2-GENERATIONS OF THE SPORADIC SIMPLE GROUPS

by

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Submitted in partial fulfilment of the
requirements for the degree of
Doctor in Philosophy
in the
Department of Mathematics and Applied Mathematics,
University of Natal,
Pietermaritzburg.

January 1997

بسم الله الرحمن الرحيم

Preface

The work described in this thesis was carried out under the supervision of Professor Jamshid Moori, Department of Mathematics and Applied Mathematics, University of Natal, Pietermaritzburg, from July 1994 to December 1996.

The thesis represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

Acknowledgements

اَلْحَمْدُ لِلّٰهِ رَبِّ الْعٰلَمِيْنَ ۝

I wish to thank my supervisor, Prof. Jamshid Moori, for the advise and encouragement he gave me in preparing this thesis. His ready assistance and love for mathematics has made a huge impression on me. I will always be indebted to him for his dedication and constant inspiration which he has given me.

I am grateful for the facilities made available to me at the department of Mathematics and Applied Mathematics, University of Natal, Pietermartizburg. I thank my colleagues at the University of the Western Cape for granting me study leave to complete my studies.

I gratefully acknowledge financial support from the Foundation for Research Development - University Development Programme, Deutscher Akademischer Austauschdienst and University of Natal (graduate assistantship and graduate scholarship).

My special thanks go to my wife Wiedaad and son Zubair for all the sacrifices they made in accompanying me in Pietermartizburg. Thanks are also due to my parents and family for their constant encouragement and support and to the Gani family in Pietermartizburg for accepting us into their home with open arms. Finally I would like to thank my room mates Eugene and Clint for creating a pleasant working environment in Room F38.

Abstract

A group G is said to be 2-generated if $G = \langle x, y \rangle$, for some non-trivial elements $x, y \in G$. In this thesis we investigate three special types of 2-generations of the sporadic simple groups.

A group G is a (l, m, n)-generated group if G is a quotient group of the triangle group $T(l, m, n) = \langle x, y, z | x^l = y^m = z^n = xyz = 1_G \rangle$. Given divisors l, m, n of the order of a sporadic simple group G, we ask the question: Is G a (l, m, n)-generated group? Since we are dealing with simple groups, we may assume that $1/l + 1/m + 1/n < 1$. Until recently interest in this type of generation had been limited to the role it played in genus actions of finite groups. The problem of determining the genus of a finite simple group is tantamount to maximizing the expression $1/l + 1/m + 1/n$ for which the group is (l, m, n)-generated.

Secondly, we investigate the nX-complementary generations of the finite simple groups. A finite group G is said to be nX-complementary generated if, given an arbitrary non-trivial element $x \in G$, there exists an element $y \in nX$ such that $G = \langle x, y \rangle$. Our interest in this type of generation is motivated by a conjecture (Brenner-Guralnick-Wiegold [18]) that every finite simple group can be generated by an arbitrary non-trivial element together with another suitable element. It was recently proved by Woldar [181] that every sporadic simple group G is pA-complementary generated, where p is the largest prime divisor of $|G|$. In an attempt to further the theory of nX-complementary generations of the finite simple groups, we pose the following problem. Which conjugacy classes nX of the sporadic simple groups are nX-complementary generated conjugacy classes. In this thesis we provide a complete solution to this problem for the sporadic simple groups HS, McL, Co_3, Co_2, J_1, J_2, J_3, J_4 and Fi_{22}. We partially answer the question on (l, m, n)-generation for the said sporadic groups.

A finite non-abelian group G is said to have spread r if for every set $\{x_1, x_2, \ldots, x_r\}$ of r non-trivial distinct elements, there is an element $y \in G$ such that $G = \langle x_i, y \rangle$, for all i. Our interest in this type of 2-generation comes from a problem by Brenner-Wiegold [19] to find all finite non-abelian groups with spread 1, but not spread 2. Every sporadic simple group has spread 1 (Woldar [181]) and we show that every sporadic simple group has spread 2.

Contents

Chapter 1

Introduction

A group G is said to be 2-*generated* if it can be generated by two suitable elements. It is well-known that every finite simple group is 2-generated. This has been known for a long time (*cf.* Miller [126]) in the case of the alternating groups. Explicit generators for the alternating groups A_n, with $n \geq 5$ (*cf.* Aschbacher-Guralnick, [4]) are:

$$a = (1,2)(n-1,n) \quad \text{and} \quad b = (1,2,\ldots,n-1), \quad \text{if } n \text{ is even}$$

$$a = (1,n)(2,n-1) \quad \text{and} \quad b = (1,2,\ldots,n-2), \quad \text{if } n \text{ is odd.}$$

For the groups of Lie type, Steinberg [155] provided a unified treatment for the 2-generation of the Chevalley groups and the Twisted groups. Steinberg's construction of a generating pair exploits the basic structure of a group of Lie type. Before this the 2-generation of certain families of Lie-groups were known (eg. $PSL(n,F)$ and $Sp(n,F)$). For the sporadic simple groups we have the following result.

Theorem 1.0.1 *(Aschbacher-Guralnick [4]) Every sporadic simple group can be generated by an involution and another suitable element.* □

Aschbacher and Guralnick were primarily concerned with applications to cohomology. They prove: *Let G be a finite group acting faithfully and irreducibly on a vector space V over the prime field $GF(p)$. Then $|H^1(G,V)| < |V|$, where $H^1(G,V)$ is the first cohomology group of G on V.* The 2-generation of the simple groups come into play in the following way. First it is proved that if G is generated by d elements then, $|H^1(G,V)| < |V|^{d-1}$, then a reduction to the case G simple is accomplished and 2-generation gives the required result.

In view of applications (as noted above), it is often important to exhibit generating pairs of some special kind, such as:

- generators carrying a geometric meaning,
- generators of some prescribed order,
- generators that offer an economical presentation of the group.

For this purpose, more subtle and detailed techniques are required. We now examine such instances.

1. Genus action: A group G is said to be $(n_1, \ldots, n_h)$-generated if G is a quotient of the group

$$\Gamma = \langle x_1, \ldots, x_h \mid {x_1}^{n_1} = {x_2}^{n_2} = \cdots = {x_h}^{n_h} = x_1 x_2 \cdots x_h = 1_G \rangle.$$

In the case where $h = 3, 4$ we call Γ a triangular group $T(n_1, n_2, n_3)$ and quadrangular group $Q(n_1, \ldots, n_4)$, respectively. The *genus* $g(G)$ of a finite group G is defined to be the smallest integer g such that some Cayley graph of G is embedded on a Riemann surface S_g with genus g. The action of groups on Riemann surfaces seeks a geometric representation theory of finite groups as automorphism groups of Riemann surfaces. The genus action plays the role of an irreducible representations in this theory.

Let Γ be a $(n_1, \ldots, n_h)$-generated finite group and let $\mathcal{H}^2$ be the hyperbolic plane. If G is a homomorphic image of Γ, then the short exact sequence

$$1_\Delta \to \Delta \to \Gamma \to G \to 1_G.$$

gives rise to an orbit space $S_g = \mathcal{H}^2/\Delta$ in the natural way of the structure of a Riemann surface on which G acts faithfully as a group of conformal mappings. Moreover, the regular branch covering $\mathcal{H}^2/\Delta \to \mathcal{H}^2/\Gamma$ has branch point orders $n_1, \ldots, n_h$.

The genus of $\mathcal{H}^2/\Delta$, hence of G, can be calculated from the genus of $\mathcal{H}^2/\Gamma$ by the well-known Riemann-Hurwitz formula

$$g(\mathcal{H}^2/\Delta) = 1 + \frac{|G|}{2}[g(\mathcal{H}^2/\Gamma) - 2 + \sum_{I=1}^{h}(1 - 1/n_i)]$$

We now restrict ourselves to finite simple groups. It is conjectured that every finite non-abelian finite simple group can be generated by an involution and another suitable element, that is, $(2, s, t)$-generated. The validity of this conjecture will simplify the calculation of the genus of finite simple groups as follows.

Proposition 1.0.2 *(Woldar [177]) Let G be a finite non-abelian $(2, s, t)$-generated group and S a Riemann surface of least genus on which G acts. Then $S/G = S^2$ (the 2-sphere) and the branch covering $\pi : S \to S/G$ has either 3 or 4 branch points.* $\square$

Thus the genus action of the $(2, s, t)$-generated finite groups arise from the short exact sequence $1_{Delta} \to \Delta \to \Gamma \to G \to 1_G$, where Γ is either a triangular group or a quadrangular group. As a consequence of the Riemann-Hurwitz equation the genus of a $(2, s, t)$-generated group G is given by

$$g(G) = 1 + \frac{|G|}{2}M,$$

where $M = 1 - 1/l - 1/m - 1/n$ or $M = 2 - 1/u - 1/v - 1/w - 1/x$, depending, respectively, on whether $\Gamma = T(l, m, n)$ or $Q(u, v, w, x)$ in the genus action. Thus the genus problem of the these groups is reduced to a problem on generations. With this in mind Moori [131] posed the following problem.

(1) *Let G be a finite simple group such that l, m, n are divisors of $|G|$ with $1/l + 1/m + 1/n < 1$. Is G a (l, m, n)-generated group?*

2. Spread: Let r be any positive integer. A finite non-abelian group G is said to have *spread* r, if for every set $\{x_1, x_2, \ldots, x_r\}$ of distinct non-trivial elements of G, there exists a complementary $y \in G$ such that $G = \langle x_i, y\rangle$ for all i. It is conjectured by Brenner-Guralnick-Wiegold [18] that every finite simple group has spread 1.

Woldar [181] proved the conjecture for all sporadic simple groups using the following definition. Let G be a finite group and nX a conjugacy class of G. The group G is called *nX-complementary generated* if, given an arbitrary $x \in G$, complementary y can always be chosen from the conjugacy class nX. Woldar proved that every sporadic simple group G is pA-complementary generated, where p is the largest prime divisor of $|G|$. In an attempt to further the theory on nX-complementary generations we pose the following problem.

(2) *Find all conjugacy classes nX of a finite simple group G such that G is nX-complementary generated.*

We say G has *exact spread* r if G has spread r but not $r+1$. An interesting question posed by Brenner-Wiegold [19] is to find all finite non-abelian groups with exact spread 1.

3. Presentations: For most group theorists studying generations, the main aim is not just to give generators, but to offer economical presentations for the groups in question. For the finite non-abelian simple groups 2-generations is an ideal starting point. Solutions to the two problems posed above will provide us with a pool of generations pairs (together with some relations) which may in time be extended to (abstract) presentations of the groups.

In this thesis we will focus on problems (1) and (2) with G one of the following sporadic simple groups: the Higman-Sims group HS; the McLaughlin group McL; the Conway groups Co_3, Co_2; the Janko groups J_1, J_2, J_3, J_4; and the Fischer group Fi_{22}. In the case of problem (1), we will restrict ourselves to the cases were l, m, n are distinct primes and some other triples (l, m, n) needed to solve problem (2). We will provide a complete answer to problem (2) for these groups.

In Chapter 2 we give a detailed account of certain types of 2-generations, their historical setting and survey known results as well as open problems associated with each type of 2-generation. The 2-generations we will consider are those that serve as a motivation for studying problems (1) and (2). Every finite 2-generated group is a homomorphic image of some triangular group and in the first section we introduce the triangular groups. There are three kinds of triangular group $T(l,m,n)$, depending on the sign of $M = 1 - 1/l - 1/m - 1/n$. The simple groups are homomorphic images of the triangular groups with $M > 0$. In the second section we discuss the genus action of a finite group. We start with maps and orientable surfaces and derive the equation for calculating the genus of a finite group first using combinatorial methods and then from a topological point of view. We then proceed to reduce the genus problem to one on generations. The smallest possible value of M is obtained when G is a $(2,3,7)$-generated group. In this case G is called a Hurwitz group. We study some properties of Hurwitz group in the second section. In the third section we introduce 2-generations with the generators having prescribed orders. We end this section by developing some theory on nX-complementary generations of the finite simple groups.

In Chapter 3 we develop a general theory that will be useful in resolving generation type questions of the finite simple groups. In this chapter we also set up the notational conventions to be used throughout the thesis. In the first section we focus on the important role between character theory and 2-generations. Given any three conjugacy classes lX, mY and nZ of a finite group G, with a fixed element $z \in nZ$, the number of ordered pairs $(x,y) \in lX \times mY$ such that $xy = z$ is given by the structure constant $\Delta_G(lX, mY, nZ)$. This value is easily calculated if the character table of G is known. We discuss techniques that will, under certain conditions, establish (l,m,n)-generation of G using the structure constant. We also look at the question of finding other generating pairs from a given pair. In the second section we consider 2-generated permutation groups. If we view the group G as a permutation group on n symbols, then Ree's theorem [140] gives a necessary condition for 2-generation of G. This result involves the number of cycles of the generating set. In the third section we derive Scott's theorem, a generalization of Ree's theorem to arbitrary modules. We also provide an easy method for applying Scott's theorem. In the last section we discuss the role of computer calculations in 2-generations. We identify two ways that the algebra packages GAP and MAGMA are useful in resolving 2-generations.

In Chapters 4 to 10 we apply the methods which were developed in Chapter 3 to nine of the sporadic simple groups, namely HS, McL, Co_3, Co_2, J_1, J_2, J_3, J_4 and Fi_{22}. We introduce each chapter with a summary of important properties of the particular group. We then continue to find all the (p,q,r)-generations, where $p < q < r$ are primes, and nX-complementary generations of these groups. The methods we used were more or less ad hoc and depend on the particular group's local structure. Some of the results appeared in Ganief-Moori [66], [67], [68], [69] and [70].

In Chapter 11 we return to the spread of the finite simple groups. In this chapter we are concerned with the question by Brenner-Wiegold to find all finite non-abelian groups with exact spread 1. The definition of the spread of a finite group G is not very useful for computational purposes. We refine this definition by showing that G has spread r if and only if for every set $\{x_1, \ldots, x_n\}$ of distinct elements of prime order, there exists an element $y \in G$ such that $G = \langle x_i, y\rangle$ for all i. Using this alternate definition we show that none of the sporadic simple groups has exact spread 1. Woldar [181] showed that every sporadic simple group has spread 1 and our main result in this chapter proves that every sporadic simple group has spread 2.

All groups under consideration will assumed to be finite non-abelian groups, unless otherwise stated. Computations were carried out with the aid of GAP [147] running on a SUN OS computer.

Chapter 2

Historical Overview

The aim of this chapter is to motivate and introduce the problems we will deal with in this thesis. We first introduce the triangular groups as the groups we will be considering are homomorphic images of the triangular groups.

2.1 Triangular Groups

We shall be concerned with the action of groups on several topological spaces. An *action of a finite group* G on a topological space X is given by an isomorphism of the group onto a subgroup of $Aut(X)$, the group of all conformal homeomorphisms on X. We will consider groups acting on the euclidean complex plane $\mathcal{E}^2$, the spherical plane $\mathcal{S}^2$ and the hyperbolic plane $\mathcal{H}^2$ with the standard topology. The groups acting on these planes can be derived from the projective special linear group $PSL(2, \mathbb{C})$ by forming subgroups and quotient groups. The proofs of the results in this section are out of the scope of this treatise and interested readers are referred to Coxeter-Moser [44], Magnus [118] and Jones-Singerman [95].

The *special linear group* $SL(2, \mathbb{C})$ is defined as the group of all 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad ad - bc = 1, \quad a, b, c, d \in \mathbb{C}.$$

The center of $SL(2, \mathbb{C})$ consist of the matrices $\pm I_2$, where I_2 denotes the 2×2 identity matrix. The quotient group of $SL(2,\mathbb{C})$ with respect to the center is the *projective special linear group* $PSL(2,\mathbb{C})$. Consider the transformations of the complex plane defined by

$$T(z)=\frac{az+b}{cz+d}, \qquad ad-bc\neq 0, \quad a,b,c,d\in\mathbb{C}.$$

Note that T does not define the coefficients a, b, c, d uniquely. If $\lambda\in\mathbb{C}-\{0\}$, then the coefficients λa, λb, λc, λd correspond to the same transformation T. Thus we can always choose coefficients a, b, c, d such that $ad-bc=1$. Transformations of the above type are known as *linear fractionals* or *Möbius transformations.* The Möbius transformations form a group $\mathcal{G}$ under composition.

There is a strong connection between Möbius transformations and matrices. If T is expressed in the above form, and

$$M_T=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the corresponding matrix, then there exists an onto homomorphism $\theta: SL(2,\mathbb{C})\to\mathcal{G}$ mapping $M\mapsto M_T$ with the kernel $K=Ker\theta=\{\pm I_2\}$. Therefore $\mathcal{G}\cong PSL(2,\mathbb{C})$. The group $PSL(2,\mathbb{C})$ is an example of a topological group.

Definition 2.1.1 *A topological group G is a topological space G which is also a group in which the group multiplication and taking of inverses are continuous maps. More precisely, the maps*

$$\begin{aligned} m: G\times G\to G, &\qquad \textit{defined by } m(g,h)=gh,\\ i: G\to G, &\qquad \textit{defined by } i(g)=g^{-1}, \end{aligned}$$

are continuous.

For example, consider the group $PGL(n,\mathbb{C})=GL(n,\mathbb{C})/Z(GL(n,\mathbb{C}))$. We obtain a topology on this group by considering the $n\times n$ matrix (λa_{ij}) to be the points $(\lambda a_{11},\lambda a_{12},\ldots,\lambda a_{1n},\lambda a_{21},\ldots,\lambda a_{2n},\ldots,\lambda a_{nn})$ in $\mathbb{C}^{n^2}$ (with the standard topology). Similarly $PGL(n,\mathbb{R})$ is a topological group.

We define a *discrete subgroup* H of a topological group G to be a subgroup with the properties that there is a neighbourhood U of the identity element 1_G of G such that $U \cap H = \{1_G\}$. It follows that H is a discrete subgroup of $SL(2, \mathbb{C})$ if and only if there is a sequence of matrices $\{A_n\}$ such that $A_n \to I_2$ implies that $A_n = I_2$ for almost all n. In general, it is difficult to describe the discrete subgroups H of an arbitrary topological group G. However, the discrete subgroups of $PSL(2, \mathbb{R})$ are those subgroups that meet each compact subset of $PSL(2, \mathbb{R})$ in only finitely many points. Also, in a compact topological space, every discrete subgroup is finite.

Let G be a group of homeomorphisms of a topological space X onto itself. Then we say G acts *discontinuously* on X if every $x \in X$ has a neighbourhood V such that $V \cap g(V) = \emptyset$, for all non-identity $g \in G$. Assume G acts discontinuously on X. It is well-known that if a subgroup G of $PSL(2, \mathbb{C})$ acts discontinuously in some non-empty open subset of $\mathbb{C}$, then G is discrete and contains countably many points. However, it is possible that a discrete subgroup of $PSL(2, \mathbb{C})$ does not act discontinuously on any open subset of $\mathbb{C}$ (*cf.* Beardon [7], pg 96). It is a difficult problem to decide whether a given discrete subgroup of $PSL(2, \mathbb{C})$ acts discontinuously. Under the following conditions discrete subgroups of the two-dimensional groups of motion act discontinuously.

Theorem 2.1.2 *Let G be a discrete subgroup of $PSL(2, \mathbb{C})$.*

(i) If D is a non-empty open G-invariant proper subset of $\mathbb{C}$, then G acts discontinuously on D.

(ii) If D is a non-empty open set such that $g(D) \cap D = \emptyset$ for all non-indentity $g \in G$, then G acts discontinuously on $\bigcup_{g \in G} g(D)$.

Proof. See Beardon [7], page 102. □

We call $\mathcal{F}$ a *fundamental region* of a group G if $\mathcal{F}$ is a closed subset such that

(i) $\bigcup_{g \in G} g(\mathcal{F}) = X$,

(ii) $\mathcal{F}^0 \cap g(\mathcal{F}^0) = \emptyset$, for all $g \in G - \{1_G\}$, where $\mathcal{F}^0$ is the interior of $\mathcal{F}$.

We define a *tessellation* of a plane as a division of the plane into non-overlapping closed regions, which we shall always assume are bounded by finite congruent polygons. Each individual polygon will be called a *tile*. The concept of congruent tiles implies the existence of a group of transformations in the plane, which allows us to define polygons of the same shape and size in different parts of the plane. By Theorem

2.1.2(ii) it follows that this group acts continuously on the plane if we choose D the interior of the tile. We now construct the discontinuous group of tessellations with a tile of triangular shape as a fundamental region.

Definition 2.1.3 *Let l, m, n be integers greater than 1, let*

$$\delta = 1/l + 1/m + 1/n - 1 \tag{2.1}$$

and let $\triangle$ be a triangle with angles π/l, π/m, π/n. If $\delta > 0$, then $\triangle$ is a spherical triangle. If $\delta = 0$, then $\triangle$ is euclidean, and if $\delta < 0$, then $\triangle$ is a hyperbolic (non-euclidean) triangle. Let $\triangle(l)$, $\triangle(m)$, $\triangle(n)$ be, respectively, the sides of $\triangle$ opposite to the angles of size π/l, π/m, π/n and let L, M, N, respectively, be the reflections of the particular plane in the straight lines (great circles in the spherical case) on which the sides $\triangle(l)$, $\triangle(m)$, $\triangle(n)$ lie. The group generated by L, M, N shall be denoted by $T^*(l,m,n)$ and called the full triangular group. The subgroup of $T^*(l,m,n)$ consisting of words of even length in the generators L, M, N shall be denoted by $T(l,m,n)$ and called the triangular group.

The group $T(l,m,n)$ consists of the orientation preserving isometries in $T^*(l,m,n)$ and is of index 2 in $T^*(l,m,n)$. We call $\triangle$ the *basic triangle* of the group $T^*(l,m,n)$. Note that the groups $T^*(l,m,n)$ and $T(l,m,n)$ are independent of the order in which l,m,n are listed. Now define a *local relation* as a relation in which all images of the fundamental region have one point in common. All relations associated with a chain of images of the fundamental region whose members do not have a point in common are called *global relations.* The important fact which we wish to establish is that the local relations define the triangular group.

Lemma 2.1.4 *(Magnus [118]) The only (unordered) triplets l,m,n of positive integers greater than 1 for which the quantity δ of (2.1) satisfy the condition $\delta = 0$ are $(2,3,6)$, $(2,4,4)$ and $(3,3,3)$. The triplets for which $\delta > 0$ are $(2,2,n)$, $n \geq 2$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$.* □

2.1.1 Euclidean Triangular groups

First consider the euclidean case ($\delta = 0$). Let us consider the individual cases.

Case 1: *The group* $T(2,3,6)$. The group $T = T(6,3,2)$ is generated by the elements $u = LM$ and $v = NL$, which, as euclidean motions, are represented by

$$u(z) = \epsilon z \quad \text{and} \quad v(z) - 1 = \epsilon^2(z-1). \qquad (\epsilon = e^{i\pi/3}) \tag{2.2}$$

The relations

$$u^6 = v^3 = (uv)^2 = 1_T \tag{2.3}$$

are the defining relations for $T(6,3,2) = T(2,3,6)$.

Case 2: *The group* $T(3,3,3)$. This is a subgroup of index 2 in $T(2,3,6)$. The generating elements u and v can be represented respectively by the rigid motions (rotations)

$$u(z) = \epsilon^2 z \qquad \text{and} \qquad v(z) - \epsilon = \epsilon^2(z - \epsilon). \tag{2.4}$$

Case 3: *The group* $T(2,4,4)$. This situation is close to the chessboard tessellation and the generating elements u and v of can be represented by

$$u(z) = 1 - z \qquad \text{and} \qquad v(z) = i\bar{z}. \tag{2.5}$$

Thus for the euclidean tessellation we have the following result.

Theorem 2.1.5 *For $\delta = 0$, the full triangular group $T^* = T^*(l,m,n)$ is defined by the local relations*

$$\begin{aligned} L^2 &= M^2 = N^2 && (2.6)\\ (LM)^n &= (MN)^l = (NL)^m = 1_{T^*}. && (2.7) \end{aligned}$$

The triangular group $T = T(l,m,n)$ of index 2 in $T^*(l,m,n)$, consisting of the orientation-preserving euclidean motions, is defined by two generators u, v, which are rotations with two of the vertices of $\triangle$ as center, and the relations

$$u^n = v^m = (uv)^l = 1_T, \tag{2.8}$$

where $u = LM$ and $v = NL$.

Proof. See Theorem 2.5, Magnus [118], page 68. □

2.1.2 Spherical Triangular groups

We now turn to the cases listed in Lemma 2.1.4, where $\delta > 0$. Details of the structure of the group of reflections L, M, N in the sides of a spherical triangle will be described in Cases 4 to 7.

Case 4: *The dihedral group* $T(2,2,n)$. Let $\epsilon = e^{i\pi/n}$. We choose the original triangle $\triangle$ on the sphere with one vertex at the south pole and the vertices $z = 1$ and $z = \epsilon$ on the equator. Stereographic projection maps the south pole onto $z = 0$. Reflection of $\triangle$ in the real axis produces the triangle $\triangle'$, which, together with $\triangle$, forms a fundamental region for $T = T(2,2,n)$. The motions u and v can be defined respectively by the matrices

$$U = \begin{bmatrix} \epsilon & 0 \\ 0 & \bar{\epsilon} \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \qquad U^n = V^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.9}$$

The matrix U defines a rotation with $z = 0$ as a fixed point and with $2\pi/n$ as the angle of rotation. This will be a generator u of $T(2,2,n)$. The other generator v is defined by V, or alternatively $v(z) = i/(iz)$. It is of order 2 and represents a rotation of the sphere with $z = 1$ as a fixed point. Thus we have

$$u^n = v^2 = (uv)^2 = 1_T, \tag{2.10}$$

where uv has the points $\pm\epsilon$ as fixed points. The group generated by u and v with defining relations (2.10) is of order $2n$, and every element can be expressed uniquely in the form $u^k v^l$ ($k = 0, \ldots n-1$; $l = 0, 1$). Thus $T(2,2,n)$ is isomorphic to the dihedral group D_{2n}.

Case 5: *The tetrahedral group* $T(2,3,3)$. A tessellation of the sphere with triangles congruent to $\triangle$ (with given angles) arises if we inscribe a regular tetrahedron in a sphere and mark the vertices together with the projections of the centers of the faces and the midpoints of the edges on the sphere, the center of projection being the center of the sphere. Repeated reflection of the sphere in the sides of the triangle $\triangle$ produces the group $T^* = T^*(2,3,3)$ with generators L, M, N and defining relations

$$L^2 = M^2 = N^2 = 1_{T^*} \quad \text{and} \quad (LM)^3 = (MN)^3 = (NL)^2 = 1_{T^*}. \tag{2.11}$$

This group is isomorphic to S_4, the symmetric group on four symbols. The group $T = T(2,3,3)$ is generated by $u = LM$ and $v = MN$ with defining relations

$$u^3 = v^3 = (uv)^2 = 1_T. \tag{2.12}$$

This is the *tetrahedral group*, isomorphic with A_4, the alternating group on four symbols. The rotations u, v of $T(2,3,3)$ may be presented as unitary matrices U, V defined by

$$U = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(-1-i) \\ \frac{1}{2}(1-i) & \frac{1}{2}(1+i) \end{bmatrix},$$

$$UV = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}. \tag{2.13}$$

Case 6: *The octahedral group* $T(2,3,4)$. The spherical triangle $\triangle$ with angles $\pi/4$, $\pi/3$, $\pi/2$ produces, under repeated reflections in its sides, a tessellation of the sphere by 48 congruent replicas of $\triangle$. The reflections L, M, N in the sides of $\triangle$ generates a group $T^* = T^*(2,3,4)$ with the defining relations

$$L^2 = M^2 = N^2 = 1_{T^*} \quad \text{and} \quad (LM)^4 = (MN)^3 = (NL)^2 = 1_{T^*}. \tag{2.14}$$

It has a subgroup $T = T(2,3,4)$ generated by the elements $u = LM$ and $v = MN$ with defining relations

$$u^4 = v^3 = (uv)^2 = 1_T, \tag{2.15}$$

where u and v are rotations of the sphere with corresponding, respectively, to Möbius transformations with matrices U, V defined by

$$U = \begin{bmatrix} (1-i)/\sqrt{2} & 0 \\ 0 & (1+i)/\sqrt{2} \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix}. \tag{2.16}$$

The matrices U, V generate a non-splitting central extension of order 48 of the group $T(4,3,2)$. The group $T(2,3,4)$ is isomorphic to S_4 and is called the octahedral group because the rotations u, v generate the group of all rotations which carry a regular octahedron, inscribed in the unit sphere, into itself.

Case 7: *The icosahedral group* $T(2,3,5)$. Let $\epsilon = e^{2i\pi/5}$. A tessellation of the sphere with triangles congruent to $\triangle$, with angles $\pi/5$, $\pi/3$, $\pi/2$, arises if we inscribe a regular icosahedron in a sphere. Define the matrices U, V by

$$U = \begin{bmatrix} -\epsilon^3 & 0 \\ 0 & -\epsilon^2 \end{bmatrix}, \quad V = \frac{-1}{\sqrt{5}} \begin{bmatrix} \epsilon^3 - \epsilon & 1 - \epsilon^4 \\ \epsilon - 1 & \epsilon^2 - \epsilon^4 \end{bmatrix}. \tag{2.17}$$

Then U, V define, respectively, fractional linear substitutions u, v which may be interpreted as rotations on the unit sphere. The group generated by u, v is a faithful representation of $T = T(2,3,5)$. It is defined by the local relations

$$u^5 = v^3 = (uv)^2 = 1_T. \tag{2.18}$$

The group generated by the matrices U, V itself is a non-splitting central extension of $T(2,3,5)$. Its center is of order 2 and generated by $U^5 = V^3$. It follows from elementary geometric arguments that $T(2,3,5)$ is of order 60 and is isomorphic to A_5.

We summarize the general result as:

Theorem 2.1.6 *The reflections L, M, N in the sides of a spherical triangle ($\delta > 0$) $\triangle$ generate a group $T^* = T^*(l,m,n)$ for which $\triangle$ is a canonical fundamental region. The local relations*

$$L^2 = M^2 = N^2 \tag{2.19}$$
$$(LM)^n = (MN)^l = (NL)^m = 1_{T^*}. \tag{2.20}$$

define the group. The group $T = T(l,m,n)$ is finite and is generated by u, v, which are rotations with two of the vertices of $\triangle$ as center, and the relations

$$u^n = v^m = (uv)^l = 1_T, \tag{2.21}$$

where $u = LM$ and $v = NL$.

Proof. See Theorem 2.6, Magnus [118], page 71. □

2.1.3 Hyperbolic Triangular groups

Recall that a hyperbolic triangle with angles α, β and γ exists provided that $\alpha+\beta+\gamma < \pi$. We shall consider triangles with angles π/l, π/m, π/n, where l, m, n are positive integers. The general results are given in the following result.

Theorem 2.1.7 *Let L, M, N be the reflections in the sides of a hyperbolic triangle $\triangle$ with angles π/l, π/m, π/n. The images of $\triangle$ under the action of the distinct elements of the group $T^* = T^*(l,m,n)$ generated by L, M, N fill the hyperbolic plane without gaps and overlapping. The group $T^*(l,m,n)$ is defined by the local relations*

$$L^2 = M^2 = N^2 = 1_{T^*}, \qquad (LM)^n = (MN)^l = (NL)^m = 1_{T^*}. \tag{2.22}$$

Proof. See Theorem 2.8, Magnus [118], page 81. $\square$

Theorem 2.1.8 *Let $T(l,m,n)$ be the subgroup of index 2 in the full triangular group $T^*(l,m,n)$ which consists of the orientation-preserving isometries of the hyperbolic plane. Then $T = T(l,m,n)$ is generated by $u = LM$ and $v = MN$ with relations defined by*

$$u^n = v^l = (uv)^m = 1_T. \tag{2.23}$$

Furthermore, the non-identity elements of finite order in $T(l,m,n)$ are conjugates of powers of u, v, or uv.

Proof. See Theorem 2.10, Magnus [118], page 87. $\square$

Although it can be shown in general that a non-euclidean triangle with angles α, β, γ exists and is uniquely determined if α, β, γ are non-negative and $\alpha+\beta+\gamma < \pi$, we shall give an explicit construction of a right-angle triangle and an explicit representation of the subgroup of proper non-euclidean motions of the group generated by reflections in the sides of the triangle. We use the unit disk as a model for non-euclidean geometry.

Theorem 2.1.9 *Let $\alpha > 0$ and $\beta \geq 0$ be angles such that $\alpha+\beta < \pi/2$. Let O, Q, P be three points in the unit disk $|z| < 1$, defined respectively by their coordinates*

$$z_O = 0, \qquad z_Q = x_Q, \qquad z_P = x_P + iy_P,$$

where

$$x_Q = (\cos\beta - \sin\alpha)/\rho, \qquad x_P = (\cos\alpha\cos(\alpha+\beta))/\rho,$$

$$y_P = (\sin\alpha\cos(\alpha+\beta))/\rho$$

and

$$\rho = (\cos^2\beta - \sin^2\alpha)^{1/2}.$$

Then O, Q, P are vertices of a non-euclidean triangle with angles α, $\pi/2$, β, respectively, at O, Q, P. The sides OQ and OP are respectively parts of the real axis and the straight euclidean line joining O and P. The side QP is part of the circle with center at $z = x_c$, where $x_c = (\cos\beta)/\rho$, and with radius $r = (\sin\alpha)/\rho$.
Let L, M, N denote, respectively, the reflections in the sides OQ, QP, PO of the triangle. Then they generate a group Γ^* which has a subgroup T of index 2 consisting of orientation-preserving non-euclidean motions and is generated by $u = LM$ and $v = LN$. To u and v there correspond respectively matrices U and V of Möbius transformations which are given by

$$U = \frac{i}{\sin\alpha}\begin{bmatrix} \cos\beta & \rho \\ -\rho & -\cos\beta \end{bmatrix}, \qquad V = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}. \tag{2.24}$$

and which map the unit circle onto itself. If $\alpha = \pi/m$ and $\beta = \pi/l$, where l and m are integers, then U and V define a group T of Möbius transformations with the defining relations

$$U^2 = V^m = (UV)^l = 1_T. \tag{2.25}$$

Proof. See Theorem 2.11, Magnus [118], page 88. □

Corollary 2.1.10 *The triangular group $T(l, m, n)$ is finite if and only if $\delta > 0$, that is, the triangle $\triangle$ is spherical.*

Proof. This is an immediate consequence of Theorems 2.1.5, 2.1.6 and 2.1.8. □

2.1.4 Fuchsian groups

All of the orientation-preserving discontinuous group of two-dimensional isometries discussed in the above sections are very special cases of the Fuchsian groups, which can be defined both geometrically and algebraically as follows.

Geometrically, a *Fuchsian group* F is defined as a finitely generated discontinuous group of orientation-preserving isometries, or equivalently, as a discontinuous group of Möbius transformations which map a circular disk onto itself. The Fuchsian groups are exactly the discrete subgroups of the group $PSL(2, \mathbb{R})$. Fuchsian groups were first studied systematically by Poincaré in 1880, although some particular examples such as the modular group and the triangular groups were investigated earlier. Poincaré was led to the Fuchsian groups after reading a paper by L. Fuchs on differential equations. The following important and difficult theorem defines the Fuchsian group algebraically. It was first established by Fricke [64] and proved with a different method by Heins [79].

Theorem 2.1.11 (Fricke) *A Fuchsian group F can be characterized by a sequence of exponents m_j $(j = 1, \ldots, t)$ where m_j are integers with $m_j \geq 2$, and an integer g (the genus). The group F has $t + 2g$ generators y_j, a_i, b_i $(i = 1, \ldots, g$ unless $g = 0)$ and defining relations*

$$y_j^{m_j} = 1_F, \qquad y_1 y_2 \ldots y_t \Pi_{i=1}^{g} [a_i, b_i] = 1_F. \tag{2.26}$$

We call the combination of the m_j and g the signature *of F and denote the group F by $F(g; m_j)$ or, explicitly, (if the m_j are given numerically) by*

$$F(g; m_1, \ldots, m_t). \tag{2.27}$$

The choice of the m_j and the g is arbitrary provided that

$$\mu(F) = 2g - 2 + \sum_{j=1}^{t} (1 - 1/m_j) > 0. \tag{2.28}$$

The number $\mu(F)$ is called the measure *of F. The area of the fundamental region of F is $\pi\mu(F)$. In a representation of F as a group of Möbius transformations, the generators y_j are represented by* elliptic *transformations. The generators a_i, b_i are always represented by* hyperbolic *transformations.* □

Fuchsian groups without elliptic transformations will be denoted by $F(g;-)$. The spherical triangular groups $T(l,m,n)$ discussed in the previous sections are Fuchsian groups $F(0;l,m,n)$. Like the free groups, the Fuchsian groups (as abstract groups) form a class of groups in which certain subgroups theorems holds. For instance, every subgroup of finite index in a Fuchsian group is again a Fuchsian group. If F is a Fuchsian group and S a subgroups of F with finite index m, then

$$\mu(S) = m\mu(F). \tag{2.29}$$

A finitely generated normal subgroup of a Fuchsian group is also of finite index (*cf.* Greenberg [74]). Furthermore, Hoare *et al.* [80] proved the relation (2.29) for the Fuchsian groups and their subgroups purely group theoretically, defining a Fuchsian group by a presentation of type (2.26). Also Curran [45], using the methods of homological algebra, showed that $\mu(F) > 0$ is a necessary and sufficient condition for F to be infinite. A remarkable new result on the Fuchsian groups (using the abstract definition) was proved algebraically by Hoare *et al.*, [81]. It states that a subgroup of infinite index in a Fuchsian group defined by (2.26) is a free product of cyclic groups.

2.2 Genus actions of the finite simple groups

An action of a finite group G on a Riemann surface is called a genus action provided G acts effectively and analytically on S but does not so act on any other Riemann surface of lesser genus (notation to be defined later). A natural question to ask is: *Why are we interested in genus actions of finite simple groups?*

One answer is that we seek a geometric representation theory of finite simple groups as automorphism groups on Riemann surfaces. The genus action play the role of irreducible representations in this theory. When these genus actions give rise to full automorphism groups of the related surface, our representation theory has a more natural form. This geometrical representation theory in turn has a faithful image in the ordinary integral representation theory of the group by means of the induced homological representation. We seek to relate these integral representations to the "number theory" of finite simple groups. The genus homology representations are minimal integral representations and as such should have a special status.

Although genus actions arise in a topological context, we will develop it using combinatorial arguments as well. We now continue to discuss graphs realised by a set of points and lines on a closed orientable surface.

2.2.1 Maps and Orientable surfaces

Maps on surfaces have been studied for two main reasons. Geometers have been interested in symmetric properties of maps, and this led to the investigation of regular maps, those possessing the greatest possible symmetry. Combinatorialists, on the other hand, have concentrated on map colourings and the graph embeddings. In this section we will concentrate on graph embeddings.

Definition 2.2.1 *A surface S is a compact topological space which has two special properties:*

(i) it is locally homeomorphic to triangles;

(ii) it is orientable.

A surface is *orientable* if it is possible to choose a consistent sense of orientation (clockwise or anti-clockwise) at every point on the surface. A decomposition of a surface into triangles is called a *triangulation*. We may think of a surface as a connected compact 2-manifold without boundary points, that is, all surfaces here are closed. The simplest example is the *sphere*, decomposed into four (spherical) triangles by projection on an inscribed tetrahedron. The next simple surface is the *torus*, obtained by identifying opposite sides of the euclidean 2-space, formed by two euclidean triangles. The Klein bottle is not allowed, since it is not orientable.

Definition 2.2.2 *A graph Γ is a pair (V, E), where V is a finite set whose members are called vertices, and E is a subset of $V^{(2)}$, the set of unordered pairs of vertices. The members of E are called* edges. *If $\{v, w\}$ is an edge of Γ, then we say that v and w are* adjacent.

A *walk of length r* is a sequence $(v_0, v_1, \ldots, v_r)$ such that $\{v_{i-1}, v_i\}$ is an edge for $1 \leq i \leq r-1$. The graph Γ is said to be *connected* if any two vertices can be joined by a walk. A *circuit* is an ordered set of distinct vertices $(a, b, \dots, f)$ such that $\{a, b\}$, $\{b, c\}$, ...,$\{f, a\}$ are edges of the graph Γ.

An *embedding* of a graph $\Gamma = (V, E)$ in a surface S is a representation of the surface by the set V and lines E' on the surface, in such a way that the lines intersect only at the points representing their end vertices, that is, a line (v, w) in E' is an edge $\{v, w\}$ in E. The lines divide the surface into connected regions, called *faces*, and the resulting configuration is a *map*. Thus a map has *points* (vertices), *lines* (edges) and *faces* (regions). An embedding of the graph Γ in a surface S is *cellular* if each face is homeomorphic to an open disc. We will assume all embeddings of Γ in S are cellular. At each vertex v, the neighbourhood of v is locally like a plane, and so the vertices adjacent to v have a cyclic ordering ρ_v corresponding to the arrangement of the edges joining them to v on the surface. We are now ready to present the formal definition of a map.

Definition 2.2.3 *A rotation of a graph $\Gamma = (V, E)$ embedded in a surface S is a set $\rho = \{\rho_v\}_{v \in V}$, where ρ_v is a cyclic permutation of the vertices adjacent to v in Γ. A map is a pair (Γ, ρ), where Γ is a connected graph and ρ is a rotation on Γ.*

For example, a rotation on the complete graph K_n is the set of all $(n-1)$-cycles in the symmetric group S_n. We turn to the formal definition of the faces of a graph embedding. Let $S\Gamma$ denote the set of sides of Γ:

$$S\Gamma = \{(v, w) | \{v, w\} \text{ is an edge of } \Gamma\}. \tag{2.30}$$

Thus each edge $\{v, w\}$ gives rise to two sides (v, w) and (w, v). A rotation ρ on Γ induces a permutation on $S\Gamma$ given by

$$\rho(v, w) = (v, \rho_v(w)). \tag{2.31}$$

This corresponds to rotating the sides pointing away from v in the order prescribed by ρ_v. The cycles of ρ on S are in one-to-one correspondence with the vertices of Γ. Define a permutation ρ^* on S as follows:

$$\rho^*(v, w) = (w, \rho_w(v)) = \rho(w, v). \tag{2.32}$$

Definition 2.2.4 *Let (Γ, ρ) be a map and suppose ρ^* is defined above. A face of (Γ, ρ) is a cyclic sequence of vertices occurring in a cycle of ρ^* on S.*

One of the oldest results in the theory of maps concerns a relationship linking the numbers of vertices, edges and faces. In order to obtain this result by combinatorial means we need some elementary lemmas.

Lemma 2.2.5 *Given any connected graph $\Gamma = (V, E)$ we can find a subset T of E such that the graph (V, T) is connected and has no circuits. Furthermore, $|T| = |V| - 1$.*

Proof. We use induction on $n = |V|$. The result is trivial if $n = 2$. Suppose that the result is true for $n - 1$ vertices, and let Γ be a connected graph with $|V| = n$. Choose $v \in V$ and let E_1 denote the set of edges not incident with v. The graph $\Gamma' = (V - \{v\}, E_1)$ is the union of disjoint connected graphs $\Gamma_\lambda = (V_\lambda, E_\lambda)$, and by the induction hypothesis, each E_λ contains a subset T_λ such that (V_λ, T_λ) is connected and has no circuits. Also, since Γ is connected, there is an edge e_λ joining v to some vertex $v_\lambda \in V_\lambda$. Let T be the union of all the edges e_λ and all the edges in the sets T_λ. We have

$$\begin{aligned} |T| &= \sum_\lambda |T_\lambda \cup \{e_\lambda\}| \\ &= \sum_\lambda |V_\lambda| \qquad \text{(by induction)} \\ &= |V - \{v\}| = |V| - 1. \end{aligned}$$

Thus T is the required subset of E. □

The set T, or the graph (V, T), is called a *spanning tree* for Γ. Suppose that $D \subseteq E$, and τ_D is the permutation of $S\Gamma$ defined by

$$\tau_D(v, w) = \begin{cases} (w, v) & \text{if } \{v, w\} \in D; \\ (v, w) & \text{otherwise.} \end{cases} \tag{2.33}$$

Then τ_D is the composition of transpositions τ_e ($e \in D$), where τ_e switches the two sides corresponding to the edge e and fixes every other side. The number of cycles of a permutation σ will be denoted by $c(\sigma)$. If σ is any permutation and $\tau = (v\ w)$ is a transposition, then it is easy to check that $c(\sigma)$ is related to $c(\sigma\tau)$ as follows:

$$c(\sigma\tau) = \begin{cases} c(\sigma)+1, & \text{if } v \text{ and } w \text{ are in the same cycle of } \sigma \\ c(\sigma)-1, & \text{otherwise.} \end{cases}$$

Lemma 2.2.6 *If (Γ,ρ) is a map and T is a spanning tree for Γ, then $\rho\tau_T$ has just one cycle on $S\Gamma$.*

Proof. Let $T = \{e_1,\ldots,e_{n-1}\}$, where n is the number of vertices of Γ. By its definition, τ_T is the composition of $n-1$ transpositions τ_i that switches the sides corresponding to e_i. Thus $\rho\tau_T = \rho\tau_1\tau_2\cdots\tau_{n-1}$. Since the two sides of e_1 are in different cycles of ρ acting on $S\Gamma$, we have $c(\rho\tau_1) = c(\rho) - 1 = n-1$. Now we may proceed inductively: suppose that $c(\rho\tau_1\cdots\tau_m) = n-m$, for some value m $(1 \le m \le n-1)$, then $c(\rho\tau_1\cdots\tau_{m+1}) = n - m \pm 1$.

If the plus sign holds, the two sides of e_{m+1} must be in the same cycle of $\rho\tau_1\cdots\tau_m$ on $S\Gamma$. This means that there is a chain of edges selected from $\{e_1,\ldots,e_m\}$ joining the two end vertices of e_{m+1}, in other words, we have a circuit in T. Thus the minus sign must hold, and the induction step is complete. It follows that

$$c(\rho\tau_T) = c(\rho\tau_1\cdots\tau_{n-1}) = n-(n-1) = 1. \qquad \square$$

Theorem 2.2.7 *Let $\Gamma = (V,E)$ be a connected graph and ρ a rotation on Γ. Let F denote the set of faces of the map (Γ,ρ) on a surface S. Then there is a non-negative integer g such that*

$$|V| - |E| + |F| = 2 - 2g. \tag{2.34}$$

Proof. Let T be a spanning tree of Γ, and let $U = E - T$. Then $\rho^* = \rho\tau_E$, since

$$\rho\tau_E(v,w) = \rho(w,v) = (w, \rho_w(v)) = \rho^*(v,w).$$

Now $|F|$ is, by definition, the number of cycles of ρ^*, and so

$$|F| = c(\rho^*) = c(\rho\tau_E) = c(\rho\tau_T\tau_U).$$

We have established in Lemma 2.2.6 that $c(\rho\tau_T) = 1$. To obtain ρ^* from $\rho\tau_T$ we form the composition with the transposition ρ_e $(e \in U)$. As each transposition is added, the number of cycles either increases or decreases by unity. Suppose that it increases h times and decreases g times. We have

$$\begin{aligned} h+g &= |U| = |E-T| = |E| - |V| + 1, \\ 1+h-g &= c(\rho^*) = |F|. \end{aligned}$$

Eliminating h gives

$$|V| - |E| + |F| = 2 - 2g,$$

as required. □

Definition 2.2.8 *The non-negative integer $g = g(S)$ occurring in Theorem 2.2.7 is called the genus of the surface S with respect to the map (Γ,ρ).*

2.2.2 Finite group actions on surfaces and Cayley graph embeddings

We have defined the genus of a surface in a purely combinatorial way. We will now discuss the action of a finite group on a closed surface from a topological point of view and show that when a group acts on an orientable surface then a Cayley graph of the group embeds in the surface.

Recall an action of a finite group G on a topological space X is given by an isomorphism of the group G onto a subgroup of the group of all homeomorphisms on X. We will not differentiate between the abstract group G and the subgroup of homeomorphisms; if $a \in G$ we will refer to the homeomorphism a. The *stabilizer* of $x \in X$ is a subgroup of G defined by

$$G_x = \{a \in G \mid a(x) = x\} \tag{2.35}$$

and

$$\mathrm{Fix}\, G = \{x \in X \mid G_x \neq 1_G\}. \tag{2.36}$$

An action of a group G on X is *free* if Fix G is empty and *pseudo-free* if Fix G is non-empty but discrete. If we assume G is finite and X is a surface, then Fix G must be finite if the action is pseudo-free (*cf.* Tucker [163]).

Given an action of the group G on the space X, the *orbit* of a point $x \in X$ is the set

$$[x] = \{y \in X \mid y = a(x) \text{ for some } a \in G\}. \tag{2.37}$$

The *quotient space* X/G is the set of all orbits with the topology that $U \subset X/G$ is open if and only if $p^{-1}(U)$ is open, where $p : X \to X/G$ is the *natural projection* $p(x) = [x]$.

Suppose G is a finite group acting pseudo-freely on a surface S. Then S/G is a surface and the natural projection $p : S \to S/G$ is a *branched covering*, that is, a local homeomorphism except at the points in $\operatorname{Fix} G$ (*cf.* Jones-Singerman [95], page 248). The set $p(\operatorname{Fix} G)$ is called a *branched set.* If $x \in \operatorname{Fix} G$, then p is locally $|G_x|$-to-one in a neighbourhood of x. If $y \in p(\operatorname{Fix} G)$, then $|G_x|$ is the same for any $x \in p^{-1}(y)$. The common number is called the *order* of the branch point y and denoted by m_y. It follows that $|p^{-1}(y)| = |G|/m_y$.

We define the *Euler characteristic* of a surface S as $\chi(S) = \alpha_0 - \alpha_1 + \alpha_2$, where α_0 is the number of vertices, α_1 the number of edges and α_2 the number of triangles in the triangulation of the surface S. The Euler characteristic of S can easily be computed from the Euler characteristic of S/G and the order of the branch points. We triangulate S/G so that every branch point is a vertex of the triangulation $\triangle$. Then $p^{-1}(\triangle)$ is a triangulation of S. Since $|p^{-1}(t)| = |G|$, if t is an edge, triangle or non-branch point vertex of $\triangle$, we have

$$\chi(S) = |G|(\chi(S/G) - \sum_y (1 - 1/m_y)), \tag{2.38}$$

where the sum is taken over the branch set (*cf.* Farkas-Kra [58], Section I.2.7). The Euler characteristic of S is also related to the genus of S.

Proposition 2.2.9 *(Jones-Singerman [95]) The Euler characteristic of a compact, connected and orientable surface S of genus $g(S)$ is given by $\chi(S) = 2 - 2g(S)$.* □

The combinatorial equivalence of this proposition is given by Theorem 2.2.7. Now from (2.38) we obtain

$$g(S) = 1 + \frac{|G|}{2}[2g(S/G) - 2 + \sum_y (1 - 1/m_y)]. \tag{2.39}$$

This equation is called the *Riemann-Hurwirtz* equation. A surface with genus g may be considered topologically as a sphere with g *handles*. A surface of genus 0 is topologically a 2-sphere and a surface of genus 1 is topologically a torus.

Proposition 2.2.10 *(Scherrer [146]) Let S be an orientable surface and $h : S \to S$ a homeomorphism of finite order. Then one of the following occur:*

(a) h has a finite number of fixed points,

(b) h is an orientation reversing involution and $S = S_1 \cup S_2$, where S_1 is connected, $h(S_1) = S_2$, and $S_1 \cap S_2$ is a finite collection of disjoint simple closed curves, at least one of which is left point-wise fixed by h. $\square$

It follows from Proposition 2.2.10 that any finite group acting on an orientable surface S acts pseudo-freely unless the action has a reflection. In particular, if G^0 denotes the subgroup of orientation preserving homeomorphisms of an action of G on an orientable surface, then G^0 always acts pseudo-freely.

Definition 2.2.11 *Let G be a finite group, with $X \subseteq G$ such that:*

(i) X generates G,

(ii) $1_G \notin X$

(iii) $x \in X$ implies $x^{-1} \in X$,

Then the Cayley graph $C(G, X)$ has vertex-set G and, for $g, h \in G$, $\{g, h\}$ is an edge of $C(G, X)$ if and only if $gh^{-1} \in X$.

For example, the complete graph K_n is a Cayley graph $C(G, X)$ for any group G of order n, where $X \in G - \{1_G\}$. Condition (i) of the definition implies that a Cayley graph is always connected. Condition (ii) ensures that no vertex is adjacent to itself, while condition (iii) guarantees that the edges are unordered pairs.

Proposition 2.2.12 *(Biggs-White [9]) The graph Γ is a Cayley graph $C(G, X)$ of a group G if and only if G acts transitively on the vertices of Γ such that no vertex of Γ is left fixed by a non-identity element of G. Edges in the same orbit of G correspond to the same generator in X.* $\square$

We will show that if a group acts on an orientable surface, then a Cayley graph of the group embeds in the surface. Along the way we will also give necessary and sufficient conditions that are satisfied by some presentation of the group in order that it acts on the given surface in a given fashion. To quell any uprisings from the many subscripts needed, we introduce some notational conveniences. In listing generators for a presentation of a Fuchsian group F, we indicate the range of a subscript in the subscript; thus a_h means $a_1, a_2, \ldots, a_h$. In giving relations, we only list a representative relation for each subscript; for example, if y_t is given in a generating set, then $y^m = 1_F$ means $y_1^{m_1} = \ldots = y_t^{m_t} = 1_F$, or if ${}_r z_{s(r)+1}$ is given in the generating set, then $(z_j z_{j+1})^{q_j}$ means $[({}_i z_j)({}_i z_{j+1})]^{q_j} = 1_F$ for all $j = 1, \ldots, s(i) + 1$ and all $i = 1, \ldots, r$. Finally, $\prod y = y_1 y_2 \cdots y_t$ and $\prod[a, b] = \prod_{i=1}^{h}[a_i, b_i]$.

Theorem 2.2.13 *(Tucker [163]) Let G be a finite group acting without reflections on the orientable surface S. Let $n = 2 - \chi(S/G) = 2 - 2h$ and let $p : S \to S/G$ have t branch points of order $m_1, \ldots, m_t$. Then there is a Cayley graph $C(G, X)$ that embeds cellularly in S, where X is the generating set in one of the following partial presentations of G:*

$$\langle a_h, b_h, y_t : \ y^m = 1_G, \quad \Pi[a, b]\Pi y = 1_G, \ldots\rangle \quad \text{if} \quad G = G^0, \tag{2.40}$$

$$\langle c_n, y_t : \ y^m = 1_G, \quad \Pi c^2 \Pi y = 1_G, \ldots\rangle \quad \text{if} \quad G \neq G^0. \tag{2.41}$$

Moreover, in case (2.41), G^0 contains y_t but not c_n. □

The situation is somewhat more complicated when G contains reflections. Then G/G^0 acts on the surface $S^0 = S/G^0$ as a single reflection τ. We seek a 2-vertex graph Γ that cellularly embeds in S^0 so that every face contains at most one branch point and such that the reflection leaves Γ invariant ($h(\Gamma) = \Gamma$) but interchanges the two vertices of Γ. The idea is to construct Γ', half of Γ, and then reflect this half to get the other half of Γ. If S_1^0 represents half of the surface S^0 under the reflection, then Γ' must take into account the genus of S_1^0, the boundary components $C_1, \ldots, C_n$ of S_1^0 (the fixed circles of τ), the branch points of G^0 lying inside S_1^0, and the branch points of G^0 lying on the fixed circles of τ. We summarize:

Theorem 2.2.14 *(Tucker [163]) Let G act on an orientable surface S with reflections. Then G/G^0 acts on S/G^0 as a single reflection. Let $C_1, \ldots, C_r$ be the fixed circles of the reflection and S_1^0 a half of the reflection. Let G^0 have t branch points in the interior of S_1^0 of orders $m_1, \ldots, m_t$, and $s(i)$ branch points of order ${}_iq_j$, $1 \leq j \leq s(i)$, on the circle C_i, $1 \leq i \leq r$. Then there is a Cayley graph $C(G, X)$ that cellularly embedded in S, where X is the generating set for the partial presentation:*

$$\langle a_h, b_h, y_t, w_r, {}_rz_{s(r)+1} \; : \; y^m = 1_G, \quad z^2 = 1_G, \quad (z_jz_{j+1})^{q_i} = 1_G,$$
$$wz_1w^{-1}z_{s(i)+1} = 1_G, \quad \Pi[a, b]\Pi y \Pi w = 1_G, \ldots\rangle$$

Moreover, G^0 contains a_h, b_h ,y_t, w_r but not ${}_rz_{s(i)+1}$. □

Observe that in the case that S/G^0 is a sphere, there are no a_i's or b_i's and since there can be only one fixed circle, the ${}_iz_j$'s need not be doubly subscripted. The partial presentation is then not so unwieldy. The presentation in the above theorem is a quotient of a non-euclidean space group. It should also be noted that the Cayley graph embedding in each of the Theorems 2.2.13 and 2.2.14 is invariant under the action of the group G on the surface S. There are converses for these theorems as well.

Theorem 2.2.15 *(Tucker [163]) Let G be a finite group having a partial presentation of type (2.40) or (2.41) in Theorem 2.2.13. Then G acts pseudo-freely on a surface S of Euler characteristic*

$$\chi(S) = |G|(\chi(S/G) - \sum_{k=1}^{t}(1 - 1/m_k)).$$

If the representation is of type (2.40), then S is orientable and the action of G is orientation preserving. If the representation is of type (2.41) and there is a subgroup of index 2 in G containing y_t but not c_n, then S is orientable but the action is not orientation preserving. Otherwise, S is non-orientable. □

Theorem 2.2.16 *(Tucker [163]) Let G be a finite group having a partial presentation given in Theorem 2.2.14, such that there is a subgroup H of index 2 in G containing*

a_h, b_h, y_t and w_r but not ${}_r z_{s(i)+1}$. Then G acts with reflections on an orientable surface S of Euler characteristic

$$\chi(S) = |G|(2 - 2h - r - \sum_k (1 - 1/m_k) - \sum_{i,j} (1 - 1/{}_i q_j)/2).$$

□

2.2.3 Reduction of the genus problem for finite simple groups

Definition 2.2.17 *The genus $g = g(G)$ of a finite group G is the smallest integer g such that some Cayley graph of G is embedded on an orientable, compact surface with genus g.*

Equivalently, the genus of a group G can be defined as the smallest integer such that G acts effectively and analytically on a compact surface with genus g. An old question, usually referred to as *Nielsen Realization Problem*, is whether every finite subgroup of the group of isotopy classes of diffeomorphisms $\pi_0 \mathit{Diff}(\mathcal{U})$ ($\mathcal{U}$ a closed hyperbolic surface) arises as a group of isometries of some hyperbolic surface. This question was answered in the affirmative by Kerckhoff.

Proposition 2.2.18 *(Kerckhoff [100]) Every finite subgroup G of $\pi_0 \mathrm{Diff}(\mathcal{U})$ can be realized as a group of isometries of a hyperbolic surface.* □

As a consequence of this result, no generality is lost if we require the action of G on S to be *conformal*, that is, analytic in some complex structure of S. Thus the surface in question can be assumed to be a *Riemann surface* which admit an effective conformal action by G. We illustrate presently how the class of hyperbolic triangular groups of such surfaces arise.

Let $\mathcal{H}^2$ be the hyperbolic plane and $T = T(l, m, n)$ be the hyperbolic triangular group acting on the triangle $\triangle$ with angles π/l, π/m, π/n.

Proposition 2.2.19 *(Jones-Singerman [95]) Let F be a Fuchsian group action on $\mathcal{H}^2$. Then the quotient space $\mathcal{H}^2/F$ is a connected Riemann surface and $\pi : \mathcal{H}^2 \to \mathcal{H}^2/F$ is a holomorphic map.* □

We may choose the region $\triangle$ as fundamental region for the group T . From Theorem 2.1.11 the area of $\triangle$ is

$$\begin{aligned} \pi\mu(T) &= \pi[2g-2+(1-1/l)+(1-1/m)+(1-1/n)] \\ &= \pi[2g+1-1/l-1/m-1/n], \end{aligned}$$

where g is the genus of $\mathcal{H}^2/T$. However, the area of a hyperbolic triangle with angles α, β, γ is $\pi-(\alpha+\beta+\gamma)$ (*cf.* Beardon [7], page 150). Thus

$$\pi[2g+1-1/l-1/m-1/n] = \pi[1-1/l-1/m-1/n],$$

and hence the genus of $\mathcal{H}^2/T$ is 0 and $\mathcal{H}^2/T = \mathcal{S}^2$, a 2-sphere.

Proposition 2.2.20 *(Tucker [163]) Let G be a group acting on an ortientable surface S. Let N be a normal subgroup of G. Then G/N acts on S/N so that $(S/N)/(G/N) = S/G$. If $N \subseteq G^0$, then S/N is an orientable surface.* □

Definition 2.2.21 *A group G is said to be $(n_1, \ldots, n_r)$-generated if, G can be generated by $x_1, \ldots, x_r$, such that*

$$x_i^{n_i} = 1_G, \quad \textit{for} \quad i = 1, \ldots, r, \quad \textit{and} \quad x_1 x_2 \ldots x_r = 1_G.$$

If $r = 3$, then G is a quotient group of the triangular group $T(n_1, n_2, n_3)$ and if $r = 4$, then G is a quotient group of a quadrangular group $Q(n_1, \ldots, n_4)$.

Now suppose G is a finite (l, m, n)-generated group. Consider the short exact sequence

$$1_\triangle \to \triangle \to T(l, m, n) \to G \to 1_G. \tag{2.42}$$

Using Proposition 2.2.19 the group $G \cong T/\triangle$ acts orientably on the closed Riemann surface $\mathcal{H}^2/\triangle$, and the branch covering $\mathcal{H}^2/\triangle \to (\mathcal{H}^2/\triangle)/(T/\triangle) = \mathcal{H}^2/T$ has 3 branch points of respective orders l, m, and n. By the Riemann-Hurwitz formula, we compute

$$g(\mathcal{H}^2/\triangle) = 1 + \frac{|G|}{2}(1-1/l-1/m-1/n). \tag{2.43}$$

Thus, if we further assume G is $(2, m, n)$-generated, then

$$g \leq 1 + \frac{|G|}{2}(1/2 - 1/m - 1/n), \tag{2.44}$$

where g is the least genus of any surface which admits an effective and analytic action by G.

Theorem 2.2.22 *(Woldar [177]) Let G be a finite non-abelian $(2, m, n)$-generated group and let S be a Riemann surface of least genus on which G acts. Then $S/G = S^2$ and $\pi : S \to S/G$ has either 3 or 4 branch points.*

Proof. From relation (2.44) and the Riemann-Hurwitz equation, we have

$$2h - 2 + \sum_{i=1}^{b}(1 - 1/n_i) \leq 1/2 - 1/m - 1/n, \tag{2.45}$$

where h is the genus of S/G and n_i denotes the order of the branch point x_i of S $(1 \leq i \leq b)$. Since $1/m + 1/n < 1/2$, it follows from (2.45) that $0 < \sum_{i=1}^{b}(1 - 1/n_i) \leq 5/2 - 2h$, and therefore $h \leq 1$. If $h = 1$, then $n_i \geq 2$ implies that $b = 0$, whence G acts fixed point free on S with orbit space the torus, a contradiction. Thus $h = 0$ and

$$\begin{aligned} & -2 + b - \sum_{i=1}^{b} 1/n_i \leq 1/2 - (1/m + 1/n) < 1/2, \\ \Rightarrow \quad & b \leq 2 + \sum_{i=1}^{b} 1/n_i < 5/2 + b/2, \qquad (n_i \geq 2) \\ \Rightarrow \quad & b < 5. \end{aligned}$$

As G cannot act as a transformation group for the regular unbranched covering $S - \pi^{-1}(x_i) \to \mathbb{C}$, we have $b \geq 3$ and the result follows. □

Let us consider the case where G is a sporadic simple group.

Proposition 2.2.23 *(Woldar [176]) Every sporadic simple group is $(2, m, n)$-generated, for some integers m and n. Moreover, all sporadic simple except M_{11}, M_{22}, M_{23} and McL are $(2, 3, n)$-generated, for some n.* □

The techniques used in the proof of the above theorem will be discussed at length in the next chapter and examples will be provided in subsequent chapters. From the foregoing discussions it is clear that in the minimal genus action for a sporadic simple group G, the surface $S = \mathcal{H}^2/\Delta$ covers the 2-sphere $\mathcal{H}^2/T(l,m,n) = \mathcal{S}^2$ and S arises from a short exact sequence

$$1_\Delta \to \Delta \to \Gamma \to G \to 1_G, \tag{2.46}$$

where Γ is either a triangular group $T(l,m,n)$ or a quadrangular group, $Q(u,v,w,x)$. The Riemann-Hurwitz equation gives

$$g(G) = 1 + \frac{|G|}{2}\mathcal{M}, \tag{2.47}$$

where $\mathcal{M} = 1 - 1/l - 1/m - 1/n$ or $\mathcal{M} = 2 - 1/u - 1/v - 1/w - 1/x$ depending, respectively, on whether Γ is triangular or quadrangular in the minimal genus action.

Thus a general methodology for determining the genus of a sporadic simple group is to minimize the quantity $\mathcal{M}$ over all triples (l,m,n) and quadruples (u,v,w,x) for which G has (l,m,n)- or (u,v,w,x)-generated. Thus the genus problem for the sporadic groups is reduced to one of generation, a decisive improvement as it enables us to bring to bear on the genus problem powerful techniques from group theory and character theory. The genus problem has been solved for 24 of the 26 sporadic simple groups and are listed in Table 2.I. The problem is as yet unresolved for the Fischer group Fi_{23} and the Fischer Monster M.

The requirement in Theorem 2.2.22 that G is $(2,m,n)$-generated is far less restrictive for the finite simple groups than it would appear at first. Indeed it is a longstanding conjecture that every finite non-abelian simple group is so generated. In particular the conjecture has been verified for the families of alternating, sporadic and a number of classes of groups of Lie type. We will discuss this problem in the next section. The group Γ in (2.46) appears with overwhelming frequency to be a triangular group. Indeed, there is no known example of a finite simple group in which Γ is non-triangular in its genus action.

TABLE 2.I

Genus of the sporadic simple groups

G	genus of G	(l, m, n)	Proof
M_{11}	631	$(2, 4, 11)$	Woldar [178], Conder [36]
M_{12}	3169	$(2, 3, 10)$	Woldar [178], Conder [36]
J_1	2091	$(2, 3, 7)$	Sah [144]
M_{22}	34849	$(2, 5, 7)$	Woldar [178], Conder [36]
J_2	7201	$(2, 3, 7)$	Finkelstein-Rudvalis [62]
M_{23}	1053361	$(2, 4, 23)$	Conder [36],
HS	1680001	$(2, 3, 11)$	Woldar [180]
J_3	1255825	$(2, 4, 5)$	Conder *et al.* [39]
M_{24}	10200961	$(3, 3, 4)$	Conder [36]
McL	78586201	$(2, 5, 8)$	Conder *et al.* [39]
He	47980801	$(2, 3, 7)$	Woldar [176]
Ru	1737216001	$(2, 3, 7)$	Woldar [176]
Suz	11208637441	$(2, 4, 5)$	Conder *et al.* [39]
$O'N$	9600323041	$(2, 3, 8)$	Conder *et al.* [39]
Co_3	5901984001	$(2, 3, 7)$	Worboys [183], Woldar [176]
Co_2	1602478080001	$(2, 3, 11)$	Conder *et al.* [39]
Fi_{22}	768592281621	$(2, 3, 7)$	Woldar [179]
HN	3250368000001	$(2, 3, 7)$	Woldar [176]
Ly	616252131000001	$(2, 3, 7)$	Woldar [176]
Th	1080308855808001	$(2, 3, 7)$	Linton [110]
Co_1	86620350136320001	$(2, 3, 8)$	Conder *et al.* [39]
J_4	1033042512453304321	$(2, 3, 7)$	Woldar [179]
Fi'_{24}	14942925109412639539201	$(2, 3, 7)$	Linton-Wilson [111]
B	86557947525550545649532928000001	$(2, 3, 7)$	Wilson [173]

2.2.4 Hurwitz groups

The automorphism group of the Riemann sphere, the unique Riemann surface of genus 0, is isomorphic to the group $PSL(2, \mathbb{C})$. The Riemann surface with genus 1 is a torus. Let $\Omega = \{m\omega_1 + n\omega_2 \,|m, n \in \mathbb{Z}\}$ be a discrete subgroup of $\mathbb{C}$ for some fixed $\omega_1, \omega_2 \in \mathbb{C}$, where $\omega_1 \neq 0 \neq \omega_2$ and $\omega_1/\omega_2 \notin \mathbb{R}$. Then $\mathbb{C}/\Omega$ is a model for the torus. Now the automorphism group of the torus $\mathbb{C}/\Omega$ is the set of all transformations $f_{a,b} : [z] \mapsto [az + b]$ such that $a, b \in \mathbb{C}$ and $a\Omega = \Omega$. Thus for Riemann surfaces with genus 0 and 1, the automorphism groups are infinite. However this is not the case for $g \geq 2$.

Suppose G is a $(n_1, \ldots, n_r)$-generated subgroup of the Fuchsian group F with signature $F(g; n_1, \ldots n_r)$, where $g \geq 2$. Consider the short exact sequence

$$1_\Delta \to \Delta \to F \to G \to 1_G.$$

The group Δ is a Fuchsian group without elliptic elements, and thus has signature $\Delta(g; -)$. Furthermore, $\mu(\Delta) = 2g - 2$. The branched covering $\mathcal{H}^2/\Delta \to \mathcal{H}^2/F$ is continuous and as $\mathcal{H}^2/\Delta$ is compact it follows that $\mathcal{H}^2/F$ is compact. Thus the fundamental region of F is compact in $\mathcal{H}^2$ and so $\mu(F)$ is finite (*cf.* Jones-Singerman [95], page 254). The group of automorphisms of $\mathcal{H}^2/\Delta$ is isomorphic to F/Δ (*cf.* Jones-Singerman [95], page 252) and by (2.29) we have

$$|F/\Delta| = \frac{\mu(\Delta)}{\mu(F)} = |Aut(\mathcal{H}^2/\Delta)|.$$

It follows easily from (2.28) that $\mu(F) \geq 1/42$. Thus we conclude:

Proposition 2.2.24 *Let S be a compact Riemann surface of genus $g \geq 2$. Then $|Aut(S)| \leq 84(g-1)$.* □

The finiteness of the automorphism group of a compact Riemann surface was first proved by Schwarz in 1878 and the upperbound given was proved by Hurwitz in 1893. We now investigate briefly the question of when the upperbound of Theorem 2.2.24 is attained. A group of $84(g - 1)$ automorphisms of a compact Riemann surface of genus $g \geq 2$ is called a *Hurwitz group*.

Theorem 2.2.25 *A finite group H is a Hurwitz group if and only if H is non-trivial and is $(2,3,7)$-generated, that is, it has two generators x, y obeying the relations*

$$x^2 = y^3 = (xy)^7 = 1_H.$$

Proof. Let H be a Hurwitz group. Then the measure $\mu(H) = 1/42$ and hence $H = T(2,3,7)/\Delta$, and Δ has signature $\Delta(g; -)$ for some integer $g \geq 2$. Let X and Y be the two generators of $T(2,3,7)$ obeying the relations $X^2 = Y^3 = (XY)^7$, so that if $\theta : T(2,3,7) \to H$ is the canonical homomorphism, then $x = \theta(X)$ and $y = \theta(Y)$ generate H and obey the relations $x^2 = y^3 = (xy)^7 = 1_H$.

Conversely, let H be a non-trivial finite group with two generators x, y obeying the relations $x^2 = y^3 = (xy)^7 = 1_H$. Then there is a homomorphism $\theta : T(2,3,7) \to H$ such that $\theta(X) = x$ and $\theta(Y) = y$. Let $\Delta = Ker\theta$. Now every elliptic element of $T(2,3,7)$ is congruent to a power of X, Y or XY, so that if Δ contains elliptic elements, then it must contain X, Y or XY. Suppose that $X \in \Delta$. Then $x = 1_H$ and hence $y^3 = y^7 = 1_H$ so that $y = 1_H$ and H is trivial. Similarly, if Δ contains Y or XY, then H is trivial. Thus Δ contains no elliptic elements. Since $T(2,3,7)$contains no parabolic elements, Δ contains no parabolic elements and so Δ has signature $\Delta(g; -)$, for some integer $g \geq 2$ by the Riemann-Hurwitz equation. Hence $T(2,3,7)/\Delta$ is the group of automorphisms of the compact Riemann surface $\mathcal{H}^2/\Delta$, of genus $g \geq 2$ and by (2.29) we have

$$|T(2,3,7)/\Delta| = \frac{2g-2}{1/42} = 84(g-1). \qquad \square$$

The generators x, y in the above theorem will be called *Hurwitz generators* of H.

Proposition 2.2.26 *Let H be a Hurwitz group and let H_1 be a non-trivial homomorphic image of H. Then H_1 is a Hurwitz group.*

Proof. Suppose H is generated by x, y with relations $x^2 = y^3 = (xy)^7 = 1_H$. Let ϕ be the homomorphism from H to H_1. If $\phi(x) = x_1$ and $\phi(y) = y_1$, then $x_1^2 = y_1^3 = (x_1 y_1)^7 = 1_{H_1}$. $\square$

Corollary 2.2.27 *A Hurwitz group of smallest order is simple.*

Proof. Let H be a Hurwitz group of smallest order. If H is not simple, then it contains a non-trivial normal subgroup N such that H/N is a Hurwitz group and $|H/N| < |H|$, a contradiction. $\square$

Proposition 2.2.28
(i) There is no Hurwitz group of order 84.

(ii) The group $PSL(2,7)$ *is a Hurwitz group of order 168.*

Proof. (i) There is no simple group of order 84.

(ii) Let $A, B \in SL(2,7)$ be defined by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Let x, y be the images of A, B respectively under the canonical homomorphism from $SL(2,7)$ to $PSL(2,7)$. Then x, y are Hurwitz generators for $PSL(2,7)$. $\square$

Thus the Hurwitz bound is not attained when $g = 2$ but is attained when $g = 3$. It is known that this bound is attained for infinitely many values of g and not attained for infinitely many values. The precise value of g for which the Hurwitz bound is attained are unknown; the first four values are $g = 3, 7, 14, 17$.

Quotients of the Hurwitz groups have importance also in the study of regular maps. A *regular map* of type $\{p,q\}$ on a surface S of Euler characteristic $\chi < 0$ is essentially a map whose automorphism group acts regularly on a set of ordered edges, so that every face of the map is surrounded by p edges, and every vertex is incident with q edges. If such a map has n vertices, then it has $nq/2$ edges and nq/p faces, and therefore $\chi = n - nq/2 + nq/p = |G|(1/q - 1/2 + 1/p)$. Hence the map with the largest possible number of automorphisms on a given surface occur only when the quantity $1/q + 1/p$ takes its largest value less than $1/2$, namely $10/21$, that is, when $\{p,q\} = \{3,7\}$. Correspondingly, G can be generated by an element of order 2 (reflection any chosen edge), and one of order 3 (permuting the edges around either a face or a vertex), such that their product is 7; in other words, G is a Hurwitz group.

A group G is called *perfect* if G and its commutator subgroup G' coincide. The Hurwitz groups are contained in the collection of perfect groups.

Proposition 2.2.29 *Suppose H is a Hurwitz group. Then H is perfect. Furthermore, H contains a maximal normal subgroup K such that H/K is a non-abelian simple Hurwitz group.*

Proof. Let x, y be Hurwitz generators for H. If H' is trivial, then H is abelian and therefore $x^2 = 1_H$ and $y^3 = 1_H$ implies $(xy)^6 = 1_H$. However, this contradicts the fact that $(xy)^7 = 1_H$. Consider the natural map $\phi : H \to H/H'$. Let $\overline{x}$, $\overline{y}$ be the respective images of x, y under ϕ. Since H/H' is abelian $(\overline{xy})^6 = 1_{H/H'}$. Furthermore, $(\overline{xy})^7 = 1_{H/H'}$, so that $\overline{x} = \overline{y} = 1_{H/H'}$. Thus H/H' is trivial and hence $H = H'$.

If K is any maximal normal subgroup of H, then H/K is simple and being also a non-trivial quotient of H, it must also be a Hurwitz group. □

These properties makes it clear that a sensible way to begin any search for Hurwitz groups is to look at finite simple groups (and in particular, those with order divisible by 84).

Relatively few non-abelian simple groups are known to be Hurwitz groups. Many small possibilities can be eliminated using ad hoc methods together with the properties outlined above, but more a sophisticated approach is likely to be required for the majority of the remaining cases. However, three infinite families of simple groups are known to be Hurwitz groups:

(1) *The alternating group A_n is a Hurwitz group, for all but finitely many positive integers n.* This was first proved by Higman, using coset diagrams for $T(2,3,7)$. The first publication on the topic was by Conder [31], showing that A_n is a Hurwitz group for all $n \geq 168$, and for all but 64 integers n in the range $3 \leq n \leq 167$. The exceptional values of n are those values of n which fail to satisfy $[n/2] + 2[n/3] + 6[n/7] \geq 2n - 2$, together with 7, 8, 9, 14, 16, 24, 30, 44 and 60.

(2) *The group $PSL(2,q)$ is a Hurwitz group when $q = 7$, and when $q = p$ for any prime $p \equiv \pm 1 \,(mod\, 7)$, and when $q = p^3$ for any prime $p \equiv \pm 2 \,(mod\, 7)$ or $\pm 3 \,(mod\, 7)$, and for no other values of q.* This result is due to MacBeath [115], who showed in fact that $PSL(2,q)$ has a Hurwitz subgroup whenever its order is divisible by 7, but all such subgroups are mutually isomorphic.

(3) *The simple Ree group ${}^2G_2(3^p)$ is a Hurwitz group for every odd prime $p > 3$.* The proof of this result is due to Sah [144].

Apart from these infinite families, twelve of the sporadic simple groups are known to be Hurwitz: the first Janko group J_1 ; the Hall-Janko group J_2; the smallest Conway group Co_3; the Held group He; the Rudvalis group Ru; the Harada-Norton group HN; the Lyons group Ly; the Fischer group Fi'_{24}; the Thompson group Th; the Fischer group Fi_{22}; the fourth Janko group J_4; and the Baby Monster B (*cf.* Table 2.I). Of the remaining sporadic simple groups, all but the Fischer Monster M (whose maximal subgroups are not yet classified) have been shown to have no Hurwitz generating pairs.

2.3 Special types of 2-Generations

In the previous section we considered generating pairs carrying a geometric meaning. In this section we will discuss various generating pairs of the finite simple groups with generating elements of a prescribed order.

2.3.1 $(2, s)$-Generations

A finite non-abelian group is said to be $(2, s)$-*generated* if it can be generated by an involution x and another suitable element y (of order s). If $o(xy) = t$, we also say G is $(2, s, t)$-*generated.* Historically, interest in such kind of generations have a geometrical motivation, namely the study of regular maps on surfaces (defined in Section 2.2.4) and their automorphisms. Brahana [13] proved that a necessary and sufficient condition for a group G to be the automorphism group of a regular map on a surface is that G is generated by two elements of which one is of order 2. This led to the conjecture: *If G is a finite non-abelian simple group, then there exist non-trivial elements $a, b \in G$ such that $G = \langle a, b\rangle$ and $a^2 = b^s = 1_G$, for some $s \geq 3$.*

This conjecture is true whenever G is a sporadic simple group (*cf.* Proposition 2.2.23). If G is an alternating group we have the following result.

Proposition 2.3.1 *For every $n \geq 5$ the alternating group A_n can be generated by an involution and another suitable element.*

Proof. Let $G = \langle a, b\rangle$ where $a = (1,2)(n-1,n)$, $b = (1,2,\ldots,n-1)$ if n is even, and $a = (1,n)(2,n-1)$, $b = (1,2,\ldots,n-2)$ if n is odd. Clearly G is transitive on $X = \{1,2,\ldots,n\}$. In fact, we will show that G is 2-transitive. This is clear if n is even, since the point stabilizer $G_{[n]}$ is transitive on the remaining points. Assume n is odd and G is not 2-transitive. Then $G_{[n]}$ is not transitive on $X - \{n\}$. Since $b \in G_{[n,n-1]} \subseteq G_{[n]}$, it follows that $G_{[n]} = G_{[n,n-1]}$. The permutation $(ba)^2$ is a n-cycle mapping n to $n-1$ so that $(ba)^2 \in N_G(G_{[n]}) \backslash G_{[n]}$. Hence $Fix(G_{[n]}) = \{n-1, n\}$ is an orbit for $N_G(G_{[n]})$ and therefore a block for G. Thus $|Fix(G_{[n]})| = 2$ divide n. But this is impossible since n is odd. Now the commutator $[a, b]$ is a cycle of length 5. However, any primitive group G of degree $n = p + h$ (p prime, $h \geq 3$) containing a cycle of length p, does contain A_n (*cf.* Wielandt [166]). Thus it follows that $G = A_n$ for all $n > 7$. Direct inspection shows that the result holds for $n = 5, 6, 7$. □

The short proof given above can be found in Aschbacher-Guralnick [4]. In fact, Miller [129] showed that, if A_n contains an element of order $s \geq 3$, then A_n is generated by an involution and a suitable element of order s. Moreover, Miller [126] had already proved that A_n can be generated by an involution and an element of order 3 except for $n = 3, 6, 7, 8$. However, Miller does not give explicit generators and such generators can be found in Dey-Wiegold [49] or Tamburini [158].

Brahana [15] also proved the conjecture for the simple groups of order less than 10^6 known at the time. We now survey simple groups of Lie type that are known to be $(2, s)$-generated.

(i) G is a group of Lie type of rank 1, that is, $G = A_1(q)$; ${}^2B_2(q)$; ${}^2A_2(q^2)$, $q \neq 2$; ${}^2G_2(q)$, $q \neq 3$ (*cf.* Aschbacher-Guralnick [4]).

(ii) G a group of type $A_n(q)$ (*cf.* Albert-Thompson [1]).

(iii) G a group of type $C_n(q)$ (*cf.* Room [141], Room-Smith [142], Stanek [154]).

(iv) G a groups of type $B_n(q)$, $D_n(q)$, ${}^2A_n(q)$ in characteristic 2. (*cf.* Weigel [165]).

(v) G is of type $B_n(q)$, q is odd (*cf.* Walter [164]). This together with (iv) settle the conjecture for all the groups of type $B_n(q)$, $n \geq 3$.

A difficult but important problem is to determine which finite simple groups are are $(2, 3)$-*generated*, that is, it can be generated by an involution and an element of order 3. This does not happen for all simple groups, for example, $PSU(3, 3^2)$ and

McL are not $(2,3)$-generated. The problem amounts to determining the homomorphic images of the group

$$\langle x, y \mid x^2 = y^3 = 1_G \rangle.$$

This is the famous modular group $PSL(2, \mathbb{Z})$, which admits such a presentation by letting x and y be the projective images of the respective matrices

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}.$$

There are many examples of such groups, in fact, from results of Schupp [148] and Mason-Pride [124], it follows that there are $2^{\aleph_0}$ isomorphism classes of simple $(2,3)$-generated groups. Indeed, every countable group can be imbedded in a simple $(2,3)$-generated group.

Recently this problem received a considerable amount of attention. A $(2,3)$-generated group is close to being perfect, and prefect groups, in particular, simple groups have been the main objects under investigation. To our knowledge the following groups are $(2,3)$-generated.

(i) The alternating groups A_n, $n \neq 6, 7, 8$ (*cf.* Miller [126]).

(ii) The projective special linear group $PSL(2,q)$, $q \neq 9$ (*cf.* Macbeath [115]).

(iii) The projective special linear group $PSL(3,q)$, $q \neq 4$ (*cf.* Garbe [71] and Cohen [30])

(iv) The special linear group $SL(4,q)$, $q > 4$ (*cf.* Tamburini-Vassello [159]).

(v) The special linear group $SL(n,q)$, $q \neq 9$, $n \geq 5$ (*cf.* DiMartino-Vavilov [53]).

(vi) The projective symplectic groups $PSp(4,q)$, $q = p^m$, $p \neq 2,3$ (*cf.* Cazzola-DiMartino [29]).

(vii) The Chevalley groups $G_2(q)$ of type G_2 and the twisted groups ${}^2G_2(q)$ (*cf.* Malle [120], [121]).

(viii) The twisted groups ${}^3D_4(q)$ and ${}^2F_4(2^{2n+1})'$ (*cf.* Malle [122]).

(ix) All sporadic simple groups, with the exception of M_{11}, M_{22}, M_{23} and McL (*cf.* Woldar [176]).

The proofs of these results suggest that, while a uniform treatment should be at hand for all classical groups when the Lie rank is large enough, the small dimensional cases require a somewhat different choice for the generators, and therefore special *ad hoc* analysis. It is conjectured that all finite simple groups of Lie type are $(2,3)$-generated, except for some groups of low rank in characteristic 2 and 3. Recently Liebeck-Shalev [106], [108], [109] have presented some new probabilistic, non-constructive, methods regarding 2-generations of finite simple groups. In the context of $(2,3)$-generations they show that all finite classical groups are $(2,3)$-generated, with the exception of $PSp(4,2^k)$ and $PSp(4,3^k)$ and finitely many other groups.

The problem of generating a group by a set of involutions of minimal size are closely related to the $(2,s)$-generation of the group. Let G be a finite group generated by a set of involutions and let $i(G) = min\{X\}$, where X runs over the set of involutions generating G. Of course, $i(G) \leq 2$ implies G is cyclic or dihedral. The problem of determining those G for which $i(G) = 3$ is much more intricate. It amounts to determining the normal subgroups of finite index of the full triangular group $T^{*}(l,m,n)$. In fact, it is reasonable to conjecture that almost all finite simple groups are so generated.

If $G = \langle a,b,c\rangle$ where $a^2 = b^2 = c^2$, then $N = \langle ab, ca\rangle$ is a normal subgroup of G such that $Na = Nb = Nc$, hence of index at most 2 in G. Now if G is prefect, then G is 2-generated.

Proposition 2.3.2 *If G is a perfect $(2,s)$-generated group, then G is generated by s conjugate involutions. Moreover, if $G = \langle a,x\rangle$ where a is an involution and there exist an involution b such that $\langle b,x\rangle$ is dihedral and $xb = c$ is an involution, then $G = \langle a,b,c\rangle$.*

Proof. Let $G = \langle a,x\rangle$ where $o(a) = 2$ and $o(x) = s$. Let $N = \langle a, a^x, a^{x^2}, \ldots, a^{x^{s-1}}\rangle$ be a normal subgroup of G and G/N is generated by $\{aN, xN\}$. Since $\langle aN, xN\rangle = \langle xN\rangle$, the group G/N is an abelian group. Thus $N \geq G'$ and since G is perfect $N = G$. Therefore G is generated by s conjugate involutions.

For the second part, it is easy to see that $x = (ca)(ab) \in \langle a, b, c\rangle$, and the result follows. $\square$

The proof of Proposition 2.2.2 for the special case $s = 3$ can be found in DiMartino-Tamburini [52]. Thus for a $(2,3)$-generated groups G, we have $i(G) = 3$. In fact, Dalla Volta [48] proved that every sporadic simple group can be generated by three involutions. The unitary group $G = PSU(3, 3^2)$ is the only known non-abelian finite simple group for which $i(G) \neq 3$. In fact, for this group $i(G) = 4$ (*cf.* DiMartino [51]).

The $(2,3)$-generation and the generation by three involution of finite simple groups seems to be intrinsically related. We have already observed that if G is a $(2,3)$-generated simple group, then G is generated by three (conjugate) involutions. A partial converse of this is the following:

Proposition 2.3.3 *Let G be a simple group generated by three involutions. Then one of the following groups are $(2,3)$-generated:*

(i) $G\wr 3$, the wreath product of G by a cyclic group of order 3.

(ii) $[G]\langle\sigma\rangle$, the extension of the holomorph of G by an automorphism $\sigma \in Aut(G)$ of order 3. $\square$

The above result can be found in Ito [85]. It was motivated by graph theoretic applications, namely the study of certain connected symmetric trivalent graphs arising from the wreath product $G\wr 3$ and the automorphism group of $G\wr 3$.

2.3.2 nX-Complementary generation of a finite group

Definition 2.3.4 *A finite non-abelian group G is said to be* 3/2-generated, *if for every non-trivial element x of G, there exists an element $y \in G$ such that $G = \langle x, y\rangle$.*

Let Γ_1 denote the collection of all finite non-abelian 3/2-generated groups. The structure of groups of this sort is very restricted. We recall that a group is *subdirectly irreducible* if the intersection of all non-trivial normal subgroups is non-trivial. This intersection is called the *monolith* and is the unique minimal normal subgroup.

Lemma 2.3.5 *[19] Let G be any finite non-abelian group. If G is $3/2$-generated, then G is subdirectly irreducible. The commutator subgroup G' is the monolith and G/G' is cyclic.*

Proof. If G is simple, then $G = G'$ and the proof is obvious. Otherwise, let H be any non-trivial normal subgroup of G and x a non-trivial element of H. Then $\langle x, y\rangle = G$ for some $y \in G$, whence $G/H = \langle yH\rangle$ is cyclic. Therefore $G' \subseteq H$. Since G is non-abelian, $G' \neq \{1_G\}$ and the result follows. □

The definition of $3/2$-generation is not very useful for computational purposes. The next result will significantly refine this definition.

Lemma 2.3.6 *A finite non-abelian group G is $3/2$-generated if and only if for every x of prime order in G, there exists an element $y \in G$ such that $G = \langle x, y\rangle$.*

Proof. The sufficiency condition is trivial. Conversely, let z be any non-trivial element of G. Then there exists positive integer m such that $z^m = x$ and $o(x) = p$, where p is a prime. Thus by assumption there exists $y \in G$ such that $G = \langle x, y\rangle = \langle z^m, y\rangle \subseteq \langle z, y\rangle \subseteq G$. Thus $G = \langle z, y\rangle$, proving the result. □

Brenner-Guralnick-Wiegold [18] conjectured that *every finite simple group is $3/2$-generated.* They prove that the conjecture holds for the for the alternating groups A_n and the projective special linear groups $PSL(2, q)$. Note that a complete verification of this conjecture would settle in the affirmative the long-standing $(2, s)$-conjecture, discussed in the previous section. Woldar [181] proved this conjecture for the sporadic simple groups using the following definition.

Definition 2.3.7 *Let G be a finite non-abelian group and nX a conjugacy class of G. We say G is nX-complementary generated if, given an arbitrary $x \in G$, there exits an element $y \in nX$ such that $G = \langle x, y\rangle$. We refer to y as complementary.*

Lemma 2.3.8 *A group G is nX-complementary generated if and only if for every conjugacy class pY of G, p prime, there exists a conjugacy class $t_{pY}Z$, depending on pY, such that the group G is $(pY, nX, t_{pY}Z)$-generated. Moreover, if G is a finite simple group, then G is not $2X$-complementary generated, for any conjugacy class of involutions.*

Proof. The first part of the result follows immediately from Lemma 2.3.6. For any positive integer n, the triangular group $T(2,2,n) \cong D_{2n}$, the dihedral group of order $2n$. Thus if G is a finite group not isomorphic to the dihedral group, then G is not $(2X, 2X, nY)$-generated, for any classes of involutions and for any conjugacy class nY. Thus by the first part it follows that G is not $2X$-complementary generated. □

Consider the conjugacy classes rX and sY of G, with $(rX)^n = sY$, for some integer n. If G is not rX-complementary generated, then there exits an element x of prime order such that $\langle x, y \rangle < G$, for all $y \in rX$. Since $x, y^n \in \langle x, y \rangle$, it follows that $\langle x, y^n \rangle \leq \langle x, y \rangle < G$, for all $y^n \in sY$. Thus we have proved the following result.

Lemma 2.3.9 *If G is sY-complementary generated and $(rX)^n = sY$, then G is rX-complementary generated.* □

Woldar [181] proved that every sporadic simple group G is pA-complementary generated where p is the largest prime dividing the order of G. It is reasonable to conjecture that *every finite simple group is nX-complementary generated for some conjugacy class nX.* In an attempt to further the theory on nX-complementary generation we pose the problem.

Problem 1: *Given a finite non-abelian simple group G, find all conjugacy classes nX of G such that G is nX-complementary generated.*

It is clear that if $s > 2$ is a fixed integer and the group G is not $(2X, sY, tZ)$-generated for any t, then G is not sY-complementary generated. Let $\Gamma_1^{(2)}$ be the collection of all finite non-abeian groups with the property that $G \in \Gamma_1^{(2)}$ if and only if every non-trivial element of G together with an element of order 2 generate the group G. Again, using arguments similar to that in Lemma 2.3.6, it follows that $G \in \Gamma_1^{(2)}$ if and only if for every element of prime order $p > 2$ there exits an element of order t_p (depending on p) such that G is $(2, p, t_p)$-generated. This together with the problems motivated in the previous sections led Moori [131] to the question.

Problem 2: *Given a non-abelian finite simple group G and l, m, n divisors of $|G|$ such that $1/l + 1/m + 1/n < 1$. Is G a (l, m, n)-generated group?*

In this treatise we will focus on solutions to Problems 1 and 2 with G one of the following sporadic simple group: the Janko groups J_1, J_2, J_3, J_4; the Higman-Sims group HS; the McLaughlin group McL; the Conway groups Co_3, Co_2 and the Fischer group Fi_{22}. We will provide a complete answer to Problem 1 for these groups. With regard to Problem 2, we will restrict ourselves to the cases were l, m, n are distinct primes and some other triples (l, m, n) needed to solve Problem 1. We believe the remaining cases can be dealt with in a similar way.

Let r be any positive integer. A finite non-abelian group G is said to have *spread* r, if for every set $\{x_1, x_2, \ldots, x_r\}$ of distinct non-trivial elements of G, there exists an element $y \in G$ such that $G = \langle x_i, y \rangle$ for all i. We say G has *exact spread* t if G has spread t but not $t + 1$. An interesting question posed by Brenner-Wiegold [19] is to find all finite simple groups with exact spread 1. In Chapter 11 we will show that none of the sporadic simple groups have exact spread 1.

Chapter 3

General Theory

The aim of this chapter is twofold, being in the first place to set up the notational conventions to be used throughout this thesis, and secondly, to provide in readily usable form a selection of results and techniques that will be useful in resolving generation type questions of finite simple groups.

Suppose that G is a (l, m, n)-generated group, that is, $G = \langle x, y \rangle$ such that $o(x) = l$, $o(y) = m$ and $o(xy) = n$. This is equivalent to saying that $G = \langle x, y, z \rangle$ with $o(x) = l$, $o(y) = m$, $o(z) = n$ and $xyz = 1_G$. If lX, mY and nZ are conjugacy classes of G that contain x, y and z, respectively, then we also say that G is (lX, mY, nZ)-*generated* and (lX, mY, nZ) is called a *generating triple* of G.

3.1 Characters and 2-Generation

The first technique we discuss to resolve 2-generation type problems is due to Woldar [178]. Let G be a finite non-abelian group and $z \in nZ$ be a fixed element. Let $\Delta_G(lX, mY, nZ)$ be the cardinality of the set

$$\mathcal{A} = \{(x, y) \in lX \times mY \mid xy = z\}. \tag{3.1}$$

The next classical result provides an explicit formula for $\Delta_G(lX, mY, nZ)$ in terms of the ordinary irreducible characters $\chi_1, \chi_2, \ldots, \chi_r$ of G.

Theorem 3.1.1 *(Burnside, 1911) With the above notation,*

$$\Delta_G(lX, mY, nZ) = \frac{|lX|\,|mY|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(x)\chi_i(y)\overline{\chi_i(z)}}{\chi_i(1_G)}, \tag{3.2}$$

where $\{\chi_1, \chi_2, \ldots, \chi_r\}$ is the set of ordinary irreducible characters of G.

The non-negative integer $\Delta_G(lX, mY, nZ)$ is called the *structure constant* and it is independent of the choice of representative z in nZ. Clearly any pair $(x, y) \in \mathcal{A}$ generates G or a proper subgroup of G. Thus by computing the relevant structure constants in appropriate subgroups of G, we can determine an upper bound for the number of pairs in $\mathcal{A}$ that generate proper subgroups of G. For the implementation of the procedure outlined above, it is necessary to have a great many character tables. For this purpose we use the character tables in the ATLAS and those stored in the GAP library. In addition, sufficient measures must be taken to ensure that pairs are not counted more than once. Thus it is important to make a judious choice of subgroups at the outset, which will prove manageable from both the character theoretic as well as group theoretic point of view. This requires a rather in-depth study of at least part of the subgroup lattice of the underlying group. We now proceed to discuss this procedure in more detail.

Let $\Delta_G^*(lX, mY, nZ)$ denote the number of pairs $(x, y) \in \mathcal{A}$ which generate the entire group G. Clearly G admits a (l, m, n)-generation if and only if there exists conjugacy classes lX, mY, nZ for which $\Delta_G^*(lX, mY, nZ) > 0$. In most instances it will be clear from the context to which conjugacy classes lX, mY, nZ we are referring. Thus we shall often suppress the conjugacy classes, using $\Delta(G)$ and $\Delta^*(G)$ as abbreviated notation for $\Delta_G(lX, mY, nZ)$ and $\Delta_G^*(lX, mY, nZ)$, respectively.

For H any subgroup of G containing the fixed element z, let $\Sigma(H)$ denote the number of pairs in $(x, y) \in \mathcal{A}$ such that $\langle x, y\rangle \leq H$. We differentiate the conjugacy classes of the subgroup H from that of G by writing nx for a general conjugacy class of H with elements of order n. Now the role of Σ in the notation $\Sigma(H)$ is to express the fundamental fact that $\Sigma(H)$ is obtain by summing the structure constants $\Delta_H(lx, my, nz)$ of H over all the H-conjugacy classes lx and my satisfying $lx \subseteq H \cap lX$ and $my \subseteq H \cap mY$. Thus in order to compute $\Sigma(H)$ we need the *fusion map* from H into G. Note that $\Sigma(H)$ is not G invariant: indeed there may be several H-classes

nz into which z can fall. When the need arises, we shall write $\Sigma(H; nz)$ in place of $\Sigma(H)$ to emphasize the conjugacy class nz of H to which z belongs. Finally, for any family $\{H_1, H_2, \ldots, H_r\}$ of subgroups of G, we denote by $\Sigma(H_1 \cup \cdots \cup H_r)$ the number of pairs in $\mathcal{A}$ which generate a subgroup of some H_i, where $i = 1, \ldots, r$. To ensure that pairs are not counted more than once in $\Sigma(H_1 \cup H_2)$ we use the formula

$$\Sigma(H_1 \cup H_2) = \Sigma(H_1) + \Sigma(H_2) - \Sigma(H_1 \cap H_2).$$

Similar formulae can be obtained for $r > 2$. We now observe that $\Delta^*(G)$ satisfy the simple relation

$$\Delta^*(G) = \Delta(G) - \Sigma(M_1 \cup \cdots \cup M_t), \tag{3.3}$$

where $\{M_1, M_2, \ldots, M_t\}$ is the family of all maximal subgroups of G which contain z. This formula is particularly useful when t is small, that is, when the intersections of maximal subgroups are manageable.

We now derive an alternative formula for $\Delta^*(G)$ which will be useful under certain conditions. Let $\{H_1, H_2, \ldots, H_k\}$ be a full set of non-conjugate (lX, mY, nZ)-generated proper subgroups of G. Let $\Sigma^*(H)$ be the number of pairs $(x, y) \in \mathcal{A}$ such that $H = \langle x, y \rangle$. Then

$$\Delta^*(G) = \Delta(G) - \sum_{i=1}^{k} h_i \Sigma^*(H_i), \tag{3.4}$$

where h_i is the number of distinct conjugates of H_i containing z. Without loss of generality, assume that $H_1, \ldots, H_{j-1}$ are subgroups of H_j, where $j \leq k$. Now if $\langle x, y \rangle = H_i^g$, for some $g \in G$ and $i < j$, then $\langle x, y \rangle \leq H_j^g$. Thus from the definition of $\Sigma(H_j)$ it follows that

$$\sum_{i=1}^{k} h_i \Sigma^*(H_i) \leq h_j \Sigma(H_j) + \sum_{i=j+1}^{k} h_i \Sigma^*(H_i).$$

Thus an understanding of the subgroup lattice of G will simplify the task of finding an lower bound for $\Delta^*(G)$ in equation (3.4). The following results allow us to compute h_i.

Theorem 3.1.2 *(Finkelstein [61]) Let K be a proper subgroup of H where H is a proper subgroup of G. The set of all conjugates of K in G which are also subgroups of*

H falls into r conjugacy classes of subgroups of H. Let $K_1, K_2, \ldots, K_r$ be representatives of these r conjugacy classes of subgroups of H. Then the number of conjugates of H in G to which K belongs is given by

$$[N_G(H) : H]^{-1} \sum_{i=1}^{r} [N_G(K) : N_H(K_i)] . \tag{3.5}$$

Proof. The number of conjugates of H in G is evidently $[G : N_G(H)]$. Each conjugate of H in G contains $\sum_{i=1}^{r}[H : N_H(K_i)]$ of conjugates of K in G. If we list the G-conjugates of K for each conjugate of H in G, then we obtain all the conjugates of K in G, each conjugate of K repeated a fixed number of times. This number is the multiplicity a fixed G-conjugate of K appearing as a subgroup of a conjugate of H in G. It follows therefore that the product of $[G : N_G(H)]$ and $\sum_{i=1}^{r}[H : N_H(K_i)]$ is the number of conjugates of K in G multiplied with h, where h the number of conjugates of H in G to which K belongs. Thus

$$\begin{aligned} h &= \frac{[G : N_G(H)]}{[G : N_G(K)]} \sum_{i=1}^{r} [H : N_H(K_i)] = \frac{|N_G(K)|}{|N_G(H)|} \sum_{i=1}^{r} \frac{|H|}{|N_H(K_i)|} \\ &= [N_G(H) : H]^{-1} \sum_{i=1}^{r} [N_G(K) : N_H(K_i)] . \qquad \square \end{aligned}$$

If H is a self normalizing subgroup of G (for instance if H is a maximal subgroup of a simple group G), then the above reduces to

$$\sum_{i=1}^{r} [N_G(K) : N_H(K_i)] .$$

On the other hand, if H has only one conjugacy class of subgroups isomorphic to K, or more generally if $r = 1$, which happens for instance if K is a Sylow p-subgroup of H for some prime p, then the above becomes

$$\frac{[N_G(K) : N_H(K)]}{[N_G(H) : H]} .$$

Finally, if both of these situations occur simultaneously, that is, H is self normalizing in G and $r = 1$, then it reduces to

$$[N_G(K) : N_H(K)] .$$

Corollary 3.1.3 *Let G be a finite group and H a subgroup of G containing a fixed element x. Then the number h of conjugates of H in G containing x is given by*

$$h = [N_G(H) : H]^{-1} \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_H(x_i)|},$$

where $x_1, \ldots, x_m$ are representatives of the H-conjugacy classes that fuse to the G-class $[x]_G$.

Proof. The number of conjugates of x in G and H are respectively $[G : C_G(x)]$ and $[H : C_H(x)]$. Moreover, H contains $\sum_{i=1}^{r}[H : C_H(x_i)]$ G-conjugates of x, where $x_1, \ldots, x_r$ are representatives of the conjugacy classes of H that fuse to the G-class $[x]_G$ containing x. The result now follows immediately from the previous theorem. □

Theorem 3.1.4 *Let G be a finite group and H a subgroup of G containing a fixed element x such that $gcd(o(x), [N_G(H){:}H]) = 1$. Then the number h of conjugates of H in G containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the conjugates of H. In particular,*

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1 \ldots x_m$ are representatives of the $N_G(H)$-conjugacy classes that fuse to the G-class $[x]_G$.

Proof. Let Ω be the set of all conjugates of the subgroup H in G. Then G acts (by conjugation) transitively on Ω and the point stabilizer G_H equals to $N_G(H)$. Thus the permutation character of G with this action on Ω is $\chi_H = (\chi_1)^G$, where χ_1 is the identity character of $N_G(H)$. By definition

$$\chi_H(x) = |\{H^g|\, (H^g)^x = H^g\}| = |\{H^g|\, x \in N_G(H^g)\}|$$

is the number of fix points of x on Ω. Let $\overline{x}$ be the image of x under the natural homomorphism $N_G(H^g) \to N_G(H^g)/H^g$. Since $(o(x), [N_G(H^g){:}H^g]) = 1$, it follows that $o(\overline{x}) = 1$ and hence $x \in H^g$. Therefore $\chi_H(x) = |\{H^g|\, x \in H^g\}|$. On the other hand,

$$\chi_H(x) = (\chi_1)^G(x) = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|},$$

where $[x]_G \bigcap N_G(H) = \bigcup_{i=1}^{m} [x_i]_{N_G(H)}$. $\square$

Whenever the permutation character of G on the conjugates of H is not known explicitly in terms of the irreducible characters of G, we use the fusion map of $N_G(H)$ into G to determine its value on the conjugacy classes of G. The centralizer of the fixed element $z \in nZ$ plays an important role in (l, m, n)-generations. We shall often write the centralizer of $z \in nZ$ as $C_G(nZ)$.

Lemma 3.1.5 *(Finkelstein-Rudvalis [62]) If $C_G(z)$ acts transitively on the set $\mathcal{A}$ defined by (3.1), then the set*

$$S = \{\langle a, b\rangle | a \in lX, b \in mY, ab \in nZ\} \tag{3.6}$$

is a conjugacy class of subgroups of G. Also, if $H = \langle a, b\rangle$ is an element in S such that $ab = z$, then $C_G(H)$ is the stabilizer of (a, b) in the action of $C_G(z)$ on $\mathcal{A}$.

Proof. We show that an arbitrary element $H_1 = \langle a_1, b_1\rangle$ of S is conjugate to H. Now $a_1 b_1 = z^g$, for some $g \in G$ by the definition of S. If $y = g^{-1}$, then $H_1^y = \langle a_1^y, b_1^y\rangle$ is a conjugate of H_1 with $a_1^y b_1^y = z$, that is $(a_1^y, b_1^y) \in \mathcal{A}$. By the transitivity of $C_G(z)$ on $\mathcal{A}$, H_1^y is conjugate to H so that H_1 is conjugate to H as claimed. The element $x \in G$ stabilizes (a, b) if and only if x centralizes both a and b and thus $H = \langle a, b\rangle$ as well. $\square$

Theorem 3.1.6 *(Finkelstein-Rudvalis [62]) If $C_G(z)$ has m orbits on $\mathcal{A}$, then the set S defined by (3.6) is the union of at most m conjugacy classes of subgroups of G. Also if (a, b) is an element of the i-th orbit $\mathcal{A}_i$, then $C_G(\langle a, b\rangle)$ is the stabilizer of (a, b) in the action of $C_G(z)$ on $\mathcal{A}_i$.*

Proof. Let (a_i, b_i) be a representative for the orbit $\mathcal{A}_i$ in the action of $C_G(z)$ on $\mathcal{A}$ and $H_i = \langle a_i, b_i\rangle$. Then as proved in the above lemma, any subgroup K in S will be conjugate to H_i, for some $1 \leq i \leq m$. Since H_i may be conjugate to H_j for distinct $i, j \leq m$, the set S is the union of at most m conjugacy classes of subgroups of G. The second part follows from the above lemma as $C_G(z)$ acts transitively on $\mathcal{A}_i$. $\square$

The following results give useful criterion for non-generation.

Lemma 3.1.7 *(Woldar [178]) Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which $\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(nZ)|$. Then $\Delta^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*

Proof. Suppose that G is a (lX, mY, nZ)-generated group, that is, $G = \langle x, y\rangle$ with $x \in lX$, $y \in mY$ and $xy = z \in nZ$. Then for any $c \in C_G(z)$, we have $G = \langle x^c, y^c\rangle$ with $x^c y^c = z$. Now if $(x^a, y^a) = (x^b, y^b)$ for $a, b \in C_G(z)$, then $ab^{-1} \in C_G(x, y) = Z(G) = \{1_G\}$, whence $a = b$. This proves that $\Delta^*(lX, mY, nZ) \geq |C_G(z)|$, and a contradiction is reached. $\square$

Lemma 3.1.8 *(Woldar [178]) Let G be a finite group and $x, y \in G$. Suppose that $\Delta(G) < |C_G(xy)|$, where $\Delta(G) = \Delta_G(lX, mY, nZ)$ with $x \in lX$, $y \in mY$ and $xy \in nZ$. Then $C_G(\langle x, y\rangle)$ is non-trivial.*

Proof. Suppose that $C_G(\langle x, y\rangle) = \{1_G\}$. Then for all $c \in C_G(xy)$, we have $x^c y^c = xy$, and moreover $(x^a, y^a) = (x^b, y^b)$ if and only if $a = b$. Thus from the structure constants we obtain

$$\Sigma(\bigcup_{g\in G} \langle x, y\rangle^g) \geq |C_G(xy)|\,.$$

But this contradicts our assumption as certainly

$$\Sigma(\bigcup_{g\in G} \langle x, y\rangle^g) \leq \Delta(G) \leq |C_G(xy)|\,.$$

$\square$

Define the *symmetric structure constant* of a finite group by

$$\xi_G(lX, mY, nZ) = \frac{|G|}{|C_G(x)|\,|C_G(y)|\,|C_G(z)|} \sum_{\chi} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1_G)}\,,$$

where the sum is taken over all irreducible characters of G.

Corollary 3.1.9 *Let G be a centerless group. Then $\Delta_G(lX, mY, nZ) < |C_G(nZ)|$ if and only if $\xi(G) = \xi_G(lX, mY, (nZ)^{-1}) < 1$. Moreover, G is not $(lX, mY, (nZ)^{-1})$-generated if $\xi(G) < 1$.*

Proof. The proof is immediate from Lemma 3.1.7 and the definitions of $\Delta(G)$ and $\xi(G)$. $\square$

Lemma 3.1.10 *(Moori [131]) Let G be a (l, m, n)-generated group with l, m, n pairwise coprime. Then G is perfect and hence has no soluble quotient.*

Proof. Assume that $G = \langle a, b \rangle$ with $o(a) = l$, $o(b) = m$ and $o(ab) = n$. Let G' be the commutator subgroup of G. Then $G/G' = \langle \overline{a}, \overline{b} \rangle$, where $\overline{a}$ and $\overline{b}$ are the images of a and b, respectively under the natural homomorphism. Since G/G' is abelian, we have $o(\overline{ab}) | lm$ and $o(\overline{ab}) | o(ab)$. But $gcd(lm, n) = 1$, and therefore $\overline{ab} = 1_{G/G'}$. Thus $\overline{a} = \overline{b}^{-1}$. Since $gcd(l, m) = 1$ we must have $gcd(o(\overline{a}), o(\overline{b})) = 1$, and therefore $\overline{a} = \overline{b} = 1_{G/G'}$. This proves that G is perfect.

If G has a soluble quotient, then it also has a non-trivial abelian quotient G/N. But then $G' \subset N$, contradicting the fact that G is perfect. $\square$

We now turn our attention to the question: Suppose G is (l, m, n)-generated. Can we deduce other generating sets, in particular generating pairs, for G? The structure constant $\Delta_G(lX, mY, nZ)$ plays a cental role in (l, m, n)-generation and for this reason we first study some of its properties.

Let G be a finite group with *exponent* s, that is, s is the least common multiple of the orders of elements in G. Let ε_s be a primitive s-th root of unity over $\mathbb{Q}$ and $\mathbb{Z}_s^*$ the multiplicative group of integers relatively prime to s. The map $\varepsilon_s \mapsto \varepsilon_s^t$, $t \in \mathbb{Z}_s^*$, defines an automorphism of $\mathbb{Q}(\varepsilon_s)$ over $\mathbb{Q}$. We adopt the following notation: given complex n_i-th roots of unity ε_{n_i} $(1 \leq i \leq k)$, we put $\mathbb{Q}(\varepsilon_{n_1}, \ldots, \varepsilon_{n_k}) = \mathbb{Q}_{n_1,\ldots,n_k}$.

Definition 3.1.11 *Let χ be a character of G and $\sigma \in Gal(\mathbb{Q}_s/\mathbb{Q})$. The Galois conjugate χ^σ of χ is the character of G defined by*

$$\chi^\sigma(g) = \sigma(\chi(g)),$$

for all $g \in G$.

There is a Galois extension $E/\mathbb{Q}$ such that E is the splitting field of G (*cf.* Karpilovsky [98], Theorem 11.1.7.(ii)). Adjoining ε_s to E, if necessary, we may assume $\varepsilon_s \in E$. Observe that $\mathbb{Q}_s/\mathbb{Q}$ is a normal extension and hence the restriction map

$$\begin{array}{rcl} Gal(E/\mathbb{Q}) & \to & Gal(\mathbb{Q}_s/\mathbb{Q}) \\ \sigma & \mapsto & \sigma|\mathbb{Q}_s \end{array}$$

is surjective. Since $\chi(g) \in \mathbb{Q}_s$ for all $g \in G$, we may replace $Gal(E/\mathbb{Q})$ by $Gal(\mathbb{Q}_s/\mathbb{Q})$, in treating Galois conjugates of χ. The proof of the following elementary properties of Galois conjugates can be found in Karpilovsky [99].

Lemma 3.1.12 *Let $C_1, C_2, \ldots, C_r$ and $\chi_1, \chi_2, \ldots, \chi_r$ be the conjugacy classes and the ordinary irreducible characters of G, respectively.*

(i) The group $Gal(\mathbb{Q}_s/\mathbb{Q})$ acts on $\{\chi_1, \chi_2, \ldots, \chi_r\}$ with $\sigma \in Gal(\mathbb{Q}_s/\mathbb{Q})$ and $\sigma(\varepsilon_s) = \varepsilon_s^m$, sending χ_i to χ_i^σ given by

$$\chi_i^\sigma(g) = \sigma(\chi_i(g)) = \chi_i(g^m),$$

for all $g \in G$.

(ii) The group $Gal(\mathbb{Q}_s/\mathbb{Q})$ acts on $\{C_1, C_2, \ldots, C_r\}$ with σ as in (i) sending C_i to $C_i^\sigma = C_i^u = \{g^u \mid g \in C_i\}$, where u is the inverse of m in $\mathbb{Z}_s^*$.

(iii) For each $\sigma \in Gal(\mathbb{Q}_s/\mathbb{Q})$ and for all $i, j \in \{1, \ldots, r\}$, we have $\chi_i(C_j) = \chi_i^\sigma(C_j^\sigma)$.

(iv) χ_i is $\mathbb{Q}$-valued if and only if $\chi_i(g) = \chi_i(g^m)$, for all $g \in G$ and $m \in \mathbb{Z}_s^*$.

(v) If $Gal(\mathbb{Q}_s/\mathbb{Q})$ is cyclic, then the number of $\mathbb{Q}$-valued characters in $Irr(G)$ is equal to the number of conjugacy classes C_i of G for which $C_i^m = C_i$, where $\varepsilon_s \mapsto \varepsilon_s^m$ is a generator for $Gal(\mathbb{Q}_s/\mathbb{Q})$. $\square$

The next lemma is a well-known result in Galois theory and the proof can be found in Karpilovsky [99], pg 901.

Lemma 3.1.13 *If m and n are positive integers with $gcd(m, n) = d$, then $\mathbb{Q}_m \cap \mathbb{Q}_n = \mathbb{Q}_d$.* $\square$

Suppose that l, m and n are integers that are pairwise coprime and divide the exponent s of the group. Let $\sigma \in Gal(\mathbb{Q}_s/\mathbb{Q})$. Then $\mathbb{Q}_{l,m,n} \subseteq \mathbb{Q}_s$ and $\sigma|\mathbb{Q}_{l,m,n} \in Gal(\mathbb{Q}_{l,m,n}/\mathbb{Q})$. It easily follows from the above lemma that

$$Gal(\mathbb{Q}_{l,m,n}/\mathbb{Q}) \cong Gal(\mathbb{Q}_l/\mathbb{Q}) \times Gal(\mathbb{Q}_m/\mathbb{Q}) \times Gal(\mathbb{Q}_n/\mathbb{Q}).$$

Thus $\sigma|\mathbb{Q}_{l,m,n}$ is completely determined by the primitive roots of unity ε_l, ε_m and ε_n.

Theorem 3.1.14 *Let G be a finite group and let l, m and n be integers that are pairwise coprime. Then for any integer t coprime to n, we have*

$$\Delta(lX, mY, nZ) = \Delta(lX, mY, (nZ)^t).$$

Moreover, G is (lX, mY, nZ)-generated if and only if G is $(lX, mY, (nZ)^t)$-generated.

Proof. Let s be the exponent of G and choose $\sigma \in Gal(\mathbb{Q}_s/\mathbb{Q})$ such that

$$\begin{aligned}
(\sigma|\mathbb{Q}_{l,m,n})(\varepsilon_l) &= \varepsilon_l \\
(\sigma|\mathbb{Q}_{l,m,n})(\varepsilon_m) &= \varepsilon_m \\
(\sigma|\mathbb{Q}_{l,m,n})(\varepsilon_n) &= \varepsilon_n^t.
\end{aligned}$$

Since $\chi_i(nZ)$ is a sum of n-*th* roots of unity, we have

$$\chi_i^\sigma(nZ) = (\sigma|\mathbb{Q}_{l,m,n})(\chi_i(nZ)) = \chi_i((nZ)^t)$$

Now from Lemma 3.1.12 we have

$$\begin{aligned}
\sum_{\chi_i \in Irr(G)} \frac{\chi_i(lX)\chi_i(mY)\overline{\chi_i((nZ)^t)}}{\chi_i(1_G)} &= \sum_{\chi_i \in Irr(G)} \frac{\chi_i(lX)\chi_i(mY)\overline{\chi_i^\sigma(nZ)}}{\chi_i(1_G)} \\
&= \sum_{\chi_i \in Irr(G)} \frac{\chi_i^\sigma(lX^\sigma)\chi_i^\sigma(mY^\sigma)\overline{\chi_i^\sigma(nZ)}}{\chi_i^\sigma(1_G^\sigma)}.
\end{aligned}$$

As $Gal(\mathbb{Q}_s/\mathbb{Q})$ acts on $Irr(G)$, it follows that σ permutes the irreducible characters of G. As the sum ranges over all irreducible characters, we have

$$\sum_{\chi_i \in Irr(G)} \frac{\chi_i(lX)\chi_i(mY)\overline{\chi_i((nZ)^t)}}{\chi_i(1_G)} = \sum_{\chi_i \in Irr(G)} \frac{\chi_i(lX^\sigma)\chi_i(mY^\sigma)\overline{\chi_i(nZ)}}{\chi_i(1_G)}. \tag{3.7}$$

Now $\chi_i(lX^\sigma) = \chi_i^{\sigma^{-1}}(lX)$ (*cf.* Lemma 3.1.12) and $\chi_i(lX) = \varepsilon_l^{\alpha_1} + \varepsilon_l^{\alpha_2} + \cdots + \varepsilon_l^{\alpha_k}$, where $k = \chi_i(1_G)$. Hence

$$\begin{aligned}\sigma(\chi_i(lX)) = \chi_i^\sigma(lX) &= [\sigma(\varepsilon_l)]^{\alpha_1} + [\sigma(\varepsilon_l)]^{\alpha_2} + \cdots + [\sigma(\varepsilon_l)]^{\alpha_k} \\ &= \varepsilon_l^{\alpha_1} + \varepsilon_l^{\alpha_2} + \cdots + \varepsilon_l^{\alpha_k} = \chi_i(lX),\end{aligned}$$

for all $\chi_i \in Irr(G)$. Thus $\chi_i(lX)$ is a fixed point of σ^{-1}. Similarly $\chi_i^{\sigma^{-1}}(mY) = \chi_i(mY)$ and from (3.7) it follows that

$$\begin{aligned}\Delta_G(lX, mY, (nZ)^t) &= \frac{|lX|\,|mY|}{|G|} \sum_{\chi_i \in Irr(G)} \frac{\chi_i(lX)\chi_i(mY)\overline{\chi_i((nZ)^t)}}{\chi_i(1_G)} \\ &= \frac{|lX|\,|mY|}{|G|} \sum_{\chi_i \in Irr(G)} \frac{\chi_i(lX)\chi_i(mY)\overline{\chi_i(nZ)}}{\chi_i(1_G)} \\ &= \Delta_G(lX, mY, nZ).\end{aligned}$$

Let $H < G$ containing a fixed element z in nZ. Then by the above argument $\Delta_H(lx, my, nz) = \Delta_H(lx, my, (nz)^t)$, for all conjugacy classes lx, my, nz of H that fuse to lX, mY, nZ, respectively. Therefore $\Sigma_H(lX, mY, nZ) = \Sigma_H(lX, mY, (nZ)^t)$. Since $\Delta^*(G)$ is completely determined by the subgroups containing z, it follows that $\Delta_G^*(lX, mY, nZ) > 0$ if and only if $\Delta_G^*(lX, mY, (nZ)^t) > 0$, proving the result. □

The condition that l, m and n are pairwise coprime is necessary. Consider the simple Ree group $R(27)$ (*cf.* Atlas [43]). The conjugacy classes $26B = (26A)^3$ and the structure constants

$$\Delta_{R(27)}(6B, 13B, 26A) = 7183566 \quad \text{and} \quad \Delta_{R(27)}(6B, 13B, 26B) = 7193043.$$

Further useful results that we shall use are:

Lemma 3.1.15 *(Conder et al. [39]) Let G be a simple $(2X, mY, nZ)$-generated group. Then G is $(mY, mY, (nZ)^2)$-generated.*

Proof. Suppose that $G = \langle x, y\rangle$ with $x \in 2X$, $y \in mY$ and $xy = z \in nZ$. Clearly, $\langle y^x, y\rangle$ is a normal subgroup of G and since G is simple, $\langle y^x, y\rangle = G$. Finally, $y^x y = xyxy = z^2$ and hence the result. □

Corollary 3.1.16 *Let G be a $(2X, 3Y, tZ)$-generated simple group. Then G is $(2X, 2X, 2X, (tZ)^3)$-generated.*

Proof. Let $x \in 2X$, $y \in 3Y$ with $G = \langle x, y\rangle$ such that $z = xy \in tZ$. Then $\langle x, x^y, x^{y^2}\rangle$ is a non-trivial normal subgroup of G, whence $G = \langle x, x^y, x^{y^2}\rangle$. Also, $xx^yx^{y^2} = (xy)^3 = z^3$, proving the result. □

3.2 2-Generated permutation groups

In this section we discuss a necessary condition on the generating set of a finite permutation groups that involving the number of cycles of the generators. Specifically, we prove the following important result, first proved by Ree [140], using the formula for the genus of a Riemann surface. The alternative proof given below is due to Conder-McKay [38].

Theorem 3.2.1 *(Ree [140]) Suppose G is a group of permutations of a set Ω of size n, and G is generated by $x_1, x_2, \ldots, x_s$, with product $x_1x_2\cdots x_s = 1_G$. If the generator x_i has exactly c_i disjoint cycles on Ω (for $1 \leq i \leq s$) and G is transitive on Ω, then*

$$c_1 + c_2 + \cdots + c_s \leq n(s-2) + 2\,.$$

Proof. Let Γ be the directed graph with vertex set Ω and each point $\alpha \in \Omega$ is joined by a directed edge labelled x_i to the corresponding point $x_i(\alpha)$, for $1 \leq i \leq s$. Then every vertex of Γ has out-degree (number of edges directed away from the vertex) s and in-degree (number of edges directed towards the vertex) s. Since G is transitive on Ω, the graph Γ is connected.

We now embed the graph Γ into an orientable surface S, by defining a rotation $\rho = \{\rho_\alpha\}$, where ρ_α is the cycle $(x_1(\alpha), x_2(\alpha), \ldots, x_s(\alpha))$. We next calculate the Euler characteristic $\chi = |V| - |E| - |F|$ of the surface S, where V, E and F denote respectively the set of vertices, edges and faces of the embedded graph. Clearly $|V| = |\Omega| = n$ and $|E| = ns$, so it remains for us to determine $|F|$.

By our choice of rotation of edges at each vertex, it is clear that there are precisely n distinct faces bounded by a directed edge sequence of the form $(x_1, x_2, \ldots, x_s)$.

Further, each cycle of each permutation x_i corresponds to a face of the embedded graph, with the elements of the cycle coinciding with the vertices of the face. That is, there are c_i faces bounded only by edges labelled x_i, for $1 \leq i \leq s$. Together these faces accounts for all the edges of Γ, hence $|F| = n + \sum_{i=1}^{s} c_i$.

If γ is the genus of the surface S, then $2 - 2\gamma = \chi = n - ns + n + \sum_{i=1}^{s} c_i$, so that

$$n(s-2) - \sum_{i=1}^{s} c_i + 2 = 2\gamma \geq 0 \, ,$$

and the result follows. □

Naturally we assume $s \geq 2$. The hypothesis $x_1 x_2 \cdots x_s = 1_G$ simply requires that x_s is the inverse of the product of the first $s-1$ generators. The generators $x_1, \ldots, x_{s-1}$ in fact generate G. A permutation x_i on Ω with c_i disjoint cycles is even if and only if $c_i \equiv n \, (mod\, 2)$. Further, the product $x_1 x_2 \cdots x_s$ of the generators of G is the identity, and therefore all but and even number of the generators will be even permutations. Without loss of generality, let $x_1, \ldots, x_{2t}$ be the be odd permutations. Then $(c_j + 1) \equiv n (mod\, 2)$, for each $j = 1, \ldots, 2t$. Therefore

$$\begin{aligned}
& \sum_{j=1}^{2t} (c_j + 1) + \sum_{2t+1}^{s} c_j \equiv n(2t)(mod\, 2) + n(s-2t)(mod\, 2) \\
\Rightarrow \quad & \sum_{j=1}^{2t} c_j + 2t + \sum_{2t+1}^{s} c_j \equiv n(2t)(mod\, 2) + n(s-2t)(mod\, 2) \\
\Rightarrow \quad & \sum_{j=1}^{2t} c_j + \sum_{2t+1}^{s} c_j \equiv ns(mod\, 2)
\end{aligned}$$

It therefore follows that $\sum_{i=1}^{s} c_i \equiv ns \, (mod\, 2)$ and hence

$$n(s-2) - \sum_{i=1}^{s} c_i + 2$$

will have even parity independent of the transitivity of G on Ω. More importantly Ree's theorem place an obvious restriction on the cycle structure of possible generators in any known transitive permutation representation for the group G. We note that the inequality can still be satisfied by the permutations which generates a proper, imprimitive, or even intransitive subgroup of the image group, in which case other means may be required to eliminate the associated type of possible generating set.

Corollary 3.2.2 *Suppose G is a group of permutations of a set Ω of size n, and G is generated by $x_1, x_2, \ldots, x_s$, with product $x_1 x_2 \cdots x_s = 1_G$. If c_i is the number of disjoint cycles of x_i (for $1 \leq i \leq s$), then*

$$c_1 + c_2 + \cdots + c_s \leq n(s-2) + 2t\,,$$

where t is the number of orbits of Ω under the action of G.

Proof. Let $\Omega_1, \ldots, \Omega_t$ be the orbits of Ω under the action of G, n_i be the size of Ω_i, where $1 \leq i \leq t$, and $c_j^{(i)}$ the number of disjoint cycles of x_j on Ω_i. Then $n_1 + \cdots + n_t = n$ and by Theorem 3.2.1 we have

$$\sum_{j=1}^{s} c_j^{(i)} \leq n_i(s-2) + 2\,,$$

for all $i = 1, \ldots, t$. Since $\sum_{j=1}^{s} c_j = \sum_{i=1}^{t}(\sum_{j=1}^{s} c_j^{(i)})$, we have

$$\sum_{j=1}^{s} c_j \leq \sum_{i=1}^{t}[n_i(s-2) + 2] = n(s-2) + 2t\,,$$

proving the result. $\square$

Corollary 3.2.3 *Suppose G is a group of permutations of a set Ω of size n, and G is generated by $x_1, x_2, \ldots, x_s$. If c_i is the number of disjoint cycles of x_i on Ω (for $1 \leq i \leq s$) and G is transitive on Ω, then*

$$c_1 + c_2 + \cdots + c_s \leq n(s-1) + 1\,.$$

Proof. Let $x_{s+1} = (x_1 x_2 \cdots x_s)^{-1}$. Then G is generated by $x_1, x_2, \ldots, x_{s+1}$ and $x_1 x_2 \cdots x_{s+1} = 1_G$. Since $\sum_{i=1}^{s} c_i + 1 \leq \sum_{i=1}^{s+1} c_i$, it follows from Theorem 3.2.1 that

$$\sum_{i=1}^{s} c_i + 1 \leq \sum_{i=1}^{s+1} c_i \leq n(s-1) + 2\,,$$

and the result follows. $\square$

For a trivial but perhaps illuminating application, suppose that $s = 2$ in Theorem 3.2.1. Then $x_1 = x_2^{-1}$, so that $c_1 = c_2$ and $G = \langle x_1, x_2 \rangle = \langle x_1 \rangle$, and the theorem gives $c_1 + c_2 \leq 2$, therefore $c_1 = c_2 = 1$. In other words, the cyclic group $\langle x_1 \rangle$ acts transitively on Ω if and only if x_1 is a single n-cycle.

Similarly, when $s = 3$ we obtain $c_1 + c_2 + c_3 \leq n + 2$. In this case an interesting result is the following corollary.

Corollary 3.2.4 *Let p, q, r be primes. If the group $\langle x, y, z \mid x^p = y^q = z^r = xyz = 1 \rangle$ has a transitive permutation representation of degree n, then*

$$(p-1)\,\frac{n}{p} + (q-1)\,\frac{n}{q} + (r-1)\,\frac{n}{r} \geq 2n - 2\,.$$

Proof. The element x can decomposed only into fixed points (1-cycles) and p-cycles. Thus if c_1 is the number of disjoint cycles of x, then

$$c_1 \geq (n - p\,\frac{n}{p}) + \frac{n}{p} = n - (p-1)\,\frac{n}{p}\,.$$

Similarly

$$\begin{aligned} c_2 \geq (n - q\,\frac{n}{q}) + \frac{n}{q} &= n - (q-1)\,\frac{n}{q}\,, \\ c_3 \geq (n - r\,\frac{n}{r}) + \frac{n}{r} &= n - (r-1)\,\frac{n}{r}\,, \end{aligned}$$

where c_2 and c_3 are the number of disjoint cycles of y and z, respectively. Therefore

$$3n - (p-1)\,\frac{n}{p} - (q-1)\,\frac{n}{q} - (r-1)\,\frac{n}{r} \leq c_1 + c_2 + c_3 \leq n(3-2) + 2\,.$$

Rearranging this relation gives the required result. $\square$

3.3 2-Generated matrix groups

The theorem by Ree discussed in the previous section can be used as a device for proving the existence of conjectured proper subgroups of a given group. Under certain circumstances it can also be used to prove a given group is not (l, m, n)-generated. In this section we will generalize Ree's theorem to matrices.

Let G be a group acting linearly on a finite dimensional vector space V over an arbitrary field F. The group G may be infinite, and we do not insist that the action is faithful on V. For X a subgroup of G, let $d(X)$ denote the codimension of the fixed-point space $C_V(X)$ of X on V, that is,

$$d(X) = dim(V/C_V(X)) = dim(V) - dim(C_V(X)).$$

The vector space V is a FG-module. We make V^*, the dual space of V, into a FG-module by defining the action of G on V^* by

$$(gf)(v) = f(gv),$$

where $g \in G$, $f \in V^*$ and $v \in V$. Also we write $d^*(X)$ for the codimension of the fixed-point space of X on V^*.

Lemma 3.3.1 *Let G be a group generated by $x_1, x_2, \ldots, x_n$ with $x_1 x_2 \cdots x_n = 1_G$. If G acts on the vector space V over a field F, then*

$$N = (1_G - x_1)V + x_1(1_G - x_2)V + \cdots + x_1 x_2 \cdots x_{n-1}(1_G - x_n)V$$

is the smallest FG-submodule of V with trivial action on the quotient space V/N. Furthermore, $dim(N) = d^*(G)$.

Proof. Let $M = (1_G - x_1)V + (1_G - x_2)V + \cdots + (1_G - x_n)V$. Then M is a subspace V. For any $m \in M$ and $1 \leq i \leq n$, we have $(1_G - x_i)m \in M$ so that $x_i m \in m + M = M$. Since G is generated by $x_1, \ldots, x_n$, M is a FG-submodule of V. Define an action of G on V/M by $g(v + M) = gv + M$. This action is well-defined as M is a submodule of V. For $i = 1, \ldots, n$ and for all $v \in V$, we know that $(1_G - x_i)v \in M$ and hence $x_i(v + M) = x_i v + M = v + M$. Since $x_1, \ldots, x_n$ generate G, the action of G on V/M is trivial. Let K be any submodule of V such that G acts trivially on V/K. Then $g(v+K) = v+K$, for all $v \in V$ and $g \in G$. In particular, $(1_G - x_i)v \in K$, for all $v \in V$, and hence $M \subseteq K$. Thus M is the smallest submodule of V such that G acts trivially on its quotient. Let $N = (1_G - x_1)V + x_1(1_G - x_2)V + \cdots + x_1 x_2 \cdots x_{n-1}(1_G - x_n)V$. Then N is a subspace of V. By definition, $(1_G - x_i)V \subseteq M$, for all $1 \leq i \leq n$. Thus $x_1 x_2 \cdots x_{i-1}(1_G - x_i)V \subseteq x_1 x_2 \cdots x_{i-1} M = M$ and hence $N \subseteq M$. Conversely, $(1_G - x_1)V \subseteq N$. For any $n' \in N$, we have $x_1 x_2 \cdots x_{i-1}(1_G - x_i)n' \in N$, so that

$x_1x_2\cdots x_{i-1}n' + N = x_1x_2\cdots x_in' + N$. If $i = 1$, then it follows that $x_1N \subseteq N$. On the other hand, $(1_G - x_1)x_1^{-1}n' \in N$ and therefore $x_1^{-1}n' \in N$ so that $N \subseteq x_1N$. Thus $x_1N = N$.

Now $x_1(1_G - x_2)V \subseteq N$, implies that $(1_G - x_2)V \subseteq x_1^{-1}N = N$. Furthermore, $x_1x_2n' + N = x_1n' + N = N$ implies that $x_2n' \in x_1^{-1}N = N$. Since this is true for all $n' \in N$ we have $x_2N \subseteq N$. Also $x_1(1_G - x_2)x_2^{-1}x_1^{-1}n' \in N$ so that $x_1x_2^{-1}x_1^{-1}n' \in N$. But then $x_2^{-1}(x_1^{-1}n') \in x_1^{-1}N = N$. As n' ranges over all elements of N so to does $x_1^{-1}n'$ and therefore $x_2^{-1}N \subseteq N$. Thus $x_2N = N$. Now $x_1x_2(1_G - x_3)V \subseteq N$ and hence $(1_G - x_3)V \subseteq x_2^{-1}x_1^{-1}N = N$. Continuing this way it follows that $(1_G - x_i)V \subseteq N$ and $x_iN = N$, for all $i = 1, \ldots, n$. Therefore $N = M$, the smallest submodule of V with trivial action on V/N.

We show that $dim(M) = d^*(G)$. Let $f \in C_{V^*}(G)$. Then $(1_G - x_i)f = 0$, that is, $f((1_G - x_i)V) = 0$, for all $1 \leq i \leq n$. Therefore, $f(M) = 0$. Conversely, if $f \in V^*$ with $f(M) = 0$, then $f((1_G - x_i)V) = 0$, for all $1 \leq i \leq n$. Thus $C_{V^*}(G) = \{f \in V^* \mid f(M) = 0\}$. Define a map $F : V^* \to M^*$ by $F(f) = f|M$. Then F is a linear map and $Ker\, F = C_{V^*}(G)$. Therefore $V^*/C_{V^*}(G) \cong M^*$ and $d^*(G) = dim(V^*/C_{V^*}(G)) = dim(M^*) = dim(M) = dim(N)$, proving the result. $\square$

Theorem 3.3.2 *(Scott [149]) Let G be a group generated by $x_1, x_2, \ldots, x_n$ with $x_1x_2\cdots x_n = 1_G$. If G acts on the vector space V over a field F, then*

$$d_1 + d_2 + \cdots + d_n \geq d(G) + d^*(G)\,, \tag{3.8}$$

where $d_i = d(\langle x_i \rangle)$, for all $1 \leq i \leq n$.

Proof. Let C be the F space of n-tuples $(v_1, v_2, \ldots, v_n)$ with $v_i \in (1_G - x_i)V$. Define a linear map $\beta : V \to C$ by

$$\beta(v) = ((1_G - x_1)v, (1_G - x_2)v, \ldots, (1_G - x_n)v)\,.$$

Also define $\delta : C \to V$ by

$$\delta(v_1, v_2, \ldots, v_n) = v_1 + x_1v_2 + x_1x_2v_3 + \cdots + x_1x_2\cdots x_{n-1}v_n\,.$$

Now

$$0 = v - x_1x_2\cdots x_nv = (1_G - x_1)v + x_1(1_G - x_2)v + \cdots + x_1\cdots x_{n-1}(1_G - x_n)v\,,$$

and hence $Im\beta \subseteq Ker\delta$. Now the image of δ is

$$(1_G - x_1)V + x_1(1_G - x_2)V + \cdots + x_1x_2\cdots x_{n-1}(1_G - x_n)V.$$

By the previous lemma $Im\delta = (1_G - x_1)V + (1_G - x_2)V + \cdots + (1_G - x_n)V$ and $C/Ker\delta$ has dimension $d^*(G)$.

On the other hand, if $v \in Ker\beta$, then $(1_G - x_i)v = 0$, that is, $x_iv = v$, for all $1 \leq i \leq n$. Since G is generated by $x_1, \ldots, x_n$, we have $Ker\beta = C_V(G)$ and by the first isomorphism theorem $V/C_V(G) \cong Im\beta$. Thus $dim(Im\beta) = d(G)$. Finally, the map $F : V \to (1_G - x_i)V$ given by $v \mapsto (1_G - x_i)v$ is an onto linear map with $Ker\ F = C_V(\langle x_i\rangle)$. Thus $dim((1_G - x_i)V) = d(\langle x_i\rangle) = d_i$, and hence $dim(C) = \sum_{i=1}^n d_i$. Thus

$$\begin{aligned}\sum_{i=1}^n d_i &= dim(C) = dim(Im\beta) + dim(Ker\delta/Im\beta) + dim(C/Ker\delta)\\ &\geq dim(Im\beta) + dim(C/Ker\delta) = d(G) + d^*(G).\end{aligned}$$

□

Ree's theorem is obtained immediately by taking G to be the group of permutation matrices. Let A be a permutation matrix of degree n, $\alpha \in S_n$ the permutation associate with the columns of A. If $x = (z_1, \ldots, z_n) \in V$, then $Ax = (z_{\alpha(1)}, z_{\alpha(2)}, \ldots, z_{\alpha(n)})$. Furthermore, let $\alpha = \alpha_1\alpha_2\cdots\alpha_s$ be a decomposition of α into disjoint cycles α_i, with α_i of length n_i. Then $Ax = x$ if and only if $z_{\alpha_j(k)} = z_{\alpha_j^2(k)} = \cdots = z_{\alpha_j^{n_j}(k)}$, for all $j = 1, \ldots, s$ and k a point in the orbit α_j. Thus the dimension of the $C_V(A)$ equals the number of orbits of α. Similarly, the codimension $d(X)$ for a subgroup X of G is just the degree minus the number of orbits of X.

Scott's theorem can be use with fields of characteristic p. The left hand side of relation (3.8) is computable in terms of the Brauer characters if $x_1, x_2, \ldots, x_n$ are all p'-elements. Indeed, Scott's theorem gives a necessary condition that a given class function be the Brauer character of an irreducible module.

Corollary 3.3.3 *(Conder* et al. *[39]) Let $x_1, x_2, \ldots, x_n$ be elements generating a group G with $x_1x_2\cdots x_n = 1_G$, and M be an irreducible module for G of dimension*

$n \geq 2$. If d_i is the codimension of the fixed-point space $C_M(\langle x_i \rangle)$ of $\langle x_i \rangle$ on M, then

$$d_1 + d_2 + \cdots + d_n \geq 2n.$$

Proof. We may, without loss of generality, replace the vector space V in Scott's theorem with a module M. Since M is irreducible $C_M(G)$ and $C_{M^*}(G)$ are zero modules and hence $d(G) = d^*(G) = n$. The result now follows from Scott's theorem. □

We now proceed to discuss methods to calculate the d_i's in Corollary 3.3.3. Let χ be an ordinary irreducible character of the group G. Then χ is afforded by an irreducible module M of dimension $\chi(1_G)$ over $\mathbb{C}$.

We think of each representation matrix x_i in its diagonal form over $\mathbb{C}$. If $o(x_i) = n_i$, then every eigenvalue of x_i is an n_i-th root of unity and the character value is nothing but the sum of eigenvalues. Thus

$$\chi(x_i) = a_1 1 + a_2\omega_1 + \cdots + a_k\omega_{k-1}\,,$$

where the ω_i is an n_i-th roots of unity, $1 \leq i \leq k-1$, and $a_1 + a_2 + \cdots + a_k = \chi(1_G)$. Now $C_M(\langle x_i \rangle) = \{m \in M \mid x_i m = m\}$ is the 1-eigenspace of x_i. Thus the dimension of $C_M(\langle x_i \rangle)$ is precisely the number a_1 (that is, the number of 1's on the diagonal of x_1). Thus the space $M/C_M(\langle x_i \rangle)$ has dimension

$$d_i = \chi(1_G) - a_1. \tag{3.9}$$

If we further assume that

(i) the conjugacy class containing x_i consist of elements of prime order p and

(ii) the character value of this class is an integer,

then

$$\chi(x_i) = a_1 + a_2\omega + \cdots + a_p\omega^{p-1}\,,$$

where $\omega \neq 1$ is a p-th root of unity. Now since we assume that $\chi(x_i) \in \mathbb{Z}$, it follows that $a_2 = a_3 = \cdots = a_p$. Also $1 + \omega + \cdots + \omega^{p-1} = 0$, and therefore

$$\begin{aligned} \chi(x_i) &= a_1 + a_2(\omega + \omega^2 + \cdots + \omega^{p-1}) \\ &= a_1 - a_2\,. \end{aligned}$$

Now $a_1+a_2+\cdots+a_p = a_1+(p-1)a_2 = \chi(1_G)$ and $\chi(1_G) = d_i - a_1$, we get $a_2 = \dfrac{d_i}{p-1}$. Hence using (3.9) once more we get

$$d_i = \frac{p-1}{p}(\chi(1_G) - \chi(x_i)).$$

Now assume that the field F has characteristic p and x_i is p-regular. If the characteristic of the field is zero, then we do not need the restrictions. Recall that the dimension of $C_M(\langle x_i \rangle)$ is the number of 1's on the diagonal of x_i, when x_i is in diagonal form. Also $\chi(x_i^j) = \chi{\downarrow_{\langle x_i\rangle}}(x_i^j)$, for all $1 \leq j \leq o(x_i)$ and the irreducible characters of $\langle x_i \rangle$ are all of degree 1. Thus the number of 1's on the diagonal of x_i is the multiplicity θ_1 (the identity character of $\langle x_i \rangle$) in $\chi{\downarrow_{\langle x_i\rangle}}$ which is given by the inner product $\langle \chi{\downarrow_{\langle x_i\rangle}}, \theta_1 \rangle$. Therefore

$$\begin{aligned} d_i &= dim(M) - dim(C_M(\langle x_i \rangle)) \\ &= dim(M) - \langle \chi{\downarrow_{\langle x_i\rangle}}, \theta_1 \rangle, \\ &= \chi(1_G) - \frac{1}{|\langle x_i \rangle|} \sum_{j=0}^{o(x_i)-1} \chi(x_i^j). \end{aligned}$$

Theorem 3.3.4 *(Brauer) Let χ be a character of G with the inner product $\langle \chi, \chi_1 \rangle = 0$, where χ_1 is the identity character of G. Let A and B be proper subgroups of G and suppose*

$$\langle \chi{\downarrow_A}, \chi_1{\downarrow_A} \rangle + \langle \chi{\downarrow_B}, \chi_1{\downarrow_B} \rangle > \langle \chi{\downarrow_{(A\cap B)}}, \chi_1{\downarrow_{(A\cap B)}} \rangle.$$

Then A and B generate a proper subgroup of G.

Proof. Let M be the $\mathbb{C}G$-module that affords χ. Then the fixed-points spaces $C_M(A)$ and $C_M(B)$ are subspaces of $C_M(A \cap B)$ and

$$\begin{aligned} dim(C_M(A)) + dim(C_M(B)) &= \langle \chi{\downarrow_A}, \chi_1{\downarrow_A} \rangle + \langle \chi{\downarrow_B}, \chi_1{\downarrow_B} \rangle \\ &> \langle \chi{\downarrow_{(A\cap B)}}, \chi_1{\downarrow_{(A\cap B)}} \rangle = dim(C_M(A \cap B)). \end{aligned}$$

It therefore follows that $C_M(A) \cap C_M(B) \neq \{0\}$. Let $0 \neq m \in C_M(A) \cap C_M(B)$. Then $gm = m$, for all $g \in \langle A, B \rangle$, so that $\langle A, B \rangle$ fixes a non-trivial point of M. Since $\langle \chi, \chi_1 \rangle = 0$, the group G fixes only the trivial element 0 of M and we conclude that $\langle A, B \rangle < G$. $\square$

3.4 2-Generations by computer

Analysis of the structure of a given finite group can often be performed with the aid of a computer. The algebra packages Cayley, and Gap together with its userpackages such as MeatAxe (developed by Richard Parker) and Smash, are at times very effective in this analysis, provided the group has a concrete representation either as a permutation group, or as a matrix group or in terms of generators and relations. Certain questions can often be answered using a mixture of theoretical and computational techniques. Even when representation is large, successful approaches to some quite difficult problems may be found. One such approach involves the use of random element generation, in the case where the production and storage of large classes of elements is restricted by the resources available.

The procedure for finding 2-generations for a group discussed in Section 3.1 requires some extreme local analysis and may be of little use when the subgroup lattice is large, or has a complicated structure that is not well-known. One way around this problem, especially suitable in the case of groups which have a permutation representation of small degree, involves the use of a computer. The following procedure was first described by Conder [35] using the Cayley package.

Using the concrete representation of the group, Cayley (or Gap) allows us to create a list of representatives of conjugacy classes for possible generating triples (lx, mY, nZ). For instance, if the elements of the class lX can be enumerated, then for any fixed $z \in nZ$, each element x in lX can be checked to see whether $x^{-1}z \in mY$, and the set of all such x can be partitioned into equivalence classes under conjugation by $C_G(z)$ (*cf.* Lemmas 3.15 and 3.1.6). Alternatively, random conjugates for a chosen representative of lX can be checked in the same way, until all the triples (with a fixed element $z \in nZ$) are accounted for. Either way, the subgroup generated by each triple can be analysed, to find its order and other properties.

An alternative approached for finding generating triples involves the use of the MeatAxe or Smash userpackages incorporated in Gap. These packages are equip to deal with algebras over finite fields and hence are particularly useful in the case of groups with matrix representations over finite fields.

Concrete matrix representations over finite field for the sporadic simple groups were found by Robert Wilson *el al.* [174] and Suleiman *et al.* [157]. For instance, $Aut(F_{22})$, the automorphism group of F_{22}, has a 78-dimensional irreducible module over $GF(2)$ (*cf.* Jansen *el al.* [91]). This representation can be used to show that $Aut(F_{22}) = \langle a, b\rangle$, where a and b are 78×78 matrices over $GF(2)$ of order 2 and 18, respectively. From the generators a and b of $Aut(F_{22})$ we can find the "standard generators" of F_{22}. The standard generators for the sporadic simple groups was introduced as a device for improving reproducibility of computational results and for avoiding duplication of work. These generators were chosen so that they are easy to reconstruct in (as far as possible) any representation.

Given a matrix representation of a group G over a finite field, the first step in the process for finding generating triples (lX, mY, nZ) is to find representatives for the conjugacy classes of G. Random generation of word (in terms of the generating matrices) will produce elements of various orders. Using *ad hoc* methods, such as power maps or the codimension of the fixed point space and the nullity of matrices, we can find representatives for the conjugacy classes. Suppose we have found $x \in lX$, $y \in mY$ such that $xy \in nZ$. Then MeatAxe allows us to generate the group generated by x and y. From the properties of the group G and its subgroups, such as the existence of prime order elements, we may be able to deduce whether $\langle x, y\rangle = G$. Once again we have to stress that these are *ad hoc* procedures and depend to a large extent on the group under consideration as well as the conjugacy classes in the triple.

Chapter 4

The Higman-Sims Group

4.1 Introduction

The Higman-Sims simple group can be constructed from the Higman-Sims graph $\mathcal{G}$. Let $\mathcal{G} = (\Omega, \mathcal{E})$ be a graph of valency 22 on the set Ω of 100 points such that any given vertex has 22 neighbours (adjacent points) and each of the remaining 77 vertices are joined to 6 of these points and may be labelled by the corresponding hexad. Two of the 77 vertices are joined only if the corresponding hexads are disjoint. The 22 points and 77 hexads (blocks) form a Steiner system $S(3,6,22)$. The Higman-Sims simple group HS is the subgroup of the even permutations of $Aut(\mathcal{G}) \cong HS{:}2$, the automorphism group of HS. The point stabilizer of $Aut(\mathcal{G})$ on Ω is $Aut(S(3,6,22)) \cong M_{22}{:}2$ and the order of the Higman-Sims group HS is $44352000 = 2^9{\cdot}3^2{\cdot}5^3{\cdot}7{\cdot}11$.

The action of HS on Ω yields a unique primitive rank-3 representation of degree 100, in which the point stabilizer is the Mathieu group M_{22}, and the orbits have length 1, 22 and 77. The group HS has two inequivalent representations of degree 176, which are doubly transitive, and the point stabilizer is isomorphic to $U_3(5){:}2$. The two conjugacy classes of $U_3(5){:}2$ are fused in $HS{:}2$. Also, HS has two primitive inequivalent representations of degree 1100, one on the set of edges of $\mathcal{G}$ with point stabilizer isomorphic to $L_3(4){:}2_1$ and the other with point stabilizer isomorphic to S_8. The subgroup S_8 is also the set stabilizer of a fixed outer automorphism of HS. The group HS acts primitively on both of its conjugacy classes of involutions $2A$ and

$2B$. The centralizer $C_{HS}(2A)$ is the split extension of a group $K \cong 4{\cdot}2^4$ of known structure by S_5 and $C_{HS}(2B) \cong 2 \times A_6{\cdot}2^2$. The set stabilizer of a nonedge of $\mathcal{G}$ is maximal in HS and is isomorphic to $2^4.S_6$, and contains the holomorph of 2^4.

We quote from Magliveras [117] the fact that the proper non-abelian simple subgroups of HS are, up to isomorphisms, A_5, A_6, A_7, A_8, $L_2(7)$, $L_2(11)$, $L_3(4)$, $U_3(5)$, M_{11} and M_{22}. There are only one class of each of M_{22}, A_8, $L_3(4)$, and $L_2(11)$, while there are two classes of $U_3(5)$ and M_{11}, interchanged by the outer automorphism. Any A_5, A_6, A_7 or $L_2(7)$ with trivial centralizer is contained in M_{22} or $U_3(5)$.

Theorem 4.1.1 *(Magliveras [116]) The Higman-Sims group HS has exactly 12 conjugacy classes of maximal subgroups, as follows:*

M_{22}	$U_3(5){:}2$ *(2 classes)*
$L_3(4){:}2_1$	S_8
$2^4.S_6$	$4^3{:}L_3(2)$
M_{11} *(2 classes)*	$4{\cdot}2^4{:}S_5$
$2 \times A_6{\cdot}2^2$	$5{:}4 \times A_5$. □

We will use the maximal subgroups of HS listed in the theorem above extensively, especially those with order divisible by 7 or 11. We list in Table 4.I the maximal subgroups and some of their properties. For any maximal subgroup M of a simple group G, the action of G on the right (or left) cosets of M is equivalent to the action of G on the conjugates of M. The permutation characters of HS on the right cosets (hence on the conjugates) of the maximal subgroups isomorphic to $4^3{:}L_3(2)$ and M_{11}, in terms of irreducible characters of HS, are not given in the ATLAS [43]. We list in Table 4.II partial fusion maps of these maximal subgroups into HS (obtained from GAP) that will enable us to evaluate the permutation characters on the relevant conjugacy classes of elements of HS (*cf.* Theorem 3.1.4).

4.2 (p, q, r)-Generations of HS

As we stated earlier, the group HS acts primitively as a rank-3 group of degree 100 on Ω. The point stabilizer under this action is isomorphic to M_{22} and the permutation

TABLE 4.I
Maximal subgroups of HS

M	$\|M\|$	Orbit Type	$\chi_M = (\chi_1{\downarrow_M})^{HS}$
M_{22}	$2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11$	[1,22,77]	$\underline{1a}+\underline{22a}+\underline{77a}$
$U_3(5){:}2$	$2^5{\cdot}3^2{\cdot}5^3{\cdot}7$	$[50^2]$	$\underline{1a}+\underline{175a}$
$U_3(5){:}2$	$2^5{\cdot}3^2{\cdot}5^3{\cdot}7$	$[50^2]$	$\underline{1a}+\underline{175a}$
$L_3(4){:}2_1$	$2^7{\cdot}3^2{\cdot}5{\cdot}7$	[2,42,56]	$\underline{1a}+\underline{22a}+\underline{77a}+\underline{175a}+\underline{825a}$
S_8	$2^7{\cdot}3^2{\cdot}5{\cdot}7$	[30,70]	$\underline{1a}+\underline{77a}+\underline{154a}+\underline{175a}+\underline{693a}$
$2^4.S_6$	$2^8{\cdot}3^2{\cdot}5$	[2, 6, 32, 60]	
$4^3{:}L_3(2)$	$2^9{\cdot}3{\cdot}7$	[8,28,64]	
M_{11}	$2^4{\cdot}3^2{\cdot}5{\cdot}11$	[12,22,66]	
M_{11}	$2^4{\cdot}3^2{\cdot}5{\cdot}11$	[12,22,66]	
$4{\cdot}2^4{:}S_5$	$2^9{\cdot}3{\cdot}5$	[20,80]	
$2\times A_6{\cdot}2^2$	$2^6{\cdot}3^2{\cdot}5$	[40,60]	
$5{:}4\times A_5$	$2^4{\cdot}3{\cdot}5^2$	$[20^5]$	

character of HS on the conjugates of M_{22} is given by $\chi_{M_{22}} = \underline{1a}+\underline{22a}+\underline{77a}$. We apply Ree's transitivity condition (*cf.* Theorem 3.2.1) to this action. Now $\chi_{M_{22}}(2A) = 20$ and therefore an element in the class $2A$, as a permutation on Ω, has 20 fixed points. Since permutations of order 2 consist of only fixed points and two cycles, the elements in class $2A$ have cycle structure $1^{20}2^{40}$. Similarly, we can find the cycle structure of elements in the other conjugacy classes of HS. We list in Table 4.III the cycle structure of the conjugacy classes that are relevant in our investigations.

If HS is (pX,qY,rZ)-generated and the elements of H are expressed as permutations on 100 points, then by Ree's theorem $c_1+c_2+c_3 \leq 102$, where c_1, c_2, c_3 are the number of disjoint cycles of a representative in pX, qY, rZ, respectively.

Lemma 4.2.1 *The group HS is not*

(i) $(2A,2A,tX)$- *or* $(2A,2B,tX)$*-generated, for any integer* t,

(ii) $(2B,2B,pX)$- *or* $(2A,3A,pX)$*-generated, for any prime* p,

(iii) $(2B,3A,7A)$*-generated.*

TABLE 4.II
Partial fusion maps into HS

$4^3{:}L_3(2)$-class	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$5a$	$7a$	
$\|C_{4^3:L_3(2)}(nx)\|$	384	192	1440	96	360	36	30	7	
$\to HS$	$2A$	$2A$	$2B$	$2B$	$3A$	$3A$	$5A$	$7A$	
M_{11}-class	$2a$	$3a$	$4a$	$5a$	$6a$	$8a$	$8b$	$11a$	$11b$
$\|C_{M_{11}}(nx)\|$	48	18	8	5	6	8	8	11	11
$\to HS$	$2A$	$3A$	$4A$	$5C$	$6A$	$8B$	$8B$	$11A$	$11B$

TABLE 4.III
Cycle type of a representative in pX

HS-class	$2A$	$2B$	$3A$	$7A$	$11A$	$11B$
$\chi_{M_{22}}(pX)$	20	0	10	2	1	1
Cycle type	$1^{20}2^{40}$	2^{50}	$1^{10}3^{30}$	$1^{2}7^{14}$	$1^{1}11^{9}$	$1^{1}11^{9}$

Proof. Consider the triple $(2A, 2A, tX)$, where tX is any conjugacy class of HS. Since the number of cycles of a representative of class $2A$ is $20 + 40 = 60$, we have $c_1 = 60 = c_2$. Thus $c_1 + c_2 + c_3 = 120 + c_3 > 102$, where c_3 is the number of disjoint cycles of a class representative of tX. This violates Ree's transitivity condition and we conclude that HS is not $(2A, 2A, tX)$-generated, for any conjugacy class tX of HS.

Similarly, an application of Ree's transitivity condition to a representative of the conjugacy classes in Table 4.III shows that all the triples in the statement of the lemma are not generating triples for HS. □

Lemma 4.2.2 *The group HS is not $(2A, 5X, 7A)$-generated, where $X \in \{A, B, C\}$.*

Proof. For the triple $(2A, 5A, 7A)$, non-generation follows immediately from the structure constant $\Delta_{HS}(2A, 5A, 7A) = 0$.

The group HS acts on a 22-dimensional irreducible complex module V. We apply

Scott's theorem (*cf.* Theorem 3.3.2) to the module V and compute

$$\begin{aligned} d_{2A} &= dim(V/C_V(2A)) &&= (22-6)/2 &&= 8 \\ d_{5B} &= dim(V/C_V(5B)) &&= 4(22-2)/5 &&= 16 \\ d_{5C} &= dim(V/C_V(5C)) &&= 4(22-2)/5 &&= 16 \\ d_{7A} &= dim(V/C_V(7A)) &&= 6(22-1)/7 &&= 18. \end{aligned}$$

Now if HS is $(2A, 5Y, 7A)$, where $Y = B$ or C, then by Scott's theorem we must have $d_{2A} + d_{5Y} + d_{7A} \geq 2 \times 22$. However, $8 + 16 + 18 < 44$, and non-generation of HS by these triples follows. $\square$

Lemma 4.2.3 *The group HS is $(2B, 5X, 7A)$-generated, where $X \in \{A, B, C\}$.*

Proof. Any maximal subgroup of HS with order divisible by $2 \times 5 \times 7$, that is containing elements of order 2, 5, and 7, is isomorphic to either M_{22}, $U_3(5){:}2$, $L_3(4){:}2_1$ or S_8. We first show that no maximal subgroup, and hence no proper subgroup, of HS is $(2B, 5X, 7A)$-generated, where $X \in \{A, B, C\}$. Now $\chi_{M_{22}}(2B) = 0$ and hence, then $2B \cap M_{22} = \emptyset$. Therefore, M_{22} and its subgroups are not $(2B, 5X, 7A)$-generated. Similarly, the permutation character values $\chi_{L_3(4):2_1}(5A) = 0 = \chi_{L_3(4):2_1}(5B)$ and $\chi_{S_8}(5A) = 0 = \chi_{S_8}(5C)$. As $\Sigma_{L_3(4):2_1}(2B, 5C, 7A) = 0$ and $\Sigma_{S_8}(2B, 5B, 7A) = 0$, it follows that $L_3(4){:}2_1$ and S_8 are not $(2B, 5X, 7A)$-generated. Also any $U_3(5){:}2$ subgroup of HS is not $(2B, 5X, 7A)$-generated since $\Sigma(U_3(5){:}2) = 0$, for all $X \in \{A, B, C\}$.

Thus we conclude that no proper subgroup of HS is $(2B, 5X, 7A)$-generated, and hence $\Delta^*(HS) = \Delta(HS)$. The result now follows from $\Delta_{HS}(2B, 5A, 7A) = 42 = \Delta_{HS}(2B, 5B, 7A)$ and $\Delta_{HS}(2B, 5C, 7A) = 490$. $\square$

Lemma 4.2.4 *The group HS is $(3A, 5X, 7A)$-generated, where $X \in \{A, B, C\}$.*

Proof. We first prove that HS is $(3A, 5A, 7A)$-generated. We calculate $\chi_{M_{22}}(5A) = \chi_{L_3(4):2_1}(5A) = \chi_{S_8}(5A) = 0$ and $\Sigma(U_3(5){:}2) = 21$, and it follows from Table 4.I that the proper subgroups of HS that admit $(3A, 5A, 7A)$-generated subgroups are contained in $U_3(5){:}2$. Now a fixed element of order 7 in $U \cong U_3(5){:}2$ is contained in $\chi_U(7A) = 1$ conjugate subgroup of U. The group HS contains two non-conjugate classes of $U_3(5){:}2$ subgroups. Let U_1 and U_2 be non-conjugate subgroups of HS isomorphic to $U_3(5){:}2$. Then

$$\begin{aligned}\Delta^*(HS) &= \Delta(HS) - \Sigma(U_1) - \Sigma(U_2) + \Sigma(U_1 \cap U_2) \\ &\geq \Delta(HS) - 2\Sigma(U_1) = 126 - 42 = 84 > 0.\end{aligned}$$

Next, we show that HS is $(3A, 5B, 7A)$-generated. We calculate $\chi_{M_{22}}(5B) = 0 = \chi_{L_3(4):2_1}(5B)$, $\Sigma(S_8) = 0$ and $\Sigma(U_3(5){:}2) = 140$. Therefore, up to isomorphisms, $U_3(5){:}2$ is the only maximal subgroups of HS that admit $(3A, 5B, 7A)$-generated subgroups. Furthermore, we calculate $\Delta(HS) = 560$, and hence $\Delta^*(HS) \geq 560 - 2(140) = 280$.

Finally we consider the triple $(3A, 5C, 7A)$. Amongst the maximal subgroups of HS with order divisible by $3 \times 5 \times 7$, the only subgroups having empty intersection with a conjugacy class in this triple are isomorphic to S_8 ($\chi_{S_8}(5C) = 0$). From the structure constants we calculate $\Sigma(M_{22}) = 2464$, $\Sigma(U_3(5){:}2) = 280$ and $\Sigma(L_3(4){:}2_1) = 882$. Also a fixed element of order 7 is contained in $\chi_{M_{22}}(7A) = 2$ conjugates of M_{22} and $\chi_{L_3(4):2_1}(7A) = 1$ conjugate of $L_3(4)$. Furthermore, $\Delta(HS) = 6720$, and hence

$$\Delta^*(HS) \geq 6720 - 2(2464) - 2(280) - 882 = 350 > 0,$$

proving the result. $\square$

Lemma 4.2.5 *The group HS is $(2B, 3A, 11Z)$-, $(2A, 5Y, 11Z)$-, $(2B, 5X_1, 11Z)$-, $(2B, 7A, 11Z)$-, $(3A, 5Y, 11Z)$-, $(5Y, 7A, 11Z)$-, $(5X_1, 5X_2, 11Z)$-generated, for distinct $X_1, X_2 \in \{A, B, C\}$ and all $Y, Z \in \{A, B\}$.*

Proof. From the list of maximal subgroups of HS we observe that, up to isomorphisms, M_{22} and M_{11} are the only maximal subgroups of HS with order divisible by 11. However, $\chi_{M_{22}}(2B) = \chi_{M_{22}}(5A) = \chi_{M_{22}}(5B) = 0$ so that the conjugacy classes $2B$, $5A$ and $5B$ have empty intersection with any M_{22} subgroups of HS. Similarly, it follows from Table 4.II that any M_{11} subgroup of HS does not meet the conjugacy classes $2B$, $5A$ and $5B$. Since all the triples in the statement of the lemma involve either a $2B$-, $5A$- or $5B$-conjugacy class, we conclude that no proper subgroup of HS is generated by these triples and hence $\Delta^*(HS) = \Delta(HS)$. From Table 4.IV we note that $\Delta(HS) > 0$ for all these triples, proving the result. $\square$

TABLE 4.IV
Structure Constants of HS

pX	$3A$	$5A$	$5B$	$5C$	$7A$
$\Delta_{HS}(2A,pX,11Z)$	11	11	11	231	825
$\Delta_{HS}(2B,pX,11Z)$	33	33	33	605	2211
$\Delta_{HS}(3A,pX,11Z)$	308	242	363	4950	17622
$\Delta_{HS}(5A,pX,11Z)$	242	176	297	3564	12672
$\Delta_{HS}(5B,pX,11Z)$	363	297	418	5907	21153

Lemma 4.2.6 *The group HS is $(2A,5C,11X)$- and $(3A,5C,11X)$-generated, where $X \in \{A,B\}$.*

Proof. We first show that HS is $(2A,5C,11X)$-generated. Let $N \leq HS$ with $N \cong M_{11}$. The subgroup N acts on Ω, the set of conjugates of $M' \cong M_{22}$ in HS, with orbits of length 12, 22 and 66 (*cf.* Table 4.I). Let Γ be the orbit of length 12 with $M \in \Gamma$ and N_M be the stabilizer of M in N. Then $[N : N_M] = 12$ and since $L_2(11)$ is the only subgroup of N with index 12, up to isomorphisms, we have $N_M \cong L_2(11)$. However, $N_M = \{g \in N | M^g = M\} = \{g \in N | g \in N_{HS}(M) = M\} = N \cap M$. We calculate $\Delta(HS) = 231$, $\Sigma(M_{22}) = 176$, $\Sigma(M_{11}) = 33$ and $\Sigma(L_2(11)) = 22$. Also if we fix an element of order 11 in N or M, then it is contained in no other conjugate of N or M, respectively. Thus $\Delta^*(HS) = \Delta(HS) - \Sigma(M \cup N_1 \cup N_2)$, where N_1 and N_2 are non-conjugate subgroups of HS isomorphic to M_{11}. Hence

$$\Delta^*(HS) \geq 231 - 176 - 2(33) + 2(22) = 33 > 0\,,$$

and the $(2A,5C,11X)$-generation of HS follows.

Next we show the $(3A,5C,11X)$-generation of HS. We calculate $\Delta(HS) = 4950$, $\Sigma(M_{22}) = 2112$ and $\Sigma(M_{11}) = 99$. Thus

$$\Delta^*(HS) \geq 4950 - 2112 - 2(99) = 2640\,,$$

and the result follows. $\square$

Lemma 4.2.7 *The group HS is $(2A,7A,11X)$-, $(3A,7A,11X)$-, and $(5C,7A,11X)$-generated, where $X \in \{A,B\}$.*

Proof. The only maximal subgroups of HS that may contain $(pX, 7A, 11X)$-generated subgroups, p a prime, are isomorphic to M_{22} (*cf.* Table 4.I). We easily calculate the structure constants $\Delta_{HS}(2A, 7A, 11X) = 825$, $\Delta_{HS}(3A, 7A, 11X) = 17622$, $\Delta_{HS}(5C, 7A, 11X) = 253440$, $\Sigma_{M_{22}}(2A, 7A, 11X) = 352$, $\Sigma_{M_{22}}(3A, 7A, 11X) = 3520$ and $\Sigma_{M_{22}}(5C, 7A, 11X) = 25344$. In all cases, $\Delta^*(HS) = \Delta(HS) - \Sigma(M_{22}) > 0$, proving the result. $\square$

We now summarize the above results in the following theorems.

Theorem 4.2.8 *The Higman-Sims group HS is (p, q, r)-generated for all $p, q, r \in \{2, 3, 5, 7, 11\}$ with $p < q < r$, except when $(p, q, r) = (2, 3, 5)$ or $(2, 3, 7)$.*

Proof. This follows from Lemmas 4.2.1 to 4.2.7 and the fact that the triangular group $T(2, 3, 5)$ is isomorphic to A_5. $\square$

Corollary 4.2.9 *The group HS is (pX, pX, qY)- generated, with $p < q$, for all $pX \in \{5A, 5B, 5C, 7A\}$, $qY \in \{7A, 11A, 11B\}$ and $(p, p, q) = (3, 3, 11)$.*

Proof. The result follows immediately from an application of Lemma 3.1.15 to Lemmas 4.2.3 and 4.2.5. $\square$

4.3 nX-Complementary generations of HS

We will now apply the techniques for finding nX-complementary generations, discussed in Section 2.3.2, to the Higman-Sims simple group. In particular, we will consider the triangular presentations of HS that allow us to deduce its nX-complementary generations (*cf.* Lemma 2.3.8). For the case where n is prime, the triangular generations in the previous section will suffice.

Lemma 4.3.1 *The group HS is not nX-complementary generated, where $nX \in \{3A, 4A, 4B\}$.*

Proof. For the conjugacy class $3A$ we use a theorem by Brauer (*cf.* Theorem 3.3.4). Let $\chi = \underline{22a} \in Irr(HS)$, $A = \langle x \rangle$ and $B = \langle y \rangle$, where $x \in 2A$ and

TABLE 4.V
Structure Constants of HS

tX	$5A$	$5B$	$5C$	$6A$	$6B$	$7A$	$8A$
$\Delta_{HS}(2A,4A,tX)$	0	0	0	15	0	0	2
$\Delta_{HS}(2A,4B,tX)$	0	0	75	27	48	35	46
$\lvert C_{HS}(tX)\rvert$	500	300	25	36	24	7	16
tX	$8BC$	$10A$	$10B$	$11AB$	$12A$	$15A$	$20AB$
$\Delta_{HS}(2A,4A,tX)$	0	0	10	0	0	0	4
$\Delta_{HS}(2A,4B,tX)$	0	0	10	22	18	15	0
$\lvert C_{HS}(tX)\rvert$	16	20	20	11	12	15	20

$y \in 3A$. Then $\langle \chi, \chi_1 \rangle = 0$ and $A \cap B = \{1_G\}$. Moreover,$\langle \chi\downarrow_A, \chi_1\downarrow_A \rangle = 14$, $\langle \chi\downarrow_B, \chi_1\downarrow_B \rangle = 10$ and $\langle \chi\downarrow_{(A\cap B)}, \chi_1\downarrow_{(A\cap B)} \rangle = 22$, and hence $\langle x, y \rangle < HS$. Thus HS is not $(2A, 3A, tX)$-generated, for any t, and it follows from Lemma 2.3.8 that HS is not $3A$-complementary generated.

It is evident from Table 4.V and Lemma 3.1.7 that HS is not $(2A, 4A, tX)$-generated, for any t, and hence not $4A$-complementary generated.

To prove that HS is not $4B$-complementary generated it suffices to show that HS is not $(2A, 4B, tX)$-generated, for any conjugacy class tX with $\Delta_{HS}(2A, 4B, tX) \geq |C_{HS}(tX)|$. We first consider the case $(2A, 4B, 5C)$. The cycle structure of elements in the conjugacy classes $2A$ and $5C$, as permutations on 100 points, are $1^{20}2^{40}$ and $1^5 5^{19}$, respectively. The conjugacy class $(4B)^2 = 2A$. As a permutation on 100 points, an element $y \in 4B$ has 8 fixed points, whilst y^2 has 20 fixed points. Thus y has 8 two cycles and consequently 20 four cycles, that is, y has cycle structure $1^8 2^8 4^{20}$. It now follows from Ree's theorem that HS is not $(2A, 4B, 5C)$-generated.

Next we consider the triple $(2A, 4B, 7A)$. Let M be a maximal subgroup of HS isomorphic to M_{22} and $x \in M$ be a fixed element of order 7. Then x is contained in 2 conjugates of M, say M and M^g. We calculate $\Sigma(M_{22}) = 28$ and thus

$$\Delta^*(HS) \leq \Delta(HS) - \Sigma(M) - \Sigma(M^g) + \Sigma(M \cap M^g) = -21 + \Sigma(M \cap M^g).$$

Now $M \cap M^g$ is a two point stabilizer on the set Ω of 100 points. The action of M on Ω gives orbits of size 1, 22 and 77. Thus $[M{:}M \cap M^g] = 22$ or 77. Since $x \in M \cap M^g$, it must follows that $[M{:}M \cap M^g] = 22$, and hence $M \cap M^g \cong L_3(4)$, the only subgroup of M_{22} with index 22, up to isomorphisms. Further we calculate $\Sigma(L_3(4)) = 21$ and therefore $\Delta^*(HS) = 0$, proving non-generation.

Next we calculate $\Delta_{HS}(2A, 4B, 6B) = 48$ and $\Sigma_{M_{22}}(2A, 4B, 6B) = 36$. Therefore $\Delta^*(HS) \leq 12 < |C_{HS}(6B)|$ and non-generation follows from Lemma 2.3.8. The same argument can be applied for the triple $(2A, 4B, 8A)$ since $\Delta(HS) = 46$ and $\Sigma(M_{22}) = 44$. Furthermore, we calculate

$$\begin{aligned} \Delta_{HS}(2A, 4B, 11X) &= 22 = \Sigma_{M_{22}}(2A, 4B, 11X) \\ \Delta_{HS}(2A, 4B, 12A) &= 18 = \Sigma_{S_8}(2A, 4B, 12A) \\ \Delta_{HS}(2A, 4B, 15A) &= 15 = \Sigma_{S_8}(2A, 4B, 15A). \end{aligned}$$

Non-generation by these triples follows once more from Lemma 2.3.8. Thus we conclude that HS is not $4B$-complementary generated, completing the proof. $\square$

Table 4.VI

Structure Constants of HS

pX	$2A$	$2B$	$3A$	$5A$	$5B$	$5C$	$7A$	$11AB$
$\Delta_{HS}(pX, 4C, 11A)$	66	198	1782	1386	2112	27720	90000	64152
$\Delta_{HS}(pX, 6A, 11A)$	132	396	3234	2442	3894	49390	175890	111914/114664
$\Delta_{HS}(pX, 6B, 11A)$	242	638	5148	3696	6160	73920	264000	167904
$\Delta_{HS}(pX, 8A, 11A)$	396	1012	7920	5544	9504	110880	396000	250272

Lemma 4.3.2 *The group HS is nX-complementary generated, where $nX \in \{4C, 6A, 6B, 8A\}$.*

Proof. Recall that the maximal subgroups of HS with order divisible by 11 are, up to isomorphims, M_{11} (two non-conjugate classes) and M_{22}. Also a fixed element of order 11 is contained in a unique conjugate of a subgroup isomorphic to M_{11} (respectively, M_{22}) in HS. We deal separately with each conjugacy class.

Table 4.VII
Structure Constants of M_{22}

pX	$2a$	$3a$	$5a$	$7ab$	$11ab$
$\Delta_{M_{22}}(pX, 4a, 11a)$	22	319	2640	1936	1364
$\Delta_{M_{22}}(pX, 4b, 11a)$	44	638	5280	3872	2728
$\Delta_{M_{22}}(pX, 6a, 11a)$	121	1155	7744	5280	3256/3124
$\Delta_{M_{22}}(pX, 8a, 11a)$	198	1760	11616	8096	4576

Table 4.VIII
Structure Constants of M_{11}

pX	$2a$	$3a$	$5a$	$11ab$
$\Delta_{M_{11}}(pX, 4a, 11a)$	11	44	198	110
$\Delta_{M_{11}}(pX, 6a, 11a)$	44	66	264	132

First we consider the class $4C$. We calculate $\Delta_{HS}(2A, 4C, 20A) = 60$. If K is a maximal subgroup of HS with non-empty intersection with each of the conjugacy classes $2A$, $4C$ and $20A$, then $K \cong U_3(5){:}2$, $4{\cdot}2^4{:}S_5$ or $5{:}4\times A_5$. However, $\Sigma(K) = 0$ for all these subgroups and therefore $\Delta^*(HS) = 60$, proving the $(2A, 4C, 20A)$-generation of HS. The fusion maps into HS give $M_{22} \cap 4C = 4b$ and $M_{11} \cap 4C = 4a$. Now for all conjugacy classes pX, with prime order representatives, other than $2A$, we observe from Tables 4.VI, 4.VII and 4.VIII that

$$\begin{aligned}\Delta^*_{HS}(pX, 4C, 11A) &\geq \Delta_{HS}(pX, 4C, 11A) - \Sigma_{M_{22}}(pX, 4C, 11A)\\ &\quad -2\Sigma_{M_{11}}(pX, 4C, 11A) > 0\ ,\end{aligned}$$

and hence HS is $(pX, 4C, 11A)$-generated. We therefore conclude that HS is $4C$-complementary generated.

The conjugacy class $6A$ does not meet any subgroup isomorphic to M_{11} or M_{22} and hence no proper subgroup of HS is $(tX, 6A, 11A)$-generated, for any t. From the structure constants in Table 4.VI, we conclude that HS is $6A$-complementary generated. The class $6B$ has non-empty intersection with all subgroups isomorphic to M_{11} and M_{22}. It is clear from Tables 4.VI, 4.VII and 4.VIII that $\Delta^*_{HS}(pX, 6B, 11A) > 0$

and $6B$-complementary generation of HS follows.

Finally, $M_{11} \cap 8A = \emptyset$ and $M_{22} \cap 8A = 8a$. Thus from the above tables we get

$$\Delta^*_{HS}(pX, 8A, 11A) = \Delta_{HS}(pX, 8A, 11A) - \Sigma_{M_{22}}(pX, 48A, 11A) > 0 \ ,$$

proving $8A$-complementary generation of HS. This completes the result. $\square$

We are now ready to prove the main result in this section.

Theorem 4.3.3 *The group HS is nX-complementary generated if and only if $nX = 4C$ or $n \geq 5$.*

Proof. From Lemma 2.3.8 it follows that HS is not $2X$-complementary generated. We proved in the previous section that the group HS is $(2X, 5A, 11Z)$–, $(3A, 5A, 11Z)$–, $(5A, 5Y, 11Z)$-, $(5A, 7A, 11Z)$-generated, $X, Z \in \{A, B\}$ and $Y \in \{A, B, C\}$. Thus we have shown that HS is $(pX, 5A, 11A)$-generated, for all conjugacy classes pX with representatives of prime order. It therefore follows from Lemma 2.3.8 that HS is $5A$-complementary generated.

Similar arguments will show that HS is $5B$-, $5C$-, $7A$- and $11X$-complementary generated, for $X \in \{A, B\}$. Furthermore, we have $(8B)^2 = 4C = (8C)^2$, $(10A)^2 = 5A$, $(10B)^2 = 5B$ and $(20A)^2 = 10A = (20B)^2$. The result now follows from Lemma 4.3.1 and an application of Lemma 2.3.9 to Lemma 4.3.2. $\square$

Chapter 5

The McLaughlin Group

5.1 Intoduction

It was shown by McLaughlin [125] that there exists a regular graph $\mathcal{G} = (\Omega, \mathcal{E})$ with 275 vertices possessing a transitive automorphism group $Aut(\mathcal{G}) \cong McL{:}2$, with McL a new simple group of order $2^7.3^6.5^3.7.11$. The McLaughlin graph $\mathcal{G}$ is a rank 3 graph of valency 112 on 275 points in which the point stabilizer U is a maximal subgroup isomorphic to $U_4(3)$. The orbits under this action are $\{x\}$, Φ and Ψ with orders 1, 112 and 162, respectively. The action of U on Φ is equivalent to the representation of $U_4(3)$ on the set of totally singular lines of the 4-dimensional unitary space V over the Galois field $GF(9)$ with the stabilizer of a point having the form $3^4{:}A_6$ and orbits of orders 1, 30 and 81, respectively. The action of U on Ψ is equivalent to the representation of $U_4(3)$ on the left cosets of a subgroup isomorphic to $L_3(4)$ with the stabilizer of a point having orbits of orders 1, 56 and 105, respectively. Thus the two point stabilizers of McL on Ω are isomorphic to either $3^4{:}A_6$ or $L_3(4)$. From this we conclude that $U \cap U^g \cong 3^4{:}A_6$ or $L_3(4)$, for any two distinct conjugate subgroups isomorphic to $U_4(3)$.

The group McL has precisely one conjugacy class of involutions and the centralizer of an involution in McL is isomorphic to $2{\cdot}A_8$, the unique perfect central extension of the alternating group A_8 by a group of order 2. Finkelstein [60] showed that the proper non-abelian simple subgroups of McL are isomorphic to A_5, A_6, A_7, $L_2(7)$, $U_4(2)$,

$U_3(3)$, $L_3(4)$, $U_3(5)$, $U_4(3)$, M_{11} and M_{22}. There are two classes of M_{22} subgroups, interchanged by the outer automorphism.

Theorem 5.1.1 *(Finkelstein [60]) The McLaughlin simple group has precisely twelve conjugacy classes of maximal subgroups. The isomorphism types in these classes are as follows:*

(i) two groups of classical type, namely, $U_4(3)$ and $U_3(5)$;

(ii) four groups of Mathieu type, namely, M_{11}, M_{22} (two classes) and $L_3(4){:}2_2$, the stabilizer the 253-dimensional representation of M_{23};

(iii) six p-local subgroups, namely, $2^4{:}A_7$ (two classes), $2{\cdot}A_8$, $3^4{:}M_{10}$, $3_+^{1+4}{:}2.S_5$ and $5_+^{1+2}{:}3{:}8$. □

In Table 5.I we list some of the properties of the maximal subgroups of McL with order divisible by 7 or 11 (the other maximal subgroups are irrelevant in this study). The permutation characters of McL on the cosets (or conjugates) of the maximal subgroups $U_4(3)$, M_{22} and $U_3(5)$ are given in the ATLAS. In Table 5.II we list partial fusion maps, obtained from GAP, of the maximal subgroups of McL (with order divisible by 7 or 11) for which the permutation character are not given in the ATLAS.

TABLE 5.I
Maximal subgroups of McL

M	$\|M\|$	Orbit Type	χ_M
$U_4(3)$	$2^7{\cdot}3^6{\cdot}5{\cdot}7$	[1,112,162]	$\underline{1a} + \underline{22a} + \underline{252a}$
M_{22}	$2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11$	[22, 77, 176]	$\underline{1a} + \underline{22a} + \underline{252a} + \underline{1750a}$
M_{22}	$2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11$	[22, 77, 176]	$\underline{1a} + \underline{22a} + \underline{252a} + \underline{1750a}$
$U_3(5)$	$2^4{\cdot}3^2{\cdot}5^3{\cdot}7$	[50^2, 175]	$\underline{1a} + \underline{22a} + \underline{252a} + \underline{1750a} + \underline{5103a}$
$L_3(4){:}2_2$	$2^7{\cdot}3^2{\cdot}5{\cdot}7$	[2,56,112,105]	
$2{\cdot}A_8$	$2^7{\cdot}3^2{\cdot}5{\cdot}7$		
$2^4{:}A_7$	$2^7{\cdot}3^2{\cdot}5{\cdot}7$	[7, 16, 112, 140]	
$2^4{:}A_7$	$2^7{\cdot}3^2{\cdot}5{\cdot}7$	[7, 16, 112, 140]	
M_{11}	$2^4{\cdot}3^2{\cdot}5{\cdot}11$	[11,12,110,132]	

Table 5.II
Partial fusion maps into McL

$L_3(4){:}2_2$-class	$2a$	$2b$	$3a$	$5a$	$7a$	$7b$			
$\lvert C_{L_3(4):2_2}(nx)\rvert$	128	336	18	5	14	14			
$\to McL$	$2A$	$2A$	$3B$	$5B$	$7A$	$7B$			
$2{\cdot}A_8$-class	$2a$	$2b$	$3a$	$3b$	$5a$	$7a$	$7b$		
$\lvert C_{2\cdot A_8}(nx)\rvert$	40320	192	360	36	30	14	14		
$\to McL$	$2A$	$2A$	$3A$	$3B$	$5A$	$7A$	$7B$		
$2^4{:}A_7$-class	$2a$	$2b$	$3a$	$3b$	$5a$	$7a$	$7b$		
$\lvert C_{2^4:A_7}(nx)\rvert$	2688	96	36	36	5	14	14		
$\to McL$	$2A$	$2A$	$3B$	$3B$	$5B$	$7A$	$7B$		
M_{11}-class	$2a$	$3a$	$4a$	$5a$	$6A$	$8a$	$8b$	$11a$	$11b$
$\lvert C_{M_{11}}(nx)\rvert$	48	18	8	5	6	8	8	11	11
$\to McL$	$2A$	$3B$	$4A$	$5B$	$6B$	$8A$	$8A$	$11A$	$11B$

TABLE 5.III
Structure constants for McL

tX	$7AB$	$8A$	$9AB$	$10A$	$11AB$	$12A$	$14AB$	$15AB$	$30AB$
$\Delta_{McL}(2A, 3A, tX)$	0	0	0	0	0	3	0	0	6
$\lvert C_{McL}(tX)\rvert$	14	8	27	30	11	12	14	30	30

5.2 (p, q, r)-Generations of McL

Lemma 5.2.1 *(Woldar [176]) The group McL is not $(2A, 3X, tY)$-generated, for any integer t, where $X \in \{A, B\}$.*

Proof. From an application of Lemma 3.1.7 to the triples in Table 5.III it immediately follows that McL is not $(2A, 3A, tY)$-generated, for any integer t.

Let $A = \langle x\rangle$ and $B = \langle y\rangle$, where $x \in 2A$ and $y \in 3B$. If χ_1 and χ are the irreducible characters of McL of degree 1 and 22, respectively, then the inner product $\langle \chi{\downarrow_A}, \chi_1{\downarrow_A}\rangle = 14$, $\langle \chi{\downarrow_B}, \chi_1{\downarrow_B}\rangle = 10$ and $\langle \chi{\downarrow_{(A\cap B)}}, \chi_1{\downarrow_{(A\cap B)}}\rangle = 22$. Thus by the result of Brauer (*cf.* Lemma 3.3.4), we have $\langle x, y\rangle < McL$, and the result follows. □

Lemma 5.2.2 *The group McL is not $(2A, 5X, 7Y)$-generated, where $X, Y \in \{A, B\}$.*

Proof. Let $H \leq McL$ with $H \cong 2^4{:}A_7$. We then calculate $\Sigma_H(2A, 5A, 7Y) = 7 = \Delta_{McL}(2A, 5A, 7Y)$ and thus McL is not $(2A, 5A, 7Y)$-generated.

The irreducible character $\underline{22a}$ of McL affords a 22-dimensional complex irreducible module V. We calculate the co-dimensions $dim(V/C_V(2A)) = 8$, $dim(V/C_V(5B)) = 16$ and $dim(V/C_V(7Y)) = 18$. But $8 + 16 + 18 = 42 < 44$, contradicting Scott's theorem and therefore McL is not $(2A, 5B, 7Y)$-generated. □

Lemma 5.2.3 *The group McL is $(3A, 5X, 7Y)$-generated, where $X, Y \in \{A, B\}$.*

Proof. The maximal subgroups of McL with non-empty intersection with the conjugacy class $3A$ are, up to isomorphisms, $U \cong U_4(3)$ and $H \cong 2{\cdot}A_8$. A fixed element of order 7 is contained in $\chi_U(7Y) = 2$ conjugate subgroups of U and in a unique conjugate subgroup of H (*cf.* Table 5.II). Also $\chi_U(5A) = 0$ and $5B \cap H = \emptyset$, thus

$$\begin{aligned}
\Delta^*_{McL}(3A, 5A, 7Y) &\geq \Delta_{McL}(3A, 5A, 7Y) - \Sigma_H(3A, 5A, 7Y) \\
&= 63 - 7 > 0, \\
\Delta^*_{McL}(3A, 5B, 7Y) &\geq \Delta_{McL}(3A, 5B, 7Y) - 2\Sigma_U(3A, 5B, 7Y) \\
&= 644 - 2(112) > 0\ ,
\end{aligned}$$

and the result follows. □

Lemma 5.2.4 *The group McL is $(3B, 5X, 7Y)$-generated, where $X, Y \in \{A, B\}$.*

Proof. We first prove the $(3B, 5A, 7Y)$-generation of McL. The maximal subgroups with non-empty intersection with the classes $5A$ and $7Y$ are isomorphic to $U_3(5)$ and $2{\cdot}A_8$. We calculate $\Delta(McL) = 595$, $\Sigma(U_3(5)) = 21 = \Sigma(2{\cdot}A_8)$. Further, a fixed element of order 7 is contained in 2 conjugates of a $U_3(5)$ subgroup and in a unique conjugagate of a $2{\cdot}A_8$ subgroup. Thus $\Delta^*(McL) \geq 595 - 2(21) - 21 = 532$ which implies the $(3B, 5A, 7Y)$-generation of McL.

Next we consider the case $(3B, 5B, 7Y)$. The only maximal subgroups of McL with order divisible by 7 and empty intersection with either class $3B$ or $5B$ are isomorphic to $2{\cdot}A_8$. For the remaining maximal subgroups M we list below $\Sigma(M)$ and the number h of conjugates of M containing a fixed element z of order 7.

M	h	$\Sigma(M)$	$h\Sigma(M)$
$U_4(3)$	2	9408	18816
M_{22}	2	2464	4928
M_{22}	2	2464	4928
$U_3(5)$	2	420	840
$L_3(4){:}2_2$	1	882	882
$2^4{:}A_7$	1	336	336
$2^4{:}A_7$	1	336	336

Thus the total number of pairs $(x, y) \in 3B \times 5B$ with $xy = z \in 7Y$, is at most 31066. The result follows since $\Delta(McL) = 50400$. □

Lemma 5.2.5 *The group McL is $(2A, 5A, 11Y)$-, $(3A, 5X, 11Y)$-, $(3B, 5A, 11Y)$-, $(3A, 7X, 11Y)$-, $(5A, 5B, 11Y)$- and $(5A, 7X, 11Y)$- generated, where $X, Y \in \{A, B\}$.*

Proof. The maximal subgroups of McL with order divisible by 11 are isomorphic to either M_{11} or M_{22}. However, $\chi_{M_{22}}(3A) = \chi_{M_{22}}(5A) = 0$ and from Table 5.II we conclude that $3A \cap M_{11} = \emptyset = 5A \cap M_{11}$. Therefore no proper subgroup of McL is $(\pi(3A), \pi(pX), \pi(11Y))$- or $(\pi(pX), \pi(5A), \pi(11Y))$-generated, for any prime p and $\pi \in S_3$. Thus for all the triples in the statement of the lemma we have $\Delta^*(McL) = \Delta(McL)$ and the result follows from Table 5.IV. □

Lemma 5.2.6 *The group McL is $(2A, 5B, 11X)$- and $(3B, 5B, 11X)$-generated, where $X \in \{A, B\}$.*

Proof. We calculate $\Delta_{McL}(2A, 5B, 11X) = 715$ and $\Delta_{McL}(3B, 5B, 11X) = 34485$. Now a fixed element of order 11 is contained in a unique conjugate of M_{11} and M_{22} subgroups of McL, respectively. Also, $\Sigma_{M_{11}}(2A, 5B, 11X) = 33$, $\Sigma_{M_{22}}(2A, 5B, 11X) =$

TABLE 5.IV
Structure Constants of McL

pX	$5A$	$5B$	$7X$
$\Delta_{McL}(2A, pX, 11Y)$	22	715	1584
$\Delta_{McL}(3A, pX, 11Y)$	44	1100	2178
$\Delta_{McL}(3B, pX, 11Y)$	1122	34485	66132
$\Delta_{McL}(5A, pX, 11Y)$	1540	47410	85536

176, $\Sigma_{M_{11}}(3B, 5B, 11X) = 99$ and $\Sigma_{M_{22}}(3B, 5B, 11X) = 2112$. Since McL contains two non-conjugate classes of M_{22} subgroups, we obtain $\Delta^{*}_{McL}(2A, 5B, 11X) \geq 715 - 33 - 2(176) = 330 > 0$ and similarly $\Delta^{*}_{McL}(2A, 5B, 11X) \geq 30162 > 0$, proving the result. $\square$

Lemma 5.2.7 *The group McL is $(2A, 7X, 11Y)$-, $(3B, 7X, 11Y)$- and $(5B, 7X, 11Y)$-generated, where $X, Y \in \{A, B\}$.*

Proof. The maximal subgroups of McL with order divisible by 7×11 are isomorphic to M_{22}. We easily calculate $\Delta_{McL}(2A, 7X, 11Y) = 1584$, $\Delta_{McL}(3B, 7X, 11Y) = 66132$, $\Delta_{McL}(5B, 7X, 11Y) = 2566080$, $\Sigma_{M_{22}}(2A, 7X, 11Y) = 176$, $\Sigma_{M_{22}}(3B, 7X, 11Y) = 1760$, $\Sigma_{M_{22}}(5B, 7X, 11Y) = 12672$. In all cases, $\Delta^{*}(McL) = \Delta(McL) - \Sigma(M_{22}) > 0$, proving the result. $\square$

We are now ready to prove the main results of this section.

Theorem 5.2.8 *The McLaughlin group McL is (p, q, r)-generated for all $p, q, r \in \{2, 3, 5, 7, 11\}$ with $p < q < r$, except when $(p, q, r) = (2, 3, 5)$, $(2, 3, 7)$ or $(2, 3, 11)$.*

Proof. The proof follows from Lemmas 5.2.1 to 5.2.7 and the fact that the triangular group $T(2, 3, 5) \cong A_5$. $\square$

Corollary 5.2.9 *The group McL is (pX, pX, qY)-generated, with $p < q$, for all $pX \in \{5A, 5B, 7A, 7B\}$ and $qY \in \{7A, 7B, 11A, 11B\}$.*

Proof. This follows immediately from an application of Lemma 3.1.15 to Lemmas 5.2.2, 5.2.5 and 5.2.7. $\square$

5.3 nX-Complementary generations of McL

We proceed as in Section 4.3. The maximal subgroups of McL with order divisible by 11, that is, containing all possible $(pX, nY, 11A)$-generated proper subgroups of McL are, up to isomorphisms, M_{11} or M_{22}.

TABLE 5.V
Structure Constants of McL

pX	$2A$	$3A$	$3B$	$5A$	$5B$	$7AB$	$11AB$
$\Delta_{McL}(pX, 4A, 11A)$	143	286	8316	12056	362780	667656	866228
$\Delta_{McL}(pX, 6A, 11A)$	44	88	2354	3300	99000	178002	227304
$\Delta_{McL}(pX, 6B, 11A)$	572	814	24992	33000	990000	1782398	2277792
$\Delta_{McL}(pX, 9A, 11A)$	759	1188	33561	44154	1332045	2376000	3037419

Lemma 5.3.1 *The group McL is nX-complementary generated, where $nX \in \{4A, 6A, 6B, 9A\}$.*

Proof. The fusion maps of the maximal subgroups into McL give $M_{11} \cap 4A = 4a$, $M_{11} \cap 6B = 6a$, $M_{22} \cap 4A = 4a \cup 4b$ and $M_{22} \cap 6B = 6a$. For $nX \in \{4A, 6B\}$, we have

$$\begin{aligned}\Delta^*_{McL}(pY, nX, 11A) \;\geq\; & \Delta_{McL}(pY, nX, 11A) - \Sigma_{M_{22}}(pY, nX, 11A) \\ & - 2\Sigma_{M_{11}}(pY, nX, 11A) > 0\ ,\end{aligned}$$

for all conjugacy classes pY with prime order representatives (*cf.* Tables 4.VII, 4.VIII and 5.V). This show that McL is $4A$- and $6B$-complementary generated.

The class $6A$ does not meet any subgroup of McL isomorphic to M_{11} or M_{22}. Also the groups M_{11} and M_{22} contain no elements of order 9. Thus if $nX \in \{6A, 9A\}$, then $\Delta^*_{McL}(pY, nX, 11A) = \Delta_{McL}(pY, nX, 11A)$, and the $6A$- and $9A$-complementary generation follows from Table 5.V. □

Theorem 5.3.2 *The group McL is nX-complementary generated if and only if $n \geq 4$.*

Proof. The group McL is not $2A$-complementary generated (*cf.* Lemma 2.3.8). Since McL is not $(2A, 3X, tY)$-generated for any integers t (*cf.* Lemma 5.2.1), it is not $3A$- or $3B$-complementary generated. We proved in the previous section that for any $pX \in \{5A, 5B, 7AB, 11AB\}$, the group McL is $(qY, pX, 11A)$-generated, for all conjugacy classes qY with elements of prime order. Therefore the group McL is pX-complementary generated.

The result now follows from an application of Lemma 2.3.9 to the conjugacy classes with prime order representatives and the classes in Lemma 5.3.1. □

Chapter 6

The Smallest Conway Group

6.1 Introduction

The *Leech lattice* is a certain 24-dimensional $\mathbb{Z}$-submodule of the 24-dimensional Euclidean space $\mathbb{R}^{24}$ discovered by John Leech. John Conway showed that the automorphism group of the Leech lattice modulo its central factor group is the Conway group Co_1. The Conway groups Co_2 and Co_3 are stabilizers of sublattices of the Leech lattice. We give a brief description of the construction of these groups, omitting detail. A comprehensive study is given by Aschbacher [2].

Let $M = M_{24}$ and $(X, \mathcal{C})$ be the Steiner system $S(24, 8, 5)$ for M. Let V be the permutation module over $GF(2)$ of M with the basis X and $V_{\mathcal{C}}$ the Golay code submodule. Let $\mathbb{R}^{24}$ be the permutation module over the reals for M with basis X and let $(\, , \,)$ be the symmetric bilinear form on $\mathbb{R}^{24}$ for which X is an orthogonal basis. Thus $\mathbb{R}^{24}$ together with $(\, , \,)$ is just the 24-dimensional Euclidean space admitting the action of M. Now for $\sum_x a_x x$ and $\sum_x b_x x$ in $\mathbb{R}^{24}$, we have

$$\left(\sum_x a_x x, \sum_x b_x x\right) = \sum_x a_x b_x.$$

For $v \in \mathbb{R}^{24}$ define $q(v) = (v, v)/16$. Thus q is a positive definite quadratic form on $\mathbb{R}^{24}$. Given $Y \subseteq X$, define $e_Y = \sum_{y \in Y} y \in \mathbb{R}^{24}$. For $x \in X$ let $\lambda_x = e_X - 4x$.

The Leech lattice is the set Λ of vectors $v = \sum_x a_x x \in \mathbb{R}^{24}$ such that:

($\Lambda 1$) $a_x \in \mathbb{Z}$ for all $x \in X$.

($\Lambda 2$) $m(v) = (\sum_x a_x)/4 \in \mathbb{Z}$.

($\Lambda 3$) $a_x \equiv m(v)(mod\, 2)$ for all $x \in X$.

($\Lambda 4$) $C(v) = \{x \in X \mid a_x \not\equiv m(v)(mod\, 4)\} \in V_C$.

The Leech lattice Λ is a $\mathbb{Z}$-submodule of $\mathbb{R}^{24}$. Let Λ_0 denote the set of vectors $v \in \Lambda$ such that $m(v) \equiv 0(mod\ 4)$. Then Λ_0 is a $\mathbb{Z}$-submodule spanned by the set $\{2e_B \mid B \subset \mathcal{C}\}$. Further, Λ as a $\mathbb{Z}$-submodule is generated by Λ_0 and λ_{x_0}, for $x_0 \in X$. Write $O(\mathbb{R}^{24})$ for the subgroup of $GL(\mathbb{R}^{24})$ preserving the bilinear form $(\ ,\ )$, or equivalently preserving the quadratic form q. Let G be the subgroup of $O(\mathbb{R}^{24})$ acting on Λ. The group G is the automorphism group of the Leech lattice. For $Y \subset X$, write ξ_Y for the element of $GL(\mathbb{R}^{24})$ such that

$$\xi_Y(x) = \begin{cases} -x & \text{,if } x \in Y, \\ x & \text{,if } x \notin Y. \end{cases}$$

Let $Q = \{\xi_Y \mid Y \in V_C\}$. Then $N = M{\cdot}Q \leq G$. Given any positive integer n, write Λ_n for the set of all vectors v in Λ with $q(v) = n$. Then $\Lambda = \cup_n \Lambda_n$. For $v = \sum_x a_x x \in \Lambda$ and i a non-negative integer, let

$$S_i(v) = \{x \mid |a_x| = i\}\ ,$$

and define the shape of v to be $(0^{n_0}, 1^{n_1}, \ldots)$, where $n_i = |S_i(v)|$. Let Λ_2^2 be the set of all vectors in Λ of shape $(2^8, 0^{16})$, Λ_2^3 the set of vectors in Λ of shape $(3, 1^{23})$, and Λ_2^4 the set vectors in Λ of shape $(4^2, 0^{22})$. Then Λ_2^i, $2 \leq i \leq 4$, are the orbits of N on Λ_2, with $|\Lambda_2^2| = 2^7{\cdot}7594$, $|\Lambda_2^3| = 2^{12}{\cdot}24$ and $|\Lambda_2^4| = 2^2{\cdot}\binom{24}{2}$. Moreover, $|\Lambda_2| = 2^4{\cdot}3^3{\cdot}5{\cdot}7{\cdot}13$ and $N = N_G(\Lambda_2^4)$. Using this information it can be shown that G acts transitively on Λ_2, Λ_3, and Λ_4. Also N is a maximal subgroup of G of order $2^{22}{\cdot}3^9{\cdot}5^4{\cdot}7^2{\cdot}11{\cdot}13{\cdot}23$. Notice that ξ_X is the scalar map on $\mathbb{R}^{24}$ determined by -1, and hence is in the center of G. Denote by Co_1 the factor group $G/\langle \xi_X \rangle$. Denote by Co_2 the stabilizer in G of a vector in Λ_2 and denote by Co_3 the stabilizer in G of a vector in Λ_3. The groups Co_1, Co_2 and Co_3 are the simple *Conway groups*, with $|Co_2| = 2^{18}{\cdot}3^6{\cdot}5^3{\cdot}7{\cdot}11{\cdot}23$ and $|Co_3| = 2^{10}{\cdot}3^7{\cdot}5^3{\cdot}7{\cdot}11{\cdot}23$.

For $v \in \Lambda$, let $\Lambda_n(v, i)$ denote the set of all $u \in \Lambda_n$ such that $(v, u) = 8i$. Let $2\Lambda = \{2v \mid v \in \Lambda\}$. Then 2Λ is a G-invariant $\mathbb{Z}$-submodule of Λ, so G acts on the factor module $\overline{\Lambda} = \Lambda/2\Lambda$. The module $\overline{\Lambda}$ is the *Leech lattice mod 2*. For $v \in \Lambda$ let $\overline{v} = v + 2\Lambda$ and for $S \subseteq \Lambda$ let $\overline{S} = \{\overline{s} \mid s \in S\}$. By construction $2\overline{v} = 0$ for all $v \in \Lambda$, so $\overline{\Lambda}$ is an elementary abelian group which we may view as a $GF(2)G$-module. Also ξ_X is trivial on $\overline{\Lambda}$, so $\overline{\Lambda}$ is also a $GF(2)$-module for $\overline{G} = G/\langle \xi_X \rangle \cong Co_1$.

Let $x_1, x_2 \in X$, and let

$$\begin{aligned} v_1 &= 4(x_1 + x_2), \\ v_2 &= -\lambda_{x_1}, \\ v_3 &= v_1 - v_2. \end{aligned}$$

Then $v_1 \in \Lambda_2^4$, $v_2 \in \Lambda_2^3$ and $v_3 \in \Lambda_3$. Thus $C_G(v_3) \cong Co_3$, the stabilizer of a vector in Λ_3. Also $\Lambda_2(v_3, 3)$ is an orbit under $C_G(v_3)$ of length 2×276, with

$$\{\{v, v_3 - v\} \mid v \in \Lambda_2(v_3, 3)\}$$

a system of imprimitivity of order 276. The set

$$\{\langle \overline{v}_3, \overline{v} \rangle \mid v \in \Lambda_2(v_3, v)\}$$

is the set of lines of $\overline{\Lambda}$ through $\overline{v}_3$ generated by elements of $\overline{\Lambda}_2$. There are 276 such lines and they form an orbit under $C_G(v_3)$. Furthermore, $C_G(v_3) \cong Co_3$ is 2-transitive on the set $\mathcal{L}$ of lines of $\overline{\Lambda}$ through $\overline{v}_3$ generated by the points of $\overline{\Lambda}_2$. Also, $\mathcal{L}$ is of order 276 and the stabilizer in $C_G(v_3)$ of a member of $\mathcal{L}$ is isomorhpic to $McL{:}2$.

L. Finkelstein [60] showed that the simple subgroups of Co_3 are, up to isomorphisms, A_5, A_6, A_7, A_8, $L_2(7)$, $L_2(8)$, $L_2(11)$, $L_3(4)$, $U_4(2)$, $U_3(3)$, $U_3(5)$, $U_4(3)$, M_{11}, M_{12}, M_{22}, M_{23}, HS and McL.

Theorem 6.1.1 *(Finkelstein [60]) The Conway simple group Co_3 has precisely fourteen conjugacy classes of maximal subgroups. The isomorphism types in these classes are as follows:*

(i) four groups of classical type, namely, $U_4(3){:}2^2$, $U_3(5){:}S_3$, $L_3(4){:}D_{12}$ and $S_3 \times L_2(8){:}3$;

(ii) one group of Mathieu type, namely, M_{23};

(iii) *two sporadic groups, namely,* HS *and* $McL{:}2$*;*

(iv) *seven p-local subgroups, namely,* $2^4{:}A_8$, $A_4 \times S_5$, $3^{1+4}_{+}{:}4.S_6$, $2^2.[2^7.3^2].S_3$ $3^5{:}(2 \times M_{11})$, $2{\cdot}S_6(2)$ *and* $2 \times M_{12}$. □

If G is a quotient group of the triangular group $T(p,q,r)$, then by the definition of the triangular group, G is also $(\pi(p),\pi(q),\pi(r))$-generated, for any $\pi \in S(p,q,r) \cong S_3$. We may therefore assume $p \leq q \leq r$. As a consequence, we only need to consider the cases $r = 7, 11, 23$ for the group Co_3. We deal separately with each case in Sections 2, 3 and 4. The nX-complementary generations of the group Co_3 will be discussed in section 5. The partial fusion maps (including all conjugacy classes with prime order representatives) of the maximal subgroups into Co_3 are given in Table 6.I. This allow us to calculate h, the number of conjugates of the maximal subgroup containing a fixed element of given order.

TABLE 6.I

Partial fusion maps into Co_3

$McL{:}2$-class	$2a$	$2b$	$3a$	$3b$	$5a$	$5b$	$7a$	$11a$	$11b$	
$\to Co_3$	$2A$	$2B$	$3A$	$3B$	$5A$	$5B$	$7A$	$11A$	$11B$	
h							3	1	1	
HS-class	$2a$	$2b$	$3a$	$5a$	$5b$	$5c$	$7a$	$11a$	$11b$	
$\to Co_3$	$2A$	$2B$	$3B$	$5A$	$5B$	$5B$	$7A$	$11A$	$11B$	
h							6	2	2	
$U_4(3){:}2^2$-class	$2a$	$2b$	$2c$	$2d$	$2e$	$3a$	$3b$	$3c$	$5a$	$7a$
$\to Co_3$	$2A$	$2A$	$2B$	$2B$	$2B$	$3A$	$3B$	$3B$	$5B$	$7A$
h										3
M_{23}-class	$2a$	$3a$	$5a$	$7a$	$7b$	$11a$	$11b$	$23a$	$23b$	
$\to Co_3$	$2A$	$3B$	$5B$	$7A$	$7A$	$11A$	$11B$	$23A$	$23B$	
h				3	3	2	2	1	1	
$3^5{:}(2\times M_{11})$-class	$2a$	$2b$	$2c$	$3a$	$3b$	$3c$	$3d$	$5a$	$11a$	$11b$
$\to Co_3$	$2A$	$2B$	$2B$	$3A$	$3B$	$3B$	$3C$	$5B$	$11A$	$11B$
h									1	1
$2{\cdot}S_6(2)$-class	$2a$	$2b$	$2c$	$3a$	$3b$	$3c$	$5a$	$7a$		
$\to Co_3$	$2A$	$2A$	$2B$	$3A$	$3A$	$3B$	$5A$	$7A$		
h								3		
$U_3(5){:}S_3$-class	$2a$	$2b$	$3a$	$3b$	$3c$	$5a$	$5b$	$7a$		
$\to Co_3$	$2A$	$2B$	$3B$	$3A$	$3C$	$5A$	$5B$	$7A$		
h								2		

TABLE 6.I (Continue)

$2^4{\cdot}A_8$-class	$2a$	$2b$	$2c$	$3a$	$3b$	$5a$	$7a$	$7b$		
$\to Co_3$	$2A$	$2B$	$2A$	$3B$	$3B$	$5B$	$7A$	$7A$		
h							3	3		
$L_3(4){:}D_{12}$-class	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$3c$	$5a$	$7a$	
$\to Co_3$	$2A$	$2B$	$2A$	$2B$	$3B$	$3B$	$3C$	$5B$	$7A$	
h									1	
$2\times M_{12}$-class	$2a$	$2b$	$2c$	$2d$	$2e$	$3a$	$3b$	$5a$	$11a$	$11b$
$\to Co_3$	$2B$	$2A$	$2B$	$2B$	$2B$	$3B$	$3C$	$5B$	$11A$	$11B$
h									1	1

6.2 $(p,q,7)$-Generations of Co_3

The group Co_3 acts as a transitive *rank*-2 group on a set Ω of 276 points. The point stabilizer of this action is isomorphic to the group $McL{:}2$ and the resulting permutation character is $\chi_{McL:2} = \underline{1a} + \underline{275a}$. The value of $\chi_{McL:2}$ on the conjugacy class pX, p a prime, will enable us to deduce the cycle type of elements in pX as a permutation of degree 276.

Lemma 6.2.1 *The group Co_3 is $(2X,3Y,7A)$-generated, for $X \in \{A,B\}$ and $Y \in \{A,B,C\}$, if and only if the ordered pair $(X,Y) = (B,C)$.*

TABLE 6.II

Cycle structure of a representative in pX

Co_3-class	$2A$	$2B$	$3A$	$3B$	$3C$	$7A$	$11AB$
$\chi_{McL:2}(pX)$	36	12	6	15	0	3	1
Cycle type	$1^{36}2^{120}$	$1^{12}2^{132}$	$1^{6}3^{90}$	$1^{15}3^{87}$	3^{92}	$1^{3}7^{39}$	$1^{1}11^{25}$

Proof. If the group Co_3 is (pX,qY,rZ)-generated, then an application of Ree's theorem to Co_3 as a permutation group on 276 points, implies that $c_1 + c_2 + c_3 \leq 278$, where c_1, c_2 and c_3 are the number of disjoint cycles of representatives in pX, qY and

rZ, respectively. From Table 6.II we conclude that Ree's theorem is violated for all triples $(2X, 3Y, 7A)$, except when $X = B$ and $Y = C$.

For the triple $(2B, 3C, 7A)$ we observe from the fusion maps into Co_3 that if M is a maximal subgroup with non-empty intersection with the classes in this triple, then M is isomophic to either, $U_3(5){:}S_3$, $L_3(4){:}D_{12}$ or $S_3 \times L_2(8){:}3$. However, we easily calculate $\Sigma(M) = 0$ for all the above subgroups and hence $\Delta^*(Co_3) = \Delta(Co_3) = 504$, proving the result. □

Lemma 6.2.2 *The group Co_3 is $(2X, 5Y, 7A)$-generated, for $X, Y \in \{A, B\}$, if and only if the ordered pair $(X, Y) \in \{(B, A), (B, B)\}$.*

Proof. We calculate $\Delta_{Co_3}(2A, 5A, 7A) = 21 < |C_{Co_3}(7A)| = 42$ and non-generation of Co_3 by this triple follows from Lamma 3.1.7. The group Co_3 acts on a 23-dimensional irreducible complex module V with

$$dim(V/C_V(2A)) = 8, \quad dim(V/C_V(5B)) = 16 \quad \text{and} \quad dim(V/C_V(7A)) = 18.$$

But $8 + 16 + 18 < 46$, and hence, by Scott's theorem, $(2A, 5B, 7A)$ is a non-generating triple of Co_3.

Next we consider the triple $(2B, 5A, 7A)$. The maximal subgroups of Co_3 with order divisible by 7 and non-empty intersection with the classes $2B$ and $5A$ are isomorphic to $McL{:}2$, HS, $2{\cdot}S_6(2)$ and $U_3(5){:}S_3$. We calculate $\Delta(Co_3) = 1512$, $\Sigma(McL{:}2) = 0 = \Sigma(U_3(5){:}S_3)$, $\Sigma(HS) = 48$ and $\Sigma(2{\cdot}S_6(2)) = 98$. A fixed element of order 7 is contained in 6 conjugate copies of HS and in 3 conjugate copies of $2{\cdot}S_6(2)$ (*cf.* Table 6.I). Thus $\Delta^*(Co_3) \geq 1512 - 6(48) - 3(98) > 0$ and therefore $(2B, 5A, 7A)$ is a generating triple for Co_3.

Finally, we show that Co_3 is $(2B, 5B, 7A)$-generated. The maximal subgroups with non-empty intersection with the classes $2B$, $5B$ and $7A$ are, up to isomorphisms, $McL{:}2$, HS, $U_4(3){:}2^2$, $U_3(5){:}S_3$, $2^4{\cdot}A_8$ and $L_3(4){:}D_{12}$. We calculate $\Delta(Co_3) = 7560$, $\Sigma(HS) = 532$, $\Sigma(2^4{\cdot}A_8) = 56$ and $\Sigma(M) = 0$ for the remaining subgroups in the above list. Also a fixed element of order 7 is contained in 3 conjugate copies of $2^4{\cdot}A_8$. Thus $\Delta^*(Co_3) \geq 7560 - 6(532) - 3(56) = 4200$, and the result follows. □

The group Co_3 acts transitively on the set Ω the set of conjugates of the subgroup $M \cong McL$. In Finkelstein [60] the lengths of the orbits of subgroups of Co_3 acting on Ω are determined. We shall use these orbit lengths to obtain information on the subgroup lattice of Co_3.

Lemma 6.2.3 *The group Co_3 is $(3X, 5Y, 7A)$-generated, for all $X \in \{A, B, C\}$ and $Y \in \{A, B\}$.*

Proof. We will treat each triple separately.

Case $(3A, 5A, 7A)$: The maximal subgroups of Co_3 that have non-empty intersection with the classes $3A$, $5A$ and $7A$ are, up to isomorphisms, $McL{:}2$, $2{\cdot}S_6(2)$ and $U_3(5){:}S_3$. We calculate $\Delta(Co_3) = 1680$, $\Sigma(McL{:}2) = 63$, $\Sigma(2{\cdot}S_6(2)) = 70$ and $\Sigma(U_3(5){:}S_3) = 0$. From Table 6.I it follows that $\Delta^*(Co_3) \geq 1680 - 3(63) - 3(70) = 1281$, and hence Co_3 is $(3A, 5A, 7A)$-generated.

Case $(3A, 5B, 7A)$: The maximal subgroups with non-empty intersection with the classes in this triple are isomorphic to $McL{:}2$, $U_4(3){:}2^2$ and $U_3(5){:}S_3$. We calculate

$$\begin{aligned}
\Sigma(McL{:}2) &= \Sigma(McL) = \Delta_{McL}(3a, 5b, 7x) = 644, \quad x \in \{a, b\}, \\
\Sigma(U_4(3){:}2^2) &= \Sigma(U_4(3)) = \Delta_{U_4(3)}(3a, 5a, 7a) = 112, \\
\Sigma(U_3(5){:}S_3) &= 0.
\end{aligned}$$

Clearly any $(3A, 5B, 7A)$-generated proper subgroup of Co_3 is contained in either McL or $U_4(3)$. From the list of maximal subgroups of $U_4(3)$ (*cf.* ATLAS [43]) we observe that, up to isomorphisms, only $L_3(4)$ and A_7 have order divisible by $3 \times 5 \times 7$. However, $\Sigma_{L_3(4)}(3a, 5a, 7a) = 0 = \Sigma_{A_7}(3a, 5a, 7a)$ and hence $\Sigma^*(U_4(3)) = \Delta_{U_4(3)}(3a, 5a, 7a) = 112$.

From Lemma 5.2.3 and the above argument it is clear that, up to isomorphisms, $U_4(3)$ is the only subgroup that contains $(3a, 5b, 7x)$-generated subgroups of McL. Furthermore, $\Sigma_{U_4(3)}(3a, 5b, 7x) = \Delta_{U_4(3)}(3a, 5a, 7a) = 112$ (the first part of the equality involves McL-classes and the second $U_4(3)$-classes). Therefore $\Sigma^*(McL) = 644 - 2(112) = 420$ (*cf.* Lemma 5.2.3). We therefore conclude that McL and $U_4(3)$ are the only $(3A, 5B, 7A)$-generated proper subgroups of Co_3. It was shown by Finkelstein [60] that Co_3 contains a unique conjugate class of subgroups isomorphic to McL and $U_4(3)$, respectively. Therefore

$$\begin{aligned}
\Delta^*(Co_3) &= \Delta(Co_3) - 3\Sigma^*(McL) - 3\Sigma^*(U_4(3)) \\
&= 1680 - 3(420) - 3(112) = 84,
\end{aligned}$$

proving generation of Co_3 by this triple.

Case $(3B, 5B, 7A)$: We calculate the structure constant $\Delta(Co_3) = 175518$. From the fusion maps of the maximal subgroups into Co_3 we note that $McL{:}2$, HS, $U_4(3){:}2^2$, M_{23}, $U_3(5){:}S_3$, $2^4{\cdot}A_8$ and $L_3(4){:}D_{12}$ are, up to isomorphisms, all the maximal subgroups that may contain $(3B, 5B, 7A)$-generated subgroups. We calculate

$$\begin{array}{lllllll}
\Sigma(McL{:}2) & = \Sigma(McL) & = 50400, & & \Sigma(HS) & = & 7280, \\
\Sigma(U_4(3){:}2^2) & = \Sigma(U_4(3)) & = 9408, & & \Sigma(M_{23}) & = & 5124, \\
\Sigma(U_3(5){:}S_3) & = \Sigma(U_3(5)) & = 420, & & \Sigma(2^4{\cdot}A_8) & = & 420, \\
\Sigma(L_3(4){:}D_{12}) & = \Sigma(L_3(4)) & = 882. & & & &
\end{array}$$

Thus any $(3B, 5B, 7A)$-generated proper subgroup of Co_3 is contained in a subgroup isomorphic to McL, HS, $U_4(3)$, M_{23}, $U_3(5)$, $2^4{\cdot}A_8$ or $L_3(4)$. By investigating the maximal subgroups of these groups and their fusions into Co_3, we find that the $(3B, 5B, 7A)$-generated proper subgroups of the above list are, up to isomorphisms, M_{22}, $2^4{:}A_7$, A_8, A_7 and (if possible) subgroups of $2^4{\cdot}A_8$, other than $2^4{:}A_7$ and A_7.

We list in Table 6.III the lengths of the orbits of the above subgroups acting on Ω. If H is any subgroup of Co_3 fixing at least one point $M' \in \Omega$, then $H \leq G_{M'} \cong McL{:}2$. Thus it follows from Table 6.III that any McL, $U_4(3)$, M_{22}, $U_3(5)$ (one fix point on Ω), $2^4{:}A_7$, $L_3(4)$ (both classes) and A_7 (both classes) subgroup of Co_3 is contained in some $McL{:}2$ subgroup of Co_3. Finkelstein [60] showed that Co_3 contains a unique conjugate class for each of the remaining subgroups in the Table 6.III.

It therefore follows from Theorem 3.1.4 that the number of pairs $(x, y) \in 3B \times 5B$, with $xy = z$ a fixed element in $7A$ and $\langle x, y \rangle < Co_3$, is at most

$$3\Sigma(McL{:}2) + 6\Sigma^*(HS) + 3\Sigma^*(M_{23}) + 2\Sigma^*(U_3(5)) + 6\Sigma^*(A_8) + 3\Sigma(2^4{\cdot}A_8). \quad (6.1)$$

We now proceed by finding an upperbound for the above equation. The groups A_7 and $L_3(4)$ contains no proper subgroups with order divisible by $3 \times 5 \times 7$ and hence $\Sigma^*(A_7) = \Sigma(A_7) = 63$ and $\Sigma^*(L_3(4)) = \Sigma(L_3(4)) = 882$. Up to isomorphisms, A_7 is the only subgroup of A_8 that admits $(3B, 5B, 7A)$-generation. Also a fixed element of order 7 is contained in a unique conjugate of a A_7 subgroup in A_8. Thus $\Sigma^*(A_8) = \Sigma(A_8) - \Sigma^*(A_7) = 84 - 63 = 21$.

For the group $U_3(5)$ we have

$$\Sigma(U_3(5)) = \Delta_{U_3(5)}(3a, 5b, 7x) + \Delta_{U_3(5)}(3a, 5c, 7x) + \Delta_{U_3(5)}(3a, 5d, 7x) \ ,$$

TABLE 6.III
Action of H on Ω

H	Length of Ω-orits	$N_{Co_3}(H)$
McL	$[1, 275]$	$McL{:}2$
HS	$[100, 176]$	HS
$U_4(3)$	$[1^2, 112, 162]$	$U_4(3){:}2^2$
M_{23}	$[23, 253]$	M_{23}
$U_3(5)$	$[50^3, 126]$	$U_3(5){:}S_3$
$U_3(5)$	$[1, 50^2, 175]$	$U_3(5){:}2$
M_{22}	$[1, 22, 77, 176]$	M_{22}
$2^4{\cdot}A_8$	$[8, 128, 140]$	$2^4{\cdot}A_8$
$2^4{:}A_7$	$[1, 7, 16, 112, 140]$	
$L_3(4)$	$[1^3, 56^3, 105]$	$L_3(4){:}D_{12}$
$L_3(4)$	$[1^2, 21^2, 56^2, 120]$	
A_8	$[8, 15^2, 70, 168]$	S_8
A_7	$[1, 7, 15^2, 35^2, 42, 126]$	S_7
A_7	$[1^2, 7, 15, 35, 42, 70, 105]$	A_7

where $x \in \{a,b\}$. We calculate $\Delta_{U_3(5)}(3a,5y,7x) = 140$, where $y \in \{b,c,d\}$. Also the maximal subgroups of $U_3(5)$ with order divisible by $3 \times 5 \times 7$ are isomorphic to A_7 (three non-conjugate types, say (i), (ii) and (iii)). The fusion map of A_7 into $U_3(5)$ yields

$$3a \to 3a \quad 3b \to 3a \quad 5a \to 5y \quad 7a \to 7a \quad 7b \to 7b \ ,$$

where $y = b,c,d$ if A_7 is of conjugate type (i), (ii), (iii), respectively. Also a fixed element of order 7 is contained in a unique A_7 subgroup of $U_3(5)$. Thus $\Delta^*_{U_3(5)}(3a,5y,7x) = 77$ and hence $\Sigma^*(U_3(5)) = 231$.

Next we consider the groups M_{22} and M_{23}. We note $\Sigma(M_{22}) = \Delta_{M_{22}}(3a,5a,7x)$, $x \in \{a,b\}$. The $(3a,5a,7x)$-generated subgroups of M_{22} are isomorphic to $L_3(4)$ and A_7 (two non-congugate copies). Using Theorem 3.1.4 we obtain $\Sigma^*(M_{22}) = 2464 - 882 - 2(63) = 1456$. The $(3B,5B,7A)$-generated maximal subgroups of M_{23} are isomorphic to M_{22}, $L_3(4){:}2_2$, $2^4{:}A_7$ and A_8. From the previous arguments it follows that if H is a $(3B,5B,7A)$-generated proper subgroup of M_{23}, then H is isomorphic to either A_7, A_8, $2^4{:}A_7$, $L_3(4)$, M_{22} or $H \leq 2^4{:}A_7$. We calculate $\Sigma(2^4{:}A_7) = 336$. Now $2^4{:}A_7$ contains a subgroup isomorphic to A_7, and from Theorem

3.1.4 we have

$$\begin{aligned}\Sigma^*(M_{23}) &\leq \Sigma(M_{23}) - 2\Sigma^*(M_{22}) - 2\Sigma^*(L_3(4)) - 2\Sigma^*(A_8) - \Sigma(2^4{:}A_7) \\ &= 70.\end{aligned}$$

For the group HS we have $\Sigma(HS) = \Delta_{HS}(3a, 5b, 7a) + \Delta_{HS}(3a, 5c, 7a)$. From Lemma 4.2.4, it follows immediately that

$$\begin{aligned}\Delta_{HS}(3a, 5b, 7a) &\leq 560 - 2(77) - 2(63) = 260 \ , \\ \Delta_{HS}(3a, 5c, 7a) &\leq 6720 - 2(1456) - 2(77) - 882 - 2(63) = 2646\end{aligned}$$

and hence $\Sigma^*(HS) \leq 2906$.

Thus an upper bound for equation (6.1) is 170694. The $(3B, 5B, 7A)$-generation of Co_3 follows from $\Delta(Co_3) = 175518 > 170694$.

Case $(3C, 5A, 7A)$: We calculate $\Delta(Co_3) = 85428$. Up to isomorphisms, $U_3(5){:}S_3$ is the only maximal subgroup of Co_3 with non-empty intersection with the classes of this triple. However, $\Sigma(U_3(5){:}S_3) = 0$ so that $(3C, 5A, 7A)$ is a generating triple for Co_3.

Case $(3C, 5B, 7A)$: The maximal subgroups of Co_3 that contain possible $(3C, 5B, 7A)$-generated subgroups are isomorphic to $U_3(5){:}S_3$ and $L_3(4){:}D_{12}$. However, we calculate $\Sigma(U_3(5){:}S_3) = 0 = \Sigma(L_3(4){:}D_{12})$ and hence $\Delta^*(Co_3) = \Delta(Co_3) = 296136$, proving the $(3C, 5B, 7A)$-generation of Co_3. □

6.3 $(p, q, 11)$-Generations of Co_3

In this section we need only to consider the maximal subgroups of Co_3 with order divisible by 11. They are, up to isomorphisms, $McL{:}2$, HS, M_{23}, $3^5{:}(2 \times M_{11})$ and $2 \times M_{12}$.

Lemma 6.3.1 *The group Co_3 is $(2X, 3Y, 11Z)$-generated, for $X, Z \in \{A, B\}$ and $Y \in \{A, B, C\}$, if and only if the ordered pair $(X, Y) = (B, C)$.*

Proof. An application of Ree's theorem to the representatives of the classes $2A$, $3B$ and $11Z$ (*cf.* Table 6.II) establishes that Co_3 is not $(2A, 3B, 11Z)$-generated. The action of Co_3 on the 23-dimensional irreducible complex module V yields

$$dim(V/C_V(2A)) = 8 \ , \ dim(V/C_V(2B)) = 12 \ , \ dim(V/C_V(3B)) = 18 \ ,$$

$$dim(V/C_V(3C)) = 12 \quad \text{and} \quad dim(V/C_V(11Z)) = 20 \ .$$

Thus the triples $(2A,3C,11Z)$ and $(2B,3B,11Z)$ violate Scott's theorem, resulting in the non-generation of Co_3 by these triples. Next we calculate the structure constants $\Delta_{Co_3}(2A,3A,11Z) = 0 = \Delta_{Co_3}(2B,3A,11Z)$ and non-generation by these triples is immediate.

Finally, we calculate $\Delta_{Co_3}(2B,3C,11Z) = 671$. The maximal subgroups of Co_3 that may contain $(2B,3C,11Z)$-generated subgroups are isomophic to $3^5{:}(2\times M_{11})$ and $2\times M_{12}$. Also $\Sigma(3^5{:}(2\times M_{11})) = 0$ and $\Sigma(2\times M_{12}) = 11$. From Table 6.I we conclude $\Delta^*(Co_3) = \Delta(Co_3) - \Sigma(2\times M_{12}) = 660$, proving the result. □

Lemma 6.3.2 *The group Co_3 is $(2X,5Y,11Z)$-generated, for all $X,Y,Z \in \{A,B\}$, except when $(2X,5Y,11Z) = (2A,5A,11Z)$.*

Proof. We treat the four cases separately.

Case $(2A,5A,11Z)$: The structure constant $\Delta(Co_3) = 44$. From the fusion maps into Co_3 we note that the $(2A,5A,11Z)$-generated proper subgroups are contained in the maximal subgroups isomorphic to $McL{:}2$ or HS. Also $\Sigma(McL{:}2) = \Sigma(McL) = 22$ and $\Sigma(HS) = 11$. It follows from Lemmas 4.2.5 and 5.2.5 that no proper subgroup of McL or HS is $(2A,5A,11Z)$-generated. Thus from the fusion maps we have

$$\Delta^*(Co_3) = \Delta(Co_3) - \Sigma^*(McL) - 2\Sigma^*(HS) = 0 \ ,$$

proving non-generation of Co_3 by this triple.

Case $(2A,5B,11Z)$: Every maximal subgroup with order divisible by 11 has non-empty intersection with each of the classes $2A$, $5B$ and $11Z$. From the structure constants we calculate

$$\Sigma(McL{:}2) = \Sigma(McL) = 715 \quad , \quad \Sigma(HS) = 242 \quad , \quad \Sigma(M_{23}) = 235 \ ,$$

$$\Sigma(3^5{:}(2\times M_{11})) = 99 \quad \text{and} \quad \Sigma(2\times M_{12}) = \Sigma(M_{12}) = 55 \ .$$

Using the ATLAS [43] and subgroup fusions into Co_3, we identify all the possible $(2A, 5B, 11Z)$-generated proper subgroups of Co_3, up to isomophisms. They are McL, HS, M_{23}, M_{22}, M_{12}, M_{11}, $L_2(11)$ and subgroups of $3^5{:}(2 \times M_{11})$. Finkelstein [60] showed that Co_3 has one conjugate class of subgroups isomorphic to M_{23}, M_{22}, M_{12}, M_{11} and $L_2(11)$, respectively. Furthermore, since $3^5{:}(2 \times M_{11})$ contains subgroups isomorphic to M_{11} and $L_2(11)$, it follows that every M_{11} and $L_2(11)$ subgroup of Co_3 is contained in some conjugate copy of a $3^5{:}(2 \times M_{11})$ subgroup. From Theorem 3.1.4, it follows that the number of pairs $(x, y) \in 2A \times 5B$, with $xy = z$ a fixed element of $11Z$ and $\langle x, y\rangle < Co_3$, is at most

$$\Sigma^*(McL) + 2\Sigma^*(HS) + 2\Sigma^*(M_{23}) + \Sigma(M_{12}) + \Sigma(3^5{:}(2 \times M_{11})). \qquad (6.2)$$

No subgroup of $L_2(11)$ has order divisible by $2 \times 5 \times 11$ and hence $\Sigma^*(L_2(11)) = \Sigma(L_2(11)) = 22$. Up to isomorphisms, $L_2(11)$ is the only proper subgroup of M_{11} that is $(2A, 5B, 11Z)$-generated and a fixed element of order 11 is contained in a unique $L_2(11)$ subgroup of M_{11}. Thus $\Sigma^*(M_{11}) = \Sigma(M_{11}) - \Sigma^*(L_2(11)) = 11$. Similarly, $\Sigma^*(M_{22}) = \Sigma(M_{22}) - \Sigma^*(L_2(11)) = 176 - 22 = 154$.

The only $(2A, 5B, 11Z)$-generated proper subgroups of each of the groups McL, HS and M_{23} are isomorphic to M_{22}, M_{11} and $L_2(11)$. A fixed element of order 11 (in M_{23}) is contained in a unique conjugate of a M_{22}, M_{11} and $L_2(11)$ subgroup, respectively. Thus $\Sigma^*(M_{23}) = 253 - 154 - 11 - 22 = 66$. From Lemmas 4.2.5 and 5.2.5 it follows immediately that $\Sigma^*(HS) = 50$ and $\Sigma^*(McL) = 374$. Thus from the equation (6.2) an upper bound for the number of pairs from $2A \times 5B$ that produce $(2A, 5B, 11Z)$-generated proper subgroups of Co_3 is 762. The $(2A, 5B, 11Z)$-generation of Co_3 follows since $\Delta(Co_3) = 1023 > 762$.

Case $(2B, 5A, 11Z)$: We calculate $\Delta(Co_3) = 2068$. Any maximal subgroup with non-empty intersection with the classes $2B$, $5A$ and $11Z$ are isomorphic to $McL{:}2$ or HS. Furthermore, $\Sigma(McL{:}2) = 0$, $\Sigma(HS) = 33$ and therefore $\Delta^*(Co_3) \geq 2068 - 2(33) = 2002$, proving the generation of Co_3 by this triple.

Case $(2B, 5B, 11Z)$: The structure constant $\Delta(Co_3) = 7513$. We observe from Table 6.I that the groups isomorphic to $McL{:}2$, HS, $3^5{:}(2 \times M_{11})$ and $2 \times M_{12}$ are the maximal subgroups of Co_3 that may contain $(2B, 5B, 11Z)$-generated subgroups. We calculate $\Sigma(McL{:}2) = 0$, $\Sigma(HS) = 638$, $\Sigma(3^5{:}(2 \times M_{11})) = 0$ and $\Sigma(2 \times M_{12}) = 33$.

Thus $\Delta^*(Co_3) \geq 7513 - 2(638) - 33 > 0$, proving that $(2B, 5B, 11Z)$ is a generating triple of Co_3. $\square$

Lemma 6.3.3 *The group Co_3 is $(2X, 7A, 11Y)$-generated, for all $X, Y \in \{A, B\}$.*

Proof. Case $(2A, 7A, 11Y)$: The structure constant $\Delta(Co_3) = 6622$. The proper subgroups of Co_3 that admit $(2A, 7A, 11Y)$-generation are contained in the maximal subgroups isomorphic to $McL{:}2$, HS and M_{23}. We also calculate $\Sigma(McL{:}2) = 3168$, $\Sigma(HS) = 825$ and $\Sigma(M_{23}) = 616$. From Table 6.I we conclude

$$\Delta^*(Co_3) \geq 6622 - 3168 - 2(825) - 2(616) > 0 \ ,$$

and generation of Co_3 by this triple follows.

Case $(2B, 7A, 11Y)$: Up to isomorphisms, $McL{:}2$ and HS are the only maximal subgroups that may admit $(2B, 7A, 11Y)$-generated subgroups. Also $\Delta(Co_3) = 57266$, $\Sigma(McL{:}2) = 0$, $\Sigma(HS) = 2211$ and hence $\Delta^*(Co_3) \geq 52844$, proving the result. $\square$

Lemma 6.3.4 *The group Co_3 is $(3X, 5Y, 11Z)$-generated, for all $X \in \{A, B, C\}$ and $Y, Z \in \{A, B\}$.*

Proof. Case $(3A, 5Y, 11Z)$: The maximal subgroups of Co_3 with order divisible by 11 and non-empty intersection with the class $3A$ are isomorphic to $McL{:}2$ and $3^5{:}(2 \times M_{11})$. Furthermore, a $3^5{:}(2 \times M_{11})$ subgroup does not meet the class $5A$ and hence

$$\begin{aligned}
\Delta^*_{Co_3}(3A, 5A, 11Z) &\geq \Delta_{Co_3}(3A, 5A, 11Z) - \Sigma_{McL:2}(3A, 5A, 11Z) \\
&= 1496 - 44 \ , \\
\Delta^*_{Co_3}(3A, 5B, 11Z) &\geq \Delta_{Co_3}(3A, 5B, 11Z) - \Sigma_{McL:2}(3A, 5B, 11Z) \\
&\quad - \Sigma_{3^5:(2\times M_{11})}(3A, 5B, 11Z) = 1232 \ ,
\end{aligned}$$

proving generation of Co_3 by these triples.

Case $(3B, 5A, 11Z)$: The $(3B, 5A, 11Z)$-generated proper subgroups of Co_3 are contained in the maximal subgroups isomorphic to $McL{:}2$ and HS. We calculate

$\Delta(Co_3) = 6380$, $\Sigma(McL{:}2) = 1122$, $\Sigma(HS) = 244$ and hence $\Delta^*(Co_3) \geq 4770$, proving generation.

Case $(3B, 5B, 11Z)$: All maximal subgroups with order divisible by 11 have non-empty intersection with all the classes in the triple. Our calculations yield

$$\Delta^*(Co_3) \geq 92070 - 34485 - 2(5313) - 2(3795) - 891 - 198 = 38280 \ ,$$

proving generation of Co_3 by the triple $(3B, 5B, 11Z)$.

Case $(3C, 5Y, 11Z)$: The maximal subgroups of Co_3 with order divisible by 11 and non-empty intersection with the class $3C$ are, up to isomorphism, $3^5{:}(2 \times M_{11})$ and $2 \times M_{12}$. However, the class $5A$ does not meet either of these subgroups. Since the structure constant $\Delta_{Co_3}(3C, 5A, 11Z) = 76472$, the $(3C, 5A, 11Z)$-generation of Co_3 is immediate. Next, $\Delta_{Co_3}(3C, 5B, 11Z) = 323081$, $\Sigma_{3^5:(2\times M_{11})}(3C, 5B, 11Z) = 1782$, $\Sigma_{2\times M_{12}}(3C, 5B, 11Z) = 253$. Thus $\Delta^*_{Co_3}(3C, 5B, 11Z) \geq 321046$ and the generation of Co_3 by this triple follows. □

Lemma 6.3.5 *The group Co_3 is $(3X, 7A, 11Y)$-generated, for all $X \in \{A, B, C\}$ and $Y \in \{A, B\}$.*

Proof. The maximal subgroups of Co_3 with order divisible by $3 \times 7 \times 11$ are, up to isomorhpisms, $McL{:}2$, HS and M_{23}. The subgroups HS and M_{23} have empty intersection with the class $3A$ and therefore

$$\begin{aligned} \Delta^*_{Co_3}(3A, 7A, 11Y) &= \Delta_{Co_3}(3A, 7A, 11Y) - \Sigma_{McL:2}(3A, 7A, 11Y) \\ &= 22000 - 4356 > 0 \ . \end{aligned}$$

Next we calculate

$$\Delta_{Co_3}(3B, 7A, 11Y) = 580800 \quad , \quad \Sigma_{McL:2}(3B, 7A, 11Y) = 132264 \ ,$$

$$\Sigma_{HS}(3B, 7A, 11Y) = 17622 \quad , \quad \Sigma_{M_{23}}(3B, 7A, 11Y) = 8272 \ ,$$

so that $\Delta^*_{Co_3}(3B, 7A, 11Y) \geq 396648$. Finally, the maximal subgroups isomorphic to $McL{:}2$, HS and M_{23} do not intersect the class $3C$ and hence $\Delta^*_{Co_3}(3C, 7A, 11Y) = \Delta_{Co_3}(3C, 7A, 11Y) = 2374614$, proving the result. □

Lemma 6.3.6 *The group Co_3 is $(5X, 7A, 11Y)$-generated, for all $X, Y \in \{A, B\}$.*

Proof. The maximal subgroups that may contain $(5,7,11)$-generated subgroups are isomorphic to $McL{:}2$, HS and M_{23}. For the triple $(5A, 7A, 11Y)$ we have $5A \cap M_{23} = \emptyset$, $\Delta(Co_3) = 6498712$, $\Sigma(McL{:}2) = 171072$ and $\Sigma(HS) = 12672$ so that $\Delta^*(Co_3) \geq 6302296$. For the remaining case $(5B, 7A, 11Y)$, we calculate $\Delta(Co_3) = 49618756$, $\Sigma(McL{:}2) = 5132160$, $\Sigma(HS) = 274593$ and $\Sigma(M_{23}) = 97192$ and hence $\Delta^*(Co_3) > 0$, and the result follows. □

6.4 $(p, q, 23)$-Generations of Co_3 and the main results

The maximal subgroups of Co_3 containing elements of order 23 are isomophic to M_{23}. It is evident from Table 6.I that a fixed element of order 23 is contained in a unique conjugate of M_{23} and such a subgroup has empty intersection with the classes $2B$, $3A$, $3C$ and $5A$. Thus whenever a triple $(pX, qY, 23Z)$ includes at least one of these classes then $\Delta^*(Co_3) = \Delta(Co_3)$. Moreover, if this triple contains none of these classes, then $\Delta^*(Co_3) = \Delta(Co_3) - \Sigma(M_{23})$.

Table 6.IV
Structure Constants of Co_3

pX	$3A$	$3B$	$3C$	$5A$	$5B$	$7A$	$11AB$
$\Delta_{Co_3}(2A, pX, 23Y)$	0	0	46	115	276	3197	7728
$\Delta_{Co_3}(2B, pX, 23Y)$	0	46	736	1955	6716	56971	120796
$\Delta_{Co_3}(3A, pX, 23Y)$	-	22	418	1380	3818	33350	66700
$\Delta_{Co_3}(3B, pX, 23Y)$	-	-	2981	10166	44160	361284	769350
$\Delta_{Co_3}(3C, pX, 23Y)$	-	-	-	70219	376372	2635317	4926278
$\Delta_{Co_3}(5A, pX, 23Y)$	-	-	-	-	817476	7893692	14954232
$\Delta_{Co_3}(5B, pX, 23Y)$	-	-	-	-	-	37913246	75202410
$\Delta_{Co_3}(7A, pX, 23Y)$	-	-	-	-	-	-	536538388

Lemma 6.4.1 *The group Co_3 is $(pX, qY, 23Z)$-generated, for primes $p \leq q$ and $pX \neq qY$, if and only if the ordered pair $(pX, qY) \notin \{(2A, 3A), (2A, 3B),$*

TABLE 6.V
Structure Constants $\Sigma(M_{23})$

pX	$5B$	$7X$	$11X$
$\Sigma_{M_{23}}(2A, pX, 23Y)$	138	368	391
$\Sigma_{M_{23}}(3B, pX, 23Y)$	2438	6624	5129
$\Sigma_{M_{23}}(5B, pX, 23Y)$	-	88320	61893
$\Sigma_{M_{23}}(7X, pX, 23Y)$	-	-	135424

$(2B, 3A)\}$.

Proof. The result is immediate from the above remarks and Tables 6.IV and 6.V. □

We summarize the main results in the following theorems and corollary.

Theorem 6.4.2 *The Conway group Co_3 is (p,q,r)-generated for all $p,q,r \in \{2,3,5,7,11,23\}$ with $p<q<r$, except when $(p,q,r) = (2,3,5)$.*

Proof. This follows from the above Lemma 6.2.1 to 6.2.3, 6.3.1 to 6.3.6 and 6.4.1 and the fact that the triangular group $T(2,3,5) \cong A_5$. □

Corollary 6.4.3 *The Conway group Co_3 is (pX, pX, qY)-generated, for all $pX \in \{3C, 5A, 5B, 7A, 11A, 11B\}$ and $qY \in \{7A, 11A, 11B, 23A, 23B\}$ with $p < q$ as well as $(pX, pX, qY) = (3B, 3B, 23X)$.*

Proof. The result follows immediately from an application of Lemma 3.1.15 to Lemmas 6.2.1, 6.2.2, 6.3.1, 6.3.2, 6.3.3 and 6.4.1. □

6.5 nX-Complementary generations of Co_3

In order to prove a group G is not nX-complementary generated it suffices to show that there is a conjugacy class pY, p a prime, such that G is not (pY, nX, tZ)-generated for all conjugacy classes tZ with $1/p + 1/n + 1/t < 1$.

Lemma 6.5.1 *The group Co_3 is not nX-complementary generated for all $nX \in \{2A, 2B, 3A, 3B, 4A, 4B\}$.*

Proof. The result for the conjugacy classes $2A$ and $2B$ is a consequence of Lemma 2.3.8. Let $\chi = \underline{23a} \in Irr(Co_3)$, $A = \langle x \rangle$ and $B = \langle y \rangle$, where $x \in 2A$ and $y \in 3B$. Then $\langle \chi\downarrow_A, \chi_1\downarrow_A \rangle = 15$, $\langle \chi\downarrow_B, \chi_1\downarrow_B \rangle = 11$ and $\langle \chi\downarrow_{(A\cap B)}, \chi_1\downarrow_{(A\cap B)} \rangle = 23$. However, $15 + 11 = 26 > 23$ and by Brauer's theorem $\langle x, y \rangle < Co_3$. Similarly, if $C = \langle z \rangle$, where $z \in 4B$, then $\langle \chi|C, \chi_1|C \rangle = 9$ and Brauer's theorem yields $\langle x, z \rangle < Co_3$. Thus Co_3 is not $3B$- or $4B$-complementary generated.

We now consider the conjugacy class $3A$. We list below the non-zero structure constants $\Delta_{Co_3}(2A, 3A, tX)$, where $t \geq 7$.

tX	$12AB$	$24A$	$30A$
$\Delta_{Co_3}(2A, 3A, tX)$	3	4	6

In all these cases $\Delta_{Co_3}(2A, 3A, tX) < |C_{Co_3}(tX)|$ and we conclude that Co_3 is not $3A$-complementary generated.

Finally, consider the class $4A$. Now $1/2 + 1/4 + 1/t < 1$ if and only if $t \geq 5$. The non-zero structure constants $\Delta_{Co_3}(2A, 4A, tX)$ are:

tX	$6E$	$8B$	$8C$	$10B$	$12A$	$12B$	$20AB$	$22AB$	$24A$	$24B$
$\Delta_{Co_3}(2A, 4A, tX)$	63	260	2	30	120	8	4	11	24	8
$\lvert C_{Co_3}(tX)\rvert$	72	192	192	20	144	48	20	22	24	24

From Lemma 3.1.7 we need only consider the conjugacy classes tX for which the structure constant $\Delta_{Co_3}(2A, 4A, tX) \geq |C_{Co_3}(tX)|$. For the class $8B$ we calculate $\Sigma_{McL:2}(2A, 4A, 8B) = 164$ and therefore $\Delta^*(Co_3) \leq 260 - 164 < |C_{Co_3}(8B)|$. Thus Co_3 is not $(2A, 4A, 8B)$-generated. Similarly,

$$\begin{aligned}
\Delta^*_{Co_3}(2A, 4A, 10B) &\leq \Delta_{Co_3}(2A, 4A, 10B) - \Sigma_{McL:2}(2A, 4A, 10B) \\
&= 30 - 25 < |C_{Co_3}(10B)|, \\
\Delta^*_{Co_3}(2A, 4A, 24A) &\leq \Delta_{Co_3}(2A, 4A, 24A) - \Sigma_{McL:2}(2A, 4A, 24A) \\
&= 24 - 24 < |C_{Co_3}(24A)|.
\end{aligned}$$

Thus Co_3 is not $4A$-complementary generated, proving the result. $\square$

We now proceed to show that Co_3 is nX-complementary generated if and only if $nX \notin \{2A, 2B, 3A, 3B, 4A, 4B\}$. We will that show for every conjugacy class pY with prime order representatives, Co_3 is $(pY, nX, 23A)$-generated. Recall that, up to isomorhpisms, M_{23} is the only maximal subgroup of Co_3 containing elements of order 23.

Lemma 6.5.2 *The group Co_3 is nX-complementary generated, for all $nX \in \{3C, 6A, 6B, 6C, 6D, 8A, 8B, 8C, 9A, 9B\}$.*

Proof. The conjugacy classes $6C$ and $8A$ have non-empty intersection with the maximal subgroup M_{23}. In particular, $6C \cap M_{23} = 6a$ and $8A \cap M_{23} = 8a$. For these cases we have

$$\Delta^*(Co_3) = \Delta(Co_3) - \Sigma(M_{23}) .$$

It follows from Table 6.VI that $\Delta^*(Co_3) > 0$, and therefore Co_3 is $6C$- and $8A$-complementary generated.

Each of remaining conjugacy classes nX in the statement of the lemma have empty intersection with M_{23}. Therefore we observe from Table 6.VI that $\Delta^*_{Co_3}(pY, nX, 23A) = \Delta_{Co_3}(pY, nX, 23A) > 0$, for all conjugacy classes pY with prime order elements. The result now follows from Lemma 2.3.8. $\square$

Theorem 6.5.3 *The group Co_3 is nX-complementary generated if and only if $nX = 3C$ or $n > 4$.*

Proof. Let $p \geq 5$ be a prime. We proved in the previous sections that for all conjugacy classes qY with elements of prime order representatives, the group Co_3 is $(qY, pX, 23A)$-generated. Thus from Lemma 2.3.8 we conclude that Co_3 is pX-complementary generated for every conjugacy class pX, where $p \geq 5$.

The power maps of Co_3 yields $(6E)^2 = 3C$, $(10A)^2 = 5A$, $(10B)^2 = 5B$, $(12A)^2 = 6A = (12B)^2$, $(12C)^2 = 6C$, $(14A)^2 = 7A$, $(15A)^3 = 5A$, $(15B)^3 = 5B$, $(18A)^3 = 6B$, $(20A)^4 = 5A = (20B)^4$, $(21A)^3 = 7A$, $(22A)^2 = 11B$, $(22B)^2 = 11A$, $(24A)^4 = 6A = (24B)^4$ and $(30A)^6 = 5A$. An application of Lemma 2.3.9 to Lemma 6.5.2 gives complementary generation of these classes. The result now follows from Lemma 6.5.1. $\square$

TABLE 6.VI

Structure constants

pX	$\Delta_{Co_3}(pX,3C,23A)$	$\Delta_{Co_3}(pX,6A,23A)$	$\Delta_{Co_3}(pX,6B,23A)$	$\Delta_{Co_3}(pX,6C,23A)$
$2A$	46	23	92	506
$2B$	736	483	2162	11316
$3A$	529	322	897	5911
$3B$	3542	2921	14168	71714
$3C$	22126	26220	80592	503976
$5A$	70219	77142	250010	1521450
$5B$	376372	348588	1295544	7372512
$7A$	2635317	2639250	9132840	53791020
$11AB$	4926278	5216400	17388000	104328000
$23A$	4683260	5071500	16763136	101004408
$23B$	4603772	5071500	16524672	100289016
pX	$\Delta_{Co_3}(pX,6D,23A)$	$\Delta_{Co_3}(pX,8A,23A)$	$\Delta_{Co_3}(pX,8B,23A)$	$\Delta_{Co_3}(pX.8C,23A)$
$2A$	1196	644	460	4692
$2B$	23736	13294	10810	77740
$3A$	14536	7383	6509	43654
$3B$	138644	81880	68264	503976
$3C$	988356	570584	580520	3429392
$5A$	3005640	1729002	1713822	10328472
$5B$	14812092	8425728	7849440	50703408
$7A$	107706240	61029856	59241376	366195328
$11AB$	208190250	117372864	117372864	704206272
$23A$	199892448	113007648	114246336	675213024
$23B$	200369376	112202832	114514608	676822656
pX	$\Delta_{Co_3}(pX,9A,23A)$	$\Delta_{Co_3}(pX,9B,23A)$	$\Sigma_{M_{23}}(pX,6C,23A)$	$\Sigma_{M_{23}}(pX,8A,23A)$
$2A$	1173	1472	322	368
$2B$	15387	30912	0	0
$3A$	7498	17572	0	0
$3B$	112700	188048	4324	7084
$3C$	691173	1330872	0	0
$5A$	2070828	4032360	0	0
$5B$	10340478	19704744	56672	83904
$7A$	73352520	143525520	121440	178112
$11AB$	139414500	277897500	77280	121072
$23A$	132908904	267934176	37536	52256
$23B$	133544808	267298272	36432	52256

Chapter 7

The Second Conway Group

7.1 Introduction

The construction of the Conway group Co_2 were discussed in the previous chapter. The subgroup structure of this group are discussed in Wilson [168], using the following information. The group Co_2 contains a 23-dimensional indecomposable representation over $GF(2)$ obtained from the complex representation of degree 23 by reducing to modulo 2 and factoring out the fixed vectors. The action of Co_2 on the vectors in this $GF(2)$-representation produces eight orbits, with point stabilizers isomorphic to Co_2, $U_6(2){:}2$, $2^{10}{:}M_{22}{:}2$, McL, $HS{:}2$, $U_4(3).D_8$, $2_+^{1+8}{:}S_8$ and M_{23}, respectively. In the following we give a complete list of non-abelian proper simple subgroups of Co_2, up to isomorphisms.

$$\begin{array}{l}
A_5,\ A_6,\ A_7,\ A_8;\\
L_2(7),\ L_2(8),\ L_2(11);\\
U_3(3),\ L_3(4),\ U_4(3),\ U_3(5);\\
U_4(2),\ U_5(2),\ U_6(2),\ S_6(2);\\
M_{11},\ M_{22},\ M_{23},\ HS,\ McL.
\end{array}$$

Any subgroup of Co_2 isomorphic to one of these groups fixes a vector in the complex 23-dimensional representation. We therefore conclude that any proper non-abelian simple subgroup must also fix a non-zero vector in the reduction modulo 2 of the

23-dimensional complex representation, and so must be contained in one of the non-trivial stabilizers above.

Theorem 7.1.1 *(Wilson [168]) The second Conway simple group Co_2 has exactly eleven conjugacy classes of maximal subgroups, as follows:*

(A) Five classes of non-local subgroups:

$U_6(2){:}2$, McL, $HS{:}2$, $U_4(3).D_8$, *and* M_{23}.

(B) Six classes of local subgroups:

$2^{10}{:}M_{22}{:}2$, $2_+^{1+8}{:}S_6(2)$, $(2_+^{1+6} \times 2^4).A_8$, $2^{4+10}.(S_5 \times S_3)$, $3_+^{1+4}{:}2_-^{1+4}.S_5$ *and* $5_+^{1+2}{:}4S_4$. $\square$

The permutation character of Co_2 on the cosets of $U_6(2){:}2$ is $\chi_{U_6(2):2} = \underline{1a} + \underline{275a} + \underline{2024a}$. For the remaining maximal subgroups with order divisible by 7, 11 or 23 we provide partial fusion maps into Co_2 in Table 7.I.

7.2 $(p,q,7)$-Generations of Co_2

The group Co_2 acts on a 23-dimensional irreducible complex module V. Let $d_{nX} = dim(V/C_V(nX))$, the co-dimension of the fix space (in V) of a representative in nX. Using the character table of Co_2 we list in Table 7.II the values of d_{pX}, for all conjugacy classes with prime order representatives.

TABLE 7.II

The co-dimensions $d_{nX} = dim(V/C_V(nX))$

d_{2A}	d_{2B}	d_{2C}	d_{3A}	d_{3B}	d_{5A}	d_{5B}	d_{7A}	d_{11A}	d_{23AB}
16	8	12	18	12	20	16	18	20	22

Lemma 7.2.1 *The group Co_2 is not $(2X,3Y,7A)$-generated, for any $X \in \{A,B,C\}$ and $Y \in \{A,B\}$.*

Proof. If the group Co_2 is (pX,qY,rZ)-generated, then by Scott's theorem $d_{pX} + d_{qY} + d_{rZ} \geq 46$. It is clear from Table 7.II that the triples $(2B,3A,7A)$, $(2B,3B,7A)$ and $(2C,3B,7A)$ violate Scott's theorem and are therefore not generating triples for Co_2.

TABLE 7.I
Partial fusion maps into Co_2

$2^10{:}M_{22}{:}2$–class	$2a$	$2b$	$2c$	$2d$	$2e$	$2f$	$2g$	$2h$	$2i$	$3a$	$5a$
$\to Co_2$	$2A$	$2B$	$2C$	$2B$	$2C$	$2A$	$2C$	$2B$	$2C$	$3B$	$5B$
$2^10{:}M_{22}{:}2$–class	$7a$	$7b$	$11a$								
$\to Co_2$	$7A$	$7A$	$11A$								
h	4	4	1								
McL–class	$2a$	$3a$	$3b$	$5a$	$5b$	$7a$	$7b$	$11a$	$11b$		
$\to Co_2$	$2B$	$3A$	$3B$	$5A$	$5B$	$7A$	$7A$	$11A$	$11A$		
h						8	8	3	2		
$2^{1+8}_+{:}S_6(2)$–class	$2a$	$2b$	$2c$	$2d$	$2e$	$2f$	$2g$	$2h$	$2i$	$2j$	$3a$
$\to Co_2$	$2A$	$2B$	$2C$	$2B$	$2A$	$2B$	$2C$	$2A$	$2C$	$2C$	$3A$
$2^{1+8}_+{:}S_6(2)$–class	$3b$	$3c$	$5a$	$7a$							
$\to Co_2$	$3B$	$3B$	$5B$	$7A$							
h				1							
$HS{:}2$–class	$2a$	$2b$	$2c$	$2d$	$3a$	$5a$	$5b$	$5c$	$7a$	$11a$	
$\to Co_2$	$2B$	$2C$	$2A$	$2C$	$3B$	$5A$	$5B$	$5B$	$7A$	$11A$	
h									4	1	
$(2^{1+6}_+ \times 2^4).A_8$–class	$2a$	$2b$	$2c$	$2d$	$2e$	$2f$	$2g$	$2h$	$2i$	$2j$	$2k$
$\to Co_2$	$2B$	$2B$	$2A$	$2A$	$2B$	$2C$	$2B$	$2C$	$2C$	$2C$	$2A$
$(2^{1+6}_+ \times 2^4).A_8$–class	$2l$	$2m$	$2n$	$3a$	$3b$	$5a$	$7a$	$7b$			
$\to Co_2$	$2B$	$2C$	$2C$	$3A$	$3B$	$5A$	$7A$	$7A$			
h							4	4			
$U4(3).D_8$–class	$2a$	$2b$	$2c$	$2d$	$2e$	$2f$	$3a$	$3b$	$3c$	$5a$	$7a$
$\to Co_2$	$2B$	$2A$	$2C$	$2A$	$2C$	$2C$	$3A$	$3B$	$3B$	$5B$	$7A$
h											2
M_{23}–class	$2a$	$3a$	$5a$	$7a$	$7b$	$11a$	$11b$	$23a$	$23b$		
$\to Co_2$	$2B$	$3B$	$5B$	$7A$	$7A$	$11A$	$11A$	$23A$	$23B$		
h				8	8	2	2	1	1		

The group Co_2 acts transitively on a set Ω of conjugates of $U_6(2)$ in Co_2 of size 2300. The point stabilizer of this action is isomorphic to the group $U_6(2){:}2$ and the resulting permutation character is $\chi_{U_6(2):2} = \underline{1a}+\underline{275a}+\underline{2024a}$. Using this permutation character we can determine the cycle structure of the elements of Co_2 as permutations on 2300 points. The elements in the conjugacy classes $2A$, $3A$, $3B$ and $7A$ have cycle structure $1^{284}2^{1008}$, $1^5 3^{765}$, $1^{41}3^{753}$ and $1^4 7^{328}$, respectively. If Co_2 is (pX,qY,rZ)-generated, then by Ree's theorem $c_1+c_2+c_3 \leq 2302$, where c_1, c_2 and c_3 are the number of disjoint cycles of representatives in pX, qY and rZ, respectively. However, this condition is violated by the triples $(2A, 3A, 7A)$ and $(2A, 3B, 7A)$ and hence they do not generate Co_2.

Finally, we calculate $\Delta_{Co_2}(2C, 3A, 7A) = 28 < |C_{Co_2}(7A)| = 56$ and non-generation of Co_2 by this triple follows from Lamma 3.1.7. This completes the proof. $\square$

Lemma 7.2.2 *The group Co_2 is $(2X, 5Y, 7A)$-generated, for $X \in \{A, B, C\}$ and $Y \in \{A, B\}$, if and only if the ordered pair $(X, Y) \in \{(C, A), (C, B)\}$.*

Proof. The structure constant $\Delta_{Co_2}(2A, 5B, 7A) = 0$ and non-generation of Co_2 by this triple follows. Next we calculate

$$\begin{aligned}
\Delta^*_{Co_{2_f}}(2A, 5A, 7A) &\leq \Delta(Co_2) - \Sigma((2^{1+6}_+ \times 2^4).A_8) \\
&= 56 - 14 < |C_{Co_2}(7A)| = 56, \\
\Delta^*_{Co_2}(2B, 5A, 7A) &\leq \Delta(Co_2) - \Sigma(McL) \\
&= 56 - 7 < |C_{Co_2}(7A)| = 56.
\end{aligned}$$

Thus non-generation of Co_2 by these triple follows from Lamma 3.1.7. From Table 7.II and Scott's theorem we conclude that Co_2 is not $(2B, 5B, 7A)$-generated.

We now consider the triple $(2C, 5A, 7A)$. The maximal subgroups of Co_2 with order divisible by 7 and non-empty intersection with the classes $2C$ and $5A$ are isomorphic to $HS{:}2$ and $(2^{1+6}_+ \times 2^4).A_8$. We calculate $\Delta(Co_2) = 9576$, $\Sigma(HS{:}2) = 42$ and $\Sigma((2^{1+6}_+ \times 2^4).A_8) = 308$. A fixed element of order 7 is contained in 4 conjugate subgroups of $HS{:}2$ and 4 conjugate copies of $(2^{1+6}_+ \times 2^4).A_8$ (*cf.* Table 7.I). Thus $\Delta^*(Co_2) \geq 9576 - 4(42) - 4(308) > 0$ and whence $(2B, 5A, 7A)$ is a generating triple for Co_2.

Finally, we show that Co_2 is $(2C, 5B, 7A)$-generated. The maximal subgroups with non-empty intersection with the classes $2C$, $5B$ and $7A$ are, up to isomorphisms, $U_6(2){:}2$, $2^{10}{:}M_{22}{:}2$, $2^{1+8}_+{:}S_6(2)$, $HS{:}2$, and $U_4(3){:}D_8$. We calculate $\Delta(Co_2) = 48580$, while

$$\Sigma(U_6(2){:}2) = 6804, \quad \Sigma(2^{10}{:}M_{22}{:}2) = 2184, \quad \Sigma(2^{1+8}_+{:}S_6(2)) = 1876,$$
$$\Sigma(HS{:}2) = 552 \quad \text{and} \quad \Sigma(U_4(3).D_8) = 0.$$

From Table 7.I we calculate

$$\Delta^*(Co_2) \geq 48580 - 4(6804) - 4(2184) - 1876 - 4(532) > 0,$$

and the result follows. $\square$

Lemma 7.2.3 *The group Co_2 is $(3X, 5Y, 7A)$-generated, for all $X, Y \in \{A, B\}$.*

Proof. We will treat each triple separately.

Case $(3A, 5A, 7A)$: The maximal subgroups of Co_2 that have non-empty intersection with the classes $3A$, $5A$ and $7A$ are, up to isomorphisms, McL, and $(2_+^{1+6} \times 2^4).A_8$. We calculate $\Delta(Co_2) = 55664$, $\Sigma(McL) = 63$, and $\Sigma((2_+^{1+6} \times 2^4).A_8) = 112$. From Table 7.I it follows that $\Delta^*(Co_2) \geq 55664 - 8(63) - 4(112) = 54712$, and hence Co_2 is $(3A, 5A, 7A)$-generated.

Case $(3A, 5B, 7A)$: The only maximal subgroups that may contain $(3A, 5B, 7A)$-generated subgroups are isomorphic to $U_6(2){:}2$, McL, $2_+^{1+8}{:}S_6(2)$ and $U_4(3).D_8$. We calculate

$$\begin{aligned} &4\Sigma(U_6(2){:}2) + 8\Sigma(McL) + \Sigma(2_+^{1+8}{:}S_6(2)) + 2\Sigma(U_4(3).D_8) \\ &= 4(3591) + 8(644) + 1848 + 2(112) = 21588. \end{aligned}$$

since $\Delta(Co_2) = 44296$, we have $\Delta^*(Co_2) > 22708$. This proves generation by this triple.

Case $(3B, 5A, 7A)$: We calculate the structure constant $\Delta(Co_2) = 27048$. The maximal subgroups of Co_2 with non-empty intersection with the classes in this triple are, up to isomorphisms, McL, $HS{:}2$ and $(2_+^{1+6} \times 2^4).A_8$ and our calculations gives

$$\begin{aligned} \Delta^*(Co_2) &\geq \Delta(Co_2) - 8\Sigma(McL) - 4\Sigma(HS{:}2) - 4\Sigma((2_+^{1+6} \times 2^4).A_8) \\ &= 27048 - 8(595) - 4(126) - 4(308) = 20552, \end{aligned}$$

and therefore Co_2 is $(3B, 5A, 7A)$-generated.

Case $(3B, 5B, 7A)$: From the fusion maps of the maximal subgroups into Co_2 we note that $U_6(2){:}2$, $2^{10}{:}M_{22}{:}2$, McL, $2_+^{1+8}{:}S_6(2)$, $HS{:}2$, $U_4(3).D_8$ and M_{23} have non-empty intersection with all the conjugacy classes in this triple. We calculate

$$\Sigma(U_6(2){:}2) = 881496, \qquad \Sigma(2^{10}{:}M_{22}{:}2) = 53536,$$
$$\Sigma(McL) = 50400, \qquad \Sigma(2_+^{1+8}{:}S_6(2)) = 6300,$$
$$\Sigma(HS{:}2) = 7280, \qquad \Sigma(M_{23}) = 5124,$$
$$\Sigma(U_4(3).D_8) = \Sigma(U_4(3)) = 9408.$$

The group Co_2 acts transitively on the set of conjugates of $U_6(2)$. The point stabilizer of this action is isomorphic to $U_6(2){:}2$. The action of the maximal subgroup McL on Ω produce two orbits Ω_1 and Ω_2 of length 275 and 2025, respectively. Let $U \in \Omega_1$. Then the point stabilizer McL_U is isomorphic to $U_4(3)$. Furthermore,

$$\begin{aligned} McL_U &= \{g \in McL | U^g = U\} = \{g \in McL | g \in N_{Co_2}(U)\} \\ &\cong McL \cap U_6(2){:}2. \end{aligned}$$

Therefore every conjugate of a $U_4(3)$ subgroup is contained in a conjugate of a McL subgroup and a conjugate of a $U_6(2){:}2$ subgroup. Consequently, $\Sigma^*(U_6(2){:}2) \leq \Sigma(U_6(2){:}2) - \Sigma(U_4(3)) = 211966$ and similarly $\Sigma^*(McL) \leq 40992$.

Thus the number of pairs $(x, y) \in 3B \times 5B$ with $xy = z$, where z is a fixed element in $7A$, that generate a proper subgroup of Co_2 is at most

$$\begin{aligned} &4\Sigma^*(U_6(2){:}2) + 4\Sigma(2^{10}{:}M_{22}{:}2) + 8\Sigma^*(McL) + \Sigma(2_+^{1+8}{:}S_6(2)) + \Sigma(HS{:}2) \\ &+ \Sigma(M_{23}) + \Sigma(U_4(3)) \\ &\leq 4(211966) + 4(53536) + 8(40966) + 6300 + 4(7280) + 8(5124) + 2(9408) \\ &= 1454429 \end{aligned}$$

The $(3B, 5B, 7A)$-generation of Co_2 follows from $\Delta(Co_2) = 1463476 > 1454429$. $\square$

7.3 $(p, q, 11)$-Generations of Co_2

For the $(p, q, 11)$-generation we need only to consider maximal subgroups of Co_2 with order divisible by 11. They are, up to isomorphisms, $U_6(2){:}2$, $2^{10}{:}M_{22}{:}2$, McL, $HS{:}2$ and M_{23}.

Lemma 7.3.1 *The group Co_2 is $(2X, 3Y, 11A)$-generated, for $X \in \{A, B, C\}$ and $Y \in \{A, B\}$, if and only if the ordered pair $(X, Y) = (C, A)$.*

Proof. We calculate the structure constants $\Delta_{Co_2}(2A, 3A, 11A) = \Delta_{Co_2}(2A, 3B, 11A) = \Delta_{Co_2}(2B, 3A, 11A) = 0$ and non-generation by these triples is immediate. It follows from Table 7.II that the triples $(2B, 3B, 11A)$ and $(2C, 3B, 11A)$ violate Scott's theorem and are therefore not generating triples for Co_2.

The only maximal subgroups of Co_2 that may contain $(2B, 3A, 11A)$-generated subgroups are isomophic to $U_6(2){:}2$. However, $\Sigma(U_6(2){:}2) = 0$ and generation by this triple follows since $\Delta^*(Co_2) = \Delta(Co_2) = 55$, proving the result. □

Lemma 7.3.2 *The group Co_2 is $(2X, 5Y, 11A)$-generated, for all $X \in \{A, B, C\}$ and $Y \in \{A, B\}$, except for $(X, Y) = (A, B)$ or (B, B).*

Proof. The structure constant $\Delta_{Co_2}(2A, 5B, 11A) = 11 = \Sigma(U_6(2){:}2)$ and consequently $\Delta^*(Co_2) = 0$. An application of Scott's theorem shows that Co_2 is not $(2B, 5B, 11A)$-generated.

The only maximal subgroups that may contain $(2A, 5A, 11A)$-generated subgroups are isomorhic to $HS{:}2$. However, $\Sigma(HS{:}2) = 0$ and generation by this triple follows since $\Delta^*(Co_2) = \Delta(Co_2) = 33$.

Next we consider the triple $(2B, 5A, 11A)$. The maximal subgroups with non-empty intersection with each of the classes in this triple are, up to isomorphisms, McL and $HS{:}2$. From the structure constants we calculate

$$\Delta^*(Co_2) \geq \Delta(Co_2) - 2\Sigma(McL) - \Sigma(HS{:}2) = 132 - 2(22) - 11 = 99,$$

proving the generation.

The subgroup isomorphic to $HS{:}2$ is the only possible $(2C, 5A, 11A)$-generated maximal subgroup of Co_2. We calculate $\Delta(Co_2) = 10428$ and $\Sigma(HS{:}2) = 33$ and therefore $\Delta^*(Co_2) \geq 10395$, proving the generation of Co_2 by this triple.

Finally we consider the triple $(2C, 5B, 11A)$. The structure constant $\Delta(Co_2) = 41712$. We observe from Table 7.I that the groups isomorphic to $U_6(2){:}2$, $2^{10}{:}M_{22}{:}2$, and $HS{:}2$ are the maximal subgroups of Co_2 that may contain $(2B, 5B, 11Z)$-generated subgroups. We calculate $\Sigma(U_6(2){:}2) = 7524$, $\Sigma(2^{10}{:}M_{22}{:}2) = 2112$ and $\Sigma(HS{:}2) = 638$. Thus $\Delta^*(Co_2) \geq 41712 - 7524 - 2112 - 638 > 0$, proving that $(2C, 5B, 11A)$ is a generating triple of Co_2. This completes the proof. $\square$

Lemma 7.3.3 *The group Co_2 is $(2X, 7A, 11A)$-generated, for all $X \in \{A, B, C\}$.*

Proof. *Case* $(2A, 7A, 11A)$: The structure constant $\Delta(Co_2) = 264$. The $(2A, 7A, 11A)$-generated proper subgroups of Co_2 are contained in the maximal subgroups isomorphic to $U_6(2){:}2$, $2^{10}{:}M_{22}{:}2$ or $HS{:}2$. We further calculate $\Sigma(U_6(2){:}2) = 99$, $\Sigma(2^{10}{:}M_{22}{:}2) = 0 = \Sigma(HS{:}2)$. From Table 7.I we conclude that $\Delta^*(Co_2) \geq 264 - 99 > 0$ and the generation of Co_2 by this triple follows.

Case $(2B, 7A, 11A)$: Every maximal subgroup of Co_2 with order divisible by 11 has non-empty intersection with all the classes in this triple. We calculate $\Delta(Co_2) = 30008$, while

$$\Sigma(U_6(2){:}2) = 8910, \quad \Sigma(2^{10}{:}M_{22}{:}2) = 2816, \quad \Sigma(McL) = 3168,$$
$$\Sigma(HS{:}2) = 825 \text{ and } \Sigma(M_{23}) = 5124.$$

It follows from Table 7.I that $\Delta^*(Co_2) \geq 9889$, proving the generation by this triple.

Case $(2C, 7A, 11A)$: Up to isomorphisms, $U_6(2){:}2$, $2^{10}{:}M_{22}{:}2$ and $HS{:}2$ are the only maximal subgroups that admit $(2C, 7A, 11A)$-generation. We calculate

$$\begin{aligned}\Delta^*(Co_2) &\geq \Delta(Co_2) - \Sigma(U_6(2){:}2) - \Sigma(2^{10}{:}M_{22}{:}2) - \Sigma(HS{:}2) \\ &= 472384 - 35640 - 8448 - 2211 \\ &= 426085,\end{aligned}$$

and the result follows. $\square$

Lemma 7.3.4 *The group Co_2 is $(3X, 5Y, 11A)$-generated, for all $X, Y \in \{A, B\}$.*

Proof. *Case* $(3A, 5Y, 11A)$: The maximal subgroups of Co_2 with order divisible by 11 and non-empty intersection with the class $3A$ are isomorphic to $U_6(2){:}2$ and McL. Further, $U_6(2){:}2 \cap 5A = \emptyset$ and hence

$$\Delta^*_{Co_2}(3A, 5A, 11A) \geq \Delta(Co_2) - 2\Sigma(McL)$$

$$\begin{aligned} &= 40755 - 2(44) > 0, \\ \Delta^*_{Co_2}(3A, 5B, 11A) &\geq \Delta(Co_2) - \Sigma(U_6(2){:}2) - 2\Sigma(McL) \\ &= 77055 - 6105 - 2(1100) > 0, \end{aligned}$$

proving the generation of Co_2 by these triples.

Case $(3B, 5A, 11Z)$: The $(3B, 5A, 11A)$-generated proper subgroups of Co_2 are contained in the maximal subgroups isomorphic to McL and $HS{:}2$. We calculate $\Delta(Co_2) = 51513$, $\Sigma(McL) = 1122$, $\Sigma(HS{:}2) = 244$ and hence $\Delta^*(Co_2) \geq 47027$, proving the generation.

Case $(3B, 5B, 11A)$: Every maximal subgroup of Co_2 with order divisible by 11 has non-empty intersection with all the classes in this triple. We calculate

$$\Sigma(U_6(2){:}2) = 149820, \quad \Sigma(2^{10}{:}M_{22}{:}2) = 33792, \quad \Sigma(McL) = 34485,$$
$$\Sigma(HS{:}2) = 5313 \text{ and } \Sigma(M_{23}) = 3795.$$

Moreover, $\Delta(Co_2) = 733117$ and therefore $\Delta^*(Co_2) \geq 265485$, proving the generation by this triple. This completes the proof. $\square$

Lemma 7.3.5 *The group Co_2 is $(3X, 7A, 11A)$-generated, for all $X \in \{A, B\}$.*

Proof. Case $(3A, 7A, 11A)$: We calculate the structure constant $\Delta(Co_2) = 1063040$. The $(3A, 7A, 11A)$-generated proper subgroups of Co_2 are contained in the maximal subgroups isomorphic to $U_6(2){:}2$ or McL. We also calculate $\Sigma(U_6(2){:}2) = 28215$ and $\Sigma(McL) = 4356$. From Table 7.I we conclude that $\Delta^*(Co_2) \geq 1026113 > 0$ and the generation of Co_2 by this triple follows.

Case $(3B, 7A, 11A)$: Every maximal subgroup of Co_2 with order divisible by 11 has non-empty intersection with all the classes in this triple. We calculate

$$\Sigma(U_6(2){:}2) = 692604, \quad \Sigma(2^{10}{:}M_{22}{:}2) = 112640, \quad \Sigma(McL) = 132264,$$
$$\Sigma(HS{:}2) = 17622 \text{ and } \Sigma(M_{23}) = 8272\,.$$

Furthermore, $\Delta(Co_2) = 6972416$ and therefore $\Delta^*(Co_2) \geq 5868478$, proving the result. $\square$

Lemma 7.3.6 *The group Co_2 is $(5X, 7A, 11A)$-generated, for all $X \in \{A, B\}$.*

Proof. The maximal subgroups that may contain $(5A, 7A, 11A)$-generated subgroups are isomorphic to McL and $HS{:}2$. We calculate $\Delta(Co_2) = 208023552$, $\Sigma(McL) = 171072$ and $\Sigma(HS{:}2) = 12672$, so that $\Delta^*(Co_2) \geq 207666736$.

Every maximal subgroup of Co_2 with order divisible by 11 has non-empty intersection with all the classes in the triple $(5B, 7A, 11A)$. We calculate

$$\Sigma(U_6(2){:}2) = 43797105, \quad \Sigma(2^{10}{:}M_{22}{:}2) = 3244032, \quad \Sigma(McL) = 5132160,$$
$$\Sigma(HS{:}2) = 274593 \text{ and } \Sigma(M_{23}) = 97134.$$

Furthermore, $\Delta(Co_2) = 1587536896$ and therefore $\Delta^*(Co_2) > 0$, and the result follows. □

7.4 $(p, q, 23)$-Generations of Co_2 and the main result

The conjugacy class $(23B)^{-1} = 23A$ and the results obtained by replacing one of these classes with the other are the same. Let $23Z$ denote the class $23A$ or $23B$. The maximal subgroups of Co_2 containing elements of order 23 are isomophic to M_{23}. It is evident from Table 7.I that M_{23} does not meet the conjugacy classes $2A$, $2C$, $3A$, and $5A$. Thus whenever a triple $(pX, qY, 23Z)$ involves one of these classes then $\Delta^*(Co_2) = \Delta(Co_2)$. Moreover, if the triple $(pX, qY, 23Z)$ contains none of these classes, then from Table 7.I we conclude that $\Delta^*(Co_2) = \Delta(Co_2) - \Sigma(M_{23})$.

Lemma 7.4.1 *The group Co_2 is $(pX, qY, 23Z)$-generated, for primes $p \leq q$ and $pX \neq qY$, if and only if the ordered pair $(pX, qY) \notin \{(2A, 3A), (2A, 3B), (2B, 3A)$ $(2B, 3B)\}$.*

Proof. The result is immediate from the above remarks and Tables 7.III and 7.IV. □

We now summarize the results in the following theorem.

Theorem 7.4.2 *The Conway group Co_2 is (p, q, r)-generated for all $p, q, r \in \{2, 3, 5, 7, 11, 23\}$ with $p < q < r$, except when $(p, q, r) = (2, 3, 5)$ or $(2, 3, 7)$.*

Table 7.IV
Structure Constants of Co_2

pX	$3A$	$3B$	$5A$	$5B$	$7A$	$11A$
$\Delta_{Co_2}(2A, pX, 23Z)$	0	0	23	23	644	5129
$\Delta_{Co_2}(2B, pX, 23Z)$	0	0	322	782	13524	93794
$\Delta_{Co_2}(2C, pX, 23Z)$	69	69	9200	37720	471960	2605624
$\Delta_{Co_2}(3A, pX, 23Z)$	-	437	31027	137471	1560320	8242625
$\Delta_{Co_2}(3B, pX, 23Z)$	-	-	90137	344701	4363008	24727875
$\Delta_{Co_2}(5A, pX, 23Z)$	-	-	4678959	23489003	251817984	1281988053
$\Delta_{Co_2}(5B, pX, 23Z)$	-	-	-	106987927	1213304832	6409940265
$\Delta_{Co_2}(7A, pX, 23Z)$	-	-	-	-	13293737984	68679139328

Table 7.V
Structure Constants $\Sigma(M_{23})$

pX	$5B$	$7A$	$11A$
$\Sigma_{M_{23}}(2B, pX, 23Y)$	138	368	782
$\Sigma_{M_{23}}(3B, pX, 23Y)$	2438	6624	10258
$\Sigma_{M_{23}}(5B, pX, 23Y)$	37582	88320	123786
$\Sigma_{M_{23}}(7A, pX, 23Y)$	-	211968	270848

Proof. This follows from the lemmas in Sections 7.2 and 7.3, Lemma 7.4.1 and the fact that the triangular $T(2,3,5)$ is isomorphic to A_5. $\square$

Corollary 7.4.3 *The Conway group Co_2 is (pX, pX, qY)-generated, for all $pX \in \{5A, 5B, 7A, 11A\}$ and $qY \in \{7A, 11A, 23A, 23B\}$ with $p < q$ as well as $(3B, 3B, 11A)$-, $(3A, 3A, 23Y)$- and $(3B, 3B, 23Y)$-generated.*

Proof. The result follows immediately from an application of Lemma 3.1.15 to Lemmas 7.2.2, 7.3.1, 7.3.2, 7.3.3 and 7.4.1. $\square$

7.5 nX-Complementary generations of Co_2

In this section we prove the following result.

Theorem 7.5.1 *The group Co_2 is nX-complementary generated if and only if $nX \in \{4G, 5A, 5B, 6A, 6B, 6E, 6F\}$ or $n \geq 7$.*

Before we prove this result, we first prove some useful lemmas. From Lemma 3.1.7 we only need to search for generating triples from amongst the triples (lX, mY, nZ) satisfying the relation $\Delta_{Co_2}(lX, mY, nZ) \geq |C_{Co_2}(nZ)|$.

Lemma 7.5.2 *The group Co_2 is not $3X$-complementary generated.*

Proof. We have $\Delta_{Co_2}(2A, 3A, tX) < |C_{Co_2}(tX)|$ for all $t \geq 7$. Therefore Co_2 is not 2-generated by any pair of elements $(x, y) \in 2A \times 3A$. From Lemma 2.3.8 we conclude that Co_2 is not 3A-complementary generated.

Let $\chi = \underline{23a} \in Irr(Co_2)$ and $A = \langle x \rangle$ and $B = \langle y \rangle$, where $x \in 2B$ and $y \in 3B$. Then $\langle \chi{\downarrow_A}, \chi_1{\downarrow_A} \rangle = 15$, $\langle \chi{\downarrow_B}, \chi_1{\downarrow_B} \rangle = 11$ and $\langle \chi{\downarrow_{(A \cap B)}}, \chi_1{\downarrow_{(A \cap B)}} \rangle = 23$. Thus by Lemma 3.3.4 we obtain $\langle x, y \rangle < Co_2$ and therefore Co_2 is not $3B$-complementary generated. $\square$

Lemma 7.5.3 *The group Co_2 is $4X$-complementary generated if and only if $X = G$.*

Proof. Let $X \in \{A, B, D\}$. Then $\Delta_{Co_2}(2A, 4X, tY) < |C_{Co_2}(tY)|$, for all t such that $1/2 + 1/4 + 1/t < 1$. Thus from Lemma 3.1.7 and Lemma 2.3.8 it follows that Co_2 is not $4A$-, $4B$- or $4D$-complementary generated.

Let $A = \langle x \rangle$ and $B = \langle y \rangle$, where $x \in 2B$ and $y \in 4C \cup 4E$. Then $\langle \chi{\downarrow_B}, \chi_1{\downarrow_B} \rangle = 9$. Again applying Lemma 3.3.4 we get $\langle x, y \rangle < Co_2$. Therefore Co_2 is not $4C$- or $4E$-complementary generated.

We now consider the class $4F$. The only conjugacy classes tY with $\Delta_{Co_2}(2A, 4F, tY) \geq |C_{Co_2}(tY)|$ are $11A$ and $18A$. However, we calculate

$$\begin{aligned} \Delta_{Co_2}(2A, 4F, 11A) &= 11 = \Sigma(U_6(2){:}2), \\ \Delta_{Co_2}(2A, 4F, 18A) &= 18 = \Sigma(U_6(2){:}2). \end{aligned}$$

Thus Co_2 is not $4F$-complementary generated.

The conjugacy class $4G$ has empty intersection with the maximal subgroup M_{23}. For every conjugacy class pY, where p is a prime, we observe from Table 7.VII that

$$\Delta^*_{Co_2}(pY, 4G, 23A) = \Delta_{Co_2}(pY, 4G, 23A) \geq |C_{Co_2}(23A)|.$$

Thus from Lemma 2.3.8 it follows that Co_2 is $4G$-complementary generated, proving the result. □

Lemma 7.5.4 *The group Co_2 is pX-complementary generated, where p is a prime, if and only if $p \geq 5$.*

Proof. It is immediate from Lemma 7.4.1 and Corollary 7.4.3 that, for $p \geq 5$ and qY any conjugacy class with prime order representatives, Co_2 is $(qY, pX, 23A)$-generated. Thus Co_2 is pX-complementary generated for all $p \geq 5$. The result now follows from Lemma 2.3.8 and Lemma 7.5.2. □

Lemma 7.5.5 *The group Co_2 is not $6C$- or $6D$-complementary generated.*

Proof. We first consider the conjugacy class $6C$. We calculate

$$\begin{array}{ll}
\Delta_{Co_2}(2A, 6C, 14A) = 84, & \Sigma(U_6(2){:}2) \geq 42, \\
\Delta_{Co_2}(2A, 6C, 16B) = 32, & \Sigma(U_6(2){:}2) = 32, \\
\Delta_{Co_2}(2A, 6C, 18A) = 24, & \Sigma(U_6(2){:}2) \geq 15, \\
\Delta_{Co_2}(2A, 6C, 24A) = 36, & \Sigma(U_6(2){:}2) \geq 24, \\
\Delta_{Co_2}(2A, 6C, 24A) = 50, & \Sigma(U_6(2){:}2) \geq 35.
\end{array}$$

In all cases $\Delta^*_{Co_2}(2A, 6C, tX) \leq \Delta(Co_2) - \Sigma(U_6(2){:}2) < |C_{Co_2}(tX)|$ and the non-generation of Co_2 by these triples follows. For the remaining triples of the form $(2A, 6C, tX)$, it follows from Lemma 3.1.7 that $\Delta(Co_2) < |C_{Co_2}(tX)|$ and the non-generation of Co_2 by these triples follows.

Next we consider the class $6D$. The only triples of the form $(2A, 6D, tX)$ that we need to consider are those for which $tX \in \{7A, 9A, 10B, 11A, 16A, 18A\}$. We calculate

$$\begin{array}{ll} \Delta_{Co_2}(2A,6D,7A) = 147, & \Sigma(U_6(2){:}2) = 105, \\ \Delta_{Co_2}(2A,6D,9A) = 57, & \Sigma(U_6(2){:}2) = 57, \\ \Delta_{Co_2}(2A,6D,10B) = 100, & \Sigma(U_6(2){:}2) \geq 70, \\ \Delta_{Co_2}(2A,6D,11A) = 33, & \Sigma(U_6(2){:}2) = 33, \\ \Delta_{Co_2}(2A,6D,16A) = 32, & \Sigma(U_6(2){:}2) = 32, \\ \Delta_{Co_2}(2A,6D,18A) = 24, & \Sigma(U_6(2){:}2) = 24. \end{array}$$

Again it follows Lemma 3.1.7 that these are not generating triples for Co_2 and hence Co_2 is not $6D$-complementary generated. This completes the proof. □

Lemma 7.5.6 *The group Co_2 is nX-complementary generated for all $nX \in \{6A, 6B, 6E, 6F, 12C\}$ or $n \in \{8, 9\}$.*

Proof. For all conjugacy classes pY, where p is a prime, we have

$$\Delta^*_{Co_2}(pY, nX, 23A) = \Delta(Co_2) - \Sigma(M_{23}).$$

It follows from Tables 7.VI and 7.VII that $\Delta^*_{Co_2}(pY, nX, 23A) > 0$, for all the conjugacy classes pY and nX in the statement of the lemma, with the exception of the triple $(2A, 6A, 23A)$.

We calculate $\Delta_{Co_2}(2A, 6A, 30B) = 30 = |C_{Co_2}(30B)|$. The only maximal subgroups, up to isomorphisms, with non-empty intersection with all the classes in this triple are $(2^{1+6}_+ \times 2^4).A_8$ and $5^{1+2}_+{:}4S_4$. However, we calculate $\Sigma((2^{1+6}_+ \times 2^4).A_8) = 0 = \Sigma(5^{1+2}_+{:}4S_4)$. Thus Co_2 is $(2A, 6A, 30B)$-generated.

Thus from Lemma 2.3.8 it follows that Co_2 is nX-complementary generated, for all the above mentioned classes. □

Proof of Theorem 7.5.1 The power maps of Co_2 yield $(12A)^2 = 6B$, $(12B)^2 = 6A$, $(12D)^2 = 6E$, $(12E)^2 = 6B$, $(12F)^2 = 6E$, $(12G)^2 = 6A$, $(12H)^2 = 6E$, $(14A)^2 = 7A$, $(14B)^2 = 7A$, $(14C)^2 = 7A$, $(15A)^3 = 5B$, $(15B)^3 = 5A$, $(15C)^3 = 5A$, $(16A)^2 = 8D$, $(16B)^2 = 8C$, $(18A)^2 = 9A$, $(20A)^2 = 10A$, $(20B)^2 = 10C$, $(24A)^2 = 12C$, $(24B)^2 = 12B$, $(28A)^4 = 7A$, $(30A)^2 = 15A$, $(30B)^2 = 15B$ and $(30C)^2 = 15C$. An application of Lemma 2.3.9 to Lemmas 7.5.6 gives complementary generation of these classes. The theorem now follows from Lemmas 7.5.2 to 7.5.6. □

TABLE 7.VI

Structure constants of Co_2

pX	$\Delta_{Co_2}(pX, 4G, 23A)$	$\Delta_{Co_2}(pX, 6A, 23A)$	$\Delta_{Co_2}(pX, 6B, 23A)$	$\Delta_{Co_2}(pX, 6E, 23A)$
$2A$	69	0	23	69
$2B$	805	69	253	1219
$2C$	22586	4071	5888	46184
$3A$	72496	14076	18952	149776
$3B$	221168	35328	60352	422096
$5A$	10969344	2455296	2696704	24488192
$5B$	56024320	11128320	14343168	117895424
$7A$	594552576	126385920	148907520	1292474880
$11A$	3004316672	667699200	741888000	6676992000
$23A$	1439286272	325317888	352114176	3228249088
$23B$	1422116864	325317888	348722688	3205356544

pX	$\Delta_{Co_2}(pX, 6F, 23A)$	$\Delta_{Co_2}(pX, 8A, 23A)$	$\Delta_{Co_2}(pX, 8B, 23A)$	$\Delta_{Co_2}(pX, 8C, 23A)$
$2A$	138	23	23	69
$2B$	2438	759	575	1265
$2C$	92368	34086	29854	48622
$3A$	299552	109296	105248	167624
$3B$	844192	298816	269008	452824
$5A$	48976384	18417664	18370560	27511680
$5B$	235790848	86765568	83568384	130251392
$7A$	2584949760	962134528	947956224	1443334272
$11A$	13353984000	5008073728	5007414272	7511451136
$23A$	6456498176	2425220096	2448536576	3635757568
$23B$	6410713088	2419496960	2431367168	3627172864

pX	$\Delta_{Co_2}(pX, 8D, 23A)$	$\Delta_{Co_2}(pX, 8E, 23A)$	$\Delta_{Co_2}(pX, 8F, 23A)$	$\Delta_{Co_2}(pX, 9A, 23A)$
$2A$	92	207	598	713
$2B$	2484	3335	13662	13938
$2C$	59064	104926	435252	489624
$3A$	179768	354752	1389568	1610897
$3B$	559176	1004272	4014144	4526515
$5A$	27617664	54935040	220588032	261146853
$5B$	140278656	270300416	1081248768	1258253433
$7A$	1486387328	2929468416	11717850112	13786398720
$11A$	7510791680	15023561728	60094246912	71221170375
$23A$	3568728576	7197420544	28812103680	34352492283
$23B$	3594482688	7214589952	28880781312	34271096571

TABLE 7.VI
Structure constants of Co_2 and M_{23}

pX	$\Delta_{Co_2}(pX, 12C, 23A)$	$\Sigma_{M_{23}}(pX, 6E, 23A)$	$\Sigma_{M_{23}}(pX, 8F, 23A)$
$2A$	230	0	0
$2B$	4370	322	368
$2C$	108376	0	0
$3A$	327520	0	0
$3B$	1032608	4324	7084
$5A$	48976384	0	0
$5B$	253596160	56672	83904
$7A$	2661258240	121480	178112
$11A$	13353984000	154560	242144
$23A$	6331860992	37536	52256
$23B$	6347122688	36432	52256

Chapter 8

The First Three Janko Groups

8.1 Introduction

The (p, q, r)-generations of the first two Janko groups J_1 and J_2, where p, q and r are prime divisors of the respective groups, were discussed by J. Moori [131]. Our interest in these two groups will be their nX-complementary generation. For detail on the construction and properties of these groups the reader is referred to Janko [86], [87], [88], Gagen [65] and Moori [131]. We now give a brief description of the third Janko group J_3.

In 1968, Z. Janko [89] announced the discovery of the two simple groups J_2 and J_3. More precisely, he proved the following result.

Let G be a finite non-abelian simple group with the following properties:

(a) The centre $Z(S)$ of a Sylow 2-subgroup S of G is cyclic.

(b) If z is an involution in $Z(S)$, then the centralizer of z in G is an extension of an extraspecial group E of order 2^5 by A_5.

Then we have the following two possibilities: If all the involutions in G are conjugates, then G is a new simple group of order 50232960 and has a uniquely determined character table. If G has more than one conjugacy classes of involutions, then G is a new simple group of order 604800 and G itself is uniquely determined.

Janko left open the existence and uniqueness of the simple group of order 50232960.

Higman-McKay [82] showed that a simple group G satisfying (a) and (b) with an unique conjugacy class of involutions does exist, and has the following additional property:

G has a subgroup H which is the extension of $L_2(16)$ by an involutory outer automorphism of $L_2(16)$.

S. K. Wong [182] proved the following result: Let G be a non-abelian simple group of order 50232960. Then G has properties (a) and (b). Thus the group constructed in this way is essentially the unique simple group of order $502329604 = 2^7{\cdot}3^5{\cdot}5{\cdot}17{\cdot}19$, known as the third Janko group J_3.

The subgroup structure of J_3 is discussed in Finkelstein-Rudvalis [62]. It follows from the character table of J_3 that J_3 has no subgroups of index less than 170, and any proper simple subgroup of J_3 must have order less than one million. It follows from M. Hall's partial classification of simple groups of order less than one million, Hall [78], that a simple proper subgroup of J_3 must be isomorphic to one of A_5, A_6, $L_2(17)$, $L_2(19)$, $L_2(16)$ or $U_4(2)$. Only the last possibility is ruled out by the fact that all elements of order 5 are conjugate in $U_2(4)$ while there are two conjugacy classes of elements of order 5 in J_3, with one class a power of the other.

Theorem 8.1.1 *(Finkelstein-Rudvalis [62]) The Janko simple group J_3 has exactly nine conjugacy classes of maximal subgroups:*

(i) Four groups of classical type, namely, $L_2(17)$, $L_2(19)$ (two classes) and $L_2(16){:}2$;

(ii) Five p-local groups, namely, $2^4{:}(3 \times A_5)$, $2_-^{1+4}{:}A_5$, $2^{2+4}{:}(3 \times S_3)$, $(3 \times A_6){:}2_2$ and $3^2.(3 \times 3^2){:}8$. $\square$

8.2 $(2,3,t)$-Generations for J_3

In this section we investigate the $(2,3,t)$-generations for the Janko group J_3. Since we may assume $1/2 + 1/3 + 1/t < 1$, it follows that $t \in \{8,9,10,12,15,17,19\}$. It is well-known that if a simple group $G = \langle x, y\rangle$, where $o(x) = 2$ and $o(y) = 3$, then $G = \langle x, x^y, x^{y^2}\rangle$. However, the order of their product $xx^yx^{y^2}$ is not known in general.

Table 8.I
Partial fusion maps of H into J_3

$2^4{:}(3\times A_5)$-class	$2a$	$2b$	$3a$	$3b$	$3c$	$3d$	$3e$	$15a$	$15b$	$15c$	$15d$	
$\lvert C_H(nx)\rvert$	192	48	180	180	36	36	9	15	15	15	15	
$\to J_3$	$2A$	$2A$	$3A$	$3A$	$3A$	$3A$	$3B$	$15A$	$15A$	$15B$	$15B$	
$L_2(17)$-class	$1a$	$2a$	$3a$	$4a$	$8a$	$8b$	$9a$	$9b$	$9c$	$17a$	$17b$	
$\lvert C_H(nx)\rvert$	2448	16	9	8	8	8	9	9	9	17	17	
$\to J_3$	$1A$	$2A$	$3B$	$4A$	$8A$	$8B$	$9A$	$9B$	$9C$	$17A$	$17B$	
$(3\times A_6){:}2_2$-class	$2a$	$2b$	$3a$	$3b$	$3c$	$3d$	$8a$	$8b$	$10a$	$10b$	$15a$	$15b$
$\lvert C_H(nx)\rvert$	48	20	1080	27	27	27	8	8	10	10	15	15
$\to J_3$	$2A$	$2A$	$3A$	$3B$	$3A$	$3A$	$8A$	$8A$	$10B$	$10A$	$15A$	$15A$
$2^{2+4}{:}(3\times S_3)$-class	$2a$	$2b$	$2c$	$3a$	$3b$	$3c$	$3d$	$3e$	$12a$	$12b$		
$\lvert C_H(nx)\rvert$	384	48	48	72	72	9	36	36	12	12		
$\to J_3$	$2A$	$2A$	$2A$	$3A$	$3A$	$3B$	$3A$	$3A$	$12A$	$12A$		

We will also investigate the $(2,2,2,p)$-generations of J_3, where p is a prime divisor of $|J_3|$.

Lemma 8.2.1 *Let H be a maximal subgroup of J_3, with $H \cong (3\times A_6){:}2_2$ and $g \in G$ such that H and H^g are distinct and $8A \cap (H \cap H^g) \neq \emptyset$. Then $H \cap H^g$ is not $(2A,3A,8A)$-generated.*

Proof. We shall show that the conjugacy class $3A$ does not meet $H \cap H^g$. First observe that $H = N_{J_3}(z)$, for some $z \in 3A$, and the fusion map of H into J_3 yields $3A \cap H = 3a \cup 3c \cup 3d$. Now $3a = \{z,\ z^{-1}\}$ and if $z \in H \cap H^g$, then $z^{g^{-1}} \in H$. Also $C_{J_3}(z) = \langle z\rangle \times A_6$ and $H - C_{J_3}(z)$ has no element of order 3. Thus $z^{g^{-1}} \in C_{J_3}(z)$ and consequently $z \in C_{J_3}(z) \cap C_{J_3}(z^g) = (\langle z\rangle \cap \langle z^g\rangle) \times (A_6 \cap A_6^g)$. If $\langle z\rangle = \langle z^g\rangle$, then $g \in N_{J_3}(z)$ and hence $H = H^g$, a contradiction. Therefore $z \in A_6 \cap A_6^g$, contrary to the fact that $\langle z\rangle \cap A_6 = \{1\}$. We therefore conclude that $3a \cap (H \cap H^g) = \emptyset$.

If $y \in (3c \cup 3d) \cap (H \cap H^g)$, then $y \in C_{J_3}(z) \cap C_{J_3}(z^g) = A_6 \cap A_6^g$, since $H - C_{J_3}(z)$ contains no element of order 3. From the fusion map of $C_{J_3}(z)$ into H, we observe that $(3c \cup 3d) \cap A_6 = \emptyset$ and therefore $y = z^i b$, for some $i = 1, 2$ and $b \in A_6$. But this is contrary to $y \in A_6 \cap A_6^g$. We therefore conclude that $H \cap H^g$ is not $(2A, 3A, 8A)$-generated. $\square$

Lemma 8.2.2 *The group J_3 is not $(2A, 3X, 8A)$-generated, where $X \in \{A, B\}$.*

Proof. (Also see Conder *et al.* [39]) We first prove that J_3 is not $(2A, 3A, 8A)$-generated. Let $x \in 8A$ be a fixed element in the maximal subgroup $H \cong (3 \times A_6){:}2_2$ of J_3. Since $|C_{J_3}(x)| = 8$, it follows from Theorem 3.1.4 and Table 8.I that x is contained in exactly 2 conjugates of H, say H and H^g. We easily calculate that $\Sigma(H) = \Sigma(H^g) = 16$ and by the previous lemma $\Sigma(H \cap H^g) = 0$. But then $\Delta^*(J_3) \leq \Delta(J_3) - \Sigma(H \cup H^g) = 36 - 32 = 4 < |C_{J_3}(x)| = 8$. From Lemma 3.1.7 it follows that $\Delta^*(J_3) = 0$ and J_3 is not $(2A, 3A, 8A)$-generated.

We apply Scott's theorem to prove that J_3 is not $(2A, 3B, 8A)$-generated. The group J_3 acts on a 85-dimensional complex irreducible module V and

$$dim(V/C_V(2A)) = 40, \qquad dim(V/C_V(3A)) = 54 \quad \text{and} \quad dim(V/C_V(8A)) = 74.$$

However, $40 + 54 + 74 = 168 < 170$, violating the conditions of Scott's theorem and the result follows. $\square$

Lemma 8.2.3 *The group J_3 is not $(2A, 3X, 9Y)$-generated, where $X \in \{A, B\}$ and $Y \in \{A, B, C\}$.*

Proof. Let $Y \in \{A, B, C\}$. From our calculations, we get $\Delta_{J_3}(2A, 3A, 9Y) = 0$ and hence J_3 is not $(2A, 3A, 9Y)$-generated.

The maximal subgroups of J_3 containing elements of order 9 are, up to isomorphisms, $L_2(19)$, two non-conjugate copies, $L_2(17)$ and $3^2.(3 \times 3^2){:}8$. The subgroups isomorphic to $3^2.(3 \times 3^2){:}8$ do not contribute to $\Delta(J_3) = \Delta_{J_3}(2A, 3B, 9Y)$ since $\Sigma(3^2.(3 \times 3^2){:}8) = 0$. The permutation character of J_3 on the conjugates of $L_2(19)$ is given by

$$\chi_{L_2(19)} = \underline{1a} + \underline{85ab} + \underline{1140aa} + \underline{1215ab} + \underline{1615a} + \underline{1902abc} + \underline{2432a}$$

and therefore a fixed element of order 9 in $L_2(19)$ is contained in $\chi_{L_2(19)}(9Y) = 3$ conjugate copies of $L_2(19)$. Using Table 8.I, a fixed element of order 9 in $L_2(17)$ is contained in 3 conjugate copies of $L_2(17)$. Further, the only maximal subgroups of $L_2(19)$ and $L_2(17)$ with order divisible by 2×9 are isomorphic to D_{18} which is not $(2,3,9)$-generated. Therefore $\Sigma(H \cap K) = 0$ for all maximal subgroups H and K of $L_2(19)$ or $L_2(17)$.

We calculate $\Sigma(L_2(19)) = \Sigma(L_2(17)) = 18$. Thus each non-conjugate copy of $L_2(19)$ and its conjugates contribute 3×18 to $\Delta(J_3)$. Also $L_2(17)$ and its conjugates contribute 3×18 to $\Delta(J_3)$. Thus

$$\begin{aligned} \Delta^*(J_3) &= \Delta(J_3) - (2 \times 3)\Sigma(L_2(19)) - 3\Sigma(L_2(17)) \\ &= 162 - (2 \times 3 \times 18) - (3 \times 18) = 0\,, \end{aligned}$$

and the result follows. □

Corollary 8.2.4 *If H is a $(2A, 3X, 9Y)$-generated subgroup of J_3, where $X \in \{A, B\}$ and $Y \in \{A, B, C\}$, then $H \cong L_2(19)$ or $L_2(17)$.*

Proof. This is an immediate consequence of the previous lemma. □

Lemma 8.2.5 *The group J_3 is $(2A, 3X, 10Y)$-generated, where $X, Y \in \{A, B\}$, if and only if $X = B$.*

Proof. We first consider the triple $(2A, 3A, 10Y)$. Let x be a fixed element of J_3 of order 10 such that $x \in L$, where $L \cong L_2(19)$. Then $\Sigma(L_2(19)) = 20$ and $\Delta^*(J_3) \leq \Delta(J_3) - \Sigma(L) = 25 - 20 = 5 < |C_{J_3}(x)| = 10$. Now the non-generation of J_3 by this triple follows from Lemma 3.1.7.

Next, we consider $(2A, 3B, 10Y)$. The maximal subgroups of J_3 with elements of order 10 and nontrivial intersection with the class $3B$ are, up to isomorphisms, $L_2(19)$ (two non-conjugate copies) and $(3 \times A_6){:}2_2$. Further, if we fix an element of order 10 in $L_2(19)$ (respectively, $(3 \times A_6){:}2_2$), then it is contained in no other conjugate of $L_2(19)$ (respectively, $(3 \times A_6){:}2_2$).

We easily calculate $\Sigma(L_2(19)) = 20$ and $\Sigma((3 \times A_6){:}2_2) = 10$. Thus

$$\Delta^*(J_3) \geq \Delta(J_3) - 2\Sigma(L_2(19)) - \Sigma((3 \times A_6){:}2_2) = 70 > 0$$

and hence J_3 is $(2A, 3B, 10X)$-generated. $\square$

The $(2A, 3B, 10Y)$-generation of J_3 was first proved by Woldar [176] who investigated the $(2,3)$-generated of the sporadic simple groups.

Lemma 8.2.6 *The group J_3 is $(2A, 3X, 12A)$-generated, where $X \in \{A, B\}$, if and only if $X = B$.*

Proof. For the case $X = A$, observe that $\Delta^*(J_3) \leq \Delta(J_3) = 11 < |C_{J_3}(12A)| = 12$. Therefore by Lemma 3.1.7 the group J_3 is not $(2A, 3A, 12A)$-generated.

The maximal subgroups of J_3 with elements of order 12 that have non-empty intersection with the conjugacy class $3B$ are, up to isomorphisms, $(3 \times A_6){:}2_2$, $3^2.(3\times 3^2){:}8$ and $2^{2+4}{:}(3\times S_3)$. We calculate $\Sigma((3\times A_6){:}2_2) = \Sigma(3^2.(3\times 3^2){:}8) = 0$ and $\Sigma(2^{2+4}{:}(3\times S_3)) = 16$. Further, a fixed element of $2^{2+4}{:}(3 \times S_3)$ of order 12 is contained in 2 conjugate copies of $2^{2+4}{:}(3 \times S_3)$. Hence $2^{2+4}{:}(3 \times S_3)$ and its conjugate contribute at most 2×16 to $\Delta(J_3)$. Since $\Delta(J_3) = 144 > 32$, we have $\Delta^*(J_3) \geq 112$. Therefore J_3 is $(2A, 3B, 12A)$-generated. $\square$

Lemma 8.2.7 *The group J_3 is $(2A, 3X, 15Y)$-generated, where $X, Y \in \{A, B\}$, if and only if $X = B$.*

Proof. Consider the triple $(2A, 3A, 15Y)$. Let x be a fixed element of J_3 of order 15 and $x \in L$, where $L \cong L_2(16){:}2$. Then we have $\Sigma(L) = 15$ such that

$$\Delta^*(J_3) \leq \Delta(J_3) - \Sigma(L) = 25 - 15 = 10.$$

Now since $|C_{J_3}(x)| = 15$, non-generation follows from Lamma 3.1.7.

Next we consider the triple $(2A, 3B, 15Y)$. The maximal subgroups of J_3 with elements of order 15 are, up to isomorphisms, $L_2(16){:}2$, $2^4{:}(3 \times A_5)$ and $(3 \times A_6){:}2_2$. We observe that the conjugacy class $3B$ fails to meet $L_2(16){:}2$. Now since $\Sigma(2^4{:}(3\times A_5)) = \Sigma((3\times A_6){:}2_2) = 0$, we have $\Delta^*(J_3) = \Delta(J_3) = 90$. Thus J_3 is $(2A, 3B, 15Y)$-generated and the result follows. $\square$

Lemma 8.2.8 *The group J_3 is $(2A, 3X, 17Y)$-generated, where $X, Y \in \{A, B\}$, if and only if $X = B$.*

Proof. We first consider the case $X = A$. The only maximal subgroups of J_3 with order divisible by 17 are, up to isomorphims, $L_2(16){:}2$ and $L_2(17)$. Let $H, K \leq J_3$ such that $H \cong L_2(16){:}2$ and $K \cong L_2(17)$. Then

$$\chi_H = \underline{1a} + \underline{323ab} + \underline{324a} + \underline{1140a} + \underline{1215ab} + \underline{1615a}.$$

It follows from Table 8.I that $\chi_K(3A) = 0$ and hence $\Sigma(K) = 0$. A fixed element of order 17 is contained in $\chi_H(3A) = 2$ distinct conjugate copies of H, namely H and H^g. Thus

$$\Delta^*(J_3) = \Delta(J_3) - 2\Sigma(H) + \Sigma(H \cap H^g).$$

We easily calculate $\Delta(J_3) = 34$ and $\Sigma(H) = 17$, so $\Delta^*(J_3) = \Sigma(H \cap H^g)$.

We will now show that $\Sigma(H \cap H^g) = 0$. The only maximal subgroup of H with order divisible by 3×17 is isomorphic to $L_2(16)$. If $H \cap H^g = L$, where $L \cong L_2(16)$, then L is a normal subgroup of H and H^g, since it has index 2 in these groups. Therefore $H,\ H^g \leq N_{J_3}(L) \leq J_3$. Since J_3 is simple, and H and H^g are maximal in J_3, so $N_{J_3}(L) = H = H^g$, contrary to the assumption that H and H^g are distinct. Therefore $\Sigma(H \cap H^g) = 0$ and hence $\Delta^*(J_3) = 0$, proving that $(2A, 3A, 17Y)$ is not a generating triple for J_3.

Next we consider $X = B$. Now $\chi_H(3B) = 0$ and consequently $\Sigma(H) = 0$. Further, a fixed element of order 17 is contained precisely in one conjugate copy of K. Thus

$$\Delta^*(J_3) = \Delta(J_3) - \Sigma(K) = 119 - 17 = 102$$

and hence J_3 is $(2A, 3B, 17X)$-generated. □

Corollary 8.2.9 *If L is a $(2A, 3A, 17X)$-generated subgroup of J_3, where $X \in \{A, B\}$, then $L \cong L_2(16)$.*

Proof. It is evident from the above lemma that if L is a $(2A, 3A, 17X)$-generated subgroup of J_3, then $L \leq H$, where $H \cong L_2(16){:}2$. Since H has a soluble quotient isomorphic to the cyclic group 2, it is not $(2A, 3A, 17X)$-generated (*cf.*, Lemma 3.1.10). Moreover, $\Sigma(H) = \Sigma(L_2(16)) = 17$ and $L_2(16)$ contains no maximal subgroup of order divisible by 3×17. Thus $L \cong L_2(16)$. □

Lemma 8.2.10 *The group J_3 is $(2A, 3X, 19Y)$-generated, where $X, Y \in \{A, B\}$.*

Proof. The maximal subgroups of J_3 with order divisible by 19 are, up to isomorphisms, $L_2(19)$, two non-conjugate copies. If $L \leq J_3$ with $L \cong L_2(19)$, then $\chi_L(3A) = 0$. Thus $\Sigma_L(2A, 3A, 19Y) = 0$ and J_3 is $(2A, 3A, 19Y)$-generated because $\Delta(J_3) = 38$.

Next we show J_3 is $(2A, 3B, 19Y)$-generated. A fixed element of order 19 is contained in exactly one conjugate copy of $L_2(19)$. Thus each non-conjugate copy of $L_2(19)$ contributes $\Sigma(L_2(19)) = 19$ to $\Delta(J_3)$. Since $\Delta(J_3) = 95$, we have $\Delta^*(J_3) \geq \Delta(J_3) - 2\Sigma(L_2(19)) = 57 > 0$, and the result follows. □

Theorem 8.2.11 *The group J_3 is $(2A, 3A, tX)$-generated if and only if $t = 19$. Furthermore, J_3 is $(2A, 3B, tX)$-generated if and only if $t \in \{10, 12, 15, 17, 19\}$.*

Proof. This follows from the lemmas proved above. □

A group G is said to be a (u, v, w, x)-generated group if it is a quotient of the quadrangle group $Q(u, v, w, x) = \langle a, b, c, d \mid a^u = b^v = c^w = d^x = abcd = 1\rangle$. If kW, lX, mY and nZ are conjugacy classes of a finite group G, and d a fixed representative of nZ, then $\Delta_G(kW, lX, mY, nZ)$ gives the number of distinct ordered triples (a, b, c) with $a \in kW$, $b \in lX$, $c \in mY$ such that $abc = d$. This number can be calculated by the formula

$$\Delta_G(kW, lX, mY, nZ) = \frac{|kW||lX||mY|}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(a)\chi(b)\chi(c)\overline{\chi(d)}}{(\chi(1))^2}.$$

The notations $\Delta^*(G)$ and $\Sigma(H)$ have analogous meaning as in the case of triples. There is a close relationship between $(2, 3, t)$-generation of a finite simple group and generation of a finite simple group by a set of three involutions as outlined in Section 2.3.1. A useful results that we shall use is:

Corollary 8.2.12 *Let G be a $(2X, 3Y, tZ)$-generated finite simple group. Then G is $(2X, 2X, 2X, (tZ)^3)$-generated.*

Proof. The result is a special case of Proposition 2.3.2. □

Lemma 8.2.13 *The group J_3 is $(2A, 2A, 2A, 3B)$-generated.*

Proof. The full list of non-conjugate maximal subgroups of J_3 containing a fixed element $x \in 3B$ are, up to isomorphisms, two non-conjugate copies of $L_2(19)$, $2^4{:}(3 \times A_5)$, $L_2(17)$, $(3 \times A_6){:}2_2$, $3^2.(3 \times 3^2){:}8$ and $2^{2+4}{:}(3 \times S_3)$. We list the values of $h = \chi_H(3B)$ and $\Sigma(H)$ in the table below.

H	h	$\Sigma(H)$	$h\Sigma(H)$
$L_2(19)$	27	1 458	39 366
$L_2(19)$	27	1 458	39 366
$2^4{:}(3 \times A_5)$	27	351	9 477
$L_2(17)$	27	1 458	39 366
$(3 \times A_6){:}2_2$	9	1458	13 122
$3^2.(3 \times 3^2){:}8$	1	0	0
$2^{2+4}{:}(3 \times S_3)$	27	243	6 561

Our calculations show that the total number of triples of involutions with product x that generate a proper subgroup of J_3 is at most 147782. The result follows since $\Delta(J_3) = 406782$. □

Remark 8.2.14 Since $(9A)^3 = (9B)^3 = (9C)^3 = 3B$, Lemmas 8.2.4 and 8.2.13 show the converse of Corollary 8.2.12 is false in general.

Theorem 8.2.15 *The group J_3 is $(2,2,2,p)$-generated, where p is a prime, if and only if $p \in \{3,5,17,19\}$.*

Proof. We use Scott's theorem to show that $p \neq 2$. The group J_3 acts on a 85-dimensional irreducible module V and $d = dim(V/C_V(2A)) = 40$. If the group J_3 is $(2A,2A,2A,2A)$-generated, then by Scott's theorem we have $4d \geq 2 \times 85$, that is $160 \geq 170$, which is a contradiction.

Now $(15A)^3 = 5B$, $(15B)^3 = 5A$, $(17A)^3 = 17B$, $(17B)^3 = 17A$, $(19A)^3 = 19B$ and $(19B)^3 = 19A$. Applying Corollary 8.2.12 to Lemmas 8.2.7, 8.2.8 and 8.2.10 we deduce that J_3 is $(2A,2A,2A,5X)$-, $(2A,2A,2A,17X)$- and $(2A,2A,2A,19X)$-generated, $X \in \{A, B\}$. □

8.3 (p, q, r)-Generations of J_3

In this section we investigate the (p, q, r)-generations of the Janko group J_3 and it consequences, where p, q and r are distinct primes satisfying the relation $p < q < r$. Since we may assume $1/p + 1/q + 1/r < 1$ it follows that $r = 17$ or 19. The maximal subgroups of J_3 with order divisible by 17 are, up to isomorphisms, $L_2(16){:}2$ and $L_2(17)$. The subgroups isomorphic to $L_2(19)$ (contained in two non-conjugate classes) are the only maximal subgroups of J_3, with order divisible by 19. Throughout this section H, K, L_1 and L_2 will denote subgroups of J_3 with $H \cong L_2(16){:}2$, $K \cong L_2(17)$, $L_1 \cong L_2(19) \cong L_2$ and $L_1 \neq L_2^g$, for all $g \in J_3$. In the previous section we dealt with the triples $(2A, 3X, 17Y)$ and $(2A, 3X, 19Y)$, where $X, Y \in \{A, B\}$.

Table 8.II

Structure constants for J_3

pX	$17A$	$17B$	$19A$	$19B$
$\Delta_{J_3}(2A, 5A, pX)$	867	867	874	874
$\Delta_{J_3}(2A, 5B, pX)$	867	867	874	874
$\Delta_{J_3}(2A, 17A, pX)$	-	-	1577	1577
$\Delta_{J_3}(2A, 17B, pX)$	-	-	1577	1577
$\Delta_{J_3}(3A, 5A, pX)$	1496	1496	1501	1501
$\Delta_{J_3}(3A, 5B, pX)$	1496	1496	1501	1501
$\Delta_{J_3}(3B, 5A, pX)$	6902	6902	6840	6840
$\Delta_{J_3}(3B, 5B, pX)$	6902	6902	6840	6840
$\Delta_{J_3}(3A, 17A, pX)$	-	-	2812	2812
$\Delta_{J_3}(3A, 17B, pX)$	-	-	2812	2812
$\Delta_{J_3}(3B, 17A, pX)$	-	-	12103	12103
$\Delta_{J_3}(3B, 17B, pX)$	-	-	12103	12103
$\Delta_{J_3}(5A, 17A, pX)$	-	-	98192	98192
$\Delta_{J_3}(5A, 17B, pX)$	-	-	98192	98192
$\Delta_{J_3}(5B, 17A, pX)$	-	-	98192	98192
$\Delta_{J_3}(5B, 17B, pX)$	-	-	98192	98192

Lemma 8.3.1 *The group J_3 is $(2A, 5X, 17Y)$-generated, where $X, Y \in \{A, B\}$.*

Proof. The only maximal subgroups of J_3 with order divisible by 17 are, up to isomorphisms, H and K. Since $|K|$ is not divisible by 5, K and its subgroups are not

$(2A, 5X, 17Y)$-generated. Now a fixed element x of order 17 is contained in $\chi_H(x) = 2$ conjugate copies of H. We calculate $\Sigma(H) = 17$ and consequently $\Delta^*(J_3) \geq \Delta(J_3) - 2\Sigma(H) = 833 > 0$ and therefore J_3 is $(2A, 5X, 17Y)$-generated. □

Lemma 8.3.2 *The group J_3 is $(2A, 5X, 19Y)$-generated, where $X, Y \in \{A, B\}$.*

Proof. A fixed element x of order 19 in L_1 is contained in $\chi_{L_1}(x) = 1$ conjugate copy of L_1. We easily calculate that $\Sigma(L_1) = 19$ and $\Delta(J_3) = 874$. Since L_1 and L_2 are the only maximal subgroups of J_3 that may contain x and since $\chi_{L_1} = \chi_{L_2}$, we have $\Delta^*(J_3) \geq \Delta(J_3) - \Sigma(L_1) - \Sigma(L_2) = 836 > 0$ and the result follows. □

Lemma 8.3.3 *The group J_3 is $(3X, 5Y, 17Z)$-generated, where $X, Y, Z \in \{A, B\}$.*

Proof. The order of the subgroup K is not divisible by 5 and therefore H is the only maximal subgroup of J_3, up to isomorphisms, that contains $(3X, 5Y, 17Z)$-generated subgroups. We will first show that J_3 is $(3A, 5Y, 17Z)$-generated. In this case $\Sigma(H) = 17$ and a fixed element of order 17 in H is contained in exactly two conjugate copies of H. Thus $\Delta^*(J_3) \geq \Delta(J_3) - 2\Sigma(H) = 1496 - 34 > 0$ and hence J_3 is $(3A, 5Y, 17Z)$-generated.

Next we show that the group J_3 is $(3B, 5Y, 17Z)$-generated. We easily calculate $\chi_H(3B) = 0$ and hence $\Sigma(H) = 0$. Therefore $\Delta^*(J_3) = \Delta(J_3) = 6902$ and the result follows. □

Corollary 8.3.4 *If M is a proper $(3A, 5Y, 17Z)$-generated subgroup of J_3, where $Y, Z \in \{A, B\}$, then $M \cong L_2(16)$.*

Proof. It is clear from the previous result that if M is $(3A, 5Y, 17Z)$-generated, then $M \leq H^g$, for some $g \in G$. Since $H^g \cong H \cong L_2(16){:}2$, H^g has a soluble quotient isomorphic to the cyclic group 2. Thus H^g is not $(3A, 5Y, 17Z)$-generated (*cf.* Lemma 3.1.10). Further, $\Sigma(H) = \Sigma(L_2(16)) = 17$ and $L_2(16)$ contains no maximal subgroup of order divisible by $3 \times 5 \times 17$. Thus $M \cong L_2(16)$. □

Lemma 8.3.5 *The group J_3 is $(3X, 5Y, 19Z)$-generated, where $X, Y, Z \in \{A, B\}$.*

Proof. The subgroups L_1 and L_2 have empty intersection with the class $3A$ since $\chi_{L_2(19)}(3A) = 0$. Thus the $(3A, 5Y, 19Z)$-generation of J_3 follows from the fact that $\Delta^*(J_3) = \Delta(J_3) = 1501$.

We now show that J_3 is $(3B, 5Y, 19Z)$-generated. A fixed element of order 19 in L_1 is contained in exactly one copy of L_1. Hence each non-conjugate copy of $L_2(19)$ contributes $\Sigma(L_1) = 38$ to $\Delta(J_3) = 6840$. Since L_1 and L_2 are the only maximal subgroups that contain elements of order 19, it follows that $\Delta^*(J_3) \geq 6840 - 2 \times 38 = 6764 > 0$ and the result follows. □

Lemma 8.3.6 *The group J_3 is $(2A, 17X, 19Y)$-, $(3X, 17Y, 19Z)$-, $(5X, 17Y, 19Z)$-, $(17X, 17Y, 19Z)$- and $(17X, 19Y, 19Z)$-generated, where $X, Y, Z \in \{A, B\}$.*

Proof. We calculate $\Delta_{J_3}(17X, 17Y, 19Z) = 175769$ and $\Delta_{J_3}(17X, 19Y, 19Z) = 155629$. Using these and the structure constants listed in Table 8.II, and the fact that the group J_3 has no maximal subgroup of order divisible by 17×19, we deduce that $\Delta(J_3) = \Delta^*(J_3) > 0$ in all cases mentioned in the statement of the lemma. Hence the result follows. □

We now summarize the above lemmas in the following theorem.

Theorem 8.3.7 *The group J_3 is (p, q, r)-generated for $p, q, r \in \{2, 3, 5, 17, 19\}$ with $p < q < r$, except when $(p, q, r) = (2, 3, 5)$.*

Proof. The triangle group $T(2, 3, 5)$ is isomorphic to A_5. The result follows from the above lemmas. □

Corollary 8.3.8 *The group J_3 is (pX, pX, qY)-generated, with $p < q$, for all $pX \in \{3A, 3B, 5A, 5B, 17A, 17B\}$ and $qY \in \{17A, 17B, 19A, 19B\}$.*

Proof. The structure constant $\Delta_{J_3}(3A, 3A, 17Y) = 68$. The subgroup $L_2(17)$ fails to meet the conjugacy class $3A$ and $\Sigma(L_2(16){:}2) = 17$. Therefore $\Delta^*(J_3) = 68 - 17 > 0$ and J_3 is $(3A, 3A, 17Y)$-generated. The result follows immediately from the Lemma 3.1.15, Lemmas 8.2.8, 8.2.9, 8.3.1, 8.3.2, 8.3.6 and the fact that $(17A)^2 = 17A$, $(17B)^2 = 17B$, $(19A)^2 = 19B$ and $(19B)^2 = 19A$. □

8.4 nX-Complementary generations of J_1

In order to apply Lemma 2.3.8 we will first find the triangular presentations of J_1 that will allow us to deduce its nX-complementary generation. The conjugacy classes of J_1 with elements of prime order are $2A$, $3A$, $5A$, $5B$, $7A$, $11A$,$19A$, $19B$ and $19C$.

Lemma 8.4.1 *(Woldar [179]) The group J_1 is $(2A, 3A, 7A)$-generated.* □

Lemma 8.4.2 *(Moori [131]) The group J_1 is $(2,5,7)$-, $(2,5,19)$-, $(2,7,11)$-, $(2,7,19)$-, $(2,11,19)$-, $(5,7,11)$-, $(5,7,19)$-, $(5,11,19)$-, $(3,5,7)$-, $(3,5,19)$-, $(3,7,11)$-, $(3,7,19)$-, $(3,11,19)$- and $(7,11,19)$-generated.* □

In Moori [131], the above result was proved for the conjugacy classes $5A$ and $19A$ (where applicable). If we replace $5A$ by $5B = (5A)^{-1}$, and $19A$ by $19B = (19A)^2$ or $19C = (19A)^4$, then the corresponding structure constants are unchanged. This is indeed a consequence of Theorem 3.1.14. Thus the result mentioned above is true for all conjugacy classes with elements of appropriate order (eg., the group J_1 is $(2A, 5X, 19Y)$-generated, for $X \in \{A, B\}$ and $Y \in \{A, B, C\}$).

Corollary 8.4.3 *The group J_1 is $(3A, 3A, 7A)$-, $(5A, 5A, 7A)$-, $(5B, 5B, 7A)$-, $(7A, 7A, 11A)$-, $(11A, 11A, 19A)$-, $(19A, 19A, 11A)$-, $(19B, 19B, 11A)$-, $(19C, 19C, 11A)$-generated.*

Proof. The proof follows from an application of Lemma 3.1.15 to the results obtained in Lemmas 8.4.1 and 8.4.2. □

Lemma 8.4.4 *The group J_1 is $(5A, 5B, 19A)$- and $(19A, 19B, 19C)$-generated.*

Proof. As argued in Moori [131], the group J_1 contains no maximal subgroup that is $(p, q, 19)$-generated, where p and q are primes. Thus for these cases $\Delta^*(J_1) = \Delta(J_1)$. We calculate $\Delta_{J_1}(5A, 5B, 19A) = 228$ and $\Delta_{J_1}(19A, 19B, 19C) = 573$ and the result follows. □

We are now ready to prove the main result of this section.

Theorem 8.4.5 *The group J_1 is nX-complementary generated if and only if $n > 2$.*

Proof. It follows from Lemma 2.3.8 that J_1 is not $2A$-complementary generated. We now show J_1 is $5A$-complementary generated. It is proved above (Lemmas 8.4.2 and 8.4.4, Corollary 8.4.3) that J_1 is $(2A, 5A, 7A)$-, $(3A, 5A, 19B)$-, $(5A, 5A, 7A)$-, $(5A, 5B, 19A)$-, $(5A, 7A, 11A)$-, $(5A, 11A, 19C)$-generated. Rearranging these triangular presentations, we in fact showed that for every conjugacy class pX, the group J_1 is $(pX, 5A, q_{pX}Y)$-generated for some conjugacy class $q_{pX}Y$. So by Lemma 2.3.8 the group J_1 is $5A$-complementary generated.

Using similar arguments we can show that J_1 is $3A$-, $5B$-, $7A$-, $11A$-, $19A$-, $19B$- and $19C$-complementary generated. Also $(6A)^2 = 3A$, $(10A)^2 = 5B$, $(10B)^2 = 5A$, $(15A)^5 = 3A = (15B)^5$ and the result follows from Lemma 2.3.9. $\square$

8.5 nX-Complemenary generations of J_2

The conjugacy classes of J_2 with elements of prime order are $2A$, $2B$, $3A$, $3B$, $5A$, $5B$, $5C$, $5D$ and $7A$.

Lemma 8.5.1 *The group J_2 is not nX-complementary generated, where $nX \in \{3A, 3B, 4A, 5A, 5B\}$.*

Proof. If the group J_2 is nX-complementary generated, then it must be $(2A, nX, tY)$-generated for some t with $1/2 + 1/n + 1/t \leq 1$. The appropriate structure constants of J_2 are listed in Table 8.III.

It is shown in Finkelstein-Rudvalis [62] and Moori [131] that J_2 is not $(2A, 3B, 7A)$-generated. Let H be a maximal subgroup of J_2 with $H \cong 2^{2+4}{:}(3 \times S_3)$. Then we calculate $\Sigma_H(2A, 3B, 8A) = 16$ and hence

$$\Delta^*_{J_2}(2A, 3B, 8A) \leq \Delta_{J_2}(2A, 3B, 8A) - \Sigma_H(2A, 3B, 8A) = 16 - 16 = 0.$$

Thus J_2 is not $(2A, 3B, 8A)$-generated. Let K be a maximal subgroup of J_2 such that $K \cong 5^2{:}D_{12}$. Then $\Sigma_K(2A, 3B, 10X) = 10$, for $X \in \{C, D\}$ and consequently

Table 8.III
Structure constants of J_2

tX	$4A$	$5A$	$5B$	$5CD$	$6A$	$6B$	$7A$	$8A$	$10AB$	$10C$	$10D$	$12A$	$15A$	$15B$
$\Delta_{J_2}(2A,3A,tX)$	-	-	-	-	-	-	0	0	0	0	0	0	0	0
$\Delta_{J_2}(2A,3B,tX)$	-	-	-	-	-	-	7	16	0	10	10	8	10	10
$\Delta_{J_2}(2A,4A,tX)$	-	0	0	0	0	0	7	4	0	0	0	8	5	5
$\Delta_{J_2}(2A,5A,tX)$	0	0	25	0	0	0	0	0	0	0	2	4	5	0
$\Delta_{J_2}(2A,5B,tX)$	0	25	0	0	0	0	0	0	0	2	0	4	0	5
$\lvert C_{J_2}(tX)\rvert$	96	300	300	50	24	24	7	8	20	10	10	12	15	15

$\Delta^*(2A,3B,10X) = 0$, proving that J_2 is not $(2A,3B,10X)$-generated. Furthermore, if $U \cong U_3(3)$ is a maximal subgroup of J_2, then $\Delta_{J_2}(2A,4A,7A) = 7 = \Sigma_U(2A,4A,7A)$ and non-generation of J_2 by this triple follows.

For the remaining triples in Table 8.III, we have $\Delta_{J_2}(2A,nX,tY) < |C_{J_2}(tX)|$, and non-generation by these triples follows from Lemma 3.1.7. Thus for every $nX \in \{3A,3B,4A,5A,5B\}$ and for any t, the group J_2 is not $(2A,nX,tY)$-generated. The result now follows immediately from Lemma 2.3.8. $\square$

Table 8.IV
Structure constants of J_2

pX	$2A$	$2B$	$3A$	$3B$	$5A$	$5B$	$5C$	$5D$	$7A$
$\Delta_{J_2}(pX,5C,7A)$	7	49	14	343	35	35	252	252	1764
$\Delta_{J_2}(pX,5D,7A)$	7	49	14	343	35	35	252	252	1764
$\Delta_{J_2}(pX,6A,7A)$	14	98	28	700	70	70	518	518	3752
$\Delta_{J_2}(pX,6B,7A)$	28	203	42	1358	175	175	1001	1001	7238
$\Delta_{J_2}(pX,8A,7A)$	42	308	70	2072	252	252	1512	1512	10752
$\Delta_{J_2}(pX,10A,7A)$	14	119	28	854	91	91	609	609	4326
$\Delta_{J_2}(pX,10B,7A)$	14	119	28	854	91	91	609	609	4326
$\Delta_{J_2}(pX,15A,7A)$	21	175	28	1113	161	161	784	784	5572
$\Delta_{J_2}(pX,15B,7A)$	21	175	28	1113	161	161	784	784	5572

Lemma 8.5.2 *The group J_2 is $6X$-complementary generated, where $X \in \{A, B\}$.*

Proof. We will first consider the case $X = A$. We calculate the structure constant $\Delta_{J_2}(2A, 6A, 10C) = 10$. The maximal subgroups of J_2 with non-empty intersection with the classes $2A$, $6A$ and $10C$ are, up to isomorphisms, $2^{1+4}{:}A_5$ and $A_5 \times D_{10}$. However, we calculate $\Sigma_M(2A, 6A, 10C) = 0$, where $M \cong 2^{1+4}{:}A_5$ or $A_5 \times D_{10}$. Thus $\Delta^*_{J_2}(2A, 6A, 10C) = 10$, and J_2 is $(2A, 6A, 10C)$-generated.

The only maximal subgroups with non-empty intersection with the classes $6A$ and $7A$ are isomorphic to $U_3(3)$. Furthermore, if $pY \in \{2B, 5A, 5B, 5C, 5D\}$, then $pY \cap U_3(3) = \emptyset$ and therefore $\Delta^*_{J_2}(pY, 6A, 7A) = \Delta_{J_2}(pY, 6A, 7A) > 0$ (*cf.* Table 8.IV), proving that J_2 is generated by these triples. Also a fixed element of order 7 is contained in 2 conjugates of a $U_3(3)$ subgroup of J_2. Thus

$$\Delta^*_{J_2}(3A, 6A, 7A) \geq \Delta_{J_2}(3A, 6A, 7A) - 2\Sigma_{U_3(3)}(3A, 6A, 7A) = 28 - 2(7) > 0.$$

Similarly, $\Delta^*_{J_2}(3B, 6A, 7A) \geq 700 - 2(56) = 588$ and $\Delta^*_{J_2}(7A, 6A, 7A) \geq 3756 - 2(168) > 0$. The $6A$-complementary generation of J_2 now follows from Lemma 2.3.8.

Next we consider the case $X = B$. The $(pX, 6B, 7A)$-generated proper subgroups of J_2 are contained in the maximal subgroups isomorphic to $L_3(2){:}2$. Also a fixed element of order 7 is contained in a unique conjugate of a $L_3(2){:}2$ subgroup. Thus

$$\Delta^*_{J_2}(pY, 6B, 7A) = \Delta_{J_2}(pY, 6B, 7A) - \Sigma_{L_3(2):2}(pY, 6B, 7A).$$

For any $pY \in \{3A, 5A, 5B, 5C, 5D\}$, we have $pY \cap L_3(2){:}2 = \emptyset$ and consequently $\Delta^*_{J_2}(pY, 6B, 7A) = \Delta_{J_2}(pY, 6B, 7A) > 0$ (*cf.* Table 8.IV). For the remaining triples we calculate $\Delta^*_{J_2}(2A, 6B, 7A) = 28$, $\Delta^*_{J_2}(2B, 6B, 7A) = 196$, $\Delta^*_{J_2}(3B, 6B, 7A) = 1358$ and $\Delta^*_{J_2}(7A, 6B, 7A) = 7238$, proving that J_2 is $6B$-complementary generated. □

Lemma 8.5.3 *The group J_2 is $8A$-complementary generated.*

Proof. The only maximal subgroups of J_2 that have non-empty intersection with the classes $7A$ and $8A$ are, up to isomorphisms, $U_3(3)$ and $L_3(2){:}2$. Since 5 does not divide the order of these groups and $\Delta_{J_2}(5X, 8A, 7A) > 0$ (*cf.* Table 8.IV), the group J_2 is $(5X, 8A, 7A)$-generated, for all $X \in \{A, B, C, D\}$. Furthermore,

$$\begin{aligned}\Delta^*_{J_2}(2A, 8A, 7A) &\geq \Delta_{J_2}(2A, 8A, 7A) - 2\Sigma_{U_3(3)}(2A, 8A, 7A) - \Sigma_{L_3(2):2}(2A, 8A, 7A) \\ &= 42 - 2(14) - 0 > 0.\end{aligned}$$

Similarly, $\Delta^*_{J_2}(2B,8A,7A) \geq 294$, $\Delta^*_{J_2}(3A,8A,7A) \geq 42$, $\Delta^*_{J_2}(3B,8A,7A) \geq 1736$ and $\Delta^*_{J_2}(7A,8A,7A) \geq 10304$. Thus we have shown that J_2 is $(pX,8A,7A)$-generated, for all classes pX with prime order representatives. Hence the result. $\square$

Lemma 8.5.4 *The group J_2 is nX-complementary generated, where $nX \in \{5C,5D, 10A,10B,15A,15B\}$.*

Proof. The subgroups $U_3(3)$ and $L_3(2){:}2$ do not contain elements of order 5, 10 or 15. Since these are the only maximal subgroups of J_2, up to isomorphisms, with order divisible by 7, it follows that $\Delta^*_{J_2}(pY,nX,7A) = \Delta_{J_2}(pY,nX,7A)$, for all $n \in \{5,10,15\}$ and all primes p dividing $|J_2|$. The result follows immediately from Lemma 2.3.8 and Table 8.IV. $\square$

We now summarize the above results in the following theorem.

Theorem 8.5.5 *The group J_2 is nX-complementary generated if and only if $nX \in \{5C,5D\}$ or $n \geq 6$.*

Proof. It is proved in Woldar [179] that J_2 is $7A$-complementary generated. The result follows from Lemmas 8.5.1, 8.5.2, 8.5.3 and 8.5.4 and the fact that $(10C)^2 = 5D$, $(10D)^2 = 5C$ and $(12A)^2 = 6A$. $\square$

8.6 nX-Complementary generations of J_3

Before we prove the main theorem of this section, we need the following result.

Lemma 8.6.1 *The group J_3 is $(3A,3B,19X)$- and $(5A,5B,19X)$-generated, where $X \in \{A,B\}$.*

Proof. The proof is similar to that of Lemma 8.3.5 with $\Sigma_{L_1}(3A,3B,19X) = 0$, $\Sigma_{L_1}(5A,5B,19X) = 38$, $\Delta_{J_3}(3A,3B,19X) = 228$ and $\Delta_{J_3}(5A,5B,19X) = 55\,898$. $\square$

Theorem 8.6.2 *The group J_3 is nX-complementary generated if and only if $n > 2$.*

Table 8.V
Structure constants of J_3

pX	$\Delta_{J_3}(pX, 4A, 19A)$
$2A$	247
$3A$	570
$3B$	2052
$5A$	17214
$5B$	17214
$17A$	31236
$17B$	31236
$19A$	31236
$19B$	25080

Proof. We proved in Section 8.2 the group J_3 is $(2A, 3B, 19Y)$-, $(3X, 3X, 19Y)$-, $(5X, 5X, 19Y)$-, $(17X, 17X, 19Y)$- and $(17X, 19Y, 19Z)$-generated, where $X, Y, Z \in \{A, B\}$. Indeed, we have showed that J_3 is $(pX, 19Y, q_{pX}Z)$-generated, for all conjugacy classes pX with elements of prime order. It therefore follows from Lemma 2.3.8 that J_3 is $19Y$-complementary generated, where $Y \in \{A, B\}$. Similar arguments will show that J_3 is $3X$-, $5X$- and $17X$-complementary generated, for $X \in \{A, B\}$. Furthermore, J_3 is not $2A$-complementary generated.

We show next J_3 is $4A$-complementary generated. The group $L_2(19)$ contains no elements of order 4 and therefore $\Delta^*_{J_3}(pX, 4A, 19A) = \Delta_{J_3}(pX, 4A, 19A) > 0$ (*cf.* Table 8.V), for all classes pX with prime order representatives, proving that J_3 is $4A$-complementary generated.

The result now follows from Lemma 2.3.9 since $(6A)^2 = 3A$, $(8A)^2 = 4A$, $(9A)^3 = (9B)^3 = (9C)^3 = 3A$, $(10A)^2 = 5A$, $(10B)^2 = 5A$, $(12A)^2 = 6A$, $(15A)^3 = 5B$ and $(15B)^3 = 5A$. $\square$

Chapter 9

The Fourth Janko Group

9.1 Introduction

In 1976, Z. Janko [90] described the properties of the fourth Janko simple group of order $2^{21}\cdot3^3\cdot5\cdot7\cdot11^3\cdot23\cdot29\cdot31\cdot37\cdot43$, denoted by J_4. Let E be an extra-special 2-subgroup of a finite group G. We say that E is "large" in G if $C_G(E) \subseteq E$ and $E = O_2(C_G(E'))$ (where $O_2(C_G(E'))$ is the normal subgroup generated by all normal 2-subgroups of $C_G(E')$). The majority of the 26 sporadic simple groups possess such large extra-special 2-subgroups. In the process of determining finite simple groups with large extra-special 2-subgroups, Janko discovered the sporadic simple group J_4. More precisely, he proved the following theorem.

Theorem 9.1.1 *(Janko [90]) Let G be a non-abelian finite simple group which possesses an involution z such that $H = C_G(z)$ satisfies the following conditions.*

(i) The subgroup $E = O_2(H)$ is an extra-special 2-group of order 2^{13} and $C_H(E) \subseteq E$.

(ii) A Sylow 3-subgroup P of $O_{2,3}(H)$ (the normal subgroup generated by all normal 2- and 3-subgroups of H) has order 3 and $C_E(P) = Z(E) = \langle z\rangle$.

(iii) The quotient $H/O_{2,3}(H) \cong Aut(M_{22})$, $N_H(P) \neq C_H(P)$, and $P \subseteq (C_H(P))'$.

Then the group G has the following properties.

(1) The order of the group G is $2^{21}{\cdot}3^3{\cdot}5{\cdot}7{\cdot}11^3{\cdot}23{\cdot}29{\cdot}31{\cdot}37{\cdot}43$.

(2) The subgroup $C_H(P)$ is isomorphic to the full covering group of M_{22} (that is, the perfect central extension of a cyclic group of order 6 by M_{22}). Moreover, $N_H(P) = N_G(P) = 6{\cdot}M_{22}{:}2$ and the group G has exactly one conjugacy class of elements of order 3.

(3) Let R be a Sylow 3*-subgroup of H. Then R is an extra-special group of order* 27 *and exponent* 3 *and we have that* $N_G(R) = N_H(R) = (2 \times 3_+^{1+2}{:}8){:}2$.

(4) Let T be a Sylow 2*-subgroup of G. Then T possesses exactly one elementary abelian subgroup V of order* 2^{11}. *We have $C_G(V) \subseteq V$ and $N_G(V) = V{:}K$, where K is isomorphic to M_{24}. The orbits of K on V have lengths* 1, $7{\cdot}11{\cdot}23$ *(with representative z') and* $4{\cdot}3{\cdot}23$ *(with representative t). Here z' is conjugate in G to z and t is not conjugate to z in G.*

(5) The group G has presicely two conjugacy classes of involutions with the representatives z and t. The centralizer $C_G(t)$ is a split extension of an elementary abelian group of order 2^{11} *by $Aut(M_{22})$. Also, $C_G(t)$ acts indecomposably on* $O_2(C_G(t))$.

(6) The group G possesses exactly one conjugacy class of self-centralizing elementary abelian group of order 2^{10} *and if A is one of them, then we have $N_G(A) = A{:}B$, where $B \cong L_5(2)$ and B acts irreducibly on A. The orbits of B on A have lengths* 1, $5{\cdot}31$ *and* $4{\cdot}7{\cdot}31$.

(7) Let Q be a Sylow 5*-subgroup of H. Then Q is also a Sylow* 5*-subgroup of G, and $C_G(Q) = Q \times J$, where J is isomorphic to the non-split extension of an elementary abelian group of order* 8 *by $L_3(2)$. Also $N_G(Q)$contains a Frobenius subgroup of order* 20. *Hence a Sylow* 5*-normalizer in G has order* $2^8{\cdot}3{\cdot}5{\cdot}7$.

(8) Let S be a Sylow 7*-subgroup of H. Then S is also a Sylow* 7*-subgroup of G, and $C_G(S) = S \times I$, where $I \cong S_5$, and $|N_G(S)| = 3{\cdot}|C_G(S)|$. Hence the Sylow* 7*-normalizer in G has order* $2^3{\cdot}3^2{\cdot}5{\cdot}7$.

(9) The group G possesses a special 2*-subgroup L of order* 2^{15} *with* $|Z(L)| = 2^3$ *so that $N_G(L)/L \cong S_5 \times L_3(2)$. Also, $N_G(L)$ contains subgroups isomorphic to S_5*

and $L_3(2)$, and it contains both a Sylow 5-normalizer and a Sylow 7-normalizer of G.

(10) A Sylow 11-normalizer in G has order $2^4{\cdot}3{\cdot}5{\cdot}11^3$ and contains a subgroup isomorphic to $GL_2(3)$. A Sylow 11-subgroup of G is extra-special of order 11^3 and exponent 11. The group G has exactly two conjugacy classes of elements of order 11.

(11) A Sylow p-subgroup is self-centralizing in G for $p = 23, 29, 31, 37$ and 43. A Sylow p-normalizer has order $23{\cdot}22$, $29{\cdot}28$, $31{\cdot}10$, $37{\cdot}12$ and $43{\cdot}14$, respectively.

(12) The group G possesses $PGL(2, 23)$ as a subgroup.

(13) The group G has exactly 62 conjugacy classes of elements. The character table of G is unique and was computed by J. Conway, S. Norton, J. G Thompson and D. Hunt (cf. ATLAS). □

The group was constructed in 1980 by D. Benson, J. Conway, S. P. Norton, R. A. Parker and J. Thackray (*cf.* [8] and [137]) as a group of 112×112 matrices over $GF(2)$. The subgroup structure of J_4 are discussed by Kleidman-Wilson [102]. The only non-abelian characteristically simple subgroups of J_4 are A_5, A_6, A_7, A_8, $L_2(7)$, $L_2(11)$, $L_2(23)$, $L_2(32)$, $L_3(4)$, $U_3(3)$, $U_3(11)$, $L_5(2)$, M_{11}, M_{12}, M_{22}, M_{23} and M_{24}.

Theorem 9.1.2 *(Kleidman-Wilson [102]) The fourth Janko simple group J_4 has exactly thirteen conjugacy classes of maximal subgroups, as follows:*

(A) Five classes of non-local subgroups:
$U_3(11){:}2$, $M_{22}{:}2$, $L_2(32){:}5$, $L_2(23){:}2$, *and* $U_3(3)$.

(B) Eight classes of lacal subgroups:
$2^{11}{:}M_{24}$, $2^{10}{:}L_5(2)$, $2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2$, $2^{3+12}.(S_5 \times L_3(2))$, $11_+^{1+2}{:}(5 \times 2S_4)$, $29{:}28$, $37{:}12$ *and* $43{:}14$. □

The (p, q, r)-generations of J_4 with $r = 7$, 11 and $r > 11$ will be discussed in Sections 9.2, 9.3 and 9.4, respectively. The nX-complementary generation of J_4 will be considered in Section 9.5.

9.2 $(p, q, 7)$-Generations of J_4

Lemma 9.2.1 *The group J_4 is $(2X, 3A, 7Y)$-generated, for $X, Y \in \{A, B\}$, if and only if $X = B$.*

Proof. We first consider the triple $(2A, 3A, 7Y)$. Let $L \cong L_2(7)$ be contained in the conjugacy class of subgroups with non-empty intersection with the class $2A$ and $N_{J_4}(L) = L$ (*cf.* Kleidman-Wilson [102], Proposition 5.4.2). Further, let $x \in L$ a fixed element of order 7. Then the fusion map of L into J_4 yields

$$2a \to 2A, \qquad 3a \to 3A, \qquad 7a \to 7A, \qquad 7b \to 7B.$$

Since $|C_L(x)| = 7$ and $|C_{J_4}(x)| = 840$, it follows from Theorem 3.1.4 that x is contained in exactly 120 conjugates of L. We note that no maximal subgroup of $L_2(7)$ has order divisible by $2 \times 3 \times 7$ and hence no proper subgroup of L is $(2, 3, 7)$-generated. Therefore

$$\begin{aligned} \Delta^*(J_4) &\leq \Delta(J_4) - 120\,\Sigma(L_2(7)) \\ &= 1435 - 120(7) = 595 < |C_{J_4}(x)|. \end{aligned}$$

Thus by Lemma 3.1.7 we conclude that J_4 is not $(2A, 3A, 7Y)$-generated.

Next we consider the triple $(2B, 3A, 7Y)$. We calculate the structure constant $\Delta(J_4) = 14889$. The maximal subgroups of J_4 with non-empty intersection with all the conjugacy classes in this triple are, up to isomorphisms, $2^{11}{:}M_{24}$, $2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2$, $2^{10}{:}L_5(2)$ and $2^{3+12}.(S_5 \times L_3(2))$. Our calculations give

$$\begin{aligned} \Delta^*(J_4) &\geq \Delta(J_4) - 5\,\Sigma(2^{11}{:}M_{24}) - 10\,\Sigma(2^{10}{:}L_5(2)) - 10\,\Sigma(2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2) \\ &\quad - \Sigma(2^{3+12}.(S_5 \times L_3(2))) \\ &= 14889 - 5(385) - 10(189) - 10(217) - 154 = 6139, \end{aligned}$$

and therefore J_4 is $(2B, 3A, 7Y)$-generated. $\square$

The $(2B, 3A, 7Y)$-generation of J_4 was first proved by Woldar [179], in order show that J_4 is a Hurwitz group. The original proof is quite extensive and the above is an alternative proof.

Lemma 9.2.2 *The group J_4 is $(2X, 5A, 7Y)$-generated, for all $X, Y \in \{A, B\}$.*

Proof. We treat the two cases $X = A$ and $X = B$ separately.

Case $(2A, 5A, 7Y)$: The maximal subgroups of J_4 with order divisible by 7 and non-empty intersection with the conjugacy classes $2A$ and $5A$ are isomorphic to $2^{11}{:}M_{24}$, $2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2$, $2^{10}{:}L_5(2)$, $2^{3+12}.(S_5 \times L_3(2))$ and $M_{22}{:}2$. We calculate

$$\begin{aligned}\Delta^*(J_4) &\geq \Delta(J_4) - 5\Sigma(2^{11}{:}M_{24}) - 10\Sigma(2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2) - 10\Sigma(2^{10}{:}L_5(2)) \\ &\quad - \Sigma(2^{3+12}.(S_5 \times L_3(2))) - 60\Sigma(M_{22}{:}2) \\ &= 517440 - 5(3360) - 10(252) - 10(2688) - 0 - 60(224) = 457800,\end{aligned}$$

and therefore J_4 is $(2A, 5A, 7Y)$-generated.

Case $(2B, 5A, 7Y)$: The maximal subgroups of J_4 with non-empty intersection with all the conjugacy classes in this triple are isomorphic to $2^{11}{:}M_{24}$, $2^{10}{:}L_5(2)$, $2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2$ and $2^{3+12}.(S_5 \times L_3(2))$. We calculate

$$\begin{aligned}\Delta^*(J_4) &\geq \Delta(J_4) - 5\Sigma(2^{11}{:}M_{24}) - 10\Sigma(2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2) - 10\Sigma(2^{10}{:}L_5(2)) \\ &\quad - \Sigma(2^{3+12}.(S_5 \times L_3(2))) \\ &= 7415800 - 5(9744) - 10(488) - 10(6720) - 0 > 0,\end{aligned}$$

and generation of J_4 by this triple follows. This completes the proof. $\square$

Lemma 9.2.3 *The group J_4 is $(3A, 5A, 7Y)$-generated, for all $Y \in \{A, B\}$.*

Proof. Up to isomorphisms, $2^{11}{:}M_{24}$, $2^{10}{:}L_5(2)$, $2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2$, $2^{3+12}.(S_5 \times L_3(2))$ and $M_{22}{:}2$ are the only maximal subgroups that may contain $(3A, 5A, 7Y)$-generated proper subgroups. Now

$$\Sigma(2^{11}{:}M_{24}) = 274176, \qquad \Sigma(2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2) = 19712, \qquad \Sigma(2^{10}{:}L_5(2)) = 84672,$$
$$\Sigma(2^{3+12}.(S_5 \times L_3(2))) = 0 \text{ and } \Sigma(M_{22}{:}2) = 2464.$$

The $(3A, 5A, 7Y)$-generation of J_4 now follows since $\Delta(J_4) = 4753175840$. $\square$

9.3 $(p, q, 11)$-Generations of J_4

The investigation of the $(p, q, 11)$-generation of J_4 will require knowledge of all the maximal subgroups of J_4 with order divisible by 11. They are, up to isomorphisms, $2^{11}{:}M_{24}$, $2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2$, $U_3(11){:}2$, $11_+^{1+2}{:}(5 \times 2S_4)$, $L_2(32){:}5$, $L_2(23){:}2$ and $M_{22}{:}2$.

Lemma 9.3.1 *The group J_4 is $(2X, 3A, 11Y)$-generated, for $X, Y \in \{A, B\}$, if and only if the ordered pair $(X, Y) = (B, B)$.*

Proof. We treat each case separately.

Case $(2A, 3A, 11A)$*:* We calculate $\Delta(J_4) = 3993 < |C_{J_4}(11A)| = 31944$. Thus from Lemma 3.1.7 it follows that J_4 is not $(2A, 3A, 11A)$-generated.

Case $(2B, 3A, 11A)$*:* The structure constant $\Delta(J_4) = 33275$ and $\Sigma(L_2(32)) = 33$. Let $H \leq J_4$ with $H \cong L_2(32)$. If $x \in H$ is a fixed element of order 11, then x is contained in 968 conjugates of H. The maximal subgroups of H with order divisible by $2 \times 3 \times 11$ are isomorphic to the dihedral group D_{66}. Futhermore, $\Sigma(D_{66}) = 0$ and therefore $\Sigma^*(L_2(32)) = 33$. Thus

$$\Delta^*(J_4) \leq \Delta(J_4) - 968\,\Sigma^*(H) = 1331 < |C_{J_4}(11A)|,$$

and non-generation of J_4 by this triple follows.

Case $(2A, 3A, 11B)$*:* As stated earlier the group J_4 possesses a 112-dimensional irreducible representation V over $GF(2)$. From this representation we can generate J_4 by two 112×112 matrices a and b (over $GF(2)$), where $o(a) = 2$, $o(b) = 4$ and $o(ab) = 37$. Using MeatAxe and GAP we calculate

$$dim(V/C_V(2A)) = 50, \qquad dim(V/C_V(3A)) = 72, \qquad dim(V/C_V(11B) = 100.$$

Since $50 + 72 + 100 = 222 < 224$, it follows from Scott's theorem that J_4 is not $(2A, 3A, 11B)$-generated.

Case $(2B, 3A, 11B)$*:* The maximal subgroups of J_4 with non-empty intersection with the classes $2B$, $3A$ and $11B$ are isomorphic to $2^{11}{:}M_{24}$, $U_3(11){:}2$, $11_+^{1+2}{:}(5 \times 2S_4)$ and $L_2(23){:}2$. We calculate $\Delta(J_4) = 18755$, $\Sigma(2^{11}{:}M_{24}) = 715$ and $\Sigma(U_3(11){:}2) = \Sigma(11_+^{1+2}{:}(5 \times 2S_4)) = \Sigma(L_2(23){:}2) = 0$. Moreover, a fixed element of order 11 is contained in exactly 11 conjugate of a $2^{11}{:}M_{24}$ subgroup. Thus $\Delta^*(J_4) \geq 18755 - 11(715) > 0$ and whence $(2B, 3A, 11B)$ is a generating triple for J_4. This completes the proof. $\square$

Lemma 9.3.2 *The group J_4 is $(2X, 5A, 11Y)$-generated, for all $X, Y \in \{A, B\}$.*

Proof. Case $(2X, 5A, 11A)$: The maximal subgroups of J_4 with non-empty intersection with all the classes in this triple are, up to isomorphisms, $2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2$, $U_3(11){:}2$, $11_+^{1+2}{:}(5\times 2S_4)$ and $L_2(32){:}5$. We calculate $\Sigma(U_3(11){:}2) = \Sigma(11_+^{1+2}{:}(5\times 2S_4)) = \Sigma(L_2(32){:}5) = 0$. Therefore

$$\begin{aligned}
\Delta^*_{J_4}(2A, 5A, 11A) &\geq \Delta(J_4) - 121\,\Sigma(2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2) \\
&= 742698 - 121(66) = 734712, \\
\Delta^*_{J_4}(2B, 5A, 11A) &\geq \Delta(J_4) - 121\,\Sigma(2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2) \\
&= 5337310 - 121(22) = 5334648,
\end{aligned}$$

and generation by these triples follows.

Case $(2X, 5A, 11B)$: The only maximal subgroups that may contain $(2X, 5A, 11B)$-generated proper subgroups are isomorphic to $2^{11}{:}M_{24}$, $U_3(11){:}2$, $11_+^{1+2}{:}(5\times 2S_4)$ and $M_{22}{:}2$. Moreover, $\Sigma(11_+^{1+2}{:}(5\times 2S_4)) = 0$ and therefore

$$\begin{aligned}
\Delta^*_{J_4}(2A, 5A, 11B) &\geq \Delta(J_4) - 11\,\Sigma(2^{11}{:}M_{24}) - 3\,\Sigma(U_3(11){:}2) - 22\,\Sigma(M_{22}{:}2) \\
&= 916696 - 11(2552) - 3(726) - 22(176) > 0, \\
\Delta^*_{J_4}(2B, 5A, 11B) &\geq \Delta(J_4) - 11\,\Sigma(2^{11}{:}M_{24}) - 3\,\Sigma(U_3(11){:}2) - 22\,\Sigma(M_{22}{:}2) \\
&= 6340884 - 11(6424) - 0 - 0 > 0.
\end{aligned}$$

Thus J_4 is $(2X, 5A, 11B)$-generated, and the result follows. $\square$

Lemma 9.3.3 *The group J_4 is $(3A, 5A, 11Y)$-generated, for all $X, Y \in \{A, B\}$.*

Proof. Case $(3A, 5A, 11A)$: The maximal subgroups with non-empty intersection with all these classes are, up to isomophisms, $2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2$, $U_3(11){:}2$, $11_+^{1+2}{:}(5\times 2S_4)$ and $L_2(32){:}5$. We calculate $\Sigma(11_+^{1+2}{:}(5\times 2S_4)) = \Sigma(L_2(32){:}5) = 0$ and

$$\begin{aligned}
&121\,\Sigma(2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2) + 3\Sigma(U_3(11){:}2) \\
&= 121(19712) + 3(2662) = 2393138.
\end{aligned}$$

Since $\Delta(J_4) = 5139555344$, it follows that $\Delta^*(J_4) > 0$ and therefore J_4 is $(3A, 5A, 11A)$-generated.

Case $(3A, 5A, 11B)$: The maximal subgroups with non-empty intersection with the classes $3A$, $5A$ and $11B$ are isomorphic to $2^{11}{:}M_{24}$, $U_3(11){:}2$, $11_+^{1+2}{:}(5 \times 2S_4)$ and $M_{22}{:}2$. Also $\Sigma(11_+^{1+2}{:}(5 \times 2S_4)) = 0$, and

$$\begin{aligned}
\Delta^*(J_4) &\geq \Delta(J_4) - 11\,\Sigma(2^{11}{:}M_{24}) - 3\,\Sigma(U_3(11){:}2) - 22\,\Sigma(M_{22}{:}2) \\
&= 5285060264 - 11(284944) - 3(26620) - 22(2144) = 5281798852,
\end{aligned}$$

and the result folows. $\square$

Lemma 9.3.4 *The group J_4 is $(2X, 7Y, 11Z)$-, $(3A, 7Y, 11Z)$- and $(5A, 7Y, 11Z)$-generated, for all $X, Y, Z \in \{A, B\}$.*

Proof. The maximal subgroups of J_4 with order divisible by 7×11 are isomorphic to $2^{11}{:}M_{24}$, $2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2$ and $M_{22}{:}2$. Also $2^{11}{:}M_{24} \cap 11A = \emptyset = M_{22}{:}2 \cap 11A$, $2_+^{1+12}{\cdot}3{\cdot}M_{22}{:}2 \cap 11B = \emptyset$ and $M_{22}{:}2 \cap 2B = \emptyset$. Therefore

$$\begin{aligned}
\Delta^*_{J_4}(2A, 7Y, 11A) &\geq 4557344 - 121(3608) = 4120776, \\
\Delta^*_{J_4}(2A, 7Y, 11B) &\geq 4365196 - 11(6600) - 22(176) = 4288724, \\
\Delta^*_{J_4}(2B, 7Y, 11A) &\geq 57925120 - 121(6952) = 57083928 \\
\Delta^*_{J_4}(2B, 7Y, 11B) &\geq 57792020 - 11(17688) - 0 = 57597452 \\
\Delta^*_{J_4}(3A, 7Y, 11A) &\geq 38254060480 - 121(223168) = 38227057152 \\
\Delta^*_{J_4}(3A, 7Y, 11B) &\geq 38232329344 - 11(749056) - 22(1760) = 38224051008, \\
\Delta^*_{J_4}(5A, 7Y, 11A) &\geq 15300853706336 - 121(538208) = 15300788583168, \\
\Delta^*_{J_4}(5A, 7Y, 11B) &\geq 15303576037904 - 11(7834112) - 22(12672) \\
&= 15303489583888.
\end{aligned}$$

Therefore these triples generate J_4, proving the result. $\square$

Table 9.I
Structure Constants

qY	$\Delta_{J_4}(2A, qY, 23A)$	$\Delta_{J_4}(2B, qY, 23A)$	$\Delta_{J_4}(3A, qY, 23A)$
$3A$	621	18055	–
$5A$	451030	7461476	4638610928
$7Y$	4915606	56620618	39137221728
$11A$	108445	1510226	987224032
$11B$	13917277	203065896	130650349632
qY	$\Sigma_{2^{11}:M_{24}}(2A, qY, 23A)$	$\Sigma_{2^{11}:M_{24}}(2B, qY, 23A)$	$\Sigma_{2^{11}:M_{24}}(3A, qY, 23A)$
$3A$	161	437	–
$5A$	2070	4002	286304
$7Y$	7820	13892	1750208
$11A$	0	0	0
$11B$	48576	110400	5581824
qY	$\Sigma_{L_2(23):2}(2A, qY, 23A)$	$\Sigma_{L_2(23):2}(2B, qY, 23A)$	$\Sigma_{L_2(23):2}(3A, qY, 23A)$
$3A$	23	0	–
$5A$	0	0	0
$7Y$	0	0	0
$11A$	0	0	0
$11B$	115	0	230
qY	$\Delta_{J_4}(5A, qY, 23A)$	$\Delta_{J_4}(7X, qY, 23A)$	$\Delta_{J_4}(11A, qY, 23A)$
$7Y$	15406480371640	–	–
$11A$	400689589224	3239401364064	–
$11B$	52881475189488	427529654927936	11156081945921
qY	$\Sigma_{2^{11}:M_{24}}(5A, qY, 23A)$	$\Sigma_{2^{11}:M_{24}}(7X, qY, 23A)$	$\Sigma_{2^{11}:M_{24}}(11A, qY, 23A)$
$7Y$	6217728	–	–
$11A$	0	0	–
$11B$	47857664	138674176	0
qY	$\Sigma_{L_2(23):2}(5A, qY, 23A)$	$\Sigma_{L_2(23):2}(7X, qY, 23A)$	$\Sigma_{L_2(23):2}(11A, qY, 23A)$
$7Y$	0	–	–
$11A$	0	0	–
$11B$	0	0	0

9.4 (p, q, r)-Generations of J_4

In this section we consider the (p, q, r)-generation of J_4, where $r > 11$ is a prime divisor of $|J_4|$.

Lemma 9.4.1 *The group J_4 is $(pX, qY, 23A)$-generated, for all distinct conjugacy classes pX and qY, where $p \leq q$ are primes and $q > 2$.*

Proof. Let pX and qY be any two distinct conjugacy classes, where $p \leq q$ are odd primes. The maximal subgroups of J_4 containing elements of order 23 are isomorphic to $2^{11}{:}M_{24}$ or $L_2(23){:}2$. Therefore

$$\Delta^*_{J_4}(pX, qY, 23A) \geq \Delta(J_4) - 2\,\Sigma(2^{11}{:}M_{24}) - \Sigma(L_2(23){:}2).$$

It now follows from this relation and Table 9.I that $\Delta^*_{J_4}(pX, qY, 23A) > 0$, for all distinct classes pX and qY containing prime order elements and $q > 2$. Thus J_4 is $(pX, qY, 23A)$-generated. □

Table 9.II
Structure Constants

qY	$\Delta_{J_4}(2A, qY, 29A)$	$\Delta_{J_4}(2B, qY, 29A)$	$\Delta_{J_4}(3A, qY, 29A)$
$3A$	348	19314	-
$5A$	458026	7470342	4638228912
$7Y$	4906916	56470540	39112667488
$11A$	105473	1538624	989959776
$11B$	13921943	203382916	130697077504
$23A$	187115511	2039767316	1439185637376
qY	$\Delta_{J_4}(5A, qY, 29A)$	$\Delta_{J_4}(7X, qY, 29A)$	$\Delta_{J_4}(11A, qY, 29A)$
$7Y$	15407517421568	-	-
$11A$	400522116224	3238688151040	-
$11B$	52878769081600	427512922845184	11158290168703
$23A$	563967038373888	4488114677481472	118462964398447
qY	$\Delta_{J_4}(11B, qY, 29A)$		
$23A$	15637110781392837		

Lemma 9.4.2 *The group J_4 is $(pX, qY, 29A)$-generated, for all distinct conjugacy classes pX and qY, where $p \leq q$ are primes and $q > 2$.*

Proof. The only maximal subgroups of J_4 with order divisible by 29 are isomorphic to 29:28. Since 29:28 has a soluble quotient, it follows from Lemma 3.1.10 that 29:28 is not $(pX, qY, 29A)$-generated and hence $\Sigma(29{:}28) = 0$ for all conjugacy classes pX and qY containing prime order elements. Therefore $\Delta^*_{J_4}(pX, qY, 29A) = \Delta_{J_4}(pX, qY, 29A) > 0$ (*cf.* Table 9.II), for all the triples in the statement of the lemma, and the result follows. □

Table 9.III
Structure Constants

qY	$\Delta_{J_4}(2A, qY, 31Z)$	$\Delta_{J_4}(2B, qY, 31Z)$	$\Delta_{J_4}(3A, qY, 31Z)$
$3A$	1240	17856	-
$5A$	585218	7101976	4850352008
$7Y$	4731158	56863114	38817663328
$11A$	123659	1496246	1020640032
$11B$	16370387	197346248	134723982272
$23A$	173062739	2039767316	1417768290752
$29A$	137252283	2076797880	1124436888576
qY	$\Sigma_{2^{10}:L_5(2)}(2A, qY, 31Z)$	$\Sigma_{2^{10}:L_5(2)}(2B, qY, 31Z)$	$\Sigma_{2^{10}:L_5(2)}(3A, qY, 31Z)$
$3A$	62	186	-
$5A$	2232	5704	80352
$7Y$	744	2232	51584
$11A$	0	0	0
$11B$	0	0	0
$23A$	0	0	0
$29A$	0	0	0
qY	$\Sigma_{L_2(32):5}(2A, qY, 31Z)$	$\Sigma_{L_2(32):5}(2B, qY, 31Z)$	$\Sigma_{L_2(32):5}(3A, qY, 31Z)$
$3A$	0	31	-
$5A$	0	0	0
$7Y$	0	0	0
$11A$	0	155	155
$11B$	0	0	0
$23A$	0	0	0
$29A$	0	0	0

Lemma 9.4.3 *The group J_4 is $(pX, qY, 31Z)$-generated, for all distinct conjugacy classes pX and qY, where $p \leq q$ are primes.*

Proof. The maximal subgroups of J_4 containing elements of order 31 are, up to isomorphisms, $2^{10}{:}L_5(2)$ and $L_2(32){:}5$. Therefore for all these triples we have

$$\Delta^*(J_4) \geq \Delta(J_4) - 2\,\Sigma(2^{10}{:}L_5(2)) - \Sigma(L_2(32){:}5).$$

It is evident from Table 9.III that $\Delta^*(J_4) > 0$, for all these triples, proving generation of J_4 by these triples. □

Table 9.III (Cont.)

qY	$\Delta_{J_4}(5A, qY, 31Z)$	$\Delta_{J_4}(7X, qY, 31Z)$	$\Delta_{J_4}(11A, qY, 31Z)$
$7Y$	15372586954808	–	–
$11A$	404233356328	3233911632480	–
$11B$	53358910283888	426876352051776	11225110434879
$23A$	561436150052976	4491489176657472	118108285918767
$29A$	445276930754048	3562215560183808	93672090722931
qY	$\Sigma_{2^{10}:L_5(2)}(5A, qY, 31Z)$	$\Sigma_{2^{10}:L_5(2)}(7X, qY, 31Z)$	$\Sigma_{2^{10}:L_5(2)}(11A, qY, 31Z)$
$7Y$	2027648	–	–
$11A$	0	0	–
$11B$	0	0	0
$23A$	0	0	0
$29A$	0	0	0
qY	$\Sigma_{L_2(32):5}(5A, qY, 31Z)$	$\Sigma_{L_2(32):5}(7X, qY, 31Z)$	$\Sigma_{L_2(32):5}(11A, qY, 31Z)$
$7Y$	0	–	–
$11A$	0	0	–
$11B$	0	0	0
$23A$	0	0	0
$29A$	0	0	0
qY	$\Delta_{J_4}(11B, qY, 31Z)$	$\Delta_{J_4}(23A, qY, 31Z)$	
$23A$	15590293070834757	–	
$29A$	12364715158080241	130098307677359105	
qY	$\Sigma_{2^{10}:L_5(2)}(11B, qY, 31Z)$	$\Sigma_{2^{10}:L_5(2)}(23A, qY, 31Z)$	
$23A$	0	–	
$29A$	0	0	
qY	$\Sigma_{L_2(32):5}(11B, qY, 31Z)$	$\Sigma_{L_2(32):5}(23A, qY, 31Z)$	
$23A$	0	–	
$29A$	0	0	

Lemma 9.4.4 *The group J_4 is $(pX, qY, 37Z)$-generated, for all distinct conjugacy classes pX and qY, where $p \leq q$ are primes and $q > 2$.*

Proof. The maximal subgroups of J_4 containing elements of order 37 are, up to isomorphisms, $U_3(11){:}2$ and $37{:}12$. Similar to our discussion in Lemma 9.4.2, we have $\Sigma(37{:}12) = 0$ and therefore for all these triples

$$\Delta^*(J_4) \geq \Delta(J_4) - 2\,\Sigma(U_3(11){:}2).$$

We calculate $\Delta_{J_4}(31X, 31Y, 37Z) = 90297160343250377$, for all $X, Y \in \{A, B, C\}$ and $\Sigma(U_3(11){:}2) = 0$. It now follows from Table 9.IV that $\Delta^*(J_4) > 0$, for all these triples, proving generation of J_4 by these triples. $\square$

Table 9.IV
Structure Constants

qY	$\Delta_{J_4}(2A, qY, 37Z)$	$\Delta_{J_4}(2B, qY, 37Z)$	$\Delta_{J_4}(3A, qY, 37Z)$
$3A$	2775	15577	-
$5A$	742516	6790686	5072686976
$7Y$	4534720	57289024	38526261248
$11A$	144633	1447884	1051593280
$11B$	19176915	191320932	138814310144
$23A$	159229315	2110989860	1396493919744
$29A$	126424375	1674630804	1107572629504
$31Y$	128404837	1540858044	1051892424704
qY	$\Sigma_{U_3(11):2}(2A, qY, 37Z)$	$\Sigma_{U_3(11):2}(2B, qY, 37Z)$	$\Sigma_{U_3(11):2}(3A, qY, 37Z)$
$3A$	111	0	-
$5A$	666	0	24642
$7Y$	0	0	0
$11A$	0	0	222
$11B$	333	0	12210
$23A$	0	0	0
$29A$	0	0	0
$31Y$	0	0	0
qY	$\Delta_{J_4}(5A, qY, 37Z)$	$\Delta_{J_4}(7X, qY, 37Z)$	$\Delta_{J_4}(11A, qY, 37Z)$
$7Y$	15337964134400	-	-
$11A$	407886901248	3229054205952	-
$11B$	53841070964736	426235155185664	11292643980331
$23A$	558909157900288	4494857809690264	117753848433883
$29A$	443272697708544	3564887889936384	93391057407791
$31Y$	416549400182784	3332395201462272	87628725367117

Table 9.IV (Cont.)

qY	$\Sigma_{U_3(11):2}(5A, qY, 37Z)$	$\Sigma_{U_3(11):2}(7X, qY, 37Z)$	$\Sigma_{U_3(11):2}(11A, qY, 37Z)$
$7Y$	0	-	-
$11A$	666	0	-
$11B$	87912	0	333
$23A$	0	0	0
$29A$	0	0	0
$31Y$	0	0	0
qY	$\Delta_{J_4}(11B, qY, 37Z)$	$\Delta_{J_4}(23A, qY, 37Z)$	$\Delta_{J_4}(29A, qY, 37Z)$
$23A$	15543510806981993	-	-
$29A$	12327611758700293	130293503609950549	-
$31Y$	11566991620054607	121704868165906751	96524550711750403
qY	$\Sigma_{U_3(11):2}(11B, qY, 37Z)$	$\Sigma_{U_3(11):2}(23A, qY, 37Z)$	$\Sigma_{U_3(11):2}(29A, qY, 37Z)$
$23A$	0	-	-
$29A$	0	0	-
$31Y$	0	0	0

Lemma 9.4.5 *The group J_4 is $(pX, qY, 43Z)$-generated, for all distinct conjugacy classes pX and qY, where $p \leq q$ are primes and $q > 2$.*

Proof. The only maximal subgroups of J_4 with order divisible by 43 are isomorphic to 43:14. Since 43:14 has a soluble quotient, it follows from Lemma 3.1.10 that $\Sigma(43{:}14) = 0$ for all conjugacy classes pX and qY containing prime order elements. We note from Table 9.V that $\Delta^*_{J_4}(pX, qY, 43A) = \Delta_{J_4}(pX, qY, 43A) > 0$, for all the triples in the statement of the lemma, and the result follows. □

We are now ready to state one of the main result in this chapter.

Theorem 9.4.6 *The Janko group J_4 is (p, q, r)-generated for all $p, q, r \in \{2, 3, 5, 7, 11, 23\}$ with $p < q < r$, except wher $(p, q, r) = (2, 3, 5)$.*

Proof. The proof follows from the lemmas proved in this chapters and the fact that the triangular group $T(2, 3, 5) \cong A_5$. □

Corollary 9.4.7 *The group J_4 is (pX, pX, qY)-generated for all odd primes $p < q$.*

Proof. This follows immediately from an application of Lemma 3.1.15 to the results in sections 9.2 to 9.4. □

Table 9.V
Structure Constants

qY	$\Delta_{J_4}(2A, qY, 43Z)$	$\Delta_{J_4}(2B, qY, 43Z)$	$\Delta_{J_4}(3A, qY, 43Z)$
$3A$	1118	19608	-
$5A$	586004	7071436	4845943136
$7Y$	4733956	56776684	38801707040
$11A$	124055	1511880	1023541728
$11B$	16385881	197575884	134774400512
$23A$	173311113	2073590892	1417341820928
$29A$	137395621	1646083516	1124351483904
$31Y$	128604271	1540975318	1051951119360
$37Y$	107623969	1290981260	881315274752
qY	$\Delta_{J_4}(5A, qY, 43Z)$	$\Delta_{J_4}(7X, qY, 43Z)$	$\Delta_{J_4}(11A, qY, 43Z)$
$7Y$	15373194571776	-	-
$11A$	404128866304	3233159261696	-
$11B$	53356774293504	426859549343744	11227855342721
$23A$	561447296270336	4491550031872000	118098018600593
$29A$	445276945022976	3562200556896265	93674297355661
$31Y$	416551275241472	3332397088145408	87628875437031
$37Y$	349000783986688	2792007308935168	73418660015849
qY	$\Delta_{J_4}(11B, qY, 43Z)$	$\Delta_{J_4}(23A, qY, 43Z)$	$\Delta_{J_4}(29A, qY, 43Z)$
$23A$	15590084680075323	-	-
$29A$	12364777698047247	130098307218469887	-
$31Y$	11567011172473581	121704868043084733	96524550809160969
$37Y$	9691263229674499	101968950013794867	80871920784987431
qY	$\Delta_{J_4}(31X, qY, 43Z)$	$\Delta_{J_4}(37X, qY, 43Z)$	
$31Y$	90297109693149531	-	
$37Y$	75654377508536629	63386103646837684	

9.5 nX-Complementary generations of J_4

We immediately prove the main result in this section.

Theorem 9.5.1 *The Janko group J_4 is nX-complementary generated if and only if $n > 2$.*

Table 9.VI
Structure Constants

pX	$\Delta_{J_4}(pX, 4A, 37A)$	$\Delta_{J_4}(pX, 4B, 37A)$	$\Delta_{J_4}(pX, 4C, 37A)$
$2A$	185	60828	66896
$2B$	10767	460576	1154400
$3A$	5062932	357396172	723809392
$5A$	2271440064	134347811632	296192040064
$7X$	19262751744	1046886709504	2407377074176
$11A$	485932544	28051642944	62591705344
$11B$	64150420032	3702833982144	8262358917120
$23A$	709326976320	38091608783552	88119142599680
$29A$	562575925824	30210548153280	69887878849536
$31X$	517728302784	28475056653120	65085843778560
$37X$	426665984256	24037623515392	54285945874432
$43X$	373243698240	20528464402880	46922413491200

Proof. Let p be any fixed odd prime dividing $|J_4|$. Then we showed in the previous sections that if qY is any conjugacy class of J_4 with prime order elements, then J_4 is $(pX, qY, 43A)$-generated. Thus by Lemma 2.3.8 we have that J_4 is pX-complementary generated, p a prime, if and only if p is an odd prime divisor of $|J_4|$.

The group J_4 contains no maximal (hence proper) subgroup with element of order 4 and 37. Therefore for any conjugacy class qY of J_4 with prime order elements $\Delta^*_{J_4}(qY, 4X, 37A) = \Delta(J_4)$, where $X \in \{A, B, C\}$. From Table 9.VI we observe that this value is positive for all cases and therefore J_4 is $4X$-complementary generated.

It is clear (*cf.* ATLAS) that if nX is any of the remaining conjugacy classes of J_4, then there is a positive integer m such that $(nX)^m$ is a class with elements of order 4 or p, where p is an odd prime. Thus the result follows from Lemma 2.3.9. □

Chapter 10

The Smallest Fischer Group

10.1 Introduction

The sporadic simple group Fi_{22} of order $2^{17}\cdot3^{9}\cdot5^{2}\cdot7\cdot11\cdot13$ was discovered by Fischer [63] in "Finite Groups Generated by Transpositions. I". It is generated by a conjugacy class D of involutions, called *3-transpositions*, any non-commuting pair of which has product of order 3. The Fischer group Fi_{22} is closely related to the Mathieu group M_{22} and Conway [41] used this relation to construct Fi_{22}.

For almost all purposes, M_{22} is best studied as a subgroup of M_{24}. The Matieu group M_{24} is a 5-transitive group on a set Ω of 24 letters. Defining the sum of two subsets of Ω as their symmetric difference, we obtain a 24-dimensional vector space over $GF(2)$, in which M_{24} leaves invariant a 12-dimensional subspace $\mathcal{C}$ (the Golay 24-code) of $\mathcal{C}$-sets, namely $\emptyset$ and Ω, together with 759 *octads* (8-element sets) and their complements, and 2576 *dodecads* (12-element sets).

We obtain the subgroup M_{22} by fixing the two points 0 and ∞ of Ω. There are only 77 special *hexads* (6-element sets) each of which together with $\{0,\infty\}$ from an octad and any one hexad is disjoint from just 16 others. Two disjoint special hexads defines a partition of Ω as $2+6+6+10$ in such a way that the union of any two parts is a $\mathcal{C}$-set, and the stabilizer of this partition in M_{24} is a group S_6. In the group S_6, which permutes 6 letters i,j,k,l,m,n, there are exactly 12 subgroups of index 6, namely the 6 subgroups S_5 fixing one letter each, and 6 further subgroups which permutes

$\{i, j, k, l, m, n\}$ in the way that $PGL_2(5)$ permutes the symbols $\{\infty, 0, 1, 2, 3, 4, \}$, by linear fractional transformations. The group $PGL_2(5)$ in which i, j, k, l, m, n play the respective roles of $\infty, 0, 1, 2, 3, 4$ will be denoted by $G(i|jklmn)$, where the last five letters is a 5-cycle which can be rotated or replaced by its powers. The group S_6 has an automorphism which interchanges the 6 subgroups S_5 with the 6 subgroups $PGL_2(5)$.

Now partition the set Ω as $2+6+6+10$ so that the union of any two parts is a $\mathcal{C}$-set, and let the 2-element part be $\{0, \infty\}$ and 6 element parts $\{i, j, k, l, m, n\}$ and $\{u, v, w, x, y, z\}$. Then the subgroup S_6 of M_{24} which fixes the partition acts on Ω as follows:

(i) Even permutations of S_6 fix 0 and ∞; odd permutations interchange them.

(ii) The stabilizer of any of u, v, w, x, y, z is one of the 6 groups like $G(i|jklmn)$.

(iii) There are 10 different $3+3$ partitions of $\{i, j, k, l, m, n\}$. The stabilizer of any of the remaining 10 points of Ω fixes one of the $3+3$ partitions of $\{i, j, k, l, m, n\}$.

A maximal commuting set of transpositions of D contains 22 elements, generating a group of order 2^{10} which is self-centralizing in Fi_{22}. We call such a maximal set of involutions the *basic set of transpositions.* The normalizer in Fi_{22} of the group they generate is a split extension $2^{10}{:}M_{22}$, whose orbits on the conjugacy class D are:

(i) $A =$ the 22 basic transpositions,

(ii) $B =$ the set of $2^5{\cdot}77$ transpositions each commuting with just 6 basic transpositions forming a hexad,

(iii) $C =$ the set of 2^{10} transpositions commuting with no one of the 22 basic transpositions.

The group Fi_{22} is generated by the 22 basic transpositions, denoted as $i, j, k, \ldots$ (typical element t), together with a further transposition s from the orbit C. Thus the action of s and the typical basic transposition on the entire set of 3510 transpositions in D will be suffice to define Fi_{22}. The element s is fixed by a group M_{22} and we now describe the orbits of M_{22} on the elements in D in more detail.

If i is a basic transposition, then the action of s yields the conjugate s^i and i is fixed under the action of a typical basic transposition t. The transpositions of the orbit C can be written as conjugates of s by products of the basic transpositions. We never need more than three basic transpositions in the product because a product of 4 distinct basic transpositions can be written as a product of either 2 or 3 basic transpositions. If i, j, k are three of the basic transpositions, we have $ijk = lmn$ for only one un-ordered triple $\{l, m, n\}$, since $\{\infty, 0, i, j, k\}$ defines a unique octad. So the involution $s^{ijk} = s^{lmn}$ has only two names of this form, where $\{i, j, k, l, m, n\}$ is a special hexad. Thus the orbit C splits into four orbits under the action of M_{22} with orbit lengths 1, 22, 231, 770. Representatives of the respective orbits are s, s^i, s^{ij}, $s^{ijk} = s^{lmn}$.

Next we consider the action of M_{22} on the orbit B. Now if $\{i, j, k, l, m, n\}$ is a special hexad, then there are exactly 32 transpositions in B which commute just with i, j, k, l, m, n. These are:

(i) 10 involutions $(ijk|lmn)$, the transform of $s^{ijk} = s^{lmn}$ by s.

(ii) 16 involutions $(ijklmn|uvwxyz)$, say, which transform by s into involutions commuting with the members of the disjoint special hexad $\{u, v, w, x, y, z\}$. These correspond one for one with the 16 such hexads disjoint from $\{i, j, k, l, m, n\}$.

(iii) 6 involutions $(i|jklmn)$, say, corresponding one for one with the 6 subgroups $G(i|jklmn)$ of the S_6 on $\{i, j, k, l, m, n\}$.

Thus the action of M_{22} on the orbit B of $2^{10}{:}M_{22}$ yields 3 orbits with representatives as above and respective orbit lengths 10·77, 16·77 and 6·77.

The action of s on the orbit representatives s, s^i, s^{ij}, $s^{ijk} = s^{lmn}$ are respectively, s, i s^{ij}, $(ijk|lmn)$ and the action of t on the respective representatives are s^t, s^{it}, s^{ijt}, $s^{ijkt} = s^{lmnt}$ (some of these may have shorter names). Using symmetry and the assertion that s fixes just 693 other transpositions, we conclude that the action of s on the orbits with representatives $(ijk|lmn)$, $(i|jklmn)$ and $(ijklmn|uvwxyz)$ are respectively $s^{ijk} = s^{lmn}$, $(i|jklmn)$ and $(uvwxyz|ijklmn)$. Moreover, we know the action of a typical basic transposition t on all the points except those of the form $(ijk|lmn)$ or $(i|jklmn)$ or $(ijklmn|uvwxyz)$, when $t \notin \{i, j, k, l, m, n\}$. Symmetry considerations now forces a unique action, which is best described by considering the various transforms of $(ijklmn|uvwxyz)$.

(i) If t is one of i, j, k, l, m, n, then $(ijklmn|uvwxyz)$ is fixed by t.

(ii) If t is one of u, v, w, x, y, z, the transform is $(i|jklmn)$, where $G(i|jklmn)$ is the subgroup fixing t.

(iii) Otherwise, the transform is $(ijk|lmn)$, where $\{i, j, k\}$, $\{l, m, n\}$ is the partition $3+3$ of $\{i, j, k, l, m, n\}$ which is fixed by the stabilizer of t in M_{22}.

This observation produces the action of t on the respective orbits

$$\begin{aligned}
(ijk|lmn) &\longrightarrow (ijk|lmn) \text{ or } (ijklmn|uvwxyz)\\
(i|jklmn) &\longrightarrow (i|jklmn) \text{ or } (ijklmn|uvwxyz)\\
(ijklmn|uvwxyz) &\longrightarrow (ijklmn|uvwxyz) \text{ or } (ijk|lmn) \text{ or } (i|jklmn).
\end{aligned}$$

This completes the construction of Fi_{22}. Enright [55] gives an alternate construction for Fi_{22} using the subgroup S_{10}. Names are assigned to all the transpositions in D which suggest how this S_{10} acts on them. It is then determine which transpositions commute with one another, and for any two that do not commute the conjugate of one by the other is found. This gives a complete transform table for Fi_{22}, that is, a table i^t, for all $i, t \in D$.

Theorem 10.1.1 *(Kleidman-Wilson [101]) The simple group Fi_{22} has exactly 14 conjugacy classes of maximal subgroups, as follows:*

$2{\cdot}U_6(2)$	$O_7(3)$ *(two classes)*
$O_8^+(2){:}S_3$	$2^{10}{:}M_{22}$
$2^6{:}S_6(2)$	$(2 \times 2_+^{1+8}{:}U_4(2)){:}2$
$S_3 \times U_4(3){:}2$	${}^2F_4(2)'$
$2^{5+8}{:}(S_3 \times A_6)$	$3_+^{1+6}{:}2^{3+4}{:}3^2{:}2$
S_{10} *(two classes)*	M_{12}. □

We will use the maximal subgroups and the permutation characters of Fi_{22} on the conjugates (right cosets) of the maximal subgroups listed in the Table 10.I extensively,

TABLE 10.I
Permutation Characters

$2{\cdot}U_6(2)$	$\underline{1a}+\underline{429a}+\underline{3080a}$
$O_7(3)$	$\underline{1a}+\underline{429a}+\underline{13650a}$
$O_7(3)$	$\underline{1a}+\underline{429a}+\underline{13650a}$
$O_8^+(2){:}S_3$	$\underline{1a}+\underline{3080a}+\underline{13650a}+\underline{45045a}$
$2^{10}{:}M_{22}$	$\underline{1a}+\underline{78a}+\underline{429a}+\underline{1430a}+\underline{3080a}+\underline{30030a}+\underline{32032a}+\underline{75075a}$
$2^6{:}S_6(2)$	$\underline{1a}+\underline{429a}+\underline{1430a}+\underline{3080a}+\underline{13650a}+\underline{30030a}+\underline{45045a}+\underline{75075a}+\underline{205920a}+\underline{320320a}$

especially those with order divisible by 7, 11 or 13. The permutation character of Fi_{22} on the conjugates of the maximal subgroup $2^6{:}S_6(2)$ is given in Moori-Mpono [136]. The permutation characters of Fi_{22} on the conjugates of the maximal subgroups isomorphic to ${}^2F_4(2)'$, S_{10} and M_{12}, in terms of the irreducible characters of Fi_{22}, are not given in the ATLAS. In Table 10.II we list the partial fusion maps of these maximal subgroups into Fi_{22} (obtained from GAP) that will enable us to evaluate the corresponding permutation characters on the different classes.

TABLE 10.II
Partial fusion maps into Fi_{22}

${}^2F_4(2)'$–class	$2a$	$2b$	$3a$	$5a$	$13a$	$13b$					
$\to Fi_{22}$	$2B$	$2C$	$3D$	$5A$	$13A$	$13B$					
h					1	1					
S_{10}–class	$2a$	$2b$	$2c$	$2d$	$2e$	$3a$	$3b$	$3c$	$5a$	$5b$	$7a$
$\to Fi_{22}$	$2A$	$2B$	$2C$	$2C$	$2B$	$3A$	$3C$	$3D$	$5A$	$5A$	$7A$
h											1
M_{12}–class	$2a$	$2b$	$3a$	$3b$	$5a$	$11a$	$11b$				
$\to Fi_{22}$	$2B$	$2C$	$3D$	$3C$	$5A$	$11A$	$11B$				
h						2	2				

10.2 $(p,q,7)$-Generations of Fi_{22}

The maximal subgroups M_i of Fi_{22} with order divisible by 7 are isomorphic to $2{\cdot}U_6(2)$, $O_7(3)$ (two non-conjugate classes), $O_8^+(2){:}S_3$, $2^{10}{:}M_{22}$, $2^6{:}S_6(2)$, $S_3 \times U_4(3){:}2$ and S_{10} (two non-conjugate classes). The number h_i of conjugates of M_i containing a fixed element of order 7 is given in Table 10.III.

Lemma 10.2.1 *(Moori [133]) The group Fi_{22} is $(2X,3Y,7A)$-generated if and only if the ordered pair $(X,Y)=(C,D)$.* □

Lemma 10.2.2 *The group Fi_{22} is $(2X,5A,7A)$-generated if and only if $X \in \{B,C\}$.*

Proof. The structure constant $\Delta_{Fi_{22}}(2A,5A,7A)=0$ and therefore non-generation of Fi_{22} by this triple follows.

We now consider the triple $(2B,5A,7A)$. The group $Aut(Fi_{22})$ has a 78-dimensional irreducible representation over $GF(2)$. We can use this representation and generate $Aut(Fi_{22})=\langle a,b\rangle$, where a and b are 78×78 matrices over $GF(2)$ with orders 2 and 18, respectively. Let $x=(bab^2)^6$, $y=(ab^9)^2$ and $c=ba$. Using MeatAxe and GAP we proved $a\in 2A$, $b\in 18E$, $x\in 5A$, $y\in 2B$ and $o(c)=42$. Now, if $t=c^{19}yc^{23}$, then $t\in 2B$ and $tx\in 7A$. Let H be a subgroup generated by t and x. Then we showed that $H\le Fi_{22}$ and there exist elements of order 7, 11 and 13 in H. Since Fi_{22} contains no proper subgroup with order divisible by $7\times 11\times 13$, we have $H=Fi_{22}$, and therefore Fi_{22} is $(2B,5A,7A)$-generated.

From Table 10.III we conclude that the number of pairs $(x,y)\in 2C\times 5A$ with $xy=z$, where z is a fixed element in $7A$ and $\langle x,y\rangle < Fi_{22}$ is at most

$$3\Sigma(2{\cdot}U_6(2))+3\Sigma(O_7(3))+\cdots+\Sigma(S_{10})=58422.$$

The result follows since $\Delta_{Fi_{22}}(2C,5A,7A)=72828$. □

Lemma 10.2.3 *The group Fi_{22} is $(3X,5A,7A)$-generated for all $X\in\{A,\ldots,D\}$.*

Proof. We first consider the case $(3A,5A,7A)$. As in the previous lemma we use the generation of $Aut(Fi_{22})$ by the 78-dimensional matrices a and b. Then we showed that $b^2 \in 2D$, $z = (ab)^{14} \in 3A$ and $x = (ab^2)^6 \in 5A$. Let $q = (ba)^{10}x^3(ba)^{32}$. Then $q \in 5A$ and $o(zq) = 7$. Let $H = \langle z, q\rangle$. Then $H \leq Fi_{22}$ and H contains elements of order 7, 11 and 13. Therefore $H = Fi_{22}$ and $(3A, 5A, 7A)$ is a generating triple for Fi_{22}.

We calculate the structure constant $\Delta_{Fi_{22}}(3B,5A,7A) = 48181$, $\Delta_{Fi_{22}}(3C,5A,7A) = 1298871$ and $\Delta_{Fi_{22}}(3D,5A,7A) = 5050836$. It is clear from Table 10.III that the number of pairs $(x,y) \in 3X \times 5A$ with $xy = z$, a fixed element in $7A$, that generates proper subgroups of Fi_{22} is less than $\Delta_{Fi_{22}}(3X,5A,7A)$, for all $X \in \{B,C,D\}$, proving the result. $\square$

TABLE 10.III
Structure constants $\Sigma(M_i)$

M_i	h_i	$\Sigma_{M_i}(2C,5A,7A)$	$\Sigma_{M_i}(3B,5A,7A)$	$\Sigma_{M_i}(3C,5A,7A)$	$\Sigma_{M_i}(3D,5A,7A)$
$2{\cdot}U_6(2)$	3	8218	1911	111531	0
$O_7(3)$	3	2548	1988	16464	12208
$O_7(3)$	3	2548	1988	16464	12208
$O_8^+(2){:}S_3$	1	588	903	2709	0
$2^{10}{:}M_{22}$	6	2688	0	39424	0
$2^6{:}S_6(2)$	6	245	0	1764	1232
$S_3 \times U_4(3){:}2$	1	0	112	8736	0
S_{10}	1	147	0	245	966
S_{10}	1	147	0	245	966

10.3 $(p, q, 11)$-Generatons of Fi_{22}

The maximal subgroups of Fi_{22} containing elements of order 11 are, up to isomorphism, $2{\cdot}U_6(2)$, $2^{10}{:}M_{22}$ and M_{12}. If we fix an element of order 11 in a subgroup $2{\cdot}U_6(2)$, $2^{10}{:}M_{22}$ and M_{12}, then it is contained in 1, 2, 2 conjugates of this subgroup, respectively. Since $(11B)^{-1} = 11A$, the results obtained by replacing one of these classes with the other are the same. Let $11Z$ denote the class $11A$ or $11B$.

Lemma 10.3.1 *(Moori [133]) The group Fi_{22} is $(2X, 3Y, 11Z)$-generated if and only if the ordered pair $(X, Y) = (C, D)$.* $\square$

Lemma 10.3.2 *The group Fi_{22} is $(2X, 5A, 11Z)$-generated if and only if $X \in \{B, C\}$.*

Proof. The structure constant $\Delta_{Fi_{22}}(2A, 5A, 11Z) = 0$ and non-generation of Fi_{22} by this triple follows.

We calculate the structure constant $\Delta_{Fi_{22}}(2B, 5A, 11Z) = 3025$, $\Sigma(2{\cdot}U_6(2)) = 935$, $\Sigma(2^{10}{:}M_{22}) = 704$ and $\Sigma(M_{12}) = 33$. Using Theorem 3.1.4 we obtain

$$\Delta^*(Fi_{22}) \geq \Delta(Fi_{22}) - (935 + 2(704) + 2(33)) = 616.$$

Therefore Fi_{22} is $(2B, 5A, 11Z)$-generated.

Similarly we obtain $\Delta_{Fi_{22}}(2C, 5A, 11Z) = 56364$, $\Sigma(2{\cdot}U_6(2)) = 8184$, $\Sigma(2^{10}{:}M_{22}) = 2112$, $\Sigma(M_{12}) = 55$ and therefore $\Delta^*(Fi_{22}) \geq 43846$, proving the result. □

Lemma 10.3.3 *The group Fi_{22} is $(2X, 7A, 11Z)$-generated if and only if $X \in \{B, C\}$.*

Proof. The order of the group M_{12} is not divisible by 7. The structure constant $\Delta_{Fi_{22}}(2A, 7A, 11Z) = 44 = \Sigma(2{\cdot}U_6(2))$, and hence $\Delta^*(Fi_{22}) = 0$, proving non-generation of Fi_{22} by this triple. We also calculate $\Delta_{Fi_{22}}(2B, 7A, 11Z) = 34199$, $\Sigma(2{\cdot}U_6(2)) = 4565$ and $\Sigma(2^{10}{:}M_{22}) = 2816$. Therefore $\Delta^*(Fi_{22}) \geq 24002$ and Fi_{22} is $(2B, 7A, 11Z)$-generated. Similarly, we calculate $\Delta_{Fi_{22}}(2C, 7A, 11Z) = 846560$, $\Sigma(2{\cdot}U_6(2)) = 40040$ and $\Sigma(2^{10}{:}M_{22}) = 8448$ and consequently $\Delta^*(Fi_{22}) \geq 789624$, proving the result. □

Lemma 10.3.4 *The group Fi_{22} is $(3X, 5A, 11Z)$-generated for all $X \in \{A, \ldots, D\}$.*

Proof. We calculate $\Delta_{Fi_{22}}(3A, 5A, 11Z) = 8437$ and $\Delta_{Fi_{22}}(3B, 5A, 11Z) = 38049$. Now for $Y \in \{A, B\}$, we have $\chi_{2^{10}:M_{22}}(3Y) = 0$ and hence $3Y \cap 2^{10}{:}M_{22} = \emptyset$. Also from Table 10.II we have $3Y \cap M_{12} = \emptyset$. Therefore

$$\begin{aligned}
\Delta^*_{Fi_{22}}(3A, 5A, 11Z) &= \Delta(Fi_{22}) - \Sigma(2{\cdot}U_6(2)) \\
&= 8437 - 2167 > 0\,, \\
\Delta^*_{Fi_{22}}(3B, 5A, 11Z) &= \Delta(Fi_{22}) - \Sigma(2{\cdot}U_6(2)) \\
&= 38049 - 3157 > 0\,,
\end{aligned}$$

proving generation of Fi_{22} by these triples.

Next we calculate $\Delta_{Fi_{22}}(3C, 5A, 11Z) = 942183$, $\Sigma(2{\cdot}U_6(2)) = 75933$, $\Sigma(2^{10}{:}M_{22}) = 33792$ and $\Sigma(M_{12}) = 253$. Thus $\Delta^*(Fi_{22}) \geq 798160$ and Fi_{22} is $(3C, 5A, 11Z)$-generated.

The conjugacy class $3D$ has empty intersections with the maximal subgroups $2{\cdot}U_6(2)$ and $2^{10}{:}M_{22}$. Since $\Delta_{Fi_{22}}(3D, 5A, 11Z) = 5445198$ and $\Sigma(M_{12}) = 198$, we obtain $\Delta^*(Fi_{22}) \geq 5444802$, proving the result. □

Lemma 10.3.5 *The group Fi_{22} is $(3X, 7A, 11Z)$-generated for all $X \in \{A, \ldots, D\}$.*

Proof. For these triples we only need to consider the maximal subgroups isomorphic to $2{\cdot}U_6(2)$ and $2^{10}{:}M_{22}$. The classes $3A$ and $3B$ have empty intersections with the subgroup $2^{10}{:}M_{22}$. Therefore, for the triple $(3A, 7A, 11Z)$, we calculate

$$\Delta^*(Fi_{22}) = \Delta(Fi_{22}) - \Sigma(2{\cdot}U_6(2)) = 93643 - 8569 = 85074,$$

and Fi_{22} is $(3A, 7A, 11Z)$-generated. Similarly for the triple $(3B, 7A, 11Z)$ we have $\Delta^*(Fi_{22}) = 586575 - 14135 > 0$, and generation by this triple follows.

We calculate the structure constants $\Delta_{Fi_{22}}(3C, 7A, 11Z) = 11828025$, $\Sigma(2{\cdot}U_6(2)) = 337755$, $\Sigma(2^{10}{:}M_{22}) = 112640$ and therefore $\Delta^*(Fi_{22}) \geq 11264990$ so that Fi_{22} is $(3C, 7A, 11Z)$-generated. Finally, $\Delta_{Fi_{22}}(3D, 7A, 11Z) = 84481650$ and the maximal subgroups $2{\cdot}U_6(2)$ and $2^{10}{:}M_{22}$ have empty intersections with the class $3D$. Thus $\Delta^*(Fi_{22}) = \Delta(Fi_{22})$, and the result follows. □

Lemma 10.3.6 *The group Fi_{22} is $(5A, 7A, 11Z)$-generated.*

Proof. From the structure constants $\Delta_{Fi_{22}}(5A, 7A, 11Z) = 2660517805$, $\Sigma(2{\cdot}U_6(2)) = 21902705$, $\Sigma(2^{10}{:}M_{22}) = 3244032$ we obtain that $\Delta^*(Fi_{22}) \geq 2632127036$, proving the result. □

10.4 $(p, q, 13)$- and (p, q, r)-Generations of Fi_{22}

The only maximal subgroups of Fi_{22} with order divisible by 13 are, up to isomorphisms, $O_7(3)$ (two non-conjugate copies) and ${}^2F_4(2)'$. Again $(13B)^{-1} = 13A$ and $13Z$ will denote the conjugacy class $13A$ or $13B$.

Lemma 10.4.1 *(Moori [133]) The group Fi_{22} is $(2X, 3Y, 13Z)$-generated if and only if the ordered pair $(X, Y) = (C, C)$ or (C, D).* □

Lemma 10.4.2 *The group Fi_{22} is $(2X, 5A, 13Z)$-generated if and only if $X \in \{B, C\}$.*

Proof. The structure constant $\Delta_{Fi_{22}}(2A, 5A, 13Z) = 0$ and non-generation of Fi_{22} by this triple follows.

Next we calculate $\Delta_{Fi_{22}}(2B, 5A, 13Z) = 1625$, $\Sigma(O_7(3)) = 104$ and $\Sigma(^2F_4(2)') = 13$. Since Fi_{22} contains two non-conjugate classes of $O_7(3)$ subgroups,

$$\Delta^*(Fi_{22}) \geq \Delta(Fi_{22}) - (2(104) + 13) = 1404.$$

Therefore Fi_{22} is $(2B, 5A, 13Z)$-generated.

Similarly $\Delta_{Fi_{22}}(2C, 5A, 13Z) = 57044$, $\Sigma(O_7(3)) = 2366$, $\Sigma(^2F_4(2)') = 338$ and therefore $\Delta^*(Fi_{22}) \geq 51974$, proving the result. □

Lemma 10.4.3 *The group Fi_{22} is $(3X, 5A, 13Z)$-generated for all $X \in \{A, \ldots, D\}$.*

Proof. The subgroup $^2F_4(2)'$ has empty intersection with each of the classes $3A$, $3B$ and $3C$ (*cf.* Table 10.II). We calculate $\Delta_{Fi_{22}}(3A, 5A, 13Z) = 4550$ and $\Sigma(O_7(3)) = 156$. Thus $\Delta^*(Fi_{22}) \geq 4238$. Similarly, $\Delta^*_{Fi_{22}}(3B, 5A, 13Z) \geq 42419 - 2(2002) > 0$ and $\Delta^*_{Fi_{22}}(3C, 5A, 13Z) \geq 743535 - 2(11856) > 0$. .

For the triple $(3D, 5A, 13Z)$ we calculate $\Delta(Fi_{22}) = 6130449$, $\Sigma(O_7(3)) = 14300$, $\Sigma(^2F_4(2)') = 3341$ and therefore $\Delta^*(Fi_{22}) \geq 6098508$, proving the result. □

Lemma 10.4.4 *The group Fi_{22} is $(2X, 7A, 13Z)$-, $(3Y, 7A, 13Z)$- and $(5A, 7A, 13Z)$-generated for all $X \in \{A, B, C\}$ and $Y \in \{A, \ldots, D\}$.*

Proof. The order of the subgroup $^2F_4(2)'$ is not divisible by 7. Thus for any triple $(pX_1, 7A, 13Z)$ in the hypothesis of this lemma, the $(pX_1, 7A, 13Z)$-generated proper subgroups will be contained in a maximal subgroup isomorphic to $O_7(3)$. From Table 10.IV we have $\Delta^*(Fi_{22}) \geq \Delta(Fi_{22}) - 2\,\Sigma(O_7(3)) > 0$, for all these triples. The result follows. □

Lemma 10.4.5 *The group Fi_{22} is $(2X_1, 11X_3, 13Z)$-, $(3X_2, 11X_3, 13Z)$-, $(5A, 11X_3, 13Z)$- and $(7A, 11X_3, 13Z)$-generated, for all $X_1 \in \{A, B, C\}$, $X_2 \in \{A, \ldots, D\}$ and $X_3 \in \{A, B\}$.*

Proof. The group Fi_{22} does not contain subgroups with order divisible by 11×13. Thus for all the triples under consideration $\Delta^*(Fi_{22}) = \Delta(Fi_{22})$. The result follows from Table 10.IV. $\square$

It is obvious that if a group G is (l, m, n)-generated, then it is $(\pi(l), \pi(m), \pi(n))$-generated, for any permutation $\pi \in S_{\{l,m,n\}} \cong S_3$. We now summarize all the above results together with the results in Section 10.2 and 10.3 in the following theorem.

Theorem 10.4.6 *The Fischer group Fi_{22} is (p, q, r)-generated for all distinct $p, q, r \in \{2, 3, 5, 7, 11, 13\}$, except when $\{p, q, r\} = \{2, 3, 5\}$.*

Proof. The triangular group $T(2, 3, 5) \cong A_5$. The theorem now follows from the lemmas in Sections 10.2, 10.3 and 10.4. $\square$

Table 10.IV

Structure constants $\Delta(Fi_{22})$ and $\Sigma(O_7(3))$

pY	$\Sigma_{O_7(3)}(pY, 7A, 13Z)$	$\Delta_{Fi_{22}}(pY, 7A, 13Z)$	$\Delta_{Fi_{22}}(pY, 11X_3, 13Z)$
$2A$	26	78	156
$2B$	1586	28717	55224
$2C$	23660	867490	1658124
$3A$	1872	77974	149760
$3B$	19240	610675	1164384
$3C$	114660	10983375	20966400
$3D$	243360	87858225	167731200
$5A$	2728960	2561905905	4891041792
$7A$			69872025600

10.5 nX-Complementary generations of Fi_{22}

The main theorem that we will prove in this section is:

Theorem 10.5.1 *The group Fi_{22} is nX-complementary generated if and only if $nX \in \{6K, 8C, 8D, 9C, 12E, \ldots, 12K\}$ or $n \in \{7, 10, 11, 13, \ldots, 30\}$.*

The proof of this theorem will follow from the lemmas proved below. As in the previous chapters, we will use Lemma 3.1.7 extensively in proving the non-complementary generations of Fi_{22}. From this result it suffices to consider only the triples for which $\Delta_G(lX, nY, nZ) \geq |C_G(nZ)|$. Recall that if G is a simple group then G is not $2X$-complementary generated, for any class of involutions.

Lemma 10.5.2 *The group Fi_{22} is not $3X$-complementary generated for any $X \in \{A, \ldots, D\}$.*

Proof. We will show that Fi_{22} can not be 2-generated by elements from the classes $2B$ and $3A$. Let $\chi = \underline{78a} \in Irr(Fi_{22})$. Let $A = \langle x \rangle$ and $B = \langle y \rangle$, where $x \in 2B$ and $y \in 3A$. Then $A \cap B = \{1_G\}$. Also $\langle \chi\downarrow_A, \chi_1\downarrow_A \rangle = 46$, $\langle \chi\downarrow_B, \chi_1\downarrow_B \rangle = 36$ and $\langle \chi\downarrow_{(A\cap B)}, \chi_1\downarrow_{(A\cap B)} \rangle = 78$. Furthermore, $46 + 36 = 82 > 78$ and the result of Brauer (*cf.* Theorem 3.3.4) yields $\langle x, y \rangle < Fi_{22}$. Thus by Lemma 2.3.8, it follows that Fi_{22} is not $3A$-complementary generated.

For $X \in \{B, C, D\}$, the structure constants $\Delta_{Fi_{22}}(2A, 3X, tY) < |C_{Fi_{22}}(tY)|$, for all conjugacy classes tY of Fi_{22}. Thus Fi_{22} is not $3X$-complementary generated, proving the result. □

Lemma 10.5.3 *The group Fi_{22} is not $4X$-complementary generated for any $X \in \{A, \ldots, E\}$.*

Proof. Once more, the structure constant $\Delta_{Fi_{22}}(2A, 4X, tY) < |C_{Fi_{22}}(tY)|$, for all conjugacy classes tY of Fi_{22}, proving the result. □

Lemma 10.5.4 *The group Fi_{22} is not $5A$-complementary generated.*

Proof. We will show Fi_{22} is not $(2A, 5A, tY)$-generated for any class tY. Applying Lemma 3.1.7, the only triple we need to consider is $(2A, 5A, 30A)$. The structure constant $\Delta_{Fi_{22}}(2A, 5A, 30A) = 36$. However, $\Sigma(2{\cdot}U_6(2)) = 30$ and hence $\Delta^*(Fi_{22}) \leq 36 - 30 < |C_{Fi_{22}}(30A)| = 30$, proving the non-generation of Fi_{22} by this triple. The result follows. □

Lemma 10.5.5 *The group Fi_{22} is not $6X$-complementary generated for any $X \in \{A, \ldots, J\}$.*

Proof. For $X \in \{A, \ldots, I\}$ we calculate $\Delta_{Fi_{22}}(2A, 6X, tY) < |C_{Fi_{22}}(tY)|$, for all conjugacy classes tY of Fi_{22}, proving non-complementary generation by these classes.

For the class $6J$ we calculate

$$\begin{aligned} \Delta_{Fi_{22}}(2A, 6J, 14A) &= 14 = \Sigma(2{\cdot}U_6(2)) , \\ \Delta_{Fi_{22}}(2A, 6J, 21A) &= 21 = \Sigma(O_8^+(2){:}S_3) , \\ \Delta_{Fi_{22}}(2A, 6J, 24A) &= 24 = \Sigma(O_8^+(2){:}S_3) . \end{aligned}$$

The other triples of the form $(2A, 6J, tY)$ we eliminate using Lemma 3.1.7. This proves the result. $\square$

Lemma 10.5.6 *The group Fi_{22} is $(2A, 6K, tY)$-generated if and only if $tY = 22Z$, where $Z \in \{A, B\}$. Furthermore, Fi_{22} is $6K$-complementary generated.*

Proof. We calculate the structure constant $\Delta_{Fi_{22}}(2A, 6K, 8D) = 32 = |C_{Fi_{22}}(8D)|$. Also $\Sigma(O_7(3)) = 16$, so that $\Delta^*(Fi_{22}) \leq 16 < |C_{Fi_{22}}(8D)|$, and non-generation follows. Next we calculate $\Delta_{Fi_{22}}(2A, 6K, 13Z) = 26$ and $\Sigma(O_7(3)) = 13$. Let O_1 and O_2 be subgroups of Fi_{22} isomorphic to $O_7(3)$ from different conjugate classes. Then $O_1 \cap O_2 \cong 3^5{:}U_4(2){:}2$, $G_2(3)$ or S_9 (*cf.* Wilson [170]). Since $2A \cap G_2(3) = \emptyset$ and 13 does not divide $|3^5{:}U_4(2){:}2|$ and $|S_9|$, we have $\Sigma(O_1 \cap O_2) = 0$ and therefore $\Delta^*(Fi_{22}) = \Delta(Fi_{22}) - 2\,\Sigma(O_7(3)) = 0$ and non-generation follows. Furthermore, the structure constant $\Delta_{Fi_{22}}(2A, 6K, 14A) = 14 = \Sigma(O_8^+(2){:}S_3)$, proving the non-generation by this triple.

We calculate the structure constant $\Delta_{Fi_{22}}(2A, 6K, 20A) = 60$. Also $\Sigma(O_7(3)) = 20$ and $\Sigma(O_8^+(2){:}S_3) = 20$. For O_1, O_2 non-conjugate copies of $O_7(3)$ in Fi_{22} with $O_1 \cap O_2 \cap 20A \neq \emptyset$, we have $O_1 \cap O_2 \cong S_9$ as neither $3^5{:}U_4(2){:}2$ nor $G_2(3)$ contains elements of order 20. But then $O_1 \cap O_2$ embeds in $O_8^+(2){:}S_3$ (*cf.* Moori [133]), so also in $O_8^+(2){:}2$. As $O_8^+(2){:}2 \cap 6K = \emptyset$, we have $\Sigma(O_1 \cap O_2) = 0$. Moreover, since $O_8^+(2){:}2$ is the only maximal subgroup of $O_8^+(2){:}S_3$ which contains elements of order 20, we have that $\Sigma^*(O_8^+(2){:}S_3) = 20$ and $\Sigma(O_7(3) \cap O_8^+(2){:}S_3) = 0$. Thus

$$\Delta^*(Fi_{22}) = \Delta(Fi_{22}) - 2\,\Sigma(O_7(3)) - \Sigma(O_8^+(2){:}S_3) = 0 .$$

The structure constant $\Delta_{Fi_{22}}(2A, 6K, 22Z) = 22 = |C_{Fi_{22}}(22Z)|$. Now the only maximal subgroups of Fi_{22} containing elements of order 22 are isomorphic to $2{\cdot}U_6(2)$. However, $6K \cap 2{\cdot}U_6(2) = \emptyset$ and therefore $\Delta^*(Fi_{22}) = \Delta(Fi_{22}) = 22$. Thus Fi_{22} is $(2A, 6K, 22A)$-generated. Finally, $\Delta_{Fi_{22}}(2A, 6K, 30A) = 30 = \Sigma(O_8^+(2){:}S_3)$. The non-generation by the other triples follows from Lemma 3.1.7.

From $6K \cap 2{\cdot}U_6(2) = \emptyset$ it follows that none of the proper subgroups of Fi_{22} is $(pY, 6K, 22A)$-generated, for all classes pY with prime order representatives. Since the structure constants are positive for all these triples (*cf.* Table 10.V), it follows from Lemma 2.3.8 that Fi_{22} is $6K$-complementary generated. This proves the result. □

We have now proved that Fi_{22} is $(2A, nX, tY)$-generated and hence $(2A, tY, nX)$-generated, for $n \leq 6$, if and only if $(2A, nX, tY) = (2A, 6K, 22Z)$. Thus to show that Fi_{22} is not $(2A, nX, tY)$-generated, for $n \geq 7$, $n \neq 22$, we only need to consider classes tY for $t \geq 7$.

Lemma 10.5.7 *The group Fi_{22} is pX-complementary generated, where p is a prime, if and only if $p \in \{7, 11, 13\}$.*

Proof. The cases for the primes $p \leq 5$ were discussed above. We showed in the previous sections that Fi_{22} is $(pX, 7A, 13Z)$-generated for classes pX, where the prime $p \leq 5$. Furthermore, the $(2A, 7A, 13Z)$- and $(7A, 11Z, 13A)$-generations of Fi_{22} also imply $(13Z, 7A, 2A)$-, $(11Z, 7A, 13A)$- and $(7A, 7A, 13Z)$-generation of Fi_{22}. Thus we have proved that Fi_{22} is $7A$-complementary generated.

We calculate the structure constants $\Delta_{Fi_{22}}(11A, 11B, 13Z) = 133391950848$ and $\Delta_{Fi_{22}}(13A, 13B, 22A) = 382022479872$. Since no proper subgroup of Fi_{22} has order divisible by 11×13, these triples generate Fi_{22}. Arguments similar to the case $p = 7$ will show that Fi_{22} is $11Z$- and $13Z$-complementary generated, proving the result. □

Lemma 10.5.8 *The group Fi_{22} is $8X$-complementary generated if and only if $X \in \{C, D\}$.*

Proof. We first consider the class $8A$. The structure constant $\Delta_{Fi_{22}}(2A, 8A, 10B) = 45$ and $\Sigma(2{\cdot}U_6(2)) = 25$. Therefore $\Delta^*(Fi_{22}) \leq 45 - 25 < |C_{Fi_{22}}(10B)| = 40$, proving non-generation. Next we calculate $\Delta_{Fi_{22}}(2A, 8A, 12K) = 36 = |C_{Fi_{22}}(12K)|$. However, $\Sigma(O_8^+(2){:}S_3) = 12$ and non-generation follows. Similarly, we calculate $\Delta_{Fi_{22}}(2A, 8A, 18D) = 36 = |C_{Fi_{22}}(18D)|$ and $\Sigma(2{\cdot}U_6(2)) = 18$, proving non-generation. Furthermore, $\Delta_{Fi_{22}}(2A, 8A, 22A) = 22 = \Sigma(2{\cdot}U_6(2))$. The remaining triples are eliminated by Lemma 3.1.7. Thus Fi_{22} is not $8A$-complementary generated.

For the class $8B$ the only triples of the form $(2A, 8B, tY)$ that we need to consider are those for which $tY \in \{11A, 11B, 14A, 21A\}$. Our calculations yield

$$\begin{aligned}
\Delta_{Fi_{22}}(2A, 8B, 11Z) &= 22 = \Sigma(2{\cdot}U_6(2)) \ , \\
\Delta_{Fi_{22}}(2A, 8B, 14A) &= 14 = \Sigma(2{\cdot}U_6(2)) \ , \\
\Delta_{Fi_{22}}(2A, 8B, 21A) &= 21 = \Sigma(O_8^+(2){:}S_3) \ .
\end{aligned}$$

Therefore Fi_{22} is not $8B$-complementary generated.

For the classes $8C$ and $8D$ it is immediate from Tables 10.V and 10.VI that

$$\Delta^*_{Fi_{22}}(pX, 8Y, 22A) = \Delta(Fi_{22}) - \Sigma(2{\cdot}U_6(2)) > 0 \ , \qquad Y \in \{C, D\}$$

for every class pX with elements of prime order. This proves the result. $\square$

Lemma 10.5.9 *The group Fi_{22} is $9X$-complementary generated if and only if $X = C$.*

Proof. We calculate $\Delta_{Fi_{22}}(2A, 9A, 20A) = 20$ and $\Sigma(O_7(3)) = 10$. Non-generation of Fi_{22} by this triple follows since $\Delta^*(Fi_{22}) \leq 20 - 10 < |C_{Fi_{22}}(20A)|$. Furthermore, $\Delta_{Fi_{22}}(2A, 9A, 22A) = 22 = \Sigma(2{\cdot}U_6(2))$. We also calculate $\Delta_{Fi_{22}}(2A, 9A, 24B) = 48 = |C_{Fi_{22}}(24B)|$. However, $\Sigma(O_8^+(2){:}S_3) = 24$ and non-generation follows. Next we calculate $\Delta_{Fi_{22}}(2A, 9A, 30A) = 70$, $\Sigma(2{\cdot}U_6(2)) = 40$ and $\Sigma(O_8^+(2){:}S_3) = 30$. Now the only maximal subgroups of $O_8^+(2){:}S_3$ containing elements of order 30 are isomorphic to $O_8^+(2){:}2$. However, $9A \cap O_8^+(2){:}2 = \emptyset$ and therefore $\Sigma^*(O_8^+(2){:}S_3) = 30$ and $\Sigma(2{\cdot}U_6(2) \cap O_8^+(2){:}S_3) = 0$. Therefore $\Delta^*(Fi_{22}) = 0$, proving non-generation. The fact that Fi_{22} is not $9A$-complementary generated now follows from Lemma 3.1.7.

We now consider the class $9B$. The structure constant $\Delta_{Fi_{22}}(2A, 8D, 9B) = 324$ and $|C_{Fi_{22}}(x)| = 162$, where $x \in 9B$. Consider the subgroup $S \cong S_9$ with $S < O_7(3)$. Then we calculate $\Sigma(S) = \Delta_S(2a, 8a, 9a) = 9$. For any maximal subgroup M_S of S (*cf.*

ATLAS), the permutation character of S on the conjugates of M_S gives $\chi_{M_S}(nx) = 0$, for $nx \in \{2a, 8a, 9a\}$. Thus no subgroup of S is $(2a, 8a, 9a)$-generated and hence $\Sigma^*(S) = 9$. Let $N = N_{Fi_{22}}(S)$. Then $S \subseteq N \subseteq M$, for some maximal subgroup M of Fi_{22}. From the fusion map of the group S_9 into the maximal subgroups of Fi_{22} we conclude that $M \cong O_7(3)$, $O_8^+(2){:}S_3$ or S_{10}. Now $9B \cap O_8^+(2){:}S_3 = \emptyset$ and consequently $M \not\cong O_8^+(2){:}S_3$. On the other hand, the S_9 subgroups of $O_7(3)$ and S_{10} are maximal in those groups, respectively. Since S is not normal in either $O_7(3)$ or S_{10}, it follows that $N = S$. From Theorem 3.1.4 we conclude that a fixed element $x \in 9B$ is contained in 18 conjugate copies of S. Since $S \not\leq 2.U_6(2)$ we obtain

$$\begin{aligned}
\Delta^*_{Fi_{22}}(2A, 8D, 9B) &\leq \Delta(Fi_{22}) - \Sigma(2.U_6(2)) - 18\Sigma^*(S) \\
&= 324 - 54 - (18 \times 9) \\
&= 108 < |C_{Fi_{22}}(x)|.
\end{aligned}$$

Thus Fi_{22} is not $(2A, 8D, 9B)$-generated and hence not $(2A, 9B, 8D)$-generated.

Next we calculate $\Delta_{Fi_{22}}(2A, 9B, 7A) = 42 = |C_{Fi_{22}}(7A)|$ and $\Sigma(2{\cdot}U_6(2)) = 14$ from which non-generation by this triple follows. Let O_1 and O_2 be subgroups isomorphic to $O_7(3)$ from different conjugate classes. We calculate $\Delta_{Fi_{22}}(2A, 9B, 13Z) = 26$, $\Sigma(O_7(3)) = 13$ and $\Sigma(O_1 \cap O_2) = 0$ (*cf.* Lemma 10.5.7), proving non-generation.

For the triple $(2A, 9B, 9C)$ we use MeatAxe and GAP on the representation of $Aut(Fi_{22})$ given by the 78-dimensional matrices a and b (*cf.* Lemma 10.2.2). We calculate $\Delta_{Fi_{22}}(2A, 9B, 9C) = 54$ and

$$\Sigma(O_7(3)) = \Delta_{O_7(3)}(2a, 9d, 9c) = 27 = |C_{O_7(3)}(9c)|.$$

Let $y = (bab^2a)^2$, and $x = a^{z^7}$, where $z = a^{y^4ay}y$. Then $o(y) = 9$ and $C_V(y) = 8$, where V is the irreducible module for Fi_{22} of dimension 78 over the field $GF(2)$. Now since $b^2 \in 9A$ and $C_V(b^2) = 10$, it follows that $y \in 9B \cup 9C$. Let $w = bab^{-1}$. Then $o(xy) = 9$ and $o(wxy) = 13$. Furthermore, if we let $K = \langle w, xy \rangle$, then $K \leq Fi_{22}$ and K contains elements of order 11 and 13. Since no proper subgroup of Fi_{22} contains elements of order 11 and 13, we have $K \cong Fi_{22}$. Now $C_V(xy) = 8$ so that $xy \in 9B \cup 9C$. But Fi_{22} is not $(2A, 9B, 13Z)$-generated and consequently $xy \in 9C$. Moreover, if $y \in 9C$, then $xy = y^h$, for some $h \in Fi_{22}$, and hence $\Delta_{Fi_{22}}(2A, 9C, 9C) > 0$. However, we calculate $\Delta_{Fi_{22}}(2A, 9C, 9C) = 0$ and therefore $(x, y) \in 2A \times 9B$ with $xy \in 9C$.

Let $H = \langle x, y\rangle$. Then $H < Fi_{22}$ and $H \cong O_7(3)$, since H contains elements of order 5, 7, and 13, whereas no proper subgroup of $O_7(3)$ contains elements with these orders. Therefore $\Sigma^*(O_7(3)) = \Sigma(O_7(3))$ and hence no proper subgroup of $O_7(3)$ is $(2a, 9d, 9c)$-generated. In particular, $\Sigma(O_1 \cap O_2) = 0$, and consequently $\Delta^*_{Fi_{22}}(2A, 9B, 9C) = 0$ and non-generation follows.

We also calculate

$$\begin{aligned}
\Delta_{Fi_{22}}(2A, 9B, 11Z) &= 22 = \Sigma(2{\cdot}U_6(2)) ,\\
\Delta_{Fi_{22}}(2A, 9B, 15A) &= 30 = \Sigma(2{\cdot}U_6(2)) ,\\
\Delta_{Fi_{22}}(2A, 9B, 20A) &= 20 = \Sigma(O_7(3)) .
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Delta_{Fi_{22}}(2A, 9B, 10A) - \Sigma(2{\cdot}U_6(2)) &= 70 - 30 < |C_{Fi_{22}}(10A)| = 60 ,\\
\Delta_{Fi_{22}}(2A, 9B, 12I) - \Sigma(2{\cdot}U_6(2)) &= 96 - 48 < |C_{Fi_{22}}(12I)| = 96 ,\\
\Delta_{Fi_{22}}(2A, 9B, 14A) - \Sigma(O_7(3)) &= 21 - 14 < |C_{Fi_{22}}(14A)| = 14 .
\end{aligned}$$

The non-generation of the remaining triples follows from Lemma 3.1.7. Thus Fi_{22} is not $9B$-complementary generated.

Using the permutation character of Fi_{22} on the conjugates of $2{\cdot}U_6(2)$ we deduce that $9C \cap 2{\cdot}U_6(2) = \emptyset$. Therefore $\Delta^*_{Fi_{22}}(pX, 9C, 22A) = \Delta(Fi_{22})$, for all classes pX. The $9C$-complementary generation of Fi_{22} follows from Table 10.V. This proves the result. $\square$

Lemma 10.5.10 *The group Fi_{22} is $10X$-complementary generated for all $X \in \{A, B\}$.*

Proof. It is immediate from Table 10.V and 10.VI that

$$\Delta^*_{Fi_{22}}(pY, 10X, 22A) = \Delta(Fi_{22}) - \Sigma(2{\cdot}U_6(2)) > 0 ,$$

for all classes pY with elements of prime order. The result follows. $\square$

Lemma 10.5.11 *The group Fi_{22} is $12X$-complementary generated if and only if $X \in \{E, \ldots, K\}$.*

Proof. The structure constant $\Delta_{Fi_{22}}(2A, 12A, tY) < |C_{Fi_{22}}(tY)|$, for all conjugacy classes tY with $t \geq 7$, and therefore Fi_{22} is not $12A$-complementary generated.

We now consider the class $12B$. Applying Lemma 3.1.7 we are left with the classes $tY \in \{20A, 30A\}$. Our calculations give

$$\begin{aligned}
\Delta_{Fi_{22}}(2A, 12B, 20A) - \Sigma(O_8^+(2){:}S_3) &= \; 25 - 20 \; < |C_{Fi_{22}}(20A)| = 20 \; , \\
\Delta_{Fi_{22}}(2A, 12B, 30A) - \Sigma(O_8^+(2){:}S_3) &= \; 50 - 30 \; < |C_{Fi_{22}}(30A)| = 30 \; .
\end{aligned}$$

Thus Fi_{22} is not $12B$-complementary generated.

For the class $12C$ we calculate $\Delta_{Fi_{22}}(2A, 12C, 14A) = 14 = \Sigma(O_8^+(2){:}S_3)$. Also $\Delta_{Fi_{22}}(2A, 12C, 18C) = 54 = |C_{Fi_{22}}(18C)|$. However, $\Sigma(O_8^+(2){:}S_3) = 18$ and non-generation follows. The fact that Fi_{22} is not $12C$-complementary generated now follows from Lemma 3.1.7.

Next we consider the class $12D$. We calculate

$$\begin{aligned}
\Delta_{Fi_{22}}(2A, 12D, 7A) - \Sigma(2{\cdot}U_6(2)) &= \; 49 - 35 \; < |C_{Fi_{22}}(7A)| = 42 \; , \\
\Delta_{Fi_{22}}(2A, 12D, 8C) - \Sigma(2{\cdot}U_6(2)) &= \; 128 - 64 \; < |C_{Fi_{22}}(30A)| = 128 \; .
\end{aligned}$$

Also

$$\begin{aligned}
\Delta_{Fi_{22}}(2A, 12D, 11Z) &= 22 = \; \Sigma(2{\cdot}U_6(2)) \; , \\
\Delta_{Fi_{22}}(2A, 12D, 14A) &= 28 = \; \Sigma(2{\cdot}U_6(2)) \; , \\
\Delta_{Fi_{22}}(2A, 12D, 18C) &= 54 = \; \Sigma(2{\cdot}U_6(2)) \; , \\
\Delta_{Fi_{22}}(2A, 12D, 21A) &= 21 = \; \Sigma(O_8^+(2){:}S_3) \; .
\end{aligned}$$

The other triples are immediately eliminated by Lemma 3.1.7. Thus Fi_{22} is not $12D$-complementary generated.

Let $X \in \{E, F\}$. Then $12X \cap 2^{10}{:}M_{22} = \emptyset$ and $12X \cap M_{12} = \emptyset$. Therefore

$$\Delta^*_{Fi_{22}}(pY, 12X, 11A) = \Delta(Fi_{22}) - \Sigma(2{\cdot}U_6(2)) \; ,$$

for every class pY.

For $X \in \{G, J\}$, we have $12X \cap 2{\cdot}U_6(2) = \emptyset$. Therefore $\Delta^*_{Fi_{22}}(pY, 12X, 22A) = \Delta(Fi_{22})$, for every class pY.

For $X \in \{H, I\}$ we have

$$\Delta^*_{Fi_{22}}(pY, 12X, 22A) = \Delta(Fi_{22}) - \Sigma(2{\cdot}U_6(2)) \ .$$

The power map yields $(12K)^2 = 6K$ and from Lemma 2.3.9 we deduce that Fi_{22} is $12K$-complementary generated. The result follows from Tables 10.V and 10.VI. □

Lemma 10.5.12 *The group Fi_{22} is $15A$-complementary generated.*

Proof. It is clear from Table 10.V and 10.VI that

$$\Delta^*_{Fi_{22}}(pX, 15A, 22A) = \Delta(Fi_{22}) - \Sigma(2{\cdot}U_6(2)) > 0 \ ,$$

for all classes pX with elements of prime order. The result follows. □

Lemma 10.5.13 *The group Fi_{22} is $18X$-complementary generated for all $X \in \{A, \ldots, D\}$.*

Proof. The subgroups $2^{10}{:}M_{22}$ and M_{12} contain no elements of order 18 and therefore

$$\Delta^*_{Fi_{22}}(pX, 18A, 11A) = \Delta(Fi_{22}) - \Sigma(2{\cdot}U_6(2)) \ ,$$

for all classes pX. Since $(18B)^{-1} = 18A$, the above relation holds if we replace $18A$ by $18B$. The intersection of the class $18C$ with the maximal subgroup $2{\cdot}U_6(2)$ is empty and therefore $\Delta^*_{Fi_{22}}(pX, 18C, 22A) = \Delta(Fi_{22})$. For the class $18D$, the following relation holds;

$$\Delta^*_{Fi_{22}}(pX, 18D, 22A) = \Delta(Fi_{22}) - \Sigma(2{\cdot}U_6(2)) \ ,$$

for all classes pX. The result now follows from Table 10.V and 10.VI. □

Lemma 10.5.14 *The group Fi_{22} is $24X$-complementary generated for all $X \in \{A, B\}$.*

Proof. The subgroup $2{\cdot}U_6(2)$ does not contain elements of order 24 and from Table IV we obtain $\Delta^*_{Fi_{22}}(pX, 24X, 22A) = \Delta(Fi_{22}) > 0$, for all classes pX with elements of prime order. The result follows. □

Proof of Theorem 5.1. The power maps of Fi_{22} yield $(14A)^2 = 7A$, $(16A)^2 = 8C = (16B)^2$, $(20A)^2 = 10B$, $(21A)^3 = 7A$, $(22A)^2 = 11B$, $(22B)^2 = 11A$ and $(30A)^2 = 15A$. An application of Lemma 2.3.9 to Lemmas 10.5.8, 10.5.9, 10.5.11 and 10.5.13 gives complementary generation of these classes. The theorem now follows from Lemma 10.5.3 to 10.5.15. □

TABLE 10.V

Structure constants of Fi_{22}

pX	$\Delta_{Fi_{22}}(pX,6K,22A)$	$\Delta_{Fi_{22}}(pX,8C,22A)$	$\Delta_{Fi_{22}}(pX,8D,22A)$	$\Delta_{Fi_{22}}(pX,9C,22A)$
$2A$	22	55	88	132
$2B$	4224	6446	36784	45144
$2C$	169290	304403	1121208	1350756
$3A$	11704	16280	101376	121968
$3B$	120340	219208	786368	948816
$3C$	1951840	3066624	14344704	17081856
$3D$	17691168	31109760	115061760	136667520
$5A$	478887552	775988224	3362596864	3985293312
$7A$	7025356800	11701352960	48037017600	56932761600
$11A$	13415124096	22335548928	91720521728	108690937344
$11B$	13416093504	22335548928	91720521728	108688884480
$13AB$	22992076800	38799129600	155196518400	183936614400

pX	$\Delta_{Fi_{22}}(pX,10A,22A)$	$\Delta_{Fi_{22}}(pX,10B,22A)$	$\Delta_{Fi_{22}}(pX,12EF,11A)$	$\Delta_{Fi_{22}}(pX,12G,22A)$
$2A$	88	44	33	44
$2B$	16555	36168	6798	5280
$2C$	624140	888140	257697	264132
$3A$	43879	98912	18304	13904
$3B$	444235	615120	183172	189992
$3C$	7086717	12431232	2949232	2703008
$3D$	63873018	88691328	26583744	27570048
$5A$	1724212545	2793824000	718502400	689762304
$7A$	25291363999	38922270464	10538035200	10401177600
$11A$	48284361983	74304378368	20119435776	19858037760
$11B$	48284361983	74304378368	20119663872	19858493952
$13AB$	82771476480	124157214720	34488115200	34488115200

pX	$\Delta_{Fi_{22}}(pX,12H,22A)$	$\Delta_{Fi_{22}}(pX,12I,22A)$	$\Delta_{Fi_{22}}(pX,12J,22A)$	$\Delta_{Fi_{22}}(pX,15A,11A)$
$2A$	55	44	44	121
$2B$	10560	10340	16632	40216
$2C$	392733	386408	502524	1217260
$3A$	27984	27808	45408	108944
$3B$	279378	274428	352704	855360
$3C$	4447080	4428864	6390912	15358464
$3D$	39960096	39869280	51194880	123019776
$5A$	1077753600	1077888064	1494484992	3586373120
$7A$	15807052800	15807077440	21349785600	51239564288
$11A$	30174629760	30178665088	40761667584	97823154176
$11B$	30174971904	30178665088	40761667584	97823154176
$13AB$	51732248832	51732172800	68976230400	165542952960

TABLE 10.V (Cont.)

pX	$\Delta_{Fi_{22}}(pX, 18AB, 11A)$	$\Delta_{Fi_{22}}(pX, 18C, 22A)$	$\Delta_{Fi_{22}}(pX, 18D, 22A)$
$2A$	55	110	66
$2B$	9504	14355	39600
$2C$	352242	703890	987822
$3A$	24816	37587	108240
$3B$	250932	506055	685212
$3C$	3957888	7214889	13782912
$3D$	35560800	73504530	98572320
$5A$	957889152	1839600477	3103588224
$7A$	14050713600	27736495875	43247001600
$11A$	26821010304	52954704099	82565163648
$11B$	26820497088	52953677667	82563624000
$13AB$	45984153600	91968307200	137952460800
pX	$\Delta_{Fi_{22}}(pX, 24A, 22A)$	$\Delta_{Fi_{22}}(pX, 24B, 22A)$	
$2A$	88	44	
$2B$	20680	29216	
$2C$	772816	734756	
$3A$	55616	80608	
$3B$	548856	509256	
$3C$	8857728	10315008	
$3D$	79738560	73846080	
$5A$	2155776128	2327678848	
$7A$	31614154880	32435201920	
$11AB$	60357330176	61927672576	
$13AB$	103464345600	1034643465600	

TABLE 10.VI

Structure constants $\Sigma(2{\cdot}U_6(2))$

pX	$\Sigma_{2\cdot U_6(2)}(pX, 8C, 22A)$	$\Sigma_{2\cdot U_6(2)}(pX, 8D, 22A)$	$\Sigma_{2\cdot U_6(2)}(pX, 10A, 22A)$
$2A$	11	44	44
$2B$	990	3960	3619
$2C$	8767	35068	36608
$3A$	1848	7392	7007
$3B$	3080	12320	12705
$3C$	73920	295680	303501
$3D$	0	0	0
$5A$	4790016	19160064	20177927
$7A$	20528640	82114560	87585905
$11A$	13060608	53242432	55337051
$11B$	13066240	53264960	55337051

TABLE 10.VI (Cont.)
Structure constants $\Sigma(2{\cdot}U_6(2))$

pX	$\Sigma_{2\cdot U_6(2)}(pX, 10B, 22A)$	$\Sigma_{2\cdot U_6(2)}(pX, 12EF, 11A)$	$\Sigma_{2\cdot U_6(2)}(pX, 12H, 22A)$
$2A$	22	11	33
$2B$	3520	1166	1936
$2C$	27610	11055	17523
$3A$	7040	2376	3080
$3B$	10252	3960	5786
$3C$	244464	95040	145376
$3D$	0	0	0
$5A$	15601696	6336000	9491328
$7A$	65700096	27371520	41057280
$11A$	42550464	17487360	25844544
$11B$	42550464	17487360	25844544
pX	$\Sigma_{2\cdot U_6(2)}(pX, 15A, 22A)$	$\Sigma_{2\cdot U_6(2)}(pX, 18AB, 11A)$	$\Sigma_{2\cdot U_6(2)}(pX, 18D, 22A)$
$2A$	33	33	44
$2B$	2024	2200	3608
$2C$	19976	5016	30492
$3A$	3344	3872	6336
$3B$	6600	7040	10560
$3C$	157344	171776	261888
$3D$	0	0	0
$5A$	10121408	11252736	17031168
$7A$	43788800	48660480	72990720
$11A$	27251840	30716928	46531584
$11B$	27251840	30716928	46531584

Chapter 11

The Spread of the Sporadic Simple Groups

It is shown by Binder in [10] and [11] that for any two non-trivial elements x_1 and x_2 of the symmetric group S_n, $n > 4$, there exists a third element y such that $S_n = \langle x_1, y\rangle = \langle x_2, y\rangle$. This work inspired the following definition by Brenner-Wiegold [19].

Definition 11.0.1 *Let r be any positive integer. A finite non-abelian group G is said to have spread r, if for every set $\{x_1, x_2, \ldots, x_r\}$ of distinct non-trivial elements of G, there exists a $y \in G$ such that $G = \langle x_i, y\rangle$ for all i. We say that G has exact spread t if G has spread t but not $t+1$.*

The element y in the definition we will refer to as *complementary*. Let Γ_r denote the collection of all non-abelian finite groups having spread r. Clearly $\Gamma_{r+1} \subseteq \Gamma_r$ for each r. We may therefore conclude from Lemma 2.3.6 that if $G \in \Gamma_r$, then G is subdirectly irreducible and G/G' is cyclic.

The content of Binder's cited work shows that the symmetric groups $S_{2n} \in \Gamma_2\backslash\Gamma_3$, while $S_{2n+1} \in \Gamma_3\backslash\Gamma_4$, apart from a few exceptions. The spread of the alternating groups are radically different to that of the symmetric groups. Brenner-Wiegold [19], [20] proved that the alternating groups $A_{2n} \in \Gamma_4\backslash\Gamma_5$, $n \geq 4$, and $A_{2n+1} \in \Gamma_3$, $n \geq 4$. Furthermore, the group A_{19} has spread $r = 17!/(3^4\, 6!) - 1$, but not spread $r+4$. This suggests that the spread of A_{2n+1} tends to infinity with n. It is also proved that for q a prime-power ($q \geq 11$, if q is odd and $q \geq 4$, otherwise) the group $PSL(2,q)$ has exact spread

$$\begin{array}{ll} q-1 & \text{if } \; q \equiv 1\,(mod\,4), \\ q-4 & \text{if } \; q \equiv 3\,(mod\,4), \\ q-2 & \text{if } \; q \text{ is a power 2.} \end{array}$$

The following result refines the definition of the spread of a finite group. A special case of this result can be found in Woldar [181].

Lemma 11.0.15 *A finite non-abelian group G has spread r if and only if for every set $\{x_1, x_2, \ldots, x_r\}$ of distinct elements of prime order in G, there exists an element $y \in G$ such that $G = \langle x_i, y\rangle$ for all i.*

Proof. The sufficiency condition is trivial. Conversely, let $\{z_1, z_2, \ldots, z_r\}$ be any set of distinct non-trivial elements of G. Then there exists positive integers m_i ($i = 1, \ldots, r$) such that $z_i^{m_i} = x_i$ and $o(x_i) = p_i$, where p_i is prime. Thus by assumption there exists $y \in G$ such that $G = \langle x_i, y\rangle = \langle z_i^{m_i}, y\rangle \subseteq \langle z_i, y\rangle \subseteq G$. Thus $G = \langle z_i, y\rangle$ for all i. $\square$

11.1 Main Result

We now restrict ourselves to the finite simple groups of spread 1 and 2. Let $\Gamma_1^{(k)}$ be the collection of all finite non-abelian groups with the property that $G \in \Gamma_1^{(k)}$ if and only if every non-trivial element of G together with an element of order k generate the group G. If $k > 2$ and G is kX-complementary generated, then $G \in \Gamma_1^{(k)}$. If G contains a unique conjugacy class with elements of order k and $G \in \Gamma_1^{(k)}$, then G is kX-complementary generated. However, this is not the case where $k = 2$. Brenner-Wiegold [19] proved that

$$\begin{array}{lll} PSL(2,q) & \in \Gamma_1^{(2)}, & \text{except when } \; q = 2,3,9, \\ PSL(n,q) & \notin \Gamma_1^{(2)}, & q \text{ odd and } n > 2, \text{ or} \\ & & q \text{ arbitrary and } n > 3, \\ PSL(3,2^s) & \in \Gamma_1^{(2)}, & \text{if and only if } s = 1. \end{array}$$

Moreover, they proved that every solvable group of spread 1 is already of spread 3. This led Brenner-Wiegold [19] to pose the following problems concerning the spread.

Problem 3: *Which groups lie in $\Gamma_1 \backslash \Gamma_2$ (that is, exact spread 1)? In particular, is this set perhaps finite?*

Problem 4: *Are almost all finite simple non-abelian groups in $\Gamma_1^{(2)}$ projective special linear groups?*

Suppose $G \in \Gamma_2$, H is a normal subgroup of G, and $\{xH, yH\}$ be a set of non-trivial distinct elements in G/H. Let $z \in G$ such that $G = \langle x, z\rangle = \langle y, z\rangle$. Then zH together with xH or yH generate the G/H. If K is a maximal normal subgroup of G, then G/K is a simple group with spread 2. Thus every group of spread 2 is an extension of a simple group of spread 2. Therefore in order to solve Problem 3, we need to find all finite non-abelian simple groups of spread 1 and 2. As noted earlier, the symmetric group S_n, the alternating group A_n and the linear group $PSL(2, q)$ do not have exact spread 1. Woldar [181] proved that the sporadic simple groups have spread 1 and we will now show that the sporadic simple groups do not have exact spread 1.

Theorem 11.1.1 *Every sporadic simple group has spread 2.*

We first set up the tools needed to prove this theorem. Let G be a sporadic simple group, and $\{x, y\} \subset G$ any two-element set of prime order elements. We hope to establish the existence of an element $z \in nX$ such that z lies outside every maximal (so every proper) subgroup of G which contains x or y. Clearly any such z will be complementary, whence G has spread 2.

For $a \in G$, define

$$nX(a) = \{z \in nX \mid G = \langle a, z\rangle\}.$$

Clearly, if

$$|nX(x)| + |nX(y)| > |nX|, \tag{11.1}$$

then there exists $z \in nX$ such that $G = \langle x, z\rangle = \langle y, z\rangle$. If this is true for every pair of elements $\{x, y\}$ of prime order, then G has spread 2. Thus our aim is to find lower bounds for $|nX(x)|$, for every prime order element.

Let $x \in qY$, where q is a prime, and let $\{M_1, \ldots, M_s\}$ denote a complete set of pairwise non-conjugate maximal subgroups of G which contain x, and set $Y_i = |M_i \cap nX|$. Let h_i be the number of conjugates of M_i containing x. Then $\sum h_i Y_i$ is an upper bound for the number of elements of nX which lie in some maximal subgroup of G which contain x. That is, $|nX - nX(x)| \leq \sum h_i Y_i$. Hence

$$|nX(x)| \geq |nX| - \sum_{i=1}^{s} h_i Y_i. \tag{11.2}$$

The values of h_i and Y_i are easily computed by the formulae

$$h_i = \sum_k \frac{|C_G(x)|}{|C_{M_i}(x_{ik})|} \quad \text{and} \quad Y_i = \sum_j \frac{|M_i|}{|C_{M_i}(y_{ij})|} \ ,$$

where $\{x_{11}, x_{12}, \ldots\}$ are representatives of the conjugacy classes of elements of M_i that fuse to qY and $\{y_{11}, y_{12}, \ldots\}$ are representatives of the conjugacy classes of elements of M_i that fuse to nX. Note that the lower bound for $|nX(x)|$ given by (11.2) is the same for all elements belonging to the conjugacy class qY. Thus to establish that G has spread 2 we only need to verify the relation (11.1) for one representative from each conjugacy class with prime order elements.

Proof of Theorem 11.1.1: Let G is a sporadic simple group, other than M_{12} and the Monster, and choose nX in relation (11.1) to be the conjugacy class pA, where p is the largest prime divisor of $|G|$. Then it is easy to check from Tables 11.I and 11.II that the relation (11.1) holds for any two representatives $\{x, y\}$ belonging to conjugacy classes with prime order elements. Thus G has spread 2. For the group M_{12} we choose nX to be the conjugacy class $10A$. Then it follows from relation (11.1) and Tables 11.I and 11.II that M_{12} has spread 2.

For the Monster M we choose nX to be the conjugacy class $71A$. The existence of the subgroup $L_2(71)$ in the Monster M is a difficult open problem. First suppose that $L_2(71)$ is not a subgroup of M. In this case the conjugates of the Sylow normalizer $N_M(\langle y \rangle) \cong 71{:}35$, where $y \in 71A$, from the unique conjugacy class of maximal subgroups of M that contains elements of order 71. Then it is easy to show (similar to the argument used for the Baby Monster B) that M has spread 2.

TABLE 11.I
Order of conjugacy classes of the sporadics

G	nX	$\lvert nX \rvert$
M_{11}	$11A$	720
M_{12}	$10A$	9504
J_1	$19A$	9240
M_{22}	$11A$	40320
J_2	$7A$	86400
M_{23}	$23A$	443520
HS	$11A$	4032000
J_3	$19A$	2643840
M_{24}	$23A$	10644480
McL	$11A$	81648000
He	$17A$	237081600
Ru	$29A$	5031936000
SuZ	$13A$	34488115200
$O'N$	$31A$	14865016320
Co_3	$23A$	21555072000
Co_2	$23A$	1839366144000
Fi_{22}	$13A$	4962885888000
HN	$19A$	14370048000000
Th	$31A$	2927288512512000
Ly	$67A$	772614612000000
Fi_{23}	$23A$	177803064056217600
Co_1	$23A$	180772904632320000
J_4	$43A$	2018036535955292160
Fi'_{24}	$29A$	4328295548933162803200
B	$47A$	88399605983540982791012352000000
M	$71A$	1138052710978187149135859021072832052120780800000000

On the other hand, assume $L_2(71)$ is a subgroup of M. Then $L_2(71)$ is a maximal subgroup of M and by the ATLAS $71{:}35 \leq L_2(71)$. Now $|C_{L_2(71)}(y)| = 71$ and therefore $Y = |M \cap 71A| = 352800$. Let $x \in qY$ be an arbitrary element of prime order . Then

$$h = \sum_k \frac{|C_M(x)|}{|C_{L_2(71)}(x_{ik})|} \leq |C_M(x)|,$$

where $\{x_{11}, x_{12}, \ldots\}$ of representatives of the conjugacy classes of elements of $L_2(71)$ that fuse to qY. Now it follows easily from the ATLAS that $hY \leq |C_M(x)|Y < \frac{1}{2}|71A|$ for all conjugacy classes qY with prime order representatives. From this observation relation (11.1) easily follows, whence M has spread 2. This completes the proof. □

TABLE 11.II

G	qY	M_i	h_i	Y_i	$\sum h_i Y_i$
M_{11}	$2A$	$L_2(11)$	4	60	240
	$3A$	$L_2(11)$	3	60	180
	$5A$	$L_2(11)$	2	60	120
	$11AB$	$L_2(11)$	1	60	60
M_{12}	$2A$	$M_{10}{:}2$	6	144	2592
		$M_{10}{:}2$	6	144	
		$2 \times S_5$	36	24	
	$2B$	$M_{10}{:}2$	10	144	3168
		$M_{10}{:}2$	10	144	
		$2 \times S_5$	12	24	
	$3A$	$M_{10}{:}2$	3	144	864
		$M_{10}{:}2$	3	144	
	$3B$	$2 \times S_5$	3	24	72
	$5A$	$M_{10}{:}2$	1	144	312
		$M_{10}{:}2$	1	144	
		$2 \times S_5$	1	24	
	$11AB$	$\emptyset$	0	0	0
J_1	$2A$	$19{:}6$	20	6	120
	$3A$	$19{:}6$	10	6	60
	$5AB$	$\emptyset$	0	0	0
	$7A$	$\emptyset$	0	0	0
	$11A$	$\emptyset$	0	0	0
	$19ABC$	$19{:}6$	1	6	6

TABLE 11.II (Cont.)

G	qY	M_i	h_i	Y_i	$\sum h_i Y_i$
M_{22}	$2A$	$L_2(11)$	32	60	1920
	$3A$	$L_2(11)$	6	60	360
	$5A$	$L_2(11)$	2	60	120
	$7A$	$\emptyset$	0	0	0
	$11AB$	$L_2(11)$	1	60	60
J_2	$2A$	$U_3(3)$	20	864	23040
		$L_3(2){:}2$	120	48	
	$2B$	$L_3(2){:}2$	20	48	960
	$3A$	$U_3(3)$	10	864	8640
	$3B$	$U_3(3)$	4	864	3744
		$L_3(2){:}2$	6	48	
	$5AB$	$\emptyset$	0	0	0
	$5CD$	$\emptyset$	0	0	0
	$7A$	$U_3(3)$	2	864	1176
		$L_3(2){:}2$	1	48	
M_{23}	$2A$	$\emptyset$	0	0	0
	$3A$	$\emptyset$	0	0	0
	$5A$	$\emptyset$	0	0	0
	$7AB$	$\emptyset$	0	0	0
	$11AB$	23:11	5	11	55
	$23AB$	23:11	1	11	11
HS	$2A$	M_{22}	20	40320	1036800
		M_{11}	160	720	
		M_{11}	160	720	
	$2B$	$\emptyset$	0	0	0
	$3A$	M_{22}	10	40320	432000
		M_{11}	20	720	
		M_{11}	20	720	

TABLE 11.II (Cont.)

G	qY	M_i	h_i	Y_i	$\sum h_i Y_i$
HS	$5A$	$\emptyset$	0	0	0
	$5B$	$\emptyset$	0	0	0
	$5C$	M_{22}	5	40320	208800
		M_{11}	5	720	
		M_{11}	5	720	
	$7A$	M_{22}	2	40320	80640
	$11AB$	M_{22}	1	40320	41760
		M_{11}	1	720	
		M_{11}	1	720	
J_3	$2A$	$L_2(19)$	96	180	34560
		$L_2(19)$	96	180	
	$3A$	$\emptyset$	0	0	0
	$3B$	$L_2(19)$	27	180	9720
		$L_2(19)$	27	180	
	$5AB$	$L_2(19)$	3	180	1080
		$L_2(19)$	3	180	
	$17AB$	$\emptyset$	0	0	0
	$19AB$	$L_2(19)$	1	180	360
		$L_2(19)$	1	180	
M_{24}	$2A$	M_{23}	8	443520	3548160
	$2B$	$L_2(23)$	320	264	84480
	$3A$	M_{23}	6	443520	2661120
	$3B$	$L_2(23)$	42	264	11088
	$5A$	M_{23}	4	443520	1174080
	$7AB$	M_{23}	3	443520	1330560
	$11A$	M_{23}	2	443520	888360
		$L_2(23)$	5	264	
	$23AB$	M_{23}	1	443520	443784
		$L_2(23)$	1	264	

TABLE 11.II (Cont.)

G	qY	M_i	h_i	Y_i	$\sum h_i Y_i$
McL	$2A$	M_{11}	840	720	9072000
		M_{22}	105	40320	
		M_{22}	105	40320	
	$3A$	$\emptyset$	0	0	0
	$3B$	M_{11}	54	720	2216160
		M_{22}	27	40320	
		M_{22}	27	40320	
	$5A$	$\emptyset$	0	0	0
	$5B$	M_{11}	5	720	406800
		M_{22}	5	40320	
		M_{22}	5	40320	
	$7AB$	M_{22}	2	40320	161280
		M_{22}	2	40320	
	$11AB$	M_{11}	1	720	81360
		M_{22}	1	40320	
		M_{22}	1	40320	
He	$2A$	$S_4(2){:}2$	154	115200	17740800
	$2B$	$S_4(2){:}2$	42	115200	4838400
	$3A$	$S_4(2){:}2$	42	115200	4838400
	$3B$	$\emptyset$	0	0	0
	$5A$	$S_4(2){:}2$	8	115200	921600
	$7A$	$\emptyset$	0	0	0
	$7B$	$\emptyset$	0	0	0
	$7C$	$\emptyset$	0	0	0
	$7D$	$\emptyset$	0	0	0
	$7E$	$\emptyset$	0	0	0
	$17AB$	$S_4(2){:}2$	1	115200	115200
Ru	$2A$	$\emptyset$	0	0	0
	$2B$	$L_2(29)$	4160	420	1747200

TABLE 11.II (Cont.)

G	qY	M_i	h_i	Y_i	$\sum h_iY_i$
Ru	$3A$	$L_2(29)$	144	420	60480
	$5A$	$\emptyset$	0	0	0
	$5B$	$L_2(29)$	40	420	16800
	$7A$	$L_2(29)$	6	420	2520
	$13A$	$\emptyset$	0	0	0
	$29AB$	$L_2(29)$	1	420	420
Suz	$2A$	$G_2(4)$	54	19353600	1104814080
		$L_3(3){:}2$	34560	864	
		$L_3(3){:}2$	34560	864	
	$2B$	$G_2(4)$	42	19353600	830753280
		$L_2(25)$	6720	1800	
		$L_3(3){:}2$	3360	864	
		$L_3(3){:}2$	3369	864	
	$3A$	$G_2(4)$	162	19353600	3135283200
	$3B$	$L_3(3){:}2$	324	864	559872
		$L_3(3){:}2$	324	864	
	$3C$	$G_2(4)$	18	19353600	349161840
		$L_2(25)$	270	1800	
		$L_3(3){:}2$	180	864	
		$L_3(3){:}2$	180	864	
	$5A$	$G_2(4)$	12	19353600	232372800
		$L_2(25)$	72	1800	
	$5B$	$G_2(4)$	2	19353600	38728800
		$L_2(25)$	12	1800	
	$7A$	$G_2(4)$	4	19353600	77414400
	$11A$	$\emptyset$	0	0	0
	$13AB$	$G_2(4)$	1	19353600	19360728
		$L_2(25)$	3	1800	
		$L_3(3){:}2$	1	864	
		$L_3(3){:}2$	1	864	

TABLE 11.II (Cont.)

G	qY	M_i	h_i	Y_i	$\sum h_i Y_i$
$O'N$	$2A$	$L_2(31)$	5040	480	4838400
		$L_2(31)$	5040	480	
	$3A$	$L_2(31)$	216	480	207360
		$L_2(31)$	216	480	
	$5A$	$L_2(31)$	24	480	23040
		$L_2(31)$	24	480	
	$7A$	$\emptyset$	0	0	0
	$7B$	$\emptyset$	0	0	0
	$11A$	$\emptyset$	0	0	0
	$19ABC$	$\emptyset$	0	0	0
	$31AB$	$L_2(31)$	1	480	690
		$L_2(31)$	1	480	
Co_3	$2A$	M_{23}	1080	443520	479001600
	$2B$	$\emptyset$	0	0	0
	$3A$	$\emptyset$	0	0	0
	$3B$	M_{23}	162	443520	71850240
	$3C$	$\emptyset$	0	0	0
	$5A$	$\emptyset$	0	0	0
	$5B$	M_{23}	20	443520	8870400
	$7A$	M_{23}	6	443520	2661120
	$11AB$	M_{23}	2	443520	887040
	$23AB$	M_{23}	1	443520	443520
Co_2	$2A$	$\emptyset$	0	0	0
	$2B$	M_{23}	15360	443520	6812467200
	$2C$	$\emptyset$	0	0	0
	$3A$	$\emptyset$	0	0	0
	$3B$	M_{23}	864	443520	383201280
	$5A$	$\emptyset$	0	0	0

TABLE 11.II (Cont.)

G	qY	M_i	h_i	Y_i	$\sum h_i Y_i$
Co_2	$5B$	M_{23}	40	443520	17740800
	$7A$	M_{23}	4	443520	1774080
	$11A$	M_{23}	2	443520	887040
	$23AB$	M_{23}	1	443520	443520
Fi_{22}	$2A$	$O_7(3)$	1408	352719360	993257717760
		$O_7(3)$	1408	352719360	
	$2B$	${}^2F_4(2)'$	5184	1382400	187758673920
		$O_7(3)$	256	352719360	
		$O_7(3)$	256	352719360	
	$2C$	${}^2F_4(2)'$	1152	1382400	91888680960
		$O_7(3)$	128	352719360	
		$O_7(3)$	128	352719360	
	$3A$	$O_7(3)$	112	352719360	79009136640
		$O_7(3)$	112	352719360	
	$3B$	$O_7(3)$	148	352719360	104404930560
		$O_7(3)$	148	352719360	
	$3C$	$O_7(3)$	49	352719360	34566497280
		$O_7(3)$	49	352719360	
	$3D$	${}^2F_4(2)'$	162	1382400	9394652160
		$O_7(3)$	13	352719360	
		$O_7(3)$	13	352719360	
	$5A$	${}^2F_4(2)'$	12	1382400	3543782400
		$O_7(3)$	5	352719360	
		$O_7(3)$	5	352719360	
	$7A$	$O_7(3)$	3	352719360	2116316160
		$O_7(3)$	3	352719360	
	$11AB$	$\emptyset$	0	0	0
	$13AB$	${}^2F_4(2)'$	1	1382400	706821120
		$O_7(3)$	1	352719360	
		$O_7(3)$	1	352719360	

TABLE 11.II (Cont.)

G	qY	M_i	h_i	Y_i	$\sum h_iY_i$
HN	$2A$	$\emptyset$	0	0	0
	$2B$	$U_3(8){:}3$	800	870912	696729600
	$3A$	$U_3(8){:}3$	1500	870912	1306368000
	$3B$	$U_3(8){:}3$	210	870912	182891520
	$5A$	$\emptyset$	0	0	0
	$5B$	$\emptyset$	0	0	0
	$5C$	$\emptyset$	0	0	0
	$5D$	$\emptyset$	0	0	0
	$5E$	$\emptyset$	0	0	0
	$7A$	$U_3(8){:}3$	20	870912	17418240
	$11A$	$\emptyset$	0	0	0
	$19AB$	$U_3(8){:}3$	1	870912	870912
Ly	$2A$	67:22	181440	22	39916800
	$3A$	$\emptyset$	0	0	0
	$3B$	$\emptyset$	0	0	0
	$5A$	$\emptyset$	0	0	0
	$5B$	$\emptyset$	0	0	0
	$7A$	$\emptyset$	0	0	0
	$11AB$	67:22	15	22	330
	$31A$	$\emptyset$	0	0	0
	$31B$	$\emptyset$	0	0	0
	$31C$	$\emptyset$	0	0	0
	$31D$	$\emptyset$	0	0	0
	$31E$	$\emptyset$	0	0	0
	$37AB$	$\emptyset$	0	0	0
	$67ABC$	67:22	1	22	22

TABLE 11.II (Cont.)

G	qY	M_i	h_i	Y_i	$\sum h_i Y_i$
Th	$2A$	$2^5{:}L_5(2)$	2169	30965760	67164733440
	$3A$	$2^5{:}L_5(2)$	3159	30965760	97820835840
	$3B$	$\emptyset$	0	0	0
	$3C$	$2^5{:}L_5(2)$	486	30965760	15049709280
		31:15	23328	15	
	$5A$	$2^5{:}L_5(2)$	100	30965760	3096588000
		31:15	800	15	
	$7A$	$2^5{:}L_5(2)$	42	30965760	1300561920
	$13A$	$\emptyset$	0	0	0
	$19A$	$\emptyset$	0	0	0
	$31AB$	$2^5{:}L_5(2)$	3	30965760	92897295
		31:15	1	15	
Fi_{23}	$2A$	$2^{11}{\cdot}M_{23}$	142152	908328960	129120778321920
	$2B$	$2^{11}{\cdot}M_{23}$	214731	908328960	195046385909760
	$2C$	$2^{11}{\cdot}M_{23}$	6507	908328960	5914000097280
		$L_2(23)$	13271040	264	
	$3A$	$\emptyset$	0	0	0
	$3B$	$\emptyset$	0	0	0
	$3C$	$2^{11}{\cdot}M_{23}$	6561	908328960	5959546306560
	$3D$	$L_2(23)$	3149280	264	831409920
	$5A$	$2^{11}{\cdot}M_{23}$	210	908328960	190749081600
	$7A$	$2^{11}{\cdot}M_{23}$	30	908328960	27249868800
	$11A$	$2^{11}{\cdot}M_{23}$	4	908328960	3633321120
		$L_2(23)$	20	264	
	$13AB$	$\emptyset$	0	0	0
	$17A$	$\emptyset$	0	0	0
	$23AB$	$2^{11}{\cdot}M_{23}$	1	908328960	908329224
		$L_2(23)$	1	264	

TABLE 11.II (Cont.)

G	qY	M_i	h_i	Y_i	$\sum h_iY_i$
Co_1	$2A$	Co_2	2280	1839366144000	5565186205286400
		Co_3	30720	21555072000	
		M_{24}	32535	21799895040	
	$2B$	M_{24}	4095	21799895040	89270570188800
	$2C$	Co_2	264	1839366144000	568606662328320
		Co_3	2048	21555072000	
		M_{24}	1783	21799895040	
	$3A$	$\emptyset$	0	0	0
	$3B$	Co_2	378	1839366144000	775817049047040
		Co_3	2016	21555072000	
		M_{24}	1701	21799895040	
	$3C$	Co_2	27	1839366144000	50438868480000
		Co_3	36	21555072000	
	$3D$	Co_3	120	21555072000	5529594470400
		M_{24}	135	21799895040	
	$5A$	$\emptyset$	0	0	0
	$5B$	Co_2	12	1839366144000	26293994496000
		Co_3	120	21555072000	
		M_{24}	75	21799895040	
	$5C$	Co_2	25	1839366144000	46199704320000
		Co_3	10	21555072000	
	$7A$	$\emptyset$	0	0	0
	$7B$	Co_2	21	1839366144000	39535429570560
		Co_3	28	21555072000	
		M_{24}	14	21799895040	
	$11A$	Co_2	6	1839366144000	11230926981120
		Co_3	6	21555072000	
		M_{24}	3	21799895040	
	$13A$	$\emptyset$	0	0	0
	$23AB$	Co_2	1	1839366144000	1882721111040
		Co_3	1	21555072000	
		M_{24}	1	21799895040	

TABLE 11.II (Cont.)

G	qY	M_i	h_i	Y_i	$\sum h_i Y_i$
J_4	$2A$	$\emptyset$	0	0	0
	$2B$	43:14	129761280	14	1816657920
	$3A$	$\emptyset$	0	0	0
	$5A$	$\emptyset$	0	0	0
	$7AB$	43:14	180	14	11760
	$11AB$	$\emptyset$	0	0	0
	$23A$	$\emptyset$	0	0	0
	$29A$	$\emptyset$	0	0	0
	$31ABC$	$\emptyset$	0	0	0
	$37ABC$	$\emptyset$	0	0	0
	$43AB$	43:14	1	14	14
Fi'_{24}	$2A$	$\emptyset$	0	0	0
	$2B$	29:14	11466178560	14	160526499840
	$3A$	$\emptyset$	0	0	0
	$3B$	$\emptyset$	0	0	0
	$3C$	$\emptyset$	0	0	0
	$3D$	$\emptyset$	0	0	0
	$3E$	$\emptyset$	0	0	0
	$5A$	$\emptyset$	0	0	0
	$7A$	$\emptyset$	0	0	0
	$7B$	29:14	882	14	12348
	$11A$	$\emptyset$	0	0	0
	$13A$	$\emptyset$	0	0	0
	$17A$	$\emptyset$	0	0	0
	$23AB$	$\emptyset$	0	0	0
	$29AB$	29:14	1	14	14

TABLE 11.II (Cont.)

G	qY	M_i	h_i	Y_i	$\sum h_i Y_i$
B	$2ABCD$	$\emptyset$	0	0	0
	$3AB$	$\emptyset$	0	0	0
	$5AB$	$\emptyset$	0	0	0
	$7A$	$\emptyset$	0	0	0
	$11A$	$\emptyset$	0	0	0
	$13A$	$\emptyset$	0	0	0
	$17A$	$\emptyset$	0	0	0
	$19A$	$\emptyset$	0	0	0
	$23AB$	47:23	22	23	506
	$31A$	$\emptyset$	0	0	0
	$47AB$	47:23	1	23	23

Bibliography

[1] A. A. Albert and J. G. Thompson, *Two-element generation of the projective unimodular group*, Illinois Math. J. **3** (1959), 421–439.

[2] M. Aschbacher, *Sporadic Groups*, Cambridge Univ. Press, London-New York, 1994.

[3] M. Aschbacher and R. Guralnick, *Solvable generation of groups and sylow subgroups of the lower central series*, J. Algebra **77** (1982), 189–201.

[4] M. Aschbacher and R. Guralnick, *Some applications of the first cohomology group*, J. Algebra **90** (1984), 446–460.

[5] M. D. Atkinson, *Computational Group Theory*, Academic Press, London-New York, 1984.

[6] A. Ballester-Bolinches and L. M. Ezquerro, *On maximal subgroups of finite groups*, Comm. Algebra **19**(8) (1991), 2373–2394.

[7] A. L. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag, New York-Heidelberg, 1983.

[8] D. J. Benson, *The Simple Group J_4*, Ph.D thesis, Cambridge, 1981.

[9] N. L. Biggs and A. T. White, *Permutations Groups and Combinatorial Structures*, Cambridge Univ. Press, London-New York, 1979.

[10] G. Ja. Binder, *The bases of the symmetric group*, Izv. Vysš. Učebn. Zaved. Matematika **78**(11) (1968), 19–25.

[11] G. Ja. Binder, *The two-element bases of the symmetric group*, Izv. Vysš. Učebn. Zaved. Matematika **90**(1) (1970), 9–11.

[12] J. Bovey and A. Williamson, *The probability of generating the symmetric group*, Bull. London Math. Soc. **10** (1978), 91–96.

[13] H. R. Brahana, *Regular maps and their groups*, Amer. J. Math. **49** (1927), 268–284.

[14] H. R. Brahana, *Certain prefect groups generated by two operators of order two and three*, Amer. J. Math. **50** (1928), 345–356.

[15] H. R. Brahana, *Pairs of generators of the known simple groups whose order are less than one million* , Ann. of Math. **31** (1930), 529–549.

[16] R. Brauer, *Representations of Finite Groups*, Lecture Notes on Mordern Math. Vol I, Wiley, New York, 1963, 133–175.

[17] R. Brauer, *A note on theorems of Burnside and Blichfieldt*, Proc. Amer. Math. Soc. **15**(1) (1964), 31–34.

[18] J. L. Brenner, R. M. Guralnick and J. Wiegold, *Two-generator groups III*, Amer. Math. Soc. **33** (1984), 82–89.

[19] J. L. Brenner and J. Wiegold, *Two-generator groups, I*, Michigan Math. J. **22** (1975), 53–64.

[20] J. L. Brenner and J. Wiegold, *Two-generator groups, II*, Bull. Austral. Math. Soc. **22** (1980), 113–124.

[21] T. Breuer, *Subgroups of J_4 inducing the same permutation character*, Comm. Algebra **23**(9) (1995), 3173–3176.

[22] N. Bryce, *On the Mathieu group M_{23}*, J. Austral. Math. Soc. **12** (1971), 385–392.

[23] G. Butler, *The maximal subgroups of the sporadic simple group of Held*, J. Algebra **69** (1981), 67–81.

[24] C. M. Campbell, M. D. E. Conder and E. F. Robertson, *Defining-Relations for Hurwitz groups*, Glasgow Math. J. **36** (1994), 363–370.

[25] J. J. Cannon, *Computers in group theory: a survey*, Comm. ACM **12**(1) (1969), 3–12.

[26] J. J. Cannon, *An introduction to the group theory language CAYLEY*, Computational Group Theory (M. Atkinson, ed.), Academic Press, San Diego, (1984), 145–183.

[27] R. D. Carmicheal, *Abstract definition of the symmetric and alternating groups and certain other permutation groups*, Quart. J. Math. **49** (1923), 226–270.

[28] R. W. Carter, *Simple Groups of Lie Type*, Wiley, London-New York, 1972.

[29] M. Cazzola and L. Di Martino, $(2,3)$-*Generation of* $PSp(4,q)$, $q = p^n$, $p \neq 2,3$, Results in Mathematics **23** (1993), 221–232.

[30] J. Cohen, *On non Hurwitz groups*, Glasgow Math. J. **22** (1981), 1–7.

[31] M. D. E. Conder, *Generators for alternating and symmetric groups* , J. London Math. Soc. **22**(2) (1980), 75–86.

[32] M. D. E. Conder, *Some results on quotients of triangle groups*, Bull. Austral. Math. Soc. **30** (1984), 73–90.

[33] M. D. E. Conder, *The genus of compact Riemann surfaces with maximal auotomorphism group*, J. Algebra **108** (1987), 204–247.

[34] M. D. E. Conder, *Hurwitz groups: A brief survey*, Bull. Amer. Math. Soc. **23**(2) (1990), 359–370.

[35] M. D. E. Conder, *Random walks in large finite groups*, Austral. J. Combin. **4** (1991), 49–57.

[36] M. D. E. Conder, *Symmetric genus of the Mathieu groups*, Bull. London Math. Soc. **23** (1991), 445–453.

[37] M. D. E. Conder, *Regular maps with small paramaters*, J. Austral. Math. Soc. (Series A) **57** (1994), 103–112.

[38] M. D. E. Conder and J. McKay, *A necessary condition for transitivity of a finite permutation group*, Bull. London Math. Soc. **20** (1988), 235–238.

[39] M. D. E. Conder, R. A. Wilson and A. J. Woldar, *The symmetric genus of sporadic groups*, Proc. Amer. Math. Soc. **116** (1992), 653-663.

[40] J. H. Conway, *A group of order 8,315,553,613,086,720,000*, Bull. London Math. Soc. **1**, (1969) 79-88.

[41] J. H. Conway, *A construction for the smallest Fischer group*, Finite Groups'72 (Gagen, Hale and Shult, eds.), North-Holland, Amsterdam, (1973), 27–35.

[42] J. H. Conway and D. B. Wales, *Matrix generators for* J_3, J. Algebra **29** (1974), 474–476.

[43] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Oxford Univ. Press (Clarendon), Oxford, 1985.

[44] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, Springer-Verlag, New York-Heidelberg, 1957.

[45] P. M. Curran, *Cohomology of finitely presented groups*, Pacific J. Math. **42** (1972), 615–620.

[46] R. T. Curtis, *On subroups of ·O. I. Lattice stabilizers*, J. Algebra **27** (1973), 549–573.

[47] R. T. Curtis, *On subroups of ·O. II. Local structure*, J. Algebra **63** (1980), 413–434.

[48] F. Dalla Volta, *Gruppi sporadici generati da tre involuzionni*, RILS **119** (1985), 65–87.

[49] I. M. S. Dey and J. Wiegold, *Generators for alternating and symmetric groups*, **12** (1971), 63–68.

[50] L. E. Dickson, *Linear groups*, Dover, New York, 1958.

[51] L. Di Martino, *Simple linear groups all of whose involutions are* 2*-reflections*, Boll. Un. Mat Ital. (5) **15**-B (1978), 509–526.

[52] L. Di Martino and M. C. Tamburini, *2-Generation of finite simple groups and some related topics*, Generators and Relations in Groups and Geometry, A. Barlotti et al., Kluwer Acad. Publ., New York (1991), 195 - 233.

[53] L. Di Martino and N. Vavilov, $(2,3)$*-Generation of* $SL(n,q)$*. I. Cases* $n = 5, 6, 7$, Comm. Algebra **22**(4) (1994), 1321–1347.

[54] D. Z. Dokovic, *Presentations of some finite simple groups*, J. Austral. Math. Soc. (Ser. A) **45** (1988), 143–168.

[55] G. M. Enright, *A description of the Fischer group* F_{22}, J. Algebra **46** (1977), 334–343.

[56] G. M. Enright, *Subgroups generated by transpositions in* F_{22} *and* F_{23}, Comm. Algebra **6** (1978), 823–837.

[57] B. Everitt, *Permutation representation of the* $(2,4,r)$ *triangle group*, Bull. Austral. Math. Soc. **49** (1994), 499–511.

[58] H. M. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag, New York-Heidelberg, 1980.

[59] B. Fine et. al., *Combinatorial Group Theory*, Contemporary Mathematics, Amer. Math. Soc., Providence, 1990.

[60] L. Finkelstein, *The maximal subgroups of Conway's group* C_3 *and McLauglin's Group*, J. Algebra **25** (1973), 58–89.

[61] L. Finkelstein, *The maximal subgroups of Janko's simple group of order 50,232,960*, J. Algebra **30** (1974), 122–143.

[62] L. Finkelstein and A. Rudvalis, *Maximal subgroups of the Hall-Janko group*, J. Algebra **24** (1973), 486–493.

[63] B. Fischer, *Finite groups generated by 3-transpositions. I*, Inventiones Math. **13** (1971), 232–246.

[64] R. Fricke and F. Klein, *Vorlesungen ueber die Theorie der Automophen Funktionen*, Vol. 1, Teubner, Leipzig, 1897.

[65] T. M. Gagen, *A charaterization of Janko's simple group*, Proc. Amer. Math. Soc., **19** (1968), 1393–1395.

[66] S. Ganief and J. Moori, $(2,3,t)$*-Generations for the Janko group* J_3, Comm. Algebra **23**(12) (1995), 4427–4437.

[67] S. Ganief and J. Moori, pX*-Complementary generations of the Janko groups* J_1, J_2 *and* J_3, Comm. Algebra **24**(3) (1996), 809–820.

[68] S. Ganief and J. Moori, (p,q,r)*-Generations and* nX*-complementary generations of the Higman-Sims group* HS *and the McLaughlin group* McL, J. Algebra, to appear.

[69] S. Ganief and J. Moori, (p,q,r)*-Generations of the smallest Conway group* Co_3, J. Algebra, to appear.

[70] S. Ganief and J. Moori, 2*-Generations of the Smallest Fischer Group* F_{22}, Nova J, Algebra and Geomerty, to appear.

[71] D. Garbe, *Über eine klasse von arthmetisch definierbaren Normalteilern der Modulgruppe*, Math. Ann. **235** (1978), 195–215.

[72] D. Gorenstein, *On the maximal subgroups of finite simple groups*, J. Algebra **1** (1964), 168–213.

[73] R. Gow, *Products of two involutions in classical groups of characteristic 2*, J. Algebra **71** (1981), 583–591.

[74] L. Greenberg , *Discrete groups of motions*, Canad. J. Math. **12** (1960), 414–425.

[75] R. M. Guralnick, *Generation of simple groups*, J. Algebra **103** (1986), 381–401.

[76] R. M. Guralnick, *On the number of generators of a finite group*, Arch. Math. **53** (1989), 521–523.

[77] R. M. Guralnick and J. G. Thompson, *Finite groups of genus zero*, J. Algebra **131** (1990), 303–341.

[78] M. Hall Jr., *A search for simple groups of order less than one million*, Computational Problems in Abstract Algebra, Pergamon, New York, 1969.

[79] M. Heins, *Fundamental polygons of Fuchsian and Fuchsoid groups*, Ann. Acad. Sci. Fenn. A **337** (1964), 30 pp.

[80] A. H. M. Hoare, A. Karrass and D. Solitar, *Subgroups of finite index of Fuchsian froups*, Math. Z. **120** (1971), 289–298.

[81] A. H. M. Hoare, A. Karrass and D. Solitar, *Subgroups of infinite index of Fuchsian groups*, Math. Z. **125** (1972), 59–69.

[82] G. Higman and J. McKay, *On Janko's group of order 50,232,960*, Bull. London Math. Soc. **1** (1969), 89–94.

[83] A. Hurwitz, *Üeber algebraische gebilde mit eindeutigen transformationen in sich*, Math. Ann. **41** (1893), 408–442.

[84] I. M. Iscaacs, *Character Theory of Finite Groups*, Dover, New-York, 1976.

[85] N. Ito, *On a certain class of connected symmetric trivalent graphs*, Math. J. Okayama Univ. **25** (1983), 145–152.

[86] Z. Janko, *A finite simple group with sbelian Sylow 2-subgroup* , Proc. Nat. acad. Sci. USA **53** (1965), 657–658.

[87] Z. Janko, *A finite simple group with abelian Sylow 2-subgroup, and its characterizations*, J. Algebra **3** (1966), 147–186.

[88] Z. Janko, *Some new simple groups of finite order*, Theory of Finite Groups, Brauer and Sah eds., Benjamin 1969, 63–64.

[89] Z. Janko, *Some new simple groups of finite order*, Symp. Math. **1** (1968), 24–64.

[90] Z. Janko, *A New simple group of order* 86, 775, 571, 046, 077, 562, 880 *which possesses* M_{24} *and the full covering group of* M_{22} *as subgroups*, J. Algebra **42** (1976), 564–596.

[91] C. Jansen, K. Lux, R. A. Parker and R. A. Wilson, *An Atlas of Brauer Characters*, Oxford Univ. Press, Oxford, 1995.

[92] G. James and M. Liebeck, *Representations and Characters of Groups*, Cambridge Univ. Press, London-New York, 1993.

[93] G. A. Jones, *Ree groups and Riemann surfaces*, J. Algebra **165** (1994), 41–62.

[94] G. A. Jones and D. Singerman, *Theory of maps on orientable surfaces*, Proc. London Math. Soc. **37**(3) (1978), 273–307.

[95] G. A. Jones and D. Singerman, *Complex Functions - An Algebraic and Geometric Viewpoint*, Cambridge Univ. Press, New York, 1987.

[96] G. A. Jones and S. A. Silver, *Suzuki groups and surfaces*, J. London Math. Soc. **48**(2) (1993), 117–125.

[97] W. M. Kantor, *Subgroups of classical groups generated by long Root elements*, Trans. Amer. Math. Soc. **248**(2) (1979), 345–379.

[98] G. Karpilovsky, *Group Representations, Voloume I, Part A: Background Material*, North-Holland, Amsterdam, 1992.

[99] G. Karpilovsky, *Group Representations, Volome I, Part B: Introduction to Group Representations and Characters*, North-Holland, Amsterdam, 1992.

[100] S. P. Kerckhoff, *The Nielsen Realization problem*, Ann. Math. **117** (1983), 235–265.

[101] P. B. Kleidman and R. A. Wilson, *The maximal subgroups of* F_{22}, Math. Proc. Camb. Phil. Soc. **102** (1987), 17–23.

[102] P. B. Kleidman and R. A. Wilson, *The maximal subgroups of* J_4, Proc. London Math. Soc. **56**(3) (1988), 484–510.

[103] W. Lempken, *The Shur multiplier of J_4 is trivial*, Arch. Math. **30** (1978), 267–270.

[104] W. Lempken, *A 2-local characterization of Janko's simple group J_4*, J. Algebra **55** (1978), 403–445.

[105] M. W. Liebeck and J. Saxl, *Groups, Combinatorics and Geometry*, London Math. Soc., Lecture Note Series **165**, Cambridge Univ. Press, London-New York, 1992.

[106] M. W. Liebeck and A. Shalev, *Classical groups, probabilistic methods and the $(2,3)$-generation problem*, to appear.

[107] M. W. Liebeck and A. Shalev, *Simple groups, probabilistic methods, and the conjecture of Kantor and Lubotzky*, to appear.

[108] M. W. Liebeck and A. Shalev, *Probabilistic methods in the generation of finite simple groups*, to appear.

[109] M. W. Liebeck and A. Shalev, *The probability of generating a finite simple group*, Geom. Dedicata **56** (1995), 103–113.

[110] S. A. Linton, *The maximal subgroups of the Thompson group*, J. London Math. Soc. **39**(2) (1989), 79–88.

[111] S. A. Linton and R. A. Wilson, *The maximal subgroups of the Fischer groups Fi_{24} and Fi'_{24}*, Proc. London Math. Soc. **63**(3) (1991), 133–164.

[112] P. Longobardi and M. Maj, *On the number of generators of a finite group*, Arch. Math. **50** (1988), 110–112.

[113] A. Lucchini, *A bound on the number of generators of a finite group*, Arch. Math. **53** (1989), 313–317.

[114] R. C. Lyndon, *Groups and Geomerty*, London Math. Soc., Lecture Note Series **101**. Cambridge Univ. Press, London-New York, 1985.

[115] A. M. Macbeath, *Generators of linear fractional groups*, Proc. Symp. Pure Math. **12** (1969), 14–32.

[116] S. S. Magliveras, *The subgroup structure of the Higman-Sims simple group*, Bull. Amer. Math. Soc. **77**(4) (1971), 535–539.

[117] S. S. Magliveras, *The Subgroup Structure of the Higman-Sims Simple Group*, Ph.D Thesis, Birmingham University, England, 1970.

[118] W. Magnus, *Noneuclidean Tesselations and Their Groups*, Academic Press, New York-London, 1974.

[119] W. Magnus et. al, *Combinatorial Group Theory*, Dover, New York, 1966.

[120] G. Malle, *Exceptional groups of Lie type as Galois groups*, J. Reine Angew. Math. **392** (1988), 70–109.

[121] G. Malle, *Hurwitz groups and G_2*, Canadian Math. Bull. **33** (1990), 349–357.

[122] G. Malle, *Small rank exceptional Hurwitz groups*, Groups of Lie type and their Geometries, Proc. Como Conference, June 1993.

[123] G. Mason, *Some remarks on the groups of type J_4*, Arch. Math. **29** (1977), 574–582.

[124] A. W. Mason and J. S. Pride, *Normal subgroups of prescribed order and zero level of the modular group and related groups*, J. London Math. Soc. **42**(2) (1990), 465–474.

[125] J. McLaughlin, *A simple group of order* 898, 128, 000, The Theory of finite groups (Brauer and Sah, eds.), Benjamin, 1969, 109–111.

[126] G. A. Miller, *On the groups generated by two operators*, Bull. Amer. Math. Soc. **7** (1901), 424–426.

[127] G. A. Miller, *On the groups generated by two operators of order two and three resprectively whose product is of order six*, Quart. J. Math. **33** (1901), 76–79.

[128] G. A. Miller, *Groups defined by the orders of two generators and the order of their product*, Bull. Amer. Math. Soc. **7** (1901), 96–100.

[129] G. A. Miller, *Possible orders of two generators of the alternating and of the symmetric group*, Trans. Amer. Math. Soc. **30** (1928), 24–32.

[130] J. Moori, *On certain groups associated with the smallest Fischer group* , J. London Math. Soc. **23**(2) (1981), 61-67.

[131] J. Moori, (p, q, r)*-Generations for the Janko groups* J_1 *and* J_2, Nova J. Algebra and Geometry **2**(3) (1993), 277–285.

[132] J. Moori, *Generating sets for* F_{22} *and its automorphism group*, J. Algebra **159** (1993), 488–499.

[133] J. Moori, $(2,3,p)$*-Generations for the Fischer group* F_{22}, Comm. Algebra **22**(11) (1994), 4597–4610.

[134] J. Moori, *Subgroups of* 3*-transposition groups generated by four* 3*-transpositions*, Quaest. Math. **17** (1994), 83–94.

[135] J. Moori, *On the ranks of the Fischer group* F_{22}, Math. Japonica , **43**(2) (1996), 365–367.

[136] J. Moori and Z. Mpono, *The Fischer-Clifford matrices of the group* $2^6{:}SP(6,2)$, submitted for publication.

[137] S. P. Norton, *The construction of* J_4, The Santa Cruz Conference on Finite Groups (eds. B. Cooperstein and G. Mason), AMS, Providence R.I., 1980, 271–278.

[138] P. M. Neumann, *The SQ-Universality of some finitely presented groups*, J. Austral. Math. Soc. **16** (1973), 1–6.

[139] A. A. H. Omar, *On the Number of Certain Permutation representations of* $\langle 2,3,n\rangle$*-Groups*, Computational Group Theory, Academic Press, 1984, 301–305.

[140] R. Ree, *A theorem on permutations*, J. Comb. Theory A **10** (1971), 174–175.

[141] T. G. Room, *The generation by two operators of the symplectic group over* $GF(2)$, J. Austral. Math. Soc **1** (1959), 38–46.

[142] T. G. Room and R. J. Smith, *A generation of the symplectic group*, Quart. J. Math. Oxford Ser. **9**(2) (1958), 177–182.

[143] J. J. Rotman, *An Introduction to the Theory of Groups - 3rd ed.*, Wm. C. Brown, Dubuque, 1988.

[144] C.-H. Sah, *Groups related to compact Riemann surfaces*, Acta. Math. **123**(2) (1969), 13–42.

[145] P. Sanchini and M. C. Tamburini, *Constructive* $(2,3)$*-generation : A permutational approach*, to appear.

[146] W. Scherrer, *Zur theorie der endlichen gruppen topologischer abbildungen von geschlossenen Flächen in sich*, Comment. Math. Helv. **1** (1929), 69–119.

[147] M. Schönert et al., *GAP—Groups, Algorithms and Programming*, Lehrstul D Für Mathematik, RWTH, Aachen, 1992.

[148] P. E. Schupp, *Embedding into simple groups*, J. London Math. Soc. **13**(2) (1976), 90–94.

[149] L. L. Scott, *Matrices and cohomology*, Ann. Math. **105**(3) (1977), 473–492.

[150] A. Sinkov, *Necessary and sufficient conditions for generating certain simple groups by two operators of period two and three*, Amer. J. Math **59** (1937), 67–76.

[151] A. Sinkov, *On the group-defining relations* $(2,3,7;p)$, Ann. of Math. **38** (1937), 577–584.

[152] D. Singerman, *Subgroups of Fuchsian groups and finite permutation groups*, Bull. London Math. Soc. **2** (1970), 319–323.

[153] F. L. Smith, *A characterization of the ·2 Conway simple group*, J. Algebra **31** (1974), 91–116.

[154] P. Stanek, *Two-element generation of the symplectic group*, Trans. Amer. Math. Soc. **108** (1963), 429–436.

[155] R. Steinberg, *Generators for simple groups*, Canad. J. Math. **14** (1962), 277–283.

[156] W. W. Stothers, *Subgroups of the* $(2,3,7)$ *triangle group*, Manuscripta Math. **20** (1977), 323–334.

[157] I. A. I. Suleiman, and R. A. Wilson, *Standard generators for* J_3, Experimental Math. **4**(1) (1995), 11–18.

[158] M. C. Tamburini, *Generation of certain simple groups by elements of small order*, Rend. Instit. Lombardo Sci (A) **121** (1987), 21–27.

[159] M. C. Tamburini and S. Vassallo, $(2,3)$-*Generazione di Gruppi Lineari*, scritti in onore di Giovanni Melzi, Vita e Pensiero (1994), 391–399.

[160] M. C. Tamburini and J. S. Wilson, *On the generation of finite simple groups by pairs of subgroups*, J. Algebra **116** (1988), 316–333.

[161] C. L. Tretkoff and M. D. Tretkoff, *Combinatorial Group Theory, Riemann Surfaces and Differential Equations*, Contemporary Mathematics, Amer. Math. Soc. **33** (1994), 467–519.

[162] T. W. Tucker, *The number of groups of a given Genus*, Trans. Amer. Math. Soc. **258**(1) (1980), 167–179.

[163] T. W. Tucker, *Finite groups acting on surfaces and the genus of a group.*, J. Combin. Theory (Series B) **34** (1983), 82–98.

[164] J. H. Walter, *Classical groups as Galois groups*, Proc. Rutgers Group Theory Year 1983–1984, ed. Ascbacher et al., Cambridge Univ. Press (1984), 357–383.

[165] T. Weigel, *Residuelle Eigenschaften freier Gruppen*, (1987) Diplomarbeit (Freiburg).

[166] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York-London, 1964.

[167] J. S. Wilson, *Economical generating sets for finite simple groups*, Groups of Lie Type and their Geometries, Cambridge Univ. Press, London, 1995.

[168] R. A. Wilson, *The maximal subgroups of the Conway group Co_2*, J. Algebra **84** (1983), 107–114.

[169] R. A. Wilson, *The maximal subgroups of the Conway group Co_1*, J. Algebra **85** (1983), 144–165.

[170] R. A. Wilson, *On the maximal subgroups of the Fischer group F_{22}*, Math. Proc. Camb. Phil. Soc. **95** (1984), 197–222.

[171] R. A. Wilson, *Maximal subgroups of automorphism groups of simple groups*, J. London Math. Soc. **32**(2) (1985), 460–466.

[172] R. A. Wilson, *The maximal subgroups of the O'Nan group*, J. Algebra **97** (1985), 467–473.

[173] R. A. Wilson, *The symmetric genus of the Baby Monster*, Quart. J. Math. Oxford **44**(2) (1993), 513–516.

[174] R. A. Wilson, *Standard generators for the sporadic simple groups*, J. Algebra (to appear).

[175] A. T. White, *On the genus of a group*, Trans. Amer. Math. Soc. **173** (1972), 203–214.

[176] A. J. Woldar, *On Hurwitz generation and genus actions of sporadic groups*, Illinois J. Math. **33**(3) (1989), 416–437.

[177] A. J. Woldar, *Genus actions of finite simple groups*, Illinois J. Math. **33**(3) (1989), 438–450.

[178] A. J. Woldar, *Representing M_{11}, M_{12}, M_{22} and M_{23} on surfaces of least genus*, Comm. Algebra **18** (1990), 15–86.

[179] A. J. Woldar, *Sporadic simple groups which are Hurwitz*, J. Algebra **144** (1991), 443–450.

[180] A. J. Woldar, *The symmetric genus of the Higman-Sims group HS and bounds for Conway's groups Co_1, Co_2*, Illinois J. Math. **36**(1) (1992), 47–52.

[181] A. J. Woldar, 3/2-*Generation of the sporadic simple groups*, Comm. Algebra **22**(2) (1994), 675–685.

[182] S. K. Wong, *On a new finite non-abelian simple group of Janko*, Bull. Austral. Math. Soc. **1** (1969), 59–79.

[183] M. F. Worboys, *Generators for the sporadic group Co_3 as a $(2,3,7)$ group*, Proc. Edinburgh Math. Soc. **25** (1982), 65–68.

Notation and conventions

Throughout this thesis all groups will be assumed to be finite non-abelian groups, unless otherwise stated. We will use the notation and terminology from the ATLAS [43].

$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
G, H, M	groups
1_G	identity element of G
$H \leq G$	H is a subgroup of G
$H < G$	H is a proper subgroup of G
$H \cong G$	H isomorphic to G
$\langle x, y \rangle$	the subgroup generated by x and y
$G{:}H$	split extention of G by H
$G{\cdot}H$	non-split extention of G by H
H^g	conjugate of the subgroup H in G
nX	a general conjugacy class of G with representatives of order n
$nXYZ$	a the conjugacy class nX or nY or nZ

nx	a general conjugacy class of a subgroup H of G with representatives of order n
$\mathcal{A}$	$\{(x,y) \in lX \times mY \mid xy = z\}$, where z is a fixed element of the class nZ
$\Delta(G) = \Delta_G(lX, mY, nZ)$	the structure constant of G
$\Delta^*(G)$	the number of pairs in the set $\mathcal{A}$ that generate G
$\Sigma(H_1 \cup \ldots \cup H_r)$	the number of pairs in $\mathcal{A}$ that generate subgroups of H_i
$\Sigma^*(H)$	the number of pairs in $\mathcal{A}$ that generate H
$o(x)$	order of $x \in G$
$C_G(x)$, $C_G(nX)$	the centralizer of $x \in nX$ in G
$N_G(H)$	the normalizer of the subgroup H in G
Γ, Ω	sets
$\|\Gamma\|$	the cardinality of the set Γ
$1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\ldots$	cycle structure of a permutation
$Irr(G)$	the set of irreducible characters of G
χ_H	the permutation character of G on the conjugates of the subgroup H
$\chi{\downarrow}_H$	the restriction of the character χ of G to the subgroup H
$\underline{na}$, $\underline{nb}$, ...	an irreducible character of G of degree n
$\langle \chi_i, \chi_j \rangle$	the innerproduct of the characters χ_i and χ_j
$dim(V/C_V(nX))$	the co-dimension of the centralizer of nX in the module V
D_{2n}	diheral group of order $2n$