Graph and Digraph Embedding Problems

by

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To my parents.

Preface

The research on which this thesis was based was carried out in the Department of Mathematics and Applied Mathematics, University of Natal, Pietermaritzburg, from January 1993 to December 1995, under the supervision of Dr. M.A. Henning.

This thesis represents original work by the author and has not been submitted in any other form to another university. Where use was made of the work of others it has been duly acknowledged.

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Abstract

This thesis is a study of the symmetry of graphs and digraphs by considering certain homogeneous embedding requirements.

Chapter 1 is an introduction to the chapters that follow. In Chapter 2 we present a brief survey of the main results and some new results in framing number theory. In Chapter 3, the notions of frames and framing numbers is adapted to digraphs. A digraph D is *homogeneously embedded* in a digraph H if for each vertex x of D and each vertex y of H, there exists an embedding of D in H as an induced subdigraph with x at y. A digraph F of minimum order in which D can be homogeneously embedded is called a *frame* of D and the order of F is called the *framing number* of D. We show that that every digraph has at least one frame and, consequently, that the framing number of a digraph is a well defined concept. Several results involving the framing number of graphs and digraphs then follow. Analogous problems to those considered for graphs are considered for digraphs.

In Chapter 4, the notions of edge frames and edge framing numbers are studied. A nonempty graph G is said to be *edge homogeneously embedded* in a graph H if for each edge e of G and each edge f of H, there is an edge isomorphism between G and a vertex induced subgraph of H which sends e to f. A graph F of minimum size in which G can be edge homogeneously embedded is called an *edge frame* of G and the size of F is called the *edge framing number* $efr(G)$ of G. We also say that G is

edge framed by F. Several results involving edge frames and edge framing numbers of graphs are presented.

For graphs G_1 and G_2, the framing number $fr(G_1, G_2)$ (edge framing number $efr(G_1, G_2)$) of G_1 and G_2 is defined as the minimum order (size, respectively) of a graph F such that G_i $(i = 1, 2)$ can be homogeneously embedded in F. In Chapter 5 we study edge framing numbers and framing number for pairs of cycles. We also investigate the framing number of pairs of directed cycles.

Contents

Chapter 1

Introduction

Essentially, this thesis is a study of the symmetry of graphs and digraphs by considering certain homogeneous embedding requirements. It was found that for certain graphs, purely group theoretic considerations give an unsatisfactory description of the symmetry of a graph. Furthermore, it was also found that a single embedding requirement alone does *not* suffice to describe graphical symmetry adequately. For example, there are graphs which are highly symmetric relative to their edges and yet lack symmetry relative to their vertices.

Chartrand, Gavlas, and Schultz [2] introduced the framing number of a graph. A graph G is *homogeneously embedded* in a graph H if for every vertex x of G and every vertex y of H, there exists an embedding of G in H as an induced subgraph with x at y. A graph F of minimum order in which G can be homogeneously embedded is called a *frame* of G, and the order of F is called the *framing number* $fr(G)$ of G.

In [2] it is shown that a frame exists for every graph, although a frame need not be unique. Results involving frames and framing numbers of graphs have been presented by, among others Chartrand, Gavlas, and Schultz [2], Chartrand, Henning, Hevia, and Jarrett [3], Gavlas, Henning, and Schultz [6], Goddard, Henning, Oellermann, and Swart [7, 8].

In Chapter 2, we present a brief survey of the main results in framing number theory. In Chapter 3, the notions of frames and framing numbers is adapted to digraphs. A digraph D is *homogeneously embedded* in a digraph H if for each vertex x of D and each vertex y of H, there exists an embedding of D in H as an induced subdigraph with x at y. A digraph F of minimum order in which D can be homogeneously embedded is called a *frame* of D and the order of F is called the *framing number* of D. Analogous problems to those considered for graphs are considered for digraphs. Results involving frames and framing numbers of digraphs have been presented by Henning and Maharaj [10].

In Chapter 4, the notions of edge frames and edge framing numbers are studied. A nonempty graph G is said to be *edge homogeneously embedded* in a graph H if for each edge e of G and each edge f of H, there is an edge isomorphism between G and a vertex induced subgraph of H which sends e to f. A graph F of minimum size in which G can be edge homogeneously embedded is called an *edge frame* of G and the size of F is called the *edge framing number* $efr(G)$ of G. We also say that G is edge framed by F. Results involving edge frames and edge framing numbers of graphs have been presented by Henning [9].

In Chapter 5 we study edge framing numbers and framing numbers for pairs of cycles. We also investigate the framing numbers of pairs of directed cycles.

1.1 Graph theory nomenclature

Throughout we shall use the terminology of [4]. Specifically, $p(G)$ and $q(G)$ denote the number of vertices (order) and edges (size), respectively, of a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v in G, the neighbourhood of v is defined by $N(v) = \{u \in V(G) \,|\, uv \in E(G)\}$. We let $\Delta(G)$ ($\delta(G)$) denote the maximum (respectively, minimum) degree among the vertices of G. Two edges e and f of a graph G are similar (or of the same type) if $\phi(e) = f$ for some edge automorphism ϕ of G. If every two edges of G are similar we say that G is edge-transitive. Similarity is an equivalence relation on the edge set of a graph, and the resulting equivalence classes are referred to as edge orbits.

Given a nonempty graph G, the line graph $L(G)$ of G is defined as that graph whose vertices can be put in a one-to-one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. Let G_1 and G_2 be two graphs with disjoint vertex sets. The *join* $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1)$ and $v \in V(G_2)\}$.

Similarly, for digraphs, $p(D)$ and $q(D)$ denote the number of vertices (order) and arcs (size), respectively, of a digraph with vertex set $V(D)$ and arc set $E(D)$. A digraph D is *symmetric* if whenever (u, v) is an arc of D, then so too is (v, u). A digraph D is *asymmetric* if whenever (u, v) is an arc of D, then (v, u) is not an arc of D. For a vertex v in D, the *out-neighbourhood* and *in-neighbourhood* of v are defined by $N^+(v) = \{u \in V(D) \,|\, (v, u) \in E(D)\}$ and $N^-(v) = \{u \in V(D) \,|\, (u, v) \in E(D)\}$, respectively. The *outdegree* of v is defined as $od\, v = |N^+(v)|$ and the *indegree* of v is $id\, v = |N^-(v)|$. The *degree* $deg\, v$ of v is defined by $deg\, v = od\, v + id\, v$. We let $\Delta_{id}(D)$ $(\delta_{id}(D))$ denote the maximum (respectively, minimum) indegree among the vertices of D. Further, we let $\Delta_{od}(D)$ $(\delta_{od}(D))$ denote the maximum (respectively, minimum) outdegree among the vertices of D. The minimum degree of D is given by $\delta(D) = \min\{deg\ v : v \in V(D)\}$, whereas the maximum degree of D is $\Delta(D) = \max\{deg\ v : v \in V(D)\}$.

For vertex disjoint digraphs G and H, the *lexicographic product* $G[H]$ has vertex set $V(G) \times V(H)$, and a vertex (g, h) is adjacent to a vertex (g', h') in $G[H]$ if and only if either g is adjacent to g' in G or $g = g'$ and h is adjacent to h' in H.

Two vertices u and v of a digraph D are called *similar* (or of the *same type*) if $\phi(u) = v$ for some automorphism ϕ of D. Every two vertices of D are similar if and only if D is vertex-transitive. Similarity is an equivalence relation on the vertex set of a digraph D, and the resulting equivalence classes are called the *orbits* of G.

Chapter 2

The framing number of a graph

2.1 Introduction

In this chapter we present a brief survey of the main results in framing number theory. We also present some new results. Results involving frames and framing numbers of graphs have been presented by, among others, Chartrand, Gavlas, and Schultz [2], Chartrand, Henning, Hevia, and Jarrett [3], Gavlas, Henning, and Schultz [6], Goddard, Henning, Oellermann, and Swart [7, 8], and Henning [9].

2.2 Basic theory

In the first book ever written in graph theory (in 1936) König proved that for every graph G with maximum degree d, there exists a d-regular graph H containing G

as an induced subgraph. For motivational purposes, we present König's technique. Let G be a graph with $\Delta(G) = d$. If G is regular, then we may take $H = G$. Otherwise, let G' be another copy of G and join corresponding vertices whose degrees are less than d, calling the resulting graph G_1. If G_1 is regular, then we may take $H = G_1$. If not, we continue this procedure until arriving at a d-regular graph G_n where $n = \Delta(G) - \delta(G)$. Chartrand, Gavlas, and Schultz [2] observed that the graph H constructed by König has the property that for every vertex v of H, there exists an induced subgraph of H containing v that is isomorphic to G. This observation motivated Chartrand, Gavlas, and Schultz [2] to define the following concept. A graph G is said to be *uniformly embedded* in a graph H if for every vertex v of H, there is an induced subgraph of H containing v that is isomorphic to G. We will deal with an even stronger embedding requirement introduced by Chartrand, Gavlas, and Schultz [2]. A graph G is *homogeneously embedded* in a graph H if for every vertex x of G and every vertex y of H, there exists an embedding of G in H as an induced subgraph with x at y. A graph F of minimum order in which G can be homogeneously embedded is called a *frame* of G, and the order of F is called the *framing number* $fr(G)$ of G. By the following theorem, all of the above notions are applicable to any graph.

Theorem 2 . 1 *(Chartrand et al. [2]) Every graph has a frame.*

However, it is also shown in [2] that a frame of a graph need not be unique.

Theorem 2 . 2 *(Chartrand et al. [2]) For a given graph G, there exists a positive integer m such that for each integer $n \geq m$, there is a graph H of order n in which G can be homogeneously embedded, while for each positive integer $n < m$, no such graph H of order n exists.*

The homogeneous embedding requirement does imply quite a number of inequalities (Chartrand et al. [2]). The first of these is an upper bound of the framing number of a graph in terms of the number of orbits and order of a graph. It is a direct consequence of the proof of Theorem 2.1.

Theorem 2 . 3 *(Chartrand et al. [2]) Let k denote the number of distinct orbits in a graph G. Then*

$$fr(G) \leq (2k-1)|V(G)|.$$

The remaining inequalities have proved to be extremely useful in attacking typical framing number problems.

Lemma 2 . 1 *(Chartrand et al. [2]) If a graph G can be homogeneously embedded in a graph H, then*

$$\Delta(G) \leq \delta(H) \leq \Delta(H) \leq |V(H)| - |V(G)| + \delta(G).$$

Two corollaries follow immediately.

Corollary 2 . 1 *(Chartrand et al. [2]) If F is a frame for a graph G, then*

$$\Delta(G) \leq \delta(F) \leq \Delta(F) \leq |V(F)| - |V(G)| + \delta(G).$$

Corollary 2 . 2 *(Chartrand et al. [2]) For a graph G,*

$$fr(G) \geq |V(G)| + \Delta(G) - \delta(G).$$

The following result of Goddard, Henning, Oellermann, and Swart [7] shows that the diameter of the frame of a connected graph cannot be too large.

Theorem 2 . 4 *(Goddard et al. [7]) If G is a connected graph and H is a frame of G, then $diam\, H \leq diam\, G + 1$.*

We present a slight improvement of this result which is a consequence of the next lemma.

Lemma 2 . 2 *Let G be a connected graph such that $\beta(G) \geq 2$. Then for each positive integer $m \geq fr(G)$, there is a graph H which homogeneously embeds G with the further property that every pair of nonadjacent vertices in H lies on an induced copy of G.*

Proof. Let $m \geq fr(G)$ be a positive integer. From among all graphs of order m which homogeneously embed G, choose one, H say, of *maximum* size. Let a and b denote a pair of nonadjacent vertices in H. Let H_1 denote the graph obtained from

H by joining the vertices a and b. By the maximality property of the graph H, the graph H_1 cannot homogeneously embed G. Thus there is a vertex x of G and a vertex y of H_1 such that there is no embedding of G in H_1 with x at y. Consider an embedding G_1 of G in H with x at y. Clearly we must have $a, b \in V(G_1)$ otherwise G_1 would be an embedding of G in H_1 with x at y which is impossible. Thus H is a graph with the desired property. □

Corollary 2 . 3 *Let G be a connected graph. Then for each positive integer $m \geq fr(G)$, there is a graph H which homogeneously embeds G such that $diam\, H \leq diam\, G$.*

Corollary 2 . 4 *Let G be a connected graph. Then G has a frame F with $diam\, F \leq diam\, G$.*

2.3 Framing ratios of graphs

The framing ratio $frr(G)$ of a graph G is defined to be the ratio $fr(G)/p(G)$ in [2]. Clearly, $frr(G) \geq 1$ for every graph G, and $frr(G) = 1$ if and only if G is vertex transitive. This graphical parameter is a certain measure of the 'symmetry' of a graph, where the closer $frr(G)$ is to 1, the more symmetric G is.

Of course, the framing ratio of every graph is a rational number. The following result shows that many rational numbers are framing ratios.

Theorem 2 . 5 *(Chartrand et al. [2]) For each rational number* $r \in [1, 2)$, *there exists a graph* G *with* $frr(G) = r$.

While it unknown whether the framing ratio of a graph can be arbitrarily large, Goddard, Henning, Oellerman and Swart [7] produced a class of graphs whose framing ratio is at least 2. By a *broom* B_n, $n \geq 5$, we mean a star $K_{1,n-2}$ with one edge subdivided once.

Theorem 2 . 6 *(Goddard et al. [7]) For* $n \geq 7$ *an integer,* $fr(B_n) \geq 2n$.

Corollary 2 . 5 *(Goddard et al. [7]) For* $n \geq 7$ *an integer,* $frr(B_n) \geq 2$.

2.4 The framing number of a graph and its complement

The following result was established by Chartrand et al. [2].

Theorem 2 . 7 *([2]) Let* G *be a graph with frame* F. *Then* $fr(G) = fr(\overline{G})$ *and* $\overline{F}$ *is a frame for* $\overline{G}$.

The next result is a consequence of the proof of Theorem 2.7.

Corollary 2 . 6 *If a graph* G *can be homogeneously embedded in a graph* H, *then* $\overline{G}$ *can be homogeneously embedded in the graph* $\overline{H}$.

2.5 The framing number of a single graph

The framing number for various classes of graphs have been established by, among others Chartrand, Gavlas, and Schultz [2], Chartrand, Henning, Hevia, and Jarrett [3], Gavlas, Henning, and Schultz [6], Goddard, Henning, Oellermann, and Swart [7, 8]. In this section we present a brief summary of these results.

The *lollipop graph* L_n is the unicyclic graph of order n containing exactly one bridge. Gavlas et al. [6] established the framing number $fr(L_n)$ for small n. The following table summarizes their results.

n	4	5	6	7	8
$fr(L_n)$	6	8	8	10	12

Table 2.1:

Gavlas et al. [6] also showed that $fr(L_n) \leq 2n - 4$ for $n \geq 6$.

Goddard et al. [7] determined the framing number of the *wheel* $W_{n+1} = C_n + K_1$ for all integers $n \geq 3$. They showed that $fr(W_4) = 4$, $fr(W_5) = 6$. More generally,

Theorem 2 . 8 *(Goddard et al. [7]) For $n \geq 5$ an integer, $fr(W_{n+1}) = 2n$.*

The next result we present is a generalisation of this theorem.

Theorem 2 . 9 *Let G be a vertex transitive graph of order n. If G is k-regular* *where* $k \leq \lceil \frac{n}{3} \rceil - 1$, *then* $fr(G + K_1) = 2n$.

Proof. Since $G+K_1$ can be homogeneously embedded in the graph $G+G$ of order $2n$, it follows that $fr(G+K_1) \leq 2n$. The desired result would follow once we have shown that there is no graph of order $2n-1$ which homogeneously embeds $G+K_1$. Suppose, to the contrary, that such a graph H exists. Let $F = G_1 + \{w\}$ be an embedding of $G+K_1$ in H where $G_1 \cong G$ and let v be a vertex in G_1. Consider a further embedding F_1 of $G + K_1$ in H with v as the central vertex. Now $F_1 - v$ must have at least one vertex, x say, in common with G_1 otherwise $|V(F_1) \cup V(F)| = 2n > p(H)$. Now $V(F_1) - \{v\}$ contains at most k vertices in common with G_1, possibly the vertex w, and a set S of at least $n-(k+1)$ other vertices. Thus the graph shown in Figure 2.1 is a subgraph of H.

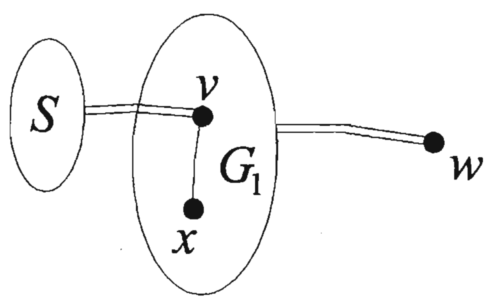


Figure 2.1:

Thus far we have accounted for at least $(n - k - 1) + (n + 1)$ vertices of H. This leaves a set T of at most $2n-1-(n-k-1)-(n+1) = k-1$ vertices. Now consider an embedding F_2 of $G+K_1$ with x as the central vertex. Then F_2 contains at most k

vertices from G_1, at most k vertices from S, possibly w and at least $n-(2k+1)$ other vertices which must come from T. Thus $|T| \geq n-(2k+1)$ whence $k-1 \geq n-(2k+1)$. Hence $k \geq \lceil \frac{n}{3} \rceil$, which is a contradiction.□

Corollary 2 . 7 *Let G be a vertex transitive graph of order n which is k-regular where $k \geq n - \lceil \frac{n}{3} \rceil$. Then $fr(G \cup K_1) = 2n$.*

Proof. Since $\overline{G \cup K_1} \cong \overline{G} + K_1$ and $\overline{G}$ is a $(n-k-1)$-regular vertex transitive graph (with $n-k-1 \leq \lceil \frac{n}{3} \rceil - 1$), it follows from Theorem 2.9 that $fr(\overline{G \cup K_1}) = fr(\overline{G} + K_1) = 2n$. By Theorem 2.7 we know that $fr(\overline{G \cup K_1}) = fr(G \cup K_1)$ so that $fr(G \cup K_1) = 2n$.□

Corollary 2 . 8 *For all integers $n \geq 1$, $fr(K_n \cup K_1) = 2n$.*

2.6 The framing number of more than one graph

Chartrand, Gavlas, and Schultz [2] extended the concept of framing numbers to more than one graph. For graphs G_1 and G_2, the framing number $fr(G_1, G_2)$ of G_1 and G_2 is defined as the minimum order of a graph F such that G_i $(i = 1, 2)$ can be homogeneously embedded in F. The graph F is called a *frame* of G_1 and G_2. Then $fr(G_1, G_2)$ exists and, in fact, $fr(G_1, G_2) \leq fr(G_1 \cup G_2)$.

Theorem 2 . 10 *(Chartrand et al. [2]) For graphs G_1 and G_2, there exists a positive integer m such that for each integer $n \geq m$, there is a graph H of order n in which*

G_1 and G_2 can be homogeneously embedded, while for each positive integer $n < m$, no such graph H of order n exists.

Much work has been done in determining the framing number $fr(S)$ where S is a set of more than one graph. For $S = \{K_{1,3}, P_n\}$ the following table summarizes the results of Gavlas et al. [6].

n	3	4	5	6	7	8
$fr(K_{1,3}, P_n)$	6	8	8	10	10	12

Table 2.2:

Gavlas et al. [6] also investigated the framing number of a claw and cycles. Tables 2.3 and 2.4 summarize these results.

n	3	4	5	6	7
$fr(K_{1,3}, C_n)$	8	6	8	8	10

Table 2.3:

(m, n)	(3,4)	(4,5)	(4,6)	(5,7)	(4,7)
$fr(K_{1,3}, C_m, C_n)$	8	8	8	10	10

Table 2.4:

Gavlas et al. [6]. also showed that $fr(K_{1,3}, C_4, C_5, C_7) = 10$.

The next result is due to Entringer et al. [5].

Theorem 2 . 11 *(Entringer et al. [5]) For integers $m, n \geq 2$,*

$$fr(K_m, \overline{K_n}) = n + m - 2 + \left\lceil 2\sqrt{(m-1)(n-1)} \right\rceil .$$

Chartrand et al. [2] investigated $fr(C_m, C_n)$ for small values of m and n. Their results are summarized in Table 2.5

(m, n)	(3,4)	(3,5)	(3,6)	(4,5)	(4,6)	(5,6)
$fr(C_m, C_n)$	6	7	8	7	8	8

Table 2.5:

Chapter 3

The framing number of a digraph

3.1 Introduction

In this chapter we adapt the concepts of frames and framing numbers to digraphs. A digraph D is *homogeneously embedded* in a digraph H if for each vertex x of D and each vertex y of H, there exists an embedding of D in H as an induced subdigraph with x at y. A digraph F of minimum order in which D can be homogeneously embedded is called a *frame* of D and the order of F is called the *framing number* of D.

Results involving frames and framing numbers of graphs are easily applicable to symmetric digraphs. If D is a symmetric digraph, then let F be a frame of the underlying graph of D. Then the (symmetric) digraph F^* obtained from F by replacing each edge uv of F by the arcs (u, v) and (v, u) is a frame of D. In all that follows, we **restrict our attention to asymmetric digraphs.**

In Section 3.2 it is shown that that every digraph has at least one frame and, consequently, that the framing number of a digraph is a well defined concept. Several results involving the framing number of graphs and digraphs then follow. In Section 3.3 bounds are established for the framing number of a digraph.

The *framing ratio* $frr(D)$ of a digraph D is defined by $frr(D) = fr(D)/|V(D)|$. This graphical parameter, studied in Section 3.4, may be considered as a certain measure of the vertex symmetry of a digraph. It is shown that every rational in the interval $[1, 3)$ is a framing ratio.

In Sections 3.5, 3.6 and 3.7, the framing number is determined for a number of classes of digraphs, including a class of digraphs whose underlying graph is a complete bipartite graph, a class of digraphs whose underlying graph is $C_n + K_1$, and the lexicographic product of a transitive tournament and a vertex transitive digraph.

Finally, in Section 3.8, a relationship between the diameters of the underlying graph of a digraph and its frame is determined. It is shown that every tournament has a frame which is also a tournament.

3.2 Existence of frames for digraphs

In [2] it is shown that every graph has a frame or, equivalently, that $fr(G)$ is defined for every graph G. We state an analogous result for digraphs, the proof of which is along similar lines as that presented in [2].

Theorem 3 . 1 *Every digraph has a frame.*

Proof. Let D be a digraph of order p. It suffices to show that there exists a digraph F in which D can be homogeneously embedded.

To construct such a digraph F, we do the following. Let $S_1, S_2, \ldots, S_k$ be the orbits of D, where $S_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$ for $1 \le i \le k$. Thus $p = \sum_{i=1}^{k} n_i$. Let $D_1, D_2, \ldots, D_{2k-1}$ be $2k-1$ copies of D. For each $i = 1, 2, \ldots, k$, we label the vertex $v_{i,j}$ in D by $v_{i,j}^m$ in D_m ($1 \le m \le 2k-1$). Take the (disjoint) union of the digraphs $D_1, D_2, \ldots, D_{2k-1}$. Then for each i, j and m, where $1 \le i \le k$, $1 \le j \le n_i$ and $1 \le m \le 2k-1$, do the following: Add the arc $(v_{i,j}^m, v)$ for each $v \in N^+(v_{\ell,1}^{m+\ell-i})$ and add the arc $(v, v_{i,j}^m)$ for each $v \in N^-(v_{\ell,1}^{m+\ell-i})$ if $i < \ell$, or add the arc $(v_{i,j}^m, v)$ for each $v \in N^+(v_{\ell,1}^{m+k+\ell-i})$ and add the arc $(v, v_{i,j}^m)$ for each $v \in N^-(v_{\ell,1}^{m+k+\ell-i})$ if $i > \ell$, for every ℓ ($1 \le \ell \le k$), where $m + \ell - i$ and $m + k + \ell - i$ are expressed modulo $2k-1$. This completes the construction of F.

It remains to show that F has the desired properties. It suffices to verify that for each ℓ ($1 \le \ell \le k$) and each vertex y of F, the digraph D can be embedded as an induced subdigraph with $v_{\ell,1}$ at y. Now y is the vertex $v_{i,j}^m$ for some i $(1 \leq i \leq k)$ and j $(1 \leq j \leq n_i)$, and m $(1 \leq m \leq 2k-1)$. If we define

$$U = \begin{cases} V(D_{m+\ell-i}) \cup \{v_{i,j}^m\} - \{v_{\ell,1}^{m+\ell-i}\} & \text{if } i < \ell \\ V(D_m) & \text{if } i = \ell \\ V(D_{m+k+\ell-i}) \cup \{v_{i,j}^m\} - \{v_{\ell,1}^{m+k+\ell-i}\} & \text{if } i > \ell \end{cases}$$

then we see that $H = \langle U \rangle \cong D$. $\square$

According to Theorem 3.1, then, for every digraph D there exists a digraph F in which D can be homogeneously embedded as an induced subdigraph. Hence, $fr(D)$ is defined for every digraph D.

Corollary 3 . 1 *For every digraph D and for every integer $n \geq fr(D)$, there exists a digraph H of order n in which D can be homogeneously embedded.*

Proof. By Theorem 3.1, there exists a frame F (of order $fr(D)$) of D. Let v be a vertex of F. Define F_1 to be the digraph of order $fr(D)+1$ obtained from F by adding a new vertex v_1 to F and inserting the arcs (v_1, w) for each $w \in N^+(v)$ and the arcs (w, v_1) for each $w \in N^-(v)$. Then v and v_1 are similar vertices, and D can be homogeneously embedded in F_1. Proceeding inductively, we see that for every integer $n \geq fr(D)$, there exists a digraph H of order n in which D can be homogeneously embedded. $\square$

Corollary 3.1 actually yields the following result.

Corollary 3 . 2 *For every digraph D, there exists a positive integer m such that for each integer $n \geq m$, there is a digraph H of order n in which D can be homogeneously embedded, while for each positive integer $n < m$, no such digraph H of order n exists.*

Proposition 3 . 1 *Let D be a digraph and let F be a frame for D. Let D' and F' be the digraphs obtained by reversing the directions of the arcs in D and F, respectively. Then $fr(D) = fr(D')$, and F' is a frame for D'.*

Proof. It is evident that F' homogeneously embeds D', so $fr(D') \leq fr(F)$. It remains to show that F' is a frame for D'. Suppose, to the contrary, that H' is a frame for D, where H' has order less than that of F'. Let H be the digraph obtained by reversing the direction of the arcs in H'. Then D can be homogeneously embedded in H, so $fr(D) \leq |V(H)| < |V(F)|$, which contradicts the fact that F is a frame for D. □

3.3 Bounds on the framing number

The construction of the digraph F in the proof of Theorem 3.1 gives an upper bound on the framing number of a digraph.

Corollary 3 . 3 *Let k denote the number of orbits in a digraph D. Then*

$$fr(D) \leq (2k - 1)|V(D)|.$$

Note that Corollary 3.3 implies that a digraph D is vertex transitive if and only if $fr(D) = |V(D)|$. Let D be a digraph and let F be a frame of D. Then it is evident that the underlying graph of D can be homogeneously embedded in the underlying graph of F. This yields the following result.

Proposition 3 . 2 *If D is a digraph and if D' is the underlying graph of D, then* $fr(D) \geq fr(D')$.

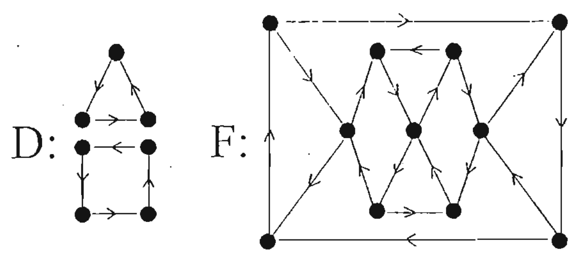


Figure 3.1: A digraph D and its frame F.

For example, consider the digraph D of Figure 3.1 which is the union of two directed cycles. Let D' denote the underlying graph of D so $D' \cong C_3 \cup C_4$. It is shown in [2] that $fr(C_3 \cup C_4) = 11$. According to Proposition 3.2, we know therefore that $fr(D) \geq 11$. However, the digraph F (of order 11) shown in Figure 3.1 has the property that D can be homogeneously embedded in F. Therefore, $fr(D) \leq 11$. Thus $fr(D) = 11$.

Let D be a digraph and let H be a digraph that homogeneously embeds D. Further, let D' and H' be the underlying graphs of D and H, respectively. Then, since D' can be homogeneously embedded in H', Theorem 3.1 yields the following result.

Lemma 3 . 1 *If a digraph D can be homogeneously embedded in a digraph H, then*

$$\Delta(D) \leq \delta(H) \leq \Delta(H) \leq |V(H)| - |V(D)| + \delta(D).$$

The following lemma will be useful in order to present a lower bound on the framing number of a digraph.

Lemma 3 . 2 *If a digraph D can be homogeneously embedded in a digraph H, then*

$$\Delta_{id}(D) \leq \delta_{id}(H) \leq \Delta_{id}(H) \leq |V(H)| - |V(D)| + \delta_{id}(D) \qquad (3.1)$$

and

$$\Delta_{od}(D) \leq \delta_{od}(H) \leq \Delta_{od}(H) \leq |V(H)| - |V(D)| + \delta_{od}(D). \qquad (3.2)$$

Proof. Necessarily, $\delta_{id}(H) \geq \Delta_{id}(D)$ and $\delta_{od}(H) \geq \Delta_{od}(D)$. Let v be a vertex of D with $id\, v = \delta_{id}(D)$. Then v is not adjacent from $|V(D)|-1-\delta_{id}(D)$ other vertices of D. Because D can be homogeneously embedded in H, every vertex of H is not adjacent from at least $|V(D)| - 1 - \delta_{id}(D)$ vertices of H. Consequently, every vertex of H is adjacent from at most $|V(H)|-1-(|V(D)|-1-\delta_{id}(D)) = |V(H)|-|V(D)|+\delta_{id}(D)$ vertices of H. This establishes (3.1). The proof of (3.2) can be obtained directly from (3.1) by reversing the directions of all arcs. □

An immediate consequence of this is the following.

Corollary 3 . 4 *If a digraph D can be homogeneously embedded in a digraph H, then*

$$|V(H)| \geq \max\{|V(D)| + \Delta(D) - \delta(D),\ |V(D)| + \Delta_{id}(D) - \delta_{id}(D),$$
$$|V(D)| + \Delta_{od}(D) - \delta_{od}(D)\}.$$

Corollary 3 . 5 *For a digraph D,*

$$fr(D) \geq \max\{|V(D)| + \Delta(D) - \delta(D),\ |V(D)| + \Delta_{id}(D) - \delta_{id}(D),$$
$$|V(D)| + \Delta_{od}(D) - \delta_{od}(D)\}.$$

Theorem 3 . 2 *If a digraph D can be homogeneously embedded in a digraph H, then*

$$Min(\Delta_{od}(H), \Delta_{id}(H)) \geq Max(\Delta_{od}(D), \Delta_{id}(D)) \tag{3.3}$$

and

$$Min(\delta_{od}(H), \delta_{id}(H)) \geq Max(\delta_{od}(D), \delta_{id}(D)). \tag{3.4}$$

Proof. Since $\Delta_{id}(D) \leq \delta_{id}(H)$ and $\Delta_{od}(D) \leq \delta_{od}(H)$, the following inequalities follow.

$$\Delta_{id}(D)p(H) \leq \textstyle\sum_{v \epsilon V(H)} id_H\, v \leq \Delta_{id}(H)p(H), \qquad \ldots(a)$$

$$\Delta_{od}(D)p(H) \leq \textstyle\sum_{v \epsilon V(H)} od_H\, v \leq \Delta_{od}(H)p(H), \qquad \ldots(b)$$

$$\delta_{id}(D)p(D) \leq \textstyle\sum_{v \epsilon V(D)} id_D\, v \leq \delta_{id}(H)p(D), \qquad \ldots(c)$$

and

$$\delta_{od}(D)p(D) \leq \textstyle\sum_{v \epsilon V(D)} od_D\, v \leq \delta_{od}(H)p(D). \qquad \ldots(d)$$

Necessarily, $\Delta_{od}(H) \geq \Delta_{od}(D)$ and $\Delta_{id}(H) \geq \Delta_{id}(D)$. Because $\sum_{v \epsilon V(H)} id_H\, v = \sum_{v \epsilon V(H)} od_H\, v$, both (a) and (b) imply that $\Delta_{id}(H) \geq \Delta_{od}(D)$ and $\Delta_{od}(H) \geq \Delta_{id}(D)$. This establishes (3.3).

Necessarily, $\delta_{od}(H) \geq \delta_{od}(D)$ and $\delta_{id}(H) \geq \delta_{id}(D)$. Because $\sum_{v \epsilon V(D)} id_D\, v = \sum_{v \epsilon V(D)} od_D\, v$, (c) and (d) imply that $\delta_{id}(H) \geq \delta_{od}(D)$ and $\delta_{od}(H) \geq \delta_{id}(D)$. This establishes (3.4). □

The proof of Theorem 3.2 yields the following results.

Corollary 3 . 6 *If a digraph D can be homogeneously embedded in a digraph H, then*

$$\left\lfloor \frac{q(H)}{p(H)} \right\rfloor \geq max(\Delta_{od}(D), \Delta_{id}(D))$$

and

$$\left\lceil \frac{q(D)}{p(D)} \right\rceil \leq min(\delta_{od}(H), \delta_{id}(H)).$$

Corollary 3 . 7 *If F is a frame for digraph D, then*

$$\left\lfloor \frac{q(F)}{p(F)} \right\rfloor \geq max(\Delta_{od}(D), \Delta_{id}(D))$$

and

$$\left\lceil \frac{q(D)}{p(D)} \right\rceil \leq min(\delta_{od}(F), \delta_{id}(F)).$$

The above result has the following interpretation. The average indegree (or outdegree) of a frame of a digraph D is at least $max(\Delta_{od}(D), \Delta_{id}(D))$. Also, the average indegree (or outdegree) of the digraph D is at most $min(\delta_{od}(F), \delta_{id}(F))$.

Theorem 3 . 3 *If a digraph D can be homogeneously embedded in a digraph H, then*

$$\Delta(H) \geq 2Max(\Delta_{od}(D), \Delta_{id}(D)) \tag{3.5}$$

and

$$\delta(H) \geq 2Max(\delta_{od}(D), \delta_{id}(D)). \tag{3.6}$$

Proof. Since $\delta_{id}(H) \geq \Delta_{id}(D)$, we have

$$\begin{aligned} \Delta(H)p(H) &\geq \textstyle\sum_{v \in V(H)} deg_H\, v \\ &= \textstyle\sum_{v \in V(H)} (id_H\, v + od_H\, v) \\ &= 2 \textstyle\sum_{v \in V(H)} id_H\, v \\ &\geq 2\Delta_{id}(D)p(H), \end{aligned}$$

whence $\Delta(H) \geq 2\Delta_{id}(D)$. Similarly, since $\delta_{od}(H) \geq \Delta_{od}(D)$, it can be shown that $\Delta(H) \geq 2\Delta_{od}(H)$. This establishes (3.5).

Since $\delta(H) \geq \Delta(D)$, we have

$$\begin{aligned} \delta(H)p(D) &\geq \textstyle\sum_{v \in V(D)} deg_D\, v \\ &= \textstyle\sum_{v \in V(D)} (id_D\, v + od_D\, v) \\ &= 2 \textstyle\sum_{v \in V(D)} id_D\, v \\ &\geq 2\delta_{id}(D)p(D), \end{aligned}$$

whence $\delta(H) \geq 2\delta_{id}(D)$. Similarly, it can be shown that $\delta(H) \geq 2\delta_{od}(D)$. This establishes (3.6). □

If a digraph D can be homogeneously embedded in a digraph H, then $fr(D) \geq \Delta(H) + 1$. Hence an immediate corollary of Theorem 3.2 now follows.

Corollary 3 . 8 *For a digraph D,*

$$fr(D) \geq 2\,max(\Delta_{od}(D), \Delta_{id}(D)) + 1.$$

For example, the digraph D of Figure 3.2 can be homogeneously embedded in the digraph F of order 5 (also shown in Figure 3.2) so that $fr(D) \leq 5$. However, $\Delta_{id}(D) = \Delta_{od}(D) = 2$. Thus, by Corollary 3.8, $fr(D) \geq 5$. Consequently, $fr(D) = 5$.

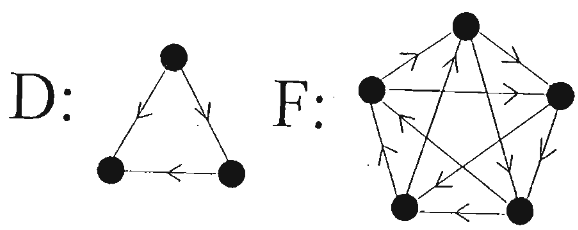


Figure 3.2: A digraph D and its frame F.

The next result includes Corollary 3.5 as a special case.

Lemma 3 . 3 *If a digraph D can be homogeneously embedded in a digraph H, then*

$$|V(H)| \geq |V(D)| + max\{\Delta(D) - \delta(D),\ max(\Delta_{od}(D), \Delta_{id}(D)) - min(\delta_{od}(D), \delta_{id}(D))\}.$$

Proof. By Corollary 3.5, we know that $|V(H)| \geq |V(D)| + \Delta(D) - \delta(D) \cdots (*)$. From Lemma 3.2, we deduce that $min(\Delta_{id}(H), \Delta_{od}(H)) \leq min\{|V(H)| - |V(D)| + \delta_{id}(D),$

$|V(H)|-|V(D)|+\delta_{od}(D)\} = |V(H)|-|V(D)|+min(\delta_{id}(D),\ \delta_{od}(D))$. It follows then by Theorem 3.2 that $max(\Delta_{id}(D), \Delta_{od}(D)) \leq |V(H)|-|V(D)|+min(\delta_{id}(D),\ \delta_{od}(D))$ or, equivalently, $|V(H)| \geq |V(D)| + max(\Delta_{id}(D), \Delta_{od}(D)) - min(\delta_{id}(D),\ \delta_{od}(D))$. This, together with inequality $(*)$, yields the desired result. □

Corollary 3 . 9 *For a digraph D,*

$$fr(D) \geq |V(D)| + max\{\Delta(D) - \delta(D),\ max(\Delta_{id}(D), \Delta_{od}(D)) - min(\delta_{id}(D), \delta_{od}(D))\}.$$

In fact the lower bound given in Lemma 3.3 can be further improved. Suppose that a digraph D can be homogeneously embedded in a digraph H. As an immediate consequence of Lemma 3.1, Lemma 3.2 and Theorem 3.2, we have the following result.

Corollary 3 . 10 *If a digraph D can be homogeneously embedded in a digraph H, then*

$$|V(H)| \geq |V(D)| + 2\,max(\Delta_{id}(D), \Delta_{od}(D)) - \delta(D).$$

Corollary 3 . 11 *For a digraph D,*

$$fr(D) \geq |V(D)| + 2\,max(\Delta_{id}(D), \Delta_{od}(D)) - \delta(D).$$

We claim that $\delta(D) \geq 2\,min(\delta_{id}(D),\ \delta_{od}(D))$ and $\Delta(D) \leq 2\,max(\Delta_{id}(D),\ \Delta_{od}(D))$. Choose a vertex $v \in V(D)$ such that $deg_D v = \delta(D)$. We have $\delta(D) = deg_D v = id_D v + od_D v \geq \delta_{id}(D) + \delta_{od}(D) \geq 2\,min(\delta_{id}(D),\ \delta_{od}(D))$. Similarly, it can be shown that $\Delta(D) \leq 2\,max(\Delta_{id}(D),\ \Delta_{od}(D))$. With these inequalities at hand it is easily checked that the lower bound presented in Corollary 3.10 is an improvement of that in Lemma 3.3.

3.4 Framing ratios of digraphs

For a digraph D, we define the *framing ratio* $frr(D)$ of D by

$$frr(D) = \frac{fr(D)}{|V(D)|}.$$

Certainly, $frr(D) \geq 1$ for every digraph D, and $frr(D) = 1$ if and only if D is vertex-transitive. The framing ratio of a digraph D produces a certain measure of the symmetry of D, where the closer $frr(D)$ is to 1, the more "symmetric" D is. For the digraph D of Figure 3.1, $frr(D) = 11/7$ while for the digraph D of Figure 3.2, $frr(D) = 5/3$.

Of course, the framing ratio of every digraph is a rational number. We show that many rational numbers are framing ratios. For the purpose of doing this, we define a digraph $\vec{K}_{p_0,p_1,p_2}$ as follows. Consider a complete 3-partite graph K_{p_0,p_1,p_2} having partite sets V_0, V_1, V_2, where $|V_i| = p_i$ for $i = 0, 1, 2$. For $i = 0, 1, 2$, replace each edge uv of D where $u \in V_i$ and $v \in V_{i+1}$ with the arc (u, v), where addition is taken modulo 3. We denote the resulting digraph by $\vec{K}_{p_0,p_1,p_2}$.

Theorem 3 . 4 *For positive integers $\ell \geq m \geq n$,*

$$fr(\vec{K}_{\ell,m,n}) = 3\ell.$$

Proof. Let $D \cong \vec{K}_{\ell,m,n}$. Since D can be homogeneously embedded in $\vec{K}_{\ell,\ell,\ell}$, it follows that $fr(D) \leq 3\ell$. We show that $fr(D) \geq 3\ell$. Let F be a frame for D. By Theorem 3.3, we know that $\delta(F) \geq 2\,max(\Delta_{id}(D), \Delta_{od}(D)) = 2\ell$. Let v be a vertex of D that belongs to the partite set of cardinality ℓ. Then v is adjacent to or from at least $\delta(F) \geq 2\ell$ other vertices in F. These vertices, together with the ℓ vertices that belong to the partite set of D that contains v, account for at least 3ℓ (distinct) vertices. Hence $fr(D) = |V(F)| \geq 3\ell$, producing the desired result. □

Theorem 3 . 5 *For each rational number $r \in [1,3)$, there exists a digraph D with $frr(D) = r$.*

Proof. Let $r \in [1,3)$ be a rational number. Then we may write $r = 2+\frac{a}{b}$, where a and b are integers with $b > 0$ and $-b \leq a < b$. Consider the digraph $D \cong \vec{K}_{4b+2a,b-a,b-a}$. By Theorem 3.2, $fr(D) = 3(4b+2a)$. Since the order of D is $6b$,

$$frr(D) = \frac{3(4b+2a)}{6b} = 2 + \frac{a}{b} = r. \,\square$$

By Corollary 3.3, if D is a digraph with k orbits, then $frr(D) \leq 2k-1$. Although this may suggest that $frr(D)$ can be arbitrarily large, we do not know whether this is the case. In fact, we do not know whether there even exists a digraph D with $frr(D) \geq 3$. On the other hand, a digraph D having a large number of orbits may have a framing ratio that is arbitrarily close to 1. For example, if D is a directed path on n vertices, then D has n orbits and is framed by a directed cycle on $n+1$ vertices.

So $frr(D) = 1 + \frac{1}{n}$. Thus it is an open question as to whether framing ratios can be arbitrarily large.

3.5 The framing number of a class of oriented complete bipartite graphs

In [2] it is shown that the framing number of the complete bipartite graph $K_{m,n}$ is $fr(K_{m,n}) = 2\max(m,n)$. Suppose that $K_{m,n}$ has partite sets V_1, V_2 where $|V_1| = m$ and $|V_2| = n$. Replace each edge uv of $K_{m,n}$ where $u \in V_1$ and $v \in V_2$ with the arc (u,v). The resulting digraph is denoted by $\vec{K}_{m,n}$. We show that $fr(\vec{K}_{m,n}) = 3\max(m,n)$. For the purpose of doing this, let $\vec{K}_{p_0,p_1,p_2}$ be the digraph defined in the paragraph immediately preceeding Theorem 3.4.

First, we establish the framing number of the digraph $\vec{K}_{1,n}$.

Proposition 3 . 3 *For any positive integer n, $fr(\vec{K}_{1,n}) = 3n$.*

Proof. Since $\vec{K}_{1,n}$ can be homogeneously embedded in the vertex transitive digraph $\vec{K}_{n,n,n}$, it follows that $fr(\vec{K}_{1,n}) \leq 3n$. However, since $\Delta_{od}(\vec{K}_{1,n}) = n$, by Corollary 3.11 it follows that $fr(\vec{K}_{1,n}) \geq 3n$. Consequently, $fr(\vec{K}_{1,n}) = 3n$. □

Proposition 3 . 4 *For positive integers m and n,*

$$fr(\vec{K}_{m,n}) = 3\,max\{m,n\}.$$

Proof. By Proposition 3.1 we may assume, without loss in generality that $m \leq n$. Since $\vec{K}_{m,n}$ can be homogeneously embedded in $\vec{K}_{n,n,n}$ it follows that $fr(\vec{K}_{m,n}) \leq 3n$. However, because $\vec{K}_{1,n} \prec \vec{K}_{m,n}$ it follows from Proposition 3.1 that $3n \leq fr(\vec{K}_{m,n})$. Consequently, $fr(\vec{K}_{m,n}) = 3n$ as required. □

3.6 The framing number of a diwheel

A directed cycle of order n in which every vertex has indegree and outdegree equal to 1, will be denoted by $\vec{C}_n$. If $\vec{C}_n$ is given by $v_1, (v_1, v_2), v_2, (v_2, v_3), v_3, \ldots, v_n, (v_n, v_1)$, v_1, then we will simply write $v_1, v_2, v_3, \ldots, v_n, v_1$. By a diwheel we mean the digraph $\vec{W}_{n+1}$ obtained from the disjoint union of $\vec{C}_n$ and K_1 by joining each vertex of $\vec{C}_n$ *to* the vertex of K_1 (which we shall call the *centre* or *central vertex* of $\vec{C}_n$). By a *rim vertex* of $\vec{W}_{n+1}$ we mean a vertex distinct from the centre of $\vec{W}_{n+1}$. In [7] the framing number of the wheel W_{n+1}, the underlying graph of $\vec{W}_{n+1}$, is established. In this section we determine the framing number of the diwheel.

The diwheel $\vec{W}_4$ can be homogeneously embedded in the digraph D of order 7 in Figure 3.3 so that $fr(\vec{W}_4) \leq 7$. However, by Corollary 3.11, $fr(\vec{W}_4) \geq 4+2\times3-3 = 7$. Thus $fr(\vec{W}_4) = 7$. The following result establishes the framing number of the diwheel $\vec{W}_{n+1}$ for all integers $n \geq 4$.

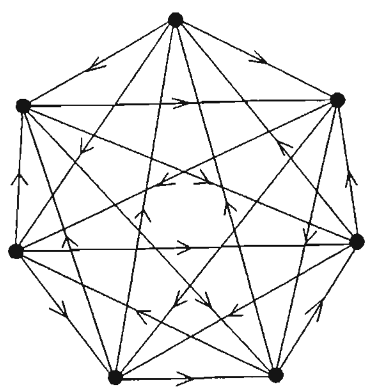


Figure 3.3: A frame for $\vec{W}_4$

Theorem 3 . 6 *For $n \geq 4$ an integer, $fr(\vec{W}_{n+1}) = 3n$.*

Proof. Since $\vec{W}_{n+1}$ can be homogeneously embedded in the vertex transitive digraph $\vec{C}_3[\vec{C}_n]$ it follows that $fr(\vec{W}_{n+1}) \leq 3n$. Employing Theorem 3.2, we show that $fr(\vec{W}_{n+1}) = 3n$ by verifying that there exists no digraph of order $3n - 1$ in which $\vec{W}_{n+1}$ can be homogeneously embedded. Suppose, to the contrary, that such digraphs do exist. From among all such digraphs, choose a digraph H of minimum size.

Before proceeding further, we introduce some notation. For each vertex x of H, let W_x denote an induced subdigraph of H that is isomorphic to $\vec{W}_{n+1}$ and that contains x as the central vertex. The set of rim vertices of W_x is denoted by $R(W_x)$. We will require a number of preliminary results.

Claim 3 . 1 $\Delta(H) \leq 2n + 1$ *and* $n \leq \Delta_{od}(H) \leq n + 1$.

Proof. By Lemma 3.1, $\Delta(H) \leq |V(H)| - |V(\vec{W}_{n+1})| + \delta(\vec{W}_{n+1}) = (3n - 1) - (n +$

$1) + 3 = 2n + 1$ and, by Theorem 3.2, $\Delta_{od}(H) \geq \Delta_{id}(\vec{W}_{n+1}) = n$. To show that $\Delta_{od}(H) \leq n + 1$, let v be a vertex in H with $od_H\, v = \Delta_{od}(H)$. Then, by Lemma 3.2, $id_H\, v \geq \delta_{id}(H) \geq \Delta_{id}(\vec{W}_{n+1}) = n$. Thus $2n + 1 \geq \Delta(H) \geq deg_H\, v = id_H\, v + od_H\, v \geq n + \Delta_{od}(H)$ whence $\Delta_{od}(H) \leq n + 1$. □

Claim 3 . 2 $\delta_{id}(H) = n$.

Proof. Let $v \in V(H)$ such that $od_H\, v = \Delta_{od}(H)$. Then, by Claim 3.1, $2n + 1 \geq \Delta(H) \geq deg_H\, v = id_H\, v + od_H\, v \geq id_H\, v + n$ so that $id_H\, v \leq n + 1$ whence $\delta_{id}(H) \leq n + 1$. By Lemma 3.2, $\delta_{id}(H) \geq \Delta_{id}(\vec{W}_{n+1}) = n$. Suppose $\delta_{id}(H) = n + 1$. Then $(n + 1)p(H) \leq \sum_{v \in V(H)} id_H\, v = \sum_{v \in V(H)} od_H\, v \leq (n + 1)p(H)$. Since all of these inequalities must be equalities, we conclude that $id_H\, v = od_H\, v = n + 1$ for all $v \in V(H)$. But this implies that $\Delta(H) = 2n + 2$, which contradicts Claim 3.1. Thus $\delta_{id}(H) = n$ as required. □

Claim 3 . 3 $\delta_{od}(H) \leq 4$.

Proof. By Claim 3.2, we may choose $b_1 \in V(H)$ such that $id\, b_1 = n$. Consider an embedding H_1 of $\vec{W}_{n+1}$ in H with b_1 as a central vertex. Since $id\, b_1 = n$, a further embedding H_2 of $\vec{W}_{n+1}$ in H with b_1 as a rim vertex yields the subdigraph of H shown in Figure 3.4 where $W_{b_1} \cong H_1 = \langle b_1, c_1, c_2, \ldots, c_n \rangle$ and $W_b \cong H_2 = \langle b, b_1, b_2, \ldots, b_n \rangle$.

Now the vertex c_1 is adjacent from c_2, b_3 and at least $n - 2$ other vertices which are not in H_1 nor H_2. These $n - 2$ vertices, together with the vertices of H_1 and H_2

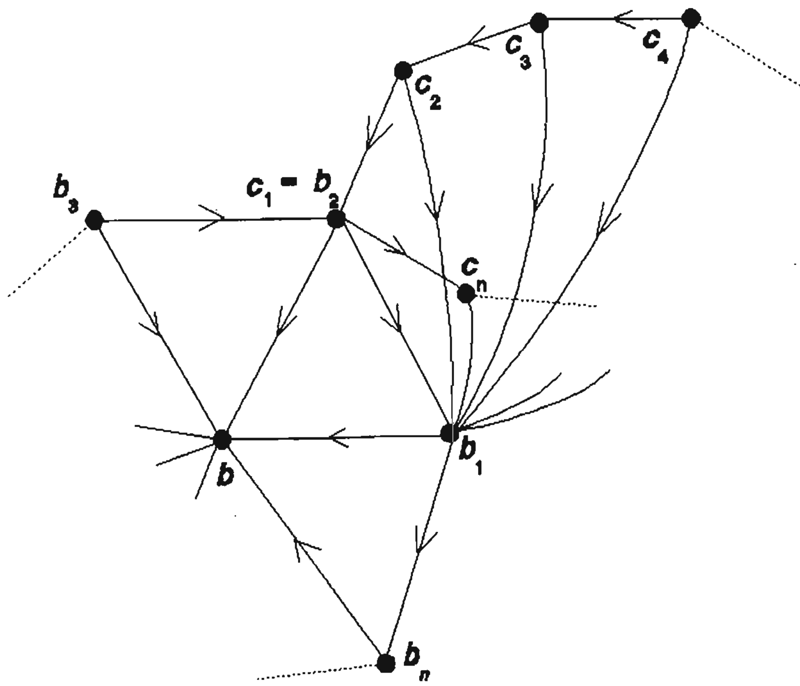


Figure 3.4: A subdigraph of H

account for $3n-2 = p(H)-1$ vertices of H. Thus c_1 is adjacent to b_1, c_n, b and at most one other vertex so that $od\, c_1 \leq 4$. Thus $\delta_{od}(H) \leq 4$. □

Before proceeding to the next claim, we introduce the following notation. For k a nonnegative integer, we let $s_k = |\{v \in V(H) : od\, v = k\}|$ and $t_k = |\{v \in V(H) : id\, v = k\}|$. Note that

$$s_{\delta_{od}(H)} + s_{\delta_{od}(H)+1} + \cdots + s_{n+1} = t_n + t_{n+1} + \cdots + t_{\Delta_{id}(H)} = 3n-1. \tag{3.7}$$

Claim 3 . 4 $\Delta_{od}(H) = n+1$.

Proof. First we prove the claim for $n \geq 5$. Using the above notation we have

$$\sum_{v \in V(H)} id\, v = nt_n + (n+1)t_{n+1} + \cdots + \Delta_{id}(H) t_{\Delta_{id}(H)}$$

and

$$\sum_{v \in V(H)} od\, v = \delta_{od}(H) s_{\delta_{od}(H)} + \cdots + n s_n + (n+1) s_{n+1}.$$

Thus

$$n t_n + (n+1) t_{n+1} + \cdots + \Delta_{id}(H) t_{\Delta_{id}(H)} = \delta_{od}(H) s_{\delta_{od}(H)} + \cdots + n s_n + (n+1) s_{n+1}. \quad (3.8)$$

Now

$$\begin{aligned}
[\text{Left hand side of } (3.8)] &\geq n t_n + (n+1)(t_{n+1} + \cdots + t_{\Delta_{id}(H)}) \\
&= n t_n + (n+1)(3n - 1 - t_n) \\
&= -t_n + (n+1)(3n-1).
\end{aligned}$$

Since $\delta_{od}(H) \leq 4 < n$,

$$\begin{aligned}
[\text{Right hand side of } (3.8)] &< (n+1) s_{n+1} + n(\delta_{od}(H) s_{\delta_{od}(H)} + \cdots + n s_n) \\
&= (n+1) s_{n+1} + n(3n - 1 - s_{n+1}) \\
&= s_{n+1} + n(3n-1).
\end{aligned}$$

Combining the above inequalities we have $s_{n+1} + t_n > 3n - 1$. Since $t_n \leq 3n - 1$, it follows that $s_{n+1} \neq 0$, that is, there is a vertex with outdegree $n+1$. Thus $\Delta_{od}(H) = n + 1$ for $n \geq 5$.

Now suppose that $n = 4$. By Claim 3.1, we know that $4 \leq \Delta_{od}(H) \leq 5$. Suppose that $\Delta_{id}(H) = 4$. By Claim 3.2, $\delta_{id}(H) = 4$, so $4\,p(H) \leq \sum_{v \in V(H)} id\, v = \sum_{v \in V(H)} od\, v \leq 4\,p(H)$. Since all these inequalities must be equalities, we conclude that $id\, v = od\, v = 4$ for all vertices v of H. Thus H is a 4-regular digraph of order

11. We remark that every vertex of H is not adjacent with exactly two other vertices; and because every vertex v of H has indegree 4, we have $R(W_v) = N^-(v)$.

Let d be a vertex of H and let F_1 be an induced subdigraph of H which is isomorphic to $\vec{W}_5$ and that contains d as a rim vertex. Let e be the vertex other than d which is common to W_d and F_1. Since H is 4-regular, e must be adjacent from two vertices not in $W_d \cup F_1$ and to another vertex not in $W_d \cup F_1$. Suppose, then, the vertices of H are labelled as in Figure 3.5, where $W_d = \langle a, b, c, d, e \rangle$ and $W_j \cong F_1 = \langle d, e, f, i, j \rangle$.

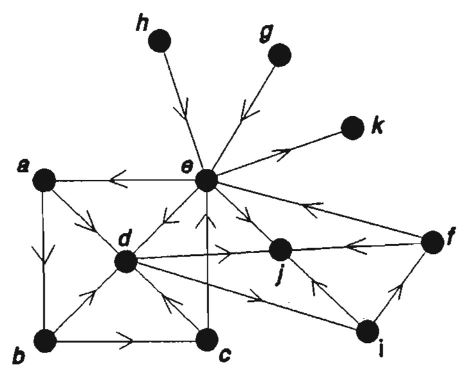


Figure 3.5: A subdigraph of H

Note that $W_e = \langle e, c, f, g, h \rangle$. Next we consider W_a. Clearly $d, b, c \notin R(W_a)$. Thus W_a must consist of the vertices e, exactly one of h, g and f, exactly one of j and k and some fourth vertex, say z, not adjacent to e. Since $b \notin R(W_a)$, $z = i$. Since $i, e \in R(W_a)$, it follows that $j \notin V(W_a)$. Thus $R(W_a)$ consists of i, e, k and one vertex from h, g and f.

Suppose that $h \in R(W_a)$. Note that h must be adjacent from i and h is not adjacent with k. Since $\langle c,f,g,h\rangle = \langle R(W_e)\rangle \cong \vec{C}_4$, h is not adjacent with one of c,g and f. Hence the two vertices of H not adjacent with h belong to the set $\{c,g,f,k\}$. Thus h must be adjacent with each of d and j. Since $id\, d = id\, j = 4$, it follows that d and j are both adjacent to h. Thus $i,d,j \in R(W_h)$. But this is clearly impossible. A similar contradiction arises if we assume that $g \in R(W_a)$. Thus $f \in R(W_a)$ and $R(W_a) = \{i,e,f,k\}$. Clearly in $R(W_a)$, and hence in H, f is not adjacent with k. Since f is also not adjacent with d, it follows that f is adjacent with every vertex of H other than d and k. In particular, f is adjacent with g, h and c. But this is impossible as $\langle c,f,g,h\rangle = \langle R(W_e)\rangle \cong \vec{C}_4$.

Thus we cannot embed $\vec{W}_5$ in H with a as the central vertex. This contradicts the fact that H homogeneously embeds $\vec{W}_5$. Hence we must conclude that $\Delta_{od}(H) = 5$. This completes the proof of Claim 3.4. $\Box$

Choose $x \in V(H)$ such that $od_H\, x = n+1$ (then $id_H\, x = n$). Consider an embedding F_1 (F_2) of $\vec{W}_{n+1}$ in H with x as a central vertex (rim vertex, respectively). Then the digraph D of Figure 3.6 is a subdigraph of H where $W_x \cong F_1 = \langle x, x_1, x_2, \ldots, x_n\rangle$ and $W_f \cong F_2 = \langle f, f_1, f_2, \ldots, f_n\rangle$.

Since $|V(F_1) \cup V(F_2)| = 2n$, there is a set S of $n-1$ vertices of H not in F_1 nor F_2. Since $od_H\, x = n+1$, x is adjacent to every vertex in S. Consider W_c, where c is the vertex shown in Figure 3.6. Clearly $R(W_c) \subseteq S \cup \{x_2, f_3\}$. Since $|R(W_c)| = n$, at least one of x_2 and f_3 belongs to $R(W_c)$.

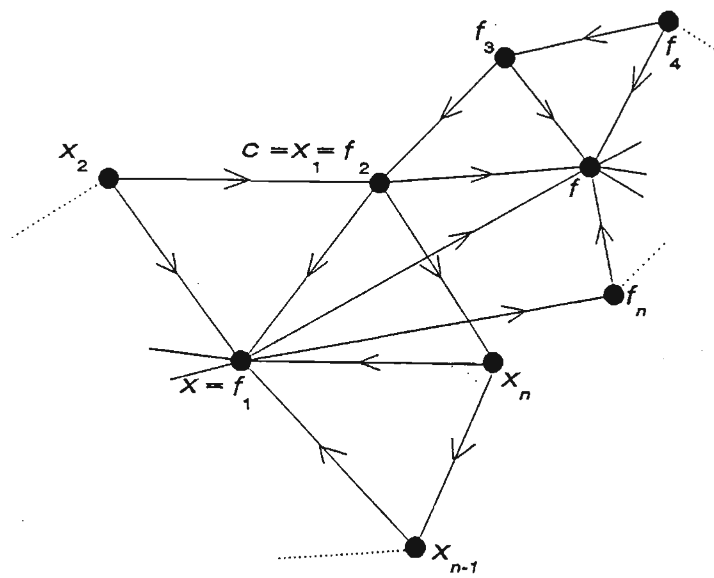


Figure 3.6: A subdigraph of H

Claim 3 . 5 *The vertex f_3 is non-adjacent to at least one vertex in $S \cap R(W_c)$.*

Proof. If $f_3 \in R(W_c)$, then, since $R(W_c)$ contains at least $n - 2$ (≥ 2) vertices of S and f_3 is adjacent to only one vertex of $R(W_c)$, the result is immediate. Assume, then, that $f_3 \notin R(W_c)$, for otherwise there is nothing left to prove. Then $R(W_c) = S \cup \{x_2\}$. Let s be the vertex of S adjacent to x_2. Since x_3 (s) is the only vertex of F_1 (S, respectively) adjacent to x_2, it follows that for $n \geq 5$ we have $R(W_{x_2}) \subset \{s, x_3, f, f_3, f_4, \ldots, f_n\}$. If $f \in R(W_{x_2})$, then at most one vertex from $f_3, f_4, \ldots, f_n$ belongs to $R(W_{x_2})$, implying that $|R(W_{x_2})| \leq 4 < 5$ which is impossible. Thus $f \notin R(W_{x_2})$; consequently, $R(W_{x_2}) = \{s, x_3, f_3, f_4, \ldots, f_n\}$. In particular, we note that $(f_3, x_2) \in E(H)$, so f_3 is adjacent to at least three vertices not in S, namely to x_2, c, and f. Hence, since $\Delta_{od}(H) = n + 1$ and $|S| = n - 1$, the result now follows for $n \geq 5$.

If $n = 4$, then H has order 11 and $R(W_{x_2}) \subset \{s, x_3, f, f_3, f_4\}$. If $f_3 \in R(W_{x_2})$, then the result follows as above. Assume, then, that $f_3 \notin R(W_{x_2})$. Then $R(W_{x_2}) = \{s, x_3, f, f_4\}$. In particular, we observe that $(f_4, x_2) \in E(H)$. If f_3 is non-adjacent to some vertex of S, then the result follows since $S \subset R(W_c)$. On the other hand, if f_3 is adjacent to all three vertices of S, then $od_H\, f_3 = 5$ $(= n+1)$. However, since x and f_3 are not adjacent and $id_H\, f_3 \geq 4$, it follows that $R(W_{f_3}) = \{x_2, x_3, x_4, f_4\}$. But this would imply that $(x_2, f_4) \in E(H)$, which produces a contradiction. This completes the proof of Claim 3.5. □

By Claim 3.5, there exists a vertex b in $S \cap R(W_c)$ that is not adjacent from f_3. Since x is adjacent to every vertex of S, $(x, b) \in E(H)$.

Claim 3 . 6 *There exists no embedding of $\vec{W}_{n+1}$ in H with b as a central vertex and x as a rim vertex.*

Proof. Assume, to the contrary, that we can embed $\vec{W}_{n+1}$ in H with b as a central vertex and x as a rim vertex. We determine $R(W_b)$. Since $b \in R(W_c)$, we know that $(b, c) \in E(H)$, so $c \notin R(W_b)$. Further, $(f_3, b) \notin E(H)$, so $f_3 \notin R(W_b)$. Since $x \in R(W_b)$, it follows that exactly one vertex from $\{x_2, x_3, \ldots, x_n\}$ belongs to $R(W_b)$. Moreover, exactly one vertex from $N^+(x) = S \cup \{f, f_n\}$ belongs to $R(W_b)$. If $f \in R(W_b)$, then no vertex from $\{f_4, \ldots, f_n\}$ belongs to $R(W_b)$, implying that $R(W_b)$ consists of only three vertices, which is impossible (since $n \geq 4$). Hence $f \notin R(W_b)$. Thus $R(W_b)$ consists of x, exactly one vertex from $\{x_2, x_3, \ldots, x_n\}$, exactly one vertex from $S \cup \{f_n\}$, and $n-3$ vertices from $\{f_4, \dots, f_{n-1}\}$. However, this is impossible as $|\{f_4, \dots, f_{n-1}\}| = n-4 < n-3$. This completes the proof of Claim 3.6. $\square$

Claim 3 . 7 *The only possible embeddings of $\vec{W}_{n+1}$ in H with both b and x as rim vertices have f as the central vertex, and as rim vertices b, x, exactly one vertex from $\{x_2, x_3, \dots, x_n\}$, and the $n-3$ vertices in $\{f_3, \dots, f_{n-1}\}$.*

Proof. Consider an embedding of $\vec{W}_{n+1}$ in H_1 with both b and x as rim vertices. Let W_y be such an embedding with central vertex y. Since x, $b \in R(W_y)$, the vertex c cannot belong to W_y. Since $x \in R(W_y)$, exactly one vertex from $\{x_2, x_3, \dots, x_n\}$ belongs to $R(W_y)$, and y must be one of the vertices in $S \cup \{f, f_n\}$.

If $y = f$, then, since x is adjacent to the vertex b on $R(W_y)$, no vertex in $S \cup \{f_n\}$ belongs to $R(W_y)$. It follows that $R(W_y)$ consists of b, x, exactly one vertex from $\{x_2, x_3, \dots, x_n\}$, and the $n-3$ vertices in $\{f_3, \dots, f_{n-1}\}$. Hence, we may assume in what follows that $y \neq f$, for otherwise there is nothing left to prove.

If $y = f_n$, then no vertex from $S \cup \{f, f_3, \dots, f_{n-1}\}$ other than b belongs to $R(W_y)$, implying that $R(W_y)$ consists of only three vertices, which is impossible. Hence $y \neq f_n$. This in turn implies that $f, f_n \notin R(W_y)$, since x is adjacent to the vertex b on $R(W_y)$.

If $y \in S$, then $R(W_y)$ consists of b, x, exactly one vertex x_ℓ (say) from $\{x_2, x_3, \dots, x_n\}$, and the $n-3$ vertices in $\{f_3, \dots, f_{n-1}\}$. Since $(f_3, b) \notin E(H)$, it follows that no vertex of $\{f_3, \dots, f_{n-1}\}$ is adjacent to b. Furthermore, we note that y is adjacent from each of b and f_3. Since $b \in R(W_c)$, it follows that one of y and f_3 does not belong to $R(W_c)$.

If $y \notin R(W_c)$, then $R(W_c) = (S - \{y\}) \cup \{x_2, f_3\}$. Since $(y, b) \notin E(H)$, it follows that there is therefore exactly one vertex in $S \cup \{x_2, f_3\}$ that is adjacent to b. This vertex, together with the vertices in $\{x, x_3, x_4, \ldots, x_n\}$, are therefore the only possible vertices adjacent to b. Since $id_H\, b \geq n$, it follows that $R(W_b)$ consists of one vertex from $S \cup \{x_2\}$ and the $n-1$ vertices from $\{x, x_3, x_4, \ldots, x_n\}$. But then $x \in R(W_b)$, which contradicts the result of Claim 3.6.

If $f_3 \notin R(W_c)$, then $R(W_c) = S \cup \{x_2\}$. Hence b is the only vertex in $S \cup \{x_2\}$ that is adjacent to y, so $x_\ell \neq x_2$. That is to say, $x_\ell \in \{x_3, \ldots, x_n\}$. Furthermore, since $(f_3, b) \notin E(H)$, there is exactly one vertex z in $S \cup \{x_2, f_3\}$ that is adjacent to b. By Claim 3.6, $x \notin R(W_b)$. Hence, $R(W_b) \subseteq \{z\} \cup \{f\} \cup (\{x_3, \ldots, x_n\} - \{x_\ell\})$, so $|R(W_b)| \leq n-1$, which is impossible. Hence $y \notin S$. This completes the proof of Claim 3.7. □

Claim 3 . 8 *For each vertex of H, there is an embedding of $\vec{W}_{n+1}$ in H with that vertex as a central or rim vertex that does not contain the arc (x, b).*

Proof. In view of Claims 3.6 and 3.7, the only vertices in doubt are f as a central vertex in some embedding of $\vec{W}_{n+1}$ in H, and the vertices b, x, $x_i\,(2 \leq i \leq n)$ and $f_j\,(3 \leq j \leq n-1)$ as rim vertices in some embedding of $\vec{W}_{n+1}$ in H. Since $W_f \cong F_2$, and $b \notin V(F_2)$, there is an embedding of $\vec{W}_{n+1}$ in H with f as a central vertex and

$f_j\,(1 \leq j \leq n)$ as a rim vertex that does not contain the arc (x, b). (Recall that $x = f_1$.) Futhermore, since $W_x \cong F_1$, and $b \notin V(F_1)$, there is an embedding of $\vec{W}_{n+1}$ in H with $x_i\,(2 \leq i \leq n)$ as a rim vertex that does not contain the arc (x, b). Finally, since $b \in R(W_c)$, and $x \notin V(W_c)$, there is an embedding of $\vec{W}_{n+1}$ in H with b as a rim vertex that does not contain the arc (x, b). □

As an immediate consequence of Claim 3.8, we have that the digraph $H - (x, b)$ obtained from H by removing the arc (x, b) homogeneously embeds $\vec{W}_{n+1}$. This, however, contradicts the minimality property of H. We deduce, therefore, that there is no digraph of order $3n - 1$ in which $\vec{W}_{n+1}$ can be homogeneously embedded. This completes the proof of Theorem 3.6. □

3.7 The framing number of a transitive tournament

In this section we determine the framing number of transitive tournaments. The following result will be useful (see [4]).

Theorem 3 . 7 *(Chartrand, Lesniak [4]) For every positive integer n, there is exactly one transitive tournament of order n.*

In fact, we will show that transitive tournaments have unique frames. For the purpose of doing this, we define two digraphs. Let n be a positive integer. Let

T_n be the transitive tournament defined by $V(T_n) = \{u_1, u_2, \ldots, u_n\}$ and $E(T_n) = \{(u_i, u_j) \mid 1 \leq i < j \leq n\}$. By Theorem 3.7, T_n is, up to isomorphism, the only transitive tournament of order n. Note that $od\,u_i = n - i$ and $id\,u_i = i - 1$ for $i = 1, \ldots, n$. Next, we define a digraph D_n with $V(D_n) = \{v_0, v_1, \ldots, v_{2n-2}\}$, where each vertex v_i $(0 \leq i \leq 2n-2)$ is adjacent to each of the vertices $v_{i+1}, v_{i+2}, \ldots, v_{n+i-1}$, where all subscripts are expressed modulo $2n - 1$. Then D_n is an $(n-1)$-regular digraph of order $2n - 1$. Furthermore, D_n is easily seen to be vertex transitive. Notice that $T_n \cong \langle\{v_0, v_2, \ldots, v_{n-1}\}\rangle \prec D_n$ so that D_n homogeneously embeds T_n.

Theorem 3 . 8 *Let T be a transitive tournament of order n and let K be a vertex transitive digraph. Then $fr(T[K]) = (2n-1)p(K)$ and the digraph $D_n[K]$ of order $(2n-1)p(K)$ is the unique frame of the digraph $T[K]$.*

Proof. By Theorem 3.7, we know that $T \cong T_n$. Thus we show that $fr(T_n[K]) = (2n-1)p(K)$ and that $T_n[K]$ is uniquely framed by $D_n[K]$. Let $D \cong T_n[K]$. Since K is vertex transitive, it is k-regular for some integer $k \geq 0$. Let H be a frame for D. Since D can be homogeneously embedded in the digraph $D_n[K]$, it follows that $|V(H)| \leq (2n-1)p(K)$. Before proceeding further, we prove three claims.

Claim 3 . 9 $\Delta_{od}(D) = \Delta_{id}(D) = k + (n-1)p(K)$, *and* $\Delta(D) = \delta(D) = 2k + (n-1)p(K)$.

Proof. A copy of D is illustrated in Figure 3.7 where $W_i \cong K$. For $i = 1, \ldots, n$, each vertex w_i of W_i is adjacent to every vertex of W_j for all j such that $n \geq j > i$, so $od\, w_i = k + (n-i)p(K)$ and $id\, w_i = k + (i-1)p(K)$. Thus each vertex of W_1 has outdegree $k + (n-1)p(K)$, and this is clearly the maximum outdegree among the vertices of D. Furthermore, each vertex of W_n has indegree $k + (n-1)p(K)$, and this is the maximum indegree among the vertices of D. Moreover, $deg\, w_i = id\, w_i + od\, w_i = 2k + (n-1)p(K)$. □

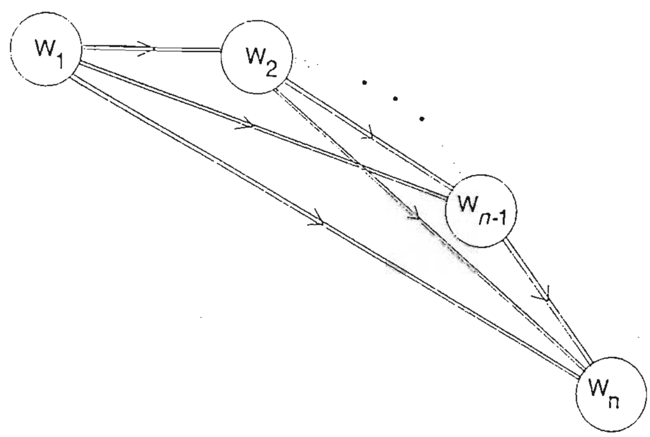


Figure 3.7: The digraph $D \cong T_n[K]$.

Claim 3 . 10 $fr(D) = |V(H)| = (2n-1)p(K)$ *and* $\Delta(H) = 2k + 2(n-1)p(K)$.

Proof. By Theorem 3.3, we know that $\Delta(H) \geq 2max(\Delta_{od}(D), \Delta_{id}(D)) = 2k + 2(n-1)p(K)$. Hence, by Lemma 3.1 and Claim 3.9, it follows that $(2n-1)p(K) \geq |V(H)| \geq |V(D)| + \Delta(H) - \delta(D) \geq (2n-1)p(K)$. Consequently, $fr(D) = |V(H)| = |V(D)| + \Delta(H) - \delta(D) = (2n-1)p(K)$ and $\Delta(H) = 2k + 2(n-1)p(K)$. □

Claim 3 . 11 *H is $(k+(n-1)p(K))$-regular.*

Proof. By Lemma 3.2 and Claim 3.9, we know that $\delta_{id}(H) \geq \Delta_{id}(D) = k+(n-1)p(K)$ and $\delta_{od}(H) \geq \Delta_{od}(D) = k+(n-1)p(K)$. Let v be an arbitrary vertex of H. Then, by Claim 3.10, $2k+2(n-1)p(K) = \Delta(H) \geq deg_H\, v = id_H\, v + od_H\, v \geq \delta_{id}(H) + \delta_{od}(H) \geq 2k+2(n-1)p(K)$. Since these inequalities must be equalities, we deduce that $id_H\, v = od_H\, v = k+(n-1)p(K)$. □

Now let w (z) be a vertex in D with $od_D\, w = \Delta_{od}(D)$ (respectively, $id_D\, z = \Delta_{id}(D)$). For any vertex x of H, let D_x^+ (D_x^-) denote an embedding of D in H as an induced subdigraph with the vertex w (z, respectively) at x. By Claims 8 and 10, it follows that in H

$$N^+[x] \subseteq V(D_x^+) \text{ and } N^-[x] \subseteq V(D_x^-). \quad \cdots(*)$$

Let $u \in V(H)$ and consider an embedding $D_u^+ = \langle U_0, U_1, \ldots, U_{n-1}\rangle$ of D in H, where each $\langle U_i\rangle$ is isomorphic to K, each vertex of U_i $(0 \leq i < n-1)$ is adjacent to to every vertex U_j for all j such that $n-1 \geq j > i$. Now let v be a vertex in U_{n-1} and consider an embedding $D_v^+ = \langle V_{n-1}, V_n, \ldots, V_{2n-2}\rangle$ of D in H, where each $\langle V_i\rangle$ is isomorphic to K, each vertex of V_i $(n-1 \leq i < 2n-2)$ is adjacent to every vertex V_j for all j such that $2n-2 \geq j > i$. Since $N^+[v] \subseteq V(D_v^+)$, it follows that $v \in V_{n-1}$.

Claim 3 . 12 $U_i \cap V_j = \emptyset$ *for* $0 \le i \le n-2$ *and* $n-1 \le j \le 2n-2$.

Proof. Since v is adjacent from all of the $(n-1)p(K)$ vertices of $\bigcup_{i=0}^{n-2} U_i$, it follows that $U_i \cap V_j = \emptyset$ for $0 \le i \le n-2$ and $n \le j \le 2n-2$. It remains for us to show that $U_i \cap V_{n-1} = \emptyset$ for $0 \le i \le n-2$. Suppose, to the contrary, that there is a vertex $x \in U_i \cap V_{n-1}$ for some i $(0 \le i \le n-2)$. Then in H, x is adjacent to k vertices of U_i and to each vertex of $(\bigcup_{j=i+1}^{n-1} U_j) \cup (\bigcup_{j=n}^{2n-2} V_j)$. Hence,

$$\begin{aligned}
|N^+(x)| &\ge k + |(\textstyle\bigcup_{j=i+1}^{n-1} U_j) \cup (\bigcup_{j=n}^{2n-2} V_j)| \\
&\ge k + |(\textstyle\bigcup_{j=i+1}^{n-2} U_j) \cup (\bigcup_{j=n}^{2n-2} V_j)| \\
&= k + (n-2-i)p(K) + (n-1)p(K) \\
&\ge k + (n-1)p(K).
\end{aligned}$$

However, by Claim 3.11, $k+(n-1)p(K) = od_H\, x = |N^+(x)|$, so the above inequalities must be equalities. In particular, this implies that $U_{n-1} \subseteq \bigcup_{j=n}^{2n-2} V_j$, which produces a contradiction since $v \in U_{n-1} \cap V_{n-1}$. Thus $U_i \cap V_{n-1} = \emptyset$ for $0 \le i \le n-2$. □

Claim 3 . 13 $V_{n-1} = U_{n-1}$.

Proof. We have

$$\begin{aligned}
|V(H)| &\ge |(\textstyle\bigcup_{j=0}^{n-1} U_j) \cup (\bigcup_{j=n-1}^{2n-2} V_j)| \\
&\ge |(\textstyle\bigcup_{j=0}^{n-2} U_j) \cup (\bigcup_{j=n-1}^{2n-2} V_j)| \\
&= (2n-1)p(K). \qquad\qquad \text{(by Claim 3.12)}
\end{aligned}$$

However, $(2n-1)p(K) = |V(H)|$, so that the above inequalities must be equalities. Consequently, $V(H) = (\cup_{j=0}^{n-2} U_j) \cup (\cup_{j=n-1}^{2n-2} V_j)$. Hence, $U_{n-1} \subseteq \cup_{j=n-1}^{2n-2} V_j$. Suppose that there is a vertex $x \in U_{n-1} \cap V_j$ for some j with $n \leq i \leq 2n-2$. Then in H, x is adjacent from k vertices of V_j and from each vertex of $(\cup_{j=0}^{n-2} U_j) \cup V_{n-1}$. Hence it follows from Claim 3.12 that x is adjacent from at least $k + np(K)$ vertices, which contradicts the result of Claim 3.11. Thus $U_{n-1} \cap V_j = \emptyset$ for $n \leq j \leq 2n-2$, implying that $U_{n-1} \subseteq V_{n-1}$. Since $|V_{n-1}| = n = |U_{n-1}|$, we must have $V_{n-1} = U_{n-1}$. □

By Claims 3.12 and 3.13, we observe that $U_i \cap V_j = \emptyset$ for $0 \leq i \leq n-1$ and $n \leq j \leq 2n-2$. For notational convenience, we set $V_j = U_j$ for $j = n, n+1, \ldots, 2n-2$. It follows then from the proof of Claim 3.13 that the digraph H has vertex set $V(H) = \cup_{j=0}^{2n-2} U_j$.

Claim 3 . 14 $H \cong D_n[W]$.

Proof. We know that each vertex of U_i $(0 \leq i < n-1)$ is adjacent to every vertex U_j for all j such that $n-1 \geq j > i$, and each vertex of U_i $(n-1 \leq i < 2n-2)$ is adjacent to every vertex U_j for all j such that $2n-2 \geq j > i$. Since H is $(k+(n-1)p(K))$-regular, it suffices for us to show that each vertex of U_i $(0 \leq i \leq 2n-2)$ is adjacent to every vertex of U_j for $j = i+1, i+2, \ldots, i+n-1$, where all subscripts are reduced modulo $2n-1$.

Let $x \in U_{2n-2}$. Then x is adjacent from each vertex of $\cup_{i=n-1}^{2n-3} U_i$, so $N^+[x] \subseteq (\cup_{i=0}^{n-2} U_i) \cup U_{2n-2}$. Since x is adjacent to exactly k vertices of U_{2n-2}, it follows that $od_H\, x = |N^+(x)| \leq (n-1)p(K)+k$. However, by Claim 3.11, $od_H\, x = (n-1)p(K)+k$.

Consequently, x must be adjacent to all of the $(n-1)p(K)$ vertices of $\cup_{i=0}^{n-2} U_i$.

Consider now a vertex y in U_{n-2}. Then y is adjacent from each vertex of $(\cup_{i=0}^{n-3} U_i) \cup U_{2n-2}$, so $N^+[y] \subseteq (\cup_{i=n-2}^{2n-3} U_i)$. Since y is adjacent to k vertices of U_{n-2}, it follows that $od_H\, y = |N^+[y]| \le (n-1)p(K)+k$. However, by Claim 3.11, $od_H\, u = (n-1)p(K)+k$. Consequently, y must be adjacent to all of the $(n-1)p(K)$ vertices of $\cup_{i=n-1}^{2n-3} U_i$.

Continuing in this way (we consider next a vertex in U_{2n-3}, and then a vertex in U_{n-3}, and so on), we may show that each vertex of U_i $(0 \le i \le 2n-2)$ is adjacent to every vertex of U_j for $j = i+1, i+2, \ldots, i+n-1$, where all subscripts are reduced modulo $2n-1$. This completes the proof of the claim and of Theorem 3.8. □

Corollary 3 . 12 *The transitive tournament T of order n is uniquely framed by the digraph D_n of order $2n-1$ so that $fr(T) = 2n-1$.*

It was noted in [10] that the framing ratio is a certain measure of symmetry. From the score sequence of the transitive tournament T of order n, we deduce that T has exactly n orbits, each consisting of a single vertex. In view of this, one would think of transitive tournaments as highly unsymmetric and hence expect them to have high framing ratios for large n. However, by Theorem 3.8, we have $fr(T) = 2 - \frac{1}{n}$. This is surprising since the digraph $\vec{K}_{m,n}$, for example, has just two orbits (irrespective of the values of m and n) and yet has framing ratios arbitrarily close to 3. In [10] it is shown that the digraph $\vec{K}_{p,q,r}$, which has just three orbits, can have framing ratios arbitrarily close to 3 for suitable values of p, q and r. Again, this is surprising as one would tend to think that $\vec{K}_{p,q,r}$ is a more *symmetric* digraph than a transitive tournament. Perhaps this can be explained by the transitivity of T which induces a certain symmetry to T and so causes the unexpected low framing ratio. Although a digraph with exactly one orbit, being vertex transitive, is highly symmetric, we must deduce that the symmetry of a digraph does not depend solely on the number of orbits. Other properties, such as the general orientation also seem to have an effect on the symmetry.

3.8 The diameter of a frame

By Theorem 2.5, the diameter of a frame of a connected graph cannot be too large. In this section we present a corresponding result for digraphs We show that the diameter of the *underlying* graph of a frame of a digraph G cannot be too large.

Theorem 3 . 9 *Let G be a connected digraph with frame H. Let G' and H' be the underlying graphs of G and H, respectively. Then $diam\, H' \leq diam\, G' + 1$.*

Proof. Set $d = diam\, G'$. Suppose $diam\, H' \geq d + 2$. Let v be a vertex of H' whose eccentricity (in H') is $D = diam\, H'$. Let V_i be the set of vertices at distance i from v in H' for $1 \leq i \leq D$. Let $u \in V_D$. Delete the vertex v from H and for each $w \in V_1$ such that $(v, w) \in E(H)$ (respectively, $(w, v) \in E(H)$), add a new arc (u, w) (respectively, (w, u)) to H. Denote the resulting digraph by H_1. Let $x \in V(G)$ and $y \in V(H_1)$.

Consider an embedding G_1 of G in H with x at y. If G_1 contains v, then replace v with u and observe that this new subdigraph of H is still induced since G' contains no vertices of V_{d+1}. If G_1 does not contain v, then G_1 is still an induced subdigraph of H_1 since G_1 cannot contain vertices from both V_1 and V_D. Thus H_1 homogeneously embeds G. Since $p(H_1) < p(H)$, this contradicts the fact that H frames G. □

Although it is not known whether the bounds in the above theorems can be attained, we do have a partial improvement of the above result. As pointed out in Section 3.1, all digraphs referred to are asymmetric digraphs.

Theorem 3 . 10 *For every connected digraph G, and for each integer $n \geq fr(G)$, there is a digraph H of order n in which G can be homogeneously embedded satisfying diam $H' \leq$ diam G' where G' and H' are the underlying graphs of G and H, respectively.*

Proof. By Theorem 2.2, we know that there exists a digraph of order n in which G can be homogeneously embedded. Among all such digraphs, let H be one of *maximal* size. If H is a tournament, then the result is immediate. Assume, then, that H is not a tournament, for otherwise there is nothing left to prove. Let u and v be nonadjacent vertices in H, and consider the digraph H_1 obtained from H by joining u to v. By the maximality property of H, the digraph G cannot be homogeneously embedded in H_1. Thus for some vertex x of G and some vertex y of H_1, there is no homogeneous embedding of G in H_1 with x at y. However, since G can be homogeneously embedded in H, there is an homogeneous embedding G_1 of G in H with x at y. Let G_1' and H' denote the underlying graphs of G_1 and H, respectively. If at most one of u and v belongs to G_1, then G_1 would be a homogeneous embedding of G in H_1 with x at y which would produce a contradiction. Hence $u, v \in V(G_1)$. It follows that in H' we have $d(u, v) \leq diam\, G_1' = diam\, G'$. Since u and v are arbitrary nonadjacent vertices in H, we conclude that $diam\, H' \leq diam\, G'$. □

Corollary 3 . 13 *Every connected digraph G has a frame whose underlying graph has diameter at most that of the underlying graph of G.*

An immediate consequence of Corollary 3.13 now follows.

Corollary 3 . 14 *Every tournament has a frame which is also a tournament.*

While it is always possible to find a frame F for a connected digraph G such that $diam\, F' \leq diam\, G'$ where G' and F' are the underlying graphs of G and F, respectively, $diam\, F'$ can be an arbitrarily amount less than $diam\, G'$. For example, the directed cycle $\vec{C}_{n+1}$ is a frame for the directed path $\vec{P}_n$ of length n and $diam\, C_{n+1} = \lfloor \frac{n+1}{2} \rfloor$ while $diam\, P_n = n - 1$.

Chapter 4

The edge framing number of a graph

4.1 Introduction

A nonempty graph G is said to be *edge homogeneously embedded* in a graph H if for each edge e of G and each edge f of H, there is an edge isomorphism between G and a vertex induced subgraph of H which sends e to f. A graph F of minimum size in which G can be edge homogeneously embedded is called an *edge frame* of G and the size of F is called the edge framing number $efr(G)$ of G. We also say that G is edge framed by F. It is shown in Section 4.2 that every graph has at least one edge frame and, consequently, that the edge framing number of a graph is a well-defined concept. In this chapter we restrict ourselves to graphs with no isolated vertices. This will not affect the generality of any of the results presented.

It is natural to ask whether the notions of edge homogeneous embedding and the usual homogeneous embedding requirement are related. In fact, as the following examples illustrate, neither of the embedding requirements directly implies the other.

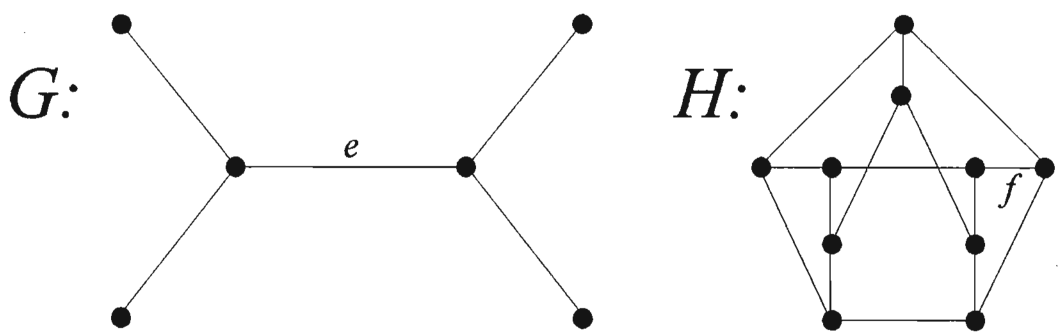


Figure 4.1:

While the graph G of Figure 4.1 can be homogeneously embedded in the graph H, G cannot be edge homogeneously embedded in H; for example, there is no embedding of G in H with the edge e at f.

The next example illustrates strikingly that edge homogeneous embedding does not directly imply homogeneous embedding in general. The complete bipartite graph $K_{1,n}$ can be edge homogeneously embedded in itself while it is obvious that $K_{1,n}$ does not homogeneously embed itself.

Although the two embedding requirements do not directly imply each other, it will be shown in Section 4.3 that they are related in a natural way through line graphs.

The *edge framing ratio* $frr(D)$ of a nonempty graph G is defined by $efrr(D) = efr(G)/|E(G)|$. This graphical parameter may be considered as a certain measure of the edge symmetry of a graph. In Section 4.4 this parameter is introduced. It is shown that every rational in the interval $[1, 3)$ is an edge framing ratio. In Section 4.5, it is shown that every nonempty connected graph G has an edge frame with diameter at most $diam\, G + 1$. Finally, in Section 4.6 the edge framing number is defined for more than one graph and the framing number is determined for pairs of cycles. Furthermore, we determine $efr(K_{1,m}, C_n)$ for all integers $m \geq 3$ and $n \geq 4$.

4.2 Existence of edge frames

Any automorphism ϕ of a nonempty graph G gives rise to an edge automorphism of G in a natural way: we define $\phi(ab) = \phi(a)\phi(b)$ for all edges ab of G. It is precisely this property of an automorphism which we use to prove that every nonempty graph has an edge frame.

Theorem 4 . 1 *Every nonempty graph has an edge frame.*

Proof. Let G be a nonempty graph. It suffices to show that there exists a graph F in which G can be edge homogeneously embedded.

Let $S_1, S_2, \ldots, S_k$ be the edge-orbits of G, where $S_i = \{e_{i,1}, e_{i,2}, \ldots, e_{i,n_i}\}$ for $1 \leq i \leq k$. Thus $q(G) = \sum_{i=1}^{k} n_i$. Set $r = max_{1 \leq i \leq k}|S_i|$. To construct F we begin with

$2k(k-1)r+1$ copies of G, denoted $G_1, G_2, ..., G_{2k(k-1)r+1}$. For each i $(1 \le i \le k)$ and for each j $(1 \le j \le n_i)$, the edge $e_{i,j}$ in G is labelled $e_{i,j}^m$ in G_m $(1 \le m \le 2k(k-1)r+1)$. Furthermore, we denote the end-vertices of the edge $e_{i,j}^m$ by $a_{i,j}^m$ and $b_{i,j}^m$ so that $e_{i,j}^m = a_{i,j}^m b_{i,j}^m$.

The vertex set of F is $\bigcup_{m=1}^{2k(k-1)r+1} V(G_m)$. Additional edges are now added as follows. Consider each edge $e_{i,j}^m = a_{i,j}^m b_{i,j}^m$ in G_m. Consider, then, also the edge $e_{\ell,1}^{\gamma}$ for $(1 \le \ell \le k)$ where

$$\gamma = \begin{cases} m + r(i-1)(k-1) + r(k+\ell-i-1) + j & \text{if } \ell < i \\ m + r(i-1)(k-1) + r(\ell-i-1) + j & \text{if } \ell > i \end{cases}$$

(where γ is expressed modulo $2k(k-1)r+1$).

Join $a_{i,j}^m$ ($b_{i,j}^m$, respectively) to each neighbour of $a_{\ell,1}^{\gamma}$ ($b_{\ell,1}^{\gamma}$, respectively) *except* to $b_{\ell,1}^{\gamma}$ ($a_{\ell,1}^{\gamma}$, respectively). Also, join $a_{\ell,1}^{\gamma}$ ($b_{\ell,1}^{\gamma}$, respectively) to each neighbour of $a_{i,j}^m$ ($b_{i,j}^m$, respectively) *except* to $b_{i,j}^m$ ($a_{i,j}^m$, respectively). Observe that the edges $a_{\ell,1}^{\gamma}$ and $a_{i,j}^m$ are similar in H. Furthermore, each of the newly adjoined edges is similar to an edge in G_m or G_{γ}. This completes the construction.

It remains to show that F has the desired properties. It suffices to verify that for each ℓ $(1 \le \ell \le k)$ and each edge e of F, the graph G can be edge-embedded as a vertex induced subgraph of F with $e_{\ell,1}$ at e. Now, by the construction of F, each edge of F not in $\bigcup_{m=1}^{2k(k-1)r+1} G_m$ is similar to an edge in $\bigcup_{m=1}^{2k(k-1)r+1} G_m$. Thus we may assume that e is $a_{i,j}^m b_{i,j}^m$ for some i, j and m where $1 \le i \le k, 1 \le j \le n_i$ and

$1 \leq m \leq 2k(k-1)r+1$. We define

$$U = \begin{cases} V(G_\beta) \cup \{a^m_{i,j}, b^m_{i,j}\} - \{a^\beta_{\ell,1}, b^\beta_{\ell,1}\} & \text{if } i \neq \ell \\ V(G_m) & \text{if } i > \ell \end{cases}$$

where

$$\beta = \begin{cases} m + r(i-1)(k-1) + r(k+\ell-i-1) + j & \text{if } i > \ell \\ m + r(i-1)(k-1) + r(\ell-i-1) + j & \text{if } i < \ell \end{cases}$$

(β is expressed modulo $2k(k-1)r+1$). Then $\langle U \rangle \cong G$. This completes the proof.

□

The next two results are interesting consequences of the proof of Theorem 4.1.

Corollary 4 . 1 *Let G be a nonempty graph. Then there is a graph H such that for each edge ab of G and each edge cd of H, there is a (vertex) embedding ϕ of G as an induced subgraph of H such that $\phi(a)\phi(b) = cd$.*

Corollary 4 . 2 *Let G be a nontrivial graph which is not complete. Then there exists a graph H such that for every pair of nonadjacent vertices a, b in G and for every pair of nonadjacent vertices c, d in H, there is an isomorphism ϕ from G onto H as an induced subgraph of H such that $\{\phi(a), \phi(b)\} = \{c, d\}$.*

Proof. Since G is not a complete graph, the complement $\bar{G}$ of G is not empty. By Corollary 4.1, there is a graph F such that that for each edge ab of $\bar{G}$ and each edge cd of F, there is a (vertex) embedding ϕ of $\bar{G}$ as an induced subgraph of F such that $\phi(a)\phi(b) = cd$. Then the graph $H \cong \bar{F}$ is a graph with the desired property. □

Corollary 4 . 3 *Let G be a nonempty graph and let v be a vertex of G. Then for each integer $m \geq 0$ there is a graph of size $efr(G) + m\, deg\, v$ which edge homogeneously embeds G.*

Proof. Let F be an edge frame for G. Form a new graph F' from F by adding a set S of m new vertices to F and joining each vertex of S with each neighbour of v. Then F' is a graph of size $efr(G) + m\, deg\, v$ which edge homogeneously embeds G. □

4.3 Lower bounds on the edge framing number

In this section we establish some lower bounds on the edge framing number.

We first show that if a nonempty graph G can be edge homogeneously embedded in a graph H, then the line graph $L(G)$ can be homogeneously embedded in the line graph $L(H)$. The following result due to Whitney [12] will be useful.

Theorem 4 . 2 *(Whitney [12]) Let ϕ be an edge isomorphism from a connected graph G to a connected graph H where G is different from the graphs G_i ($i = 1, 2, 3, 4, 5$) shown in Figure 4.2. Then ϕ is induced by an isomorphism from G to H so that $G \cong H$.*

We present a slight improvement of the above result which will be useful for our purposes.

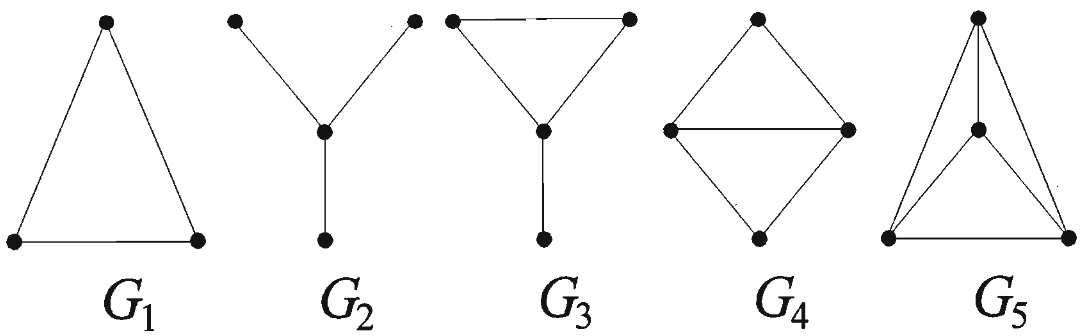


Figure 4.2:

Theorem 4 . 3 *Let G and H be connected edge isomorphic graphs where G is different from C_3 and $K_{1,3}$. Then G and H are isomorphic.*

Proof. By Theorem 4.2, we know the result to be true if G is different from the graphs G_3 ,G_4 and G_5 shown in Figure 4.2. Assume then that G is isomorphic to G_i for some $i = 3, 4, 5$. Since $q(G_i) = i + 1$ for $i = 3, 4, 5$, it follows that G_i is not edge isomorphic to G_j for $3 \leq i < j \leq 5$. Thus H is different from the graphs G_j $(j \neq i)$. Let $\phi : E(G) \longrightarrow E(H)$ be an edge isomorphism between G and H. Suppose that G and H are not isomorphic. Then ϕ^{-1} is an edge isomorphism from a graph different from the graphs G_i $(i = 1, 2, 3, 4, 5)$ shown in Figure 4.2 onto G. By Theorem 4.2 it follows that G and H are isomorphic. As this is contrary to hypothesis we must conclude that G and H are isomorphic. This completes the proof. □

Corollary 4 . 4 *Let G and H be edge isomorphic graphs where the components of G are different from C_3 and $K_{1,3}$. Then G and H are isomorphic.*

We will also require the following result due to Whitney [12].

Theorem 4 . 4 *(Whitney [12]) Let G and H be non-trivial connected graphs. Then $L(G) \cong L(H)$ if and only if $G \cong H$ or one of G and H is the graph C_3 and the other is $K_{1,3}$.*

Theorem 4 . 5 *Let G be a nonempty graph which is different from C_3 and $K_{1,3}$. If G can be edge homogeneously embedded in a graph H then the line graph $L(G)$ of G can be homogeneously embedded in the line graph $L(H)$ of H. Consequently*

$$efr(G) \geq fr(L(G)).$$

Proof. Let $x \in V(L(G))$ and $y \in V(L(H))$ and suppose that x and y correspond to edges e_x and e_y of G and H respectively. Since G can be edge homogeneously embedded in H, there is an edge-embedding G' of G as an induced subgraph of H with e_x at e_y. By Corollary 4.4, G and G' are isomorphic. Consequently, by Theorem 4.4, the line graphs $L(G)$ and $L(G')$ are isomorphic. Since $L(G')$ is an induced subgraph of $L(H)$, it follows that $L(G')$ is an embedding of $L(G)$ in $L(H)$ with x at y. Thus $L(G)$ can be homogeneously embedded in $L(H)$. Let F be an edge frame for G. Then, since $L(F)$ homogeneously embeds $L(G)$, we have $efr(G) = q(F) = p(L(F)) \geq fr(L(G))$ and the desired inequality follows.□

Since an edge symmetric graph gives rise to a vertex symmetric line graph, it follows that $efr(G) = fr(L(G))$ whenever G is edge symmetric. For the graph G

of Figure 4.4, it will be shown that $efr(G) = 12$ while in [8] it was established that $fr(L(G)) = 6$. Thus there exist graphs for which $efr(G) > fr(L(G))$.

Corollary 4 . 5 *Let G be a nonempty graph which is different from C_3 and $K_{1,3}$. Then there exists a graph H such that the line graph $L(G)$ of G can be homogeneously embedded in the line graph $L(H)$.*

Before proceeding, we digress slightly to show that for a large class of graphs, the edge homogeneous embedding requirement is stronger than the homogeneous embedding requirement in a sense which will become clear in what follows. We consider the following problem : given a pair of graphs G and H, when can G be homogeneously embedded in H? We show that this problem can be reduced to a problem of edge homogeneous embedding for a large class of graphs. The following result due to Beineke [1] will be useful.

Theorem 4 . 6 *(Beineke [1]) A graph H is a line graph if and only if none of the graphs G_i $(1 \le i \le 9)$ of Figure 4.3 is an induced subgraph of H.*

Let P denote the class of graphs G different from C_3 and with the property that none of the graphs of Figure 4.3 is an induced subgraph of G. By Theorem 4.6, each graph in P is a line graph so that for each $G \in P$, there exists a graph G' such that $G \cong L(G')$. We denote such a graph G' by $L^{-}(G)$. Theorem 4.5 yields the following result.

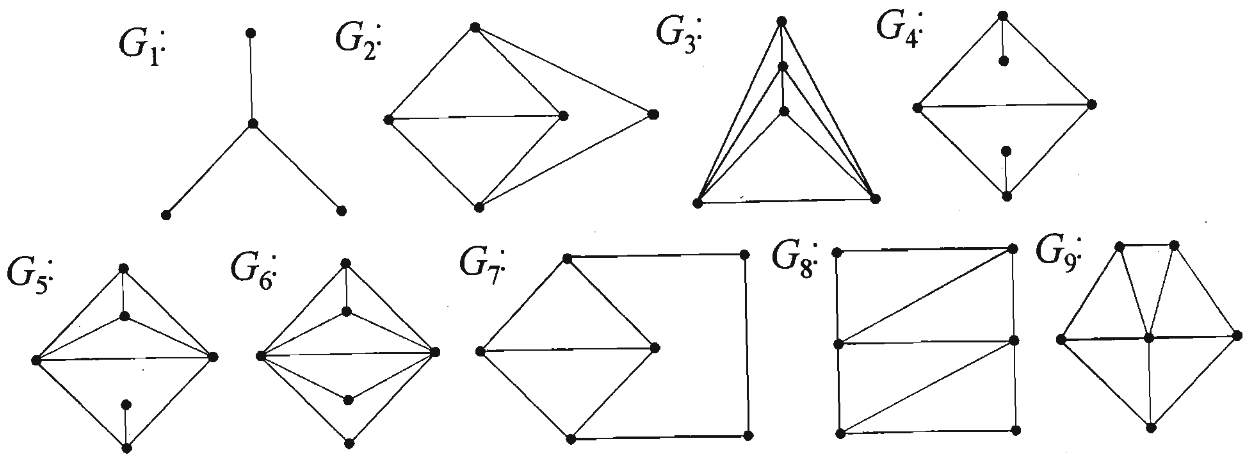


Figure 4.3:

Theorem 4 . 7 *For a given pair of graphs G and H in P, if $L^-(G)$ can be edge homogeneously embedded in $L^-(H)$, then G can be homogeneously embedded in H.*

Thus, for a large class of graphs, homogeneous embedding reduces to edge homogeneous embedding. In this sense, the edge homogeneous embedding requirement is a stronger embedding requirement than the usual homogeneous embedding requirement.

Let $e = ab$ be an edge of a nonempty graph G. We define the edge degree of e in G to be $edg_G(e) = deg_G a + deg_G b - 2$. If v is the vertex of the line graph $L(G)$ corresponding to e, then $edg_G(e) = deg_{L(G)} v$. Since the sum of the edge degrees in G is just the sum of the degrees of the vertices of the line graph $L(G)$, which is an even number, it follows that there are always an even number of edges in G of odd edge degree. We denote the maximum (minimum) edge degree of G by $\Delta_{edg}(G)$ ($\delta_{edg}(G)$, respectively). Note that $\Delta_{edg}(G) = \Delta(L(G))$ and $\delta_{edg}(G) = \delta(L(G))$. Next we present the edge analogue to Lemma 2.1.

Theorem 4 . 8 *If a nonempty graph G can be edge homogeneously embedded in a graph H, then*

$$\Delta_{edg}(G) \leq \delta_{edg}(H) \leq \Delta_{edg}(H) \leq |E(H)| - |E(G)| + \delta_{edg}(G).$$

Proof. Necessarily $\Delta_{edg}(G) \leq \delta_{edg}(H)$. The result is easily seen to be true if G is C_3 or $K_{1,3}$. Assume, then, that G is different from C_3 and $K_{1,3}$. By Theorem 4.5 we know that $L(G)$ can be homogeneously embedded in $L(H)$. Thus, by Lemma 2.1, it follows that $\Delta(L(G)) \leq \delta(L(H)) \leq \Delta(L(H)) \leq |V(L(G))| - |V(L(G))| + \delta(L(G))$. By the remarks preceding the theorem, the desired inequality follows. □

Corollary 4 . 6 *If a nonempty graph G can be edge homogeneously embedded in a graph H, then*

$$|E(H)| \geq |E(G)| + \Delta_{edg}(G) - \delta_{edg}(G).$$

Corollary 4 . 7 *For any nonempty graph G*

$$efr(G) \geq |E(G)| + \Delta_{edg}(G) - \delta_{edg}(G).$$

Let G be a graph which is different from C_3 and $K_{1,3}$. If G can be homogeneously embedded in a graph H, then by Lemma 2.1 we know that $\Delta(G) \leq \delta(H)$. This result is not necessarily true if H edge homogeneously embeds G. However, we do have the following result.

Theorem 4 . 9 *Let G be a nonempty graph which is different from C_3 and $K_{1,3}$. If G can be edge homogeneously embedded in a graph H, then*

$$\delta(H) \geq \max\{\min\{deg_G\, a, deg_G\, b\} : ab \in E(G)\}$$

Corollary 4 . 8 *Let G be a nonempty graph which is different from C_3 and $K_{1,3}$. Suppose also that G has two vertices of maximum degree which are adjacent. If G can be edge homogeneously embedded in a graph H, then $\delta(H) \geq \Delta(G)$.*

Corollary 4 . 9 *Let G be a nonempty graph which is different from C_3 and $K_{1,3}$. Suppose also that G has two vertices of maximum degree which are adjacent. If F is an edge frame of G, then $\delta(F) \geq \Delta(G)$.*

Theorem 4 . 10 *If a graph G can be edge homogeneously embedded in a graph H, then*

$$\Delta(G) \leq \Delta(H) \leq |V(H)| - |V(G)| + \Delta(G).$$

Proof. Necessarily, $\Delta(G) \leq \Delta(H)$. Let v be a vertex of G. Then v is not adjacent to at least $|V(G)|-\Delta(G)-1$ vertices in H. Since H edge homogeneously embeds G, every vertex of H is not adjacent to at least $|V(G)|-\Delta(G)-1$ vertices in H. Consequently, every vertex of H is adjacent with at most $|V(H)| - 1 - (|V(G)| - 1 - \Delta(G)) = |V(H)| - |V(G)| + \Delta(G)$ vertices. That is, $\Delta(H) \leq |V(H)| - |V(G)| + \Delta(G)$. □

Corollary 4 . 10 *If F is an edge frame of a graph G, then*

$$\Delta(G) \le \Delta(F) \le |V(F)| - |V(G)| + \Delta(G).$$

Theorem 4 . 11 *Let G be a nonempty graph which is different from C_3 and $K_{1,3}$. Suppose also that G has two vertices of minimum degree which are adjacent. If G can be edge homogeneously embedded in a graph H, then*

$$\delta(G) \le \delta(H) \le |V(H)| - |V(G)| + \delta(G).$$

Proof. Necessarily $\delta(G) \le \delta(H)$. The last inequality follows from an argument similar to that used in Theorem 4.10. □

Corollary 4 . 11 *Let G be a nonempty graph which is different from C_3 and $K_{1,3}$. Suppose also that G has two vertices of minimum degree which are adjacent. If F is an edge frame of a graph G, then*

$$\delta(G) \le \delta(F) \le |V(F)| - |V(G)| + \delta(G).$$

In order to illustrate the concepts described above, we determine the edge framing numbers of the graph $P_3 \times K_2$ and the graph G shown in Figure 4.4.

First we consider the graph $R \cong P_3 \times K_2$. Since R can be edge homogeneously embedded in the graph $C_4 \times K_2$ of size 12, $efr(R) \le 12$. Let F be an edge frame for R. By Corollary 4.9 and Corollary 4.11 we have $3 = \Delta(R) \le \delta(F) \le |V(F)| - |V(R)| + \delta(R) = |V(F)| - 4 \cdots (\star)$ whence $|V(F)| \ge 7$. If $|V(F)| = 7$ then all the inequalities in $(\star)$ are equalities and F is a 3-regular graph of order 7 which is impossible. Thus $|V(F)| \geq 8$ and $2q(F) \geq \delta(F)|V(F)| \geq 3 \times 8$ whence $efr(R) = q(F) \geq 12$. Consequently $efr(R) = 12$.

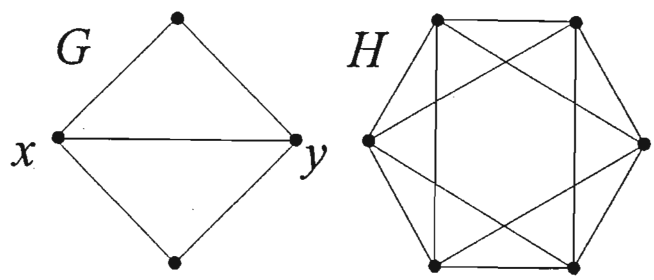


Figure 4.4: A graph G and its edge frame.

Next we consider the graph G shown in Figure 4.4. Since G can be edge homogeneously embedded in the graph H of size 12 shown in Figure 4.4, it follows that $efr(G) \leq 12$. We next show that $efr(G) \geq 12$. Let F be an edge frame for G. Let a be a vertex of minimum degree $\delta(F)$ in F and let $b \in N(a)$. Consider an embedding G' of G in F with edge xy at ab, say $G' \cong \langle a, b, c, d \rangle$ as shown in Figure 4.5.

An embedding of G in F with edge xy at ad implies the existence of a vertex, not in $\{a, b, c, d\}$, which is adjacent with a and d. Thus $\delta(F) = deg\, a \geq 4$. Hence $p(F) \geq \delta(F) + 1 = 5$. If $p(F) = 5$ then $F \cong K_5$ which contradicts the fact that F edge homogeneously embeds G. Thus $p(F) \geq 6$. Consequently, $2q(F) \geq 4p(F) \geq 24$, so $efr(G) = q(F) \geq 12$. Hence $efr(G) = 12$.

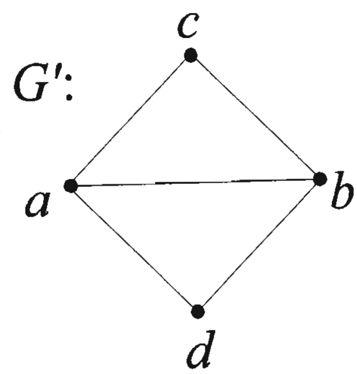


Figure 4.5:

4.4 Edge framing ratios of graphs

For a nonempty graph G, we define the *edge framing ratio* $efrr(G)$ of G by $effr(G) = efr(G)/q(G)$. Certainly, $effr(G) \geq 1$ for every nonempty graph G, and $effr(G) = 1$ if and only if G is edge transitive. The edge framing ratio of a graph G produces a certain measure of the 'edge symmetry' of G, where the closer $effr(G)$ is to 1, the more "edge symmetric" G is.

For the graph G of Figure 4.4, $efrr(G) = 12/5$ while the path P_n of length $n-1$ is edge framed by the cycle C_n so $efrr(P_n) = \frac{n}{n-1} = 1 + \frac{1}{n-1}$ which can be arbitrarily close to 1.

While a graph G may be very symmetric relative to its edges, it may be unsymmetric relative to its vertices. For example, the star $K_{1,m}$ always has edge framing ratio 1, while it is shown in [2] that $K_{1,m}$ can have framing ratios arbitrarily close to 2. The following result establishes a relationship between these two graphical parameters.

Theorem 4 . 12 *For a graph G,*

$$efrr(G) \geq frr(L(G)).$$

Proof. The result is easily seen to be true if G is C_3 or $K_{1,3}$. Assume, then, that G is different from C_3 and $K_{1,3}$. By Theorem 4.5 we know that $efr(G) \geq fr(L(G))$. Thus, since $q(G) = p(L(G))$, it follows that $efrr(G) = \frac{efr(G)}{q(G)} \geq \frac{fr(L(G))}{p(L(G))} = frr(L(G))$. □

Of course, the edge framing ratio of every nonempty graph is a rational number. We show that many rational numbers are edge framing ratios. We will require the following result.

Theorem 4 . 13 *For positive integers $m \geq n$,*

$$efr(K_{m,m,n}) = 3m^2.$$

Proof. Since $K_{m,m,n}$ can be edge homogeneously embedded in the graph $K_{m,m,m}$ of size $3m^2$, it follows that $efr(K_{m,m,n}) \leq 3m^2$. Let F be an edge frame for $K_{m,m,n}$. Let e be an edge of $K_{m,m,n}$ which joins the two partite sets of orders m and n and let $f = ab$ be an arbitrary edge of F. Then an edge embedding of $K_{m,m,n}$ in F with e at f implies the existence of an independent set S of m vertices such that each vertex of S is adjacent to each of a and b. We denote the set of $2m$ edges joining the vertices of S with a and b by S_{ab}.

Consider an edge embedding G of $K_{m,m,n}$ in F. Denote the two partite sets of G of order m by $U = u_1, u_2, \ldots, u_m$ and $V = v_1, v_2, \ldots, v_m$. The edges $e_i = u_iv_i$

$(1 \leq i \leq m)$ are independent. Moreover, since U and V are independent sets, the sets $S_{u_iv_i}$ $(1 \leq i \leq m)$ are pairwise disjoint. The $2m^2$ edges of $\bigcup_{i=1}^{m} S_{u_iv_i}$ together with the m^2 edges joining U and V account for $3m^2$ edges in F. Hence $efr(K_{m,m,n}) = q(F) \geq 3m^2$. Thus $efr(K_{m,m,n}) = 3m^2$ as required. $\square$

Theorem 4 . 14 *For each rational number $r \in [1,3)$, there exists a graph G with $efrr(G) = r$.*

Proof. Let $r \in [1,3)$ be a rational number. Then we may write $r = 2+\frac{a}{b}$, where a and b are integers with $b > 0$ and $-b \leq a < b$. Consider the graph $G \cong K_{4b+2a,4b+2a,b-a}$. By Theorem 4.13, $efr(G) = 3(4b+2a)^2 = 12(2b+a)^2$. Since the size of G is $12b(2b+a)$,

$$efrr(G) = \frac{12(2b+a)^2}{12b(2b+a)} = 2 + \frac{a}{b}. \quad \square$$

4.5 The diameter of an edge frame

In this section we prove a partial edge analogue to Theorem 2.5.

Theorem 4 . 15 *Let G be a nonempty connected graph, different from C_3 and $K_{1,3}$, with diameter d. Let S be the set of all integers q such that there is a graph of size q which edge homogeneously embeds G. Then for each $q \in S$, there is a graph H of size q which edge homogeneously embeds G with the property that diam $H \leq d+1$.*

Proof. Let $q \in S$. From among all graphs of size q which edge homogeneously embed G, choose one, call it H, of minimum order. Let v be a vertex of H with eccentricity $D = diam\, H$. Suppose that $D \geq d+2$. Let V_i be the set of vertices at distance i from v $(1 \leq i \leq D)$ and let $u \in V_D$. Let H' be obtained from H by joining u to every vertex of V_1 and deleting v. Let $e \in E(G)$ and $f \in E(H')$. If f is not one of the newly adjoined edges in H', then an edge embedding of G in H is also an edge embedding of G in H' with e at f because such an embedding cannot contain vertices from both V_1 and V_D. Suppose, then, that f is one of the newly adjoined edges of H'. Then $f = uv'$ for some $v' \in V_1$. Let G_1 be an edge embedding of G in H with e at uv'. Then the subgraph induced by $[V(G_1) - \{v\}] \cup \{u\}$ in H' is an edge embedding of G in H' with e at f. Thus G can be edge homogeneously embedded in H'. Since $p(H) < p(H')$, this contradicts the minimality property of H. Thus $D \leq d+1$ and H is a graph with the desired property. □

Corollary 4 . 12 *Let G be a nonempty connected graph with diameter d. Then G has an edge frame F with diameter at most $d+1$.*

Proof. If G is C_3 or $K_{1,3}$ then we may take F to be G itself. If G is different from C_3 and $K_{1,3}$, then the result is an immediate consequence of Theorem 4.15. □

While it is always possible to find an edge frame F for a nonempty connected graph G such that $diam\, F \leq diam G + 1$, $diam\, F$ can be an arbitrarily large amount less than $diam\, G$. For example, the cycle C_{n+1} is an edge frame for the path P_n of length

n and $diam\, C_{n+1} = \lfloor \frac{n+1}{2} \rfloor$ while $diam\, P_n = n - 1$.

4.6 The edge framing number of two or more graphs

The concept of edge framing numbers can be extended to more than one graph. For graphs G_1 and G_2, the edge framing number $efr(G_1, G_2)$ of G_1 and G_2 is defined as the minimum size of a graph F such that G_i $(i = 1, 2)$ can be edge homogeneously embedded in F. The graph F is called an *edge frame* of G_1 and G_2. Notice that $efr(G_1, G_2)$ exists and, in fact, $efr(G_1, G_2) \leq efr(G_1 \cup G_2)$.

In this section, we determine $efr(K_{1,m}, C_n)$ for all integers $m \geq 3$ and $n \geq 4$.

Theorem 4 . 16 *For integers $m \geq 3$ and $n \geq 4$,*

$$efr(K_{1,m}, C_n) = \begin{cases} (m-2)\lceil \frac{n}{2} \rceil + n & \text{if } n \equiv 0, 3 \,(mod\, 4) \\ & \text{or if } m \text{ is even and} \\ & n \equiv 1 \,(mod\, 4)\ (n \geq 9) \text{ or } n \equiv 2 \,(mod\, 4) \\ (m-2)\lceil \frac{n}{2} \rceil + n + 1 & \text{if } m \text{ is odd and} \\ & n \equiv 1 \,(mod\, 4)\ (n \geq 9) \text{ or } n \equiv 2 \,(mod\, 4) \\ 4m-3 & \text{if } n = 5 \end{cases}$$

Proof. First we present upper bounds for $efr(K_{1,m}, C_n)$ by constructing graphs which edge homogeneously embed $K_{1,m}$ and C_n. Thereafter we will proceed to show that these constructions are optimal.

Construction 1 $n \equiv 0, 3\ (mod\,4)$:

Let $v_1, v_2, \ldots, v_n, v_1$ be a cycle of length n and let $S_1, S_2, \ldots, S_{\lceil\frac{n}{4}\rceil}$ be $\lceil\frac{n}{4}\rceil$ pairwise disjoint sets of independent vertices each of cardinality $m-2$. Let D_1 be the graph obtained by joining each vertex of S_i with the vertices v_{4i-3} and v_{4i-1} $(1 \leq i \leq \lceil\frac{n}{4}\rceil)$ where all subscripts are reduced modulo n. Then D_1 is a graph of size $2(m-2)\lceil\frac{n}{4}\rceil + n = (m-2)\lceil\frac{n}{2}\rceil + n$ which edge homogeneously embeds $K_{1,m}$ and C_n. Thus $efr(K_{1,m}, C_n) \leq (m-2)\lceil\frac{n}{2}\rceil + n$.

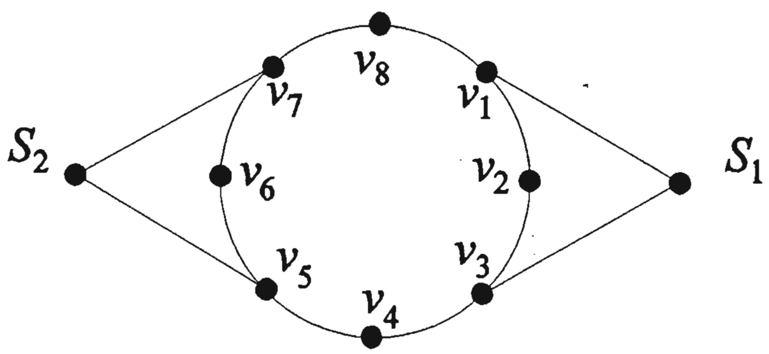


Figure 4.6: An edge frame for $K_{1,3}$ and C_8.

Construction 2.1 $n \equiv 1\,(mod\,4)$ $(n \geq 9)$ and m is even:

Let $v_1, v_2, \ldots, v_n, v_1$ be a cycle of length n and let $S_1, S_2, \ldots, S_{\frac{n+1}{2}}$ be $(n+1)/2$ pairwise disjoint sets of independent vertices such that $|S_i| = \frac{m-2}{2}$ $(1 \leq i \leq \frac{n+1}{2})$. Join each vertex of S_i with the vertices v_{2i-3} and v_{2i+1} $(2 \leq i \leq \frac{n+1}{2} - 1)$. Also join

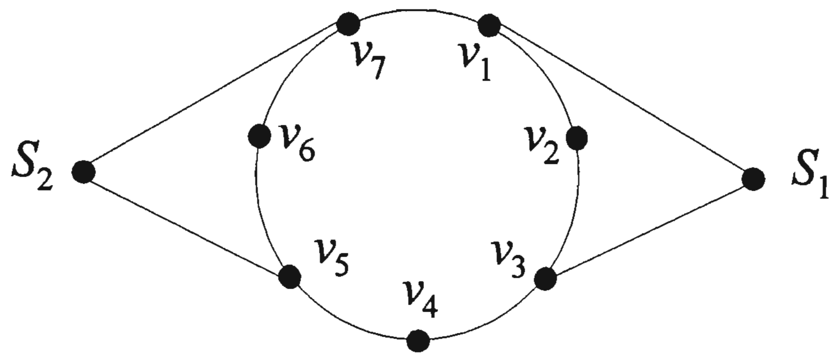


Figure 4.7: An edge frame for $K_{1,3}$ and C_7.

each vertex of S_1 ($S_{\frac{n+1}{2}}$) with the vertices v_1,v_3 (v_{n-2},v_n, respectively). Let D_2 denote the resulting graph.

Choose $u_i \in S_i$ ($1 \leq i \leq \frac{n+1}{2}$). It is clear that every edge in D_2 lies on an induced $K_{1,n}$ and that each edge of C' and each of the edges u_1v_1, u_1v_3, $u_{\frac{n+1}{2}}v_{n-2}$, $u_{\frac{n+1}{2}}v_n$ lies on an induced C_n in D_2. Observe that $C_n \cong \langle [V(C') - \{v_{2i-2}, v_{2i+2}\}] \cup \{u_i, u_{i+1}\} \rangle$ for $2 \leq i \leq \frac{n+1}{2} - 2$. It is now clear that each of the remaining edges of D_2 lies on an induced C_n. Thus D_2 is a graph of size $\frac{mn}{2} + \frac{m}{2} - 1 = (m-2)\lceil\frac{n}{2}\rceil + n$ which edge homogeneously embeds $K_{1,m}$ and C_n so that $efr(K_{1,m}, C_n) \leq (m-2)\lceil\frac{n}{2}\rceil + n$.

Construction 2.2 $n \equiv 1\,(mod\,4)$ $(n \geq 9)$ and m is odd:

Let $v_1, v_2, \ldots, v_n, v_1$ be a cycle of length n and let $S_1, S_2, \ldots, S_{\frac{n+1}{2}}$ be $\frac{n+1}{2}$ pairwise disjoint sets of independent vertices such that $|S_{\frac{n+1}{2}}| = \frac{m-1}{2}$ and for $1 \leq i \leq \frac{n+1}{2} - 1$,

$$|S_i| = \begin{cases} \frac{m-3}{2} & \text{if } i \equiv 1, 4\,(mod\,4) \\ \frac{m-1}{2} & \text{if } i \equiv 2, 3\,(mod\,4) \end{cases}$$

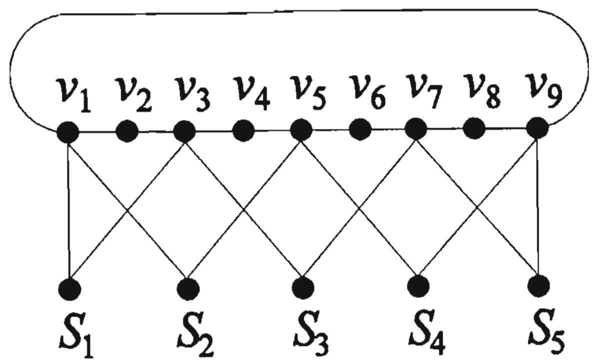


Figure 4.8: An edge frame for $K_{1,4}$ and C_9.

For $(2 \leq i \leq \frac{n+1}{2} - 1)$, join each vertex of S_i with the vertices v_{2i-3} and v_{2i+1}. Also join each vertex of S_1 ($S_{\frac{n+1}{2}}$) with the vertices v_1,v_3 (v_{n-2},v_n, respectively). Let D_3 denote the resulting graph.

Then D_3 is a graph of size $\frac{mn}{2} + \frac{m}{2} = (m-2)\lceil\frac{n}{2}\rceil + n + 1$ which edge homogeneously embeds $K_{1,m}$ and C_n so that $efr(K_{1,m}, C_n) \leq (m-2)\lceil\frac{n}{2}\rceil + n + 1$.

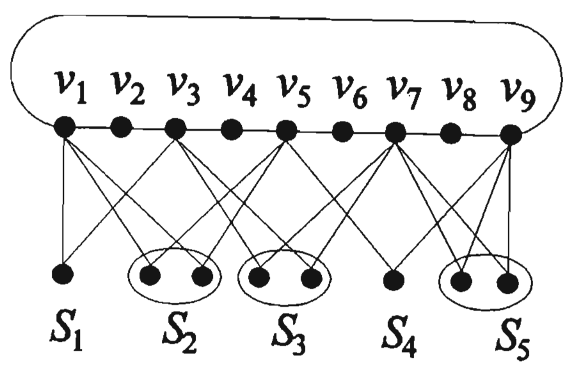


Figure 4.9: An edge frame for $K_{1,5}$ and C_9.

Construction 3.1 $n \equiv 2\,(mod\,4)$ and m is even:

Let $v_1, v_2, \ldots, v_n, v_1$ be a cycle of length n and let $S_1, S_2, \ldots, S_{\frac{n}{2}}$ be $\frac{n}{2}$ pairwise disjoint sets of independent vertices such that $|S_i| = \frac{m-2}{2}$ $(1 \leq i \leq \frac{n}{2})$. Let D_4 be the graph obtained by joining each vertex of S_i with the vertices v_{2i-1} and v_{2i+1} $(1 \leq i \leq \frac{n}{2})$ where all subscripts are reduced modulo n. Then D_4 is a graph of size $\frac{mn}{2} = (m-2)\lceil\frac{n}{2}\rceil + n$ which edge homogeneously embeds $K_{1,m}$ and C_n. Thus $efr(K_{1,m}, C_n) \leq (m-2)\lceil\frac{n}{2}\rceil + n$.

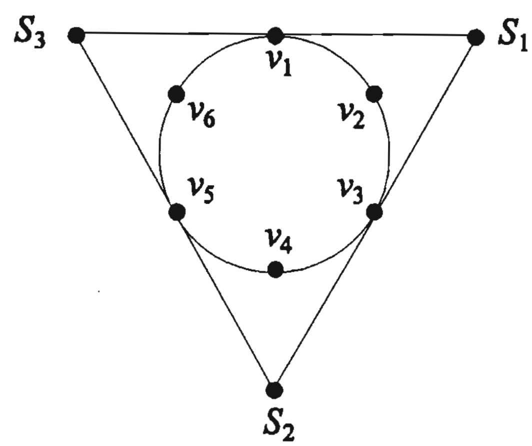


Figure 4.10: An edge frame for $K_{1,4}$ and C_6.

Construction 3.2 $n \equiv 2\,(mod\,4)$ and m is odd:

Let $v_1, v_2, \ldots, v_n, v_1$ be a cycle of length n and let $S_1, S_2, \ldots, S_{\frac{n}{2}}$ be $\frac{n}{2}$ pairwise disjoint sets of independent vertices such that for $1 \leq i \leq \frac{n}{2}$, $|S_i| = \frac{m-3}{2}$ if i is even and $|S_i| = \frac{m-1}{2}$ if i is odd. Let D_5 be the graph obtained by joining each vertex of S_i with the vertices v_{2i-1} and v_{2i+1} $(1 \leq i \leq \frac{n}{2})$ where all subscripts are reduced modulo

n. Then D_5 is a graph of size $\frac{mn}{2}+1 = (m-2)\lceil\frac{n}{2}\rceil+n+1$ which edge homogeneously embeds $K_{1,m}$ and C_n. Thus $efr(K_{1,m}, C_n) \leq (m-2)\lceil\frac{n}{2}\rceil + n + 1$.

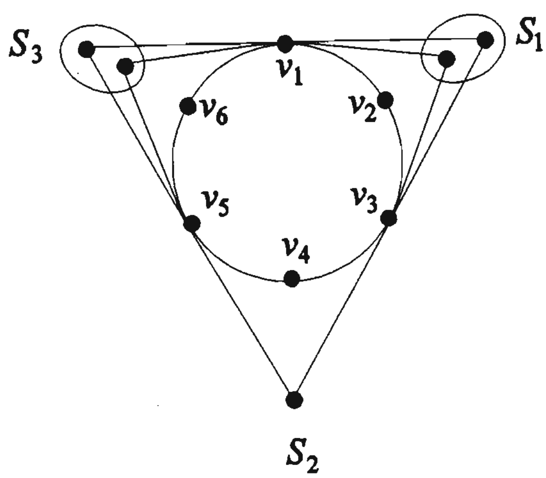


Figure 4.11: An edge frame for $K_{1,5}$ and C_6.

Construction 4 $n \equiv 5$:

Let $v_1, v_2, v_3, v_4, v_5, v_1$ be a cycle of length 5 and let S_1, S_2 be pairwise disjoint sets of independent vertices such that $|S_i| = m-2$ $(i = 1, 2)$. Let D_6 be the graph obtained by joining each vertex of S_1 (S_2) with the vertices v_1,v_3 (v_1,v_4, respectively). Then D_6 is a graph of size $4m-3$ which edge homogeneously embeds $K_{1,m}$ and C_5. Thus $efr(K_{1,m}, C_5) \leq 4m-3$.

Next we show that the upper bounds given above are also lower bounds. In what follows, we refer to a vertex v of a graph as a *central vertex* if v lies on an induced $K_{1,m}$ in which v has degree m. Before proceeding with the proof, we establish the following preliminary results.

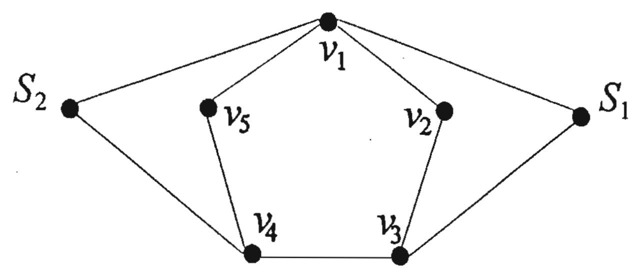


Figure 4.12: An edge frame for $K_{1,3}$ and C_5.

Claim 4 . 1 *Suppose that H is an edge frame for $K_{1,3}$ and C_n. If $n \neq 5$, then no edge of H lies on a C_3. If $n = 5$ and some edge of H lies on an induced C_3, then $q(H) \geq 4m - 3 = 9$.*

Proof Suppose that some edge, v_1v_2 say, of H lies on an induced C_3. Let C' : $v_1, v_2, \ldots, v_n, v_1$ be an induced C_n which contains the edge v_1v_2. Since each of the edges $e_i = v_{2i}v_{2i+1}$ $(1 \leq i \leq \lfloor \frac{n}{2} \rfloor)$ lies on an induced $K_{1,3}$ or C_3, e_i is incident with at least one other edge, say f_i, not on C'. Observe that, since C' is an induced C_n in H, $f_i \neq f_j$ for $i \neq j$. Thus $efr(K_{1,3}, C_n) = q(H) \geq 2 + \lfloor \frac{n}{2} \rfloor + n \cdots (\star)$. Since $2 + \lfloor \frac{n}{2} \rfloor + n > \lceil \frac{n}{2} \rceil + n$, inequality $(\star)$ contradicts the upper bounds given in Constructions 1 and 2.2. If n is even then $2 + \lfloor \frac{n}{2} \rfloor + n = \frac{3n}{2} + 2$ and inequality $(\star)$ contradicts the upper bound given in Construction 3.2. Thus if $n \neq 5$ then no edge of H lies on a C_3. If $n = 5$ then inequality $(\star)$ becomes $q(H) \geq 9$ as required. □

Claim 4 . 2 *If H is an edge frame for $K_{1,m}$ and C_n $((m,n) \neq (3,5))$, then every induced C_n in H contains at least $\lceil \frac{n}{2} \rceil$ central vertices and $q(H) \geq (m-2)\lceil \frac{n}{2} \rceil + n$.*

Proof. Let H' be an induced C_n in H. By Claim 4.1 no edge of H' lies on a C_3. Let c denote the number of vertices of H' which are central vertices. Since every edge e of H must lie on an induced $K_{1,m}$, it follows that at least one end-vertex of e must be a central vertex. Each central vertex of H' lies on a $K_{1,m}$ which contains at most two (consecutive) edges of H'. Thus, since every edge of H' lies on a $K_{1,m}$, it follows that $2c \geq n$, whence $c \geq \lceil \frac{n}{2} \rceil$. Now each of the central vertices on H' is incident with at least $m-2$ edges none of which lie on H'. Thus $efr(K_{1,m}, C_n) = q(H) \geq c(m-2) + n \geq (m-2)\lceil \frac{n}{2} \rceil + n$ as required. □

The lower bound given by Claim 4.2 coincides with the upper bounds given in Constructions 1, 2.1 and 3.1. Thus it remains for us to consider the cases $n \equiv 2 \,(mod\, 4)$ (m odd), $n \equiv 1 \,(mod\, 4)$ (where $n \geq 9$ and m is odd) and $n = 5$.

Case 1 m is odd and $n \equiv 1 \,(mod\, 4)$ $(n \geq 9)$ or $n \equiv 2 \,(mod\, 4)$:

Let H be an edge frame for $K_{1,m}$ and C_n. We must show that $q(H) \geq (m-2)\lceil \frac{n}{2} \rceil + n + 1$. Suppose, to the contrary, that $q(H) \leq (m-2)\lceil \frac{n}{2} \rceil + n$. By Claim 4.2, $q(H) \geq (m-2)\lceil \frac{n}{2} \rceil + n$. Thus $q(H) = (m-2)\lceil \frac{n}{2} \rceil + n$. By Claim 4.1 we know that no edge of H lies on an induced C_3.

Claim 4 . 3 *Let H' be an induced C_n in H. Then H' contains exactly $\lceil \frac{n}{2} \rceil$ central vertices and the only edges in H are those incident with the central vertices of H on*

H'. Furthermore, the central vertices of H on H' all have degree m and the remaining vertices of H' all have degree 2.

Proof. Suppose that H' is the cycle $v_1, v_2, \ldots, v_n, v_1$. Let c denote the number of vertices of H' which are central vertices. By Claim 4.2, $c \geq \lceil \frac{n}{2} \rceil$. Each of the central vertices on H' is incident with at least $m-2$ edges none of which lie on H'. Thus $q(H) = (m-2)\lceil \frac{n}{2} \rceil + n \geq c(m-2) + n \cdots (\star)$ whence $c \leq \lceil \frac{n}{2} \rceil$. Consequently $c = \lceil \frac{n}{2} \rceil$ and inequality $(\star)$ is an equality. Furthermore, it follows that the only edges in H are those incident with the central vertices of H'. Consequently, the central vertices of H on H' all have degree m and the remaining vertices of H' all have degree 2. □

Since H edge homogeneously embeds C_n, we have the following result.

Corollary 4 . 13 *Every vertex in H has either degree 2 or m. Furthermore, every vertex of degree m is a central vertex.*

Corollary 4 . 14 *If H' is an induced C_n in H, then at most two consecutive vertices on H' are central vertices.*

Now let K be an induced C_n in H. Then, by Claim 4.3, K contains exactly $\lceil \frac{n}{2} \rceil$ central vertices and the only edges in H are those incident with the central vertices of K. If v is any vertex outside K then, since it lies on an induced C_n and the only vertices adjacent to it are central vertices, v cannot be a central vertex. By Corollary 4.13 we deduce that every vertex outside K has degree 2. Thus the central vertices of K have degree m and all the remaining vertices of H have degree 2. But this implies that there are an odd number of vertices, namely the $\lceil \frac{n}{2} \rceil$ central vertices of K, of odd degree in H which is impossible. Thus $efr(K_{1,m}, C_n) = q(H) \geq (m-2)\lceil \frac{n}{2} \rceil + n + 1$ and consequently $efr(K_{1,m}, C_n) = (m-2)\lceil \frac{n}{2} \rceil + n + 1$. □

Case 2 $n = 5$:

Let H be an edge frame for $K_{1,m}$ and C_5. We must show that $q(H) \geq 4m - 3$. If $m = 3$ and some edge of H lies on a C_3, then by Claim 4.1 we have $efr(K_{1,3}, C_5) = q(H) \geq 4m - 3 = 9$ and we are done. Thus, in what follows, we may assume that no edge of H lies on a C_3 if $m = 3$. Let $H' : a_1, a_2, a_3, a_4, a_5, a_1$ be an induced C_5 in H. By Claim 4.2 there are at least three central vertices on H'. Also, by Corollary 4.8, $\delta(H) \geq 2$.

If there are at least four central vertices on H' then, since each of these central vertices lies on an induced $K_{1,m}$ which contains at most two (consecutive) edges of H', $efr(K_{1,m}, C_5) = q(H) \geq 4(m-2) + 5 = 4m - 3$ and we are done. Assume then that H' contains exactly three central vertices. Since each edge of H' lies on an induced $K_{1,m}$ we may, without loss of generality, assume that a_1, a_2 and a_4 are the central vertices of H on H'.

Suppose that there is one other central vertex , say v, in H (i.e. not on H'). Let K_v be an induced $K_{1,m}$ in H containing v as a central vertex. Then, in K_v, v is adjacent with at most two vertices from H' (possibly from a_1, a_2 and a_4). Consequently,

$efr(K_{1,m}, C_5) = q(H) \geq (m-2) + 3(m-2) + 5 = 4m - 3$ and we are done. Assume then that a_1, a_2 and a_4 are the only central vertices in H otherwise there is nothing to prove.

Since a_1 (a_2) is a central vertex, there is an independent set T_1 (T_2, respectively) of $m-2$ vertices in H, none of which lie on H', such that $\langle T_i \cup a_i \rangle \cong K_{1,m-2}$ for $i = 1, 2$. Let $a \in T_1$ and $b \in T_2$. Since no edge of H lies on an induced C_3 it follows that $a \neq b$. Furthermore, since a and b are not central vertices, a and b cannot be adjacent. Then, since $\delta(H) \geq 2$, there are at least $2|T_1 \cup T_2| = 4(m-2)$ edges incident with the vertices in $T_1 \cup T_2$. These edges, together with the five edges of H', account for at least $4m-3$ edges in H. Consequently, $efr(K_{1,m}, C_3) = q(H) \geq 4m - 3$ as required. □

Chapter 5

Homogeneous embeddings of cycles in graphs

5.1 Introduction

In this chapter we investigate the framing number and edge framing number of pairs of cycles. We also investigate the framing number of pairs of directed cycles.

In Section 5.2 we determine the framing number $fr(G_1, G_2)$ for several pairs G_1, G_2 of cycles. We extend the results of Chartrand et al. [2]. For $n > m \geq 3$, we show that $fr(C_m, C_n) \geq n + 2$ and we characterize all those pairs of cycles C_m and C_n which have framing number $n + 2$. Furthermore, for each such pair (m, n), we determine all the nonisomorphic frames of C_m and C_n. For $m = 3$ or 4 and $n = 7, 8, 9,$

or for $9 \leq m+2 \leq n \leq 2m-5$, we establish that $fr(C_m, C_n) = n+3$. Furthermore, in Section 5.3, for all integers $n > m \geq 3$, we establish upper bounds on $fr(C_m, C_n)$. We show that $fr(C_m, C_n)$ is at most $n + \lceil n/3 \rceil$ if $m = 3$ or 4, at most $n + n/(m-1)$ if $m - 1 \,|\, n$ and $m > 4$, and at most $n + \lceil n/(m-1) \rceil + 1$ otherwise.

In Section 5.4 we investigate the edge framing number $efr(G_1, G_2)$ for several pairs G_1, G_2 of cycles. We show that $efr(C_m, C_n) = n+4$ if $n = 2m-4$ and $m \geq 5$, $efr(C_m, C_n) = n+5$ if $n = 2m-6$ and $m \geq 7$ and $efr(C_m, C_n) = n+6$ if $n = 2m-8$ $(m \geq 10)$ or $m = n-1$ (where $n \geq 5$ and $n \notin \{6, 8\}$) or $m = n-2$ ($n = 6$ or $n \geq 9$). It is also shown that $efr(C_m, C_n) \geq n+6$ for $n > m \geq 4$ with $n \neq 2m-4$ or $2m-6$ and $(m, n) \neq (5, 7)$. Furthermore, for the cases $n = 2m-4$ $(m \geq 5)$ and $n = 2m-6$ $(m \geq 7)$ we show that C_m and C_n are uniquely edge framed.

Chartrand, Gavlas, and Schultz [2] extended the concept of framing numbers to more than one graph. Framing numbers of two or more digraphs can be defined similarly. For digraphs D_1 and D_2, the framing number $fr(D_1, D_2)$ of D_1 and D_2 is defined as the minimum order of a digraph F such that D_i $(i = 1, 2)$ can be homogeneously embedded in F. The digraph F is called a *frame* of D_1 and D_2. Notice that $fr(D_1, D_2)$ exists and, in fact, $fr(D_1, D_2) \leq fr(D_1 \cup D_2)$. A directed cycle of order n in which every vertex has indegree and outdegree equal to 1, will be denoted by $\vec{C}_n$. If $\vec{C}_n$ is given by $v_1, (v_1, v_2), v_2, (v_2, v_3), v_3, \ldots, v_n, (v_n, v_1), v_1$, then we will simply write $v_1, v_2, v_3, \ldots, v_n, v_1$. In Section 5.5 we investigate the framing number $fr(G_1, G_2)$ for several pairs G_1, G_2 of directed cycles. We characterize all those pairs of directed cycles $\vec{C}_m$ and $\vec{C}_n$ which have framing number $n+2$. Furthermore, for each such pair (m,n), we determine all the nonisomorphic frames of $\vec{C}_m$ and C_n. For $m = 3$ or 4 and $n = 7, 8, 9$, or for $9 \leq m+2 \leq n \leq 2m-5$, we establish that $fr(\vec{C}_m, \vec{C}_n) = n+3$. Furthermore, in Section 5.6, for all integers $n > m \geq 3$, we establish upper bounds on $fr(\vec{C}_m, \vec{C}_n)$. We show that $fr(\vec{C}_m, \vec{C}_n)$ is at most $n + \lceil n/2 \rceil$ if $m = 3$ or 4, at most $n + n/(m-1)$ if $m-1 \mid n$ and $m > 4$, and at most $n + \lceil n/(m-1) \rceil + 1$ otherwise.

5.2 The framing number of pairs of cycles

Chartrand et al. [2] investigated the framing number $fr(G_1, G_2)$ for several pairs G_1, G_2 of cycles. For small m and n, they established the values of $fr(C_m, C_n)$. Their results are summarized in Table 2.5 in Section 2.2. In this section, we extend the results of [2]. For $n > m \geq 3$ we characterize all those pairs of cycles C_m and C_n which have framing number $n+2$. Furthermore, for each such pair (m,n), we determine all the nonisomorphic frames of C_m and C_n. The following lemma will prove to be useful.

Lemma 5 . 1 *For integers $n > m \geq 3$, $fr(C_m, C_n) \geq n+2$.*

Proof. By Theorem 2.11, it suffices to show that there is no graph of order $n+1$ which homogeneously embeds C_n and C_m. Assume, to the contrary, that such a graph H exists. Let $C' : a_1, a_2, \ldots, a_n, a_1$ be an induced C_n in H, and let x be the name of the vertex of H not in C'. Let C_x be an induced C_n containing x. Without loss of generality, we may assume that C_x is given by $x, a_2, a_3, \ldots, a_n, x$. Hence $deg\, a_2 = deg\, a_n = 3$ and $deg\, a_i = 2$ for $i = 3, \ldots, n-1$. However there is then no induced C_m containing the vertex v_i $(3 \le i \le n-1)$. This produces a contradiction.

□

Let $S = \{(3,5),(3,6)\} \cup \{(m,n) \,|\, n = m+1 \text{ and } m \ge 3\} \cup \{(m,n) \,|\, n = 2m-4 \text{ and } m \ge 6\} \cup \{(m,n) \,|\, n = 2m-3 \text{ and } m \ge 5\} \cup \{(m,n) \,|\, n = 2m-2 \text{ and } m \ge 4\}$. For each $(m,n) \in S$, we define a set $\mathcal{F}_{m,n}$ of graphs as follows. For $m = 3$ and for $i \in \{4,5,6\}$, or for $m = 4$ and $i = 5$, let $\mathcal{F}_{m,i}$ be the set of all nonisomorphic graphs obtainable from the graph $F_{m,i}$ in Figure 5.1 by adding any combination (the presence or absence) of the dotted edges, provided that if uw is an edge of $F_{4,5}$, then so too are uv and wx. Let $\mathcal{F}_{4,6}$ be the set of all nonisomorphic graphs obtainable from the graph $F_{4,6}$ or $G_{4,6}$ in Figure 5.2 or the graph $H_{4,6}$ in Figure 5.1 by adding any combination (the presence or absence) of the dotted edges. Let $\mathcal{F}_{6,8}$ be the set of all nonisomorphic graphs obtainable from the graph $G_{6,8}$ or $H_{6,8}$ in Figure 5.1 or the graph $F_{6,8}$ in Figure 5.2 by adding any combination (the presence or absence) of the dotted edges. For $m \ge 5$ and $i = m+1$, or for $m = 5$ or $m \ge 7$ and $i = 2m-3$, or for $m \ge 7$ and $i = 2m-4$, let $\mathcal{F}_{m,i}$ be the set of all nonisomorphic graphs obtainable from the graph $F_{m,i}$ in Figure 5.2 by adding any combination (the presence or absence) of the dotted edges, provided that if uw is an edge of $F_{m,2m-3}$, then so too is vw.

Let $\mathcal{F}_{6,9}$ be the set of all nonisomorphic graphs obtainable from the graph $F_{6,9}$ in Figure 5.2 by adding any combination (the presence or absence) of the dotted edges. For $m = 5$ or $m \geq 7$, let $\mathcal{F}_{m,2m-2}$ be the set of all nonisomorphic graphs obtainable from the graph $F_{m,2m-2}$ or $G_{m,2m-2}$ in Figure 5.2 by adding any combination (the presence or absence) of the dotted edges. Let $\mathcal{F}_{6,10}$ be the set of all nonisomorphic graphs obtainable from the graph $H_{6,10}$ in Figure 5.1 or the graph $F_{6,10}$ or $G_{6,10}$ in Figure 5.2 by adding any combination (the presence or absence) of the dotted edges. We are now in a position to present our next result.

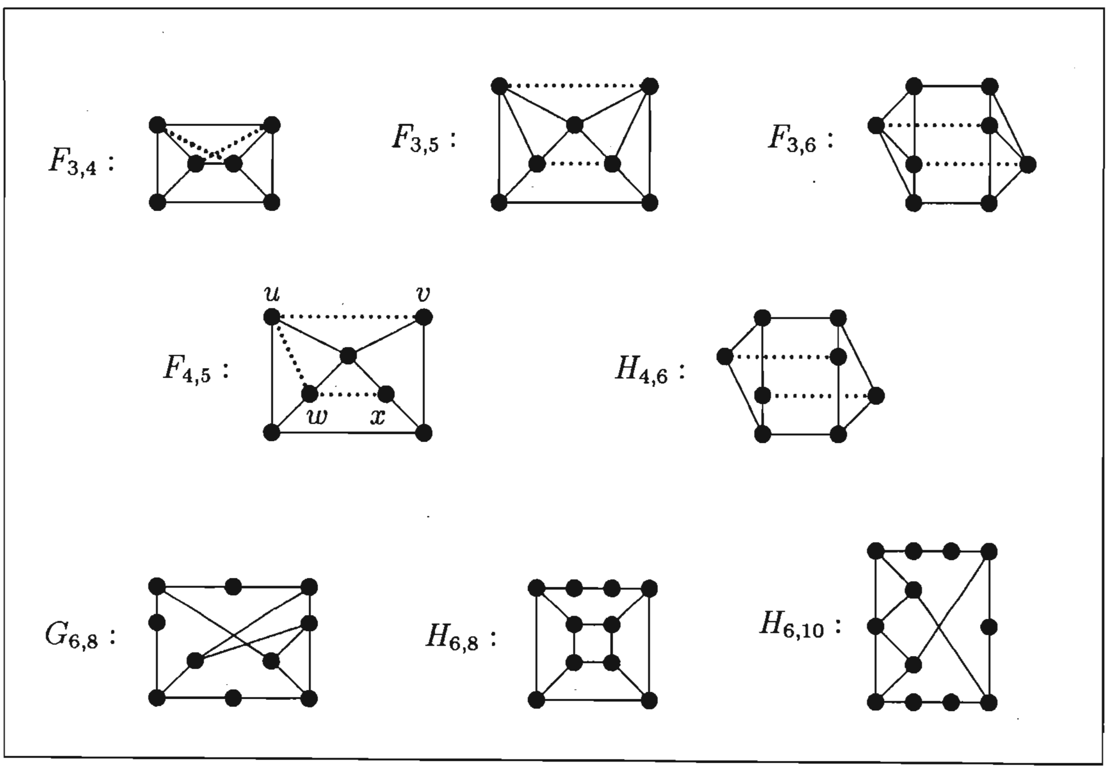


Figure 5.1:

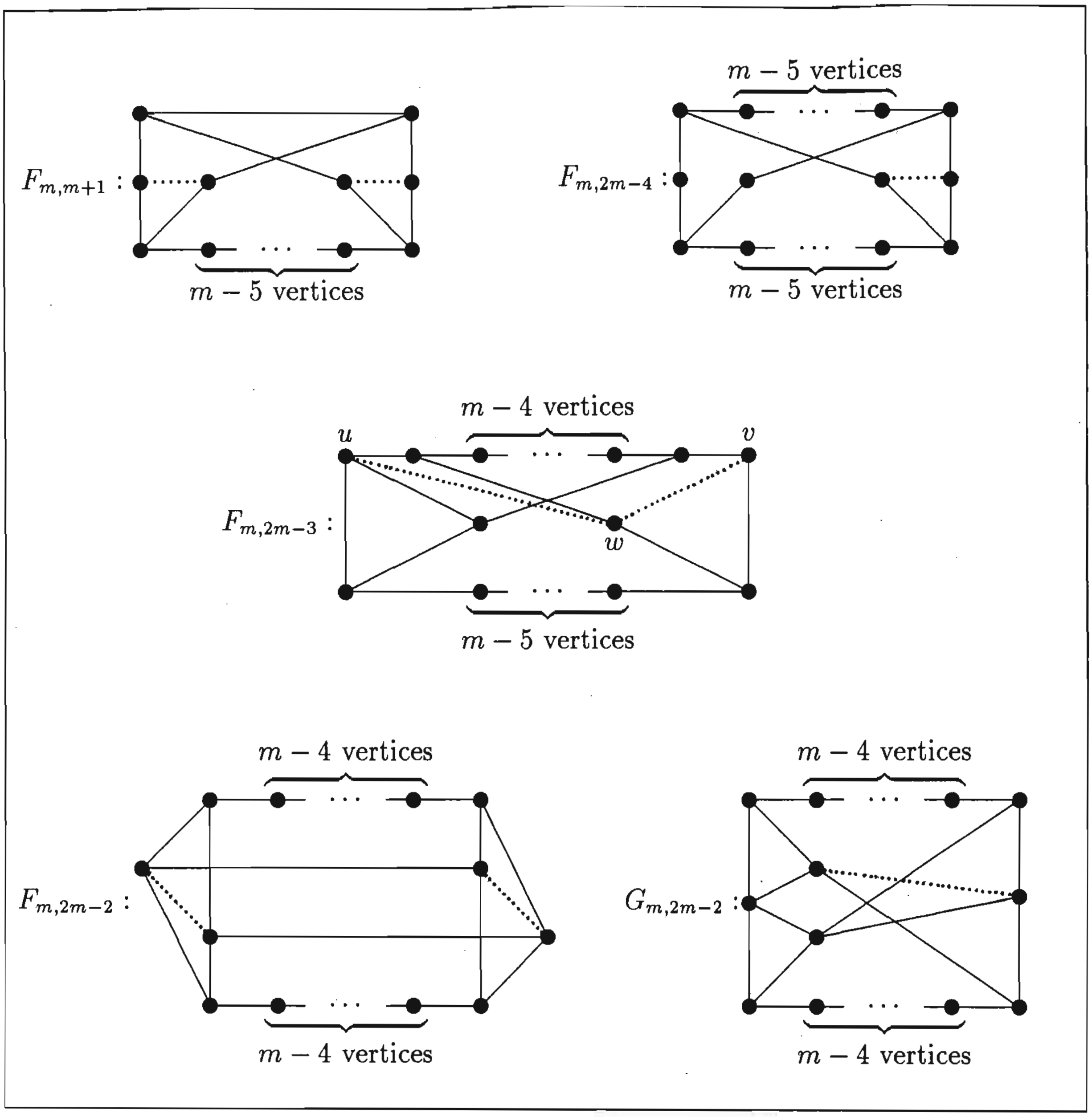


Figure 5.2:

Theorem 5 . 1 *For integers $n > m \geq 3$, $fr(C_m, C_n) = n+2$ if and only if $(m,n) \in S$. Furthermore, if $(m,n) \in S$, then the set of all nonisomorphic frames of C_m and C_n is given by $\mathcal{F}_{m,n}$.*

Proof. If $(m,n) \in S$, then C_m and C_n can be homogeneously embedded in each graph (of order $n+2$) from the set $\mathcal{F}_{m,n}$, so $fr(C_m, C_n) \leq n+2$. However, by Lemma 5.1, $fr(C_m, C_n) \geq n+2$. Hence $fr(C_m, C_n) = n+2$. This establishes the sufficiency.

Next we consider the necessity. Let m and n be integers satisfying $n > m \geq 3$ and assume that $fr(C_m, C_n) = n+2$. Let H be a frame for C_m and C_n. Then $p(H) = n+2$ and Lemma 2.1 implies that $2 \leq \delta(H) \leq \Delta(H) \leq (n+2) - n + 2 = 4$.

First we assume that for any induced n-cycle C' in H, the two vertices of H not in C' do not belong to a common C_n. Let $v_1, v_2, \ldots, v_n, v_1$ be an induced C_n in H. Let a and b be the names of the two remaining vertices of H. Further, let C_a (C_b) be an induced C_n that contains the vertex a (b, respectively). By hypothesis, a and b do not belong to a common induced C_n. Without loss of generality, we may assume that C_a is $v_1, v_2, \ldots, v_{n-1}, a, v_1$. Since the vertices v_n and b do not belong to C_a, our assumption implies that v_n and b do not belong to a common induced C_n. Hence C_b must contain the vertices $v_1, v_2, \ldots, v_{n-1}$. Hence C_b is given by $v_1, v_2, \ldots, v_{n-1}, b, v_1$. Thus $deg\, v_1 = deg\, v_{n-1} = 3$ and $deg\, v_i = 2$ for $i = 2, 3, \ldots, n-2$. Hence H has the subgraph shown in Figure 5.3. However, there is then no induced C_m containing the vertex v_i $(2 \leq i \leq n-2)$.

Thus there exists an induced C_n in H, say $C' : v_1, v_2, \ldots, v_n, v_1$, such that the two vertices of H outside C', call them g and h, belong to a common induced C_n, say C_g. Then C_g contains the vertices g and h and $n-2$ vertices of C'. If g and h are adjacent vertices on C_g, then, without loss of generality, we may assume that C_g is given by

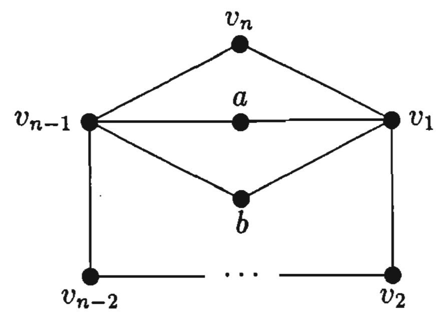


Figure 5.3: A subgraph of H.

$v_3, v_4, \ldots, v_n, g, h, v_3$. Hence $deg\, v_3 = deg\, v_n = 3$ and $deg\, v_i = 2$ for $i = 4, \ldots, n-1$. Thus the graph shown in Figure 5.4 is a subgraph of H. If $n > 4$, then there is no induced C_m containing the vertex v_i $(4 \leq i \leq n-1)$, which produces a contradiction. Hence $n = 4$, so $m = 3$. Furthermore, if $v_1 g$ or $v_2 h$ is not an edge of H, then H does not homogeneously embed C_3. Hence $v_1 g$ and $v_2 h$ are both edges of H. Thus there are three possibilities for H, depending on the presence or absence of the edges $v_2 g$ and $v_1 h$. This yields the set $\mathcal{F}_{3,4}$ of three nonisomorphic frames for C_3 and C_4. Next we assume that g and h are nonadjacent vertices.

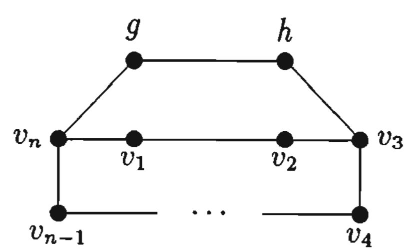


Figure 5.4: A subgraph of H.

If g and h are joined by a path of length 2 in C_g, then, without loss of generality, we may assume that C_g is given by $v_2, h, v_4, v_5, \ldots, v_n, g, v_2$. Hence H has the subgraph shown in Figure 5.5. Then $n \geq 5$, $deg\, v_2 = 4$, $deg\, v_4 = deg\, v_n = 3$ and $deg\, v_i = 2$ $(5 \leq i \leq n-1)$. Hence any cycle contains either all or no vertex from the set $\{v_5, \ldots, v_{n-1}\}$. Suppose $n \geq 6$. Then any induced C_m containing the vertex v_i $(5 \leq i \leq n-1)$ would contain the $n-3$ vertices from the set $\{v_4, v_5, \ldots, v_n\}$, exactly one vertex from each of $\{v_1, g\}$ and $\{v_3, h\}$ and therefore would have length at least $n-1$. Thus $m = n-1 \geq 5$. However there is then no induced C_m containing the vertex v_2. Hence $n = 5$, so $m = 3$ or 4. If $m = 3$, then since each of v_4 and v_5 belongs to a C_3, both v_1g and v_3h must be edges of H. Hence there are four possibilities for H, depending on the presence or absence of the edges v_1h and v_3g. This yields the set $\mathcal{F}_{3,5}$ of three nonisomorphic frames for C_3 and C_5. If $m = 4$, then for v_2 to belong to a C_4, at most one of v_1g and v_3h is edges of H. If exactly one of v_1g and v_3h is an edge, then both v_1h and v_3g must be edges of H. If neither v_1g nor v_3h is an edge, then there are four possibilities for H, depending on the presence or absence of the edges v_1h and v_3g. This yields the set $\mathcal{F}_{4,5}$ of four nonisomorphic frames for C_4 and C_5.

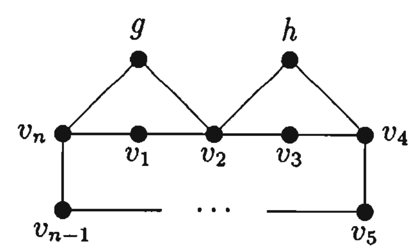


Figure 5.5: A subgraph of H.

Next we assume that g and h are at distance at least 3 apart on C_g. Then $n \geq 6$ and, without loss of generality, we may assume that C_g is given by either $v_1, g, v_i, v_{i-1}, \ldots, v_3, h, v_{i+2}, v_{i+3}, \ldots, v_n, v_1$ $(4 \leq i \leq n-2)$, in which case H has the subgraph shown in Figure 5.6(i), or $v_1, v_2, \ldots, v_k, h, v_{k+2}, v_{k+3}, \ldots, v_{n-1}, g, v_1$ $(2 \leq k \leq n-4)$, in which case H has the subgraph shown in Figure 5.6(ii).

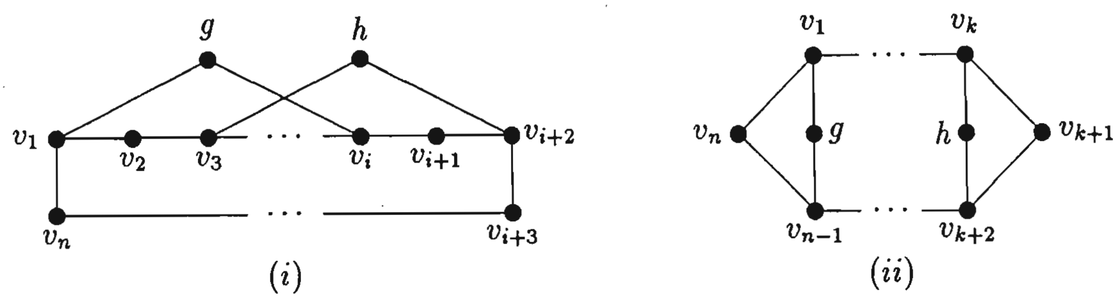


Figure 5.6:

Suppose that H has the subgraph shown in Figure 5.6(i). Then each of the vertices v_1, v_3, v_i and v_{i+2} has degree 3 while the remaining vertices of C' have degree 2, except possibly for v_2 and v_{i+1}. For notational convenience, we write $u \perp v$ if u and v are adjacent vertices, and $u \pm v$ if u and v are not adjacent. We consider two possibilities.

Case 1. $i > 4$.

Then the vertex v_4 belongs to induced cycles of only three possible lengths, namely, i, $i+1$ and n depending on the presence or absence of the edges v_2g, v_2h, $v_{i+1}g$ and $v_{i+1}h$. Since v_4 belongs to an induced C_m, we must have $i = m-1$ or m. We consider the two possibilities in turn.

Case 1.1. $i = m - 1$.

Then $m = i + 1 \geq 6$ and $v_2 \pm g$ or $v_m \pm h$ for otherwise v_4 belongs to no induced C_m. Without loss of generality, we may assume that $v_m \pm h$. Suppose $n = m + 1\,(= i + 2)$. If $v_2 \perp g$, then the vertex v_1 belongs to induced cycles of only four possible lengths, namely, 3, 4, 5, and n. Since v_1 belongs to an induced C_m, and $m \geq 6$, this produces a contradiction. Hence $v_2 \pm g$. Then there are four possibilities for H, depending on the presence or absence of the edges v_2h and v_mg. This yields the set $\mathcal{F}_{m,m+1}$ of three nonisomorphic frames for C_m and C_{m+1} $(m \geq 6)$. Hence in what follows in Case 1.1, we may assume that $n \geq m + 2$ for otherwise there is nothing left to prove. Then the vertex v_n belongs to induced cycles of only three possible lengths, namely, $n - m + 3$, $n - m + 4$, and n. Since v_n belongs to an induced C_m, it follows that $n = 2m - 4$ or $2m - 3$. We now consider four cases.

Case 1.1.1. $v_2 \pm g$ and $n = 2m - 4$.

Then $v_2 \pm h$ or $v_m \pm g$. Without loss of generality, we may assume that $v_m \pm g$. Then there are two possibilities for H, depending on the presence or absence of the edge v_2h. This yields the set $\mathcal{F}_{m,2m-4}$ of two nonisomorphic frames for C_m and C_{2m-4} $(m \geq 6)$.

Case 1.1.2. $v_2 \pm g$ and $n = 2m - 3$.

Then $v_2 \perp h$ or $v_m \perp g$. Without loss of generality, we may assume that $v_2 \perp h$. Then there are two possibilities for H, depending on the presence or absence of the edge $v_m g$. This yields two nonisomorphic frames for C_m and C_{2m-3} ($m \geq 6$), namely the graph $F_{m,2m-3}$ in Figure 5.2 and the graph obtained from $F_{m,2m-3}$ by adding the edge vw.

Case 1.1.3. $v_2 \perp g$ and $n = 2m - 4$.

Then $v_2 \pm h$ or $v_m \pm g$. If $v_2 \pm h$ and $v_m \pm g$, then this yields the graph obtained from $F_{m,2m-4}$ ($m \geq 6$) in Figure 5.2 by adding the dotted edge. If $v_2 \perp h$ and $v_m \pm g$, then $m = 6$ for otherwise v_2 belongs to no induced C_m, while if $v_2 \pm h$ and $v_m \perp g$, then $m = 6$ for otherwise g belongs to no induced C_m. Both cases yield the graph $G_{6,8}$ of Figure 5.1.

Case 1.1.4. $v_2 \perp g$ and $n = 2m - 3$.

Then $v_2 \perp h$ or $v_m \perp g$. If $v_2 \perp h$ and $v_m \perp g$, then this yields the graph obtained from $F_{m,2m-3}$ ($m \geq 6$) in Figure 5.2 by adding the two dotted edges. If $v_2 \perp h$ and $v_m \pm g$, then $m = 6$ for otherwise g belongs to no induced C_m, while if $v_2 \pm h$ and $v_m \perp g$, then $m = 6$ for otherwise v_2 belongs to no induced C_m. Both cases yield the graph obtained from the graph $F_{6,9}$ in Figure 5.2 by adding the edge uw.

Case 1.2. $i = m$.

Then $m \geq 5$ and $v_2 \perp g$ or $v_{m+1} \perp h$ for otherwise v_4 belongs to no induced C_m. Without loss of generality, we may assume that $v_2 \perp g$. Then the vertex v_1 belongs to induced cycles of only four possible lengths, namely, 3, $n - m + 2$, $n - m + 3$, and n. Since v_n belongs to an induced C_m, and $m \geq 5$, it follows that $n = 2m - 3$ or

$2m-2$. We now consider four cases.

Case 1.2.1. $v_{m+1} \pm h$ and $n = 2m-3$.

Then $v_2 \pm h$ or $v_{m+1} \pm g$. If $v_2 \pm h$ and $v_{m+1} \pm g$, then then this yields the graph $F_{m,2m-3}$ $(m \geq 5)$ in Figure 5.2. If $v_2 \perp h$ and $v_{m+1} \pm g$, then $m = 6$ for otherwise h belongs to no induced C_m, while if $v_2 \pm h$ and $v_{m+1} \perp g$, then $m = 6$ for otherwise v_{m+1} belongs to no induced C_m. Both cases yield the graph obtained from the graph $F_{6,9}$ in Figure 5.2 by adding the edge uw.

Case 1.2.2. $v_{m+1} \pm h$ and $n = 2m-2$.

Then $v_2 \perp h$ or $v_{m+1} \perp g$. If $v_2 \perp h$ and $v_{m+1} \perp g$, then then this yields the graph $F_{m,2m-2}$ $(m \geq 5)$ in Figure 5.2 by adding any combination (the presence or absence) of the dotted edges. If $v_2 \perp h$ and $v_{m+1} \pm g$, then $m = 6$ for otherwise v_{m+1} belongs to no induced C_m, while if $v_2 \pm h$ and $v_{m+1} \perp g$, then $m = 6$ for otherwise h belongs to no induced C_m. Both cases yield the graph $H_{6,10}$ in Figure 5.1.

Case 1.2.3. $v_{m+1} \perp h$ and $n = 2m-3$.

Then $v_2 \pm h$ or $v_{m+1} \pm g$. Without loss of generality, we may assume that $v_{m+1} \pm g$. Then there are two possibilities for H, depending on the presence or absence of the edge v_2h. This yields two nonisomorphic frames for C_m and C_{2m-3} $(m \geq 5)$, namely the graph obtained from $F_{m,2m-3}$ in Figure 5.2 by adding the edge vw and the graph obtained from $F_{m,2m-3}$ by adding the edges uw and vw.

Case 1.2.4. $v_{m+1} \perp h$ and $n = 2m - 2$.

Then $v_2 \perp h$ or $v_{m+1} \perp g$. Without loss of generality, we may assume that $v_2 \perp h$. Then there are two possibilities for H, depending on the presence or absence of the edge $v_{m+1}g$. This yields two nonisomorphic frames for C_m and C_{2m-2} ($m \geq 5$), namely the graph obtained from $G_{m,2m-2}$ in Figure 5.2 by adding or omitting the dotted edge.

Case 2. $i = 4$.

Then the vertex v_n belongs to induced cycles of only three possible lengths, namely, $n-2$, $n-1$ and n depending on the presence or absence of the edges v_2g, v_2h, v_5g and v_5h. Since v_n belongs to an induced C_m, we must have $m = n-2$ or $m = n-1$. We consider the two possibilities in turn.

Case 2.1. $m = n - 2$.

Then $v_2 \perp h$ or $v_5 \perp g$. Without loss of generality, we may assume that $v_5 \perp g$. Then the vertex v_4 belongs to induced cycles of only four possible lengths, namely, 3, 4, 5, and n. Since v_4 belongs to an induced C_m, and $m = n - 2 \geq 4$, it follows that $m = 4$ or $m = 5$. We consider two cases in turn.

Case 2.1.1. $m = 4$.

Then $n = 6$ and $v_2 \perp g$ or $v_5 \perp h$, for otherwise v_4 belongs to no induced C_4. If $v_2 \perp g$ and $v_5 \perp h$, then there are two possibilities for H, depending on the presence or absence of the edge v_2h. This yields two nonisomorphic frames for C_4 and C_6, namely the graph $G_{4,6}$ in Figure 5.2 or the graph obtained from $G_{4,6}$ by adding the dotted edge. If $v_2 \perp g$ and $v_5 \pm h$, then $v_2 \perp h$ for otherwise h belongs to no induced C_4, while if $v_2 \pm g$ and $v_5 \perp h$, then $v_2 \perp h$ for otherwise v_2 belongs to no induced C_4. Both cases yield the graph $G_{4,6}$ in Figure 5.2.

Case 2.1.2. $m = 5$.

Then $n = 7$ and $v_2 \pm g$ or $v_5 \pm h$, for otherwise v_4 belongs to no induced C_5. If $v_2 \pm g$ and $v_5 \pm h$, then there are two possibilities for H, depending on the presence or absence of the edge v_2h. If $v_2 \perp g$ and $v_5 \pm h$, then $v_2 \perp h$ for otherwise v_2 belongs to no induced C_5. If $v_2 \pm g$ and $v_5 \perp h$, then $v_2 \perp h$ for otherwise h belongs to no induced C_5. This yields the set $\mathcal{F}_{5,7}$ of three nonisomorphic frames for C_5 and C_7.

Case 2.2. $m = n - 1$.

Then $m \geq 5$ and $v_2 \pm h$ or $v_5 \pm g$. Without loss of generality, we may assume that $v_2 \pm h$. If $v_5 \pm g$, then there are four possibilities for H, depending on the presence or absence of the edges v_2g and v_5h. This yields the set $\mathcal{F}_{m,m+1}$ of three nonisomorphic frames for C_m and C_{m+1} $(m \geq 5)$. Suppose that $v_5 \perp g$. Then the vertex v_4 belongs to induced cycles of only four possible lengths, namely, 3, 4, 5, and n. Since v_4 belongs to an induced C_m, and $m = n - 1 \geq 5$, it follows that $m = 5$, so $n = 6$. Thus $v_5 \pm h$, for otherwise v_5 belongs to no induced C_5. Furthermore, $v_2 \pm g$, for otherwise g belongs to no induced C_5. This yields the graph obtained from $F_{5,6}$ in Figure 5.2 by adding exactly one of the dotted edges.

Suppose next that H has the subgraph shown in Figure 5.6(ii). Then each of v_1, v_k, v_{k+2} and v_{n-1} has degree 3 while the remaining vertices of C' have degree 2, except possibly for v_{k+1} and v_n.

Suppose, firstly, that $n = 6$, i.e., C_g is given by $g, v_1, v_2, h, v_4, v_5, g$. Then no vertex of H belongs to an induced C_5 irrespective of the presence or absence of the edges v_3g, v_3h, v_6g and v_6h. Hence $m = 3$ or 4. If $m = 3$, then we must have $v_3 \perp h$ and $v_6 \perp g$. Thus H homogeneously embeds C_3 and C_6 and this does not depend on the presence or absence of the edges v_3g or v_6h. This yields the set $\mathcal{F}_{3,6}$ of three nonisomorphic frames for C_3 and C_6. Suppose $m = 4$. If $v_3 \pm h$ and $v_6 \pm g$, then H homogeneously embeds C_4 and C_6 and this does not depend on the presence or absence of the edges v_3g or v_6h. This yields three nonisomorphic frames for C_4 and C_6, namely the nonisomorphic graphs obtained from $H_{4,6}$ in Figure 5.1 by adding any combination (the presence or absence) of the dotted edges. If $v_3 \perp h$ or $v_6 \perp g$, then without loss of generality, we may assume that $v_6 \perp g$. Since v_6 (g) belongs to an induced C_4, $v_6 \perp h$ ($v_3 \perp g$, respectively). Thus H homogeneously embeds C_4 and C_6 and this does not depend on the presence or absence of the edge v_3h. This yields two nonisomorphic frames for C_4 and C_6, both of which are obtainable from $F_{4,6}$ in Figure 5.2 by adding either one or both of the dotted edges.

Suppose, next, that $n \geq 7$. Then $k \geq 3$ or $k \leq n-5$; that is, there must exist an internal vertex on the v_1-v_k path or the v_{k+2}-v_{n-1} path on C' that does not contain v_n. Such a vertex belongs to no C_3 or C_4. Hence $m \geq 5$. Let C'_m be an induced C_m

containing the vertex v_1.

If v_n and g belong to C'_m, then, since $m \geq 5$, it follows that $v_n \pm g$ and C'_m must contain the vertices v_{k+1} and h (so $v_n h$ and $v_{k+1} g$ are edges on C'_m). If $v_{k+1} \pm h$, then C'_m is of length 6, so $m = 6$. It is readily seen that if $n \neq 8$, then there exists an internal vertex on the v_1-v_k path or the v_{k+2}-v_{n-1} path on C' that does not contain v_n that belongs to no 6-cycle. Hence $n = 8$. Then there exists an internal vertex on the v_1-v_k path or the v_{k+2}-v_{n-1} path on C' that does not contain v_n that belongs to no induced 6-cycle unless one of these paths have length 3 and the other has length 1. Without loss of generality, we may assume that $k = 4$. This yields the graph $H_{6,8}$ in Figure 5.1 which frames C_6 and C_8. On the other hand, if $v_{k+1} \perp h$, then C'_m is of length 5, so $m = 5$. If $n = 7$, then either $k = 2$, in which case v_2 belongs to no induced C_5, or $k = 3$, in which case v_5 belongs to no induced C_5. If $n \geq 9$, then there exists an internal vertex on the v_1-v_k path or the v_{k+2}-v_{n-1} path on C' that does not contain v_n that belongs to no 5-cycle. Hence $n = 8$. Then there exists an internal vertex on the v_1-v_k path or the v_{k+2}-v_{n-1} path on C' that does not contain v_n that belongs to no induced 5-cycle unless both of these paths have length 2. i.e., unless $k = 2$. Hence H is the graph shown in Figure 5.7. However, this graph is isomorphic to the graph obtained from $F_{5,8}$ in Figure 5.2 by adding exactly one of the dotted edges.

Next we assume that v_n and g do not both belong to C'_m. Then C'_m contains the k vertices from the set $\{v_1, v_2, \ldots, v_k\}$, exactly one vertex from each of $\{v_n, g\}$ and

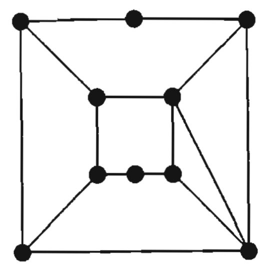


Figure 5.7: A frame for C_5 and C_8.

$\{v_{k+1}, h\}$, and either all or no vertex from the set $\{v_{k+2}, v_{k+3}, \ldots, v_{n-1}\}$. Since C'_m has length m, it follows that C'_m contains no vertex from the set $\{v_{k+2}, v_{k+3}, \ldots, v_{n-1}\}$. Therefore C'_m has length $k+2 = m$. Consequently, $k = m-2$. However, if we consider an induced C_m containing the vertex v_{k+2}, then we may show that this cycle contains the $n-k-2$ vertices from the set $\{v_{k+2}, v_{k+3}, \ldots, v_{n-1}\}$, exactly one vertex from each of $\{v_n, g\}$ and $\{v_{k+1}, h\}$, and no vertex from the set $\{v_1, v_2, \ldots, v_k\}$. This shows that $n - k = m$, or, equivalently, $k = n - m$. Consequently, $n = 2m - 2$ and $k = m - 2$. Without loss of generality, we may assume that $v_n \perp h$. If $v_{m-1} \pm g$, then v_{m-1} belongs to no induced $C_m\,(5 \leq m < n)$. Hence $v_{m-1} \perp g$. Thus H homogeneously embeds C_m and C_{2m-2} and this does not depend on the presence or absence of the edges $v_{m-1}h$ or $v_{2m-2}g$. This yields three nonisomorphic frames for C_m and $C_{2m-2}\,(m \geq 5)$, namely the nonisomorphic graphs obtained from $F_{m,2m-2}$ in Figure 5.2 by adding any combination (the presence or absence) of the dotted edges. This completes the proof of the theorem. $\square$

As a corollary of Theorem 5.1, we may determine exactly how many nonisomorphic frames of C_m and C_n exist. Let $S_2 = \{(m,n) \mid n = 2m-4 \text{ and } m \geq 7\}$, $S_3 = \{(3,4),(3,5),(3,6)\} \cup \{(m,n) \mid n = m+1 \text{ and } m \geq 5\} \cup \{(m,n) \mid n = 2m-3 \text{ and } m = 5 \text{ or } m \geq 7\}$, $S_4 = \{(4,5),(6,8),(6,9)\}$, $S_5 = \{(m,n) \mid n = 2m-2 \text{ and } m = 5 \text{ or } m \geq 7\}$, $S_6 = \{(6,10)\}$, and $S_7 = \{(4,6)\}$. Then $S = \cup_{i=3}^{7} S_i$. The following result follows immediately from Theorem 5.1.

Corollary 5 . 1 *If $(m,n) \in S$, then C_m and C_n have exactly i nonisomorphic frames of order $n+2$ if and only if $(m,n) \in S_i$ for some i with $3 \leq i \leq 7$.*

The next result is an immediate consequence of Lemma 5.1 and Theorem 5.1.

Corollary 5 . 2 *For positive integers $n > m \geq 3$, if $(m,n) \notin S$, then $fr(C_m, C_n) \geq n+3$.*

Proposition 5 . 1 *For $m \geq 7$, $fr(C_m, C_{m+2}) = m+5$.*

Proof. Since C_m and C_{m+2} ($m \geq 7$) can be homogeneously embedded in the graph of order $m+5$ shown in Figure 5.8, it follows that $fr(C_m, C_{m+2}) \leq m+5$. However, by Corollary 5.2, for $m \geq 7$, $fr(C_m, C_{m+2}) \geq m+5$. Hence for $m \geq 7$, $fr(C_m, C_{m+2}) = m+5$. $\square$

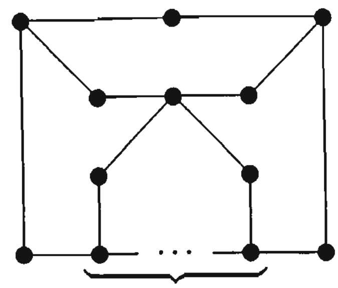


Figure 5.8: A frame for C_m and C_{m+2} $(m \geq 7)$.

5.3 Upper bounds on $fr(C_m, C_n)$

In this section, we establish upper bounds on $fr(C_m, C_n)$ for all integers $n > m \geq 3$.

Theorem 5 . 2 *For integers $n > m \geq 3$,*

$$fr(C_m, C_n) \leq \begin{cases} n + \lceil \frac{n}{3} \rceil & \textit{if } m = 3 \textit{ or } 4 \\ n + \frac{n}{m-1} & \textit{if } m-1 \,|\, n \textit{ and } m > 4 \\ n + \lceil \frac{n}{m-1} \rceil + 1 & \textit{otherwise} \end{cases}$$

Proof. Suppose firstly that $m = 3$. Let $k = \lceil n/3 \rceil$. Let G be the graph obtained from the induced n-cycle $C' : v_0, v_1, v_2, \ldots, v_{n-1}, v_0$ by adding k new vertices $w_0, w_1, \ldots, w_{k-1}$ and, for $i = 0, 1, \ldots, k-1$, joining w_i to the three vertices v_{3i}, v_{3i+1} and v_{3i+2} where addition is taken modulo n. Then each vertex of G clearly belongs to a C_3. Furthermore, the cycle obtained from C' by replacing the vertex v_{3i+1} with the vertex w_i (and the edges $w_i v_{3i}$ and $w_i v_{3i+2}$) is an induced C_n containing w_i $(0 \leq i \leq k-1)$. Hence C_3 and C_n can be homogeneously embedded in the graph G

of order $n+k = n + \lceil n/3 \rceil$. Thus $fr(C_3, C_n) \le n + \lceil \frac{n}{3} \rceil$. If $m = 4$, then let G' be the graph obtained from G by deleting the edges $w_i v_{3i+1}$ for $i = 0, 1, \ldots, k-1$. Then C_4 and C_n can be homogeneously embedded in the graph G' of order $n + k = n + \lceil n/3 \rceil$. Thus $fr(C_4, C_n) \le n + \lceil \frac{n}{3} \rceil$.

Suppose next that $m \ge 5$. Let $\ell = \lceil n/(m-1) \rceil$. Let $G_{m,n}$ be the graph obtained from the induced n-cycle $C' : v_0, v_1, v_2, \ldots, v_{n-1}, v_0$ by adding ℓ new vertices $w_0, w_1, \ldots, w_{\ell-1}$ and, for $i = 0, 1, \ldots, \ell-1$, joining w_i to the three vertices $v_{i(m-1)-1}$, $v_{i(m-1)+1}$ and $v_{(i+1)(m-1)}$ where addition is taken modulo n.

Case 1. $m - 1 \mid n$.

Thus $n = \ell(m-1)$. (The graph $G_{5,16}$ is shown in Figure 5.9.) Then C_m and C_n can be homogeneously embedded in the graph $G_{m,n}$ of order $n + \ell = n + n/(m-1)$. To see this, observe that for $i = 0, 1, \ldots, \ell-1$, each vertex w_i belongs to an induced C_m, namely $C_m^{(i)} : w_i,\ v_{i(m-1)+1},\ v_{i(m-1)+2},\ \ldots,\ v_{(i+1)(m-1)},\ w_i$. Furthermore, replacing the vertex $v_{i(m-1)}$ on C' with the vertex w_i for all $i = 0, 1, \ldots, \ell-1$ produces an induced C_n containing each w_i. Furthermore, each vertex of C' belongs to $C_m^{(i)}$ for exactly one i $(0 \le i \le \ell - 1)$. Consequently, $G_{m,n}$ homogeneously embeds C_m and C_n. Thus, $fr(C_m, C_n) \le n + n/(m-1)$.

Case 2. $m - 1 \mid n + 1$.

Thus $n = \ell(m-1) - 1$. Let $F_{m,n}$ be the graph obtained from $G_{m,n}$ by deleting the edge $w_{\ell-1}v_1$ and adding a new vertex w_ℓ and joining it to v_0, v_2 and $w_{\ell-1}$. (The graph

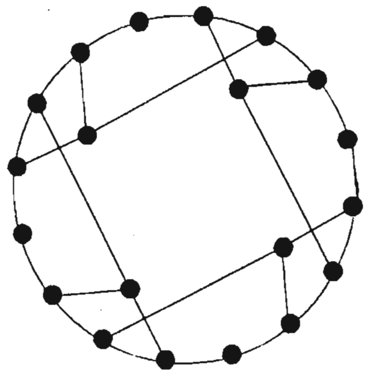


Figure 5.9: The graph $G_{5,16}$.

$F_{5,15}$ is shown in Figure 5.10.) Then C_m and C_n can be homogeneously embedded in the graph $F_{m,n}$ of order $n + \ell + 1 = n + \lceil n/(m-1) \rceil + 1$. Thus, $fr(C_m, C_n) \leq n + \lceil n/(m-1) \rceil + 1$.

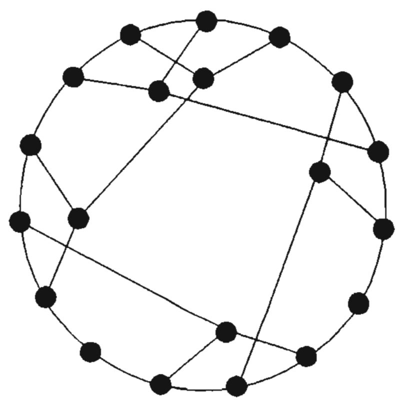


Figure 5.10: The graph $F_{5,15}$.

Case 3. $m-1 \mid n-1$.

Thus $n=(\ell-1)(m-1)+1$. Let $H_{m,n}$ be the graph obtained from $G_{m,n}$ as follows: Delete the edge $w_{\ell-2}v_{n-1}$ and add the edge $w_{\ell-2}v_{(\ell-3)(m-1)}$; delete the three edges incident with $w_{\ell-1}$ and join $w_{\ell-1}$ to $v_{(\ell-2)(m-1)+1}$, v_{n-1} and v_1; add a new vertex w_ℓ and join it to $v_{(\ell-2)(m-1)}$, $v_{(\ell-2)(m-1)+2}$ and v_0. (The graph $H_{6,16}$ is shown in Figure 5.11.) Then C_m and C_n can be homogeneously embedded in the graph $H_{m,n}$ of order $n+\ell+1=n+\lceil n/(m-1)\rceil+1$. Thus, $fr(C_m,C_n)\le n+\lceil n/(m-1)\rceil+1$.

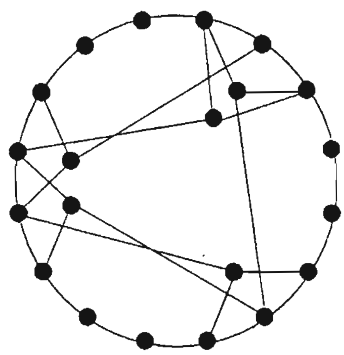


Figure 5.11: The graph $H_{6,16}$.

Case 4. $m-1$ *does not divide* $n-1$ *or* n *or* $n+1$.

Thus $n=(\ell-1)(m-1)+r$ for some r satisfying $1<r<m-2$. Let $I_{m,n}$ be the graph obtained from $G_{m,n}$ by adding a new vertex w_ℓ and joining it to $v_{(\ell-1)(m-1)}$, $v_{\ell(m-1)-1}$ and $v_{\ell(m-1)+1}$ where addition is taken modulo n; that is, w_ℓ is joined to v_{n-r}, v_{m-r-2} and v_{m-r}. (The graph $I_{5,14}$ is shown in Figure 5.12.) Then C_m and C_n can be homogeneously embedded in the graph $I_{m,n}$ of order $n+\ell+1=n+\lceil n/(m-1)\rceil+1$.

Thus, $fr(C_m, C_n) \leq n + \lceil n/(m-1) \rceil + 1$. □

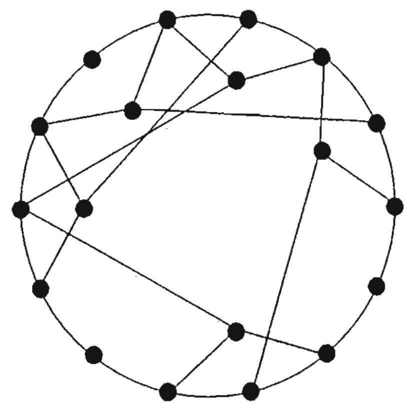


Figure 5.12: The graph $I_{5,14}$.

Two immediate corollaries of Theorems 5.1 and 5.2 and Corollary 5.2 now follow.

Corollary 5 . 3 *For* $m = 3$ *or* $m = 4$ *and* $n = 7, 8, 9$, *or for* $7 \leq m+2 \leq n \leq 2m-5$, $fr(C_m, C_n) = n + 3$.

Corollary 5 . 4 *For* $m \geq 4$ *and* $n = 2(m-1)$ *or* $n = 3(m-1)$,

$$fr(C_m, C_n) = n + \frac{n}{m-1}.$$

5.4 The edge framing number of pairs of cycles

Since $K_{1,3}$ and C_3 are edge isomorphic, the following result is an immediate consequence of Theorem 4.16.

Proposition 5 . 2 *For any integer $n > 3$,*

$$efr(C_3, C_n) = \begin{cases} \lceil \frac{n}{2} \rceil + n & \textit{if } n \equiv 0 \textit{ or } 3 \,(mod\, 4) \\ \lceil \frac{n}{2} \rceil + n + 1 & \textit{if } n \equiv 1 \textit{ or } 2 \,(mod\, 4) \end{cases}$$

Hence in this section we consider integers $n > m \geq 4$. For such integers, every graph that edge homogeneously embeds C_n and C_m also vertex homogeneously embeds C_n and C_m. Hence we have the following corollary of Lemma 5.1.

Corollary 5 . 5 *For integers $n > m \geq 4$, if H is a graph that edge homogeneously embeds C_n and C_m, then $p(H) \geq n + 2$.*

The following lemmas will prove to be useful.

Lemma 5 . 2 *Let G and H be graphs with no induced C_4, and let F be an edge frame of G and H. If u and v are two vertices of degree 2 in F, then $N(u) \neq N(v)$.*

Proof. Assume, to the contrary, that $N(u) = N(v)$. We show then that $F - u$ edge homogeneously embeds G and H. Let $e \in E(G)$ and let $f \in E(F - u)$. Let G_e be an edge embedding of G in F with e at f. If $u \notin V(G_e)$, then G_e is in $F - u$. If $u \in V(G_e)$, then, since $C_4 \not\prec G$, $v \notin V(G_e)$ and therefore $\langle (V(G_e) - \{u\}) \cup \{v\} \rangle$ is an edge embedding of G in $F - u$ with e at f. Hence $F - u$ edge homogeneously embeds G. Similarly, $F - u$ edge homogeneously embeds H. This, however, contradicts the fact that F is an edge frame of G and H. □

Lemma 5 . 3 *For integers $n > m \geq 4$, if H is a graph that edge homogeneously embeds C_n and C_m, then H contains at least three vertices of degree at least* 3.

Proof. Let $C' : v_0, v_1, \ldots, v_{m-1}, v_0$ be an induced C_m in H, and let C'' be an induced C_n in H which contains the edge v_0v_1. Further, let $v_i, v_{i+1}, \ldots, v_0, v_1, \ldots, v_{j-1}, v_j$ $(j < i)$ where addition is taken modulo m, be a longest path common to C' and C'' that contains the edge v_0v_1. Since v_{i-1} and v_{j+1} do not belong to C'', it follows that each of v_i and v_j has degree at least 3. We deduce, therefore, that every induced C_m and C_n contains at least two vertices of degree at least 3.

Suppose that H has exactly two vertices, a and b say, of degree at least 3. Since every induced C_m and C_n contains at least two vertices of degree at least 3, the vertices a and b must lie on every induced C_m and C_n in H. Consequently, the graph H consists of the vertices a and b and a set S of internally disjoint paths joining a and b. Observe that any induced cycle containing an edge of a path from S must contain all the edges of this path. Hence we may denote an induced C_m or C_n containing a path $P \in S$ by $C_m(P)$ or $C_n(P)$, respectively. Let P' be a shortest a-b path, and let $P^{(1)}$ denote the a-b path of length $n - d(a,b)$ on $C_n(P')$ which is disjoint from P'. Furthermore, let $P^{(2)}$ denote the a-b path of length $m - (n - d(a,b))$ on $C_m(P^{(1)})$ which is disjoint from $P^{(1)}$. Then $P^{(2)}$ is an a-b path of length less than $d(a,b)$, which is impossible. The desired result now follows. □

Proposition 5 . 3 *For $m \geq 5$, $efr(C_m, C_{2m-4}) = 2m$. Furthermore, C_m and C_{2m-4} are uniquely edge framed by the graph shown in Figure 5.13.*

Proof. Since C_m and C_{2m-4} can be edge homogeneously embedded in the graph of size $2m$ shown in Figure 5.13, it follows that $efr(C_m, C_{2m-4}) \leq 2m$. Now let F be an edge frame for C_{2m-4} and C_m. By Corollary 5.5, $p(F) \geq 2m - 2$. Applying Theorem 4.8, we have $\delta(F) \geq 2$. Let k be the number of vertices of H of degree at least 3. By Lemma 5.3, $k \geq 3$. Hence $2(2m) \geq 2q(F) \geq 3k + 2(p(F) - k) = 2p(F) + k \geq 2p(F) + 3$ whence $p(F) \leq 2m - 2$. Thus $p(F) = 2m - 2 = fr(C_m, C_{2m-4})$. By Theorem 5.1, the only graph of order $2m - 2$ which both frames C_m and C_{2m-4} and edge homogeneously embeds C_m and C_{2m-4} is the graph shown in Figure 5.13. Consequently, $efr(C_m, C_{2m-4}) = 2m$, and C_m and C_{2m-4} are uniquely edge framed by the graph shown in Figure 5.13. □

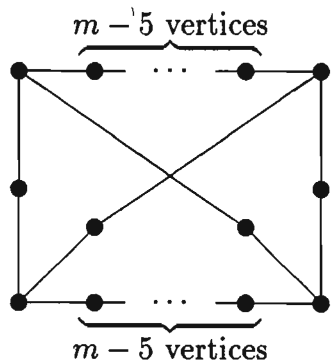


Figure 5.13: An edge frame for C_m and C_{2m-4} for $m \geq 5$.

Lemma 5 . 4 *Let $n > m \geq 4$ where $n \neq 2m - 4$ and $(m, n) \neq (5, 7)$. If a graph H edge homogeneously embeds C_m and C_n, then $p(H) \geq n + 3$.*

Proof. Let H be a graph which edge homogeneously embeds C_m and C_n. By Corollary 5.5, $p(H) \geq n + 2$. Suppose that $p(H) = n + 2$. Then by Lemma 5.1 we deduce that H frames C_m and C_n. By Theorem 5.1 it follows that $(m, n) \in S$, where S is the set of ordered pairs defined in section 5.2 For $(m, n) \in S$ the frames for C_m and C_n have been completely determined in Theorem 5.1 and in each case it is easily checked that H does not edge homogeneously embed C_m and C_n unless $n = 2m - 4$ (in which case H is the graph shown in Figure 5.13) or $n = 2m - 3$ and $m = 5$ (in which case H is the graph shown in Figure 5.14). This produces a contradiction and we deduce that $p(H) \geq n + 3$. □

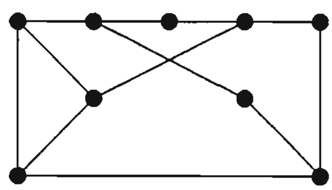


Figure 5.14: An edge frame for C_5 and C_7.

Proposition 5 . 4 *For* $m \geq 7$, $efr(C_m, C_{2m-6}) = 2m - 1$.

Proof. Since C_m and C_{2m-6} can be edge homogeneously embedded in the graph of size $2m - 1$ shown in Figure 5.15, it follows that $efr(C_m, C_{2m-6}) \leq 2m - 1$. We show that $efr(C_m, C_{2m-6}) = 2m - 1$ by verifying that there is no graph of size $2m - 2$ or less which edge homogeneously embeds C_m and C_{2m-6}. Suppose, to the contrary, that such a graph H exists. By Lemma 5.4, $p(H) \geq 2m - 3$. Applying Theorem 4.8, we have $\delta(H) \geq 2$. Let k be the number of vertices of H of degree at least 3. By

Lemma 5.3, $k \geq 3$. Hence $4m - 4 \geq 2q(H) \geq 3k + 2(p(H) - k) = 2p(H) + k \geq 2(2m - 3) + 3 = 4m - 3$, which is impossible. □

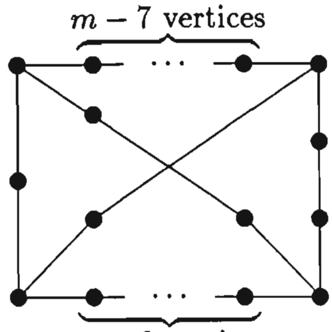


Figure 5.15: An edge frame for C_m and C_{2m-6} for $m \geq 7$.

Lemma 5 . 5 *For $n > m \geq 4$ where $n \neq 2m - 4$ or $n = 2m - 6$, there is no graph of order $n + 3$ and size at most $n + 5$ that edge homogeneously embeds C_m and C_n.*

Proof. Assume, to the contrary, that such a graph H exists. Applying Theorem 4.8, we have $\delta(H) \geq 2$. Let k be the number of vertices of H of degree at least 3. Hence $2n + 10 \geq 2q(H) \geq 3k + 2(p(H) - k) = 2p(H) + k = 2n + 6 + k$, so $k \leq 4$. By Lemma 5.3, $k \geq 3$. Thus $k = 3$ or 4.

Case 1. $k = 3$.

Since every graph contains an even number of vertices of odd degree, at least one vertex of H has degree 4 or more. Thus $2n + 10 \geq 2q(H) \geq 10 + 2(p(H) - 3) = 2p(H) + 4 = 2n + 10$. Since all these inequalities must be equalities, it follows that $q(H) = n + 5$ and H contains two vertices of degree 3, one of degree 4, and n of degree 2. Let w denote the vertex of degree 4. Since no vertex of degree 2 in H can lie on a K_3, and since $q(H) = n + 5$ and $\delta(H) = 2$, it follows that every induced C_n in H must contain the vertex w. Let $C_w : w = w_1, w_2, \ldots, w_n, w_1$ be an induced C_n containing w, and let a, b, and c be the names of the three vertices of H not in C_w. Without loss of generality, we may assume that w is adjacent to a and b. Since $q(H) = n + 5$ and $\delta(H) = 2$, at most one of a and b is adjacent to a vertex of C_w different from w. Without loss of generality, we may assume that b is adjacent to no vertex of C_w other than w. Since no vertex of degree 2 in H can lie on a K_3, and since $q(H) = n+5$, the vertices a and b cannot be adjacent. Hence b is adjacent only to c and w.

Suppose firstly that a is adjacent to c. If $deg\, c = 2$, then c belongs to no induced C_ℓ for $\ell \geq 5$. Hence $deg\, c = 3$. Then a and b are vertices of degree 2 with $N(a) = N(b)$. Thus we must have $m = 4$ otherwise by Lemma 5.2 we have a contradiction. Now c is adjacent with w_j for some j $(2 \leq j \leq n)$. Thus H is the graph shown in Figure 5.16.

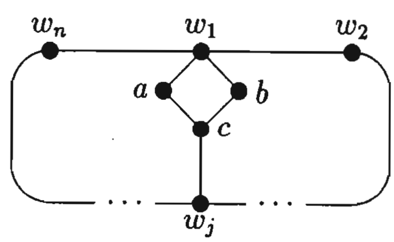


Figure 5.16: The graph H.

Then $deg\, w_j = deg\, c = 3$, $deg\, w_1 = 4$, and the remaining vertices of H have degree 2. Thus any induced C_4 containing the edge w_1w_2 must contain the vertices w_1, w_j,

c and either a or b. Consequently $j = 2$. Similarly, by considering the edge w_1w_n we get $j = n$. Thus $n = 2$, a contradiction. Thus a and c are not adjacent. Since $q(H) = n + 5$, $deg\, a = deg\, c = 2$. Since no vertex of degree 2 belongs to a K_3, the vertex a is not adjacent to w_2 or w_n. Furthermore, the vertex c is not adjacent to w_2 or w_n, for otherwise c belongs to no induced C_n for $n \geq 5$. Without loss of generality, we may assume that a is adjacent to w_r and c is adjacent to w_s where $3 \leq s < r \leq n-1$. The graph H is shown in Figure 5.17.

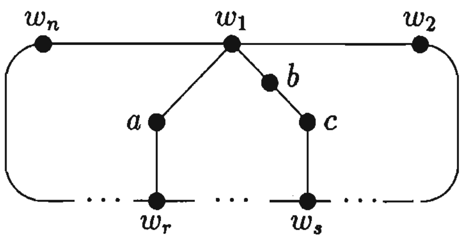


Figure 5.17: The graph H.

Since the vertex b belongs to no C_4, we must have $m \geq 5$. If $r = n-1$, then a and w_n are vertices of degree 2 with $N(a) = N(w_n)$ which contradicts Lemma 5.2. Hence $r \leq n-2$. We now consider the vertex a. The vertex a belongs to three cycles, namely, $C^{(1)} : a, w_r, w_{r+1}, \ldots, w_n, w_1, a$ (of length $n-r+3$), $C^{(2)} : a, w_1, w_2, \ldots, w_r, a$ (of length $r+1$) and $C^{(3)} : a, w_1, b, c, w_s, w_{s+1}, \ldots, w_r, a$ (of length $r-s+5$). At least one of these cycles is of length n. If $C^{(1)}$ has length n, then $r = s = 3$ contradicting $r > s$. If $C^{(2)}$ has length n, then $r = n-1$ contradicting $r \leq n-2$. Therefore $C^{(3)}$ must be of length n, implying that $n-2 \geq r = n+s-5$, so $s \leq 3$. Thus $s = 3$ and $r = n-2$. But then the vertex w_n belongs to three cycles of lengths 5, n and $n+1$.

Hence $m = 5$. However the edge w_3w_4 then belongs to no C_5, a contradiction. Hence Case 1 produces a contradiction.

Case 2. $k = 4$.

Then $2n + 10 \geq 2q(H) \geq 2n + 6 + k = 2n + 10$. Since all these inequalities must be equalities, it follows that $q(H) = n + 5$ and H contains four vertices of degree 3 and $n - 1$ vertices of degree 2. The following claim will prove to be useful.

Claim 5 . 1 *If C' is an induced C_n in H and U the set of three vertices of H that do not belong to C', then $\langle U \rangle \cong K_1 \cup K_2$ or P_3. Furthermore, if $\langle U \rangle \cong K_1 \cup K_2$, then each vertex of U has degree 2 in H. If $\langle U \rangle \cong P_3$, then the central vertex of this P_3 has degree 3 in H and the two end-vertices have degree 2 in H.*

Proof. Since $q(H) = n + 5$, there are exactly five edges incident with the vertices of U. Since $\delta(H) = 2$, and no vertex of degree 2 belongs to a K_3, a simple counting argument shows that $q(\langle U \rangle) = 1$ or 2. Hence $\langle U \rangle \cong K_1 \cup K_2$ or P_3. If $\langle U \rangle \cong K_1 \cup K_2$, then, since $q(H) = n + 5$, each vertex of U has degree 2 in H. If $\langle U \rangle \cong P_3$, then three of the five edges incident with vertices of U are also incident with vertices of C'. It follows that exactly three of the four vertices of degree 3 belong to C' and the remaining vertex of degree 3 is in U. Hence one vertex of U has degree 3 and the remaining two vertices have degree 2. Suppose $\langle U \rangle$ is the path a, b, c, and C' is the (induced) cycle $v_1, v_2, \ldots, v_n, v_1$. We show that $deg\, b = 3$. If this is not the case, then we may assume that $deg\, a = 3$ and $deg\, b = deg\, c = 2$. Without loss of generality, we may assume av_1, av_i and cv_j are edges of H where $2 \leq i < j \leq n$. The graph H is shown in Figure 5.18.

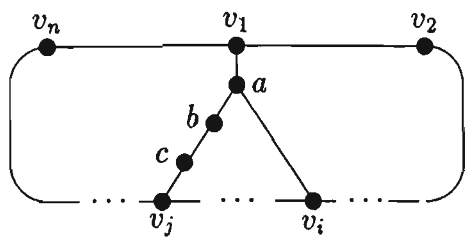


Figure 5.18: The graph H.

Since the vertex b belongs to no 4-cycle, we may assume here that $n > m \geq 5$. Now there are only two induced cycles containing the edge v_1v_2, namely C' and the cycle $C'' : v_1, v_2, \ldots, v_i, a, v_1$. Since C' has length n, C'' must have length m so that $i = m - 1$. We now consider the edge av_1. The edge av_1 belongs to three induced cycles, namely, C'' (of length m), $v_1, a, v_{m-1}, v_m, v_{m+1}, \ldots, v_n, v_1$ (of length $n-m+4$) and $C''' : v_1, a, b, c, v_j, \ldots, v_n, v_1$ (of length $n - j + 5$). Thus $n = n - m + 4$ or $n = n - j + 5$. If $n = n - m + 4$, then $m = 4$ contradicting $m \geq 5$. Thus C''' has length n and $j = 5$. Hence $m - 1 = i \leq j - 1 = 4$, so $m \leq 5$, i.e., $m = 5$. But then the edge av_4 belongs to no C_n, a contradiction. We deduce, therefore, that $deg\, b = 3$ and $deg\, a = deg\, c = 2$. This completes the proof of the claim. □

We now return to the proof of Case 2. Let u and v be two (distinct) vertices of degree 3 for which $d(u, v)$ is a *minimum*, and let P be a shortest u-v path. Then all interior vertices (if any) of P have degree 2. Let $C_P : v_1, v_2, \ldots, v_n, v_1$ be an induced C_n containing an edge of P. Necessarily, C_P contains all edges of P. Let a, b, c be the three vertices of H that do not belong to C_P. By Claim 5.1, $\langle\{a,b,c\}\rangle \cong K_1 \cup K_2$ or P_3. We consider the two possibilities in turn.

Case 2.1 $\langle\{a,b,c\}\rangle \cong P_3$.

Without loss of generality, we may assume that a, b, c is a path. By Claim 5.1, $deg\, b = 3$ and $deg\, a = deg\, c = 2$. Since b is adjacent to a vertex of degree 3 of C_P, our choice of u and v implies that $d(u,v) = 1$, so u and v are adjacent vertices on C_P. Without loss of generality, we may assume that $u = v_1$ and $v = v_2$. If b is adjacent to either u or v, then, without loss of generality, H is then the graph shown in Figure 5.19(i). Since the vertex a belongs to induced cycles of only two possible lengths, namely, 4 and n, we must have $m = 4$. But then the edge v_1v_n belongs to no C_m, a contradiction. Hence b is adjacent to neither u nor v, so bv_i is an edge for some i ($3 \le i \le n$).

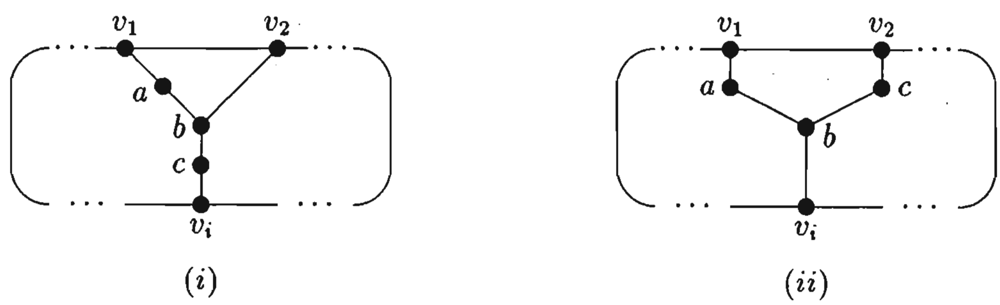


Figure 5.19: The graph H.

Without loss of generality, H is then the graph shown in Figure 5.19(ii). Since the edge v_1v_2 belongs to no 4-cycle, we must have $m \ge 5$. The edge bc belongs to three cycles, namely b, c, v_2, v_1, a, b (of length 5), $b, c, v_2, v_3, \ldots, v_i, b$ (of length $i+1$) and $c, b, v_i, v_{i+1}, \ldots, v_n, v_1, v_2, c$ (of length $n-i+5$). Since $n > 5$, we must have $n = i+1 \, or \, n-i+5$. Suppose $n = n-i+5$. Then $i = 5$ and the edge v_1v_n lies on cycles of only two possible lengths, namely, $n-1$ and n. Hence $m = n-1$. Now the edge v_1v_2 (v_2v_3) lies on cycles of length $5, 7$ and n $(6, 7$ and n, respectively). We deduce that $m = 7$ and $n = 8$. However, then, $n = 2m - 6$ which is contrary to our choice of m and n. Thus $n = i+1$, i.e., $i = n-1$. The edge v_2v_3 then lies only on cycles of length n and $n+1$ so that v_2v_3 does not lie on any cycle of length m. This produces a contradiction.

Case 2.2 $\langle\{a, b, c\}\rangle \cong K_1 \cup K_2$.

Without loss of generality, we may assume that a is the isolated vertex in $\langle U\rangle$, so bc is an edge. By Claim 5.1, each of a, b and c has degree 2. Let C_a be an induced C_n containing the vertex a. We show that the edge bc belongs to C_a. If this is not the case, then, without loss of generality, we may assume that C_a is $a, v_2, v_3, \ldots, v_n, a$. By Claim 5.1, the three vertices v_1, b and c that do not belong to C_a induce either a P_3 or $K_1 \cup K_2$. If $\langle\{v_1, b, c\}\rangle \cong P_3$, then, since $\Delta(H) = 3$, the vertex v_1 must be an end-vertex of $\langle\{v_1, b, c\}\rangle \cong P_3$. But then v_1 has degree 3 in H which contradicts Claim 5.1. Thus $\langle\{v_1, b, c\}\rangle \cong K_1 \cup K_2$ and v_1 has degree 2 in H. Hence a and v_1 are two nonadjacent vertices of degree 2 in H with $N(a) = N(v_1)$. This, however, contradicts Lemma 5.2 if $m \geq 5$. Hence $m = 4$. Without loss of generality, we may assume that the vertex b (c) is adjacent with the vertex v_i (v_j, respectively) where

$3 \leq i < j \leq n-1$. Since the edge bc must lie on an induced C_4, it follows that $j = i+1$. However the edge bc then belongs to no cycle of length 5 or more. This produces a contradiction. We deduce, therefore, that the edge bc must belong to C_a.

Let S be the set of three vertices of C_P that do not belong to C_a. By Claim 5.1, $\langle S \rangle \cong K_1 \cup K_2$ or P_3. Clearly, $\langle S \rangle \cong K_1 \cup K_2$. Without loss of generality, we may assume that $S = \{v_2, v_i, v_{i+1}\}$ where $5 \leq i \leq n-2$. Hence $n \geq 7$, and v_1, v_3, v_{i-1} and v_{i+2} are the four vertices of degree 3 in H. If $N(a) = N(v_2)$, then, since the edge bc belongs to cycles only of length 6 and n, it follows that $m = 6$. However, the vertex a belongs to cycles only of length 4 and n, so $m = 4$, a contradiction. Hence $N(a) \neq N(v_2)$.

If C_a is given by $v_1, b, c, v_3, v_4 \ldots, v_{i-1}, a, v_{i+2}, \ldots, v_n, v_1$, then H is the graph shown in Figure 5.20(i). Now the edge v_1v_n belongs to cycles of length $n-1, n, n+1$. Thus $m = n-1$. However, the edge bc belongs to no induced C_{n-1} ($n \geq 7$). Hence we may assume, without loss of generality, that C_a is given by either $C_a^{(1)}$: $v_1, a, v_{i-1}, v_{i-2}, \ldots, v_3, b, c, v_{i+2}, \ldots, v_n, v_1$, in which case H is the graph shown in Figure 5.20(ii), or $C_a^{(1)}$: $v_1, b, c, v_{i-1}, v_{i-2}, \ldots, v_3, a, v_{i+2}, \ldots, v_n, v_1$, in which case H is the graph shown in Figure 5.20(iii). If C_a is $C_a^{(1)}$, then the edge v_1v_n belongs to cycles of length $n-i+4$ and n. Thus $m = n-i+4$. Furthermore, the edge v_3v_4 belongs to cycles of length $i, i+2$ and n. Thus $m = i$ or $i+2$. If $m = i$, then $n = 2m-4$ and if $m = i+2$, then $n = 2m-6$. In either case we contradict our choice of m and n. A similar argument shows that C_a cannot be $C_a^{(2)}$. This completes the proof of Case 2.2, and therefore of

Lemma 5.5. □

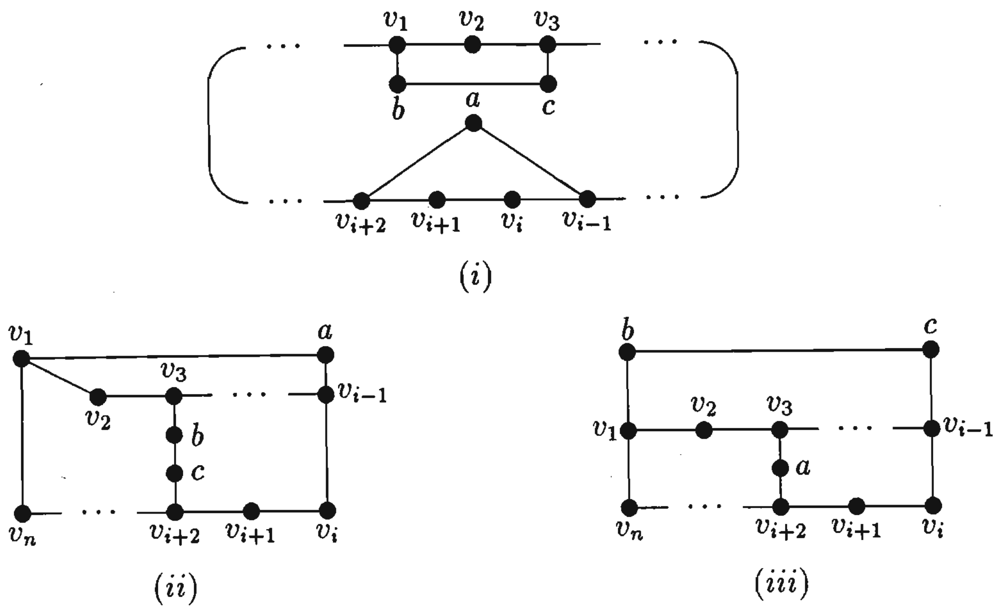


Figure 5.20:

Corollary 5 . 6 *For $n > m \geq 4$ with $n \neq 2m-4$ or $n \neq 2m-6$ and $(m,n) \neq (5,7)$,*

$$efr(C_m, C_n) \geq n + 6.$$

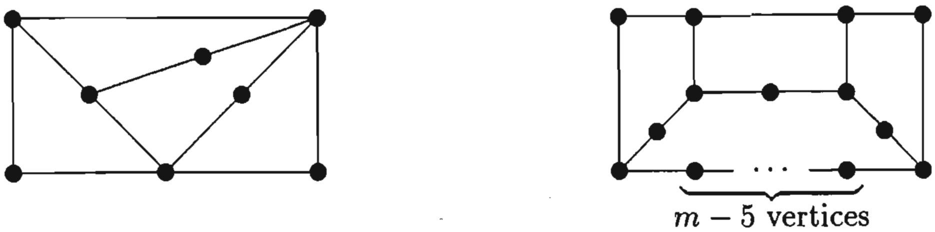
Proof. We show that $efr(C_m, C_n) \geq n + 6$ by verifying that there is no graph of size $n + 5$ or less which edge homogeneously embeds C_m and C_n. Suppose, to the contrary, that such a graph H exists. By Lemma 5.4, $p(H) \geq n + 3$, and by Lemma 5.5, $p(H) \neq n + 3$; consequently, $p(H) \geq n + 4$. Applying Theorem 4.8, we have $\delta(H) \geq 2$. Let k be the number of vertices of H of degree at least 3. By Lemma 5.3, $k \geq 3$. Hence $2n+10 \geq 2q(H) \geq 3k+2(p(H)-k) = 2p(H)+k \geq 2n+11$, which is impossible. □

Corollary 5 . 7 *For $m \geq 7$, C_m and C_{2m-6} are uniquely edge framed by the the graph of size $2m-1$ shown in Figure 5.15.*

Proof. Let F be an edge frame for C_m and C_{2m-6}. Then by Proposition 5.4, $q(F) = 2m-1$. By Corollary 5.5, $p(F) \geq 2m-4$. Applying Theorem 4.8, we have $\delta(F) \geq 2$. Let k be the number of vertices of F of degree at least 3. By Lemma 5.3, $k \geq 3$. Hence $4m-2 = 2q(F) \geq 3k + 2(p(F)-k) = 2p(F) + k \geq 2p(F) + 3$, whence $p(F) \leq 2m-3$. Thus $2m-4 \leq p(F) \leq 2m-3$. If $p(F) = 2m-4$, then $p(F) = fr(C_m, C_{2m-6})$ and so F frames C_m and C_{2m-6}. However, by Theorem 5.1, there is no graph of order $2m-4$ which edge homogeneously embeds C_m and C_{2m-6} for $m \geq 7$. Thus $p(F) = 2m-3 = (2m-6)+3$. From the proof of Lemma 5.5 we deduce that C_m and C_{2m-6} have at most one edge frame. We conclude that C_m and C_{2m-6} are uniquely edge framed. $\square$

Proposition 5 . 5 *For $m \geq 4$ and $m \notin \{5,7\}$, $efr(C_m, C_{m+1}) = m+7$.*

Proof. Since C_m and C_{m+1} can be edge homogeneously embedded in the graph of size $m+7$ shown in Figure 5.21(i) for $m = 4$ and in Figure 5.21(ii) for $m = 6$ or $m \geq 8$, it follows that $efr(C_m, C_{m+1}) \leq m+7$. By Corollary 5.6, $efr(C_m, C_{m+1}) \geq m+7$. Consequently $efr(C_m, C_{m+1}) = m+7$ as required. $\square$



(i) (ii)

Figure 5.21:

Proposition 5 . 6 *For* $m \geq 10$, $efr(C_m, C_{2m-8}) = 2m - 2$.

Proof. Since C_{2m-8} and C_m can be edge homogeneously embedded in the graph of size $2m-2$ shown in Figure 5.22, it follows that $efr(C_{2m-8}, C_m) \leq 2m-2$. By Corollary 5.6, $efr(C_{2m-8}, C_m) \geq 2m-2$. Consequently $efr(C_{2m-8}, C_m) = 2m-2$ as required.□

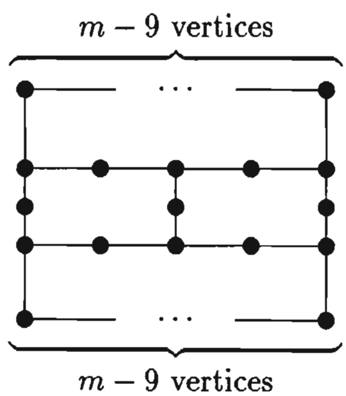


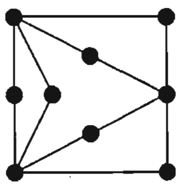
Figure 5.22: An edge frame for C_m and C_{2m-8} for $m \geq 10$.

Proposition 5 . 7 $efr(C_5, C_7) = 12$.

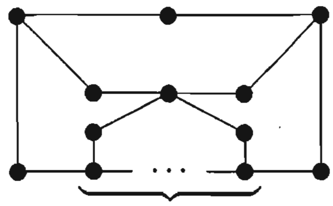
Proof. Since C_5 and C_7 can be edge homogeneously embedded in the graph $F_{5,7}$ (without the dotted edges) of size 12 shown in Figure 5.2, it follows that $efr(C_5, C_7) \leq 12$. We show that $efr(C_5, C_7) = 12$ by verifying that there is no graph of size at most 11 which edge homogeneously embeds C_5 and C_7. Suppose, to the contrary, that such a graph H exists. By Corollary 5.5, $p(H) \geq 9$. Applying Theorem 5.8, we have $\delta(H) \geq 2$. Let k be the number of vertices of H of degree at least 3. By Lemma 5.3, $k \geq 3$. Hence $22 \geq 2q(H) \geq 3k + 2(p(H) - k) = 2p(H) + k \geq 2p(H) + 3$ whence $p(H) \leq 9$. Consequently, $p(H) = 9 = fr(C_5, C_7)$ and so H frames C_5 and C_7. However, from Theorem 5.1, the frames for C_5 and C_7 all have sizes greater than 11. This produces a contradiction. □

Proposition 5 . 8 *For* $m = 4$ *or* $m \geq 7$, $efr(C_m, C_{m+2}) = m + 8$.

Proof. Since C_m and C_{m+2} can be edge homogeneously embedded in the graph of size $m + 8$ shown in Figure 5.23(i) for $m = 4$ and in Figure 5.23(ii) for $m \geq 7$, it follows that $efr(C_m, C_{m+2}) \leq m + 8$. By Corollary 5.6, $efr(C_m, C_{m+2}) \geq m + 8$. Consequently $efr(C_m, C_{m+2}) = m + 8$ as required. □



(i)



(ii)

Figure 5.23:

5.5 Framing numbers of pairs of directed cycles

In this we investigate the framing number $fr(G_1, G_2)$ for several pairs G_1, G_2 of directed cycles.

The proof of the following result is very similar to the proof of Corollary 3.2 and is therefore omitted.

Theorem 5 . 3 *For digraphs D_1 and D_2, there exists a positive integer m such that for each integer $n \geq m$, there is a digraph H of order n in which D_1 and D_2 can be homogeneously embedded, while for each positive integer $n < m$, no such digraph H of order n exists.*

Proposition 5 . 9 $fr(\vec{C}_3, \vec{C}_4) = 6$.

Proof. The digraph F of order 6 shown in Figure 5.24 has the property that $\vec{C}_3$ and $\vec{C}_4$ can be homogeneously embedded in F. Therefore, $fr(\vec{C}_3, \vec{C}_4) \leq 6$. However, it is

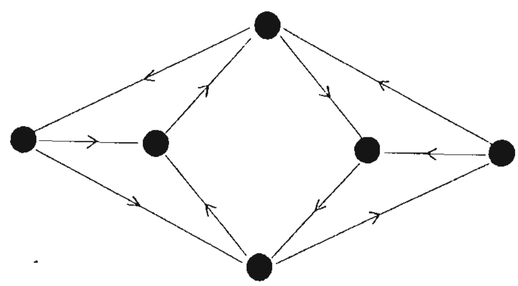


Figure 5.24: A frame for $\vec{C}_3$ and $\vec{C}_4$.

shown in [2] that $fr(C_3, C_4) = 6$. Hence, according to Proposition 3.2, $fr(\vec{C}_3, \vec{C}_4) \geq 6$. Thus $fr(\vec{C}_3, \vec{C}_4) = 6$. □

Proposition 5 . 10 $fr(\vec{C}_3, \vec{C}_5) = 8$.

Proof. The digraph F of order 8 shown in Figure 5.25 has the property that $\vec{C}_3$ and $\vec{C}_5$ can be homogeneously embedded in F. Therefore, $fr(\vec{C}_3, \vec{C}_5) \leq 8$. By Theorem 5.3, it will follow that $fr(\vec{C}_3, \vec{C}_5) = 8$ once we show that there does not exist a digraph H of order 7 in which $\vec{C}_3$ and $\vec{C}_5$ can be homogeneously embedded.

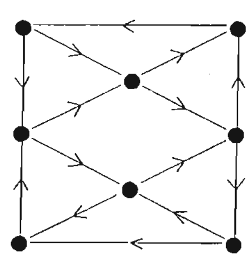


Figure 5.25: A frame for $\vec{C}_3$ and $\vec{C}_5$.

Suppose, to the contrary, that there exists such a digraph H. Each vertex of H must belong to a $\vec{C}_3$ and an induced $\vec{C}_5$, so $\delta(H) \geq 3$. Futher, since H homogeneously embeds $\vec{C}_5$, Lemma 3.2 implies that $\Delta(H) \leq 7 - 5 + 2 = 4$. First we claim that H does not contain two disjoint copies of $\vec{C}_3$. Suppose, to the contrary, that H contains two disjoint copies F_1 and F_2 of $\vec{C}_3$. Let $V(F_1) = \{a, b, c\}$ and let $V(F_2) = \{d, e, f\}$ and let g be the vertex of H not belonging to F_1 and F_2. Then every induced $\vec{C}_5$ of H must contain the vertex g and exactly two vertices from each of F_1 and F_2. Without loss of generality, we may assume that the digraph shown in Figure 5.26 is a subdigraph of H. Now let H_c be an induced subdigraph of H that is isomorphic to $\vec{C}_5$ and that contains the vertex c. Since g belongs to H_c, the vertex a cannot belong to H_c. This in turn implies that b belongs to H_c, and therefore that d does not belong to H_c. It follows that $V(H_c) = \{b, c, e, f, g\}$. However, then the vertex e has outdegree at least two in H_c, which produces a contradiction. Thus, as claimed, H does not contain two disjoint copies of $\vec{C}_3$.

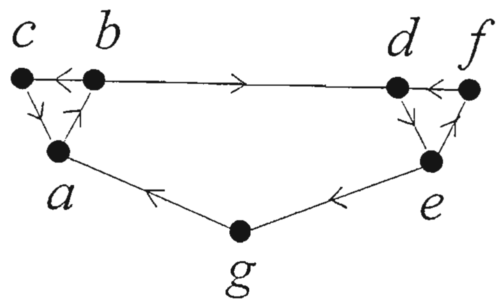


Figure 5.26: A subdigraph of H.

Next we show that H does not contain two $\vec{C}_3$'s having exactly one vertex in common, for suppose that it does. Then H has the subdigraph shown in Figure 5.27. Then $deg\, a = 4$. Let T_f (T_g) be a $\vec{C}_3$ that contains the vertex f (g, respectively). Since the vertex a does not belong to T_f, neither can the vertex g, for otherwise this would produce two disjoint copies of $\vec{C}_3$. Similarly, $f \notin V(T_g)$. For the same reason, T_f and T_g have at least one vertex in common. Without loss of generality, we may therefore assume that the vertex b belongs to T_f and to T_g. Then T_f consists of b, f and exactly one of d and e. This implies, however, that $deg\, b \geq 5$, which contradicts that fact that $\Delta(H) = 4$.

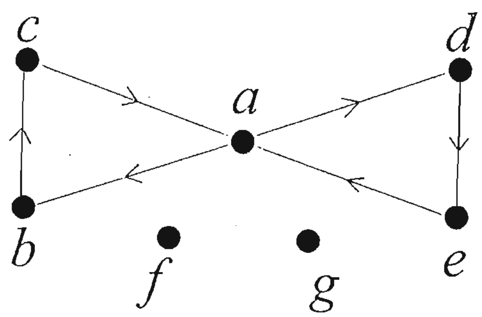


Figure 5.27: A subdigraph of H.

Hence every two $\vec{C}_3$'s of H share a common arc. But this implies that some arc (u, v) lies on every $\vec{C}_3$ of H. However, since every vertex of H belongs to a $\vec{C}_3$, both u and v have degree 6 in H, which is impossible. □

The proof of the next result is similar to that of Proposition 5.10, and is therefore omitted.

Proposition 5 . 11 $fr(\vec{C}_3, \vec{C}_6) = 9$.

A frame for $\vec{C}_3$ and $\vec{C}_6$ is shown in Figure 5.28.

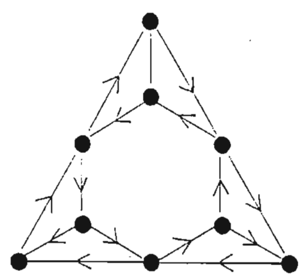


Figure 5.28: A frame for $\vec{C}_3$ and $\vec{C}_6$

Proposition 5 . 12 $fr(\vec{C}_4, \vec{C}_5) = 8$.

Proof. The digraph F of order 8 shown in Figure 5.29 has the property that $\vec{C}_4$ and $\vec{C}_5$ can be homogeneously embedded in F. Therefore, $fr(\vec{C}_4, \vec{C}_5) \leq 8$. By Theorem 5.3, it will follow that $fr(\vec{C}_4, \vec{C}_5) = 8$ once we show that there does not exist a digraph H of order 7 in which $\vec{C}_4$ and $\vec{C}_5$ can be homogeneously embedded. Suppose, to the contrary, that such a digraph H exists. Then $2 \leq \delta(H) \leq \Delta(H) \leq 4$. Before proceeding further, we prove the following claim.

Claim 5 . 2 *If H' is an induced $\vec{C}_5$ in H, then the two vertices of H not in H' do not belong to a common $\vec{C}_5$.*

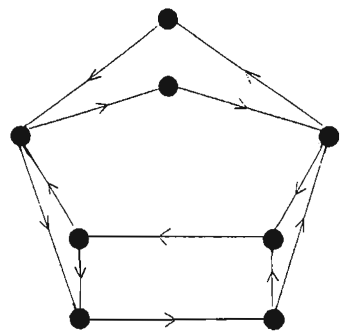
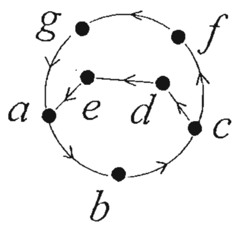


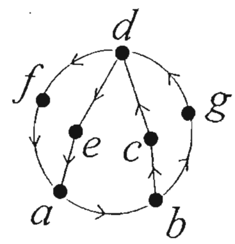
Figure 5.29: A frame for $\vec{C}_4$ and $\vec{C}_5$.

Proof. Let H' be a, b, c, d, e, a and let f and g be the names of the two remaining vertices of H. Assume, to the contrary, that f and g belong to a common induced $\vec{C}_5$, say T_f. If f and g are adjacent vertices on T_f, then, without loss of generality, we may assume that T_f is a, b, c, f, g, a. Hence H has the subdigraph shown in Figure 5.30(a). Then $deg\, a = deg\, c = 3$ and $deg\, b = 2$. However, there is then no induced $\vec{C}_4$ containing the vertex b.

On the other hand, if f and g are not adjacent vertices on T_f, then, without loss of generality, we may assume that T_f is a, b, g, d, f, a. Hence H has the subdigraph shown in Figure 5.30(b). Then $deg\, a = deg\, b = 3$ and $deg\, d = 4$. However, there is then no induced $\vec{C}_4$ containing the vertex d. (This is evident since such a $\vec{C}_4$ would contain exactly one vertex from each of $\{e, f\}$ and $\{c, g\}$, and therefore a vertex $x \in \{a, b\}$. But then the vertex x would have degree 1 in such a $\vec{C}_4$, which is impossible.) $\square$



(a)



(b)

Figure 5.30: A subdigraph of H.

The digraph H must contain $\vec{C}_5$ as an induced subdigraph, say a, b, c, d, e, a. Let f and g be the names of the two remaining vertices of H. Further, let T_f (T_g) be an induced $\vec{C}_5$ that contains the vertex f (g, respectively). By Claim 5.2, f and g do not belong to a common induced $\vec{C}_5$. We may assume, without loss of generality, that T_f is a, b, c, d, f, a. Since the vertices e and g do not belong to T_f, Claim 5.2 implies that e and g do not belong to a common induced $\vec{C}_5$. Hence T_g contains the vertices a, b, c and d. Thus H has the subdigraph shown in Figure 5.31. Then $deg\, a = deg\, d = 4$ and $deg\, b = deg\, c = 2$. However, there is then no induced $\vec{C}_4$ containing the vertex b.

□

Proposition 5 . 13 $fr(\vec{C}_4, \vec{C}_6) = 8$.

Proof. The digraph F of order 8 shown in Figure 5.32 has the property that $\vec{C}_4$ and $\vec{C}_6$ can be homogeneously embedded in F. Therefore, $fr(\vec{C}_4, \vec{C}_6) \leq 8$. However, it is

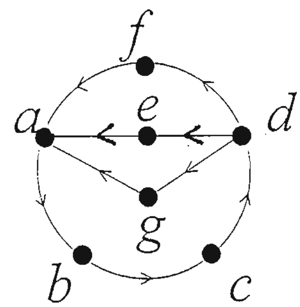


Figure 5.31: A subdigraph of H.

shown in [2] that $fr(C_4, C_6) = 8$. Hence, according to Proposition 5.2, $fr(\vec{C}_4, \vec{C}_6) \geq 8$. Thus $fr(\vec{C}_4, \vec{C}_6) = 8$. $\square$

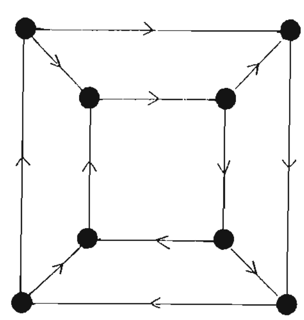


Figure 5.32: A frame for $\vec{C}_4$ and $\vec{C}_6$.

We are now in a position to characterize all those pairs of dicycles $\vec{C}_m$ and $\vec{C}_n$ $(n > m \geq 3)$ which have framing number $n + 2$.

Theorem 5 . 4 *For integers $n > m \geq 3$, $fr(\vec{C}_m, \vec{C}_n) = n+2$ if and only $n = 2m-2$ where $m \geq 4$. Furthermore $\vec{C}_m$ and $\vec{C}_{2m-2}$ have exactly five nonisomorphic frames.*

Proof. First suppose that $n = 2m - 2$ where $m \geq 4$. Then, by Theorem 5.1 and Proposition 3.2, $fr(\vec{C}_m, \vec{C}_n) \geq fr(C_m, C_n) = n + 2$. However, since $\vec{C}_m$ and $\vec{C}_n$ can be homogeneously embedded in the digraph of order $n + 2$ shown in Figure 5.33, we have $fr(\vec{C}_m, \vec{C}_n) \leq n + 2$. Consequently, $fr(\vec{C}_m, \vec{C}_n) = n + 2$.

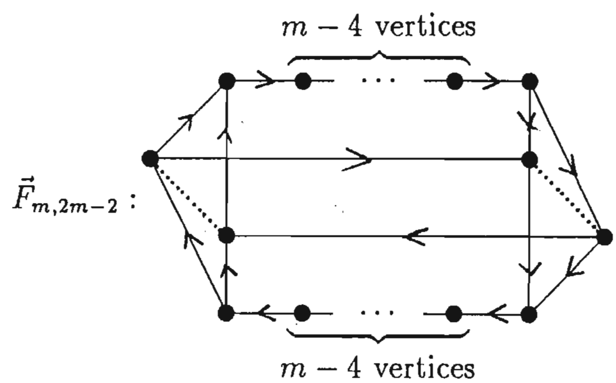


Figure 5.33:

Next we consider the necessity. Let m and n be integers satisfying $n > m \geq 3$ and assume that $fr(\vec{C}_m, \vec{C}_n) = n + 2$. Let D be a frame for $\vec{C}_m$ and $\vec{C}_n$ and let D' be the underlying graph of D. Since $fr(\vec{C}_m, \vec{C}_n) = n + 2$, by Proposition 5.2 and Lemma 5.1 we conclude that $fr(C_m, C_n) = n + 2$. By Theorem 5.1 it follows that (m, n) must belong to the set S (defined in section 5.2) and that the graph D' must belong to $\mathcal{F}_{m,n}$ (also defined in section 5.2). By Propositions 5.9, 5.10, 5.11, 5.12 and 5.13, with the possible exception of the graphs $H_{4,6}$ and $H_{6,10}$ shown in Figure 5.1, the graph D' cannot be any of the graphs shown in Figure 5.1. It is easily checked that D' cannot be the graph $H_{6,10}$. Futhermore, it is easily checked that D' cannot be any of the graphs $F_{m,m+1}$, $F_{m,2m-4}$, $F_{m,2m-3}$ and $G_{m,2m-2}$ of Figure 5.2. Thus the possibilities for D' includes the graph $H_{4,6}$ in Figure 5.1 and the graph $F_{m,2m-2}$ of Figure 5.2. Thus $n = 2m-2$ and the possible frames for $\vec{C}_m$ and $\vec{C}_n$ are obtainable from the digraph $\vec{F}_{m,2m-2}$ by adding and orienting any combination (the presence or absence) of the dotted edges. This yields the five nonisomorphic frames for $\vec{C}_m$ and $\vec{C}_n$. □

Corollary 5 . 8 *For integers $n > m \geq 3$, if $n \neq 2m-2$, then $fr(\vec{C}_m, \vec{C}_n) \geq n+3$.*

5.6 Upper bounds on $fr(\vec{C}_m, \vec{C}_n)$

In this section, we establish upper bounds on $fr(\vec{C}_m, \vec{C}_n)$ for all integers $n > m \geq 3$.

Theorem 5 . 5 *For integers $n > m \geq 3$,*

$$fr(\vec{C}_m, \vec{C}_n) \leq \begin{cases} n + \lceil \frac{n}{2} \rceil & \text{if } m = 3 \text{ or } 4 \\ n + \frac{n}{m-1} & \text{if } m-1 \mid n \text{ and } m > 4 \\ n + \lceil \frac{n}{m-1} \rceil + 1 & \text{otherwise} \end{cases}$$

Proof. Suppose firstly that $m = 3$. Let $k = \lceil n/2 \rceil$. Let G be the digraph obtained from the induced n-cycle $C' : v_0, v_1, v_2, \ldots, v_{n-1}, v_0$ by adding k new vertices $w_0, w_1, \ldots, w_{k-1}$ and, for $i = 0, 1, \ldots, k-1$, joining w_i to v_{2i}, to v_{2i+1} and from v_{2i+2} where addition is taken modulo n. If n is odd, then join w_0 to w_{k-1}. Then each vertex of G clearly belongs to a $\vec{C}_3$. Let $S = \{w_0, w_1, \ldots, w_{k-1}\} \cup \{v_2, v_4, \ldots v_{2k-2}\}$. If n is even then add the vertex v_0 to S. Then the subdigraph induced by the vertices of S is isomorphic to $\vec{C}_n$. Hence $\vec{C}_3$ and $\vec{C}_n$ can be homogeneously embedded in the digraph G of order $n + k = n + \lceil n/2 \rceil$. Thus $fr(\vec{C}_3, \vec{C}_n) \leq n + \lceil \frac{n}{2} \rceil$.

If $m = 4$, then let G' be the digraph obtained from G by deleting the arcs (w_i, v_{2i+1}) for $i = 0, 1, \ldots, k-1$. Then $\vec{C}_4$ and $\vec{C}_n$ can be homogeneously embedded in the digraph G' of order $n + k = n + \lceil n/3 \rceil$. Thus $fr(\vec{C}_4, \vec{C}_n) \leq n + \lceil \frac{n}{2} \rceil$.

Suppose next that $m \geq 5$. Let $\ell = \lceil n/(m-1) \rceil$. Let $\vec{G}_{m,n}$ be the digraph obtained from the induced n-cycle $C' : v_0, v_1, v_2, \ldots, v_{n-1}, v_0$ by adding ℓ new vertices $w_0, w_1, \ldots, w_{\ell-1}$ and, for $i = 0, 1, \ldots, \ell-1$, joining w_i from $v_{i(m-1)-1}$, to $v_{i(m-1)+1}$ and from $v_{(i+1)(m-1)}$ where addition is taken modulo n.

Case 1. $m - 1 \mid n$.

Thus $n = \ell(m-1)$. (The digraph $\vec{G}_{5,16}$ is shown in Figure 5.34.) Then $\vec{C}_m$ and $\vec{C}_n$ can be homogeneously embedded in the digraph $\vec{G}_{m,n}$ of order $n + \ell = n + n/(m-1)$. To see this, observe that for $i = 0, 1, \ldots, \ell - 1$, each vertex w_i belongs to an induced $\vec{C}_m$, namely $C_m^{(i)} : w_i,\ v_{i(m-1)+1},\ v_{i(m-1)+2},\ \ldots,\ v_{(i+1)(m-1)},\ w_i$. Furthermore, replacing the vertex $v_{i(m-1)}$ on C' with the vertex w_i for all $i = 0, 1, \ldots, \ell - 1$ produces an induced $\vec{C}_n$ containing each w_i. Furthermore, each vertex of C' belongs to $C_m^{(i)}$ for exactly one i $(0 \leq i \leq \ell - 1)$. Consequently, $\vec{G}_{m,n}$ homogeneously embeds $\vec{C}_m$ and $\vec{C}_n$. Thus, $fr(\vec{C}_m, \vec{C}_n) \leq n + n/(m-1)$.

Case 2. $m - 1 \mid n + 1$.

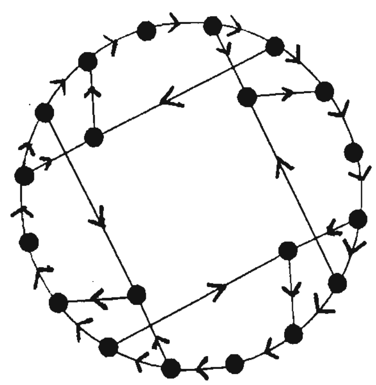


Figure 5.34: The digraph $\vec{G}_{5,16}$.

Thus $n = \ell(m-1) - 1$. Let $\vec{F}_{m,n}$ be the digraph obtained from $\vec{G}_{m,n}$ by deleting the arc $(v_1, w_{\ell-1})$ and adding a new vertex w_ℓ and joining it from v_0, to v_2 and to $w_{\ell-1}$. (The graph $\vec{F}_{5,15}$ is shown in Figure 5.35.) Then $\vec{C}_m$ and $\vec{C}_n$ can be homogeneously embedded in the digraph $\vec{F}_{m,n}$ of order $n + \ell + 1 = n + \lceil n/(m-1) \rceil + 1$. Thus, $fr(\vec{C}_m, \vec{C}_n) \leq n + \lceil n/(m-1) \rceil + 1$.

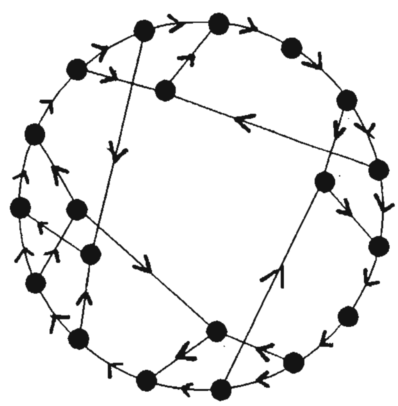


Figure 5.35: The graph $\vec{F}_{5,15}$.

Case 3. $m-1 \mid n-1$.

Thus $n = (\ell-1)(m-1)+1$. Let $\vec{H}_{m,n}$ be the digraph obtained from $\vec{G}_{m,n}$ as follows: Delete the arc $(v_{n-1}, w_{\ell-2})$ and add the arc $(w_{\ell-2}, v_{(\ell-3)(m-1)})$; delete the three arcs incident with $w_{\ell-1}$ and join $w_{\ell-1}$ to $v_{(\ell-2)(m-1)+1}$, from v_{n-1} and to v_1; add a new vertex w_ℓ and join it from $v_{(\ell-2)(m-1)}$, to $v_{(\ell-2)(m-1)+2}$ and from v_0. (The digraph $\vec{H}_{6,16}$ is shown in Figure 5.36.) Then $\vec{C}_m$ and $\vec{C}_n$ can be homogeneously embedded in the digraph $\vec{H}_{m,n}$ of order $n+\ell+1 = n + \lceil n/(m-1) \rceil + 1$. Thus, $fr(\vec{C}_m, \vec{C}_n) \le n + \lceil n/(m-1) \rceil + 1$.

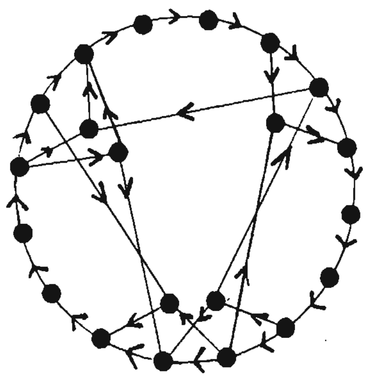


Figure 5.36: The graph $\vec{H}_{6,16}$.

Case 4. $m-1$ *does not divide* $n-1$ *or* n *or* $n+1$.

Thus $n = (\ell-1)(m-1)+r$ for some r satisfying $1 < r < m-2$. Let $\vec{I}_{m,n}$ be the digraph obtained from $\vec{G}_{m,n}$ by adding a new vertex w_ℓ and joining it to $v_{(\ell-1)(m-1)}$, from $v_{\ell(m-1)-1}$ and to $v_{\ell(m-1)+1}$ where addition is taken modulo n; that is, w_ℓ and joined to v_{n-r}, from v_{m-r-2} and to v_{m-r}. (The digraph $\vec{I}_{5,14}$ is shown in

Figure 5.37.) Then $\vec{C}_m$ and $\vec{C}_n$ can be homogeneously embedded in the digraph $\vec{I}_{m,n}$ of order $n+\ell+1 = n+\lceil n/(m-1)\rceil+1$. Thus, $fr(\vec{C}_m, \vec{C}_n) \leq n+\lceil n/(m-1)\rceil+1$. □

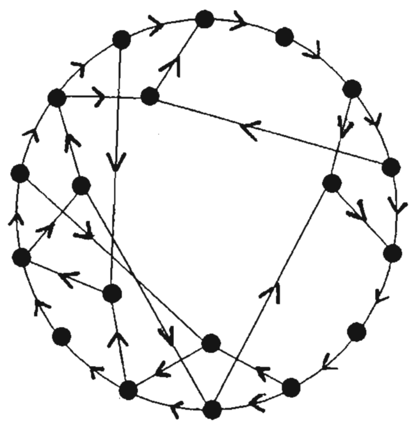


Figure 5.37: The digraph $\vec{I}_{5,14}$.

Two immediate corollaries of Theorems 5.1 and 5.5 and Corollary 5.2 now follow.

Corollary 5 . 9 *For $m = 3$ or 4 and $n = 7, 8, 9$, or for $7 \leq m+2 \leq n \leq 2m-5$, $fr(\vec{C}_m, \vec{C}_n) = n+3$.*

Corollary 5 . 10 *For $m \geq 4$ and $n = 2(m-1)$ or $3(m-1)$, $fr(\vec{C}_m, \vec{C}_n) = n + \frac{n}{m-1}$.*

Bibliography

[1] L.W. Beineike, Derived graphs and digraphs. *Beiträge zur Graphentheorie.* Teubner, Leipzig (1968), 17–33.

[2] G. Chartrand, H. Gavlas, and M. Schultz, FRAMED! A graph embedding problem. *Bull. Inst. Combin. Applic.* **4** (1992), 35–50.

[3] G. Chartrand, M.A. Henning, H. Hevia, and E. Jarrett, A new characterization of the Petersen graph. *J. Combin. Inf. Syst. Sci.* **20** (1995), 219–227.

[4] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Second Edition, Wadsworth and Brooks/Cole, Monterey, CA (1986).

[5] R.C. Entringer, W. Goddard, and M.A. Henning, A note on cliques and independent sets. Submitted for publication.

[6] H. Gavlas, M.A. Henning, and M. Schultz, On graphs and their frames. *Vishwa International J. Graph Theory* **1**(2) (1992), 111–131.

[7] W. Goddard, M.A. Henning, O.R. Oellermann, and H.C. Swart, Some general results on the framing number of a graph. *Quaestiones Math.* **16(3)** (1993), 289–300.

[8] W. Goddard, M.A. Henning, O.R. Oellermann, and H.C. Swart, Which trees are uniquely framed by the Heawood graph. *Quaestiones Math.* **16(3)** (1993), 237–251.

[9] M.A. Henning, On edge cliques and edge independent sets. To appear in the *Bulletin of the ICA.*

[10] M.A. Henning and H. Maharaj, Frames of digraphs: A digraph embedding problem. *Utilitas Mathematica* **47** (1995), 3–19.

[11] D. König, Theorie der endlichen und unendlichen Graphen, Leipzig, 1936. Reprinted, Chelsea, New York (1950).

[12] H. Whitney, Congruent graphs and the connectivity of graphs. *Amer. J. Math.* **54** (1932), 150–168