Exact Models for Radiating Relativistic Stars

by

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Abstract

In this thesis, we seek exact solutions for the interior of a radiating relativistic star undergoing gravitational collapse. The spherically symmetric interior spacetime, when matched with the exterior radiating Vaidya spacetime, at the boundary of the star, yields the governing equation describing the gravitational behaviour of the collapsing star. The investigation of the model hinges on the solution of the governing equation at the boundary. We first examine shear-free models which are conformally flat. The boundary condition is transformed to an Abel equation and several new solutions are generated. We then study collapse with shear in geodesic motion. Two classes of solutions are generated which are regular at the stellar centre. Our treatment extends the results of Naidu *et al* (2006) which had the undesirable feature of a singularity at the centre of the star. In an attempt to find more general models, we transform the fundamental equation to a Riccati equation. Two general classes of solution are found and are used to study the thermal evolution in the causal theory of thermodynamics. These solutions are shown to reduce to the Friedmann dust solution in the absence of heat flow. Furthermore, we obtain new categories of solutions for the case of gravitational collapse with expansion, shear and acceleration of the stellar fluid. This is achieved by transforming the boundary condition into a Riccati equation. In special cases the Bernoulli equation is regained. The solutions are given in terms of elementary functions and they permit the investigation of the physical features of radiative stellar collapse.

Dedicated to

Narisha and Manash

Preface and Declaration

The study described in this thesis was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban. This thesis was completed under the supervision of Professor S D Maharaj.

The research contained in this thesis represents original work by the author and has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

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Contents

Chapter 1

Introduction

In theories of gravitation we attempt to describe observable and measurable effects of gravity in a scientific framework. The failure of Newtonian theory, to explain the perihelion advance of Mercury and other gravitational phenomena, became the motivational platform for the adoption of a new theory which revolutionized our ideas of space and time. Einstein's general relativity theory is founded on the notion of a four-dimensional spacetime manifold in which the metric is a dynamical, symmetric and nonsingular tensorial field. The metric tensor field acts as the carrier of the gravitational field. Locally the metric resembles that of special relativity but globally it is described by a curved pseudo-Riemannian manifold. The Einstein tensor, which arises from quantities associated with the Riemann tensor, describes the geometry of the spacetime manifold. The dynamics of the gravitational field is accounted for by the Einstein field equations which couple the energy momentum tensor to the Einstein tensor.

The successful application of general relativity theory to relativistic astrophysics and cosmology includes the prediction and description of phenomena such as the bending of light, gravitational lensing, gravitational red shift, the cosmic microwave background, black holes and gravitational waves. Compact stars and their gravitational collapse have received particular attention in general relativity due to evidence from astronomical observations of supernova, neutron stars and pulsars (Duyvendak 1942, Hewish *et al* 1968). Baade and Zwicky (1934) proposed the view that a supernova represented the transition of an ordinary star to a neutron star, consisting mainly of neutrons. Using the static spherical solutions of Tolman (1939), Oppenheimer and Volkoff (1939) subsequently laid the groundwork for the study of superdense collapsing stars within the framework of general relativity theory. Oppenheimer and Snyder (1939) then argued that the continued gravitational collapse of a massive spherical star would eventually lead to the formation of a black hole.

Although radiative processes were understood to be intrinsic to collapse, these pioneering models were formulated with the unrealistic Schwarzschild exterior solution which is static. The interior spacetime was shear-free, comprising a homogenous matter distribution, with radiation in equilibrium with the barotropic matter distribution. In the absence of a radiating exterior spacetime, the problem of energy dissipation during the evolution of massive stars therefore remained unresolved. This difficulty was removed when the solution for the exterior radiating metric was obtained by Vaidya (1951). Misner and Sharp (1964) pioneered the shift from static perfect fluids by deriving the quasi-static equations in relativistic adiabatic processes. Note that the intriguing cosmic censorship conjecture of Penrose (1969) was drawn from considerations of the radiative processes in gravitational collapse.

The complete investigation of the gravitational behaviour of the collapsing star, however, depended on the formulation of the junction conditions, matching the interior metric with the exterior radiating Vaidya metric at the boundary of the star. In his seminal paper, Santos (1985) completed the derivation of the junction conditions for a radiating stellar fluid. This treatment paved the way for the examination of significant stellar physical features such as the surface luminosity, dynamical stability, relaxation effects and temperature profiles in the context of causal thermodynamical processes. The nonvanishing of the pressure at the boundary of the star was the critical insight in the analysis of Santos (1985); this formed the basis of a variety of exact radiative collapse models. Since this thesis focuses on exact solutions in radiative collapse, we provide an overview of the significant advances that followed the formulation of the boundary conditions for radiative gravitational collapse.

The Einstein field equations describing radiating stellar fluid models are a system of nonlinear coupled partial differential equations, which are difficult to solve in closed form. Since our understanding of the gravitational behaviour of these models hinges on exact solutions, we need to solve the field equations. In the integration process innovative and even ad hoc techniques have been utilised to obtain solutions. The field equations are considerably simplified by assuming shear-free collapse and pressure isotropy, but the junction condition remains highly nonlinear. However, under particular assumptions, exact models for shear-free collapse have been found. De Oliveira *et al* (1985) proposed the model of an initial static Schwarzschild configuration which was then allowed to undergo slow gravitational collapse. Note that it was earlier shown by Raychaudhuri (1955) that the slowest collapse arises in the case of shear-free fluid interiors. The solution obtained by De Oliveira *et al* (1985) was used to investigate the effect of the initial mass and radius on the formation of the event horizon (De Oliveira *et al* 1988). Another approach suggested by Kolassis *et al* (1988) generated exact solutions by assuming geodesic fluid trajectories. The interior Friedmann solution describing a compact dust ball was recovered when the heat flow was switched off. The Kolassis *et al* (1988) models formed the basis for the study of physical features such as the surface luminosity and temperature profiles in causal thermodynamics. Kramer (1992) and Maharaj and Govender (1997) formulated a nonstatic model by allowing one of the parameters of the static Schwarzschild interior solution, in isotropic coordinates, to become a function of time. This approach produced an interior spacetime which was then matched with the exterior Vaidya metric to yield exact solutions. A perturbative approach by Govender *et al* (2003), where an initial static solution was perturbed with a nonstatic separable perturbative functional form in the metric coefficients, was also successful in obtaining closed form solutions. This class of solutions served as realistic models for the formation of cold compact stars. Classes of solutions, with a homothetic symmetry, describing self-similar shear-free fluids with heat flow, have been studied by Wagh *et al* (2001). Further investigations by Chan *et al* (2003) showed that the collapse of a self-similar fluid with dissipation always led to the formation of naked singularities. It is interesting to note that earlier solutions obtained by Som and Santos (1981) were later shown to be self-similar. Recently, Brandt *et al* (2006) assumed an equation of state to study the evolution of a gravitating anisotropic self-similar fluid, of the second kind in terms of the kinetics, and obtained a class of exact solutions of the Einstein field equations. It was found that, depending on the self-similar parameter, these solutions may represent a black hole or a naked singularity.

The simplest radiating models have vanishing shear. In a recent treatment by Herrera *et al* (2004a), spherical radiating collapse was studied by imposing conformal flatness on the shear-free model with isotropic pressure. This led to a simplification of the Einstein field equations. However, the junction conditions generated a highly nonlinear ordinary differential equation in terms of the metric functions. It was believed that an exact solution to this model was not possible. By motivating simple forms for an approximate solution, Herrera *et al* (2004a) then studied the influence of relaxation effects on the model. Maharaj and Govender (2005), and later Herrera *et al* (2006), solved the nonlinear differential equation exactly by linearising the boundary condition. An objective of this thesis is to extend the shear-free conformally flat model with nonlinear boundary conditions for greater generality. The inherent nonlinear behaviour of the collapsing shear-free fluid requires a fully nonlinear model for proper investigations of the physical features.

For a complete treatment of radiative collapse we should include the effects of shear. The extension of the shear-free model, to include nonzero shear and pressure anisotropy, is important for the investigation of the physical processes that occur during particular phases of collapse. Martinez (1996) considered the collapse of a radiating viscous star and showed that the process, in the presence of shear, can be altered by neutrino trapping at high nuclear densities. A qualitative analysis using dynamical systems techniques by Di Prisco *et al* (2001), demonstrated that the process of collapse is driven by the bulk viscous pressures. The Einstein field equations describing shearing dissipative collapse are highly nonlinear, and exceedingly difficult to solve in closed form. There have been numerous attempts to construct exact models: Barreto *et al* (1992), Herrera and Martinez (1998), Chan (2000), and Nogueira and Chan (2004), amongst others. Most of these treatments utilise numerical techniques to cope with the high degree of nonlinearity. Recently, Naidu *et al* (2006) obtained the first analytical model describing a radiating, collapsing fluid with shear. In their paper, a particular solution in terms of elementary functions was found for geodesic motion. A limitation of their model is the presence of singularities at the stellar core. We seek to obtain nonsingular solutions in closed form which describe the complete gravitational behaviour of the shearing and radiating stellar fluid. Such solutions, if they exist in closed form, may be used for investigations of important physical features of collapsing stars, such as the thermal evolution, luminosity and dynamical stability. In this thesis, we show that general nonsingular classes of solutions exist which contain the solution of Naidu *et al* (2006).

Investigation of the thermodynamical behaviour of the collapsing relativistic fluid employs the truncated version of the Maxwell-Catteneo heat transport equation as derived from the Israel-Stewart theory. For reviews of causal thermodynamics and applications to cosmology the reader is referred to Israel and Stewart (1976), Maartens (1995) and Muller and Ruggeri (1992). An objective of this study is to systematically study the radiating model in order to analyse temperature profiles when the Eckart theory is no longer valid. Relaxational effects play a significant role in causal thermodynamics (Govender *et al* 1998, Govender *et al* 1999) and they should be included in a relativistic radiating model. It is desirable to obtain wider categories of solutions, which permit the study of the thermal behaviour with a varying relaxation parameter, to supplement existing toy models. We demonstrate that new radiating stellar models in the presence of shear can be found.

We observe that in most previous treatments of shearing collapse the particle acceleration was assumed to be zero. A challenging step forward is to derive exact models which include nonzero acceleration with expansion and shear. This is the most general case for radiative collapse and has thus far presented a formidable problem to solve analytically. Such models may be used to study a variety of realistic physical processes, e.g. particle producion at the stellar surface. In this thesis we attempt to extend the analysis to include acceleration, expansion and shear. We exhibit several classes of exact solutions by transforming the junction condition into a Riccati equation.

The thesis is organised as follows:

- Chapter 1: Introduction

- Chapter 2: In this chapter, we consider the shear-free case with vanishing Weyl tensor. The governing equation is transformed to an Abel equation and several classes of nonlinear exact solutions are presented. The results of this chapter extend earlier treatments by a systematic analysis of the nonlinear boundary condition.

- Chapter 3: We study shearing models with vanishing acceleration. New solutions which are nonsingular are shown to exist. These contain the Naidu *et al* (2006) solution. An important feature of our solutions is that they reduce to the Friedmann dust solutions in the absence of heat flow.

- Chapter 4: A systematic study of the governing equation for shearing collapse, assuming geodesic motion, is carried out. We rewrite the equation as a Riccati equation and generate two new classes of solutions by choosing specific temporal dependencies for one of the metric coefficients. This choice has the advantage of transforming the Riccati equation to a separable form which simplifies the integration process. In the relevant limit, the models reduce to the Friedmann dust model.

- Chapter 5: In this chapter we extend previous treatments to include expansion, shear and acceleration. By assuming separable forms for two metric functions, several classes of new exact solutions are obtained. We observe that the boundary condition is transformable to the Riccati equation for the models generated.

- Chapter 6: Conclusion

Chapter 2

Radiative nonlinear shear-free collapse

2.1 Introduction

From observations of supernova and through theoretical investigations of radiative models, we have reason to believe that the gravitational collapse of a star is a highly dissipative process (Herrera *et al* 2004b, Herrera and Santos 2004, Mitra 2006). The negative binding energy of the resulting compact star may be accounted for by neutrino emission. Exact analytical models, describing gravitational collapse with dissipation, are therefore very important for the investigation of the behaviour of stars in the final stages of collapse. In this chapter, we consider shear-free radiative collapse. In a recent treatment, Herrera *et al* (2004a) proposed a model in which the form of the Weyl tensor was highlighted when studying shear-free radiative collapse. This approach has the advantage of simplifying the Einstein field equations. However, Herrera *et al* (2004a) were not able to solve the junction conditions; only an approximate solution was found. Maharaj and Govender (2005) showed that it is possible to solve the field equations and the junction conditions exactly. Their solution is expressible in terms of elementary functions and contains the Friedmann dust solution as a special case. It is interesting to note that Herrera *et al* (2006) showed that other classes of solutions in terms of the elementary functions are possible. The exact solutions, in both the Maharaj and Govender (2005) and Herrera *et al* (2006) models, depend upon the introduction of a transformation that linearises the governing equation. However, physically realistic models are nonlinear by nature. The previous investigations of the conformally flat model have effectively linearised the boundary condition to obtain exact solutions. In this chapter, we demonstrate that it is possible to obtain other models by transforming the boundary condition to an Abel equation which is necessarily nonlinear. The main objective is to show that we can generate radiating relativistic stellar models without having to eliminate the nonlinearity at the boundary. In §2.2, we describe the basic features of the model for a radiating star and present the relevant equations which lead to the formulation of the governing equation. Results generated in previous investigations are briefly discussed in §2.3. These have been obtained by introducing transformations which lead to linear boundary conditions. In §2.4, we introduce a new transformation and generate a nonlinear Abel equation. We explicitly exhibit exact solutions to the Abel equation, under particular assumptions, and thereby demonstrate that conformally flat radiating stars contain a richer structure than previously suspected. Consequently, a variety of new models for radiating relativistic stars, with vanishing Weyl stresses, are possible. The physical features of the solutions are briefly considered in §2.5. The results of this chapter have been published in Misthry *et al* (2008).

2.2 The model

We consider a spherically symmetric radiating star undergoing shear-free gravitational collapse. The line element for shear-free matter, interior to the boundary of the radiating star, is given by

$$ds^2 = -A^2dt^2 + B^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{2.2.1}$$

where $A = A(t, r)$ and $B = B(t, r)$ are the metric functions. The energy momentum tensor, including radiation, for the interior spacetime has the form

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a \tag{2.2.2}$$

where the energy density ρ, the pressure p and the heat flow vector q_a are measured relative to the timelike fluid four-velocity $u^a = \frac{1}{A}\delta^a_0$. The heat flow vector assumes the form $q^a = (0, q, 0, 0)$ since $q^a u_a = 0$ for radially directed heat flow.

The nonzero components of the Einstein field equations, for the line element (2.2.1) and the energy momentum (2.2.2), can be written as

$$\rho = \frac{3}{A^2}\frac{\dot{B}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right) \tag{2.2.3a}$$

$$\begin{aligned} p &= \frac{1}{A^2}\left(-2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right) \\ &\quad + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + 2\frac{A'}{A}\frac{B'}{B} + \frac{2}{r}\frac{A'}{A} + \frac{2}{r}\frac{B'}{B}\right) \end{aligned} \tag{2.2.3b}$$

$$\begin{aligned} p &= -2\frac{1}{A^2}\frac{\ddot{B}}{B} + 2\frac{\dot{A}}{A^3}\frac{\dot{B}}{B} - \frac{1}{A^2}\frac{\dot{B}^2}{B^2} + \frac{1}{r}\frac{A'}{A}\frac{1}{B^2} \\ &\quad + \frac{1}{r}\frac{B'}{B^3} + \frac{A''}{A}\frac{1}{B^2} - \frac{B'^2}{B^4} + \frac{B''}{B^3} \end{aligned} \tag{2.2.3c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B}\right) \tag{2.2.3d}$$

which govern the gravitational field in the stellar interior. It is interesting to observe that the Weyl tensor has all components proportional to

$$C_{2323} = \frac{r^4}{3}B^2\sin^2\theta\left[\left(\frac{A'}{A} - \frac{B'}{B}\right)\left(\frac{1}{r} + 2\frac{B'}{B}\right) - \left(\frac{A''}{A} - \frac{B''}{B}\right)\right]$$

according to

$$\begin{aligned} C_{2323} &= -r^4\left(\frac{B}{A}\right)^2\sin^2\theta C_{0101} = 2r^2\left(\frac{B}{A}\right)^2\sin^2\theta C_{0202} \\ &= 2r^2\left(\frac{B}{A}\right)^2 C_{0303} = -2r^2\sin^2\theta C_{1212} = -2r^2 C_{1313} \end{aligned}$$

The Weyl tensor represents the effect of tidal forces. For conformal flatness these components must all vanish so that $C_{2323} = 0$. This leads to the nonlinear partial differential equation

$$\left(\frac{A'}{A} - \frac{B'}{B}\right)\left(\frac{1}{r} + 2\frac{B'}{B}\right) - \left(\frac{A''}{A} - \frac{B''}{B}\right) = 0 \tag{2.2.4}$$

By introducing the transformation,

$$u = A'B - B'A$$

(2.2.4) can be written as the integrable form

$$\frac{u'}{u} = \frac{1}{r} + 2\frac{B'}{B}$$

so that

$$A = (C_1(t)r^2 + 1)B \tag{2.2.5}$$

In (2.2.5) the function $C_1(t)$ arises from integration and we set the second constant of integration to unity without any loss in generality. Now from (2.2.3b) and (2.2.3c) we have

$$\frac{A'}{A}\left(\frac{1}{rB^2} + 2\frac{B'}{B^3}\right) - \frac{A''}{A}\frac{1}{B^2} + \frac{B'}{B}\left(\frac{1}{rB^2} + 2\frac{B'}{B^3}\right) - \frac{B''}{B^3} = 0 \tag{2.2.6}$$

which is the condition of pressure isotropy. We can eliminate the function A from (2.2.6), with the help of (2.2.5), to obtain

$$\frac{B''}{B'} - 2\frac{B'}{B} - \frac{1}{r} = 0 \tag{2.2.7}$$

Equation (2.2.7) is integrable and we get

$$B = \frac{1}{C_2(t)r^2 + C_3(t)}$$

where the integration constants $C_2(t)$ and $C_3(t)$ are functions of time. The forms for the metric functions A and B given above generate an exact solution to the Einstein field equations (2.2.3).

The interior spacetime (2.2.1) has to be matched, across the boundary $r = b$, to the exterior Vaidya spacetime

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right) dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.2.8}$$

The hypersurface at the boundary is denoted by Σ. The junction conditions at Σ have the form

$$(Adt)_\Sigma = \left[\left(1 - \frac{m}{R} + 2\frac{dR}{dv}\right)^{1/2} dv\right]_\Sigma \tag{2.2.9a}$$

$$(rB)_\Sigma = R_\Sigma \tag{2.2.9b}$$

$$(p_r)_\Sigma = (qB)_\Sigma \tag{2.2.9c}$$

$$m(v)_\Sigma = \left[\frac{r^3}{2}\left(\frac{\dot{B}^2 B}{A^2} - \frac{B'^2}{B}\right) - r^2 B'\right]_\Sigma \tag{2.2.9d}$$

on matching (2.2.1) and (2.2.8). For our model the junction conditions (2.2.9) reduce to the following nonlinear ordinary differential equation

$$\ddot{C}_2 b^2 + \ddot{C}_3 - \frac{3}{2}\frac{(\dot{C}_2 b^2 + \dot{C}_3)^2}{C_2 b^2 + C_3} - \frac{\dot{C}_1 b^2(\dot{C}_2 b^2 + \dot{C}_3)}{C_1 b^2 + 1} - 2(\dot{C}_3 C_1 - \dot{C}_2)b$$

$$+2\frac{C_1 b^2 + 1}{C_2 b^2 + C_3}\left[C_2(C_2 - 2C_1C_3)b^2 + C_3(C_1C_3 - 2C_2)\right] = 0 \qquad (2.2.10)$$

resulting from the (nonvanishing) pressure gradient across the hypersurface Σ. Equation (2.2.10) governs the evolution of a radiating star with vanishing Weyl stresses. To complete the description of this radiating model we need to solve the remaining junction condition (2.2.10).

2.3 Elementary solutions

The governing equation (2.2.10) is a highly nonlinear equation which presents a formidable mathematical task to solve exactly in general. Note that in previous attempts to integrate (2.2.10), assumptions were made that effectively linearised this boundary condition. We briefly summarise the known results.

Herrera *et al* (2004a) assumed the following approximate (first order) forms for the temporal functions

$$C_1 = \epsilon c_1(t), \quad C_2 = 0, \quad C_3 = \frac{a}{t^2} \qquad (2.3.1)$$

where $0 < \epsilon << 1$ and $a > 0$, is a constant. With the assumptions contained in (2.3.1), (2.2.10) yields the approximate solution

$$C_1 \approx C_1(0) \exp\left(\frac{-t^2}{2b^2} - \frac{2t}{b}\right)$$

Note that on setting $C_1 = 0$ the solution reduces to a collapsing Friedmann dust sphere.

Maharaj and Govender (2005) were the first to determine a closed form solution for (2.2.10). They assumed that

$$C_1 = C \;\; \text{(a constant)}, \quad C_2 = 0$$

and then introduced the transformation

$$C_3(t) = u^{-2} \tag{2.3.2}$$

so that (2.2.10) takes the linear form

$$\ddot{u} - 2Cb\dot{u} - (Cb^2 + 1)Cu = 0$$

in the new variable u. Three categories of closed form solutions, in terms of the elementary functions, were obtained depending on the nature of the roots of the characteristic equation.

Herrera *et al* (2006) extended this treatment to obtain a wider class of solutions. They set

$$C_1 = C \;\; \text{(a constant)}, \quad C_2 = \alpha C_3 \;\; (\alpha, \; \text{a constant})$$

and introduced the transformation

$$C_3(t) = u^{-2}$$

so that (2.2.10) can be written as

$$\ddot{u} - \frac{2(C-\alpha)b}{\alpha b^2+1}\dot{u} - \frac{(Cb^2+1)}{(\alpha b^2+1)^2}[\alpha(\alpha-2C)b^2+(C-2\alpha)]u = 0 \tag{2.3.3}$$

which is linear in u. Then equation (2.3.3) admits three classes of solution depending on the nature of the roots of the characteristic equation. These solutions are given by

Case 1: $(C-\alpha)^2b^2+(Cb^2+1)[\alpha(\alpha-2C)b^2+(C-2\alpha)] > 0$

$$C_3(t) =$$

$$\left[\beta_1 \exp\left(\frac{(C-\alpha)b+\sqrt{(C-\alpha)^2b^2+(Cb^2+1)[\alpha(\alpha-2C)b^2+(C-2\alpha)]}}{\alpha b^2+1}\right)t\right.$$

$$\left.+\beta_2 \exp\left(\frac{(C-\alpha)b-\sqrt{(C-\alpha)^2b^2+(Cb^2+1)[\alpha(\alpha-2C)b^2+(C-2\alpha)]}}{\alpha b^2+1}\right)t\right]^{-2}$$

Case 2: $(C-\alpha)^2b^2+(Cb^2+1)[\alpha(\alpha-2C)b^2+(C-2\alpha)] < 0$

$C_3(t) =$

$$\left[\exp\left(\frac{(C-\alpha)b}{\alpha b^2+1}t\right)\left(\beta_1\cos\left(\frac{\sqrt{(C-\alpha)^2b^2+(Cb^2+1)[\alpha(\alpha-2C)b^2+(C-2\alpha)]}}{(\alpha b^2+1)}\right)t\right.\right.$$

$$\left.\left.+\beta_2\sin\left(\frac{\sqrt{(C-\alpha)^2b^2+(Cb^2+1)[\alpha(\alpha-2C)b^2+(C-2\alpha)]}}{(\alpha b^2+1)}\right)t\right)\right]^{-2}$$

Case 3: $(C-\alpha)^2b^2+(Cb^2+1)[\alpha(\alpha-2C)b^2+(C-2\alpha)] = 0$

$$C_3(t) = (\beta_1+\beta_2 t)^{-2}\exp\left(\frac{-2(C-\alpha)b}{\alpha b^2+1}t\right)$$

where β_1 and β_2 are constants of integration. The solutions given above reduce to the Maharaj and Govender (2005) model when $\alpha = 0$. Note that other transformations, different from (2.3.2), that linearise (2.2.10) are possible as indicated in Herrera *et al* (2006).

2.4 Abel equation

The nonlinearity and complexity in the boundary condition (2.2.10) is clearly evident. It is therefore remarkable that closed form solutions in terms of elementary functions have been shown to exist, as shown in §2.3. These particular closed form solutions have been generated from linearised forms of the governing equation (2.2.10). A natural extension would be a study of the existence of nonlinear solutions to the differential equation (2.2.10). Such classes of solutions, if they exist, are important in the study of the nonlinear behaviour of the shear-free, conformally flat model for a radiating star. Retaining the nonlinear feature of the governing gravitational equation may help to identify physical features not present in the corresponding linear case. Consequently, we seek classes of solutions which retain the inherent nonlinear structure of (2.2.10). These have not been found in the past due to the inherent difficulties of coping with nonlinearity.

Here, we consider a particular nonlinear transformation which leads to exact solutions. It is convenient to replace the function $C_1(t)$ with

$$U = C_1 b^2 + 1 \tag{2.4.1}$$

Then the governing equation (2.2.10) may be written, after some rearrangement, as

$$\dot{U}(\dot{C}_2 b^2 + \dot{C}_3) + U\left[\frac{3}{2}\frac{(\dot{C}_2 b^2 + \dot{C}_3)^2}{C_2 b^2 + C_3} - \frac{2}{b}(\dot{C}_2 b^2 + \dot{C}_3) - (\ddot{C}_2 b^2 + \ddot{C}_3)\right]$$
$$+2U^2\left[\frac{\dot{C}_3}{b} - \frac{1}{C_2 b^2 + C_3}(C_2^2 b^2 - \frac{C_3^2}{b^2})\right] + 2U^3\frac{2C_2 b^2 - C_3}{C_2 b^2 + C_3}.\frac{C_3}{b^2} = 0 \tag{2.4.2}$$

Equation (2.4.2) is complicated, but has the generic structure

$$\mathcal{A}\dot{U} + \mathcal{B}U + \mathcal{C}U^2 + \mathcal{D}U^3 = 0 \tag{2.4.3}$$

where we have set

$$\mathcal{A} = \dot{C}_2 b^2 + \dot{C}_3$$

$$\mathcal{B} = \frac{3}{2}\frac{(\dot{C}_2 b^2 + \dot{C}_3)^2}{C_2 b^2 + C_3} - \frac{2}{b}(\dot{C}_2 b^2 + \dot{C}_3) - (\ddot{C}_2 b^2 + \ddot{C}_3)$$

$$\mathcal{C} = 2\left(\frac{\dot{C}_3}{b} - \frac{1}{C_2 b^2 + C_3}(C_2^2 b^2 - \frac{C_3^2}{b^2})\right)$$

$$\mathcal{D} = 2\left(\frac{2C_2 b^2 - C_3}{C_2 b^2 + C_3} \cdot \frac{C_3}{b^2}\right)$$

The transformed equation (2.4.3) is an Abel equation of the first kind in the variable U. Abelian equations are difficult to solve in general; only special cases admit exact solutions, as pointed by Polyanin and Zaitsev (2003). However, the advantage of utilising the transformation (2.4.1) is that (2.4.3) becomes a first order differential equation in U. In the following, we present a comprehensive mathematical treatment of (2.4.3) and derive several classes of solutions.

2.4.1 Case 1: $\mathcal{A} = 0$

The restriction $\mathcal{A} = 0$ immediately gives

$$C_2 b^2 + C_3 = \alpha$$

where α is a constant of integration. Then (2.4.2) becomes

$$2U^2\left[\frac{\dot{C}_3}{b} - \frac{1}{\alpha}\left(C_2^2 b^2 - \frac{C_3^2}{b^2}\right)\right] + 2U^3 \frac{2C_2 b^2 - C_3}{\alpha} \cdot \frac{C_3}{b^2} = 0 \qquad (2.4.4)$$

which is an algebraic equation in U.

Two cases arise: $U = 0$ or $U \neq 0$ in (2.4.4). We easily find that

$$C_1 = \begin{cases} -\frac{1}{b^2} & , \ U = 0 \\ \frac{\alpha}{C_3(2\alpha - 3C_3)} \left(\frac{\alpha}{b^2} - \frac{4C_3}{b^2} + \frac{3C_3^2}{\alpha b^2} - \frac{\dot{C}_3}{b} \right) & , \ U \neq 0 \end{cases} \tag{2.4.5a}$$

$$C_2 = \frac{\alpha - C_3}{b^2} \tag{2.4.5b}$$

$$C_3 = \text{arbitrary function of time} \tag{2.4.5c}$$

This solution is particularly attractive since we have an infinite choice of C_3 and no integration is required.

2.4.2 Case 2: $\mathcal{D} = 0$

With $\mathcal{D} = 0$ we have two possibilities: either $2C_2b^2 - C_3 = 0$ or $C_3 = 0$.

We first consider $2C_2b^2 - C_3 = 0$. Then (2.4.2) becomes

$$\dot{U} + U \left[\frac{3}{2} \frac{\dot{C}_3}{C_3} - \frac{2}{b} - \frac{\ddot{C}_3}{\dot{C}_3} \right] = -U^2 \left[\frac{4}{3b} + \frac{2}{3} \frac{C_3}{b^2 \dot{C}_3} \right]$$

This is a Bernoulli equation with solution

$$U = \frac{\dot{C}_3 C_3^{-3/2} e^{2t/b}}{K_1 - \frac{8}{3b} e^{2t/b} C_3^{-1/2} + \frac{6}{b^2} \int C_3^{-1/2} e^{2t/b} dt}$$

where K_1 is a constant of integration. Hence, for this first case we have the solution

$$C_1 = \frac{1}{b^2}\left(\frac{\dot{C}_3 C_3^{-3/2} e^{2t/b}}{K_1 - \frac{8}{3b} e^{2t/b} C_3^{-1/2} + \frac{6}{b^2}\int C_3^{-1/2} e^{2t/b} dt} - 1\right) \quad (2.4.6a)$$

$$C_2 = \frac{C_3}{2b^2} \quad (2.4.6b)$$

$$C_3 = \text{arbitrary function of time} \quad (2.4.6c)$$

This is an infinite class of solutions depending on C_3.

Secondly, we consider $C_3 = 0$. The Abel equation (2.4.2) becomes

$$\dot{U} + U\left(\frac{3}{2}\frac{\dot{C}_2}{C_2} - \frac{2}{b} - \frac{\ddot{C}_2}{\dot{C}_2}\right) = 2U^2 \frac{C_2}{\dot{C}_2 b^2}$$

This is again a Bernoulli equation with solution

$$U = \frac{\dot{C}_2 C_2^{-3/2} e^{2t/b}}{K_2 - \frac{2}{b^2}\int e^{2t/b} C_2^{-1/2} dt}$$

where K_2 is a constant of integration. Therefore for the second case we have the solution

$$C_1 = \frac{1}{b^2}\left(\frac{\dot{C}_2 C_2^{-3/2} e^{2t/b}}{K_2 - \frac{2}{b^2}\int e^{2t/b} C_2^{-1/2} dt} - 1\right) \quad (2.4.7a)$$

$$C_2 = \text{arbitrary function of time} \quad (2.4.7b)$$

$$C_3 = 0 \quad (2.4.7c)$$

Again we have generated an infinite class of solutions depending on C_2.

2.4.3 Case 3: $\mathcal{C} = 0$

Upon setting $\mathcal{C} = 0$ we obtain the equation

$$\frac{\dot{C}_3}{b} - \frac{1}{C_2 b^2 + C_3}\left(C_2^2 b^2 - \frac{C_3^2}{b^2}\right) = 0$$

This equation is a first order differential equation in C_3 and difficult to integrate. However, we can write the equation in the form

$$b^2 C_2^2 - \dot{C}_3 b C_2 - C_3\left(\frac{C_3}{b^2} + \frac{\dot{C}_3}{b}\right) = 0$$

which is quadratic in the quantity C_2. Consequently, we can find C_2 in the form

$$C_2 = \frac{\dot{C}_3 b \pm \sqrt{\dot{C}_3{}^2 b^2 - 4C_3(C_3 + \dot{C}_3 b)}}{2b^2}$$

so that C_2 is a known quantity if the function C_3 is specified.

The Abelian equation (2.4.2), with $\mathcal{C} = 0$, has the form

$$(\dot{C}_2 b^2 + \dot{C}_3)\dot{U} + \left[\frac{3}{2}\frac{(\dot{C}_2 b^2 + \dot{C}_3)^2}{C_2 b^2 + C_3} - \frac{2}{b}(\dot{C}_2 b^2 + \dot{C}_3) - (\ddot{C}_2 b^2 + \ddot{C}_3)\right] U$$

$$= -2\left[\frac{2C_2 b^2 - C_3}{C_2 b^2 + C_3} \cdot \frac{C_3}{b^2}\right] U^3$$

The equation is complicated, but may be written concisely as

$$\alpha\dot{U} + \beta U = -\gamma U^3 \qquad (2.4.8)$$

where

$$\alpha = \dot{C}_2 b^2 + \dot{C}_3$$

$$\beta = \frac{3}{2}\frac{(\dot{C}_2 b^2 + \dot{C}_3)^2}{C_2 b^2 + C_3} - \frac{2}{b}(\dot{C}_2 b^2 + \dot{C}_3) - (\ddot{C}_2 b^2 + \ddot{C}_3)$$

$$\gamma = 2\frac{2C_2 b^2 - C_3}{C_2 b^2 + C_3}.\frac{C_3}{b^2}$$

The simpler equation (2.4.8) has the form of a Bernoulli equation with solution

$$\begin{aligned} U &= \frac{1}{e^{\int(\beta/\alpha)dt}\left(\int \frac{2\gamma}{\alpha} e^{-\int(2\beta/\alpha)dt} dt\right)^{1/2}} \\ &= \frac{e^{(2t/b)}(\dot{C}_2 b^2 + \dot{C}_3)}{(C_2 b^2 + C_3)^{3/2}\left[K_3 + \frac{4}{b^2}\int \frac{e^{(4t/b)} C_3 (2C_2 b^2 - C_3)(\dot{C}_2 b^2 + \dot{C}_3)}{(C_2 b^2 + C_3)^4} dt\right]^{1/2}} \end{aligned}$$

where K_3 is a constant of integration. Consequently, for this case we have the solution

$$C_1 = \frac{1}{b^2}\left(\frac{e^{2t/b}(\dot{C}_2 b^2 + \dot{C}_3)}{(C_2b^2+C_3)^{3/2}\left[K_3 + \frac{4}{b^2}\int \frac{e^{(4t/b)}C_3(2C_2b^2-C_3)(\dot{C}_2b^2+\dot{C}_3)}{(C_2b^2+C_3)^4}dt\right]^{1/2}}\right) - \frac{1}{b^2} \tag{2.4.9a}$$

$$C_2 = \frac{\dot{C}_3 b \pm \sqrt{\dot{C}_3{}^2 b^2 - 4C_3(C_3+\dot{C}_3 b)}}{2b^2} \tag{2.4.9b}$$

$$C_3 = \text{arbitrary function of time} \tag{2.4.9c}$$

Again an infinite class of solutions is possible.

2.4.4 Case 4: $\mathcal{A} \neq 0$, $\mathcal{B} \neq 0$, $\mathcal{C} \neq 0$, $\mathcal{D} \neq 0$

This is the most general case and corresponds to the situation for which all of the coefficients $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are nonzero. Equation (2.4.3) can be written as

$$\dot{U} = -\frac{\mathcal{B}}{\mathcal{A}}U - \frac{\mathcal{C}}{\mathcal{A}}U^2 - \frac{\mathcal{D}}{\mathcal{A}}U^3$$

so that a variable separable equation is possible if $\frac{\mathcal{B}}{\mathcal{A}}$, $\frac{\mathcal{C}}{\mathcal{A}}$ and $\frac{\mathcal{D}}{\mathcal{A}}$ are constants. Then the solution may be written as the quadrature

$$t - t_0 = \int \frac{dU}{\frac{\mathcal{B}}{\mathcal{A}}U + \frac{\mathcal{C}}{\mathcal{A}}U^2 + \frac{\mathcal{D}}{\mathcal{A}}U^3} \tag{2.4.10}$$

It is important to emphasize that the additional constraints generated by $\frac{\mathcal{B}}{\mathcal{A}}$, $\frac{\mathcal{C}}{\mathcal{A}}$ and $\frac{\mathcal{D}}{\mathcal{A}}$ being constant simultaneously are not easy to simplify, and, in fact, may not be consistent. Note that when $\mathcal{B} = 0$ and $C_2b^2 + C_3 = \alpha$, we regain Case 1 with $\mathcal{A} = 0$, and (2.4.2) becomes a Bernoulli equation. Then solution (2.4.5) is applicable. However, in general, when $\mathcal{B} = 0$, the quadrature (2.4.10) is applicable.

2.5 Discussion

Herrera *et al* (2004a) obtained the equation (2.2.10) governing the gravitational behaviour of a radiating spherical star undergoing shear-free gravitational collapse by imposing the condition of conformal flatness on the model. Investigations of this model have thus far been confined to exact solutions of linearised forms of this equation. The nonlinear behaviour of relativistic stellar models is an inherent part of realistic stars undergoing radiative gravitational collapse. A study of the physical features of these models hinges on the solution of the governing nonlinear equations. In this chapter, we have presented exact solutions of the governing equation in which the nonlinearity has been explicitly preserved. This feature has been effected by transforming the equation (2.2.10) into the Abel equation (2.4.3). We have found several classes of exact solutions given in (2.4.5), (2.4.6), (2.4.7) and (2.4.9), and retained the nonlinearity of the model. Note that these classes generate an infinite family of solutions which allow for a systematic study of radiating relativistic spheres in different scenarios.

It is important to observe that simple particular cases can be generated from our nonlinear models in §2.4. For example with $\mathcal{A} = 0$ and $U \neq 0$ we obtain the line element

$$ds^2 = B^2[-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{2.5.1}$$

where (2.5.1) is in conformally flat form. Here, for the simple case $C_3 = \alpha$ in (2.4.5) we obtain $C_1 = 0$, $C_2 = 0$ and then B^2 is a constant; the Minkowski spacetime is regained. It is interesting to observe that the case $C_1 = 0$ in (2.4.5) also arises when C_3 takes the value

$$C_3 = \frac{2\alpha^2\beta e^{-2t/b} - \alpha}{2\alpha\beta e^{-2t/b} - 3}$$

where β is a constant of integration and $C_2 = (\alpha - C_3)/b^2$. Then we can write

$$B^2 = \left[\frac{b^2(2\alpha\beta e^{-2t/b} - 3)}{\alpha b^2 - 2\alpha r^2 - 2b^2\alpha^2\beta e^{-2t/b}}\right]^2$$

This simple analytic form for the potential B facilitates the analysis of the physical features of the model.

We consider now some physical features which may be investigated in future work. With suitable choices of the arbitrary time functions, the luminosity radius

$$L = (rB)_\Sigma$$

may be easily found. The quantity

$$\Gamma = \frac{d\ln p}{d\ln \rho}$$

gives a measure of the dynamical instability of the stellar configuration at any given instant in time. We can use this result to confirm that the centre of the star is more unstable than the outer regions. Of particular importance is the thermal evolution of the fluid. The causal transport equation in the absence of rotation and viscous stress is

$$\tau h_a^b \dot{q}_b + q_a = -\kappa(h_a^b \nabla_b T + T\dot{u}_a) \qquad (2.5.2)$$

where $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space, T is the local equilibrium temperature, $\kappa\,(\geq 0)$ is the thermal conductivity, and $\tau\,(\geq 0)$ is the relaxation time-scale which gives rise to the causal and stable behaviour of the theory. As shown in the treatment of Maharaj and Govender (2005), for a physically reasonable radiative stellar model, (2.5.2) becomes

$$\beta(qB)\dot{}\,T^{-\sigma} + A(qB) = -\alpha\frac{T^{3-\sigma}(AT)'}{B} \qquad (2.5.3)$$

where A and B are the metric functions. Both the causal and noncausal solutions of (2.5.3) may be investigated in a simple model. The temperature profiles resulting from (2.5.3) have been investigated by Govender *et al* (1998, 1999, 2003) for specific models. The general solution of (2.5.3) can be found explicitly as demonstrated by Govender and Govinder (2001). A review of the causal theory of thermodynamics, with particular application to dissipative cosmological models, was completed by Maartens (1995) and Zimdahl (1996).

Chapter 3

Nonsingular models for shearing collapse

A natural extension of the shear-free fluid model of Chapter 2 is to include the effects of shear, and to incorporate the effects of anisotropy. The origins and effects of an anisotropic gravitating fluid for relativistic stellar objects have been analysed by several authors, including Chan (2000), Herrera *et al* (2004b) and Herrera and Santos (1997). In particular, Martinez (1996) considered the collapse of a radiating viscous star and showed that the process can be altered by neutrino trapping at nuclear densities. Barreto *et al* (1992) and Herrera and Martinez (1998) have attempted to construct radiating stellar models with shear. Most of these treatments are approximate and the authors utilised numerical techniques to cope with the high degree of nonlinearity. Recently Noguiera and Chan (2004) reduced the boundary condition to a nonlinear ordinary differentiable equation by assuming separable forms for the metric functions; again only a numerical treatment was possible. Until a recent paper by Naidu *et al* (2006), an exact solution to the governing equation describing a collapsing fluid model with shear was not known. In their paper a particular solution with elementary functions for geodesic motion was found. An undesirable feature of their solution is the presence of singularities at the stellar centre. The main objective of this chapter is to generate exact nonsingular models which describe the gravitational history of a radiating anisotropic stellar fluid in the presence of nonzero shear. In §3.1, we present the defining equations of the model and generate the governing boundary condition by considering the junction conditions at the stellar surface. In §3.2, we obtain two new classes of exact solutions which are generally nonsingular at the stellar centre. The Naidu *et al* (2006) result is regained as a special case and we demonstrate that the Friedmann dust model results when the heat flux is absent. An expression for the temperature is found, in §3.3, by solving the Maxwell-Cattaneo equation. The simple form of our solution will help in a detailed physical analysis. The results of this chapter have been submitted for publication (Maharaj and Misthry 2008).

3.1 Formulation of the Model

We model a spherically symmetric star undergoing radiative gravitational collapse with nonzero shear. The line element is taken to be of the form

$$ds^2 = -dt^2 + B^2 dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (3.1.1)$$

where the metric functions B and Y are functions of t and r. For this model, the fluid four-velocity, $u^a = \delta^a_0$ is comoving. The fluid four-acceleration vector $\dot{u}^a$, the expansion scalar Θ, and the magnitude of the shear scalar σ, respectively, are given by

$$\dot{u}^a = 0 \tag{3.1.2a}$$

$$\Theta = \left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right) \tag{3.1.2b}$$

$$\sigma = \frac{1}{3}\left(\frac{\dot{Y}}{Y} - \frac{\dot{B}}{B}\right) \tag{3.1.2c}$$

for the line element (3.1.1). From (3.1.2) we observe that the particle trajectories of the collapsing fluid are geodesics because $\dot{u}^a = 0$. However, the fluid expansion and the shear may be nonzero in general. In the spherical case with vanishing shear the temporal evolution of the model can be found explicitly as shown by Kolassis *et al* (1988). In this case the model has a Friedmann limit and the dust cosmological model is regained. It is interesting to note that studying the behaviour of the temperature in causal thermodynamics for geodesic motion gives higher central temperatures than the Eckart theory as established by Govender *et al* (1998). In this study we seek to incorporate the effects of shear in the model.

The energy momentum tensor for the interior spacetime is given by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} \tag{3.1.3}$$

where ρ is the density of the gravitating fluid, p is the isotropic pressure, q_a is the heat flux vector and π_{ab} is the stress tensor. These quantities are measured relative to the four-velocity $\boldsymbol{u}$. The stress tensor can be written in the form

$$\pi_{ab} = (p_r - p_t)\left(n_a n_b - \frac{1}{3}h_{ab}\right) \tag{3.1.4}$$

where p_r is the radial pressure, p_t is the tangential pressure, and $\boldsymbol{n}$ is the unit radial vector orthogonal to $\boldsymbol{u}$ which is given by $n^a = \frac{1}{B}\delta^a_1$. The isotropic pressure

$$p = \frac{1}{3}(p_r + 2p_t)$$

relates the radial pressure and the tangential pressure.

The existence of local anisotropy represented by the expansion (3.1.4) is important in modelling astrophysical processes in radiating stars. The presence of the anisotropic term (3.1.4) has been numerically studied in dissipative, spherically symmetric gravitational collapse with shear by Herrera *et al* (2004b) and Di Prisco *et al* (1997). This term affects the phases of intense dynamical activity in the collapse of massive stars or the rapid collapse phase preceding the formation of a neutron star. The appearance of anisotropy can arise from a number of different physical reasons: the existence of type 3A superfluid (Kippenhahn 1990), different kinds of phase transitions (Sokolov 1980), pion condensation (Sawyer 1972), effects of slow rotation in stars (Herrera and Santos 1995) or a mixture of gases such as ionized hydrogen and electrons (Letelier 1980). Consequently, the existence of anisotropy can alter the evolution and physical properties of astrophysical objects.

The Einstein field equations are given by

$$\rho = 2\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{1}{Y^2} + \frac{\dot{Y}^2}{Y^2} - \frac{1}{B^2}\left(2\frac{Y''}{Y} + \frac{Y'^2}{Y^2} - 2\frac{B'}{B}\frac{Y'}{Y}\right) \quad (3.1.5a)$$

$$p_r = \left(-2\frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} + 2\frac{\dot{Y}}{Y}\right) + \frac{1}{B^2}\left(\frac{Y'^2}{Y^2}\right) - \frac{1}{Y^2} \quad (3.1.5b)$$

$$p_t = \frac{1}{B^2}\left(-\frac{B'}{B}\frac{Y'}{Y} + \frac{Y''}{Y}\right) - \left(\frac{\ddot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y}\right) \quad (3.1.5c)$$

$$q = -\frac{2}{B^2}\left(-\frac{\dot{Y}'}{Y} + \frac{\dot{B}}{B}\frac{Y'}{Y}\right) \quad (3.1.5d)$$

for the spacetime (3.1.1) and the matter distribution (3.1.3). The heat flux $q^a = (0, q, 0, 0)$ has only a nonvanishing radial component. We observe that if the gravitational potentials B and Y are specified then the matter variables ρ, p_r, p_t and q immediately follow from (3.1.5).

The exterior spacetime is described by the Vaidya metric which is given by

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right)dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (3.1.6)$$

where $m(v)$ represents the mass of the fluid as measured by an observer at infinity. The metric (3.1.6) represents coherent null radiation flowing in the radial direction outwards from the hypersurface Σ which represents the boundary of the star. Matching the line elements and the extrinsic curvature components of the exterior spacetime with the interior spacetime leads to the following set of junction conditions on the hypersurface Σ:

$$dt = \left(1 - \frac{2m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv}\right)^{1/2} dv \tag{3.1.7a}$$

$$Y(r_\Sigma, t) = R_\Sigma(v) \tag{3.1.7b}$$

$$m(v)_\Sigma = \left[\frac{Y}{2}\left(1 + \dot{Y}^2 - \frac{Y'^2}{B^2}\right)\right]_\Sigma \tag{3.1.7c}$$

$$(p_r)_\Sigma = (qB)_\Sigma \tag{3.1.7d}$$

The nonvanishing of the radial pressure, p_r, at Σ was established by Santos (1985) for shear-free spacetimes, and extended to spacetimes with nonzero shear by Glass (1989) and Maharaj and Govender (2000), amongst others. For a recent generalisation involving nonadiabatic charged fluids, applicable to shearing spherically symmetric gravitational collapse with dissipation, see Di Prisco *et al* (2007).

The junction condition $(p_r)_\Sigma = (qB)_\Sigma$ yields

$$2Y\ddot{Y} + \dot{Y}^2 - \frac{Y'^2}{B^2} + \frac{2}{B}Y\dot{Y}' - 2\frac{\dot{B}}{B^2}YY' + 1 = 0 \tag{3.1.8}$$

which follows from (3.1.5). To complete the model we need to integrate (3.1.8) and find the metric functions B and Y. An exact solution to (3.1.8) was first found by Naidu *et al* (2006). The explicit form of the solution is

$$Y = rt^{2/3} \tag{3.1.9a}$$

$$B = \frac{1 + f(r)\exp(3t^{1/3}/r)}{1 - f(r)\exp(3t^{1/3}/r)}t^{2/3} \tag{3.1.9b}$$

where $f(r)$ is an arbitrary function.

3.2 New exact models

The exact solution of Naidu *et al* (2006) has a simple form and makes possible a physical analysis of the thermal evolution of a radiating star with anisotropic pressure and nonzero shear. The solution (3.1.9) has the undesirable feature of a singularity at the stellar centre. With reference to the existence of singularities in the metric functions at $t = 0$, Naidu *et al* (2006) suggest that the singularity at $t = 0$ can be avoided by noting that the life of the star is taken to start at $t = -\infty$ and ends at $t = 0$. The singularity at $r = 0$ can be avoided if the solution represents an envelope describing the region of spacetime closer to the boundary; another solution representing the core of the star has to match with the envelope. It would be preferable, if possible, to generate a model to fully describe the gravitational history of a radiating anisotropic matter distribution without singularities in the metric functions. We show in this investigation that other solutions to (3.1.8) are possible, and explicitly give two exact solutions in terms of elementary functions. Both models avoid the singularities at $t = 0$ and $r = 0$ without having to place additional constraints on the evolution of the model.

3.2.1 First Solution

We observe that a replacement of the temporal and radial coordinates by

$$t \to t + a, \qquad r \to r + b$$

respectively, leaves the form of equation (3.1.8) invariant. This immediately leads to an exact solution

$$Y = (r+b)(t+a)^{2/3} \tag{3.2.1a}$$

$$B = \frac{1+g(r)\exp[3(t+a)^{1/3}/(r+b)]}{1-g(r)\exp[3(t+a)^{1/3}/(r+b)]}(t+a)^{2/3} \tag{3.2.1b}$$

where a and b are arbitrary constants and the quantity $g(r)$ is related to a function of integration. If $a = 0$ and $b = 0$ then (3.2.1) reduces to (3.1.9), and we regain the Naidu *et al* (2006) solution. When $t = 0$ and $r = 0$ we note that the metric functions in (3.2.1) remain regular at the origin for nonvanishing a and b. Hence we have found a new nonsingular solution for an anisotropic star which is radiating and has nonzero shear. The matter variables corresponding to the gravitational potentials (3.2.1) are given by

$$
\begin{aligned}
\rho &= \frac{4}{3\tilde{t}^{\,2}}\left[1+\frac{3\tilde{t}^{\,2/3}}{4\tilde{r}^{\,2}}\right]-\left[\frac{1-g\exp[3\tilde{t}^{\,1/3}/\tilde{r}]}{\tilde{r}\tilde{t}^{\,2/3}(1+g\exp[3\tilde{t}^{\,1/3}/\tilde{r}])}\right]^2 \\
&\quad +\frac{8g\exp[3\tilde{t}^{\,1/3}/\tilde{r}]}{3\tilde{t}^{\,5/3}\tilde{r}(1-g^2\exp 2[3\tilde{t}^{\,1/3}/\tilde{r}])} \\
&\quad +\frac{4\exp[3\tilde{t}^{\,1/3}/\tilde{r}](1-g\exp[3\tilde{t}^{\,1/3}/\tilde{r}])}{\tilde{r}\tilde{t}(1+g[\exp 3\tilde{t}^{\,1/3}/\tilde{r}])^3}\left[\frac{g'}{\tilde{t}^{\,1/3}}-\frac{3g}{\tilde{r}^2}\right] \qquad (3.2.2a)
\end{aligned}
$$

$$
p_r = -\frac{4g\exp[3\tilde{t}^{\,1/3}/\tilde{r}]}{\tilde{t}^{\,4/3}\tilde{r}^{\,2}(1+g\exp[3\tilde{t}^{\,1/3}/\tilde{r}]^2} \qquad (3.2.2b)
$$

$$
\begin{aligned}
p_t &= -\frac{2g\exp[3\tilde{t}^{\,1/3}/\tilde{r}]}{\tilde{r}^{\,2}\tilde{t}^{\,4/3}(1-g^2\exp 2[3\tilde{t}^{\,1/3}/\tilde{r}])}\left[1+\frac{2g\exp[3\tilde{t}^{\,1/3}/\tilde{r}]}{(1-g\exp[3\tilde{t}^{\,1/3}/\tilde{r}])}+\frac{4\tilde{r}}{3\tilde{t}^{\,1/3}}\right] \\
&\quad -\frac{2\exp[3\tilde{t}^{\,1/3}/\tilde{r}](1-g\exp[3\tilde{t}^{\,1/3}/\tilde{r}])}{\tilde{r}\tilde{t}(1+g\exp[3\tilde{t}^{\,1/3}/\tilde{r}])^3}\left[\frac{g'}{\tilde{t}^{\,1/3}}-\frac{3g}{\tilde{r}^{\,2}}\right] \qquad (3.2.2c)
\end{aligned}
$$

$$
q = -\frac{4g\exp[3\tilde{t}^{\,1/3}/\tilde{r}](1-g\exp[3\tilde{t}^{\,1/3}/\tilde{r}])}{\tilde{r}^{\,2}\tilde{t}^{\,2}(1+g\exp[3\tilde{t}^{\,1/3}/\tilde{r}])^3} \qquad (3.2.2d)
$$

where we have set $\tilde{t}=t+a$ and $\tilde{r}=r+b$. It is interesting to note that when $g=0$ then (3.2.1) and (3.2.2) yield the metric

$$
ds^2=-dt^2+t^{4/3}[dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)]
$$

with

$$
\rho=\frac{4}{3t^2}, \quad p_r=p_t=0
$$

Thus we have regained the Friedmann solution representing a dust sphere. When $q=0$

(since $g = 0$) the pressure vanishes which allows for geodesic motion.

3.2.2 Second Solution

Clearly other solutions to (3.1.8) are possible but these are difficult to find explicitly because of the nonlinearity in the potentials B and Y. Here, we demonstrate the existence of another exact solution by inspection. We observe from (3.2.1a) in the solution above that the separable functional form in Y leads to a simplification of the boundary condition (3.1.8). If we take the radial dependence of Y to be of exponential form, and retain the same temporal form as in (3.2.1a), then another exact solution can be found. We obtain the result

$$Y = [(t+a)\exp(r+b)]^{2/3} \tag{3.2.3a}$$

$$B = \frac{2}{3}[(t+a)\exp(r+b)]^{2/3}\left(\frac{1+h(r)\exp\left[\frac{3(t+a)^{1/3}}{\exp(2(r+b)/3)}\right]}{1-h(r)\exp\left[\frac{3(t+a)^{1/3}}{\exp(2(r+b)/3)}\right]}\right) \tag{3.2.3b}$$

where a and b are arbitrary constants. The function $h(r)$ is related to the function of integration. It is clear that (3.2.3) is a new solution to (3.1.8). When $t = 0$ and $r = 0$ the metric functions remain regular at the origin for nonvanishing a and b. Therefore we have generated another nonsingular solution for an anisotropic star whose interior has nonvanishing shear and the matter is radiating. The matter variables corresponding to the gravitational potentials (3.2.3) are given by

$$\rho = \frac{4}{3\tilde{t}^{\,2}}\left[1+\frac{3\tilde{t}^{\,2/3}}{4\exp{(4\tilde{r}/3)}}\right]-\left[\frac{1-h\exp[3\tilde{t}^{\,1/3}/\tilde{r}]}{\tilde{t}^{\,2/3}\exp{(2\tilde{r}/3)}(1+h\exp[3\tilde{t}^{\,1/3}/\tilde{r}])}\right]^2$$

$$+\frac{8h\exp[3\tilde{t}^{\,1/3}/\tilde{r}]}{3\tilde{t}^{\,5/3}\exp{(2\tilde{r}/3)}(1-h^2\exp 2[3\tilde{t}^{\,1/3}/\tilde{r}])}$$

$$+\frac{6\exp[3\tilde{t}^{\,1/3}/\tilde{r}](1-h\exp[3\tilde{t}^{\,1/3}/\tilde{r}])}{\tilde{t}\exp{(4\tilde{r}/3)}(1+h\exp[3\tilde{t}^{\,1/3}/\tilde{r}])^3}\left[\frac{h'}{\tilde{t}^{\,1/3}}-\frac{2h}{\exp{(2\tilde{r}/3)}}\right]$$

$$p_r = \left[\frac{1-h\exp[3\tilde{t}^{\,1/3}/\tilde{r}]}{\tilde{t}^{\,2/3}\exp{(2\tilde{r}/3+(1+h\exp[3\tilde{t}^{\,1/3}/\tilde{r}])}}\right]^2-\frac{1}{\tilde{t}^{\,4/3}\exp{(4\tilde{r}/3)}}$$

$$p_t = -\frac{2h\exp[3\tilde{t}^{\,1/3}/\tilde{r}]}{\tilde{t}^{\,4/3}\exp{(2\tilde{r}/3)}(1-h^2\exp 2[3\tilde{t}^{\,1/3}/\tilde{r}])}\left[\frac{4}{3\tilde{t}^{\,1/3}}+\frac{1}{\exp{(2\tilde{r}/3)}}\right.$$

$$\left.+\frac{2h\exp[3\tilde{t}^{\,1/3}/\tilde{r}]}{\exp{(2\tilde{r}/3)}(1-h\exp[3\tilde{t}^{\,1/3}/\tilde{r}])}\right]$$

$$-\frac{3\exp[3\tilde{t}^{\,1/3}/\tilde{r}](1-h\exp{[3\tilde{t}^{\,1/3}/\tilde{r}]})}{\tilde{t}\exp{(4\tilde{r}/3)}(1+h\exp{[3\tilde{t}^{\,1/3}/\tilde{r}]})^3}\left[\frac{h'}{\tilde{t}^{\,1/3}}-\frac{2h}{\exp{(2\tilde{r}/3)}}\right]$$

$$q = -\frac{6h\exp[3\tilde{t}^{\,1/3}/\tilde{r}](1-h\exp{[3\tilde{t}^{\,1/3}/\tilde{r}]})}{\tilde{t}^{\,2}\exp{(2\tilde{r})}(1+h\exp{[3\tilde{t}^{\,1/3}/\tilde{r}]})^3}$$

where we have again set $\tilde{t}=t+a$ and $\tilde{r}=r+b$. When $h=0$ the heat flow vanishes with

$$\rho=\frac{4}{3t^2},\quad p_r=p_t=0$$

If we make the transformation $\bar{r}\to\exp[(r+b)/3]$ in the metric coefficients (3.2.3) then the line element (3.1.1) can be written in the form

$$ds^s = -dt^2 + t^{4/3}[d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)].$$

Hence we have again regained the Friedmann dust solution when the heat flux vanishes.

3.3 Temperature

The simple form of the exact solutions (3.2.1) and (3.2.3) makes it possible to study the physical features of a radiating anisotropic star with shear. To study the temperature we consider the line element in the form

$$ds^2 = -d\tilde{t}^{\,2} + \tilde{t}^{\,4/3}\left[\left(\frac{1 + g\exp[3(\tilde{t}^{\,1/3}/\tilde{r}]}{1 - g\exp[3\tilde{t}^{\,1/3}/\tilde{r}]}\right)^2 d\tilde{r}^{\,2} + \tilde{r}^{\,2}(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3.3.1}$$

The total luminosity for an observer at rest is of the form

$$L_\infty(v) = -\left(\frac{dm}{dv}\right)_\Sigma = \frac{(p_R)_\Sigma}{2}\left[Y^2\left(\frac{Y'}{B} + \dot{Y}\right)^2\right]_\Sigma \tag{3.3.2}$$

where $dm/dv \leq 0$ since $L_\infty > 0$. The Maxwell-Cattaneo heat transport equation, for causal thermodynamical processes in relativistic astrophysics, is given by

$$\tau h_a^b \dot{q}_b + q_a = -\kappa(h_a^b \nabla_b T + T\dot{u}_a) \tag{3.3.3}$$

where τ is the relaxation time, κ is the thermal conductivity, and $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space. When $\tau = 0$ we regain the acausal Fourier heat transport equation. Martinez (1996) suggests that if the velocity of thermal dissipative signals is comparable to the adiabatic sound speed then the relaxational time is proportional to the adiabatic sound speed. With this assumption and the requirement that the mean collision time is constant, the heat transport equation (3.3.3) becomes

$$(AT)^4 = -\frac{4}{\alpha}\left[\beta \int B(qB)^{\cdot} dr + \int qB^2 dr\right] + F(t) \tag{3.3.4}$$

where α, β are constants. The noncausal temperature is regained when $\beta = 0$ in (3.3.4). From (3.3.1), (3.3.2) and (3.3.4) we can generate the result

$$\begin{aligned}
T^4 \;=\; & \frac{L_\infty}{(4\pi\delta Y^2)_\Sigma} \\
& + \frac{16\beta g}{3\alpha\tilde{t}^{\,5/3}}\left[\frac{\exp\left(3\tilde{t}^{\,1/3}/R+b\right)}{(R+b)(1+g\exp\left(3\tilde{t}^{\,1/3}/R+b\right))} - \frac{\exp\left(3\tilde{t}^{\,1/3}/\tilde{r}\right)}{\tilde{r}(1+g\exp\left(3\tilde{t}^{\,1/3}/\tilde{r}\right))}\right] \\
& + \frac{16}{9\alpha\tilde{t}^{\,2}}\left(\log\left[\left(\frac{-1+g\exp\left(3\tilde{t}^{\,1/3}/R+b\right)}{-1+g\exp\left(3\tilde{t}^{\,1/3}/R+b\right)}\right)^2 \left(\frac{1+g\exp\left(3\tilde{t}^{\,1/3}/\tilde{r}\right)}{1+g\exp\left(3\tilde{t}^{\,1/3}/\tilde{r}\right)}\right)^3\right]\right) \\
& + \frac{16}{3\alpha\tilde{t}}\left[\tanh^{-1}\left(g\exp\left(2\tilde{t}^{\,1/3}/R+b\right)\right) - \tanh^{-1}\left(g\exp\left(2\tilde{t}^{\,1/3}/R+b\right)\right)\right]
\end{aligned}$$

where R is the stellar radius. We note that at the boundary

$$T(t,R) = \tilde{T}(t,R) \tag{3.3.5}$$

so that the causal temperature T and the noncausal temperature $\tilde{T}$ coincide. However, at all interior points of the radiating star the causal and noncausal temperatures are different; in particular the causal temperatures have greater values. The temperature profiles are similar for small values of β but for large values of β the relaxational effects are significant. Note that at the centre $r = 0$ the causal temperature $T(t,0)$ has a finite value for appropriate choices of the parameter values; this is different from the solution of Naidu *et al* (2006) which contains a singularity at $r = 0$.

Chapter 4

A Riccati equation in shearing radiative collapse

4.1 Introduction

The existence of an exact solution is an important requirement in the analysis of the physical features of the radiating model. Due to the difficulty in obtaining closed form solutions to the Einstein field equations and the boundary condition, the complete investigation of the model's gravitational history is severely restricted. In the previous chapter, the spatial components of the solutions have been restricted to particular forms. This restriction limits the investigation of the relativistic effects such as the relaxation time scales on the model. It has been shown (Govender *et al* 1999) that relaxation effects have a significant influence on temperature profiles in the causal theory. Consequently, it is desirable, if possible, to obtain a wider class of solutions. The main objective of this chapter is to carry out a systematic study of the governing equation, at the boundary, for the shearing collapse of a compact radiating stellar fluid model. We seek to obtain a general class of nonsingular exact solutions which allows for flexibility in the choice of physical parameters required to investigate the features of the model. It is not desirable to eliminate the inherent nonlinearity at the boundary; instead we seek to transform the governing equation to a familiar form, namely the Riccati equation. We show that the Riccati equation admits two classes of new solutions in closed form. Particular models, obtained in previous investigations are regained, as special cases. A significant feature of these solutions is the general spatial dependence in the metric functions. Using the first solution, we then examine the behaviour of the causal temperature in inhomogeneous spacetimes. It is demonstrated that our model is suitable for a proper qualitative investigation of the thermodynamical behaviour of gravitational collapse; in particular, the inclusion of a realistic relaxation parameter. In §4.2, we formulate the model using the Einstein field equations together with the junction conditions. In §4.3, we first transform the governing equation into a Riccati equation, and then propose two transformations which lead to separable equations. Two new classes of exact solutions are found. It is shown that solutions found earlier with nonzero shear are contained in our models. In §4.4, we integrate the truncated form of the Maxwell-Cattaneo heat transport equation and obtain an explicit form for the causal temperature. We present profiles for the causal and acausal temperatures and briefly discuss the features of the graphs. The results of this chapter have been accepted for publication (Rajah and Maharaj 2008).

4.2 Formulation of the Model

In Chapter 3 we found particular solutions to the boundary condition by inspection. In this chapter we analyse the boundary condition systematically in an attempt to obtain a deeper insight of the gravitational collapse of a radiating star. We seek to model a spherically symmetric star undergoing radiative gravitational collapse with nonzero shear in the context of general relativity. The line element describing the gravitational field for the interior spacetime is taken to be

$$ds^2 = -dt^2 + B^2 dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.2.1}$$

where the metrics B and Y are functions of both the temporal coordinate t and radial coordinate r. For this model, the fluid four-velocity $u^a = \delta^a_0$ is comoving. The fluid four-acceleration vector $\dot{u}^a$, the expansion scalar Θ, and the magnitude of the shear scalar σ, respectively, are given by

$$\dot{u}^a = 0 \tag{4.2.2a}$$

$$\Theta = \left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right) \tag{4.2.2b}$$

$$\sigma = \frac{1}{3}\left(\frac{\dot{Y}}{Y} - \frac{\dot{B}}{B}\right) \tag{4.2.2c}$$

for the line element (4.2.1). We observe that the particle trajectories of the collapsing fluid are geodesics because $\dot{u}^a = 0$. However, note from (4.2.2) that the fluid expansion and the shear may be nonzero in general. A similar analysis was performed by Kolassis *et al* (1988) when the shear is vanishing; in that case it was possible to solve the boundary condition and the field equations and obtain an exact solution. The Kolassis *et al* (1988) model has a Friedmann limit and the dust cosmological model is regained. It would be interesting to compare the temporal evolution of the model when shear is present. Investigation of the behaviour of the temperature in causal thermodynamics for geodesic motion has revealed higher central temperatures than the Eckart theory, as established by Govender *et al* (1998). In this study we seek to incorporate the effects of shear in the model.

The energy momentum tensor for the interior spacetime has the form

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} \tag{4.2.3}$$

where ρ is the density of the gravitating fluid, p is the isotropic pressure, q_a is the heat flux vector and π_{ab} is the stress tensor. These quantities are measured relative to the four-velocity $\boldsymbol{u}$. The stress tensor can be written explicitly as

$$\pi_{ab} = (p_r - p_t)\left(n_a n_b - \frac{1}{3}h_{ab}\right)$$

where p_r is the radial pressure, p_t is the tangential pressure, and $\boldsymbol{n}$ is the unit radial vector orthogonal to $\boldsymbol{u}$. Hence we have $n^a = \frac{1}{B}\delta^a_1$. The isotropic pressure

$$p = \frac{1}{3}(p_r + 2p_t)$$

relates the radial pressure and the tangential pressure.

It is possible to write the Einstein field equations as the set

$$\rho = 2\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{1}{Y^2} + \frac{\dot{Y}^2}{Y^2} - \frac{1}{B^2}\left(2\frac{Y''}{Y} + \frac{Y'^2}{Y^2} - 2\frac{B'}{B}\frac{Y'}{Y}\right) \tag{4.2.4a}$$

$$p_r = \left(-2\frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} + 2\frac{\dot{Y}}{Y}\right) + \frac{1}{B^2}\left(\frac{Y'^2}{Y^2}\right) - \frac{1}{Y^2} \tag{4.2.4b}$$

$$p_t = \frac{1}{B^2}\left(-\frac{B'}{B}\frac{Y'}{Y} + \frac{Y''}{Y}\right) - \left(\frac{\ddot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y}\right) \tag{4.2.4c}$$

$$q = -\frac{2}{B^2}\left(-\frac{\dot{Y}'}{Y} + \frac{\dot{B}}{B}\frac{Y'}{Y}\right) \tag{4.2.4d}$$

for the spacetime (4.2.1) and the matter distribution (4.2.3). The fluid pressure is anisotropic and the heat flux $q^a = (0, q, 0, 0)$ has only a nonvanishing radial component. We observe that if functional forms for the gravitational potentials B and Y are given, then expressions for the matter variables ρ, p_r, p_t and q immediately follow from (4.2.4).

The exterior spacetime, describing the region outside the stellar boundary, is described by the Vaidya metric

$$ds^2 = -\left(1 - \frac{2m(v)}{\mathrm{R}}\right) dv^2 - 2dvd\mathrm{R} + \mathrm{R}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.2.5}$$

where $m(v)$ denotes the mass of the fluid as measured by an observer at infinity. The metric (4.2.5) describes coherent null radiation which is flowing in the radial direction relative to the hypersurface Σ which represents the boundary of the star. The matching of the exterior spacetime with the interior spacetime leads to the following set of junction conditions on the hypersurface Σ:

$$dt = \left(1 - \frac{2m}{\mathrm{R}_\Sigma} + 2\frac{d\mathrm{R}_\Sigma}{dv}\right)^{1/2} dv \tag{4.2.6a}$$

$$Y(r_\Sigma, t) = \mathrm{R}_\Sigma(v) \tag{4.2.6b}$$

$$m(v)_\Sigma = \left[\frac{Y}{2}\left(1 + \dot{Y}^2 - \frac{Y'^2}{B^2}\right)\right]_\Sigma \tag{4.2.6c}$$

$$(p_r)_\Sigma = (qB)_\Sigma \tag{4.2.6d}$$

The nonvanishing of the radial pressure p_r at Σ leads to an additional differential equation, namely the boundary condition (4.2.6d), which has to be satisfied together with the field equations (4.2.4). This condition was first established by Santos (1985) for shear-free spacetimes, and extended to spacetimes with nonzero shear by Glass (1989) and Maharaj and Govender (2000), amongst others. In a recent investigation by Di Prisco *et al* (2007), the matching conditions applicable to spherically symmetric gravitational collapse with dissipation and nonzero shear have been generalised to include nonadiabatic charged fluids.

4.3 Solution of the governing equation

The junction condition $(p_r)_\Sigma = (qB)_\Sigma$ becomes

$$2Y\ddot{Y} + \dot{Y}^2 - \frac{Y'^2}{B^2} + \frac{2}{B}Y\dot{Y}' - 2\frac{\dot{B}}{B^2}YY' + 1 = 0 \qquad (4.3.1)$$

which follows from (4.2.4). Equation (4.3.1) governs the gravitational behaviour of a radiating star with anisotropic pressure and nonzero shear. To complete the description of the gravitational behaviour of the model we need to integrate the junction condition (4.3.1); this will lead to functional forms for the metric functions $B(r,t)$ and $Y(r,t)$. Exact solutions for the junction condition (4.3.1) have been extremely difficult to obtain due to the nonlinear nature of the equation. In an earlier study of a shearing radiating model, Noguiera and Chan (2004) used numerical techniques to obtain approximate solutions. Ideally an exact solution is desirable in terms of elementary or special functions. An exact solution for this physical model was obtained by Naidu *et al* (2006) in terms of the elementary functions. This class of solution is singular at the stellar centre. In Chapter 3, we extended the Naidu *et al* (2006) model and showed that a wider category of solutions is possible; the singularities at the centre were shown to be avoidable.

In this treatment, we seek to obtain a general class of nonsingular solutions which will allow for an investigation of the physical features of the model. Previous treatments were ad hoc. Our objective is to write (4.3.1) in a generic form, and then obtain solutions systematically. It is shown that the model leads to the formation of a Riccati equation where the potential B is the dependent variable. We present, in the following, a method of solving (4.3.1) exactly, and find several classes of solutions depending on the form of Y used. We rewrite (4.3.1) in the form

$$\dot{B} = \left(\frac{\ddot{Y}}{Y'} + \frac{\dot{Y}^2}{2YY'} + \frac{1}{2YY'}\right) B^2 + \frac{\dot{Y}'}{Y'} B - \frac{Y'}{2Y} \tag{4.3.2}$$

which is a Riccati equation in the potential B. We demonstrate that two classes of solutions can be found for this Riccati equation.

4.3.1 The first solution

We seek solutions where Y is a separable function of the form

$$Y = R(r)(t+a)^{2/3} \tag{4.3.3}$$

so that the temporal evolution of the model is specified. It is convenient at this point to introduce the transformation

$$B = Z(t+a)^{2/3}$$

Then (4.3.2) can be written in the form

$$(t+a)^{2/3}\dot{Z} = \frac{1}{2RR'}(Z^2 - R'^2) \tag{4.3.4}$$

Equation (4.3.4) is simple and separable with solution

$$Z = R'\left(\frac{1 + g(r)\exp[3(t+a)^{1/3}/R]}{1 - g(r)\exp[3(t+a)^{1/3}/R]}\right)$$

where $g(r)$ is related to an arbitrary function of integration. Consequently, the potential B can be obtained explicitly in the form

$$B = R' \left(\frac{1 + g(r) \exp[3(t+a)^{1/3}/R]}{1 - g(r) \exp[3(t+a)^{1/3}/R]} \right) (t+a)^{2/3} \tag{4.3.5}$$

From (4.3.3) and (4.3.5) we may write the interior metric (4.2.1) as

$$\begin{aligned} ds^2 &= -dt^2 + (t+a)^{4/3} \left[R'^2 \left(\frac{1 + g(r) \exp[3(t+a)^{1/3}/R]}{1 - g(r) \exp[3(t+a)^{1/3}/R]} \right)^2 dr^2 \right. \\ & \left. + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \end{aligned} \tag{4.3.6}$$

which describes the interior spacetime of the radiating star.

The matter variables for the model are given by

$$\rho = \frac{4}{3\tilde{t}^{\,2}}\left[1+\frac{3\tilde{t}^{\,2/3}}{4R^2}\right] - \left[\frac{1-g\exp[3\tilde{t}^{\,1/3}/R]}{R\tilde{t}^{\,2/3}(1+g\exp[3\tilde{t}^{\,1/3}/R])}\right]^2$$

$$+\frac{8g\exp[3\tilde{t}^{\,1/3}/R]}{3\tilde{t}^{\,5/3}R(1-g^2\exp 2[3\tilde{t}^{\,1/3}/R])}$$

$$+\frac{4\exp[3\tilde{t}^{\,1/3}/R](1-g\exp[3\tilde{t}^{\,1/3}/R])}{RR'\tilde{t}(1+g[\exp 3\tilde{t}^{\,1/3}/R])^3}\left[\frac{g'}{\tilde{t}^{\,1/3}}-\frac{3g}{R^2}\right]$$

$$p_r = -\frac{4g\exp[3\tilde{t}^{\,1/3}/R]}{\tilde{t}^{\,4/3}R^2(1+g\exp[3\tilde{t}^{\,1/3}/R]^2}$$

$$p_t = -\frac{2g\exp[3\tilde{t}^{\,1/3}/R]}{R^{\,2}\tilde{t}^{\,4/3}(1-g^2\exp 2[3\tilde{t}^{\,1/3}/R])}\left[1+\frac{2g\exp[3\tilde{t}^{\,1/3}/R]}{(1-g\exp[3\tilde{t}^{\,1/3}/R])}+\frac{4R}{3\tilde{t}^{\,1/3}}\right]$$

$$-\frac{2\exp[3\tilde{t}^{\,1/3}/R](1-g\exp\,[3\tilde{t}^{\,1/3}/R])}{RR'\tilde{t}(1+g\exp\,[3\tilde{t}^{\,1/3}/R])^3}\left[\frac{g'}{\tilde{t}^{\,1/3}}-\frac{3g}{R^{\,2}}\right]$$

$$q = -\frac{4g\exp[3\tilde{t}^{\,1/3}/R](1-g\exp\,[3\tilde{t}^{\,1/3}/R])}{R^{\,2}R'\tilde{t}^{\,2}(1+g\exp\,[3\tilde{t}^{\,1/3}/R])^3}$$

which satisfies the Einstein equations (4.2.4). For simplicity we have set $\tilde{t} = t + a$.

We have obtained an exact solution to the Einstein field equations (4.2.4) that satisfies the boundary condition (4.3.1) for a radiating relativistic star. This is a new class of exact solutions where the spatial dependence in the function R is arbitrary. Consequently, particular solutions found in the past can be shown to be contained in this class. We observe that when $R = r + b$ the metric (4.3.6) becomes

$$ds^2 = -dt^2 + (t+a)^{4/3}\left[\left(\frac{1+g(r)\exp[3(t+a)^{1/3}/(r+b)]}{1-g(r)\exp[3(t+a)^{1/3}/(r+b)]}\right)^2 dr^2\right.$$

$$\left. + (r+b)^2(d\theta^2 + \sin^2\,\theta d\phi^2)\right]$$

Thus we have regained the model of Chapter 3 which is regular at the stellar origin. The Naidu *et al* (2006) model is regained from (4.3.6) when $a = 0$ and $b = 0$:

$$ds^2 = -dt^2 + t^{4/3}\left[\left(\frac{1 + g(r)\exp[3t^{1/3}/r]}{1 - g(r)\exp[3t^{1/3}/r]}\right)^2 dr^2 + r^2(d\theta^2 + \sin^2\,\theta d\phi^2)\right]$$

Consequently, the Naidu *et al* (2006) solution is central to the class of models with this particular choice for the radial function R.

Other forms for the function R are clearly possible: the choice should be such that the model remains regular at the centre and is well behaved in the interior. We further observe that the model yields the Friedmann dust model when $g = 0$. In this case we can find coordinates such that

$$ds^2 = -dt^2 + t^{4/3}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

for which the heat flux vector q_a vanishes and

$$p_r = p_t = 0, \quad \rho = \frac{4}{3t^2}$$

in geodesic motion.

4.3.2 The second solution

Other solutions to the Riccati equation (4.3.2) exist in closed form but these are difficult to find in practice. It is possible to find a second class of solutions to (4.3.2) by assuming that

$$Y = R(r)(t + a) \tag{4.3.8}$$

In this case we introduce the transformation

$$B = (t+a)Z$$

Then (4.3.8) can be written in the form

$$(t+a)\dot{Z} = \frac{R^2+1}{2RR'}\left(Z^2 - \frac{R'^2}{R^2+1}\right) \tag{4.3.9}$$

Equation (4.3.9) is a simple equation, separable in the variables Z and t, with solution

$$Z = \frac{R'}{\sqrt{R^2+1}}\left(\frac{1+h(r)(t+a)^{\sqrt{\frac{R^2+1}{R^2}}}}{1-h(r)(t+a)^{\sqrt{\frac{R^2+1}{R^2}}}}\right)$$

where $h(r)$ is an arbitrary function of integration. Therefore we can express the metric potential in the form

$$B = \frac{R'}{\sqrt{R^2+1}}\left(\frac{1+h(r)(t+a)^{\sqrt{\frac{R^2+1}{R^2}}}}{1-h(r)(t+a)^{\sqrt{\frac{R^2+1}{R^2}}}}\right)(t+a) \tag{4.3.10}$$

Then from (4.3.8) and (4.3.10) we obtain the interior metric

$$\begin{aligned} ds^2 &= -dt^2 + (t+a)^2\left[\frac{R'^2}{R^2+1}\left(\frac{1+h(r)(t+a)^{\sqrt{\frac{R^2+1}{R^2}}}}{1-h(r)(t+a)^{\sqrt{\frac{R^2+1}{R^2}}}}\right)^2 dr^2 \right. \\ &\quad \left. + R^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \end{aligned}$$

which describes the stellar interior. The Einstein field equations (4.2.4) then imply the matter variables which are given by

$$\rho = \frac{2}{\tilde{t}^{\,2}}\left[1+\frac{1}{2R^2}\right]-(3R^2+1)\left[\frac{1-h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}{R\tilde{t}\left(1+h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)}\right]^2$$

$$+\frac{2h\sqrt{R^2+1}}{RR'}\frac{\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}{\tilde{t}^{\,2}\left(1-h^2\,\tilde{t}^{\,2\sqrt{\frac{R^2+1}{R^2}}}\right)}$$

$$+\frac{4\,(R^2+1)^2\;\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}{R'R\tilde{t}^{\,2}\left(1+h\,\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)^2}\left[\frac{h'}{R^2+1}-\frac{h\ln\tilde{t}}{R^9}\right]$$

$$p_r = \frac{1}{\tilde{t}^{\,2}}\frac{R^2+1}{R^2}\left[\left(\frac{1-h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}{1+h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}\right)^2-1\right]$$

$$p_t = -\frac{2h\sqrt{R^2+1}\tilde{t}^{\,2\sqrt{\frac{R^2+1}{R^2}}}}{R\tilde{t}^{\,2}\left(1-h^2\,\tilde{t}^{\,2\sqrt{\frac{R^2+1}{R^2}}}\right)}\left[1+\frac{\sqrt{R^2+1}}{RR'}-\frac{1}{R'}\right]-\frac{1}{\tilde{t}^{\,2}}$$

$$+\frac{1}{\tilde{t}^{\,2}}\left(\frac{1-h\,\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}{1+h\,\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}\right)^2-\frac{4h^2(R^2+1)\,\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}{R^2R'\tilde{t}^{\,2}\left(1-h^2\,\tilde{t}^{\,2\sqrt{\frac{R^2+1}{R^2}}}\right)\left(1-h\,\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)}$$

$$-\frac{2(R^2+1)\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\left(1-h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)}{RR'\tilde{t}^{\,2}\left(1+h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)^2}\left[h'-\frac{h(R^2+1)\ln\tilde{t}}{R^9\left(1+h\,\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)}\right]$$

$$q = -\frac{4h(R^2+1)^{3/2}\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\left(1-h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)}{R^2R'^2\left(1+h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)^3}$$

where we have set $\tilde{t} = t + a$ for convenience.

We have generated another exact solution to the field equations (4.2.4) that satisfies the boundary condition (4.3.1). Again the spatial dependence in the function is arbitrary and the model is regular at the centre. We observe that the solution contains an exponential temporal dependence on the spatial function $R(r)$ of the form $t^{\sqrt{(R^2+1)/R^2}}$. Such solutions are difficult to interpret but may be relevant in describing new physical models in the strong gravity regime for gravitational collapse.

4.4 Causal temperature

The simple forms of the solutions found in this chapter, in particular the first solution, permit a study of their physical features. We consider briefly the relativistic effects on the temperature. For a shearing superdense matter distribution, we employ the Maxwell-Cattaneo heat transport equation to investigate the causal thermodynamical behaviour of the model. In the absence of rotation and viscous stress this is given by the truncated version

$$\tau h_a^b \dot{q}_b + q_a = -\kappa(h_a^b \nabla_b T + T\dot{u}_a) \tag{4.4.1}$$

where τ is the relaxation time, κ is the thermal conductivity, and $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space. When $\tau = 0$ we regain the acausal Fourier heat transport equation. For our model equation (4.4.1) may be written as

$$T = -\frac{\tau}{\kappa}\int (qB)^{\cdot} B dr - \frac{1}{\kappa}\int qB^2 dr \tag{4.4.2}$$

describing the evolution of the causal temperature. It has been shown in previous investigations of relativistic stellar models that the effect of the relaxation time τ on the thermal evolution plays a significant role in the latter stages of collapse (Di Prisco *et al* 1996, Govender *et al* 1998, Govender *et al* 1999, Martinez 1996). Naidu *et al* (2006) showed that in the presence of shear stress, the relaxation time decreases as the collapse proceeds and the central temperature increases. The particular form of the relaxational time τ is dependent on the physical constraints of the model during the latter phases of collapse. We observe that since our solutions have elementary functions with an arbitrary form for the spatial component, it is possible to integrate (4.4.2) for different choices of τ. In particular, modelling the effect of decreasing relaxation time with decreasing radius and higher central temperature is possible by incorporating a varying function τ.

In this study, we set τ and κ to be constant in order to examine the causal and acausal behaviour in the first solution (4.3.6). We need to choose particular forms for the arbitrary function $R(r)$ to complete the integration. As a first example we set $R = r + b$ to obtain for the temperature

$$\begin{aligned} T &= \frac{4\tau}{9\kappa\tilde{t}^{\,2}}\left(\mathrm{Log}\left[\frac{(1+g\exp[3\tilde{t}^{\,1/3}/r+b])^3}{(-1+g\exp[3\tilde{t}^{\,1/3}/r+b])^2}\right]\right) \\ &\quad -\frac{4\tau}{3\kappa\tilde{t}^{\,5/3}}\left(\frac{g\exp[3\tilde{t}^{\,1/3}/r+b]}{(r+b)(1+g\exp[3\tilde{t}^{\,1/3}/r+b])}\right) \\ &\quad -\frac{4}{3\kappa\tilde{t}}\mathrm{Tanh}^{-1}(g\exp[3\tilde{t}^{\,1/3}/r+b])+f(t) \end{aligned} \tag{4.4.3}$$

where we have kept g constant, and $f(t)$ is a constant of integration related to the luminosity as observed by a distant observer. As a second example we set $R = e^r$ to obtain the temperature

$$
\begin{aligned}
T &= \frac{4\tau}{9\kappa\tilde{t}^2}\left(\mathrm{Log}\left[\frac{(1+g\exp[3\tilde{t}^{\,1/3}/e^r])^3}{(-1+g\exp[3\tilde{t}^{\,1/3}/e^r])^2}\right]\right) \\
&\quad -\frac{4\tau}{3\kappa\tilde{t}^{\,5/3}}\left(\frac{g\exp[3\tilde{t}^{\,1/3}/e^r]}{e^r(1+g\exp[3\tilde{t}^{\,1/3}/e^r])}\right) \\
&\quad -\frac{4}{3\kappa\tilde{t}}\mathrm{Tanh}^{-1}(g\exp[3\tilde{t}^{\,1/3}/e^r]) + f(t) \qquad (4.4.4)
\end{aligned}
$$

Consequently, it is possible to find analytic forms for the causal temperature in terms of elementary functions as shown in (4.4.3) and (4.4.4). We regain the noncausal (Eckart) temperature profiles when $\tau = 0$. Our simple forms for T assist in studying the evolution of a radiating star in different time intervals. These models provide examples of temperatures where inhomogeneity is directly related to dissipation.

It is possible to qualitatively distinguish the causal and acausal temperatures for the region between the centre and the surface of the star. In Figure 4.1, we provide plots of the causal (solid line) and Eckart (broken line) temperatures against the radial coordinate on the interval $0 \leq r \leq 1$, where we have selected $\tau = 1$ for simplicity. This figure has been generated with the help of Mathematica. We observe that the temperature is a monotonically decreasing function as we approach the boundary from the stellar centre. Also, it is immediately clear that the causal temperature is everywhere greater than the acausal temperature in the interior of the star. At the boundary Σ, however

$$
T(t, r_\Sigma)_{causal} = T(t, r_\Sigma)_{acausal}
$$

This simple figure has been generated by assuming a particular constant value for the relaxation time τ and the thermal conductivity κ. Changing the magnitude of these values would produce a change in the separation of the curves but the results do not change qualitatively. For example, in Figure 4.2, we provide a plot of the causal (solid line) and Eckart (broken line) temperatures for $\tau > 1$. We note that in this case both temperatures decrease more rapidly as we approach the boundary; the value of τ affects the gradient of the temperature. As indicated previously, it is possible for the relaxation time τ to vary. The choice for τ should be dictated on physical grounds, e.g. rate of particle production at the stellar surface, and should be variable in general.

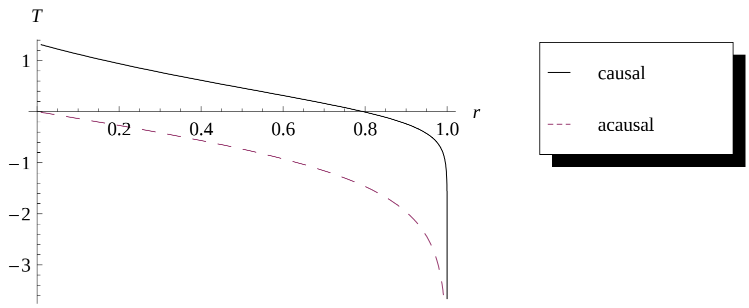

Figure 4.1: Temperature T vs radial coordinate r ($\tau = 1$)

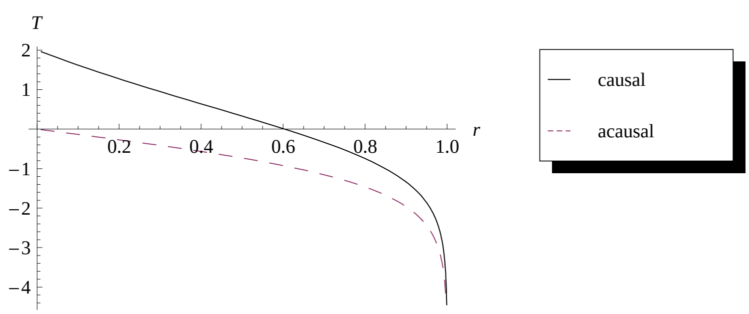

Figure 4.2: Temperature T vs radial coordinate r ($\tau > 1$)

Chapter 5

Shearing and accelerating collapse

5.1 Introduction

Incorporating into the model real features of a gravitating star, such as dissipation and shear viscosity, leads to a governing equation which presents a formidable mathematical problem to solve in closed form. Until the recent advances of exact models by Naidu *et al* (2006) and the results of Chapters 3 and 4, the realistic model of shearing collapse with dissipation could not be investigated analytically. We observe that the nonlinearity of these pioneering models is reduced considerably by the assumption of geodesic fluid trajectories so that the acceleration vanishes. With these models, the physical features of dissipative collapse may be confirmed qualitatively; however, the nonlinear contribution of accelerating fluid particles to the collapse process is absent. Hence, these models may not be suitable for phases of rapid collapse in the evolution of the self-gravitating star. The objective of this chapter is to extend the treatments of Chapters 3 and 4 to include nonzero acceleration in shearing gravitational collapse. The existence of exact models for this important and realistic scenario would be a significant step forward, since these models would then form the basis of a deeper analysis of the physical processes involved in gravitational collapse with dissipation. In §5.2, we set up the model for radiative collapse including expansion, shear and acceleration of the stellar fluid and obtain the junction condition that governs the stellar collapse. The governing equation is transformed into the familiar Riccati equation in §5.3. We then propose forms for the metric functions for which three distinct classes of solutions, under particular assumptions, are shown to exist. Insights gained in the previous chapter with no acceleration, lead us to a second model which is also shown to exhibit three classes of solutions described in §5.4. We summarise our results in §5.5 and demonstrate, for a particular solution, that it is possible to plot the causal and Eckart temperature profiles. Note that the results of this chapter are original work, and that they are being prepared for publication.

5.2 The model

In this chapter, we model a spherically symmetric star undergoing gravitational collapse with nonzero expansion, shear and acceleration. Our intention is to generalise the results of Chapters 3 and 4 to include nongeodesic motion in the presence of shear. The interior spacetime is described by the line element

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.2.1}$$

in comoving coordinates. The metric functions A, B and Y are functions of both t and r in general.

The fluid four-velocity is given by $u^a = \frac{1}{A}\delta^a_0$ which is necessarily comoving and timelike. The four-acceleration vector $\dot{u}^a$, the expansion scalar Θ, and the magnitude of the shear scalar σ, respectively, are given by

$$\dot{u}^a = \left(0, \frac{A'}{AB^2}, 0, 0\right) \tag{5.2.2a}$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right) \tag{5.2.2b}$$

$$\sigma = \frac{1}{3A}\left(\frac{\dot{Y}}{Y} - \frac{\dot{B}}{B}\right) \tag{5.2.2c}$$

for the line element (5.2.1). The kinematical quantities in (5.2.2) represent a collapsing stellar fluid in which the particle trajectories experience shear and expansion in accelerated motion. In Chapters 3 and 4, investigations of radiative collapse with nonzero shear have constrained the particle motion to be geodesic motion where $\dot{u}^a = 0$. This assumption simplifies the formulation of the model considerably; however, realistic physical processes involve accelerated fluid particle motion which could be significant in particular phases of collapse. In this section, we extend earlier treatments of radiative collapse to the general case ($\dot{u}^a \neq 0$, $\Theta \neq 0$, $\sigma \neq 0$) and formulate the relevant junction condition.

The energy momentum tensor for the interior spacetime is given by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} \tag{5.2.3}$$

where ρ is the density of the fluid, p is the isotropic pressure, q_a is the heat flux vector and π_{ab} is the stress tensor. These quantities are measured relative to the four-velocity $\boldsymbol{u}$. It is convenient to write the stress tensor in the form

$$\pi_{ab} = (p_r - p_t)\left(n_a n_b - \frac{1}{3}h_{ab}\right)$$

where p_r is the radial pressure and p_t is the tangential pressure. The unit vector $\boldsymbol{n}$ is orthogonal to $\boldsymbol{u}$; in terms of components it is given by $n^a = \frac{1}{B}\delta^a_1$. The isotropic pressure

$$p = \frac{1}{3}(p_r + 2p_t)$$

relates the radial pressure and the tangential pressure.

The Einstein field equations for the interior matter distribution are given by

$$\begin{aligned}
\rho &= \frac{2}{A^2}\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{1}{Y^2} + \frac{1}{A^2}\frac{\dot{Y}^2}{Y^2} \\
&- \frac{1}{B^2}\left(2\frac{Y''}{Y} + \frac{Y'^2}{Y^2} - 2\frac{B'}{B}\frac{Y'}{Y}\right) \qquad (5.2.4a) \\
p_r &= \frac{1}{A^2}\left(-2\frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} + 2\frac{\dot{A}}{A}\frac{\dot{Y}}{Y}\right) \\
&+ \frac{1}{B^2}\left(\frac{Y'^2}{Y^2} + 2\frac{A'}{A}\frac{Y'}{Y}\right) - \frac{1}{Y^2} \qquad (5.2.4b) \\
p_t &= -\frac{1}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y} - \frac{\dot{A}}{A}\frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y}\right) \\
&+ \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} + \frac{A'}{A}\frac{Y'}{Y} - \frac{B'}{B}\frac{Y'}{Y} + \frac{Y''}{Y}\right) \qquad (5.2.4c) \\
q &= -\frac{2}{AB^2}\left(-\frac{\dot{Y}'}{Y} + \frac{\dot{B}}{B}\frac{Y'}{Y} + \frac{A'}{A}\frac{\dot{Y}}{Y}\right) \qquad (5.2.4d)
\end{aligned}$$

for the interior metric (5.2.1), with the matter content (5.2.3). Note that the heat flux vector $q^a = (0, q, 0, 0)$ has only a radial component. If we set $B = Y$ then the shear vanishes and (5.2.4) becomes (2.2.3) given in Chapter 2. If we set $A = 1$ then the acceleration vanishes and (5.2.4) becomes (3.1.5), which is the case studied in Chapter 3, and (4.2.4) which is considered in Chapter 4. The system (5.2.4) is the most general case for spherically symmetric gravitational fields. The field equations (5.2.4) describe the gravitational interaction of a shearing matter distribution with heat flux and anisotropic pressure. We need to find explicit functional forms for the metric functions A, B and Y to demonstrate an exact solution to the field equations (5.2.4).

The exterior radiating spacetime is described by the outgoing Vaidya metric given by

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right) dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.2.5}$$

The quantity $m(v)$ represents the mass of the fluid as measured by an observer at infinity. The metric (5.2.5) is used to model the exterior spacetime as it describes coherent null radiation flowing in the radial direction relative to Σ which is the stellar boundary. The matching of the exterior spacetime with the interior spacetime leads to the following set of junction conditions on the hypersurface Σ:

$$A(r_\Sigma, t)dt = \left(1 - \frac{2m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv}\right)^{1/2} dv \tag{5.2.6a}$$

$$Y(r_\Sigma, t) = R_\Sigma(v) \tag{5.2.6b}$$

$$m(v)_\Sigma = \left[\frac{Y}{2}\left(1 + \frac{\dot{Y}^2}{A^2} - \frac{Y'^2}{B^2}\right)\right]_\Sigma \tag{5.2.6c}$$

$$(p_r)_\Sigma = (qB)_\Sigma \tag{5.2.6d}$$

For further information on junction conditions in the presence of shear see Di Prisco *et al* (2007), Glass (1989) and Maharaj and Govender (2000). The junction conditions (5.2.6) are similar to the junction conditions (2.2.9), (3.1.7) and (4.2.6) of previous chapters. However, in this chapter, the shear is nonzero and nongeodesic motion is possible. This allows for greater generality and leads to more complicated behaviour of the model.

The condition (5.2.6d), which arises from the nonvanishing of the pressure p_r on the hypersurface Σ representing the boundary of the star, leads to the master equation

$$
\begin{aligned}
&2Y\ddot{Y} + \dot{Y}^2 - 2\left(\frac{\dot{A}}{A} + \frac{A'}{B}\right)Y\dot{Y} \\
&+2\frac{A}{B}Y\dot{Y}' - 2\frac{A}{B^2}(A' + \dot{B})YY' - \frac{A^2}{B^2}Y'^2 + A^2 = 0 \qquad (5.2.7)
\end{aligned}
$$

which is highly nonlinear. As far as we are aware this junction condition has not been studied before, and no solution in closed form has been found. It is important to find exact solutions to perform a physical analysis of the model. An initial study of (5.2.7) suggests that a general solution in closed form may be exceedingly difficult to obtain. A useful approach to simplifying the junction condition (5.2.7) is to propose particular forms for some of the metric functions that are in terms of elementary functions or special functions. This simplifies the integration procedure and exact solutions in closed form are possible. These exact solutions allow for simpler analysis of the physical features of the model such as the temperature evolution, dynamical stability and luminosity profiles. To this end, we propose simple forms for the metric functions A and Y. We then carry out a systematic study of the transformed governing equation in the remaining potential B and demonstrate that several classes of solutions exist in closed form.

5.3 A Riccati equation

We can demonstrate a class of exact solutions by choosing separable forms for the metric functions A and Y. We are motivated to choose the temporal dependence of Y by the specific solutions for geodesic motion obtained in previous chapters. For this first model, we propose the following forms for the metric functions

$$A = \alpha(r)f(t) \tag{5.3.1a}$$

$$B = B(r,t) \tag{5.3.1b}$$

$$Y = \gamma(r)(t+a)^{2/3} \tag{5.3.1c}$$

where $\alpha(r)$ and $\gamma(r)$ are arbitrary functions of the radial coordinate, $f(t)$ is an arbitrary function of time, and a is a constant. With the assumptions contained in (5.3.1), equation (5.2.7) may be written as

$$\begin{aligned}\dot{B} &= \left(\frac{\alpha f}{2\gamma\gamma'}\right)\frac{B^2}{\tilde{t}^{\,4/3}} - \left(\frac{2}{3}\frac{\dot{f}}{\alpha f^2}\frac{\gamma}{\gamma'}\right)\frac{B^2}{\tilde{t}} \\ &\quad + \frac{2}{3}\left(1 - \frac{\alpha'}{\alpha}\frac{\gamma}{\gamma'}\right)\frac{B}{\tilde{t}} - \frac{\alpha f}{2}\left(\frac{\gamma'}{\gamma} + \frac{2\alpha'}{\alpha}\right)\end{aligned} \tag{5.3.2}$$

where we have set $\tilde{t} = (t+a)$. It is interesting to observe that equation (5.3.2) may be written in the generic form

$$\dot{B} = \mathcal{A}B^2 + \mathcal{B}B + \mathcal{C} \tag{5.3.3}$$

where the coefficients have the following particular forms

$$\begin{aligned}
\mathcal{A} &= \left(\frac{\alpha f}{2\gamma\gamma'}\right)\frac{1}{\tilde{t}^{\,4/3}} - \left(\frac{2}{3}\frac{\dot{f}}{\alpha f^2}\frac{\gamma}{\gamma'}\right)\frac{1}{\tilde{t}} \\
\mathcal{B} &= \frac{2}{3}\left(1 - \frac{\alpha'}{\alpha}\frac{\gamma}{\gamma'}\right)\frac{1}{\tilde{t}} \\
\mathcal{C} &= -\frac{\alpha f}{2}\left(\frac{\gamma'}{\gamma} + \frac{2\alpha'}{\alpha}\right)
\end{aligned}$$

The transformed equation (5.3.3) is a Riccati equation in the metric function B. Analytical approaches of solving Riccati equations with arbitrary coefficients usually depend on the forms of the coefficients, in this case $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$. These Riccati equations are generally difficult to solve by quadrature methods; however, specific forms have been found to be integrable, as pointed out by Polyanin and Zaitsev (2003). It is also observed that a linear equation is obtained for $\mathcal{A} = 0$; for $\mathcal{C} = 0$ the reduced equation is a Bernoulli equation. In the next section we present a systematic treatment of the Riccati equation (5.3.3) and demonstrate that, under particular assumptions, several classes of solutions for the transformed governing equation (5.3.3) may be derived in closed form.

5.3.1 Case 1: $\mathcal{A} = 0$

For this case we require

$$\left(\frac{\alpha f}{2\gamma\gamma'}\right)\frac{1}{\tilde{t}^{\,4/3}} - \left(\frac{2}{3}\frac{\dot{f}}{\alpha f^2}\frac{\gamma}{\gamma'}\right)\frac{1}{\tilde{t}} = 0$$

This can be written in the form

$$\frac{3}{4}\frac{\alpha^2}{\gamma^2} = \frac{\dot{f}}{f^3}\tilde{t}^{1/3} = k_1$$

since the variables t and r separate, and $k_1 > 0$ is a constant. Hence, we obtain the restrictions

$$\gamma = \sqrt{\frac{3}{4k_1}}\alpha \tag{5.3.4}$$

$$f = \frac{1}{\sqrt{k_2 - 3k_1\tilde{t}^{2/3}}} \tag{5.3.5}$$

and k_2 is a constant that arises from integration. The Riccati equation (5.3.2) is then reduced to the simple linear form

$$\dot{B} = \frac{3}{2}\alpha'\left(\frac{1}{\sqrt{k_2 - 3k_1\tilde{t}^{2/3}}}\right) \tag{5.3.6}$$

with the help of (5.3.4) and (5.3.5). Then (5.3.6) can be written as the quadrature

$$B = \frac{3}{2}\frac{\alpha'}{\sqrt{3k_1}}\int\left(\frac{1}{\sqrt{\frac{k_2}{3k_1} - \tilde{t}^{2/3}}}\right)dt \tag{5.3.7}$$

The integration of equation (5.3.7) yields the result

$$B = \frac{\sqrt{3}}{4}\frac{\alpha'}{\sqrt{k_1}}\left[-3\tilde{t}^{\,1/3}\sqrt{\frac{k_2}{3k_1} - \tilde{t}^{\,2/3}}\right.$$
$$\left. + \frac{k_2}{k_1}\tan^{-1}\left(\frac{\tilde{t}^{1/3}}{\sqrt{\frac{k_2}{3k_1} - \tilde{t}^{\,2/3}}}\right)\right] + \beta(r)$$

where $\beta(r)$ is a function of integration.

Hence, for this first case we have generated the exact solution

$$A = \alpha(r)\frac{1}{\sqrt{k_2 - 3k_1\tilde{t}^{2/3}}} \tag{5.3.8a}$$

$$B = \frac{\sqrt{3}}{4}\frac{\alpha'}{\sqrt{k_1}}\left[-3\tilde{t}^{\,1/3}\sqrt{\frac{k_2}{3k_1} - \tilde{t}^{\,2/3}}\right.$$
$$\left. + \frac{k_2}{k_1}\tan^{-1}\left(\frac{\tilde{t}^{1/3}}{\sqrt{\frac{k_2}{3k_1} - \tilde{t}^{\,2/3}}}\right)\right] + \beta(r) \tag{5.3.8b}$$

$$Y = \sqrt{\frac{3}{4k_1}}\alpha(r)\tilde{t}^{\,2/3} \tag{5.3.8c}$$

This is an example of an expanding, accelerating and shearing fluid which models the interior of an anisotropic radiating star. We cannot regain geodesic motion as a limiting case in this class of solution as $A \neq 1$. This is a simple class of solutions and the gravitational potentials are given in terms of elementary functions. The temporal dependence of the model is fully specified and essentially follows from the choice (5.3.1c). The function $\alpha(r)$ is an arbitrary function of the radial coordinate; we should choose physically reasonable forms to study the physical features of the model.

5.3.2 Case 2: $\mathcal{B} = 0$

In this case we set

$$\frac{2}{3}\left(1 - \frac{\alpha'}{\alpha}\frac{\gamma}{\gamma'}\right)\frac{1}{\tilde{t}} = 0$$

This condition immediately implies

$$\gamma = k_3\alpha$$

where $k_3 > 0$ is a constant of integration. Then (5.3.2) takes the form

$$\dot{B} = \frac{1}{\alpha'}\left(\frac{f}{2k_3^2\tilde{t}^{\,4/3}} - \frac{2}{3}\frac{\dot{f}}{f^2\tilde{t}}\right)B^2 - \frac{3}{2}\alpha' f \tag{5.3.9}$$

Equation (5.3.9) is a nonhomogenous Riccati equation which is difficult to solve in general for arbitrary values of f. However, for specific choices of the function $f(t)$ it may be possible to demonstrate a solution.

5.3.3 Case 3: $\mathcal{C} = 0$

Upon setting $\mathcal{C} = 0$, we obtain the condition

$$\frac{\gamma'}{\gamma} + \frac{2\alpha'}{\alpha} = 0$$

This generates the restriction

$$\gamma = \frac{k_4}{\alpha^2}$$

where k_4 is a constant of integration. Then the master equation (5.3.2) is reduced to an equation in the metric function B given by

$$\dot{B} - \frac{2}{3}\left(1 + \frac{1}{2k_4}\right)\frac{B}{\tilde{t}} = -\left(\frac{\alpha^6 f}{4k_4^2\alpha'}\right)\frac{B^2}{\tilde{t}^{\,4/3}} + \left(\frac{1}{3}\frac{\dot{f}}{f^2}\right)\frac{B^2}{k_4\alpha'\tilde{t}}$$

This is an example of a Bernoulli equation which can be integrated with solution

$$B = \frac{\tilde{t}^n}{\frac{\alpha^6}{4k_4^2\alpha'}\int f\tilde{t}^{n-\frac{4}{3}}dt - \frac{1}{3k_4\alpha'}\int \frac{\dot{f}}{f^2}\tilde{t}^{n-1}dt + K(r)} \tag{5.3.10}$$

where we have set $n = \frac{2}{3}(1 - \frac{1}{2k_4})$ and $K(r)$ is a function of integration. If a form for $f(t)$ is specified, then the integration in (5.3.10) can be completed.

Thus we have obtained an infinite class of solutions in closed form given by

$$A = \alpha(r)f(\tilde{t}) \tag{5.3.11a}$$

$$B = \frac{\tilde{t}^n}{\frac{\alpha^6}{4k_4^2\alpha'}\int f\tilde{t}^{n-\frac{4}{3}}dt - \frac{1}{3k_4\alpha'}\int \frac{\dot{f}}{f^2}\tilde{t}^{n-1}dt + K(r)} \tag{5.3.11b}$$

$$Y = \frac{k_4}{\alpha^2}\tilde{t}^{2/3} \tag{5.3.11c}$$

A variety of particular models are possible. To illustrate a simple model, we let

$$f = \tilde{t}^{-1/3}, \quad k_4 = 1, \qquad K(r) = 0$$

in (5.3.10). Then we obtain the particular solution

$$A = \alpha(r)\tilde{t}^{-1/3}$$

$$B = \beta(r)\tilde{t}^{2/3}$$

$$Y = \alpha(r)^{-2}\tilde{t}^{2/3}$$

where $\beta(r) = \frac{12\alpha'}{9\alpha^6+4}$. This particular example leads to a separable function for the potential B which is restrictive. This is an example of an expanding and accelerating fluid; the shear is a vanishing quantity. For nonzero shear another choice of $f(t)$ must be made, and B should not be separable.

5.4 Another Riccati equation

We can demonstrate a second class of exact solutions by choosing a different separable form for the metric function Y. We are guided to this choice by our solutions for geodesic motion which were obtained in previous chapters. We propose the following forms for the metric functions

$$A = \alpha(r)f(t) \tag{5.4.1a}$$

$$B = B(r,t) \tag{5.4.1b}$$

$$Y = \gamma(r)(t+a) \tag{5.4.1c}$$

With the assumptions (5.4.1), the governing equation (5.2.7) then takes the form

$$\dot{B} + \left(\frac{\alpha'}{\alpha}\frac{\gamma}{\gamma'} - 1\right)\frac{B}{\tilde{t}} = \left(\frac{\alpha f}{2\gamma\gamma'} + \frac{\gamma}{2\alpha\gamma' f}\right)\frac{B^2}{\tilde{t}^{\,2}} - \left(\frac{\dot{f}}{\alpha f^2}\frac{\gamma}{\gamma'}\right)\frac{B^2}{\tilde{t}} - \frac{\alpha f}{2}\left(\frac{\gamma'}{\gamma} + \frac{2\alpha'}{\alpha}\right) \tag{5.4.2}$$

where we have again set $(t + a) = \tilde{t}$. It is interesting to observe that (5.4.2) also has the generic form of a Riccati equation

$$\dot{B} = \mathcal{A}B^2 + \mathcal{B}B + \mathcal{C}$$

where the coefficients for this model have the following particular forms

$$\mathcal{A} = \left(\frac{\alpha f}{2\gamma\gamma'} + \frac{\gamma}{2\alpha\gamma' f}\right)\frac{1}{\tilde{t}^{\,2}} - \left(\frac{\dot{f}}{\alpha f^2}\frac{\gamma}{\gamma'}\right)\frac{1}{\tilde{t}}$$

$$\mathcal{B} = \left(1 - \frac{\alpha'}{\alpha}\frac{\gamma}{\gamma'}\right)\frac{1}{\tilde{t}}$$

$$\mathcal{C} = -\frac{\alpha f}{2}\left(\frac{\gamma'}{\gamma} + \frac{2\alpha'}{\alpha}\right)$$

We systematically repeat the analysis for this model as was done in §5.3. We seek the integrable cases for this Riccati equation as discussed in Polyanin and Zaitsev (2003).

5.4.1 Case 1: $\mathcal{A} = 0$

For this case we require

$$\left(\frac{\alpha f}{2\gamma\gamma'} + \frac{\gamma}{2\alpha\gamma' f}\right)\frac{1}{\tilde{t}^{\,2}} - \left(\frac{\dot{f}}{\alpha f^2}\frac{\gamma}{\gamma'}\right)\frac{1}{\tilde{t}} = 0 \tag{5.4.3}$$

For consistency we require that

$$\gamma = k_1 \alpha$$

where k_1 is a constant. With this restriction we find that (5.4.3) can be written as

$$\frac{1}{2}\left(\frac{1+f^2}{f}\right)\frac{1}{\tilde{t}} - \left(\frac{\dot{f}}{f^2}\right) = 0$$

The variables separate so that the above equation has solution

$$f = \sqrt{\frac{\tilde{t}}{k_2 - \tilde{t}}} \tag{5.4.4}$$

where k_2 is a constant of integration. Substituting (5.4.4) into the Riccati equation, (5.4.2), yields the simple linear equation

$$\dot{B} = \frac{3}{2}\alpha' \sqrt{\frac{\tilde{t}}{k_2 - \tilde{t}}}$$

Hence we obtain the result

$$B = \frac{3}{2}\alpha' \sqrt{k_2 \tilde{t} - \tilde{t}^{\,2}} + \frac{3}{4}\alpha' k_2 \sin^{-1}\left(\frac{k_2 - 2\tilde{t}}{k_2}\right) + K(r)$$

where $K(r)$ is a function of integration.

For this first case we have generated the class of exact solutions

$$A = \alpha(r)\sqrt{\frac{\tilde{t}}{k_2 - \tilde{t}}} \tag{5.4.5a}$$

$$B = \frac{3}{2}\alpha'\sqrt{k_2\tilde{t} - \tilde{t}^{\,2}} + \frac{3}{4}\alpha' k_2 \sin^{-1}\left(\frac{k_2 - 2\tilde{t}}{k_2}\right) + K(r) \tag{5.4.5b}$$

$$Y = k_1\alpha(r)\tilde{t} \tag{5.4.5c}$$

The solution (5.4.5) is a second example of an expanding, accelerating and shearing fluid for an anisotropic, radiating stellar interior. This model is similar to the result (5.3.8); however, geodesic motion cannot be regained as $A \neq 1$. The temporal dependence is fully specified and is a consequence of the choice (5.4.1c). The radial dependence of the model is arbitrary. The solution is given completely in terms of elementary functions.

5.4.2 Case 2: $\mathcal{B} = 0$

In this case we set

$$1 - \frac{\alpha'}{\alpha}\frac{\gamma}{\gamma'} = 0$$

This condition implies

$$\gamma = k_3\alpha$$

where k_3 is a constant of integration. Then the equation (5.4.2) takes the form

$$\dot{B} = \left(\frac{f}{2k_3^2\alpha'} + \frac{1}{2\alpha' f}\right)\frac{B^2}{\tilde{t}^{\,2}} - \left(\frac{\dot{f}}{\alpha' f^2}\right)\frac{B^2}{\tilde{t}} - \frac{3}{2}\alpha' f \tag{5.4.6}$$

We observe that (5.4.6) is also a nonhomogenous Riccati equation which is difficult to solve in general for arbitrary forms of $f(t)$. It may be possible to demonstrate a solution for specific choices of the function f.

5.4.3 Case 3: $\mathcal{C} = 0$

For this case we set

$$\gamma = \frac{k_4}{\alpha^2}$$

where k_4 is a constant. Then the Riccati equation (5.4.2) is reduced to an equation in the metric function B given by

$$\dot{B} - \frac{3}{2}\frac{B}{\tilde{t}} = -\left(\frac{\alpha^6 f}{4k_4^2\alpha'} + \frac{1}{4\alpha' f}\right)\frac{B^2}{\tilde{t}^{\,2}} + \left(\frac{\dot{f}}{2\alpha' f^2}\right)\frac{B^2}{\tilde{t}}$$

This is a Bernoulli equation which can be integrated to yield the result

$$B = \frac{\tilde{t}^{3/2}}{\int\left(\frac{\alpha^6 f}{4k_4^2\alpha'} + \frac{1}{4\alpha' f}\right)\frac{1}{\tilde{t}^{1/2}}dt - \int\frac{\dot{f}}{2\alpha' f^2}\tilde{t}^{1/2}dt + K(r)} \tag{5.4.7}$$

for the potential B and $K(r)$ is a constant of integration.

We have generated an infinite class of solutions in closed form given by

$$A = \alpha(r)f(t) \tag{5.4.8a}$$

$$B = \frac{\tilde{t}^{3/2}}{\int\left(\frac{\alpha^6 f}{4k_4^2\alpha'} + \frac{1}{4\alpha' f}\right)\frac{1}{\tilde{t}^{1/2}}dt - \int\frac{\dot{f}}{2\alpha' f^2}\tilde{t}^{1/2}dt + K(r)} \tag{5.4.8b}$$

$$Y = \frac{k_4}{\alpha(r)^2}\tilde{t} \tag{5.4.8c}$$

We may obtain simple forms for the metric functions. For example, if we let

$$f = \tilde{t}^{1/2}, \quad k_4 = 1 \text{ and } \quad K(r) = 0$$

in (5.4.7), we obtain

$$A = \alpha\tilde{t}^{1/2} \tag{5.4.9a}$$

$$B = \frac{4\alpha'}{\alpha^6}\tilde{t}^{1/2} \tag{5.4.9b}$$

$$Y = \gamma\tilde{t} \tag{5.4.9c}$$

as a particular solution.

5.5 Discussion

In this chapter, we have generated several classes of exact solutions for an expanding, accelerating and shearing fluid. The functional forms of the solutions are given in

(5.3.8), (5.3.11), (5.4.5), and (5.4.8). The solutions may be used to model the gravitational behaviour of a collapsing star with dissipation and shear. These new expanding, accelerating and shearing solutions represent an advancement of the theory of gravitational collapse since the first elementary dust model of Oppenheimer and Snyder (1939). The master equation which had to be solved was a nonhomogenous Riccati equation with arbitrary coefficients. In some cases, the solutions could be found in general. In other cases, specific solutions were demonstrated. It is interesting to note that in particular models, Bernoulli equations were obtainable; all such cases could be solved in general.

These solutions are of particular importance for the analysis of the physical features in dissipative gravitational collapse. This will be pursued in future work. However, here we briefly comment on the behaviour of the temperature. The simple forms for the metric functions may be used to obtain the temperature profiles in the context of irreversible causal thermodynamical theory. For a shearing superdense matter distribution, we use the Maxwell-Cattaneo heat transport equation.

$$\tau h_a^b \dot{q}_b + q_a = -\kappa(h_a^b \nabla_b T + T\dot{u}_a) \tag{5.5.1}$$

where τ is the relaxation time, κ is the thermal conductivity, and $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space. When $\tau = 0$ we regain the acausal Fourier heat transport equation. For the spacetime (5.2.1), the heat transport equation (5.5.1) may be written as

$$T = -\frac{\tau}{2A\kappa}\int (qB)^{\cdot} B dr - \frac{1}{2A\kappa}\int qB^2 dr \tag{5.5.2}$$

When $A = 1$, the temperature (5.5.2) reduces to the expression for geodesic motion. We can integrate (5.5.2) for the particular model (5.4.9) and plot the temperature profiles. In Figure 5.1, the temperature is plotted against a radial coordinate. The causal temperature (solid line) is everywhere greater than the Eckart temperature (dashed line). The temperature is a decreasing function as we approach the boundary.

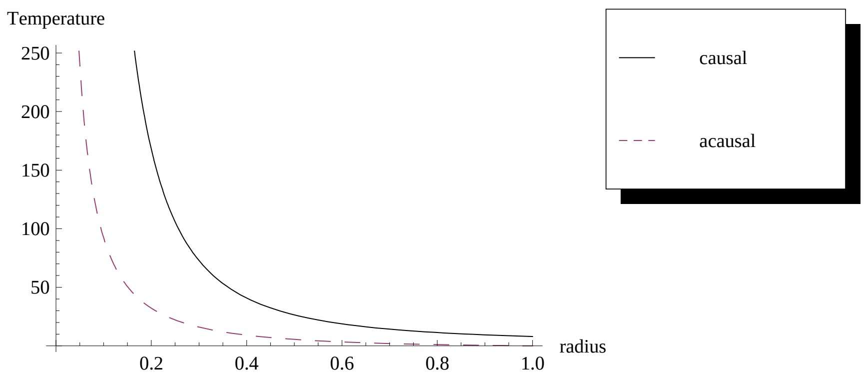

Figure 5.1: Temperature T vs radial coordinate r

Chapter 6

Conclusion

The primary objective of this thesis was to obtain new exact models which may be used to extend the physical analysis of the gravitational collapse of spherically symmetric radiating stars. Considering the complicated, nonlinear structure of the Einstein field equations which govern the process of collapse for various spacetimes, we sought to transform the master equation for the boundary condition, in each case, into a familiar form; such as the Abel equation and the Riccati equation. We then carried out a systematic investigation of the transformed governing equation. Several new classes of exact solutions, in terms of elementary functions, were generated. For the case of shear-free radiative collapse, with vanishing Weyl stresses, we obtained nonlinear models by transforming the governing equation into an Abel equation. Then, we examined the master equation for shearing collapse with dissipation for particles in geodesic motion. Our intention was to eliminate the singularities in the stellar centre contained in the models of Naidu *et al* (2006). Our investigation yielded two classes of new solutions which exhibit physically reasonable features. However, the fixed forms of the metric functions in the particular solutions obtained, restricted the study of the contribution of relaxational effects to the thermal evolution of the model. We sought to generalise the solutions found in our earlier work by allowing for greater freedom in the gravitational potentials. The governing equation was transformed into a Riccati equation. New categories of exact solutions with arbitrary spatial dependences were obtained. To the best of our knowledge, the incorporation of acceleration with expansion and shear has not produced exact models in radiative gravitational collapse before. For this general case, we transformed the master equation, governing the boundary condition, into a Riccati equation. A systematic investigation of the transformed equation yielded several new classes of exact solutions. In addition to the application of our work to analytical astrophysical investigations, we note that these exact solutions are also important for the benchmarking of algorithms utilised in numerical investigations.

We present a synopsis of the main results obtained during the course of this project:

- Our intention in Chapter 2 was to obtain new solutions for the conformally flat, shear-free model of Herrera *et al* (2004a) by retaining the nonlinearity of the boundary condition. We demonstrated the existence of several classes of solutions which may be used to realistically study the nonlinear behaviour that is intrinsic to gravitational collapse. This was effected by transforming the governing equation into an Abel equation of the first kind, given by

$$\dot{U}(\dot{C}_2 b^2 + \dot{C}_3) + U\left[\frac{3}{2}\frac{(\dot{C}_2 b^2 + \dot{C}_3)^2}{C_2 b^2 + C_3} - \frac{2}{b}(\dot{C}_2 b^2 + \dot{C}_3) - (\ddot{C}_2 b^2 + \ddot{C}_3)\right]$$

$$+2U^2\left[\frac{\dot{C}_3}{b} - \frac{1}{C_2 b^2 + C_3}(C_2^2 b^2 - \frac{C_3^2}{b^2})\right] + 2U^3 \frac{2C_2 b^2 - C_3}{C_2 b^2 + C_3}.\frac{C_3}{b^2} = 0$$

  and then systematically analysing the equation. Several classes of exact models, in terms of elementary functions, were obtained by integrating the Abel equation. It is interesting to observe that for particular parameter values, the Bernoulli equation is regained. The advantage of the solutions obtained in this chapter is that the boundary condition remains nonlinear, unlike the treatments of Maharaj and Govender (2005) and Herrera *et al* (2006).

- Our objective in Chapter 3 was to derive new models which may be utilised to describe the gravitational behaviour of shearing radiative collapse in geodesic motion. We demonstrated that the gravitational evolution of the model is governed by the equation

$$2Y\ddot{Y} + \dot{Y}^2 - \frac{Y'^2}{B^2} + \frac{2}{B}Y\dot{Y}' - 2\frac{\dot{B}}{B^2}YY' + 1 = 0$$

  We found two new exact solutions to this equation by inspection. The model of Naidu *et al* (2006) was regained as a special case. Their model had the undesirable feature of a singularity at $t = 0$ and at $r = 0$. Our models are regular at the stellar centre and are consequently physically reasonable. An interesting feature of our models was that they reduced to the Friedmann dust model, in the absence of heat flow. The simple forms of our solutions facilitated a study of the temperature of the models using causal thermodynamical theory.

- Our intention in Chapter 4 was to systematically investigate the junction condition for dissipative collapse with nonzero shear and geodesic fluid particle trajectories. The junction condition was transformed to the master equation

$$\dot{B} = \left(\frac{\ddot{Y}}{Y'} + \frac{\dot{Y}^2}{2YY'} + \frac{1}{2YY'}\right)B^2 + \frac{\dot{Y}'}{Y'}B - \frac{Y'}{2Y}$$

  which is a Riccati equation in the metric function B. We obtained two general classes of solutions in closed form. An important feature of the solutions is the arbitrary radial dependence, which is suitable for a wider study of the physical features of gravitational collapse. This lays the groundwork for future invesigations in which it may be possible to include a varying relaxation parameter in the Maxwell-Cattaneo heat equation for the causal temperature. The causal and acausal temperature profiles were generated for particular parameters. It was seen that the causal temperature is everywhere greater than the acausal temperature.

- The objective of Chapter 5 was to extend the study to the model of radiative collapse with expansion, shear and acceleration for the interior gravitating stellar fluid. The fundamental equation governing the behaviour of the model is given by

$$2Y\ddot{Y} + \dot{Y}^2 - 2\left(\frac{\dot{A}}{A} + \frac{A'}{B}\right)Y\dot{Y}$$

$$+2\frac{A}{B}Y\dot{Y}' - 2\frac{A}{B^2}(A' + \dot{B})YY' - \frac{A^2}{B^2}Y'^2 + A^2 = 0$$

We believe that this equation has not been studied before. We obtained two distinct exact models which contained several classes of solutions which are given in terms of elementary functions. This was effected by assuming particular forms for some of the metric functions and then transforming the governing equation into the familiar Riccati equation. The specific forms for the metric $A \neq 1$ do not allow the geodesic case ($A = 1$) to be recovered. It was shown that the causal temperature profile in the stellar interior is everywhere higher than the acausal temperature counterpart by considering a particular metric. The results of this chapter is the first complete treatment, including $\dot{u}^a \neq 0$, $\Theta \neq 0$, $\sigma \neq 0$, for spherically symmetric gravitational fields in radiative gravitational collapse, since Oppenheimer and Snyder (1939) first proposed the idea of the gravitational collapse of a star.

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