PARTIAL EXCHANGEABILITY AND RELATED TOPICS

by

Delia Elizabeth North

Submitted in partial fulfilment of the
requirements for the degree of
Doctor of Philosophy,
in the
Department of Mathematical Statistics,
University of Natal
1991

Durban 1991

PREFACE

The research described in this thesis was carried out in the Department of Mathematical Statistics, University of Natal, Durban, from February, 1984 to December, 1985 and from February, 1987 to December, 1991, under the supervision of Doctor A.I. Dale.

These studies represent original work by the authoress and have not been submitted in any form to another University. Where use was made of the work of others it has been duly acknowledged in the text.

ACKNOWLEDGEMENTS

First and foremost I wish to thank most gratefully my supervisor, Doctor A.I. Dale for his continued interest, constructive advice and tremendous patience throughout the preparation of this thesis.

I am greatly indebted to the Council for Scientific and Industrial Research for the award of a Special Merit Bursary which gave me the financial assistance required to complete this research.

I also wish to thank Mrs Jackie de Gaye for her excellent typing of this work and Professor L. Troskie for the use of facilities in his department.

Finally, a very special thanks is due to my husband, Lindsay, and our three little boys for the interest, encouragement and sympathy when needed.

ABSTRACT

Partial exchangeability is the fundamental building block in the subjective approach to the probability of multi-type sequences which replaces the independence concept of the objective theory.

The aim of this thesis is to present some theory for partially exchangeable sequences of random variables based on well-known results for exchangeable sequences.

The reader is introduced to the concepts of partially exchangeable events, partially exchangeable sequences of random variables and partially exchangeable σ-fields, followed by some properties of partially exchangeable sequences of random variables.

Extending de Finetti's representation theorem for exchangeable random variables to hold for multi-type sequences, we obtain the following result to be used throughout the thesis:

There exists a σ-field, conditional upon which, an infinite partially exchangeable sequence of random variables behaves like an independent sequence of random variables, identically distributed within types.

Posing (i) a stronger requirement (spherical symmetry) and (ii) a weaker requirement (the selection property) than partial exchangeability on the infinite multi-type sequence of random variables, we obtain results related to de Finetti's representation theorem for partially exchangeable sequences of random variables.

Regarding partially exchangeable sequences as mixtures of independent and identically distributed (within types) sequences, we (i) give three possible expressions for the directed random measures of the partially exchangeable sequence and (ii) look at three possible expressions for the σ-field mentioned in de Finetti's representation theorem.

By manipulating random measures and using de Finetti's representation

theorem, we point out some concrete ways of constructing partially exchangeable sequences.

The main result of this thesis follows by extending de Finetti's representation theorem in conjunction with the Chatterji principle to obtain the following result:

Given any *a.s.* limit theorem for multi-type sequences of independent random variables, identically distributed within types, there exists an analogous theorem satisfied by all partially exchangeable sequences and by all sub-subsequences of some subsequence of an arbitrary dependent infinite multi-type sequence of random variables, tightly distributed within types.

We finally give some limit theorems for partially exchangeable sequences of random variables, some of which follow from the above mentioned result.

CONTENTS

CHAPTER 1

INTRODUCTION

The phrase *events are independent with unknown probability p* is of no value in the subjective approach to probability, since *learning through experience* is of prime importance. To the objectivist independence is of fundamental importance. Clearly the subjectivist needed a basic concept to replace the independence concept of the objective approach.

Should independence be relaxed the next simplest thing is to continue to regard the order of the events as irrelevant, i.e. exchangeable (also called symmetric, permutable) events. Exchangeability, from a probabilistic point of view, was first introduced by J. Haag (1924a, 1928). He gave a slight indication (1924b, 1928) of the well-known representation theorem for exchangeable events (see Chapter 3). A precise statement and rigorous proof of this theorem was given by de Finetti (1932), following a paper he published in 1931 characterizing all stochastic processes which could be regarded as mixtures of coin tossing processes. This result was generalized by de Finetti (1937) so as to characterize mixtures of sequences of independent, identically distributed random variables. Not until the publication of this paper was the importance of the concept of exchangeability in the subjective approach to probability first noted. (A translation of this paper appears in Kyburg & Smokler (1964).)

Exchangeability is the fundamental building block in the subjective approach to probability which replaces the independence concept of the objective theory. Good background reading can be found in Link (1980), Hamaker (1977) and Goldstein (1986).

In 1938 de Finetti introduced the idea of partial exchangeability (see Chapter 2), often more appropriate than exchangeability. Partial exchangeability has not, to date, received anywhere near as much attention as exchangeability has. This thesis aims to provide some theory for partially exchangeable sequences of random variables, leading to limit theorems by extending well-known results for exchangeable sequences.

In Chapter 2 examples, definitions and properties of partially exchangeable sequences will be given.

Chapter 3 deals with de Finetti's theorem, extended to hold for partially exchangeable sequences. The proof, limitations and implications of this important theorem are thoroughly discussed.

Viewing partially exchangeable sequences as mixtures in Chapter 4 we obtain many more properties. Towards the end of this chapter, we construct some partially exchangeable sequences.

The definition of statutes in Chapter 5 leads to a whole new approach to partially exchangeable sequences. In this chapter we obtain an extension of Chatterji's principle (to allow multi-type sequences) and then give some limit theorems for partially exchangeable sequences.

TERMINOLOGY AND NOTATION

The following terminology and notation will be used throughout this thesis:

$\mathbf{R}$	$=$	$\{\text{reals}\}$	
$\mathbf{N}$	$=$	$\{1,2,3,\ldots\}$	
$[k]$	$=$	$\{1,2,3,\ldots,k\}$	$\forall k \in \mathbf{N}$
$B(\mathbf{R})$		denotes the class of Borel sets in $\mathbf{R}$.	

Unless otherwise indicated S will denote a separable metrisable space. Topological spaces will be equipped with the σ-field generated by the open sets and product spaces will be given product topologies and product σ-fields.

Let $C(S)$ $(L^{\infty}(S))$ be the set of continuous (measurable), bounded, real valued functions on S.

$P(S)$ will denote the space of probability measures on S, equipped with the weak topology

$$\lambda_n \Rightarrow \lambda$$

if and only if

$$\lim_n \int f \, d\lambda_n = \int f \, d\lambda \qquad \forall f \in C(S) \, . \tag{1.1}$$

Note that $P(S)$ is itself a metrisable space (Parthasarathy (1967), Theorem 6.2). For measurable $A \subset \mathbf{R}$, the map $\lambda \to \lambda(A)$ from $P(\mathbf{R}) \to \mathbf{R}$ is measurable. Conversely (Jagers (1974)), if $\mathbf{R}_0$ is a dense subset of $\mathbf{R}$, then the collection of maps

$$\lambda \to \lambda((-\infty, x]) \qquad x \in \mathbf{R}$$

generates the σ-field on $P(\mathbf{R})$.

Throughout this thesis $(\Omega, \mathcal{A}, P)$ will denote a probability space. A measurable function $T : \Omega \to S$ will be called a *random measure* when $S = P(\mathbf{R})$; a *random variable* when $S = \mathbf{R}$ and in general a *random map.*

We shall denote $\{\omega; X(\omega) \in A\}$ by $[X \in A]$ for arbitrary random variable X and Borel set A.

Let $L(\mathbf{T})$ denote the joint distribution of a countable sequence of random maps $\mathbf{T} = \{T_{ij};\ i \in [g],\ j \in \mathbf{N}\}$, for some finite $g \in \mathbf{N}$. Write $T_{in} \xrightarrow{D} T_i$ for $L(T_{in}) \Rightarrow L(T_i)$, $\forall i \in [g]$. Write $\mathcal{F}(T)$ for the σ-field generated by T and $I(A)$ for the indicator function of event A. For $s \in S$, write δ_s for the measure $\delta_s(A) = I(s \in A)$.

Let $g \in \mathbf{N}$ be an arbitrary, finite number. By the term *g-fold infinite sequence* we shall mean a sequence which contains an infinite number of items of each of g types. A *g-IID sequence* shall mean a g-fold infinite sequence of independent random variables, identically distributed within types.

Vector X and matrix A will be denoted by $\underline{X}$ and $\underset{\sim}{A}$ respectively.

Statements of theorems and various other results referred to in the thesis will be given in Appendix A1. Conditional independence is discussed in Appendix A2.

CHAPTER 2

PARTIALLY EXCHANGEABLE SEQUENCES

The aim of this chapter is to introduce the concept of partially exchangeable sequences for use in further chapters. This is done by giving definitions and examples of sequences of exchangeable events, partially exchangeable events, partially exchangeable random variables, the partially exchangeable σ-field, proving properties of partially exchangeable sequences of random variables and looking at sequences related to partially exchangeable sequences of random variables.

2.1 DEFINITIONS AND EXAMPLES

The following example gives an idea of what is meant by *exchangeable events*, an idea first introduced by J. Haag (1924a, 1928).

Consider a sequence of tosses of a single coin. There are many cases in which probabilities within each frequency group are equal. Suppose, for example, the coin is tossed 9 times and $HTHHTHTTH$ is obtained (H = head, T = tail). The judgement of the probability of the event *head on the* 10^{th} *toss* is likely to be affected by the frequency of heads and tails in the previous 9 tosses and not by the particular order in which the heads and tails were obtained. We are thus saying that all different sequences of 5 heads and 4 tails have the same probability and will all result in the same influence on the 10^{th} toss. This leads to the definition of exchangeable events.

Definition 2.1 (subjective viewpoint)
A sequence of events is said to be *exchangeable* if the events are symmetric in relation to our judgement of probability, i.e. the probability that we assign to a particular n of these events occurring depends only on n and not on the particular events chosen.

The corresponding definition in the objective theory is:

Definition 2.2 (objective viewpoint)
A sequence of events is *exchangeable* if the probability that any n of these events occur depends only on n and not on the particular events considered.

Definitions, theorems, etc. will be considered in an objective context, the corresponding subjective results follow by the obvious changes as demonstrated by Definitions 2.1 and 2.2.

Now suppose g coins to be tossed. There are three different cases to be considered:

(i) If the coins are perfectly equal, then an exchangeable sequence of events will be generated.

(ii) The extreme opposite case is that in which the g coins are all completely different. Each of the g coins will generate a sequence of exchangeable events, with complete independence between the g sequences.

(iii) Between these two cases lies an intermediate case: The outcomes of trials with one coin will influence the probability with respect to trials with other coins, but this influence is in a less direct manner than in (i), i.e. we have g exchangeable sequences as in (ii) but with some interdependence between the sequences.

Case (iii) above leads to the definition of a partially exchangeable sequence of events.

Definition 2.3
A sequence of events is said to be *g-fold partially exchangeable* if the sequence of events splits into g types (i.e. a g-fold sequence of events) and events of the same type are exchangeable (permutable, symmetric) in relation to probabilities, i.e. the joint probability of the occurrence of a particular n_i events of type i, $i \in [g]$, depends only on the n_i's, and not on the particular events chosen.

Remark
Note that a g-fold partially exchangeable sequence reduces to an exchangeable sequence in the $g = 1$ case.

Definition 2.4
A g-fold sequence of random variables, $\mathbf{Z} = \{Z_{ij}; i \in [g], j \in \mathbf{N}\}$, is said to be *g-fold partially exchangeable* if the joint distribution of any n_i random variables of type i, $i \in [g]$, depends only on the n_i's, and not on the particular random variables chosen.

Note that the term "g-fold" may be dropped in future discussions.

Attention will be restricted to a study of partially exchangeable sequences of random variables rather than partially exchangeable events since partially exchangeable events can be found as a special case of partially exchangeable random variables, by making use of indicator functions as demonstrated below:

Suppose that $\mathbf{E} = \{E_{ij}; i \in [g], j \in \mathbf{N}\}$ is an infinite g-fold sequence of events. Let

$$Z_{ij} = I(E_{ij}) \qquad \forall i \in [g], \ \forall j \in \mathbf{N} \ .$$

Then $\mathbf{E}$ is a partially exchangeable sequence of events if the corresponding g-fold infinite sequence of random variables, $\mathbf{Z} = \{Z_{ij}; i \in [g], j \in \mathbf{N}\}$, is partially exchangeable.

Now let us give a detailed definition of a g-fold partially exchangeable sequence of random variables for the finite as well as the infinite case.

Definition 2.5
$\mathbf{Z} = \{Z_{ij}; j \in [n_i], i \in [g]\}$ is said to be a *(g-fold) finite partially exchangeable* sequence of random variables if

$$L(\mathbf{Z}) = L(\mathbf{Z}_\pi)$$

where

$$\mathbf{Z}_\pi = \left\{Z_{i\pi_i(j)}; j \in [n_i], i \in [g]\right\}$$

$$n_i < \infty, \qquad n_i \in \mathbf{N} \qquad \forall i \in [g]$$

$$\pi_i \text{ is any permutation of } \{1, 2, \ldots, n_i\} \qquad \forall i \in [g] \ .$$

In order to get the corresponding result for an infinite sequence of random variables we need the following definition.

Definition 2.6
A *finite permutation* of $\{1, 2, \ldots\}$ is a map $\pi : \mathbf{N} \to \mathbf{N}$ such that

$$\#\{i; \pi(i) \neq i\} < \infty$$

i.e. π only permutes finitely many elements of the sequences, infinitely many retaining their original positions.

Definition 2.7
$\mathbf{Z} = \{Z_{ij} \; ; \; i \in [g], \; j \in \mathbf{N}\}$ is said to be a *(g-fold) infinite partially exchangeable* sequence of random variables if

$$L(\mathbf{Z}) = L(\mathbf{Z}_\pi)$$

where

$$\mathbf{Z}_\pi = \left\{Z_{i\pi_i(j)}; i \in [g], j \in \mathbf{N}\right\}$$

for each finite permutation π_i of $\{1, 2, \ldots\}, i \in [g]$.

Suppose that there are g urns, urn i containing n_i balls, labelled $z_{i1}, z_{i2}, \ldots, z_{in_i}$, $\forall i \in [g]$. Let Z_{ij} denote the result of the jth draw from urn i, then

(i) an infinite partially exchangeable sequence of random variables is formed by $\{Z_{ij}; i \in [g], j \in \mathbf{N}\}$ when drawing with replacement from the urns.

(ii) a finite partially exchangeable sequence of random variables is formed by $\{Z_{ij}; j \in [n_i], i \in [g]\}$ when drawing without replacement from the urns.

Case (i) follows immediately from Definition 2.7. In order to obtain the desired result for case (ii) note that $\mathbf{Z} = \{Z_{ij}; j \in [n_i], i \in [g]\}$ can have any of $\prod_{i=1}^{g} n_i!$ outcomes, assigning a different $\mathbf{z} = \{z_{ij}; j \in [n_i], i \in [g]\}$ to $\mathbf{Z}$. Now let Π_i^* be a variable, denoting a permutation of $\{1, 2, \ldots, n_i\}$ $\forall i \in [g]$, so that $\Pi^* = \prod_{i=1}^{g} \Pi_i^*$ denotes a random variable which has $\prod_{i=1}^{g} n_i!$ outcomes, all equally probable. We may thus write

$$\mathbf{Z} = \left\{z_{i\pi_i^*(j)}; j \in [n_i], i \in [g]\right\}$$

and have indexing by means of a variable.

Now let π_i be any of the $n_i!$ permutations of $\{1, 2, \ldots, n_i\}$, $i \in [g]$, then

$$\begin{aligned} L(\mathbf{Z}) &= L\left(\left\{z_{i\pi_i^*(j)}; j \in [n_i], i \in [g]\right\}\right) \\ &= L\left(\left\{z_{i\pi_i[\pi_i^*(j)]}; j \in [n_i], i \in [g]\right\}\right) \\ &= L\left(\{\mathbf{Z}_\pi\}\right) \end{aligned}$$

where Z_π is as in Definition 2.5. Hence the desired result follows from Definition 2.5.

For the time being attention will be restricted to infinite partially exchangeable sequences of random variables; asymptotic results will be discussed in Chapter 5.

On extending the notion of an exchangeable σ-field (Aldous (1985) and Taylor, Daffer and Patterson (1985)) a definition for a partially exchangeable σ-field is obtained for use in further chapters. The following preliminary definition is required:

Definition 2.8
Call a subset B of $\overset{g}{\underset{i=1}{\mathrm{X}}} \overset{\infty}{\underset{j=1}{\mathrm{X}}} \mathbf{R}$ *partially exchangeable* if

$$\mathbf{x} = \{x_{ij}; i \in [g], j \in \mathbf{N}\} \in B \Rightarrow x_\pi = \left\{x_{i\pi_i(j)}; i \in [g], j \in \mathbf{N}\right\} \in B$$

for each finite permutation π_i of $\{1, 2, \ldots\}$, $i \in [g]$.

The corresponding *B-partially exchangeable event* is then

$$[\mathbf{X} \in B] = \{\omega; \mathbf{X}(\omega) \in B\} = \{\omega; \{X_{ij}(\omega); i \in [g], j \in \mathbf{N}\} \in B\} \quad .$$

Definition 2.9
Given a g-fold infinite sequence of random variables, $\mathbf{X} = \{X_{ij}; i \in [g], j \in \mathbf{N}\}$ the *partially exchangeable σ-field*, $E_{\mathbf{X}}$, is the set of all B-partially exchangeable events.

Remark
The exchangeable σ-field for a sequence of random variables $\mathbf{X} = \{X_i; i \in \mathbf{N}\}$ may be found from the partially exchangeable σ-field by taking the special case $g = 1$.

2.2 SOME PROPERTIES OF PARTIALLY EXCHANGEABLE RANDOM VARIABLES

In this section some basic properties of partially exchangeable sequences are presented for use in further chapters. To simplify the notation we consider the $g = 2$ case only, the general theory follows by the obvious adjustments to the results. Throughout this section $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$ will denote a 2-fold infinite sequence of random variables.

Theorem 2.1
If $\mathbf{Z}$ is partially exchangeable then $\mathbf{Z}$ is identically distributed within types.

Proof
$\forall i \in [2]$, $\mathbf{Z}_i = \{Z_{ij}; j \in \mathbf{N}\}$ is exchangeable, and hence $\forall a, b \in \mathbf{R}$, $\forall j \in \mathbf{N}$,

$$\begin{aligned} P(Z_{i1} \leq a) &= \lim_{b \to \infty} P(Z_{i1} \leq a, Z_{ij} \leq b) \\ &= \lim_{b \to \infty} P(Z_{ij} \leq a, Z_{i1} \leq b) \\ &= P(Z_{ij} \leq a) \quad . \end{aligned}$$

The following counter example demonstrates that a partially exchangeable sequence need not be independent. The example is numbered for future reference.

Example 2.1
Let $\mathbf{Y} = \{Y_{ij}; i \in [2], j \in \mathbf{N}\}$ be a 2-IID sequence of random variables, and let Y_1 and Y_2 be two random variables, independent of $\mathbf{Y}$ and also of each other. Let

$$Z_{ij} = Y_{ij} + Y_i \quad . \tag{2.1}$$

where $E(Y_{ij}) = 0 = E(Y_i) \qquad \forall i \in [2], \forall j \in \mathbf{N}$.

Then $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$ is a partially exchangeable sequence of random variables, and hence (Theorem 2.1) is identically distributed within types, so that, $\forall i \in [2]$, $\forall j \in \mathbf{N}$,

$$\operatorname{Var}(Z_{ij}) = \operatorname{Var}(Y_{i1}) + \operatorname{Var}(Y_i) \quad . \tag{2.2}$$

$$\operatorname{Cov}(Z_{i1}, Z_{i2}) = E(Z_{i1}Z_{i2}) - E^2(Z_{i1}) \quad .$$

Using (2.1) and (2.2) we thus obtain

$$\begin{aligned} \operatorname{Cov}(Z_{i1}, Z_{i2}) &= E(Y_i^2) - E^2(Y_i) \\ &= \operatorname{Var}(Y_i) \quad . \end{aligned} \tag{2.3}$$

From this last result we clearly only have independence between $\mathbf{Z}$ variables of the same type if $\operatorname{Var}(Y_i) = 0$ $\forall i \in [2]$, which from the construction of the partially exchangeable sequence, $\mathbf{Z}$, in (2.1) need not generally be true. We are thus able to construct a partially exchangeable sequence which does not have independent random variables since the random variables need not even be independently distributed within types.

Remark

It follows immediately from the basic definitions that a 2-IID sequence is partially exchangeable: that the reverse need not hold follows from Theorem 2.1 and Example 2.1. The identical distribution within types of a partially exchangeable sequence follows from Theorem 2.1 but the independence of the random variables cannot be obtained in general. There is, however, a partial solution to this problem as will be seen in §3.1.

Partially exchangeable sequences do however partake of some of the properties of 2-IID sequences as the following example shows.

Let $\{Z_{ij}; i \in [2], j \in \mathbf{N}\}$ be a partially exchangeable sequence and let

$$\mathbf{Z} = \{Z_{ij}; j \in [n_i], i \in [2]\}$$

$$\mathbf{Z}_\pi = \left\{Z_{i\pi_i(j)};\ j \in [n_i], i \in [2]\right\}$$

where $\forall i \in [2]$, $n_i \in \mathbf{N}$, π_i denotes any one of the $n_i!$ permutations of $\{1, 2, \ldots, n_i\}$.

Suppose that $X = \phi(\mathbf{Z})$ is a function of $\{Z_{ij}; j \in [n_i], i \in [2]\}$. Then clearly

$$L(X) = L(X_\pi)$$

where

$$X_\pi = \phi(\mathbf{Z}_\pi) \ .$$

We thus obtain

$$\begin{aligned} E(X) &= (n_1!n_2!)^{-1} \sum_\pi E(X_\pi) \\ &= E[\psi(\mathbf{Z})] \end{aligned}$$

where $\psi(\mathbf{Z}) = (n_1!n_2!)^{-1} \sum_\pi \phi(\mathbf{Z}_\pi)$ and the summation extends over all $n_1!n_2!$ possible permutations Π_i of $\{1, 2, \ldots, n_i\}$, $i \in [2]$.

It immediately follows that if $\psi(\mathbf{Z}) = 0$ then $E[\phi(\mathbf{Z})] = 0$, a result which is well-known when $\mathbf{Z}$ is a 2-IID sequence. The relationship between partially exchangeable and 2-IID sequences will receive much more attention in further chapters, especially in §3.3.

Another interesting result which may be used to generalize the results of the following chapters is given in the next theorem.

Theorem 2.2
Let $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$ be a partially exchangeable sequence of random variables. If $h_i : \mathbf{R} \to \mathbf{R}$, $i \in [2]$, is a Borel measurable function, then $\{h_i(Z_{ij}); i \in [2], j \in \mathbf{N}\}$ is a partially exchangeable sequence of random variables.

Proof
For any finite permutation Π_i of $\{1, 2, \ldots\}$, $i \in [2]$ and Borel subsets $\{B_{ij}; i \in [2], j \in \mathbf{N}]\}$,

$$P\left\{\bigcap_{i=1}^{2}\bigcap_{j=1}^{\infty}\left[h_i(Z_{i\pi_i(j)})\in B_{ij}\right]\right\}$$

$$=P\left\{\bigcap_{i=1}^{2}\bigcap_{j=1}^{\infty}\left[Z_{i\pi_i(j)}\in h_i^{-1}(B_{ij})\right]\right\}$$

$$=P\left\{\bigcap_{i=1}^{2}\bigcap_{j=1}^{\infty}\left[Z_{ij}\in h_i^{-1}(B_{ij})\right]\right\}$$

$$=P\left\{\bigcap_{i=1}^{2}\bigcap_{j=1}^{\infty}\left[h_i(Z_{ij})\in B_{ij}\right]\right\}\quad .$$

Remark

Note that by taking $h_i = h$ for $i \in [2]$ in Theorem 2.2 we see that a Borel function of a partially exchangeable sequence preserves the partial exchangeability.

2.3 SEQUENCES RELATED TO PARTIALLY EXCHANGEABLE SEQUENCES OF RANDOM VARIABLES

A brief discussion of the selection property, spherical symmetry and partially exchangeable arrays of random variables now follows.

The selection property for a sequence of random variables (Kingman (1978)) is extended below to permit 2-fold sequences.

Definition 2.10

A 2-fold sequence of random variables, $\mathbf{X} = \{X_{ij}; i \in [2], j \in \mathbf{N}\}$, has the *selection property* if, for all integers

$$1 \le m_1 < m_2 < \cdots < m_{k_i} \quad , \qquad \forall i \in [2],\ k_i \in \mathbf{N}\ .$$

$$L\left\{X_{im_j}; j \in [k_i], i \in [2]\right\} = L\left\{X_{ij}; j \in [k_i], i \in [2]\right\}$$

where it is to be understood that the values of m_j in X_{1m_j} and X_{2m_j} might differ, for any $j \in$ minimum $\{k_1, k_2\}$.

The following theorem presents the relationship between partial exchangeability and the selection property.

Theorem 2.3
Suppose $\mathbf{X}$ to be a sequence of random variables which splits into two types.

(i) If $\mathbf{X}$ is a finite sequence, then $\mathbf{X}$ has the selection property if $\mathbf{X}$ is partially exchangeable. The reverse need not hold.

(ii) If $\mathbf{X}$ is an infinite sequence (infinitely many of each type, i.e. a 2-fold infinite sequence), then $\mathbf{X}$ has the selection property if and only if $\mathbf{X}$ is partially exchangeable.

Proof
It immediately follows from Definitions 2.5, 2.7 and 2.10 that $\mathbf{X}$ has the selection property if it is partially exchangeable (for finite and infinite sequences $\mathbf{X}$). It thus remains to be shown that

(a) for $\mathbf{X}$ a finite sequence, $\mathbf{X}$ need not be a partially exchangeable if it has the selection property

(b) for $\mathbf{X}$ an infinite sequence, $\mathbf{X}$ is partially exchangeable if it has the selection property.

Consider case (a), i.e. let $\mathbf{X} = \{X_{ij}; j \in [n_i], n_i \in \mathbf{N}, i \in [2]\}$ have the selection property. For $m_i < n_i \ \forall i \in [2]$, let Π_i be any permutation of $\{1, 2, \ldots, n_i\}$ where

$$1 \leq \Pi_i(1) < \Pi_i(2) < \cdots < \Pi_i(m_i) \qquad \forall i \in [2]$$

does not hold. Then $\{X_{ij}; j \in [m_i], i \in [2]\}$ need not have the same distribution as $\left\{X_{i\pi_i(j)}; j \in [m_i], i \in [2]\right\}$, i.e. $\mathbf{X}$ need not be partially exchangeable.

Case (b) will be proved in Chapter 3 (Theorem 3.4) since it follows on from an extension of the representation theorem as proved by Kingman (1978).

Remark
Comparing Definitions 2.7 and 2.10 it would seem that the requirement for $\mathbf{X}$, a 2-fold sequence of random variables, to have the selection property is a weaker requirement than partial exchangeability. This is in fact true if $\mathbf{X}$ is finite (Theorem 2.3 (i)), but if $\mathbf{X}$ has infinitely many variables of

each type then extending the concept of spherical symmetry (discussed by Kingman (1972 and 1978)) to permit 2-fold sequences, we obtain a stronger requirement than partial exchangeability.

Let $\mathbf{X}$ and $\mathbf{Y}$ be two-fold infinite sequences of random variables and let $\mathbf{X}_m(\mathbf{Y}_m)$ denote finite sequences of random variables, containing m_i random variables of type i, $m_i \in \mathbf{N}, i \in [2]$, from $\mathbf{X}(\mathbf{Y})$, i.e.

$$\mathbf{X}_m = \{X_{ij}; j \in [m_i], i \in [2]\}$$

$$\mathbf{Y}_m = \{Y_{ij}; j \in [m_i], i \in [2]\} \quad .$$

Suppose $\underset{\sim}{A} = \{a_{ij}; i \in [m_1 + m_2], j \in [m_1 + m_2], a_{ij} \in \mathbf{R}\}$ to be an array. If

$$\underline{\mathbf{Y}}_m = \underline{\mathbf{X}}_m \underset{\sim}{A}$$

where

$$\underset{\sim}{A} = \begin{pmatrix} \underset{\sim}{A_1} & 0 \\ 0 & \underset{\sim}{A_2} \end{pmatrix}$$

for $\underset{\sim}{A_1} = \{a_{ij}; i \in [m_1], j \in [m_1]\}$

$\underset{\sim}{A_2} = \{a_{ij}; i \in [m_2], j \in [m_2]\}$

then $\mathbf{Y}_m$ is a *linear transformation* of $\mathbf{X}_m$.

Assume the above terminology until further notice.

Definition 2.11
In the above notation, let $\mathbf{Y}_m$ be a linear transformation of $\mathbf{X}_m$. If $\underset{\sim}{A}$ is orthogonal with determinant 1 then $\underset{\sim}{A}$ is a *rotation array* and $\mathbf{Y}_m$ is a *rotation* of $\mathbf{X}_m$.

Definition 2.12
$\mathbf{X}_m$ is said to have *(2-fold) spherical symmetry* if the joint distribution of $\mathbf{X}_m$ is the same as the joint distribution of $\mathbf{Y}_m$ whenever $\mathbf{Y}_m$ is a rotation of $\mathbf{X}_m$.

Note that the term *2-fold* may be dropped in further discussions.

$\mathbf{X}$ is said to have *spherical symmetry* if $\mathbf{X}_m$ has spherical symmetry $\forall m_i \in \mathbf{N}$, $i \in [2]$.

Remark
It immediately follows from Definition 2.12 that if $\mathbf{X}$ has spherical symmetry, then $\mathbf{X}$ also has spherical symmetry within types, i.e. $\{X_{ij}; j \in \mathbf{N}\}$ has spherical symmetry $\forall i \in [2]$.

The following theorem shows that spherical symmetry implies partial exchangeability.

Theorem 2.4
If $\mathbf{X}$ has spherical symmetry, then $\mathbf{X}$ is partially exchangeable.

Proof:
Let Π_i be any permutation of $\{1, 2, \ldots, m_i\}$, $m_i \in \mathbf{N}$, $\forall i \in [2]$. Then the linear mapping, Π, defined by

$$\Pi : \mathbf{R}^{m_1+m_2} \rightarrow \mathbf{R}^{m_1+m_2} \;\; ; \;\; \Pi\left(\mathbf{X}_{ij}\right) = \mathbf{X}_{i\pi_i(j)} \qquad \forall j \in [m_i], \forall i \in [2]$$

may be viewed as a linear transformation with a rotation matrix and hence $\{X_{ij}; j \in [m_i], i \in [2]\}$ has the same joint distribution as $\left\{X_{i\pi_i(j)}; j \in [m_i], i \in [2]\right\}$. Since this holds for any $m_i \in \mathbf{N}$, we thus have $\mathbf{X}$ to be partially exchangeable.

In Chapter 3 (Theorem 3.5) a representation theorem for spherically symmetric sequences is given.

Aldous (1981) dealt with partially exchangeable arrays of random variables. Even though it is beyond the scope of this thesis to deal with arrays, the definitions below are given as interesting extensions of partial exchangability.

Definition 2.13
For the random array $\underset{\sim}{\mathbf{X}} = \{X_{ij}; i \in \mathbf{N}, j \in \mathbf{N}\}$, say $\underset{\sim}{\mathbf{X}}$ is

(i) *row-exchangeable*, if the joint distribution of $\underset{\sim}{\mathbf{X}}$ is unaltered by permutations within rows

(ii) *column-exchangeable*, if the joint distribution of $\underset{\sim}{\mathbf{X}}$ is unaltered by permutations within columns

(iii) *row-and-column-exchangeable* if the joint distribution of $\underset{\sim}{\mathbf{X}}$ is unaltered by permutations within rows and columns.

Remark
Partial exchangeability clearly follows as a special case of row-exchangeability where only finitely many rows are allowed (the $i \in [2]$ case is considered in this thesis).

Next we define triangular arrays of exchangeable random variables, a very interesting type of exchangeability.

Definition 2.14
An array of random variables $\underset{\sim}{\mathbf{X}} = \{X_{ij}; j \in [i],\ i \in \mathbf{N}\}$ is a *triangular array of exchangeable random variables* if each row of $\underset{\sim}{\mathbf{X}}$ forms an exchangeable sequence.

Remark
The above definition is taken from Taylor, Daffer and Patterson (1985). Chernoff and Teicher (1958) and Weber (1980) presented results for triangular arrays of exchangeable random variables.

CHAPTER 3

A REPRESENTATION THEOREM FOR PARTIALLY EXCHANGEABLE RANDOM VARIABLES

This chapter is devoted to proving de Finetti's representation theorem for partially exchangeable sequences which gives a relationship between a partially exchangeable sequence, $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$, and a 2-IID sequence of random variables. Conditions under which the representation theorem holds are discussed and finally further results concerning the tie-up between partially exchangeable and 2-IID sequences of random variables are given.

3.1 DE FINETTI'S REPRESENTATION THEOREM FOR MULTI-TYPE SEQUENCES OF RANDOM VARIABLES

Let $X_1, X_2, \ldots$ be a sequence of random variables taking values in $\{0,1\}$. De Finetti's representation theorem (de Finetti (1932)) shows that $\{X_1, X_2, \ldots\}$ is exchangeable iff $\{X_1, X_2, \ldots\}$ is a mixture of coin tossing processes, i.e. $\forall n \in \mathbf{N}$, and all strings $x_1, x_2, \ldots, x_n$ of 0's and 1's,

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \int p^t (1-p)^{n-t} \mu(dp)$$

where $t = \sum_{i=1}^{n} x_i$, p is the probability of obtaining a 1 and μ is a probability on $[0,1]$, uniquely determined by P.

This theorem has been generalized in several directions, the most famous of which undoubtedly is de Finetti's representation theorem for a partially exchangeable sequence of events (de Finetti (1937)) (see Appendix A1), which leads to the following statement of de Finetti's representation theorem for partially exchangeable random variables:

an infinite partially exchangeable sequence of random variables, $\{Z_{ij}, i \in [g], j \in \mathbf{N}\}$, *is a mixture of* g*-IID sequences.*

Precisely what is meant by this mixture will be discussed in Chapter 4, but loosely, it means that there exists a σ-field conditional upon which the g-fold partially exchangeable sequence behaves like a g-IID sequence.

A precise statement and proof of this fundamental theorem for the $g = 2$ case follows in Theorem 3.1 (results reduce to work by Kingman (1978) for the $g = 1$ case).

Throughout this chapter let $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$ be an infinite 2-fold sequence of random variables.

Definition 3.1
A random variable is (n,m)-*symmetric*, for $n, m \in \mathbf{N}$, if it is a function of $\mathbf{Z}$ which is unchanged if the first n variables of type 1 and the first m variables of type 2 are permuted in any way (within types).

Remark:
Let ϕ be a function of $\mathbf{Z}$ and let $n, m \in \mathbf{N}$. From Definition 3.1 it immediately follows that if ϕ is (n,m)-symmetric, then it is also $(n-k, m-l)$-symmetric for any $k \in \{0, 1, \ldots, n-1\}$ and any $l \in \{0, 1, \ldots, m-1\}$. Similarly, if ϕ is not (n,m)-symmetric, then it is not $(n+k, m+l)$-symmetric, for any $k, l \in \{0, 1, 2, \ldots\}$, k and l not both zero.

Example

(i) $$\psi = (Z_{11} + Z_{12} + Z_{13})\, Z_{21} + Z_{14} Z_{22}$$

is (3,1)-symmetric but not (4,1)- or (3,2)-symmetric.

(ii) $$\phi = Z_{11} Z_{12}\, (Z_{21} + Z_{22} + Z_{23} + Z_{24}) - Z_{25} Z_{13}$$

is (2,4)-symmetric but not (3,1)- or (2,5)-symmetric.

Definition 3.2
Let $\mathcal{F}_{nm}$ denote the σ-field generated by all (n,m)-symmetric random variables and let

$$\mathcal{F}_\infty = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{F}_{nm} \quad . \tag{3.1}$$

Note that (i) $\mathcal{F}_{n+1\, m} \subset \mathcal{F}_{nm}$ and $\mathcal{F}_{n\, m+1} \subset \mathcal{F}_{nm}$ $\quad \forall n, m \in \mathbf{N}$ (3.2)

(ii) $\mathcal{F}_\infty = \lim_{n\to\infty} \lim_{m\to\infty} \mathcal{F}_{nm}$. (3.3)

The following lemma presents a strong law of large numbers which will be used to prove a representation theorem for partially exchangeable random variables.

Lemma 3.1
Let $\mathbf{Z}$ be an infinite sequence of partially exchangeable random variables and let f be a measurable function for which $E|f(Z_{i1})| < \infty$, $\forall i \in [2]$. Then

$$\lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} f(Z_{1j}) = E\left[f(Z_{11})/\mathcal{F}_{\bullet m}\right] \ a.s. \ \forall m \in \mathbf{N} \tag{3.4}$$

and

$$\lim_{m\to\infty} m^{-1} \sum_{j=1}^{m} f(Z_{2j}) = E\left[f(Z_{21})/\mathcal{F}_{n\bullet}\right] \ a.s. \ \forall n \in \mathbf{N} \tag{3.5}$$

where

$$\left.\begin{aligned} \mathcal{F}_{\bullet m} &= \lim_{n\to\infty} \mathcal{F}_{nm} = \bigcap_{n=1}^{\infty} \mathcal{F}_{nm} \quad \forall m \in \mathbf{N} \\ \mathcal{F}_{n\bullet} &= \lim_{m\to\infty} \mathcal{F}_{nm} = \bigcap_{m=1}^{\infty} \mathcal{F}_{nm} \quad \forall n \in \mathbf{N} \ . \end{aligned}\right\} \tag{3.6}$$

Proof:
Let $Y = g\left(Z_{11}, Z_{12}, \ldots, Z_{21}, Z_{22}, \ldots\right\}$ be a bounded function which is (n,m)-symmetric for some $n, m \in \mathbf{N}$. Now $\{Z_{1j}; j \in \mathbf{N}\}$ is exchangeable and hence, $\forall j \leq n$,

$$\begin{aligned} E\left[f(Z_{1j})Y\right] &= E\left[f(Z_{1j})g(Z_{11}, Z_{12}, \ldots, Z_{21}, Z_{22}, \ldots)\right] \\ &= E\left[f(Z_{11})g(Z_{1j}, Z_{12}, \ldots, Z_{1j-1}, Z_{11}, Z_{1j+1}, \ldots, Z_{21}, Z_{22}, \ldots)\right] \\ &= E\left[f(Z_{11})Y\right] \ . \end{aligned}$$

Hence

$$\sum_{j=1}^{n} E\left[f(Z_{1j})Y\right] = nE\left[f(Z_{11})Y\right] \ .$$

Now take Y to be the indicator function of $A \in \mathcal{F}_{nm}$, so that

$$\int_A n^{-1} \sum_{j=1}^{n} f(Z_{1j}) dP = \int_A f(Z_{11}) dP$$

and hence

$$E\left[n^{-1}\sum_{j=1}^{n} f(Z_{1j})/\mathcal{F}_{nm}\right] = E\left[f(Z_{11})/\mathcal{F}_{nm}\right] \quad a.s. \ .$$

Since $\sum_{j=1}^{n} f(Z_{1j})$ is measurable w.r.t. $\mathcal{F}_{nm}$,

$$n^{-1}\sum_{j=1}^{n} f(Z_{1j}) = E\left[f(Z_{11})/\mathcal{F}_{nm}\right] \quad a.s. \ .$$

For any $m \in \mathbf{N}$, $\{\mathcal{F}_{nm}; n \in \mathbf{N}\}$ is a decreasing sequence of σ-fields, so that an elementary martingale convergence theorem (Appendix A1) yields

$$\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} f(Z_{1j}) = E\left[f(Z_{11})/\mathcal{F}_{\bullet m}\right] \quad a.s.$$

which is the desired result, (3.4). Using variables of type 2, (3.5) follows similarly.

We are now in a position to prove a representation theorem for partially exchangeable random variables by following the line of thought adopted by Kingman (1978) but adapting his work to allow two types of random variables.

Theorem 3.1
Let $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$ be a partially exchangeable sequence of random variables. Then

$$P\left[\bigcap_{i=1}^{2}\bigcap_{j=1}^{m_i}[Z_{ij} \le z_{ij}]/\mathcal{F}_{\infty}\right]$$

$$= \prod_{i=1}^{2}\prod_{j=1}^{m_i} P\left[Z_{ij} \le z_{ij}/\mathcal{F}_{\infty}\right] \qquad a.s. \tag{3.7}$$

where $m_i \in \mathbf{N}, z_{ij} \in \mathbf{R}$, $\forall i \in [2]$, $\forall j \in [m_i]$.

Proof:
Let f be a bounded measurable function on $\mathbf{R}^{m_1+m_2}$ and let $\mathbf{Y} = g(\mathbf{Z})$ be (n_1, n_2)-symmetric for $n_i \geq m_i$, $n_i, m_i \in \mathbf{N}$, $\forall i \in [2]$. Now choose any m_i variables from the first n_i variables of type i, $i \in [2]$. Denote the chosen variables by $\left\{Z_{1\theta(1)}, \ldots, Z_{1\theta(m_1)}, Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)}\right\}$. There are clearly $\binom{n_1}{m_1}\binom{n_2}{m_2}$ ways of choosing the variables, the summation over all the possible choices will be denoted by $\sum_C$.

Due to the partial exchangeability of $\mathbf{Z}$,

$$E\left[f(Z_{1\theta(1)}, \ldots, Z_{1\theta(m_1)}, Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)})g(\mathbf{Z})\right]$$

$$= E\left[f(Z_{11}, \ldots Z_{1m_1}, Z_{21}, \ldots, Z_{2m_2})g'(\mathbf{Z})\right] \tag{3.8}$$

where $g'(\mathbf{Z})$ is just $g(\mathbf{Z})$ altered by interchanging $Z_{1\theta(k)}$ and Z_{1k} $\forall k \in [m_1]$ and $Z_{2\alpha(k)}$ and Z_{2k} $\forall k \in [m_2]$.

Since $g(\mathbf{Z})$ is (n_1, n_2)-symmetric, we thus obtain

$$E\left[f(Z_{1\theta(1)}, \ldots, Z_{1\theta(m_1)}, Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)})\mathbf{Y}\right]$$

$$= E\left[f(Z_{11}, \ldots, Z_{1m_1}, Z_{21}, \ldots, Z_{2m_2})\mathbf{Y}\right] \ . \tag{3.9}$$

Any one of the $\binom{n_1}{m_1}\binom{n_2}{m_2}$ ways of choosing the variables would clearly satisfy (3.9). Now take Y to be the indicator function of $A \in \mathcal{F}_{n_1 n_2}$, so that,

$$\int_A f\left(Z_{11}, \ldots, Z_{1m_1}, Z_{21}, \ldots, Z_{2m_2}\right) dP = \int_A \left[\binom{n_1}{m_1}\binom{n_2}{m_2}\right]^{-1} \sum_C f\left(Z_{1\theta(1)}, \ldots, Z_{1\theta(m_1)}, Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)}\right) dP$$

and immediately then

$$E\left[f\left(Z_{11}, \ldots, Z_{1m_1}, Z_{21}, \ldots, Z_{2m_2}\right) / \mathcal{F}_{n_1 n_2}\right]$$

$$= E\left\{\left[\binom{n_1}{m_1}\binom{n_2}{m_2}\right]^{-1} \sum_C f\left(Z_{1\theta(1)}, \ldots, Z_{1\theta(m_1)}, Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)}\right) / \mathcal{F}_{n_1 n_2}\right\} \ a.s.$$

$$= \left[\binom{n_1}{m_1}\binom{n_2}{m_2}\right]^{-1} \sum_C f\left(Z_{1\theta(1)}, \ldots, Z_{1\theta(m_1)}, Z_{2\alpha(1)}, \ldots, Z_{2\alpha(m_2)}\right) \ a.s. \ . \tag{3.10}$$

Using a martingale convergence theorem (Appendix A1), we get

$$\lim_{n_1,n_2\to\infty} E\left[f\left(Z_{11},\ldots,Z_{1m_1},Z_{21},\ldots,Z_{2m_2}\right)/\mathcal{F}_{n_1n_2}\right]$$

$$= \lim_{n_1}\left\{E\left[f\left(Z_{11},\ldots,Z_{1m_1},Z_{21},\ldots,Z_{2m_2}\right)/\mathcal{F}_{n_1\bullet}\right]\right\} \ a.s.$$

$$= E\left[f\left(Z_{11},\ldots,Z_{1m_1},Z_{21},\ldots,Z_{2m_2}\right)/\mathcal{F}_{\infty}\right] \ a.s.$$

and hence, from (3.10),

$$\lim_{n_1,n_2\to\infty}\left[\binom{n_1}{m_1}\binom{n_2}{m_2}\right]^{-1}\sum_{C} f\left(Z_{1\theta(1)},\ldots,Z_{1\theta(m_1)},Z_{2\alpha(1)},\ldots,Z_{2\alpha(m_2)}\right)$$

$$= E\left[f\left(Z_{11},\ldots,Z_{1m_1},Z_{21},\ldots,Z_{2m_2}\right)/\mathcal{F}_{\infty}\right] \ a.s. \ . \tag{3.11}$$

If the m_i variables chosen from n_i for type i, $i \in [2]$, allow repeats, i.e. need not all be distinct, then (3.11) becomes

$$\lim_{n_1,n_2\to\infty} n_1^{-m_1}n_2^{-m_2}\sum_{D} f\left(Z_{1\theta(1)},\ldots,Z_{1\theta(m_1)},Z_{2\alpha(1)},\ldots,Z_{2\alpha(m_2)}\right)$$

$$= E\left[f\left(Z_{11},\ldots,Z_{1m_1},Z_{21},\ldots,Z_{2m_2}\right)/\mathcal{F}_{\infty}\right] \ a.s. \tag{3.12}$$

where

$$\sum_{D} = \sum_{\theta(1)=1}^{n_1}\cdots\sum_{\theta(m_1)=1}^{n_1}\sum_{\alpha(1)=1}^{n_2}\cdots\sum_{\alpha(m_2)=1}^{n_2} \ .$$

In particular if

$$f\left(z_{11},\ldots,z_{1m_1},z_{21},\ldots,z_{2m_2}\right) = \prod_{i=1}^{2}\prod_{j=1}^{m_i} f_{ij}(z_{ij}) \ ,$$

where

$$f_{ij}(z) = I\left[Z_{ij} \le z\right] \quad \forall z \in \mathbf{R}, \quad \forall j \in [m_i], \quad \forall i \in [2] \ ,$$

then using (3.12),

$$P\left[\bigcap_{i=1}^{2}\bigcap_{j=1}^{m_i}\left[Z_{ij}\le z_{ij}\right]/\mathcal{F}_\infty\right]$$

$$=E\left[I\left(\bigcap_{i=1}^{2}\bigcap_{j=1}^{m_i}\left[Z_{ij}\le z_{ij}\right]\right)/\mathcal{F}_\infty\right]\quad a.s.$$

$$=\lim_{n_1,n_2\to\infty}n_1^{-m_1}n_2^{-m_2}\sum_{D}I\left[\left(\bigcap_{j=1}^{m_1}\left[Z_{1\theta(j)}\le z_{1j}\right]\right)\cap\left(\bigcap_{j=1}^{m_2}\left[Z_{2\alpha(j)}\le z_{2j}\right]\right)\right]\quad a.s.$$

$$=\lim_{n_1,n_2\to\infty}\left\{\prod_{j=1}^{m_1}\left(n_1^{-1}\sum_{\theta(j)=1}^{n_1}I\left[Z_{1\theta(j)}\le z_{1j}\right]\right)\prod_{j=1}^{m_2}\left(n_2^{-1}\sum_{\alpha(j)=1}^{n_2}I\left[Z_{2\alpha(j)}\le z_{2j}\right]\right)\right\}\ a.s.$$

$$=\lim_{n_1,n_2\to\infty}\left\{\prod_{j=1}^{m_1}\left(n_1^{-1}\sum_{j=1}^{n_1}I\left[Z_{1j}\le z_{1j}\right]\right)\prod_{j=1}^{m_2}\left(n_2^{-1}\sum_{j=1}^{n_2}I\left[Z_{2j}\le z_{2j}\right]\right)\right\}\ a.s.\ .\ (3.13)$$

Using Lemma 3.1, we thus see that the right hand side of (3.13) may be replaced by

$$\prod_{j=1}^{m_1}\left\{\lim_{n_2\to\infty}E\left[I\left(Z_{11}\le z_{1j}\right)/\mathcal{F}_{\bullet n_2}\right]\right\}\prod_{j=1}^{m_2}\left\{\lim_{n_1\to\infty}E\left[I\left(Z_{21}\le z_{2j}\right)/\mathcal{F}_{n_1\bullet}\right]\right\}$$

with probability one, and hence, using the martingale convergence theorem again (Appendix A1), we find that (3.13) becomes

$$P\left[Z_{11}\le z_{11},\ldots,Z_{1m_1}\le z_{1m_1},Z_{21}\le z_{21},\ldots,Z_{2m_2}\le z_{2m_2}/\mathcal{F}_\infty\right]$$

$$=\prod_{j=1}^{m_1}E\left[I\left(Z_{11}\le z_{1j}\right)/\mathcal{F}_\infty\right]\prod_{j=1}^{m_2}E\left[I\left(Z_{21}\le z_{2j}\right)/\mathcal{F}_\infty\right]\quad a.s.$$

$$=\prod_{i=1}^{2}\prod_{j=1}^{m_i}P\left[Z_{i1}\le z_{ij}/\mathcal{F}_\infty\right]\quad a.s.\ .$$

Remark

Note that Theorem 3.1 shows that a partially exchangeable sequence $\mathbf{Z}=\{Z_{ij};i\in[2],j\in\mathbf{N}\}$ is conditionally 2-IID given $\mathcal{F}_\infty$, i.e. conditional on $\mathcal{F}_\infty$ the partially exchangeable sequence behaves like a 2-IID sequence.

Corollary 3.1
Let $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$ be a partially exchangeable sequence with random distribution functions

$$F_i(x) = P[Z_{ij} \leq x / \mathcal{F}_\infty] \qquad x \in \mathbf{R}, i \in [2], j \in \mathbf{N} . \tag{3.14}$$

If $\mathcal{G}$ is any sub-σ-field of $\mathcal{F}_\infty$ with respect to which $F_1(x)$ and $F_2(x)$ are measurable $\forall x \in \mathbf{R}$, then

$$P\left\{\bigcap_{i=1}^{2}\bigcap_{j=1}^{m_i}[Z_{ij} \leq z_{ij}] / \mathcal{G}\right\} = \prod_{i=1}^{2}\prod_{j=1}^{m_i} F_i(z_{ij}) \tag{3.15}$$

$\forall m_i \in \mathbf{N}$, $\forall z_{ij} \in \mathbf{R}$, $\forall j \in \mathbf{N}$, $\forall i \in [2]$.

Proof:
Using Theorem 3.1,

$$E\left\{P\left(\bigcap_{i=1}^{2}\bigcap_{j=1}^{m_i}[Z_{ij} \leq z_{ij}] / \mathcal{F}_\infty\right) / \mathcal{G}\right\}$$

$$= E\left[\prod_{i=1}^{2}\prod_{j=1}^{m_i} F_i(z_{ij}) / \mathcal{G}\right]$$

$$= \prod_{i=1}^{2}\prod_{j=1}^{m_i} F_i(z_{ij}) \ .$$

Remark
Taking $\mathcal{G} = \mathcal{F}(F_1, F_2)$ in Corollary 3.1 we obtain

$$P\left(\bigcap_{i=1}^{2}\bigcap_{j=1}^{m_i}[Z_{ij} \leq z_{ij}] / F_1, F_2\right) = \prod_{i=1}^{2}\prod_{j=1}^{m_i} F_i(z_{ij}) \tag{3.16}$$

$\forall m_i \in \mathbf{N}, \forall z_{ij} \in \mathbf{R}, \forall j \in [m_i], \forall i \in [2]$, a form of de Finetti's representation theorem for partially exchangeable random variables which will be referred to in Chapter 4.

Using (3.16) we have a general method for constructing partially exchangeable sequences. First construct a sequence of 2-IID random variables having common distribution functions within types F_1 and F_2. Then allow F_1 and

F_2 to vary randomly. The randomisation destroys the independence while preserving the partial exchageability (see §4.3). Every infinite partially exchangeable sequence can be constructed in this way (see §4.3).

We now present a lemma and definition which we shall use in order to obtain an alternative representation theorem for partially exchangeable random variables.

Lemma 3.2
Let Y be a random variable and let $\mathcal{F}, \mathcal{G}$ be two σ-fields such that $\mathcal{F} \subset \mathcal{G}$. If

$$E\left[E(Y/\mathcal{G})\right]^2 = E\left[E(Y/\mathcal{F})\right]^2$$

then

$$E[Y/\mathcal{G}] = E[Y/\mathcal{F}] \ \ a.s. \ .$$

Proof

$$E[E(Y/\mathcal{G}) - E(Y/\mathcal{F})]^2$$

$$= E[E(Y/\mathcal{G})]^2 - 2E[E(Y/\mathcal{G})E(Y/\mathcal{F})] + E[E(Y/\mathcal{F})]^2$$

$$= 0$$

since

$$\begin{aligned} E[E(Y/\mathcal{G})E(Y/\mathcal{F})] &= E\{E[E(Y/\mathcal{G})E(Y/\mathcal{F})/\mathcal{F}]\} \\ &= E\{E(Y/\mathcal{F})E[E(Y/\mathcal{G})/\mathcal{F}]\} \\ &= E[E(Y/\mathcal{F})E(Y/\mathcal{F})] \\ &= E[E(Y/\mathcal{F})]^2 \ . \end{aligned}$$

Definition 3.3
Let $\mathbf{Z}$ be a 2-fold infinite sequence of random variables. The *tail σ-field* of $\mathbf{Z}$ is defined by

$$\tau = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{F}_{nm}^{\infty}$$

where

$$\mathcal{F}_{nm}^{\infty} = \mathcal{F}\left(Z_{1\,n+1}, Z_{1\,n+2}, \ldots, Z_{2\,m+1}, Z_{2\,m+2}, \ldots\right) \qquad \forall n, m \in \mathbf{N} \ . \tag{3.17}$$

Theorem 3.2
A partially exchangeable sequence of random variables, $\mathbf{Z}$, is conditionally 2-IID given τ.

Proof
Let

$$\mathcal{F}_{n\bullet}^{\infty} = \lim_{m} \mathcal{F}_{nm}^{\infty} \qquad \forall n \in \mathbf{N}$$

and

$$\mathcal{F}_{\bullet m}^{\infty} = \lim_{n} \mathcal{F}_{nm}^{\infty} \qquad \forall m \in \mathbf{N}$$

Using Theorem 2.3 (ii) we see that, for $m, n \in \mathbf{N}$,

$$L\left(Z_{11}, Z_{12}, Z_{13}, \ldots, Z_{2m+1}, Z_{2m+2}, \ldots\right) = L\left(Z_{11}, Z_{1n+1}, Z_{1n+2}, \ldots, Z_{2m+1}, Z_{2m+2}, \ldots\right) \tag{3.18}$$

and hence

$$L\left[E\left(Z_{11}/\mathcal{F}_{1m}^{\infty}\right)\right] = L\left[E\left(Z_{11}/\mathcal{F}_{nm}^{\infty}\right)\right] \ . \tag{3.19}$$

Using the martingale convergence theorem (Appendix A1) and noting that (3.19) holds $\forall m, n \in \mathbf{N}$, we obtain the following

$$L\left[E\left(Z_{11}/\mathcal{F}_{1\bullet}^{\infty}\right)\right] = L\left[E\left(Z_{11}/\tau\right)\right] \tag{3.20}$$

by taking the limit over all $m \in \mathbf{N}$ and then over all $n \in \mathbf{N}$. Since τ is contained in $\mathcal{F}_{1\bullet}^{\infty}$ we may use Lemma 3.2 to obtain

$$E\left(Z_{11}/\mathcal{F}_{1\bullet}^{\infty}\right) = E\left(Z_{11}/\tau\right) \quad a.s. \tag{3.21}$$

and thus

$$E\left(Z_{11}/\tau, \mathcal{F}_{1\bullet}^{\infty}\right) = E\left(Z_{11}/\tau\right) \quad a.s. \ .$$

From this last result we conclude (Appendix A2, point A_4 plus the remark below it) that Z_{11} and $\mathcal{F}_{1\bullet}^{\infty}$ are conditionally independent given τ.

Applying the same argument to Z_{1s}, Z_{1s+1}, in place of $Z_{11}, Z_{12}, \ldots$ one obtains, $\forall s \in \mathbf{N}$,

Z_{1s} and $\mathcal{F}_{s\bullet}^{\infty}$ are conditionally independent given τ. (3.22)

Using similar arguments we see that $\forall t \in \mathbf{N}$,

Z_{2t} and $\mathcal{F}_{\bullet t}^{\infty}$ are conditionally independent given τ. (3.23)

Let $s \in \mathbf{N}$. Then

$$\mathcal{F}_{sm}^{\infty} \supset \mathcal{F}_{sm+1}^{\infty} \quad \forall m \in \mathbf{N} \tag{3.24}$$

so that

Z_{1s} and $\mathcal{F}_{sm}^{\infty}$ are conditionally independent given τ $\forall m \in \mathbf{N}$ (3.25)

follows from (3.22). Similarly, from (3.23) we see that, for $t \in \mathbf{N}$,

Z_{2t} and $\mathcal{F}_{nt}^{\infty}$ are conditionally independent given τ $\forall n \in \mathbf{N}$. (3.26)

From (3.25), (3.26) and a property of conditional independence (Appendix A2, point A_5) we see that $\mathbf{Z}$ is conditionally independent given τ.

We shall now show that $\mathbf{Z}$ is identically distributed within types given τ. Due to partial exchangeability, $\forall m, n \in \mathbf{N}$,

$$\begin{aligned} & L\left(Z_{11} Z_{1m+1}, Z_{1m+2}, \ldots Z_{2n}, Z_{2n+1}, Z_{2n+2}, \ldots\right) \\ = \; & L\left(Z_{1m}, Z_{1m+1}, Z_{1m+2}, \ldots, Z_{2n}, Z_{2n+1}, Z_{2n+2}, \ldots\right) \end{aligned} \tag{3.27}$$

which leads to

$$E\left[\phi(Z_{11})/\mathcal{F}_{m\bullet}^{\infty}\right] = E\left[\phi(Z_{1m})/\mathcal{F}_{m\bullet}^{\infty}\right] \quad a.s. \tag{3.28}$$

for any bounded, measurable function, ϕ.

Now

$$E\left\{E\left[\phi(Z_{11})/\mathcal{F}_{m\bullet}^{\infty}\right]/\tau\right\} = E\left\{E\left[\phi(Z_{1m})/\mathcal{F}_{m\bullet}^{\infty}\right]/\tau\right\} \quad a.s. \tag{3.29}$$

and since $\tau \subset \mathcal{F}_{m\bullet}^{\infty}$, (3.29) becomes

$$E\left[\phi(Z_{11})/\tau\right] = E\left[\phi(Z_{1m})/\tau\right] \quad a.s. \;\; \forall m \in \mathbf{N} \;\; . \tag{3.30}$$

Similarly

$$E[\phi(Z_{21})/\tau] = E[\phi(Z_{2n})/\tau] \quad a.s. \ \forall n \in \mathbf{N} \tag{3.31}$$

and hence, taking ϕ to be an indicator function, (3.30) and (3.31) show that $\mathbf{Z}$ is identically distributed within types after conditioning on τ. Together with the conditional independence already shown, we thus get $\mathbf{Z}$ to be 2-IID conditional on τ.

Remark
Theorems 3.1 and 3.2 are both representation theorems for partially exchangeable random variables and are extensions of de Finetti's representation theorem for exchangeable random variables as found in Kingman(1978) and Aldous (1985) respectively, the difference between these two statements of the same theorem lying in the σ-field upon which the space is conditioned. An extension of de Finetti's representation theorem for exchangeable sequences to be applicable to a 2-fold partially exchangeable sequence would state the existence of a σ-field conditional upon which the partially exchangeable sequence behaves like a 2-IID sequence, whether the conditioning σ-field is $\mathcal{F}_\infty$ (as in Theorem 3.1) or τ (as in Theorem 3.2) being irrelevant at this stage but to receive much attention in Chapter 4.

It is important to note that Theorems 3.1 and 3.2 cannot be applied to finite sequences of partially exchangeable random variables. It is beyond the scope of this thesis to consider finite sequences of random variables but for the interested reader a finite form of de Finetti's representation theorem for exchangeable random variables is to be found in Kendall (1967) and Heath and Sudderth (1976).

A more general result than Theorems 3.1 and 3.2 is the following:

Theorem 3.3 de Finetti's Theorem
Let $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$ be a partially exchangeable sequence of random variables. Then there exists a σ-field conditional upon which $\mathbf{Z}$ behaves like a 2-IID sequence.

The above result follows immediately from either Theorem 3.1 or Theorem 3.2.

Remarks

1) Note that this theorem is an extension of de Finetti's representation theorem for exchangeable sequences of random variables and thus not really de Finetti's theorem, but it will be referred to as such since the $g = 1$ case is commonly referred to as de Finetti's theorem.

2) Looking at Corollary 3.1 we find that there are many forms, other than Theorems 3.1 and 3.2, of de Finetti's representation theorem for partially exchangeable random variables.

3) The obvious extension of Theorem 3.3 to multi-type sequences shows that for a partially exchangeable sequence $\mathbf{Z} = \{Z_{ij}, i \in [g], j \in \mathbf{N}\}$, there exists a σ-field conditional upon which $\mathbf{Z}$ behaves like a g-IID sequence or, as will be discussed in Chapter 4, "$\mathbf{Z}$ is a mixture of g-IID sequences".

3.2 SOME LIMITATIONS AND IMPLICATIONS OF DE FINETTI'S THEOREM

De Finetti's theorem (Theorem 3.3) is a very useful result as it shows how the possible lack of independence in a partially exchangeable sequence of random variables (see Chapter 2, Example 2.1) can be partially overcome. The corresponding fundamental result for the $g = 1$ case has attracted a variety of generalizations, extensions and analogues as will be discussed here and in further chapters.

De Finetti's theorem holds in compact metric spaces, S, but breaks down in the absence of strong enough topological assumptions on S. In this regard Dubins and Freedman (1979) gave an example of a separable space S where de Finetti's theorem for exchangeable random variables cannot be applied. It is thus possible to find a separable space S where Theorem 3.3 does not hold.

Further work on the exchangeable case was done by Freedman (1980) and Dubins (1983).

Dacunha-Castelle (1974) and Kingman (1978) discussed the sufficiency of

weaker requirements than exchangeability of $\mathbf{Z}$ to obtain de Finetti's theorem. Improving on these results, Aldous (1982) gave a minimal requirement.

We now prove de Finetti's theorem (Theorem 3.3) for a 2-fold infinite sequence which has the selection property (see Definition 2.10 in Chapter 2) rather than partial exchangeability. The following result is given in Kingman [(1978), p.188] and will be used in Theorem 3.4.

Lemma 3.3
Let $k \in [n]$ for any $n \in \mathbf{N}$. If, $\forall r \leq k$, the bounded sequence of variables $\{a_r(j), j \geq 1\}$ has Cesàro limit

$$\alpha_r = \lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} a_r(j)$$

then

$$\lim_{n\to\infty} \binom{n}{k}^{-1} \sum_{M} \prod_{s=1}^{k} a_s(j_s) = \prod_{s=1}^{k} \alpha_s$$

where

$$M = \{(j_1, j_2, \ldots, j_k) \; ; \; j_1 < j_2 < \cdots < j_k \leq n\} \quad .$$

Theorem 3.4
Let $\mathbf{Z} = \{Z_{ij};\ i \in [2], j \in \mathbf{N}\}$ be a 2-fold infinite sequence of random variables, with $E(Z_{i1}) < \infty \ \forall i \in [2]$, which has the selection property (Definition 2.10). Then there exists a σ-field conditional upon which $\mathbf{Z}$ behaves like a 2-IID sequence.

Proof:
It is sufficient to show that if $\mathbf{Z}$ has the selection property then $\mathbf{Z}$ is partially exchangeable as the desired result then follows immediately from Theorem 3.3.

For all $i \in [2]$ let

$$C_i = \bigcap_{n=1}^{\infty} C_{in}$$

where

$$C_{in} = \mathcal{F}(Z_{in}, Z_{i\,n+1}, Z_{i\,n+2}, \ldots) \qquad \forall n \in \mathbf{N} \quad .$$

Since $\mathbf{Z}$ has the selection property, $\mathbf{Z}_i = \{Z_{ij}\ ;\ j \in \mathbf{N}\}$ is stationary (Loève (1978), Vol. II, p.83) $\forall i \in [2]$, and hence by the Birkhoff ergodic theorem (Appendix A1) we have

$$\lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} Z_{ij} = E(Z_{i1}/\mathcal{C}_i) \ \ a.s., \ \ \forall i \in [2] \ . \tag{3.32}$$

Taking f to be the indicator function of $[-\infty, x]$ for some $x \in \mathbf{R}$, one finds that $\{f(Z_{ij});\ i \in [2], j \in \mathbf{N}\}$ has the selection property, so that

$$\lim_{n\to\infty} n^{-1} \# [j \le n;\ Z_{ij} \le x] = F_i(x) \ \ a.s. \tag{3.33}$$

where

$$\begin{aligned} F_i(x) &= E\left(I[Z_{i1} \le x]/\mathcal{C}_i\right) \\ &= P\left[Z_{i1} \le x/\mathcal{C}_i\right] \qquad\qquad \forall i \in [2], \ \ x \in \mathbf{R} \end{aligned}$$

follows from (3.32).

Now, $\forall r \in \mathbf{N}$, $\forall j \in \mathbf{N}$ let $a_r(j) = f_r(Z_{1j})$, $f_r(Z_{1j}) = I\left(Z_{1j} \le x_r\right)$, then (3.33) and Lemma 3.3 with $\alpha_i = F_1(x_i) = P(Z_{11} \le x_i)$ show that

$$\lim_{n\to\infty} \binom{n}{k}^{-1} \sum_{M} \prod_{s=1}^{k} f_s(Z_{1j_s}) = \prod_{s=1}^{k} F_1(x_s) \tag{3.34}$$

where

$$M = \{(j_1, j_2, \ldots, j_k); j_1 < j_2 < \cdots < j_k \le n\} \ .$$

Similarly, for any $m \in \mathbf{N}$ and any $l \in [m]$,

$$\lim \binom{m}{l}^{-1} \sum_{M'} \prod_{s=1}^{l} I\left[Z_{2i_l} \le x_l\right] = \prod_{s=1}^{l} F_2(x_s)$$

where

$$M' = \{(i_1, i_2, \ldots, i_l);\ i_1 < i_2 < \cdots < i_l \le m\} \ .$$

Combining (3.34) and this previous result we obtain

$$\begin{aligned} &\lim_{\substack{n\to\infty \\ m\to\infty}} \left[\binom{n}{k}\binom{m}{l}\right]^{-1} \sum_{M} \sum_{M'} I\left\{\left(\bigcap_{s=1}^{k} [Z_{1j_s} \le x_{j_s}]\right) \cap \left(\bigcap_{s=1}^{l} [Z_{2i_s} \le x_{i_s}]\right)\right\} \\ &= \prod_{s=1}^{k} F_1(x_{j_s}) \prod_{r=1}^{l} F_2(x_{i_r}). \end{aligned} \tag{3.35}$$

There are $\binom{m}{l}\binom{n}{k}$ ways of choosing $i_1 < i_2 < \cdots < i_l \leq m$ and $j_1 < j_2 < \cdots < j_k \leq n$ from $(1, 2, \ldots, m)$ and $(1, 2, \ldots, n)$ respectively. By the selection property of $\mathbf{Z}$, these choices will all result in

$$P\left\{\left(\bigcap_{s=1}^{k}[Z_{1j_s} \leq x_{1s}]\right) \bigcap \left(\bigcap_{s=1}^{l}[Z_{2i_s} \leq x_{2s}]\right)\right\}$$

being equal to

$$P\left\{\left(\bigcap_{s=1}^{k}[Z_{1s} \leq x_{1s}]\right) \bigcap \left(\bigcap_{s=1}^{l}[Z_{2s} \leq x_{2s}]\right)\right\}$$

so that, using (3.35) and noting that l and k depend on m and n respectively,

$$\lim_{\substack{m\to\infty \\ n\to\infty}} P\left\{\left(\bigcap_{s=1}^{k}[Z_{1s} \leq x_{1s}]\right) \bigcap \left(\bigcap_{s=1}^{l}[Z_{2s} \leq x_{2s}]\right)\right\}$$

$$= E\left[\prod_{s=1}^{k} F_1(x_{1s}) \prod_{r=1}^{l} F_2(x_{2r})\right] \quad .$$

The symmetry (permutability) of this last expression shows that $\mathbf{Z}$ is partially exchangeable.

Following the line of thought adopted by Kingman (1972) in the $g = 1$ case we now present a theorem that demonstrates how a more specific result than de Finetti's representation theorem may be obtained by posing a stronger condition than partial exchangeability on the infinite 2-fold sequence of random variables.

Theorem 3.5

Let $\mathbf{Z} = \{Z_{ij};\ i \in [2],\ j \in \mathbf{N}\}$ be an infinite 2-fold sequence of random variables such that for all $n_i \in \mathbf{N}$, $i \in [2]$, the distribution of the sequence of random variables $(Z_{11}, Z_{12}, \ldots, Z_{1n_1}, Z_{21}, Z_{22}, \ldots, Z_{2n_2})$ has spherical symmetry (Definition 2.12). Then there exist random variables V_1 and V_2, real and non-negative, such that, conditional on $\mathcal{F}(V_1, V_2)$, $\mathbf{Z}$ is independent and identically distributed, within types as $N(0, V_i)$.

Proof

$\mathbf{Z}$ is partially exchangeable (Theorem 2.4) and hence conditionally 2-IID given $\mathcal{F}_\infty$ (Theorem 3.1), so that we may define a joint random distribution function for one variable of each type, as follows:

$$F(x,y) = P\left[Z_{1k} \leq x, Z_{2j} \leq y / \mathcal{F}_\infty\right] \qquad \forall k,j \in \mathbf{N},\ x,y \in \mathbf{R}$$

($\mathcal{F}_\infty$ as in Definition 3.2). Now define the corresponding joint characteristic function

$$\begin{aligned} \phi(s,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[i(sx+ty)]dF(x,y) \\ &= E\left\{\exp\left[i(sZ_{1k}+tZ_{2j})\right]/\mathcal{F}_\infty\right\} \end{aligned} \tag{3.36}$$

$\forall k,j \in \mathbf{N}$ and $s,t \in \mathbf{R}$.

Now $\phi(s,t)$ is thus a random, $\mathcal{F}_\infty$-measurable, continuous function. Using the conditional independence of $\mathbf{Z}$ (Theorem 3.1) and (3.36) it follows that

$\forall s_j,t_j \in \mathbf{R},\ \ \forall n \in \mathbf{N},\ \ j \in [n]$,

$$E\left[\exp i \sum_{j=1}^{n}(s_jZ_{1j}+t_jZ_{2j})/\mathcal{F}_\infty\right] = \prod_{j=1}^{n}\phi(s_j,t_j) \tag{3.37}$$

and hence

$$\begin{aligned} E\left[\prod_{j=1}^{n}\phi(s_j,t_j)\right] &= E\left\{E\left[\exp i\sum_{j=1}^{n}(s_jZ_{1j}+t_jZ_{2j})/\mathcal{F}_\infty\right]\right\} \\ &= E\left[\exp i\sum_{j=1}^{n}(s_jZ_{1j}+t_jZ_{2j})\right] \ . \end{aligned} \tag{3.38}$$

Using a result from Lord (1954) (see Appendix A1) it follows that the right hand side of (3.38) is a function of $\sum_{i=1}^{n}(t_i^2+s_i^2)$ only.

For any $u_i,v_i \in \mathbf{R}$, $i \in [2]$, define

$$s = \left(u_1^2+v_1^2\right)^{\frac{1}{2}} \qquad\qquad t = \left(u_2^2+v_2^2\right)^{\frac{1}{2}} \ . \tag{3.39}$$

Then

$$E\left[|\phi(s,t) - \phi(u_1,u_2)\,\phi(v_1,v_2)|^2\right]$$

$$= E\left\{[\phi(s,t) - \phi(u_1,u_2)\phi(v_1,v_2)]\left[\overline{\phi(s,t) - \phi(u_1,u_2)\phi(v_1,v_2)}\right]\right\}$$

$$= E[\phi(s,t)\phi(-s,-t)] - E[\phi(s,t)\phi(-u_1,-u_2)\phi(-v_1,-v_2)]$$

$$-E[\phi(u_1,u_2)\phi(v_1,v_2)\phi(-s,-t)]$$

$$+E[\phi(u_1,u_2)\phi(v_1,v_2)\phi(-u_1,-u_2)\phi(-v_1,-v_2)] \ .$$

Using (3.39) and applying the result from Lord (1954) to each of the four terms in this last expression, it follows that they are all equal, so that

$$E\left[|\phi(s,t) - \phi(u_1,u_2)\phi(v_1,v_2)|^2\right] = 0$$

and hence

$$\phi\left((u_1^2+v_1^2)^{\frac{1}{2}},(u_2^2+v_2^2)^{\frac{1}{2}}\right) = \phi(u_1,u_2)\phi(v_1,v_2) \quad a.s. \tag{3.40}$$

follows from (3.39).

Now let $u_1 = v_1$ and $u_2 = v_2$ in (3.40). Then

$$\phi(s,t) = \phi^2\left(2^{-\frac{1}{2}}s, 2^{-\frac{1}{2}}t\right) \tag{3.41}$$

Now let

$$\psi(s,t) = \ln\phi(s,t) \ . \tag{3.42}$$

Then (3.41) and (3.42) show that

$$\begin{aligned}\psi(s,t) &= 2\psi\left(2^{-\frac{1}{2}}s, 2^{-\frac{1}{2}}t\right)\\ &\vdots\\ &= 2^n\psi\left(2^{-\frac{n}{2}}s, 2^{-\frac{n}{2}}t\right)\end{aligned}$$

so that

$$\psi(s,t)(s^2+t^2)^{-1} = \psi\left(2^{-\frac{n}{2}}s, 2^{-\frac{n}{2}}t\right)\left[(2^{-\frac{n}{2}}s)^2 + (2^{-\frac{n}{2}}t)^2\right]^{-1} \ . \tag{3.43}$$

Now let

$$m_i = E(Z_{ij}/\mathcal{F}_\infty)$$
$$V_i = E(Z_{ij}/\mathcal{F}_\infty)^2 - m_i^2$$
$$Y_{ij} = (Z_{ij} - m_i)(V_i)^{-\frac{1}{2}} \qquad \forall i \in [2],\ \forall j \in \mathbf{N}\ . \tag{3.44}$$

Since $\mathbf{Z}$ behaves like a 2-IID sequence once conditioned on $\mathcal{F}_\infty$ (Theorems 2.4 and 3.1), it follows that

$$E(Y_{ij}/\mathcal{F}_\infty) = 0 \text{ and } \operatorname{Var}(Y_{ij}/\mathcal{F}_\infty) = 1 \quad \forall i \in [2],\ \forall j \in \mathbf{N}\ . \tag{3.45}$$

Now let

$$\phi_{Y_{1k}Y_{2j}}(s,t) = E\left[\exp i(sY_{1k} + tY_{2j})/\mathcal{F}_\infty\right] \tag{3.46}$$

$s,t \in \mathbf{R}$, $\forall k,j \in \mathbf{N}$.

Using (3.37) and (3.44), we have

$$\phi(s,t) = \exp i(sm_1 + tm_2)\phi_{Y_{1k},Y_{2j}}\left(sV_1^{\frac{1}{2}}, tV_2^{\frac{1}{2}}\right) \tag{3.47}$$

$\forall s,t \in \mathbf{R}$, $\forall k,j \in \mathbf{N}$. Now let

$$\psi_{Y_{1k},Y_{2j}}(s,t) = \ln \phi_{Y_{1k}Y_{2j}}(s,t)\ . \tag{3.48}$$

Replacing $\mathbf{Z}$ by $\mathbf{Y} = \{Y_{ij}; i \in [2], j \in \mathbf{N}\}$ [as in (3.44)] in (3.43) and using L'Hôspital's rule, (3.45) and (3.48), it follows that

$$\psi_{Y_{1k}Y_{2j}}(s,t)(s^2+t^2)^{-1} = -\frac{1}{2}$$

and hence

$$\phi_{Y_{1k}Y_{2j}}(s,t) = \exp\left[-\frac{1}{2}(s^2+t^2)\right]$$

so that, from (3.47),

$$\phi(s,t) = \exp i(sm_1 + tm_2)\exp\left[-\frac{1}{2}(s^2V_1 + t^2V_2)\right]\ . \tag{3.49}$$

Since $\mathbf{Z}$ has spherical symmetry the joint distribution of (Z_{1k}, Z_{2j}) is identical to the joint distribution of $(-Z_{1k}, -Z_{2j})$ $\forall j,k \in \mathbf{N}$, and hence from (3.36) $\forall s,t \in \mathbf{R}$, $\phi(s,t) = \phi(-s,-t)$, so that

$$\phi(s,t) = \frac{1}{2}\left[\phi(s,t) + \phi(-s,-t)\right]\ .$$

From (3.49),

$$\exp\left[i(sm_1+tm_2)\right]\exp\left[-\tfrac{1}{2}(s^2V_1+t^2V_2)\right]$$

$$= \tfrac{1}{2}\exp\left[-\tfrac{1}{2}(s^2V_1+t^2V_2)\right]\left\{\exp[i(sm_1+tm_2)]+\exp[i(-sm_1,-tm_2)]\right\} \ .$$

This last equation only holds $\forall s,t \in \mathbf{R}$, if $m_1 = m_2 = 0$, and hence from (3.44), conditional on $\mathcal{F}_\infty$, $\mathbf{Z}$ elements have zero mean. Using this last result and (3.49),

$$\phi(s,t) = \exp\left[-\frac{1}{2}(s^2V_1+t^2V_2)\right] \tag{3.50}$$

Since $\overline{\phi(s,t)} = \phi(-s,-t)$, (3.50) yields

$$\exp\left[-\frac{1}{2}(s^2V_1+t^2V_2)\right] = \overline{\exp\left[-\frac{1}{2}(s^2V_1+t^2V_2)\right]} \ .$$

We thus conclude that V_1 and V_2 are real valued. Since $|\phi(s,t)| \leq 1$ it follows from (3.50) that V_1 and V_2 are non-negative.

From (3.50),

$$\phi(1,0) = \exp\left(-\frac{1}{2}V_1^2\right)$$

so that

$$V_1 = -2\ln\phi(1,0)$$

and similarly

$$V_2 = -2\ln\phi(0,1)$$

and hence (using (3.36)) it follows that V_1 and V_2 are $\mathcal{F}_\infty$-measurable random variables.

For any random variable X,

$$E\left(X/V_1,V_2\right) = E\left[E(X/\mathcal{F}_\infty)/V_1,V_2\right] \ . \tag{3.51}$$

Substituting $X = \exp\left[i\sum_{r=1}^{n}(s_rZ_{1r}+t_rZ_{2r})\right]$ into (3.51) and using (3.50) and (3.37), one has

$$E\left\{\exp\left[i\sum_{r=1}^{n}(s_rZ_{1r}+t_rZ_{2r})\right]/V_1,V_2\right\}$$

$$=E\left[E\left\{\exp\left[i\sum_{r=1}^{n}(s_rZ_{1r}+t_rZ_{2r})\right]/\mathcal{F}_\infty\right\}/V_1,V_2\right]$$

$$=E\left[\prod_{r=1}^{n}\phi(s_r,t_r)/V_1,V_2\right]$$

$$=\prod_{r=1}^{n}\exp\left[-\tfrac{1}{2}\left(V_1s_r^2+V_2t_r^2\right)\right]\quad.$$

For any $s_r,t_r\in\mathbf{R}$, $\exp\left[-\frac{1}{2}(V_1s_r^2+V_2t_r^2)\right]$ is the characteristic function of a $N\left(\binom{0}{0},\binom{(V_1\ 0)}{0\ V_2}\right)$ distribution, so that, conditional on $\mathcal{F}_\infty$, $\mathbf{Z}$ is 2-IID as $N(0,V_i)$ for variables of type i, $i\in[2]$.

Remarks:

1. Theorem 3.5 has a close resemblance, on the one hand to de Finetti's theorem (Theorem 3.3) and on the other hand to Maxwell's theorem (Appendix A1) which gives a characterization of the normal distribution.

2. Theorem 3.3 ensures the existence of some σ-field, conditional upon which the partially exchangeable sequence behaves like a 2-IID sequence, no mention being made of the particular type of distribution. Theorem 3.5 shows that posing the stronger requirement of spherical symmetry on the random variables enables us to know the exact distribution of the 2-IID sequence mentioned in de Finetti's theorem.

3. Ressel (1985) looked at the necessary and sufficient conditions required for an exchangeable sequence of random variables to behave like an IID sequence of a particular type (normal, exponential, Poisson, gamma) after conditioning on a σ-field. Freedman (1962) also gave many interesting examples in this regard.

3.3 ERGODICITY AND THE RELATIONSHIP BETWEEN PARTIALLY EXCHANGEABLE AND 2-IID SEQUENCES OF RANDOM VARIABLES

We have established that even though partially exchangeable sequences need not be 2-IID (Example 2.1), it is possible (Theorem 3.3) to regard the partially exchangeable sequence as a 2-IID sequence by making use of conditional probabilities. It is thus not surprising that the relationship between partial exchangeability and 2-IID leads to further results. In Chapter 5 we shall prove that all a.s. limit theorems for 2-IID random variables have an analogue for partially exchangeable random variables.

The class of partially exchangeable sequences can be viewed as the class of distributions invariant under certain transformations. Based on this idea of invariant distributions and using ergodic theory we prove an extension of the Hewitt-Savage zero-one law (Appendix A1) which gives a further connection between the concepts of partial exchangeability and independent identical distributions within types.

Assume the following general setting throughout this section:

Let S be a Polish space, let K denote a countable group of measurable maps $T : S \to S$ and let $P(S)$ be the set of probability measures on S.

Definition 3.4
For arbitrary map T define the *induced map* $\tilde{T}$ as follows:

(i) If $T \in K$ then $\tilde{T} : P(S) \to P(S)$ where

$$\tilde{T}[L(X)] = L[T(X)] \ . \tag{3.52}$$

(ii) If T is any measurable map from $S \times S$ to S, then $\tilde{T} : S \times P(S) \to P(S)$ where

$$\tilde{T}[s, L(X)] = L[T(s, X)] \tag{3.53}$$

for any random element X and any $s \in S$.

X, a random element of S, is *invariant* if

$$L[T(X)] = L(X) \qquad \forall T \in K \tag{3.54}$$

μ, a distribution on S, is an *invariant distribution* if

$$\tilde{T}(\mu) = \mu \qquad \forall T \in K \ . \tag{3.55}$$

Let M denote the set of invariant distributions and suppose that M is non-empty. A subset $A \subset S$ is an *invariant subset* if

$$T(A) = A \qquad \forall T \in K \ . \tag{3.56}$$

J is the *invariant σ-field* made up by the family of invariant subsets.

An invariant distribution μ is *ergodic* if

$$\mu(A) = 0 \text{ or } 1 \qquad \forall A \in J \ .$$

Definition 3.5
For $\mathbf{X} = \{X_{ij} \ ; \ i \in [2], \ j \in \mathbf{N}\}$ an infinite two-fold sequence of random variables, $\{\omega; \mathbf{X}(\omega) \in B\}$ is a *partially exchangeable event* if B is a subset of $\mathbf{R}^\infty \times \mathbf{R}^\infty$ such that

$$(x_{11}, x_{12}, \ldots, x_{21}, x_{22}, \ldots) \in B \Rightarrow \left(x_{1,\pi_1(1)}, x_{1,\pi_1(2)}, \ldots, x_{2,\pi_2(1)}, x_{2,\pi_2(2)}, \ldots\right) \in B$$

where π_i is any finite permutation of $\{1, 2, \ldots\}$, $\forall i \in [2]$.

$E_{\mathbf{X}}$, *the partially exchangeable σ-field* is the set of all partially exchangeable events.

Remark
In Chapter 4 (Theorem 4.9) we shall show that for a partially exchangeable sequence $\mathbf{X}$ the tail σ-field $\tau_{\mathbf{X}}$ and $E_{\mathbf{X}}$ coincide, i.e. we may condition on $E_{\mathbf{X}}$ in Theorem 3.2.

If $S = \mathbf{R}^\infty \times \mathbf{R}^\infty, K = \{T_\pi; T_\pi$ finitely permutes x_{ij} to $x_{i\pi_i(j)}$, $i \in [2], \ j \in \mathbf{N}\}$ then $\mathbf{X} = \{X_{ij}; \ i \in [2], \ j \in \mathbf{N}\}$ is partially exchangeable if and only if $\mathbf{X}$ is invariant under K. The invariant σ-field is the partially exchangeable

σ-field E_X and the next theorem shows that the ergodic processes are 2-IID sequences.

We now present a result based on the Hewitt-Savage zero-one law (Appendix A1) but differing from the latter by allowing multi-type sequences.

Theorem 3.6
Let $\mathbf{X} = \{X_{ij};\ i \in [2],\ j \in \mathbf{N}\}$ be a 2-IID sequence of random variables. Then $P(A) = 0$ or 1 for all $A \in E_{\mathbf{X}}$.

Proof
Let π_i be any finite permutation of $\{1, 2, \ldots\}$, $i \in [2]$. Let

$$\mathbf{X}_\pi = \left\{X_{i\pi_i(j)}; i \in [2],\ j \in \mathbf{N}\right\} \tag{3.57}$$

For $B \in B\left(\mathbf{R}^\infty \times \mathbf{R}^\infty\right)$, let

$$A = [\mathbf{X} \in B] \tag{3.58}$$

$$\pi(A) = [\mathbf{X}_\pi \in B] \quad . \tag{3.59}$$

For $n \in \mathbf{N}$, $m \in \mathbf{N}$ and

$$B_{nm} \in B\left(\mathbf{R}^n \times \mathbf{R}^m\right) \tag{3.60}$$

let

$$A_{nm} = [\{X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, X_{22}, \ldots, X_{2m}\} \in B_{nm}] \quad . \tag{3.61}$$

For π_1 (π_2) any permutation of $[n]$ ($[m]$) respectively, let

$$\pi_{nm}(\mathbf{X}) = \left\{X_{1\pi_1(1)}, X_{1\pi_1(2)}, \ldots, X_{1\pi_1(n)}, X_{1n+1}, \ldots, X_{2\pi_2(1)}, X_{2\pi_2(2)}, \ldots,\right.$$

$$\left. X_{2\pi_2(m)}, X_{2m+1}, \ldots\right\} \tag{3.62}$$

$$\pi_{nm}(A) = [\pi_{nm}(\mathbf{X}) \in B] \quad . \tag{3.63}$$

Suppose that $A \in E_{\mathbf{X}}$. Then A is a partially exchangeable event and hence

$$\pi(A) = A \tag{3.64}$$

for all $\pi = \{\pi_i(j) \ ; \ i \in [2], j \in \mathbf{N}\}$ where π_i is a finite permutation of $\{1, 2, \ldots\}$, $\forall i \in [2]$.

For $n, m \in \mathbf{N}$, choose sets $B_{nm} \in B\left(\mathbf{R}^n \times \mathbf{R}^m\right)$ such that

$$\lim_{n,m \to \infty} P\left(A \Delta A_{nm}\right) = 0 \ . \tag{3.65}$$

For $B \in B\left(\mathbf{R}^\infty \times \mathbf{R}^\infty\right)$, let

$$P_{\mathbf{X}}(B) = P[\mathbf{X} \in B] \tag{3.66}$$

and for $n \in \mathbf{N}, m \in \mathbf{N}$, let

$$P_{\pi_{nm}(\mathbf{X})}(B) = P\left[\pi_{nm}(\mathbf{X}) \in B\right] \ . \tag{3.67}$$

Since $\mathbf{X}$ is 2-IID, for any $B \in B\left(\mathbf{R}^\infty \times \mathbf{R}^\infty\right)$, any $n \in \mathbf{N}, m \in \mathbf{N}$,

$$P_{\mathbf{X}}(B) = P_{\pi_{nm}(\mathbf{X})}(B) \ . \tag{3.68}$$

From (3.58) and (3.61),

$$\begin{aligned} P\left(A \Delta A_{nm}\right) &= P\left\{[\mathbf{X} \in B] \Delta [\{X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, X_{22}, \ldots, X_{2m}\} \in B_{nm}]\right\} \\ &= P\left\{[\mathbf{X} \in B \Delta B_{nm}]\right\} \\ &= P_{\mathbf{X}}(B \Delta B_{nm}) \ . \end{aligned} \tag{3.69}$$

Using (3.68) we thus have

$$P\left(A \Delta A_{nm}\right) = P_{\pi_{nm}(\mathbf{X})}\left(B \Delta B_{nm}\right) \ . \tag{3.70}$$

Since A is a partially exchangeable event,

$$A = \pi_{nm}(A)$$

i.e.

$$[\mathbf{X} \in B] = [\pi_{nm}(\mathbf{X}) \in B]$$

so that (3.70) becomes

$$\begin{aligned} P(A\Delta A_{nm}) &= P[\pi_{nm}(\mathbf{X}) \in B\Delta B_{nm}] \\ &= P\{[\pi_{nm}(\mathbf{X}) \in B]\,\Delta\,[\pi_{nm}(\mathbf{X}) \in B_{nm}]\} \\ &= P\{[\mathbf{X} \in B]\Delta\pi_{nm}(A_{nm})\} \\ &= P\{A\Delta\pi_{nm}(A_{nm})\} \quad . \end{aligned}$$

Using (3.65) we thus have that

$$\lim_{n,m\to\infty} P[A\Delta\pi_{nm}(A_{nm})] = 0 \quad . \tag{3.71}$$

Now

$$P\{A\Delta[A_{nm} \cap \pi_{nm}(A_{nm})]\}$$

$$= P\{(A\Delta A_{nm}) \cap [A\Delta\pi_{nm}(A_{nm})]\} + P\{(A \cap A_{nm})\Delta[A \cap \pi_{nm}(A_{nm})]\}$$

so that (3.65) and (3.71) show that

$$\lim_{n,m\to\infty} P\{A\Delta[A_{nm} \cap \pi_{nm}(A_{nm})]\} = 0$$

and hence

$$\lim_{n,m\to\infty} P[A_{nm} \cap \pi_{nm}(A_{nm})] = P(A) \quad . \tag{3.72}$$

Since $\mathbf{X}$ is 2-IID,

$$\begin{aligned} &P[A_{nm} \cap \pi_{nm}(A_{nm})] \\ &= P\{[(X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, \ldots, X_{2m}) \in B_{nm}] \\ &\qquad \cap \left[(X_{1\pi_1(1)}, \ldots, X_{1\pi_1(n)}, X_{2\pi_2(1)}, \ldots, X_{2\pi_2(m)}) \in B_{nm}\right]\} \\ &= P\{[(X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, \ldots, X_{2m}) \in B_{nm}] \\ &\qquad \cap [(X_{1n+1}, \ldots, X_{12n}, X_{2m+1}, \ldots, X_{22m}) \in B_{nm}]\} \\ &= P[(X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, \ldots, X_{2m}) \in B_{nm}] \\ &\quad P[(X_{1n+1}, X_{1n+2}, \ldots, X_{12n}, X_{2m+1}, X_{2m+2}, \ldots, X_{22m}) \in B_{nm}] \\ &= P(A_{nm})P[\pi_{nm}(A_{nm})] \quad . \end{aligned} \tag{3.73}$$

From (3.71) and (3.65) respectively

$$\lim_{n,m\to\infty} P\left[\pi_{nm}(A_{nm})\right] = P(A)$$

and

$$\lim_{n,m\to\infty} P(A_{nm}) = P(A)$$

follows, so that (3.72) and (3.73) shows that

$$P(A) = [P(A)]^2 \ . \tag{3.74}$$

The desired result follows immediately since only $P(A) = 0$ or $P(A) = 1$ satisfies (3.74).

Remark

Considering $\{X_i,\ i \in \mathbf{N}\}$ (i.e. $g = 1$ case) Aldous and Pitman (1979) gave a weaker assumption than IID of $\mathbf{X}$ which is sufficient for the exchangeable σ-field to be trivial (i.e. all events in this σ-field have measure zero or one). Sendler (1975) also published work on this topic but the Aldous and Pitman results are more powerful.

CHAPTER 4

ANALYSIS OF PARTIALLY EXCHANGEABLE SEQUENCES

De Finetti's representation theorem for exchangeable random variables is commonly stated as:

an infinite exchangeable sequence is a mixture of IID sequences.

In this chapter we shall investigate exactly what is meant by a mixture; use this theory to draw conclusions about a 2-fold partially exchangeable sequence (since it turns out to be a mixture of 2-IID sequences) and finally construct partially exchangeable sequences.

4.1 MIXTURES AND DIRECTION

Definition 4.1
Given a family $\{\mu_\gamma; \gamma \in \Gamma\}$ of distributions on a space S, a distribution ν is said to be a *mixture* of $\{\mu_\gamma; \gamma \in \Gamma\}$ if

$$\nu(\cdot) = \int_\Gamma \mu_\gamma(\cdot)\Theta(d\gamma)$$

for some distribution Θ on Γ.

A special case of the above definition describes the distribution of a sequence which is a mixture of 2-IID sequences:

Definition 4.2
Let $\mathbf{Y} = \{Y_{ij};\ i \in [2],\ j \in \mathbf{N}\}$ be an infinite 2-fold sequence of random variables. Suppose that Θ is a distribution on $P(\mathbf{R}) \times P(\mathbf{R})$. Then the *distribution of* $\mathbf{Y}$ *is a mixture of 2-IID sequences* if

$$P(\mathbf{Y} \in A) = \int_{P(\mathbf{R})\times P(\mathbf{R})} \theta^\infty(A)\Theta(d\theta) \qquad \forall A \in B(\mathbf{R}^\infty \times \mathbf{R}^\infty) \qquad (4.1)$$

where

$$\theta = \theta_1 \times \theta_2$$

$$A = \overset{2}{\underset{i=1}{\mathrm{X}}} \overset{\infty}{\underset{j=1}{\mathrm{X}}} A_{ij} \qquad A_{ij} \subset \mathbf{R}, \quad \forall i \in [2], \forall j \in \mathbf{N}$$

$$\theta^{\infty} = \overset{2}{\underset{i=1}{\mathrm{X}}} \overset{\infty}{\underset{j=1}{\mathrm{X}}} \theta_i \,, \qquad \text{the distribution on } \mathbf{R}^{\infty} \times \mathbf{R}^{\infty} \text{ of a 2-IID sequence,}$$

variables of type i having distribution θ_i, $\forall i \in [2]$.

$$\theta^{\infty}(A) = \prod_{i=1}^{2} \prod_{j=1}^{\infty} \theta_i(A_{ij}).$$

Remark
Definition 4.2 merely gives the Bayesian idea that $\mathbf{Y}$ is 2-IID, variables of type i having common distribution θ_i $\forall i \in [2]$, where $\theta = \theta_1 \times \theta_2$ has prior distribution Θ.

We now discuss random measures and regular conditional distributions: this will enable us to deal with mixtures of random variables rather than mixtures of distributions.

Definition 4.3
A *random measure* α, is a $P(\mathbf{R})$-valued random variable, viewed as a function of two variables as follows:

$$\alpha(\omega, \cdot) : B(\mathbf{R}) \to \mathbf{R}, \text{ is a probability measure for fixed } \omega \in \Omega \tag{4.2}$$

$$\alpha(\cdot, A) : \Omega \to \mathbf{R}, \text{ is a random variable for fixed } A \in B(\mathbf{R}). \tag{4.3}$$

Random measures α_1 and α_2 are *a.s. equivalent* if they are *a.s.* equivalent when viewed as random variables in $P(\mathbf{R})$, i.e.

$$\alpha_1(\cdot, A) = \alpha_2(\cdot, A) \qquad a.s. \qquad \forall A \in B(\mathbf{R}) \tag{4.4}$$

or equivalently,

$$P\{\omega; \alpha_1(\omega, A) = \alpha_2(\omega, A)\} = 1 \qquad \forall A \in B(\mathbf{R}) \ .$$

Definition 4.4
A *regular conditional distribution* (r.c.d.) for a random variable Y given a σ-field $\mathcal{F}$ is a random measure α such that

$$\alpha(\cdot, A) = P(Y \in A/\mathcal{F}) \ \ a.s. \qquad \forall A \in B(\mathbf{R}) \ . \tag{4.5}$$

Remark
An r.c.d. is thus a special type of random measure, one which is obtained by conditioning on a σ-field (as the name suggests).

It is well known that r.c.d.'s exist and satisfy the fundamental property stated in the following lemma. This property will frequently be used to enable versions of conditional expectations to be computed.

Lemma 4.1 (Shirayayev (1984), page 231, exercise 3)
Let X and Y be random variables and suppose that α is an r.c.d. for Y given a σ-field $\mathcal{F}$. If X is $\mathcal{F}$-measurable and $E|g(X,Y)| < \infty$, then

$$E\left[g(X,Y)/\mathcal{F}\right] = \int g(X,y)\alpha(\omega, dy) \quad a.s. \ . \tag{4.6}$$

We are now in a position to define a sequence which is a mixture of 2-IID random variables.

Definition 4.5
Let α_1 and α_2 be random measures and let $\mathbf{Y} = \{Y_{ij};\ i \in [2],\ j \in \mathbf{N}\}$ be an infinite 2-fold sequence of random variables. Then $\mathbf{Y}$ is a *mixture of 2-IID random variables, directed by* α_1 *and* α_2 if

$$\overset{2}{\underset{i=1}{\mathrm{X}}} \alpha_i^{\infty} \ \text{ is an r.c.d. for } \mathbf{Y} \text{ given } \ \mathcal{F}\,(\alpha_1, \alpha_2) \tag{4.7}$$

or equivalently, if

$$P\left(\mathbf{Y} \in A/\alpha_1, \alpha_2\right) = \prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}) \ \ a.s. \tag{4.8}$$

where $A = \overset{2}{\underset{i=1}{\mathrm{X}}} \overset{\infty}{\underset{j=1}{\mathrm{X}}} A_{ij}$, $A \in B(\mathbf{R}^{\infty} \times \mathbf{R}^{\infty})$.

Remark
Definition 4.2 differs from Definition 4.5 in that it defines the *distribution* of a sequence which is a mixture of 2-IID sequences.

Definition 4.6
Let $\mathbf{Y} = \{Y_{ij};\ i \in [2],\ j \in \mathbf{N}\}$ be an infinite 2-fold sequence of random variables and let $\mathcal{F}$ be a σ-field. Then $\mathbf{Y}$ is *conditionally independent* given $\mathcal{F}$ if

$$P[\mathbf{Y} \in A/\mathcal{F}] = \prod_{i=1}^{2} \prod_{j=1}^{\infty} P[Y_{ij} \in A_{ij}/\mathcal{F}] \tag{4.9}$$

and $\mathbf{Y}$ is *conditionally identically distributed within types* given $\mathcal{F}$ if,

$$P[Y_{ij} \in A_{ij}/\mathcal{F}] = P[Y_{ik} \in A_{ij}/\mathcal{F}] \qquad \forall k, j \in \mathbf{N}, i \in [2]. \tag{4.10}$$

where $A = \underset{i=1}{\overset{2}{\mathrm{X}}} \underset{j=1}{\overset{\infty}{\mathrm{X}}} A_{ij}$, $A \in B(\mathbf{R}^\infty \times \mathbf{R}^\infty)$.

$\mathbf{Y}$ is *conditionally 2-IID given* $\mathcal{F}$ if both (4.9) and (4.10) hold.

Remark
Using Appendix A2 we may give various results, equivalent to (4.9), which would result in $\mathbf{Y}$ being conditionally independent given $\mathcal{F}$.

Remark 4.1 (numbered for future reference)
Definition 4.5 may be re-stated as:

$\mathbf{Y}$ is a mixture of 2-IID random variables directed by α_1 and α_2 if $\mathbf{Y}$ is conditionally 2-IID given $\mathcal{F}(\alpha_1, \alpha_2)$ and

$$P(Y_{i1} \in A_{ij}/\alpha_1, \alpha_2) = \alpha_i(\cdot, A_{ij}) \quad \forall j \in \mathbf{N},\ \forall i \in [2],\ \forall A_{ij} \in B(\mathbf{R}) \ .$$

Theorem 4.1
Let $\mathbf{Y} = \{Y_{ij}; i \in [2],\ j \in \mathbf{N}\}$ be an infinite 2-fold sequence of random variables and let $\mathcal{F}$ be an arbitrary σ-field. Suppose that $\mathbf{Y}$ is conditionally 2-IID given $\mathcal{F}$ and let α_i be an r.c.d. for Y_{i1} given $\mathcal{F}$ $\forall i \in [2]$. Then

(i) $\mathbf{Y}$ is a mixture of 2-IID sequences of random variables directed by α_1 and α_2

(ii) $\mathbf{Y}$ and $\mathcal{F}$ are conditionally independent given $\mathcal{F}(\alpha_1, \alpha_2)$.

Proof

Let $A = \underset{i=1}{\overset{2}{\mathrm{X}}} \underset{j=1}{\overset{\infty}{\mathrm{X}}} A_{ij}$ for $A \in B(\mathbf{R}^\infty \times \mathbf{R}^\infty)$. Clearly

$$\alpha_i(\cdot, A_{ij}) = P\,(Y_{i1} \in A_{ij}/\mathcal{F}) \qquad \forall i \in [2],\ \forall j \in \mathbf{N} \tag{4.11}$$

and hence, using (4.9), (4.10) and (4.11),

$$\begin{aligned} P[\mathbf{Y} \in A/\mathcal{F}] &= \prod_{i=1}^{2} \prod_{j=1}^{\infty} P\,[Y_{ij} \in A_{ij}/\mathcal{F}] \\ &= \prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}). \end{aligned} \tag{4.12}$$

From (4.11) we see that α_i is $\mathcal{F}$-measurable $\forall i \in [2]$ and hence $\mathcal{F}(\alpha_i) \subset \mathcal{F}$ $\forall i \in [2]$. Using this and

$$\mathcal{F}(\alpha_1, \alpha_2) = \mathcal{F}\,[\mathcal{F}(\alpha_1) \cup \mathcal{F}(\alpha_2)] \tag{4.13}$$

we see that

$$\mathcal{F}(\alpha_1, \alpha_2) \subset \mathcal{F}\ . \tag{4.14}$$

From (4.12) and (4.14),

$$\begin{aligned} P\,[\mathbf{Y} \in A/\alpha_1, \alpha_2] &= E\,[P(\mathbf{Y} \in A/\mathcal{F})/\alpha_1, \alpha_2] \\ &= \prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}) \end{aligned} \tag{4.15}$$

which proves (i).

For any $A \in B(\mathbf{R}^\infty \times \mathbf{R}^\infty)$, (4.12), (4.14) and (4.15) yield

$$\begin{aligned} P\,[\mathbf{Y} \in A/\mathcal{F}, \mathcal{F}(\alpha_1, \alpha_2)] &= P[\mathbf{Y} \in A/\mathcal{F}] \\ &= \prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}) \\ &= P\,[\mathbf{Y} \in A/\alpha_1, \alpha_2] \end{aligned}$$

which is the desired result for (ii).

Suppose that $\mathbf{Y} = \{Y_{ij}; i \in [2],\ j \in \mathbf{N}\}$ is a mixture of 2-IID sequences for some unspecified random measures α_1 and α_2. The next theorem shows that we can determine the directed random measures, but we first need to prove the following Lemma.

Lemma 4.2
Let $\mathbf{Y} = \{Y_{ij}; i \in [2],\ j \in \mathbf{N}\}$ be a 2-IID sequence. Suppose, $\forall i \in [2]$, that random variables of type i have distribution function F_i. Then there exist empirical distribution functions $\{F_{in_i}\ ;\ n_i \in \mathbf{N}\}$, based on n_i samples from F_i, such that

$$\lim_{n_i \to \infty} F_{in_i}(\cdot, \omega) = F_i(\cdot) \quad a.s. \quad .$$

Proof
Let $i \in [2]$. Fix $n_i \in \mathbf{N}$ and for each $\omega \in \Omega$, let the n_i real numbers $\{Y_{ij}(\omega);\ j \in [n_i]\}$ be arranged in non-decreasing order, denoted by $\{x_{ij}; j \in [n_i]\}$ where

$$X_{i1}(\omega) \leq X_{i2}(\omega) \leq \ldots \leq X_{in_i}(\omega) \quad .$$

Now define a distribution function $F_{in_i}(\cdot, \omega)$ as follows:
For $x \in \mathbf{R}$,

$$F_{in_i}(x, \omega) = \begin{cases} 0 & \text{if} \quad x < X_{i1}(\omega) \\ \frac{k}{n_i} & \text{if} \quad X_{ik}(\omega) \leq x < X_{ik+1}(\omega) \qquad \text{for } 1 \leq k < n_i \\ 1 & \text{if} \quad x \geq X_{in_i} \end{cases}$$

or equivalently

$$F_{in_i}(x, \omega) = \frac{1}{n_i} \sum_{j=1}^{n_i} \xi_{ij}(x, \omega)$$

where

$$\xi_{ij}(x, \omega) = I\left[Y_{ij}(\omega) \leq x\right] \quad .$$

Call $F_{in_i}(\cdot,\omega)$ the empirical distribution, based on n_i samples from F_i. From the Glivenko-Cantelli theorem (Appendix A1) we thus have

$$\lim_{n_i\to\infty} F_{in_i}(\cdot,\omega) = F_i(\cdot) \quad a.s. \ .$$

Theorem 4.2
Let $\mathbf{Y} = \{Y_{ij}; i \in [2],\ j \in \mathbf{N}\}$ be an infinite two-fold sequence of random variables. If $\mathbf{Y}$ is a mixture of 2-IID sequences, then $\mathbf{Y}$ is directed by α_1 and α_2, where

$$\alpha_i = \Lambda_i\left(\{Y_{ij}; j \in \mathbf{N}\}\right) \qquad a.s. \qquad \forall i \in [2] \tag{4.16}$$

for some function Λ_i from $\mathbf{R}^\infty$ to $P(\mathbf{R})$.

Proof:
Fix $n_i \in \mathbf{N} \quad \forall i \in [2]$ and define the following empirical distribution function for $x \in \mathbf{R}, \omega \in \Omega$:

$$\Lambda_{in_i} : \mathbf{R}^{n_i} \to P(\mathbf{R})$$

by

$$\Lambda_{in_i}\left[Y_{i1}(\omega),\ldots,Y_{in_i}(\omega)\right] = n_i^{-1}\sum_{j=1}^{n_i} \xi_{Y_{ij}}(\omega) \tag{4.17}$$

where

$$\xi_{Y_{ij}(\omega)}(A) = I\left[Y_{ij}(\omega) \in A\right]$$

for $A \subset \mathbf{R}, \quad \forall j \in [n_i]$. Also define

$$\Lambda_i : \mathbf{R}^\infty \to P(\mathbf{R})$$

by

$$\Lambda_i\left(\{Y_{ij}(\omega); j \in \mathbf{N}\}\right) = \lim_{n_i\to\infty} \Lambda_{in_i}\left(Y_{i1}(\omega),\ldots,Y_{in_i}(\omega)\right) \ . \tag{4.18}$$

Now let

$$A \in B(\mathbf{R}^\infty \times \mathbf{R}^\infty),\ A = \mathop{\mathrm{X}}_{i=1}^{2}\mathop{\mathrm{X}}_{j=1}^{\infty} A_{ij}\ ,\ A_{ij} \subset \mathbf{R} \qquad \forall i \in [2],\ \forall j \in \mathbf{N}.$$

Then, from (4.8),

$$P\left(\mathbf{Y} \in A/\alpha_1,\alpha_2\right) = \prod_{i=1}^{2}\prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}) \quad a.s. \tag{4.19}$$

Now let $\alpha^{\infty}(\cdot, A) = \prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij})$.

From (4.19) and Lemma 4.1,

$$\prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}) = E\{I[\mathbf{Y} \in A]/\alpha_1, \alpha_2\} \quad a.s.$$

$$= \int I[\mathbf{Y}(\omega) \in A]\, \alpha^{\infty}(\omega, d\mathbf{Y}(\omega))] \quad a.s. \ . \tag{4.20}$$

From Lemma 4.2, (4.17), (4.18) and (4.20) we thus have

$$\alpha_i = \Lambda_i(\{Y_{ij}; j \in \mathbf{N}\}) \quad a.s.\ \forall i \in [2].$$

Remark
If $\mathbf{Y} = \{Y_{ij}; i \in [2],\ j \in \mathbf{N}\}$ is a mixture of 2-IID sequences then we have an expression for the directing random measures.

So we may speak of *the set of directing random measures* as Theorem 4.2 gives an expression for the directing random measures which holds with probability one.

Corollary 4.1
Let $\mathbf{Y} = \{Y_{ij}; i \in [2],\ j \in \mathbf{N}\}$ be a mixture of 2-IID random variables. Then the directing random measures α_1 and α_2 are τ-measurable, where τ is the tail σ-field of $\mathbf{Y}$.

Proof
From Theorem 4.2 we have

$$\alpha_i = \Lambda_i(\{Y_{ij}; j \in \mathbf{N}\}) \quad a.s. \qquad \forall i \in [2]$$

where $\Lambda_i : \mathbf{R}^{\infty} \to P(\mathbf{R})$ is as defined in Theorem 4.2. Let $i \in [2]$. Changing a finite number of random variables in $\{Y_{ij}; j \in \mathbf{N}\}$ will not change $\Lambda_i(\{Y_{ij}, j \in \mathbf{N}\})$, hence α_i is measurable w.r.t. τ, since

$$\tau = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{F}(Y_{1n+1}, Y_{1n+2}, \ldots, Y_{2m+1}, Y_{2m+2}, \ldots) \ .$$

The following very useful result will often be referred to.

Theorem 4.3
Let $\mathbf{Y} = \{Y_{ij}, i \in [2],\ j \in \mathbf{N}\}$ be a mixture of 2-IID sequences and let α_1 and α_2 be the directing random measures. Let τ denote the tail σ-field of $\mathbf{Y}$. Then the following are all a.s. equal:

(i) $P\left[Y_{ij} \in A_{ij}; j \in [n_i], \forall i \in [2] / Y_{1m_1}, Y_{1m_1+1}, \ldots, Y_{2m_2}, Y_{2m_2+1}, \ldots\right]$
$m_i > n_i\ ,\ \forall i \in [2]$

(ii) $P\left[Y_{ij} \in A_{ij};\ j \in [n_i],\ \forall i \in [2] / Y_{1m_1}, Y_{1m_1+1}, \ldots, Y_{2m_2}, Y_{2m_2+2}, \ldots, \alpha_1, \alpha_2\right]$
$m_i > n_i\ ,\ \forall i \in [2]$

(iii) $P\left[Y_{ij} \in A_{ij};\ j \in [n_i],\ \forall i \in [2]/\tau\right]$

(iv) $P\left[Y_{ij} \in A_{ij};\ j \in [n_i],\ \forall i \in [2]/\alpha_1, \alpha_2\right]$

(v) $\prod_{i=1}^{2} \prod_{j=1}^{n_i} \alpha_i(\cdot, A_{ij})$

where

$$A_{ij} \in B(\mathbf{R}),\ \ \forall i \in [2],\ \ \forall j \in \mathbf{N},\ m_i \in \mathbf{N},\ n_i \in \mathbf{N},\ \ \forall i \in [2]\ .$$

Proof:
From the definition of the tail σ-field of $\mathbf{Y}$ (Definition 3.3) and Corollary 4.1 it follows that (i) and (ii) are equal.

Remark (4.1) shows that $\mathbf{Y}$ is conditionally independent given $\mathcal{F}(\alpha_1, \alpha_2)$, so that $\{Y_{ij}; j \in [n_i],\ \ \forall i \in [2]\}$ and $\{Y_{ij};\ j \in \{m_i+1, m_i+2, \ldots\},\ \forall i \in [2]\}$ are conditionally independent given $\mathcal{F}(\alpha_1, \alpha_2)$ (i.e. once conditioned on $\mathcal{F}(\alpha_1, \alpha_2)$, any random variable from the first sequence is independent of any one from the second sequence). By a standard property of conditional independence (Appendix A2, property A3) we see that (ii) = (iv). Also (iv) = (v) follows immediately from Definition 4.5.

Now

$$\mathcal{F}(\alpha_1, \alpha_2) = \mathcal{F}\left[\mathcal{F}(\alpha_1) \cup \mathcal{F}(\alpha_2)\right]$$

so that

$$\mathcal{F}(\alpha_1, \alpha_2) \subset \tau \tag{4.21}$$

since α_1 and α_2 are both τ-measurable (see Corollary 4.1). From the definition of τ it follows that

$$\tau \subset \mathcal{F}\left(Y_{1m_1}, Y_{1m_1+1}, \ldots, Y_{2m_2}, Y_{2m_2+1}, \ldots, \alpha_1, \alpha_2\right) \quad , m_i \in \mathbf{N}, \ \forall i \in [2] \ . \tag{4.22}$$

Using (4.21), (4.22) and the fact that (ii) = (iv),

$$P\left[Y_{ij} \in A_{ij} \ , \ j \in [n_i] \ , \ \forall i \in [2]/\tau\right]$$

$$= E\left\{E[I[Y_{ij} \in A_{ij}, j \in [n_i], \ \forall i \in [2]]/Y_{1m_1}, Y_{1m_1+1}, \ldots, Y_{2m_2}, Y_{2m_2+2}, \ldots, \alpha_1, \alpha_2]/\tau\right\}$$

$$= E\left\{E[I[Y_{ij} \in A_{ij}, \ j \in [n_i], \ \forall i \in [2]]/\alpha_1, \alpha_2]/\tau\right\}$$

$$= E\left\{I[Y_{ij} \in A_{ij}; \ j \in [n_i], \ \ \forall i \in [2]]/\alpha_1, \alpha_2\right\}$$

so that (iii) = (iv).

Theorem 4.4
Let $\mathbf{Y} = \{Y_{ij}; \ i \in [2], \ j \in \mathbf{N}\}$ be an infinite 2-fold sequence of random variables. If $\mathbf{Y}$ is a mixture of 2-IID random variables then $\mathbf{Y}$ is conditionally 2-IID given τ, the tail σ-field of $\mathbf{Y}$.

Proof:
Let $A_{ij} \in B(\mathbf{R})$ $\forall j \in [n_i]$, for $n_i \in \mathbf{N}$, $\forall i \in [2]$. Then from Definition 4.5,

$$P\left[Y_{ij} \in A_{ij}; \ j \in [n_i], \ \forall i \in [2]/\alpha_1\alpha_2\right] = \prod_{i=1}^{2} \prod_{j=1}^{n_i} \alpha_i(\cdot, A_{ij}) \ . \tag{4.23}$$

Let $j \in [n_i], n_i \in \mathbf{N}$, $\forall i \in [2]$. Then from Remark 4.1 and Theorem 4.3 ((iii) = (iv)), $\forall A_{ij} \in B(\mathbf{R})$,

$$\begin{aligned} \alpha_i(\cdot, A_{ij}) &= P\left(Y_{i1} \in A_{ij}/\alpha_1, \alpha_2\right) \\ &= P\left(Y_{ij} \in A_{ij}/\alpha_1, \alpha_2\right) \\ &= P\left(Y_{ij} \in A_{ij}/\tau\right) \qquad \forall j \in [n_i], \ \forall i \in [2] \ . \end{aligned} \tag{4.24}$$

So that

$$\prod_{i=1}^{2}\prod_{j=1}^{n_i}\alpha_i(\cdot, A_{ij}) = \prod_{i=1}^{2}\prod_{j=1}^{n_i} P\left[Y_{ij} \in A_{ij}/\tau\right] \ a.s. \ . \tag{4.25}$$

Now consider the right hand side of (4.23). Using (iii) = (iv) in Theorem 4.3 we obtain

$$P\left[Y_{ij} \in A_{ij};\ j \in [n_i],\ \ \forall i \in [2]/\alpha_1, \alpha_2\right] = P[Y_{ij} \in A_{ij}\ ;\ j \in [n_i],\ \forall i \in [2]/\tau]$$

and substituting this last result and (4.25) with (4.23) we see that

$$P\left[Y_{ij} \in A_{ij};\ j \in [n_i],\ \ \forall i \in [2]/\tau\right] = \prod_{i=1}^{2}\prod_{j=1}^{n_i} P\left[Y_{ij} \in A_{ij}/\tau\right] \ .$$

So that $\mathbf{Y}$ is conditionally independent given τ. The identical distributions within types given τ comes from (4.24) since

$$P\left[Y_{ij} \in A_{ij}/\tau\right] = P\left[Y_{ij} \in A_{ij}/\alpha_1, \alpha_2\right] \ \ a.s.$$

and the right hand side of this equation is independent of j as far as Y_{ij} is concerned (see Remark 4.1).

Corollary 4.2
Let $\mathbf{Y} = \{Y_{ij}; i \in [2],\ j \in \mathbf{N}\}$ be an infinite 2-fold sequence of random variables. If $\mathbf{Y}$ is a mixture of 2-IID random variables, then it is directed by random measures α_1 and α_2, and we have three possible expressions for them

(i) $\alpha_i = \Lambda_i\left(\{Y_{ij}; j \in \mathbf{N}\}\right)$ a.s., where Λ_i is as in Theorem 4.2.

(ii) α_i is an r.c.d. for Y_{i1} given τ.

(iii) α_i is an r.c.d. for Y_{i1} given $\mathcal{F}\left(Y_{1m_1}, Y_{1m_1+1}, \ldots, Y_{2m_2}, Y_{2m_2+1}, \ldots\right)$
$m_i \in \mathbf{N}$, $\forall i \in [2]$.

Proof:
Expression (i) for the d.r.m.'s comes from Theorem 4.2. From (4.24),

$$\alpha_i\left(\cdot, A_{ij}\right) = P\left[Y_{ij} \in A_{ij}/\tau\right] \qquad \forall j \in [n_i]$$

so that

$$\alpha_i(\cdot, A_{ij}) = P\left[Y_{i1} \in A_{ij}/\tau\right] \tag{4.26}$$

and hence (ii) follows.

Taking $n_i = 1$, $\forall i \in [2]$, using Theorem 4.3 (i) = (iii) and (4.26) gives (iii).

Remarks
Corollary 4.2 provides three expressions for the d.r.m.'s, they are all a.s. equal (as the d.r.m.'s are a.s. unique). Each of the expressions is useful in some circumstances.

4.2 PARTIALLY EXCHANGEABLE RANDOM VARIABLES, MIXTURES AND DIRECTION

In this section we obtain results for partially exchangeable sequences by applying the theory of §4.1.

Throughout this section $\mathbf{Z} = \{Z_{ij}; i \in [2],\ j \in \mathbf{N}\}$ will denote an infinite (2-fold) sequence of random variables.

De Finetti's theorem for 2-fold partially exchangeable random variables (Theorem 3.3) may be restated as follows:

Theorem 4.5
If $\mathbf{Z}$ is partially exchangeable then it is a mixture of 2-IID sequences, directed by α_1 and α_2, where

$$\alpha_i(\cdot, A_{ij}) = P\left[Z_{i1} \in A_{ij}/\tau\right] \qquad \forall i \in [2], \forall j \in \mathbf{N}, \forall A_{ij} \in B(\mathbf{R}) \tag{4.27}$$

and τ denotes the tail σ-field of $\mathbf{Z}$.

Proof:
The desired result follows immediately from Theorems 3.2 and 4.1 (i).

Remark
Using Theorems 3.1 and 4.1 we could similarly show that a partially ex-

changeable sequence $\mathbf{Z}$ is a mixture of 2-IID sequences, directed by

$$\alpha_i(\cdot, A_{ij}) = P\left[Z_{i1} \in A_{ij}/\mathcal{F}_\infty\right], \quad i \in [2] \tag{4.28}$$

where $\mathcal{F}_\infty$ is as in Theorem 3.1.

Corollary 4.3
$\mathbf{Z}$ is a mixture of 2-IID sequences directed by α_1 and α_2 (as in (4.27) or (4.28)) if and only if $\mathbf{Z}$ is partially exchangeable.

Proof:
Suppose that $\mathbf{Z}$ is a mixture of 2-IID sequences, directed by α_1 and α_2 and let

$$\mathcal{F}^\infty(\cdot, A) = \prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}) \tag{4.29}$$

where

$$A = \mathop{\mathrm{X}}_{i=1}^{2} \mathop{\mathrm{X}}_{j=1}^{\infty} A_{ij}, \text{ i.e. } A \in B(\mathbf{R}^\infty \times \mathbf{R}^\infty) \ .$$

Then, using Lemma 4.1 and Definition 4.5,

$$\prod_{i=1}^{2} \prod_{j=1}^{\infty} \alpha_i(\cdot, A_{ij}) = \int I\left[\mathbf{Z}(\omega) \in A\right] F^\infty(\omega, d\mathbf{Z}(\omega)) \ . \tag{4.30}$$

The right hand side of (4.30) shows that $\mathbf{Z}$ is partially exchangeable. The reverse implication follows from Theorem 4.5.

Remark
Partially exchangeable sequences are usually defined in terms of the invariance of their joint distribution under permutations of the variables (see Definition 2.7). In view of Corollary 4.3 however, Definition 2.7 may be replaced by the following:

> $\mathbf{Z}$ is (2-fold) partially exchangeable if there exist random measures α_1 and α_2 such that $\mathbf{Z}$ is a mixture of 2-IID sequences, directed by α_1 and α_2.

Aldous ((1977), page 61) defines an exchangeable sequence in this manner.

Theorem 4.6
If $\mathcal{F}$ is an arbitrary σ-field such that $\mathbf{Z}$ is conditionally 2-IID given $\mathcal{F}$ then

(i) $\mathbf{Z}$ is partially exchangeable

(ii) $\mathbf{Z}$ and $\mathcal{F}$ are conditionally independent given $\mathcal{F}(\alpha_1, \alpha_2)$ where

$$\alpha_i(\cdot, A) = P[Z_{i1} \in A/\mathcal{F}] \quad \forall i \in [2] \quad \forall A \in B(\mathbf{R}) \; . \tag{4.31}$$

Proof:
The desired result follows immediately from Theorem 4.1 and Corollary 4.3.

Remark 4.2 (numbered for future reference)
If there exists a σ-field $\mathcal{F}$ such that $\mathbf{Z}$ is conditionally 2-IID given $\mathcal{F}$, then we may refer to $\mathbf{Z}$ as being partially exchangeable without particular reference to the two d.r.m's (bearing in mind that they are of the form given in (4.31)).

Similarly, de Finetti's theorem can be loosely stated as:

Partially exchangeable sequences are mixtures of 2-IID sequences

i.e. no reference to the two d.r.m.'s.

The next theorem however, gives various expressions for the d.r.m.'s of partially exchangeable sequences.

Theorem 4.7
If $\mathbf{Z}$ is a partially exchangeable sequence, then it is directed by random measures α_1 and α_2 and we have three possible expressions for them.

(i) $\alpha_i = \Lambda_i(\{Z_{ij}; j \in \mathbf{N}\})$ a.s. where Λ_i is as in Theorem 4.2.

(ii) α_i is an r.c.d. for Z_{i1} given τ the tail σ-field of $\mathbf{Z}$.

(iii) α_i is an r.c.d. for Z_{i1} given $\mathcal{F}(Z_{1m_1}, Z_{1m_1+1}, \cdots, Z_{2m_2}, Z_{2m_2+1}, \cdots)$
$\forall i \in [2], \quad m_1 \in \mathbf{N}, \; m_2 \in \mathbf{N}.$

Proof:
The desired result follows immediately from Theorem 4.5 and Corollary 4.2.

Theorem 4.4 leads us to the following very useful result:

Theorem 4.8
If $\mathbf{Z}$ is partially exchangeable, then $\mathbf{Z}$ is conditionally 2-IID given τ, the tail σ-field of $\mathbf{Z}$.

Proof:
If $\mathbf{Z}$ is partially exchangeable then it is a mixture of 2-IID sequences (Remark 4.2) and the result follows on application of Theorem 4.4.

The following definitions will be used to obtain other σ-fields which can replace the tail σ-field in Theorem 4.8.

Definition 4.7
Let $\mathbf{Z}$ be a partially exchangeable sequence of random variables and let V be any random variable. Then $\mathbf{Z}$ is *partially exchangeable over* V if

$$L(V, \mathbf{Z}) = L(V, \mathbf{Z}_\pi) \tag{4.32}$$

where

$$\mathbf{Z}_\pi = \left\{ Z_{i\pi_i(j)};\ \ i \in [2],\ j \in \mathbf{N} \right\}$$

for π_i a finite permutation of $\{1, 2, \ldots\}$, $i \in [2]$.

Definition 4.8
Let $\mathbf{Z}$ be a partially exchangeable sequence of random variables and let $\mathcal{G}$ be any σ-field. Then $\mathbf{Z}$ is *partially exchangeable over* $\mathcal{G}$ if (4.32) holds for each random variable V which is measurable with respect to $\mathcal{G}$.

Lemma 4.3
Let $\mathbf{Z}$ be a partially exchangeable sequence of random variables and let V be a random variable such that $\mathbf{Z}$ is partially exchangeable over V. Then

(i) $\mathbf{Z}$ is conditionally 2-IID given (V, α_1, α_2) where α_1 and α_2 are the d.r.m.'s for $\mathbf{Z}$

(ii) $\mathbf{Z}$ and V are conditionally independent given $\mathcal{F}(\alpha_1, \alpha_2)$.

Proof:
Define a new 2-fold infinite sequence of random variables as follows:

$$\hat{\mathbf{Z}} = \left\{\hat{Z}_{ij};\ i \in [2],\ j \in \mathbf{N}\right\}$$

where

$$\hat{Z}_{ij} = (V, Z_{ij}) \qquad \forall i \in [2],\ \forall j \in \mathbf{N}\ .$$

Then $\hat{\mathbf{Z}}$ is clearly partially exchangeable. If we denote the d.r.m.'s of $\hat{\mathbf{Z}}$ by $\hat{\alpha}_1$ and $\hat{\alpha}_2$ then $\hat{\mathbf{Z}}$ is conditionally 2-IID given $\mathcal{F}(\hat{\alpha}_1, \hat{\alpha}_2)$ (see (4.28)). In particular $\mathbf{Z}$ is conditionally 2-IID given $\mathcal{F}(\hat{\alpha}_1, \hat{\alpha}_2)$.

Applying Theorem 4.7 (i) to $\hat{\mathbf{Z}}$ we see that for each $i \in [2]$

$$\begin{aligned}\hat{\alpha}_i &= \Lambda_i\left(\{\hat{Z}_{ij};\ j \in \mathbf{N}\}\right) \\ &= \lim_{n_i \to \infty} n_i^{-1} \sum_{j=1}^{n_i} \delta_{\hat{Z}_{ij}} \qquad (4.33)\end{aligned}$$

where, $\forall A \in B(\mathbf{R}^2)$,

$$\delta_{\hat{Z}_{ij}}(A) = \begin{cases} 1 & \hat{Z}_{ij} = (V, Z_{ij}) \in A \\ 0 & \text{elsewhere.} \end{cases} \qquad (4.34)$$

If $A = A_1 \times A_2$ for $A_i \in B(\mathbf{R})$ $\forall i \in [2]$ then

$$\delta_{\hat{Z}_{ij}}(A) = \delta_{Z_{ij}}(A_1)\delta_V(A_2) \qquad \forall j \in \mathbf{N}$$

where

$$\delta_{Z_{ij}}(A_1) = I\left[Z_{ij} \in A_1\right] \qquad \forall j \in \mathbf{N} \text{ and } \delta_V(A_2) = I[V \in A_2]$$

so that (4.33) becomes

$$\begin{aligned}\hat{\alpha}_i &= \lim_{n_i \to \infty} n_i^{-1} \sum_{j=1}^{n_i} \delta_{Z_{ij}}\delta_V \\ &= \delta_V \lim_{n_i \to \infty} n_i^{-1} \sum_{j=1}^{n_i} \delta_{Z_{ij}}\end{aligned}$$

and hence, from Theorem 4.7 (i),

$$\hat{\alpha}_i = \delta_V \alpha_i \quad . \tag{4.35}$$

We thus have a partially exchangeable sequence $\hat{\mathbf{Z}}$ with d.r.m.'s $\delta_V \alpha_1$ and $\delta_V \alpha_2$. It thus follows that $\mathcal{F}(\hat{\alpha}_i) = \mathcal{F}(V, \alpha_i)\ \forall i \in [2]$, and hence

$$\mathcal{F}(\hat{\alpha}_1, \hat{\alpha}_2) = \mathcal{F}(V, \alpha_1, \alpha_2) \quad .$$

We have thus shown (i) since $\mathbf{Z}$ is conditionally 2-IID given $\mathcal{F}(\hat{\alpha}_1, \hat{\alpha}_2)$.

From (i) and Theorem 4.6 we see that $\mathbf{Z}$ and $\mathcal{F}(V, \alpha_1, \alpha_2)$ are conditionally independent given $\mathcal{F}(\alpha_1, \alpha_2)$, i.e.

$$P[\mathbf{Z} \in A / \mathcal{F}(V, \alpha_1, \alpha_2), \mathcal{F}(\alpha_1, \alpha_2)] = P[\mathbf{Z} \in A / \mathcal{F}(\alpha_1, \alpha_2)]$$

so that

$$P[\mathbf{Z} \in A / \mathcal{F}(V), \mathcal{F}(\alpha_1, \alpha_2)] = P[\mathbf{Z} \in A / \mathcal{F}(\alpha_1, \alpha_2)]$$

and hence $\mathbf{Z}$ and V are conditionally independent given $\mathcal{F}$ (see Appendix A2, point A3) so that we have shown (ii).

We are now in a position to prove a very important result.

Theorem 4.9
Let $\mathbf{Z}$ be a partially exchangeable sequence of random variables, directed by α_1 and α_2. Then

$$E_{\mathbf{Z}} = \tau_{\mathbf{Z}} = \mathcal{F}(\alpha_1, \alpha_2) \quad a.s. \tag{4.36}$$

where $E_{\mathbf{Z}}$ and $\tau_{\mathbf{Z}}$ are the partially exchangeable and tail σ-fields of $\mathbf{Z}$ respectively, (see Definitions 3.3 and 3.5).

Proof:
Suppose that $A \in \tau_{\mathbf{Z}}$. Then $A \in \mathcal{F}_{nm}^{\infty}$ for infinitely many values of $n, m \in \mathbf{N}$, where

$$\mathcal{F}_{nm}^{\infty} = \mathcal{F}(Z_{1n+1}, Z_{1n+2}, \ldots, Z_{2m+1}, Z_{2m+2}, \ldots)$$

and hence $A \in E_{\mathbf{Z}}$ so that

$$\tau_{\mathbf{Z}} \subset E_{\mathbf{Z}} \quad . \tag{4.37}$$

For $B \in E_{\mathbf{Z}}$, define random variable

$$I_B(\omega) = \begin{cases} 1 & \omega \in B \\ 0 & \omega \notin B \, . \end{cases} \tag{4.38}$$

Then B is a subset of Ω which $\mathbf{Z}$ maps to $\mathbf{R}^\infty \times \mathbf{R}^\infty$ such that $\forall \omega \in B$,

$$\mathbf{Z}(\omega) = \mathbf{Z}_\pi(\omega)$$

where

$$\mathbf{Z}_\pi = \left\{ Z_{i\pi_i(j)};\ i \in [2],\ j \in \mathbf{N} \right\}$$

and π_i is any finite permutation of $\{1, 2, \ldots\}$, $\forall i \in [2]$.

Clearly then, since $\mathbf{Z}$ is partially exchangeable we obtain

$$L\left(I_B, \mathbf{Z}\right) = L\left(I_B, \mathbf{Z}_\pi\right) \quad .$$

Definition 4.7 shows that $\mathbf{Z}$ is partially exchangeable over I_B for $B \in E_{\mathbf{Z}}$. Lemma 4.3 then yields the following:

$\mathbf{Z}$ and I_B are conditionally independent given $\mathcal{F}(\alpha_1, \alpha_2)$.

Using Definition 4.8, we thus have the following:

$\mathbf{Z}$ and $E_{\mathbf{Z}}$ are conditionally independent given $\mathcal{F}(\alpha_1, \alpha_2)$.

Clearly $E_{\mathbf{Z}} \subset \mathcal{F}(\mathbf{Z})$ so that $E_{\mathbf{Z}}$ is conditionally independent of itself given $\mathcal{F}(\alpha_1, \alpha_2)$. Hence

$$E_{\mathbf{Z}} \subset \mathcal{F}(\alpha_1, \alpha_2) \ \ a.s. \tag{4.39}$$

follows from A6 in Appendix A2. From Corollary 4.1 and Theorem 4.5,

$$\mathcal{F}(\alpha_1, \alpha_2) \subset \tau_{\mathbf{Z}} \quad a.s. \tag{4.40}$$

Equations (4.39) and (4.40) thus yield

$$E_{\mathbf{Z}} \subset \mathcal{F}(\alpha_1, \alpha_2) \subset \tau_{\mathbf{Z}} \quad a.s. \tag{4.41}$$

Combining this with (4.37) we have

$$E_{\mathbf{Z}} = \mathcal{F}(\alpha_1, \alpha_2) = \tau(\mathbf{Z}) \quad a.s.$$

as required.

Remarks
From Theorem 4.8, Corollary 4.3 and Theorem 4.9 we see that $\mathbf{Z}$ is partially exchangeable, directed by α_1 and α_2, if and only if $\mathbf{Z}$ is conditionally 2-IID given either

(i) $\tau_{\mathbf{Z}}$, the tail σ-field of $\mathbf{Z}$

or

(ii) $E_{\mathbf{Z}}$, the partially exchangeable σ-field of $\mathbf{Z}$

or

(iii) $\mathcal{F}(\alpha_1, \alpha_2)$, where α_1 and α_2 are the d.r.m.'s of $\mathbf{Z}$.

Theorem 4.6 thus gives the following three possible expressions for d.r.m.'s α_i , $i \in [2]$.

Let $A \in B(\mathbf{R})$, then $\alpha_i(\cdot, A)$, the d.r.m., is any one of the following three expressions:

(i) $P\,(Z_{i1} \in A/\tau_{\mathbf{Z}})$ (see Theorem 4.7 (ii))

(ii) $P\,(Z_{i1} \in A/E_{\mathbf{Z}})$

(iii) $P\,(Z_{i1} \in A/\alpha_1, \alpha_2)$.

Remark
The three expressions for the set of directed random variables are a.s. equal as can be seen from Theorem 4.9. We may thus speak of *the* set of directed random measures.

4.3 CONSTRUCTION OF PARTIALLY EXCHANGEABLE RANDOM VARIABLES

The purpose of this section is to point out some concrete ways of constructing partially exchangeable sequences. Most of the results are direct consequences of de Finetti's theorem (Theorem 3.3) and show how to manipulate random measures.

Throughout this section $\mathbf{Z} = \{Z_{ij}; i \in [2],\ j \in \mathbf{N}\}$ will denote an infinite 2-fold sequence of random variables. If $\mathbf{Z}$ is partially exchangeable, then we denote the d.r.m.'s by α_i, $i \in [2]$.

The next two examples may be considered to be *degenerate* partially exchangeable sequences:

(i) If $\mathbf{Z}$ is 2-IID, variables of type i having distribution θ_i , $\forall i \in [2]$, then $\mathbf{Z}$ may be viewed as a partially exchangeable sequence directed by θ_i, $\forall i \in [2]$.

(ii) If $\mathbf{Z}$ has $Z_{i1} = Z_{i2} = \cdots$ a.s. , $\forall i \in [2]$, then $\mathbf{Z}$ may be viewed as a partially exchangeable sequence with $\alpha_i(\cdot, A) = I(X_i \in A)$ $\forall A \in B(\mathbf{R})$ for some fixed random variable X_i , $\forall i \in [2]$.

A natural way to construct a partially exchangeable sequence (this was hinted at after Corollary 3.1) is to take a parametric family of distributions, choose the parameters randomly and then take a 2-IID sequence which has a distribution with these random parameters. The next example demonstrates this method.

Example
Denote the Normal (θ, σ^2) distribution by $\mu_{\theta,\sigma}$. Let $\mathbf{X} = \{X_{ij}; i \in [2],\ j \in \mathbf{N}\}$ be a 2-IID sequence, $X_{ij} \sim N(0,1)$, $\forall i \in [2],\ \forall j \in \mathbf{N}$.

If $\mathbf{Z}$ is defined by

$$Z_{ij} = \Theta_i + S_i X_{ij} \quad \forall i \in [2],\ \forall j \in \mathbf{N}$$

where Θ_i and S_i are random variables denoting the mean and standard deviation of Z_{ij}, then $\mathbf{Z}$ is a partially exchangeable sequence of random variables, directed by $\alpha_i = \mu_{\Theta_i, S_i}$, $\forall i \in [2]$ (conditioning on $\Theta_i = \theta_i$, $S_i = \sigma_i$, $\forall i \in [2]$, clearly gives a 2-IID sequence).

In general however, $\mathbf{Z}$ is a more complicated mixture of parametric families as the following examples demonstrate.

Example
Let $\mathbf{Y} = \{Y_{ij};\ i \in [2],\ j \in \mathbf{N}\}$ and $\mathbf{X} = \{X_{ij};\ i \in [2],\ j \in \mathbf{N}\}$ be 2-fold infinite sequences of random variables. Suppose that $\mathbf{X}$ is a 2-IID sequence, independent of $\mathbf{Y}$, such that $\{X_{ij};\ j \in \mathbf{N}\}$ takes distinct values from $\{1, 2, \ldots\}$, $\forall i \in [2]$. If $\mathbf{Z}$ is defined by

$$Z_{ij} = Y_{iX_{ij}} \quad \forall i \in [2],\ \forall j \in \mathbf{N}\ .$$

then $\mathbf{Z}$ is partially exchangeable.

Using Theorem 4.7 (i) we next find expressions for the d.r.m.'s.

Let $i \in [2]$,

$$\begin{aligned}
\alpha_i &= \Lambda_i\left(\{Z_{ij};\ j \in \mathbf{N}\}\right) \\
&= \lim_{n_i \to \infty} \sum_{j=1}^{n_i} \delta_{Y_{iX_{ij}}} P(X_{i1} = j) \\
&= \lim_{n_i \to \infty} \sum_{j=1}^{n_i} \delta_{Y_{ij}} P(X_{i1} = j) \\
&= \sum_{j=1}^{\infty} \delta_{Y_{ij}} P_{ij}
\end{aligned}$$

where

$$P_{ij} = P(X_{i1} = j) \quad \forall j \in \mathbf{N}.$$

The following simple method of construction also yields a partially exchangeable sequence.

Example 4.1 (numbered for future reference)
Let $\mathbf{X} = \{X_{ij};\ i \in [2],\ j \in \mathbf{N}\}$ be a 2-fold infinite sequence of random variables. Suppose that $\mathbf{X}$ is a 2-IID sequence, variables of type i having distribution θ_i , $\forall i \in [2]$. Let Y be an arbitrary random variable, which is independent of $\mathbf{X}$, with distribution ϕ. If $\mathbf{Z}$ is defined by

$$Z_{ij} = f(Y, X_{ij}) \quad \forall i \in [2],\ \forall j \in \mathbf{N}$$

for $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ an arbitrary Borel-measurable function, then $\mathbf{Z}$ is partially exchangeable. In order to find the d.r.m.'s of this sequence we recall the induced map as defined in Definition 3.4 for the special case $S = \mathbf{R}$.

If f has induced map $\tilde{f} : \mathbf{R} \times P(\mathbf{R}) \to P(\mathbf{R})$ (see Definition 3.4 (ii)) then Corollary 4.2 (i) shows that $\mathbf{Z}$ is directed by $\tilde{f}(Y, \theta_1)$ and $\tilde{f}(Y, \theta_2)$.

Remark

Let $\mathbf{Z}$ be a partially exchangeable sequence of random variables directed by α_1 and α_2 and suppose that $f : \mathbf{R} \to \mathbf{R}$ is an arbitrary Borel measurable function. If $f(\mathbf{Z}) = \{f(\mathbf{Z}_{ij}); i \in [2], j \in \mathbf{N}\}$ then $f(\mathbf{Z})$ is a partially exchangeable sequence of random variables (see Theorem 2.2) and using Corollary 4.2 (i) we see that $f(\mathbf{Z})$ is directed by $\tilde{f}(\alpha_1)$ and $\tilde{f}(\alpha_2)$.

CHAPTER 5

LIMIT THEOREMS

To conclude this thesis we present some limit theorems for partially exchangeable random variables. These limit theorems are immediate consequences of the main result of this chapter:

Given any a.s. limit theorem for 2-IID sequences of random variables, there exists an analogous theorem satisfied by all (2-fold) partially exchangeable sequences of random variables and by all sub-subsequences of some infinite subsequence of an arbitrary dependent 2-fold infinite sequence of random variables, tightly distributed within types.

Section 5.1 is devoted to notation, definitions, results and a brief discussion of the above mentioned theorem. This shows the curious link between some of the limit theorems of partially exchangeable sequences and those of sub-subsequences of arbitrary 2-fold infinite sequences.

The proof of the above mentioned main result appears in §5.2 while §5.3 is devoted to limit theorems for partially exchangeable sequences which follow from the main result.

5.1 NOTATION, DEFINITIONS AND DISCUSSION OF RESULTS

Komlós (1967) showed that if $\{X_i,\ i \in \mathbf{N}\}$ is a sequence of random variables such that $\sup_i E|X_i| < \infty$ then there exists an increasing sequence of integers $\{n_1, n_2, \ldots\}$ and a random variable V such that

$$\lim_N N^{-1} \sum_1^N X_{n_i} = V \quad a.s. .$$

This result was the prototype for the well-known Chatterji principle (Chatterji (1974a, 1974b, 1985 and 1986)):

> *For every a.s. limit theorem for IID random variables under certain moment conditions, there exists an analogous theorem such*

that an arbitrarily-dependent sequence (under the same moment conditions) always contains a subsequence satisfying this analogous theorem.

Note that Komlós' result demonstrates Chatterji's principle where the limit theorem in question is the strong law of large numbers.

Chatterji's principle in conjunction with the relationship between IID and exchangeable random variables (see Chapter 3) led to the following conjecture of Kingman (1978):

Every limit property enjoyed by all exchangeable sequences is shared by some subsequence of every tight sequence.

Aldous provided a counter example to the above conjecture (see Kingman (1978), p.190) and proved the following result in place of it:

For every a.s. limit theorem for IID random variables there exists an analogous theorem satisfied by all exchangeable sequences and by all sub-subsequences of some subsequence of an arbitrarily dependent tight sequence of random variables.

Following what Kingman calls "a brilliant display" by Aldous to prove the previous result (see Aldous (1977)) we extend these results in §5.2 to an infinite 2-fold sequence of random variables, a very powerful result which is an extension of Chatterji's principle and forms the main result of this chapter.

Unless otherwise indicated the following notation will be used here and in §5.2:

For $\lambda \in P(\mathbf{R})$, let $\lambda^* \in P(\mathbf{R}^\infty)$ be the infinite product measure $\lambda \times \lambda \times \cdots$. For a random measure μ, let μ^* be the random map into $P(\mathbf{R}^\infty)$ such that $\mu^*(\omega) = \mu(\omega) \times \mu(\omega) \times \cdots$.

We shall be dealing with 2-fold infinite sequences and we shall use $\mu(\nu)$ to denote the random measure associated with random variables of type 1 (2).

Remark

Clearly $\mathbf{X} = \{X_{ij}; i \in [2], j \in \mathbf{N}\}$ *is a 2-fold infinite sequence of random*

variables with $L(\mathbf{X}) = \lambda_1^* \times \lambda_2^*$, $\lambda_i \in P(\mathbf{R})$ $\forall i \in [2]$, is a concise way of saying that $\mathbf{X}$ is a 2-IID sequence of random variables.

Introduce a special notation for subsequences of 2-fold sequences. Write $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2)$ and $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2)$ where $\mathbf{n}_1 = (n_{11}, n_{12}, n_{13}, \ldots)$ and $\mathbf{m}_1 = (m_{11}, m_{12}, m_{13}, \ldots)$ and these sequences denote strictly increasing sequences of positive integers (similarly $\mathbf{n}_2$ and $\mathbf{m}_2$). Write $\mathbf{m} \subset \mathbf{n}$ to mean that $\mathbf{m}_i$ is a subsequence of $\mathbf{n}_i$ $\forall i \in [2]$. Write $\mathbf{X}_n$ for the subsequence $\{X_{1n_{11}}, X_{1n_{12}}, \ldots, X_{2n_{21}}, X_{2n_{22}}, \ldots\}$ of $\mathbf{X}$, a 2-fold infinite sequence of random variables.

The following result will often be referred to in §5.2.

Theorem 5.1
A 2-fold infinite sequence of random variables, $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$, is partially exchangeable if there exist random measures μ and ν such that

$$\mu^* \times \nu^* \text{ is a r.c.d. for } \mathbf{Z} \text{ given } \mathcal{F}(\mu, \nu) \ . \tag{5.1}$$

Proof
The result follows immediately from Corollary 4.3.

Remark
Call the pair (μ, ν) of the above theorem the *2-fold canonical random measure (2-c.r.m.)* associated with the partially exchangeable sequence $\mathbf{Z}$.

Remark 5.1
Let $\mathbf{X} = \{X_i; i \in \mathbf{N}\}$ be a sequence of random variables. Then $X_i \to X\sigma(L', L^\infty)$ for random variable X means that

$$\lim_i E(X_i/B) = E(X/B)$$

for each $B \in B(\mathbf{R})$, $P(B) > 0$.

Remark 5.2
It is well known (Meyer (1966), T23) that for $\mathbf{X} = \{X_i, i \in \mathbf{N}\}$ to be $\sigma(L', L^\infty)$ relatively sequentially compact it is necessary and sufficient that $\mathbf{X}$ be uniformly integrable, and in particular it is sufficient that $\mathbf{X}$ be L^p-bounded for some $p > 1$.

Definition 5.1
For $\lambda \in P(\mathbf{R})$, write

$$|\lambda|^p = \int_{-\infty}^{\infty} |x|^p \lambda(dx) \qquad 0 < p < \infty \qquad (5.2)$$

$$|\lambda|_1 = \int_{-\infty}^{\infty} x\lambda(dx)$$

$$|\lambda|_2 = \int_{-\infty}^{\infty} x^2\lambda(dx) - (|\lambda|_1)^2 \ .$$

We refer to $|\lambda|_1$ and $|\lambda|_2$ as the *mean* and the *variance* of λ respectively. For definiteness write $|\lambda|_1 = \infty$ if $|\lambda|^1 = \infty$ and $|\lambda|_2 = \infty$ if $|\lambda|^2 = \infty$.

Remark
Tjur (1980) discusses distributions of random measures on the metrisable space (Parthasarathy (1967), T6.2) $P(\mathbf{R})$. Billingsley ((1968), p.238) defines moments of measures which are consistent with the definitions for $|\lambda|_1$ and $|\lambda|_2$ as given in Definition 5.1.

For the remainder of this section and §5.2 (unless otherwise indicated) assume $\mathbf{X} = \{X_{ij}; i \in [2], j \in \mathbf{N}\}$ to be a 2-fold infinite sequence of random variables such that $\{L(X_{ij}); j \in \mathbf{N}\}$ is tight $\forall i \in [2]$ (see Appendix A1).

The next two lemmas are extensions of work done by Aldous ((1977), pp.61-62, Lemma 2 and equations (3.2) and (3.3)). We re-state his result for each type of random variable separately, making the obvious adjustments.

Lemma 5.1

$$E|(\mu(\omega))|^p \leq \limsup_j E|X_{1j}|^p \quad , \quad 0 < p < \infty$$

$$E|(\nu(\omega))|^p \leq \limsup_j E|X_{2j}|^p \quad , \quad 0 < p < \infty$$

where

$$E|X_{ij}|^p = \int_{\Omega} |X_{ij}(\omega)|^p dP(\omega) \quad , \quad i \in [2], \ 0 < p < \infty \ .$$

Lemma 5.2
If

$$\alpha_1(\omega) = |\mu(\omega)|_1, \quad \alpha_2(\omega) = |\nu(\omega)|_1$$

$$\beta_1(\omega) = |\mu(\omega)|_2, \quad \beta_2(\omega) = |\nu(\omega)|_2$$

then

$$\sup_j E|X_{1j}| < \infty \text{ implies } \alpha_1(\omega) < \infty \text{ a.s.}$$

$$\sup_j E|X_{2j}| < \infty \text{ implies } \alpha_2(\omega) < \infty \text{ a.s.}$$

$$\sup_j E|X_{1j}|^2 < \infty \text{ implies } \beta_1(\omega) < \infty \text{ a.s.}$$

$$\sup_j E|X_{2j}|^2 < \infty \text{ implies } \beta_2(\omega) < \infty \text{ a.s. .}$$

Remark
The previous two lemmas provide us with some technical results which random measures μ and ν, which we construct in §5.2, satisfy.

Results in §5.2 are stated in terms of $\mathbf{X}$, μ and ν. When applying these results to specific theorems α_i and β_i , $i \in [2]$ play the role of means and variances in the 2-IID case.

Consider the following special property ß which $\mathbf{X}$ might possess:

Definition 5.2
$\mathbf{X}$ possesses *property* ß if there exists a (2-fold) partially exchangeable sequence of random variables $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$ such that

(i) (μ, ν) is the 2-c.r.m. for $\mathbf{Z}$

(ii) $\sum_{i=1}^{2} \sum_{j=1}^{\infty} \left| X_{im_{ij}} - Z_{ij} \right| < \infty$ a.s. for some $\mathbf{m}$.

Remark
In §5.2 the results will be formulated so as to be almost obvious if $\mathbf{X}$ possesses property ß. However, $\mathbf{X}$ need not generally possess property ß (see

Aldous (1977), p.81 for the trivial case where only one type of random variable is considered).

In the introduction to Chapter 5 mention was made of the main result of §5.2 which concerns *a.s. limit theorems for 2-IID random variables.* Exactly what is meant by this statement will be made clear by the following definition.

Definition 5.3
A *statute* A is a measurable subset of $P(\mathbf{R}) \times P(\mathbf{R}) \times \mathbf{R}^\infty \times \mathbf{R}^\infty$ such that for each $(\lambda_1, \lambda_2) \in P(\mathbf{R}) \times P(\mathbf{R})$,

$$(\lambda_1, \lambda_2, \mathbf{X}(\omega)) \in A \ \ a.s. \ \text{when} \ \ L(\mathbf{X}) = \lambda_1^* \times \lambda_2^* \ . \tag{5.3}$$

This is equivalent to

$$(\lambda_1^* \times \lambda_2^*) \{\mathbf{X}(\omega) \in \mathbf{R}^\infty \times \mathbf{R}^\infty; (\lambda_1, \lambda_2, \mathbf{X}(\omega)) \in A\} = 1 \tag{5.4}$$

or

$$P\{\omega; (\lambda_1, \lambda_2, \mathbf{X}(\omega)) \in A\} = 1 \tag{5.5}$$

when $\mathbf{X}$ is a 2-IID sequence of random variables, λ_i being the common distribution of random variables of type i, $\forall i \in [2]$.

The following examples of statutes representing some well-known a.s. limit theorems for IID random variables (i.e. for the $g = 1$ case where only one type of random variable is considered) appear in Aldous (1977):

Example 5.1
Let $\mathbf{X} = \{X_i; i \in \mathbf{N}\}$ be an infinite sequence of IID random variables, then Kolmogorov's SLLN and the Law of the Iterated Logarithm (Shirayayev (1984), pp.366, 372 respectively) may be represented by the following statutes:
For each $\lambda \in P(\mathbf{R})$,

$$A_1 = \left\{(\lambda, \mathbf{X}(\omega)); \lim_{n\to\infty} n^{-1} \sum_1^n X_i(\omega) = |\lambda|_1\right\} \cup \left\{(\lambda, \mathbf{X}(\omega)); |\lambda|_1 = \infty\right\} \tag{5.6}$$

$$A_2 = \left\{(\lambda, \mathbf{X}(\omega)); \limsup_{N\to\infty} \frac{\sum_1^N X_i(\omega) - N|\lambda|_1}{(2N \log\log N)^{\frac{1}{2}}} = (|\lambda|_2)^{\frac{1}{2}}\right\} \cup \{(\lambda, \mathbf{X}(\omega)); |\lambda|_2 = \infty\} . \tag{5.7}$$

Remark
The above two statutes will be extended in §5.3 to allow two types of random variables, i.e. an analogue of Kolmogorov's SLLN and the Law of the Iterated Logarithm as a.s. limit theorems for 2-IID sequences will be presented in statute form.

Definition 5.4
A statute A is said to be a *limit statute* if

$$(\lambda_1, \lambda_2, \mathbf{X}(\omega)) \in A \text{ and } \sum_{i=1}^{2} \sum_{j=1}^{\infty} \left| X'_{ij}(\omega) - X_{ij}(\omega) \right| < \infty \tag{5.8}$$

implies

$$\left(\lambda_1, \lambda_2, \mathbf{X}'(\omega)\right) \in A \ . \tag{5.9}$$

5.2 PARTIAL EXCHANGEABILITY AND SUBSEQUENCES OF ARBITRARY DEPENDENT 2-FOLD INFINITE SEQUENCES OF RANDOM VARIABLES

As mentioned in the introduction to this chapter and in §5.1, this section contains a very powerful result which shows the link between a.s. limit theorems for 2-IID sequences of random variables and those of (i) partially exchangeable sequences and (ii) sub-subsequences of some subsequence of an arbitrary 2-fold infinite sequence of random variables which have a tight distribution (Appendix A1) for each type of random variable.

Any a.s. limit theorem for 2-IID random variables may be represented in statute form (see §5.3) and it thus remains to be shown that partially exchangeable sequences and the particular subsequences mentioned may be used in any statute.

For the remainder of this section, unless otherwise indicated, assume $\mathbf{X} = \{X_{ij}; i \in [2], j \in \mathbf{N}\}$ to be a 2-fold infinite sequence of random variables, tightly distributed (Appendix A1) within types. To briefly describe the technique used we extract a subsequence $\mathbf{Y} = \{Y_{ij}; i \in [2], j \in \mathbf{N}\}$ from $\mathbf{X}$ and associate with it a partially exchangeable sequence $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$.

We will then show that certain types of properties of $\mathbf{Z}$ are shared by $\mathbf{Y}$ (in particular some a.s. limit theorems).

Remark
Here follows a brief survey of related work for the $g = 1$ case:

(1) Given a sequence $\mathbf{X} = \{X_i; i \in \mathbf{N}\}$ of random variables with a tight distribution, we can find a subsequence $\mathbf{Y} = \{Y_i; i \in \mathbf{N}\}$ which is asymptotically exchangeable, i.e. for some exchangeable sequence of random variables $\mathbf{Z} = \{Z_i; i \in \mathbf{N}\}$,

$$\lim_j L(Y_{j+1}, Y_{j+2}, \ldots) = L(Z_1, Z_2, \ldots) \ .$$

This result was given independently by Dacunha-Castelle (1974) and Figiel and Sucheston (1976).

(2) Berkes (1982) gave a survey of results concerning the structure of subsequences of random variables. He showed that the strongest exchangeability property that can be guaranteed for suitable subsequences of general sequences of random variables is a certain form of asymptotic exchangeability which he calls *strong exchangeability at infinity.*

Theorem 5.2
Let $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$ be a partially exchangeable sequence of random variables. If $\mathbf{Z}$ has 2-c.r.m. (μ, ν) then

$$(\mu(\omega), \nu(\omega), \mathbf{Z}(\omega)) \in A \quad a.s. \tag{5.10}$$

for any statute A.

Proof
From Lemma 4.1, using $\mathbf{z}$ for $\mathbf{Z}(\omega)$,

$$\int_{\mathbf{R}^\infty \times \mathbf{R}^\infty} I[(\mu(\omega), \nu(\omega), \mathbf{z}) \in A] \, \mu^* \times \nu^*(\omega, d\mathbf{z})$$

$$= E\{I[(\mu, \nu, \mathbf{Z}) \in A] / \mathcal{F}(\mu, \nu)\} \quad a.s. \ . \tag{5.11}$$

Also, from (5.4),

$$\int_{\mathbf{R}^\infty \times \mathbf{R}^\infty} I\left[(\mu(\omega), \nu(\omega), \mathbf{z}) \in A\right] \mu^* \times \nu^*(\omega, d\mathbf{z})$$

$$= \mu^* \times \nu^* \{\mathbf{z}; (\mu(\omega), \nu(\omega), \mathbf{z}) \in A\}$$

$$= 1 \ . \tag{5.12}$$

From (5.11) and (5.12) we thus have

$$E\left(E\{I[(\mu, \nu, \mathbf{Z}) \in A]/\mathcal{F}(\mu, \nu)\}\right) = 1$$

and hence

$$E\left\{I(\mu, \nu, \mathbf{Z}) \in A\right\} = 1$$

so that

$$(\mu(\omega), \nu(\omega), \mathbf{Z}(\omega)) \in A \quad a.s. \ .$$

Remark 5.3
Theorem 5.2 shows that a.s. limit theorems for 2-IID sequences of random variables extend immediately to (2-fold) partially exchangeable random variables, a result which is not surprising in view of our earlier work in Chapter 3 (in particular Theorem 3.3).

The following theorem shows that a.s. limit theorems for 2-IID sequences of random variables, which can be represented by a limit statute, extend immediately to all sub-subsequences of some particular subsequence of $\mathbf{X}$.

Theorem 5.3
Let A be any limit statute. Then there exists $\mathbf{m}$ such that for each $\mathbf{n} \subset \mathbf{m}$,

$$(\mu(\omega), \nu(\omega), \mathbf{X}_n(\omega)) \in A \quad a.s. \ . \tag{5.13}$$

Proof
The proof of this theorem will be given towards the end of this chapter as it relies on many results that still need to be presented.

Corollary 5.1
Theorem 5.3 is immediate if $\mathbf{X}$ has property ß.

Proof
From Definition 5.2 it follows that there exists a partially exchangeable sequence of random variables $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$ with 2-c.r.m. (μ, ν) and $\mathbf{m}$ such that

$$\sum_{i=1}^{2}\sum_{j=1}^{\infty}\left|X_{im_{ij}} - Z_{ij}\right| < \infty \quad a.s. \ . \tag{5.14}$$

Theorem 5.2, (5.14) and Definition 5.4 show that

$$(\mu(\omega), \nu(\omega), \mathbf{X}_m(\omega)) \in A \quad a.s. \tag{5.15}$$

for any limit statute A.

Now suppose that $\mathbf{n} \subset \mathbf{m}$ and choose $\mathbf{Z}'$ to be a subsequence of $\mathbf{Z}$ indexed by the same subscripts as subsequence X_n is from X_m.

From (5.14),

$$\sum_{i=1}^{2}\sum_{j=1}^{\infty}\left|X_{in_{ij}} - Z'_{ij}\right| \leq \sum_{i=1}^{2}\sum_{j=1}^{\infty}\left|X_{im_{ij}} - Z_{ij}\right| < \infty \quad a.s.$$

and hence

$$(\mu(\omega), \nu(\omega), \mathbf{X}_n(\omega)) \in A \quad a.s.$$

follows by using the same argument as was used to obtain (5.15) (note that $\mathbf{Z}'$ is partially exchangeable if $\mathbf{Z}$ is).

Combining the results from Theorem 5.2 and Theorem 5.3 we thus have the main result of this section:

> *Given any a.s. limit theorem for 2-IID random variables, there exists an analogous theorem satisfied by (2-fold) partially exchangeable sequences and by all sub-subsequences of some subsequence of an arbitrarily dependent 2-fold infinite sequence of random variables, tightly distributed within types.*

In order to prove Theorem 5.3 we need to construct the random measures μ and ν for variables of type 1 and 2 respectively. The method of construction reduces to a construction in Révész (1967, Theorem 6.1.1) when only one type of random variable is considered (i.e. the $g = 1$ case).

Using the Relative Compactness Criterion (Appendix A1) we see that $\{L(X_{ij}); j \in \mathbf{N}\}$ is relatively compact $\forall i \in [2]$, and hence

$$L(X_{ij}) \Rightarrow \gamma_i \qquad \forall i \in [2] \tag{5.16}$$

where $\gamma_i \in P(\mathbf{R}), \quad \forall i \in [2]$.

The following notation will be used throughout this section:

Results will be stated for random variables of type i where $i \in [2]$. Let $D_{i\infty}$ be a countably dense set of continuity points of γ_i (see (5.16)). For each $k \in \mathbf{N}$ choose a finite set
$D_{ik} = \left\{x_{ik_j}; j \in [q_k + 1]\right\}$ such that

$$x_{ik_j} < x_{ik_{j+1}} < x_{ik_j} + 2^{-k} \qquad j \leq q_k \tag{5.17}$$

$$P\left[x_{ik_1} < X_{in} < x_{ik_{q_k}}\right] \geq 1 - 2^{-k} \qquad n \in \mathbf{N} \tag{5.18}$$

$$D_{ik} \subset D_{ik+1} \tag{5.19}$$

$$D_{i\infty} = \bigcup_{k=1}^{\infty} D_{ik} \ . \tag{5.20}$$

Let J_{ik} be the set of intervals $(-\infty, x_{ik_1}], (x_{ik_1}, x_{ik_2}], \ldots, \left(x_{ik_{q_k}}, \infty\right)$ and let

$$J_i = \bigcup_k J_{ik} \ . \tag{5.21}$$

Define $P_{ik} : \mathbf{R} \to D_{ik}$ by

$$P_{ik}(x) = \begin{cases} \inf\left\{x_{ik_j} \in D_{ik}; x_{ik_j} \geq x\right\} & \text{for } x \leq x_{kq_k} \\ x_{ik_{q_k+1}} & \text{else} \end{cases} \tag{5.22}$$

for $x \in \mathbf{R}$.

Note that $\mathcal{P}_{ik}$ is a constant on each $J \in \mathcal{J}_{ik}, i \in [2], k \in \mathbf{N}$. For future reference note that (5.17) and (5.18) imply both the next two results:

$$P\left[|\mathcal{P}_{ik}(X_{in}) - X_{in}| \geq 2^{-k}\right] \leq 2^{-k} \quad k, n \in \mathbf{N}, i \in [2] \tag{5.23}$$

and for any $\mathbf{V} = \{V_{ij}; i \in [2], j \in \mathbf{N}\}$ with $L(V_{ij}) = \gamma_i$, $\gamma_i \in P(\mathbf{R})$, $j \in \mathbf{N}, i \in [2]$,

$$P\left[|\mathcal{P}_{ik}(V_{ij}) - V_{ij}| \geq 2^{-k}\right] \leq 2^{-k} \quad \forall j \in \mathbf{N}, k \in \mathbf{N}, i \in [2] \ . \tag{5.24}$$

The following result follows immediately from work published by Aldous ((1977), p.69) by applying Aldous' results to each type of random variable separately.

Theorem 5.4
There exists a subsequence $\mathbf{Y} = \mathbf{X}_n$ and random measures μ and ν such that

$$\text{for each } J \in \mathcal{J}_1, \lim_{k\to\infty} E\left[I(Y_{1k} \in J)/\mathcal{F}_{1k-1}\right](\omega) = \mu(\omega, J) \tag{5.25}$$

$$\text{for each } J \in \mathcal{J}_2, \lim_{k\to\infty} E\left[I(Y_{2k} \in J)/\mathcal{F}_{2k-1}\right](\omega) = \nu(\omega, J) \tag{5.26}$$

where μ is $\mathcal{F}_1$-measurable and ν is $\mathcal{F}_2$-measurable. For

$$\mathcal{F}_{ik-1} = \mathcal{F}\left(\mathcal{P}_{i1}(Y_{i1}), \mathcal{P}_{i2}(Y_{i2}), \ldots, \mathcal{P}_{ik-1}(Y_{ik-1})\right) \ i \in [2], k \in \mathbf{N} \tag{5.27}$$

$$\mathcal{F}_i = \bigcup_k \mathcal{F}_{ik} \qquad i \in [2]. \tag{5.28}$$

We now construct a partially exchangeable sequence $\mathbf{Z}$ whose 2-c.r.m. is (μ, ν) and then compare properties of $\mathbf{Y}$ and $\mathbf{Z}$ (where $\mathbf{Y}, \mu, \nu$ are as in Theorem 5.4).

From (5.27) and (5.28) we note that

$$\mathcal{F}_i \subset \mathcal{F}(\mathbf{Y}) \qquad i \in [2]. \tag{5.29}$$

For the remainder of this section let $\mathbf{Z} = \{Z_{ij}; i \in [2], j \in \mathbf{N}\}$ be a sequence of random variables such that

$$\mu^* \times \nu^* \text{ is a r.c.d. for } \mathbf{Z} \text{ given } \mathcal{F}(\mathbf{Y}, \mu, \nu). \tag{5.30}$$

Since μ is $\mathcal{F}_1$-measurable and ν is $\mathcal{F}_2$-measurable (see Theorem 5.4), it follows that (5.29) and (5.30) yield

$$\mu^* \times \nu^* \text{ is a r.c.d. for } \mathbf{Z} \text{ given } \mathcal{F}(\mathbf{Y}). \tag{5.31}$$

Using (5.30) and (5.31) we see that

$$\mu^* \times \nu^* \text{ is a r.c.d. for } \mathbf{Z} \text{ given } \mathcal{F}(\mu, \nu)$$

so that (μ, ν) is the 2-c.r.m. for the (2-fold) partially exchangeable sequence, $\mathbf{Z}$.

Theorem 5.5
Let V be an $\mathcal{F}(\mathbf{Y})$-measurable map into some separable, metrisable space, S. For all $j \in \mathbf{N}$,

$$\text{(i)} \quad \lim_{m\to\infty} L(V, Y_{1m}, Y_{2n}) = L(V, Z_{1j}, Y_{2n}) \qquad \forall n \in \mathbf{N} \tag{5.32}$$

$$\text{(ii)} \quad \lim_{n\to\infty} L(V, Y_{1m}, Y_{2n}) = L(V, Y_{1m}, Z_{2j}) \qquad \forall m \in \mathbf{N}. \tag{5.33}$$

Moreover suppose that $f \in L^\infty(S \times \mathbf{R} \times \mathbf{R})$ is such that, for each $s \in S$, $f(s, \cdot, Y_{2n})$ is a constant on each $J \in J_{1j}$, then

$$\text{(iii)} \quad \lim_{m\to\infty} E[f(V, Y_{1m}, Y_{2n})] = E[f(V, Z_{1j}, Y_{2n})] \qquad \forall j \in \mathbf{N} \tag{5.34}$$

and if $f(s, Y_{1m}, \cdot)$ is a constant on each $J \in J_{2j}$, then

$$\text{(iv)} \quad \lim_{n\to\infty} E[f(V, Y_{1m}, Y_{2n})] = E[f(V, Y_{1m}, Z_{2j})]. \qquad \forall j \in \mathbf{N} \tag{5.35}$$

Proof
The proof of this theorem follows from Aldous' related result (Aldous (1977), Lemma 12, p.72) for one type of random variable by making the obvious changes to his proof.

Corollary 5.2
V and $\mathbf{Z}_i = \{Z_{ij}; j \in \mathbf{N}\}$ are conditionally independent given $\mathcal{F}_i$ $\qquad \forall i \in [2]$.

Proof
For $i \in [2]$, let $J \in J_i$ and let A be a measurable subset of S. Then, using (5.29), we see that

$$P[V \in A, Z_{ij} \in J/\mathcal{F}_i]$$

$$= E[I(V \in A, Z_{ij} \in J)/\mathcal{F}_i]$$

$$= E\{E[I(V \in A, Z_{ij} \in J)/\mathbf{Y}]/\mathcal{F}_i\}$$

$$= E\{I(V \in A)E[I(Z_{ij} \in J)/\mathbf{Y}]/\mathcal{F}_i\}. \tag{5.36}$$

Consider the $i = 1$ case and recall that μ is $\mathcal{F}_1$-measurable (Theorem 5.4). From (5.31) and (5.36) then, $\forall j \in \mathbf{N}$,

$$P[V \in A, Z_{1j} \in J/\mathcal{F}_1]$$

$$= E[I[V \in A]\mu(\omega, J)/\mathcal{F}_1]$$

$$= \mu(\omega, J)E[I[V \in A]/\mathcal{F}_1]$$

$$= E[\mu(\omega, J)/\mathcal{F}_1]\, E(I[V \in A]/\mathcal{F}_1)$$

$$= E[E(I[Z_{1j} \in J]/\mathbf{Y})/\mathcal{F}_1]\, E(I[V \in A]/\mathcal{F}_1).$$

From (5.29) then,

$$P[V \in A, Z_{1j} \in J/\mathcal{F}_1] = P[Z_{1j} \in J/\mathcal{F}_1]\, P[V \in A/\mathcal{F}_1]$$

the $i = 2$ case follows similarly.

Remark

Since $\mathbf{Y}$ is an infinite subsequence of $\mathbf{X}$, (5.16) shows that for $i \in [2]$,

$$L(Y_{ij}) \Rightarrow \gamma_i \qquad \forall j \in \mathbf{N}\,.$$

Hence, from Theorem 5.5 we obtain, for $i \in [2]$,

$$L(Z_{ij}) = \gamma_i \qquad \forall j \in \mathbf{N}. \tag{5.37}$$

Consider a function $g : P(\mathbf{R}) \times P(\mathbf{R}) \times \mathbf{R}^\infty \times \mathbf{R}^\infty \to \mathbf{R}$. We aim to use the *asymptotic conditional independence* property given by Theorem 5.5 to

show that $E[g(\mu,\nu,\mathbf{Y}_n)]$ is *close* to $E[g(\mu,\nu,\mathbf{Z})]$ whenever $\mathbf{n}$ *increases sufficiently rapidly* (exactly what is meant by this will be stated below). The situation is simplest when g is continuous, but the alternative conditions below will sometimes be needed.

For each $\mathbf{x}\in\mathbf{R}^\infty\times\mathbf{R}^\infty, \mathbf{y}\in\mathbf{R}^\infty\times\mathbf{R}^\infty$ and $\lambda_i\in P(\mathbf{R})\quad i\in[2]$,

$$g(\lambda_1,\lambda_2,\mathbf{x})\le \liminf_{i_1,i_2} g(\lambda_1,\lambda_2,x_{11},\ldots,x_{1i_1},x_{21},\ldots,x_{2i_2},y_{1i_1+1},y_{1i_1+2},\ldots, y_{2i_2+1},y_{2i_2+2},\ldots)\quad \forall i_j\in\mathbf{N},\ j\in[2]. \tag{5.38}$$

If

$$P_{ij}(x_{ij}) = P_{ij}(y_{ij}) \qquad \forall i\in[2],\forall j\in\mathbf{N}$$

then

$$g(\lambda_1,\lambda_2,\mathbf{x}) = g(\lambda_1,\lambda_2,\mathbf{y}) \tag{5.39}$$

where P_{ij} is as in (5.22).

Definition 5.5
Let Q denote an assertion applicable to increasing sequences (within types) $\mathbf{n}=\{n_{ij};i\in[2],j\in\mathbf{N}\}$ of positive integers (i.e. $\mathbf{n}_i=\{n_{ij};j\in\mathbf{N}\}$ is increasing $\forall i\in[2]$) and let Q' denote the set of $\mathbf{n}\in\mathbf{N}^\infty\times\mathbf{N}^\infty$ for which Q is true. If there exist functions L_1 and L_2,

L_i : {finite sequences in $\mathbf{N}$} $\to\mathbf{N}$ $\forall i\in[2]$ such that, $\forall j\in\mathbf{N}, i\in[2]$,

$$n_{ij}\ge L_i(n_{i1},n_{i2},\ldots,n_{ij-1}) \text{ implies } \mathbf{n}\in Q', \tag{5.40}$$

then we say that Q holds for all $\mathbf{n}$ *increasing sufficiently rapidly.* (Note that we require the sequence to be increasing sufficiently rapidly for each type of variable, separately).

Theorem 5.6
Let $P_0^{(1)}(\mathbf{R})$ and $P_0^{(2)}(\mathbf{R})$ be two measurable subsets of $P(\mathbf{R})$ such that

$$\mu(\omega)\in P_0^{(1)}(\mathbf{R}) \text{ a.s. and } \nu(\omega)\in P_0^{(2)}(\mathbf{R}) \text{ a.s.}. \tag{5.41}$$

Let $P_0^{(i)}(\mathbf{R})$ be equipped with the separable, metrisable topology, τ_i such that

$$\lambda_{ij}\xrightarrow{\tau_i}\lambda_i \text{ implies } \lambda_{ij}\Rightarrow\lambda_i \qquad \forall i\in[2]\ . \tag{5.42}$$

The weak topology and τ_i generate the same σ-field $\forall i \in [2]$. (5.43)

Let $\xi > 0$ be given. Let $g : P_0^{(1)}(\mathbf{R}) \times P_0^{(2)}(\mathbf{R}) \times \mathbf{R}^\infty \times \mathbf{R}^\infty \to \mathbf{R}$ satisfy either
(i) g is bounded and continuous
or
(ii) g is bounded, measurable and satisfies (5.38) and (5.39).

Then

$$E[g(\mu, \nu, \mathbf{Y}_n)] \leq E[g(\mu, \nu, \mathbf{Z})] + \xi \tag{5.44}$$

for all $\mathbf{n}$ increasing sufficiently rapidly.

Proof
Suppose, inductively, that we have chosen the first k_i variables of type i, $i \in [2]$, i.e. $n_{i1} < n_{i2} < \ldots < n_{ik_i}$ have been specified $\forall i \in [2]$.

Define random variables

$$G_{00} = g(\mu, \nu, \mathbf{Z}) \tag{5.45}$$

$$\begin{aligned} G_{ij} \;=\; & g(\mu, \nu, Y_{1n_{11}}, Y_{1n_{12}}, \ldots, Y_{1n_{1i}}, Y_{2n_{21}}, Y_{2n_{22}}, \ldots, Y_{2n_{2j}}, \\ & Z_{1i+1}, Z_{1i+2}, \ldots, Z_{2j+1}, Z_{2j+2}, \ldots), \quad i \in [k_1], j \in [k_2], k_i \in \mathbf{N} \\ & \forall i \in [2]. \end{aligned} \tag{5.46}$$

For $m_i > n_{ik_i}$ $\forall i \in [2]$, define

$$\begin{aligned} B_{m_1 m_2} \;=\; & g(\mu, \nu, Y_{1n_{11}}, Y_{1n_{12}}, \ldots, Y_{1n_{1k_1}}, Y_{2n_{21}}, Y_{2n_{22}}, \ldots, Y_{2n_{2k_2}}, \\ & Y_{1m_1}, Y_{2m_2}, Z_{1k_1+2}, Z_{1k_1+3}, \ldots, Z_{2k_2+2}, Z_{2k_2+3}, \ldots). \end{aligned} \tag{5.47}$$

We shall prove

$$\lim_{m_1, m_2} E(B_{m_1, m_2}) = E(G_{k_1 k_2}) \tag{5.48}$$

in order to prove the theorem. We now proceed to show that if (5.48) holds then the desired result follows. We thus accept (5.48) for the time being.

From (5.46), (5.47) and (5.48) we see that there exist functions L_1 and L_2,

$$L_i : \{\text{finite sequences in } \mathbf{N}\} \to \mathbf{N} \quad \forall i \in [2], \text{ such that}$$

if

$$n_{ik_i+1} \geq L_i(n_{i1}, n_{i2}, \ldots, n_{ik_i}) \qquad \forall i \in [2]$$

then $E(G_{k_1+1k_2})$ and $E(G_{k_1k_2+1})$ can be brought *as close to* $E(G_{k_1k_2})$ as desired, i.e. from Definition 5.5, we may say that for all $\mathbf{n}$ increasing sufficiently rapidly, there exist $\xi_i > 0$, $\forall i \in [2]$, such that

$$E(G_{k_1+1k_2}) \leq E(G_{k_1k_2}) + \xi_1 2^{-k_1-1} \tag{5.49}$$

and

$$E(G_{k_1k_2+1}) \leq E(G_{k_1k_2}) + \xi_2 2^{-k_2-1}. \tag{5.50}$$

Now suppose that $\mathbf{n}$ increases sufficiently rapidly, from (5.49) and (5.50) there exist $\xi > 0$ such that

$$\begin{aligned} E(G_{k_1k_2}) &\leq E(G_{k_1-1k_2}) + \xi_1 2^{-k_1} \\ &\vdots \\ &\leq E(G_{0k_2}) + \xi_1 \left(2^{-k_1} + 2^{-k_1+1} + \cdots + 2^{-1}\right) \\ &\leq E(G_{0k_2}) + \xi_1 \\ &\leq E(G_{0k_2-1}) + \xi_2 2^{-k_2} + \xi_1 \\ &\vdots \\ &\leq E(G_{0,0}) + \xi \end{aligned} \tag{5.51}$$

where $\xi = \xi_1 + \xi_2$. Since (5.51) holds $\forall k_i \in \mathbf{N}$, $\forall i \in [2]$, we may use (5.45) to obtain the following:

> If $\mathbf{n}$ increases sufficiently rapidly then there exists $\xi > 0$ such that
>
> $$\sup_{k_1,k_2} E(G_{k_1k_2}) \leq E[g(\mu, \nu, \mathbf{Z})] + \xi. \tag{5.52}$$

At the outset of the theorem it was assumed that g either satisfies (i) or (ii). Assume that g satisfies (ii). From (5.38) with $\mathbf{x} = \mathbf{Y}_n = \{Y_{1n_{11}}, Y_{1n_{12}}, \ldots, Y_{2n_{21}}, Y_{2n_{22}}, \ldots\}$, $\mathbf{y} = \mathbf{Z} = \{Z_{11}, Z_{12}, \ldots, Z_{21}, Z_{22}, \ldots\}$, $i = k_1$, $j = k_2$, $\lambda_1 = \mu$ and $\lambda_2 = \nu$, we see that

$$\begin{aligned} g(\mu, \nu, \mathbf{Y}_n) \leq \liminf_{k_1,k_2} g(\mu, \nu, &Y_{1n_{11}}, Y_{1n_{12}}, \ldots, Y_{1n_{1k_1}}, Y_{2n_{21}}, Y_{2n_{22}}, \ldots, Y_{2n_{2k_2}}, \\ &Z_{1k_1+1}, Z_{1k_1+2}, \ldots, Z_{2k_2+1}, Z_{2k_2+2}, \ldots) \end{aligned}$$

and hence, from (5.46),

$$g(\mu, \nu, \mathbf{Y}_n) \leq \liminf_{k_1, k_2} G_{k_1 k_2}. \tag{5.53}$$

Since (5.53) follows immediately if g is continuous (which is assumed if g satisfies (i)) we thus have g to satisfy (5.53).

From (5.53) and Fatou's lemma (Appendix A1),

$$\begin{aligned} E[g(\mu, \nu, \mathbf{Y}_n)] &\leq E\left(\liminf_{k_1,\, k_2} G_{k_1 k_2}\right) \\ &\leq \liminf_{k_1,\, k_2} E(G_{k_1 k_2}). \end{aligned} \tag{5.54}$$

Using (5.52) and (5.54) we thus see that for all $\mathbf{n}$ increasing sufficiently rapidly, there exists $\xi > 0$ such that (5.44) follows, since

$$\begin{aligned} E\left[g(\mu, \nu, \mathbf{Y}_n)\right] &\leq \liminf_{k_1,\, k_2} E(G_{k_1 k_2}) \\ &\leq \sup_{k_1, k_2} E(G_{k_1 k_2}) \\ &\leq E\left[g(\mu, \nu, \mathbf{Z})\right] + \xi\,. \end{aligned}$$

The proof will thus be complete if (5.48) is shown to hold. For all $i, j \in \mathbf{N}$, define functions $g_{ij} : P_0^{(1)}(\mathbf{R}) \times P_0^{(2)}(\mathbf{R}) \times \mathbf{R}^i \times \mathbf{R}^j \to \mathbf{R}$ by

$$\begin{aligned} g_{ij} &= (\lambda_1, \lambda_2, y_{11}, y_{12}, \ldots, y_{1i}, y_{21}, y_{22}, \ldots, y_{2j}) \\ &= \int_{\mathbf{R}} g(\lambda_1, \lambda_2, y_{11}, y_{12}, \ldots, y_{1i}, y_{21}, y_{22}, \ldots, y_{2j}, x_{1i+1}, x_{1i+2}, \ldots, x_{2j+1}, x_{2j+2}, \ldots) \\ &\qquad\qquad \lambda_1^* \times \lambda_2^* d(\mathbf{x}) \end{aligned} \tag{5.55}$$

where $R = \left(\overset{\infty}{\underset{i+1}{\mathbf{X}}} \mathbf{R}\right) \times \left(\overset{\infty}{\underset{j+1}{\mathbf{X}}} \mathbf{R}\right)$.

From (5.55) and the assumptions made on g it immediately follows that g_{ij} is bounded and measurable $\forall i, j \in \mathbf{N}$.

Define a random map V into $P_0^{(1)}(\mathbf{R}) \times P_0^{(2)}(\mathbf{R}) \times \mathbf{R}^{k_1} \times \mathbf{R}^{k_2}$ for $k_i \in \mathbf{N}$,

$i \in [2]$ by

$$V = \left(\mu, \nu, Y_{1n_{11}}, Y_{1n_{12}}, \ldots, Y_{1n_{1k_1}}, Y_{2n_{21}}, Y_{2n_{22}}, \ldots, Y_{2n_{2k_2}}\right) \tag{5.56}$$

From (5.29) and the fact that μ is $\mathcal{F}_1$-measurable we conclude that

$$\mu \text{ is } \mathcal{F}(\mathbf{Y})-\text{measurable.} \tag{5.57}$$

Similarly for ν, so that (5.56) shows that

$$V \text{ is } \mathcal{F}(\mathbf{Y})-\text{measurable.} \tag{5.58}$$

For random variables W_1 and W_2 and $k_i \in \mathbf{N}$, $i \in [2]$, we may regard (V, W_1, W_2) as a random map into $P_0^{(1)}(\mathbf{R}) \times P_0^{(2)}(\mathbf{R}) \times \mathbf{R}^{k_1+1} \times \mathbf{R}^{k_2+1}$. Using Lemma 4.1, the definitions of the functions involved and (5.31), we see that

$$E\left(G_{k_1k_2}/\mathbf{Y}\right) = g_{k_1k_2}(V) \text{ a.s.} \tag{5.59}$$

$$E\left(B_{m_1m_2}/\mathbf{Y}\right) = g_{k_1+1k_2+1}\left(V, Y_{1m_1}, Y_{2m_2}\right) \text{ a.s.} \tag{5.60}$$

$$E\left[g_{k_1+1k_2+1}\left(V, Z_{1k_1+1}, Z_{2k_2+1}/\mathbf{Y}\right)\right] = g_{k_1k_2}(V) \text{ a.s. .} \tag{5.61}$$

We now show that if

$$\lim_{m_1,m_2} E\left[g_{k_1+1k_2+1}(V, Y_{1m_1}, Y_{2m_2})\right] = E\left[g_{k_1+1k_2+1}(V, Z_{1k_1+1}, Z_{2k_2+1})\right] \ a.s. \tag{5.62}$$

holds, then (5.48) follows.

From (5.60) and (5.62),

$$\begin{aligned} & \lim_{m_1,m_2} E\left[E(B_{m_1m_2}/\mathbf{Y})\right] \\ = & \lim_{m_1,m_2} E\left[g_{k_1+1k_2+1}(V, Y_{1m_1}, Y_{2m_2})\right] \quad a.s. \\ = & E\left[g_{k_1+1k_2+1}(V, Z_{1k_1+1}, Z_{2k_2+1})\right] \ ; \ a.s. \ . \end{aligned} \tag{5.63}$$

From a basic property of conditional expectations, (5.59), (5.61) and (5.63),

it follows that (5.48) holds, since

$$\begin{aligned}
\lim_{m_1,m_2} E(B_{m_1m_2}) &= E\left[g_{k_1+1k_2+1}(V, Z_{1k_1+1}, Z_{2k_2+1})\right] \quad a.s. \\
&= E\left\{E\left[g_{k_1+1k_2+1}(V, Z_{1k_1+1}, Z_{2k_2+1})/\mathbf{Y}\right]\right\} \quad a.s. \\
&= E\left[g_{k_1k_2}(V)\right] \quad a.s. \\
&= E\left[E(G_{k_1k_2}/\mathbf{Y})\right] \quad a.s. \\
&= E\left(G_{k_1k_2}\right) \quad a.s. \ .
\end{aligned}$$

The proof of the theorem is thus complete if (5.62) can be shown to hold. Once again, we consider the two cases separately.

Suppose that g satisfies (ii). Let $k_1, k_2 \in \mathbf{N}$. From (5.55), for arbitrary $m, n \in \mathbf{N}$,

$$\begin{aligned}
&g_{k_1+1k_2+1}\left(\lambda_1, \lambda_2, y_{11}, y_{12}, \ldots, y_{1k_1}, y_{21}, y_{22}, \ldots, y_{2k_2}, y_{1m}, y_{2n}\right) \\
&= \int_R g\left(\lambda_1, \lambda_2, y_{11}, y_{12}, \ldots, y_{1k_1+1}, y_{21}, y_{22}, \ldots y_{2k_2+1}, x_{1k_1+2}, x_{1k_1+3}, \ldots, x_{2k_2+2}, x_{2k_2+3}, \ldots\right) \\
&\qquad\qquad \lambda_1^* \times \lambda_2^*(d\mathbf{x}) \qquad\qquad (5.64)
\end{aligned}$$

where $R = \left(\overset{\infty}{\underset{k_1+2}{\mathbf{X}}} \mathbf{R}\right) \times \left(\overset{\infty}{\underset{k_2+2}{\mathbf{X}}} \mathbf{R}\right)$.

Now $\mathcal{P}_{ik_i+1}$ is a constant on each $J \in J_{ik_i+1}$ $\forall i \in [2]$ (this was noted just below the definition of $\mathcal{P}_{ik}(x)$, see (5.22)), so that (5.39) and (5.55) immediately show that if we fix $(\lambda_1, \lambda_2, y_{11}, \ldots, y_{1k_1}, y_{21}, \ldots, y_{2k_2}, y_{2n})$ and consider $g_{k_1+1k_2+1}\left(\lambda_1, \lambda_2, y_{11}, y_{12}, \ldots, y_{1k_1}, y_{21}, \ldots, y_{2k_2}, y_{1m}, y_{1n}\right)$ as a function of y_{1m} only, then the right hand side of (5.64) is a constant on each $J_1 \in J_{k_1+1}$. We may thus use Theorem 5.5 (iii) to obtain

$$\begin{aligned}
&\lim_{m\to\infty} \quad E\left[g_{k_1+1k_2+1}\left(\lambda_1, \lambda_2, Y_{11}, \ldots, Y_{1k_1}, Y_{21}, \ldots, Y_{2k_2}, Y_{1m}, Y_{1n}\right)\right] \\
&= \qquad E\left[g_{k_1+1k_2+1}\left(\lambda_1, \lambda_2, Y_{11}, \ldots, Y_{1k_1}, Y_{21}, \ldots, Y_{2k_2}, Z_{1i}, Y_{1n}\right)\right] \quad \forall i \in \mathbf{N}.
\end{aligned}$$

Using similar arguments we thus see that, $\forall i, j \in \mathbf{N}$,

$$\lim_{m,n\to\infty} E\left[g_{k_1+1k_2+1}\left(\lambda_1,\lambda_2,Y_{11},\ldots,Y_{1k_1},Y_{21},\ldots,Y_{2k_2},Y_{1m},Y_{1n}\right)\right]$$

$$= E\left[g_{k_1+1k_2+1}\left(\lambda_1,\lambda_2,Y_{11},\ldots,Y_{1k_1},Y_{21},\ldots,Y_{2k_2},Z_{1i},Z_{2j}\right)\right] \qquad (5.65)$$

so that (5.62) holds as it is just a special case of (5.65).

We now show that (5.62) holds if g satisfies (i). For V as in (5.56), V is an $\mathcal{F}(\mathbf{Y})$-measurable map into $S = P_0^{(1)}(\mathbf{R}) \times P_0^{(2)}(\mathbf{R}) \times \mathbf{R}^{k_1} \times \mathbf{R}^{k_2}$, and we may thus use Theorem 5.5 (i) to see that

$$\lim_{m\to\infty} L(V,Y_{1m},Y_{2n}) = L(V,Z_{1i}Y_{2n}) \qquad \forall i \in \mathbf{N} \quad (5.66)$$

and

$$\lim_{n\to\infty} L(V,Y_{1m},Y_{2n}) = L(V,Y_{1m},Z_{2j}) \qquad \forall j \in \mathbf{N} \quad (5.67)$$

so that (5.66) and (5.67) yield

$$\lim_{m,n\to\infty} L(V,Y_{1m},Y_{2n}) = L(V,Z_{1i},Z_{2j}). \qquad \forall i,j \in \mathbf{N} \quad (5.68)$$

The continuity of $g_{k_1+1k_2+1}$ $\forall k_i \in \mathbf{N}$, $i \in [2]$, follows from the continuity of g and hence (5.62) is obtained from (5.68). The proof of Theorem 5.6 is thus complete.

Remark
Theorem 5.6 reduces to a result by Aldous ((1977), Proposition 13, p.74) when only variables of one type are considered.

We are almost in a position to prove Theorem 5.3 but we first need establish the following lemmas:

Lemma 5.3
Let $\left\{Q_{(j_1,j_2)(k_1,k_2)};\, 1 \le k_i \le q_{j_i}, i \in [2], (j_1,j_2) \in \mathbf{N}\times\mathbf{N}\right\}$ be a collection of properties, each of which holds for all $\mathbf{n}$ increasing sufficiently rapidly (see Definition 5.5). Then there exists $\mathbf{m}$ which satisfies the following:

For each $\mathbf{n} \subset \mathbf{m}$ and each $(j_1,j_2) \in \mathbf{N}\times\mathbf{N}$, there exists

$$\mathbf{n}^{(j_1,j_2)} = \left\{n_{11}^{j_1}, n_{12}^{j_1}, \ldots, n_{21}^{j_2}, n_{22}^{j_2}, \ldots\right\} \text{ such that}$$

$$\mathbf{n}^{(j_1,j_2)} \text{ satisfies properties } Q_{(j_1,j_2)(k_1,k_2)} \qquad 1 \leq k_i \leq q_{j_i}, i \in [2] \quad (5.69)$$

$$n_{ik} = n_{ik}^{j_i} \qquad \text{for all } k > j_i, i \in [2]. \quad (5.70)$$

Proof
The desired result follows from Aldous' related result (Aldous (1977), Lemma 14, p.76) since $\mathbf{n}$ increasing sufficiently rapidly requires $\mathbf{n}_i = \{n_{ij}; j \in \mathbf{N}\}$ to increase sufficiently rapidly $\forall i \in [2]$ (see Definition 5.5).

Lemma 5.4
Let A be any limit statute, with A' as its complement in $P(\mathbf{R}) \times P(\mathbf{R}) \times \mathbf{R}^\infty \times \mathbf{R}^\infty$. Let $j_i \in \mathbf{N}$ $\forall i \in [2]$ be given. Then

$$P\left[(\mu, \nu, \mathbf{Y}_n) \in A'\right] \leq 2^{-j_1-j_2} \quad (5.71)$$

for all $\mathbf{n}$ increasing sufficiently rapidly.

Proof
We start off by showing that A' is a limit statute.

In this regard suppose that

$$(\lambda_1, \lambda_2, \mathbf{x}) \in A' \text{ and } \sum_{i=1}^{2} \sum_{j=1}^{\infty} \left| x_{ij} - x'_{ij} \right| < \infty \quad (5.72)$$

for some $\mathbf{x}, \mathbf{x}' \in \mathbf{R}^\infty \times \mathbf{R}^\infty$ and $\lambda_i \in P(\mathbf{R})$, $\forall i \in [2]$.

Suppose that $(\lambda_1, \lambda_2, \mathbf{x}')$ is not in A'. Then $(\lambda_1, \lambda_2, \mathbf{x}') \in A$, and from the second part of (5.72), since A is a limit statute, we conclude that $(\lambda_1, \lambda_2, \mathbf{x}) \in A$. Clearly then $(\lambda_1, \lambda_2, \mathbf{x}) \notin A'$, which contradicts the first part of (5.72). We may thus conclude that A' is a limit statute if A is.

We cannot apply Theorem 5.6 directly to $I(A')$ since (5.39) is not satisfied, so we need a more sophisticated approach.

Consider the functions P_{ij} as defined in (5.22). From (5.37) and (5.24), $\forall j \in \mathbf{N}$, we have that

$$P\left[|P_{ik}(Z_{ij}) - Z_{ij}| \geq 2^{-k}\right] \leq 2^{-k} \quad \forall k \in \mathbf{N}, \ \forall i \in [2]$$

and hence, $\forall j \in \mathbf{N}$,

$$P\left[|Z_{ij} - \mathcal{P}_{ij}(Z_{ij})| \geq 2^{-j}\right] \leq 2^{-j} \quad \forall j \in \mathbf{N}, \ \forall i \in [2] \tag{5.73}$$

so that

$$\sum_{i=1}^{2}\sum_{j=1}^{\infty} |Z_{ij} - \mathcal{P}_{ij}(Z_{ij})| < \infty \quad a.s.. \tag{5.74}$$

From Theorem 5.5, it follows that, $\forall \mathbf{n} \subset \mathbf{N}^{\infty} \times \mathbf{N}^{\infty}$, $\forall i \in [2]$, $\forall j \in \mathbf{N}$,

$$P\left[\left|Y_{in_{ij}} - \mathcal{P}_{ij}(Y_{in_{ij}})\right| \geq 2^{-j}\right] = P\left[\left|Z_{ij} - \mathcal{P}_{ij}(Z_{ij}) \geq 2^{-j}\right]\right.$$

so that

$$\sum_{i=1}^{2}\sum_{j=1}^{\infty} \left|Y_{in_{ij}} - \mathcal{P}_{ij}(Y_{in_{ij}})\right| < \infty \quad a.s. \tag{5.75}$$

follows from (5.73) in the same way that (5.74) did.

Now define $\mathcal{P} : \mathbf{R}^{\infty} \times \mathbf{R}^{\infty} \to \mathbf{R}^{\infty} \times \mathbf{R}^{\infty}$ by

$$\mathcal{P}(\mathbf{x}) = (\mathcal{P}_{11}(x_{11}), \mathcal{P}_{12}(x_{12}), \ldots, \mathcal{P}_{21}(x_{21}), \mathcal{P}_{22}(x_{22}), \ldots) . \tag{5.76}$$

Due to the symmetry of (5.74) and the fact that A' is a limit statute, we conclude that

$$(\mu, \nu, \mathbf{Z}(\omega)) \in A' \text{ if and only if} (\mu, \nu, \mathcal{P}(\mathbf{Z}(\omega))) \in A'$$

for $\mathbf{Z}$ a partially exchangeable sequence with 2-c.r.m. (μ, ν), so that

$$P\left[(\mu, \nu, \mathbf{Z}) \in A'\right] = P\left[(\mu, \nu, \mathcal{P}(\mathbf{Z})) \in A'\right] . \tag{5.77}$$

Similarly, from (5.75),

$$P\left[(\mu, \nu, \mathbf{Y}_n) \in A'\right] = P\left[(\mu, \nu, \mathcal{P}(\mathbf{Y}_n) \in A'\right] . \tag{5.78}$$

However, from Theorem 5.2 and (5.77),

$$P\left[(\mu, \nu, \mathcal{P}(\mathbf{Z})) \in A\right] = 1$$

so that

$$P\left[(\mu, \nu, \mathcal{P}(\mathbf{Z})) \in A'\right] = 0. \tag{5.79}$$

Since a probability measure on a metric space is regular, there exists an open set $G \supset A'$ such that (5.79) leads us to

$$P\left[(\mu, \nu, P(\mathbf{Z})) \in G\right] \leq 2^{-j_1 - j_2 - 1} \qquad \forall j_i \in \mathbf{N},\ i \in [2]. \quad (5.80)$$

Now let

$$H = \{(\lambda_1, \lambda_2, \mathbf{x}); (\lambda_1, \lambda_2, P(\mathbf{x})) \in G \text{ for } \lambda_i \in P(\mathbf{R}) \quad \forall i \in [2]\ , \mathbf{x} \in \mathbf{R}^\infty \times \mathbf{R}^\infty\}\ . \quad (5.81)$$

For $\lambda_i \in P(\mathbf{R})\ \forall i \in [2]\ , \mathbf{x} \in \mathbf{R}^\infty \times \mathbf{R}^\infty$, let

$$g(\lambda_1, \lambda_2, \mathbf{x}) = I(H) = \begin{cases} 1 & \text{if} \quad (\lambda_1, \lambda_2 P(\mathbf{x})) \in G \\ 0 & \text{if} \quad (\lambda_1, \lambda_2 P(\mathbf{x})) \notin G. \end{cases} \quad (5.82)$$

From (5.82) it follows that $g(\lambda_1, \lambda_2, \mathbf{x})$ is bounded and measurable. Since $g(\lambda_1, \lambda_2, \mathbf{x})$ satisfies (5.38) and (5.39) we may use Theorem 5.6 (ii) to see that, for all $\mathbf{n}$ increasing sufficiently rapidly,

$$E\left[g(\mu, \nu, \mathbf{Y}_n)\right] \leq E\left[g(\mu, \nu, \mathbf{Z})\right] + \xi\ .$$

From (5.82) we thus have,

$$E\left\{I\left[(\mu, \nu, P(\mathbf{Y}_n)) \in G\right]\right\} \leq E\left\{I\left[(\mu, \nu, P(\mathbf{Z})) \in G\right]\right\} + \xi$$

and hence

$$P\left[(\mu, \nu, P(\mathbf{Y}_n)) \in G\right] \leq P\left[(\mu, \nu, P(\mathbf{Z})) \in G\right] + \xi\ . \quad (5.83)$$

From (5.80) and (5.83) we thus have that for all $\mathbf{n}$ increasing sufficiently rapidly,

$$P\left[(\mu, \nu, P(\mathbf{Y}_n)) \in G\right] \leq 2^{-j_1 - j_2}\ . \quad (5.84)$$

The lemma thus follows immediately from (5.78), (5.84) and the fact that $A' \subset G$.

We are finally in a position to prove Theorem 5.3.

Proof of Theorem 5.3

Let A be any limit statute. From Lemma 5.3 we know that $\exists \mathbf{m} \in \mathbf{N}^{\infty} \times \mathbf{N}^{\infty}$, such that, for all $\mathbf{n} \subset \mathbf{m}$ and each $(j_1 \times j_2) \in \mathbf{N} \times \mathbf{N}$, we can find a $\mathbf{n}^{(j_1,j_2)}$, as in (5.70), which satisfies a particular set of properties for sequences which increase sufficiently rapidly. From Lemma 5.4 then,

$$P\left[(\mu, \nu, \mathbf{Y}_{\mathbf{n}^{(j_1,j_2)}}) \in A'\right] \leq 2^{-j_1-j_2} \tag{5.85}$$

for $j_i \in \mathbf{N}$, $\forall i \in [2]$.

From the construction of $\mathbf{n}^{(j_1,j_2)}$ it follows that

$$\sum_{i=1}^{2} \sum_{k=1}^{\infty} \left| Y_{in_{ik}} - Y_{in_{ik}^{j_i}} \right| < \infty \, . \tag{5.86}$$

From the symmetry of (5.86) and the fact that A' is a limit statute it immediately follows that

$$(\mu, \nu, \mathbf{Y}_n) \in A' \text{ if and only if } (\mu, \nu, \mathbf{Y}_{\mathbf{n}^{(j_1,j_2)}}) \in A'$$

so that

$$P\left[(\mu, \nu, \mathbf{Y}_n) \in A'\right] = P\left[(\mu, \nu, \mathbf{Y}_{\mathbf{n}^{(j_1,j_2)}}) \in A'\right] \, . \tag{5.87}$$

From (5.85) and (5.87) then, for all $j_i \in \mathbf{N}$, $\forall i \in [2]$,

$$P\left[(\mu, \nu, \mathbf{Y}_n) \in A'\right] \leq 2^{-j_1-j_2}$$

so that

$$P\left[(\mu, \nu, \mathbf{Y}_n) \in A\right] = 1$$

and hence the desired result follows.

5.3 LIMIT THEOREMS FOR PARTIALLY EXCHANGEABLE SEQUENCES

We start off by extending some well-known limit theorems for IID random variables to hold for 2-IID sequences. Limit theorems for partially exchangeable sequences follow immediately (see Remark 5.3).

Let $\mathbf{X} = \{X_{ij}; i \in [2], j \in \mathbf{N}\}$ be a 2-IID sequence of random variables,

and let $\mathbf{S} = \left\{ S_{n_1 n_2} = \sum_{i=1}^{2} \sum_{j=1}^{n_i} X_{ij}; n_i \in \mathbf{N}, \forall i \in [2] \right\}$.

Definition 5.6
A *tail event* on $\mathbf{S}$ is an event which is independent of $\{S_{kl}; 1 \leq k \leq n_1, 1 \leq l \leq n_2\}$ for any finite $n_i \in \mathbf{N}$, $i \in [2]$.

The following corollary to the Hewitt-Savage zero-one law for 2-IID sequences (Theorem 3.6) reduces to a result in Breiman ((1968), p.64) when only variables of one type are considered.

Corollary 5.3
Let $\mathbf{X} = \{X_{ij}; i \in [2], j \in \mathbf{N}\}$ be a 2-IID sequence and $\mathbf{S}$ as defined above. Then every tail event on $\mathbf{S}$ has probability zero or one.

Proof
Let A be a tail event on $\mathbf{S}$. Then A is unaffected by finite permutations (within types) of $\mathbf{X}$, and hence $A \in E_{\mathbf{X}}$ (see Definition 3.5). The desired result follows from Theorem 3.6.

Example
Let $\mathbf{X}$ and $\mathbf{S}$ be as in Corollary 5.3, and let $I_{n_1 n_2} \in B(\mathbf{R}^2)$ $\forall n_i \in \mathbf{N}$, $i \in [2]$. Then

$$A = [S_{n_1 n_2} \in I_{n_1 n_2} \text{ for infinitely many } n_1 \text{ and } n_2 \text{ from } \mathbf{N}]$$

$$B = \left[\lim_{n_1, n_2} S_{n_1 n_2} < \infty \right]$$

are both tail events on $\mathbf{S}$ and hence, by Corollary 5.3, have measure zero or one.

The following limit theorem for 2-IID sequences is an extension of Kolmogorov's inequality (Appendix A1).

Theorem 5.7
Let $\mathbf{X} = \{X_{ij}; i \in [2], j \in \mathbf{N}\}$ be a 2-IID sequence of random variables with

$$E(X_{ij}) = 0, \quad E(X_{ij}^2) < \infty \qquad \forall i \in [2], \ \forall j \in \mathbf{N}$$

Let

$$E(X_{ij}^2) = V_i \qquad \forall j \in \mathbf{N}, \quad i \in [2].$$

Then, $\forall \xi > 0$,

$$P\left[\max_{\substack{1 \le k \le n_i \\ i \in [2]}} \left| \sum_{i=1}^{2} \sum_{j=1}^{k_i} X_{ij} \right| \ge \xi \right] \le 4\xi^{-2} \sum_{i=1}^{2} n_i V_i.$$

Proof

For all $k_i \in [n_i]$, $i \in [2]$

$$\left| \sum_{i=1}^{2} \sum_{j=1}^{k_i} X_{ij} \right| \le \left| \sum_{j=1}^{k_1} X_{1j} \right| + \left| \sum_{j=1}^{k_2} X_{2j} \right|$$

and hence

$$\max_{\substack{1 \le k_i \le n_i \\ i \in [2]}} \left| \sum_{i=1}^{2} \sum_{j=1}^{k_i} X_{ij} \right| \ge \xi$$

$$\Rightarrow \max_{1 \le k_1 \le n_1} \left| \sum_{j=1}^{k_1} X_{1j} \right| + \max_{1 \le k_2 \le n_2} \left| \sum_{j=1}^{k_2} X_{2j} \right| \ge \xi$$

so that

$$P\left[\max_{\substack{1 \le k_i \le n_i \\ i \in [2]}} \left| \sum_{i=1}^{2} \sum_{j=1}^{k_i} X_{ij} \right| \ge \xi \right]$$

$$\le P\left\{ \left[\max_{1 \le k_1 \le n_1} \left| \sum_{j=1}^{k_1} X_{1j} \right| \ge \xi/2 \right] \cup \left[\max_{1 \le k_2 \le n_2} \left| \sum_{j=1}^{k_2} X_{2j} \right| \ge \xi/2 \right] \right\}$$

$$\le P\left[\max_{1 \le k_1 \le n_1} \left| \sum_{j=1}^{k_1} X_{1j} \right| \ge \xi/2 \right] + P\left[\max_{1 \le k_2 \le n_2} \left| \sum_{j=1}^{k_2} X_{2j} \right| \ge \xi/2 \right]. \qquad (5.88)$$

Now applying Kolmogorov's inequality (Appendix A1) to each term on the right hand side of (5.88), we see that

$$P\left[\max_{\substack{1 \le k_i \le n_i \\ i \in [2]}} \left| \sum_{i=1}^{2} \sum_{j=1}^{k_i} X_{ij} \right| \ge \xi \right] \le 4\xi^{-2}(n_1 V_1 + n_2 V_2).$$

Lemma 5.5
Let $\mathbf{X}$ be a 2-IID sequence with zero means, finite second moments, where $\sum_{i=1}^{2}\sum_{j=1}^{\infty} V_{ij}$ is convergent ($V_{ij} = \mathrm{Var}(X_{ij})$). Then $\sum_{i=1}^{2}\sum_{j=1}^{\infty} X_{ij}$ is a.s. convergent.

Proof:
Since $\sum_{i=1}^{2}\sum_{j=1}^{\infty} V_{ij} < \infty$, it clearly follows that

$$\sum_{j=1}^{\infty} V_{ij} < \infty \qquad \forall i \in [2] ,$$

and hence, from Clarke (1975, Lemma 3, p.179) we see that $\sum_{j=1}^{\infty} X_{ij}$ is a.s. convergent $\forall i \in [2]$. The desired result now follows immediately.

An idea obtained in a paper by Aldous ((1981), p.590) led to the following extension of Kolmogorov's SLLN (see Appendix A1):

Theorem 5.8
Let $\mathbf{X}$ be a 2-IID sequence of random variables with $E|X_{i1}| < \infty$, $\forall i \in [2]$ and let $E(X_{i1}) = \mu_i$, $\forall i \in [2]$. Then

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} X_{1k}X_{2k} = \mu_1\mu_2 \quad a.s.$$

Proof
Clearly $\{X_{1k}X_{2k}; k \in \mathbf{N}\}$ is an IID sequence. Now

$$E|X_{1k}X_{2k}| \leq E|X_{1k}|E|X_{2k}| < \infty$$

so that Kolmogorov's above mentioned result yields

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} X_{1k}X_{2k} = \mu_1\mu_2 \quad a.s.$$

Using the notation of §5.1 and §5.2, in particular Definition 5.3, we now present the previous theorem in statute form.

$$A_3 = \left\{(\lambda_1, \lambda_2, \mathbf{x}); \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} x_{1k}x_{2k} = |\lambda_1|_1 \cdot |\lambda_2|_1\right\} \cup$$

$$\left\{(\lambda_1, \lambda_2, \mathbf{x}); |\lambda_1|_1 = \infty \text{ or } |\lambda_2|_1 = \infty\right\}$$

Similarly a statute representing an extension of the Law of the Iterated Logarithm (see Example 5.1) allowing 2-type sequences follows from statute A_2 in Example 5.1 in exactly the same way as A_3 above generalizes A_1:

$$A_4 = \left\{(\lambda_1, \lambda_2, \mathbf{x}); \quad \limsup_{N\to\infty} \frac{\left(\sum_{k=1}^{N} x_{1k}x_{2k} - N|\lambda_1|_1 \cdot |\lambda_2|_1\right)}{(2N \log\log N)^{\frac{1}{2}}}\right.$$

$$\left. = [|\lambda_1|_2 \cdot |\lambda_2|_2 + (|\lambda_1|_1)^2|\lambda_2|_2 + (|\lambda_2|_1)^2|\lambda_1|_2]^{\frac{1}{2}} \right\} \cup$$

$$\{(\lambda_1, \lambda_2, \mathbf{x}); \quad |\lambda_1|_2 \cdot |\lambda_2|_2 + (|\lambda_1|_1)^2|\lambda_2|_2 + (|\lambda_2|_1)^2|\lambda_1|_2 = \infty\},$$

where we have used the fact (see Clarke (1975), p.107) that, for any two random variables X and Y,

$$\mathrm{Var}(XY) = \mathrm{Var}(X)\mathrm{Var}(Y) + (E(X))^2\mathrm{Var}(Y) + (E(Y))^2\mathrm{Var}(X) .$$

Remark

Theorems 5.2 and 5.3 and statutes A_3 and A_4 above demonstrate how a.s. limit theorems for 2-IID sequences of random variables extend to partially exchangeable sequences and to sub-subsequences of an arbitrary tight sequence of random variables.

REFERENCES

Aldous, D.J. (1977) Limit Theorems for Subsequences of Arbitrarily-Dependent Sequences of Random Variables, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 40, 59-82.

——— (1981) Representations for Partially Exchangeable Arrays of Random Variables, *J. Multivariate Anal.* 11, 581-598.

——— (1982) On Exchangeability and Conditional Independence, *Exchangeability in Probability and Statistics*, North Holland, Amsterdam, New York, 165-170.

Aldous, D.J., Ibragimov, I.A. & Jacod, J. (1985) Exchangeability and Related Topics, *Lecture Notes in Mathematics*, Springer-Verlag, New York, 1117, 1-199.

Aldous, D.J. & Pitman, J. (1979) On the Zero-One Law for Exchangeable Events, *Ann. Probability* 7, No. 4, 704-723.

Berkes, I. (1982) Exchangeability and Limit Theorems for Subsequences of Random Variables, *Limit Theorems in Probability and Statistics*, Vol. I (Veszprém), Colloq. Math. Soc. János Bolyai 36, North Holland, Amsterdam, New York, 109-151.

Billingsley, P. (1968) *Convergence of Probability Measures*, Wiley, New York.

Breiman, L. (1968) *Probability*, Addison-Wesley, New York.

Chatterji, S.D. (1974a) A Subsequence Principle in Probability Theory, *Bull. Amer. Math. Soc.* 80, 495-497.

——— (1974b) A Principle of Subsequences in Probability Theory, *Advances in Math.* 13, 31-54.

——— (1985) A Subsequence Principle in Probability Theory, *Jahresber. Deutsch. Math. Verein.* 87, no. 3, 91-107.

——— (1986) Some Remarks on the Subsequence Principle in Probability Theory, *Stochastic Processes in Classical and Quantum Systems (Ascona, 1985), Lecture Notes in Physics* 262, Springer, Berlin, New York, 154-159.

Chernoff, H. & Teicher, H. (1958) A Central Limit Theorem for Sums of Interchangeable Random Variables, *Ann. Math. Stat.* 29, 118-130.

Chow, Y.S. & Teicher, H. (1978) *Probability Theory*, Springer-Verlag, New York.

Chung, K.L. (1974) *A Course in Probability Theory*, Academic Press, New York, London.

Clarke, L.E. (1975) *Random Variables*, Longman, New York, London.

Dacunha-Castelle, D. (1974) Indiscernability and Exchangeability in L^P-Spaces, *Proceedings of the Seminar on Random Series, Convex Sets and Geometry of Banach Spaces*, Mat. Inst. Aarhus Univ., Aarhus, 24, 50-56.

Dale, A.I. (1984) Some results in partial exchangeability, *Studia Math. Sci. Hung.* 19, 233-244.

de Finetti, B. (1931) Funzione caratteristica di un fenomeno aleatorio, *Atti della R. Academia Nazionale dei Lincei, Ser. 6, Memorie Classe di Scienze, Fisiche, Mathematiche e Naturali* 4, 251-299.

——— (1932) Funzione caratteristica di un fenomeno aleatorio, *Att. Congr. Int. Math., Bologna 1928* 6, 179-190.

——— (1937) La prévision: ses lois logiques, ses sources subjectives, *Ann. Inst. H. Poincaré* 7, 1-68 (translated in Kyburg and Smokler (1964)).

——— (1938) Sur la condition d' "équivalence partielle", *Act. Sci. Ind.* 739, 5-18. Hermann et cie, Paris.

Doob, J.L. (1953) *Stochastic Processes*, Wiley, New York.

Dubins, L.E. (1983) Some exchangeable probabilities are singular with respect to all presentable probabilities, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 64, 1-5.

Dubins, L.E. & Freedman, D.A. (1979) Exchangeable processes need not be mixtures of independent, identically distributed random variables, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 48, no. 2, 115-132.

Feller, W. (1971) *An Introduction to Probability Theory and Its Applications, Vol. II*, Wiley, New York, London.

Freedman, D.A. (1962) Invariants under mixing which generalize de Finetti's Theorem, *Ann. Math. Statist.* 33, 916-923.

——— (1980) A mixture of IID random variables need not admit a regular conditional probability given the exchangeable σ-field, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 51, 239-248.

Figiel, T. & Sucheston, L. (1976) An application of Ramsay sets in analysis, *Advances in Math.* 20, 103-105.

Goldstein, M. (1986) Exchangeable Belief Structures, *J. Amer. Statist. Assoc.* 81, 971-976.

Haag, J. (1924a) Sur un problème de probabilités, *C.R. Acad. Sci. Paris* 178, 838-840.

——— (1924b) Sur une question de probabilités, *C.R. Acad. Sci. Paris* 178, 1140-1142.

——— (1928) Sur un problème général de probabilités et ses diverses applications, *Proc. Int. Math. Congress Toronto 1924*, Toronto, 659-674.

Hamaker, H.C. (1977) Subjective probability and exchangeability from an objective point of view, *Int. Statist. Rev.* 45, 223-231.

Heath, D. & Sudderth, W. (1976) De Finetti's theorem on exchangeable variables, *Amer. Statist.* 30, 188-189.

Jagers, P. (1974) Aspects of random measures and point processes, *Advances in Probability 3*, Marcel Dekker, New York.

Kendall, D.G. (1967) On finite and infinite sequences of exchangeable events, *Studia Math. Sci. Hung.* 2, 319-327.

Kingman, J.F.C. (1972) On random sequences with spherical symmetry, *Biometrika* 59, 492-494.

——— (1978) Uses of exchangeability, *Ann. Probability* 6, No. 2, 183-197.

Komlós, J. (1967) A generalisation of a problem of Steinhaus, *Acta. Math. Acad. Sci. Hungar.* 18, 217-229.

Kyburg, H.E. & Smokler, H.E. (1964) *Studies in Subjective Probability*, Wiley, New York.

Link, G. (1980) Representation Theorems of de Finetti Type for (Partially) Symmetric Probability Measures, p.207-231 in R.C. Jeffrey (ed.), *Studies in Inductive Logic and Probability, Vol. II*, University of California Press, Berkeley.

Loève, M. (1977) *Probability Theory I*, Springer-Verlag, New York.

——— (1978) *Probability Theory II*, Springer-Verlag, New York.

Lord, R.D. (1954) The use of the Hankel transform in Statistics, *Biometrika* 41, 44-55.

Meyer, P.A. (1966) *Probability and Potentials*, Waltham, Mass. Blaisdell.

Parthasarathy, K.P. (1967) *Probability Measures on Metric Spaces*, Academic Press, New York, London.

Pfeiffer, P.E. (1979) *Conditional Independence in Applied Probability*, UMAP, Educational Development Center, Newton, Mass.

Ressel, P. (1985) De Finetti-type theorems — an analytical approach, *Ann. Probability* 13, 898-922.

Révész, P. (1967) *The Law of Large Numbers*, Academic Press, New York, London.

Sendler, W. (1975) A note on the proof of the Zero-One Law of Blum and Pathak, *Ann. Probability* 3, 1055-1058.

Shirayayev, A.N. (1984) *Probability*, Springer- Verlag, New York, Berlin, Heidelberg, Tokyo.

Taylor, R.L., Daffer, P.Z. & Patterson, R.F. (1985) *Limit Theorems for Sums of Exchangeable Random Variables*, Rowman & Allanheld Publishers, U.S.A.

Tjur, T. (1980) *Probability based on Radon Measures*, Wiley, New York.

Weber, N.C. (1980) A martingale approach to central limit theorems for exchangeable random variables, *J. Appl. Prob.* 17, 662-673.

APPENDIX A1

De Finetti's representation theorem for partially exchangeable events (Dale (1984), p.234)
For every infinite sequence of g-fold partially exchangeable events, there corresponds a (unique) g-dimensional distribution function F on $G \equiv [0,1]^g$ such that

$$m_{r_1,r_2,\ldots,r_g}^{(n_1,n_2,\ldots,n_g)} = \int_G \prod_{i=1}^{g} \binom{n_i}{r_i} x_i^{r_i}(1-x_i)^{s_i} dF(x_1,x_2,\ldots,x_g)$$

where $r_i + s_i = n_i \qquad \forall i \in [g]$

$m_{r_1,r_2,\ldots,r_g}^{(n_1,n_2,\ldots,n_g)}$ denotes the probability of $r_1, r_2, \ldots, r_g$ occurences in the $n_1, n_2, \ldots, n_g$ events respectively.

x_i denotes the probability of an occurence of event of type i.

Birkhoff ergodic theorem for stationary processes (Loève (1978), p.76)
Let $\mathbf{X} = \{X_i, i \in \mathbf{N}\}$ be a stationary sequence of random variables. If $E(X_1)$ exists, then

$$\frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow[\infty]{n} E(X_1/\mathcal{C}) \qquad a.s.$$

where $\mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}_n$

$$\mathcal{C}_n = \mathcal{F}\{X_n, X_{n+1}, \ldots\} \qquad \forall n \in \mathbf{N}.$$

A result from Lord ((1954) p.45, result 2.2).
If $\mathbf{X} = \{X_i, i \in \mathbf{N}\}$ has spherical symmetry, then

$$\phi(T) = E\left[\exp i\left(\sum_{j=1}^{n} t_j X_j\right)\right]$$

is a function of ρ only, where $\rho = |T| = |(t_1, t_2, \ldots, t_n)|$ ($|\cdot|$ denotes modulus).

Maxwell's theorem (Feller, (1971), Vol. II, Chapter 3, §4)
An independent spherically symmetric sequence, $\{X_i; i \in \mathbf{N}\}$, has a $N(0, \sigma^2)$ distribution for some $\sigma^2 > 0$.

Hewitt-Savage zero-one law (Loève (1977), Vol. I, p.374)
Exchangeable events have a probability of 0 or 1 and exchangeable functions are degenerate on a sequence of IID random variables.

Glivenko-Cantelli theorem (Chung (1974), p.132)
Let $\{X_n, n \in \mathbf{N}\}$ be a sequence of IID random variables, with common distribution function F. Take $\{X_j(\omega);\ j \in [n]\}$ for some $n \in \mathbf{N}$ and arrange them in non-decreasing order, denoted by

$$Y_{n_1}(\omega) \leq Y_{n_2}(\omega) \leq \cdots \leq Y_{n_n}(\omega).$$

Now define a discrete distribution function $F_n(\cdot, \omega)$ as follows:
For $x \in \mathbf{R}$,

$$F_n(x, \omega) = \begin{cases} 0 & \text{if} \quad x < Y_{n_1}(\omega) \\ \frac{k}{n} & \text{if} \quad Y_{n_k}(\omega) \leq x < Y_{n_{k+1}}(\omega) \quad \forall k \in [n-1] \\ 1 & \text{if} \quad x \geq Y_{n_n}(\omega). \end{cases}$$

Call $F_n(\cdot, \omega)$ the empiric distribution function based on n samples from F.

Now introduce indicator random variables,

$$\xi_j(x, \omega) = I\left[X_j(\omega) \leq x\right]$$

Then

$$F_n(x, \omega) = \frac{1}{n}\sum_{j=1}^{n} \xi_j(x, \omega)$$

and hence, from the SLLN (Chung (1974), Theorem 5.1.2),

$$\lim_{n\to\infty} F_n(x,\omega) = F(x) \qquad a.s. \ .$$

Martingale convergence theorem
(Doob (1953), Chapter 7, Theorem 4.3)
Let Z be a random variable with $E|Z| < \infty$, and let $\cdots \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ be Borel fields of measurable ω sets. Let $\mathcal{F}_{-\infty} = \bigcap_n \mathcal{F}_n$ and let $\mathcal{F}_\infty$ be the smallest Borel field of sets with $\mathcal{F}_\infty \supset \bigcup_n \mathcal{F}_n$. Then

$$\lim_{n\to-\infty} E(Z/\mathcal{F}_n) = E(Z/\mathcal{F}_{-\infty}) \quad a.s.$$

$$\lim_{n\to\infty} E(Z/\mathcal{F}_n) = E(Z/\mathcal{F}_\infty) \qquad a.s. \ .$$

Tight distributions of sequences of random variables
(Loève (1977), Vol. 1 p.194 or Shirayayev (1984), p.315)
A family $\mathcal{P}$ of probabilities on $\mathcal{A}$ is *tight* if $\forall \xi > 0, \exists$ compact K_ξ such that $P\left(K'_\xi\right) < \xi \quad \forall P \in \mathcal{P}$, where $\mathcal{A} = \mathcal{F}$ (open sets of S) and S is a metric space.

Relative Compactness (Loève (1977), Vol. I, p.195)
Let S be a separable, complete, metric space with Borel field S. A family $\mathcal{P}$ of probabilities on S is *relatively compact* if every sequence of members of $\mathcal{P}$ contains a subsequence which converges weakly to a probability on S.

Relative Compactness Criterion (Loève (1977), Vol. I, p.195)
For the notation above, $\mathcal{P}$ is relatively compact if and only if P is tight. In fact the "if" part holds for general metric spaces S.

Fatou's Lemma (Feller (1971), Vol. II, p.110)
Suppose $\{U_n, n \in \mathbf{N}\}$ to be a sequence of non-negative, integrable functions. Then

$$E\left(\lim_n \inf U_n\right) \leq \lim_n \inf E(U_n) \ .$$

Kolmogorov's inequality (Clarke (1975), p.177)
Suppose that $\{X_i; i \in [n],\ n \in \mathbf{N}\}$ are independent random variables with zero means and finite second moments, and let

$$V_k = \text{variance of } X_k, \qquad k \in [n].$$

Then, for any $\xi < 0$,

$$P\left[\max_{1 \le k \le n} \left|\sum_{i=1}^{k} X_i\right| \ge \xi\right] \le \xi^{-2} \sum_{k=1}^{n} V_k \,.$$

Kolmogorov's SLLN (Shirayayev (1984), p.366)
For $\{X_i; i \in \mathbf{N}\}$ an IID sequence of random variables with $E|X_i| < \infty$ $\forall i \in \mathbf{N}$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i = E(X_1) \ \ a.s.$$

APPENDIX A2

Listed below are properties of conditional independence taken from Aldous (1985), p.183.

An excellent elementary verification of these properties is given by Pfeiffer (1979) and a measure theoretic account is given by Chow and Teicher (1978).

In what follows sets are measurable and functions ϕ_i are bounded and measurable $\forall i \in [2]$.

Measurable, real valued random variables X and Y are conditionally independent given σ-field $\mathcal{F}$ if

A1 $\qquad P[X \in A,\ Y \in B/\mathcal{F}] = P[X \in A/\mathcal{F}]P[Y \in B/\mathcal{F}] \ \ \forall A, B \in \mathcal{B}(\mathbf{R})$

Each of the following is equivalent to A1

A2 $\qquad E\left[\phi_1(X)\phi_2(Y)/\mathcal{F}\right] = E\left[\phi_1(X)/\mathcal{F}\right] E\left[\phi_2(Y)/\mathcal{F}\right], \ \ \forall \phi_1, \phi_2$

A3 $\quad P[X \in A/\mathcal{F}, Y] = P[X \in A/\mathcal{F}]$

A4 $\quad E[\phi_i(X)/\mathcal{F}, Y] = E[\phi_i(X)/\mathcal{F}], \ \forall i \in [2].$

In these definitions we can replace a random variable X by a σ-field $\mathcal{G}$, by replacing events $[X \in A]$ with events G $(G \in \mathcal{G})$, and replacing functions $\phi(X)$ with bounded random variables $V \in \mathcal{G}$.

A sequence of random variables, $\{X_i; i \in \mathbf{N}\}$ is conditionally independent given σ-field $\mathcal{F}$ if the product form in A1 holds for each finite subset of $\{X_i, \ i \in \mathbf{N}\}$.

Here are some properties. Let $\mathcal{F}$ and $\mathcal{C}$ denote arbitrary σ-fields.

A5 Suppose that for each $j \geq 1$, X_j and $\mathcal{F}\{X_i; i > j\}$ are conditionally independent given $\mathcal{F}$. Then $\{X_i; \ i \in \mathbf{N}\}$ are conditionally independent given $\mathcal{F}$.

A6 If X is conditionally independent of itself given $\mathcal{F}$ then $X \in \mathcal{F}$ a.s.

A7 Suppose X and $\mathcal{F}$ are conditionally independent given $\mathcal{C}$ and suppose that X and $\mathcal{C}$ are conditionally independent given H where $H \subset \mathcal{C}$, then X and $\mathcal{F}$ are conditionally independent given H.