \( T \) is a tree, then \( \gamma_{ns}^{-101}(T) \geq 1 \). Furthermore, equality holds if and only if \( T \) is a star.

The value of \( \gamma_{ks}^{-101}(G) \), where \( G \) is a comet, is calculated in [23].
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Theorem 2.37 [23] Let \( n, s \) and \( t \) be positive integers such that \( n = s + t \), let \( k \) be an integer such that \( 1 \leq k \leq n - 1 \) and let \( G = C_{s,t} \). If \( t \geq 2 \) and \( s \geq 2 \), then

\[
\gamma^{-101}_{ks}(G) = \begin{cases} 
  k - n + 2 & \text{if } 1 \leq k \leq s \\
  \left\lfloor \frac{k-t+1}{s} \right\rfloor + k - n + 1 & \text{if } s + 1 \leq k \leq n.
\end{cases}
\]

Note that \( \gamma^{-}(C_{s,t}) = \left\lfloor \frac{t+1}{3} \right\rfloor \), where \( s \) and \( t \) are positive integers.

The value of \( \gamma^{-101}_{ks}(C_n) \) is calculated in [21].

Theorem 2.38 [21] If \( n \geq 3 \) and \( 1 \leq k \leq n - 1 \), then

\[
\gamma^{-101}_{ks}(C_n) = \begin{cases} 
  \left\lfloor \frac{n-2}{3} \right\rfloor & \text{if } k = n - 1 \text{ and } (k \equiv 0 \text{ or } k \equiv 1 \text{ (mod 3) }) \\
  2\left\lfloor \frac{2k+4}{3} \right\rfloor - n & \text{otherwise}.
\end{cases}
\]

This result supplements the following result.

Theorem 2.39 [10] If \( n \geq 3 \), then \( \gamma^{-101}_{ns}(C_n) = \left\lfloor \frac{n}{3} \right\rfloor \).

Lastly, consider the decision problem

PARTIAL MINUS DOMINATING FUNCTION (PMDF)

INSTANCE: A graph \( G \), positive rational number \( r \leq 1 \) (in its simplest form) and an integer \( \ell \).

QUESTION: Is there a function \( f : V(G) \to \{-1, 0, 1\} \) of weight \( \ell \) or less for \( G \) such that \( |C_f| \geq r|V(G)| \)?

Hattingh, McRae and Ungerer [20] showed that PMDF is NP-complete by describing a polynomial transformation from the following NP-complete problem [12]:

EXACT COVER BY 3-SETS (X3C)
CHAPTER 2. A LITERATURE SURVEY

INSTANCE: A set $X = \{x_1, \ldots, x_3\}$ and a set $C = \{C_1, \ldots, C_t\}$ where $C_j \subseteq X$ and $|C_j| = 3$ for $j = 1, \ldots, t$.

QUESTION: Does $C$ have a pairwise disjoint $q$-subset of $C$ whose union is $X$ (i.e. an exact cover)?

If $r = 1$, then PMDF is the NP-complete problem MINUS DOMINATING FUNCTION [9]. Hence, we also assume that $r < 1$.

Theorem 2.40 [20] PMDF is NP-complete, even for bipartite graphs.
Chapter 3

COMPLETE MULTIPARTITE GRAPHS

3.1 Introduction

We devote this chapter to the signed and minus \( k \)-subdomination numbers of certain complete multipartite graphs and their complements. In Section 3.2, we compute the values of \( \gamma_{kS}^{-11}(K_n) \) and \( \gamma_{kS}^{-101}(K_n) \). In Section 3.3, we determine the values of \( \gamma_{kS}^{-11}(K_{m,n}) \) and \( \gamma_{kS}^{-101}(K_{m,n}) \), where \( K_{m,n} \) is the complete bipartite graph with partite sets \( V_m \) and \( V_n \), where \(|V_m|=m\) and \(|V_n|=n\). In Section 3.4, we compute the values of \( \gamma_{kS}^{-11}(K_m \cup K_n) \) and \( \gamma_{kS}^{-101}(K_m \cup K_n) \). Note that \( K_m \cup K_n = \overline{K}_{m,n} \). In Section 3.5, we determine \( \gamma_{kS}^{-11}(S_{m,n}) \) and \( \gamma_{kS}^{-101}(S_{m,n}) \). The results of this chapter have been published in [17].

The notation that we will need in this chapter is as follows. Suppose \( f \) is a signed or a minus \( kSF \) of a graph \( G \). The set of positive vertices (zero vertices, negative vertices, respectively) is defined as \( P_f = \{v|f(v)=1\} \) \((Z_f = \{v|f(v)=0\}\), \( N_f = \{v|f(v)=-1\}\), respectively). Note that \( f \) is a signed \( kSF \), then \( Z_f = \emptyset \).
For $v \in V(G)$, we will abbreviate $f(N[v])$ as $f[v]$.

### 3.2 Complete graphs

In this section we compute the values of $\gamma_{k^s}^{-11}(G)$ and $\gamma_{k^s}^{-101}(G)$ for a complete graph $G$.

**Proposition 3.1** If $1 \leq k \leq n$, then

$$
\gamma_{k^s}^{-11}(K_n) = \begin{cases} 
1 & \text{if } n \text{ is odd} \\
2 & \text{if } n \text{ is even}
\end{cases}
$$

**Proof.** Since $k \geq 1$, there is a vertex $v \in V(K_n) \cap C_f$. If $n$ is odd, $1 \leq f[v] = f(V(K_n))$. If $n$ is even then $1 \leq f[v] = f(V(K_n))$ implies that $|P_f| \geq \frac{n}{2} + 1$. Thus, $f(V(K_n)) \geq \left(\frac{n}{2} + 1\right) + (-1)(\frac{n}{2} - 1) = 2$.

Let $U \subseteq V(K_n)$ such that $|U| = \left\lceil \frac{n}{2} \right\rceil + 1$. The function $f$ defined by

$$
f(v) = \begin{cases} 
1 & \text{if } v \in U \\
-1 & \text{otherwise}
\end{cases}
$$

is a $kSF$ with

$$
f(V(K_n)) = \begin{cases} 
1 & \text{if } n \text{ is odd} \\
2 & \text{if } n \text{ is even}
\end{cases}
$$

**Proposition 3.2** If $1 \leq k \leq n$, then $\gamma_{k^s}^{-101}(K_n) = 1$.

**Proof.** Since $k \geq 1$, there is a vertex $v \in V(K_n) \cap C_f$. Thus, $1 \leq f[v] = f(V(K_n))$. The function $f$ that assigns 1 to exactly one vertex of $K_n$ and 0 to all other vertices of $K_n$ is a $kSF$ of $K_n$ such that $f(V(K_n)) = 1$. $\Diamond$
3.3 Complete bipartite graphs

In this section we compute the value of $\gamma_{k,s}^{-11}(G)$ and $\gamma_{k,s}^{-101}(G)$ for complete bipartite graphs $G$. The partite sets of $K_{m,n}$ will be denoted by $V_m$ and $V_n$, respectively.

**Proposition 3.3** Suppose $n \geq m \geq 2$. If $1 \leq k \leq n$, then

$$\gamma_{k,s}^{-11}(K_{m,n}) = \begin{cases} 3 - n & \text{if } m \text{ is odd} \\ 2 - n & \text{if } m \text{ is even} \end{cases}$$

**Proof.** Before proceeding further, we prove the following claim:

**Claim.** If $v \in V_n \cap C_f$, then $f(V) \geq \begin{cases} 3 - n & \text{if } m \text{ is odd} \\ 2 - n & \text{if } m \text{ is even} \end{cases}$

**Proof.** On the one hand, if $f(v) = -1$, then

$$f(V) = f(V_m) + f(V_n) \geq \left(\left\lceil \frac{m}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor + 2 \right) + (-1)n = \begin{cases} 3 - n & \text{if } m \text{ is odd} \\ 2 - n & \text{if } m \text{ is even} \end{cases}.$$ 

On the other hand, if $f(v) = 1$, then

$$f(V) = f(V_m) + f(V_n) \geq \left(\left\lceil \frac{m}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor + (-1)(n - 1) + 1 \right) = \begin{cases} 3 - n & \text{if } m \text{ is odd} \\ 2 - n & \text{if } m \text{ is even} \end{cases}.$$ 

Since $k \geq 1$, there is a covered vertex, say $v$. If $v \in V_n$, then

$$f(V) \geq \begin{cases} 3 - n & \text{if } m \text{ is odd} \\ 2 - n & \text{if } m \text{ is even} \end{cases}$$
Note that
\[
\begin{cases}
\min\{3 - m, 3 - n\} & \text{if } n \text{ is odd and } m \text{ is odd} \\
\min\{2 - m, 3 - n\} & \text{if } n \text{ is even and } m \text{ is odd} \\
\min\{3 - m, 2 - n\} & \text{if } n \text{ is odd and } m \text{ is even} \\
\min\{2 - m, 2 - n\} & \text{if } n \text{ is even and } m \text{ is even}
\end{cases}
\]

\[
\begin{cases}
3 - n & \text{if } n \text{ is odd and } m \text{ is odd} \\
3 - n & \text{if } n \text{ is even and } m \text{ is odd} \\
2 - n & \text{if } n \text{ is odd and } m \text{ is even} \\
2 - n & \text{if } n \text{ is even and } m \text{ is even}
\end{cases}
\]

If \( v \in V_m \), then, by the above,
\[
f(V) \geq \begin{cases}
3 - m & \text{if } n \text{ is odd} \\
2 - m & \text{if } n \text{ is even}
\end{cases}
\]
\[
g(V) \geq \begin{cases}
3 - n & \text{if } m \text{ is odd} \\
2 - n & \text{if } m \text{ is even}
\end{cases}
\]

Let \( U \subseteq V_m \) such that \( |U| = \lceil \frac{m}{2} \rceil + 1 \). The function \( g \) defined by
\[
g(v) = \begin{cases}
1 & \text{if } v \in U \\
-1 & \text{otherwise}
\end{cases}
\]
is a kSF with
\[
g(V) = \begin{cases}
3 - n & \text{if } m \text{ is odd} \\
2 - n & \text{if } m \text{ is even}
\end{cases}
\]
The result now follows. \( \diamond \)

**Lemma 3.4** If \( v \in V_m \cap C_f \) and \( f(v) = -1 \), then \( f(V_n) \geq \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor + 2 \). Moreover, if \( f(V_n) = \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor + 2 \), then \( |V_n \cap P_f| = \lceil \frac{n}{2} \rceil + 1 \).
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Proof. Since $f(v) = -1$ and $f[v] \geq 1$, then $|V_n \cap P_f| = \left[ \frac{n}{2} \right] + 1$. Thus, $f(V_n) \geq (\left[ \frac{n}{2} \right] + 1) - (\left[ \frac{n}{2} \right] - 1) = \left[ \frac{n}{2} \right] - \left[ \frac{n}{2} \right] + 2$. \hfill \(\lozenge\)

Lemma 3.5 If $v \in V_m \cap C_f$ and $f(v) = 1$, then $f(V_n) \geq \left[ \frac{n}{2} \right] - \left[ \frac{n}{2} \right]$. Moreover, if $f(V_n) = \left[ \frac{n}{2} \right] - \left[ \frac{n}{2} \right]$, then $|V_n \cap P_f| = \left[ \frac{n}{2} \right]$. 

Proof. Since $f(v) = 1$ and $f[v] \geq 1$, then $|V_n \cap P_f| = \left[ \frac{n}{2} \right]$. Thus, $f(V_n) \geq \left[ \frac{n}{2} \right] - \left[ \frac{n}{2} \right]$. \hfill \(\lozenge\)

We henceforth assume $k > n$, $V_m \cap C_f \neq \emptyset$ and $V_n \cap C_f \neq \emptyset$. Let $u \in V_m \cap C_f$ and let $v \in V_n \cap C_f$.

Proposition 3.6 Suppose $n \geq m \geq 2$. If $n < k \leq \left[ \frac{n}{2} \right] + \left[ \frac{m}{2} \right]$, then

$$\gamma_{k, \{1\}}(K_{m,n}) = \begin{cases} 0 & \text{if } m \text{ and } n \text{ are even} \\ 1 & \text{if } m \text{ and } n \text{ have different parities} \\ 2 & \text{if } m \text{ and } n \text{ are odd} \end{cases}$$

Proof. By Lemmas 3.4 and 3.5,

$$f(V) = f(V_m) + f(V_n) \geq \left[ \frac{n}{2} \right] - \left[ \frac{n}{2} \right] + \left[ \frac{m}{2} \right] - \left[ \frac{m}{2} \right] \begin{cases} 0 & \text{if } m \text{ and } n \text{ are even} \\ 1 & \text{if } m \text{ and } n \text{ have different parities} \\ 2 & \text{if } m \text{ and } n \text{ are odd} \end{cases}$$

Let $U_1 \subseteq V_m$ such that $|U_1| = \left[ \frac{m}{2} \right]$ and let $U_2 \subseteq V_n$ such that $|U_2| = \left[ \frac{n}{2} \right]$. The function $f$ defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U_1 \cup U_2 \\ -1 & \text{otherwise} \end{cases}$$
is a $kSF$ with

$$f(V) = \begin{cases} 
0 & \text{if } m \text{ and } n \text{ are even} \\
1 & \text{if } m \text{ and } n \text{ have different parities} \\
2 & \text{if } m \text{ and } n \text{ are odd}
\end{cases}$$

The result now follows. ◊

Lemma 3.7 If $f(u) = f(v) = 1$, then

$$f(V) \geq \begin{cases} 
0 & \text{if } m \text{ and } n \text{ are even} \\
1 & \text{if } m \text{ and } n \text{ have different parities} \\
2 & \text{if } m \text{ and } n \text{ are odd}
\end{cases}$$

Moreover, if equality holds, then $k \leq \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil$.

Proof. By Lemma 3.5,

$$f(V) \geq \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor$$

$$= \begin{cases} 
0 & \text{if } m \text{ and } n \text{ are even} \\
1 & \text{if } m \text{ and } n \text{ have different parities} \\
2 & \text{if } m \text{ and } n \text{ are odd}
\end{cases}$$

If $f(V)$ equals this lower bound, then $|V_n \cap P_f| = \left\lceil \frac{n}{2} \right\rceil$ and $|V_m \cap P_f| = \left\lceil \frac{m}{2} \right\rceil$. Then $N_f \cap C_f = \emptyset$, so $k \leq |C_f| \leq \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor$. ◊

Lemma 3.8 If $f(u) = 1$ and $f(v) = -1$, then

$$f(V) \geq \begin{cases} 
2 & \text{if } m \text{ and } n \text{ are even} \\
3 & \text{if } m \text{ and } n \text{ have different parities} \\
4 & \text{if } m \text{ and } n \text{ are odd}
\end{cases}$$

Moreover, if equality holds, then $k \leq n + \left\lceil \frac{m}{2} \right\rceil + 1$. 
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Proof. By Lemmas 3.4 and 3.5,

\[ f(V) \geq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m}{2} \right\rfloor + 2 = \]

\[ = \begin{cases} 
2 & \text{if } m \text{ and } n \text{ are even} \\
3 & \text{if } m \text{ and } n \text{ have different parities} \\
4 & \text{if } m \text{ and } n \text{ are odd} 
\end{cases} \]

If \( f(V) \) equals this lower bound, then \( |V_n \cap P_f| = \left\lfloor \frac{n}{2} \right\rfloor \) and \( |V_m \cap P_f| = \left\lfloor \frac{m}{2} \right\rfloor \). Then \( (N_f \cap V_m) \cap C_f = \emptyset \), so \( k \leq |C_f| \leq \left\lfloor \frac{m}{2} \right\rfloor + 1 + n \). \( \diamond \)

Similarly one may prove

Lemma 3.9 If \( f(u) = -1 \) and \( f(v) = 1 \), then

\[ f(V) \geq \begin{cases} 
2 & \text{if } m \text{ and } n \text{ are even} \\
3 & \text{if } m \text{ and } n \text{ have different parities} \\
4 & \text{if } m \text{ and } n \text{ are odd} 
\end{cases} \]

Moreover, if equality holds, then \( k \leq m + \left\lfloor \frac{n}{2} \right\rfloor + 1 \).

Lemma 3.10 If \( f(u) = -1 \) and \( f(v) = -1 \), then

\[ f(V) \geq \begin{cases} 
4 & \text{if } m \text{ and } n \text{ are even} \\
5 & \text{if } m \text{ and } n \text{ have different parities} \\
6 & \text{if } m \text{ and } n \text{ are odd} 
\end{cases} \]

Proof. By Lemma 3.4, \( f(V) \geq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 2 + \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m}{2} \right\rfloor + 2 = \)

\[ = \begin{cases} 
4 & \text{if } m \text{ and } n \text{ are even} \\
5 & \text{if } m \text{ and } n \text{ have different parities} \\
6 & \text{if } m \text{ and } n \text{ are odd} 
\end{cases} \]
Proposition 3.11 Suppose $n \geq m \geq 2$. If $\max\{\lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil + 1, n + 1\} \leq k \leq n + \lceil \frac{m}{2} \rceil + 1$, then

$$\gamma_{k, 1}^{-1}(K_{m,n}) = \begin{cases} 
2 & \text{if } m \text{ and } n \text{ are even} \\
3 & \text{if } m \text{ and } n \text{ have different parities} \\
4 & \text{if } m \text{ and } n \text{ are odd}
\end{cases}$$

Proof. Since $k \geq \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil + 1$, Lemmas 3.5 and 3.7 imply that if $f(u) = f(v) = 1$, then

$$f(V) = \begin{cases} 
2 & \text{if } m \text{ and } n \text{ are even} \\
3 & \text{if } m \text{ and } n \text{ have different parities} \\
4 & \text{if } m \text{ and } n \text{ are odd}
\end{cases}$$

If $f(u) = 1$ and $f(v) = -1$, $f(u) = -1$ and $f(v) = 1$ or $f(u) = f(v) = -1$, the result follows from Lemmas 3.8, 3.9 and 3.10.

Let $U_1 \subseteq V_m$ such that $|U_1| = \lceil \frac{m}{2} \rceil + 1$ and let $U_2 \subseteq V_n$ such that $|U_2| = \lceil \frac{n}{2} \rceil$. The function $f$ defined by $f(v) = \begin{cases} 
1 & \text{if } v \in U_1 \cup U_2 \\
-1 & \text{otherwise}
\end{cases}$ is a $k$SF with

$$f(V) = \begin{cases} 
2 & \text{if } m \text{ and } n \text{ are even} \\
3 & \text{if } m \text{ and } n \text{ have different parities} \\
4 & \text{if } m \text{ and } n \text{ are odd}
\end{cases}$$

Similarly,

Proposition 3.12 Suppose $n \geq m \geq 2$. If $\max\{\lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil + 1, m + 1\} \leq k \leq m + \lceil \frac{n}{2} \rceil + 1$, then

$$\gamma_{k, 1}^{-1}(K_{m,n}) = \begin{cases} 
2 & \text{if } m \text{ and } n \text{ are even} \\
3 & \text{if } m \text{ and } n \text{ have different parities} \\
4 & \text{if } m \text{ and } n \text{ are odd}
\end{cases}$$
Proposition 3.13 Suppose \( n \geq m \geq 2 \). If \( \max\{n + \lceil \frac{m}{2} \rceil + 1, m + \lceil \frac{n}{2} \rceil + 1\} + 1 \leq k \leq m + n \), then

\[
\gamma_{k}^{-11}(K_{m,n}) = \begin{cases} 
4 & \text{if } m \text{ and } n \text{ are even} \\
5 & \text{if } m \text{ and } n \text{ have different parities} \\
6 & \text{if } m \text{ and } n \text{ are odd} 
\end{cases}
\]

Proof. Since \( k \geq \max\{n + \lceil \frac{m}{2} \rceil + 1, m + \lceil \frac{n}{2} \rceil + 1\} + 1 \), Lemmas 3.4 and 3.5 imply that if \( f(u) = 1 \) and \( f(v) = -1 \) or if \( f(u) = -1 \) and \( f(v) = 1 \), then

\[
f(V) \geq \begin{cases} 
4 & \text{if } m \text{ and } n \text{ are even} \\
5 & \text{if } m \text{ and } n \text{ have different parities} \\
6 & \text{if } m \text{ and } n \text{ are odd} 
\end{cases}
\]

If \( f(u) = f(v) = -1 \), the result follows from Lemma 3.10. If \( f(u) = f(v) = 1 \), then we may assume that \( C_{f} \cap N_{f} = \emptyset \), or else we have a previous case. This fact and Lemma 3.5 imply \( |V_{u} \cap P_{f}| = \lfloor \frac{n}{2} \rfloor \) and \( |V_{m} \cap P_{f}| = \lfloor \frac{m}{2} \rfloor \). Thus, \( f(V) = \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor - \lfloor \frac{m}{2} \rfloor \). Lemma 3.7 implies that \( k \leq \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor \), which is a contradiction.

Let \( U_{1} \subseteq V_{m} \) such that \( |U_{1}| = \lfloor \frac{m}{2} \rfloor + 1 \) and let \( U_{2} \subseteq V_{n} \) such that \( |U_{2}| = \lfloor \frac{n}{2} \rfloor + 1 \). The function \( f \) defined by

\[
f(v) = \begin{cases} 
1 & \text{if } v \in U_{1} \cup U_{2} \\
-1 & \text{otherwise} 
\end{cases}
\]

is a \( kSF \) with

\[
f(V) = \begin{cases} 
4 & \text{if } m \text{ and } n \text{ are even} \\
5 & \text{if } m \text{ and } n \text{ have different parities} \\
6 & \text{if } m \text{ and } n \text{ are odd} 
\end{cases}
\]

The result now follows. \( \diamond \)
Proposition 3.14 Let $n \geq 1$ be an integer. Then

$$\gamma_{kf}^{11}(K_{1,n}) = \begin{cases} 2k - n + 1 & \text{if } k \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 2k - n - 1 & \text{if } k \geq \left\lceil \frac{n}{2} \right\rceil + 1 \end{cases}.$$  

Proof. Let $V(K_{1,n}) = \{v, v_1, \ldots, v_n\}$ with $v$ being the central vertex. Let $V' = \{v_1, \ldots, v_n\}$.

Case 1. $k = 1$. If $v_i \in C_f$, then $f(v) = f(v_i) = 1$, and $f(V) \geq 2 + (n - 1)(-1) = 3 - n$. If $V' \cap C_f = \emptyset$, then $v \in C_f$, and

$$f(V) \geq 1 + \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{otherwise} \end{cases}.$$  

Thus, $f(V) \geq 3 - n = 2 \cdot 1 - n + 1 = 2k - n + 1$. The function $f$ defined by $f(v) = f(v_1) = 1$ and $f(v) = -1$ otherwise is a $kSF$ with $f(V) = 2k - n + 1$.

Case 2. $2 \leq k$.

Since $k \geq 2$, $f(v) = 1$.

Case 2.1 $k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

If $v \not\in C_f$, then $f(V) \geq 1 + k + (n - k)(-1) = 2k - n + 1$. Suppose, therefore, $v \in C_f$. Then $|P_f \cap V'| \geq \left\lfloor \frac{n}{2} \right\rfloor$. Assume $\{v_1, \ldots, v_{\left\lfloor \frac{n}{2} \right\rfloor}\} \subseteq P_f$. Then the function $f^*$ defined by $f^*(v) = 1$ for all $v \in P_f - \{v_{\left\lfloor \frac{n}{2} \right\rfloor}\}$ and $f^*(v) = -1$ otherwise is a $kSF$ of $K_{1,n}$ such that $f^*(V) < f(V)$, which is a contradiction. Thus, $f(V) \geq 2k - n + 1$. The function $f$ defined by $f(v) = 1$, $f(v_i) = 1$ for $i = 1, \ldots, k$ and $f(v) = -1$ otherwise is a $kSF$ of $K_{1,n}$ such that $f(V) = 2k - n + 1$.

Case 2.2 $k = \left\lfloor \frac{n}{2} \right\rfloor$.

If $v \not\in C_f$, then $f(V) \geq 2k - n + 1$. Suppose, therefore, that $v \in C_f$. Then
The function $f$ defined by $f(v) = 1$, $f(v_i) = 1$ for $i = 1, \ldots, k$ and $f(v) = -1$ otherwise is a $kSF$ of $K_{1,n}$ such that $f(V) = 2k - n + 1$.

**Case 2.3** $k \geq \left[ \frac{n}{2} \right] + 1$.

If $v \notin C_f$, then $f(V) \geq 2k - n + 1$. Suppose, therefore, that $v \in C_f$. Then

$$f(V) \geq 1 + \left[ \frac{n}{2} \right] - \left[ \frac{n}{2} \right] = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{otherwise} \end{cases}.$$ 

Also, $f(V) \geq k + (n - k + 1)(-1) = 2k - n - 1$. Thus,

$$f(V) \geq \begin{cases} \max\{2, 2k - n - 1\} & \text{if } n \text{ is odd} \\ \max\{1, 2k - n - 1\} & \text{otherwise} \end{cases} = 2k - n - 1.$$ 

Thus, $f(V) \geq 2k - n - 1$. The function $f$ defined by $f(v) = 1$, $f(v_i) = 1$ for $i = 1, \ldots, k-1$ and $f(v) = -1$ otherwise is a $kSF$ of $K_{1,n}$ such that $f(V) = 2k - n - 1$. The result follows. \(\diamondsuit\)

The value of $\gamma_{ks}^{-11}(K_{m,n})$ is completely determined by Propositions 3.3, 3.6, 3.11, 3.12, 3.13 and 3.14. This result generalizes a result of [4].

We now turn our attention to the computation of $\gamma_{ks}^{-101}(K_{m,n})$.

**Lemma 3.15** If $v \in V_m \cap C_f$, then

$$f(V_n) \geq 1 - f(v).$$
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**Proof.** Since \(1 \leq f[v] = f(v) + f(V_n)\), it follows that \(f(V_n) \geq 1 - f(v)\). \(\Box\)

**Proposition 3.16** Suppose \(n \geq m \geq 2\). If \(1 \leq k \leq n\), then

\[
\gamma_{ks}^{101}(K_{m,n}) = 2 - n.
\]

**Proof.** Before proceeding further, we prove

**Claim 1** If \(v \in V_n \cap C_f\), then \(f(V) \geq 2 - n\).

**Proof.** If \(f(v) = -1\), then, by Lemma 3.15, \(f(V_m) \geq 2\). Thus, \(f(V) = f(V_m) + f(V_n) \geq 2 + (-1)n = 2 - n\). If \(f(v) = 0\), then, by Lemma 3.15, \(f(V_m) \geq 1\). Thus, \(f(V) = f(V_m) + f(V_n) \geq 1 + 0 + (-1)(n - 1) = 2 - n\). If \(f(v) = 1\), then, by Lemma 3.15, \(f(V_m) \geq 0\). Thus, \(f(V) = f(V_m) + f(V_n) \geq 0 + 1 + (-1)(n - 1) = 2 - n\). \(\Box\)

Since \(k \geq 1\), there is a covered vertex, say \(v\). On the one hand, if \(v \in V_n\), then \(f(V) \geq 2 - n\). On the other hand, if \(v \in V_m\), then \(f(V) \geq 2 - m \geq 2 - n\).

Let \(u \in V_m\) and let \(U \subseteq V_m - \{u\}\) such that \(|U| = \lceil \frac{m}{2} \rceil\). If \(m\) is odd, the function \(f\) defined by

\[
\begin{cases}
1 & \text{if } v \in U \\
0 & \text{if } v = u \\
-1 & \text{otherwise}
\end{cases}
\]

is a \(kSF\) with \(f(V) = 2 - n\). If \(m\) is even, then the function \(f\) defined by

\[
\begin{cases}
1 & \text{if } v \in U \cup \{u\} \\
-1 & \text{otherwise}
\end{cases}
\]
is a \( kSF \) with \( f(V) = 2 - n \). The result follows. \( \diamondsuit \)

We henceforth assume \( k > n, V_m \cap C_f \neq \emptyset \) and \( V_n \cap C_f \neq \emptyset \). Let \( u \in V_m \cap C_f \) and let \( v \in V_n \cap C_f \).

**Proposition 3.17** Suppose \( n \geq m \geq 2 \). If \( n < k \leq n + \lfloor \frac{m}{2} \rfloor \), then

\[ \gamma_{k^*}^{101}(K_{m,n}) = 1. \]

**Proof.** If \( f(u) \leq 0 \), then, by Lemma 3.15, \( f(V) = f(V_m) + f(V_n) \geq 0 + 1 = 1 \).

Similarly, if \( f(v) \leq 0 \), then \( f(V) = f(V_m) + f(V_n) \geq 1 + 0 = 1 \). We assume therefore that \( f(u) = f(v) = 1 \). Let \( V'_n = V_n - \{v\} \). Let \( V'_m = V_m - \{u\} \).

Before proceeding further, we prove

**Claim 2** \( f(V'_n) \geq 0 \) or \( f(V'_m) \geq 0 \).

**Proof.** Suppose, to the contrary, that \( f(V'_n) < 0 \) and \( f(V'_m) < 0 \). Since \( 1 \leq f[u] = f(u) + f(v) + f(V'_n) = 1 + 1 + f(V'_n) \), we have \( f(V'_n) \geq -1 \), whence \( f(V'_n) = -1 \) and \( f(V_n) = 0 \). Similarly, \( f(V_m) = 0 \). Let \( \ell = |V_n \cap P_f| \). Then \( 0 = f(V_n) \geq \ell + (n - \ell)(-1) = 2\ell - n \), so that \( \ell \leq \frac{n}{2} \). Thus, \( |V_n \cap P_f| \leq \frac{n}{2} \), and, similarly, \( |V_m \cap P_f| \leq \frac{m}{2} \). Hence, \( |P_f| \leq \frac{n}{2} + \frac{m}{2} \leq \frac{3n}{2} = n \).

If \( x \in C_f \), then, without loss of generality, we may assume that \( x \in V_m \), so that \( 1 \leq f[x] = f(x) + f(V_n) = f(x) \leq 1 \), which implies that \( x \in P_f \). Thus, \( C_f \subseteq P_f \).

We conclude that \( n + 1 \leq k \leq |C_f| \leq |P_f| \leq n \), which is a contradiction, and the claim follows. \( \diamondsuit \)

So either \( f(V'_n) \geq 0 \) or \( f(V'_m) \geq 0 \). If \( f(V'_n) \geq 0 \), then \( f(V) = f(v) + f(u) + f(V'_n) + f(V'_m) \geq 1 + 1 + 0 - 1 = 1 \). Similarly, if \( f(V'_m) \geq 0 \), then \( f(V) \geq 1 \). Thus, \( f(V) \geq 1 \).
Let \( u \in V_m \) and let \( U \subseteq V_m \setminus \{u\} \) such that \(|U| = \lceil \frac{m}{2} \rceil\). If \( m \) is odd, the function \( f \) defined by

\[
  f(v) = \begin{cases} 
    1 & \text{if } v \in U \\
    0 & \text{if } v \in V_n \\
    -1 & \text{otherwise}
  \end{cases}
\]

is a kSF with \( f(V) = 1 \). If \( m \) is even, then the function \( f \) defined by

\[
  f(v) = \begin{cases} 
    1 & \text{if } v \in U \\
    0 & \text{if } v \in V_n \cup \{u\} \\
    -1 & \text{otherwise}
  \end{cases}
\]

is a kSF with \( f(V) = 1 \). The result follows.

Proposition 3.18 Suppose \( n \geq m \geq 2 \). If \( n + \lceil \frac{m}{2} \rceil + 1 \leq k \leq n + m \), then

\[
  \gamma_{K_m,n}^{-101}(K_{m,n}) = 2.
\]

Proof. Let \( f \) be a kSF of \( K_{m,n} \) with \( f(V) = \gamma_{K_m,n}^{-101}(K_{m,n}) \) such that \(|Z_f|\) is maximized. By Lemma 3.15, \( f(V_m) \geq 0 \) and \( f(V_n) \geq 0 \).

Suppose \( f(V_n) = 0 \). If \(|V_m \cap C_f| \leq \lceil \frac{m}{2} \rceil\), then \( k \leq |C_f| \leq n + \lceil \frac{m}{2} \rceil\), which is a contradiction. Thus, \(|V_m \cap C_f| \geq \lceil \frac{m}{2} \rceil + 1\). Let \( x \in V_m \cap C_f \). Then, since \( 1 \leq f(x) = f(x) + f(V_n) = f(x) \leq 1 \), we have \( f(x) = 1 \). Thus, \( V_m \cap C_f \subseteq V_m \cap P_f \), so that \(|V_m \cap P_f| \geq |V_m \cap C_f| \geq \lceil \frac{m}{2} \rceil + 1\). Hence, \( f(V) = f(V_m) + f(V_n) \geq (\lceil \frac{m}{2} \rceil + 1) - (\lceil \frac{m}{2} \rceil - 1) + 0 = \lceil \frac{m}{2} \rceil - \frac{m}{2} + 2 \geq 2\), as desired.

Assume, therefore, that \( f(V_n) \geq 1 \). If \( f(V_n) \geq 2 \), then \( f(V) = f(V_m) + f(V_n) \geq 0 + 2 \geq 2\), as required. Thus, assume \( f(V_n) = 1 \). We are done if \( f(V_m) \geq 1 \), and so we assume that \( f(V_m) = 0 \).

We show that \( V_m \subseteq Z_f \). For suppose to the contrary that \( x \in V_m \cap P_f \) and \( y \in V_m \cap N_f \). Define \( f^*: V \to \{-1, 0, 1\} \) by \( f^*(z) = 0 \) if \( z \in \{x, y\} \) and \( f^*(z) = f(z) \)
if \( z \notin \{x, y\} \). Then \( f^* \) is a kSF of \( K_{m,n} \) with \( f^*(V) = f(V) \) and \( |Z_{f^*}| > |Z_f| \), which contradicts the choice of \( f \).

We conclude that \( V_m \subseteq C_f \), so that \( |V_n \cap C_f| \geq k - m \geq n + \lceil \frac{m}{2} \rceil + 1 - m = n - \lfloor \frac{m}{2} \rfloor + 1 \).

Let \( y \in V_n \cap C_f \). Then, since \( 1 \leq f[y] = f(y) + f(V_m) = f(y) \leq 1 \), we have \( f(y) = 1 \). Thus, \( V_n \cap C_f \subseteq V_n \cap P_f \), so that \( |V_n \cap P_f| \geq |V_n \cap C_f| \geq n - \lfloor \frac{m}{2} \rfloor + 1 \). Hence,
\[
f(V_n) \geq (n - \lfloor \frac{m}{2} \rfloor + 1) - (n - (n - \lceil \frac{m}{2} \rceil + 1)) = (n - \lfloor \frac{m}{2} \rfloor + 1) - (\lfloor \frac{m}{2} \rfloor - 1) = n - 2 \lfloor \frac{m}{2} \rfloor + 2 \geq m - 2 \lceil \frac{m}{2} \rceil + 2 \geq 2 ,
\]
a contradiction.

Let \( u \in V_m \) and let \( v \in V_n \). The function \( f \) defined by
\[
f(x) = \begin{cases} 
1 & \text{if } x \in \{u, v\} \\
0 & \text{otherwise}
\end{cases}
\]
is a kSF of \( K_{m,n} \) such that \( f(V) = 2 \). The result now follows.

Hattingh and Ungerer [21] obtained the following lower bound on \( \gamma_{ks}^{-101}(T) \) for a tree \( T \).

**Proposition 3.19** If \( T \) is a tree of order \( n \geq 2 \) and \( 1 \leq k \leq n - 1 \), then
\[
\gamma_{ks}^{-101}(T) \geq k - n + 2.
\]

As a consequence we obtain

**Proposition 3.20** If \( n \geq 1 \) is an integer and \( k \leq n \), then
\[
\gamma_{ks}^{-101}(K_{1,n}) = k - n + 1.
\]

**Proof.** By Proposition 3.19, \( \gamma_{ks}^{-101}(K_{1,n}) \geq k - (n + 1) + 2 = k - n + 1 \). The function \( f \) that assigns 1 to the central vertex, 0 to \( k \) leaves and \(-1\) to the remaining leaves of \( K_{1,n} \) is a kSF with \( f(V) = k - n + 1 \). The result follows. \( \diamond \)
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Proposition 3.21 If $n \geq 1$ is an integer, then

$$\gamma^{-101}_{(n+1)}(K_{1,n}) = 1.$$ 

Proof. If $v$ is the central vertex of $K_{1,n}$, then $f(V) = f[v] \geq 1$. The function $f$ that assigns 1 to $v$ and 0 to the remaining vertices of $K_{1,n}$ is a $kSF$ of $K_{1,n}$ with $f(V) = 1$. The result follows. ◆

The value of $\gamma^{-101}_{ks}(K_{m,n})$ is completely determined by Propositions 3.16, 3.17, 3.18, 3.20, 3.21.

3.4 The disjoint union of two complete graphs

In this section we compute the values of $\gamma^{-11}_{ks}(G)$ and $\gamma^{-101}_{ks}(G)$ where $G$ is the disjoint union of two complete graphs. Let $V_m = V(K_m)$ and $V_n = V(K_n)$.

Proposition 3.22 If $1 \leq k \leq m \leq n$, then

$$\gamma^{-11}_{ks}(K_m \cup K_n) = \begin{cases} 1 - n & \text{if } m \text{ is odd} \\ 2 - n & \text{if } m \text{ is even} \end{cases}.$$ 

Proof. Since $k \geq 1$, there is a covered vertex, say $v \in V_m \cup V_n$. If $v \in V_m$, then by Proposition 3.1, $f(V_m) \geq 1$ if $m$ is odd and $f(V_m) \geq 2$ if $m$ is even. Thus, if $m$ is odd, $f(V_m \cup V_n) = f(V_m) + f(V_n) \geq 1 + (-1)n = 1 - n$. On the other hand, if $m$ is even, $f(V_m \cup V_n) = f(V_m) + f(V_n) \geq 2 + (-1)n = 2 - n$. Similarly it can be shown that if $v \in V_n$, then $f(V_m \cup V_n) \geq (-1)m + 1 = 1 - m$ if $n$ is odd and $f(V_m \cup V_n) \geq (-1)m + 2 = 2 - m$ if $n$ is even.
Note that

\[
\begin{cases}
\min\{1 - m, 1 - n\} & \text{if } n \text{ is odd and } m \text{ is odd} \\
\min\{2 - m, 1 - n\} & \text{if } n \text{ is even and } m \text{ is odd} \\
\min\{1 - m, 2 - n\} & \text{if } n \text{ is odd and } m \text{ is even} \\
\min\{2 - m, 2 - n\} & \text{if } n \text{ is even and } m \text{ is even}
\end{cases}
\]

Thus, \( f(V_m \cup V_n) \geq \begin{cases}
1 - n & \text{if } m \text{ is odd} \\
2 - n & \text{if } m \text{ is even}
\end{cases} \)

Let \( U \subseteq V_m \) such that \( |U| = \lfloor \frac{m}{2} \rfloor + 1 \). The function \( f \) defined by

\[
f(v) = \begin{cases}
1 & \text{if } v \in U \\
-1 & \text{otherwise}
\end{cases}
\]

is a kSF with

\[
f(V_m \cup V_n) = \begin{cases}
1 - n & \text{if } m \text{ is odd} \\
2 - n & \text{if } m \text{ is even}
\end{cases}
\]

**Proposition 3.23** Suppose \( n \geq m \geq 1 \). If \( m < k \leq n \) then

\[
\gamma_{ks}^{-11}(K_m \cup K_n) = \begin{cases}
1 - m & \text{if } n \text{ is odd} \\
2 - m & \text{if } n \text{ is even}
\end{cases}
\]

**Proof.** Since \( k > m \), there is a covered vertex, \( v \in V_n \). By Proposition 3.1, \( f(V_n) \geq 1 \) if \( n \) is odd and \( f(V_n) \geq 2 \) if \( n \) is even. Thus, if \( n \) is odd,
$f(V_m \cup V_n) = f(V_m) + f(V_n) \geq (-1)m + 1 = 1 - m$. On the other hand, if $n$ is even, $f(V_m \cup V_n) = f(V_m) + f(V_n) \geq (-1)m + 2 = 2 - m$.

Let $U \subseteq V_n$ such that $|U| = \lceil \frac{n}{2} \rceil + 1$. The function $f$ defined by

$$f(v) = \begin{cases} 
1 & \text{if } v \in U \\
-1 & \text{otherwise}
\end{cases}$$

is a $kSF$ such that

$$f(V_m \cup V_n) = \begin{cases} 
1 - m & \text{if } n \text{ is odd} \\
2 - m & \text{if } n \text{ is even}
\end{cases} \quad \Diamond$$

**Proposition 3.24** Suppose $n \geq m \geq 1$. If $n < k \leq n + m$ then

$$\gamma_{kS}^{11}(K_m \cup K_n) = \begin{cases} 
2 & \text{if } m \text{ and } n \text{ are odd} \\
3 & \text{if } m \text{ and } n \text{ have different parities} \\
4 & \text{if } m \text{ and } n \text{ are even}
\end{cases}$$

**Proof.** Since $k \geq n$, there is a covered vertex, $v \in V_m$ and a covered vertex, $u \in V_n$.

By Proposition 3.1,

$$f(V_m) \geq \begin{cases} 
1 & \text{if } m \text{ is odd} \\
2 & \text{if } m \text{ is even}
\end{cases}$$

and

$$f(V_n) \geq \begin{cases} 
1 & \text{if } n \text{ is odd} \\
2 & \text{if } n \text{ is even}
\end{cases}$$

Thus,

$$f(V_m \cup V_n) \geq \begin{cases} 
2 & \text{if } m \text{ and } n \text{ are odd} \\
3 & \text{if } m \text{ and } n \text{ have different parities} \\
4 & \text{if } m \text{ and } n \text{ are even}
\end{cases}$$
Let $U_1 \subseteq V_m$ such that $|U_1| = \lceil \frac{m}{2} \rceil + 1$. Let $U_2 \subseteq V_n$ such that $|U_2| = \lceil \frac{n}{2} \rceil + 1$. The function $f$ defined by

$$f(v) = \begin{cases} 
1 & \text{if } v \in U_1 \cup U_2 \\
-1 & \text{otherwise}
\end{cases}$$

is a $k$SF such that

$$f(V_m \cup V_n) = \begin{cases} 
2 & \text{if } m \text{ and } n \text{ are odd} \\
3 & \text{if } m \text{ and } n \text{ have different parities} \\
4 & \text{if } m \text{ and } n \text{ are even}
\end{cases}$$

These results generalize a result of [4].

We now turn our attention to computing $\gamma_{k_2}^{-101}(K_m \cup K_n)$.

**Proposition 3.25** If $1 \leq k \leq m \leq n$, then

$$\gamma_{k_2}^{-101}(K_m \cup K_n) = 1 - n.$$

**Proof.** Since $k \geq 1$, $C_f \neq \emptyset$, say $v \in C_f$. If $v \in V_m$, then by Proposition 3.2, $f(V_m) \geq 1$. Thus, $f(V_m \cup V_n) = f(V_m) + f(V_n) \geq 1 + (-1)n = 1 - n$. Similarly, if $v \in V_n$, then $f(V_m \cup V_n) \geq 1 - m \geq 1 - n$. Let $u \in V_m$. The function $f$ defined by

$$f(v) = \begin{cases} 
1 & \text{if } v = u \\
0 & \text{if } v \in V_m - \{u\} \\
-1 & \text{otherwise}
\end{cases}$$

is a $k$SF with $f(V_m \cup V_n) = 1 - n$. ◊

**Proposition 3.26** If $1 \leq m < k \leq n$, then

$$\gamma_{k_2}^{-101}(K_m \cup K_n) = 1 - m.$$
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Proof. Since \( k > m \), there is a vertex \( v \), say, such that \( v \in V_n \cap C_f \). As before, 
\( f(V_n) \geq 1 \) and \( f(V_m \cup V_n) \geq 1 - m \). Let \( u \in V_n \). The function \( f \) defined by

\[
f(v) = \begin{cases} 
1 & \text{if } v = u \\
0 & \text{if } v \in V_n - \{u\} \\
-1 & \text{otherwise}
\end{cases}
\]

is a \( kSF \) with \( f(V_m \cup V_n) = 1 - m \). \( \diamond \)

Proposition 3.27 Suppose \( n \geq m \geq 1 \). If \( n < k \leq n + m \) then

\[
\gamma_{k}^{-101}(K_m \cup K_n) = 2.
\]

Proof. Since \( k > n \), there is a covered vertex, \( v \in V_m \) and a covered vertex, \( u \in V_n \).
By Proposition 3.2, \( f(V_m) \geq 1 \) and \( f(V_n) \geq 1 \), so that \( f(V_m \cup V_n) \geq 2 \). Let \( u_1 \in V_m \)
and \( u_2 \in V_n \). The function \( f \) defined by

\[
f(v) = \begin{cases} 
1 & \text{if } v \in \{u_1, u_2\} \\
0 & \text{otherwise}
\end{cases}
\]

is a \( kSF \) with \( f(V_m \cup V_n) = 2 \). \( \diamond \)

3.5 Split graphs

In this section we compute \( \gamma_{k}^{-11}(G) \) and \( \gamma_{k}^{-101}(G) \) of a split graph \( G \). The set \( V_n \) will
denote the vertices in the clique and the set \( V_m \) will denote the set of independent
vertices. Since \( S_{1,n} \) is the complete graph of order \( n + 1 \), we may assume that \( m \geq 2 \).

Lemma 3.28 If \( V_n \cap C_f \neq \emptyset \), then

\[
f(V(S_m,n)) \geq \begin{cases} 
2 & \text{if } m + n \text{ is even} \\
1 & \text{if } m + n \text{ is odd}
\end{cases}
\]
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Proof. If \( v \in V_n \cap C_f \), then \( 1 \leq f[v] = f(V(S_{m,n})) \), which implies that 
\[ |P_f \cap (V_m \cup V_n)| \geq \left\lceil \frac{m+n}{2} \right\rceil + 1 \]. Therefore, if \( m+n \) is even, then 
\[ f(V(S_{m,n})) \geq \left( \frac{m+n}{2} + 1 \right) - \left( \left\lceil \frac{m+n}{2} \right\rceil - 1 \right) = 2. \]
If \( m+n \) is odd, then 
\[ f(V(S_{m,n})) \geq \left( \left\lceil \frac{m+n}{2} \right\rceil + 1 \right) - \left( \left\lfloor \frac{m+n}{2} \right\rfloor - 1 \right) = 1. \]
\[ \Box \]

Proposition 3.29 Suppose \( n \geq 1 \) and \( m \geq 1 \). If \( 1 \leq k \leq m \), then

\[ \gamma_k^{-1}(S_{m,n}) = \begin{cases} 2 - m & \text{if } n \text{ is even} \\ 3 - m & \text{if } n \text{ is odd} \end{cases} \]

Proof. Since \( k \geq 1 \), there is a covered vertex, \( v \) (say). On the one hand, if \( v \in V_n \), then, by Lemma 3.28,

\[ f(V(S_{m,n})) \geq \begin{cases} 2 & \text{if } m+n \text{ is even} \\ 1 & \text{if } m+n \text{ is odd} \end{cases} \]

On the other hand, suppose \( v \in V_m \). If \( f(v) = -1 \), then \( f[v] = -1 + f(V_n) \geq 1 \), and so \( f(V_n) \geq 2 \), which implies that \( f(V(S_{m,n})) = f(V_m) + f(V_n) \geq 2 - m \). Moreover, if \( n \) is odd, then 
\[ |P_f \cap V_n| \geq \left\lceil \frac{n}{2} \right\rceil + 1 \] and \( f(V_n) \geq \left( \left\lceil \frac{n}{2} \right\rceil + 1 \right) - \left( \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) = 3. \) Therefore, 
\[ f(V(S_{m,n})) = f(V_m) + f(V_n) \geq 3 - m. \]
If \( f(v) = 1 \), then \( f[v] = 1 + f(V_n) \geq 1 \), and so \( f(V_n) \geq 0 \), which implies that \( f(V(S_{m,n})) = f(V_m) + f(V_n) \geq 1 - (m-1) + 0 = 2 - m. \) Moreover, if \( n \) is odd, 
\[ |P_f \cap V_n| \geq \left\lceil \frac{n}{2} \right\rceil \] and \( f(V_n) \geq \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor = 1. \) Thus, 
\[ f(V(S_{m,n})) = f(V_m) + f(V_n) \geq 1 - (m-1) + 1 = 3 - m. \]
We conclude that if \( v \in V_m \),

\[ f(V(S_{m,n})) = f(V_m) + f(V_n) \]

\[ \geq \begin{cases} 3 - m & \text{if } n \text{ is odd} \\ 2 - m & \text{if } n \text{ is even} \end{cases} \]
Notice if \( m \geq 2 \), then \( 2 - m < 3 - m < 1 < 2 \). Therefore

\[
f(V(S_{m,n})) \geq \begin{cases} 
2 - m & \text{if } n \text{ is even} \\
3 - m & \text{if } n \text{ is odd}
\end{cases}
\]

Let \( U \subseteq V_n \) such that \( |U| = \left\lfloor \frac{n}{2} \right\rfloor + 1 \). The function \( f \) defined by

\[
f(v) = \begin{cases} 
1 & \text{if } v \in U \\
-1 & \text{otherwise}
\end{cases}
\]

is a \( kSF \) with

\[
f(V(S_{m,n})) = \begin{cases} 
3 - m & \text{if } n \text{ is odd} \\
2 - m & \text{if } n \text{ is even}
\end{cases}
\]

**Proposition 3.30** Suppose \( n \geq 1 \) and \( m \geq 1 \). If \( m < k \leq n + m \) then

\[
\gamma^{-11}_{kS}(S_{m,n}) = \begin{cases} 
2 & \text{if } m + n \text{ is even} \\
1 & \text{if } m + n \text{ is odd}
\end{cases}
\]

**Proof.** Since \( k > m \), there is a covered vertex \( v \in V_n \). By Lemma 3.28,

\[
f(V(S_{m,n})) \geq \begin{cases} 
2 & \text{if } m + n \text{ is even} \\
1 & \text{if } m + n \text{ is odd}
\end{cases}
\]

Let \( U_1 \subseteq V_n \) be such that \( |U_1| = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and let \( U_2 \subseteq V_m \) such that

\[
|U_2| = \begin{cases} 
\left\lfloor \frac{m}{2} \right\rfloor & \text{if } m \text{ and } n \text{ are both even} \\
\left\lfloor \frac{m}{2} \right\rfloor - 1 & \text{otherwise}
\end{cases}
\]

Then the function \( f \) defined by

\[
f(v) = \begin{cases} 
1 & \text{if } v \in U_1 \cup U_2 \\
-1 & \text{otherwise}
\end{cases}
\]
is a $k$SF with

$$f(V(S_{m,n})) = \begin{cases} 
2 & \text{if } m + n \text{ is even} \\
1 & \text{if } m + n \text{ is odd}
\end{cases}. \diamondsuit$$

We now turn our attention to computing $\gamma_{kS}^{-101}(G)$ of a split graph $G$.

**Proposition 3.31** Suppose $n \geq 1$ and $m \geq 2$. If $1 \leq k \leq m$ then

$$\gamma_{kS}^{-101}(S_{m,n}) = 2 - m.$$

**Proof.** Since $k \geq 1$, there is a covered vertex, $v \in V_m \cup V_n$. If $v \in V_n$, then $1 \leq f[v] = f(V(S_{m,n}))$. Now suppose $v \in V_m$. If $f(v) = -1$, then $f(V_n) \geq 2$ and $f(V(S_{m,n})) = f(V_m) + f(V_n) \geq 2 - m$. If $f(v) = 0$, then $f(V_n) \geq 1$ and $f(V(S_{m,n})) = f(V_m) + f(V_n) \geq 0 - (m - 1) + 1 = 2 - m$. If $f(v) = 1$, then $f(V_n) \geq 0$ and $f(V(S_{m,n})) = f(V_m) + f(V_n) \geq 0 + 1 - (m - 1) = 2 - m$. Since $m \geq 2$, $2 - m \leq 1$. Hence, $f(V(S_{m,n})) \geq 2 - m$.

The function $f$ that assigns 1 to exactly two vertices of $V_n$, 0 to the remaining vertices of $V_n$ and $-1$ to all vertices of $V_m$ is a $k$SF with $f(V(S_{m,n})) = 2 - m$. $\diamondsuit$

**Proposition 3.32** Suppose $n \geq 1$ and $m \geq 2$. If $m < k \leq n + m$ then

$$\gamma_{kS}^{-101}(S_{m,n}) = 1.$$

**Proof.** Since $k > m$, there is a covered vertex $v \in V_n$. Thus, $1 \leq f[v] = f(V(S_{m,n}))$.

The function that assigns 1 to exactly one vertex of $V_n$ and 0 to all remaining vertices of $S_{m,n}$ is a $k$SF with $f(V(S_{m,n})) = 1$. $\diamondsuit$
Chapter 4

TOTAL SIGNED

$k$-SUBDOMINATION NUMBERS

4.1 Introduction

In this chapter, we focus on total signed $k$-subdomination, or, as it will not cause any confusion, total $k$-subdomination. In Section 4.2, we give a lower bound on the total $k$-subdomination number in terms of the minimum degree, maximum degree and the order of the graph. A lower bound in terms of the degree sequence is also given. In Section 4.3, we compute the total $k$-subdomination number of a cycle. In Section 4.4, we present a characterization of graphs $G$ with equal total $k$-subdomination and total $\ell$-subdomination numbers. In the final section, we establish a sharp upper bound on the total $k$-subdomination number of a tree in terms of its order $n$ and $k$ where $1 \leq k < n$. Moreover, we characterize trees attaining these bounds for certain values of $k$. For this purpose, we first establish the total $k$-subdomination number of simple structures, including paths and spiders.
4.2 Lower bounds

Our aim in this section is to give lower bounds on the total $k$-subdomination number of a graph. We first establish such a lower bound in terms of its minimum degree, maximum degree and its order. The second lower bound is in terms of the degree sequence of the graph. We begin with the following observation. (For a vertex $v \in V$, we denote $f(N(v))$ by $f[v]$.)

**Observation 4.1** Let $f$ be a $T_k$SF of $G$ and let $v \in C_f$. If $\deg v$ is even, then $f[v] \geq 2$, while if $\deg v$ is odd, then $f[v] \geq 1$.

**Theorem 4.2** Let $G = (V, E)$ be a graph of order $n$ with minimum degree $\delta$ and maximum degree $\Delta$. For $1 \leq k \leq n$, let $f$ be a $\gamma^{-1}_{tk}$-function, and let $\ell$ denote the number of vertices with even degree in $C_f$. Then,

$$\gamma^{-1}_{tk}(G) \geq \frac{2k(1 + \Delta) + \delta n - 3n\Delta + 2\ell}{\Delta + \delta}.$$

**Proof.** We consider the sum $N = \sum \sum f(u)$, where the outer sum is over all $v \in V$ and the inner sum is over all $u \in N(v)$. This sum counts the value $f(u)$ exactly $\deg u$ times for each $u \in V$, so $N = \sum (\deg u) \cdot f(u)$, over all $u \in V$. Let $V_{\text{even}}$ denote the set of all vertices with even degree in $C_f$. Then, by Observation 4.1, $N = \sum f[v]$ over all $v \in V$ satisfies

$$\sum f[v] = \sum (\deg v) \cdot f[v].$$
\( N = \sum_{v \in \text{Even}} f[v] + \sum_{v \in C_f - \text{Even}} f[v] + \sum_{v \in C_f} f[v] \)

\[ \geq 2\ell + |C_f| - \ell + (n - |C_f|)(-\Delta) \]

\[ = \ell + |C_f| (1 + \Delta) - n\Delta \]

\[ \geq \ell + k(1 + \Delta) - n\Delta. \quad (4.1) \]

Let \( P \) and \( M \) be the sets of those vertices in \( G \) which are assigned under \( f \) the values +1 and -1, respectively. Then, \( \gamma_{k\Delta}^{11}(G) = f(V) = |P| - |M| = n - 2|M| \). We now write \( V \) as the disjoint union of six sets. Let \( P = P_\Delta \cup P_\delta \cup P_\lambda \) where \( P_\Delta \) and \( P_\delta \) are sets of all vertices of \( P \) with degree equal to \( \Delta \) and \( \delta \), respectively, and \( P_\lambda \) contains all other vertices in \( P \), if any. Let \( M = M_\Delta \cup M_\delta \cup M_\lambda \) where \( M_\Delta \), \( M_\delta \), and \( M_\lambda \) are defined similarly. Further, for \( i \in \{\Delta, \delta, \lambda\} \), let \( V_i \) be defined by \( V_i = P_i \cup M_i \). Thus, \( n = |V_\Delta| + |V_\delta| + |V_\lambda| \).

If \( u \in V_\lambda \), then \( \delta + 1 \leq \deg u \leq \Delta - 1 \). Therefore, writing the sum in \((4.1)\) as the sum of six summations and replacing \( f(u) \) with the corresponding value of 1 or -1 yields

\[ \sum_{u \in P_\Delta} \Delta + \sum_{z \in P_\delta} \delta + \sum_{z \in P_\lambda} (\Delta - 1) - \sum_{z \in M_\Delta} \Delta - \sum_{z \in M_\delta} \delta - \sum_{z \in M_\lambda} (\delta + 1) \geq \ell + k(1 + \Delta) - n\Delta. \]

Replacing \(|P_i|\) with \(|V_i| - |M_i|\) for \( i \in \{\Delta, \delta, \lambda\} \), yields

\[ \Delta|V_\Delta| + \delta|V_\delta| + (\Delta - 1)|V_\lambda| - 2\Delta|M_\Delta| - 2\delta|M_\delta| - (\Delta + \delta)|M_\lambda| \geq \ell + k(1 + \Delta) - n\Delta. \quad (4.2) \]
We now simplify the left hand side of (4.2) as follows. Replacing $|V_\delta|$ with $|P_\delta| + |M_\delta|$, and $|M_\delta| + |M_\lambda|$ with $|M| - |M_\Delta|$, we have

$$\delta |V_\delta| - 2\delta |M_\delta| - \delta |M_\lambda| = \delta |P_\delta| - \delta |M_\delta| - \delta |M_\lambda| = \delta |P_\delta| - \delta |M| + \delta |M_\Delta|. \quad (4.3)$$

Further, replacing $|V_\Lambda|$ with $n - |V_\delta| - |V_\lambda|$, we have

$$\Delta |V_\Delta| + \Delta |V_\lambda| - 2\Delta |M_\Delta| - \Delta |M_\lambda|$$

$$= n\Delta - \Delta |V_\delta| - 2\Delta |M_\Delta| - \Delta |M_\lambda|$$

$$= n\Delta - \Delta |P_\delta| - \Delta |M| - \Delta |M_\Delta|. \quad (4.4)$$

Using (4.3) and (4.4), the left hand side of (4.2) can be written as

$$n\Delta - |V_\lambda| - (\Delta - \delta) |P_\delta| - (\Delta + \delta) |M| - (\Delta - \delta) |M_\Delta|.$$ 

Thus (4.2) becomes

$$2n\Delta - k(1 + \Delta) - \ell \geq |V_\lambda| + (\Delta - \delta) |P_\delta| + (\Delta + \delta) |M| + (\Delta - \delta) |M_\Delta|$$

$$\geq (\Delta + \delta) |M|. \quad (4.5)$$

Hence, since $\gamma_{tkd}^{\text{11}}(G) = n - 2|M|$, it follows from (4.5) that

$$\gamma_{tkd}^{\text{11}}(G) \geq n - 2 \left( \frac{2n\Delta - k(1 + \Delta) - \ell}{\Delta + \delta} \right) = \frac{2k(1 + \Delta) + \delta n - 3n\Delta + 2\ell}{\Delta + \delta},$$

as desired. ♦

The next result gives a lower bound on the total $k$-subdomination number of a graph in terms of its degree sequence.
Theorem 4.3  Let \( G = (V, E) \) be a graph of order \( n \) where the degrees \( d_i \) of vertices \( v_i \) satisfy \( d_1 \leq d_2 \leq \cdots \leq d_n \), let \( f \) be a \( \gamma_{tk^1}(G) \)-function, and let \( \ell \) denote the number of vertices of even degree in \( C_f \). Then,

\[
\gamma_{tk^1}(G) \geq \left( \frac{\ell + k + \sum_{i=1}^{k} d_i}{d_n} \right) - n.
\]

Proof. Let \( f \) be a \( \gamma_{tk^1}(G) \)-function. Let \( V_{\text{even}} \) denote the set of all vertices with even degree in \( C_f \). Let \( g: V \to \{0, 1\} \) be the function defined by \( g(v) = (f(v) + 1)/2 \) for all vertices \( v \in V \). We consider the sum \( N = \sum \sum g(u) \), where the outer sum is over all \( v \in C_f \) and the inner sum is over all \( u \in N(v) \). Then,

\[
N = \sum_{v \in C_f} \sum_{u \in N(v)} \frac{1}{2}(f(u) + 1) = \sum_{v \in C_f} \frac{1}{2}(f(v) + \deg v) = \frac{1}{2} \left( \sum_{v \in C_f} f(v) + \sum_{v \in C_f} \deg v \right)
\]

\[
\geq \frac{1}{2} \left( \sum_{i=1}^{k} d_i + \sum_{v \in V_{\text{even}}} \deg v + \sum_{v \in C_f - V_{\text{even}}} \deg v \right) \geq \frac{1}{2} \left( 2 \ell + |C_f| - \ell + \sum_{i=1}^{k} d_i \right)
\]

\[
\geq \frac{1}{2} (\ell + k + \sum_{i=1}^{k} d_i).
\]

On the other hand,

\[
N \leq \sum_{v \in V} \sum_{u \in N(v)} g(u) = \sum_{v \in V} (\deg v \cdot g(v) \leq d_n g(V),
\]

and so

\[
g(V) \geq \frac{(\ell + k + \sum_{i=1}^{k} d_i)}{2d_n}
\]

The desired result now follows since \( \gamma_{tk^1}(G) = f(V) = 2g(V) - n \). \( \diamond \)
As an immediate consequence of Theorem 4.2 or Theorem 4.3, we have the following result.

**Corollary 4.4** For $r \geq 1$, if $G$ is an $r$-regular graph of order $n$, then

$$
\gamma_{\text{tk}s}^{-11}(G) \geq \begin{cases} 
k \left(\frac{r+1}{r}\right) - n & \text{if } r \text{ is odd} \\
k \left(\frac{r+2}{r}\right) - n & \text{if } r \text{ is even} \end{cases}
$$

**Corollary 4.5** If $G$ is a graph of order $n$, size $m$ and maximum degree $\Delta$, then

$$
\gamma_{\text{tk}s}^{-11}(G) \geq k - 2n + \frac{k + 2m}{\Delta}.
$$

**Proof.** Let the degrees $d_i$ of the vertices of $G$ satisfy $d_1 \leq d_2 \leq \cdots \leq d_n = \Delta$. It follows from Theorem 4.3 that

$$
\gamma_{\text{tk}s}^{-11}(G) \geq \frac{1}{\Delta} \left(k + \sum_{i=1}^{k} d_i\right) - n
$$

$$
= \frac{1}{\Delta} \left(k + 2m - \sum_{i=k+1}^{n} d_i\right) - n
$$

$$
\geq \frac{1}{\Delta} (k + 2m - (n - k)\Delta) - n
$$

$$
= k - 2n + \frac{k + 2m}{\Delta}. \quad \diamond
$$
CHAPTER 4. TOTAL SIGNED $k$-SUBDOMINATION NUMBERS

4.3 Cycles

Our aim in this section is to determine the total $k$-subdomination number of a cycle. As a special case of Corollary 4.4, we have that $\gamma^{-11}_{tk_s}(C_n) \geq 2k - n$. If $k \in \{n/2, n\}$, we show this lower bound is sharp. We shall prove:

**Proposition 4.6** For $n \geq 3$ and $1 \leq k \leq n$,

$$\gamma^{-11}_{tk_s}(C_n) = \begin{cases} 2k - n & \text{if } k \in \{n/2, n\} \\ 2k + 2 - n & \text{otherwise.} \end{cases}$$

**Proof.** We show first that $\gamma^{-11}_{tk_s}(C_n) \geq 2k + 2 - n$ except when $k = n/2$ or $k = n$, in which case $\gamma^{-11}_{tk_s}(C_n) = 2k - n$. Let $f$ be a $\gamma^{-11}_{tk_s}(C_n)$-function. Let $M = \{v \in V(C_n) \mid f(v) = -1\}$ and $P = \{v \in V(C_n) \mid f(v) = +1\}$. Note that, since $k \geq 1$, $P \neq \emptyset$. Let $M_c = C_f \cap M$, $P_c = C_f \cap P$, $M_{uc} = M - M_c$ and $P_{uc} = P - P_c$. Let $H = G[M_c \cup P]$, i.e., $H$ is the subgraph of $G$ induced by $M_c \cup P$ where $G = C_n$. The two vertices adjacent to a vertex in $M_c$ are in $P_{uc}$, while the two vertices adjacent to a vertex in $P_c$ are in $P$. It follows that

$$2m(G[P]) = \sum_{v \in P} \deg_{G[P]} v \geq \sum_{v \in P_c} \deg_{G[P]} v = 2|P_c|,$$

whence $m(G[P]) \geq |P_c|$. Thus $m(H) = 2|M_c| + m(G[P]) \geq 2|M_c| + |P_c|$. Further if $m(G[P]) = |P_c|$, then $\deg_{G[P]}(v) = 0$ for all $v \in P_{uc}$ and, since $C_n$ is connected and none of the vertices in $P_c$ are adjacent to any of the vertices of $M \cup P_{uc}$, either $V = P_c$ or $P_c = \emptyset$. So, if $m(G[P]) = |P_c|$, either $V = P_c$ or $P = P_{uc}$ and $m(G[P]) = 0$.

**Case 1.** $M_{uc} = \emptyset$. Then $H \cong C_n$, so $|M_c| + |P| = m(H) \geq 2|M_c| + |P_c|$. Thus, $|P| \geq |M_c| + |P_c| = |C_f| \geq k$ and so $\gamma^{-11}_{tk_s}(C_n) \geq 2k - n$. If we have strict
inequality in any of the above inequalities or if \(|C_f| \geq k + 1\), then \(|P| \geq k + 1\), and \(\gamma_{tk}^{11}(C_n) = 2|P| - n \geq 2(k + 1) - n = 2k + 2 - n\). Hence, suppose we have equality throughout in the above inequalities and \(|M_c| + |P_c| = k\). Then, by our remarks above, either \(V = P_c\), in which case \(|P_c| = k = n\), or \(P_c = \emptyset\), in which case \(|M_c| = k\) and \(n = m(H) = 2|M_c| = 2k\) and so \(k = n/2\).

**Case 2.** \(M_{uc} \neq \emptyset\). In this case \(H\) consists of a disjoint union of \(\ell \geq 1\) paths. Then, \(|M_c| + |P| - \ell = m(H) \geq 2|M_c| + |P_c|\). Thus, \(|P| \geq |M_c| + |P_c| + \ell \geq |C_f| + 1 \geq k + 1\), and so \(\gamma_{tk}^{11}(C_n) \geq 2(k + 1) - n = 2k + 2 - n\).

We have shown that \(\gamma_{tk}^{11}(C_n) \geq 2(k + 1) - n = 2k + 2 - n\) except when \(k = n/2\) or \(k = n\), in which case \(\gamma_{tk}^{11}(C_n) \geq 2k - n\). We now show that \(\gamma_{tk}^{11}(C_n) \leq 2k - n\) if \(k = n/2\) or \(k = n\) and that \(\gamma_{tk}^{11}(C_n) \leq 2k + 2 - n\) otherwise. For this purpose, we denote the vertex set of the cycle \(C_n\) by \(\{0, 1, \ldots, n - 1\}\). We now define a function \(f(V(C_n)) \rightarrow \{-1, 1\}\) as follows:

For \(1 \leq k < n/2\), let \(f(v_i) = 1\) if \(i \in \{0, 2, \ldots, 2k\}\) and \(f(v_i) = -1\) otherwise. Then, \(f(V) = 2(k + 1) - n\), and \(\{v_1, v_3, \ldots, v_{2k-1}\} \subseteq C_f\), so that \(|C_f| \geq k\).

For \(k = n/2\), let \(f(v_i) = 1\) if \(i\) is even and \(f(v_i) = -1\) otherwise. Then, \(f(V) = 0\) and \(\{v_1, v_3, \ldots, v_{n-1}\} \subseteq C_f\), so that \(|C_f| \geq k\).

For \((n + 2)/2 \leq k \leq n - 1\) and \(n\) even, let \(f(v_i) = 1\) if \(i\) is even or \(i \in \{1, 3, \ldots, 2k-n+1\}\) and \(f(v_i) = -1\) otherwise. Then, \(f(V) = 2|P| - n = 2k - n + 2\), and \(\{v_1, v_3, \ldots, v_{n-1}\} \cup \{v_2, v_4, \ldots, v_{2k-n}\} \subseteq C_f\) so that \(|C_f| \geq n/2 + (k - n/2) = k\).

For \((n+1)/2 \leq k \leq n - 1\) and \(n\) odd, let \(f(v_i) = 1\) if \(i\) is even or \(i \in \{1, 3, \ldots, 2k-n\}\) and \(f(v_i) = -1\) otherwise. Then, \(f(V) = 2|P| - n = 2k - n + 2\), and \(\{v_1, v_3, \ldots, v_{n-2}\} \cup \{v_0, v_2, \ldots, v_{2k-n-1}\} \subseteq C_f\) so that \(|C_f| \geq (n - 1)/2 + (2k - n + 1)/2 = k\).
For $k = n$, the function that assigns 1 to every vertex of the cycle is the desired function.

In all the above cases, $f$ is a $TkSF$ of $C_n$. Thus, $\gamma_{tk_{11}}(C_n) \leq f(V) = 2k - n$ if $k = n/2$ or $k = n$, while $\gamma_{tk_{11}}(C_n) \leq f(V) = 2k + 2 - n$ otherwise. 

4.4 Graphs with equal total $k$- and $\ell$-subdomination numbers

Our aim in this section is to give a characterization of graphs $G$ with equal total $k$-subdomination and total $\ell$-subdomination numbers where $1 \leq k < \ell \leq |V(G)|$.

Our proof is along similar lines to that presented in [34].

**Theorem 4.7** Let $G = (V, E)$ be a graph of order $n$ and let $1 \leq k < \ell \leq n$ be integers. Then $\gamma_{tk_{11}}(G) = \gamma_{t\ell_{11}}(G)$ if and only if there exists a partition $(P, M)$ of $V$ for which

1. $|N(x) \cap P| - |N(x) \cap M| \geq 1$ for at least $\ell$ of the vertices of $G$, and

2. for any $P' \subseteq P$ and any $M' \subseteq M$ satisfying $|P'| > |M'|$, we have

   $$|\{x \in V \mid 2(|N(x) \cap P' - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|\}| > n - k.$$

**Proof.** Suppose $\gamma_{tk_{11}}(G) = \gamma_{t\ell_{11}}(G)$. Let $f$ be a $TlSF$ of $G$ such that $f(V) = \gamma_{tk_{11}}(G) = \gamma_{t\ell_{11}}(G)$. Let $P = \{x \in V \mid f(x) = 1\}$ and $M = \{x \in V \mid f(x) = -1\}$. Then $(P, M)$ constitutes a partition of $V$. For each $x \in C_f$, we have $f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1$. Since $|C_f| \geq \ell$, Condition (1) holds.

To verify that Condition (2) holds, consider any $P' \subseteq P$ and $M' \subseteq M$ such that $|P'| > |M'|$. Let $X = (P \setminus P') \cup M'$ and $Y = (M \setminus M') \cup P'$. Define a function
$g : V \to \{-1, 1\}$ as follows: $g(x) = 1$ for every $x \in X$ and $g(x) = -1$ for every $x \in Y$. Then $g(V) = |X| - |Y| = (|P| - |P'| + |M'|) - (|M| - |M'| + |P'|) = |P| - |M| - 2(|P'| - |M'|) < |P| - |M| = f(V) = \gamma_{tk}^{-11}(G)$. Thus, $g$ is not a $TkSF$ of $G$, and so $|C_g| < k$. Consequently,

$$|\{x \in V \mid g(x) \leq 0\}| = |V - C_g| = n - |C_g| > n - k. \quad (4.6)$$

Note that

$$g[x] = |N(x) \cap X| - |N(x) \cap Y|$$

$$= |N(x) \cap ((P \setminus P') \cup M')| - |N(x) \cap ((M \setminus M') \cup P')|$$

$$= |N(x) \cap (P \setminus P')| + |N(x) \cap M'| - |N(x) \cap (M \setminus M')|$$

$$- |N(x) \cap P'|$$

$$= |N(x) \cap P| - |N(x) \cap P'| + |N(x) \cap M'| - |N(x) \cap M|$$

$$+ |N(x) \cap M'| - |N(x) \cap P'|$$

$$= |N(x) \cap P| - |N(x) \cap M| - 2(|N(x) \cap P'| - |N(x) \cap M'|). \quad (4.7)$$

Combining (4.6) and (4.7), we obtain Condition 2.

For the sufficiency, suppose there is a partition $(P, M)$ of $V$ such that Conditions (1) and (2) hold. Define a function $f : V \to \{-1, 1\}$ as follows: $f(x) = 1$ for every $x \in X$ and $f(x) = -1$ for every $x \in M$. Then $f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1$ for at least $\ell$ vertices of $G$ (by Condition (1)). Thus, $f$ is a $T\ell SF$ of $G$, so that $\gamma_{tk}^{-11}(G) \leq |P| - |M|$. We now show that $\gamma_{tk}^{-11}(G) \geq |P| - |M|$: Suppose, to the contrary, $\gamma_{tk}^{-11}(G) < |P| - |M|$. Let $g$ be a $TkSF$ of $G$ such that $\gamma_{tk}^{-11}(G) = g(V)$. Let $X = \{x \in V \mid$
Chapter 4

TOTAL SIGNED
k-SUBDOMINATION NUMBERS

4.1 Introduction

In this chapter, we focus on total signed $k$-subdomination, or, as it will not cause any confusion, total $k$-subdomination. In Section 4.2, we give a lower bound on the total $k$-subdomination number in terms of the minimum degree, maximum degree and the order of the graph. A lower bound in terms of the degree sequence is also given. In Section 4.3, we compute the total $k$-subdomination number of a cycle. In Section 4.4, we present a characterization of graphs $G$ with equal total $k$-subdomination and total $\ell$-subdomination numbers. In the final section, we establish a sharp upper bound on the total $k$-subdomination number of a tree in terms of its order $n$ and $k$ where $1 \leq k < n$. Moreover, we characterize trees attaining these bounds for certain values of $k$. For this purpose, we first establish the total $k$-subdomination number of simple structures, including paths and spiders.
4.2 Lower bounds

Our aim in this section is to give lower bounds on the total \( k \)-subdomination number of a graph. We first establish such a lower bound in terms of its minimum degree, maximum degree and its order. The second lower bound is in terms of the degree sequence of the graph. We begin with the following observation. (For a vertex \( v \in V \), we denote \( f(N(v)) \) by \( f[v] \).)

**Observation 4.1** Let \( f \) be a \( T_kSF \) of \( G \) and let \( v \in C_f \). If \( \deg v \) is even, then \( f[v] \geq 2 \), while if \( \deg v \) is odd, then \( f[v] \geq 1 \).

**Theorem 4.2** Let \( G = (V, E) \) be a graph of order \( n \) with minimum degree \( \delta \) and maximum degree \( \Delta \). For \( 1 \leq k \leq n \), let \( f \) be a \( \gamma_{tk}^{-11}(G) \)-function, and let \( \ell \) denote the number of vertices with even degree in \( C_f \). Then,

\[
\gamma_{tk}^{-11}(G) \geq \frac{2k(1 + \Delta) + \delta n - 3n\Delta + 2\ell}{\Delta + \delta}.
\]

**Proof.** We consider the sum \( N = \sum \sum f(u) \), where the outer sum is over all \( v \in V \) and the inner sum is over all \( u \in N(v) \). This sum counts the value \( f(u) \) exactly \( \deg u \times \deg v \) times for each \( u \in V \), so \( N = \sum (\deg u) \cdot f(u) \), over all \( u \in V \). Let \( V_{\text{even}} \) denote the set of all vertices with even degree in \( C_f \). Then, by Observation 4.1, \( N = \sum f[v] \) over all \( v \in V \) satisfies
Let \( P \) and \( M \) be the sets of those vertices in \( G \) which are assigned under \( f \) the values +1 and -1, respectively. Then, \( \gamma_{tk}^{11}(G) = f(V) = |P| - |M| = n - 2|M| \). We now write \( V \) as the disjoint union of six sets. Let \( P = P_{\Delta} \cup P_{\delta} \cup P_{\lambda} \) where \( P_{\Delta} \) and \( P_{\delta} \) are sets of all vertices of \( P \) with degree equal to \( \Delta \) and \( \delta \), respectively, and \( P_{\lambda} \) contains all other vertices in \( P \), if any. Let \( M = M_{\Delta} \cup M_{\delta} \cup M_{\lambda} \) where \( M_{\Delta}, M_{\delta}, \) and \( M_{\lambda} \) are defined similarly. Further, for \( i \in \{\Delta, \delta, \lambda\} \), let \( V_i \) be defined by \( V_i = P_i \cup M_i \). Thus, \( n = |V_{\Delta}| + |V_{\delta}| + |V_{\lambda}| \).

If \( u \in V_{\lambda} \), then \( \delta + 1 \leq \deg u \leq \Delta - 1 \). Therefore, writing the sum in (4.1) as the sum of six summations and replacing \( f(u) \) with the corresponding value of 1 or -1 yields

\[
\sum_{u \in P_{\Delta}} \Delta + \sum_{z \in P_{\delta}} \delta + \sum_{x \in M_{\Delta}} (\Delta - 1) - \sum_{x \in M_{\delta}} \delta - \sum_{x \in M_{\lambda}} (\delta + 1) \geq \ell + k(1 + \Delta) - n\Delta.
\]

Replacing \( |P_i| \) with \( |V_i| - |M_i| \) for \( i \in \{\Delta, \delta, \lambda\} \), yields

\[
\Delta|V_{\Delta}| + \delta|V_{\delta}| + (\Delta - 1)|V_{\lambda}| - 2\delta|M_{\delta}| - (\Delta + \delta)|M_{\lambda}| \geq \ell + k(1 + \Delta) - n\Delta. \tag{4.2}
\]
We now simplify the left hand side of (4.2) as follows. Replacing $|V_\delta|$ with $|P_\delta| + |M_\delta|$, and $|M_\delta| + |M_\lambda|$ with $|M| - |M_\Delta|$, we have
\[ \delta|V_\delta| - 2\delta|M_\delta| - \delta|M_\lambda| = \delta|P_\delta| - \delta|M_\delta| - \delta|M_\lambda| = \delta|P_\delta| - \delta|M| + \delta|M_\Delta|. \quad (4.3) \]
Further, replacing $|V_\Delta|$ with $n - |V_\delta| - |V_\lambda|$, we have
\[
\Delta|V_\Delta| + \Delta|V_\lambda| - 2\Delta|M_\Delta| - \Delta|M_\lambda| \\
= n\Delta - \Delta|V_\delta| - 2\Delta|M_\Delta| - \Delta|M_\lambda| \\
= n\Delta - \Delta|P_\delta| - \Delta|M| - \Delta|M_\Delta|. \quad (4.4)
\]
Using (4.3) and (4.4), the left hand side of (4.2) can be written as
\[ n\Delta - |V_\lambda| - (\Delta - \delta)|P_\delta| - (\Delta + \delta)|M| - (\Delta - \delta)|M_\Delta|. \]
Thus (4.2) becomes
\[
2n\Delta - k(1 + \Delta) - \ell \geq |V_\lambda| + (\Delta - \delta)|P_\delta| + (\Delta + \delta)|M| + (\Delta - \delta)|M_\Delta| \\
\geq (\Delta + \delta)|M|. \quad (4.5)
\]
Hence, since $\gamma_{tk}^{-11}(G) = n - 2|M|$, it follows from (4.5) that
\[
\gamma_{tk}^{-11}(G) \geq n - 2\left(\frac{2n\Delta - k(1 + \Delta) - \ell}{\Delta + \delta}\right) = \frac{2k(1 + \Delta) + \delta n - 3n\Delta + 2\ell}{\Delta + \delta},
\]
as desired. ∘

The next result gives a lower bound on the total $k$-subdomination number of a graph in terms of its degree sequence.
Theorem 4.3 Let $G = (V, E)$ be a graph of order $n$ where the degrees $d_i$ of vertices $v_i$ satisfy $d_1 \leq d_2 \leq \cdots \leq d_n$, let $f$ be a $\gamma_{tk}$-function, and let $\ell$ denote the number of vertices of even degree in $C_f$. Then,

$$\gamma_{tk}^{-11}(G) \geq \left( \frac{\ell + k + \sum_{i=1}^{k} d_i}{d_n} \right) - n.$$ 

Proof. Let $f$ be a $\gamma_{tk}^{-11}(G)$-function. Let $V_{\text{even}}$ denote the set of all vertices with even degree in $C_f$. Let $g : V \to \{0, 1\}$ be the function defined by $g(v) = (f(v) + 1)/2$ for all vertices $v \in V$. We consider the sum $N = \sum_{v \in V} g(u)$, where the outer sum is over all $v \in C_f$ and the inner sum is over all $u \in V(v)$. Then,

$$N = \sum_{v \in C_f} \sum_{u \in N(v)} \frac{1}{2} (f(u) + 1) = \sum_{v \in C_f} \frac{1}{2} (f[v] + \deg v) = \frac{1}{2} \left( \sum_{v \in C_f} f[v] + \sum_{v \in C_f} \deg v \right)$$

$$\geq \frac{1}{2} \left( \sum_{i=1}^{k} d_i + \sum_{v \in V_{\text{even}}} \deg v + \sum_{v \in C_f - V_{\text{even}}} \deg v \right) \geq \frac{1}{2} (2\ell + |C_f| - \ell + \sum_{i=1}^{k} d_i)$$

$$\geq \frac{1}{2} (\ell + k + \sum_{i=1}^{k} d_i).$$

On the other hand,

$$N \leq \sum_{v \in V} \sum_{u \in N(v)} g(u) = \sum_{v \in V} (\deg v) \cdot g(v) \leq d_n g(V),$$

and so

$$g(V) \geq \frac{(\ell + k + \sum_{i=1}^{k} d_i)}{2d_n}.$$ 

The desired result now follows since $\gamma_{tk}^{-11}(G) = f(V) = 2g(V) - n$. ◇
As an immediate consequence of Theorem 4.2 or Theorem 4.3, we have the following result.

**Corollary 4.4** For \( r \geq 1 \), if \( G \) is an \( r \)-regular graph of order \( n \), then

\[
\gamma_{tks}^{-11}(G) \geq \begin{cases} 
  k \left( \frac{r + 1}{r} \right) - n & \text{if } r \text{ is odd} \\
  k \left( \frac{r + 2}{r} \right) - n & \text{if } r \text{ is even}
\end{cases}
\]

**Corollary 4.5** If \( G \) is a graph of order \( n \), size \( m \) and maximum degree \( \Delta \), then

\[
\gamma_{tks}^{-11}(G) \geq k - 2n - \frac{k + 2m}{\Delta}.
\]

**Proof.** Let the degrees \( d_i \) of the vertices of \( G \) satisfy \( d_1 \leq d_2 \leq \cdots \leq d_n = \Delta \). It follows from Theorem 4.3 that

\[
\gamma_{tks}^{-11}(G) \geq \frac{1}{\Delta} \left( k + \sum_{i=1}^{k} d_i \right) - n
\]

\[
= \frac{1}{\Delta} \left( k + 2m - \sum_{i=k+1}^{n} d_i \right) - n
\]

\[
\geq \frac{1}{\Delta} \left( k + 2m - (n-k)\Delta \right) - n
\]

\[
= k - 2n + \frac{k + 2m}{\Delta}.
\]

\( \diamond \)
4.3 Cycles

Our aim in this section is to determine the total $k$-subdomination number of a cycle. As a special case of Corollary 4.4, we have that $\gamma_{\text{tk}}^{-11}(C_n) \geq 2k - n$. If $k \in \{n/2, n\}$, we show this lower bound is sharp. We shall prove:

**Proposition 4.6** For $n \geq 3$ and $1 \leq k \leq n$,

$$\gamma_{\text{tk}}^{-11}(C_n) = \begin{cases} 2k - n & \text{if } k \in \{n/2, n\} \\ 2k + 2 - n & \text{otherwise.} \end{cases}$$

**Proof.** We show first that $\gamma_{\text{tk}}^{-11}(C_n) \geq 2k + 2 - n$ except when $k = n/2$ or $k = n$, in which case $\gamma_{\text{tk}}^{-11}(C_n) = 2k - n$. Let $f$ be a $\gamma_{\text{tk}}^{-11}(C_n)$-function. Let $M = \{v \in V(C_n) \mid f(v) = -1\}$ and $P = \{v \in V(C_n) \mid f(v) = +1\}$. Note that, since $k \geq 1$, $P \neq \emptyset$. Let $M_c = C_f \cap M$, $P_c = C_f \cap P$, $M_{uc} = M - M_c$ and $P_{uc} = P - P_c$. Let $H = G[M_c \cup P]$, i.e., $H$ is the subgraph of $G$ induced by $M_c \cup P$ where $G = C_n$. The two vertices adjacent to a vertex in $M_c$ are in $P_{uc}$, while the two vertices adjacent to a vertex in $P_c$ are in $P$. It follows that

$$2m(G[P]) = \sum_{v \in P} \deg_{G[P]}(v) \geq \sum_{v \in P_c} \deg_{G[P]}(v) = 2|P_c|,$$

whence $m(G[P]) \geq |P_c|$. Thus $m(H) = 2|M_c| + m(G[P]) \geq 2|M_c| + |P_c|$. Further if $m(G[P]) = |P_c|$, then $\deg_{G[P]}(v) = 0$ for all $v \in P_{uc}$ and, since $C_n$ is connected and none of the vertices in $P_c$ are adjacent to any of the vertices of $M \cup P_{uc}$, either $V = P_c$ or $P_c = \emptyset$. So, if $m(G[P]) = |P_c|$, either $V = P_c$ or $P = P_{uc}$ and $m(G[P]) = 0$.

**Case 1.** $M_{uc} = \emptyset$. Then $H \cong C_n$, so $|M_c| + |P| = m(H) \geq 2|M_c| + |P_c|$. Thus, $|P| \geq |M_c| + |P_c| = |C_f| \geq k$ and so $\gamma_{\text{tk}}^{-11}(C_n) \geq 2k - n$. If we have strict
inequality in any of the above inequalities or if $|C_f| \geq k + 1$, then $|P| \geq k + 1$, and $\gamma_{tk}^{11}(C_n) = 2|P| - n \geq 2(k + 1) - n = 2k + 2 - n$. Hence, suppose we have equality throughout in the above inequalities and $|M_c| + |P_c| = k$. Then, by our remarks above, either $V = P_c$, in which case $|P_c| = k = n$, or $P_c = \emptyset$, in which case $|M_c| = k$ and $n = m(H) = 2|M_c| = 2k$ and so $k = n/2$.

Case 2. $M_{uc} \neq \emptyset$. In this case $H$ consists of a disjoint union of $\ell \geq 1$ paths. Then, $|M_c| + |P| - \ell = m(H) \geq 2|M_c| + |P_c|$. Thus, $|P| \geq |M_c| + |P_c| + \ell \geq |C_f| + 1 \geq k + 1$, and so $\gamma_{tk}^{11}(C_n) \geq 2(k + 1) - n = 2k + 2 - n$.

We have shown that $\gamma_{tk}^{11}(C_n) \geq 2(k + 1) - n = 2k + 2 - n$ except when $k = n/2$ or $k = n$, in which case $\gamma_{tk}^{11}(C_n) \geq 2k - n$. We now show that $\gamma_{tk}^{11}(C_n) \leq 2k - n$ if $k = n/2$ or $k = n$ and that $\gamma_{tk}^{11}(C_n) \leq 2k + 2 - n$ otherwise. For this purpose, we denote the vertex set of the cycle $C_n$ by $\{0, 1, \ldots, n-1\}$. We now define a function $f(V(C_n)) \rightarrow \{-1, 1\}$ as follows:

For $1 \leq k < n/2$, let $f(v_i) = 1$ if $i \in \{0, 2, \ldots, 2k\}$ and $f(v_i) = -1$ otherwise. Then, $f(V) = 2(k + 1) - n$, and $\{v_1, v_3, \ldots, v_{2k-1}\} \subseteq C_f$, so that $|C_f| \geq k$.

For $k = n/2$, let $f(v_i) = 1$ if $i$ is even and $f(v_i) = -1$ otherwise. Then, $f(V) = 0$ and $\{v_1, v_3, \ldots, v_{n-1}\} \subseteq C_f$, so that $|C_f| \geq k$.

For $(n + 2)/2 \leq k \leq n - 1$ and $n$ even, let $f(v_i) = 1$ if $i$ is even or $i \in \{1, 3, \ldots, 2k - n + 1\}$ and $f(v_i) = -1$ otherwise. Then, $f(V) = 2|P| - n = 2k - n + 2$, and $\{v_1, v_3, \ldots, v_{n-1}\} \cup \{v_2, v_4, \ldots, v_{2k-n}\} \subseteq C_f$ so that $|C_f| \geq n/2 + (k - n/2) = k$.

For $(n + 1)/2 \leq k \leq n - 1$ and $n$ odd, let $f(v_i) = 1$ if $i$ is even or $i \in \{1, 3, \ldots, 2k - n\}$ and $f(v_i) = -1$ otherwise. Then, $f(V) = 2|P| - n = 2k - n + 2$, and $\{v_1, v_3, \ldots, v_{n-2}\} \cup \{v_0, v_2, \ldots, v_{2k-n-1}\} \subseteq C_f$ so that $|C_f| \geq (n - 1)/2 + (2k - n + 1)/2 = k$. 
For \( k = n \), the function that assigns 1 to every vertex of the cycle is the desired function.

In all the above cases, \( f \) is a \( T_kSF \) of \( C_n \). Thus, \( \gamma^{-11}_{tks}(C_n) \leq f(V) = 2k - n \) if \( k = n/2 \) or \( k = n \), while \( \gamma^{-11}_{tks}(C_n) \leq f(V) = 2k + 2 - n \) otherwise. ◊

### 4.4 Graphs with equal total \( k \)- and \( \ell \)-subdomination numbers

Our aim in this section is to give a characterization of graphs \( G \) with equal total \( k \)-subdomination and total \( \ell \)-subdomination numbers where \( 1 \leq k < \ell \leq |V(G)| \).

Our proof is along similar lines to that presented in [34].

**Theorem 4.7** Let \( G = (V, E) \) be a graph of order \( n \) and let \( 1 \leq k < \ell \leq n \) be integers. Then \( \gamma^{-11}_{tks}(G) = \gamma^{-11}_{tls}(G) \) if and only if there exists a partition \((P, M)\) of \( V \) for which

1. \(|N(x) \cap P| - |N(x) \cap M| \geq 1\) for at least \( \ell \) of the vertices of \( G \), and

2. for any \( P' \subseteq P \) and any \( M' \subseteq M \) satisfying \(|P'| > |M'|\), we have

\[ |\{x \in V \mid 2(|N(x) \cap P' - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|\}| \geq n - k. \]

**Proof.** Suppose \( \gamma^{-11}_{tks}(G) = \gamma^{-11}_{tls}(G) \). Let \( f \) be a \( TlSF \) of \( G \) such that \( f(V) = \gamma^{-11}_{tks}(G) = \gamma^{-11}_{tls}(G) \). Let \( P = \{x \in V \mid f(x) = 1\} \) and \( M = \{x \in V \mid f(x) = -1\} \).

Then \((P, M)\) constitutes a partition of \( V \). For each \( x \in C_f \), we have \( f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1 \). Since \( |C_f| \geq \ell \), Condition (1) holds.

To verify that Condition (2) holds, consider any \( P' \subseteq P \) and \( M' \subseteq M \) such that \(|P'| > |M'|\). Let \( X = (P \setminus P') \cup M' \) and \( Y = (M \setminus M') \cup P' \). Define a function
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9 : V \to \{-1, 1\} as follows: \(g(x) = 1\) for every \(x \in X\) and \(g(x) = -1\) for every \(x \in Y\). Then \(g(V) = |X| - |Y| = (|P| - |P'| + |M'|) - (|M| - |M'| + |P'|) = |P| - |M| - 2(|P'| - |M'|) < |P| - |M| = f(V) = \gamma_{l_{k+1}}(G)\). Thus, \(g\) is not a \(TkSF\) of \(G\), and so \(|C_g| < k\). Consequently,

\[|\{x \in V \mid g(x) \leq 0\}| = |V - C_g| = n - |C_g| > n - k. \quad (4.6)\]

Note that

\[
g[x] = |N(x) \cap X| - |N(x) \cap Y| \\
= |N(x) \cap (P \setminus P') \cup M'| - |N(x) \cap (M \setminus M') \cup P'| \\
= |N(x) \cap (P \setminus P')| + |N(x) \cap M'| - |N(x) \cap (M \setminus M')| \\
- |N(x) \cap P'| \\
= |N(x) \cap P| - |N(x) \cap P'| + |N(x) \cap M'| - |N(x) \cap M| \\
+ |N(x) \cap M'| - |N(x) \cap P'| \\
= |N(x) \cap P| - |N(x) \cap M| - 2(|N(x) \cap P'| - |N(x) \cap M'|). \quad (4.7)\]

Combining (4.6) and (4.7), we obtain Condition 2.

For the sufficiency, suppose there is a partition \((P, M)\) of \(V\) such that Conditions (1) and (2) hold. Define a function \(f : V \to \{-1, 1\}\) as follows: \(f(x) = 1\) for every \(x \in P\) and \(f(x) = -1\) for every \(x \in M\). Then \(f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1\) for at least \(\ell\) vertices of \(G\) (by Condition (1)). Thus, \(f\) is a \(T\ell SF\) of \(G\), so that \(\gamma_{l_{k+1}}(G) \leq |P| - |M|\).

We now show that \(\gamma_{l_{k+1}}(G) \geq |P| - |M|\): Suppose, to the contrary, \(\gamma_{l_{k+1}}(G) < |P| - |M|\). Let \(g\) be a \(TkSF\) of \(G\) such that \(\gamma_{l_{k+1}}(G) = g(V)\). Let \(X = \{x \in V \mid g(x) = -1\} = X_g\). We have

\[\gamma_{l_{k+1}}(G) = g(V) = |X = X_g| = |X| = |X - X_g| = n - |X_g| > n - k. \quad (4.6)\]
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\[ g(x) = 1 \] and \( Y = \{ x \in V \mid g(x) = -1 \}. \] Let \( P' = P \setminus X \) and \( M' = M \setminus Y \). Then \( P' \subseteq P, M' \subseteq M, X = (P \setminus P') \cup M' \) and \( Y = (M \setminus M') \cup P' \). Moreover, \( |P| - |M| + 2(|M'| - |P'|) = |P| - |P'| + |M'| - |M| + |M'| - |P'| = |X| - |Y| = \gamma^{\text{ts}}_{k}(G) < |P| - |M| \), so that \( |P'| > |M'| \). By Condition (2), \( |V - C_g| = |\{ x \in V \mid g(x) \leq 0 \}| = |\{ x \in V \mid 2(|N(x) \cap P'| - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|} \mid > n - k \). Thus, \( |C_g| < k \), contradicting the fact that \( g \) is TkSF of \( G \). Hence, \( \gamma^{\text{ts}}_{k}(G) \geq |P| - |M| \).

We conclude that \( |P| - |M| \leq \gamma^{\text{ts}}_{k}(G) \leq \gamma^{\text{ts}}_{k}(G) \leq |P| - |M| \), so that \( \gamma^{\text{ts}}_{k}(G) = \gamma^{\text{ts}}_{k}(G) \).

\[ \square \]

**Theorem 4.8** Let \( G = (V, E) \) be a graph of order \( n \) and let \( 1 \leq k \leq n \) be integers. Then \( \gamma^{\text{ts}}_{k}(G) = a \) if and only if there exists a partition \((P, M)\) of \( V \) for which

1. \( |N(x) \cap P| - |N(x) \cap M| \geq 1 \) for at least \( k \) of the vertices of \( G \),

2. \( |P| - |M| = a \), and

3. for any \( P' \subseteq P \) and any \( M' \subseteq M \) satisfying \( |P'| > |M'| \), we have

\[ |\{ x \in V \mid 2(|N(x) \cap P'| - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|} \mid > n - k \).

**Proof.** Suppose \( \gamma^{\text{ts}}_{k}(G) = a \). Let \( f \) be a TkSF of \( G \) such that \( f(V) = \gamma^{\text{ts}}_{k}(G) = a \). Let \( P = \{ x \in V \mid f(x) = 1 \} \) and \( M = \{ x \in V \mid f(x) = -1 \} \). Conditions (1) and (3) follows as in the proof of Theorem 4.7. Moreover, \( f(V) = |P| - |M| \), so Condition (2) holds.

For the sufficiency, suppose there is a partition \((P, M)\) of \( V \) such that Conditions (1), (2) and (3) hold. Define a function \( f: V \to \{-1, 1\} \) as follows: \( f(x) = 1 \) for every \( x \in P \) and \( f(x) = -1 \) for every \( x \in M \). Then \( f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1 \) for at least \( k \) vertices of \( G \) (by Condition (1)). Thus, \( f \) is a TkSF of \( G \), so that \( \gamma^{\text{ts}}_{k}(G) \leq |P| - |M| = a \) (by Condition (2)). As in the proof of Theorem 4.7,
\[ \gamma_{ks}(G) \geq |P| - |M|. \] Hence, \(|P| - |M| \leq \gamma_{ks}(G) \leq |P| - |M| = a\), so that \(\gamma_{ks}(G) = a\). ∎

### 4.5 Trees

An opinion function on \(G\) is a function \(f: V \rightarrow \{-1, +1\}\); \(f(v)\) is the opinion of the vertex \(v\). For an opinion function \(f\), we say that a vertex \(v\) votes aye if \(f(v) \geq 1\) and nay otherwise. Thus, \(\gamma_{tks}^{-11}(G) = \min\{f(V) | f\) is an opinion function of \(G\) in which at least \(k\) vertices vote aye \}. 

By giving a positive opinion to the center of a star of order \(n \geq 3\) and negative opinions to all the leaves we obtain a \(TkSF\) of the star. Thus

**Proposition 4.9** For \(n \geq 3\) and \(1 \leq k < n\), \(\gamma_{tks}^{-11}(K_{1,n-1}) = 2 - n\).

Hence the total \(k\)-subdomination number of a tree can be arbitrarily large negative if \(k\) is less than the order of the tree.

When \(k = n\), the total \(k\)-subdomination number is the total signed domination number. In [27], lower and upper bounds on the total signed domination number of a tree in terms of its order are given and the trees attaining these bounds are characterized.

**Theorem 4.10** [27] If \(T\) is a tree of order \(n \geq 2\), then

\[ 2 \leq \gamma_{tns}^{-11}(T) \leq n. \]

Furthermore, \(\gamma_{tns}^{-11}(T) = 2\) if and only if every vertex \(v \in V(T) - L(T)\) has odd degree and is adjacent to at least \((\deg v - 1)/2\) leaves, while \(\gamma_{tns}^{-11}(T) = n\) if and only if every vertex of \(T\) is a support vertex or is adjacent to a vertex of degree \(2\).
Our aim in this section is to establish a sharp upper bound on the total \( k \)-subdomination number of a tree in terms of its order \( n \) and \( k \) when \( 1 \leq k < n \), and to characterize trees attaining these bounds for certain values of \( k \). For this purpose, we first establish the total \( k \)-subdomination number of simple structures, including paths and spiders.

### 4.5.1 Paths

In this subsection, we establish the total \( k \)-subdomination number of a path. We begin with the following lemma.

**Lemma 4.11** For \( n \geq 3 \) and \( 1 \leq k < n \), there exists a \( \gamma_{tk}^{-11}(P_n) \)-function that assigns to one of its leaves a negative opinion and to its neighbor a positive opinion.

**Proof.** Let \( T \) be the path \( v_1, v_2, \ldots, v_n \) and let \( f \) be a \( \gamma_{tk}^{-11}(T) \)-function. Let \( i \) be the smallest subscript such that \( f(v_i) = -1 \). If \( i \geq 2 \), then the function obtained from \( f \) by interchanging the values of \( v_1 \) and \( v_i \) is an opinion function having the same weight as \( f \) and with at least as many vertices voting aye as under \( f \). Hence, we can choose \( f \) so that \( f(v_1) = -1 \). Now let \( j \) be the smallest subscript such that \( f(v_j) = 1 \). If \( j \geq 3 \), then the function obtained from \( f \) by interchanging the values of \( v_2 \) and \( v_j \) is an opinion function having the same weight as \( f \) and with at least as many vertices voting aye as under \( f \). Hence, we can choose \( f \) so that \( f(v_2) = 1 \). □

**Proposition 4.12** For \( n \geq 2 \) and \( 1 \leq k \leq n \),

\[
\gamma_{tk}^{-11}(P_n) = \begin{cases} 
-1 & \text{if } k = \frac{1}{2}(n + 1) \\
2k - n & \text{otherwise.}
\end{cases}
\]
Proof. We proceed by induction on the order $n \geq 2$ of a path $P_n$. If $n = 2$, then $\gamma_{tks}^{-11}(P_2) = 2k - n$ for $k = 1$ or $k = 2$. Suppose $n = 3$. If $k = 3$, then $\gamma_{tks}^{-11}(P_3) = 3 = 2k - n$, while for $1 \leq k \leq 2$, $\gamma_{tks}^{-11}(P_3) = -1$ by Proposition 4.9 and the desired result follows. This proves the base cases when $n = 2$ or $n = 3$.

Suppose that $n \geq 4$ and that for every nontrivial path $P_{n'}$ of order $n' < n$, and any integer $k'$ with $1 \leq k' \leq n'$, $\gamma_{tks}^{-11}(P_{n'}) \leq -1$ if $k' = (n' + 1)/2$ and $\gamma_{tks}^{-11}(P_{n'}) \leq 2k' - n'$ otherwise. Let $T$ be a path $P_n$ of order $n$. Let $u$ be a leaf of $T$ and let $v$ be the vertex adjacent to $u$.

If $k = 1$, then giving a positive opinion to $v$ and negative opinions to all other vertices of $T$ we obtain a $TkSF$ of $T$ of weight $2 - n$. Since $\gamma_{tks}^{-11}(G) \geq 2 - n$ for all graphs $G$ with no isolated vertex, $\gamma_{tks}^{-11}(P_n) = 2 - n = 2k - n$. Hence we may assume $k \geq 2$. Furthermore, if $k = n$, then the result follows from Theorem 4.10. Hence we may assume that $k < n$. Let $T' = T - u - v$. Then, $T'$ is a path of order $n' = n - 2$.

Let $k' = k - 1$. Since $2 \leq k \leq n - 1$, it follows that $1 \leq k' \leq n'$. Let $f'$ be a $\gamma_{tks}^{-11}(T')$-function. Let $f : V(T) \to \{-1, 1\}$ be the function defined by $f(w) = f'(w)$ if $w \in V(T')$, $f(v) = 1$ and $f(u) = -1$. Every vertex that votes aye in $T'$ also votes aye in $T$, while $u$ votes aye in $T$. Hence at least $k' + 1 = k$ vertices of $T$ vote aye, and so $f$ is a $TkSF$ of $T$. Thus, $\gamma_{tks}^{-11}(T) \leq f(V(T)) = f'(V(T')) = \gamma_{tks}^{-11}(T')$. On the other hand, by Lemma 4.11 there exists a $\gamma_{tks}^{-11}(T)$-function $g$ that assigns to $u$ a negative opinion and to $v$ a positive opinion. Let $g'$ be the restriction of $g$ to $V(T')$. Then, $g'$ is a $Tk'sF$ of $T'$. Thus, $\gamma_{tks}^{-11}(T') \leq g'(V(T')) = g(V(T)) = \gamma_{tks}^{-11}(T)$. Consequently, $\gamma_{tks}^{-11}(T) = \gamma_{tks}^{-11}(T')$.

Suppose $k' = (n' + 1)/2$. Then, $k = (n + 1)/2$ and by the inductive hypothesis, $f'(V(T')) = -1$, and so $\gamma_{tks}^{-11}(T) = f'(V(T')) = -1$. Suppose $k' \neq (n' + 1)/2$. Then, $k \neq (n + 1)/2$ and by the inductive hypothesis, $f'(V(T')) = 2k' - n' = 2k - n$, and so $\gamma_{tks}^{-11}(T) = f'(V(T')) = 2k - n$. \(\Diamond\)
4.5.2 Spiders

In this subsection, we establish the total \( k \)-subdomination number of an even spider. We begin with the following lemma.

**Lemma 4.13** Let \( T \) be an even spider of (odd) order \( n \geq 3 \). For \( \frac{n+1}{2} \leq k < n \), there exists a \( \gamma^{s-k}_{tks}(T) \)-function that assigns to a leaf at maximum distance from the head of \( T \) a negative opinion and to its neighbor a positive opinion.

**Proof.** If \( T \) is a path, then the result follows from Lemma 4.11. Hence we may assume that \( \Delta(T) \geq 3 \). Let \( v \) be the head of the spider and let \( f \) be a \( \gamma^{s-k}_{tks}(T) \)-function. Let \( x \) be a leaf at maximum distance from \( v \). For two distinct vertices \( a \) and \( b \) of \( T \), we denote by \( f_{a,b} \) the function obtained from \( f \) by interchanging the values of \( a \) and \( b \) and leaving the values of all other vertices unchanged.

Suppose \( f(x) = 1 \). We construct a new opinion function \( g \) having the same weight as \( f \) and with at least as many vertices voting aye as under \( f \) but with \( g(x) = -1 \). If some vertex on the \( v-x \) path has a negative opinion, then let \( w \) be such a vertex at maximum distance from \( v \) (possibly, \( v = w \)) and take \( g = f_{w,x} \). On the other hand, suppose every vertex on the \( v-x \) path has a positive opinion. Since \( k < n \), at least one vertex of \( T \) has a negative opinion. Hence there exists a leaf \( z \) of \( T \) such that the \( v-z \) path contains at least one vertex with a negative opinion. Let \( v, v_1, v_2, \ldots, v_{2r} = z \) denote the \( v-z \) path. If \( f(v_i) = -1 \) for some \( i \) with \( i \) odd, then take \( g = f_{v_j,x} \) where \( j \) is the largest odd integer such that \( f(v_j) = -1 \). On the other hand, if \( f(v_i) = 1 \) for all odd \( i \), then take \( g = f_{v_i,x} \) where \( i \) is the smallest (even) integer such that \( f(v_i) = -1 \). Hence we can choose \( f \) so that \( f(x) = -1 \).

Let \( w \) be the vertex adjacent to \( x \). Suppose \( f(w) = -1 \). We construct a new opinion function \( h \) having the same weight as \( f \) and with at least as many vertices voting aye as under \( h \) but with \( h(w) = 1 \) and \( h(x) = -1 \). If every vertex different from \( v \) that is at even distance from \( v \) has a negative opinion, then every vertex at odd distance
from $v$ votes nay. Further, since $x$ votes nay, this would imply that $k \leq (n - 1)/2$, a contradiction. Hence $f(d) = 1$ for at least one vertex $d \neq v$ at even distance from $v$. We can now take $h = f_{d,w}$ where $d$ is a vertex at maximum even distance from $v$ with $f(d) = 1$. Hence we can choose $f$ so that $f(w) = 1$ and $f(x) = -1$, as desired. ♦

**Proposition 4.14** Let $T$ be an even spider of (odd) order $n \geq 3$. For $(n + 1)/2 \leq k \leq n$,

$$\gamma^{-11}_{tks}(T) = \begin{cases} -1 & \text{if } k = \frac{1}{2}(n + 1) \\ 2k - n & \text{otherwise}, \end{cases}$$

**Proof.** We proceed by induction on the order $n$ of the even spider $T$. If $T$ is a path, then the desired result follows from Proposition 4.12. In particular, this proves the base cases when $n = 3$ and $n = 5$. Suppose that $n \geq 7$ (and so, $k \geq 4$) and that for every even spider $T'$ of (odd) order $n' < n$, and any integer $k'$ with $(n' + 1)/2 \leq k' \leq n'$, $\gamma^{-11}_{tks}(T') = -1$ if $k' = (n' + 1)/2$ and $\gamma^{-11}_{tks}(T') = 2k' - n'$ otherwise. Let $T$ be an even spider of order $n$. We may assume $\Delta(T) \geq 3$, for otherwise the result follows from Proposition 4.12. Let $v$ be the head of $T$. Let $x$ be a leaf at maximum distance from $v$ in $T$ and let $w$ be the support vertex adjacent to $x$.

If $k = n$, then the result follows from Theorem 4.10. Hence we may assume that $k \leq n - 1$. Let $T' = T - w - x$. Then, $T'$ is an even spider of order $n' = n - 2$. Let $k' = k - 1$, and so $(n' + 1)/2 \leq k' \leq n'$.

Any $Tk'sF$ of $T'$ can be extended to a $TkSF$ of $T$ by assigning a positive opinion to the vertex $w$ and a negative opinion to the vertex $x$. It follows that $\gamma^{-11}_{tks}(T) \leq \gamma^{-11}_{tks}(T')$. On the other hand, by Lemma 4.13 there exists a $\gamma^{-11}_{tks}(T)$-function $g$ that assigns to $x$ a negative opinion and to $w$ a positive opinion. Let $g'$
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be the restriction of \( g \) to \( V(T') \). If \( d(v, x) \geq 4 \), then \( g' \) is a \( T_k'sF \) of \( T' \). Suppose \( d(v, x) = 2 \). Then, \( T \) is an even spider with every leg of length 2. Since \( k \geq (n+1)/2 \), we can choose \( g \) so that every neighbor of \( v \) has a positive opinion and still \( g(x) = -1 \) (if some neighbor \( a \) of \( v \) has a negative opinion, then there must be a leaf \( b \) with a positive opinion, and we can simply take \( g^* = g_{a,b} \)). Hence, once again \( g' \) is a \( T_k'sF \) of \( T' \). Thus, \( \gamma_{T_k'sF}^{-11}(T') \leq g'(V(T')) = g(V(T)) = \gamma_{T_k'sF}^{-11}(T) \). Consequently, \( \gamma_{T_k'sF}^{-11}(T) = \gamma_{T_k'sF}^{-11}(T') \).

Suppose \( k' = (n' + 1)/2 \). Then, \( k = (n + 1)/2 \) and by the inductive hypothesis, \( \gamma_{T_k'sF}^{-11}(T') = -1 \), and so \( \gamma_{T_k'sF}^{-11}(T) = -1 \). Suppose \( k' \geq (n' + 3)/2 \). Then, \( k \geq (n + 3)/2 \) and by the inductive hypothesis, \( \gamma_{T_k'sF}^{-11}(T') = 2k' - n' = 2k - n \), and so \( \gamma_{T_k'sF}^{-11}(T) = 2k - n \).

4.5.3 Upper Bounds

We now present an upper bound on the total \( k \)-subdomination number of a tree in terms of its order and \( k \).

**Theorem 4.15** For any tree \( T \) of order \( n \geq 2 \), and any integer \( k \) with \( 1 \leq k \leq n \),

\[
\gamma_{T_k'sF}^{-11}(T) \leq \begin{cases} 
-1 & \text{if } k = \frac{1}{2}(n + 1) \\
2k - n & \text{otherwise.}
\end{cases}
\]

and these bounds are sharp.

**Proof.** We proceed by induction on the order \( n \geq 2 \) of a tree \( T \). If \( n \in \{2, 3\} \), then \( T = P_n \) and the result follows from Proposition 4.12. This proves the base cases when \( n = 2 \) or \( n = 3 \).
Suppose that \( n \geq 4 \) and that for every nontrivial tree \( T' \) of order \( n' < n \), and any integer \( k' \) with \( 1 \leq k' \leq n' - 1 \), \( \gamma_{tk^s}(T') \leq -1 \) if \( k' = (n' + 1)/2 \) and \( \gamma_{tk^s}(T') \leq 2k' - n' \) otherwise. Let \( T \) be a tree of order \( n \).

If \( T \) is a star, then, by Proposition 4.9, \( \gamma_{tk^s}(T) = 2 - n < -1 \). Thus, \( \gamma_{tk^s}(T) = 2k - n \) if \( k = 1 \), while \( \gamma_{tk^s}(T) < 2k - n \) if \( 2 \leq k \leq n \). Hence the desired result follows if \( T \) is a star. Thus we may assume that \( \text{diam}(T) \geq 3 \).

If \( k = n \), then, by Theorem 4.10, \( \gamma_{tk^s}(T) \leq n = 2k - n \). Hence we may assume \( k < n \).

Let \( T \) be rooted at a leaf \( r \) of a longest path. Let \( v \) be a vertex at distance \( \text{diam}(T) - 1 \) from \( r \) on a longest path starting at \( r \), and let \( w \) be the parent of \( v \). Let \( |N(v) - \{w\}| = m \). Then, \( m \geq 1 \). If \( k \leq m \), then giving a positive opinion to \( v \) and negative opinions to all the other vertices we obtain a \( TkSF \) of \( T \) of weight \( 2 - n \), and the desired result follows. Hence we may assume \( k > m \).

Let \( T' = T - (N[v] - \{w\}) \). Then, \( T' \) has order \( n' = n - m - 1 \). Since \( \text{diam}(T) \geq 3 \), \( n' \geq 2 \). Let \( k' = k - m \). Since \( m + 1 \leq k \leq n - 1 \), we have \( 1 \leq k' \leq n' \).

Let \( f' \) be a \( \gamma_{tk^s}(T') \)-function. Let \( f : V(T) \to \{-1,1\} \) be the function defined by \( f(w) = f'(w) \) if \( w \in V(T') \), \( f(v) = 1 \) and \( f(u) = -1 \) for every child of \( v \). Every vertex that votes \( \text{aye} \) in \( T' \) also votes \( \text{aye} \) in \( T \), while each child of \( v \) votes \( \text{aye} \) in \( T \). Hence at least \( k' + m = k \) vertices of \( T \) vote \( \text{aye} \), and so \( f \) is a \( TkSF \) of \( T \). Thus, \( \gamma_{tk^s}(T) \leq f(V(T)) = f'(V(T')) + 1 - m \).

Suppose \( k' = (n' + 1)/2 \). Then, \( k = (n + m)/2 \). By the inductive hypothesis, \( \gamma_{tk^s}(T') \leq -1 \), and so \( \gamma_{tk^s}(T) \leq -m \). Thus if \( m = 1 \), then \( k = (n + 1)/2 \) and \( \gamma_{tk^s}(T) \leq -1 \), while if \( m \geq 2 \), then \( k \geq (n + 2)/2 \) and \( \gamma_{tk^s}(T) \leq -2 < 2 \leq 2k - n \).

In any event, the result follows.

On the other hand, suppose \( k' \neq (n' + 1)/2 \). By the inductive hypothesis, \( \gamma_{tk^s}(T') \leq 2k' - n' = 2k - n + 1 - m \), and so \( \gamma_{tk^s}(T) \leq 2k - n + 2(1 - m) \).
Suppose \( k = (n + 1)/2 \). Then, \( k' = (n' - m + 2)/2 \). Since \( k' \neq (n' + 1)/2 \), it follows that \( m \geq 2 \), and so \( \gamma_{\text{tsk}}^{-11}(T) \leq 2k - n + 2(1 - m) \leq -1 \). Suppose \( k \neq (n + 1)/2 \). Then, since \( m \geq 1 \), \( \gamma_{\text{tsk}}^{-11}(T) \leq 2k - n \). Once again, the desired result follows.

That the bounds are sharp, follows from Proposition 4.12.

As an immediate consequence of Theorem 4.8, we have the following result.

**Corollary 4.16** Let \( T = (V, E) \) be a tree of order \( n \) and let \( 1 \leq k \leq n \) be an integer. Then, \( \gamma_{\text{tsk}}^{-11}(T) = 2k - n \) if and only if there exists a partition \( (P, M) \) of \( V \) for which

1. \( |N(x) \cap P| - |N(x) \cap M| \geq 1 \) for at least \( k \) of the vertices of \( T \),
2. \( |P| - |M| = 2k - n \), and
3. for any \( P' \subseteq P \) and any \( M' \subseteq M \) satisfying \( |P'| > |M'| \), we have
   \[ |\{x \in V \mid 2(|N(x) \cap P' - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|\}| > n - k. \]

### 4.5.4 Extremal Trees

The trees of order \( n \) with maximum total \( k \)-subdomination number when \( k = n \) are characterized in Theorem 4.10. Our aim in this subsection is to characterize those trees of order \( n \) achieving the maximum possible total \( k \)-subdomination number when (i) \( n \) is even and \( n/2 \leq k \leq n/2 + 3 \) and when (ii) \( n \) is odd and \( (n + 3)/2 \leq k \leq (n + 5)/2 \), i.e., we characterize those trees achieving equality in the upper bounds in Theorem 4.15 for \( k = n/2 \) and \( n/2 + 1 \leq k \leq n/2 + 3 \). We first consider the case when \( n \) is even.

**Theorem 4.17** For any tree \( T \) of even order \( n \geq 2 \), and any integer \( k \) with \( n/2 \leq k \leq n/2 + 3 \), \( \gamma_{\text{tsk}}^{-11}(T) = 2k - n \) if and only if \( T \) is a path.
**Proof.** The sufficiency follows from Proposition 4.12. To prove the necessity, we proceed by induction on the order $n$ of a tree $T$, where $n \geq 2$ is even, $n/2 \leq k \leq n$, $k = n/2 + i$ where $0 \leq i \leq 3$, and $\gamma_{\text{ks}}^{-11}(T) = 2k - n$. If $n = 2$, then $T = P_2$. If $n = 4$, then $k \in \{2, 3, 4\}$. If $T = K_{1,3}$, then it follows from Theorem 4.10 (if $k = 4$) or Proposition 4.9 (if $k \in \{2, 3\}$) that $\gamma_{\text{ks}}^{-11}(T) < 2k - n$, a contradiction. Hence, $T = P_4$. Thus if $n \in \{2, 4\}$, then $T$ is a path. This proves the base cases when $n = 2$ and $n = 4$.

Suppose that $n \geq 4$ is even and that for every nontrivial tree $T'$ of even order $n' < n$, and any integer $k'$ with $n'/2 \leq k' \leq n'$ and $k' = n'/2 + i$ where $0 \leq i \leq 3$, that if $\gamma_{\text{ks}}^{-11}(T') = 2k' - n'$, then $T'$ is a path. Let $T$ be a tree of order $n$.

If $k = n$, then since $k \leq (n + 6)/2$, $n = k = 6$, and so it follows from Theorem 4.10 that $\gamma_{\text{ks}}^{-11}(T) \leq n < 2k - n$ unless $T = P_6$. Hence the desired result follows if $k = n$. Thus we may assume $k < n$. In particular, if $k = n/2 + 3$, then $n \geq 8$.

Following the notation used in paragraph 5 and 6 of the proof of Theorem 4.15, $\gamma_{\text{ks}}^{-11}(T) \leq f(V(T)) - f'(V(T')) + 1 - m$. If $k' = (n' + 1)/2$, then $k = (n + m)/2$.

By Theorem 4.15, $\gamma_{\text{ks}}^{-11}(T') \leq -1$, and so $\gamma_{\text{ks}}^{-11}(T) \leq -m$, a contradiction. Hence, $k' \neq (n' + 1)/2$. By Theorem 4.15, $\gamma_{\text{ks}}^{-11}(T') \leq 2k' - n' = 2k - n + 1 - m$, and so $\gamma_{\text{ks}}^{-11}(T) \leq 2k - n + 2(1 - m)$. If $m \geq 2$, then $\gamma_{\text{ks}}^{-11}(T) \leq 2k - n + 2(1 - m) \leq 2(k - 1) - n$, a contradiction. Hence, $m = 1$, and so $k' = k - m = k - 1$, $n' = n - 2$ and $\gamma_{\text{ks}}^{-11}(T) \leq f'(V(T')) = \gamma_{\text{ks}}^{-11}(T')$. Furthermore, $n'$ is even and $n'/2 \leq k' \leq n'$ and $k' = n'/2 + i$ where $0 \leq i \leq 3$.

By Theorem 4.15, $\gamma_{\text{ks}}^{-11}(T') \leq 2k' - n'$. If $\gamma_{\text{ks}}^{-11}(T') \leq 2(k' - 1) - n'$, then $\gamma_{\text{ks}}^{-11}(T) \leq 2(k - 1) - n$, a contradiction. Hence, $\gamma_{\text{ks}}^{-11}(T') = 2k' - n'$. Applying the inductive hypothesis to $T'$, $T'$ is a path. Let $u$ denote the child of $v$.

Suppose $w$ is neither a leaf nor a support vertex of $T'$. Let $v'$ be the child of $w$ different from $v$, and let $u'$ be the child of $v'$. Assign a positive opinion to $w$ and its two children and to all vertices of degree 2 at even distance from $w$. Assign a
negative opinion to all remaining vertices. For \( k = n/2 + 1 \) this is a \( TkSF \). If \( k = n/2 \), reassign to \( w \) a negative opinion. If \( k = n/2 + 2 \), reassign to \( u \) a positive opinion, while if \( k = n/2 + 3 \), reassign to each of \( u \) and \( u' \) a positive opinion. In all cases, this produces a \( TkSF \) of weight \( 2(k - 1) - n \), and so \( \gamma_{tk}^{-1}(T) \leq 2(k - 1) - n \), a contradiction. Hence, \( w \) is either a leaf or a support vertex of \( T' \).

Suppose that \( w \) is a support vertex of \( T' \). Let \( v' \) be the child of \( w \) different from \( v \). Assign a positive opinion to \( w \) and to all vertices different from \( v' \) whose distance from \( w \) in \( T \) is odd. Assign a negative opinion to all remaining vertices. For \( k = n/2 + 1 \) this is a \( TkSF \). If \( k = n/2 \), reassign to \( w \) a negative opinion. If \( k = n/2 + 2 \), reassign to \( u \) a positive opinion, while if \( k = n/2 + 3 \), reassign to each of the two vertices at distance 2 from \( w \) positive opinion. In all cases, this produces a \( TkSF \) of weight \( 2(k - 1) - n \), and so \( \gamma_{tk}^{-1}(T) \leq 2(k - 1) - n \), a contradiction. Thus, \( w \) is a leaf in \( T' \), whence \( T \) is a path of even order.

Theorem 4.17 is not true for \( n/2 + 4 \leq k \leq n - 1 \). For example, let \( \ell \geq 2 \) be an integer and let \( T \) be a spider of order \( n = 2(\ell + 3) \) with three legs, two of length 2 and one of length \( 2\ell + 1 \). Then for \( n/2 + 4 \leq k \leq n - 1 \), \( \gamma_{tk}^{-1}(T) = 2k - n \). Thus if \( \gamma_{tk}^{-1}(T) = 2k - n \) and \( n/2 + 4 \leq k \leq n - 1 \), then \( T \) is not necessarily a path.

Next we characterize those trees of order \( n \) achieving the maximum possible total \( k \)-subdomination number when \( n \) is odd and \( (n + 3)/2 \leq k \leq (n + 5)/2 \).

**Theorem 4.18** For any tree \( T \) of odd order \( n \geq 3 \), and for any integer \( k \) with \( (n + 3)/2 \leq k \leq (n + 5)/2 \), \( \gamma_{tk}^{-1}(T) = 2k - n \) if and only if \( T \) is an even spider.

**Proof.** The sufficiency follows from Proposition 4.14. To prove the necessity, we proceed by induction on the order \( n \) of a tree \( T \), where \( n \geq 3 \) is odd, \( k = (n+1)/2+i \).
where \(1 \leq i \leq 2\) and \(\gamma_{tk}^{-11}(T) = 2k - n\). If \(n = 3\), then \(T = P_3\). If \(n = 5\), then \(k \in \{4, 5\}\). If \(T\) is not a path, then it is easy to check that \(\gamma_{tk}^{-11}(T) < 2k - n\), a contradiction. Hence, \(T = P_5\). Thus if \(n \in \{3, 5\}\), then \(T\) is an even spider. This proves the base cases when \(n = 3\) and \(n = 5\).

Suppose that \(n \geq 7\) is odd and that for every tree \(T'\) of odd order \(n'\), where \(3 \leq n' < n\), and for \(k' = (n' + 1)/2 + i\) where \(1 \leq i \leq 2\), that if \(\gamma_{tk'}^{-11}(T') = 2k' - n'\), then \(T'\) is an even spider. Let \(T\) be a tree of order \(n\) with \(\gamma_{tk}^{-11}(T) = 2k - n\), where \(k = (n + 1)/2 + i\) and \(1 \leq i \leq 2\). Since \(n \geq 7\), \(k < n\). If \(T\) is a star, then by Proposition 4.9, \(\gamma_{tk}^{-11}(T) = 2 - n < -1\), a contradiction. Hence, \(\text{diam}(T) \geq 3\).

Following the notation used in the proof of Theorem 4.15, let \(T\) be rooted at a leaf \(r\) of a longest path, let \(v\) be a vertex at distance \(\text{diam}(T) - 1\) from \(r\) on a longest path starting at \(r\) and let \(w\) be the parent of \(v\). Let \(|N(v) - \{w\}| = m\). If \(m \geq 2\), then proceeding as in the proof of Theorem 4.17, \(\gamma_{tk}^{-11}(T) < 2k - n\), a contradiction. Hence, \(m = 1\). Let \(u\) be the child of \(v\). Then, \(T'\) has odd order \(n' = n - 2\). Let \(k' = k - 1\), and so \(k' = (n' + 1)/2 + i\). By Theorem 4.15, \(\gamma_{tk'}^{-11}(T') \leq 2k' - n'\).

If \(\gamma_{tk'}^{-11}(T') < 2k' - n'\), then any \(\gamma_{tk'}^{-11}(T')\)-function can be extended to a TKSF of \(T\) by assigning a positive opinion to \(v\) and a negative opinion to \(u\), whence \(\gamma_{tk}^{-11}(T) < 2k - n\), a contradiction. Hence, \(\gamma_{tk}^{-11}(T') = 2k' - n'\). Applying the inductive hypothesis to \(T'\), \(T'\) is an even spider.

Let \(g\) be the opinion function that assigns to the head of \(T'\) and all vertices at even distance from the head in \(T'\) a negative opinion and to all other vertices a positive opinion. Then the head of \(T'\) and all vertices at even distance from the head in \(T'\) vote aye under \(g\), while all other vertices vote nay. Hence, \((n' + 1)/2 = (n - 1)/2\) vertices of \(T'\) vote aye under \(g\). Further, \(g(V(T')) = -1\), and all leaves of \(T'\) have a negative opinion under \(g\) while all support vertices of \(T'\) have a positive opinion under \(g\).
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Suppose $w$ is a support vertex of $T'$. Let $v'$ be the child of $w$ in $T'$. If $k = (n+3)/2$, then the function $g$ can be extended to a $TkSF$ of $T$ by assigning to both $u$ and $v$ a positive opinion. If $k = (n+5)/2$, then the function $g$ can be extended to a $TkSF$ of $T$ by assigning to both $u$ and $v$ a positive opinion and by reassigning to $v'$ a positive opinion. In both cases, we produce a $TkSF$ of weight $2(k - 1) - n$, a contradiction. Hence $w$ is a leaf of $T'$ or at distance 2 from a leaf in $T'$. It follows that if $w$ is a leaf of $T'$ or if $w$ is the head of $T'$, then $T$ is an even spider. Hence we may assume that $w$ is at distance 2 from a leaf in $T'$ but that $w$ is neither a leaf of $T'$ nor the head of $T'$.

Let $x$ be the head of $T'$. By assumption, $x \neq w$. Suppose that $T'$ is not a path. Then, $x$ has degree at least 3. Since $T'$ is an even spider, $d(x,w)$ is even. Let $P$ denote the $x$-$w$ path and let $h$ be the opinion function of $T$ defined as follows: let $h(v) = 1$ and $h(u) = -1$, let $h(y) = -g(y)$ for all vertices $y \in V(P)$, and let $h(y) = g(y)$ for all remaining vertices $y \in V(T') - V(P)$. Then, $h(V(T)) = g(V(T)) + 2 = 1$. If $y \in V(T')$ and $y \notin V(P) - \{w, x\}$ and $y$ votes aye in $T'$ under $g$, then $y$ also votes aye in $T$ under $h$. If $y \in V(P) - \{w, x\}$ and $y$ is at odd distance from $x$, then $y$ votes nay in $T'$ under $g$ but aye in $T$ under $h$. On the other hand, if $y \in V(P) - \{w, x\}$ and $y$ is at even distance from $x$, then $y$ votes aye in $T'$ under $g$ but nay in $T$ under $h$. Since $|V(P)|$ is odd, it follows that the number of vertices in $V(T')$ that vote aye in $T$ is one more than the number of vertices in $V(T')$ that vote aye in $T'$, i.e., there are $(n' + 3)/2 = (n+1)/2$ vertices in $V(T')$ that vote aye in $T$ under $h$. Therefore, if $k = (n+3)/2$, then $h$ is a $TkSF$ of $T$, whence $\gamma_{tksf}^{-1}(T) \leq h(V(T)) = 1 < 2k - n$, a contradiction. If $k = (n+5)/2$, then reassigning a positive opinion to the leaf in $T'$ at distance 2 from $w$ produces a $TkSF$ of $T$ of weight $2(k - 1) - n$, a contradiction. Hence, $T'$ is a path, and so $T$ is an even spider (with $w$ as its head).
Theorem 4.18 is not true for \((n+7)/2 \leq k \leq n-1\). For example, let \(\ell \geq 3\) be an integer and let \(T\) be a tree of order \(n = 2\ell + 3\) obtained from an even spider with \(\ell\) legs each of length 2 by adding a path of length 2 to one of the support vertices of the spider. Then, for \((n+7)/2 \leq k \leq n-1\), \(\gamma_{tks}^{-11}(T) = 2k - n\). Thus, if \(\gamma_{tks}^{-11}(T) = 2k - n\) and \((n+7)/2 \leq k \leq n-1\), then \(T\) is not necessarily an even spider.
Chapter 5

COMPLEXITY OF TOTAL MINUS AND SIGNED DOMINATION

5.1 Introduction

This chapter is devoted to complexity issues of total minus and total signed domination. In Section 5.2, we discuss the complexity of the decision problems corresponding to the computation of $\gamma^{-101}_t(G)$ and $\gamma^{-11}_t(G)$ of a graph, where $\gamma^{-101}_t(G) = \gamma^{-101}_{t=1}(G)$, $\gamma^{-11}_t(G) = \gamma^{-11}_{t=1}(G)$ and $n = n(G)$. In Sections 5.3 and 5.4, we present linear algorithms for finding $\gamma^{-101}_t(T)$ and $\gamma^{-11}_t(T)$ of a nontrivial tree $T$. A total minus dominating function will be abbreviated by $TMDF$, while a total signed dominating function will be abbreviated by $TSDF$.  

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5.2 Complexity issues

In this section we discuss complexity issues regarding the computation of $\gamma_t^{-101}(G)$ and $\gamma_t^{-11}(G)$ for a graph $G$.

The following decision problem corresponding to the computation of the total domination number is known to be NP-complete, even when restricted to bipartite graphs or chordal graphs [33].

**TOTAL DOMINATION (TD)**

*Instance:* A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

*Question:* Does $G$ have a total dominating set of cardinality $k$ or less?

We will demonstrate a polynomial time reduction from this problem to the following decision problem:

**TOTAL MINUS DOMINATION (TMD)**

*Instance:* A graph $H = (V, E)$ and a positive integer $\ell \leq |V|$.

*Question:* Does $H$ have a TMDF of weight $\ell$ or less?

**Theorem 5.1** TMD is NP-complete, even when restricted to bipartite or chordal graphs.

**Proof.** It is obvious that TMD is a member of NP since we can, in polynomial time, guess a function $f : V \rightarrow \{-1, 0, 1\}$ and verify that $f$ has weight at most $\ell$ and is a TMDF. We next show how a polynomial time algorithm for TMD could be used to solve TD. Given a graph $G = (V, E)$ and a positive integer $k$, construct the graph $H$ by adding to each vertex $v_i$ of $G$ a path of length four, consisting of the consecutive vertices $v_i, w_i, x_i, y_i$ and $z_i$. It is easy to see that the graph $H$ can be constructed in polynomial time, and that if $G$ is a bipartite or chordal graph, then so too is $H$. 
Lemma 5.2 $\gamma_t^{-101}(H) = \gamma_t(H) = \gamma_t(G) + 2|V(G)|$.

Proof. Let $v_i \in V(G)$ and let $f$ be a $\gamma_t^{-101}(H)$-function. Since $N(v_i) = \{y_i\}$ and $f(N(v_i)) \geq 1$, we have $f(y_i) = 1$. Also, $1 \leq f(N(v_i)) = f(z_i) + f(x_i)$, so that $f(z_i) \geq 0$ and $f(x_i) \geq 0$. Similarly, using the facts that $1 \leq f(N(x_i))$ and $1 \leq f(N(w_i))$, we have $f(w_i) \geq 0$ and $f(v_i) \geq 0$.

Thus, $\text{Im}(f) \subseteq \{0, 1\}$, and so $f$ is a TDF of $H$. Consequently, $\gamma_t(H) \leq f(V(H)) = \gamma_t^{-101}(H)$. On the other hand, if $S$ is a $\gamma_t(H)$-set, then the characteristic function $h$ of $S$ is a TMDF of $H$, so $\gamma_t^{-101}(H) \leq h(V(H)) = \gamma_t(H)$. Consequently, $\gamma_t^{-101}(H) = \gamma_t(H)$.

Let $n = |V(G)|$ and let $S$ be a $\gamma_t(G)$-set. Then $S \cup \bigcup_{i=1}^n \{x_i, y_i\}$ is a total dominating set of $H$. Thus, $\gamma_t(H) \leq \gamma_t(G) + 2n$.

To see that the reverse inequality holds, let $S$ be a $\gamma_t(H)$-set for which $|S \cap \bigcup_{i=1}^n \{w_i, x_i, y_i, z_i\}|$ is minimized.

We may assume, without loss of generality, $z_i \notin S$ and $\{x_i, y_i\} \subseteq S$. For suppose $z_i \in S$. It follows $y_i \in S$. If $x_i \in S$, then $S - \{z_i\}$ is a total dominating set, contradicting the minimality of $S$. Thus, $x_i \notin S$, and $S' = (S - \{z_i\}) \cup \{x_i\}$ is a $\gamma_t(H)$-set such that $z_i \notin S'$ and $\{x_i, y_i\} \subseteq S'$.

We next show that $w_i \notin S$ for all $1 \leq i \leq n$. For suppose, to the contrary, $w_i \in S$ for some $1 \leq i \leq n$. Since $S - \{w_i\}$ is not a total dominating set, $v_i$ is uniquely (open) dominated by $w_i$. Let $v_j$ be any vertex adjacent to $v_i$. Then $v_j \notin S$. If $v_i \in S$, then $S' = (S - \{w_i\}) \cup \{v_j\}$ is a $\gamma_t(H)$-set with $|S' \cap \bigcup_{i=1}^n \{w_i, x_i, y_i, z_i\}| < |S \cap \bigcup_{i=1}^n \{w_i, x_i, y_i, z_i\}|$, which is a contradiction.

We may, therefore, assume $v_i \notin S$. If $v_j$ is dominated by some vertex $v_k \in S$, then $(S - \{w_i\}) \cup \{v_j\}$ is a $\gamma_t(H)$-set, contradicting our choice of $S$, as before. Thus, $v_j$ must be uniquely dominated by $w_j$. But then $(S - \{w_i, w_j\}) \cup \{v_i, v_j\}$ is a $\gamma_t(H)$-set, again contradicting our choice of $S$.  


Since \( w_i \notin S \) for all \( 1 \leq i \leq n \), \( S - \bigcup_{i=1}^{n} \{x_i, y_i\} \) is a total dominating set of \( G \), so \( \gamma_t(G) \leq |S| - 2n = \gamma_t(H) - 2n \). It now follows that \( \gamma_t(H) = \gamma_t(G) + 2|V(G)| \). \( \diamond \)

Lemma 5.2 implies that if we let \( \ell = k + 2|V(G)| \), then \( \gamma_t(G) \leq k \) if and only if \( \gamma_{t^{-10}}(H) \leq \ell \), and our proof is complete. \( \diamond \)

Problem TD is polynomial for fixed \( k \). To see this, let \( G = (V, E) \) be a graph with \( |V| = n \). If \( k \geq n \), then \( V \) is a total dominating set of \( G \) of cardinality at most \( k \). On the other hand, if \( k < n \), then consider all the \( r \)-subsets of \( V \), where \( r = 1, \ldots, k \). There are \( \sum_{r=1}^{k} \binom{n}{r} \) of these subsets, which is bounded above by the polynomial \( \sum_{r=1}^{k} n^r \). It takes a polynomial amount of time to verify that a set is or is not a total dominating set. These remarks show that it takes a polynomial amount of time to verify whether \( G \) has a total dominating set of cardinality at most \( k \) when \( k \) is fixed. Hence for fixed \( k \), TD \( \in \mathbb{P} \).

In contrast, we now show that for a fixed \( k \), TMD may be NP-complete. To see this, we will demonstrate a polynomial time reduction of TMD to the following decision problem.

**ZERO TOTAL MINUS DOMINATION (ZTMD)**

**Instance:** A graph \( G = (V, E) \).

**Question:** Does \( G \) have a \( TMDF \) of weight at most 0?

**Theorem 5.3** ZTMD is NP-complete, even when restricted to bipartite or chordal graphs.

**Proof.** It is obvious that ZTMD is a member of \( \text{NP} \) since we can, in polynomial time, guess at a function \( f : V(G) \rightarrow \{-1, 0, 1\} \) and verify that \( f \) has weight at most 0 and is a \( TMDF \).
We next show how a polynomial time algorithm for ZTMD could be used to solve TMD in polynomial time. Before proceeding further, we use the above figure to prove the following helpful result.

**Lemma 5.4** $\gamma_{t}^{-101}(G_i) = \gamma_{t}^{-11}(G_i) = -1$ for $i = 1, 2$.

**Proof.** Suppose $f$ is a $\gamma_{t}^{-101}(G_i)$-function ($\gamma_{t}^{-11}(G_i)$-function, respectively). Every vertex adjacent to an end-vertex must receive 1 under $f$, since otherwise that end-vertex would not have an open neighborhood sum of at least 1 under $f$. If any end-vertex has a value other than $-1$ assigned to it by $f$, we may reassign $-1$ to it and the resulting function will still be a TMDF (TSDF, respectively) of $G_i$, which is a contradiction. Thus, each end-vertex of $G_i$ is assigned $-1$ by $f$. It now follows that $\gamma_{t}^{-101}(G_i) = \gamma_{t}^{-11}(G_i) = -1$. ◇

Note that $G_1$ is bipartite, while $G_2$ is chordal.

Given a graph $H = (V, E)$ and a positive integer $\ell$, let $J_1 = H \cup \bigcup_{j=1}^{\ell} H_{1,j}$, where $H_{1,j} \cong G_1$ for $j = 1, \ldots, \ell$ ($J_2 = H \cup \bigcup_{j=1}^{\ell} H_{2,j}$, where $H_{2,j} \cong G_2$ for $j = 1, \ldots, \ell$, respectively). It is clear that $J_1$ ($J_2$, respectively) can be constructed in polynomial
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We now show that \( \gamma_t^{-101}(H) \leq \ell \) if and only if \( \gamma_t^{-101}(J_i) \leq 0 \) for \( i = 1, 2 \). Let \( 1 \leq i \leq 2 \). Suppose first \( \gamma_t^{-101}(H) \leq \ell \) and \( f \) is a \( \gamma_t^{-101}(H) \)-function. Let \( f_j \) be any TMDF of weight \(-1\) for \( H_{i,j} \) for \( j = 1, \ldots, \ell \). Define \( g : V(G) \to \{-1, 0, 1\} \) where \( G = J_i \) by \( g(x) = f_j(x) \) if \( x \in V(H_{i,j}) \), \( (j = 1, \ldots, \ell) \), while \( g(x) = f(x) \) for \( x \in V(H) \). Then \( g \) is a TMDF of \( G \) of weight \( \gamma_t^{-101}(H) + \ell(-1) \leq \ell - \ell = 0 \). Conversely, suppose \( \gamma_t^{-101}(J_i) \leq 0 \) and \( g \) is a \( \gamma_t^{-101}(J_i) \)-function. Let \( f \) be the restriction of \( g \) on \( V(H) \) and let \( f_j \) be the restriction of \( g \) on \( V(H_{i,j}) \) for \( j = 1, \ldots, \ell \). Then \( \gamma_t^{-101}(H) + \ell(-1) = \gamma_t^{-101}(H) + \sum_{j=1}^{\ell} \gamma_t^{-101}(H_{i,j}) \leq f(V(H)) + \sum_{j=1}^{\ell} f_j(V(H_{i,j})) = g(V(J_i)) = \gamma_t^{-101}(J_i) \leq 0 \), so that \( \gamma_t^{-101}(H) \leq \ell \). ⊓⊔

Henning [27] showed that the following decision problem is NP-complete.

TOTAL SIGNED DOMINATION (TSD)

Instance: A graph \( H = (V, E) \) and a positive integer \( \ell \leq |V| \).

Question: Does \( H \) have a TSDF of weight \( \ell \) or less?

Theorem 5.5 TSD is NP-complete, even when restricted to bipartite or chordal graphs.

As before, by using Lemma 5.4, one may show that the following decision problem is NP-complete, even for bipartite and chordal graphs.

ZERO TOTAL SIGNED DOMINATION (ZTSD)

Instance: A graph \( G = (V, E) \).

Question: Does \( G \) have a TSDF of weight at most 0?
5.3 A linear algorithm for $\gamma_{t}^{-101}(T)$

Next we present a linear algorithm for finding a $\gamma_{t}^{-101}(T)$-function in a nontrivial tree $T$. The variable $OpenSum$ denotes the sum of the values assigned to the open neighborhood of $v$.

**Algorithm: TOTAL MINUS DOMINATION(TMD).** Given a nontrivial tree $T$ on $n$ vertices, root the tree $T$ and label the vertices of $T$ from 1 to $n$ so that $\text{label}(w) > \text{label}(y)$ if the level of vertex $w$ is less than the level of vertex $y$. Note the root of $T$ will be labeled $n$.

for $i \leftarrow 1$ to $n$ do
  $f(i) \leftarrow -1$
for $i \leftarrow 1$ to $n$ do
  begin
    1. if vertex $i$ is a leaf and $i < n$
      then begin
        $OpenSum \leftarrow 1$
        $f(\text{parent}(i)) \leftarrow 1$
      end
    else $OpenSum \leftarrow f(N(i))$
    2. if $i < n$
      then while ($OpenSum < 1$) and ($f(\text{parent}(i)) < 1$) do
      begin
        $\text{parent}(i) \leftarrow f(\text{parent}(i)) + 1$
        $OpenSum \leftarrow OpenSum + 1$
      end
    3. while $OpenSum < 1$ do
      begin
        Choose a child of $i$, say $v$, for which $f(v) < 1$
while \((\text{OpenSum} < 1) \text{ and } (f(v) < 1)\) do \\
begin \\
\hspace{1cm} f(v) \leftarrow f(v) + 1 \\
\hspace{1cm} \text{OpenSum} \leftarrow \text{OpenSum} + 1 \\
end \\

**Theorem 5.6** Algorithm TMD produces a \(\gamma_r^{-101}(T)\)-function in a nontrivial tree \(T\).

**Proof.** Let \(T = (V, E)\) be a nontrivial tree of order \(n\) and let \(f\) be the function produced by the Algorithm TMD. Then \(f : V \rightarrow \{-1, 0, 1\}\). For convenience, the variable \(\text{OpenSum}\) which was used by Algorithm TMD when it considered the vertex \(v\), will be denoted by \(\text{OpenSum}(v)\).

**Lemma 5.7** The function \(f\) produced by Algorithm TMD is a TMDF.

**Proof.** First consider the case when \(v\) is a leaf. The algorithm assigns, in Step 1, the value 1 to the parent of \(v\), and since values are never decreased by the algorithm, the open neighborhood sum of \(v\) is at least one.

Next consider the case when \(v\) is not a leaf. If \(\text{OpenSum}(v) \geq 1\), we are done. If not, then Steps 2 and 3 of the algorithm increase the value of vertices in the open neighborhood of \(v\) such that \(\text{OpenSum}(v) \geq 1\), as required. \(\diamondsuit\)

To show that the function \(f\) obtained by Algorithm TMD is a \(\gamma_r^{-101}(T)\)-function, let \(g\) be any \(\gamma_r^{-101}(T)\)-function for the rooted tree \(T\). If \(f \neq g\), then we will show that \(g\) can be transformed into a new \(\gamma_r^{-101}(T)\)-function \(g'\) that will differ from \(f\) in fewer values than \(g\) did. This process will continue until \(f = g'\). Suppose, then, that \(f \neq g\). Let \(v\) be the lowest labeled vertex for which \(f(v) \neq g(v)\). Then all descendants of \(v\) are assigned the same value under \(g\) as under \(f\).
Lemma 5.8 If \( g(v) < f(v) \), then the initial value assigned to the vertex \( v \) was increased in Step 3 of Algorithm TMD.

Proof. Suppose the value of \( v \) was increased in Step 1. Then \( v \) is the parent of some leaf, say \( u \). Since \( g(v) < f(v) \), we have \( g(v) \leq 0 \). But then \( g(N(v)) = g(v) \leq 0 \), contradicting the fact that \( g \) is a TMDF of \( T \).

Suppose the value of \( v \) was increased in Step 2. This occurred when the algorithm was processing a vertex, say \( u \), whose parent is \( v \). Then \( f(N(u)) \leq 1 \) and \( g(N(u)) = g(N(u) - \{v\}) + g(v) = f(N(u) - \{v\}) + g(v) = f(N(u)) - f(v) + g(v) < f(N(u)) \leq 1 \), which contradicts the fact that \( g \) is a TMDF for \( T \).

Lemma 5.9 If \( g(v) < f(v) \), then the function \( g' \) defined by \( g'(u) = f(u) \) if \( u \in N(parent(v)) \) and \( g'(u) = g(u) \) if \( u \notin N(parent(v)) \) is a \( \gamma_{T101}(T) \)-function that differs from \( f \) in fewer values than does \( g \).

Proof. By Lemma 5.8, the initial value of \( v \) is increased in Step 3 of Algorithm TMD, which occurs when the parent of \( v \) was being processed. Let \( w \) be the parent of \( v \). So \( g' \) is defined by \( g'(u) = f(u) \) if \( u \in N(w) \) and \( g'(u) = g(u) \) for all remaining vertices in \( V \).

The algorithm ensures that \( f(N(w)) = 1 \). Also, since \( g \) is a TMDF of \( T \), \( f(N(w)) = 1 \leq g(N(w)) \). Furthermore, \( g'(V) = g'(V - N(w)) + g'(N(w)) = g(V - N(w)) + f(N(w)) \leq g(V - N(w)) + g(N(w)) = g(V) \). Thus, \( g'(V) \leq g(V) \).

Since all the descendants of \( w \), other than its children, have the same values under \( g \) as under \( f \), \( g'(N(u)) = f(N(u)) \) if \( u = w \) or if \( u \) is a descendant of \( w \), other than a child of \( w \). Moreover, since the value of \( v \) was increased in Step 3, then, if \( w \) had a parent, its value was either already 1 or otherwise it was increased to 1 in Step 2. Thus, \( g'(N(u)) \geq g(N(u)) \) for all vertices \( u \) different from \( w \) or a descendant of \( w \), other than a child of \( w \). Thus, since \( f \) and \( g \) are TMDFs of \( T \), so too is \( g' \). Since
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\[ g'(V) \leq g(V), \] \( g' \) is a \( \gamma_{-1}^{-1}(T) \)-function of \( T \) that differs from \( f \) in fewer values than does \( g \). ◇

We now consider the case where \( f(v) < g(v) \). We will need the following result.

**Lemma 5.10** A TMDF on a graph \( G = (V, E) \) is minimal if and only if for every vertex \( v \in V \) with \( f(v) \in \{0, 1\} \), there exists a vertex \( u \in N(v) \) with \( f(N(u)) = 1 \).

**Proof.** Let \( f \) be a minimal TMDF of \( G \). Suppose there is a vertex \( v \in V \) with \( f(v) \in \{0, 1\} \) and \( f(N(u)) \geq 2 \) for every vertex \( u \in N(v) \). Define a function \( g : V \rightarrow \{-1, 0, 1\} \) by \( g(v) = f(v) - 1 \) and \( g(w) = f(w) \) for all \( w \neq v \). Thus \( g(N(w)) = f(N(w)) \geq 1 \) for all \( w \notin N(v) \) and \( g(N(w)) = f(N(w)) - 1 \geq 1 \) for all \( w \in N(v) \). So \( g \) is a TMDF with \( g < f \), contradicting the minimality of \( f \).

Conversely, let \( f \) be a TMDF such that for every vertex \( v \in V \) with \( f(v) \in \{0, 1\} \), there exists a vertex \( u \in N(v) \) with \( f(N(u)) = 1 \). Suppose \( f \) is not minimal. Then there exists a TMDF \( g \) with \( g < f \). Thus, \( g(w) \leq f(w) \) for all \( w \in V \) and there exists a vertex \( v \in V \) such that \( g(v) < f(v) \). Therefore \( f(v) \in \{0, 1\} \) and by the assumption there is a vertex \( u \in N(v) \) with \( f(N(u)) = 1 \). So \( g(N(u)) \leq f(N(u)) - 1 = 0 \), which contradicts the fact that \( g \) is a TMDF. ◇

If the vertex \( v \) is the root then \( f(V) < g(V) = \gamma_{-1}^{-1}(T) \) which is a contradiction. Thus, we may assume that \( v \) is not the root of \( T \).

Since the labeling at each level is arbitrary, if any vertex \( x \) at the same level as \( v \) has \( g(x) < f(x) \), we can proceed as before to find a TMDF \( g' \) that agrees with \( f \) in more values than \( g \) does. Thus we may assume that every vertex \( x \) at the same level as \( v \) has \( f(x) \leq g(x) \).

Since \( f(v) < g(v) \), we know that \( g(x) \in \{0, 1\} \). By Lemma 5.10, there must be a vertex \( x \in N(v) \) such that \( g(N(x)) = 1 \). Let \( w \) be the parent of \( v \) and \( u \) be the parent of \( w \). If \( f(u) \leq g(u) \), then \( f(N(x)) = f(N(x) - \{v\}) + f(v) \leq \)
g(N(x) - \{v\}) + g(v) - 1 = g(N(x)) - 1 = 0, which contradicts the fact that $f$ is a TMDF.

Thus $f(u) > g(u)$. Suppose $f(u) = g(u) + r$ and $f(v) = g(v) - s$ where $r, s \in \{1, 2\}$. Define $g': V \rightarrow \{-1, 0, 1\}$ as follows: $g'(y) = g(y)$ for all vertices $y \in V - \{u, v\}$,

$$g'(u) = \begin{cases} 
    f(u) - 1 & \text{if } r = 2 \text{ and } s = 1 \\
    f(u) & \text{otherwise}
\end{cases}$$

and

$$g'(v) = \begin{cases} 
    f(v) + 1 & \text{if } r = 1 \text{ and } s = 2 \\
    f(v) & \text{otherwise}
\end{cases}.$$

Then

$$g'(u) = \begin{cases} 
    f(u) - 1 & \text{if } r = 2 \text{ and } s = 1 \\
    f(u) & \text{otherwise}
\end{cases}$$

$$= \begin{cases} 
    g(u) + r - 1 & \text{if } r = 2 \text{ and } s = 1 \\
    g(u) + r & \text{otherwise}
\end{cases}$$

$$\geq g(u) + 1.$$

It follows that the only vertex with possibly a smaller value under $g'$ than under $g$ is $v$. For each child $x$ of $v$, we have $g'(N(x)) = g'(N(x) - \{v\}) + g'(v) \geq f(N(x) - \{v\}) + f(v) = f(N(x)) \geq 1$.

Furthermore,

$$g'(u) + g'(v) = \begin{cases} 
    f(u) + f(v) + 1 & \text{if } r = 1 \text{ and } s = 2 \\
    f(u) - 1 + f(v) & \text{if } r = 2 \text{ and } s = 1 \\
    f(u) + f(v) & \text{otherwise}
\end{cases}$$
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\[
\begin{align*}
&= \begin{cases} 
(g(u) + 1) + (g(v) - 2) + 1 & \text{if } r = 1 \text{ and } s = 2 \\
(g(u) + 2) - 1 + (g(v) - 1) & \text{if } r = 2 \text{ and } s = 1 \\
g(u) + g(v) & \text{otherwise}
\end{cases} \\
&= g(u) + g(v).
\end{align*}
\]

Thus, \( g'(N(w)) = g'(N(w) - \{u, v\}) + g'(u) + g'(v) = g(N(w) - \{u, v\}) + g(u) + g(v) = g(N(w)) \geq 1 \) and \( g'(V) = g'(V - \{u, v\}) + g'(u) + g'(v) = g(V - \{u, v\}) + g(u) + g(v) = g(V) \). This shows that \( g' \) is a \( \gamma_t^{-101}(T) \)-function which differs from \( f \) in fewer values than does \( g \).

5.4 A linear algorithm for \( \gamma_t^{-11}(T) \)

In our final section, we present a linear algorithm for finding a minimum total signed dominating function in a nontrivial tree \( T \). The algorithm roots the tree \( T \) and associates various variables with the vertices of \( T \) as it proceeds. For any vertex \( v \), the variable \( \text{MinSum} \) denotes the minimum possible sum of values that may be assigned to the open neighborhood of \( v \). So \( \text{MinSum} = 1 \) or 2 depending on whether \( v \) has odd or even degree, respectively. The variable \( \text{OpenSum} \) denotes the sum of the values assigned to the open neighborhood of \( v \).

Algorithm: TOTAL SIGNED DOMINATION (TSD). Given a nontrivial tree \( T \) on \( n \) vertices, root the tree \( T \) and relabel the vertices of \( T \) from 1 to \( n \) so that \( \text{label}(w) > \text{label}(y) \) if the level of vertex \( w \) is less than the level of vertex \( y \). Note the root of \( T \) will be labeled \( n \).

\begin{verbatim}
for i ← 1 to n do 
    f(i) ← −1 
for i ← 1 to n do
\end{verbatim}
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begin

1. \( \text{deg } i \leftarrow \text{degree of the vertex } i \text{ in } T \)
2. if \( \text{deg } i \) is odd
   then \( \text{MinSum} \leftarrow 1 \)
   else \( \text{MinSum} \leftarrow 2 \)
3. if vertex \( i \) is a leaf and \( i < n \)
   then begin
      \( \text{OpenSum} \leftarrow 1 \)
      3.1. \( f(\text{parent}(i)) \leftarrow 1 \)
   end
   else \( \text{OpenSum} \leftarrow f(N(i)) \)
4. if \( \text{OpenSum} < \text{MinSum} \)
   then begin
      if \( i < n \) and \( f(\text{parent}(i)) = -1 \)
      then begin
      4.1. \( f(\text{parent}(i)) = 1 \)
           \( \text{OpenSum} \leftarrow \text{OpenSum} + 2 \); 
      end
   while \( \text{OpenSum} < \text{MinSum} \) do
begin
4.2. increase the value of one of the children of \( i \);
     \( \text{OpenSum} \leftarrow \text{OpenSum} + 2 \)
end
end

We now verify the validity of Algorithm TSD.

**Theorem 5.11** Algorithm TSD produces a \( \gamma^{-11}_t(T) \)-function in a nontrivial tree \( T \).
Proof. Let $T = (V, E)$ be a nontrivial tree of order $n$, and let $f$ be the function produced by Algorithm TSD. Then $f : V \rightarrow \{-1, 1\}$. For convenience, the variables $MinSum$ and $OpenSum$, which were used by Algorithm TSD when it considered the vertex $v$, will be denoted by $MinSum(v)$ and $OpenSum(v)$, respectively.

Lemma 5.12 The function $f$ produced by Algorithm TSD is a TSDF for $T$.

Proof. First consider the case when $v$ is a leaf. The algorithm assigns, in Step 3, the value 1 to the parent of $v$, and since values are never decreased by the algorithm, the open neighborhood sum of $v$ is at least one.

Next consider the case when $v$ is not a leaf. If $OpenSum(v) \geq MinSum(v) \geq 1$, we are done. If not, then Step 4 of the algorithm increases the value of vertices in the open neighborhood of $v$ such that $OpenSum(v) \geq MinSum(v) \geq 1$, as required. 

To show that the TSDF $f$ obtained by Algorithm TSD is minimum, let $g$ be any $\gamma_{t}^{-11}(T)$-function for the rooted tree $T$. If $f \neq g$, then we will show that $g$ can be transformed into a new $\gamma_{t}^{-11}(T)$-function $g'$ that will differ from $f$ in fewer values than $g$ did. This process will continue until $f = g'$. Suppose, then, that $f \neq g$. Let $v$ be the lowest labeled vertex for which $f(v) \neq g(v)$. Then all descendants of $v$ are assigned the same value under $g$ as under $f$.

Lemma 5.13 If $g(v) < f(v)$, then the initial value assigned to the vertex $v$ was increased in Step 4.2 of Algorithm TSD.

Proof. Suppose the value of $v$ was increased in Step 3.1. Then $v$ is the parent of some leaf, say $u$. But then $g(N(u)) = g(v) = -1$, contradicting the fact that $g$ is a TSDF of $T$.

Suppose the value of $v$ was increased in Step 4.1. This occurred when the algorithm was processing a vertex, say $u$, whose parent is $v$. Then $f(N(u)) = MinSum(u) \leq 2$
and $g(N(u)) = g(N(u) - \{v\}) + g(v) = f(N(u) - \{v\}) - 1 = f(N(u)) - f(v) - 1 = f(N(u)) - 2 \leq 0$, which is a contradiction.

Thus, the value of $v$ was increased in Step 4.2. of Algorithm TSD. $\diamond$

**Lemma 5.14** If $g(v) < f(v)$, then the function $g'$ defined by $g'(u) = f(u)$ if $u \in N(parent(v))$ and $g'(u) = g(u)$ if $u \notin N(parent(v))$ is a $\gamma^{-11}_t(T)$-function of $T$ that differs from $f$ in fewer values than does $g$.

**Proof.** By Lemma 5.13, the initial value assigned to the vertex $v$ was increased in Step 4.2 of Algorithm TSD and this occurs when the parent of $v$ was being processed. Let $w$ be the parent of $v$. Thus $g'$ is defined by $g'(u) = f(u)$ if $u \in N(w)$ and $g'(u) = g(u)$ for all remaining vertices $u$ in $V$.

Then $f(N(w)) = \text{MinSum}(w)$. If $\deg w$ is even, then $\text{MinSum}(w) = 2$, so $g(N(w)) \geq 2 = \text{MinSum}(w) = f(N(w))$. If $\deg w$ is odd, then $g(N(w)) \geq 1 = \text{MinSum}(w) = f(N(w))$. Hence, $f(N(w)) \leq g(N(w))$. Furthermore, $g'(V) = g'(V - N(w)) + g'(N(w)) = g(V - N(w)) + f(N(w)) \leq g(V - N(w)) + g(N(w)) = g(V)$. Since all the descendants of $w$, other than its children, have the same values under $g$ as under $f$, $g'(N(u)) = f(N(u))$ if $u = w$ or if $u$ is a descendant of $w$, other than a child of $w$. Moreover, since the value of $v$ was increased in Step 4.2, then, if $w$ had a parent, its value was either already 1 or otherwise it was increased to 1 in Step 4.1. Thus, $f(parent(w)) = 1$, so that $g'(N(u)) \geq g(N(u))$ for all vertices $u$ different from $w$ or a descendant of $w$, other than a child of $w$. Thus, since $f$ and $g$ are TSDFs of $T$, so too is $g'$. Since $g'(V) \leq g(V)$, $g'$ is a $\gamma^{-11}_t(T)$-function of $T$ that differs from $f$ in fewer values than does $g$. $\diamond$

It remains for us to consider the case where $f(v) < g(v)$. We will need the following result from [27].

**Lemma 5.15** A TSDF $f$ on a graph $G = (V, E)$ is minimal if and only if for every vertex $v \in V$ with $f(v) = 1$, there exists a vertex $u \in N(v)$ with $f(N(u)) \in \{1, 2\}$. 


Here the vertex \( v \) is not the root of \( T \), for otherwise \( f(V) < g(V) = \gamma^{-1}_t(T) \), which is impossible. Since the labeling of the vertices was arbitrary at each level, if any vertex \( x \) at the same level as \( v \) has \( g(x) < f(x) \), we can proceed as before to find a TSDF \( g' \) that agrees with \( f \) in more values than under \( g \). So we may assume in what follows that every vertex \( x \) at the same level as \( v \) has \( f(x) \leq g(x) \).

Since \( f(v) < g(v) \), it follows that \( f(v) = -1 \) and \( g(v) = 1 \). By the minimality of \( g \) (cf. Lemma 5.15), there exists a vertex \( x \in N(v) \) such that \( g(N(x)) \in \{1, 2\} \). Let \( w \) be the parent of \( v \) and let \( u \) be the parent of \( w \). If \( f(u) \leq g(u) \), then
\[
\begin{align*}
f(N(x)) &= f(N(x) - \{v\}) + f(v) \\
&\leq g(N(x) - \{v\}) + g(v) - 2 = g(N(x)) - 2 \leq 0,
\end{align*}
\]
which is a contradiction.

Hence \( f(u) > g(u) \), i.e., \( f(u) = 1 \) and \( g(u) = -1 \). Define a function \( g' : V \rightarrow \{-1, 1\} \) by \( g'(y) = g(y) \) if \( y \in V - \{v, u\} \), \( g'(v) = -1 \) and \( g'(u) = 1 \). Note that \( f(v) = g'(v) = -1 \) and \( f(u) = g'(u) = 1 \). The only vertices whose neighborhood sums are decremented under \( g' \) are the children of \( v \). However, these open neighborhood sums under \( g' \) are at least as large as under \( f \). Thus, since \( g \) and \( f \) are TSDFs, so too is \( g' \). Furthermore, \( g'(V) = g(V) \), so that \( g' \) is a \( \gamma^{-1}_t(T) \)-function which differs from \( f \) in fewer values than does \( g \). $\Diamond$
Chapter 6

COMPLEXITY OF TOTAL $k$-SUBDOMINATION

6.1 Introduction

In this chapter, we focus on the algorithmic complexity of $k$-subdomination. In Section 6.2, we show that the decision problem corresponding to the computation of the total signed $k$-subdomination number is NP-complete. In Section 6.3, we present a cubic time algorithm to compute the total signed $k$-subdomination number of a tree. In Section 6.4, we discuss an algorithm that appears in [35]. The algorithm is omitted and the complexity analysis seems to be incorrect. We correct this by providing a detailed cubic time algorithm to compute $\gamma^{-11}_{ks}(T)$ of a tree $T$. In Section 6.5, we show that the decision problem corresponding to the computation of the total minus $k$-subdomination number is NP-complete, even for bipartite graphs. In Section 6.6, we present a cubic time algorithm to compute the total minus $k$-subdomination number of a tree. Finally, in Section 6.7, we provide a cubic time algorithm to compute $\gamma^{-101}_{ks}(T)$ of a tree $T$. 

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6.2 Complexity of total signed domination

In this section, we show that the decision problem

**TOTAL SIGNED SUBDOMINATING FUNCTION (TSSF)**

**INSTANCE:** A graph $G = (V, E)$, positive integers $c, d$ such that $\gcd(c, d) = 1$ and $0 < \frac{c}{d} \leq 1$ and an integer $t$.

**QUESTION:** Is there a total signed subdominating function $f$ such that $f(V) \leq t$ and $|C_f| \geq \left\lceil \frac{|V|}{d} \right\rceil$?

is NP-complete by describing a polynomial transformation from the following problem:

**TOTAL DOMINATING SET, RESTRICTED TO 4-REGULAR GRAPHS (TDS)**

**INSTANCE:** A 4-regular graph $G = (V, E)$ and a positive integer $k \leq \frac{|V|}{2}$.

**QUESTION:** Is there a total dominating set of cardinality $k$ or less for $G$?

We first show that TDS is NP-complete by describing a polynomial transformation from the decision problem **DOMINATING SET**.

**DOMINATING SET, RESTRICTED TO PLANAR CUBIC GRAPHS (DS)**

**INSTANCE:** A planar cubic graph $G = (V, E)$ and a positive integer $k \leq \frac{|V|}{2}$.

**QUESTION:** Is there a total dominating set of cardinality $k$ or less for $G$?

Starting with the graph $G$, take two copies of the vertex set of $G$ (which will be independent sets), and join a vertex to all vertices in the other copy that are in its closed neighborhood in $G$. The resulting graph has total domination number equal to twice the domination number of $G$. This construction transforms a cubic graph into a 4-regular graph. Since DS is NP-complete [12], TDS is NP-complete.

If $\frac{c}{d} = 1$, then TSSF is the NP-complete problem **TOTAL SIGNED**
CHAPTER 6. COMPLEXITY OF TOTAL K-SUBDOMINATION

DOMINATION (see [13], [27], and Theorem 5.5). Hence, we also assume that \(0 < \frac{q}{d} < 1\). For convenience, we set \(q = \frac{s}{d}\), and denote \(\min\{f(V(G)) \, | \, f\) is a total signed subdominating function with \(|C_f| \geq [q|V(G)|]\} \) by \(\gamma_q(G)\).

We will need the following lemma.

Lemma 6.1 If \(c, d, p\) are positive integers such that \(0 < q = \frac{s}{d} < 1\), then there exist positive integers \(\ell\) and \(r\) such that \(8 \leq \ell \leq d^2\left(\left\lceil \frac{p}{2} \right\rceil + 4\right)\), \(r < d^2\left(\left\lceil \frac{p}{2} \right\rceil + 4\right)\) and \(q = \frac{p + r}{2p + r + \ell}\).

Proof. Since \(c < d\), we have \(c \geq 1\), \(d \geq 2\) and \(d - c \geq 1\). Let \(t = \left\lceil \frac{p}{2} \right\rceil + 4\). Then \(dt(d - c) \geq 2t\) and \(cdt \geq 2t\). However, \(2t \geq p + 8\), whence \(dt(d - c) \geq p + 8\) and \(cdt > p\). Let \(t\) be the smallest positive integer such that \(dt(d - c) \geq p + 8\) and \(cdt > p\). It follows that \(\ell \leq \left\lceil \frac{p}{2} \right\rceil + 4\). Let \(r = cdt - p\) and \(\ell = ddt - cdt - p\). Note that \(r\) and \(\ell\) are both positive integers such that \(r, \ell < ddt \leq d^2\left(\left\lceil \frac{p}{2} \right\rceil + 4\right)\). Furthermore, \(\ell \geq 8\) and \(q = \frac{p + r}{2p + r + \ell}\).

Theorem 1 The decision problem TSSF is NP-complete.

Proof. Obviously, TSSF is in NP.

Let \(G\) be a 4-regular graph, \(p = n(G)\) and \(k\) be an integer such that \(k \leq p/2\). By Lemma 6.1, there exists positive integers \(r, \ell\) such that \(\ell \geq 8\) and \(q = \frac{p + r}{2p + r + \ell}\). Let \(H\) be the graph constructed from \(G\) as follows: Take a complete graph \(F\) on \(p + \ell\) vertices, a fixed subset \(U \subseteq V(F)\) with \(|U| = 3\) and an empty graph \(L\) on \(r\) vertices, and let \(H\) be obtained from the disjoint union of \(F, G,\) and \(L\) by joining each vertex of \(U\) to every vertex in \(V(G) \cup V(L)\). Since \(n(H) = 2p + r + \ell < 2(p + d^2\left(\left\lceil \frac{p}{2} \right\rceil + 4\right))\), the graph \(H\) can be constructed from \(G\) in polynomial time.

We will use the abbreviations TDS for a total dominating set and TSSF for a total signed dominating function. We start by showing that if \(S\) is a TDS of \(G\) of
cardinality at most \( k \), then there is a TSSF \( f \) of \( H \) of weight at most \( 2k - 2p - r - \ell + 6 \). Define \( f : V(H) \to \{-1, 1\} \) by \( f(v) = 1 \) if \( v \in S \cup U \), while \( f(v) = -1 \) otherwise.

Let \( v \in V(G) \). Since \( S \) is a TDS of \( G \), \( v \) is adjacent to some vertex \( u \in S \) for which \( f(u) = 1 \). Since \( G \) is 4-regular and \( f(U) = 3 \), we have \( f[v] \geq 1 \). It is clear that \( f[w] = 3 \) for each vertex \( w \in V(L) \), so that \( f[v] \geq 1 \) for at least \( p + r = q(2p + r + \ell) = qn(H) \) vertices. This shows that \( f \) is a TSSF of \( H \) of weight \( 2|S| - 2p - r - \ell + 6 \leq 2k - 2p - r - \ell + 6 \).

For the converse, assume that \( \gamma_q(H) \leq 2k - 2p - r - \ell + 6 \). Among all the minimum TSSF’s of \( H \), let \( f \) be one that assigns the value +1 to as many vertices of \( U \) as possible. Let \( P \) and \( M \) be the sets of vertices in \( H \) that are assigned the values +1 and \( -1 \), respectively, under \( f \). Then \(|P| + |M| = 2p + r + \ell\), and \(|P| - |M| = \gamma_q(H)\).

Before proceeding further we prove three claims.

Claim 1 \( |P| \leq k + 3 \).

**Proof.** Suppose \( |P| \geq k + 4 \). Then \( |M| \leq 2p + r + \ell - k - 4 \), so that \( \gamma_q(H) = |P| - |M| \leq 2k - 2p - r - \ell + 8 \), which contradicts the fact that \( \gamma_q(H) \leq 2k - 2p - r - \ell + 6 \). \( \diamond \)

Claim 2 \( f[v] \leq 0 \) for all \( v \in V(F) \).

**Proof.** Suppose there exists a \( v \in V(F) \) such that \( f[v] \geq 1 \). If \( v \in U \), then, since \( v \) dominates \( H \), it follows that \( 0 = 1 - 1 \leq f[v] + f(v) = f(V(H)) = \gamma_q(H) \leq 2k - 2p - r - \ell + 6 \), whence \( p + \frac{r}{2} < k \), which is a contradiction. Hence \( v \in V(F) - U \).

Since \( N(v) = V(F) - \{v\} \) and \( f[v] \geq 1 \), it follows that more than half of the vertices of the set \( V(F) - \{v\} \) have the value 1 assigned to them under \( f \). This implies that \( |P| \geq \frac{p + \frac{r}{2}}{2} = \frac{p}{2} + \frac{r}{2} \geq \frac{p}{2} + 4 \). By Claim 1 and the fact that \( k \leq \frac{p}{2} \), it follows that \( |P| \leq \frac{p}{2} + 3 \), which is a contradiction. \( \diamond \)

By Claim 2, it follows that \( f[v] \geq 1 \) for all \( v \in V(G) \cup V(L) \).

Claim 3 \( f(U) = 3 \).
Proof. Suppose that \( f(u) = -1 \) for some \( u \in U \). If \( f(v) = -1 \) for all \( v \in V(G) \), then \( f[v] \leq -3 \) for all \( v \in V(G) \), which is a contradiction. It follows that there exists a \( v \in V(G) \) such that \( f(v) = 1 \). Define \( g : V(H) \to \{-1, 1\} \) by \( g(w) = f(w) \) if \( w \in V(H) - \{u, v\} \), \( g(v) = -1 \) and \( g(u) = 1 \), and consider a vertex \( x \in V(G) \cup V(L) \). Note that if \( x \not\in N(v) \) or \( x = v \), then \( g[x] = f[x] + 2 \), while if \( x \in N(v) \), then \( g[x] = f[x] \). It follows that \( g[v] \geq 1 \) for at least \( q \) of the vertices of \( H \) while the weights of \( g \) and \( f \) are equal. Hence \( g \) is a TSSF of \( H \) of weight \( \gamma_q(H) \) that assigns the value +1 to more vertices of \( U \) than does \( f \), contradicting our choice of \( f \). 

Let \( S = P \cap V(G) \). Since \( f[v] \geq 1 \) for all \( v \in V(G) \), it follows that every \( v \in V(G) \) is adjacent to some vertex in \( S \), which shows that \( S \) is a TDS of \( G \). Since \( f(U) = 3 \), Claim 1 implies that \( |S| \leq k \), which completes the proof.

6.3 Computing \( \gamma_{tkS}^{-11}(T) \) for a tree \( T \)

In this section, we will present a cubic time algorithm to compute the total signed \( k \)-subdomination number of a tree.

The tree \( T \) will be rooted and represented by the resulting parent array \( \text{parent}[1 \ldots n] \). We make use of the well-known fact that the tree \( T \) can be constructed recursively from the single vertex \( K_1 \) using only one rule of composition, which combines two trees \( (G, x) \) and \( (H, y) \), by adding an edge between \( x \) and \( y \) and calling \( x \) the root of the larger tree \( F \). We express this as follows: \((F, x) = (G, x) \circ (H, y)\). With each such subtree \((F, x)\), we associate the following data structure:

1. \( \text{table}[x].\text{numvertices} \): the number of vertices in the subtree \((F, x)\).

2. \( \text{table}[x].\text{degree} \): \( \deg_F(x) \).

3. \( \text{table}[x].\text{sum}[f(x), t, k] \): the minimum weight of a function \( f : V(F) \to \{-1, 1\} \) such that \( x \) is assigned \( f(x) \), \( |t| \leq \deg_T(x) - \deg_F(x) \) (representing all
possible sums of assignments of $-1$ and $+1$ to the vertices of $N_T(x) - N_F(x)$ and $|\{v | f(N_F(v)) + t \geq 1 \text{ when } v = x \text{ and } f(N_F(v)) \geq 1 \text{ when } v \neq x\}| \geq k$, where $1 \leq k \leq \text{table}[x].\text{numvertices}$.}

Our input consist of the order of the tree $T$, say $n$, and the parent array of the tree, rooted at a certain vertex. The root of the tree $T$ is labeled with 1, the vertices on the next level are labeled with 2 through 2 plus the number of vertices on level 2, and so on. Using the parent array, we compute $\text{deg}_T(x)$ for each vertex $x$, $x = 1, \ldots, n$. We then initialize the variable $\text{table}[x]$ for each vertex $x$, where $x = 1, \ldots, n$. Let $x$ be an arbitrary vertex of $T$. Initially, $(F, x) = (K_1, x)$, whence $\text{table}[x].\text{numvertices} = 1$ and $\text{table}[x].\text{degree} = 0$. Suppose $t$ is an integer such that $|t| \leq \text{deg}_T(x) - \text{deg}_F(x) = \text{deg}_T(x)$, representing all possible sums of assignments of $-1$ and $+1$ to the vertices of $N_T(x) - N_F(x) = N_T(x)$. Then $t \in \{-\text{deg}_T(x), -\text{deg}_T(x) + 2, \ldots, \text{deg}_T(x)\}$. The only way for $f(N_F(x)) + t = t \geq 1$, is for $t \geq 1$ if $\text{deg}_T(x)$ is odd and for $t \geq 2$ if $\text{deg}_T(x)$ is even. Thus, we have the following initializations:

**Case 1:** $\text{deg}_T(x)$ is odd and $t \in \{1, 3, \ldots, \text{deg}_T(x)\}$.
Then $\text{table}[x].\text{sum}[f(x), t, 1] = \text{table}[x].\text{sum}[f(x), t, 0] = f(x)$ where $f(x) \in \{-1, 1\}$.

**Case 2:** $\text{deg}_T(x)$ is odd and $t \in \{-\text{deg}_T(x), -\text{deg}_T(x) + 2, \ldots, -1\}$.
Then $\text{table}[x].\text{sum}[f(x), t, 1]$ is undefined, and $\text{table}[x].\text{sum}[f(x), t, 0] = f(x)$ where $f(x) \in \{-1, 1\}$.

**Case 3:** $\text{deg}_T(x)$ is even and $t \in \{2, 4, \ldots, \text{deg}_T(x)\}$.
Then $\text{table}[x].\text{sum}[f(x), t, 1] = \text{table}[x].\text{sum}[f(x), t, 0] = f(x)$ where $f(x) \in \{-1, 1\}$.

**Case 4:** $\text{deg}_T(x)$ is even and $t \in \{-\text{deg}_T(x), -\text{deg}_T(x) + 2, \ldots, 0\}$.
Then $\text{table}[x].\text{sum}[f(x), t, 1]$ is undefined, and $\text{table}[x].\text{sum}[f(x), t, 0] = f(x)$
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where \( f(x) \in \{-1, 1\} \).

The following code implements the aforementioned discussion.

**Algorithm: To compute \( \gamma_{ik}^{-11}(T) \) for a tree \( T \).**

```plaintext
for vertex ← 1 to n do
    degree[vertex] ← 0
for vertex ← 2 to n do
    begin
        degree[vertex] ← degree[vertex]+1
        degree[parent[vertex]] ← degree[parent[vertex]]+1
    end
for vertex ← 1 to n do
    begin
        table[vertex].numvertices ← 1
        table[vertex].degree ← 0

        if degree[vertex] is odd
        then startvalue ← 1
        else startvalue ← 0

        for excessvalue ← startvalue to degree[vertex] step 2 do
            begin
                table[vertex].sum[1,excessvalue,1] ← 1
                table[vertex].sum[-1,excessvalue,1] ← -1
                table[vertex].sum[1,excessvalue,0] ← 1
                table[vertex].sum[-1,excessvalue,0] ← -1
            end

        for excessvalue ← -degree[vertex] to startvalue-2 step 2 do
            begin
```
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\[ \text{table[vertex].sum}[1, \text{excessvalue}, 1] \leftarrow 10000 \]
\[ \text{table[vertex].sum}[-1, \text{excessvalue}, 1] \leftarrow 10000 \]
\[ \text{table[vertex].sum}[1, \text{excessvalue}, 0] \leftarrow 1 \]
\[ \text{table[vertex].sum}[-1, \text{excessvalue}, 0] \leftarrow -1 \]
end
end

Inputting the parent array takes \( O(n) \) steps, while computing the degree array from the parent array also takes \( O(n) \) steps. Initializing the array table takes

\[ O\left( \sum_{\text{vertex}=1}^{n} \text{deg}_{T}(\text{vertex}) \right) = O(2m(T)) = O(2(n - 1)) = O(n) \]

steps. Thus, the overall complexity here is \( O(n^2) \).

Our next result shows that our algorithm is correct.

**Theorem 6.2** Suppose \( (G, x) \) and \( (H, y) \) are two disjoint rooted subtrees, and let \( (F, x) = (G, x) \circ (H, y) \). Let \( s \in \{-1, 1\}, \ t \) be an integer such that \( |t| \leq \text{deg}_{T}(x) - \text{deg}_{F}(x) \) with \( t \equiv \text{deg}_{T}(x) - \text{deg}_{F}(x) \pmod{2} \), and \( k \) be an integer with \( 0 \leq k \leq |V(F)| \). Then

\[ \text{table}[x].\text{sum}[s, t, k] = \min\{\text{table}[x].\text{sum}[s, t + s', j] + \text{table}[y].\text{sum}[s', s, k - j] \mid s' \in \{-1, 1\}, 0 \leq j \leq k\} = \min\{\text{table}[x].\text{sum}[s, t + s', j] + \text{table}[y].\text{sum}[s', s, k - j] \mid s' \in \{-1, 1\}, \max\{0, k - |V(H)|\} \leq j \leq \min\{k, |V(G)|\}\}. \]

Moreover, \( |t| \leq \text{deg}_{T}(x) - \text{deg}_{F}(x) \) if and only if \( -(\text{deg}_{T}(x) - \text{deg}_{G}(x) - 1) \leq t \leq \text{deg}_{T}(x) - \text{deg}_{G}(x) - 1 \).
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Proof. Suppose $f : V(F) \to \{-1, 1\}$ such that

$$f(V(F)) = \text{table}[x].\text{sum}[s, t, k].$$

Let $g$ (respectively, $h$) be the restriction of $f$ on $V(G)$ (respectively, $V(H)$) and $s^* = h(y) = f(y)$. Note that $f(N_F(x)) + t = g(N_G(x)) + t + s^*$ and $f(N_F(v)) = g(N_G(v))$ for all $v \in V(G) - \{x\}$, while $f(N_F(y)) = h(N_H(y)) + s$ and $f(N_F(v)) = g(N_H(v))$ for all $v \in V(H) - \{y\}$. Thus, $k \leq |\{v \mid f(N_F(v)) + t \geq 1$ when $v = x$ and $f(N_F(v)) \geq 1$ when $v \neq x\}| = |\{v \mid g(N_G(v)) + t + s^* \geq 1$ when $v = x$ and $g(N_G(v)) \geq 1$ when $v \neq x\}| + |\{v \mid h(N_H(v)) + s \geq 1$ when $v = y$ and $h(N_H(v)) \geq 1$ when $v \neq y\}|$. If $j = |\{v \mid g(N_G(v)) + t + s^* \geq 1$ when $v = x$ and $g(N_G(v)) \geq 1$ when $v \neq x\}|$, then $k - j \leq |\{v \mid h(N_H(v)) + s \geq 1$ when $v = y$ and $h(N_H(v)) \geq 1$ when $v \neq y\}|$. It now follows that $\text{table}[x].\text{sum}[s, t + s^*, j] + \text{table}[y].\text{sum}[s^*, s, k - j] \leq g(V(G)) + h(V(H)) = \text{table}[x].\text{sum}[s, t, k]$. Hence, $\min\{\text{table}[x].\text{sum}[s, t + s', j] + \text{table}[y].\text{sum}[s', s, k - j] \mid s' \in \{-1, 1\}, 0 \leq j \leq k\} \leq \text{table}[x].\text{sum}[s, t, k]$.

On the other hand, suppose $g : V(G) \to \{-1, 1\}$ such that

$$g(V(G)) = \text{table}[x].\text{sum}[s, t + s', j]$$

and $h : V(H) \to \{-1, 1\}$ such that

$$h(V(H)) = \text{table}[y].\text{sum}[s', s, k - j].$$

Define $f : V(F) \to \{-1, 1\}$ by $f(v) = g(v)$ if $v \in V(G)$ and $f(v) = h(v)$ for all $v \in V(H)$. As before, $f(N_F(x)) + t = g(N_G(x)) + t + s'$ and $f(N_F(v)) = g(N_G(v))$ for all $v \in V(G) - \{x\}$, while $f(N_F(y)) = h(N_H(y)) + s$ and $f(N_F(v)) = g(N_H(v))$ for all $v \in V(H) - \{y\}$. Thus, $|\{v \mid f(N_F(v)) + t \geq 1$ when $v = x$ and $f(N_F(v)) \geq 1$ when $v \neq x\}| = |\{v \mid g(N_G(v)) + t + s' \geq 1$ when $v = x$ and $g(N_G(v)) \geq 1$ when $v \neq x\}| + |\{v \mid h(N_H(v)) + s \geq 1$ when $v = y$ and $h(N_H(v)) \geq 1$ when $v \neq y\}| \geq j + (k - j) = k$. Hence, $\text{table}[x].\text{sum}[s, t, k] \leq f(V(F)) = \text{table}[x].\text{sum}[s, t, k].$
\( g(V(G)) + h(V(H)) = \text{table}[x].\text{sum}[s, t + s', j] + \text{table}[y].\text{sum}[s', s, k - j] \). Thus, \( \text{table}[x].\text{sum}[s, t, k] \leq \min\{\text{table}[x].\text{sum}[s, t + s', j] + \text{table}[y].\text{sum}[s', s, k - j] \mid s' \in \{-1, 1\}, 0 \leq j \leq k\} \).

Since \( 0 \leq j \leq |V(G)| \) and \( j \leq k \), we have \( 0 \leq k - j \leq |V(H)| \), so that \( 0 \geq j - k \geq -|V(H)| \), whence \( j \geq k - |V(H)| \). We conclude that \( \max\{0, k - |V(H)|\} \leq j \leq \min\{k, |V(G)|\} \).

Lastly, \( |t| \leq \deg_T(x) - \deg_F(x) \) if and only if \( -\deg_T(x) + \deg_G(x) + 1 \leq t \leq \deg_T(x) - \deg_G(x) - 1 \), since \( \deg_F(x) = \deg_G(x) + 1 \). ◇

At the conclusion of our algorithm, \( T = F \), and so \( t = 0 \). Clearly, \( \gamma_{tks}^{-11}(T) = \min\{\text{table}[1].\text{sum}[1, 0, k], \text{table}[1].\text{sum}[-1, 0, k]\} \).

We are now in a position to present the remainder of the algorithm.

**Algorithm:** To compute \( \gamma_{tks}^{-11}(T) \) for a tree \( T \) (continued).

for oldroot ← n downto 2 do
begin

\( \text{resulttable.numvertices} \leftarrow \text{table}[oldroot].\text{numvertices} + \text{table}[\text{parent}[oldroot]].\text{numvertices} \)

\( \text{resulttable.degree} \leftarrow \text{table}[\text{parent}[oldroot]].\text{degree} + 1 \)

\( \text{range} \leftarrow \text{degree}[\text{parent}[oldroot]] - \text{resulttable.degree} \)

for newrootvalue ← -1 to 1 step 2 do
begin

for newrootexcess ← -range to range step 2 do
begin

for \( k \leftarrow 0 \) to resulttable.numvertices do
begin

minimum ← 1000

\( \text{startvalue} \leftarrow \max(0, k - \text{table}[oldroot].\text{numvertices}) \)

\( \text{stopvalue} \leftarrow \min(k, \text{table}[\text{parent}[oldroot]].\text{numvertices}) \)

for \( j \leftarrow \text{startvalue} \) to \( \text{stopvalue} \) do
begin
for oldrootvalue ← -1 to 1 step 2 do 
begin 
  number ← degree[parent[oldroot]] - table[parent[oldroot]].degree - 1
  if -number ≤ newrootexcess ≤ number then 
  begin 
    summand1 ← table[parent[oldroot]].
    sum[newrootvalue, newrootexcess + oldrootvalue, j]

    summand2 ← table[oldroot].
    sum[oldrootvalue, newrootvalue, k-j]

    temp ← summand1 + summand2 
  end 
  if (temp < minimum) 
  then minimum ← temp 
end 
resulttable.sum[newrootvalue, newrootexcess, k] ← minimum 
end 
table[parent[oldroot]] ← resulttable 
end 
for k ← 0 to n do 
output (k, min(table[1].sum[1, 0, k],table[1].sum[-1, 0, k]))

The complexity of the above part of the algorithm, excluding the output phase, is
\[ O((n-2) \sum_{n-oldroot=0} 2 \times \text{deg}_T[\text{parent}[\text{oldroot}]] \times n \times n \times 2) \]
\[ = O(4n^2 \sum_{v \in V(T)} \text{deg}_T(v)) \]
\[ = O(4n^2 2m(T)) = (4n^2 \times 2 \times (n - 1)) \]
\[ = O(n^3), \]
while the complexity of the output phase is \( O(n) \). Thus, the overall complexity of
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the algorithm is \( O(n^3) \).

6.4 A cubic algorithm to compute \( \gamma_{ks}^{-11}(T) \) of a tree \( T \)

A "quadratic" time algorithm to compute the total signed \( k \)-subdomination number of a tree appears in [35]. Unfortunately, the initialization phase of the algorithm is omitted and other aspects of the algorithm are not clear either. Also, the complexity analysis of the algorithm seems to be incorrect. In this section, we present a cubic algorithm to compute \( \gamma_{ks}^{-11}(T) \) of a tree \( T \). The approach here is similar to what we described in the previous section. Here we have the following data structure, associated with the subtree \((F, x)\).

1. \( \text{table}[x].\text{numvertices} \): the number of vertices in the subtree \((F, x)\).

2. \( \text{table}[x].\text{degree} \): \( \deg_F(x) \).

3. \( \text{table}[x].\text{sum}[f(x), t, k] \): the minimum weight of a function \( f : V(F) \to \{ -1, 1 \} \) such that \( x \) is assigned \( f(x) \), \( |t| \leq \deg_T(x) - \deg_F(x) \) (representing all possible sums of assignments of \(-1\) and \(+1\) to the vertices of \( N_T(x) - N_F(x) \) and \(|\{v \mid f(N_F[v]) + t \geq 1 \text{ when } v = x \text{ and } f(N_F[v]) \geq 1 \text{ when } v \neq x\}| \geq k \), where \( 1 \leq k \leq \text{table}[x].\text{numvertices} \).

The initialization phase here proceeds as follows.

Let \( x \) be an arbitrary vertex of \( T \). Initially, \((F, x) = (K_1, x)\), whence \( \text{table}[x].\text{numvertices}=1 \) and \( \text{table}[x].\text{degree}=0 \). Suppose \( t \) is an integer such that \( |t| \leq \deg_T(x) - \deg_F(x) = \deg_T(x) \), representing all possible sums of assignments of \(-1\) and \(+1\) to the vertices of \( N_T(x) - N_F(x) = N_T(x) \). Then
\( t \in \{ -\deg_T(x), -\deg_T(x)+2, \ldots, \deg_T(x) \} \). The only way for \( f(N_F(x))+f(x)+t = f(x)+t \geq 1 \), is for \( t \geq 2-f(x) \) if \( \deg_T(x) \) is odd and for \( t \geq 1-f(x) \) if \( \deg_T(x) \) is even. Thus, we have the following initializations:

**Case 1:** \( \deg_T(x) \) is odd and \( t \in \{ 2-f(x), 4-f(x), \ldots, \deg_T(x) \} \).

Then \( \text{table}[x].\text{sum}[f(x), t, 1] = \text{table}[x].\text{sum}[f(x), t, 0] = f(x) \) where \( f(x) \in \{-1, 1\} \).

**Case 2:** \( \deg_T(x) \) is odd and \( t \in \{ -\deg_T(x), -\deg_T(x)+2, \ldots, -f(x) \} \).

Then \( \text{table}[x].\text{sum}[f(x), t, 1] \) is undefined, and \( \text{table}[x].\text{sum}[f(x), t, 0] = f(x) \) where \( f(x) \in \{-1, 1\} \).

**Case 3:** \( \deg_T(x) \) is even and \( t \in \{ 1-f(x), 3-f(x), \ldots, \deg_T(x) \} \).

Then \( \text{table}[x].\text{sum}[f(x), t, 1] = \text{table}[x].\text{sum}[f(x), t, 0] = f(x) \) where \( f(x) \in \{-1, 1\} \).

**Case 4:** \( \deg_T(x) \) is even and \( t \in \{ -\deg_T(x), -\deg_T(x)+2, \ldots, -1-f(x) \} \).

Then \( \text{table}[x].\text{sum}[f(x), t, 1] \) is undefined, and \( \text{table}[x].\text{sum}[f(x), t, 0] = f(x) \) where \( f(x) \in \{-1, 1\} \).

A result analogous to Theorem 6.2 appears in [35].

We are now in a position to state the algorithm. Note that the initialization phase of the algorithm has complexity

\[
O\left( \sum_{v \in V(T)} 2 \times \deg_T(v) \right) = O(2m(T)) = O(2(n-1)) = O(n).
\]

Thus, the overall complexity of the algorithm is also \( O(n^3) \).

**Algorithm:** To compute \( \gamma_{k^s_{11}}(T) \) for a tree \( T \).

for vertex \( \leftarrow 1 \) to \( n \) do
    degree[vertex] \( \leftarrow 0 \)
for vertex \( \leftarrow 2 \) to \( n \) do
    begin
degree[vertex] ← degree[vertex]+1
degree[parent[vertex]] ← degree[parent[vertex]]+1
end
for vertex ← 1 to n do
  if degree[vertex] is odd then
    begin
      for rootvalue ← -1 to 1 step 2 do
        begin
          for excessvalue ← 2 - rootvalue to degree[vertex] step 2 do
            begin
              table[vertex].sum[rootvalue,excessvalue,1] ← rootvalue
              table[vertex].sum[rootvalue,excessvalue,0] ← rootvalue
            end
          for excessvalue ← -degree[vertex] to -rootvalue step 2 do
            begin
              table[vertex].sum[rootvalue,excessvalue,1] ← 10000
              table[vertex].sum[rootvalue,excessvalue,0] ← rootvalue
            end
        end
    end else
    begin
      for rootvalue ← -1 to 1 step 2 do
        begin
          for excessvalue ← 1 - rootvalue to degree[vertex] step 2 do
            begin
              table[vertex].sum[rootvalue,excessvalue,1] ← rootvalue
              table[vertex].sum[rootvalue,excessvalue,0] ← rootvalue
            end
          for excessvalue ← -degree[vertex] to -1-rootvalue step 2 do
begin
  table[vertex].sum[rootvalue,excessvalue,1] ← 10000
  table[vertex].sum[rootvalue,excessvalue,0] ← rootvalue
end
end
end

for oldroot ← n downto 2 do
begin
  resulttable.numvertices ← table[oldroot].numvertices +
  table[parent[oldroot]].numvertices
  resulttable.degree ← table[parent[oldroot]].degree + 1
  range ← degree[parent[oldroot]] - resulttable.degree
  for newrootvalue ← -1 to 1 step 2 do
    for newrootexcess ← -range to range step 2 do
      for k ← 0 to resulttable.numvertices do
        begin
          minimum ← 1000
          startvalue ← max(0, k - table[oldroot].numvertices)
          stopvalue ← min(k, table[parent[oldroot]].numvertices)
          for j ← startvalue to stopvalue do
            begin
              for oldrootvalue ← -1 to 1 step 2 do
                begin
                  number ← degree[parent[oldroot]] - table[parent[oldroot]].degree - 1
                  if -number ≤ newrootexcess ≤ number then
                    begin
                      summand1 ← table[parent[oldroot]].
                      sum[newrootvalue, newrootexcess + oldrootvalue, j]
6.5 Complexity result for total minus domination

In this section we will show that the decision problem corresponding to the computation of the total minus $k$-subdomination number is NP-complete by describing a polynomial transformation from the NP-complete problem EXACT COVER BY 3-SETS.

Let $r = \frac{a}{b} \leq 1$ be a fixed positive rational number (in lowest terms). Consider the decision problem

TOTAL MINUS SUBDOMINATING FUNCTION (TMSF)

INSTANCE: A graph $G$ and an integer $\ell$. 

```plaintext
summand2 ← table[oldroot].
    sum[oldrootvalue, newrootvalue, k-j]

    temp ← summand1 + summand2
    end
    if (temp < minimum)
        then minimum ← temp
    end
    end
    resulttable.sum[newrootvalue, newrootexcess, k] ← minimum
    end
    table[parent[oldroot]] ← resulttable
end
for k ← 0 to n do
    output (k, min(table[1].sum[-1, 0, k], table[1].sum[1, 0, k]))
```
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QUESTION: Is there a function \( f : V(G) \to \{-1, 0, 1\} \) of weight \( \ell \) or less for \( G \) such that \( |C_f| \geq r|V(G)| \)?

In this section we show that TMSF is \textbf{NP}-complete by describing a polynomial transformation from the following \textbf{NP}-complete problem (see [12]):

\textbf{EXACT COVER BY 3-SETS (X3C)}

INSTANCE: A set \( X = \{x_1, \ldots, x_{3q}\} \) and a set \( C = \{C_1, \ldots, C_m\} \) where \( C_j \subseteq X \) and \( |C_j| = 3 \) for \( j = 1, \ldots, m \).

QUESTION: Does \( C \) have a pairwise disjoint \( q \)-subset of \( C \) whose union is \( X \) (i.e. an exact cover)?

If \( r = 1 \), then TMSF is the \textbf{NP}-complete problem \textbf{TOTAL MINUS DOMINATING FUNCTION} (see Theorem 5.1). Hence, we also assume that \( r < 1 \). For two real numbers \( a \) and \( b \), we say that \( a \) divides \( b \) if there is an integer \( k \) such that \( b = ka \).

\textbf{Theorem 6.3} TMSF is \textbf{NP}-complete, even for bipartite graphs.

\textbf{Proof}. It is obvious that TMSF is in \textbf{NP}. To show that TMSF is an \textbf{NP}-complete problem, we will establish a polynomial transformation from the \textbf{NP}-complete problem \textbf{X3C}. Let \( X = \{x_1, \ldots, x_{3q}\} \) and \( C = \{C_1, \ldots, C_m\} \) be an arbitrary instance of \textbf{X3C} where \( C_j \subseteq X \) and \( |C_j| = 3 \) for \( 1 \leq j \leq m \). We will construct a bipartite graph \( G \) and an integer \( \ell \) such that this instance of \textbf{X3C} will have an exact cover if and only if

there is a function \( f : V(G) \to \{-1, 0, 1\} \) of weight at most \( \ell \) such that \( |C_f| \geq r|V(G)| \).

Corresponding to each \( x_i \in X \), associate the graph constructed from the path \( P_5 \), with vertices labeled \( x_i, y_i, z_i, v_i, w_i \), and the path \( P_2 \), with vertices labeled \( u_i \) and \( t_i \), by joining the vertices \( u_i \) and \( v_i \). Corresponding to each \( C_j \), associate the
graph constructed from the path $P_4$, with vertices labeled $c_j, d_j, e_j, f_j$, and the path $P_2$, with vertices labeled $g_j, h_j$, by joining the vertices $g_j$ and $e_j$. Add the edges \( \{x_i c_j \mid x_i \in C_j \} \) and call the resulting graph $H$. Note that $n(H) = 6m + 21q$. Let

$$
\mu = \begin{cases} 
0 & \text{if } r \text{ divides } 6m + 21q \\
 a - (6m + 21q) \bmod a & \text{otherwise}
\end{cases}
$$

Then $\mu$ is the smallest nonnegative integer that may be added to $6m + 21q$ so that $r$ divides $6m + 21q + \mu$ evenly. Construct the (bipartite) graph $G = (V, E)$ as follows. Take the disjoint union of two copies of $H$, say $H_1$ and $H_2$, add a set $S$ of $\alpha := 2\left(\frac{6m + 21q + \mu}{r} - (6m + 21q)\right)$ vertices, and, with $S = \{s_1, s_2, \ldots, s_\alpha\}$, add the edges $s_k s_{k+1}$, where $k = 1, 3, \ldots, \alpha - 1$. The graph $G$ has order $2\left(\frac{6m + 21q + \mu}{r}\right)$, and, since $0 \leq \mu \leq a - 1$, $G$ can be constructed from the input in polynomial time. Lastly, let $\ell = 2(8m + 28q + 2\mu - \left(\frac{6m + 21q + \mu}{r}\right))$. We will denote a vertex $\nu_i$ or $\nu_j$ of $H$ in $H_\beta$ by $\nu_{i,\beta}$ or $\nu_{j,\beta}$, for $\beta = 1, 2$.

Suppose $C' \subseteq C$ is an exact cover for $X$. Let $P = \bigcup_{i=1}^{3q}\{u_{i,1}, u_{i,2}, v_{i,1}, v_{i,2}, z_{i,1}, z_{i,2}\} \cup \bigcup_{j=1}^{m}\{g_{j,1}, g_{j,2}, e_{j,1}, e_{j,2}, d_{j,1}, d_{j,2}\} \cup \bigcup_{k=1}^{2\mu}\{s_k\} \cup \{c_{j,1}, c_{j,2} \mid C_j \in C'\}$ and $M = \bigcup_{i=1}^{3q}\{w_{i,1}, w_{i,2}\} \cup \bigcup_{j=1}^{m}\{f_{j,1}, f_{j,2}\} \cup \bigcup_{k=2\mu+1}^{2\mu}\{s_k\}$.

Define $f : V \to \{-1, 0, 1\}$ by

$$
f(x) = \begin{cases} 
1 & \text{if } x \in P \\
-1 & \text{if } v \in M \\
0 & \text{otherwise}
\end{cases}
$$

Notice that

$$
f(V) = \sum_{i=1}^{3q} 6 + \sum_{j=1}^{m} 6 + \sum_{k=1}^{2\mu} 1 + 2q - \left(\sum_{i=1}^{3q} 2 + \sum_{j=1}^{m} 2 + \sum_{k=2\mu+1}^{\alpha} 1\right)
$$

$$
= 14q + 4m + 4\mu - \alpha
$$

$$
= 2(8m + 28q + 2\mu - \frac{6m + 21q + \mu}{r}).
$$
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Note that \( x \in C_f \) for all \( x \in V(H_1) \cup V(H_2) \cup \bigcup_{k=1}^{2\mu} \{ s_k \} \). Thus, \( |C_f| \geq 2(6m + 21q) + 2\mu = 2(6m + 21q + \mu) = r^2 \frac{6m + 21q + \mu}{r} = r|V| \).

We now prove the converse. Let \( L \) be the set of all leaves of \( G \). Among all functions \( f : V \to \{-1, 0, 1\} \) for which \( f(V) \leq \ell \) and \( |C_f| \geq r|V(G)| \), choose one, say \( f \), for which \( f(L) \) is as small as possible. This implies that \( f(x) \in \{-1, 1\} \) for all \( x \in S \).

Note that \( |C_f| \geq r^2 \frac{6m + 21q + \mu}{r} = 12m + 42q + 2\mu \).

The function \( g_{r_1, \rho_1, r_2, \rho_2} \) is the function obtained from \( f \) by assigning some vertex \( r_1 \) in \( V(H_1) \cup V(H_2) \) the value \( \rho_1 \), some vertex \( r_2 \) in \( S \) the value \( \rho_2 \), while all other vertices are assigned the same value as under \( f \), where \( \rho_1, \rho_2 \in \{-1, 0, 1\} \). In all cases, a neighbor of \( r_1 \) will become covered, while the neighbor of \( r_2 \) will no longer be covered, so \( |C_g| \geq |C_f| \). Moreover, \( g(V) \leq f(V) \leq \ell \).

Let \( i \in \{1, \ldots, 3q\} , j \in \{1, \ldots, m\} \) and \( \beta \in \{1, 2\} \).

**Fact 1.** \( f(g_{j, \beta}) = 1 \) (and, similarly, \( f(e_{j, \beta}) = f(u_{i, \beta}) = f(v_{i, \beta}) = 1 \)).

**Proof.** For suppose, to the contrary, \( f(g_{j, \beta}) \leq 0 \). Then \( h_{j, \beta} \notin C_f \), and since \( |C_f| \geq 12m + 42q \), there is \( s_k \in S \cap C_f \). This implies \( f(s_{k-1}) = 1 \) or \( f(s_{k+1}) = 1 \) — assume the latter. Then \( g = g_{g_{j, \beta}, 1, s_{k+1}, f(g_{j, \beta})} \) has \( g(L) < f(L) \), which is a contradiction. \( \diamond \)

**Fact 2.** \( f(h_{j, \beta}) = 0 \) (and, similarly, \( f(t_{i, \beta}) = 0 \)).

**Proof.** For suppose, to the contrary, \( f(h_{j, \beta}) = -1 \). Then \( g_{j, \beta} \notin C_f \). Since \( |C_f| \geq 12m + 42q \), there is \( s_k \in S \cap C_f \). This implies \( f(s_{k-1}) = 1 \) or \( f(s_{k+1}) = 1 \) — assume the latter. Then \( g = g_{h_{j, \beta}, 0, s_{k+1}, -1} \) has \( g(L) < f(L) \), which is a contradiction. Furthermore, since \( f(L) \) is a minimum, \( f(h_{j, \beta}) = f(t_{i, \beta}) = 0. \) \( \diamond \)

**Fact 3.** \( e_{j, \beta} \in C_f \) (and, similarly, \( u_{i, \beta} \in C_f \)).

**Proof.** For suppose, to the contrary, that \( e_{j, \beta} \notin C_f \). Then \( f(d_{j, \beta}) + f(f_{i, \beta}) \leq -1 \). This implies that \( f(d_{j, \beta}) \leq 0 \). Since \( |C_f| \geq 12m + 42q \), there is \( s_k \in S \cap C_f \). This implies \( f(s_{k-1}) = 1 \) or \( f(s_{k+1}) = 1 \) — assume the latter. Then \( g = g_{d_{j, \beta}, 1, s_{k+1}, f(d_{j, \beta})} \).
has $g(L) < f(L)$, which is a contradiction. ◊

Fact 4. $f(f_{j,\beta}) = -1$ and $f(d_{j,\beta}) = 1$ (and, similarly, $f(z_{i,\beta}) = 1$ and $f(w_{i,\beta}) = -1$).

**Proof.** Since $e_{j,\beta} \in C_f$, $f(d_{j,\beta}) + f(g_{j,\beta}) + f(f_{j,\beta}) = f(d_{j,\beta}) + 1 + f(f_{j,\beta}) \geq 1$, which implies $f(d_{j,\beta}) + f(f_{j,\beta}) \geq 0$.

If $f(d_{j,\beta}) + f(f_{j,\beta}) \geq 1$, then $f(f_{j,\beta}) \geq 0$. Then $g : V \to \{-1,0,1\}$ defined by

$$g(x) = \begin{cases} f(x) - 1 & \text{if } x = f_{j,\beta} \\ f(x) & \text{otherwise} \end{cases}$$

is a function such that $|C_g| = |C_f| \geq r|V|$, $g(V) \leq \ell$ and $g(L) < f(L)$, which is a contradiction.

Hence, $f(d_{j,\beta}) + f(f_{j,\beta}) = 0$. If $f(f_{j,\beta}) \geq 0$, then the function $g : V \to \{-1,0,1\}$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \not\in \{d_{j,\beta}, f_{j,\beta}\} \\ f(x) + 1 & \text{if } x = d_{j,\beta} \\ f(x) - 1 & \text{otherwise} \end{cases}$$

is a function such that $|C_g| = |C_f| \geq r|V|$, $g(V) \leq \ell$ and $g(L) < f(L)$, which is a contradiction. ◊

Fact 5. $d_{j,\beta} \in C_f$ and $f(c_{j,\beta}) \geq 0$ (and, similarly, $z_{i,\beta} \in C_f$ and $f(y_{i,\beta}) \geq 0$).

**Proof.** If $d_{j,\beta} \not\in C_f$, then, since $f(e_{j,\beta}) = 1$, $f(c_{j,\beta}) = -1$. Since $|C_f| \geq 12m + 42q$, there is $s_k \in S \cap C_f$. This implies $f(s_{k-1}) = 1$ or $f(s_{k+1}) = 1$—assume the latter. Then $g = g_{c_{j,\beta}, 1, s_{k+1}, -1}$ has $g(L) < f(L)$, which is a contradiction. ◊

In a similar way, one can show that

Fact 6. $y_{i,\beta} \in C_f$ and $f(c_{i,\beta}) \geq 0$. ◊

Fact 7. $c_{j,\beta} \in C_f$ and $x_{i,\beta} \in C_f$.

**Proof.** Since $c_{j,\beta}$ is adjacent to $d_{j,\beta}$, which is assigned $+1$ under $f$, and three
vertices in \{x_{1,\beta}, \ldots, x_{3q,\beta}\}, all of which are assigned at least 0 under \(f\), we have that \(c_{i,\beta} \in C_f\).

Suppose, to the contrary, that \(x_{i,\beta} \not\in C_f\). Then all vertices adjacent to \(x_{i,\beta}\) are assigned the value 0 under \(f\) – particularly \(f(y_{i,\beta}) = 0\). Since \(|C_f| \geq 12m + 42q\), there is \(s_k \in S \cap C_f\). This implies \(f(s_{k-1}) = 1\) or \(f(s_{k+1}) = 1\) – assume the latter.

Then \(g = g_{y_{i,\beta}, s_{k+1}, -1}\) has \(g(L) < f(L)\), which is a contradiction. \(\diamondsuit\)

Combining the facts above, we have \(V(H_1) \cup V(H_2) \subseteq C_f\). Since \(n(H_1) + n(H_2) = 12m + 42q\) and \(|C_f| \geq 12m + 42q + 2\mu\), \(|S \cap C_f| \geq 2\mu\).

Let \(X_\beta = \{x_{1,\beta}, \ldots, x_{3q,\beta}\}, Y_\beta = \{y_{1,\beta}, \ldots, y_{3q,\beta}\}, C_\beta = \{y_{1,\beta}, \ldots, y_{m,\beta}\}, c_{x_\beta} = |X_\beta \cap P_f|, c_{y_\beta} = |Y_\beta \cap P_f|\) and \(c_{c_\beta} = |C_\beta \cap P_f|\).

Since \(f(V(H_1 \cup H_2) - X_1 - X_2 - Y_1 - Y_2 - C_1 - C_2) = 2(2(3q) + 2m) = 12q + 4m, f(V) \geq 12q + 4m + cx_1 + cx_2 + cy_1 + cy_2 + cc_1 + cc_2 + 2\mu + (-1)(2\frac{6m+21q+\mu}{r} - 2(6m+21q) - 2\mu) = 54q + 16m + 4\mu + cx_1 + cx_2 + cy_1 + cy_2 + cc_1 + cc_2 - 2(\frac{6m+21q+\mu}{r})\). But \(f(V) \leq 56q + 16m + 4\mu - 2(\frac{6m+21q+\mu}{r})\), and so \(cx_1 + cx_2 + cy_1 + cy_2 + cc_1 + cc_2 \leq 2q\).

Hence \(cc_1 + cc_2 \leq 2q - (cx_1 + cx_2 + cy_1 + cy_2)\), and so at most \(3(c_1 + c_2) \leq 6q - 3(x_1 + x_2 + y_1 + y_2)\) vertices of \(X_1 \cup X_2\) are adjacent to vertices of \((C_1 \cup C_2) \cap P_f\), \(cx_1\) vertices in \(X_1\) are assigned a +1 under \(f\), \(cx_2\) vertices in \(X_2\) are assigned a +1 under \(f\), \(cy_1\) vertices in \(Y_1\) are assigned a +1 under \(f\), and \(cy_2\) vertices in \(Y_2\) are assigned a +1 under \(f\). Thus, at most \(6q - 3(cx_1 + cx_2 + cy_1 + cy_2) + cx_1 + cx_2 + cy_1 + cy_2 = 6q - 2(cx_1 + cx_2 + cy_1 + cy_2)\) of \(X_1 \cup X_2\) are either adjacent to a vertex of \((Y_1 \cup Y_2 \cup C_1 \cup C_2) \cap P_f\) or assigned a +1 under \(f\). If \(cx_1 + cx_2 + cy_1 + cy_2 > 0\), then there is a vertex in \(X_1 \cup X_2\), say \(x\), such that \(x \not\in C_f\), which is a contradiction. Thus, \(cx_1 + cx_2 + cy_1 + cy_2 = 0\), and \(c_1 + c_2 \leq 2q\). Since \(x_{i,\beta} \in C_f\) for \(i = 1, \ldots, 3q\) and \(\beta = 1, 2, c_1 = q\) and \(c_2 = q\). It now follows that \(C' = \{C_j | f(c_{j,1}) = 1\}\) is an exact three cover for \(X\). \(\diamondsuit\)
6.6 A cubic algorithm to compute $\gamma_{tks}^{-101}(T)$ of a tree $T$

In this section, we will present a cubic time algorithm to compute the total minus $k$-subdomination number of a tree. The tree $T$ will be rooted and represented by the resulting parent array parent[1 ... n]. We make use of the well-known fact that the tree $T$ can be constructed recursively from the single vertex $K_1$ using only one rule of composition, which combines two trees $(G, x)$ and $(H, y)$, by adding an edge between $x$ and $y$ and calling $x$ the root of the larger tree $F$. We express this as follows: $(F, x) = (G, x) \circ (H, y)$. With each such subtree $(F, x)$, we associate the following data structure:

1. $\text{table}[x].\text{numvertices}$: the number of vertices in the subtree $(F, x)$.
2. $\text{table}[x].\text{degree}$: $\deg_F(x)$.
3. $\text{table}[x].\text{sum}(f(x), t, k)$: the minimum weight of a function $f : V(F) \to \{-1, 0, 1\}$ such that $x$ is assigned $f(x)$, $|t| \leq \deg_T(x) - \deg_F(x)$ (representing all possible sums of assignments of $-1$, $0$ and $+1$ to the vertices of $N_T(x) - NF(x)$ and $|\{v | f(N_F(v)) + t \geq 1 \text{ when } v = x \text{ and } f(N_F(v)) \geq 1 \text{ when } v \neq x\}| \geq k$, where $1 \leq k \leq \text{table}[x].\text{numvertices}$.

Our input consist of the order of the tree $T$, say $n$, and the parent array of the tree, rooted at a certain vertex. The root of the tree $T$ is labeled with $1$, the vertices on the next level are labeled with $2$ through $2$ plus the number of vertices on level $2$, and so on. Using the parent array, we compute $\deg_T(x)$ for each vertex $x$, $x = 1, \ldots, n$. We then initialize the variable $\text{table}[x]$ for each vertex $x$, where $x = 1, \ldots, n$. Let $x$ be an arbitrary vertex of $T$. Initially, $(F, x) = (K_1, x)$, whence $\text{table}[x].\text{numvertices}=1$ and $\text{table}[x].\text{degree}=0$. Suppose $t$ is an integer
such that $|t| \leq \deg_T(x) - \deg_F(x) = \deg_T(x)$, representing all possible sums of assignments of $-1, 0$ and $+1$ to the vertices of $N_T(x) - N_F(x) = N_T(x)$. Then $t \in \{-\deg_T(x), \ldots, \deg_T(x)\}$. The only way for $f(N_F(x)) + t = t \geq 1$, is for $t \geq 1$. Thus, we have the following initializations:

Case 1: $t \in \{1, \ldots, \deg_T(x)\}$.
Then $\text{table}[x].\text{sum}[f(x), t, 1] = \text{table}[x].\text{sum}[f(x), t, 0] = f(x)$ where $f(x) \in \{-1, 0, 1\}$.

Case 2: $t \in \{-\deg_T(x), \ldots, 0\}$.
Then $\text{table}[x].\text{sum}[f(x), t, 1]$ is undefined, and $\text{table}[x].\text{sum}[f(x), t, 0] = f(x)$ where $f(x) \in \{-1, 0, 1\}$.

The following code implements the aforementioned discussion.

Algorithm: To compute $\gamma^{\text{tk}_1}_T(T)$ for a tree $T$

```c
for vertex ← 1 to n do
    degree[vertex] ← 0
for vertex ← 2 to n do
    begin
        degree[vertex] ← degree[vertex] + 1
        degree[parent[vertex]] ← degree[parent[vertex]] + 1
    end
for vertex ← 1 to n do
    begin
        table[vertex].numvertices ← 1
        table[vertex].degree ← 0
        for excessvalue ← 1 to degree[vertex] do
            for rootvalue ← -1 to 1 do
                begin
                    table[vertex].sum[rootvalue, excessvalue, 1] ← rootvalue
                    table[vertex].sum[rootvalue, excessvalue, 0] ← rootvalue
```
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end

for excessvalue ← -degree[vertex] to 0 do
    for rootvalue ← -1 to 1 do
        begin
            table[vertex].sum[rootvalue, excessvalue, 1] ← 10000
            table[vertex].sum[rootvalue, excessvalue, 0] ← rootvalue
        end
    end

Inputting the parent array takes $O(n)$ steps, while computing the degree array from the parent array also takes $O(n)$ steps. Initializing the array table takes

$$O \left( \sum_{\text{vertex}=1}^{n} (2 \deg_T(\text{vertex}) + 1) \times 3 \right)$$

$$= O(6 \times 2m(T)) + O(3n)$$

$$= O(12(n - 1)) + O(n)$$

$$= O(n)$$

steps. Thus, the overall complexity here is $O(n^2)$.

**Theorem 6.4** Suppose $(G, x)$ and $(H, y)$ are two disjoint rooted subtrees, and let $(F, x) = (G, x) \circ (H, y)$. Let $s \in \{-1, 0, 1\}$, $t$ be an integer such that $|t| \leq \deg_T(x) - \deg_F(x)$ and $k$ be an integer with $0 \leq k \leq |V(F)|$. Then

$$\text{table}[x].\text{sum}[s, t, k] =$$

$$\min\{\text{table}[x].\text{sum}[s, t + s', j] + \text{table}[y].\text{sum}[s', s, k - j] \mid s' \in \{-1, 0, 1\}, 0 \leq j \leq k\} = \min\{\text{table}[x].\text{sum}[s, t + s', j] + \text{table}[y].\text{sum}[s', s, k - j] \mid s' \in \{-1, 0, 1\},$$

$$\max\{0, k - |V(H)|\} \leq j \leq \min\{k, |V(G)|\}\}.$$
Moreover, \(|t| \leq \deg_T(x) - \deg_F(x)\) if and only if
\(-\deg_T(x) - \deg_G(x) - 1) \leq t \leq \deg_T(x) - \deg_G(x) - 1.

**Proof.** Suppose \(f : V(F) \to \{-1, 0, 1\}\) such that

\[ f(V(F)) = \text{table}[x].\text{sum}[s, t, k]. \]

Let \(g\) (respectively, \(h\)) be the restriction of \(f\) on \(V(G)\) (respectively, \(V(H)\)) and \(s^* = h(y) = f(y)\). Note that \(f(N_F(x)) + t = g(N_G(x)) + t + s^*\) and
\(f(N_F(v)) = g(N_G(v))\) for all \(v \in V(G) - \{x\}\), while \(f(N_F(y)) = h(N_H(y)) + s\) and \(f(N_F(v)) = g(N_H(v))\) for all \(v \in V(H) - \{y\}\). Thus, \(k \leq \{|v| f(N_F(v)) + t \geq 1\}
when \(v = x\) and \(f(N_F(v)) \geq 1\) when \(v \neq x\)|\(|\{v| g(N_G(v)) + t + s^* \geq 1\}\) when \(v = x\) and \(g(N_G(v)) \geq 1\) when \(v \neq x\). If \(j = |\{v| g(N_G(v)) + t + s^* \geq 1\}\) when \(v = x\) and \(g(N_G(v)) \geq 1\) when \(v \neq x\)|, then \(k - j \leq |\{v| h(N_H(v)) + s \geq 1\}\) when \(v = y\) and \(h(N_H(v)) \geq 1\) when \(v \neq y\)|. It now follows that \(\text{table}[x].\text{sum}[s, t + s^*, j] + \text{table}[y].\text{sum}[s^*, s, k - j] \leq g(V(G)) + h(V(H)) = \text{table}[x].\text{sum}[s, t, k]\). Hence, \(\min\{\text{table}[x].\text{sum}[s, t + s', j] + \text{table}[y].\text{sum}[s', s, k - j] | s' \in \{-1, 1\}, 0 \leq j \leq k\} \leq \text{table}[x].\text{sum}[s, t, k]\).

On the other hand, suppose \(g : V(G) \to \{-1, 0, 1\}\) such that

\[ g(V(G)) = \text{table}[x].\text{sum}[s, t + s', j] \]

and \(h : V(H) \to \{-1, 0, 1\}\) such that

\[ h(V(H)) = \text{table}[y].\text{sum}[s', s, k - j]. \]

Define \(f : V(F) \to \{-1, 0, 1\}\) by \(f(v) = g(v)\) if \(v \in V(G)\) and \(f(v) = h(v)\) for all \(v \in V(H)\). As before, \(f(N_F(x)) + t = g(N_G(x)) + t + s'\) and \(f(N_F(v)) = g(N_G(v))\) for all \(v \in V(G) - \{x\}\), while \(f(N_F(y)) = h(N_H(y)) + s\) and \(f(N_F(v)) = g(N_H(v))\) for all \(v \in V(H) - \{y\}\). Thus, \(|\{v| f(N_F(v)) + t \geq 1\}\) when \(v = x\) and
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$f(N_{F}(v)) \geq 1$ when $v \neq x$, $|\{v \mid g(N_{G}(v)) + t + s' \geq 1 \text{ when } v = x \text{ and } g(N_{G}(v)) \geq 1 \text{ when } v \neq x\}| + |\{v \mid h(N_{H}(v)) + s \geq 1 \text{ when } v = y \text{ and } h(N_{H}(v)) \geq 1 \text{ when } v \neq y\}| \geq j + (k - j) = k$. Hence, $\text{table}[x].\text{sum}[s, t, k] \leq f(V(F)) = g(V(G)) + h(V(H)) = \text{table}[x].\text{sum}[s, t + s', j] + \text{table}[y].\text{sum}[s', s, k - j]$. Thus, $\text{table}[x].\text{sum}[s, t, k] \leq \min\{\text{table}[x].\text{sum}[s, t + s', j] + \text{table}[y].\text{sum}[s', s, k - j] \mid s' \in \{-1, 1\}, 0 \leq j \leq k\}$.

Since $0 \leq j \leq |V(G)|$ and $j \leq k$, we have $0 \leq k - j \leq |V(H)|$, so that $0 \geq j - k \geq -|V(H)|$, whence $j \geq k - |V(H)|$. We conclude that $\max\{0, k - |V(H)|\} \leq j \leq \min\{k, |V(G)|\}$.

Lastly, $|t| \leq \deg_{T}(x) - \deg_{F}(x)$ if and only if $-\deg_{T}(x) + \deg_{G}(x) + 1 \leq t \leq \deg_{T}(x) - \deg_{G}(x) - 1$, since $\deg_{F}(x) = \deg_{G}(x) + 1$. \(\triangle\)

At the conclusion of our algorithm, $T = F$, and so $t = 0$. Clearly, $\gamma_{tk}^{101}(T) = \min\{\text{table}[1].\text{sum}[-1, 0, k], \text{table}[1].\text{sum}[0, 0, k] \text{ table}[1].\text{sum}[1, 0, k]\}$.

We are now in a position to present the remainder of the algorithm.

**Algorithm:** To compute $\gamma_{tk}^{-101}(T)$ for a tree $T$ (continued).

for oldroot ← n downto 2 do begin

resulttable.numvertices ← table[oldroot].numvertices + table[parent[oldroot]].numvertices

resulttable.degree ← table[parent[oldroot]].degree + 1

range ← degree[parent[oldroot]] - resulttable.degree

for newrootvalue ← -1 to 1 do

for newrootexcess ← -range to range do

for k ← 0 to resulttable.numvertices do begin

minimum ← 1000

startvalue ← max(0, k - table[oldroot].numvertices)

end
stopvalue ← min(k, table[parent[oldroot]].numvertices)
for j ← startvalue to stopvalue do
begin
    for oldrootvalue ← -1 to 1 do
    begin
        number ← degree[parent[oldroot]] - table[parent[oldroot]].degree - 1
        if -number ≤ newrootexcess ≤ number then
            begin
                summand1 ← table[parent[oldroot]].
                sum[newrootvalue, newrootexcess + oldrootvalue, j]
                summand2 ← table[oldroot].
                sum[oldrootvalue, newrootvalue, k-j]
                temp ← summand1 + summand2
                if (temp < minimum)
                    then minimum ← temp
            end
        end
    end
    resulttable.sum[newrootvalue, newrootexcess, k] ← minimum
end

for k ← 0 to n do
  output (k, min(table[1].sum[1, 0, k], table[1].sum[0, 0, k], table[1].sum[-1, 0, k]))
The complexity of the above part of the algorithm, excluding the output phase, is
\[
O\left(\sum_{n=0}^{(n-2)} \times 3 \times (2 \times \deg_T[\text{parent}[\text{oldroot}]] + 1) \times n \times n \times 3\right)
\]
\[
= O(18n^2 \sum_{v \in V(T)} \deg_T(v)) + O(n \times 9n^2)
\]
\[
= O(18n^2 m(T)) + O(n^3)
\]
\[
= (18n^2 \times 2 \times (n - 1)) + O(n^3)
\]
\[
= O(n^3),
\]
while the complexity of the output phase is \(O(n)\). Thus, the overall complexity of the algorithm is \(O(n^3)\).

### 6.7 A cubic algorithm to compute \(\gamma_{ks}^{-101}(T)\) of a tree \(T\)

In this section, we present a cubic algorithm to compute \(\gamma_{ks}^{-101}(T)\) of a tree \(T\). The approach here is similar to what we described in the previous section. Here we have the following data structure, associated with the subtree \((F, x)\).

1. \texttt{table}[x].numvertices: the number of vertices in the subtree \((F, x)\).
2. \texttt{table}[x].degree: \(\deg_F(x)\).
3. \texttt{table}[x].sum\([f(x), t, k]\): the minimum weight of a function \(f: V(F) \to \{-1, 0, 1\}\) such that \(x\) is assigned \(f(x)\), \(|t| \leq \deg_T(x) - \deg_F(x)\) (representing all possible sums of assignments of \(-1, 0\) and \(+1\) to the vertices of \(N_T(x) - N_F(x)\) and \(|\{v \mid f(N_F[v]) + t \geq 1 \text{ when } v = x \text{ and } f(N_F[v]) \geq 1 \text{ when } v \neq x\}| \geq k\), where \(1 \leq k \leq \text{table}[x].numvertices\).

The initialization phase here proceeds as follows.
Let $x$ be an arbitrary vertex of $T$. Initially, $(F, x) = (K_1, x)$, whence $table[x].numvertices=1$ and $table[x].degree=0$. Suppose $t$ is an integer such that $|t| \leq \deg_T(x) - \deg_F(x) = \deg_T(x)$, representing all possible sums of assignments of $-1$, 0 and $+1$ to the vertices of $N_T(x) - N_F(x) = N_T(x)$. Then $t \in \{-\deg_T(x), \ldots, \deg_T(x)\}$. The only way for $f(N_F(x)) + f(x) + t = f(x) + t \geq 1$, is for $t \geq 1 - f(x)$. Thus, we have the following initializations:

**Case 1:** $t \in \{1 - f(x), \ldots, \deg_T(x)\}$.
Then $table[x].sum[f(x), t, 1] = table[x].sum[f(x), t, 0] = f(x)$ where $f(x) \in \{-1, 0, 1\}$.

**Case 2:** $t \in \{-\deg_T(x), \ldots, -f(x)\}$.
Then $table[x].sum[f(x), t, 1]$ is undefined, and $table[x].sum[f(x), t, 0] = f(x)$ where $f(x) \in \{-1, 0, 1\}$.

One may prove a result analogous to Theorem 6.2

We are now in a position to state the algorithm. Note that the initialization phase of the algorithm has complexity

$$O(\sum_{v \in V(T)} 3 \times (2 \deg_T(v) + 1))$$

$$= O(6 \times 2m(T)) + O(3n)$$

$$= O(12(n - 1)) + O(n)$$

$$= O(n).$$

Thus, the overall complexity of the algorithm is also $O(n^3)$.

**Algorithm:** To compute $\gamma_{K_4}^{101}(T)$ for a tree $T$.

```plaintext
for vertex ← 1 to n do
    degree[vertex] ← 0
for vertex ← 2 to n do
    begin
    degree[vertex] ← degree[vertex]+1
```
degree[parent[vertex]] ← degree[parent[vertex]] + 1
end

for vertex ← 1 to n do
    for rootvalue ← -1 to 1 do
        begin
            for excessvalue ← 1 - rootvalue to degree[vertex] do
                begin
                    table[vertex].sum[rootvalue, excessvalue, 1] ← rootvalue
                    table[vertex].sum[rootvalue, excessvalue, 0] ← rootvalue
                end
            for excessvalue ← -degree[vertex] to -rootvalue do
                begin
                    table[vertex].sum[rootvalue, excessvalue, 1] ← 10000
                    table[vertex].sum[rootvalue, excessvalue, 0] ← rootvalue
                end
        end
    end
for oldroot ← n downto 2 do
    begin
        resulttable.numvertices ← table[oldroot].numvertices +
                                  table[parent[oldroot]].numvertices
        resulttable.degree ← table[parent[oldroot]].degree + 1
        range ← degree[parent[oldroot]] - resulttable.degree
        for newrootvalue ← -1 to 1 do
            for newrootexcess ← -range to range do
                for k ← 0 to resulttable.numvertices do
                    begin
                        minimum ← 1000
                        startvalue ← max(0, k - table[oldroot].numvertices)
                        stopvalue ← min(k, table[parent[oldroot]].numvertices)
                        for j ← startvalue to stopvalue do
begin
for oldrootvalue ← -1 to 1 do
begin
number ← degree[parent[oldroot]] - table[parent[oldroot]].degree - 1
if -number ≤ newrootexcess ≤ number then
begin
summand1 ← table[parent[oldroot]].
    sum[newrootvalue, newrootexcess + oldrootvalue, j]

summand2 ← table[oldroot].
    sum[oldrootvalue, newrootvalue, k-j]

    temp ← summand1 + summand2
end
if (temp < minimum)
then minimum ← temp
end
end
resulttable.sum[newrootvalue, newrootexcess, k] ← minimum
end
table[parent[oldroot]] ← resulttable
end
for k ← 0 to n do
    output (k, min(table[1].sum[1, 0, k], table[1].sum[0, 0, k], table[1].sum[-1, 0, k]))

All the above algorithms are implemented in C++. 
Bibliography


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