NOETHER'S THEOREM AND FIRST INTEGRALS OF ORDINARY DIFFERENTIAL EQUATIONS

by

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On Noether's theorem, its relationship with the Lie theory of extended groups and first integrals of ordinary differential equations

by

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Abstract

The Lie theory of extended groups is a practical tool in the analysis of differential equations, particularly in the construction of solutions. A formalism of the Lie theory is given and contrasted with Noether's theorem which plays a prominent role in the analysis of differential equations derivable from a Lagrangian. The relationship between the Lie and Noether approach to differential equations is investigated. The standard separation of Lie point symmetries into Noetherian and nonNoetherian symmetries is shown to be irrelevant within the context of nonlocality. This also emphasises the role played by nonlocal symmetries in such an approach.

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Declaration

I, Sibusiso Moyo, affirm that the material presented in this dissertation has not to my knowledge been published elsewhere except where due reference has been made in the text and that this dissertation has not been used for the award of any other degree or diploma in any university or institution.

S Moyo

November 1997

Preface

Symmetries of differential equations can be used to find first integrals. The determination of a sufficient number of first integrals is the major part of determining integrable systems. Apart from the Lie method of extended groups which is based on the invariance of the differential equations under infinitesimal transformations the 'direct' method or Noether's theorem can be used to determine first integrals. Noether's theorem is based on the invariance of the Action Integral. The relationship between Lie and Noether symmetries and the extension of the latter to a higher dimensional problem is of particular interest and has motivated this study.

Dedication

A special dedication to my parents for their constant support and to my sisters (Mary and Maureen) and my brother (Sabelo) for their love and inspiration. Finally, to Tom, I say thank you for your love and encouragement.

Contents

Chapter 1

Introduction

I think a mathematician is well suited to be in prison. S Lie

1.1 Structure of the dissertation

In this Chapter we discuss the evolution of symmetry and give a brief autobiography of Emmy Noether. The Lie theory of extended groups is introduced and its connection with Noether's theorem is stated. In Chapter 2 the Lie theory of extended groups is introduced with particular reference to examples which illustrate the different aspects of the theory. This introduces the basic concepts of Lie symmetries (which can be point, contact, generalised or nonlocal), Lie algebras and first integrals. Chapter 3 introduces Noether's theorem. The groundwork is laid with emphasis on the different aspects of the theorem. The theorem is postulated in both the Lagrangian and Hamiltonian contexts. Particular attention is paid to some developments in the literature since the enunciation of Noether's theorem in 1918. In Chapter 4 the relationship between the two methods introduced in Chapters 2 and 3 is established. The importance of nonlocal symmetries in Noether's theorem is illustrated by showing the relationship between Lie and Noether symmetries. The conclusions of the dissertation are presented in Chapter 5.

1.2 Symmetry

A symmetry can be defined as an exact correspondence in position or form about a given point, line, or plane and has its origins from the Latin word *symmetria* [12]. In other words a symmetry is an operation that leaves invariant that upon which it acts. The idea of symmetry has its applications in art, biology, chemistry and physics. The works of the Dutch designer Maurits Cornelius Escher [21] display the role played by symmetry in arts. The evolution of the idea of symmetry can be associated with two mathematicians, the German Felix Klein and the Norwegian Sophus Lie, who played a significant role in developing the necessary mathematical tools needed to explain the notion. The early provenances of mathematics involved the idea of symmetries as is observed by the first mathematical contemplation dating back to the Paleolithic period and due to the Cro-Magnons[1] [83, p v]. The importance of symmetry is also seen in the Ionic school of Thales of Miletus and the southern Italian school of the Pythagoreans. The theorems ascribed to Thales that the diameter divides a circle into congruent parts and that the angle subtended by a diameter is a right angle show that the proofs as given by the Ionic school are based on symmetry. The Lie group analysis was developed in the 1870s by Sophus Lie (1842–1899), a leading mathematician of the 19th century. Lie used group-theoretical methods to provide a classification of all ordinary differential equations of arbitrary order in terms of their symmetry groups and hence described the whole set integrable by group-theoretical methods. Historically, Lie's theory for differential equations was a match to the differential Galois theory of Vessiot and Picard [34], later developed by Ritt [76] and Kolchin [49]. A contemporary treatment of the differential Galois theory can be found in Kaplansky [47]. The concept of invariance of differential equations under infinitesimal transformations which involves various techniques makes it possible to construct solutions of differential equations. Another technique that

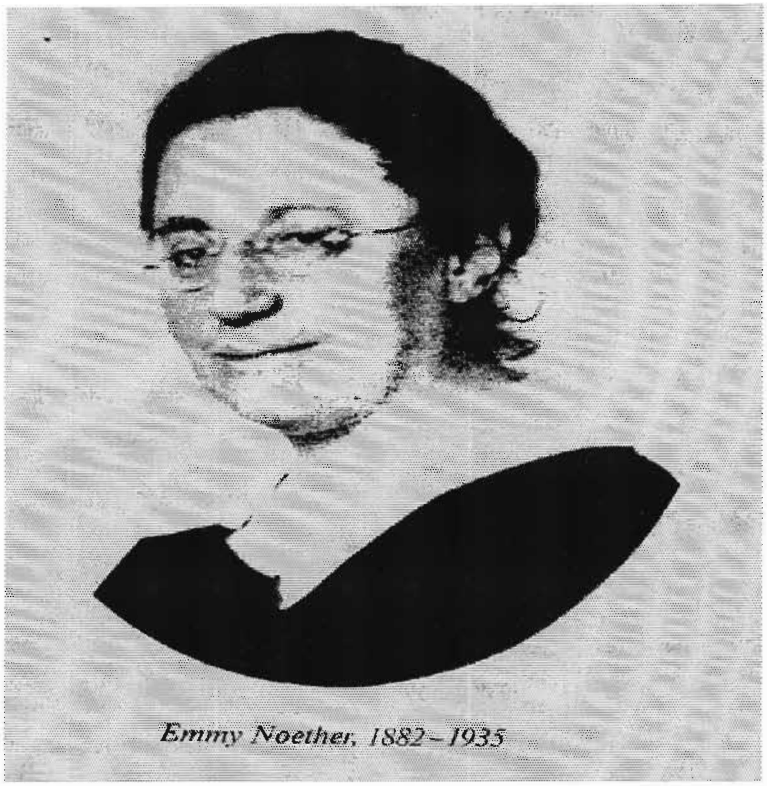
[1]The first anatomically modern humans.

stands out and is closely associated with Lie's work is that of Emmy Noether, who is currently better known for her work in abstract algebra. While Emmy Noether's work was mathematical, the applications of her work in theoretical physics led to her recognition as one of the leading women in physics [48]. Noether worked on the theory of invariants all her life, but, under the influence of Ernst Fischer, she focused on the field of algebraic invariant theory [11]. In an address by Alexandrov [41] on September 5, 1935, Emmy Noether was described as one of the foremost mathematicians of modern times.

1.3 Emmy Noether's history

Emmy Noether [41] was born in Erlangen Germany in 1882. Her father, Max Noether, was a famous mathematician and a professor at the Royal University of Erlangen. Emmy was the eldest of four children, but one of only two that survived childhood. Her brother Fritz was also a mathematician.

As a child Emmy did not concentrate on Mathematics. Her childhood was spent in school studying languages, with a concentration on French and English. She graduated from the gymnasium after passing a test that allowed her to teach both French and English at schools for young women.



Taylor [81].

Emmy Noether later decided to take classes in Mathematics at the University of Erlangen (1900–1903) where her brother Fritz was a student and her father a professor. Unfortunately because she was a woman the university refused to let Emmy Noether take classes. Instead she was permitted to audit classes. A few years later she took the exam that would permit her to be a doctoral student in Mathematics which she passed and finally became a student in good standing at the University. She spent the winter semester 1903/1904 studying in Göttingen and since the autumn of 1904 she was enrolled at Erlangen and had Fischer as one of her professors. Klein and Hilbert were her teachers in Göttingen. At Erlangen Emmy Noether studied as Gordan's student and subsequently defended her thesis on Gordan's formal computational invariant theory in 1907. She was granted the degree which was the second doctorate to be offered to a woman in the field of Mathematics.

Noether spent her early years after graduating helping her father at the Mathematics Institute in Erlangen as she could not be hired by the University of Erlangen. Noether soon began to publish papers on her work. In 1918 after the first world war in Germany, Noether and all the other women were given a right to vote for the very first time but even then she was not paid for her teaching. Felix Klein and David Hilbert were working on further defining one of Einstein's theories at the University of Göttingen. They felt that Noether's expertise could help them in their work and asked her to join them. Later she was given a job as a lecturer at the university and began receiving a salary three years later. In 1933 Noether left for the United States. She was offered a teaching position at Bryn Mawr College where she taught until her death in 1935.

Emmy Noether made many contributions to the field of mathematics. She spent her time studying abstract algebra, with special attention to rings, groups and fields. Her insight enabled her to establish useful relationships in algebra. In her lifetime she published over 40 papers. Noether's celebrated theorem of 1918 [69] states that *a dynamical system described by an action invariant under a Lie group with n parameters admits n invariants (conserved quantities) that remain constant in time during the evolution of the system.* The theorem is very general being applicable to both discrete and continuous, classical and quantum mechanical systems, although it was originally derived in the classical sense. Noether's theorem tells us explicitly how to construct conserved quantities (integral invariants) in classical field theories if the symmetry group of the differential equations of the field is known [48]. The theorem associates each element of the Lie algebra (generator of a one-parameter transformation of the group) with a corresponding conserved quantity, for instance, the association of invariance under time translation with energy conservation. Similarly, invariance under spatial translation and rotation imply the conservation of linear momentum and angular momentum respectively. This is the key to the relation of symmetry to conservations laws in Physics. In this dissertation we look at the derivation of Noether's theorem and some of its applications relating to recent developments since it was first published in 1918.

1.4 Differential operators

The differential operator

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{1.4.1}$$

where ξ and η are infinitesimal transformations of the variables x and y is said to be a symmetry of the function $f = f(x,y)$ if

$$\xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y} = 0. \tag{1.4.2}$$

The associated Lagrange's system

$$\frac{\mathrm{d}x}{\xi(x,y)} = \frac{\mathrm{d}y}{\eta(x,y)} \tag{1.4.3}$$

is used to determine the functions ξ and η. The inverse problem involves the determination of the function invariant under a given transformation. Under the infinitesimal transformation [10, p 56]

$$\bar{x} = x + \varepsilon\xi \qquad \bar{y} = y + \varepsilon\eta \tag{1.4.4}$$

the first and second derivatives transform as

$$\bar{y}' = y' + \varepsilon(\eta' - y'\xi')$$
$$\bar{y}'' = y'' + \varepsilon(\eta'' - 2y''\xi' - y'\xi'')$$

to the first order in ε. (These results have been generalised to higher order terms in the literature [10, 74].)

A differential equation

$$E(x,y,y',\ldots,y^{(n)}) = 0 \tag{1.4.5}$$

is said to admit a symmetry (1.4.1) if it remains invariant under the operation of the nth extension of G, *i.e.*

$$G^{[n]}E_{|_{E=0}} = 0. \tag{1.4.6}$$

We can then define a symmetry group of a differential equation as a transformation group that maps every solution of the equation under consideration into a solution of the same equation [55]. Once the symmetry group of a system of differential equations has been determined, the defining property of such a group can be used to construct new solutions to the system from known ones. Hence the symmetry group can be used in the classification of different symmetry classes of solutions. The concept of invariance undoubtedly plays an important role in the search for symmetries and invariants for differential equations arising from various processes and models. Both the Lie theory and Noether's theorem involve determining the symmetries of a system and then finding the corresponding invariants or constants of motion. While in the case of the Lie theory the differential equation is to be left invariant, Noether's theorem requires the invariance of the Action Integral. If a system of ordinary differential equations is derived from a variational principle, say as the Euler-Lagrange equations of some functional, then symmetry group reduction becomes more fruitful since a one-parameter group of variational symmetries allows one to reduce the order of the system by two. The different aspects of the theory will become clearer in the following chapters.

Chapter 2

Lie Group Formalism for Differential Equations

This chapter introduces some basic ideas of the classical Lie theory of extended groups as applied to differential equations with simple illustrations to clarify some of the aspects of the Lie approach.

The Lie theory of extended groups is one of the group-theoretic tools used for efficiently solving differential equations [10, p 124]. We recall here that, once the symmetry group of a differential equation is found, it can be used for many purposes [40]. The symmetry group can be used to perform symmetry reduction [10], that is, reduce the order of an ordinary differential equation or the number of variables in a partial differential equation [10]. Isometry between symmetry groups of differential equations can be used to identify equivalent equations, especially in determining whether a nonlinear equation can be linearised by a point transformation [10]. Symmetry groups can also be used to show integrability [15, 33] by Lax pair techniques. Such an approach is motivated by the fact that differential equations are increasingly important in modelling physical phenomena and other applications. Several different approaches exist in the literature [55, 56, 57, 58, 59, 71, 72].

2.1 Transformation groups

The following definitions are intrinsic to the Lie approach.

Group: A group is a set of elements G with a law of composition ϕ between the elements satisfying the following axioms [10, p 31]:

(i) **Closure property:** For any element x and y of G $\phi(x, y)$ is an element of G.

(ii) **Associative property:** For any elements x, y and z of G

$$\phi(x, \phi(y, z)) = \phi(\phi(x, y), z).$$

(iii) **Identity element:** There exists a unique identity element I of G such that, for any element x of G,

$$\phi(x, I) = \phi(I, x) = x.$$

(iv) **Inverse element:** For any element x of G there exists a unique inverse element x^{-1} in G such that

$$\phi(x, x^{-1}) = \phi(x^{-1}, x) = I.$$

Subgroup: We define a subgroup of G as a subset of G with the same law of composition ϕ.

Abelian group: A group G is Abelian if $\phi(x, y) = \phi(y, x)$ holds for all elements x and y in G.

Groups of transformations: The set of transformations

$$\bar{\mathbf{x}} = \mathbf{X}(\mathbf{x}; \varepsilon)$$

defined for each $\mathbf{x}$ in the space $D \subset R^n$, depending on the parameter ε lying in the set $S \subset R$ with $\phi(\varepsilon, \delta)$ defining a law of composition of parameters ε and δ in S, forms a group of transformations on D if:

(i) For each parameter ε in S the transformations are one-to-one onto D. In particular $\bar{\mathbf{x}}$ lies in D.

(ii) S with the law of composition ϕ forms a group G.

(iii) $\bar{\mathbf{x}} = \mathbf{x}$ when $\varepsilon = I$, *i.e.*

$$\mathbf{X}(\mathbf{x}; I) = \mathbf{x}.$$

(iv) If $\bar{\mathbf{x}} = \mathbf{X}(\mathbf{x};\varepsilon)$ and $\bar{\bar{\mathbf{x}}} = \mathbf{X}(\bar{\mathbf{x}};\delta)$, then

$$\bar{\bar{\mathbf{x}}} = \mathbf{X}(\mathbf{x}; \phi(\varepsilon, \delta)).$$

One-parameter Lie group of transformations: A **one-parameter Lie group of transformations** is a group of transformations which satisfies the additional conditions:

(i) ε is a continuous parameter, *i.e.* S is an interval in R. Without loss of generality $\varepsilon = 0$ corresponds to the identity element I.

(ii) $\mathbf{X}$ is infinitely differentiable with respect to $\mathbf{x}$ in D and an analytic function of ε in S.

(iii) $\phi(\varepsilon, \delta)$ is an analytic function of ε and δ, $\varepsilon \in S$, $\delta \in S$.

The group consisting of all nonsingular complex $n \times n$ matrices is called **complex general linear group** $GL(n, C)$ and the **real general linear group** $GL(n, R)$ comprises all nonsingular real $n \times n$ matrices. The $GL(n, R)$ is a subgroup of the $GL(n, C)$. The **complex special linear group** $SL(n, C)$ is the subgroup of $GL(n, C)$ consisting of matrices with determinant one. The **real special linear group** $SL(n, R)$ is the intersection of these two subgroups

$$SL(n, R) = SL(n, C) \cap GL(n, R).$$

Rotation group: The **rotation group** $SO(n, R)$ is the special or proper real orthogonal group given by the intersection of the group of orthogonal matrices $O(n, R)$ and the complex special linear group, *i.e.*

$$SO(n, R) = O(n, R) \cap SL(n, C).$$

Lie algebra: We define a Lie bracket (commutator) $[x, y]$ of operators x and y by the formula

$$[x, y] = xy - yx.$$

A vector space $\mathcal{L}$ on which is defined a commutator $[x, y]$ that

(i) is BILINEAR, *i.e.*

$$[x, k_1 y + k_2 z] = k_1[x, y] + k_2[x, z] \qquad [k_1 y + k_2 z, x] = k_1[y, x] + k_2[z, x]$$

(ii) is ANTISYMMETRIC

$$[x, y] = -[y, x]$$

(iii) satisfies the JACOBI IDENTITY

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all vectors x, y, z in $\mathcal{L}$ is said to be a **Lie algebra**.

Abelian algebra: A Lie algebra $\mathcal{L}$ is said to be **Abelian** (equivalently commutative) [83, p 214] if $[x, y] = 0 \,\forall\, x, y \in \mathcal{L}$.

Solvable algebra: A Lie algebra $\mathcal{L}$ is called **solvable** if the derived series

$$\begin{aligned} \mathcal{L} \supseteq \mathcal{L}' &= [\mathcal{L}, \mathcal{L}] \\ \supseteq \mathcal{L}'' &= [\mathcal{L}', \mathcal{L}'] \\ \supseteq \cdots & \\ \supseteq \mathcal{L}^{(k)} &= [\mathcal{L}^{(k-1)}, \mathcal{L}^{(k-1)}] \end{aligned}$$

is such that $\mathcal{L}^{(k)} = 0, k > 0$. Every Abelian algebra and two-dimensional Lie algebra is solvable.

Remark: A Lie algebra is usually defined over real and complex fields. A differential equation that admits the operators x and y also admits their commutator $[x, y]$. A continuous Lie group can be assigned a corresponding Lie algebra. For example the real special linear group $SL(n, R)$ has the corresponding Lie algebra $s\ell(n, R)$.

2.2 Lie Analysis Algorithm

This section formalises the Lie analysis of differential equations. More detailed accounts can be found in [8, 10, 16, 46, 55, 70, 72, 79]. The algorithm for finding the Lie group of point transformations leaving a system of differential equations invariant has been computerised. `PROGRAM LIE` by Head [42] can be used to find the Lie symmetries and corresponding Lie brackets. Suppose that an nth order ordinary differential equation

$$E(x, y, y', \dots, y^{(n)}) = 0 \tag{2.2.1}$$

admits the one-parameter Lie group of point transformations

$$\bar{x} = x + \varepsilon\xi \tag{2.2.2}$$

$$\bar{y} = y + \varepsilon\eta \tag{2.2.3}$$

with infinitesimal generator

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}. \tag{2.2.4}$$

Then equation (1.4.6) given by

$$G^{[n]}E\big|_{E=0} = 0 \tag{2.2.5}$$

is satisfied where $G^{[n]}$ is the nth extension of G needed to transform the derivatives in (2.2.1) given by [62]

$$G^{[n]} = G + \sum_{i=1}^{n}\left\{\eta^{(i)} - \sum_{j=0}^{i-1}\begin{pmatrix} i \\ j \end{pmatrix} y^{(j+1)}\xi^{(i-j)}\right\}\frac{\partial}{\partial y^{(i)}}, \tag{2.2.6}$$

where the indices in (2.2.6) denote total differentiation. For partial differential equations and systems, ξ, η, x and y are given relevant subscripts. We say that (2.2.1) possesses the symmetry (group generator) (2.2.4) iff (2.2.5) holds. For point symmetries the coefficient functions ξ and η depend on x and y only. In that case the operation of (2.2.6) on (2.2.1) produces an overdetermined system of linear partial differential equations, the solution of which gives ξ and η. Consider as an example the equation

$$y'' = y^{-3}. \tag{2.2.7}$$

Since

$$G^{[2]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + (\eta' - y'\xi')\frac{\partial}{\partial y'} + (\eta'' - 2y''\xi' - y'\xi'')\frac{\partial}{\partial y''}, \tag{2.2.8}$$

its operation on (2.2.7) leads to the equation

$$\eta'' - 2y''\xi' - y'\xi'' + 3\eta y^{-4} = 0 \tag{2.2.9}$$

which has to be solved subject to the condition (2.2.7). The consequence of this is the expansion

$$\begin{aligned} &\frac{\partial^2\eta}{\partial x^2} + 2y'\frac{\partial^2\eta}{\partial x\partial y} + y'^2\frac{\partial^2\eta}{\partial y^2} + y^{-3}\frac{\partial\eta}{\partial y} - 2y^{-3}\frac{\partial\xi}{\partial x} \\ -2y^{-3}y'\frac{\partial\xi}{\partial y} - y'&\left(\frac{\partial^2\xi}{\partial x^2} + 2y'\frac{\partial^2\xi}{\partial x\partial y} + y'^2\frac{\partial^2\xi}{\partial y^2} + y^{-3}\frac{\partial\xi}{\partial y}\right) \\ &+3\eta y^{-4} = 0. \end{aligned} \tag{2.2.10}$$

Since ξ and η are functions of x and y only we can separate by different powers of y' to obtain a system of linear partial differential equations. Thus the equation (2.2.10) becomes

$$\begin{aligned} y'^3: &\quad \frac{\partial^2\xi}{\partial y^2} &= 0 \\ y'^2: &\quad \frac{\partial^2\eta}{\partial y^2} - 2\frac{\partial^2\xi}{\partial x\partial y} &= 0 \\ y'^1: &\quad 2\frac{\partial^2\eta}{\partial x\partial y} - \frac{3}{y^3}\frac{\partial\xi}{\partial y} - \frac{\partial^2\xi}{\partial x^2} &= 0 \\ y'^0: &\quad \frac{\partial^2\eta}{\partial x^2} + \frac{1}{y^3}\frac{\partial\eta}{\partial y} - \frac{2}{y^2}\frac{\partial\xi}{\partial x} &= -\frac{3\eta}{y^4}. \end{aligned} \tag{2.2.11}$$

We solve (2.2.11) to obtain

$$\begin{aligned} \xi &= A_0 + A_1x + A_2x^2 \\ \eta &= \left(\tfrac{1}{2}A_1 + A_2x\right)y \end{aligned} \tag{2.2.12}$$

which gives the three symmetries of equation (2.2.7) as

$$\begin{aligned} G_1 &= \frac{\partial}{\partial x} \\ G_2 &= 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ G_3 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \end{aligned} \tag{2.2.13}$$

with corresponding Lie brackets

$$\begin{aligned} [G_1, G_2] &= 2G_1 \\ [G_1, G_3] &= G_2 \\ [G_2, G_3] &= 2G_3. \end{aligned}$$

The Lie bracket relations of the symmetries (2.2.13) form the Lie algebra $s\ell(2, R)$ [62].

2.3 Reduction of order

Symmetries of a differential equation can be used to reduce the order of the equation. If an equation

$$E = E(x, y, y', ...y^{(n)}) \tag{2.3.1}$$

is invariant under the symmetry

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}, \tag{2.3.2}$$

then the variables for the reduction of order are obtained by requiring

$$G^{[1]}E = 0, \tag{2.3.3}$$

where E is an arbitrary function of its arguments. The operation (2.3.3) results in the equation

$$\xi\frac{\partial E}{\partial x} + \eta\frac{\partial E}{\partial y} + (\eta' - y'\xi')\frac{\partial E}{\partial y'} = 0 \tag{2.3.4}$$

which has the associated Lagrange's system

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}y}{\eta} = \frac{\mathrm{d}y'}{\eta' - y'\xi'}. \tag{2.3.5}$$

The integration of the first two terms gives the differential invariant of zeroth order denoted by u and that of the second and third terms the first order differential invariant denoted by v.

To illustrate this consider the equation [80]

$$y^{iv} + y'y'' - yy''' = 0. \tag{2.3.6}$$

This equation has two Lie point symmetries

$$\begin{aligned} G_1 &= \frac{\partial}{\partial x} \\ G_2 &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \end{aligned} \tag{2.3.7}$$

with Lie bracket

$$[G_1, G_2] = G_1. \tag{2.3.8}$$

In the case of G_1 (2.3.3) leads to

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}y'}{0} \tag{2.3.9}$$

from which

$$u = y \qquad v = y'. \tag{2.3.10}$$

To find the reduced equation we substitute the following terms

$$\begin{aligned} vv' &= y'' \\ vv'^2 + v^2v'' &= y''' \\ v^3v''' + 4v'v^2v'' + vv'^3 &= y^{iv}, \end{aligned} \tag{2.3.11}$$

into the original equation which leads to the reduced equation

$$v^3v''' + 4v'v^2v'' + vv'^3 + v^2v' - u\left(vv'^2 + v^2v''\right) = 0. \tag{2.3.12}$$

Direct integration of the reduced equation leads to

$$I = v^2v'' + vv'^2 + v^2 - uvv'. \tag{2.3.13}$$

In this case the reduced equation is not easily solved. The existence of a symmetry does not guarantee that one can find an explicit expression for the solution of the reduced equation.

Consider equation (2.2.7) with the three point symmetries (2.2.13)

$$\begin{aligned} G_1 &= \frac{\partial}{\partial x} \\ G_2 &= 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ G_3 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \end{aligned}$$

In the reduction via G_1 we obtain the associated Lagrange's system [10]

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}y'}{0} \tag{2.3.14}$$

from which u and v are given as

$$u = y \qquad\qquad v = y'. \tag{2.3.15}$$

Equation (2.2.7) therefore reduces to

$$v' = u^{-3}/v, \tag{2.3.16}$$

where

$$\begin{aligned} v' &= \frac{\mathrm{d}v}{\mathrm{d}u} \\ &= \frac{\mathrm{d}v/\mathrm{d}x}{\mathrm{d}u/\mathrm{d}x} \\ &= \frac{y''}{y'} \\ &= \frac{y^{-3}}{y'} \\ &= \frac{u^{-3}}{v} \end{aligned} \tag{2.3.17}$$

and the variables are separable. Similarly G_2 has the characteristics

$$u = y/x^{\frac{1}{2}} \qquad\qquad v = y'x^{\frac{1}{2}} \tag{2.3.18}$$

obtained from

$$\frac{\mathrm{d}x}{2x}=\frac{\mathrm{d}y}{y}=-\frac{\mathrm{d}y'}{y'}. \tag{2.3.19}$$

Thus equation (2.2.7) becomes

$$\frac{\mathrm{d}v}{\mathrm{d}u}=\frac{u^{-3}+\frac{1}{2}v}{v-\frac{1}{2}u}, \tag{2.3.20}$$

which is an Abel's equation of the second kind. G_3 results in the characteristics

$$u=\frac{y}{x} \qquad v=xy'-y. \tag{2.3.21}$$

The reduced equation is now

$$\frac{\mathrm{d}v}{\mathrm{d}u}=\frac{u^{-3}}{v}, \tag{2.3.22}$$

which has separable variables.

In general, if $[G_1, G_2]=\lambda G_1$, reduction by G_1 will result in the descendant of G_2 being a point symmetry of the reduced equation. On the other hand reduction by G_2 will result in the loss of a descendant of G_1 as a point symmetry of the reduced equation [70].

It is also important to mention that it is possible to determine the finite (global) transformation from the generator of the infinitesimal transformation

$$G=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y} \tag{2.3.23}$$

by integrating

$$\frac{\mathrm{d}\bar{x}}{\xi}=\frac{\mathrm{d}\bar{y}}{\eta}=\mathrm{d}a \tag{2.3.24}$$

subject to the initial conditions

$$\bar{x}=x \qquad \bar{y}=y \tag{2.3.25}$$

at $a=0$. If $G_1=\frac{\partial}{\partial x}$ and $G_2=x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}$, then we have in the case of G_1 and G_2 the finite transformations

$$\bar{x}=x+a \qquad \bar{y}=y \tag{2.3.26}$$

and

$$\bar{x}=x\mathrm{e}^{-a} \qquad \bar{y}=y\mathrm{e}^{a} \tag{2.3.27}$$

respectively which can be combined to show that (2.3.6) is invariant under the finite transformations

$$\bar{x} = (x + a)\mathrm{e}^{-a} \qquad \bar{y} = y\mathrm{e}^{a}. \tag{2.3.28}$$

2.4 First integrals

One of the utilities of symmetries is to determine first integrals (which can be interpretted as conserved quantities such as energy and momentum in Classical Mechanics) of systems of equations. The differential equation (2.2.1) possesses a first integral

$$I = f(x, y, y', \ldots, y^{(n-1)}) \tag{2.4.1}$$

in which the dependence on $y^{(n-1)}$ is nontrivial if

$$\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{E=0} = 0. \tag{2.4.2}$$

To calculate a first integral, I, associated with a symmetry, G, two linear partial differential equations need to be solved. The first condition ensures that the integral satisfies the requirement of annihilation under the action of the $(n-1)$th extension of the symmetry and the second its vanishing total derivative with respect to the independent variable, x. If the first integral is (2.4.1), we require

$$G^{[n-1]}f = 0. \tag{2.4.3}$$

There are n characteristics of (2.4.3) which we denote by u_i, $i = 1, \ldots, n$ and f is given as

$$f = g(u_i). \tag{2.4.4}$$

The second partial differential equation results from the further requirement

$$\frac{\mathrm{d}g}{\mathrm{d}x}\Big|_{E=0} = 0 \tag{2.4.5}$$

for which there are $n-1$ characteristics denoted by v_i, $i = 1, \ldots, n-1$. The first integral is now

$$I = h(v_i), \tag{2.4.6}$$

where h is an arbitrary function of its arguments. In particular each of the v_i is a first integral. In some cases it is favourable to take combinations of first integrals. The equation (2.3.6) has a first integral

$$I = f(x, y, y', y'', y''') \tag{2.4.7}$$

if

$$G^{[3]} I \Big|_{y^{iv} + y'y'' - yy''' = 0} = 0. \tag{2.4.8}$$

Now G_1 becomes

$$G_1^{[3]} = \frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + 0\frac{\partial}{\partial y'} + 0\frac{\partial}{\partial y''} + 0\frac{\partial}{\partial y'''}. \tag{2.4.9}$$

Therefore we have by condition (2.4.3) that

$$u_1 = y, \quad u_2 = y', \quad u_3 = y'', \quad u_4 = y'''. \tag{2.4.10}$$

Hence by condition (2.4.5) we have that

$$u_2\frac{\partial I}{\partial u_1} + u_3\frac{\partial I}{\partial u_2} + u_4\frac{\partial I}{\partial u_3} + (u_1u_4 - u_2u_3)\frac{\partial I}{\partial u_4} = 0. \tag{2.4.11}$$

The associated Lagrange's system is given by

$$\frac{\mathrm{d}u_1}{u_2} = \frac{\mathrm{d}u_2}{u_3} = \frac{\mathrm{d}u_3}{u_4} = \frac{\mathrm{d}u_4}{u_1u_4 - u_2u_3} \tag{2.4.12}$$

so that

$$\frac{2u_2\mathrm{d}u_2}{2u_2u_3} = \frac{-u_1\mathrm{d}u_3}{-u_1u_4} = \frac{u_4}{u_1u_4 - u_2u_3} = \frac{-u_3\mathrm{d}u_1}{-u_3u_2} \tag{2.4.13}$$

which leads to

$$\frac{2u_2\mathrm{d}u_2 - u_1\mathrm{d}u_3 + \mathrm{d}u_4 - u_3\mathrm{d}u_1}{0}. \tag{2.4.14}$$

Therefore we have the first integral $I = g(p)$, where

$$p = u_2^2 - u_1u_3 + u_4. \tag{2.4.15}$$

A similar procedure can be followed for G_2.

2.5 Generalised and Contact Symmetries

We discuss the concept of generalised and contact symmetries. We started with Lie point symmetries of the form

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}, \tag{2.5.1}$$

where the coefficient functions ξ and η depended on x and y only. In the case of generalised symmetries we extend the dependence of the coefficient functions to derivatives of the dependent variable. Generalised symmetries have the form [70, p 288]

$$G = \xi(x, y, y', \ldots)\frac{\partial}{\partial x} + \sum_{i=0}^{n}\eta_i(x, y, y', \ldots)\frac{\partial}{\partial y^{(i)}}. \tag{2.5.2}$$

The other term used for symmetries of the form (2.5.2) is Lie-Bäcklund symmetries [8, 10]. That the transition from contact symmetries to the consideration of generalised symmetries was not made by both Lie and Bäcklund is mentioned by Olver [70, p 286].

The procedure for the determination of generalised symmetries is basically the same as that of point symmetries except for its degree of complexity which arises due to the dependence of the coefficient functions. The dependence of the coefficient functions can go up to the nth derivative of y. In practice the determination of these symmetries is not easy and hence one has to place a restriction on the order of the derivatives. Examples of generalised symmetries can be found in [70, p 295]. A case of these is the fifth order generalised symmetries of the Korteweg-de Vries equation [10]

$$u_t + uu_x + u_{xxx} = 0. \tag{2.5.3}$$

In the determination of generalised symmetries great simplification is achieved if the coefficient functions become free of time derivatives making it possible to proceed with the usual technique for point symmetries. Such an approach of replacing time derivatives is applicable to both partial differential equations and ordinary differential equations [7].

Contact symmetries are a subset of generalised symmetries. These are important in the study of higher order ordinary differential equations [60].[1] A scalar nth order ordinary differential equation

$$E(x, y, y', \ldots, y^{(n)}) = 0 \tag{2.5.4}$$

possesses the contact symmetry

$$G = \xi(x, y, y')\frac{\partial}{\partial x} + \eta(x, y, y')\frac{\partial}{\partial y} + \zeta(x, y, y')\frac{\partial}{\partial y'} \tag{2.5.5}$$

provided there exists a characteristic function W [60] such that

$$\xi = \frac{\partial W}{\partial y'} \tag{2.5.6}$$

$$\eta = y'\frac{\partial W}{\partial y'} - W \tag{2.5.7}$$

$$\zeta = -\frac{\partial W}{\partial x} - y'\frac{\partial W}{\partial y}. \tag{2.5.8}$$

Note that W is defined by [10]

$$W = \xi y' - \eta. \tag{2.5.9}$$

From (2.5.6)–(2.5.8) we have that

$$\frac{\partial \eta}{\partial y'} - y'\frac{\partial \xi}{\partial y'} = 0. \tag{2.5.10}$$

`PROGRAM LIE` [42] can be used to obtain point, contact and generalised symmetries. Second order ordinary differential equations have an infinite number of contact symmetries [60]. These symmetries have a form and relationship to first integrals which is of some considerable interest [63], but their existence for second order equations is analogous to that of point symmetries for first order equations. The role of contact symmetries is seen by their application in third order equations [5, 6, 26, 27].

[1] The link between contact transformations and the theory of waves has been pointed out in [46, 60].

2.6 Nonlocal symmetries

Nonlocal transformations are infinitesimal transformations whose coefficient functions depend on integrals of the dependent and independent variables. The nonlocal symmetries obtained enlarge the class of symmetries and are connected with integrable models [3]. The algorithm found in [37] essentially involves finding a 'useful' nonlocal symmetry for a differential equation which has one Lie point symmetry. The symmetry is said to be useful if, when the order of the equation is reduced using the existing point symmetry, the nonlocal symmetry becomes a Lie point symmetry. This means that the nonlocal symmetry becomes a Type II hidden symmetry for the reduced equation [1, 2, 3, 4]. The equation

$$F(y, y', y'') = y'' - g(y, y') = 0 \tag{2.6.1}$$

with sole Lie point symmetry

$$G_1 = \frac{\partial}{\partial x} \tag{2.6.2}$$

was analysed for the existence of nonlocal symmetries in [37]. In the case of (2.6.1) having two point symmetries the Lie bracket relationship

$$[G_1, G_2] = \lambda G_1 \tag{2.6.3}$$

guarantees G_2 as a point symmetry of the reduced equation. If G_1 is as defined in (2.6.2) and

$$G_2 = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2.6.4}$$

equation (2.6.3) implies that

$$\frac{\partial \xi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial y} = \lambda\frac{\partial}{\partial x}. \tag{2.6.5}$$

Therefore

$$\xi = \lambda x + k(y) \qquad \eta = a(y) \tag{2.6.6}$$

and G_2 takes the form

$$G_2 = (\lambda x + k(y))\frac{\partial}{\partial x} + a(y)\frac{\partial}{\partial y}. \tag{2.6.7}$$

The reduction of (2.6.1) by G_1 *viz.*

$$u = y \qquad v = y' \tag{2.6.8}$$

results in the first-order equation

$$vv' = g(u, v). \tag{2.6.9}$$

By the use of (2.6.8) and the first extension of (2.6.7), G_2 reduces to

$$X = a(u)\frac{\partial}{\partial u} + v(a'(u) - \lambda - k'(u)v)\frac{\partial}{\partial v}. \tag{2.6.10}$$

The structure of a nonlocal symmetry is

$$G_{nl} = \xi(x, y, I)\frac{\partial}{\partial x} + \eta(x, y, I)\frac{\partial}{\partial y}, \tag{2.6.11}$$

where

$$I = \int f(x, y)\mathrm{d}x \tag{2.6.12}$$

and by the use of (2.6.10) we conclude that

$$\eta = a(y) \qquad \xi = \lambda x + k(y, I). \tag{2.6.13}$$

This comes from the requirement that

$$[G_1, G_{nl}] = \lambda G_1 \tag{2.6.14}$$

and the nonlocal symmetry of (2.6.1) is now

$$G_{nl} = (\lambda x + k(y, I))\frac{\partial}{\partial x} + a(y)\frac{\partial}{\partial y}. \tag{2.6.15}$$

The requirement that ξ be free of y' and ξ' be free of x and I gives

$$G_{nl} = (\lambda x + I)\frac{\partial}{\partial x} + a(y)\frac{\partial}{\partial y}, \tag{2.6.16}$$

where

$$I = \int c(y)\mathrm{d}x. \tag{2.6.17}$$

The coefficient functions in (2.6.16) are determined by using the requirement that

$$G_{nl}{}^{[2]}F\Big|_{F=0} = 0 \tag{2.6.18}$$

and solving the associated Lagrange's system [37] for g to give

$$g = e^{-\int \phi \mathrm{d}y}\left[\int \Phi e^{\int \phi \mathrm{d}y}\mathrm{d}y + L(u)\right], \tag{2.6.19}$$

where

$$\phi = \frac{a' - 2(\lambda + c)}{a} \tag{2.6.20}$$

$$\Phi = \frac{a(c' - a'')}{u^2}\exp\left(-2\int \frac{\lambda + c}{a}\mathrm{d}y\right) \tag{2.6.21}$$

$$u = \frac{a}{y'}\exp\left(-\int \frac{\lambda + c}{a}\mathrm{d}y\right) \tag{2.6.22}$$

and L is an arbitrary function of its argument. Nonlocal symmetries can be used in the context of Noether's theorem to produce useful first integrals [36].

Chapter 3

Noether's Theorem

In this chapter we introduce Noether's theorem. The theorem was enunciated by Emmy Noether in her paper of 1918 [69] on invariant variational problems, but there have been some developments since the paper. There have been a wide range of contributions to the subject aimed at generalising some aspects of the original theorem, re-examining its methodological aspects or, sadly to relate, reducing it.[1] For a detailed review of the theorem see Sarlet and Cantrijn [1981]. We highlight some aspects of the theorem and illustrate them by means of specific examples. We give a brief introduction to the Calculus of Variations to aid us in the subsequent analysis.

[1]The interpretation of what a Noetherian integral is by Garcia-Sucre *et al* [31] is an instance of a reduction in the theorem. Flessas *et al* [25] clarified what a Noetherian integral is and pointed out that velocity-dependent transformations were in fact introduced by Noether in her paper. This is contrary to the claim that the use of velocity-dependent transformations was a development of Noether's theorem subsequent to the formulation of it in the 1918 *festschrift* for Felix Klein on the occasion of the fiftieth anniversary of his doctorate [69]. In yet another paper by Flessas *et al* [24] is the correction of the impression that Lagrangian systems possess first integrals of motion which are nonNoetherian. It has been decades since Emmy Noether's 1918 paper and yet the differences in opinion as to what is to be termed a Noether invariant still persist.

3.1 Calculus of variations

Let A be a functional defined by

$$A = \int_{x_0}^{x_1} \mathcal{L}(x, y, y')\mathrm{d}x, \tag{3.1.1}$$

where x is the independent variable, y the dependent variable and $\mathcal{L}$ is an analytic function of x, y and y'. The value of A depends upon the functional dependence of y upon x. The problem is to find $y = y(x)$ such that A takes a stationary value. Suppose that y is varied infinitesimally as

$$\bar{y} = y + \varepsilon\eta(x), \tag{3.1.2}$$

where ε is the infinitesimal parameter. Then the variation of A is

$$\delta A = \int_{x_0}^{x_1} \mathcal{L}(x, \bar{y}, \bar{y}')\mathrm{d}x - \int_{x_0}^{x_1} \mathcal{L}(x, y, y')\mathrm{d}x, \tag{3.1.3}$$

where $\eta(x_0) = 0$ and $\eta(x_1) = 0$ (that is, there is no variation at the endpoints; this ensures that the variation in y does not affect the endpoints). By Taylor's expansion we have

$$\begin{aligned}\mathcal{L}(x, \bar{y}, \bar{y}') &= \mathcal{L}(x, y + \varepsilon\eta, y' + \varepsilon\eta') \\ &= \mathcal{L}(x, y, y') + \varepsilon\left(\eta\frac{\partial \mathcal{L}}{\partial y} + \eta'\frac{\partial \mathcal{L}}{\partial y'}\right) + O(\varepsilon^2),\end{aligned} \tag{3.1.4}$$

where the higher order terms are neglected since ε is an infinitesimal. Therefore equation (3.1.3) becomes

$$\delta A = \int_{x_0}^{x_1} \varepsilon\left(\eta\frac{\partial \mathcal{L}}{\partial y} + \eta'\frac{\partial \mathcal{L}}{\partial y'}\right)\mathrm{d}x. \tag{3.1.5}$$

This expression is not accessible to further analysis in its present form because $\eta(x)$ and $\eta'(x)$ are not necessarily independent of each other. To remove this difficulty we apply the method of integration by parts. This gives us

$$\begin{aligned}\int_{x_0}^{x_1}\left(\eta\frac{\partial \mathcal{L}}{\partial y} + \eta'\frac{\partial \mathcal{L}}{\partial y'}\right)\mathrm{d}x &= \int_{x_0}^{x_1} \eta\frac{\partial \mathcal{L}}{\partial y}\mathrm{d}x + \left(\eta\frac{\partial \mathcal{L}}{\partial y'}\right)\Bigg|_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial y'}\mathrm{d}x \\ &= \int_{x_0}^{x_1} \eta\left(\frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial y'}\right)\mathrm{d}x.\end{aligned} \tag{3.1.6}$$

The middle term vanishes since $\eta(x_0) = \eta(x_1) = 0$. For A to take a stationary value we must have that $\delta A = 0$, *i.e.* equation (3.1.3) can be rewritten as

$$\varepsilon \int_{x_0}^{x_1} \eta \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathcal{L}}{\partial y'} \right) \mathrm{d}x = 0. \tag{3.1.7}$$

Since η is an arbitrary C^1 function, subject to $\eta(x_0) = 0 = \eta(x_1)$, this means that

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathcal{L}}{\partial y'} = 0. \tag{3.1.8}$$

This equation is called the Euler-Lagrange equation of the Calculus of Variations. It is important to mention here that the equations follow from the requirement that $\delta A = 0$ while the nature of the stationary value is not specified.[2] The nature of the stationary value is determined by the second order term in the variation. This term is given by

$$\delta^2 A = \varepsilon^2 \int_{x_0}^{x_1} \left(\eta^2 \frac{\partial^2 \mathcal{L}}{\partial y^2} + 2\eta\eta' \frac{\partial^2 \mathcal{L}}{\partial y \partial y'} + \eta'^2 \frac{\partial^2 \mathcal{L}}{\partial y'^2} \right) \mathrm{d}x. \tag{3.1.9}$$

The sign of $\delta^2 A$ depends upon the nature of the Hessian matrix

$$\begin{pmatrix} \dfrac{\partial^2 \mathcal{L}}{\partial y^2} & \dfrac{\partial^2 \mathcal{L}}{\partial y \partial y'} \\[2ex] \dfrac{\partial^2 \mathcal{L}}{\partial y \partial y'} & \dfrac{\partial^2 \mathcal{L}}{\partial y'^2} \end{pmatrix}.$$

If it is positive definite, then the stationary value is a minimum. It is a maximum if the Hessian is negative definite. A saddle point occurs if the matrix is indefinite.

For a functional

$$A = \int_{x_0}^{x_1} \mathcal{L}(x, y, y', y'')\mathrm{d}x \tag{3.1.10}$$

[2]Hamilton's Principle requires that the variation in the functional be zero under zero endpoint variation. Thus η is chosen to conform to Hamilton's principle. Marpetius believed that it was the functional A which nature sought to minimise and wrote [18] 'It is the quantity of action which is Nature's storehouse, and which it economises as much as possible in the motion of light.'

the Euler-Lagrange equation can be determined using the same methodology. Let y be transformed as in (3.1.2), that is

$$\bar{y} = y + \varepsilon\eta(x), \tag{3.1.11}$$

where ε is the parameter of smallness and $\eta(x)$ is an arbitrary C^2 function and $\eta'(x_0) = 0$ and $\eta'(x_1) = 0$ (the functions $y(x)$ and $\bar{y}(x)$ are tangential at the endpoints). This means that the variation in A becomes

$$\begin{aligned}
\delta A &= \int_{x_0}^{x_1} \mathcal{L}(x, \bar{y}, \bar{y}', \bar{y}'')\mathrm{d}x - \int_{x_0}^{x_1} \mathcal{L}(x, y, y', y'')\mathrm{d}x \\
&= \int_{x_0}^{x_1} \mathcal{L}(x, y + \varepsilon\eta, y' + \varepsilon\eta', y'' + \varepsilon\eta'')\mathrm{d}x - \int_{x_0}^{x_1} \mathcal{L}(x, y, y', y'')\mathrm{d}x \\
&= \int_{x_0}^{x_1} \varepsilon \left(\eta\frac{\partial\mathcal{L}}{\partial y} + \eta'\frac{\partial\mathcal{L}}{\partial y'} + \eta''\frac{\partial\mathcal{L}}{\partial y''}\right)\mathrm{d}x. \qquad (3.1.12)
\end{aligned}$$

Application of integration by parts twice leads to

$$\begin{aligned}
\delta A = \varepsilon &\left(\int_{x_0}^{x_1} \eta\frac{\partial\mathcal{L}}{\partial y}\mathrm{d}x + \left(\eta\frac{\partial\mathcal{L}}{\partial y'}\right)\Bigg|_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\mathcal{L}}{\partial y'}\right)\mathrm{d}x\right) \\
+\varepsilon &\left(\left(\eta'\frac{\partial\mathcal{L}}{\partial y''}\right)\Bigg|_{x_0}^{x_1} - \left(\eta\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\mathcal{L}}{\partial y''}\right)\right)\Bigg|_{x_0}^{x_1} + \int_{x_0}^{x_1} \eta\frac{d^2}{dx^2}\left(\frac{\partial\mathcal{L}}{\partial y''}\right)\mathrm{d}x\right).
\end{aligned}$$

Therefore

$$\delta A = \varepsilon\int_{x_0}^{x_1} \eta\left(\frac{\partial\mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial\mathcal{L}}{\partial y''}\right). \tag{3.1.13}$$

Since we require that $\delta A = 0$, this means that

$$\frac{\partial\mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial\mathcal{L}}{\partial y''} = 0. \tag{3.1.14}$$

In general for an nth order Lagrangian of the form

$$\mathcal{L} = \mathcal{L}(x, y, y', ..., y^{(n)}) \tag{3.1.15}$$

the corresponding Euler-Lagrange equation is given by

$$\frac{\partial\mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial\mathcal{L}}{\partial y''} - \ldots + (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{\partial\mathcal{L}}{\partial y^{(n)}} = 0. \tag{3.1.16}$$

3.2 The classical Noether's theorem

We introduce the formulation of the classical Noether's theorem. Many other versions appear in text books. For a sampling of these versions see [9, 10, 14, 70].

In formulating the theorem we start by considering an infinitesimal transformation in the (x, y)-space defined by

$$\bar{x} \longrightarrow x + \varepsilon\xi \tag{3.2.1}$$

$$\bar{y} \longrightarrow y + \varepsilon\eta, \tag{3.2.2}$$

where ξ and η are functions of x and y only (these can be chosen to be velocity dependent as was the case in Noether's paper). We therefore have that each curve $x \longrightarrow y(x)$ defined on the interval $[a, b]$ is transformed into a parameter dependent curve $\bar{x} \longrightarrow \bar{y}(\bar{x})$ in the new variables. The transformation in the first derivative consequent upon the transformation in x and y is

$$\begin{aligned}\bar{y}' = \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} &= \frac{\mathrm{d}(y + \varepsilon\eta)}{d(x + \varepsilon\xi)} \\ &= \frac{y' + \varepsilon\eta'}{1 + \varepsilon\xi'} \\ &= y' + \varepsilon(\eta' - y'\xi') + O(\varepsilon^2).\end{aligned}$$

Similarly for the second and third derivatives we have

$$\begin{aligned}\bar{y}'' &= y'' + \varepsilon\left(\eta'' - 2y''\xi' - y'\xi''\right) \\ \bar{y}''' &= y''' + \varepsilon\left(\eta''' - 3y'''\xi' - 3y''\xi'' - y'\xi'''\right).\end{aligned}$$

We have the general formula

$$\bar{y}^{(n)} = y^{(n)} + \varepsilon\phi_n \tag{3.2.3}$$

with

$$\phi_n = \phi_{n-1}' - \xi' y^{(n)} \qquad n \geq 1 \tag{3.2.4}$$

and

$$\phi_0 = \eta.$$

Consider the effect of the infinitesimal transformation on the Action Integral. We find ξ and η so that the Action Integral is changed by an infinitesimal constant. Hence, we have

$$\int_{\bar{x}_0}^{\bar{x}_1} \mathcal{L}(\bar{x}, \bar{y}, \bar{y}')\mathrm{d}\bar{x} = \int_{x_0}^{x_1} \mathcal{L}(x, y, y')\mathrm{d}x + \varepsilon K, \tag{3.2.5}$$

where εK represents an infinitesimal constant so that the transformation is continuously deformable from the identity. We have

$$\begin{aligned}\mathcal{L}(\bar{x}, \bar{y}, \bar{y}')\mathrm{d}\bar{x} &= \mathcal{L}(x+\varepsilon\xi, y+\varepsilon\eta, y'+\varepsilon\zeta)\mathrm{d}(x+\varepsilon\xi) \\ &= \left\{\mathcal{L}(x,y,y') + \varepsilon\xi\frac{\partial\mathcal{L}}{\partial x} + \varepsilon\eta\frac{\partial\mathcal{L}}{\partial y} + \varepsilon\zeta\frac{\partial\mathcal{L}}{\partial y}\right\}(\mathrm{d}x + \varepsilon\xi'\mathrm{d}x) \\ &= \mathcal{L}\mathrm{d}x + \varepsilon\left(\xi\frac{\partial\mathcal{L}}{\partial x} + \eta\frac{\partial\mathcal{L}}{\partial y} + \zeta\frac{\partial\mathcal{L}}{\partial y'} + \xi'\mathcal{L}\right)\mathrm{d}x\end{aligned}$$

to $O(\varepsilon)$ with $\zeta = \eta' - y'\xi'$. Then equation (3.2.5) becomes

$$\int_{x_0}^{x_1}\left(\mathcal{L} + \varepsilon\left(\xi\frac{\partial\mathcal{L}}{\partial x} + \eta\frac{\partial\mathcal{L}}{\partial y} + \zeta\frac{\partial\mathcal{L}}{\partial y'} + \xi'\mathcal{L}\right)\right)\mathrm{d}x - \int_{x_0}^{x_1}\mathcal{L}\mathrm{d}x = \varepsilon K, \tag{3.2.6}$$

i.e.

$$\int_{x_0}^{x_1}\varepsilon\left(\xi\frac{\partial\mathcal{L}}{\partial x} + \eta\frac{\partial\mathcal{L}}{\partial y} + \zeta\frac{\partial\mathcal{L}}{\partial y'} + \xi'\mathcal{L}\right)\mathrm{d}x = \varepsilon K. \tag{3.2.7}$$

This means that

$$\xi\frac{\partial\mathcal{L}}{\partial x} + \eta\frac{\partial\mathcal{L}}{\partial y} + \zeta\frac{\partial\mathcal{L}}{\partial y'} + \xi'\mathcal{L} = f', \tag{3.2.8}$$

where $' = \mathrm{d}/\mathrm{d}x$ and f is a C^1 function such that

$$\int_{x_0}^{x_1} f'\mathrm{d}x = K \iff f(x_1) - f(x_0) = K. \tag{3.2.9}$$

The function f is called a gauge function. Its appearance in the expression is due to the fact that the Euler-Lagrange equation is unaffected by the addition of a total time derivative to the Lagrangian. The equation (3.2.8) is referred to as a Killing-type equation by Sarlet and Cantrijn [77]. The Killing-type equation[3] (3.2.8) for a Lagrangian, $\mathcal{L}(x, y, y')$ is

$$f' = \xi'\mathcal{L} + \xi\frac{\partial\mathcal{L}}{\partial x} + \eta\frac{\partial\mathcal{L}}{\partial y} + \left((\eta' - y'\xi')\frac{\partial\mathcal{L}}{\partial y'}\right). \tag{3.2.10}$$

[3]This terminology has been used in [77] to refer to the set of partial differential equations from which Noether transformations have to be determined.

To obtain the expression for the corresponding first integral we aim to put a total derivative under one bracket.[4] Thus

$$(f-\xi\mathcal{L})' = f' - \xi'\mathcal{L} - \xi\left(\frac{\partial\mathcal{L}}{\partial x} + y'\frac{\partial\mathcal{L}}{\partial y} + y''\frac{\partial\mathcal{L}}{\partial y'}\right). \tag{3.2.11}$$

We then substitute for $f' - \xi'\mathcal{L}$ in (3.2.10) so that we obtain

$$\begin{aligned}(f-\xi\mathcal{L})' &= -\xi\left(\frac{\partial\mathcal{L}}{\partial x} + y'\frac{\partial\mathcal{L}}{\partial y} + y''\frac{\partial\mathcal{L}}{\partial y'}\right) + \xi\frac{\partial\mathcal{L}}{\partial x} + \eta\frac{\partial\mathcal{L}}{\partial y} + \left((\eta' - y'\xi')\frac{\partial\mathcal{L}}{\partial y'}\right)\\ &= -\xi\left(y'\frac{\partial\mathcal{L}}{\partial y} + y''\frac{\partial\mathcal{L}}{\partial y'}\right) + \eta\frac{\partial\mathcal{L}}{\partial y} + \left((\eta' - y'\xi')\frac{\partial\mathcal{L}}{\partial y'}\right)\\ &= \left\{(\eta - y'\xi)\frac{\partial\mathcal{L}}{\partial y'}\right\}' - (\eta - y'\xi)\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'} + (\eta - y'\xi)\frac{\partial\mathcal{L}}{\partial y'}\end{aligned} \tag{3.2.12}$$

which can be rearranged as

$$\begin{aligned}\left\{f - \left(\xi\mathcal{L} + (\eta - y'\xi)\frac{\partial\mathcal{L}}{\partial y'}\right)\right\}' &= (\eta - y'\xi)\left(\frac{\partial\mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'}\right)\\ &= 0\end{aligned} \tag{3.2.13}$$

when the Euler-Lagrange equation is invoked. Integration of (3.2.13) gives us

$$I = f - \left(\xi\mathcal{L} + (\eta - y'\xi)\frac{\partial\mathcal{L}}{\partial y'}\right) \tag{3.2.14}$$

which is a first integral for the first order Lagrangian. Therefore a given Lagrangian $\mathcal{L} = \mathcal{L}(x, y, y', \ldots., y^{(n)})$ has a Noether symmetry

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \tag{3.2.15}$$

if

$$G^{[n]}\mathcal{L} + \xi'\mathcal{L} = f', \tag{3.2.16}$$

where ξ and η are functions of their arguments and f is a gauge function.

Noether transformation: We define an infinitesimal transformation (3.2.1) and (3.2.2) satisfying (3.2.10) for a given Lagrangian system $\mathcal{L}$ and some gauge function f a **classical Noether transformation** corresponding to $\mathcal{L}$ [77].

[4] This simplifies the route to integration.

Noether's theorem states that to each Noether transformation (3.2.1) and (3.2.2) there corresponds a constant of the motion

$$I(x,y,y') = f(x,y) - \left(\xi\mathcal{L} + (\eta - y'\xi)\frac{\partial \mathcal{L}}{\partial y'}\right).$$

Note that this is the theorem for a first order Lagrangian. The theorem can be extended to higher order Lagrangians as we shall see below.

Consider as an illustration the equation

$$y'' + y = \frac{1}{y^3}, \tag{3.2.17}$$

a particular case of the Ermakov-Pinney equation [20]. In this case the Lagrangian $\mathcal{L}$ is taken as

$$\mathcal{L} = -\frac{y'^2}{2} + \frac{y^2}{2} + \frac{1}{2y^2}. \tag{3.2.18}$$

Then condition (3.2.8) yields

$$f' = \eta\left(y - \frac{1}{y^3}\right) - y'(\eta' - y'\xi') + \xi'\left(-\frac{y'^2}{2} + \frac{y^2}{2} + \frac{1}{2y^2}\right) \tag{3.2.19}$$

which implies that

$$\begin{aligned}\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} = {} & \eta\left(y - \frac{1}{y^3}\right) - y'\left(\frac{\partial \eta}{\partial x} + y'\frac{\partial \eta}{\partial y}\right) \\ & + \left(\frac{y^2}{2} + \frac{y'^2}{2} + \frac{1}{2y^2}\right)\left(\frac{\partial \xi}{\partial x} + y'\frac{\partial \xi}{\partial y}\right).\end{aligned} \tag{3.2.20}$$

As ξ, η and f do not contain y', we can separate by coefficients of y' to obtain the system of partial differential equations:

$$\begin{array}{llcl} y'^3: & \dfrac{\partial \xi}{\partial y} & = & 0 \\ y'^2: & \dfrac{\partial \eta}{\partial y} & = & \dfrac{1}{2}a' \\ y'^1: & \dfrac{1}{2y^2}\dfrac{\partial \xi}{\partial y} + \dfrac{y^2}{2}\dfrac{\partial \xi}{\partial y} - \dfrac{\partial \eta}{\partial x} & = & \dfrac{\partial f}{\partial y} \\ y'^0: & \eta\left(y - \dfrac{1}{y^3}\right) + \dfrac{y^2}{2}\dfrac{\partial \xi}{\partial x} + \dfrac{1}{2y^2}\dfrac{\partial \xi}{\partial x} & = & \dfrac{\partial f}{\partial x}, \end{array} \tag{3.2.21}$$

Noether Symmetries	Noether Integral	Gauge function
$G_1 = \dfrac{\partial}{\partial x}$	$I_1 = \dfrac{1}{2}y'^2 + \dfrac{y^2}{2} + \dfrac{1}{2y^2}$	$f = 0$
$G_2 = \cos 2x\dfrac{\partial}{\partial x} - y\sin 2x\dfrac{\partial}{\partial y}$	$I_2 = \dfrac{1}{2}y^2\cos 2x - \dfrac{1}{2y^2}\cos 2x$ $+yy'\sin 2x - \dfrac{1}{2}y'^2\cos 2x$	$f = y^2\cos 2x$
$G_3 = \sin 2x\dfrac{\partial}{\partial x} + y\cos 2x\dfrac{\partial}{\partial y}$	$I_3 = \dfrac{1}{2}y^2\sin 2x - \dfrac{1}{2y^2}\sin 2x$ $+yy'\cos 2x - \frac{1}{2}y'^2\sin 2x$	$f = y^2\sin 2x$

Table 3.1: Noether symmetries, their corresponding first integrals and gauge functions for the Ermakov-Pinney equation $y'' + y = \frac{1}{y^3}$.

from the first three of which

$$\begin{aligned} \xi &= a(x) \\ \eta &= \tfrac{1}{2}a'y + b(x) \\ f &= -\frac{1}{4}a''y^2 - b'y + g(x). \end{aligned} \tag{3.2.22}$$

The fourth equation in system (3.2.21) gives

$$\begin{aligned} a &= A_0 + A_1\cos 2x + A_2\sin 2x \\ b &= 0 \\ g &= constant. \end{aligned} \tag{3.2.23}$$

The g is ignored as it appears as an additive constant. Table 3.1 gives the Noether symmetries and the corresponding integrals and gauge functions. In this case we obtain three point symmetries which is the same number obtained when the Lie analysis is applied to the differential equation.

3.3 Noether's theorem for the functional $\mathcal{L}(x, y, y', y''$

Let A be the Action Integral $\int_{x_0}^{x_1} \mathcal{L}(x, y, y', y'')\mathrm{d}x$. The transformed functional A becomes

$$\bar{A} = \int_{\bar{x}_0}^{\bar{x}_1} \mathcal{L}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'')\mathrm{d}\bar{x}. \tag{3.3.1}$$

Use of the Taylor expansion leads us to

$$\begin{aligned}\bar{A} &= \int_{\bar{x}_0}^{\bar{x}_1} \mathcal{L}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'')\mathrm{d}\bar{x} \\ &= \int_{\bar{x}_0}^{\bar{x}_1} \mathcal{L}(x + \varepsilon\xi, y + \varepsilon\eta, y' + \varepsilon\zeta_1, y'' + \varepsilon\zeta_2)\mathrm{d}(x + \varepsilon\xi),\end{aligned}$$

with

$$\begin{aligned}\zeta_1 &= \eta' - y'\xi' \\ \zeta_2 &= \eta'' - 2y''\xi' - y'\xi''.\end{aligned}$$

The requirement that $A - \bar{A} = \varepsilon K$, where the difference is an infinitesimal constant leads to

$$\bar{A} = \int_{x_0}^{x_1} \mathcal{L}(x, y, y', y'')\mathrm{d}x + \int_{x_0}^{x_1} \varepsilon \left(\xi\frac{\partial\mathcal{L}}{\partial x} + \eta\frac{\partial\mathcal{L}}{\partial y} + \zeta_1\frac{\partial\mathcal{L}}{\partial y'} + \zeta_2\frac{\partial\mathcal{L}}{\partial y''} + \xi'\mathcal{L}\right)\mathrm{d}x \tag{3.3.2}$$

so that

$$f' = \xi\frac{\partial\mathcal{L}}{\partial x} + \eta\frac{\partial\mathcal{L}}{\partial y} + \zeta_1\frac{\partial\mathcal{L}}{\partial y'} + \zeta_2\frac{\partial\mathcal{L}}{\partial y''} + \xi'\mathcal{L}. \tag{3.3.3}$$

The expression for the first integral is obtained by manipulating the above expression. There is no specific method for doing this. To do this we aim to group total derivatives on one side so that the expression is easily integrable.

3.3.1 The first integral for a functional $\mathcal{L}(x, y, y', y'')$

To find the expression for the first integral of a second order Lagrangian we recall that for the first order Lagrangian the integral is given by

$$I = f - \left(\xi\mathcal{L} + (\eta - y'\xi)\frac{\partial\mathcal{L}}{\partial y'}\right). \tag{3.3.4}$$

We also have for the functional $\mathcal{L}(x, y, y', y'')$ the equation

$$f' = \xi\frac{\partial\mathcal{L}}{\partial x} + \eta\frac{\partial\mathcal{L}}{\partial y} + (\eta' - y'\xi')\frac{\partial\mathcal{L}}{\partial y'} + (\eta'' - 2y''\xi' - y'\xi'')\frac{\partial\mathcal{L}}{\partial y''} + \xi'\mathcal{L}. \quad (3.3.5)$$

We add the term

$$\left((\eta - y'\xi)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\mathcal{L}}{\partial y''}\right)\right)'$$

to both sides of equation (3.3.5). Since

$$\left[f - \left(\xi\mathcal{L} + (\eta - y'\xi)\frac{\partial\mathcal{L}}{\partial y'}\right)\right]' = f' - \xi'\mathcal{L} - \xi\mathcal{L}' - (\eta' - y''\xi - y'\xi')\frac{\partial\mathcal{L}}{\partial y'} - (\eta - y'\xi)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\mathcal{L}}{\partial y'}\right),$$

(3.3.5) becomes

$$\begin{aligned}\left\{f - \left[\xi\mathcal{L} + (\eta - y'\xi)\frac{\partial\mathcal{L}}{\partial y'} - (\eta - y'\xi)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\mathcal{L}}{\partial y''}\right)\right]\right\}' &= -\xi y''\frac{\partial\mathcal{L}}{\partial y'} - \xi y'''\frac{\partial\mathcal{L}}{\partial y''} \\ &+ \frac{\mathrm{d}}{\mathrm{d}x}\left[(\eta' - y''\xi - y'\xi')\frac{\partial\mathcal{L}}{\partial y''}\right] + y'''\xi\frac{\partial\mathcal{L}}{\partial y''} \\ &+ (\eta' - y'\xi')\frac{\partial\mathcal{L}}{\partial y'} - (\eta' - y''\xi - y'\xi')\frac{\partial\mathcal{L}}{\partial y'}.\end{aligned} \quad (3.3.6)$$

Thus

$$\left(f - \left[\xi\mathcal{L} + (\eta - y'\xi)\left(\frac{\partial\mathcal{L}}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\mathcal{L}}{\partial y''}\right)\right)\right]\right)' = \frac{\mathrm{d}}{\mathrm{d}x}\left[(\eta' - y''\xi - y'\xi')\frac{\partial\mathcal{L}}{\partial y''}\right] \quad (3.3.7)$$

so that

$$I = f - \left[\xi\mathcal{L} + (\eta - y'\xi)\left(\frac{\partial\mathcal{L}}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\mathcal{L}}{\partial y''}\right)\right) + (\eta' - y''\xi - y'\xi')\frac{\partial\mathcal{L}}{\partial y''}\right] \quad (3.3.8)$$

is a first integral. As an illustration of the theorem consider the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}e^x y''^2 \quad (3.3.9)$$

with Euler-Lagrange equation

$$\begin{aligned} &e^x\left(y^{iv} + 2y''' + y''\right) = 0 \\ \iff\ &y^{iv} + 2y''' + y'' = 0.\end{aligned}$$

It follows from (3.3.3) that

$$\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+y''\frac{\partial f}{\partial y'}=\frac{\xi'}{2}e^x y''^2+\frac{1}{2}\xi e^x y''^2+\left(\eta''-2y''\xi'-y'\xi''\right)e^x y''. \quad (3.3.10)$$

In the case of point transformations we obtain

$$\begin{aligned}\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+y''\frac{\partial f}{\partial y'}=&\frac{e^x}{2}y''^2\left(\frac{\partial \xi}{\partial x}+y'\frac{\partial \xi}{\partial y}\right)\\&+\tfrac{1}{2}\xi y''^2+e^x y''\left(\frac{\partial^2\eta}{\partial x^2}+2y'\frac{\partial^2\eta}{\partial x\partial y}+y'^2\frac{\partial^2\eta}{\partial y^2}+y''\frac{\partial \eta}{\partial y}\right)\\&-2e^x y''^2\left(\frac{\partial \xi}{\partial x}+y'\frac{\partial \xi}{\partial y}\right)\\&-e^x y' y''\left(\frac{\partial^2\xi}{\partial x^2}+2y'\frac{\partial^2\xi}{\partial x\partial y}+y'^2\frac{\partial^2\xi}{\partial y^2}+y''\frac{\partial \xi}{\partial y}\right).\end{aligned}$$

Since ξ and η are functions of x and y only, we can separate as follows:

$$\begin{aligned}
y'y''^2: &\quad \frac{e^x}{2}\frac{\partial \xi}{\partial y}-2e^x\frac{\partial \xi}{\partial y}-e^x\frac{\partial \xi}{\partial y} &= 0\\
y''^2: &\quad \frac{e^x}{2}\frac{\partial \xi}{\partial x}+\frac{e^x}{2}\xi+e^x\frac{\partial \eta}{\partial y}-2e^x\frac{\partial \xi}{\partial x} &= 0\\
y''^1: &\quad e^x\frac{\partial^2\eta}{\partial x^2}+2e^x y'\frac{\partial^2\mathcal{L}}{\partial x\partial y}+y'^2\frac{\partial^2\eta}{\partial y^2}-e^x y'\frac{\partial^2\xi}{\partial x^2}-2e^x y'^2\frac{\partial^2\xi}{\partial x\partial y}-y'^3\frac{\partial^2\xi}{\partial y^2} &= \frac{\partial f}{\partial y'}\\
y''^0: &\quad \frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y} &= 0.
\end{aligned} \quad (3.3.11)$$

From this system of partial differential equations we obtain the following

$$\xi=a(x)$$

$$\eta=\left(\frac{3}{2}a'-\frac{a}{2}\right)y+b(x)$$

$$\begin{aligned}f=&e^x y'\left(\left(\frac{3}{2}a'''-\frac{a''}{2}\right)y+b''\right)+e^x y'^2\left(\frac{3}{2}a''-\frac{a'}{2}\right)\\&-\frac{e^x}{2}y'^2 a''+g(x,y)\end{aligned}$$

$$
\begin{aligned}
0 = e^x y' &\left(\left(\frac{3}{2}a''' - \frac{a''}{2}\right) y + b''\right) \\
&+ e^x y' \left(\left(\frac{3}{2}a^{iv} - \frac{a'''}{2}\right) y + b'''\right) + e^x y'^2 \left(\frac{3}{2}a'' - \frac{a'}{2}\right) + e^x y'^2 \left(\frac{3}{2}a''' - \frac{a''}{2}\right) \\
&- \frac{e^x}{2} y'^2 \left(a'' + a'''\right) + \frac{\partial g}{\partial x} + y' \frac{\partial g}{\partial y} + y'^2 e^x \left(\frac{3}{2}a''' - \frac{a''}{2}\right). \qquad (3.3.12)
\end{aligned}
$$

The y' terms in (3.3.12d) lead us to the following

$$
\begin{aligned}
y'^2: &\qquad \frac{5}{2}a''' - \frac{a'}{2} &= 0 \\
y'^1: &\quad e^x \left(\left(\frac{3}{2}a''' - \frac{a''}{2}\right) y + b''\right) + e^x \left(\left(\frac{3}{2}a^{iv} - \frac{a'''}{2}\right) y + b'''\right) &= \frac{\partial g}{\partial y} \\
&\Longrightarrow h(x) - e^x \left(\frac{1}{4}\left(3a^{iv} + 2a''' - a''\right) y^2 + \left(b''' + b''\right) y\right) &= g \\
y'^0: &\qquad \frac{\partial g}{\partial x} &= 0
\end{aligned}
\qquad (3.3.13)
$$

Thus from (3.3.13b) and (3.3.13c) we have

$$
\begin{aligned}
y^2: &\quad 3a^v + 5a^{iv} + a''' - a'' &= 0 \\
y^1: &\quad b^{iv} + 2b''' + b'' &= 0 \\
y^0: &\quad h' &= 0.
\end{aligned}
\qquad (3.3.14)
$$

Therefore, (3.3.13a) in combination with (3.3.14a) leads to

$$a = A_0 \qquad (3.3.15)$$

while (3.3.14b) leads to

$$b = B_0 + B_1 x + B_2 e^{-x} + x B_3 e^{-x}. \qquad (3.3.16)$$

Hence

$$\xi = A_0 \qquad (3.3.17)$$

and

$$\eta = -\frac{A_0}{2} y + B_0 + B_1 x + B_2 e^{-x} + x B_3 e^{-x}. \qquad (3.3.18)$$

The five-parameter symmetry G of the Lagrangian in question is

$$G = A_0\frac{\partial}{\partial x} + \left(-\frac{A_0}{2}y + B_0 + B_1 x + B_2 e^{-x} + xB_3 e^{-x}\right)\frac{\partial}{\partial y}. \tag{3.3.19}$$

The five one-parameter Noether symmetries are given by

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial x} - \tfrac{1}{2}y\frac{\partial}{\partial y}\\
G_2 &= \frac{\partial}{\partial y}\\
G_3 &= x\frac{\partial}{\partial y}\\
G_4 &= e^{-x}\frac{\partial}{\partial y}\\
G_5 &= xe^{-x}\frac{\partial}{\partial y}.
\end{aligned}$$

The last four symmetries are the solution symmetries.[5] Note that the obvious integral follows from G_2. The first integral corresponding to the symmetry G_1 is

$$I = \tfrac{1}{2}e^x y''^2 - \tfrac{1}{2}yy''e^x - \tfrac{1}{2}y'y''e^x - \tfrac{1}{2}e^x yy''' - y'y'''e^x. \tag{3.3.20}$$

3.4 The third order Lagrangian

Let $A = \int_{x_0}^{x_1}\mathcal{L}(x,y,y',y'',y''')\mathrm{d}x$ be the Action Integral for the third order Lagrangian $\mathcal{L}$. If x, y, y', y'' and y''' are allowed to undergo transformations the transformed functional A becomes

$$\bar{A} = \int_{\bar{x}_0}^{\bar{x}_1}\mathcal{L}(\bar{x},\bar{y},\bar{y}',\bar{y}'',\bar{y}''')\mathrm{d}\bar{x}. \tag{3.4.1}$$

By the Taylor expansion of the functional we have that

$$\begin{aligned}
\bar{A} &= \int_{\bar{x}_0}^{\bar{x}_1}\mathcal{L}(\bar{x},\bar{y},\bar{y}',\bar{y}'',\bar{y}''')\mathrm{d}\bar{x}\\
&= \int_{x_0}^{x_1}\mathcal{L}(x+\varepsilon\xi, y+\varepsilon\eta, y'+\varepsilon\zeta_1, y''+\varepsilon\zeta_2, y'''+\varepsilon\zeta_3)\mathrm{d}(x+\varepsilon\xi),
\end{aligned}$$

[5] The symmetries G_i, $i=2,..,5$ are a particular type of symmetry called solution symmetries since the solution set of $e^x\left(y^{iv}+2y'''+y''\right)=0$ is $\{x^i, x^i e^{-x}, i=0,1\}$. The solution symmetries are point symmetries and so are expected when one calculates point or contact symmetries of the equation.

where

$$
\begin{aligned}
\zeta_1 &= \eta' - y'\xi' \\
\zeta_2 &= \eta'' - 2y''\xi' - y'\xi'' \\
\zeta_3 &= \eta''' - 3y'''\xi' - 3y''\xi'' - y'\xi'''.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\bar{A} = &\int_{x_0}^{x_1} \mathcal{L}(x, y, y', y'', y''')\mathrm{d}x \\
&+ \int_{x_0}^{x_1} \varepsilon \left(\xi \frac{\partial \mathcal{L}}{\partial x} + \eta \frac{\partial \mathcal{L}}{\partial y} + \zeta_1 \frac{\partial \mathcal{L}}{\partial y'} + \zeta_2 \frac{\partial \mathcal{L}}{\partial y''} + \zeta_3 \frac{\partial \mathcal{L}}{\partial y'''} + \xi' \mathcal{L} \right) \mathrm{d}x
\end{aligned}
$$

to first order in ε. We use the requirement that $\bar{A} - A = \varepsilon K$, where K is a constant, so that

$$
f' = \xi \frac{\partial \mathcal{L}}{\partial x} + \eta \frac{\partial \mathcal{L}}{\partial y} + \zeta_1 \frac{\partial \mathcal{L}}{\partial y'} + \zeta_2 \frac{\partial \mathcal{L}}{\partial y''} + \zeta_3 \frac{\partial \mathcal{L}}{\partial y'''} + \xi' \mathcal{L}. \tag{3.4.2}
$$

3.4.1 Construction of the first integral for the third order Lagrangian $\mathcal{L}(x, y, y', y'', y''')$

We now construct the expression for the first integral using the Killing-type equation for a third order functional.

To obtain the first integral expression we put the total derivative on the right hand side of equation (3.4.2). We use the equation for the first integral of a second order Lagrangian as a starting point. We observe that

$$
\Phi = \left\{ f - \left[\xi \mathcal{L} + (\eta - y'\xi) \left(\frac{\partial \mathcal{L}}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial \mathcal{L}}{\partial y''} \right) \right) + (\eta' - y''\xi - y'\xi') \frac{\partial \mathcal{L}}{\partial y''} \right] \right\}', \tag{3.4.3}
$$

if

$$
\begin{aligned}
\Phi = f' - \xi' \mathcal{L} - \xi &\left(\frac{\partial \mathcal{L}}{\partial x} + y' \frac{\partial \mathcal{L}}{\partial y} + y'' \frac{\partial \mathcal{L}}{\partial y'} + y''' \frac{\partial \mathcal{L}}{\partial y''} + y^{iv} \frac{\partial \mathcal{L}}{\partial y'''} \right) \\
&- (\eta - y'\xi)' \left(\frac{\partial \mathcal{L}}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathcal{L}}{\partial y''} \right) - (\eta - y'\xi) \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathcal{L}}{\partial y'} - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{\partial \mathcal{L}}{\partial y''} \right) \\
&+ (\eta - y'\xi)' \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathcal{L}}{\partial y''} - (\eta - y'\xi')'' \frac{\partial \mathcal{L}}{\partial y''}.
\end{aligned}
$$

We then substitute for $f' - \xi'\mathcal{L}$ into equation (3.4.2) to obtain

$$\left\{f - \left[\xi\mathcal{L} + (\eta - y'\xi)\left(\frac{\partial\mathcal{L}}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\mathcal{L}}{\partial y''}\right)\right) + (\eta' - y''\xi - y'\xi')\frac{\partial\mathcal{L}}{\partial y''}\right]\right\}' = \Phi_1, \tag{3.4.4}$$

where

$$\Phi_1 = \left[(\eta - y'\xi)'\left(\frac{\partial\mathcal{L}}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y''}\right)\right]$$
$$+ \left[(\eta - y'\xi)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial\mathcal{L}}{\partial y'''}\right]' + \left[(\eta - y'\xi)''\frac{\partial\mathcal{L}}{\partial y'''}\right]'$$
$$+ (\eta - y'\xi)\left(\frac{\partial\mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial\mathcal{L}}{\partial y''} - \frac{\mathrm{d}^3}{\mathrm{d}x^3}\frac{\partial\mathcal{L}}{\partial y'''}\right)$$

so that

$$\begin{aligned} 0 = &\left\{f - \left(\xi\mathcal{L} + (\eta - y'\xi)\left(\frac{\partial\mathcal{L}}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y''} + \frac{d^2}{dx^2}\frac{\partial\mathcal{L}}{\partial y'''}\right)\right)\right\}' \\ &-\left\{(\eta - y'\xi)'\left(\frac{\partial\mathcal{L}}{\partial y''} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'''}\right)\right\}' \\ &-\left\{(\eta - y'\xi)''\left(\frac{\partial\mathcal{L}}{\partial y'''}\right)\right\}'. \end{aligned} \tag{3.4.5}$$

The required expression for the integral is given by

$$\begin{aligned} I = &f - \xi\mathcal{L} \\ &- (\eta - y'\xi)\left(\frac{\partial\mathcal{L}}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y''} + \frac{d^2}{dx^2}\frac{\partial\mathcal{L}}{\partial y'''}\right) \\ &- (\eta - y'\xi)'\left(\frac{\partial\mathcal{L}}{\partial y''} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'''}\right) \\ &- (\eta - y'\xi)''\left(\frac{\partial\mathcal{L}}{\partial y'''}\right). \end{aligned}$$

We now apply Noether's theorem to the third order Lagrangian

$$\mathcal{L} = \frac{y'''^2}{2} \tag{3.4.6}$$

with associated Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial\mathcal{L}}{\partial y''} - \frac{\mathrm{d}^3}{\mathrm{d}x^3}\frac{\partial\mathcal{L}}{\partial y'''} = 0.$$

which gives

$$y^{vi} = 0. \tag{3.4.7}$$

We want to calculate the Noether point symmetries of $\mathcal{L} = y'''^2/2$. By means of (3.4.2) we have that

$$\begin{aligned}
\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + y'''\frac{\partial f}{\partial y''} = {} & y'''\left(\frac{\partial^3\eta}{\partial x^3} + 3y'\frac{\partial^3\eta}{\partial x^2\partial y} + 3y'^2\frac{\partial^3\eta}{\partial x\partial y^2}\right) \\
& + y'''\left(3y'y''\frac{\partial^2\eta}{\partial y^2} + 3y''\frac{\partial^2\eta}{\partial x\partial y} + y'^3\frac{\partial^3\eta}{\partial y^3} + y'''\frac{\partial\eta}{\partial y}\right) \\
& - 3y''y'''\left(\frac{\partial^2\xi}{\partial x^2} + 2y'\frac{\partial^2\xi}{\partial x\partial y} + y'^2\frac{\partial^2\xi}{\partial y^2} + y''\frac{\partial\xi}{\partial y}\right) \\
& - 3y'''^2\left(\frac{\partial\xi}{\partial x} + y'\frac{\partial\xi}{\partial y}\right) \\
& - y'y'''\left(\frac{\partial^3\xi}{\partial x^3} + 3y'\frac{\partial^3\xi}{\partial x^2\partial y} + 3y'^2\frac{\partial^2\xi}{\partial x\partial y^2}\right) \\
& - y'y'''\left(3y'y''\frac{\partial^2\xi}{\partial y^2} + 3y''\frac{\partial^2\xi}{\partial x\partial y} + y'^3\frac{\partial^3\xi}{\partial y^3} + y'''\frac{\partial\xi}{\partial y}\right) \\
& + \frac{y'''^2}{2}\left(\frac{\partial\xi}{\partial x} + y'\frac{\partial\xi}{\partial y}\right).
\end{aligned} \tag{3.4.8}$$

The coefficient of y'''^2 is:

$$\frac{\partial\eta}{\partial y} - 3\left(\frac{\partial\xi}{\partial x} + y'\frac{\partial\xi}{\partial y}\right) + \tfrac{1}{2}\left(\frac{\partial\xi}{\partial x} - y'\frac{\partial\xi}{\partial y}\right) = 0. \tag{3.4.9}$$

Since ξ and η are functions of x and y only,

$$\begin{aligned}
y'^1: \quad & \frac{\partial\xi}{\partial y} = 0 \\
y'^0: \quad & \frac{-5}{2}\frac{\partial\xi}{\partial x} = \frac{\partial\eta}{\partial y}
\end{aligned} \tag{3.4.10}$$

so that

$$\begin{aligned}
\xi &= a(x) \\
\eta &= \frac{5}{2}a'y + b(x).
\end{aligned} \tag{3.4.11}$$

Substituting (3.4.11) into the coefficient of y''' in (3.4.8) yields

$$\frac{\partial^3\eta}{\partial x^3}+3y'\frac{\partial^3\eta}{\partial x^2\partial y}+3y''\frac{\partial^2\eta}{\partial x\partial y}-3y''\frac{\partial^2\xi}{\partial x^2}-y'\frac{\partial^3\xi}{\partial x^3}=\frac{\partial f}{\partial y''}$$

$$\Longleftrightarrow\quad \frac{\partial f}{\partial y''}=\frac{5}{2}a^{iv}y+b'''+\frac{13}{2}a'''y'+\frac{13}{2}y''a'' \tag{3.4.12}$$

$$\Longleftrightarrow\quad f=g(x,y,y')+y''\left(\frac{5}{2}a^{iv}y+b'''+\frac{13}{2}a'''y'\right)+\frac{13}{4}y''^2a''.$$

We also have for the coefficient of y'''^0 in (3.4.8) the expression

$$\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+y''\frac{\partial f}{\partial y'}=0 \tag{3.4.13}$$

which, taking equation (3.4.12c) becomes

$$\begin{aligned}\frac{\partial g}{\partial x}+y'\frac{\partial g}{\partial y}+y''\frac{\partial g}{\partial y'}+y''y'\left(\frac{5}{2}a^{iv}\right)+y''^2\left(\frac{13}{2}a'''\right)&\\ +y''\left(\frac{5}{2}a^{v}y+b^{iv}+\frac{13}{2}a^{iv}y'\right)+\frac{13}{4}y''^2a'''=0.&\end{aligned} \tag{3.4.14}$$

Upon collecting the powers of y'' in the previous equation we have

$$\begin{aligned}y''^2:&\qquad & a''' &= 0\\ y''^1:&\qquad & \frac{\partial g}{\partial y'}+\frac{5}{2}a^{v}y+\frac{13}{2}a^{iv}y'+b^{iv}+\frac{5}{2}a^{iv}y' &= 0\\ y''^0:&\qquad & \frac{\partial g}{\partial x}+y'\frac{\partial g}{\partial y} &= 0\end{aligned} \tag{3.4.15}$$

from which

$$\begin{aligned}-b^{iv}y'+h(x,y)&=g\\ -b^{v}y'+\frac{\partial h}{\partial x}+y'\frac{\partial h}{\partial y}&=0.\end{aligned}$$

From the coefficient of y' in the last equation

$$\begin{aligned}\frac{\partial h}{\partial y}&=b^{v}\\ \Longrightarrow h&=b^{v}y+c(x).\end{aligned}$$

The y'^0 term gives

$$\frac{\partial h}{\partial x} = 0$$

$$\Longrightarrow b^{vi}y + c' = 0. \tag{3.4.16}$$

For the y^1 term we have

$$b^{vi} = 0 \tag{3.4.17}$$

and the y^0 term gives

$$c' = 0. \tag{3.4.18}$$

It is evident that

$$\begin{aligned}
a &= A_0 + A_1x + A_2x^2 \\
b &= B_0 + B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 \\
c &= C_0,
\end{aligned} \tag{3.4.19}$$

where C_0 is ignored since it appears as an additive constant in the expression for the gauge function f. Therefore

$$\xi = A_0 + A_1x + A_2x^2 \tag{3.4.20}$$

and

$$\eta = \left(B_0 + B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5\right) + \frac{5}{2}\left(A_1y + 2A_2xy\right). \tag{3.4.21}$$

The Noether symmetries are

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial y} \\
G_2 &= x\frac{\partial}{\partial y} \\
G_3 &= x^2\frac{\partial}{\partial y} \\
G_4 &= x^3\frac{\partial}{\partial y} \\
G_5 &= x^4\frac{\partial}{\partial y}
\end{aligned}$$

$$
\begin{aligned}
G_6 &= x^5\frac{\partial}{\partial y}\\
G_7 &= \frac{\partial}{\partial x}\\
G_8 &= x\frac{\partial}{\partial x} + \frac{5}{2}y\frac{\partial}{\partial y}\\
G_9 &= x^2\frac{\partial}{\partial x} + 5xy\frac{\partial}{\partial y}.
\end{aligned}
\tag{3.4.22}
$$

The first six form the algebra $6A_1$ (solution symmetries) and the last three form the algebra $sl(2,R)$. The missing symmetry is the homogeneity symmetry[6] which is a Lie symmetry of the equation. The corresponding first integrals are given as follows

$$
\begin{aligned}
I_1 &= y^v\\
I_2 &= y^{iv} - xy^v\\
I_3 &= 2xy^{iv} - 2y''' - x^2y^v\\
I_4 &= 6y'' - x^3y^v + 3x^2y^{iv} - 6xy'''\\
I_5 &= 24xy'' - 24y' - x^4y^v + 4x^3y^{iv} - 12x^2y'''\\
I_6 &= 60x^2y'' - 120xy' + 120y - x^5y^v + 5x^4y^{iv} - 20x^3y'''\\
I_7 &= \tfrac{1}{2}y'''^2 + y'y^v - y''y^{iv}\\
I_8 &= \frac{-5}{2}yy^v + y'y^vx - xy''y^{iv} + \frac{3}{2}y'y^{iv} - \frac{1}{2}y''y''' + \frac{1}{2}xy'''^2\\
I_9 &= y^{iv}(5y + 3xy' - y''x^2) - y'''(8y' + xy'') - 5xyy^v + x^2y'y^v + \tfrac{1}{2}x^2y'''^2 + \frac{9}{2}y''^2.
\end{aligned}
$$

3.5 Generalised symmetries

We have thus far dealt with cases in which the infinitesimal transformations ξ and η are functions of x and y only. Noether's original presentation of the theorem [69] also allows for dependency on derivatives of higher order.[7] If we

[6] This is not surprising as the Lagrangian is not homogeneous of degree zero.

[7] Emmy Noether first recognised that one could significantly extend the application of symmetry group methods by including derivatives of dependent variables in the transformations or their infinitesimal generators. Generalised symmetries are useful in the study of

have a functional $\mathcal{L} = \mathcal{L}(x, y, y')$, then ξ and η can include terms in y', y'', y''', *etc.* The fact that the derivatives $y'', y''', \ldots$ are not independent due to their corresponding Euler-Lagrange equation does not cause a problem since the Euler-Lagrange equation is only imposed in the the determination of the first integral. Hence Noether symmetries can be determined with no prior knowledge of the Euler-Lagrange equation. The same comments apply *mutatis mutandis* in the case of higher order Lagrangians.

While the theory remains the same the practical aspect of the procedure has a slight variation from that of calculating point symmetries. For instance, while for point symmetries we use separation by powers by y' (in the case of a first order Lagrangian), this is no longer possible as both ξ and η have a functional dependence on y'. In this case we can separate terms by powers of y''. To clarify this consider

$$\xi = \xi(x, y, y')$$

and

$$\eta = \eta(x, y, y').$$

Then the Killing-type equation for this case becomes

$$\begin{aligned}
\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} &= \xi\frac{\partial \mathcal{L}}{\partial x} + \eta\frac{\partial \mathcal{L}}{\partial y} \\
&+ \left(\frac{\partial \eta}{\partial x} + y'\frac{\partial \eta}{\partial y} + y''\frac{\partial \eta}{\partial y'} - y'\frac{\partial \xi}{\partial x} - y'^2\frac{\partial \xi}{\partial y} - y'y''\frac{\partial \xi}{\partial y'}\right)\frac{\partial \mathcal{L}}{\partial y'} \\
&+ \left(\frac{\partial \xi}{\partial x} + y'\frac{\partial \xi}{\partial y} + y''\frac{\partial \xi}{\partial y'}\right)\mathcal{L}. \qquad (3.5.1)
\end{aligned}$$

Separation of powers is now by y'' and not y'. The coefficient of y'' is

$$\frac{\partial f}{\partial y'} = \frac{\partial \eta}{\partial y'}\frac{\partial \mathcal{L}}{\partial y'} - y'\frac{\partial \xi}{\partial y'}\frac{\partial \mathcal{L}}{\partial y'} + \mathcal{L}\frac{\partial \xi}{\partial y'} \qquad (3.5.2)$$

---

nonlinear wave equations where the possession of an infinite number of such symmetries is a characteristic property of solvable equations such as the Kortweg de-Vries equation, which have soliton solutions and are linearisable directly or via the inverse scattering method [70].

and that of y''^0

$$\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}=\xi\frac{\partial \mathcal{L}}{\partial x}+\eta\frac{\partial \mathcal{L}}{\partial y}+\frac{\partial \mathcal{L}}{\partial y'}\left(\frac{\partial \eta}{\partial x}+y'\frac{\partial \eta}{\partial y}\right)$$
$$-y'\frac{\partial \mathcal{L}}{\partial y'}\left(\frac{\partial \xi}{\partial x}+y'\frac{\partial \xi}{\partial y}\right)+\mathcal{L}\left(\frac{\partial \xi}{\partial x}+y'\frac{\partial \xi}{\partial y}\right). \tag{3.5.3}$$

To make any progress some assumptions about the y' dependence have to be made. We assume that ξ and η are linear in y' *i.e.*

$$\xi = a(x,y)+b(x,y)y'$$
$$\eta = c(x,y)+d(x,y)y'.$$

As an example consider the Lagrangian $\mathcal{L}$ given by

$$\mathcal{L}=\frac{1}{2}\left(y'^2-y^2\right). \tag{3.5.4}$$

We obtain the Noether transformations from the equation

$$f'=\xi\frac{\partial \mathcal{L}}{\partial x}+\eta\frac{\partial \mathcal{L}}{\partial y}+(\eta'-y'\xi')\frac{\partial \mathcal{L}}{\partial y'}+\xi'\mathcal{L}. \tag{3.5.5}$$

Equation (3.5.2) now becomes

$$\begin{aligned}\frac{\partial f}{\partial y'} &= y'd-\frac{1}{2}\left(y'^2+y^2\right)b\\ \Longrightarrow f &= \frac{y'^2}{2}d-\frac{1}{6}y'^3b-\frac{y'y^2}{2}b+f_0(x,y)\end{aligned} \tag{3.5.6}$$

and equation (3.5.3) results in

$$\frac{y'^2}{2}\frac{\partial d}{\partial x}-\frac{1}{6}y'^3\frac{\partial b}{\partial x}-\frac{1}{2}y^2y'\frac{\partial b}{\partial x}+\frac{\partial f_0}{\partial x}+y'\left(\frac{y'^2}{2}\frac{\partial d}{\partial y}-\frac{1}{6}y'^3\frac{\partial b}{\partial y}-y'\frac{y^2}{2}\frac{\partial b}{\partial y}-y'yb+\frac{\partial f_0}{\partial y}\right)=$$

$$-y\left(c+dy'\right)+y'\left(\frac{\partial c}{\partial x}+y'\frac{\partial d}{\partial x}+y'\frac{\partial c}{\partial y}+y'^2\frac{\partial d}{\partial y}\right)$$

$$-\frac{1}{2}y'^2\left(\frac{\partial a}{\partial x}+y'\frac{\partial b}{\partial x}+y'\frac{\partial a}{\partial y}+y'^2\frac{\partial b}{\partial y}\right)$$

$$-\frac{1}{2}y^2\left(\frac{\partial a}{\partial x}+y'\frac{\partial b}{\partial x}+y'\frac{\partial a}{\partial y}+y'^2\frac{\partial b}{\partial y}\right). \tag{3.5.7}$$

From the y'^4 coefficient in (3.5.7) we have

$$\frac{\partial b}{\partial y} = 0, \tag{3.5.8}$$

which means that

$$b = b_0(x). \tag{3.5.9}$$

The y'^3 term gives

$$d - a = \frac{2}{3} b_0{}' y + a_0(x). \tag{3.5.10}$$

From y'^2 we obtain

$$c = c_0(x) - \frac{1}{2} a_0' y - \frac{1}{6} b_0'' y^2 - \frac{y^2}{2} b_0 \tag{3.5.11}$$

and y'^1 leads to

$$\frac{\partial f_0}{\partial y} = \frac{\partial c}{\partial x} - \frac{1}{2} y^2 \frac{\partial a}{\partial y} + \frac{1}{2} y^2 \frac{\partial b}{\partial x} - dy. \tag{3.5.12}$$

Using the y'^0 term we have

$$\frac{\partial f_0}{\partial x} = -cy - \frac{1}{2} y^2 \frac{\partial a}{\partial x}. \tag{3.5.13}$$

The last two equations are manipulated to yield

$$\frac{\partial^2}{\partial y \partial x}\left(f_0 + \frac{1}{2} y^2 a\right) = -c_0 + a_0' y + \frac{1}{2} b_0'' y^2 + \frac{3}{2} y^2 b_0 \tag{3.5.14}$$

and

$$\frac{\partial^2}{\partial x \partial y}\left(f_0 + \frac{1}{2} y^2 a\right) = c_0'' - \frac{y^2}{2} b_0'' - \frac{1}{2} a_0''' y - \frac{1}{6} b_0{}^{iv} y^2 - y a_0' - \frac{2}{3} b_0{}'' y^2 \tag{3.5.15}$$

from which we have (after separating by coefficients of powers of $y^{(n)}$)

$$c_0'' + c_0 = 0$$

$$b_0^{iv} + 10 b_0'' + 9 b_0 = 0$$

$$a_0''' + 2a_0' = 0.$$

Therefore

$$c_0 = c_1 \sin x + c_2 \cos x$$

$$b_0 = b_1 \cos x + b_2 \sin x + b_3 \cos \sqrt{3}x + b_4 \sin \sqrt{3}x$$

$$a_0 = a_1 + a_2 \sin \sqrt{2}x + a_3 \cos \sqrt{2}x.$$

We also have that

$$f_0 + \tfrac{1}{2}y^2 a = -\frac{7}{18}b_0' y^3 - \tfrac{1}{2}a_0 y^2 - \frac{1}{18}b_0''' y^3 - \frac{1}{4}a_0'' y^2 + c_0' y + g(x). \tag{3.5.16}$$

Since

$$\frac{\partial}{\partial x}\left(f_0 + \tfrac{1}{2}ay^2\right) = \frac{1}{6}b_0'' y^3 + \tfrac{1}{2}a_0' y^2 + \tfrac{1}{2}y^3 b_0 - y c_0, \tag{3.5.17}$$

differentiation of (3.5.16) once with respect to x and equating it to (3.5.17) yields $g' = 0$ which means that g can be ignored. As a result of this (3.5.16) becomes

$$\begin{aligned} f_0 + \tfrac{1}{2}y^2 a = & -(c_1 \cos x + c_2 \sin x)\, y - \frac{1}{2}a_1 - \frac{7}{18}y^3\left(-b_3\sqrt{3}\sin\sqrt{3}x + b_4\sqrt{3}\cos\sqrt{3}x\right) \\ & - \frac{1}{12}\left(9b_3 \sin\sqrt{3}x - 9b_4 \cos\sqrt{3}x\right) + \frac{11}{36}\left(b_2 \sin x - b_3 \cos x\right). \end{aligned}$$

The first integral is given as

$$I = f_0 + \tfrac{1}{2}ay^2 - cy' - \tfrac{1}{2}(d-a)\, y'^2 - \frac{1}{6}y'^3 b - \tfrac{1}{2}by'^3 - by'^2 - y'c. \tag{3.5.18}$$

3.6 Higher dimensional systems

In this section we discuss Noether's theorem concerning Lagrangians of systems of more than one degree of freedom. We can then say that the symmetry

$$G = \xi\frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial y_i}, \tag{3.6.1}$$

is a Noether symmetry if it satisfies the general formula

$$f' = \xi\frac{\partial \mathcal{L}}{\partial x} + \eta_i\frac{\partial \mathcal{L}}{\partial y_i} + \left(\eta' - y'_i\xi'\right)\frac{\partial \mathcal{L}}{\partial y'_i}, \tag{3.6.2}$$

where

$$\xi = \xi(x, y_1, ..., y_n, y_1', ..., y_n')$$

and

$$\eta_i = \eta_i(x, y_1, ..., y_n, y_1', ..., y_n').$$

The first integral is determined from the expression

$$I = f - \left[\xi\mathcal{L} + (\eta_i - y_i'\xi)\frac{\partial \mathcal{L}}{\partial y_i'}\right], \tag{3.6.3}$$

and the repeated index i denotes summation.[8] Note that we use the standard notation in the case of a second order Lagrangian. Given a Lagrangian for the free particle in two dimensions

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) \tag{3.6.4}$$

we can apply the theorem to determine the Noether point symmetries and their associated first integrals. From equation (3.6.2)

$$\begin{aligned}\frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y} = &\left(\frac{\partial \eta}{\partial t} + \dot{x}\frac{\partial \eta}{\partial x} + \dot{y}\frac{\partial \eta}{\partial y} - \dot{x}\left(\frac{\partial \xi}{\partial t} + \dot{x}\frac{\partial \xi}{\partial x} + \dot{y}\frac{\partial \xi}{\partial y}\right)\right)\dot{x}\\ &+ \left(\frac{\partial \tau}{\partial t} + \dot{x}\frac{\partial \tau}{\partial x} + \dot{y}\frac{\partial \tau}{\partial y} - \dot{y}\left(\frac{\partial \xi}{\partial t} + \dot{x}\frac{\partial \xi}{\partial x} + \dot{y}\frac{\partial \xi}{\partial y}\right)\right)\dot{y},\end{aligned}$$

with $\eta_1 = \eta$ and $\eta_2 = \tau$. We can then separate by powers of $\dot{x}$ and $\dot{y}$ to obtain

$$\begin{aligned} \dot{x}^3 : \quad -\frac{\partial \xi}{\partial x} &= 0 \\ \dot{x}^2\dot{y} : \quad -\frac{\partial \xi}{\partial y} &= 0 \\ \dot{x}\dot{y}^2 : \quad -\frac{\partial \xi}{\partial x} &= 0 \\ \dot{y}^3 : \quad -\frac{\partial \xi}{\partial y} &= 0. \end{aligned} \tag{3.6.5}$$

This system gives ξ as $\xi = a(t)$. The coefficient of $\dot{x}^2$ gives η as

$$\eta = \dot{a}x + b(t, y), \tag{3.6.6}$$

and that of $\dot{x}\dot{y}$ gives τ as

$$\tau = -x\frac{\partial b}{\partial y} + c(t, y). \tag{3.6.7}$$

For $\dot{y}^2$ we have

$$c = \dot{a}y + d(t) \tag{3.6.8}$$

[8]In fact while the theory is easily applicable the degree of complexity in the expressions obtained increases during the manipulation of the problem.

and

$$b = ye(t) + h(t). \tag{3.6.9}$$

Hence

$$\begin{aligned} \xi &= a(t) \\ \eta &= \dot{a}x + ey + h(t) \\ \tau &= -ex + \dot{a}y + d(t). \end{aligned} \tag{3.6.10}$$

From the coefficient of $\dot{x}$ we observe that

$$f = \frac{1}{2}\ddot{a}x^2 + \dot{e}xy + \dot{h}x + j(t, y). \tag{3.6.11}$$

The coefficient of $\dot{y}$ requires

$$\dot{e}x + \frac{\partial j}{\partial y} = -\dot{e}x + \ddot{a}y + \dot{d} \tag{3.6.12}$$

which means that

$$\dot{e} = 0 \tag{3.6.13}$$

and

$$j = \frac{1}{2}\ddot{a}y^2 + \dot{d}y + s(t). \tag{3.6.14}$$

The last term gives the equation

$$\frac{1}{2}\dddot{a}\,x^2 + \ddot{h}x + \frac{1}{2}\dddot{a}\,y^2 + \ddot{d}y + \dot{s} = 0 \tag{3.6.15}$$

from which it is evident that

$$\begin{aligned} a &= a_0 + a_1 t + a_2 t^2 \\ h &= h_0 + h_1 t \\ d &= d_0 + d_1 t \\ s &= s_0. \end{aligned} \tag{3.6.16}$$

Therefore

$$\begin{aligned} \xi &= a_0 + a_1 t + a_2 t^2 \\ \eta &= (a_1 + 2a_2 t)x + e_0 y + h_0 + h_1 t \\ \tau &= -e_0 x + (a_1 + 2a_2 t)y + d_0 + d_1 t \end{aligned}$$

and the gauge function is

$$f = a_2\dot{x}^2 + h_1 x + a_2 y^2 + d_1 y. \tag{3.6.17}$$

The eight point symmetries obtained are

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial t} \\
G_2 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\
G_3 &= t^2\frac{\partial}{\partial t} + 2t\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \\
G_4 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \\
G_5 &= \frac{\partial}{\partial x} \\
G_6 &= t\frac{\partial}{\partial x} \\
G_7 &= \frac{\partial}{\partial y} \\
G_8 &= t\frac{\partial}{\partial y},
\end{aligned}$$

where the first three symmetries form the algebra $sl(2,R)$, the fourth one $so(2)$ and the last four are the 'solution' symmetries and form the Lie algebra $4A_1$.

3.7 The gauge function

In the literature there are some instances in Noether's theorem in which the gauge function is taken to be zero. We consider the simple Lagrangian $\mathcal{L} = \frac{1}{2}y'^2$ when we take the gauge function as zero. In this case the Killing-type equation $f' = G^{[1]}\mathcal{L} + \xi'\mathcal{L}$ takes the form

$$0 = \xi\frac{\partial \mathcal{L}}{\partial x} + \eta\frac{\partial \mathcal{L}}{\partial y} + (\eta' - y'\xi')\frac{\partial \mathcal{L}}{\partial y'} + \xi'\mathcal{L}. \tag{3.7.1}$$

We therefore have

$$0 = \left(\frac{\partial \eta}{\partial x} + y'\frac{\partial \eta}{\partial y}\right)y' - y'^2\left(\frac{\partial \xi}{\partial x} + y'\frac{\partial \xi}{\partial y}\right) + \frac{y'^2}{2}\left(\frac{\partial \xi}{\partial x} + y'\frac{\partial \xi}{\partial y}\right). \tag{3.7.2}$$

We obtain the following system of equations

$$y'^3: \qquad -\frac{1}{2}\frac{\partial \xi}{\partial y} = 0$$

$$y'^2: \qquad \frac{\partial \eta}{\partial y} - \frac{1}{2}\frac{\partial \xi}{\partial x} = 0 \tag{3.7.3}$$

$$y'^1: \qquad \frac{\partial \eta}{\partial x} = 0$$

from which

$$\begin{aligned} \xi &= a(x) \\ \eta &= \frac{1}{2}a'y + b(x) \\ 0 &= \frac{1}{2}a''y + b'. \end{aligned}$$

The coefficient of y gives

$$\frac{1}{2}a'' = 0 \Longrightarrow a = A_0 + A_1 x \tag{3.7.4}$$

and the y^0 coefficient gives

$$b' = 0 \Longrightarrow b = B_0. \tag{3.7.5}$$

The Noether point symmetries are therefore

$$\begin{aligned} G_1 &= \frac{\partial}{\partial x} \\ G_2 &= y\frac{\partial}{\partial y} \\ G_3 &= x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y}. \end{aligned}$$

The non-zero commutators are

$$[G_1, G_2] = G_1$$

$$[G_1, G_3] = 0$$

$$[G_2, G_3] = 0$$

which is the algebra $A_1 \oplus A_2$. In this case we have three Noether symmetries instead of the expected five which are obtained by using Noether's theorem [64]. In cases where the gauge function is set to zero the number of Noether symmetries obtained is less than the maximal number one can get when the gauge is not set to zero. This causes a great reduction of terms in the determination of Noether integrals.

3.8 The Hamiltonian formulation of Noether's theorem

Under the Legendre transformation

$$H = p_i \dot{q}_i - L, \qquad p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{3.8.1}$$

we move from the Lagrangian formulation based on $L(q, \dot{q}, t)$ and the second order Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = 0 \tag{3.8.2}$$

to the Hamiltonian formulation based on $H(q, p, t)$ and the first order Hamilton's equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}. \tag{3.8.3}$$

If we apply the transformation (3.8.1) to (3.6.3), the formula for the Noetherian integral becomes

$$I = f + \tau H - p\eta \tag{3.8.4}$$

as Stan [78] has reported. The method we have used to obtain the result (3.8.4) is not a derivation of Noether's theorem in the Hamiltonian formalism, but a rewriting of the last line, as it were, of the proof in the Lagrangian context to present the result in Hamiltonian terms. In the case that τ and η are the coefficient functions of a generalised symmetry we must replace in them, and

f, any appearance of $\dot{q}$ by its inversion in terms of q, p, and t through (3.8.1a). In the Lagrangian formulation the independent variable is t and the dependent variable q. In the Hamiltonian formulation time is again the independent variable, but now there are two dependent variables q and p. Consequently we must revise the derivation of Noether's theorem if we wish to place it in the Hamiltonian context.

Let

$$\bar{t} = t + \varepsilon\tau \qquad \bar{q} = q + \varepsilon\eta \qquad \bar{p} = p + \varepsilon\zeta \tag{3.8.5}$$

be an infinitesimal transformation generated by the differential operator

$$G = \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial q} + \zeta\frac{\partial}{\partial p} \tag{3.8.6}$$

in which the variable dependences of the coefficient functions are, as above, not specified. The Action Integral is

$$A = \int_{t_0}^{t_1} \{p\dot{q} - H(q,p,t)\}\mathrm{d}t \tag{3.8.7}$$

in which we take $\dot{q} = \dot{q}(q,p,t)$. Under the infinitesimal transformation we obtain

$$\begin{aligned}\bar{A} &= \int_{\bar{t}_0}^{\bar{t}_1} \{\bar{p}\dot{\bar{q}} - H(\bar{q},\bar{p},\bar{t})\}\mathrm{d}\bar{t} \\ &= \int_{t_0}^{t_1} \{p\dot{q} - H + \varepsilon[p\psi + \dot{q}\zeta - \eta\frac{\partial H}{\partial q} - \zeta\frac{\partial H}{\partial p} - \tau\frac{\partial H}{\partial t} + \dot{\tau}(p\dot{q} - H)]\}\mathrm{d}t \\ &\quad + \varepsilon\{(p_1\dot{q}_1 - H_1)\tau_1 - (p_0\dot{q}_0 - H_0)\tau_0\},\end{aligned} \tag{3.8.8}$$

where the subscripts 1 and 0 denote evaluation at t_1 and t_0 respectively and $\varepsilon\psi$ is the infinitesimal change in $\dot{q}$ produced by the infinitesimal transformation (3.8.5). We have set

$$\psi = \tau\frac{\partial\dot{q}}{\partial t} + \eta\frac{\partial\dot{q}}{\partial q} + \zeta\frac{\partial\dot{q}}{\partial p}. \tag{3.8.9}$$

The differential operator G is a Noether symmetry of the Action Integral if $\bar{A} = A$, *i.e.*

$$\dot{f} = p\psi + \dot{q}\zeta - \eta\frac{\partial H}{\partial q} - \zeta\frac{\partial H}{\partial p} - \tau\frac{\partial H}{\partial t} + \dot{\tau}(p\dot{q} - H) \tag{3.8.10}$$

in which we have written

$$(p_1\dot{q}_1 - H_1)\tau_1 - (p_0\dot{q}_0 - H_0)\tau_0 = -\int_{t_0}^{t_1} \dot{f}\mathrm{d}t. \tag{3.8.11}$$

Again we remark that (3.8.10) applies independently of the application of Hamilton's Principle.

In (3.8.9) we have written the change in $\dot{q}$ due to the infinitesimal transformation (3.8.5) in terms of τ, η and ζ. However, the transformation in $\dot{q}$ is also a differential consequence of the transformations in q and τ. Hence we have also that

$$\psi = \dot{\eta} - \dot{q}\dot{\tau}. \tag{3.8.12}$$

In (3.8.10) this and Hamilton's equation of motion for q give

$$\dot{f} = p\dot{\eta} - \eta\frac{\partial H}{\partial q} - \tau\frac{\partial H}{\partial t} - \dot{\tau}H \tag{3.8.13}$$

from which the first integral

$$I = f + \tau H - p\eta \tag{3.8.14}$$

follows by inspection. A consequence of (3.8.9) and (3.8.12) is that the infinitesimal transformation in p is not independent of that in t and $\dot{q}$. We have

$$p = \frac{1}{\dfrac{\partial^2 H}{\partial p^2}}\left\{\dot{\eta} - \dot{\tau}\frac{\partial H}{\partial p} - \tau\frac{\partial^2 H}{\partial p\partial t} - \eta\frac{\partial^2 H}{\partial p\partial q}\right\}. \tag{3.8.15}$$

This illustrates the fact that underlying the Hamiltonian formalism in $2n+1$ variables there is a basic space of $n+1$ dimensions. In the general case (3.8.13) and (3.8.14) are

$$\dot{f} = p_i\dot{\eta}_i - \eta_i\frac{\partial H}{\partial q_i} - \tau\frac{\partial H}{\partial t} - \dot{\tau}H \tag{3.8.16}$$

and

$$I = f + \tau H - p_i\eta_i. \tag{3.8.17}$$

3.9 Noether symmetries as generators of generalised canonical transformations.

A generalised canonical transformation is given by [22, 23, 50]

$$Q = \tilde{Q}(q, p, t) \quad P = \tilde{P}(q, p, t) \quad T = \tilde{T}(t), \tag{3.9.1}$$

in which we distinguish between the variable, say Q, and $\tilde{Q}$ by means of an overtilde, where

$$\left[\tilde{Q}, \tilde{P}\right]_{PBqp} = 1 \tag{3.9.2}$$

as usual. (The generalisation to more pairs of canonical variables should be obvious.) A generalised canonical transformation is a standard canonical transformation coupled with a reparametrisation of the time. As an example of its use consider the Hamiltonian of the time-dependent oscillator

$$H = \frac{1}{2}(p^2 + \omega^2(t)q^2). \tag{3.9.3}$$

Under the canonical transformation [50]

$$Q = \frac{q}{\rho} \qquad P = \rho p - \dot{\rho} q, \tag{3.9.4}$$

where $\rho(t)$ is a solution of the Ermakov-Pinney equation [20, 75]

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3}, \tag{3.9.5}$$

which has the Type II generating function

$$F_2(q, P, t) = \frac{qP}{\rho} - \frac{1}{2}\frac{\dot{\rho}}{\rho}q^2, \tag{3.9.6}$$

the transformed Hamiltonian is

$$\bar{H} = \frac{1}{2\rho^2}(P^2 + Q^2). \tag{3.9.7}$$

We introduce new time through

$$T = \int \frac{\mathrm{d}t}{\rho^2} \tag{3.9.8}$$

to obtain the new Hamiltonian

$$\bar{\bar{H}} = \frac{1}{2}(P^2 + Q^2) \tag{3.9.9}$$

which, when we express $\bar{\bar{H}}$ in terms of the original variables, is

$$\bar{\bar{\bar{H}}} = \frac{1}{2}\left[(\rho p - \dot{\rho} q)^2 + \frac{q^2}{\rho^2}\right], \tag{3.9.10}$$

the Ermakov-Lewis invariant [20, 52, 53] for the time dependent oscillator. The generalised canonical transformation taking H (3.9.3) to $\bar{\bar{H}}$ (3.9.9) is

$$Q = \frac{q}{\rho} \qquad P = \rho p - \dot{\rho} q \qquad T = \int \rho^{-2}\mathrm{d}t \tag{3.9.11}$$

subject to (3.9.5).

We have illustrated the concept of generalised canonical transformations in some detail to correct a possible misconception [68] which would be inferred in Stan's paper [78]. The paper by Stan [78] presents a simple proof of Noether's theorem in the framework of the Hamiltonian formalism of Classical Mechanics commencing from the usual Lagrangian formalism. Unfortunately the sources cited [17, 43, 66] failed to give a proper statement of Noether's theorem. A necessary condition for a Noether symmetry to be the generator of an infinitesimal generalised canonical transformation is that τ in (3.8.5) be a function of time only. We also require that the Poisson Bracket of the canonical variable be unity, *i.e.*

$$\begin{aligned} [\bar{q}, \bar{p}]_{PBqp} &= [q + \varepsilon\eta, p + \varepsilon\zeta]_{PBqp} \\ &= 1 \end{aligned} \tag{3.9.12}$$

which, to the first order in ε, means that

$$[\eta, p]_{PBqp} + [q, \zeta]_{PBqp} = 0$$

$$\Longleftrightarrow \frac{\partial \eta}{\partial q} + \frac{\partial \zeta}{\partial p} = 0. \tag{3.9.13}$$

Equation (3.9.13) implies the existence of a function, $\Gamma(q,p,t)$, such that

$$\eta = \frac{\partial \Gamma}{\partial p} \qquad \zeta = -\frac{\partial \Gamma}{\partial q} \tag{3.9.14}$$

so that the infinitesimal generalised canonical transformation generated by

$$G = \tau(t)\frac{\partial}{\partial t} + \frac{\partial \Gamma}{\partial p}\frac{\partial}{\partial q} - \frac{\partial \Gamma}{\partial q}\frac{\partial}{\partial p}, \tag{3.9.15}$$

is

$$\bar{t} = \tau + \varepsilon\tau(t) \qquad \bar{q} = q + \varepsilon\frac{\partial \Gamma}{\partial p} \qquad \bar{p} = p - \varepsilon\frac{\partial \Gamma}{\partial q}. \tag{3.9.16}$$

The finite transformation follows by exponentiation of (3.9.16).[9]

In the case of several variables (3.9.12) becomes

$$[\bar{q}_i, \bar{q}_j]_{PB_{qp}} = 0 \qquad [\bar{q}_i, \bar{p}_j]_{PB_{qp}} = \delta_{ij} \qquad [\bar{p}_i, \bar{p}_j]_{PB_{qp}} = 0 \tag{3.9.17}$$

which at the infinitesimal level is

$$\begin{aligned} [\eta_i, q_j]_{PB_{qp}} + [q_i, \eta_j]_{PB_{qp}} &= 0 \\ [\eta_i, p_j]_{PB_{qp}} + [q_i, \zeta_j]_{PB_{qp}} &= 0 \\ [\zeta_i, p_j]_{PB_{qp}} + [p_i, \zeta_j]_{PB_{qp}} &= 0. \end{aligned} \tag{3.9.18}$$

Equations (3.9.18) are fulfilled if there exists a function Γ such that

$$\eta_i = -\frac{\partial \Gamma}{\partial p_i} \qquad \zeta_i = -\frac{\partial p}{\partial q_i} \tag{3.9.19}$$

which is the direct generalisation of (3.9.14) and agrees with the conclusion of Stan [78]. However, here we have not had to make any assumptions about the independence of the coefficient functions because we used the Poisson Bracket requirement that a transformation be canonical.

We recall that the basis for canonical transformations is the maintenance of Hamilton's Principle which is based on the Action Integral

$$A = \int_{t_0}^{t_1} (p\dot{q} - H)\mathrm{d}t. \tag{3.9.20}$$

[9] In the example of the transformation of the time-dependent oscillator to autonomous form by the generalised canonical transformation (3.9.11) there is a connection between the time function in the new canonical variables and in the new time. This connection was useful in the example under consideration, but it is not a general requirement [23].

The process of making a generalised canonical transformation has two steps. In the first the canonical variables are transformed with the same time. In the second step the transformation to new time is made. Under the first step the Action Integral becomes

$$A = \int_{t_0}^{t_1} \left(P\dot{Q} - \bar{H} \right) \mathrm{d}t \tag{3.9.21}$$

and under the second step

$$\bar{A} = \int_{T_0}^{T_1} \left(P\frac{\mathrm{d}Q}{\mathrm{d}T} - \tilde{H} \right) \mathrm{d}T, \tag{3.9.22}$$

where $\tilde{H} = \bar{H}\dfrac{\partial \tau}{\partial T}$. We have

$$\begin{aligned} p\dot{q} - H - \dot{F} &= \left(P\frac{\mathrm{d}Q}{\mathrm{d}T} - \tilde{H} \right) \dot{T} \\ \Longleftrightarrow \quad p\dot{q} - H - \dot{F} &= P\dot{Q} - \bar{H}. \end{aligned} \tag{3.9.23}$$

The usual generating function formalism [82] follows from (3.9.23). We can express $\bar{H}$ and so H in terms of a generating function $S(q, p, t)$ [54] as

$$\bar{H} = H + p\frac{\partial Q}{\partial t} + \frac{\partial S}{\partial t}, \tag{3.9.24}$$

where

$$\frac{\partial S}{\partial q} = p - P\frac{\partial Q}{\partial q} \qquad \frac{\partial S}{\partial p} = -P\frac{\partial Q}{\partial p}. \tag{3.9.25}$$

(In passing we note that the usual requirement on the Poisson Bracket for the transformation to be canonical follows from requiring $\partial^2 S/\partial q \partial p = \partial^2 S/\partial p \partial q$.) From (3.9.16)

$$\frac{\partial S}{\partial q} = \varepsilon \left(\frac{\partial \Gamma}{\partial q} - p\frac{\partial^2 \Gamma}{\partial q \partial p} \right) \qquad \frac{\partial S}{\partial p} = -\varepsilon p \frac{\partial^2 \Gamma}{\partial p^2} \tag{3.9.26}$$

to the first order term in ε. Consequently the generating function is given by

$$S = \varepsilon \left(\Gamma - p\frac{\partial \Gamma}{\partial p} \right) \tag{3.9.27}$$

up to the usual additive arbitrary function of time. From (3.9.24)

$$\bar{H} = H + \varepsilon\frac{\partial}{\partial t}\left[\Gamma + p\left(\frac{\partial \Gamma}{\partial q} - \frac{\partial \Gamma}{\partial p} \right) \right] \tag{3.9.28}$$

so that

$$\tilde{H} = \bar{H}\left(1 - \varepsilon\dot{\tau}\right). \tag{3.9.29}$$

3.10 Discussion

We have presented Noether's theorem deriving it for first order and higher order Lagrangians. This enables us to apply all that has been developed to third and higher order Lagrangians. A generalisation of the theorem has been made recently [73, 74]. The formulation of Noether's theorem presented in both the Lagrangian and Hamiltonian formalism is consistent with original statement of the theorem by Noether [69]. The approach taken takes into account some of the developments in the concept of symmetry which have occured since Noether [69] first presented the theorem. Treatments of the theorem in both Lagrangian and Hamiltonian formalisms complementing the results in a recent paper by Stan [78] and correcting misconceptions of the theorem have been presented in [68]. The original generality of the theorem is oftenly understated in many texts, but we hope that the approach taken assists in addressing the problem.

Chapter 4

NonNoetherian Lie symmetries are Noether symmetries

We discuss the different provenances of the Lie symmetries and Noether symmetries [67]. We recall that a symmetry

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \tag{4.1}$$

is a Lie symmetry of a differential equation

$$E_n(x, y, y', \ldots, y^{(n)}) = 0, \tag{4.2}$$

where $y^{(i)} := \mathrm{d}^i y/\mathrm{d}x^i$, if

$$G^{[n]}E_n\Big|_{E_n=0} = 0, \tag{4.3}$$

where $G^{[n]}$ is the nth extension of G (required to determine the effect of the infinitesimal transformation induced by G on derivatives up to the nth) given by [62]

$$G^{[n]} = G + \sum_{i=1}^{n}\left\{\eta^{(i)} - \sum_{j=1}^{i}\left(\begin{array}{c} i \\ j \end{array}\right) y^{(i+1-j)}\xi^{(j)}\right\}\frac{\partial}{\partial y^{(i)}}. \tag{4.4}$$

The symmetry G is a Noether symmetry of the Lagrangian $L_n(x, y, y', y'', \ldots, y^{(n)})$ if

$$G^{[n]}L_n + \xi' L_n = f', \tag{4.5}$$

where f is a gauge function. Although our discussion can be applied to differential equations and Lagrangians of all orders, the essential features can be found in second order ordinary differential equations and first order Lagrangians.

In the case of a scalar second order ordinary differential equation there is a first integral, I, associated with G. It is the solution of the dual system

$$\begin{aligned} G^{[1]}I_{|_{E_2=0}} &= 0 \\ \frac{\mathrm{d}I}{\mathrm{d}x}_{|_{E_2=0}} &= 0. \end{aligned} \tag{4.6}$$

In practice the method of the solution of the system (4.5) is not always transparent [35]. On the other hand, if one imposes the requirement that the first variation of the Action Functional be zero — thereby leading to the Euler-Lagrange equation — on (4.5), the Noether symmetry leads naturally to a first integral given by [77]

$$I = f - \left[\xi L + (\eta - \xi y')\frac{\mathrm{d}L}{\mathrm{d}y'}\right]. \tag{4.7}$$

For higher order Lagrangians terms additional to those in (4.7) are required. They become increasingly complicated as the order is increased.

It is important to realise that the Noether symmetry exists independently of the Euler-Lagrange equation. Consequently the latter cannot be used to simplify expressions in Noether symmetries. However, as it is invoked to obtain the expression for the first integral, the Euler-Lagrange equation can be used to simplify the expression for the first integral. In the context of point and velocity-dependent transformations for which the coefficient functions ξ and η are functions of x, y and y' only this is of no importance, but it does have relevance for higher order and generalised transformations and also for nonlocal transformations for which the utility of integration by parts can be enhanced by the use of a differential equation to remove higher order terms.

Nonlocal transformations have been found to be of use in both the Lie and Noether analyses. As far as the Lie analysis is concerned, the observation that a nonlocal symmetry can become local under reduction of order (equally increase of order) [4] has been gainfully employed by the introduction of the idea of 'useful' nonlocal symmetries which do just that [37]. Such a symmetry can be used to further reduce the order of the equation under consideration. Exponential nonlocal symmetries fulfil this role in another sense [32]. In the case of Noether's theorem the connection of a nonlocal Lagrangian and a nonlocal Noether symmetry to produce a local first integral has been reported [38]. In that work it was noted that a particular nonlocal Lie symmetry was also a nonlocal Noether symmetry.

Despite the recent exception noted at the end of the last paragraph there has been a long and continuing discrimination between Lie and Noether symmetries. Indeed even within the class of Noether symmetries there has been discrimination following from the concept of nonequivalent Lagrangians. Thus, for example in the class of first order Lagrangians considered here, the Noether point symmetries of a Lagrangian for the free particle ($y'' = 0$) may range from zero to five. The maximum of five Noether point symmetries is found in the 'natural' Lagrangian

$$L = \tfrac{1}{2}y'^2. \tag{4.8}$$

The Lie point symmetries of the corresponding Euler-Lagrange equation

$$y'' = 0 \tag{4.9}$$

are eight in number. Note that here we confine ourselves to point symmetries in both cases. Why the difference in number? The point symmetries are

$$
\begin{array}{ll}
\text{Lie} & \text{Noether} \\
G_1 = \dfrac{\partial}{\partial y} & G_1 = \dfrac{\partial}{\partial y} \\
G_2 = x\dfrac{\partial}{\partial y} & G_2 = x\dfrac{\partial}{\partial y} \\
G_3 = \dfrac{\partial}{\partial x} & G_3 = \dfrac{\partial}{\partial x} \\
G_4 = 2x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} & G_4 = 2x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} \\
G_5 = x^2\dfrac{\partial}{\partial x} + xy\dfrac{\partial}{\partial y} & G_5 = x^2\dfrac{\partial}{\partial x} + xy\dfrac{\partial}{\partial y}, \\
G_6 = y\dfrac{\partial}{\partial y} & \\
G_7 = y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y} & \\
G_8 = yx\dfrac{\partial}{\partial x} + y^2\dfrac{\partial}{\partial y} &
\end{array}
\tag{4.10}
$$

in which the symmetries are listed in the order: solution symmetries; $s\ell(2,R)$ symmetries; homogeneity symmetry and nonCartan symmetries. That the homogeneity symmetry does not persist in the Noether symmetries is not surprising as the Lagrangian (4.8) is not of degree zero in y and is not in an equation, which would allow homogeneities of other degrees. No similar comment is available in the case of the nonCartan symmetries. It is true that the transformations obtained by the exponentiation of G_7 and G_8 are not of the form

$$X = F(x) \qquad Y = G(y,x) \tag{4.11}$$

which preserves Cartan symmetries [45] (*cf* the restriction to homographic transformations for the preservation of the Painlevé Property [13]). There is no constraint in the statement of Noether's theorem [69] which suggests that Noether symmetries of nonCartan type should not exist.

The comments after (4.10) do not really answer the question, but merely offer soothing phrases. We believe that the question is ill-posed. It is not so much that there is a difference in the number of symmetries *à la Lie ou à la Noether* but that the method of calculation of the Noether symmetries is incomplete. The standard method of calculation of Noether point symmetries is to separate by powers of y' with the coefficients of those powers being explicitly functions of y and x. This analytic procedure certainly recovers those symmetries given above. In the calculation of velocity-dependent symmetries the same principle is employed. To enable separation by powers of y' a specific y' dependence in the coefficient functions is assumed and *mutatis mutandis* the computation is no different to that described above.

Where does this leave the excluded three point symmetries? They are not in either scenario. However, the situation changes if one is open to the introduction of nonlocal terms. We illustrate this in the case of those point symmetries for the free particle. Consider L as given by (4.8) and G_6, G_7 and G_8 of (4.10) in turn in (4.5) for $n = 1$.

For G_6 (4.5) gives

$$\begin{aligned} f' &= y'^2, \\ \textit{i.e.,} \qquad & \\ f &= \textstyle\int y'^2 \mathrm{d}x. \end{aligned} \tag{4.12}$$

Note that it is improper to do anything at this stage with the integral in (4.12) in terms of simplification since we are not in the context of the Euler-Lagrange equation which follows *after* the imposition of the requirement that the variation of the Action Integral for (4.5) be stationary. Consequently G_6 is a Noether symmetry corresponding to a nonlocal gauge function which must be interpreted as a function of time – recall the reason for the introduction of the gauge function – although its precise form as a function of time becomes known only after the solution of (4.12) is obtained. The expression for the integral in (4.7) comes after the application of the variational principle and for G_6 gives

$$I_6 = \int y'^2 \mathrm{d}x - yy'. \tag{4.13}$$

We may simplify (4.13) in one of two ways. The first, which is rather loose, is to evaluate the integral in (4.7) by parts and to use the Euler-Lagrange equation for the Lagrangian (4.8) as appropriate. Thus

$$I_6 = yy' - \int yy'' \mathrm{d}x - yy'$$

$$= 0 \tag{4.14}$$

which is true, albeit disconcerting. Alternatively we can be more precise and include limits. Then

$$\begin{aligned} I_6 &= \int_0^x y'^2 \mathrm{d}x - yy' \\ &= yy'|_0^x - \int_0^x yy'' \mathrm{d}x - yy' \\ &= -y(0)y'(0) \end{aligned} \tag{4.15}$$

which makes somewhat more sense in light of our foreknowledge of the two integrals y' and $y'x - y$ since

$$\begin{aligned} I_6 &= -y(0)y'(0) \\ &= (y'x - y)y' \end{aligned} \tag{4.16}$$

so that (4.16) is a Noether integral to associate with G_6. If, as we have in this case, there is foreknowledge of some integrals, we may as well make use of that knowledge immediately in (4.13). Thus (4.13) becomes

$$\begin{aligned} I_6 &= \int y'^2 \mathrm{d}x - yy' \\ &= y'^2 \int \mathrm{d}x - yy' \\ &= y'(y'x - y) \end{aligned} \tag{4.17}$$

which recovers (4.16) as one would expect.

Similarly for G_7 and G_8 we find that

$$\begin{aligned} I_7 &= \tfrac{1}{2}y'^2(y - y'x) \qquad & f_7 &= \tfrac{1}{2}\textstyle\int (2y' - y'^3)\, \mathrm{d}x \\ I_8 &= -y'(y - y'x)^2 \qquad & f_8 &= \tfrac{1}{2}\textstyle\int (3yy'^2 - y'^3 x)\, \mathrm{d}x. \end{aligned} \tag{4.18}$$

Consequently by an admission of a broader class of expressions for the gauge function, f, than is habitually assumed – even by some writers (for example Olver [70]) – we see that the standard Lie point symmetries do play rôles as Noether symmetries.

We have considered the example of the free particle at length to enable a clear view of the proposition being made in this chapter [67]. However, the free particle is a trivially integrable problem. It suffices for the ideas we wish to promote save one which is that of utility [40].

We do well to demonstrate that our ideas are not mere frivolous playthings by considering a problem of less friendly aspect. A good enough candidate is to be found in the generalised Emden-Fowler equation, *viz.*

$$y'' = f(x)y^n, \tag{4.19}$$

which has attracted attention by various writers over a number of years (for a sampling see [19, 28, 29, 30, 39, 44, 51, 61] and references cited therein).

If the Emden-Fowler equation is of order two, *viz.*

$$y'' = f(x)y^2, \tag{4.20}$$

it is well-known [65] that the equation has the Lie point symmetry

$$G = a(x)\frac{\partial}{\partial x} + [(\tfrac{1}{2}a'(x) + \alpha)y + d(x)]\frac{\partial}{\partial y} \tag{4.21}$$

for the case that $f(x)$ has the form

$$f(x) = Ka(x)^{-5/2}\exp\left[-\frac{\alpha}{2}\int^x \frac{\mathrm{d}x'}{a(x')}\right] \tag{4.22}$$

in which K and α are arbitrary constants, $a(x)$ is a solution of the integro-differential equation

$$a''' = 4Kd(x)a^{-5/2}\exp\left[-\alpha\int^x \frac{\mathrm{d}x'}{a(x')}\right] \tag{4.23}$$

and $d(x)$ a solution of the somewhat simpler

$$d'' = 0. \tag{4.24}$$

For the more general equation (4.19) with $n \neq -3, 0, 1, 2$ the symmetry has the slightly simpler form

$$G = a(x)\frac{\partial}{\partial x} + (\tfrac{1}{2}a'(x) + \alpha)y\frac{\partial}{\partial y} \tag{4.25}$$

provided

$$f(x) = Ka(x)^{-(n+3)/2}\exp\left[-(n-1)\alpha\int^x \frac{\mathrm{d}x'}{a(x')}\right] \tag{4.26}$$

and $a(x)$ is a solution of the equation

$$a'''(x) = 0. \tag{4.27}$$

(As above K and α are constants.)

Equations (4.19) and (4.20) are not obviously integrable. Apart from certain specific values of α they are not known to be integrable. The 'standard' version of Noether's theorem (*i.e.* the one restricted to point symmetries) yields a symmetry and so a first integral only in the case $\alpha = 0$.

How then does the generalised Emden-Fowler equation respond to our more general approach to Noether's theorem? A Lagrangian for (4.19) is

$$L = \tfrac{1}{2}y'^2 + \frac{1}{n+1}f(x)y^{n+1}. \tag{4.28}$$

Using (4.25) in (with the n in *it* set at one and the gauge function written as F to avoid confusion with the function, $f(x)$, (4.19))

$$F' = \tfrac{1}{2}a''yy' + \alpha y'^2 + \frac{a'f + af'}{n+1}y^{n+1} + \left(\tfrac{1}{2}a' + \alpha\right)fy^{n+1} \tag{4.29}$$

so that

$$F = \int\left[\tfrac{1}{2}a''yy' + \alpha y'^2 + \frac{a'f + af'}{n+1}y^{n+1} + \left(\tfrac{1}{2}a' + \alpha\right)fy^{n+1}\right]\mathrm{d}x. \tag{4.30}$$

The associated integral given by (4.7) is

$$\begin{aligned} I = {} & \tfrac{1}{2}ay'^2 - \left(\tfrac{1}{2}a' + \alpha\right)yy' - \frac{af}{n+1}y^{n+1} \\ & + \int\left[\tfrac{1}{2}a''yy' + \alpha y'^2 + \frac{a'f + af'}{n+1}y^{n+1} + \left(\tfrac{1}{2}a' + \alpha\right)fy^{n+1}\right]\mathrm{d}x. \end{aligned} \tag{4.31}$$

The simplification of the integral in (4.31) is made by integration by parts and use of (4.26) and (4.27). After some routine algebra (4.31) reduces to

$$I = \tfrac{1}{2}ay'^2 - \tfrac{1}{2}a'yy' + \frac{1}{4}a''y^2 - \frac{af}{n+1}y^{n+1} - \alpha\frac{n-1}{n+1}\int yy''\mathrm{d}x. \tag{4.32}$$

The Noetherian first integral is nonlocal except when $\alpha = 0$ ($n = 1$ is an excluded value). Thus we see that the first integral exists for all values of α, but it is only a useful integral when $\alpha = 0$. This answers the question of completeness of the results obtained using only point symmetrie [51]. However, the first integral does exist for all values of the parameter α, *i.e.* for all Lie point symmetries of (4.19) in the sense of (4.6b).

We have demonstrated how the distinction between Lie symmetries and Noether symmetries can be removed when nonlocal considerations are taken into account. The advantage in using Noether's theorem is that once one has found a (Noether) symmetry the first integral is easily determined. In the case of scalar nth order equations Noether's theorem yields one integral for each symmetry. On the other hand the Lie method yields $(n-1)$ integrals for each symmetry. The gain in using the Lie approach is that it gives more point symmetries than Noether's theorem. However, Noether's theorem can be used to obtain velocity-dependent symmetries of any order as there is no restriction in its use [69] except the existence of a functional for the equation at hand. This is not the case with the Lie approach as it is restricted to point or contact symmetries when applied to the equations of motion.

Chapter 5

Conclusion

We have shown that the idea of invariance plays a significant role in the search for symmetries and first integrals of ordinary differential equations arising from different models and processes. The Lie theory is one of the methods used for this analysis. In this dissertation our concern has been on Noether's theorem and first integrals of ordinary differential equations. We have given a derivation of Noether's theorem for first and higher order Lagrangians taking into account the main aspects of the theorem. In particular Noether's theorem for a third order Lagrangian $\frac{1}{2}y'''^2$, which has the sixth order Euler-Lagrange equation $y^{vi} = 0$, was derived. The corresponding first integrals were determined. The third order Lagrangian $\frac{1}{2}y'''^2$ was found to have nine Noether point symmetries all of which agreed with the Lie point symmetries except for the missing homogeneity symmetry. Noether's theorem has been proposed in both the Lagrangian and Hamiltonian contexts the latter of which is consistent with the original statement of the theorem by Noether [69]. The concept of generalised canonical transformations was illustrated. This process involved two steps, the first one being the transformation of the canonical variables with the same time and in the second step the transformation to new time was made. This was done in some detail to correct a possible misunderstanding which would be inferred in Stan's paper [78]. The connection between a nonlocal Lagrangian and a nonlocal Noether symmetry to produce a local first integral and that between a nonlocal Lie symmetry and a nonlocal Noether symmetry has been reported in [38]. It is with this perception that we have discussed the different provenances of the Lie and Noether symmetries and pointed out some of the differences between the two approaches. When nonlocal considerations are taken into account, the distinction between Lie symmetries and Noether symmetries is removed. The advantage in using Noether's theorem is that once one has found a (Noether) symmetry the first integral is easily determined. In the case of scalar nth order ordinary differential equations Noether's theorem yields one integral for each symmetry while the Lie method yields $(n-1)$ integrals. Noether's theorem can be used to obtain generalised symmetries of any order as there is no restriction in its use [69] except the existence of a functional for the equation at hand.

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