THE EFFECT OF ANTISYMMETRIZATION IN DIQUARK MODELS OF BARYONS

BY

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ABSTRACT

In this thesis the effect of antisymmetrization in diquark models of baryons composed of light (u and d) quarks is investigated. The diquark in this study is considered alternately as a point-like and as a composite particle. The wave functions for both diquark and diquark-quark systems have been determined in a non-relativistic approximation by using the radial Schrodinger equation and a range of central potentials. The ground state masses of the diquark-quark system have been calculated in three distinct ways:

(1) The ground state energy eigenvalues and the wave functions for the diquark and diquark-quark, each being treated as a two-body system, have been calculated by using the generalized Runge-Kutta and search methods.

(2) The expectation values for the potential energy and kinetic energy have been calculated by using the wave functions derived in (1) for the two-body system without antisymmetrization. These results have been checked by applying the virial theorem in parallel calculations.

(3) The potential and kinetic energy expectation values have also been determined by taking antisymmetrization into account via operator kernels namely, norm, potential and kinetic energy which have been derived by using the non-local Generator Coordinate Method (GCM). The expectation values of these operator kernels have been calculated with respect to the wave functions produced in (1). For the purpose of performing the integrations the wave functions , expanded in terms of cubic splines, and Gaussian quadrature have been employed.

Lastly the diquark and diquark-quark ground state masses were calculated for each approach, (1) - (3), and compared with

(a) each other,

(b) the results for a two-body system,

(c) the results for a full three-body treatment and,

(d) the average mass of N-Δ.

The form factors and root mean square radii of the baryon for the four central potentials have been calculated with antisymmetrization for each approach (1) - (3) and compared with

(a) each other,

(b) the results for the baryon without antisymmetrization

(c) the results for the baryon with antisymmetrization including the meson cloud

(d) the experimental data.

The trends found are striking and it can be concluded that there is a strong dynamical effect due to the presence of antisymmetrization in diquark models of baryons.

PREFACE

The work described in this thesis was carried out in the Department of Physics, University of Natal, Pietermaritzburg, from March 1994 to November 1997, under the supervision of Doctor Robert M. Adam of the Department of Arts, Culture, Science and Technology, Pretoria and Professor Clive Graham of the Department of Physics, Pietermaritzburg.

These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any University. Where use has been made of the work of others it is duly acknowledged in the text.

LIST OF CONTENTS

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1 INTRODUCTION

1.1 WHY ANTISYMMETRIZATION?

The diquark system is formed as a bound state of two quarks and regarded as elementary when it interacts with the third quark (spectator) to form a baryon. The computational labour involved in baryon calculations using a diquark model is reduced since the dynamics are mediated by the interactions within an effective two-body rather than three-body system. According to Lichtenberg [1] many authors have assumed that diquarks can be treated as elementary and as a result no attempt has been made to antisymmetrize the wave function under the interchange of any two quarks in the system, that is, of a diquark pair and a spectator quark. However, although quarks are fermions, correct antisymmetrization of the wave function is often neglected in diquark calculations [1]. Antisymmetrization is necessary because of the exchange of cluster quarks with the spectator inside baryons. Lichtenberg [1] pointed out that if an attempt is made to symmetrize the wave function between a diquark pair and a quark outside the diquark, then the wave function is more complicated, which results in losing some of the value of the simple diquark picture (see also [2]). He furthermore pointed out that the fact that some physicists obtain good predictions by using diquarks without completely antisymmetrizing the wave function, shows that interference terms which result from antisymmetrization are often less important. The only way to show that these interference terms are indeed small, is to antisymmetrize the wave functions [1].

In this work a diquark is assumed not to be an elementary particle but rather a correlated state of two quarks. Therefore the aim of this study is to investigate what effect antisymmetrization has on the baryon mass in diquark models. It is with this in mind that a rigorous determination of the effects of antisymmetrization in diquark-quark models of baryons within

the non-relativistic constituent quark model has been undertaken. The formalism has been developed by Adam and Bakker [3]. The use of the non-local Generator Coordinate Method (GCM) [4,5] in exchange operator kernels is proposed to theoretically study the effect of antisymmetrization in diquark models of baryons. GCM was invented by Griffin, Hill and Wheeler [6,7,8]. The GCM formalism has been applied to bound state problems and has proved to be very powerful and general. Griffin and Wheeler [7] proposed that the distance between the colliding particles must be the most important degree of freedom which must also be kept as the only generator coordinate. GCM is considered in this study and the required operator kernels to be derived in chapter 4 are based on this method. In this work a non-relativistic constituent quark formalism is applied to baryonic systems. In the model the state of the system is completely characterized by the time independent diquark-quark wave function. Within this approach, a systematic study of antisymmetrization in the ground state of a baryon built from light (u and d) quarks is presented.

1.2 DIQUARK MODELS IN ACTION

A model in which a baryon is a bound state of a diquark-quark [9-13] is employed in this work. The fact that [13] quarks cluster in pairs in baryons leading to a two-body structure is a central idea in the diquark simplification of the quark model. The diquark is as old as the quark model itself and was suggested by Gell-Mann in his pioneering paper on quarks [14]. He suggested that a bound state of two quarks or diquark may be stable. Since then a number of authors [9-13;15-19] have considered models containing diquarks. In particular, a model of excited baryon states in which one quark acquires orbital angular momentum while the other two quarks remain as an S-wave diquark, has been useful [17].

Leinwenberg [18] and Fleck et al [19], performed theoretical investigations of diquark clustering which produced encouraging results. Leinwenberg employed a gauge invariant method to investigate the clustering of the scalar diquark. Fleck et al.searched for the diquark clustering in baryons by performing non-relativistic three-body calculations where ground states and orbital excitations for different flavour combinations have been considered. A diquark is regarded by Fleck et al [19] as a correlated pair of up (u) and down (d) quarks having the right quantum numbers to combine with the third quark to form a baryon.

A diquark is defined by Anselmino [2] in two ways; first as any system of two quarks considered collectively and second, as a two-quark correllation in a hadron containing more than two quarks. According to a diquark model,if one quark is excited, the other two remain unexcited relative to each other inside a baryon. The diquark may release a quark to be captured by the spectator quark, which then becomes an unexcited diquark.

Close [20] proposed that one quark feels an attraction to two quarks in a baryon and that this pair (diquark) can have the same effective colour charge as a single antiquark and in this way attract and neutralize the third quark. Diquarks have played a role in the description of many exclusion processes.

(a) NN elastic scattering [21]

(b) $p\bar{p}$ annihilation into hyperon-antihyperon pairs [22]

(c) $p\bar{p}$ creation in photon-photon interactions [23]

(d) $p\gamma$ elastic scattering [24]

(e) Charmonium decay into baryon-antibaryon pairs [25,26].

Lately a simplified diquark model has been used in the description of $J/\Psi \rightarrow p\bar{p}\gamma$ [27]. It has been also used to describe a violation of the Gottfried sum rule [28]. Diquarks were also

used to explain the abundant large - P_T deuteron observed in 70-GeV pp interactions [29]. Concerning the baryon decays, in 1972 and 1973, Ono [30] used a diquark-quark model to computerize the baryon decay rates where he found good agreement with experiment. By using a diquark-quark model Bediaga et al [31] have been able to estimate the life-time of the charmed baryon, Λ_C and its value was found to be in agreement with the experimental results. To date about 300 papers on diquarks have been written [32].

1.3 OUTLINE OF STUDY

In chapter 2 the wave function for the diquark-quark system is discussed with respect to symmetries:

(a) spatial,

(b) spin,

(c) flavour and ,

(d) colour.

Chapter 3 is devoted to a discussion of the Schrodinger equation for the diquark-quark system with four different central potentials:

(a) Bhaduri,

(b) Martin,

(c) Cornell and,

(d) Quigg.

A more detailed description of the operator kernels, namely

(a) norm kernel,

(b) potential energy kernel,

(c) kinetic energy kernel,

(d) form factor kernel,

(e) charge density and

(f) root mean square radius,

is given in chapter 4. Chapter 4 is devoted to operator kernels in the single channel case whereas chapter 5 is devoted to operator kernels, namely

(a) norm kernel,

(b) potential energy kernel and

(c) kinetic energy kernel

in the coupled channel case. The results are given in chapter 6. Concluding remarks and the way forward are given in chapter 7.

2 TOTAL WAVE FUNCTION FOR THE DIQUARK-QUARK SYSTEM

2.1 INTRODUCTION

The total wave function of the diquark-quark ($\bar{D}q$) system is constructed such that it is antisymmetric upon the exchange of any two quarks. The relation $\vec{J} = \vec{L} + \vec{S}$, where $\vec{J}$ is the total angular momentum, $\vec{L}$ is the total orbital angular momentum and $\vec{S}$ is the total spin angular momentum, will be used in this chapter in the discussion of spatial and spin parts of the total wave function. The details of how the antisymmetric $\bar{D}q$ wave function is constructed are given in the following sections.

2.2 THE PAULI EXCLUSION PRINCIPLE

The Pauli exclusion principle is equivalent to the rule [33] that:

pairs of identical fermions can exist only in states whose total wave functions are antisymmetric (change sign) under interchange of the two particles.

In the case where all the three quarks of the baryon have the same flavour, the flavour factor becomes symmetric under the interchange of any two quarks. The spin factor is also symmetric under the interchange of any two quarks since all the quarks have the same spins and they point in the same direction. The fact that the spins of the quarks couple together to form the total spin of the baryon, implies that there is no orbital angular momentum (that is, L=0) for the three quarks. This leads to the conclusion that the quarks are positioned in a symmetric way which causes the space factor to be symmetric too, under the interchange of any two quarks. All three factors are symmetric, and this causes the product of these factors to be symmetric as well. This violates the Pauli exclusion principle. This contradiction between the quark model and the Pauli exclusion principle was resolved in 1964 by Greenberg [34] by suggesting that quarks

must possess colour in addition to space, spin and isospin (flavour). This leads to the generalized Pauli exclusion principle [33] which states that:

the total wave function which consists of space, spin, isospin and colour must be antisymmetric for a pair of particles.

Mathematically the generalization of this principle can be expressed as (see also [35,36])

$$\Psi_B = \Psi_{\bar{D}q}(\vec{\xi},\vec{\eta})\,\Psi(\vec{S},\vec{\sigma},m_s)\,\Psi(\vec{T},\vec{t},m_T)\,\Psi_c \qquad (2.2.1)$$

where $\Psi_{\bar{D}q}(\vec{\xi},\vec{\eta})$ describes an orbital motion, $\Psi(\vec{S},\vec{\sigma},m_S)$ is the spin state, $\Psi(\vec{T},\vec{t},m_T)$ is the isospin part and Ψ_C represents the totally antisymmetric colour singlet state. Therefore the total wave function (Ψ_B) is antisymmetric on the interchange of each diquark constituent with the spectator. The Pauli exclusion principle is an important tool of understanding fermion systems and because the baryons are fermions they are therefore subject to the Pauli exclusion principle.

In chapter 4 the spatial wave function which fully takes into account the antisymmetrization will be discussed.

2.3 THE SPACE WAVE FUNCTION

For the diquark-quark model there are two orbital angular momenta $\vec{L}_{12}$ and $\vec{L}_3$ where $\vec{L}_{12}$ is the orbital angular momentum of the chosen pair of quarks in their mutual centre-of-mass frame and $\vec{L}_3$ is the orbital angular momentum of the spectator quark about the center-of-mass of the pair in the total centre-of-mass frame. The total orbital angular momentum can therefore be written in the form

$$\vec{L} = \vec{L}_{12} + \vec{L}_3 \qquad (2.3.2)$$

in the diquark-quark model.

For the lowest lying state (S-state):

$$\vec{L}=0 \tag{2.3.3}$$

and furthermore it is assumed, in a single channel approach, that

$$\vec{L}_{12} = \vec{L}_3 = 0 \tag{2.3.4}$$

Denoting the wave function of the diquark by $\Phi_{\bar{D}}$ and that of the diquark-quark by $\Psi_{\bar{D}q}$, the unsymmetrized baryon wave function is then given by

$$\Psi_B(\vec{\xi},\vec{\eta}) = \Phi_{\bar{D}}(\vec{\xi}) \otimes \Psi_{\bar{D}q}(\vec{\eta}) \tag{2.3.5}$$

where $\vec{\xi}$ is the relative coordinate of the pair of quarks and $\vec{\eta}$ is the relative coordiante of the spectator quark about the centre-of-mass of the diquark in the total centre-of-mass frame. The requirements that any bound state wave function and its derivative [37] must meet are

(a) $\Phi_{\bar{D}}$ and $\Psi_{\bar{D}q}$ must vanish at least at $\vec{\xi} = \pm\infty$ and $\vec{\eta} = = \pm\infty$ respectively.

(b) $\Phi_{\bar{D}}$, $\frac{d\Phi_{\bar{D}}}{d\xi}$, $\Psi_{\bar{D}q}$ and $\frac{d\Psi_{\bar{D}q}}{d\eta}$ must be finite and continuous.

The symmetrized space wave function of baryon will be discussed explicitly in chapter 4.

2.4 THE SPIN WAVE FUNCTION

Spin is a quantum degree of freedom. It is a form of angular momentum, an intrinsic angular momentum which is not associated with orbital motion. The spin angular momentum is represented by the vector $\vec{S}$ and the quantum number for the spin angular momentum by the letter s. The total spin angular momentum can be expressed in the form (see also [38]):

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 \tag{2.4.6}$$

for all the three quarks inside baryon.

A baryon is a bound triquark state with either spin-$\frac{1}{2}$ or spin-$\frac{3}{2}$. Because in this work only the baryon with the flavour configuration uud or udd is treated, spin-$\frac{3}{2}$ will not be treated as

it is suitable for flavour configuration uuu. When the spins of two quarks in a baryon couple they either form a scalar diquark, $s_{12} = 0$ or a vector diquark, $s_{12} = 1$. Kobzarev et al. [39] noted that in the ground state, the baryon has an equal probability of having either the scalar or the vector diquark. The scalar diquark spin state is antisymmetric whereas the vector diquark state is symmetric. In the case of the scalar diquark, the two quarks have opposite spin whereas in the case of the vector diquark the spins are parallel. For the three-quark system there are two kinds of mixed symmetric states, the Π_1-type and Π_2-type constructed as follows (see, for example [40]):

(i) the Π_1-type state is constructed from coupling spins of quarks 1 and 2 to form a scalar diquark which after coupling with spin of quark 3 the total spin S becomes

$$S = s_{12} \otimes s_3 = 0 \otimes \frac{1}{2} = \frac{1}{2} \tag{2.4.7}$$

In this state the spin wave functions are antisymmetric between quarks 1 and 2 but under the interchange of quarks 1 and 3 or 2 and 3, they are neither symmetric nor antisymmetric.

(ii) The Π_2-type is constructed from coupling spins of quarks 1 and 2 to form a vector diquark where after coupling with quark 3, the total spin S is

$$S = s_{12} \otimes s_3 = 1 \otimes \frac{1}{2} = \frac{1}{2} \tag{2.4.8}$$

In this state the wave functions are only symmetric in quarks 1 and 2 and there is no possibility for a totally antisymmetric spin-state for the bound triquark state taking place. By considering

$$2 \times 2 \times 2 = 8$$

$$= 4 + 2 + 2, \tag{2.4.9}$$

it is clear that there are 2^3 states in a baryon made of identical quarks because each quark can be labelled as spin up or spin down. Equation (2.4.9) means that there are four symmetric states, two mixed symmetric Π_1-type and two mixed symmetric Π_2-type states.

For the coupling of quarks 2 and 3 as well as 1 and 3 to form s_{23} and s_{13} respectively, the same procedure is followed.

The spin-1 state has the symmetric triplet states:

$$\Phi_D^{spin} = \begin{cases} \chi_{\frac{1}{2}}(1)\chi_{\frac{1}{2}}(2), & S=1, \quad M_s=1 \\ \frac{1}{\sqrt{2}}\left(\chi_{\frac{1}{2}}(1)\chi_{-\frac{1}{2}}(2) + \chi_{-\frac{1}{2}}(1)\chi_{\frac{1}{2}}(2)\right), & S=1, \quad M_s=0 \\ \chi_{-\frac{1}{2}}(1)\chi_{-\frac{1}{2}}(2), & S=1, \quad M_s=-1 \end{cases} \tag{2.4.10}$$

where $\chi_{\frac{1}{2}}$ represents the spin up and $\chi_{-\frac{1}{2}}$ represents the spin down. The spin-0 state has the antisymmetric singlet state:

$$\Phi_D^{spin} = \frac{1}{\sqrt{2}}\left(\chi_{\frac{1}{2}}(1)\chi_{-\frac{1}{2}}(2) - \chi_{-\frac{1}{2}}(1)\chi_{\frac{1}{2}}(2)\right), \qquad S=0, \quad M_s=0 \tag{2.4.11}$$

In spin formalism using the Clebsch-Gordan coefficients and $\vec{L} = 0$, eqns (2.4.10) and (2.4.11) can generally be written as

$$\Phi_D = \sum_{s_1 s_2} <\frac{1}{2}\frac{1}{2}s_1 s_2 | s\sigma>\chi_{s_1}(1)\chi_{s_2}(2) \tag{2.4.12}$$

where $\sigma = s_1 + s_2$, $s_1 = s_2 = +\frac{1}{2}$ or $-\frac{1}{2}$ and χ is the spin wave function of a quark. Therefore the total spin wave function of the baryon is given in the form:

$$\Psi_B = \sum_{s_1 s_2 s_3 \sigma} <\frac{1}{2}\frac{1}{2}s_1 s_2 | s\sigma><s\frac{1}{2}\sigma s_3 | SM_s>\chi_{s_1}(1)\chi_{s_2}(2)\chi_{s_3}(3) \tag{2.4.13}$$

where

$$s_1 + s_2 + s_3 = \sigma + s_3 = M_s \text{ and } s_3 = \frac{1}{2} \tag{2.4.14}$$

Since the diquark is either a scalar or a vector, eqn. (2.4.13) becomes

$$\Psi_B^{spin}(12,3)_1 = \frac{1}{\sqrt{6}}[2\chi_{\frac{1}{2}}(1)\chi_{\frac{1}{2}}(2)\chi_{-\frac{1}{2}}(3) - \chi_{\frac{1}{2}}(1)\chi_{-\frac{1}{2}}(2)\chi_{\frac{1}{2}}(3) - \chi_{-\frac{1}{2}}(1)\chi_{\frac{1}{2}}(2)\chi_{\frac{1}{2}}(3)] \tag{2.4.15}$$

where the subscript 1 refers to the vector diquark and the state (12,3) refers to the diquark-quark system comprised of the diquark, (12), and a quark 3. For the scalar diquark eqn. (2.4.13) becomes

$$\Psi_B^{spin}(12,3)_0 = \frac{1}{\sqrt{2}}[\chi_{\frac{1}{2}}(1)\chi_{-\frac{1}{2}}(2)\chi_{\frac{1}{2}}(3) - \chi_{-\frac{1}{2}}(1)\chi_{\frac{1}{2}}(2)\chi_{\frac{1}{2}}(3)] \tag{2.4.16}$$

where the subscript 0 refers to the scalar diquark. On exchange of the diquark constituent, that is, quark 2 with quark 3, eqn. (2.4.13) becomes

$$\Psi_B^{spin}(13,2)_1 = \frac{1}{\sqrt{6}}[2\chi_{\frac{1}{2}}(1)\chi_{\frac{1}{2}}(3)\chi_{-\frac{1}{2}}(2) - \chi_{\frac{1}{2}}(1)\chi_{-\frac{1}{2}}(3)\chi_{\frac{1}{2}}(2) - \chi_{-\frac{1}{2}}(1)\chi_{\frac{1}{2}}(3)\chi_{\frac{1}{2}}(2)] \tag{2.4.17}$$

when the vector diquark is taken into account and

$$\Psi_B^{spin}(13,2)_0 = \frac{1}{\sqrt{2}}[\chi_{\frac{1}{2}}(1)\chi_{-\frac{1}{2}}(3)\chi_{\frac{1}{2}}(2) - \chi_{-\frac{1}{2}}(1)\chi_{\frac{1}{2}}(3)\chi_{\frac{1}{2}}(2)] \tag{2.4.18}$$

when the scalar diquark is considered. The next exchange of the diquark constituent, that is, quark 1 with the external quark 2 in a baryon changes eqn. (2.4.13) into the form

$$\Psi_B^{spin}(32,1)_1 = \frac{1}{\sqrt{6}}[2\chi_{\frac{1}{2}}(3)\chi_{\frac{1}{2}}(2)\chi_{-\frac{1}{2}}(1) - \chi_{\frac{1}{2}}(3)\chi_{-\frac{1}{2}}(2)\chi_{\frac{1}{2}}(1) - \chi_{-\frac{1}{2}}(3)\chi_{\frac{1}{2}}(2)\chi_{\frac{1}{2}}(1)] \tag{2.4.19}$$

for the vector diquark and

$$\Psi_B^{spin}(32,1)_0 = \frac{1}{\sqrt{2}}[\chi_{\frac{1}{2}}(3)\chi_{-\frac{1}{2}}(2)\chi_{\frac{1}{2}}(1) - \chi_{-\frac{1}{2}}(3)\chi_{\frac{1}{2}}(2)\chi_{\frac{1}{2}}(1)] \tag{2.4.20}$$

for the scalar diquark.

2.5 THE FLAVOUR WAVE FUNCTION

Baryons occur in groups or multiplets of like masses in which the particles differ with respect to charge. The isospin is denoted by T. The z-component of the isospin, M_T, is quantized according to

$$M_T = T, T-1, ..., -T \tag{2.5.21}$$

The main purpose of isospin formalism is to allow u and d quarks to be considered as identical particles with respect to mass (see also[41]). According to elementary group theory, the three fundamental multiplets (representations) of $SU_F(3)$ (F for flavour) can be given by (see also [42]:

$$3 \times 3 \times 3 = 10 + 8 + 8 + 1 \tag{2.5.22}$$

The relation (2.5.22) indicates that there are 10 symmetric states, 8 mixed symmetric π_1-type and 8 mixed symmetric π_2-type states. The π_1-type is antisymmetric and π_2-type is symmetric under the exchange of quarks 1 and 2. In eqn. (2.5.22) 1 represents a totally antisymmetric state under the exchange of any pair.

Therefore

$$\Phi_D^{flavour} = \begin{cases} uu \\ \frac{1}{\sqrt{2}}(ud+du) \\ dd \end{cases} \tag{2.5.23}$$

for the vector diquark and

$$\Phi_D^{flavour} = \frac{1}{\sqrt{2}}(ud - du) \tag{2.5.24}$$

for the scalar diquark.

For the π_1-and π_2-types

$$\Psi_B^{flavour} = \frac{1}{\sqrt{2}}(ud - du)u \tag{2.5.25}$$

and

$$\Psi_B^{flavour} = \frac{1}{\sqrt{6}}[(ud + du)u - 2uud] \tag{2.5.26}$$

respectively. By analogy with spin wave function the following, for the bound states (12), (13) and (32), can be found:

$$\Psi_B^{flavour}(12,3)_1 = \frac{1}{\sqrt{6}}[2\zeta_{\frac{1}{2}}(1)\zeta_{\frac{1}{2}}(2)\zeta_{-\frac{1}{2}}(3) - \zeta_{\frac{1}{2}}(1)\zeta_{-\frac{1}{2}}(2)\zeta_{\frac{1}{2}}(3) - \zeta_{-\frac{1}{2}}(1)\zeta_{\frac{1}{2}}(2)\zeta_{\frac{1}{2}}(3)] \tag{2.5.27}$$

$$\Psi_B^{flavour}(12,3)_0 = \frac{1}{\sqrt{2}}[\zeta_{\frac{1}{2}}(1)\zeta_{-\frac{1}{2}}(2)\zeta_{\frac{1}{2}}(3) - \zeta_{-\frac{1}{2}}(1)\zeta_{\frac{1}{2}}(2)\zeta_{\frac{1}{2}}(3)] \tag{2.5.28}$$

$$\Psi_B^{flavour}(13,2)_1 = \frac{1}{\sqrt{6}}[2\zeta_{\frac{1}{2}}(1)\zeta_{\frac{1}{2}}(3)\zeta_{-\frac{1}{2}}(2) - \zeta_{\frac{1}{2}}(1)\zeta_{-\frac{1}{2}}(3)\zeta_{\frac{1}{2}}(2) - \zeta_{-\frac{1}{2}}(1)\zeta_{\frac{1}{2}}(3)\zeta_{\frac{1}{2}}(2)] \tag{2.5.29}$$

$$\Psi_B^{flavour}(13,2)_0 = \frac{1}{\sqrt{2}}[\zeta_{\frac{1}{2}}(1)\zeta_{-\frac{1}{2}}(3)\zeta_{\frac{1}{2}}(2) - \zeta_{-\frac{1}{2}}(1)\zeta_{\frac{1}{2}}(3)\zeta_{\frac{1}{2}}(2)] \tag{2.5.30}$$

$$\Psi_B^{flavour}(32,1)_1 = \frac{1}{\sqrt{6}}[2\zeta_{\frac{1}{2}}(3)\zeta_{\frac{1}{2}}(2)\zeta_{-\frac{1}{2}}(1) - \zeta_{\frac{1}{2}}(3)\zeta_{-\frac{1}{2}}(2)\zeta_{\frac{1}{2}}(1) - \zeta_{-\frac{1}{2}}(3)\zeta_{\frac{1}{2}}(2)\zeta_{\frac{1}{2}}(1)] \tag{2.5.31}$$

and

$$\Psi_B^{flavour}(32,1)_0 = \frac{1}{\sqrt{2}}[\zeta_{\frac{1}{2}}(3)\zeta_{-\frac{1}{2}}(2)\zeta_{\frac{1}{2}}(1) - \zeta_{-\frac{1}{2}}(3)\zeta_{\frac{1}{2}}(2)\zeta_{\frac{1}{2}}(1)] \tag{2.5.32}$$

2.6 THE PROJECTIONS IN SPIN-FLAVOUR SPACE

The projections in spin-flavour space are calculated as follows

$$P = \langle \Psi_B^s(1,2,3) \otimes \Psi_B^t(1,2,3) | \Psi_B^s(3,1,2) \otimes \Psi_B^{t'}(3,1,2) \rangle$$

$$= \langle \Psi_B^s(1,2,3) | \Psi_B^{s'}(3,1,2) \rangle \langle \Psi_B^t(1,2,3) | \Psi_B^{t'}(3,1,2) \rangle$$

$$= \hat{s}\hat{s}'\hat{t}\hat{t}' \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & s \\ S & \frac{1}{2} & s' \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & t \\ T & \frac{1}{2} & t' \end{Bmatrix} \delta_{ss'}\delta_{tt'} \tag{2.6.33}$$

where

$$\langle \Psi_B^S(1,2,3) | \Psi_B^{S'}(3,1,2) \rangle = -\hat{s}\hat{s}' \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & s \\ S & \frac{1}{2} & s' \end{Bmatrix} \delta_{SS'} \tag{2.6.34}$$

and

$$\langle \Psi_B^t(1,2,3) | \Psi_B^{t'}(3,1,2) \rangle = -\hat{t}\hat{t}' \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & t \\ T & \frac{1}{2} & t' \end{Bmatrix} \delta_{tt'} \tag{2.6.35}$$

where the Wigner 6j [43] symbols are defined by

$$\langle j_1 j_2 (j_{12}) j_3 j m | j_1 j_2 j_3 (j_{23}) j' m' \rangle = \delta_{jj'}\delta_{mm'}(-1)^{j_1+j_2+j_3+j} \sqrt{(2j_{12}+1)(2j_{23}+1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \tag{2.6.36}$$

For the scalar and the vector diquark basis functions, the following matrix elements can be

obtained by using eqns. (2.4.15) through (2.4.20):

$$\langle \Psi_B^s(12,3)_1 | \Psi_B^{s'}(31,2)_1 \rangle = -\frac{1}{2}\delta_{ss'} \tag{2.6.37}$$

$$\langle \Psi_B^s(12,3)_1 | \Psi_B^{s'}(31,2)_0 \rangle = \frac{\sqrt{3}}{2}\delta_{ss'} \tag{2.6.38}$$

$$\langle \Psi_B^s(12,3)_0 | \Psi_B^{s'}(31,2)_0 \rangle = \frac{1}{2}\delta_{ss'} \tag{2.6.39}$$

$$\langle \Psi_B^s(12,3)_0 | \Psi_B^{s'}(31,2)_1 \rangle = \frac{\sqrt{3}}{2}\delta_{ss'} \tag{2.6.40}$$

with analogous expressions holding for the isospin functions and for the (231) permutation. In the case of central spin-independent potentials, the spin/isospin wave function will be a symmetric mixture of vector and scalar basis functions. That is

$$\Psi_B^{(s,t)}(1,2,3) = \frac{1}{\sqrt{2}}\{\Psi_B^{(s)}(1,2,3)_1 \Psi_B^{(t)}(1,2,3)_1 + \Psi_B^{(s)}(1,2,3)_0 \Psi_B^{(t)}(1,2,3)_0\} \tag{2.6.41}$$

Using eqns. (2.6.37) through (2.6.40) and the identical results for the isospin matrix elements it is easy to show that

$$\langle \Psi_B^{(s,t)}(1,2,3) | \Psi_B^{(s,t)}(3,1,2) \rangle = 1 \tag{2.6.42}$$

Indeed, this must be so, because for a symmetric basis function any cyclic permutation maps the basis function onto itself.

2.7 THE COLOUR WAVE FUNCTION

The force between quarks originates as a result of the interaction between colour charges. Colour theory assumes that any quark inside a baryon can take on one of the three different colours, namely red (R), green (G) and blue (B). The baryons are colour singlet, that is, they are colourless. The colour part of the baryon ($\bar{D}q$) wave function is therefore [44]

$$\Psi_{colour} = \frac{1}{\sqrt{3}}(\bar{R}R + \bar{G}G + \bar{B}B) \qquad (2.7.43)$$

This equation means that the diquark colour wave function is an anti-triplet. There are six ways of making a symmetric combination of two colours and three ways of forming an antisymmetric combination (anti-triplet) (see also [20]). In other words any pair is in the $\bar{3}$ and is antisymmetric. The six symmetric combinations are

$$RB + BR;\ RG + GR;\ BG + GB;\ RR;\ GG;\ \text{and}\ BB \qquad (2.7.44)$$

and the three antisymmetric combinations are

$$BR - RB;\ RG - GR \text{ and } GB - BG. \qquad (2.7.45)$$

In qq system the state $\bar{3}$ indicates that indeed a diquark acts like an antiquark in colour space. In qqq system

$$qqq = 10 + 8 + 8 + 1 \qquad (2.7.46)$$

The singlet state 1 is an indication that qqq system is the physical system of baryons and it must be borne in mind that there are the only physically observed states. It must be noted that any pair of quarks can have spin = 0 or 1 and $SU(3)_{flavour} = \bar{3}$ or 6. Here 0 and $\bar{3}$ are antisymmetric and 1 and 6 are symmetric under interchange of the labels. If the qq structure of a diquark $\bar{D}$ is considered, eqn.(2.7.43) becomes (see also [20, 44])

$$\Psi^{qqq}_{Colour} = \frac{1}{\sqrt{3}}\left\{\frac{1}{\sqrt{2}}(GB-BG)R + \frac{1}{\sqrt{2}}(BR-RB)G + \frac{1}{\sqrt{2}}(RG-GR)B\right\} \qquad (2.7.47a)$$

This equation is allowed and required by colour confinement and can be generalized as [45]

$$\Psi_C = \frac{1}{\sqrt{6}}\sum_{ikl}\varepsilon_{ikl}\,q_{1i}\,q_{2k}\,q_{3l} \qquad (2.7.48b)$$

where ε_{ikl} is the permutation symbol defined by $\varepsilon_{ikl} = +1$ if i, k, l is an even permutation of 1, 2, 3 or $\varepsilon_{ikl} = -1$ if any two of the subscripts i, k, l is an odd permutation of 1, 2, 3 or $\varepsilon_{ikl} = 0$ if any two of the subscripts i, k, l are equal. For example

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$ and $\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{231} = -1$. This colour singlet consists of a triplet combined with an anti-triplet. This equation is totally antisymmetric under exchange of any two identical quarks and totally symmetric under a cyclic permutation. Any pair inside the baryon is in the $\bar{3}$ and is antisymmetric. The remainder of the wave function must therefore be totally symmetric under exchange. The colour R, G or B of a quark can be represented by the so called colour spinors, namely (see also [38], [45])

$$C_1 = R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad C_2 = G = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad C_3 = B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.7.49}$$

It must be noted that the colour wave functions are acted upon by the so-called colour operators. Eight independent colour operators are identified as follows (see also [38], [45]):

$$\hat{C} = \frac{1}{2}\lambda_i \qquad (i = 1, 2, 3, \ldots, 8) \tag{2.7.50}$$

with

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{2.7.51}$$

3 SCHRODINGER EQUATION FOR THE DIQUARK-QUARK SYSTEM.

3.1 INTRODUCTION.

The Schrodinger equation is required in this work to generate the data for the space wave function and to determine the energy eigenvalue by the search method for both diquark and diquark-quark systems. The treatment of the motion of quarks inside baryons is an example of a few-body problem. In such systems each quark interacts with all the others. If the forces acting between these quarks are known then the Schrodinger equation of their motion can be written down.

A quark of constituent mass, m_q, moving in a field of spherical symmetry, obeys the Schrodinger equation for a stationary state:

$$H\Psi(r,\theta,\vartheta) = E\Psi(r,\theta,\vartheta) \tag{3.1.1}$$

where H is the Hamiltonian operator written as

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(r) \tag{3.1.2}$$

and $\Psi(r,\theta,\vartheta)$ is the wave function of the quark dependent on r, θ and ϑ.

For three particles, equation (3.1.1) generalizes to

$$H = -\frac{\hbar^2}{2}\sum_i^3 \frac{\nabla_i^2}{m_i} + \sum_{1=i<j}^3 V(r_{ij}) \tag{3.1.3}$$

where $V(r_{ij})$ is a central potential which is dependent on the relative distance only. The Jacobi coordinates for three bodies are given by (see also [40, 46, 47]):

$$\vec{\xi} = \vec{r}_2 - \vec{r}_1 \tag{3.1.4}$$

$$\vec{\eta} = \frac{2}{\sqrt{3}}\left(\vec{r}_3 - \frac{\vec{r}_1+\vec{r}_2}{2}\right) \tag{3.1.5}$$

and

$$\vec{R} = \frac{1}{3}(\vec{r}_1+\vec{r}_2+\vec{r}_3) \tag{3.1.6}$$

where $\vec{\xi}$ is the separation vector between any two chosen particles, $\vec{\eta}$ is the separation vector

between spectator quark and the centre-of-mass of the diquark system and $\vec{R}$ is the relative coordinate of the centre-of-mass of the diquark-quark system.

3.2 CENTRAL POTENTIALS

A range of central potentials have been used in this work to balance the opposing requirements of mathematical convenience and physical realism. They depend on the coordinates of the system only. In other words, consideration is given to the motion of three quarks inside a single baryon which are subject to forces that depend on the positions between two quarks and between diquark and a quark. In the three-body problem, the central potential energy in eqn.(3.1.3) for the three identical quarks, is written in the form (see also [47]):

$$\begin{aligned}\hat{V} &= \sum_{1=i<j}^{3} V(r_{ij}) \\ &= V(r_{12}) + V(r_{13} + V(r_{23}) \\ &= V(\xi) + V(\xi') + V(\xi'')\end{aligned} \tag{3.2.7}$$

where (see also [46, 48])

$$\vec{\xi} = \vec{r}_1 - \vec{r}_2 \tag{3.2.8}$$

$$\begin{aligned}\vec{\xi}' &= \vec{r}_3 - \vec{r}_1 \\ &= -\frac{1}{2}\vec{\xi} + \frac{\sqrt{3}}{2}\vec{\eta}\end{aligned} \tag{3.2.9}$$

and

$$\begin{aligned}\vec{\xi}'' &= \vec{r}_2 - \vec{r}_3 \\ &= -\frac{1}{2}\vec{\xi} - \frac{\sqrt{3}}{2}\vec{\eta}\end{aligned} \tag{3.2.10}$$

Four central potentials are employed in this study:

(a) Bhaduri potential [49],

(b) Cornell potential [50],

(c) Martin potential [51] and

(d) Quigg potential [52].

These potentials are discussed in the subsections below.

3.2.1 Bhaduri potential.

The central potential that will be used with the Schrodinger equation is given by

$$V_{q\bar{q}} = -\frac{A}{r_{ij}} + Br_{ij} - D \tag{3.2.11}$$

This potential has been used by Bhaduri, Cohler and Nogami. It is sometimes called a " Coulomb- plus-linear" potential [53] because it rises linearly in r at large distances and it is coulombic in r at short distances .The potential was proposed to generate the spectra of mesons. From the Lipkin rule [54,55,56], the qq interaction is half as strong as the $V_{q\bar{q}}$ potential in mesons [57, 58, 59]. This assumption is not rigorously justifiable although compatible with the experimental results on baryons. That is,

$$V_{qq} = \frac{1}{2}V_{q\bar{q}} \tag{3.2.12}$$

where $V_{q\bar{q}}$ is given by eqn. (3.2.11).

Equation (3.2.12) gives the relation between the quark-antiquark and the quark-quark potentials. The shape of a typical qq potential is shown [60] in fig. 3.1.

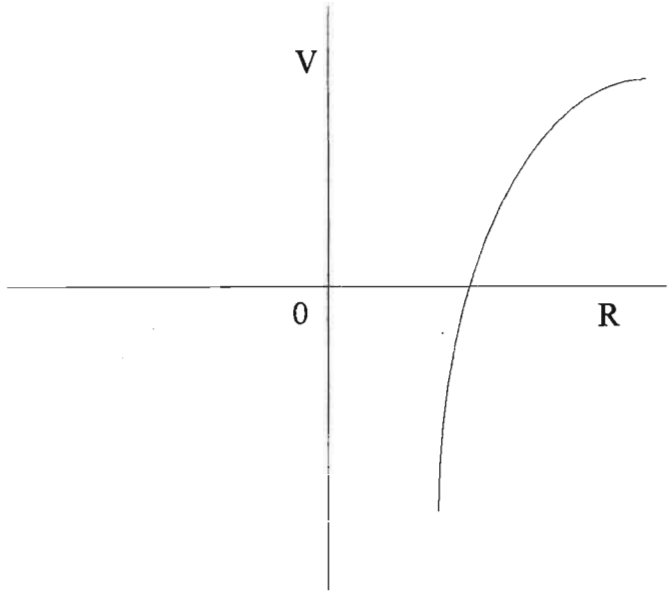

Fig.3.1 A funnel-like potential.

where V denotes the potential and R denotes the relative coordinate of either the diquark or the diquark-quark system. This potential is named a funnel-like potential because of its shape. The relation between the diquark-quark potential and the quark-quark potential is analogous to eq. (3.2.12):.

$$V_{\bar{D}q} = 2V_{qq} = V_{q\bar{q}} \tag{3.2.13}$$

where V_{qq} and $V_{q\bar{q}}$ are given by eqns. (3.2.11) and (3.2.12) respectively.

Equation (3.2.13) describes a two-body potential.

3.2.2 CORNELL POTENTIAL

This potential has the same form as that used by Bhaduri, that is, eqn.(3.2.11), and the difference is only in the parameters [50]. Eqns.(3.2.12) and (3.2.13) are also used for this potential.

3.2.3 MARTIN POTENTIAL

The power law potential

$$V(r) = A + Br^{\beta} \tag{3.2.14}$$

has been proposed by Martin for mesons [51]. Later on, Richard and Taxil [59] applied this potential to baryon spectroscopy. In this work the parameters which were used by Richard and Taxil have been adopted and are discussed in chapter 6. Eqns. (3.2.12) and (3.2.13) are still applicable to this potential.

3.2.4 QUIGG POTENTIAL

Quigg and Rosner proposed the potential of the form [52]

$$V_{q\bar{q}} = A\ln(Br) \tag{3.2.15}$$

where A and B are the adjustable parameters. This equation was also used by Lipkin [53] in his analysis. Eqns.(3.2.12) and (3.2.13) are similarly used for this potential.

3.3 KINETIC ENERGY OPERATOR.

The laplacian operator for the three identical quarks in eqn. (3.1.3) can be written as

$$\sum_{i=1}^{3} \nabla_i^2 = \nabla_1^2 + \nabla_2^2 + \nabla_3^2 \tag{3.3.16}$$

for the three-body problem. By using the Jacobi coordinates eqns. (3.1.4) through (3.1.6) and the chain rule method, the following is obtained

$$\vec{\nabla}_1 = \frac{\partial\vec{\xi}}{\partial r_1}\cdot\vec{\nabla}_\xi + \frac{\partial\vec{\eta}}{\partial r_1}\cdot\vec{\nabla}_\eta + \frac{\partial\vec{R}}{\partial r_1}\cdot\vec{\nabla}_R$$

$$= \vec{\nabla}_\xi - \frac{1}{\sqrt{3}}\vec{\nabla}_\eta + \frac{1}{3}\vec{\nabla}_R, \tag{3.3.17a}$$

$$\vec{\nabla}_2 = -\vec{\nabla}_\xi - \frac{1}{\sqrt{3}}\vec{\nabla}_\eta + \frac{1}{3}\vec{\nabla}_R \tag{3.3.17b}$$

and

$$\vec{\nabla}_3 = \frac{2}{\sqrt{3}}\vec{\nabla}_\eta + \frac{1}{3}\vec{\nabla}_R \tag{3.3.17c}$$

On squaring eqns. (3.3.17a) through (3.3.17c) the following are obtained

$$\nabla_1^2 = \nabla_\xi^2 + \frac{1}{3}\nabla_\eta^2 + \frac{1}{9}\nabla_R^2 , \tag{3.3.18}$$

$$\nabla_2^2 = \nabla_\xi^2 + \frac{1}{3}\nabla_\eta^2 + \frac{1}{9}\nabla_R^2 \tag{3.3.19}$$

and

$$\nabla_3^2 = \frac{4}{3}\nabla_\eta^2 + \frac{1}{9}\nabla_R^2 \tag{3.3.20}$$

Therefore on adding eqns. (3.3.18) through (3.3.20), eqn. (3.3.16) becomes

$$\sum_{i=1}^{3}\nabla_i^2 = 2\nabla_\xi^2 + 2\nabla_\eta^2 + \frac{1}{3}\nabla_R^2 \tag{3.3.21}$$

In this work, the motion of the baryon is ignored but the motion of the quarks inside a single baryon is taken into account. Therefore the factor $\frac{1}{3}\nabla_R^2$ is dropped in eqn.(3.3.21) and this leaves

$$\sum_{i=1}^{3}\nabla_i^2 = 2\,(\nabla_\xi^2 + \nabla_\eta^2) \tag{3.3.22}$$

This operator gives the kinetic energy operator of the baryon as

$$T_B = -\frac{\hbar^2}{2}\sum_{i=1}^{3}\frac{\nabla_i^2}{m_i} = -\frac{\hbar^2}{m}\,(\nabla_\xi^2 + \nabla_\eta^2) \tag{3.3.23}$$

where $m = m_1 = m_2 = m_3$ is the mass of the quark.

3.4 VIRIAL THEOREM

For the stationary state, the virial theorem is written in the form (see also [61])

$$\langle T\rangle = \frac{1}{2}\langle \vec{r}\,.\,\vec{\nabla}V\rangle \tag{3.4.24}$$

The quantity $\frac{1}{2}\langle \vec{r}\,.\,\vec{\nabla}V\rangle$ is called the virial of the particle(s). When the forces are central and conservative, eqn.(3.4.24) becomes

$$\langle T \rangle = \frac{1}{2} \langle r \frac{dV}{dr} \rangle \tag{3.4.25}$$

The virial theorem is applied in the calculations of both the diquark and diquark-quark systems without effecting antisymmetrization.

3.5 RADIAL SCHRODINGER EQUATION

The Schrodinger equation is used with a range of central potentials discussed in section 3.2 to determine the wave functions for both diquark and diquark-quark systems in a non-relativistic approximation. For spherically symmetric potentials, the angular part of the Schrodinger equation is separable from the radial part. In this section the radial equation for the central potentials is discussed. Eqn (3.1.2) can be written as

$$\nabla^2 \Psi + \frac{2m}{\hbar^2} (E - V(r)) \Psi = 0 \tag{3.5.26}$$

for a spherically symmetric potential where ∇^2 is the Laplacian operator, E is the energy eigenvalue and V(r) is the central potential. By letting Ψ to be the function, $\Psi(r, \theta, \vartheta)$, the following can be obtained.

$$\nabla^2 \Psi = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\Psi) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \frac{\partial \Psi}{\partial \theta}) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Psi}{\partial \vartheta^2} \tag{3.5.27}$$

The solution of the Schrodinger equation can be separated by standard techniques [62], as follows.

If

$$\Psi(r, \theta, \vartheta) = R(r) Q(\theta, \vartheta) \tag{3.5.28a}$$

and

$$F(r) = \frac{2m}{\hbar^2} [E - V(r)] \tag{3.5.28b}$$

then

$$r^2\frac{1}{rR}\frac{\partial^2}{\partial r^2}(rR)+r^2F(r)=-\frac{1}{Q}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial Q}{\partial\theta})+\frac{1}{\sin^2\theta}\frac{\partial^2 Q}{\partial\theta^2}\right\} \tag{3.5.29}$$

Eqn.(3.5.29) leads to [62]

$$\frac{1}{r}\frac{d^2}{dr^2}(rR)+[F(r)-\frac{\beta}{r^2}]R=0 \tag{3.5.30}$$

and

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial Q}{\partial\theta})-\frac{1}{\sin^2\theta}\frac{\partial^2 Q}{\partial\vartheta^2}=\beta Q \tag{3.5.31}$$

Eqs (3.5.30) and (3.5.31) are called the radial and the angular equations respectively. In order to yield permissible solutions

$$\beta=l(l+1) \qquad ,l=0,1,2,... \tag{3.5.32}$$

where l is the orbital angular momentum which determines the angular part of the wave function. The values of l do not depend on the shape of the central potential, V(r). Therefore eqn.(3.5.30) can also be written as [62]

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}(rR)+V(r)(rR)+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}(rR)=E(rR) \tag{3.5.33}$$

If U(r) = r R(r), which is called a reduced wave function, and l=0, eqn.(3.5.33) is much simpler to solve than eqn (3.1.1) and has the form of one-dimensional Schrodinger equation but differs as follows [63]:

(a) the variable r, ranges from zero to infinity whereas x is considered from the negative infinity to the positive infinity.

(b) the radial wave function, R(r) is finite at the origin following the condition that U(0) = 0, where $R(r)=\frac{U(r)}{r}$.

4 THE SINGLE CHANNEL CASE

4.1 INTRODUCTION

This chapter treats the single channel case only, the coupled channel case being discussed in chapter 5. In the single channel case only one spin state of the diquark is considered, that is, either the scalar diquark or vector diquark. These are respectively the spin-0 and spin-1 channnels. If the two channels are simultaneously considered then the coupled channel case is obtained. In this study numerical calculations are performed only in the single channel case. However the algebraic expressions for the coupled channel case are derived in chapter 5.

It has been shown in chapter 2 that the total baryon wave function in a non-relativistic independent quark model consisting of four factors, namely colour, flavour, spin and spatial, must be totally antisymmetric under an exchange of any two quarks. The main aim of this chapter is to discuss and derive the spatial component $\Psi_B(\vec{\xi},\vec{\eta})$ of the operator kernels effecting the antisymmetrization.

4.2 MEAN SQUARE RADIUS

If the volume density of a system is ρ, the mean square radius is given by

$$\langle r^2 \rangle = \frac{\int \rho r^2 d^3r}{\int \rho d^3r} \tag{4.2.1}$$

where r is measured relative to the centre-of-mass of the system.

For a two-quark system, $\rho(\vec{r})$ is expressed in terms of the wave function Φ. Therefore

$$\rho(\vec{r}) = \int |\Phi(\vec{r}_1-\vec{r}_2)|^2 \delta(\vec{r}-\vec{r}_1)\,\delta(\vec{r}_1+\vec{r}_2)\,d^3r_1 d^3r_2 + \int |\Phi(\vec{r}_1-\vec{r}_2)|^2 \delta(\vec{r}-\vec{r}_2)\,\delta(\vec{r}_1+\vec{r}_2)\,d^3r_1 d^3r_2 \tag{4.2.2}$$

Since the condition $\delta(\vec{r}_1+\vec{r}_2) = 0$, is imposed to remove the centre-of-mass, the equation $\vec{r}_1 = -\vec{r}_2$, holds. Therefore

$$\rho(\vec{r}) = 2|\Phi(2\vec{r})|^2 \tag{4.2.3}$$

By substituting eqn. (4.2.3) into eqn. (4.2.1), the following is obtained

$$\langle r^2 \rangle = \frac{\int |\Phi(2\vec{r})|^2 r^2 d^3r}{\int |\Phi(2\vec{r})|^2 d^3r} \tag{4.2.4}$$

If $\vec{r}\,' = 2\vec{r}$ then

$$\langle (\vec{r}\,')^2 \rangle = \langle (2\vec{r})^2 \rangle = 4 \langle r^2 \rangle \tag{4.2.5}$$

Therefore the root mean square (r.m.s) radius can be written in the form

$$\langle r^2 \rangle^{\frac{1}{2}} = \frac{1}{2}\left(\frac{\int |\Phi(\vec{r}\,')|^2 \vec{r}^{\,2} d^3r'}{\int |\Phi(\vec{r}\,')|^2 d^3r'} \right)^{\frac{1}{2}} \tag{4.2.6}$$

4.3 EXPECTATION VALUES WITHOUT ANTISYMMETRIZATION

In Quantum Physics, mathematical operators play a central role. An operator is considered to be some instruction that when applied to a function, changes it into another function. In this thesis an operator $\hat{O}$ is designated by the small circumflex over it. In order to calculate the mass of the baryon the expectation value of the operator $\hat{O}$ must be defined. Two postulates of quantum mechanics [63] are:

(i) the average of a large number of measurements of the observable $\hat{O}$ of which $\hat{O}$ is the operator and Φ is the wave function is given by

$$\langle \hat{O} \rangle = \frac{\int \Phi^* \hat{O} \Phi d^3r}{\int \Phi^* \Phi d^3r} \tag{4.3.7}$$

(ii) the product of $\Phi^*\Phi$ is a measure of the probability that the particle with state function Φ can be found at the position r of configuration space, that is, $\Phi^*\Phi$ is a measure of particle density at r.

Eqn. (4.3.7) is called an expectation value of operator $\hat{O}$. Therefore from eqn. (4.3.7) the expectation value of the potential energy for both the diquark and the diquark-quark systems can be respectively given by

$$\langle V_{\bar{D}} \rangle = \frac{\int \Phi^*_{\bar{D}}(\vec{\xi})\, V_{\bar{D}}(\xi)\, \Phi_{\bar{D}}(\vec{\xi})\, d^3\xi}{\int \Phi^*_{\bar{D}}(\vec{\xi})\, \Phi_{\bar{D}}(\vec{\xi})\, d^3\xi} \tag{4.3.8}$$

and

$$\langle V_{\bar{D}q} \rangle = \frac{\int \Psi^*_{\bar{D}q}(\vec{\eta})\, V_{\bar{D}q}(\eta)\, \Psi_{\bar{D}q}(\vec{\eta})\, d^3\eta}{\int \Psi^*_{\bar{D}q}(\vec{\eta})\, \Psi_{\bar{D}q}(\vec{\eta})\, d^3\eta} \tag{4.3.9}$$

where $\vec{\xi}$ and $\vec{\eta}$ are defined in eqns.(3.1.4) and (3.1.5).

The expectation value of the kinetic energy for both the diquark and the diquark-quark systems are also given by means of the Virial theorem

$$\langle T_{\bar{D}} \rangle = \frac{\frac{1}{2}\int \Phi^*_{\bar{D}}(\vec{\xi})\, \xi \frac{dV_{\bar{D}}(\xi)}{d\xi} \Phi_{\bar{D}}(\vec{\xi})\, d^3\xi}{\int \Phi^*_{\bar{D}}(\vec{\xi})\, \Phi_{\bar{D}}(\vec{\xi})\, d^3\xi} \tag{4.3.10}$$

and

$$\langle T_{\bar{D}q} \rangle = \frac{\frac{1}{2}\int \Psi^*_{\bar{D}q}(\vec{\eta})\, \eta \frac{dV_{\bar{D}q}(\eta)}{d\eta} \Psi_{\bar{D}q}(\vec{\eta})\, d^3\eta}{\int \Psi^*_{\bar{D}q}(\vec{\eta})\, \Psi_{\bar{D}q}(\vec{\eta})\, d^3\eta} \tag{4.3.11}$$

where $\langle T_{\bar{D}} \rangle$ and $\langle T_{\bar{D}q} \rangle$ are generally defined in eqn. (3.4.25).

4.4 EXPECTATION VALUES WITH ANTISYMMETRIZATION

4.4.1 Operator kernels for diquark models of baryons

If the coordinates of the identical quarks 1,2 and 3 in a baryon are specified as $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ then the coordinates of the centre-of-mass of the two-body substates 12, 13, and 23 are

given by

$$\vec{R}_{12} = \frac{\vec{r}_1 + \vec{r}_2}{2},$$

$$\vec{R}_{13} = \frac{\vec{r}_1 + \vec{r}_3}{2}$$

and

$$\vec{R}_{23} = \frac{\vec{r}_2 + \vec{r}_3}{2} \tag{4.4.12}$$

In the centre-of-mass the coordinates $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ obey the condition that

$$\vec{r}_1 + \vec{r}_2 + \vec{r}_3 = 0 \tag{4.4.13}$$

This condition represents the centre-of-mass at rest. In the diquark model the spatial wave function must be in the following form:

$$\Psi_B(\vec{\xi},\vec{\eta}) = \Phi_{\bar{D}}(\vec{\xi}) \otimes \Psi_{\bar{D}q}(\vec{\eta}) \tag{4.4.14}$$

where $\vec{\xi}$ and $\vec{\eta}$ are given by eqns. (3.1.4) and (3.1.5). Therefore the expectation value for the operator $\hat{O}$ as given by eqn. (4.3.7) becomes

$$\langle\hat{O}\rangle = \frac{\int \Psi_B^*(\vec{\xi},\vec{\eta})\,\hat{O}\Psi_B(\vec{\xi},\vec{\eta})\,d^3\xi d^3\eta}{\langle\hat{1}\rangle} \tag{4.4.15}$$

for the baryon where $\langle\hat{1}\rangle$ is the normalization. By taking into account the effects of exchange between the diquark pair and the external quark, the expectation value of the operator $\hat{O}$ is defined as

$$\langle\hat{O}\rangle = \frac{\int \Psi_{\bar{D}q}^*(\vec{X})\,\Phi_{\bar{D}}^*(\vec{\xi})\,\delta(\vec{X}-\vec{\eta})\,\hat{O}_{sym}A[\Phi_{\bar{D}}(\vec{\xi})\,\delta(\vec{X}'-\vec{\eta})]\,\Psi_{\bar{D}q}(\vec{X}')\,d^3\xi d^3\eta d^3X d^3X'}{\langle\vec{1}\rangle} \tag{4.4.16}$$

If $K_{\hat{O}}(\vec{X},\vec{X}')$ is defined by

$$K_{\hat{O}}(\vec{X},\vec{X}') = \int \Phi_{\bar{D}}^{*}(\vec{\xi})\,\delta(\vec{X}-\vec{\eta})\,\hat{O}_{sym} A[\Phi_{\bar{D}}(\vec{\xi})\,\delta(\vec{X}-\vec{\eta})]\,d^3\xi d^3\eta \qquad (4.4.17)$$

then

$$\langle\hat{O}\rangle = \frac{\int \Psi_{Dq}^{*}(\vec{X})\,K_{\hat{O}}(\vec{X},\vec{X}')\,\Psi_{\bar{D}q}(\vec{X}')\,d^3X d^3X'}{\langle\hat{1}\rangle} \qquad (4.4.18)$$

where $K_{\hat{O}}(\vec{X},\vec{X}')$ is the operator kernel, A is the antisymmetrization operator and $\hat{O}_{sym}$ is the symmetrized form of the operator $\hat{O}$.

To make eqns.(4.4.15) and (4.4.18) equivalent, eqn.(4.4.18) needs to be divided by the expectation value of the norm kernel $K_{\hat{1}}(\vec{X},\vec{X}')$. There are two parts to any operator kernels - direct and exchange. The capital letters D and E will be used for the direct and exchange parts respectively.

4.4.2 The norm kernel

4.4.2.1 The direct part of the norm kernel

Equation (4.4.17) becomes

$$K_{\hat{1}}^{D}(\vec{X},\vec{X}') = \int \Phi_{\bar{D}}^{*}(\vec{\xi})\,\delta(\vec{X}-\vec{\eta})\,\Phi_{\bar{D}}(\vec{\xi})\,\delta(\vec{X}'-\vec{\eta})\,d^3\xi d^3\eta$$
$$= \delta(\vec{X}-\vec{X}')\int \Phi_{\bar{D}}^{*}(\vec{\xi})\,\Phi_{\bar{D}}(\vec{\xi})\,d^3\xi \qquad (4.4.19)$$

for the unnormalized $\Phi_{\bar{D}}(\vec{\xi})$.

4.4.2.2 The exchange part of the norm kernel

Equation (4.4.17) becomes

$$K_{\hat{1}}^{E}(\vec{X},\vec{X}') = 2\int \Phi_{\bar{D}}^{*}(\vec{\xi})\,\delta(\vec{X}-\vec{\eta})\,\Phi_{\bar{D}}(\vec{\xi}')\,\delta(\vec{X}'-\vec{\eta}')\,d^3\xi d^3\eta$$
$$= 2\left(\frac{2}{\sqrt{3}}\right)^3 \Phi_{\bar{D}}^{*}\left(\left|\frac{2}{\sqrt{3}}\vec{X}'+\frac{1}{\sqrt{3}}\vec{X}\right|\right)\Phi_{\bar{D}}\left(\left|\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right|\right) \qquad (4.4.20)$$

where the factor 2 arises from the equivalence of the (13) and (23) permutations in even parity states. The variables $\vec{\xi}'$and $\vec{\eta}'$

$$\vec{\xi}' = \vec{r}_3 - \vec{r}_1$$
$$= -\frac{1}{2}\vec{\xi} + \frac{\sqrt{3}}{2}\vec{\eta} \qquad (4.4.21)$$

and

$$\vec{\eta}' = \frac{2}{\sqrt{3}}(\vec{r}_2 - \frac{\vec{r}_1 + \vec{r}_3}{2})$$

$$= -\frac{\sqrt{3}}{2}\vec{\xi} - \frac{1}{2}\vec{\eta} \qquad (4.4.22)$$

These coordinates describe the exchange between quark 2 and quark 3 in a diquark and a diquark-quark respectively.

4.4.2.3 The expectation values of the norm kernel

The expectation values of the direct and exchange parts of the norm kernel are respectively

$$<K_1^D(\vec{X},\vec{X}')> = \int \Psi^*_{\bar{Dq}}(\vec{X})\,\Phi^*_{\bar{D}}(\vec{\xi})\,\Phi_{\bar{D}}(\vec{\xi})\,\Psi_{\bar{Dq}}(\vec{X})\,d^3\xi d^3X \qquad (4.4.23)$$

and

$$<K_1^E(\vec{X},\vec{X}')> = \frac{128\pi^2}{3\sqrt{3}}\int \Psi^*_{\bar{Dq}}(\vec{X})\,\Phi^*_{\bar{D}}(|\frac{2}{\sqrt{3}}\vec{X}' + \frac{1}{\sqrt{3}}\vec{X}|)\,\Phi_{\bar{D}}(|\frac{1}{\sqrt{3}}\vec{X}' + \frac{2}{\sqrt{3}}\vec{X}|)\,\Psi_{\bar{Dq}}(\vec{X}')$$

$$\vec{X}^2(\vec{X}')^2 dX dX' d\mu \qquad (4.4.24)$$

where

$$\mu = \frac{\vec{X}.\vec{X}'}{XX'} \qquad (4.4.25)$$

4.4.3 The potential energy kernel

4.4.3.1 The direct part of the potential energy kernel

Equation (4.4.17) must now be written in the form

$$K_V^D(\vec{X},\vec{X}') = \int \Phi_D^*(\vec{\xi})\,\delta(\vec{X}-\vec{\eta}) \sum_{i<j} V(r_{ij})\,\Phi_D(\vec{\xi})\,\delta(\vec{X}'-\vec{\eta})\,d^3\xi d^3\eta$$

$$= \delta(\vec{X}-\vec{X}')\left\{\int \Phi_D^*(\vec{\xi})\,V(\xi)\,\Phi_D(\vec{\xi})\,d^3\xi + 2\int \Phi_D^*(\vec{\xi})\,V\left(\left|\frac{\sqrt{3}}{2}\vec{X}+\frac{1}{2}\vec{\xi}\right|\right)\Phi_D(\vec{\xi})\,d^3\xi\right\} \qquad (4.4.26)$$

or

$$K_V^D(\vec{X},\vec{X}') = \delta(\vec{X}-\vec{X}')\left\{\int \Phi_D^*(\vec{\xi})\,V_D(\xi)\,\Phi_D(\vec{\xi})\,d^3\xi + 2\int \Phi_D^*(\vec{\xi})\,V\left(\left|\frac{\sqrt{3}}{2}\vec{X}-\frac{1}{2}\vec{\xi}\right|\right)\Phi_D(\vec{\xi})\,d^3\xi\right\} \qquad (4.4.27)$$

which implies that eqns (4.4.26) and (4.4.27) are equivalent.

4.4.3.2 The exchange part of the potential energy kernel

Equation (4.4.17) takes the form:

$$K_V^E(\vec{X},\vec{X}') = 2\int \Phi_D^*(\vec{\xi})\,\delta(\vec{X}-\vec{\eta}) \sum_{i<j} V(r_{ij})\,\Phi_D(\vec{\xi})\,\delta(\vec{X}'-\vec{\eta}')\,d^3\xi d^3\eta$$

$$= 2\left(\frac{2}{\sqrt{3}}\right)^3 \Phi_D^*\left(\left|\frac{2}{\sqrt{3}}\vec{X}'+\frac{1}{\sqrt{3}}\vec{X}\right|\right)\Phi_D\left(\left|\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right|\right)$$

$$\left\{V\left(\left|\frac{2}{\sqrt{3}}\vec{X}'+\frac{1}{\sqrt{3}}\vec{X}\right|\right)+V\left(\left|\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right|\right)+V\left(\left|\frac{1}{\sqrt{3}}\vec{X}'-\frac{1}{\sqrt{3}}\vec{X}\right|\right)\right\} \qquad (4.4.28)$$

for the exchange term for the potential energy kernel and the factor 2 has the same meaning as in section 4.4.2.2.

4.4.3.3 The expectation values of the potential energy kernel.

The expectation values for the direct and exchange terms are respectively expressed in the following forms

$$<K_V^D(\vec{X},\vec{X}')> = \int\Psi^*_{\bar{D}q}(\vec{X})\,\Phi^*_{\bar{D}}(\vec{\xi})\,V_{\bar{D}}(\xi)\,\Phi_{\bar{D}}(\vec{\xi})\,\Psi_{\bar{D}q}(\vec{X})\,d^3\xi d^3X$$
$$+ 8\pi^2\int\Psi^*_{\bar{D}q}(\vec{X})\,\Phi^*_{\bar{D}}(\vec{\xi})\,\Phi_{\bar{D}}(\vec{\xi})\,\Psi_{\bar{D}q}(\vec{X})$$
$$\left\{V\left(\left|\frac{\sqrt{3}}{2}\vec{X}-\frac{1}{2}\vec{\xi}\right|\right)+V\left(\left|\frac{1}{2}\vec{\xi}+\frac{\sqrt{3}}{2}X\right|\right)\right\}\xi^2X^2d\xi dXd\mu \qquad (4.4.29a)$$

or

$$<K_V^D(\vec{X},\vec{X}')> = \int\Psi^*_{\bar{D}q}(\vec{X})\,\Phi^*_{\bar{D}}(\vec{\xi})\,V_{\bar{D}}(\xi)\,\Phi_{\bar{D}}(\vec{\xi})\,\Psi_{\bar{D}q}(\vec{X})\,d^3\xi d^3X$$
$$+ 16\pi^2\int\Psi^*_{\bar{D}q}(\vec{X})\,\Phi^*_{\bar{D}}(\vec{\xi})\,V\left(\left|\frac{1}{2}\vec{\xi}+\frac{\sqrt{3}}{2}\vec{X}\right|\right)\Phi_{\bar{D}}(\vec{\xi})\,\Psi_{\bar{D}q}(\vec{X})$$
$$\xi^2X^2d\xi dXd\mu \qquad (4.4.29b)$$

or

$$<K_V^D(\vec{X},\vec{X}')> = \int\Psi^*_{\bar{D}q}(\vec{X})\,\Phi^*_{\bar{D}}(\vec{\xi})\,V_{\bar{D}}(\xi)\,\Phi_{\bar{D}}(\vec{\xi})\,\Psi_{\bar{D}q}(\vec{X})\,d^3\xi d^3X$$
$$+ 16\pi^2\int\Psi^*_{\bar{D}q}(\vec{X})\,\Phi^*_{\bar{D}}(\vec{\xi})\,V\left(\left|\frac{1}{2}\vec{\xi}-\frac{\sqrt{3}}{2}\vec{X}\right|\right)\Phi_{\bar{D}}(\vec{\xi})\,\Psi_{\bar{D}q}(\vec{X})$$
$$\xi^2X^2d\xi dXd\mu \qquad (4.4.29c)$$

where $\mu = \dfrac{\vec{\xi}.\vec{X}}{\xi X}$ and

$$<K_V^E(\vec{X},\vec{X}')> = 8\pi^2\left(\frac{2}{\sqrt{3}}\right)^3\int\Psi^*_{\bar{D}q}(\vec{X})\,\Phi^*_{\bar{D}}\left(\left|\frac{2}{\sqrt{3}}\vec{X}'+\frac{1}{\sqrt{3}}\vec{X}\right|\right)\Phi_{\bar{D}}\left(\left|\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right|\right)\Psi_{\bar{D}q}(\vec{X}')$$
$$\left\{V\left(\left|\frac{2}{\sqrt{3}}\vec{X}'+\frac{1}{\sqrt{3}}\vec{X}\right|\right)+V\left(\left|\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right|\right)+V\left(\left|\frac{1}{\sqrt{3}}\vec{X}'-\frac{1}{\sqrt{3}}\vec{X}\right|\right)\right\}$$
$$X^2(\vec{X}')^2dXdX'd\mu \qquad (4.4.30)$$

4.4.4 The kinetic energy kernel

4.4.4.1 The direct part of the kinetic energy kernel

The direct term for the kinetic energy kernel is given by

$$K_T^D(\vec{X},\vec{X}') = -\frac{\hbar^2}{2m_q}\int\Phi_D^*(\vec{\xi})\,\delta(\vec{X}-\vec{\eta})\,(2\nabla_\xi^2+2\nabla_\eta^2)\,\Phi_D(\vec{\xi})\,\delta(\vec{X}'-\vec{\eta})\,d^3\xi d^3\eta$$

$$= -\frac{\hbar^2}{m_q}\delta(\vec{X}-\vec{X}')\left\{\int\Phi_D^*(\vec{\xi})\,\nabla_\xi^2\,\Phi_D(\vec{\xi})\,d^3\xi - \nabla_X^2\int\Phi_D^*(\vec{\xi})\,\Phi_D(\vec{\xi})\,d^3\xi\right\} \tag{4.4.31}$$

where the expression $2\nabla_\xi^2+2\nabla_\eta^2$ is defined in eqn.(3.3.22).

4.4.5.2 The exchange part of the kinetic energy kernel

The exchange term of the kinetic operator kernel is given by

$$K_T^E(\vec{X},\vec{X}') = -2\frac{\hbar^2}{m_q}\int\Phi_D^*(\vec{\xi})\,\delta(\vec{X}-\vec{\eta})\,(\nabla_\xi^2+\nabla_\eta^2)\,\Phi_D(\vec{\xi}')\,\delta(\vec{X}-\vec{\eta}')\,d^3\xi d^3\eta$$

$$= -\frac{\hbar^2}{m_q}\int\Phi_D^*(\vec{\xi})\,\delta(\vec{X}-\vec{\eta})\,(\nabla_\xi^2+\nabla_\eta^2+\nabla_{\xi'}^2+\nabla_{\eta'}^2)\,\Phi_D(\vec{\xi}')\,\delta(\vec{X}'-\vec{\eta}')\,d^3\xi d^3\eta \tag{4.4.32}$$

After some algebra, eqn. (4.4.32) becomes

$$K_T^E(\vec{X},\vec{X}') = -\frac{\hbar^2}{m_q}\left(\frac{2}{\sqrt{3}}\right)^3\left\{\Phi_D^*\left(\left|\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right|\right)\nabla^2\Phi_D\left(\left|\frac{1}{\sqrt{3}}\vec{X}+\frac{2}{\sqrt{3}}\vec{X}'\right|\right)\right.$$

$$+\Phi_D^*\left(\left|\frac{1}{\sqrt{3}}\vec{X}+\frac{2}{\sqrt{3}}\vec{X}'\right|\right)\nabla^2\Phi_D\left(\left|\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right|\right)$$

$$+\nabla_X^2\Phi_D^*\left(\left|\frac{1}{\sqrt{3}}\vec{X}+\frac{2}{\sqrt{3}}\vec{X}'\right|\right)\Phi_D\left(\left|\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right|\right)$$

$$\left.+\nabla_{X'}^2\Phi_D^*\left(\left|\frac{1}{\sqrt{3}}\vec{X}+\frac{2}{\sqrt{3}}\vec{X}'\right|\right)\Phi_D\left(\left|\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right|\right)\right\} \tag{4.4.33}$$

4.4.4.3 The expectation values of the kinetic energy operator kernel

The expectation values for eqns. (4.4.31) and (4.4.33) are respectively

$$<K_T^D(\vec{X},\vec{X}')> = -\frac{\hbar^2}{m_q}\int\Psi^*_{\bar{D}q}(\vec{X})\,\Phi^*_{\bar{D}}(\vec{\xi})\,\nabla^2_{\xi}\,\Phi_{\bar{D}}(\vec{\xi})\,\Psi_{\bar{D}q}(\vec{X})\,d^3\xi d^3X$$

$$= -\frac{\hbar^2}{m_q}\int\Psi^*_{\bar{D}q}(\vec{X})\,\nabla^2_{X}\,\Psi_{\bar{D}q}(\vec{X})\,\Phi^*_{\bar{D}}(\vec{\xi})\,\Phi_{\bar{D}}(\vec{\xi})\,d^3\xi d^3X \qquad (4.4.34)$$

and

$$<K_T^E(\vec{X},\vec{X}')> = 8\pi^2\int\Psi^*_{\bar{D}q}(\vec{X})\,K_T^E(\vec{X},\vec{X}',\mu)\,\Psi_{\bar{D}q}(\vec{X}')\,X^2\,(X')^2 dX dX' d\mu \qquad (4.4.35)$$

where $K_T^E(\vec{X},\vec{X}',\mu)$ is given by eqn. (4.4.33).

4.5 FORM FACTOR

4.5.1 INTRODUCTION

The form factor is defined as [40]

$$F(\vec{q}) = \int\rho(\vec{r})\,e^{i\vec{q}.\vec{r}}d^3r \qquad (4.5.36)$$

where $\rho(\vec{r})$ is the charge density normalized to unity and $\vec{q}$ is the wave number vector, that is

$$\int\rho(\vec{r})\,d^3r = 1 \qquad (4.5.37)$$

For a point distribution,

$$F(\vec{q}) = 1 \qquad (4.5.38)$$

For a spherically symmetric uniform charge distribution, $\rho(\vec{r}) = \rho_0$, with a charge edge at r=R the form factor, eqn.(4.5.36), becomes [40]

$$F(q) = \frac{4\pi\rho_0}{q^3}(\sin(qR) - qR\cos(qR)) \qquad (4.5.39)$$

For a smooth charge distribution [40]

$$\rho(\vec{r}) = \frac{\mu^3}{8\pi}e^{-\mu r} \qquad (4.5.40)$$

, where μ is a constant, the form factor, eqn. (4.5.36) becomes

$$F(q) = (1+\frac{q^2}{\mu^2})^{-2} \tag{4.5.41}$$

Equation (4.5.41) is called the dipole form factor. Taking $\mu^2 = 0.71\text{GeV}^2$ [64] gives good agreement with the proton experimental form factor, 0.83 fm [64]. To take into account the low-q^2 contribution of the mesons, the monopole form factor is defined as [64]

$$F(q) = \frac{\Omega^2}{\Omega^2+q^2} \tag{4.5.42}$$

where $\Omega = 0.795\text{GeV}$ [64].

To account for the dressing by a meson cloud, the monopole meson term [64]

$$F(q) = \frac{\Omega^2}{\Omega^2+q^2} F_E^{quark}(q) \tag{4.5.43}$$

can be considered, where $F_E^{quark}(q)$ is the electric form factor to be numerically calculated in chapter 6. Equation (4.5.43) describes the bare-constituent quark model with meson cloud.

4.5.2 FORM FACTOR WITHOUT ANTISYMMETRIZATION

To calculate the form factor for the diquark-quark system, the spherical Bessel function [65] will be used as the operator. This is defined by

$$\begin{aligned} j_l(x) &= \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x) \\ &= (-x)^l (\frac{1}{x}\frac{d}{dx})^l \frac{\sin x}{x} \end{aligned} \tag{4.5.44}$$

In this study l=0 and $x = q(\vec{r}_1 - \vec{R})$ and therefore eqn. (4.5.44) becomes

$$j_o(q(\vec{r}_1 - \vec{R})) = \frac{\sin(q|\vec{r}_1 - \vec{R}|)}{q|\vec{r}_1 - \vec{R}|} \tag{4.5.45}$$

From eqns. (3.1.5) and (3.1.6) the following can be deduced

$$\vec{r}_1 - \vec{R} = \frac{1}{\sqrt{3}}\vec{\eta} \tag{4.5.46}$$

The form factor for the diquark-quark system is therefore defined by the following equation:

$$F_{\bar{D}q}(q) = \frac{\int \Psi^{*}_{\bar{D}q}(\vec{\eta})\, j_o\left(\frac{q}{\sqrt{3}}\vec{\eta}\right) \Psi_{\bar{D}q}(\vec{\eta})\, d^3\eta}{\int \Psi^{*}_{\bar{D}q}(\vec{\eta})\, \Psi_{\bar{D}q}(\vec{\eta})\, d^3\eta} \tag{4.5.47}$$

4.5.3 FORM FACTOR WITH ANTISYMMETRIZATION

4.5.3.1 The norm kernel

The direct and exchange norm kernels are respectively given by eqns. (4.4.23) and (4.4.24).

4.5.3.2 The form factor operator kernel

Here the operator kernel is given by

$$\frac{\sum_i^3 \sin q|\vec{r}_i - \vec{R}|}{q|\vec{r}_i - \vec{R}|} = \frac{\sin\frac{q}{\sqrt{3}}\eta}{\frac{q}{\sqrt{3}}\eta} + \frac{\sin\frac{q}{\sqrt{3}}\eta'}{\frac{q}{\sqrt{3}}\eta'} + \frac{\sin\frac{q}{\sqrt{3}}\eta''}{\frac{q}{\sqrt{3}}\eta''} \tag{4.5.48}$$

4.5.3.2.1 The direct term

The direct term for the form factor operator kernel and its expectation value are respectively given by

$$K_F^D(\vec{X}, \vec{X}') = \int \Phi^{*}_{\bar{D}}(\vec{\xi})\, \frac{\sin\frac{q}{\sqrt{3}}\eta}{\frac{q}{\sqrt{3}}\eta}\, \Phi_{\bar{D}}(\vec{\xi})\, \delta(\vec{X} - \vec{\eta})\, \delta(\vec{X}' - \vec{\eta})\, d^3\xi\, d^3\eta \tag{4.5.49}$$

and

$$\langle K_F^D(\vec{X}, \vec{X}') \rangle = 16\pi^2 \int \Psi^{*}_{\bar{D}q}(\vec{X})\, \Phi^{*}_{\bar{D}}(\vec{\xi})\, \frac{\sin\frac{q}{\sqrt{3}}X}{\frac{q}{\sqrt{3}}X}\, \Phi_{\bar{D}}(\vec{\xi})\, \Psi_{\bar{D}q}(\vec{X})\, \xi^2 X^2\, dX\, d\xi \tag{4.5.50}$$

4.5.3.2.2 The exchange term

The exchange term of the form factor operator kernel and its expectation value are calculated as follows:

$$K_F^E(\vec{X},\vec{X}') = \int \Phi_D^*(\vec{\xi}) \left(\frac{\sin\frac{q}{\sqrt{3}}\eta'}{\frac{q}{\sqrt{3}}\eta'} + \frac{\sin\frac{q}{\sqrt{3}}\eta''}{\frac{q}{\sqrt{3}}\eta''} \right) \Phi_D(\vec{\xi}')\,\delta(\vec{X}-\vec{\eta})\,\delta(\vec{X}'-\vec{\eta}')\,d^3\xi d^3\eta \tag{4.5.51}$$

and

$$\langle K_F^E(\vec{X},\vec{X}')\rangle = \frac{64\pi^2}{3\sqrt{3}} \int \Psi_{Dq}^*(\vec{X})\,\Phi_D^*\left(\frac{2}{\sqrt{3}}\vec{X}'+\frac{1}{\sqrt{3}}\vec{X}\right) \left(\frac{\sin\frac{q}{\sqrt{3}}X'}{\frac{q}{\sqrt{3}}X} + \frac{\sin\frac{q}{\sqrt{3}}|\vec{X}+\vec{X}'|}{\frac{q}{\sqrt{3}}|\vec{X}+\vec{X}'|} \right)$$

$$\Phi_D\left(\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right)\Psi_{Dq}(\vec{X}')\,X^2(X')^2 dX dX' d\mu \tag{4.5.52}$$

By integrating eqn. (4.5.36) over angles, it can be found that [66]

$$F(q) = 4\pi\int_0^\infty \rho(r)\,\frac{\sin qr}{qr}\,r^2 dr \tag{4.5.53}$$

where q is equal to the 3-momentum transfer.

$F(q)$ is a three-dimensional Fourier transform of a charge distribution in space. The charge density $\rho(r)$ may be obtained from the inverse Fourier transform [66]:

$$\rho(r) = \frac{1}{2\pi^2}\int_0^\infty F(q)\,\frac{\sin qr}{qr}\,q^2 dq \tag{4.5.54}$$

Equation (4.5.36) can again be simplified to give [66]

$$F(q^2) = 1 - \frac{1}{6}\langle r^2\rangle q^2 + \text{---} \tag{4.5.55}$$

Differentiating eqn. (4.5.55) with respect to q^2 and taking the limit as $q^2 \to 0$, the following is obtained

$$\lim_{q^2 \to 0} \frac{dF(q^2)}{dq^2} = -\frac{1}{6}\langle r^2 \rangle \tag{4.5.56}$$

Therefore [40]

$$\langle r^2 \rangle = -6 \lim_{q^2 \to 0} \frac{dF(q^2)}{dq^2} \tag{4.5.57}$$

For a very small q, the mean square radius is therefore

$$\langle r^2 \rangle \approx -6 \frac{F(q^2) - F(0)}{q^2 - 0}$$

$$= -6 \frac{F(q^2) - 1}{q^2} \tag{4.5.58}$$

From this equation the root mean square radius can be calculated by taking the limit of q tends to zero, namely $q = 0.1$, and substitute into eqn. (4.5.58). The experimental value of the root mean square radius for the proton ,(uud), is 0.83 fm [64].

4.6 THE HARMONIC OSCILLATOR

The harmonic oscillator used in this work is just to test whether the diquark model behaves properly or not. If the size for the diquark is assumed and the symmetrical oscillator Hamiltonian is used, then the following are true (see, for example [67, 68]):

$$V(\vec{\xi}, \vec{\eta}) = \frac{3}{2} \frac{\hbar^2}{mb^4} (\xi^2 + \eta^2) \tag{4.6.59}$$

and

$$\Psi_{000} = \Phi_D(\vec{\xi}) \otimes \Psi_{Dq}(\vec{\eta}) = \frac{\alpha_1^{\frac{3}{2}}}{\pi^{\frac{3}{4}}} \exp\left(-\frac{\alpha_1^2}{2}\xi^2\right) \frac{\alpha_2^{\frac{3}{2}}}{\pi^{\frac{3}{4}}} \exp\left(-\frac{\alpha_2^2}{2}\eta^2\right) \tag{4.6.60}$$

These equations imply that

$$\Phi_D(\vec{\xi}) = \frac{\alpha_1^{\frac{3}{2}}}{\pi^{\frac{3}{4}}} \exp\left(-\frac{\alpha_1^2}{2}\xi^2\right), \tag{4.6.61}$$

with $$V(\vec{\xi}) = \frac{3}{2}\frac{\hbar^2}{mb^4}\xi^2 \tag{4.6.62}$$

and

$$V(\vec{\eta}) = \frac{3}{2}\frac{\hbar^2}{mb^4}\eta^2, \tag{4.6.63}$$

with $$\Psi_{\bar{D}q} = \frac{\alpha_2^{\frac{3}{2}}}{\pi^{\frac{3}{4}}}\exp\left(-\frac{\alpha_2^2}{2}\eta^2\right) \tag{4.6.64}$$

where α_1 is free and $\alpha_2^4 = \frac{3}{2}\frac{1}{b^4}$ with b being the size of the diquark and has the dimension of length. The direct and exchange terms for the harmonic oscillator are calculated in the following subsections:

4.6.1 The direct term of the harmonic oscillator

The direct term can be calculated as

$$V_{\bar{D}}^{D}(\vec{X},\vec{X}') = \delta(\vec{X}-\vec{X}')\frac{\alpha_1^3}{\pi^{\frac{3}{2}}}\int\frac{3}{2}\frac{\hbar^2}{mb^4}(\xi^2+X^2)\exp(-\alpha_1^2\xi^2)\,d^3\xi$$

$$= \frac{3}{2}\frac{\hbar^2}{mb^4}\delta(\vec{X}-\vec{X}')\left(X^2+\frac{3}{2}\frac{1}{\alpha_1}\right) \tag{4.6.65}$$

If the diquark size is small then α_1 is large and so $\frac{1}{\alpha_1^2}$ term contributes very little. Therefore eqn. (4.6.65) becomes

$$V_{\bar{D}}^{D}(\vec{X},\vec{X}') = \frac{3}{2}\frac{\hbar^2}{mb^4}X^2\delta(\vec{X}-\vec{X}') \tag{4.6.66}$$

4.6.2 The exchange term of the harmonic oscillator

The exchange can be deduced as follows:

$$V_{\bar{D}}^{E}(\vec{X},\vec{X}') = \left(\frac{2}{\sqrt{3}}\right)^{3}\frac{\alpha_1^3}{\pi^{\frac{3}{2}}}\frac{3}{2}\frac{\hbar^2}{mb^4}\int\delta(\vec{X}-\vec{X}')\,\delta\left[\vec{\xi}-\left(-\frac{2}{\sqrt{3}}\vec{X}'-\frac{1}{\sqrt{3}}\vec{X}\right)\right](\xi^2+X^2)$$

$$\exp(-\alpha_1^2\xi^2)\,d^3\xi d^3X \tag{4.6.67}$$

After a little algebra and the use of L'Hospital's rule, eqn. (4.6.67) reduces to

$$V_{\bar{D}}^{E}(\vec{X},\vec{X}') = 0 \tag{4.6.68}$$

The expectation value for $\hat{H}_B$ as a function of a diquark size is therefore determined as

$$\langle H_{\bar{D}}\rangle = 4\frac{\alpha_1^3}{\pi^{\frac{3}{2}}}\pi\int_0^{\infty}\left[\frac{\hbar^2}{m}(3\alpha_1-\alpha_1^4\xi^2)+\frac{3}{2}\frac{\hbar^2}{mb^4}\xi^2\right]\exp(-\alpha_1^2\xi^2)\,\xi^2 d\xi$$

$$= \frac{3}{2}\alpha_1^2\frac{\hbar^2}{m}+\frac{9}{4b^4}\frac{\hbar^2}{m}\frac{1}{\alpha_1^2} \tag{4.6.69}$$

Similarly for the diquark-quark potential the following is obtained

$$\langle H_{\bar{D}q}\rangle = \frac{3}{2}\alpha_2^2\frac{\hbar^2}{m}+\frac{9}{4b^4}\frac{\hbar^2}{m}\frac{1}{\alpha_2^2} \tag{4.6.70}$$

Therefore from eqns. (4.6.69) and (4.6.70) the expectation value of the Hamiltonian for the baryon can be calculated as

$$\langle H_B\rangle = \langle H_{\bar{D}}\rangle + \langle H_{\bar{D}q}\rangle = \frac{3}{2}\frac{\hbar^2}{m}\alpha_1^2+\frac{3}{2}\frac{\hbar^2}{m}\alpha_2^2$$

$$= \frac{3}{2}\frac{\hbar^2}{m}(\alpha_1^2+\alpha_2^2) \tag{4.6.71}$$

for $\alpha_1 \to \infty$ and $\alpha_2 \to \infty$.

Eqn. (4.6.71) is also the expectation value of the direct part of the kernel.

5 THE COUPLED CHANNEL CASE

5.1 INTRODUCTION

In the case of a coupled channel approach, both possibilities ,that is, $s_{12}=0$ and $s_{12}=1$ are simultaneously taken into account. In such a model it is possible to distinguish, for example, between different members of the $N-\Delta$ multiplet as opposed to averaging over the multiplet. The aim of this chapter is to discuss the effect of antisymmetrization in the diquark models of baryons when both possibilities are simultaneously considered. The emphasis here is on the development of the algebraic expressions of the operator kernels for the diquark-quark system. In this chapter the single and double primes will denote the single and double exchanges of quarks respectively.

5.2 DIQUARK-QUARK WAVE FUNCTION

The unsymmetrized wave function in the coupled channel state is given by

$$\Psi_{\bar{D}q}(\vec{X},\vec{\xi},\vec{\eta}) = \Phi_0 X_0 X_0 + \Phi_1 X_1 X_1 \tag{5.2.1}$$

with

$$\Phi_i = \phi_i(\vec{\xi})\,\psi_i(\vec{\eta})\,\delta(\vec{X}-\vec{\eta}) \tag{5.2.2}$$

where $\vec{\xi}$ and $\vec{\eta}$ are defined in eqns.(3.1.4) and (3.1.5).

The symmetrized wave function is given by

$$\Psi^S_{Dq}(\vec{X}',\vec{\xi},\vec{\eta}) = (e+P+P^2)\,\Psi_{\bar{D}q}(\vec{X}',\vec{\xi},\vec{\eta})$$

$$= \Phi_0 X_0 X_0 + \Phi_1 X_1 X_1 + \Phi_0'\left\{\frac{1}{4}X_0X_0 + \frac{3}{4}X_1X_1 + \frac{\sqrt{3}}{4}(X_0X_1 + X_1X_0)\right\}$$

$$+ \Phi_0''\left\{\frac{1}{4}X_0X_0 + \frac{1}{4}X_1X_1 - \frac{\sqrt{3}}{4}(X_0X_1 + X_1X_0)\right\}$$

$$+ \Phi_1' \left\{ \frac{3}{4} X_0 X_0 + \frac{1}{4} X_1 X_1 - \frac{\sqrt{3}}{4} (X_0 X_1 + X_1 X_0) \right\}$$

$$+ \Phi_1'' \left\{ \frac{3}{4} X_0 X_0 + \frac{1}{4} X_1 X_1 + \frac{\sqrt{3}}{4} (X_0 X_1 + X_1 X_0) \right\} \tag{5.2.3}$$

for

$$\Phi_i' = \phi_i (\vec{\xi}') \psi_i (\vec{\eta}') \delta (\vec{X}' - \vec{\eta}') , \tag{5.2.4}$$

where

$$\vec{\xi}' = -\frac{1}{2} \vec{\xi} + \frac{\sqrt{3}}{2} \vec{\eta} \tag{5.2.5a}$$

and

$$\vec{\eta}' = -\frac{\sqrt{3}}{2} \vec{\xi} - \frac{1}{2} \vec{\eta} \tag{5.2.5b}$$

and for

$$\Phi_i'' = \phi_i (\vec{\xi}'') \psi_i (\vec{\eta}'') \delta (\vec{X}' - \vec{\eta}'') \tag{5.2.6}$$

with

$$\vec{\xi}'' = -\frac{1}{2} \vec{\xi} - \frac{\sqrt{3}}{2} \vec{\eta} \tag{5.2.7}$$

and

$$\vec{\eta}'' = \frac{\sqrt{3}}{2} \vec{\xi} - \frac{1}{2} \vec{\eta} \tag{5.2.8}$$

If the following basis vectors

$$\Phi_\alpha^S = \Phi_\alpha (\vec{\xi}) + \Phi_\alpha (\vec{\xi}') + \Phi_\alpha (\vec{\xi}'') \tag{5.2.10}$$

$$\Phi_\alpha^1 = \frac{\sqrt{3}}{2} \{ \Phi_\alpha (\vec{\xi}') - \Phi_\alpha (\vec{\xi}'') \} \tag{5.2.11}$$

$$\Phi_\alpha^2 = \frac{1}{2} \{ 2\Phi_\alpha (\vec{\xi}) - \Phi_\alpha (\vec{\xi}') - \Phi_\alpha (\vec{\xi}'') \} \tag{5.2.12}$$

$$X^S = X_0 X_0 + X_1 X_1 \tag{5.2.13}$$

$$X^1 = X_0 X_1 + X_1 X_0 \tag{5.2.14}$$

$$\chi^2 = \chi_0\chi_0 - \chi_1\chi_1 \tag{5.2.15}$$

$$\chi^A = \chi_0\chi_1 - \chi_1\chi_0 \tag{5.2.16}$$

are considered then

$$\Psi^S_{\tilde{Dq}} = \frac{1}{4}\left\{(\Phi^S_0 + \Phi^S_1)\,\chi^S + (\Phi^1_0 - \Phi^1_1)\,\chi^1 + (\Phi^2_0 - \Phi^2_1)\,\chi^2\right\} \tag{5.2.17}$$

and

$$\Psi_{\tilde{Dq}} = \frac{1}{2}(\Phi_0 + \Phi_1)\,\chi^S + \frac{1}{2}(\Phi_0 - \Phi_1)\,\chi^2 \tag{5.2.18}$$

If the normalization expression is defined as

$$<\chi^\alpha|\chi^\beta> = 2\delta_{\alpha\beta} \tag{5.2.19}$$

then

$$<\Psi_{\tilde{Dq}}\,|\,\Psi^S_{\tilde{Dq}}> = \frac{1}{4}\left\{<\Phi_0 + \Phi_1\,|\,\Phi^S_0 + \Phi^S_1> + <\Phi_0 - \Phi_1\,|\,\Phi^2_0 - \Phi^2_1>\right\} \tag{5.2.20}$$

gives the overlap of ψ and ψ^S.

5.3 OPERATOR KERNELS

5.3.1 The norm kernel

The operator kernels are 2 x 2-matrices in Ψ-space. The dependencies are denoted by subscripts $\alpha\beta$, as in $K(\vec{X},\vec{X}')_{\alpha\beta}$.

5.3.1.1 The direct norm kernel.

Following the discussion in chapter 4, the direct norm kernel can be given by

$$K^D_{\hat{1}}(\vec{X},\vec{X}')_{\alpha\beta} = \delta(\vec{X} - \vec{X}')\,\delta_{\alpha\beta} \tag{5.3.21}$$

5.3.1.2 The exchange norm kernel.

The exchange norm kernel for the coupled channel can be calculated as follows:

$$K_{\hat{I}}^{E}(\vec{X},\vec{X}')_{00} = 2\left(\frac{2}{\sqrt{3}}\right)^{3}\frac{1}{4}\Phi_0^*\left(\left|\frac{1}{\sqrt{3}}\vec{X}+\frac{2}{\sqrt{3}}\vec{X}'\right|\right)\Phi_0\left(\left|\frac{2}{\sqrt{3}}\vec{X}+\frac{1}{\sqrt{3}}\vec{X}'\right|\right) \tag{5.3.22}$$

$$K_{\hat{I}}^{E}(\vec{X},\vec{X}')_{01} = 2\left(\frac{2}{\sqrt{3}}\right)^{3}\frac{3}{4}\Phi_0^*\left(\left|\frac{1}{\sqrt{3}}\vec{X}+\frac{2}{\sqrt{3}}\vec{X}'\right|\right)\Phi_1\left(\left|\frac{2}{\sqrt{3}}\vec{X}+\frac{1}{\sqrt{3}}\vec{X}'\right|\right) \tag{5.3.23}$$

$$K_{\hat{I}}^{E}(\vec{X},\vec{X}')_{10} = 2\left(\frac{2}{\sqrt{3}}\right)^{3}\frac{3}{4}\Phi_1^*\left(\left|\frac{1}{\sqrt{3}}\vec{X}+\frac{2}{\sqrt{3}}\vec{X}'\right|\right)\Phi_0\left(\left|\frac{2}{\sqrt{3}}\vec{X}+\frac{1}{\sqrt{3}}\vec{X}'\right|\right) \tag{5.3.24}$$

$$K_{\hat{I}}^{E}(\vec{X},\vec{X}')_{11} = 2\left(\frac{2}{\sqrt{3}}\right)^{3}\frac{1}{4}\Phi_1^*\left(\left|\frac{1}{\sqrt{3}}\vec{X}+\frac{2}{\sqrt{3}}\vec{X}'\right|\right)\Phi_1\left(\left|\frac{2}{\sqrt{3}}\vec{X}+\frac{1}{\sqrt{3}}\vec{X}'\right|\right) \tag{5.3.25}$$

5.3.2 Potential energy kernel

5.3.2.1 The central potential

The central potential is given by

$$V^{C} = V^{C}(\xi) + V^{C}(\xi') + V^{C}(\xi'') \tag{5.3.26}$$

where ξ' and ξ'' are given by eqns. (5.2.5) and (5.2.7) respectively.

As in chapter 4 the direct and exchange terms under central potential can be obtained.

(a) The direct term

The direct term is given by

$$K_{V^c}^{D}(\vec{X},\vec{X}')_{\alpha\beta} = \delta_{\alpha\beta}\delta(\vec{X}-\vec{X}')\int|\phi_\alpha(\vec{\xi})|^2\left\{V^{C}(\xi) + V^{C}\left(\left|\frac{1}{2}\vec{\xi}-\frac{\sqrt{3}}{2}\vec{X}\right|\right)\right.$$

$$\left.+V^{C}\left(\left|\frac{1}{2}\vec{\xi}+\frac{\sqrt{3}}{2}\vec{X}\right|\right)d^3\xi\right\} \tag{5.3.27}$$

where V^{C} denotes the central potential.

(b) The exchange term

The exchange term for the central potential is given by

$$K_{V^c}^{E}(\vec{X},\vec{X}')_{\alpha\beta} = K_{1}^{E}(\vec{X},\vec{X}')_{\alpha\beta} \left\{ V^{c}\left(\left|\frac{1}{\sqrt{3}}\vec{X}+\frac{2}{\sqrt{3}}\vec{X}'\right|\right) + V^{c}\left(\left|\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right|\right) + V^{c}\left(\left|\frac{1}{\sqrt{3}}\vec{X}-\frac{1}{\sqrt{3}}\vec{X}'\right|\right)\right\} \tag{5.3.28}$$

where $K_{1}^{E}(\vec{X},\vec{X}')_{\alpha\beta}$ are given by eqns. (5.3.22) through (5.3.25).

5.3.2.2 The spin-spin potential

The interaction is defined algebraically as follows:

$$V^{S,S} = V(\xi)\,\sigma_1.\sigma_2 + V(\xi')\,\sigma_2.\sigma_3 + V(\xi'')\,\sigma_3.\sigma_1 \tag{5.3.29}$$

where σ_1, σ_2 and σ_3 are the Pauli's spinors.

In spin-space eqn. (5.3.29) is a 2 x 2 - matrix given by the following equations:

$$V^{S,S}\chi_0 = \{-V^{S}(\xi)\}\chi_0 + \sqrt{3}\{V^{S}(\xi'') - V^{S}(\xi')\}\chi_1 \tag{5.3.30}$$

and

$$V^{S,S} = \sqrt{3}\{V^{S}(\xi'') - V^{S}(\xi')\}\chi_0 + \{V^{S} - 2V^{S}(\xi') - 2V^{S}(\xi'')\}\chi_1 \tag{5.3.31}$$

(a) The direct kernel

The direct operator kernels under potential are calculated as follows:

$$K_{V^s}^{D}(\vec{X},\vec{X}')_{00} = \delta(\vec{X}-\vec{X}')\int|\phi_0(\vec{\xi})|^2\{-3V^{s}(\xi)\}\,d^3\xi \tag{5.3.32}$$

$$K_{V^s}^{D}(\vec{X},\vec{X}')_{01} = 0 \tag{5.3.33}$$

$$K_{V^s}^{D}(\vec{X},\vec{X}')_{10} = 0 \tag{5.3.34}$$

$$K_{V^s}^{D}(\vec{X},\vec{X}')_{11} = \delta(\vec{X}-\vec{X}')\int|\phi_1(\vec{\xi})|^2\{V^{S}(\xi) - 4V^{S}\left(\left|\frac{1}{2}\vec{\xi}-\frac{\sqrt{3}}{2}\vec{X}\right|\right)\}\,d^3\xi \tag{5.3.35}$$

(b) The exchange kernel

The calculations are performed as follows:

$$K_{V^S}^{E}(\vec{X},\vec{X}')_{00} = 2\left(\frac{2}{\sqrt{3}}\right)^3 \phi_0^*\left(\left|\frac{2}{\sqrt{3}}\vec{X}' + \frac{1}{\sqrt{3}}\vec{X}\right|\right)\phi_0\left(\left|\frac{2}{\sqrt{3}}\vec{X} + \frac{1}{\sqrt{3}}\vec{X}'\right|\right)$$
$$\frac{3}{2}\left\{V^S\left(\left|\frac{2}{\sqrt{3}}\vec{X}' + \frac{1}{\sqrt{3}}\vec{X}\right|\right) - V^S\left(\left|\frac{2}{\sqrt{3}}\vec{X} + \frac{1}{\sqrt{3}}\vec{X}'\right|\right) + V^S\left(\left|\frac{1}{\sqrt{3}}\vec{X} - \frac{1}{\sqrt{3}}\vec{X}'\right|\right)\right\}$$

(5.3.36)

$$K_{V^S}^{E}(\vec{X},\vec{X}')_{01} = 2\left(\frac{2}{\sqrt{3}}\right)^3 \phi_0^*\left(\left|\frac{2}{\sqrt{3}}\vec{X}' + \frac{1}{\sqrt{3}}\vec{X}\right|\right)\phi_1\left(\left|\frac{2}{\sqrt{3}}\vec{X} + \frac{1}{\sqrt{3}}\vec{X}'\right|\right)$$

$$\left\{-\frac{9}{2}V^S\left(\left|\frac{2}{\sqrt{3}}\vec{X}' + \frac{1}{\sqrt{3}}\vec{X}\right|\right) + \frac{3}{2}V^S\left(\left|\frac{2}{\sqrt{3}}\vec{X} + \frac{1}{\sqrt{3}}\vec{X}'\right|\right) + \frac{3}{2}V^S\left(\left|\frac{1}{\sqrt{3}}\vec{X} - \frac{1}{\sqrt{3}}\vec{X}'\right|\right)\right\}$$

(5.3.37)

$$K_{V^S}^{S}(\vec{X},\vec{X}')_{10} = 2\left(\frac{2}{\sqrt{3}}\right)^3 \phi_1^*\left(\left|\frac{2}{\sqrt{3}}\vec{X}' + \frac{1}{\sqrt{3}}\vec{X}\right|\right)\phi_0\left(\left|\frac{2}{\sqrt{3}}\vec{X} + \frac{1}{\sqrt{3}}\vec{X}'\right|\right)$$

$$\left\{\frac{3}{2}V^S\left(\left|\frac{2}{\sqrt{3}}\vec{X}' + \frac{1}{\sqrt{3}}\vec{X}\right|\right) - \frac{9}{2}V^S\left(\left|\frac{2}{\sqrt{3}}\vec{X} + \frac{1}{\sqrt{3}}\vec{X}'\right|\right) - \frac{3}{2}V^S\left(\left|\frac{1}{\sqrt{3}}\vec{X} - \frac{1}{\sqrt{3}}\vec{X}'\right|\right)\right\}$$

(5.3.38)

$$K_{V^S}^{E}(\vec{X},\vec{X}')_{11} = 2\left(\frac{2}{\sqrt{3}}\right)^3 \phi_1^*\left(\left|\frac{2}{\sqrt{3}}\vec{X}' + \frac{1}{\sqrt{3}}\vec{X}\right|\right)\phi_1\left(\left|\frac{2}{\sqrt{3}}\vec{X} + \frac{1}{\sqrt{3}}\vec{X}'\right|\right)$$

$$\left\{\frac{1}{2}V^S\left(\left|\frac{2}{\sqrt{3}}\vec{X}' + \frac{1}{\sqrt{3}}\vec{X}\right|\right) + \frac{3}{2}V^S\left(\left|\frac{2}{\sqrt{3}}\vec{X} + \frac{1}{\sqrt{3}}\vec{X}'\right|\right) - \frac{5}{2}V^S\left(\left|\frac{1}{\sqrt{3}}\vec{X} - \frac{1}{\sqrt{3}}\vec{X}'\right|\right)\right\}$$

(5.3.39)

5.3.3 The kinetic energy kernel

5.3.3.1 The direct term

The direct term is the same as that in the single channel and is given by:

$$K_T^D(\vec{X},\vec{X}') = -\frac{\hbar^2}{m_q}\delta(\vec{X}-\vec{X}')\left\{\nabla_X^2\int\phi_{\bar{D}}^*(\vec{\xi})\,\phi_{\bar{D}}(\vec{\xi})\,d^3\xi + \int\phi_{\bar{D}}^*(\vec{\xi})\,T_{\bar{D}}\,\phi_{\bar{D}}(\vec{\xi})\,d^3\xi\right\} \tag{5.3.40}$$

5.3.3.2 The exchange term

The exchange is given by

$$K_T^E(\vec{X},\vec{X}') = -\frac{\hbar^2}{m_q}\left[\frac{1}{2}K_{\hat{1}}^E(\vec{x},\vec{X}')\left\{\nabla_X^2+\nabla_{X'}^2\right\}\right.$$

$$+\left(\frac{2}{\sqrt{3}}\right)^3\phi_{\bar{D}}^*\left(\left|\frac{1}{\sqrt{3}}\vec{X}+\frac{2}{\sqrt{3}}\vec{X}'\right|\right)\nabla^2\phi_{\bar{D}}\left(\left|\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right|\right)$$

$$\left.+\left(\frac{2}{\sqrt{3}}\right)^3\phi_{\bar{D}}^*\left(\left|\frac{1}{\sqrt{3}}\vec{X}'+\frac{2}{\sqrt{3}}\vec{X}\right|\right)\nabla^2\phi_{\bar{D}}\left(\left|\frac{1}{\sqrt{3}}\vec{X}+\frac{2}{\sqrt{3}}\vec{X}'\right|\right)\right] \tag{5.3.41}$$

where $K_{\hat{1}}^E(\vec{X},\vec{X}')$ is given by eqns. (5.3.22) through (5.3.25).

6 NUMERICAL ANALYSIS

6.1 INTRODUCTION

The calculations which include and exclude antisymmetrization have been calculated and the results are given below. In order to obtain the well-behaved wave functions and the corresponding eigenvalues, the radial Schrodinger equation has been integrated numerically using the Runge-Kutta-Nystrom method [69]. The spline approximation and the Gauss quadrature have been employed in this thesis. This is a reliable and a robust self-starting method. To investigate the effect of antisymmetrization, the data for the diquark and diquark-quark wave functions has been generated by solving the Schrodinger equation. To demonstrate that convergence was achieved, the data obtained has been used to draw the graphs given in figs. 1-4 and 5-8 for the diquark and the diquark-quark respectively. The calculations of the antisymmetrized and non-antisymmetrized models follow below.

6.2 GENERALIZED RUNGE-KUTTA METHOD.

In order to obtain the well-behaved wave functions and the corresponding eigenvalues the radial Schrodinger equation will be integrated numerically by using standard integration techniques, in this case the generalized Runge-Kutta method. Solving the Schrodinger equation is an initial-value problem and therefore the Runge-Kutta method is useful. The initial-value problem considered is of the form

$$Y'' = F(X, Y, Y') \tag{6.2.1}$$

where $Y(X_0) = Y_0$ and $Y'(X_0) = Y_0'$.

F is assumed in such a way that the problem has unique solution on some interval containing X_0. The aim here is to find the approximate values of $Y_1, Y_2, \ldots,$ at uniformly spaced

points of distance H, that is, $X_1 = X_0 + H$, $X_2 = X_0 + 2H$, ..., respectively. The values Y_1', Y_2', ..., correspond to the uniformly spaced points X_1', X_2', ..., respectively. A generalization of the Runge-Kutta method is called Runge-Kutta-Nystrom (see, for example [69]) and is the fourth-order method. Four auxiliary quantities A_N, B_N, C_N and D_N are evaluated and then used to approximate the new values Y_{N+1} and Y_{N+1}' of the solutions Y and Y' respectively. The algorithm for solving eqn (6.2.1) is adopted from Kreyszig [69] as follows:

$$K = \frac{1}{2}H$$

$$N = 0,1,2,\ldots,n-1$$

where H and n are given.

$$A_N = KF(X_N, Y_N, Y_N')$$

$$B_N = KF(X_N + K, Y_N + \beta_N, Y_N' + A_N)$$

where $\beta_N = K(Y_N' + \frac{1}{2}A_N)$

$$C_N = KF(X_N + K, Y_N + \beta_N, Y_N' + B_N)$$

$$D_N = KF(X_N + H, Y_N + \alpha_N, Y_N' + 2C_N)$$

where $\alpha_N = H(Y_N' + C_N)$

$$Y_{N+1} = Y_N + H\left(Y_N' + \frac{1}{3}(A_N + B_N + C_N)\right)$$

and

$$Y_{N+1}' = Y_N' + \frac{1}{3}(A_N + 2B_N + 2C_N + D_N) \qquad (6.2.2)$$

It can be noticed that Y_{N+1} and Y_{N+1}' are the approximations to Y and Y' respectively at

$$X_{N+1} = X_0 + (N+1)H \qquad (6.2.3)$$

6.3 SPLINE FUNCTIONS

In this thesis the most commonly used spline function called the cubic spline is employed.. This function can be thought of as an analytical French curve. Its four parameters can be chosen to cause the curve to go through two adjacent points and the first and second derivatives to be continuous(see also [69]). In such a case a cubic polynomial is considered and is called a cubic spline function. A cubic spline is defined as a continuous function which has continuous first and second derivatives and is represented by a third degree polynomial. The method of splines was originated in 1946 by Schoenberg [70]. The discussion and the algorithm of the spline functions have been adopted from Schwarz and Waldvogel [71], Kreyszig, [69], Ebert et al, [72] Daniels, [73] Maron et al. [74]. The problem here is to approximate a function f(x). The (n+1) pairwise different support abscissae are given by $x_0<x_1<\ldots<x_n$ and the corresponding function values are given by $y_0, y_1, y_2, \ldots, y_n$. The condition imposed on an interpolating function s(x) is that it must be at least continuously differentiable. Therefore the bending curve of the spline must be taken as the solution s(x) of the interpolation problem. Three properties uniquely define the interpolating spline function s(x).

(a) s(x) is a cubic polynomial in the interval $[x_i, x_{i+1}]$ and is piecewise composed of polynomials of degree three.

(b) The first and second derivatives of s(x) are continuous at the interior support abscissae.

(c) The second derivative is zero at the end-points x_0 and x_n, that is,

$$s''(x_0) = s''(x_n) = 0 \tag{6.3.4}$$

This property is called the natural condition. Therefore the resulting s(x) is called a natural cubic spline function. The algorithm of the cubic spline function s(x) is as follows:

(i) $$h_i = x_{i+1} - x_i \tag{6.3.5}$$

for the subinterval $[x_i, x_{i+1}]$ where h_i are the i^{th}.

(ii) The i^{th} cubic polynomials $s_i(x)$ are given by

$$s_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i \tag{6.3.6}$$

(iii) The values of $s_i(x)$ and the first two derivatives at the end-points of the interval are as follows:

$$s_i(x_i) = d_i = y_i \tag{6.3.7}$$

$$s_i(x_{i+1}) = a_i h_i^3 + b_i h_i^2 + c_i h_i + d_i = y_{i+1} \tag{6.3.8}$$

$$s_i'(x_i) = c_i \tag{6.3.9}$$

$$s_i'(x_{i+1}) = 3a_i h_i^2 + 2b_i h_i + c_i \tag{6.3.10}$$

$$s_i''(x_i) = 2b_i = y_i'' \tag{6.3.11}$$

$$s_i''(x_{i+1}) = 6a_i h_i + 2b_i = y_{i+1}'' \tag{6.3.12}$$

(iv) Therefore from (iii) the following can be derived

$$a_i = \frac{1}{6h_i}(y_{i+1}'' - y_i'') \tag{6.3.13}$$

$$b_i = \frac{1}{2}y_i'' \tag{6.3.14}$$

$$c_i = \frac{1}{h_i}(y_{i+1} - y_i) - \frac{1}{6}h_i(y_{i+1}'' + 2y_i'') \tag{6.3.15}$$

$$d_i = y_i \tag{6.3.16}$$

(v) By substituting for a_i, b_i and c_i, the following can be derived from (iii)

$$s_i'(x_{i+1}) = \frac{1}{h_i}(y_{i+1} - y_i) + \frac{1}{6}h_i(2y_{i+1}'' + y_i'') \tag{6.3.17}$$

(vi) On decreasing the index i by one, the following can be obtained from (iii), (iv) and (v):

$$\frac{1}{h_{i-1}}(y_i - y_{i-1}) + (2y_i'' + y_{i-1}'') = \frac{1}{h_i}(y_{i+1} - y_i) - \frac{1}{h_i}(y_{i+1}'' + 2y_i'') \tag{6.3.18}$$

(vii) By multiplying eqn. (6.3.18) by 6, the following can be derived

$$h_{i-1}y''_{i-1} + 2(h_{i-1}+h_i)y''_i + h_i y''_{i+1} - \frac{6}{h_i}(y_{i+1}-y_i) + \frac{6}{h_{i-1}}(y_i - Y_{i-1}) = 0 \quad (6.3.19)$$

6.4 GAUSS QUADRATURE

The word quadrature is used in the place of numerical integration. The discussion of Gauss quadrature in this section is adopted from Ref. [75]. The evaluation of a definite integral over a finite interval is given by [75]

$$G = \int_a^b f(x)\,dx \quad (6.4.20)$$

with a and b being finite and f(x) being a continuous function of x for the interval $a \le x \le b$. It can be shown that by choosing the location of the two points carefully, an exact formula for the integral of a cubic polynomial can be obtained. If the limits of integration a to b can be replaced by -1 and +1 respectively, and a new variable α can be defined as [75]:

$$\alpha = \frac{2x-(b+a)}{b-a} \quad (6.4.21)$$

which reduces to

$$x = \frac{1}{2}(b-a)\alpha + \frac{1}{2}(b+a) \quad (6.4.22)$$

then eqn. (6.4.20) becomes [75]

$$G = \int_{-1}^{+1} \Psi(\alpha)\,d\alpha \quad (6.4.23)$$

with

$$\Psi(\alpha) = \frac{1}{2}(b-a)\, f\left[\frac{1}{2}(b-a)\alpha + \frac{1}{2}(b+a)\right] \quad (6.4.24)$$

Here a linear function

$$y = \mu_0 + \mu_1\alpha \quad (6.4.25)$$

is defined such that [75]

$$\int_{-1}^{+1} (\mu_0 + \mu_1\alpha)\,d\alpha = \int_{-1}^{+1} \Psi(\alpha)\,d\alpha \quad (6.4.26)$$

Let

$$G_I = B_0\Psi(\alpha_0) + B_1\Psi(\alpha_1) \tag{6.4.27}$$

with B_0, B_1, α_0 and α_1 being arbitrary. These four parameters can be chosen such that an exact formula for a cubic integrand is obtained [75]:

$$\Psi(\alpha) = b_0 + b_1\alpha + b_2\alpha^2 + b_3\alpha^3 \tag{6.4.28}$$

which reduces to [75]

$$\Psi(\alpha) = \mu_0 + \mu_1\alpha + (\alpha - \alpha_0)(\alpha - \alpha_1)(\mu_0 + \mu_1\alpha) \tag{6.4.29}$$

After integration the following is obtained

$$\alpha_1 = -\alpha_0 = \frac{1}{\sqrt{3}} \tag{6.4.30}$$

and

$$B_0 = B_1 = 1 \tag{6.4.31}$$

Therefore

$$G_I = \Psi\left(-\frac{1}{\sqrt{3}}\right) + \Psi\left(\frac{1}{\sqrt{3}}\right) \tag{6.4.32}$$

Eqn. (6.4.26) is the Gauss quadrature formula for two points. Note that for lower polynomials the truncation error is zero.

For higher degree polynomials the truncation error is therefore given by

$$e_T = C\Psi^{iv}(\beta), \qquad -1<\beta<1 \tag{6.4.33}$$

For the higher order Gauss quadrature, formulas are obtained from using more points and different weights (B_i) [75]:

$$\int_{-1}^{+1}\Psi(\alpha)\,d\alpha = \sum_{i=0}^{n-1} B_i\Psi(\alpha_i) \tag{6.4.34}$$

An exact formula for a polynomial of degree 2n-1 can be obtained with n points as in eqn.

(6.4.34). The α_i in eqn. (6.4.34) are also the roots of the Legendre polynomial of degree n and the method discussed above is called Legendre-Gauss quadrature. The Legendre polynomials and the weights are defined by [75]

$$P_0(\alpha) = 1 \tag{6.4.34a}$$

$$P_1(\alpha) = \alpha \tag{6.4.34b}$$

$$P_n = \frac{1}{n}\left[(2n-1)\alpha P_{n-1}(\alpha) - (n-1)P_{n-2}(\alpha)\right] \tag{6.4.34c}$$

and

$$B_i = \frac{2}{(1-\alpha_i^2)\left[P_n'(\alpha_i)\right]^2} \tag{6.4.34d}$$

The general truncation error is given by [75]

$$e_T = \frac{\Psi^{(2n)}(\beta)}{(2n)!}\left(\frac{2}{2n+1} - \sum_{i=0}^{n} B_i \alpha_i^{2n}\right) \tag{6.4.35}$$

It should be appreciated that the α_i are symmetric about the origin and that the B_1 for α_1 is the same as that for $-\alpha_1$,

6.5 DIQUARK AND DIQUARK-QUARK WAVE FUNCTIONS

The parameters for the various central potentials used to construct the D and Dq wave functions shown in figs. 1-12 have been given as follows:

(a) Bhaduri potential

$$V(r) = -Ar + \frac{B}{r} - D$$

where $A = 520.3\,\hbar c$

$B = 185.7/\hbar c$

$\hbar c = 0.197239$ MeV fm

$$D = 913.5 \text{ MeV}$$

$$m_q = 337 \text{ MeV} \qquad (6.5.36)$$

(b) Cornell potential

$$V(r) = -Ar + \frac{B}{r} - D$$

where $A = 520.0\ \hbar c$

$$B = 182.62/\hbar c$$

$$\hbar c = 0.197239 \text{ MeV fm}$$

$$D = 975.0 \text{ MeV}$$

$$m_q = 330 \text{ MeV} \qquad (6.5.37)$$

(c) **Martin potential**

$$V(r) = A + B r^{\beta}$$

where $A = -8337$ MeV

$$B = 6992.3 \text{ MeV}$$

$$\beta = 0.1$$

$$m_q = 300 \text{ MeV} \qquad (6.5.38)$$

(d) **Quigg potential**

$$V(r) = A\ln(Br)$$

where $A = 744$ MeV

$$B = 0.87 \text{ fm}^{-1}$$

$$m_q = 330 \text{ MeV} \qquad (6.5.39)$$

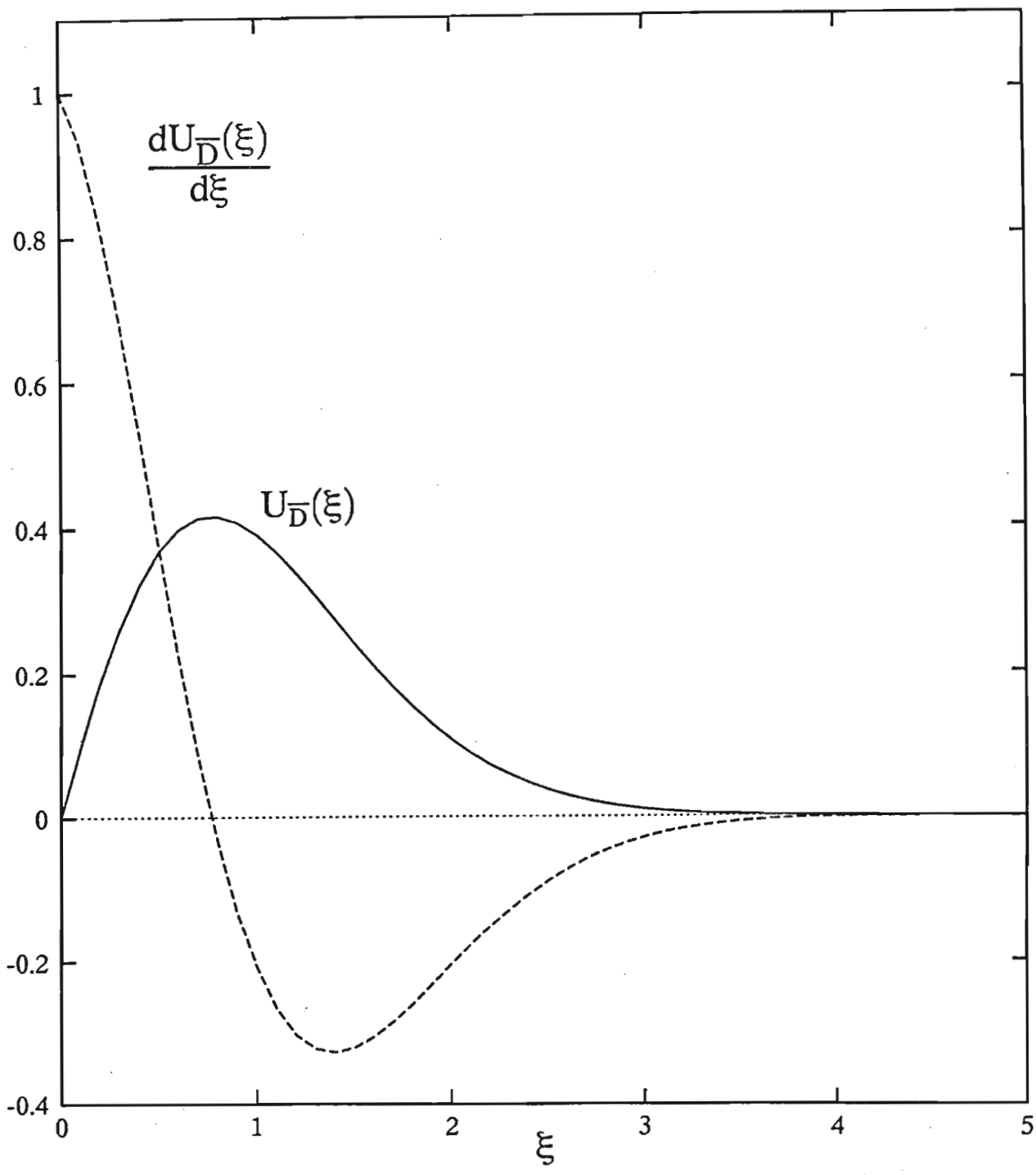

Fig. 1. The reduced diquark wave function and its derivative for Bhaduri potential.

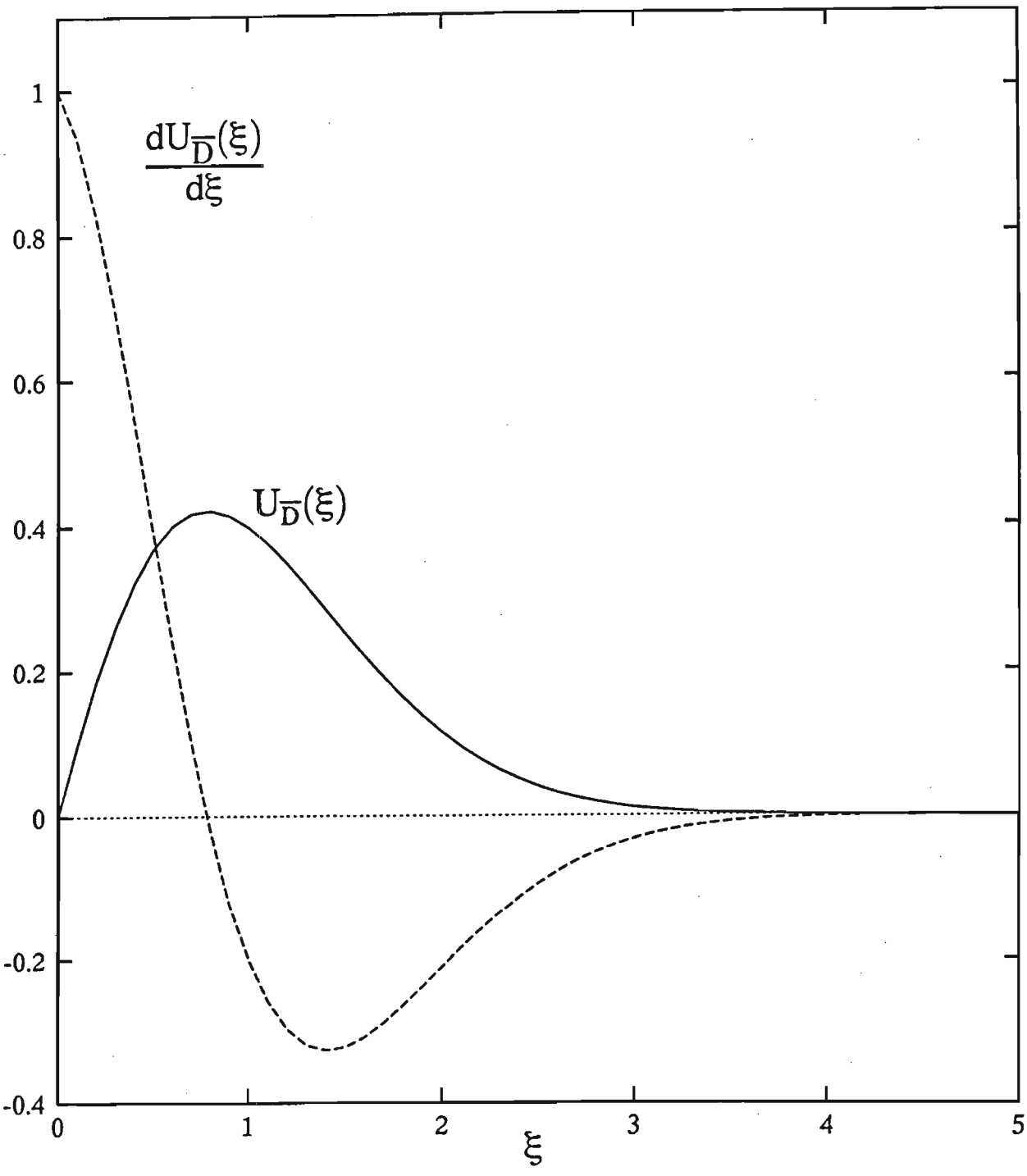

Fig. 2. The reduced diquark wave function and its derivative for the Cornell potential.

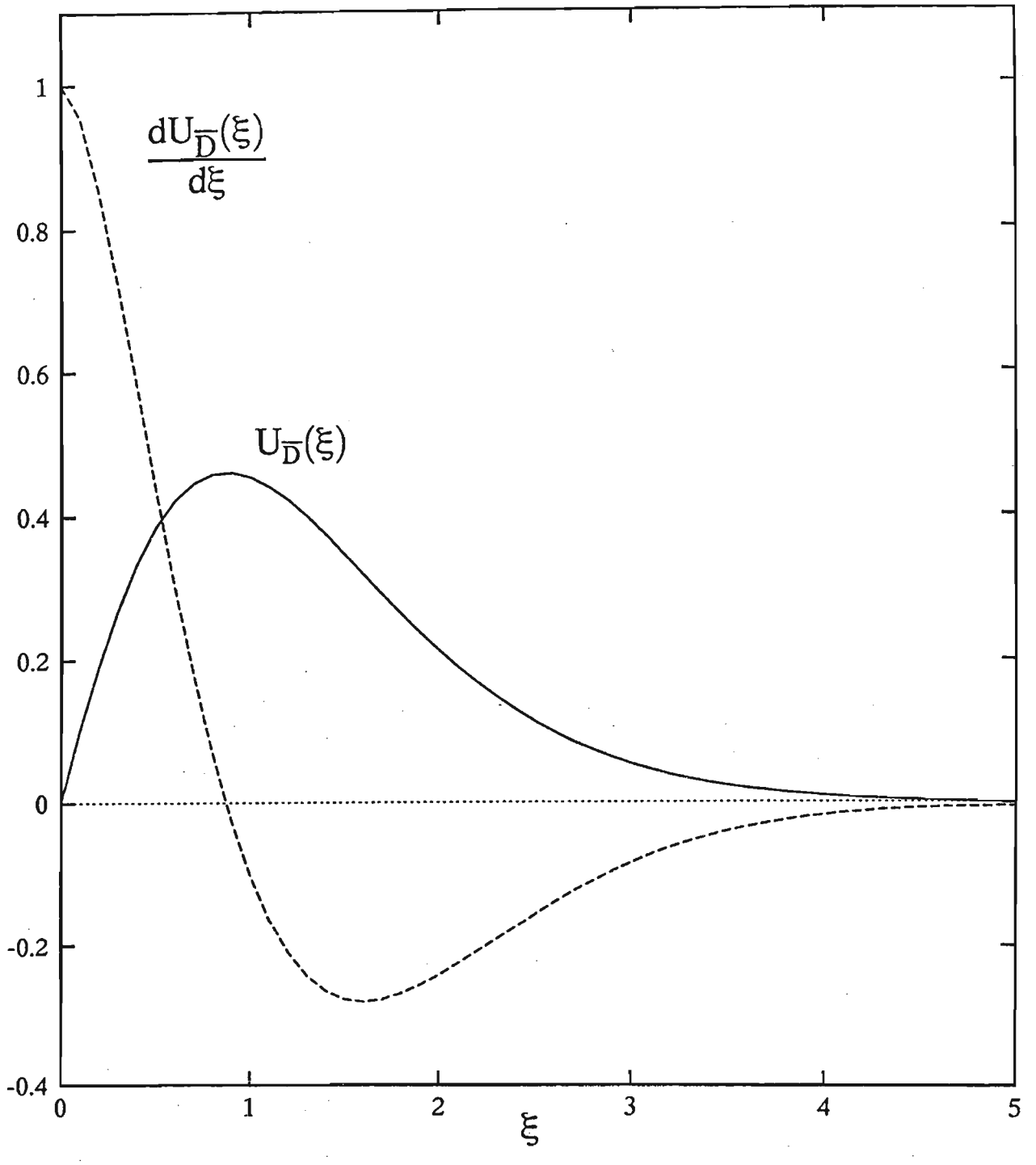

Fig. 3. The reduced diquark wave function and its derivative for Martin potential.

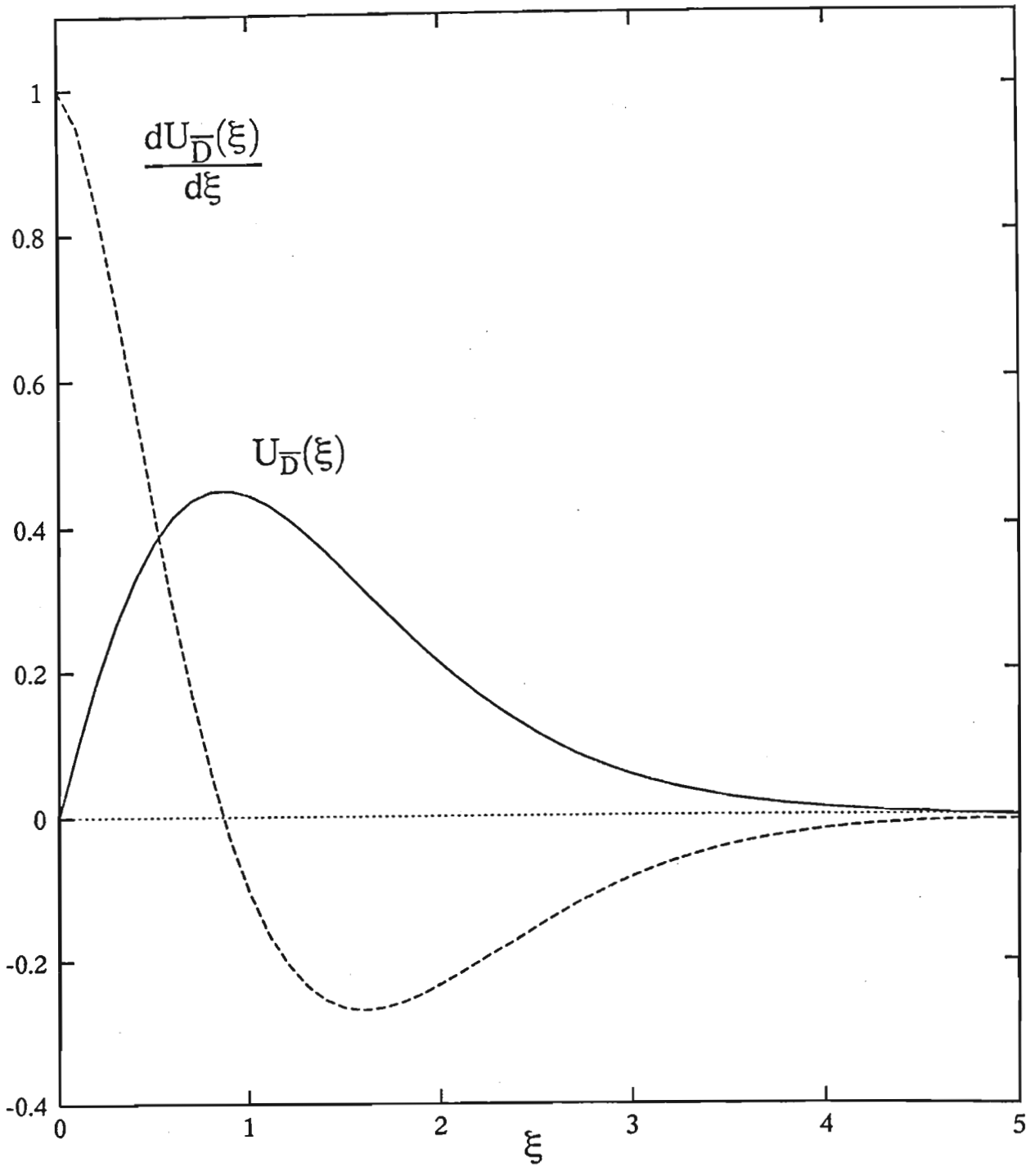

Fig. 4. The reduced diquark wave function and its derivative for the Quigg potential.

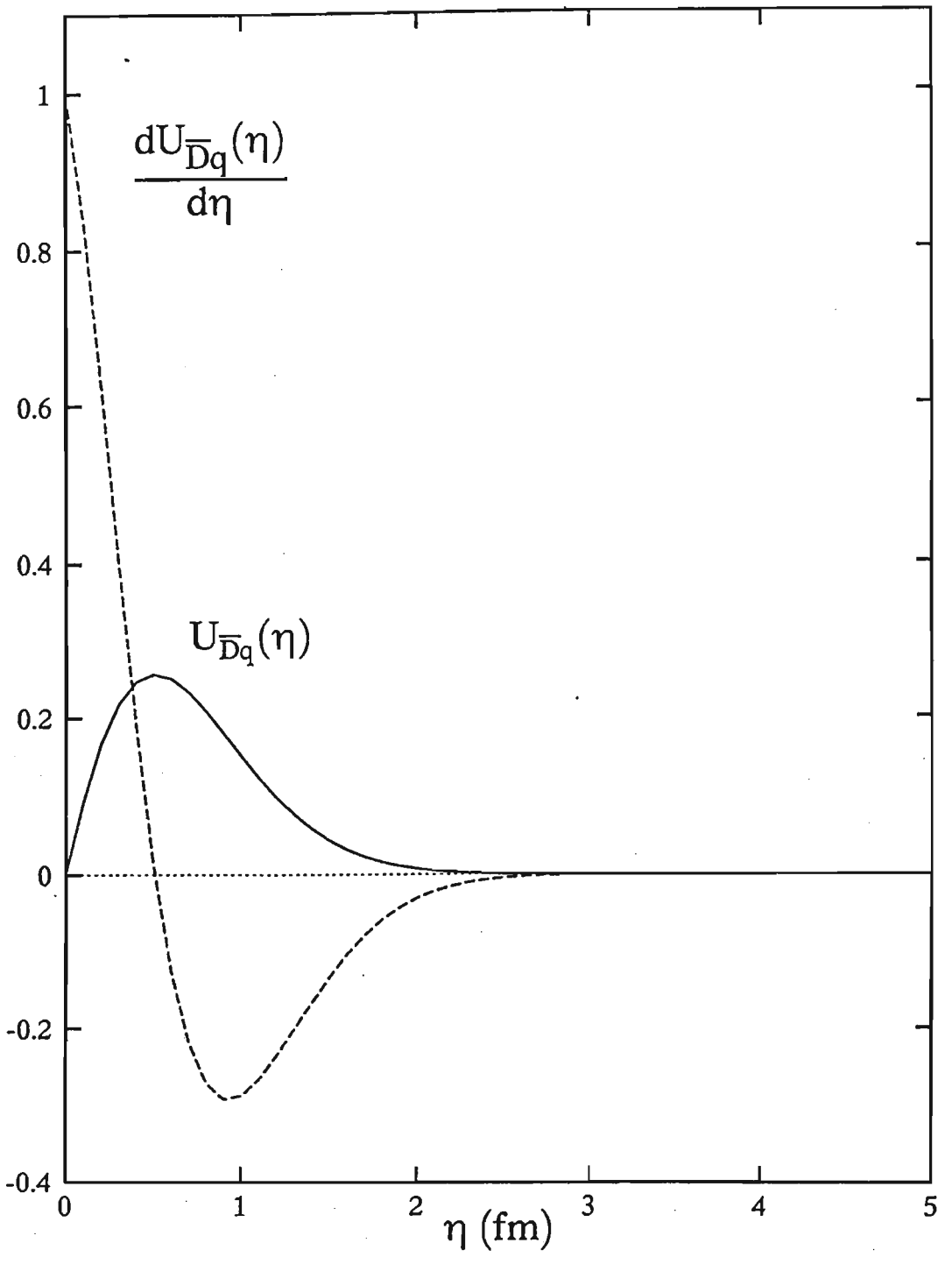

Fig. 5. The reduced diquark-quark wave function and its derivative for the Bhaduri potential.

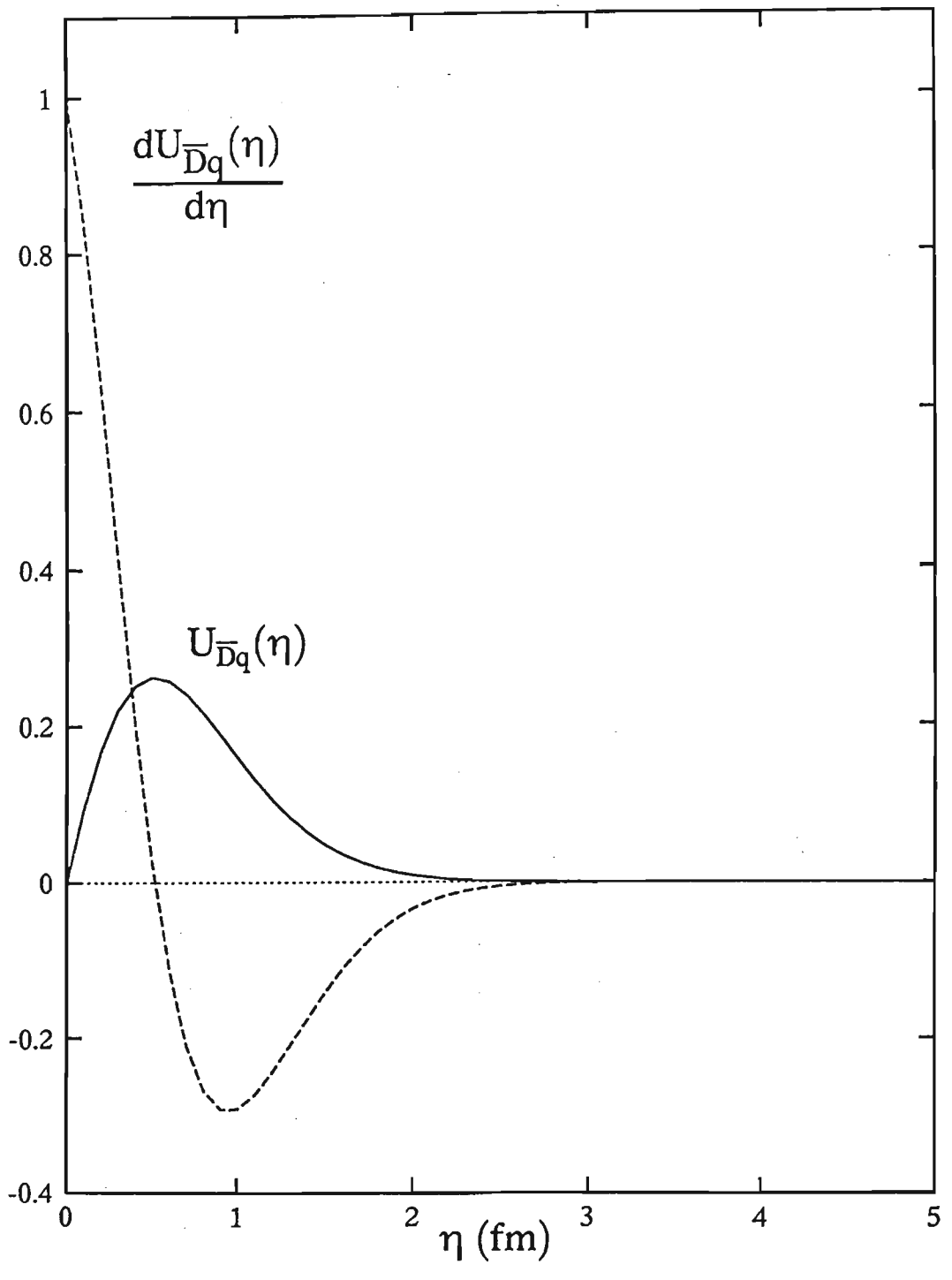

Fig. 6. The diquark-quark wave function and its derivative for the Cornell potential.

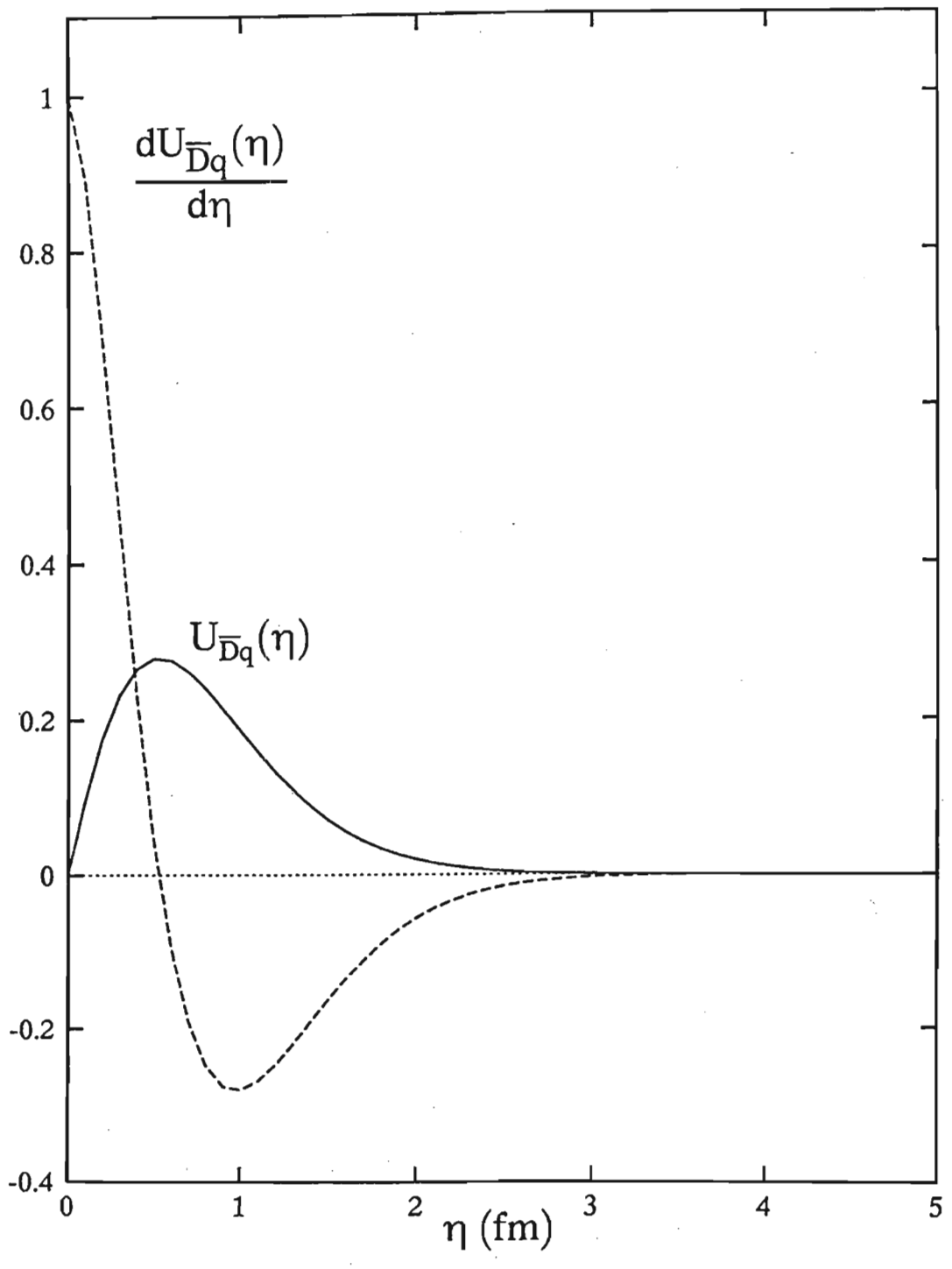

Fig. 7. The diquark-quark wave function (U) and its derivative for the Martin potential.

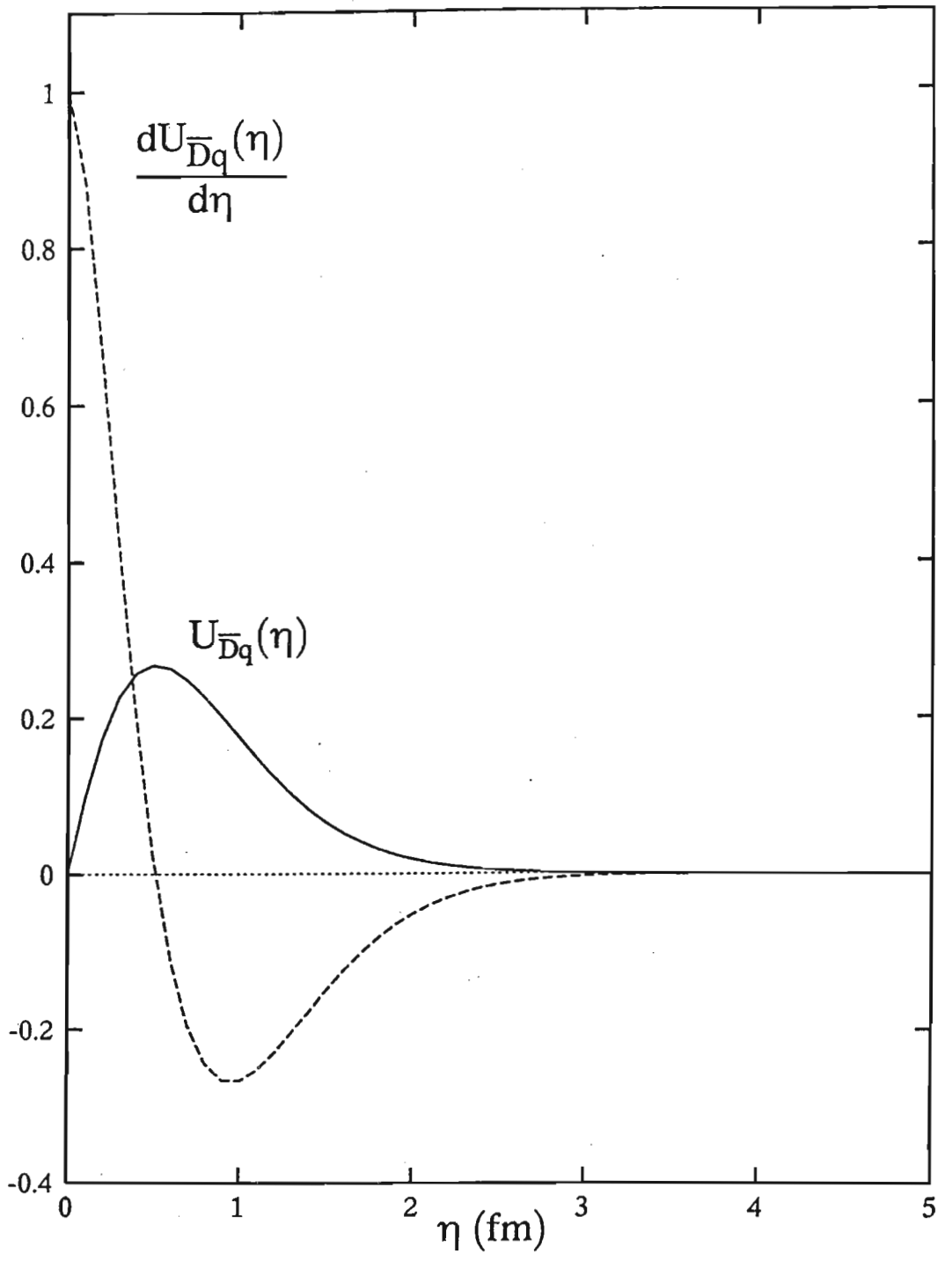

Fig. 8. The diquark-quark wave function (U) and its derivative for the Quigg potential.

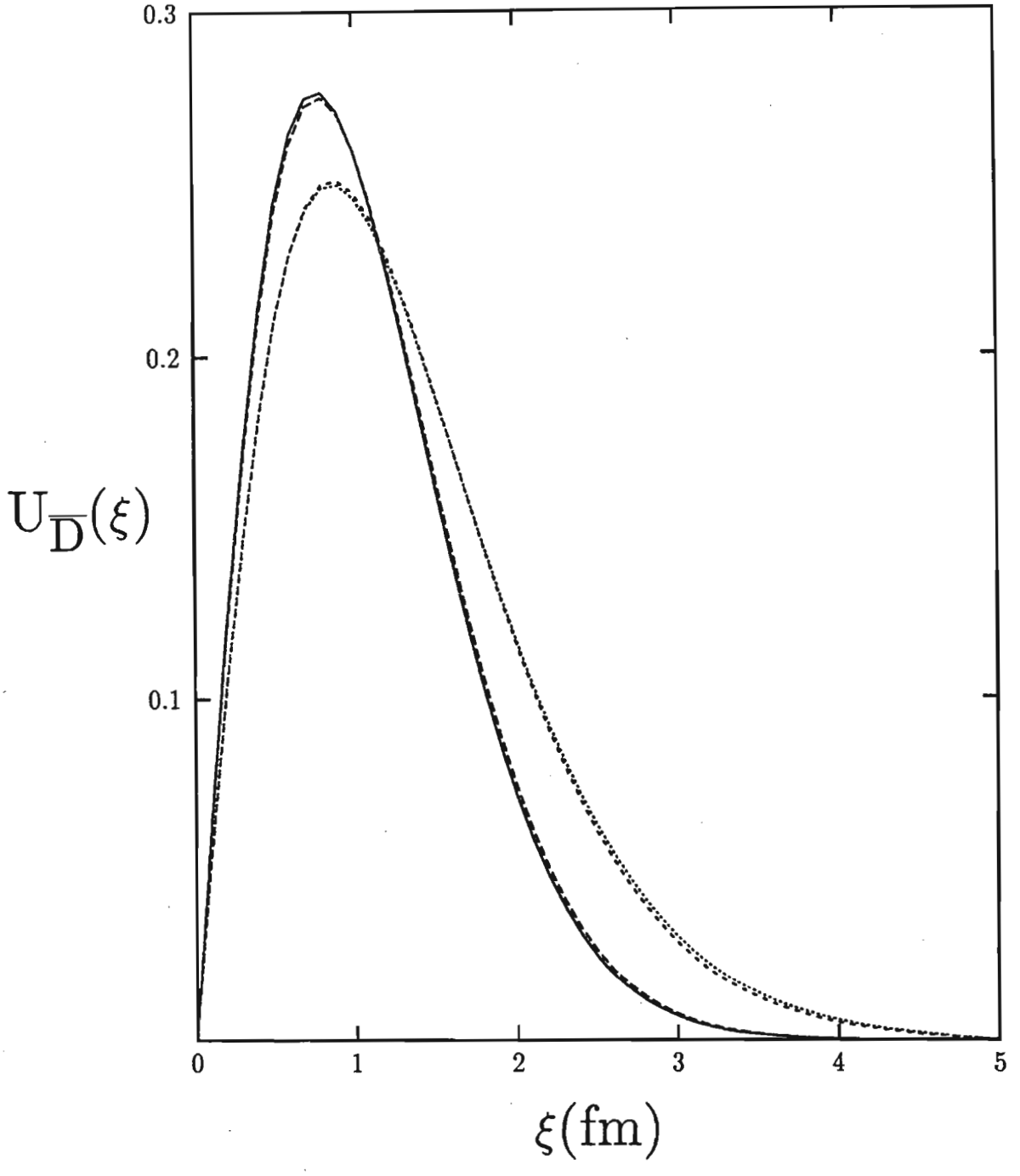

Fig.9. Comparison of the normalized diquark wave functions for the different central potentials The solid curve indicates the results for the Bhaduri potential, the dashed curve for the Cornell potential, the broken curve for the Martin potential and the dotted curve for the Quigg potential.

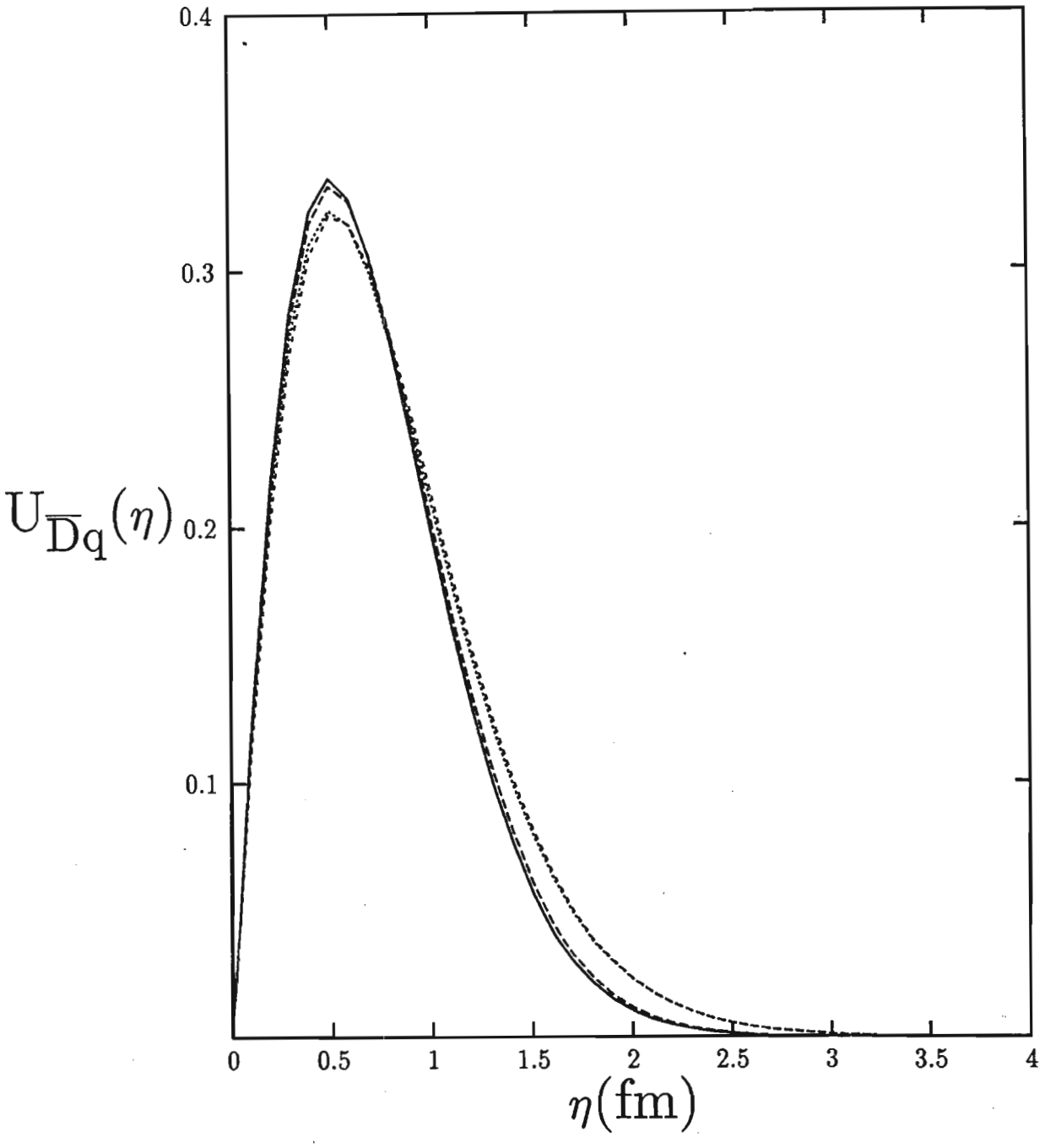

Fig.10. Comparison of the normalized diquark-quark wave functions for the different central potentials . The solid curve indicates the results for the Bhaduri potential, the dashed curve for the Cornell potential, the broken curve for the Martin potential and the dotted curve for the Quigg potential.

The normalized diquark and diquark-quark wave functions of Bhaduri and Cornell potentials as well as Martin and Quigg potentials look similar (see figs. 9 -10) because the potentials are of the same nature.

6.6 CALCULATING THE MASS OF A BARYON WITHOUT ANTISYMMETRIZATION

The cubic spline method (section 6.3), together with the Gaussian quadrature (section 6.4), were used to calculate the potential and kinetic energy for both the diquark and diquark-quark systems without antisymmetrization. A consistency check between the calculation of the mass of the baryon approach using both eigenvalue and expectation values was performed in a FORTRAN code and the results were identical in both cases. Tables I and II give the binding energies and the masses of the diquark and diquark-quark systems when Bhaduri, Cornell, Martin and Quigg potentials are used in the radial Schrodinger equation without antisymmetrization.

Table I: The values of the potential energy, the kinetic energy, the binding energy and the mass of the diquark system without antisymmetrization for L=0.

Potential	$V_{\bar{D}}$ MeV	$T_{\bar{D}}$ MeV	$E_{\bar{D}}$ MeV	$M_{\bar{D}}$ MeV
Bhaduri	-93.1	254.7	161.7	835.7
Cornell	-124.0	253.6	129.6	789.6
Martin	-63.2	205.3	142.1	742.1
Quigg	-63.0	186.0	123.0	783.0

The binding energies in tables I and II are used for calculating the masses of the diquark and diquark-quark systems from the equations:

$$M_{\bar{D}} = 2m_q + E_{\bar{D}} \tag{6.4.21}$$

for the diquark system and

$$M_{\bar{D}q} = 3m_q + E \tag{6.4.22}$$

for the diquark-quark system, where

$$E = E_{\bar{D}} + E_{\bar{D}q} \tag{6.4.23}$$

Using a constituent quark mass of 337 MeV with Bhaduri potential, 330 MeV with the Cornell potential and 300 MeV with Martin potential in the Schrodinger equation the results obtained in the centre-of-mass frame are comparable to those obtained by Silvestre-Brac et al [19]. These results are given in Table II.

Table II: The values of the potential and kinetic energies, the binding energy and the baryon mass of the diquark-quark without antisymmetrization for L=0.

Potential	$V_{\bar{D}q}$ MeV	$T_{\bar{D}q}$ MeV	$E_{\bar{D}q}$ MeV	$M_{\bar{D}q}$ MeV
Bhaduri	-550.9	402.3	-148.6	1024.1
Cornell	-606.9	400.7	-206.2	913.4
Martin	-525.6	390.6	-135.0	907.1
Quigg	-510.8	372.0	-138.8	974.2

6.7 CALCULATING THE MASS OF BARYON INCLUDING ANTISYMMETRIZATION

To demonstrate the validity of eqns.(4.4.29a) through (4.4.29c) a consistency check was performed in a FORTRAN code by exchanging quark labels and the results were identical in all the cases. In Table III, the antisymmetrization results of the kinetic and potential energy operator kernels are given.

Table III: The calculated values of the kinetic and the potential energies, binding energy and the mass of the baryon including antisymmetrization with L=0.

Potential	$V_{\bar{Dq}}$ MeV	$T_{\bar{Dq}}$ MeV	$E_{\bar{Dq}}$ MeV	$M_{\bar{Dq}}$ MeV
Bhaduri	-559.2	783.2	224.0	1235.0
Cornell	-647.5	775.3	127.8	1117.8
Martin	-480.7	703.6	223.1	1123.1
Quigg	-463.8	652.2	188.4	1178.4

In Table IV, the baryon mass for the operator kernels was compared with the baryon mass obtained from the two- and three-body calculations as well as the average mass of $N-\Delta$.

Table IV: Comparison of the GCM diquark-quark ground state masses for the four central potentials with (a) each other, (b) results for the two-body treatment, (c) the results for a full three-body treatment and (d) the average mass of N-Δ.

Potential	GCM calculations MeV	Two-body calculations MeV	Three-body calculations MeV	Average mass MeV
Bhaduri	1235.0	1024.1	1204.0	1173.0
Cornell	1117.8	913.5	1089.5	1173.0
Martin	1122.9	907.1	1086.0	1173.0
Quigg	1178.3	974.2	---------	1173.0

6.8 THE CALCULATIONS OF THE BARYON FORM FACTORS

The form factors for the diquark-quark have been calculated with and without antisymmetrization as well as the inclusion of meson cloud for the various potentials as indicated in figures 13 through 18. The wave numbers have been selected as 1.0 -15.0 with the unit fm^{-1}.

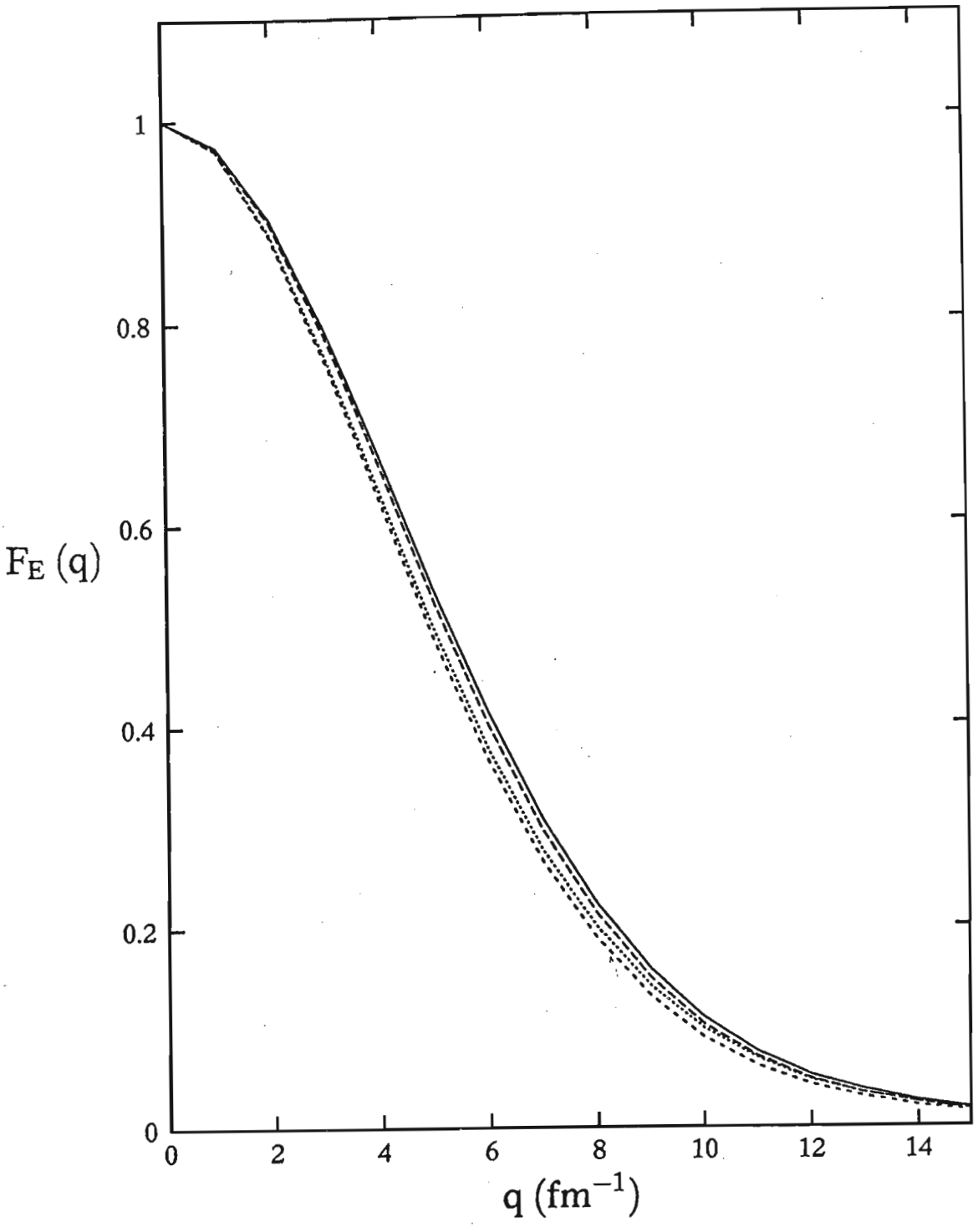

Fig.11. Comparison of the constituent quark elastic electric form factors for the Bhaduri, Cornell, Martin and Quigg central potentials without antisymmetrization. The solid curve indicates the results for the Bhaduri potential, the broken curve for the Cornell Potential, the dotted curve for the Martin potential and the dashed curve for the Quigg potential.

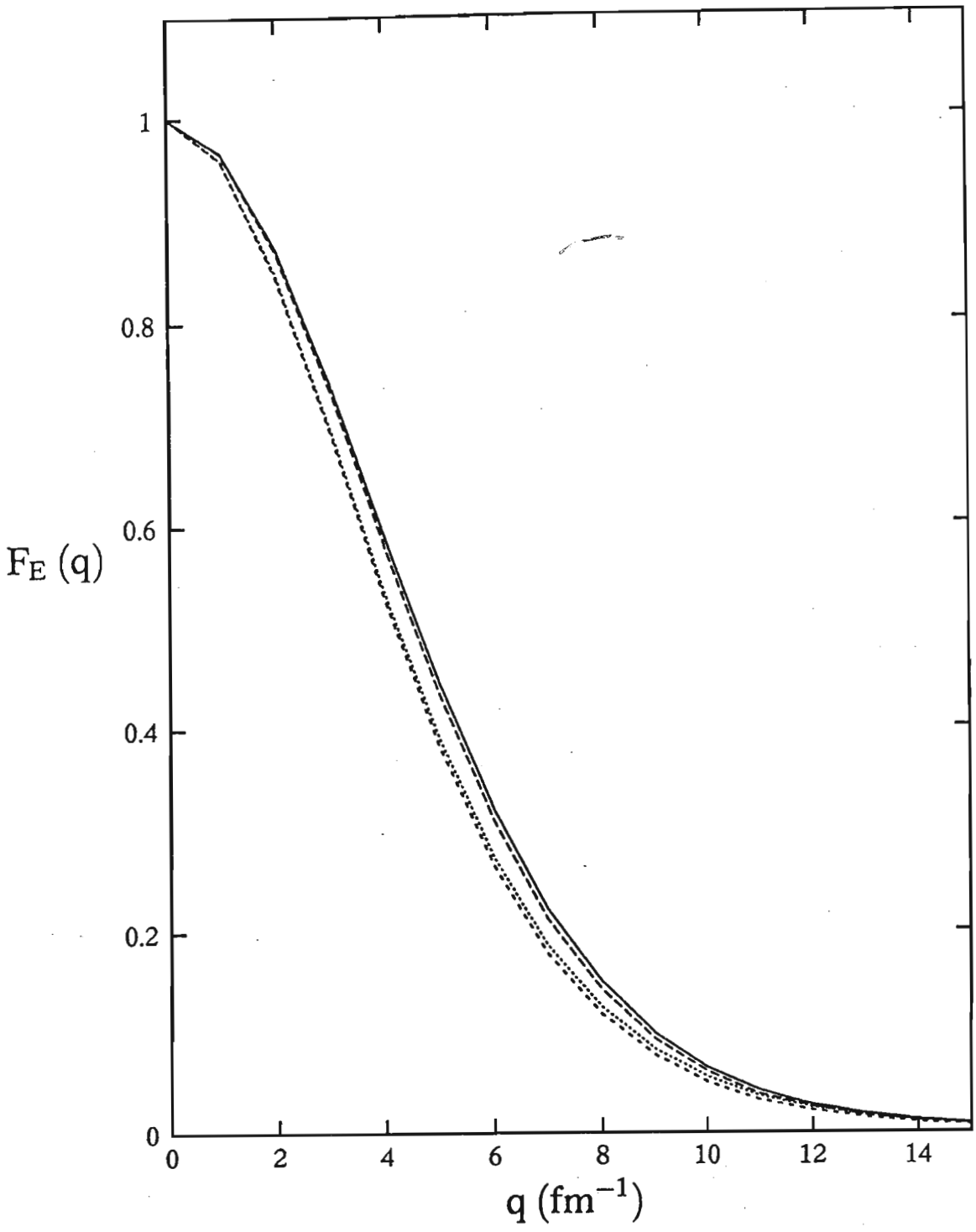

Fig. 12. Comparison of the constituent quark elastic electric form factors for the Bhaduri, Cornell, Martin and Quigg central potentials with antisymmetrization. The solid curve indicates the results for the Bhaduri potential, the broken curve for the Cornell potential, the dotted curve for the Martin potential and the dashed curve for the Quigg potential.

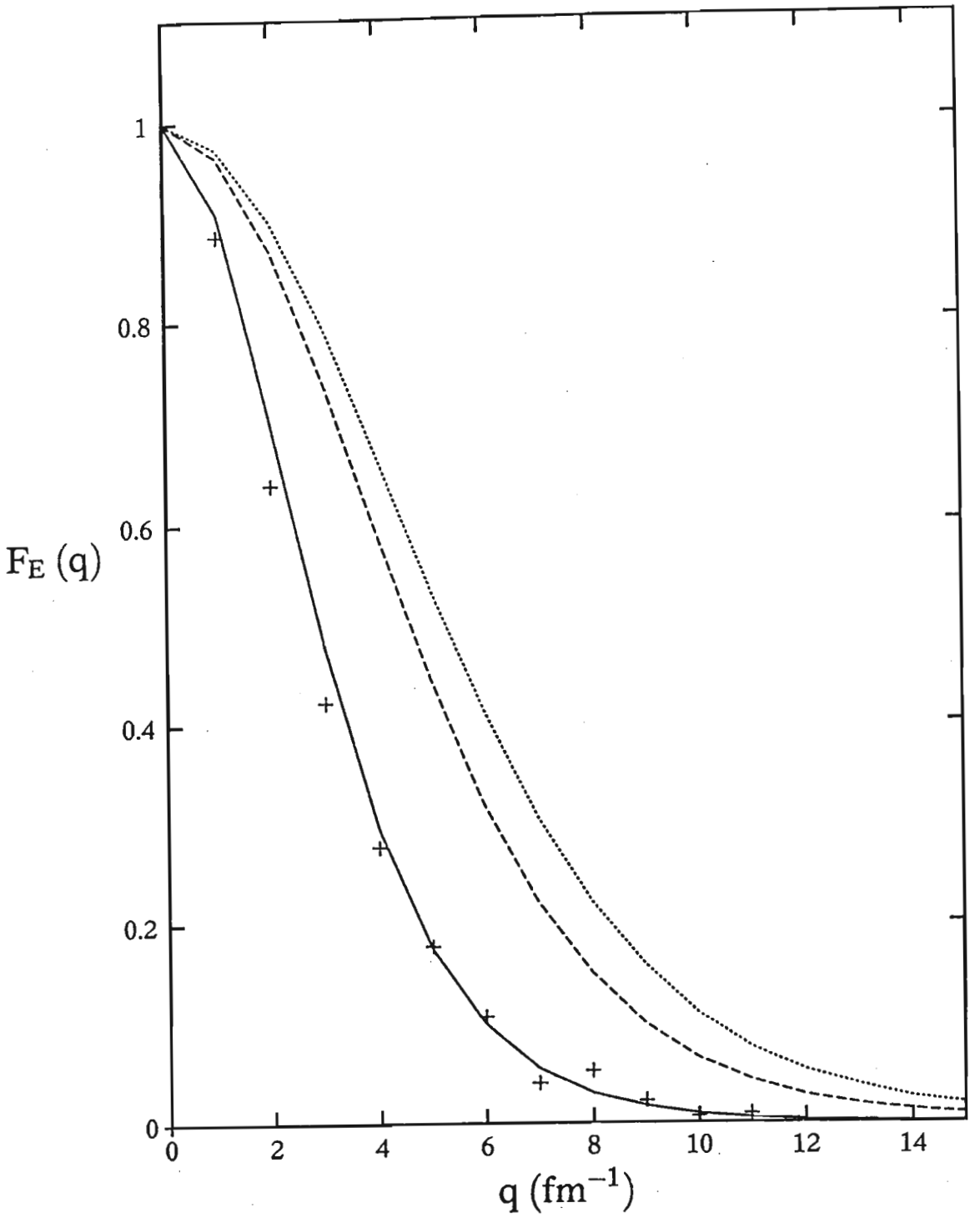

Fig. 13. The baryon form factors for the Bhaduri Potential. The dotted curve indicates results for the form factor without antisymmetrization, the dashed curve for including antisymmetrization and the solid curve for including antisymmetrization and the meson cloud. The experimental data are adopted from ref.[76].

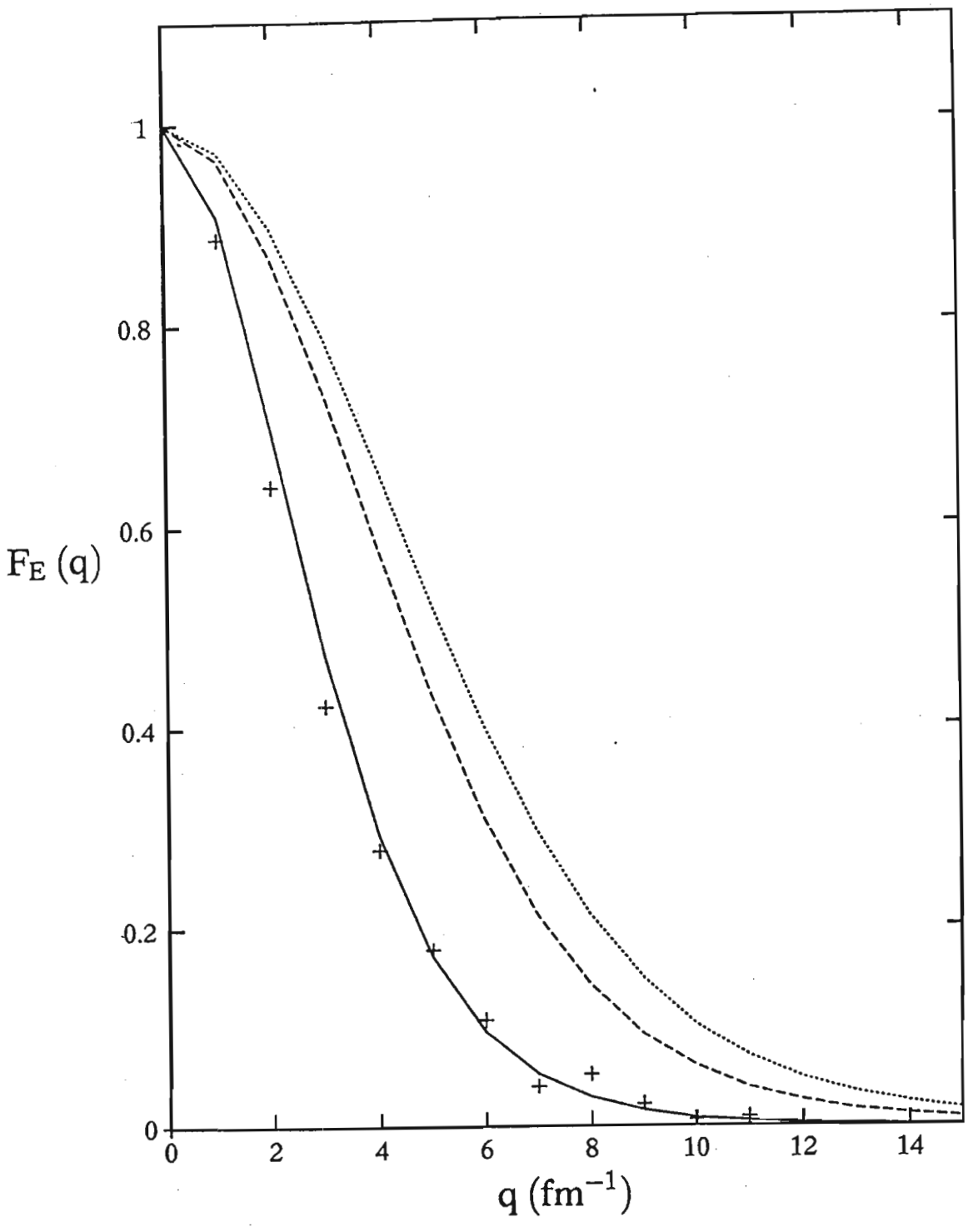

Fig. 14. The baryon form factors for the Cornell potential. The dotted curve indicates results for the form factor without antisymmetrization, the dashed curve for including antisymmetrization and the solid curve for including antisymmetrization and the meson cloud. The experimental data are adopted from ref.[76].

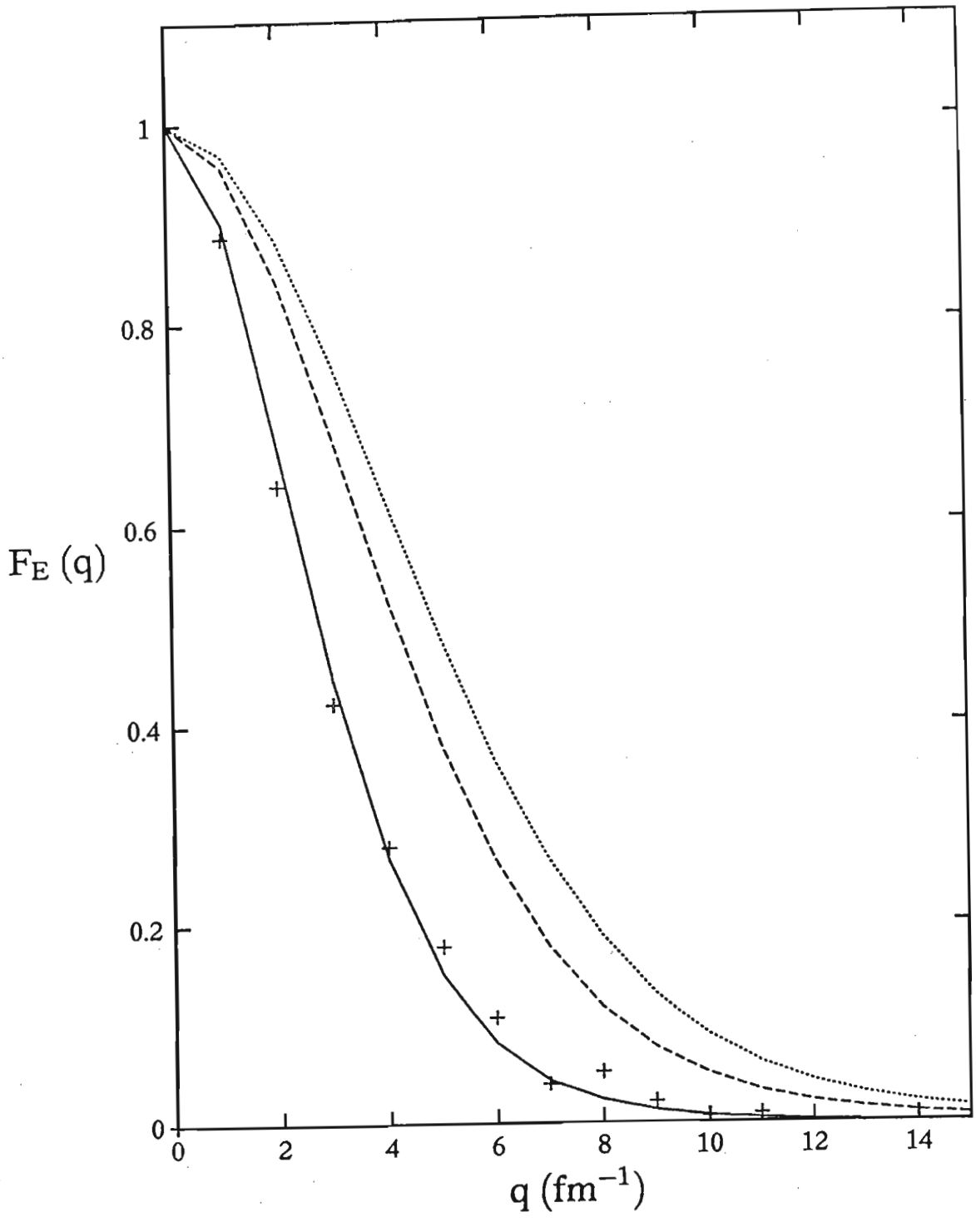

Fig. 15. The baryon form factors for the Martin potential. The dotted curve indicates results for the form factor without antisymmetrization, the dashed curve for including antisymmetrization and the solid curve for including antisymmetrization and the meson cloud. The experimental data are adopted from ref.[76].

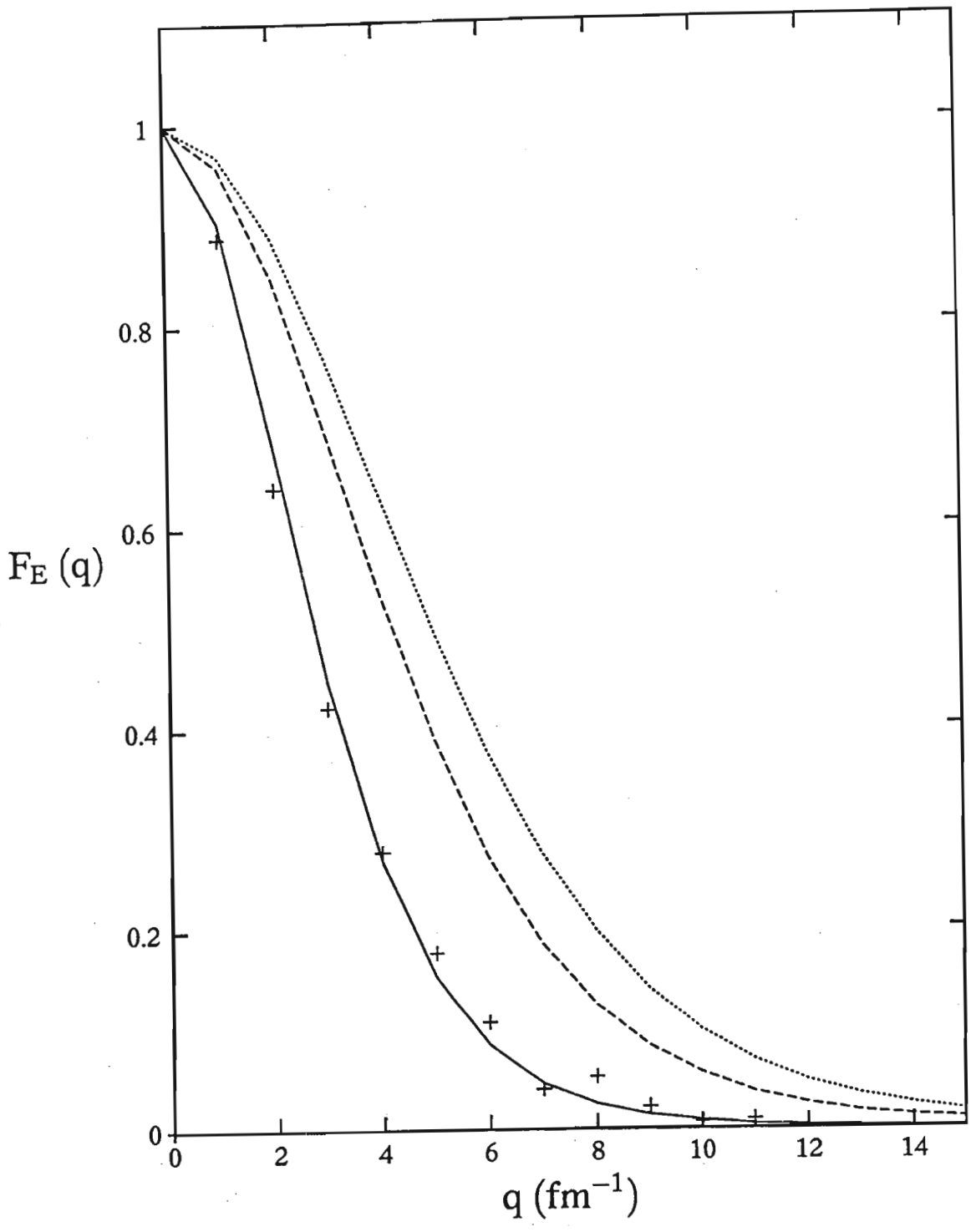

Fig. 16. The baryon form factors for the Quigg potential. The dotted curve indicates results for the form factor without antisymmetrization, the dashed curve for including antisymmetrization and the solid curve for including antisymmetrization and the meson cloud. The experimental data are adopted from ref.[76].

Table V: The calculated root mean-square radii for the baryon for the four central potentials.

Method	Bhaduri potential	Cornell potential	Martin potential	Quigg potential
	$\langle r^2 \rangle^{\frac{1}{2}}$ fm	$\langle r^2 \rangle^{\frac{1}{2}}$ fm	$\langle r^2 \rangle^{\frac{1}{2}}$ fm	$\langle r^2 \rangle^{\frac{1}{2}}$ fm
Non-antisymmetrization	0.39	0.40	0.43	0.42
Non-antisymmetrization + meson cloud	0.72	0.73	0.74	0.74
Antisymmetrization	0.45	0.46	0.51	0.50
Antisymmetrization + meson cloud	0.76	0.76	0.79	0.79
Dipole	0.81	0.81	0.81	0.81
Experiment	0.83	0.83	0.83	0.83

7 CONCLUSION

The motivation for undertaking these calculations was to examine the effect of antisymmetrization in diquark models of baryons in the ground state within a well-defined diquark model. The Generator Coordinate Method (GCM) operator kernels provide a transparent and explicit way for including antisymmetrization within such a model. The expectation values for several observables have been calculated in a pure diquark model and compared with corresponding results where antisymmetrization has been included via the GCM kernel. The results are also compared with those for three-body models of baryons and with experiment. The observables studied were the masses, the elastic electric form factors and the charge densities. The diquark model used was a simple non-relativistic one involving central interactions and a single channel.

7.1 COMPARISON OF THE DIFFERENT CALCULATIONS OF THE Δ-N MASS

In the case of central potential the implicit averaging over spins necessitates a comparison with the averaged Δ-N measurements rather than with those of a specific baryon. In general, two-body models of baryons produce unphysically deep binding due to the large reduced mass of the diquark-quark system. The resulting Δ-N masses are about 180 MeV lower than the corresponding three-body results for the same potentials - see Table IV. This trend can be understood in terms of the fact that both the inclusion of the binding energy in the diquark mass and the ``reduction'' in the number of particles (from three to two) in the system are non-variational procedures. For the central potentials used the two-body masses range between 907.1 MeV and 1024.1 MeV (the experimental value is 1173 MeV). The corresponding three-body results vary from 1086 MeV to 1204 MeV and were calculated by means of the

Integrodifferential Equation Approach (IDEA). For three bodies the IDEA is an augmented version of the S-projected Faddeev Equation and takes higher partial waves into account in an average way. For interactions with no hard core, such as those encountered in quark systems, the IDEA is essentially exact.

There is much better agreement between the three-body IDEA masses and equivalent two-body GCM kernel results than when antisymmetrization is not taken into account. The agreement for all four potentials considered is within about four percent. In all cases, the three-body masses are somewhat lower - see Table IV. Given the fact that the kinetic and potential energies are appreciable fractions of the masses this concordance is noteworthy.

Comparing results in Tables II and III it can be seen that the antisymmetrization procedure has relatively little influence on the expectation values of the potential energy. The large positive shifts in the masses are entirely due to the approximate doubling of kinetic energy expectation values. Are these shifts believable? Gavin et al [64], using the hypervirial approach [77] in a three-body system, have determined that the expectation value of the kinetic energy is about 70 % of the baryon ground state mass for quark potential models. This result corresponds very well with the GCM diquark values in Table III.

The relative model independence of the potential energy (two-body results including and excluding antisymmetrization are not very different from corresponding three-body values) is interesting. The explanation for this insensitivity is probably that the motion of the third quark is mediated by an interaction which is well-approximated by an aggregate static potential resulting from the other two quarks. This explanation is closely related to the reason for the success of the non-relativistic quark potential model:- the colour fields are essentially frozen with respect to the motion of the quarks.

The explanation for the low kinetic energy expectation values when antisymmetrization is not taken into account via the GCM kernels must be that the large values of the diquark-quark reduced masses (see Table II) produce unphysically deep binding for the effective two-body system. This effect is offset when the GCM kinetic energy operator, which is a function of constituent quark masses rather than diquark masses (see equations (4.4.31) and (4.4.32)), is used.

7.2 THE FORM FACTORS

The elastic electric form factor results (see Figures 11 - 16) confirm the pattern of better reproduction of experimental data when the antisymmetrization kernels are introduced . When the effect of the meson cloud around the constituent quarks is introduced by means of the appropriate monopole factor, the correspondence agreement between the data and the GCM kernel results is almost perfect.

The agreement between the results of calculations of r.m.s radii (see Table V) is also improved by the introduction of antisymmetrization. When the meson cloud contribution is included in a standard way the r.m.s radii agree to within 2-5% of the experimental proton value of 0.83 fm [64]. This is further evidence for the usefulness of the diquark model improved by antisymmetrization.

7.3 SUMMARY AND WAY FORWARD

It is clear from this study that diquark wave functions may be used to extract physical information which is of a surprisingly high quality. This affirms the wide applicability of the GCM and more generally the resonating group model approaches. These were developed more

specifically for few-nucleon systems (see [4] and [78]) but it has been shown here that the GCM is equally applicable to quark systems.

A logical generalization of this work would be to extend the analysis to systems of coupled channels (see chapter 5). The single channel model implies a comparison with a fictitious Δ-N system rather than a proton or neutron. Analysis of other baryons such as Λ and of excited states would also be possible.

Another interesting way to build on the formalism developed here would be to calculate relativistic corrections to the masses. This could be done by transforming the diquark and diquark-quark wave functions to momentum space and calculating the corresponding kinetic energy expectation values using the appropriate relativistic operators. To include relativistic effects in coordinate space is not convenient because of the square root form of the kinetic energy operator. However, the approach described above would also not give the full relativistic correction, which arises also from a consideration of retardation effects and of Lorentz invariance of the appropriate phase space. These effects are most easily treated in momentum space too. Nevertheless, confining potentials (which occur is of necessity in quark calculations) provide some difficulty in momentum space. Recently, however, some progress has been made in applying the fact that the Fourier transform of a linearly increasing potential does exist in a distributional sense [79] and it would be interesting to apply this full formalism (hitherto restricted to mesons) to diquark models of baryons too.

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APPENDIX

A.1 DERIVATION OF AN OPERATOR KERNEL

The total baryon wave function appears as

$$\Psi_B = \Psi_{\bar{D}q}(\vec{\xi}, \vec{\eta})\, \Psi_S \Psi_T \Psi_C \tag{A.1.1}$$

where $\Psi_{\bar{D}q}(\vec{\xi}, \vec{\eta})$ is the spatial wave function

Ψ_S is the spin wave function

Ψ_T is the isospin wave function, and

Ψ_C is the colour wave function.

The expectation value of the operator $\hat{O}$ in the pure diquark model is of the form

$$\langle \hat{O} \rangle = \int \Psi_B^*(\vec{r})\, \hat{O} \Psi_B(\vec{r})\, d^3r \tag{A.1.2}$$

where $\vec{r}$ is the relative coordinate.

To take into account the effects of exchange between the diquark pair and the external quark, $\langle \hat{O} \rangle$ in eqn. (A.1.2) is redefined as

$$\langle \hat{O} \rangle = \int \Psi^*_{\bar{D}q}\left(\vec{r}_3 - \frac{\vec{r}_1 + \vec{r}_2}{2}\right) \Phi^*_{\bar{D}}(\vec{r}_1 - \vec{r}_2)\, \delta\left(\vec{r} - \left(\vec{r}_3 - \frac{\vec{r}_1 + \vec{r}_2}{2}\right)\right) \hat{O} A \left[\Phi_{\bar{D}}(\vec{r}_1 - \vec{r}_2)\right.$$

$$\left. \delta\left(\vec{r}\,' - \left(\vec{r}_3 - \frac{\vec{r}_1 + \vec{r}_2}{2}\right)\right)\right] \Psi_{\bar{D}q}\left(\vec{r}_3 - \frac{\vec{r}_1 + \vec{r}_2}{2}\right) \delta\left(\frac{\vec{r}_1 + \vec{r}_2 + \vec{r}_3}{3}\right)$$

$$d^3r_1 d^3r_2 d^3r_3 d^3r d^3r' \tag{A.1.3}$$

where A is the antisymmetrization operator. In the account of eqn. (A.1.2) the following can be written:

$$\langle \hat{O} \rangle = \int \Psi^*_{\bar{D}q}(\vec{r})\, K_{\hat{O}}(\vec{r}, \vec{r}\,')\, \Psi_{\bar{D}q}(\vec{r}\,')\, d^3r d^3r' \tag{A.1.4}$$

where $K_{\hat{O}}(\vec{r}, \vec{r}\,')$ is defined as

$$K_{\hat{O}}(\vec{r},\vec{r}\,') = \int \Phi^{*}_{\bar{D}}(\vec{r}_1-\vec{r}_2)\,\delta\left(\vec{r}-(\vec{r}_3-\frac{\vec{r}_1+\vec{r}_2}{2})\right)\hat{O}A\left[\Phi_{\bar{D}}(\vec{r}_1-\vec{r}_2)\right.$$

$$\left.\delta\left(\vec{r}\,'-(\vec{r}_3-\frac{\vec{r}_1+\vec{r}_2}{2})\right)\delta\left(\frac{\vec{r}_1+\vec{r}_2+\vec{r}_3}{3}\right)d^3r_1 d^3r_2 d^3r_3 \qquad \text{(A.1.5)}$$

In using the Jacobi coordinates

$$\vec{\xi} = \vec{r}_1 - \vec{r}_2,$$

$$\vec{\eta} = \frac{2}{\sqrt{3}}\left(\vec{r}_3 - \frac{\vec{r}_1+\vec{r}_2}{2}\right)$$

and

$$\vec{R} = \frac{1}{3}\left(\vec{r}_1+\vec{r}_2+\vec{r}_3\right)$$

with

$$J\left(\frac{\vec{r}_1,\vec{r}_2,\vec{r}_3}{\vec{\xi},\vec{\eta},\vec{R}}\right) = 1$$

the following is then obtained

$$K_{\hat{O}}(\vec{X},\vec{X}\,') = \int \Phi^{*}_{\bar{D}}(\vec{\xi})\,\delta(\vec{X}-\vec{\eta})\,A\Phi_{\bar{D}}(\vec{\xi})\,\delta(\vec{X}\,'-\vec{\eta})\,\delta(\vec{R})\,d^3\xi d^3\eta d^3R \qquad \text{(A.1.6)}$$

For eqns. (A.1.2) and (A.1.4) to be equivalent, eqn. (A.1.4) must be divided by the expectation value of the norm kernel $K_{\hat{1}}(\vec{X},\vec{X}\,')$, that is

$$\langle\hat{O}\rangle = \frac{\int \Psi^{*}_{\bar{D}q}(\vec{X})\,K_{\hat{O}}(\vec{X},\vec{X}\,')\,\Psi_{\bar{D}q}(\vec{X}\,')\,d^3X d^3X'}{\langle K_{\hat{1}}(\vec{X},\vec{X}\,')\rangle} \qquad \text{(A.1.7)}$$

A.2 FORTRAN CODE

PROGRAM RAYZA

```
IMPLICIT REAL*8(A-H, O-Z)
PARAMETER(EIN=-300.0, EFIN=300.0)
PARAMETER(ETOL=0.0000001, YTOL=0.00000001)
PARAMETER(RIN=0.00000001, RFIN=5.D0)
PARAMETER(N=500, NLIM=120)
PARAMETER(NIN=20)
PARAMETER(NPTS=50, NMU=20)
PARAMETER(LTP=1)
DIMENSION R(0:N),Y(0:N), Z(0:N)
DIMENSION CD(4, N+1),CDQ (4, N+1)
DIMENSION T(NIN), W(NIN), WMU (NMU), TMU (NMU)
COMMON/CHM/HM
COMMON/CMASS/QM,DMASS
QM=QMASS
IF (LTP.EQ.1)THEN
QMASS=330.D0
WRITE(3,*)'CORNELL POTENTIAL'
ENDIF
IF (LTP.EQ.2)THEN
QMASS=337.D0
```

```
WRITE(3,*)'BHADURI POTENTIAL '
ENDIF
IF (LTP.EQ.3)THEN
QMASS=300D0
WRITE (3,*)'MARTIN POTENTIAL'
END IF
IF (LTP.EQ.4)THEN
QMASS=330D0
WRITE (3,*)'QUIGG POTENTIAL'
ENDIF
PI=4.0*ATAN(1.D0)
P=16.0*PI**2
P1=4.0*PI
P2=3.D0*SQRT(3.0)
P3=128.D0*PI**2/P2
P4=64.D0*PI**2/P2
P5=8.*PI**2
HM=3.892168D4/QMASS
D1=3.892168D4/QMASS
DO 300 K=1,2
WRITE(3,*)'HM=',HM
H=(RFIN-RIN)/FLOAT(N-1)
E1=EIN
```

```
CALL RAY(R, Y, Z, H, N, E1, LTP, K)
IF (Y(N).LT.0.0) GO TO 15
E2=EFIN
CALL RAY(R, Y, Z, H, N, E2, LTP, K)
IF (Y(N).GT.0.0) STOP
15    CONTINUE
DO 20 J=1, NLIM
E3=(E1+E2)/2.D0
CALL RAY(R, Y, Z, H, N, E3, LTP, K)
IF (ABS(E1-E2)/2.D0.LE.ETOL) THEN
IF (ABS(Y(N)).LE.YTOL) THEN
CONTINUE
ENDIF
ENDIF
IF (Y(N).GT.0.0) THEN
E1=E3
ELSE
E2=E3
ENDIF
20    CONTINUE
IF (K.EQ.1) EDIQUARK = E3
WRITE(3,*)'ENERGY=',E3
WRITE(3,*) ' '
```

```
WRITE(3,205)
DO 250 I=0, N, 10
WRITE(3,203) R(I),Y(I),Z(I)
250   CONTINUE
WRITE(3,*) '     '
RM=RIN
RP=RFIN
IF(K.EQ.1)THEN
CALL CUBS3(R, Y, CD, N+1)
ELSE
CALL CUBS3(R, Y, CDQ, N+1)
ENDIF
CALL GAUSSPT(1, NIN, NIN, RM, RP, T, W)
CALL GAUSSPT(1, NMU, NMU,-1.D0, 1.D0, TMU, WMU)
SUM1=0.D0
SUM2=0.D0
SUM3=0.D0
SUM3A=0.D0
SUM4=0.D0
SUM5=0.D0
SUM6=0.D0
SUM6A=0.D0
SUM7=0.D0
```

```
SUM7A=0.D0
SUM7B=0.D0
SUM8A=0.D0
SUM8B=0.D0
SUM8C=0.D0
SUM9A=0.D0
SUM9B=0.D0
SUM10A=0.D0
SUM10B=0.D0
SUM11A=0.D0
SUM11B=0.D0
SUM11C=0.D0
SUM11D=0.D0
DO 500 I=1,NIN
C..... CALCULATIONS OF THE BARYON MASS WITHOUT ANTISYMMETRIZATION
IF (K.EQ.1)THEN
SUM1=SUM1+P1*CINT(CD, R, T(I),N+1)**2*W(I)
SUM2=SUM2+P1*CINT(CD, R, T(I),N+1)**2*V(T(I),LTP, K)*W(I)
SUM3=SUM3+P1*CINT(CD, R, T(I),N+1)**2*VC1(T(I),K)*W(I)
SUM3A=SUM3A+P1*CINT(CD, R, T(I),N+1)**2*F*(T(I),K)*W(I)
VD=SUM2/SUM1
TD=SUM3/SUM1
FD=SUM3A/SUM1
```

```
ELSE
SUM4=SUM4+P1*CINT(CDQ, R, T(I),N+1)**2*W(I)
SUM5=SUM5+P1*CINT(CDQ, R, T(I),N+1)**2*VC1(T(I),K)*W(I)
SUM6=SUM6+P1*CINT(CDQ, R, T(I),N+1)**2*V(T(I),LTP, K)*W(I)
SUM6A=SUM6A+P1*CINT(CDQ, R, T(I),N+1)**2*F(T(I),LTP, K)*W(I)
  TDQ=SUM5/SUM4
VDQ=SUM6/SUM4
FDQ=SUM6A/SUM4
ENDIF
C.... CALCULATIONS OF BARYON MASS EFFECTING ANTISYMMETRIZATION
DO 600 J=1, NIN
SUM9A=SUM9A-D1*P*CINT(CDQ, R, T(I),N+1)**2*CINT(CD, R, T(J),N+1)
#      *CINTT(CD, R, T(J),N+1)*W(I)*W(J)
SUM9B=SUM9B-D1*P*CINT(CDQ, R, T(I),N+1)*CINT(CD, R, T(J),N+1)**2
#      *CINTT(CDQ, R, T(I),N+1)*W(I)*W(J)
SUM7A=SUM7A+P*CINT(CDQ, R, T(I),N+1)**2* CINT(CD, R, T(J),N+1)**2 *W(I)*W(J)
SUM10A=SUM10A+P*CINT(CDQ, R, T(I),N+1)**2* CINT(CD, R, T(J),N+1)**2
#      *F1(T(J))*W(I)*W(J)
SUM8A=SUM8A+P*CINT(CDQ, R, T(I),N+1)**2* CINT(CD, R, T(J),N+1)**2
#      *VC(T(J))*W(I)*W(J)
DO 700 KK=1,NMU
XA=SQRT((T(J)**2-2.*SQRT(3.)*T(I)*T(J)*TMU(KK)+3.*T(I)**2)/4.)
XB=SQRT((T(J)**2+2.*SQRT(3.)*T(I)*T(J)*TMU(KK)+3.*T(I)**2)/4.)
```

```
SUM8B=SUM8B+P5*CINT(CDQ, R, T(I),N+1)**2* CINT(CD, R, T(J),N+1)**2
#       *(VC(XA)+VC(XB))*W(I)*W(J)*WMU(KK)
XC=SQRT((T(I)**2+4.*T(I)*T(J)*TMU(KK)+4.*T(J)**2)/3.)
XD=SQRT((T(J)**2+4.*T(I)*T(J)*TMU(KK)+4.*T(I)**2)/3.)
SUM7B=SUM7B+P3*CINT(CDQ, R, T(I),N+1)*CINT(CDQ, R, T(J),N+1)
#     *CINT(CD, R, XC, N+1)*CINT(CD, R, XD, N+1)
#     *T(I)*T(J)/(XC*XD)*W(I)*W(J)*WMU(KK)
SUM11A=SUM11A-P4*D1*CINT(CDQ, R, T(I),N+1)*CINT(CDQ, R, T(J),N+1)
#       *CINT(CD, R, XD, N+1)*CINTT(CD, R, XC, N+1)
#       *T(I)*T(J)/(XC*XD)*W(I)*W(J)*WMU(KK)
SUM11B=SUM11B-P4*D1*CINT(CDQ, R, T(I),N+1)*CINT(CDQ, R, T(J),N+1)
#       *CINT(CD, R, XC, N+1)*CINTT(CD, R, XD, N+1)
#       *T(I)*T(J)/(XC*XD)*W(I)*W(J)*WMU(KK)
SUM11C=SUM11C-P4*D1*CINT(CDQ, R, T(I),N+1)*CINTT(CDQ, R, T(J),N+1)
#       *CINT(CD, R, XC, N+1)*CINT(CD, R, XD, N+1)
#       *T(I)*T(J)/(XC*XD)*W(I)*W(J)*WMU(KK)
SUM11D=SUM11D-P4*D1*CINT(CDQ, R, T(J), N+1)*CINTT(CDQ, R, T(I),N+1)
#       *CINT(CD, R, XC, N+1)*CINT(CD, R, XD, N+1)
#       *T(I)*T(J)/(XC*XD)*W(I)*W(J)*WMU(KK)
XE=SQRT((T(I)**2-2.*T(I)*T(J)*TMU(KK)+T(J)**2)/3.)
SUM8C=SUM8C+P3*CINT(CDQ, R, T(I),N+1)*CINT(CDQ, R, T(J),N+1)
#     *(VC(XC)+VC(XD)+VC(XE))*CINT(CD, R, XC, N+1)
#     *CINT(CD,R,XD,N+1)*T(I)*T(J)/(XC*XD)*W(I)*W(J)*WMU(KK)
```

```
      XF=SQRT(T(I)**2+2*T(I)*T(J)*TMU(KK)+T(J)**2)
      SUM10B=SUM10B+P4*CINT(CDQ, R, T(I),N+1)*CINT(CDQ, R, T(J),N+1)
     #    *CINT(CD, R, T(J),N+1)*CINT(CD, R, XF, N+1)*(F1(T(J)+F1(XF))
     #    *T(I)*T(J)*W(I)*W(J)*WMU(KK)
  700 CONTINUE
  600 CONTINUE
  500 CONTINUE
      WRITE(3,*)'CALCULATING BARYON MASS WITHOUT ANTISYMMETRIZATION'
      IF (K.EQ.) THEN
      ED=VD+TD
      WRITE(3,*)'ENERGY OF THE DIQUARK=',ED
C     (a) Potential and kinetic energies of the diquark
      WRITE(3,*)'VD=',VD
      WRITE(3,*)'TD=',TD
      WRITE(3,*)'SUM1=',SUM1
      WRITE(3,*)'SUM2=',SUM2
      WRITE(3,*)'SUM3=',SUM3
C     (b) Potential and kinetic energies of the diquark-quark
      ELSE
      EDQ=VDQ+TDQ
      WRITE(3,*) 'ENERGY OF THE DIQUAR-QUARK=',EDQ
      WRITE(3,*)'VDQ=',VDQ
      WRITE(3,*)'TDQ=',TDQ
```

```
WRITE(3,*)'FORM FACTOR WITHOUT ANTISYMMETRIZATION=',FDQ
ENDIF
WRITE(3,*) '    '
DQMASS=(2*QMASS+E3)*QMASS/(3*QMASS+E3)
DMASS=2*QMASS+EDIQUARK
HM=3.892168D4/DQMASS/2.D0
300   CONTINUE
C------------NORM KERNEL-----------------------------
        SUNA=SUM7A+SUM7B
C-----------POTENTIAL ENERGY-----------------------
          SUNB=(SUM8A+SUM8B+SUM8C)/SUNA
C-------------KINETIC ENERGY--------------------------
SUNC=(SUM9A+SUM9B+SUM11A+SUM11B+SUM11C+SUM11D)/SUNA
C------TOTAL ENERGY FROM KINETIC AND POTENTIAL ENERGY KERNELS-----
SUND=SUNB+SUNC
C-----------THE MASS OF A BARYON---------------------------
SUNE=3.*QMASS+SUND
C---------------------FORM FACTOR THROUGH OPERATOR KERNELS---------------
SUNFF=(SUM10A+SUM10B)/SUNA
WRITE(3,*)'FORM FACTOR WITH ANTISYMMETRIZATION=', SUNFF
WRITE(3,*)'CALCULATIONS OF BARYON MASS WITH ANTISYMMETRIZATION'
WRITE(3,*)'SUM4=',SUM4
WRITE(3,*)'SUM5=',SUM5
```

```
WRITE(3,*)'SUM6=',SUM6
WRITE(3,*)'SUM6A=',SUM6A
WRITE(3,*)'SUM7A=',SUM7A
WRITE(3,*)'SUM7B=',SUM7B
WRITE(3,*)'SUM8A=',SUM8A
WRITE(3,*)'SUM8B=',SUM8B
WRITE(3,*)'SUM8C=',SUM8C
WRITE(3,*)'SUM9A=',SUM9A
WRITE(3,*)'SUM9B=',SUM9B
WRITE(3,*)'SUM10A=',SUM10A
WRITE(3,*)'SUM10B=',SUM10B
WRITE(3,*)'SUM11A=',SUM11A
WRITE(3,*)'SUM11B=',SUM11B
WRITE(3,*)'SUM11C=',SUM11C
WRITE(3,*)'SUM11D=',SUM11D
WRITE(3,*)'NORM KERNEL=',SUNA
WRITE(3,*)'POTENTIAL KERNEL =',SUNB
WRITE(3,*)'KINETIC KERNEL =',SUNC
WRITE(3,*)'ENERGY OF A DIQUARK-QUARK FROM KERNELS=',SUND
WRITE(3,*)'DIQUARK -QUARK MASS USING KERNELS=',SUNE
BMASS=3.*QMASS+EDIQUARK+E3
```

```
WRITE(3,*) '       '
WRITE(3,*) 'DIQUARK MASS=',DMASS
WRITE(3,*) '       '
WRITE(3,*) 'BARYON MASS=',BMASS
203     FORMAT(T5,F6.3,5X,2(F10.7,5X))
205     FORMAT(T8,'R',12X,'U',13X,'U1',/,T5,6('='),8X,6('='),8X,6('='))
STOP
END
SUBROUTINE RAY(R,Y,Z,H,N,E,LTP,K)
IMPLICIT REAL*8(A-H,O-Z)
PARAMETER(RIN=0.00000001, YA=0.0, ZA=1.0)
DIMENSION  R(0:N), Y(0:N), Z(0:N)
R(0)=RIN
Y(0)=YA
Z(0)=ZA
DO 15 I= 0, N-1
X=R(I)
C1= H/2.*FUNC(X, Y(I),Z(I), E, LTP, K)
C2= H/2.*FUNC(X+H/2.,Y(I)+H/2.*(Z(I)+C1/2.),Z(I)+ C1, E, LTP, K)
C3= H/2.*FUNC(X+H/2.,Y(I)+H/2.*(Z(I)+C1/2.),Z(I)+C2, E, LTP, K)
C4= H/2.*FUNC(X+H,Y(I)+H*(Z(I)+C3),Z(I)+2.*C3, E, LTP, K)
R(I+1)= R(I) + H
Y(I+1) = Y(I) + H*Z(I) +H*(C1+C2+C3)/3.
```

```
Z(I+1) = Z(I) + (C1+2.*C2+2.*C3+C4)/3.
15    CONTINUE
RETURN
END
FUNCTION FUNC(X, Y, Z, E, LTP, K)
IMPLICIT REAL*8(A-H, O-Z)
COMMON/CHM/HM
FUNC=Y*(V(X, LTP, K) - E)/HM
RETURN
END
C..... POTENTIAL ENERGIES OF CORNELL, BHADURI, MARTIN AND QUIGG AND
C     THEIR  DERIVATIVES FROM THE VIRIAL THEOREM.....
FUNCTION V(X, LTP, K)
IMPLICIT REAL*8(A-H, O-Z)
COMMON/CHM/HM
COMMON/CMASS/QM,DMASS
GO TO(1,2,3,4), LTP
C     CORNELL POTENTIAL
1     HC=0.197D0
A=5.2D2*HC
B=1.8262D2/HC
D=975.D0
V= -A/X + B*X -D
```

```
V= V/2.D0 *K
RETURN
C     BHADURI CENTRAL POTENTIAL
2     HC=0.197239D0
A=5.203D2*HC
B=1.857D2/HC
D=913.5D0
V= -A/X + B*X -D
V= V/2.D0 *K
RETURN
C     MARTIN POTENTIAL
3     A=-8.337D3
B=6.9923D3
C=0.1D0
V=A+B*(5.07*X)**C
V=V/2.D0*K
RETURN
C     QUIGG POTENTIAL
4     A=744.D0
B=0.87D0
V=A*LOG(B*X)
V=V/2.D0*K
RETURN
```

```
END
FUNCTION VM1(X, K)
IMPLICIT REAL*8(A-H, O-Z)
A=-8.337D3
B=6.9923D3
C=0.1D0
VM1=0.5*5.07*B*C*X*((5.07*X)**(C-1.D0))
VM1=VM1/2.D0*K
RETURN
END
FUNCTION VM(X)
IMPLICIT REAL*8(A-H, O-Z)
A=-8.337D3
B=6.9923D3
C=0.1D0
VM=0.5*(A+B*(5.07*X)**C)
RETURN
END
FUNCTION VQ(X)
IMPLICIT REAL*8(A-H, O-Z)
A=744.D0
B=0.87D0
VQ=0.5*A*LOG(B*X)
```

```
RETURN
END
FUNCTION VQ1(X, K)
IMPLICIT REAL*8(A-H, O-Z)
A=744.D0
B=0.87D0
VQ1=0.5*A
VQ1=VQ1/2.D0*K
RETURN
END
FUNCTION VC(X)
IMPLICIT REAL*8(A-H, O-Z)
HC=0.197D0
A=5.2D2*HC
B=1.8262D2/HC
D=975.D0
VC=0.5*(-A/X+B*X-D)
RETURN
END
FUNCTION VC1(X, K)
IMPLICIT REAL*8(A-H, O-Z)
HC=0.197D0
A=5.2D2*HC
```

```
B=1.8262D2/HC
VC1=0.5*(A/X+B*X)
VC1=VC1/2.D0*K
RETURN
END
FUNCTION VB(X)
IMPLICIT REAL*8(A-H, O-Z)
HC=0.197239D0
A=520.3*HC
B=185.7/HC
D=913.5D0
VB=0.5*(-A/X+B*X-D)
RETURN
END
FUNCTION F(X, K)
IMPLICIT REAL*8(A-H, O-Z)
Q=1.DQ
R1=Q/SQRT{3}
F=SIN(R1*X)/(R1*X)
F=F/2.D0*K
RETURN
END
FUNCTION F1(X)
```

```
IMPLICIT REAL*8(A-H, O-Z)
Q=1.D0
R1=Q/SQRT{3}
F1=SIN(R1*X)/R1*X
RETURN
END
FUNCTION VB1 (X, K)
IMPLICIT REAL*8(A-H ,O-Z)
HC=0.197239D0
A=520.3*HC
B=185.7/HC
VB1=0.5*(A/X+B*X)
VB1=VB1/2.D0*K
RETURN
END
SUBROUTINE GAUSSPT(NLOW, NDIM, NGAUSS, A, B, X, W)
IMPLICIT REAL*8(A-H, O-Z)
DIMENSION X(NLOW:NDIM),W(NLOW:NDIM)
GN=0.5/NGAUSS
EXTRA=1.0/(0.4*NGAUSS*NGAUSS+5.0)
XZ =-GN
NT=0
NTEKEN=0
```

```
5     PNM2=1.0
PNM1=XZ
DO 10 I=2,NGAUSS
PNM1XZ=PNM1*XZ
PN=2.0*PNM1XZ-PNM2-(PNM1XZ-PNM2)/I
PNM2=PNM1
10    PNM1=PN
MTEKEN=1
IF (PN.LE.0.0) MTEKEN=-1
IF((MTEKEN+NTEKEN).EQ.0) GO TO 15
GO TO 20
15    NT=NT+1
X(NT)=XZ
20    NTEKEN=MTEKEN
IF ((1.0-XZ).LE.EXTRA) GO TO 30
XZ=XZ+(1.0-XZ*XZ)*GN+EXTRA
GO TO 5
30    CONTINUE
DO 60 I=1,NT
XZ=X(I)
DELTA2=1.0
35    PNM2=1.0
PNM1=XZ
```

```
PNM1AF=1.0
T=0.5+1.5*XZ*XZ
DO 40 K=2,NGAUSS
PNM1XZ=PNM1*XZ
PN=2.0*PNM1XZ-PNM2-(PNM1XZ-PNM2)/K
PNAF=XZ*PNM1AF+K*PNM1
T=T+(K+0.5)*PN*PN
PNM2=PNM1
PNM1=PN
PNM1AF=PNAF
40    CONTINUE
DELTA1=PN/PNAF
XZ=XZ-DELTA1
IF (DELTA1.LT.0.0) DELTA1=-DELTA1
IF ((DELTA1.GE.DELTA2).AND.(DELTA2.LT.1.E-6)) GO TO 50
DELTA2=DELTA1
GO TO 35
50    X(I)=XZ
W(I)=1.0/T
60    CONTINUE
NGHALF=NGAUSS/2
NGP1=NGAUSS+1
NTP1=NT+1
```

```
APB=A+B
BMAG2=(B-A)/2.0
DO 90 I=1,NGHALF
X(NGP1-I)=B-BMAG2*(1.0-X(NTP1-I))
90    W(NGP1-I)=BMAG2*W(NTP1-I)
IF (NGHALF.NE.NT) GO TO 100
GO TO 110
100   X(NT)=APB/2.0
W(NT)=W(1)*BMAG2
110   DO 120 I=1,NGHALF
X(I)=APB-X(NGP1-I)
120   W(I)=W(NGP1-I)
RETURN
END
C     DOUBLE PRECISION FUNCTION  CINT(C, X, R, N)
C     ==========================================
IMPLICIT REAL*8(A-H, O-Z)
DIMENSION C(4, N),X(N)
RR=R
IF(RR.LT. X(1))THEN
C     RR=X(1)
CINT=0.D0
ENDIF
```

```
DO 1 K=2, N
K1=K-1
IF(X(K1). LE. RR. AND. RR. LT. X(K))THEN
DX=RR-X(K1)
CINT=C(1, K1)+C(2, K1)*DX+C(3, K1)/2.D0*DX**2+C(4, K1)/6.D0*DX**3
ENDIF
1 CONTINUE
RETURN
END
DOUBLE PRECISION FUNCTION  CINTT(C, X, R, N)
C  ==========================================
IMPLICIT REAL*8(A-H, O-Z)
DIMENSION C(4, N),X(N)
RR=R
IF(RR. LT. X(1))THEN
C     RR=X(1)
CINT=0.D0
ENDIF
DO 1 K=2, N
K1=K-1
IF(X(K1). LE. RR. AND. RR. LT. X(K))THEN
DX=RR-X(K1)
CINTT=C(3, K1)+C(4, K1)*DX
```

```
ENDIF
1 CONTINUE
RETURN
END
SUBROUTINE CUBS3 (X, FX, C, N)
C       ==========================
IMPLICIT REAL*8(A-H, O-Z)
DIMENSION C(4, N),X(N),FX(N)
C      SEE COMMENTS IN 'CARL DE BOOR:  A PRACTICAL GUIDE TO SPLINES
C                              PAGE 57
C    X-MESH, FX: FUNCTION AT X
C    C=CUBIC COEFFICIENTS
C    F(S)=C(1, I)+H*(C(2, I)+H*(C(3, I)+H*C(4, I)/3.)/2.)
C    WHERE H=S-X(I)
L=N-1
IG=0
IE=0
DO 1 J=1, N
1    C(1, J)=FX(J)
DO 10 M=2, N
C(3, M)=X(M)-X(M-1)
10   C(4, M)=(C(1, M)-C(1, M-1))/C(3, M)
IF(IG-1)11,15,16
```

```
11   IF(N. GT.2)GO TO 12
C(4,1)=1.D0
C(3,1)=1.D0
C(2,1)=2.D0*C(4,2)
GO TO 25
12   C(4,1)=C(3,3)
C(3,1)=C(3,2)+C(3,3)
C(2,1)=((C(3,2)+2.*C(3,1))*C(4,2)*C(3,3)+C(3,2)**2*C(4,3))/C(3,1)
15   C(4,1)=1.D0
C(3,1)=0.D0
GO TO 18
16   C(4,1)=1.D0
C(3,1)=1.D0
C(2,1)=3.D0*C(4,2)-C(3,2)/2.D0*C(2,1)
18   IF(N. EQ.2)GO TO 25
19   DO 20 M=2, L
G=-C(3, M+1)/C(4, M-1)
C(2, M)=G*C(2, M-1)+3.D0*(C(3, M)*C(4, M+1)+C(3, M+1)*C(4, M))
20   C(4, M)=G*C(3, M-1)+2.D0*(C(3, M)+C(3, M+1))
IF(IE-1)21,30,24
21   IF(N. EQ. 3.AND. IG. EQ.0)GO TO 22
G=C(3, N-1)+C(3, N)
C(2, N)=((C(3, N)+2.D0*G)*C(4, N)*C(3, N-1)
```

```
*      +C(3, N)**2*(C(1, N-1)-C(1, N-2))/C(3, N-1))/G
G=-G/C(4, N-1)
C(4, N)=C(3, N-1)
GO TO 29
22   C(2, N)=2.D0*C(4, N)
C(4, N)=1.D0
GO TO 28
24   C(2, N)=3.D0*C(4, N)+C(3, N)/2.D0*C(2, N)
C(4, N)=2.D0
GO TO 28
25   IF(IE-1)26,30,24
26   IF(IG. GT.0)GO TO 22
C(2, N)=C(4, N)
GO TO 30
28   G=-1.D0/C(4, N-1)
29   C(4, N)=G*C(3, N-1)+C(4, N)
C(2, N)=(G*C(2, N-1)+C(2, N))/C(4, N)
30   DO 40 J=L, 1,-1
40   C(2, J)=(C(2, J)-C(3, J)*C(2, J+1))/C(4, J)
DO 50 I=2, N
S=C(3, I)
SS=(C(1, I)-C(1, I-1))/S
SSS=C(2, I-1)+C(2, I)-2.D0*SS
```

```
C(3, I-1)=2.D0*(SS-C(2, I-1)-SSS)/S

50   C(4, I-1)=SSS/S*6.D0/S

RETURN

END
```