THE ROLE OF VISUAL LITERACY ON GRADE 11 LEARNERS' CONCEPTUAL UNDERSTANDING OF GRAPHICAL FUNCTIONAL RELATIONSHIPS

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The Role of Visual Literacy on Grade 11 Learners' Conceptual Understanding of Graphical Functional Relationships

BY

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Thesis

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DEDICATION

To

The glory of God and Our Divine Master His Holiness Sri Swami Sivananda and His Holiness Sri Swami Sahajananda.

DECLARATION

I, Rajesh Rampersad, declare that the research involved in my dissertation submitted in partial fulfilment for the M.Ed Degree in Mathematics Education, entitled ***The Role of Visual Literacy on Grade 11 Learners' Conceptual Understanding of Graphical Functional Relationships,*** represents my own and original work.

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RAJESH RAMPERSAD

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DATE

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DR. V. MUDALY

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ABSTRACT

This study intends to foreground visual literacy within the wider context of visualisation and visual thinking in mathematics teaching and learning. Visualisation in general has been receiving attention in mathematics education research. I distinguish visual literacy from visualisation by referring to visual literacy as the combination of visualisation and logical thought. Visual literacy emphasises construction of meaning through the process of decomposition, comprehension and analysis of visual representations. The section on functional relationships is located in the National Curriculum Statement (NCS) for mathematics in the Further Education and Training (FET) phase for Grades 10-12 (Department of Education, 2003). Graphical functional relationships, which form an integral part of functions and algebra in the FET phase for Grades 10-12, demand visual literacy, which includes graphical interpretation and comprehension skills. Therefore, the conceptual understanding associated with graphs is dependent on the way graphs are presented.

This study examines learners' and educators' procedural and conceptual understanding of the graphs they sketch and interpret in the FET curriculum. The data analysis contributes towards the fast growing body of knowledge on visualisation in mathematics with the significant impact visual literacy has on the conceptual understanding of mathematical graphs. The analysis reveals that the overarching theoretical framework of constructivism embracing the Process-Object, Visualizer-Analyzer and Semiotic models are useful in illustrating and justifying the link between visual literacy and the conceptual understanding of learners.

In examining the visual understanding of graphical representations of ten Grade 11 learners and the two mathematics educators that teach them, the data reveals that learners display a somewhat skewed understanding of the nature of the Cartesian plane, the characteristics of graphs, functional notation and graphical terminology. In fact their educators, in some instances, displayed similar understandings. Learners display procedural understanding of graphical representations to a large extent. The educators’ visual understanding does suggest that learners’ interpretation of graphs is in some way influenced by the way they teach. The overriding contribution of the research study is that visual literacy plays a significant role in the conceptual understanding of functional graphical relationships. The relationship between graphical representations and logical thought is central to visual literacy.

Key concepts: visual literacy, conceptual understanding, graphical representations, visualisation, analytical thinking, constructivism, process-object, Visualizer / Analyzer, semiotics and vehicles of reasoning.

CHAPTER ONE

INTRODUCTION AND OVERVIEW

1.1 Preamble to the study

This study serves to locate visual literacy as a learning strategy for mathematics teaching and learning. Visual literacy is an emerging heuristic discipline that establishes itself within the context of visualisation, visual thinking and analytical thinking. The emphasis is on the demand for higher order skills in thinking. Therefore, the focus of the study is on the exploration of the role of visual literacy on the conceptual understanding of Grade 11 mathematics learners and their mathematics educators of graphical functional relationships. The National Curriculum Statement for Grades 10-12 (NCS) in mathematics stipulates functional relationships as a key learning outcome. Learners are expected to apply higher order investigative, analytical and descriptive skills in the section on graphical representations (Department of Education, 2003).

As a mathematics educator, the researcher has found that learners in Grades 10-12 encounter difficulties with graphical skills and graphical interpretation. Frequently, learners engage in graphical tasks without much understanding of what they are doing and why they do such tasks. As a result of this, the limited understanding they acquire in Grade 10 affects them in the future grades. Consequently, the emphasis in this research study is on the learners' interpretation of graphs. The researcher claims that graphs through their visual structure of patterns, symbols, lines and geometric curves influences learners' conceptual understanding of graphical functional relationships. Therefore the main purpose is to explore the relationship between visual literacy, graphs and learners' conceptual understanding of graphical concepts. In doing so, the researcher probes the meaning Grade 11 learners have of the characteristics of graphs (major study) and examines the impact of the educator's knowledge of graphs (minor study) on learners' construction of meaning.

1.2 Background of the study

The thrust of the study originates with the significance accorded to visual literacy and visualisation as essential proponents of teaching and learning in mathematics classrooms. The current trend in mathematical inquiry is the predominant theme of the role of visual representations depicted through data representations, pictures, diagrams, graphs, symbols, words and patterns in the development of mathematical thought. Visual – spatial learning has been a major component of the mathematical discourse. The role of visual literacy in mathematics is emphasised through the following quotations:

- "No soul thinks without a mental image", by Aristotle cited in Zazkis, Dubinsky and Dautermann (1996).
- "The element of thought is visual", by Einstein cited in Thornton (2002).
- "A picture is worth a thousand words" by Thornton (2002).
- "Seeing the unseen" by Arcavi (2003).
- "Seeing the big picture" by Silverman (2002).

Presmeg (1997) notes that visualisation has been "devalued" in mathematics, while Arcavi (2003) reports that visual representations have been considered as "second-class citizens". This might be due to the prominence of traditional teaching methodologies intended for the 'auditory-sequential learner' stressing logical, sequential steps in problem solving and computations which are enforced by the educator's 'verbal-logic' approach to teaching (Thornton, 2002). A paradigm shift from this traditional view entails a transformation to visualisation skills such as multiple perceptions, pattern finding, graphical and creative thinking (Silverman, 2002). Since visual literacy has been neglected, the need to integrate the visual approach into the traditional curriculum and teaching and learning methods is essential. The prevalence of the proliferation of visual presentations in various media and resource materials is certain to influence mathematics education in the 21st century. We are faced with a visual culture that demands our attention to consider, generate, communicate, and understand images. However Arcavi (2003) affirms that from a socio-cultural perspective, we are faced with obstacles to visualisation. Some views tend to portray visualisation as an easier method in contrast to the view that visualisation requires high cognitive demands. The other difficulty is the dilemma of cultures that are more visually exposed to resources as opposed to disadvantaged visual cultures, with inadequate visual resources.

The paradigm shift in the way we respond to visual representations and in particular to graphical representations is highlighted by van Dyke (2002) whereby the emphasis is on adopting a "visual approach to the teaching of graphs". Romberg, Fennema and Carpenter (1993) maintain that research on graphical representations of functions is insufficient. Williams (1993) on the other hand insists that functions and graphs are distinct from number concepts and computational skills and claims that research on graphs has not investigated the content domain of graphs appropriately.

Graphs have the potential to provide deeper insight and revelation through its symbolic nature and geometrical formation of points, lines, arrows, curves and axes. Functional relationships are regarded as the most influential and valuable notation in mathematics. While there are numerous studies in other countries on improving graphical skills through the use of computers and graphic calculators, this is not the case in the South African education system. Students rely on paper and pencil images and graphical representations that appear in study material such as in text books. The question that reflects this situation is as Williams (1993) portrays: What kinds of expertise with graphical representations are really needed?

The traditional curriculum on graphical representations included the use of tables, formula and graphs. Kieran (1993) suggests that using algebraic and graphical representations present a new psychological impression in approaching functions. Therefore, the transformation from the traditional approach includes the assimilation of the numerical, algebraic and graphical representations of functions through the emphasis on global features of graphs, unscaled axes, exploration of relationships within the fundamental families of graphs and the utilisation of graphs in modelled contexts. It is within this framework and paradigm shift that the researcher consigns the use of visual literacy in graphical interpretation as an emerging discipline of cognitive development.

Arcavi (2003) enlists the various visualisation skills necessary for graphical analysis. These include transcending beyond the procedural application of rules and procedures to the conceptual application of visual, algebraic and analytical reasoning which are supported by symbols and verbalization. Consequently, the researcher employs Zimmermann and Cunningham's (1991) description of the study of mathematics as "the visual metaphor of patterns" to identify visual literacy as a heuristic instrument in studying graphical representations.

Therefore the visual literacy processes involving graphical interpretation may be conceptualised in a progressive manner commencing with the formation of visual images, both physical and mental images. Due to visual perception, objects are perceived through the sense of sight. At this stage visualisation occurs, where the nature of what is seen enters the process of meaning construction. Visualisation associates the mental image with the physical image. Through visualisation, visual thinking ensues with the formation of thought originating from visual images. The eventual cognitive process of analytical thinking occurs. As a result visual literacy may be located within the broad framework of associated visualisation disciplines and represented as follows:

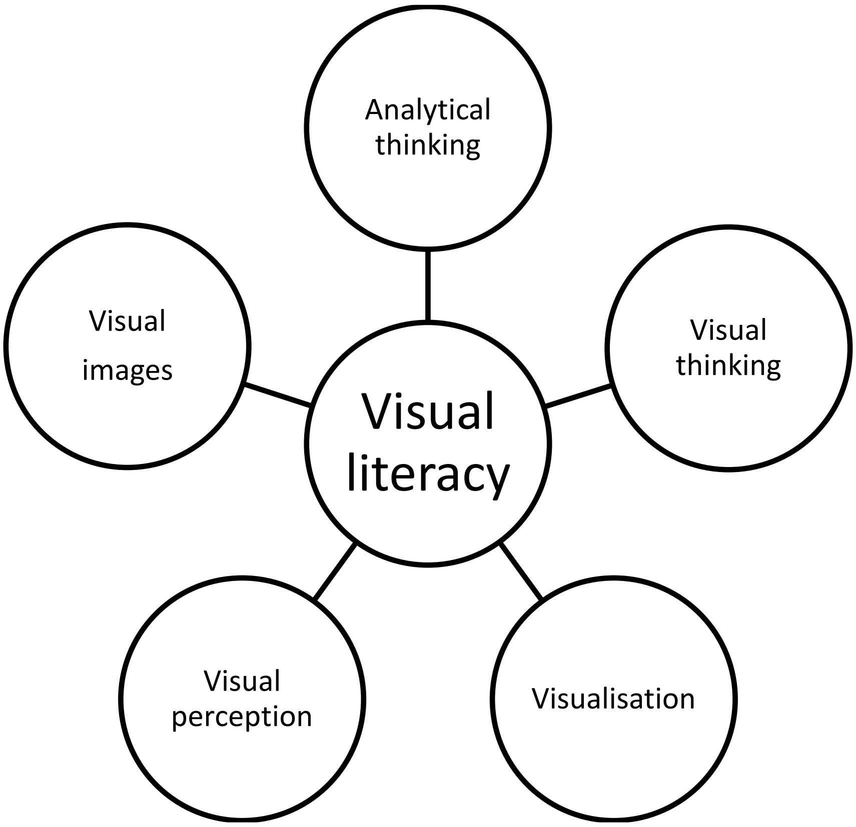


Figure 1: Related visual literacy concepts

1.3 Aim and objectives of the study

- **Aim**: To explore the role of visual literacy on Grade 11 learners' conceptual understanding of graphical functional relationships.

- **Objectives**:
    - To examine the relationship between visual literacy and learners' conceptual understanding of graphical functional relationships.
    - To explore the learners' and educators' visual understanding of graphs.
    - To ascertain to what extent learners' understanding of graphical representations is procedural or conceptual.
    - To determine if there is any relationship between the educators' and learners' understanding of graphs.
    - To relate the understanding of graphs to a visualisation-based theoretical framework.

- To highlight the importance of using visual literacy skills in the teaching and learning of graphical representations in Grades 10-12.

1.4 Chapter outline

Chapter one presents the overview of the study. It provides a description of the motivation for the research with the emphasis on visual literacy and graphical functional relationships as it appears in the secondary school curriculum. The background of the study traces the role of visual literacy in mathematics and the impact of graphical representations in mathematical learning. It also sets out the aims and objectives of the study.

Chapter two commences with a critique of visual literacy within the broad conceptual framework of visualisation and visual thinking; and provides definitions of the related visualisation concepts. It then proceeds to identify visual literacy as it appears in the teaching and learning situation with special emphasis on mathematics. The significance and development of graphical representations is portrayed and evidence of related mathematical research is provided focussing on learners' understanding of graphs. The effect of technology related activities is examined in terms of the effect it has on graphical skills. Graphical representations are then examined within the Grades 10-12 curriculum detailing the various types and characteristics of graphs. However to engage in graphical tasks, requires the implementation of graphical and visualisation skills, which are discussed in this chapter. The understanding of graphs is discussed from the procedural versus conceptual dichotomy, outlining the significance and use of both methods of understanding graphs. The role of analogies, imagery and metaphors in mathematics is detailed with reference to graphs. Lastly, the chapter ends with a brief outline on the educators' mathematical knowledge of graphical representations as they appeared in past research.

Chapter three focuses on related visualisation theoretical frameworks that the researcher bases the study and findings on. Visual literacy is positioned within the overarching theory of constructivism on the basis that knowledge is constructed by learners and educators in relation to the social environment. An exposition of the various visualisation theories follow thereafter. It commences with the process-object perspective highlighting the APOS theory as a means of advanced mathematical thinking. Then it demonstrates the applicability of the Visualizer/Analyzer model which involves the cyclic process of using visualisation and analysis. As graphical representations constitute signs, symbols and tools, an examination of the role of semiotics as exemplified by Peirce is conducted and illustrated through the inter-relationship between semiotics and vehicles of reasoning. The chapter concludes with Vygotsky's semiotic mediation as a means of constructing knowledge.

Chapter four discusses the research design and methodologies used in the study. The study is located within the qualitative research paradigm. An explanation of qualitative data collection methods and measures of validity and reliability are provided. The chapter concludes with how the researcher intends analysing the data and the key research questions are identified.

Chapter five focuses on the analysis of the data. Firstly, a statistical analysis of learners' responses to the worksheets is conducted, which classifies learners understanding as poor or good. Secondly, a qualitative analysis ensues, whereby a detailed and in-depth analysis is presented on learners and educators responses to the interview questions. Their responses are classified according to research sub questions and are further critiqued from the theoretical perspectives. A semiotic analysis is conducted as well as the role of semiotic mediation is considered. The chapter concludes with the analysis of the educators' responses.

Chapter six reports on conclusions gathered from the research process and makes recommendations for further research. The conclusions are based on the key research questions and provide insight into the relationship between visual literacy and conceptual understanding of graphical representations for Grades 10-12.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

The investigation of visual literacy and graphical functional relationships is influenced by a plethora of factors. The study of graphical representations, visual literacy and conceptual understanding indicate their inter-relationships and importance to mathematical teaching and learning. It is therefore necessary to explore the roles of visual literacy, visualisation, graphs, computer technology, graphical skills, conceptual understanding and vehicles of reasoning. In this way, an overall perspective of visual literacy and graphical representations would be obtained.

2.2 A critique of Visual Literacy

Literacy has always been associated with language and linguistic forms, especially with being able to read and write. The demand for literacy as a learning tool has broadened due to acquisition of basic skills, knowledge and competencies in various other fields. Stokes (2002) identified print literacy, aural literacy, media literacy, computer literacy, social literacy and eco-literacy as other forms of literacy. In recent developments, mathematical literacy has emerged as a basic necessity in society and is therefore included in the school curriculum. Visual literacy gained prominence with the advent of technological media, computers, movies and television (Silverman, 2002). Burns (2006) refers to the world as consisting of a variety of graphics or multimedia displayed by images on billboards, television, films, computers and the internet. This paradigm shift can be described as the change from typography to graphics (Burns, 2006), from the Age of Literacy to the Age of Information (Silverman, 2002) , from the reign of the left hemisphere of the brain to the right hemisphere (Williams, 1983) and from print culture to visual culture (Arcavi, 2003).

Visual literacy is related to our sense of vision or seeing, with visualisation and visual thinking being related constructs. On a very superficial or literal level, seeing involves the use of vision which we engage in our daily experiences. Seeing can be a physical or mental process, such as formation of a mental picture of a parabola (Mudaly, 2008). However, seeing forms an aspect of our visual perception. It is not uniquely related to the sense of sight and extends beyond the concrete given information to include interpretation and analysis of visual images (Arcavi, 2003; Elkins, 2003; Pylyshyn, 2003). Zimmerman and Cunningham (1991) appropriately describe seeing as "to see is not necessarily to understand". Imagistic processes form a significant aspect of visual reasoning whereby pictures, sketches and diagrams result in the formation of images (Wheatley, 1997). This now leads to the concept of visualisation which is a very commonly used teaching and learning strategy.

Visualisation is described in many different ways. Zazkis et al (1996) summarise the issues and debate around visual thinking in terms of the dichotomies of "visual versus verbal, visualized versus actual, memory images versus present perceptions, spatial versus abstraction, static images versus dynamic images and visualisation versus analytical". In the psychological field visualisation is related to human reasoning through the construction of an image (Zimmermann & Cunningham, 1991). The image may be regarded as a mental representation in the mind's eye and is therefore not concrete or physical. Pylyshyn (2003) maintains that one can think with or without visualisation. He makes reference to visualising an office by picturing the appearance of the room, but thinking about the room without visualising entails focusing on selected components of the office and is not necessarily thinking. Therefore, in this case visualisation excludes thinking.

However, Arcavi (2003) links visual thinking to visualisation and defines visualisation as "the ability, process and the product of creation, interpretation, use of and reflection upon pictures, images and diagrams in our minds". According to Williams (1993) visual thinking consists of seeing, representing information graphically and visualisation. This is similar to visualisation perceived as the ability to "recall and construct visual images within the mind" (Williams, 1983). Visual thinking is mathematical thought derived through visual images (Zimmermann & Cunningham, 1991). According to Zimmermann and Cunningham (1991) visualisation in mathematics focuses on understanding, interpretation and self discovery resulting from the formation of mental images, using pencil and paper or technological means. Zazkis et al (1996) define visualisation as "an act of connection between an internal construct and an external object or event gained through the senses". Presmeg (1997) also adopts this interpretation and refers to visualisation as "the process of constructing and transforming visual mental images, as well as those in drawings, diagrams or computer screens".

In describing visual thinking, Piaget distinguishes between static mental images (figurative knowledge) and the active visualisation, manipulation and transformation of the image in the mind (operative knowledge) (Fisher, 2005). Visual literacy is based on the transformation and manipulation of images in the mind. Mudaly (2008) distinguishes between visual literacy and visualisation by emphasising that visual literacy is "visualisation combined with logical thought". Visual literacy is very often literally or mechanically understood. For example, the numerous road signs symbolise and convey meaning to pedestrians and drivers thereby forcing them to react in certain ways (Mudaly, 2008). The pictures and concrete objects we see from as early as childhood are visual stimuli that influences our visual perception and due to some past experiences we think , feel and act in appropriate ways. Visual literacy goes beyond this basic understanding of utilising visual materials to promote recognition, recall or retention of concepts.

The following definitions locate visual literacy as an emerging heuristic instrument where it is defined as the ability:

- to interpret images as well as to generate images for communicating ideas and concepts (Stokes, 2002).
- to decode, comprehend and analyse images in order to construct meaning from visual representations of ideas and concepts (Burns, 2006).
- to understand, use (write), think and learn in images (Horton in Cuoco, 2001)

These definitions are connected to Mudaly's (2008) view of the reference of visual literacy to "internal processes of the mind" as a result of physical or mental visual stimuli. Visualisation skills are related to the distinction in the cognitive function between the right and left hemisphere of the brain (Silverman, 2002; Stokes, 2002; Thornton, 2002; Williams, 1983). Accordingly, while the left hemisphere of the brain processes sequentially, analytically and is verbally oriented, the right hemisphere operates spatially and is visually-spatially oriented.

In summarizing the main tenets of visual literacy, the following features are central:

- Visual literacy entails interpreting concrete visual representations such as in art, language, posters, advertisements, propaganda, cartoons, pictures, photography, models, films, diagrams, graphs, images and maps.
- It is associated with the right side of the brain.
- Visualisation and analytical thinking are key components of it.
- It depends on the visual perception and visual-spatial reasoning capacity.
- It is associated with diagram literacy.
- It is a visual and mental process.

2.3 The role of visual literacy in teaching and learning

The emerging significance of visual representations is due to the continuous past emphasis on the textual and verbal modes of learning (Fennema & Romberg, 1999). Learning through visualisation techniques increases the capacity to learn (Stokes, 2002). The use of pictures and other visual representations assists in the learning process (Kadunz & Sträßer, 2004). From the South African education perspective visual literacy is also entrenched in the curriculum. The policy document by the Department of Education (2003) stipulates "critical and creative thinking, analysis of information, communicating effectively using visual, symbols and language and demonstrating an understanding of the world and problem solving" as critical outcomes of educational practices.

While learning is still largely text based (Burns, 2006), children are exposed to visual thinking through words, numbers, pictures, images, patterns, signs and symbols from a very early age (Fisher, 2005). The use of pictures, maps, diagrams and charts are visual strategies that make understanding easier (Williams, 1983). Venn diagram representations are useful visual aids that results in the conversion from abstract to spatial orientation and promotes reasoning (Pylyshyn, 2003).

The following figure (Figure 2), an adaptation of the example that appeared in a Grade 11 Mathematics Paper 3 (Department of Education, 2007), demonstrates this. However visual skills pertaining to intersection and union of sets, as well as knowledge of universal sets are required pre-requisites.

In the Venn diagram, C stands for the set of learners who play cricket, S for those who play soccer and R stands for those who play rugby. The numbers shown indicate the preferences of 160 Grade 11 learners for the three codes of sport. Determine the value of x?

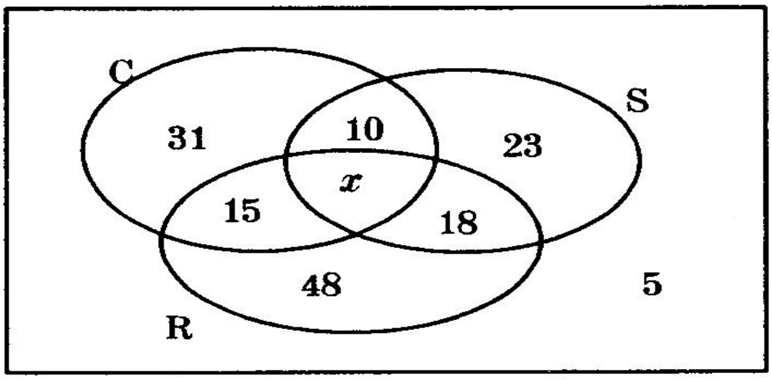


Figure 2: Venn diagram

A distinction is usually made between visual and verbal thinkers (Silverman, 2002; Pylyshyn, 2003; Presmeg, 2006). Silverman (2002) identifies the learning skills of verbal learners as auditory-sequential and those of visual thinkers as visual-spatial. Auditory sequential proficiencies include rote learning, verbal-logic, listening and reading. Visual-spatial skills include complex concept formation, mathematical reasoning and problem solving.

Arcavi (2003) identifies visualisation as being central in mathematical thinking through concept formation, graphical analysis, mathematical proofs, problem solving and visual reasoning. Through graphical representation and data analysis, information that is readily accessible can be obtained by introspecting on factors beyond what is seen. Arcavi (2003) refers to this as “seeing the unseen”, a process of conceptual development when encountering visual representations. In visual interpretation the use of symbols and words are applicable and complementary (Arcavi, 2003). The use of visual proofs in mathematics has been a debatable issue and is often looked down upon in preference for analytical methods of proving (Thornton, 2002; Arcavi, 2003). For Presmeg (1997), this type of reaction is symbolic of a decreasing value accorded to visualisation.

Barwise and Etchemendy (1991) demonstrated visual proofs when they constructed the proof of the Pythagorean Theorem using diagrams and visual analysis like looking at a diagram and then deriving in the mind the properties of the given facts. This is shown in figure 3 below:

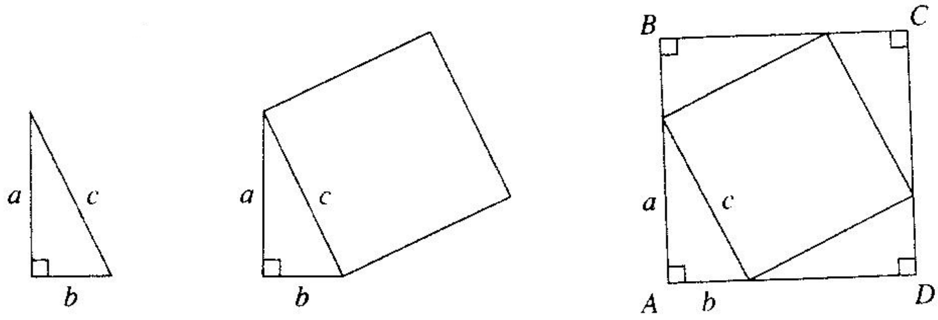


Figure 3: The Pythagorean Theorem

By constructing a square and congruent triangles to the first figure, the area of square ABCD is shown to be $(a + b)^2$ since AB = a + b. However, the area of ABCD can also be expressed as $c^2 + 4(\frac{1}{2}\, a\, b)$ which is equal to $c^2 + 2a\,b$. Therefore $(a + b)^2 = c^2 + 2a\,b$ which becomes $a^2 + b^2 = c^2$. This example depicts the use of geometrical and algebraic techniques, and is largely visually oriented.

Arcavi (2003) presents a student's visual solution (Figure 4) to an arithmetic sequence problem. When given the 10^{th} term = 20 (T_{10}=20) and the sum of the first ten terms is 65 (S_{10}=65), the student was able to determine the ten terms and the common difference by forming visual patterns of arcs and jumps and not through analytical methods.

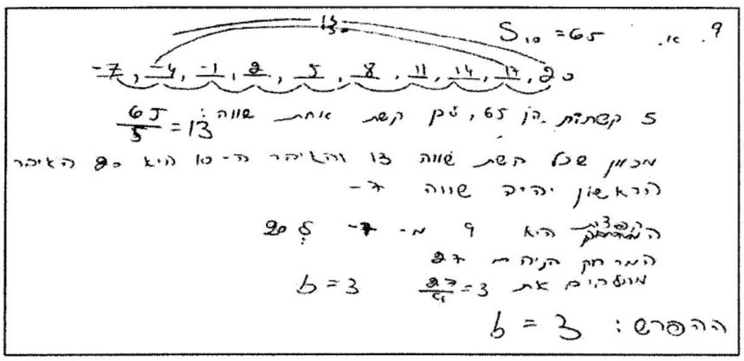


Figure 4: From Arcavi (2003) – arithmetic sequence

The formation of a gestalt image or solution is related to visual analysis as displayed by Thornton (2002) through the use of diagrams and visual clues in problem solving. Thornton (2002) exemplifies the significant role patterns play in inductive and deductive reasoning. Through visual thinking, learners engage in pattern generalisation activities in algebraic and geometric patterns. The derivation of proofs in geometry and the comprehension of algebraic concepts are supplemented through diagrams and visual pattern analysis. The use of dots, squares and arrays facilitate the interpretation of patterns and formulation of algebraic concepts (Thornton, 2002). Rivera (2007) highlights that through visualisation, patterns can lead to generalizations.

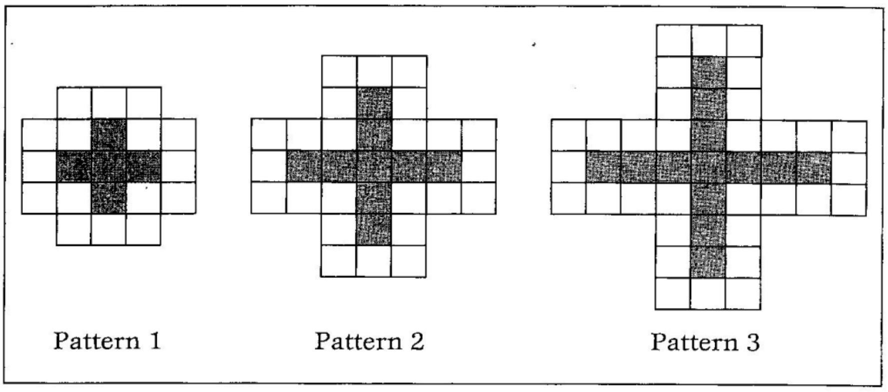


Figure 5: Tiling square problem from Rivera (2007).

Students used visualisation strategies such as the figural additive strategy (adding 4 each time), figural cues and concentric visual counting to arrive at the solution $4n + 1$ as the general term for the number of shaded tiles in the n^{th} pattern.

Arcavi (2003) highlights the role of visualisation in a problem described as an ‘array of matches’ and found that educators utilised visual processes such as decomposition and symbol sense in deriving solutions for the problem. Decomposition is demonstrated as a process of breaking up the whole diagram into parts and rearranging to determine possible patterns. Bergsten (2000, p.123) describes symbol sense as “mathematical thinking facilitated by observing the figurative characteristics of symbols and supported by image schemata”. This is evident when students respond visually to the patterns in algebra such as

$x^2 - y^2 = (x + y)(x - y)$ by manipulating the given symbols (Kirshner & Awtry, 2004). Kruteski and Moses cited in Thornton (2002) show the success of using visual stimulation and creativity in solving analytical geometry problems. Other uses of visual representations in mathematical reasoning include the use of images, metaphors and diagrams (Kadunz & Sträßer, 2004) and the use of concept maps in making learning meaningful (Afamasaga-Fuata'i, 2004).

2.4 Graphical Visual Representations

2.4.1 Related mathematical research

Historically, graphs were used in mathematics in the study of the physical and natural science and thereafter developed as a component of functional relationship through the introduction of the x and y variables, depicting independent and dependent variables (Kieran, 1993). Anderson (2008) traces the historical development of graphs as pictorial representations as way back as 300 BCE. Nicole Oresme (1323-1382) had connected a table of values to a graph. Graphs had progressed to be used as scientific tools depicting the relationship between two quantities. This influenced René Descartes (1596-1650) to connect geometry with algebra and formulate a co-ordinate axis which today is known as the Cartesian plane. However, it was due to the efforts and contributions of mathematicians such as Fermat, Newton and Leibniz that led to the modern graphs that we encounter today.

Kleiner (1993) describes a function as represented by a formula, a rule, a correspondence, a relation between variables, a table of values, a graph, a mapping, a transformation, an operation or a set of ordered pairs. The visual and symbolic nature of functional relationships is portrayed through tables, formulae and graphs. The usual instructional technique of learning functions is characterised by the three core representation system (O'Callaghan, 1998). This is referred to as NAG- numerical (table), algebraic (formula) and graphical (Verhage et al., 2000). The Cartesian graph depicts a functional relationship usually drawn by first tabulating values that satisfy the equation, corresponding points are plotted and the points are eventually joined freehand (Skemp, 1986). Janvier (1987) refers to diagrams, tables, Cartesian graphs, letters and brackets as abstract modes of representations. Visual dimensions, referred to as specifiers, such as lines, marks on the graphs and the labels are part of graphical representations (Friel, Curcio & George, 2001).

The Cartesian plane, its representation including the number line, the x and y axes, co-ordinates and arrowheads is a complex metaphorical structure (English, 1997) and a meaningful and structured representational system for discovery of patterns (Goldin & Shteingold, 2001). Goldin & Shteingold (2001) differentiates between the external representational systems comprising the conventions of algebraic notation, the number line and Cartesian co-ordinates and the internal representational system comprising of the student's own spatial understanding of the Cartesian plane. According to Dörfler (2000), the number line being a prototype for the set of integers (*Z*), is regarded as an operative prototype because actions needs to be performed on the number line in order to obtain rational numbers. Likewise, the Cartesian graph is classified as a relational prototype whereby the graph on paper has to be used for further operations. The number line and the Cartesian graph can therefore be regarded as carriers of meaning.

A graph is defined as "information transmitted by the position of a point, line or area of a two dimensional surface" (Friel et al, 2001). It is referred to as a diagrammatic representation by Kadunz and Sträßer (2004), as a geometric representation and pictures of relationships by Romberg et al (1993) and very commonly referred to as a collection of points in the Cartesian plane. The Cartesian graphs of linear functions are referred to as "an underlying representation of conceptual structures" (Arcavi, 2003). Graphs are meant for visual interpretation and their symbolic features possess meaning which have to be 'read' (Pimm, 1995).

Graphical interpretation has posed challenges for many students and they have displayed numerous misconceptions and alternate conceptions. Initial research evidence relate to students confusing the physical aspect of lines on graphs depicting real-life situations (Kieran, 1993). This is similar to Pimm's (1995) reference to research where students made literal interpretations of graphs such as positive gradients depicting uphill walks in a distance versus time graph. This is referred to as figurative association where the visual feature of the shape of the graph is related to the problem. Arcavi (2003) refers to this as "pictorial distraction" where visually salient information is interpreted and the underlying meanings are not considered. Kieran (1993) also cites research when students were asked to justify whether the co-ordinates (4; 5) existed on $y = x^2 - 3x + 1$. The students displayed finite orientation by their failure to associate the graphical with the algebraic and resulted in the misconception of not believing that any number can be substituted into the formula.

According to Norman (1992), students favour the symbolic rather than the graphical form due to an inadequate visual understanding of the aspects entrenched in graphs. This relationship between the graph and the equation is referred to as "Cartesian connection" and always poses difficulty for learners (Anderson, 2007). The metaphorical structure of the Cartesian plane appears complex to students, especially with difficulties on the recognition of rational and irrational numbers on the number line where students indicated that there are no numbers between two whole numbers, as well as the infinite extension of the number line (English, 1997). According to Williams (1993) the "topology of the real line" as represented in graphs, refers to the understanding of the correlation between the curve and a point, and continuity of curves. In a study conducted by Williams (1993) students gave alternate conceptions of the meaning of graphs such as "there is like a point and in that point there are many more points".

The ability to transfer knowledge of graphical functional relationships to real world situations is of great significance (Van Streun, 2004). Verhage et al (2000) explored the interrelationships between graphs, formulae and tables in mathematical modelling tasks relating graphs to real-life contexts. They ascertained that learners experienced difficulties in discussing the behaviour of graphs when describing maximum, minimum, increasing and decreasing functions. Similar research on real-life contexts referred to as "Global Graphs" found that learners' misconceptions were related to their interpretation and understanding of the qualitative features such as the shape of the graph, rise, fall and the x and y axes, all of which are visually interpreted (Julie et al., 1998).

Research studies confirm that learners lack graphing skills in mathematics and even in other subjects like physical science (Asli, 2001). This might be related to the order in which graphs are introduced to learners referred to as the translation process of going from one mode to another such as from table to graph sketching (Janvier, 1987). Graphs are usually taught after the numerical and symbolic stages (Romberg et al, 1993). However the use of qualitative graphs (graphs that depict relationships between a graph and a verbal statement or practical situation) should be done first (Van Dyke, 2002). Thereafter, the quantitative graphs (tables or algebraic expressions) should be taught and lastly the equation (abstract) must be presented. Graphs are usually taught as an end in itself and the global meanings are ignored in mathematics classrooms (Asli, 2001). Van Dooren (2008) cites research examples when students often depict a fixation with the linearity concept. These include students drawing a straight line when asked to draw a graph of any function through two points and searching for straight line relationships when viewing a parabolic curve.

Since Cartesian graphs form part of functional relationships, the understanding of the function concept is relevant. Sierpinska (1992) emphasised that the function concept depicted relationships of change where the x and the y represent changing objects which are visible numerically and geometrically through activities such as rotation, translation and reflection. A change in one quantity sometimes producing a change in the other in a pattern-like manner is typical of visual graphical actions. English (1997) further describes the formation of the Cartesian function as a blend of the numerical and geometrical, whereby the ordered pairs become a curved or a straight line trajectory. According to Sierpinska (1992) students displayed difficulties in understanding the connection among the different modes of functional representations. To the students f(x) meant substituting into the formula and then computing its value.

Visual representations have played a significant role in the curriculum reform process where understanding and interpretation of calculus concepts and applications depended on graphical or diagrammatical representations (Zimmermann & Cunningham, 1991). Calculus concepts such as continuity, derivatives, increasing and decreasing values of a function, convexity, local maximum and minimum and the definite integral can be taught using visual forms (Eisenberg & Dreyfus, 1991; Tall, 1991). Students were found to follow the algorithmic method of solving calculus problems rather than employing visual techniques. This might be the reason for students performing well in differential and integral calculus, but failing to internalise relationships among first and second derivatives, limits and continuity(Cooley, Trigueros, & Baker, 2007).

Other difficulties include the derivative of a curve at a point in relation to the tangent concept (Tall, 1991). Asiala et al (1997) introduced a visual instructional treatment where students formed mental constructions and improved their understanding of derivative, tangent and slope. Further studies included the diagrammatical representations high school students formulated when answering calculus problems (Bremigan, 2005). The significant role of visual representations, as well as the effect of the symbolic and verbal representations on mathematical thinking was identified. Through modification and reconstruction of diagrams, students were able to make progress when concentrating on important aspects of their diagrams.

2.4.2 Graphs and computer technology

Graphic calculators and computer technology have contributed to students understanding of graphical functional relationships (Romberg et al, 1993;

Zimmermann & Cunningham, 1991). Pimm (1995) however shows his dissatisfaction with computers and graphic calculators as they might prevent students' engagement and development of understanding in mathematics. This is in line with Mudaly's (2008) argument that whilst one can see an object, understanding only occurs when 'seeing' requires some active mental reaction to what is seen. The calculator, for example, enables learners to derive the salient points that learners need in order to sketch a graph of the function $f(x) = \sin 2x$. By using particular buttons on the calculator the student can draw a fairly accurate graph of the function $f(x)$, but with little understanding of the actual function itself. By rote, they memorise the basic shape of graphs and are able to construct transformations of $f(x)$ by using the calculator, without any knowledge of the transformation involved.

The process of visualisation is related to computers, and as such computers are regarded as visualisation tools. Computer-based visualisation may be static, dynamic or interactive, transforming the symbolic into geometric (Zimmermann & Cunningham, 1991). While technology primarily generates a visual picture graphically of the function, Romberg et al (1993) maintain that there is the potential to allocate secondary significance to the use of formula. The use of numeric, symbolic and graphical representations is essential for visualisation. The use of technology should not be seen as an end in itself, but when used together with pencil and paper graphical activities, provides a viable means to problem solving , understanding and development of visualisation skills (Zimmermann & Cunningham, 1991).

According to Dugdale in Romberg et al (1993) computer software such as *Eureka*, *Pathfinder* and *Green Globs* influence students understanding of the global meanings of graphs beyond merely plotting of points and reading of values. Research on children's interpretation of graphs found that they could comprehend computer generated graphs even without being taught about graphs (Pimm, 1995). Students lacked the conceptual understanding of the qualitative features of graphs and comprehension was limited to the physical features of graphs (Cates, 2002).

Kieran (1993) displays the role of computer technology in performing transformations of graphs such as in $y = -2(x - 3)^2 + 2$. The transformation process starts with the graph of $y = -x^2$, and then shifts three units to the right and two units up. The Green Globs provides dynamic exposure to basic graph concepts in a concrete form such as transforming $y = x^2$ into $y = x^2 + 3$ as well as the gradient and y-intercept of $y = mx + c$ focusing on straight lines such as $y = 3x + 1$, $y = 2x + 1$ and $y = 4x + 1$ (Beigie, 2005). However in a study conducted by Magidson (as reported in Arcavi, 2003), where these linear functions were used to analyse student's interpretation, the students focussed on other visual descriptions other than the y-intercept concept. The intended connection between the equation and the graph was not derived.

The effects of the parameters on graphs can be investigated through the dynamic display of graphs using the Geometer's Sketchpad and Autograph (Boshoff, 2007). These effects such as m, and c in $y = mx + c$; a, h and k in $y = a(x - h)^2 + k$; a, b and c in $y = ax^2 + bx + c$ and a and q in $y = a\sin(x) + q$ can be deduced through interactive procedures. Research conducted by Mudaly (2008) showed that learners improved their understanding of geometrical problems after undergoing visual investigations on the Geometers Sketchpad. In this study the Geometers Sketchpad allowed learners to visualise, analyse, discover and make conjectures about their observations.

2.4.3 The National Curriculum Statement

The National Curriculum Statement Grade 10-12 (Department of Education, 2003) policy document for mathematics stipulates four learning outcomes which indicate what learners are to achieve in each learning outcome. According to Learning Outcome Two (LO2), stated as Functions and Algebra:
"The learner is able to investigate, analyse, describe and represent a wide range of functions and solve related problems".

The assessment standards which prescribe competences for graphical functional relationships for Grade 10 and Grade 11 include:

- the understanding of patterns and functions.
- use of mathematical models in describing real-life context graphs.
- recognising relationships in tabular, symbolic and graphical representations.
- converting between the above forms.

Learners are expected to generate the following graphs first through point-by-point plotting and then through generalisations and conjecturing about the parameters of *a*, *q*, *k* and *p*:

Grade 10	**Grade 11**	**Description**
$y = a x + q$		Straight line
$y = a x^2 + q$	$y = a (x + p)^2 + q$	Parabola
$y = \frac{a}{x} + q$	$y = \frac{a}{x + p} + q$	Hyperbola
$y = a b^{x} + q$; $b > 0$	$y = a b^{x+p} + q$; $b > 0$	Exponential
$y = a \sin(x) + q$	$y = \sin(k x)$ and $y = \sin(x + p)$	Trigonometric
$y = a \cos(x) + q$	$y = \cos(k x)$ and $y = \cos(x + p)$	
$y = a \tan(x) + q$	$y = \tan(k x)$ and $y = \tan(x + p)$	

Table 1: The equations and types of graphical representations

Learners are expected to identify the following characteristics and use these to sketch the graphs of the above functions:

- domain and range.
- intercepts with the axes.
- turning points, maxima and minima.

- asymptotes.
- shape and symmetry.
- intervals on which increasing and decreasing functions occur.
- discrete and continuous nature of graphs.
- periodicity and amplitude.

Learners are expected to perform operations such as drawing a sketch graph given the equation, determining the equation of a graph given the sketch of the graph, apply knowledge of above characteristics, investigate the properties of graphs and perform transformations of graphs. These expectations of learners demand higher order interpretive skills. The parameters together with the graphical concepts have a visual starting point and are hence associated with visual literacy skills.

2.5 Graphical actions and visualisation skills

Particular graphical actions and skills are necessary components for visual learning or visual literacy to occur. Tabulation, plotting of points, curve sketching and reading off values are the usual graphical skills emphasised in the past by textbooks and educators (Kieran, 1993). According to Kieran (1993), this traditional skill has undergone change with the emphasis now on interpreting qualitative features of graphs, examining the role of parameters and the application of graphs to problem solving, contextual situations. The interpretive and comprehension skills are inherently linked to the understanding of graphs. Friel, Curcio and George (2001) in their study with statistical graphs refer to graph comprehension and graph sense as abilities to gain meaning from graphs. They emphasise comprehension activities such as translation (descriptive level), interpretation (seeking relationships), and extrapolation and intrapolation (consequences of graphs). This exemplifies that graphs have the potential to bypass symbolic notation.

O’ Callaghan (1998) refers to translation as a procedural skill in advancing from tabular to symbolic to graphical. However Janvier (1987) regards the translation process of computing values and plotting of points as a traditional form. He suggests a problem based ideology where the focus is on the interpretative aspect of graphs. In alternating between the visual form of tables, words, symbols and graphs various skills and processes are identified. These are pattern finding, parameter recognition, data generation and qualitative graphical interpretation. Many authors proclaim that the interpretation of graphical representations is more superior to merely sketching of graphs (Dugdale in Romberg et al, 1993). Yerushalmy and Schwartz (1993) utilise operations such as translating, dilation and reflection, while O’Callaghan (1998) highlights four competencies in modelling, interpreting, translating and reifying. These conceptual skills have consequences for linking the visual skills to higher mental operations.

Zimmermann and Cunningham (1991) identify various graphical and visual skills pertinent to calculus understanding in reinforcing visualisation as a mathematical skill. These include:

- Focusing on pertinent mathematics detail in the Cartesian plane, connecting information and thoughts and the removal of conflicting information.
- Elementary function sketching skills and intuitive plotting of functions.
- The recognition of the visual structure in graphs.
- A geometrical awareness to be able to understand symmetry, transformations and similarity, as well as engaging in pattern recognition.
- To move between a graphical (geometric) and an analytical (algebraic) approach.

Perhaps the visual index skill proposed by Pylyshyn (2003) whereby diagrams could be visually targeted by focussing on parts of the representation, could be applied to graphical representations. This is similar to decomposing diagrams as exemplified by Arcavi (2003). However pre-knowledge is an important requirement before engaging in a graphical procedure such as cited by Tall in Zimmerman and Cunningham (1991) whereby students' intuitive understanding of limits depended on the gradient concept. Furthermore, the rules, conventions and notations of a graph must be understood well for visual understanding to occur. This implies that symbol sense is essential for visual analysis to occur.

Goldenberg in Zimmerman and Cunningham (1991) relates graphical skills to computer technology, which are applicable to pencil and paper graphs as they reinforce a visual focal orientation. These skills include the modification of functions by changing the parameters such as the *c in* y = m x + *c*, observing the effect of *p* in $y = (x - p)^2$, comparing functions such as f(x) = sin x and g(x) = sin x + 2. Further skills mentioned is the identification of regions and markings of graphs, such as the vertex, x and y intercepts, and the turning point, all of which are significant visual starting points. Perhaps the most important skill for learners to possess is to have a sense for functions and to be able to visualise graphs (Eisenberg, 1992). Eisenberg (1992) affirms that to have function sense is to be able to sketch the graphs of f (k x), f (x ± k), f' (x) and f^{-1} (x) when given the graph of f(x). In this way reversing a particular action is a graphical skill that will test the level of understanding.

Arcavi (2003) demonstrates the importance of contexts in which diagrams or images are presented. In this respect the visual implications when presented with three parallel lines on the Cartesian plane are highlighted. These include that the lines are linear functions, belong to the equation y = a x + b, have the same gradient and that there is no simultaneous equation. When examining the following example cited by Eisenberg and Dreyfus (1991) and used by Asiala et al (1997) in their research, the task of identifying the intended visual implications and graphical skills is pertinent:

If L is the tangent to the curve y = f(x) at a point (5; 3), determine f (5) and f ′ (5).

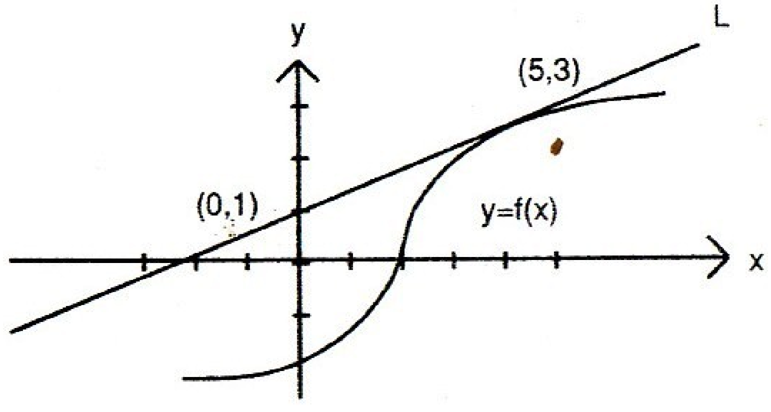


Figure 6: Visual skill-from Eisenberg and Dreyfus (1991).

Their (Asiala et al, 1997) main interests in the above example were the students' visual understanding of y = f(x), the co-ordinate system and the derivative concept relating f′(x) to the gradient of the tangent at the point (x; f(x)). This association of connecting symbolism to imagery is a visual, graphical expectation (Pimm, 1995).

The second example, an adaptation of the Grade 11 Mathematics Paper Two question paper (Department of Education, 2008) also depicts visual expectations:
The diagram below represents the graphs of the functions f and g for x ϵ [0°; 270°]
f and g intersect at A (30°; 0,87) and at B.

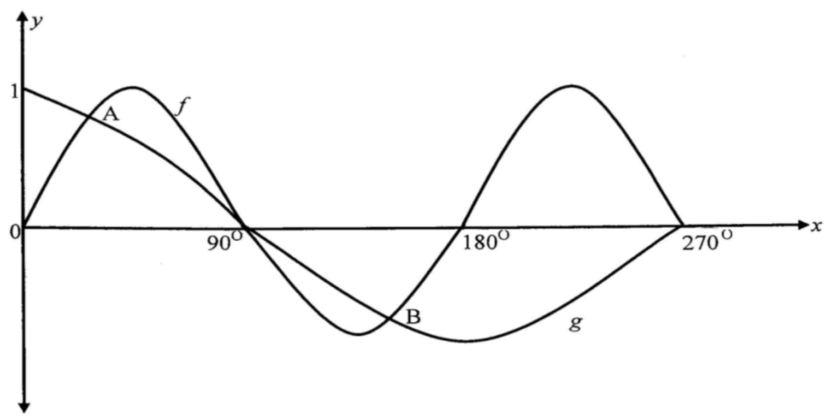


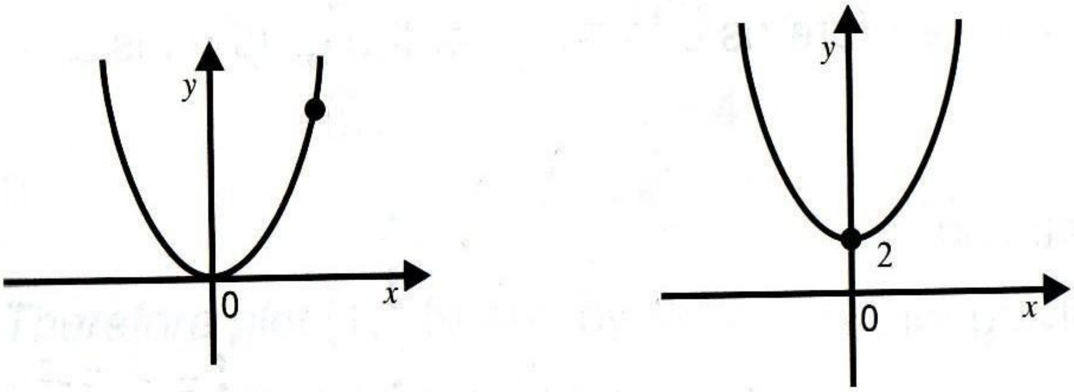
Figure 7: Trigonometric functions.

- Determine the equations of f and g.
- What is the range of g?
- For which values of x is $f(x) \geq g(x)$?
- For which values of x is $f(x) \geq 0$?
- For which values of x is g(x) decreasing as x increases?
- Write down the new equation of g if it is shifted 30° horizontally to the right.
- Write down the co-ordinates of B.
- Determine the equation of h(x) if $h(x) = g(x) + 1$

The intended visual implications for this example include the visual and mental images of the trigonometric functions, the recognition of the parameters and patterns, connecting symbolism to imagery and the use of graphical terminology. The sketched graph itself assumes a symbol and if the learners understood their symbols well, then finding the equations for f and g would be easy. Each of the questions asked requires visual and spatial orientation. A similar question could be formulated by sketching the graphs on an axes, without indicating which is f and which is g. Two equations with variables are given and learners are expected to determine which of the equations is appropriate to which of the given graphs. This further emphasises the role played by symbolic notation in mathematics.

The transformation of graphs is a common feature in graphical interpretation. Translation, reflection and rotation skills are essential graphical skills that require visual thinking.

The following example, extracted from a Grade 10 mathematics text book, depicts the translation skill:



$f(x) = x^2$ → $g(x) = f(x) + 2$

Figure 8: Translation of graphs

2.6. Developing a conceptual understanding of graphical representations

2.6.1 Introduction

The development of functional relationships is stipulated in the National Curriculum Statement (NCS) for Grades 7-9 and incorporated in the learning outcome Patterns, Functions and Algebra (Department of Education, 2002). In Grade 7, learners are expected to represent relationships through verbal descriptions, flow diagrams and tables, as well as interpreting and drawing graphs depicting real-life situations. In Grade 8 similar skills are prescribed, but trends and features such as linear, non-linear, increasing, decreasing, maximum and minimum, the use of equations, expressions, symbols and variables are included. In Grade 9, learners must draw and interpret linear graphs on the Cartesian plane. This foundation from Grades 7-9 forms the basis for the development, interpretation and assimilation of graphical concepts required in Grades 10-12. Most often functional relationships are represented by arbitrary association, using only formulae and graphs, but the emphasis in the senior phase (Grade 10-12) is on notation, relationships between various representations and, analytical and graphical skills. This approach to graphical functional relationships is similar to the Zimbabwe school system as demonstrated by Kwari (2007).

However, linearity and linear reasoning commences at a very early age in children and students were found to continuously rely on linearity in their future years (Van Doreen et al, 2008). In the early stages of developing the function concept, learners perform on the operational level through different stages of thought levels (Sfard, 1992). According to Kleiner (1993) the historical development of the function concept since the 18th century has influenced the way we approach the various representations of functions. He identifies the relationship between algebra and geometry that has been instrumental in our present approach to functional relationships. This is depicted as a shift from:

- Euclidean geometry to the analytical and graphical aspects of functions.
- dependence relationships expressed as equations and formula to rules of correspondence.
- procedural approach to a structural approach.
- geometric to symbolic.
- real life to arbitrary correspondence.

With the emphasis shifting to symbolic and spatial characteristics, understanding therefore has now evolved to include analytical and visualisation skills. Understanding is viewed as a mental experience and graphical representations can be transformed, interpreted and constructed through enactive, iconic and symbolic mental representations in the process of understanding (Sierpinska, 1994). Pimm (1995) regards the relationship between symbols and images as being central to developing understanding in mathematics and the derivation of mathematical meaning. Deriving meaning, making sense, reasoning, solving, perceiving and being insightful constitute understanding in mathematics (Sierpinska, 1994). Graphs which serve as display representations are meant for visual interpretation

and this leads to the formation of images where understanding occurs through connections and associations (Pimm, 1995).

It is significant to ascertain what it really means to understand graphical representations. Williams (1993) interrogates what it really means to do and know graphical functional representations. He regards understanding as being inclusive of interpreting qualitative features of graphs, engaging in understanding functions as a process and as an object, and lastly translating among the three modes of representations. These include the meaning inferred by:

- Describing functions as defined, non-defined, increasing, decreasing, continuous and smooth.
- Extracting the components of a function such as zeros, derivatives and infinity.
- Performing operations such as plotting of points and calculating values such as f (2).
- Utilising functions for representing relationships, modelling, predicting, interpolating and approximating.
- Understanding that $y = 2x + 3$ represents a linear function.

Yerushalmy in Romberg et al (1993) advocates pre-requisites for graphical understanding and these resemble visual skills. These include the understanding of the three mode representational system, the translation process, classification of graphs, the role of parameters and geometrical spatial transformation.

2.6.2. Conceptual versus procedural understanding

Although this study focuses on the conceptual understanding of graphical representations, the distinction between conceptual and procedural understanding has always dominated mathematical inquiry in the pedagogical domain. These two dimensions of learning have always been interpreted and represented in different ways. Kilpatric, Swafford and Findell (2001) identify conceptual understanding and procedural fluency as two mathematical proficient strands. They differentiate between these where the former pertains to comprehension of and relationships among concepts, while the latter refers to the application of rules, procedures and skills in computations. This distinction is similar to Skemp's (1976) reference to relational understanding as a means of knowing how to perform mathematical actions with providing reasons, and instrumental understanding as a means to the application of rules and procedures without providing reasons. The dichotomy between these two constructs of understanding is described as conceptual knowledge and procedural knowledge (Eisenhart et al, 1993; White & Mitchelmore, 1996; Long, 2005).

Conceptual knowledge is described as constructing and creating relationships between mathematical objects that exist within the given mathematical problem or it might incorporate new forms of knowledge. This aspect of linking and connecting mathematical ideas is the underlying structure of mathematics according to Eisenhart et al (1993) and is the cornerstone of conceptual understanding. Procedural knowledge implies knowing the formal aspect of mathematics, algorithms, rules, definitions and strategies for performing tasks and is regarded as a computational skill.

The debate and tension around these two theories of understanding include whether conceptual or procedural understanding should occur first and whether these two are linked (Long, 2005; Pimm, 1995). According to Eisenhart et al (1993) teaching for understanding should include both procedural and conceptual knowledge. Pimm (1995) emphasises "understanding before doing" or understanding before mastery. Kilpatric et al (2001), in recognising the significance of procedural proficiency in mathematics, discusses the link between conceptual understanding and procedural fluency where procedural fluency can actually support conceptual understanding and advocates a combination of these two constructs referred to as proceptual understanding. Long (2005) motivates for

the inter-relationship and correlation of the two as well as the integration of procedures with the understanding of concepts.

The development of conceptual understanding is shown to be influenced by introducing teaching and learning activities in the development of a graphical understanding of the derivative and function concepts (Asiala et al, 1997). White and Mitchelmore (1996) identified students' failure to identify relationships and deriving meaning when working with maximising functions. These students displayed manipulations and procedural techniques when using x and y symbols. Conceptual understanding therefore depends on the meaning that is derived or transmitted. Noss, Healy and Hoyles (1997) point out the disadvantages in 'pattern-spotting' exercises in algebra and geometry where students draw tables to derive generalisations, but with little or no conceptual understanding. Here Thornton (2002) emphasises the advantages of visualising the relationships as a means to conceptual understanding. Very often, rules and definitions can be memorized mechanically or by rote with little or no conceptual understanding (Goldin & Shteingold, 2001).

However, in this process learners may experience cognitive obstacles when faced with the conflict between their personal, internal psychological meaning of symbols, notations and graphs and the conventional mathematical meaning taught to them (Goldin & Shteingold, 2001). Sierpinska (1994) highlighted the effect of cultural constraints on understanding. The cultural background, language, beliefs and communication may lead to the formation of epistemological obstacles in the process of understanding. Williams (1993) identified the role of the student's "phenomenal world" incorporating the social and cultural practices as influencing their cultural understanding of graphs.

2.7 The role of imagery, analogies, metaphors and diagrams.

Our conceptual understanding of graphs is dependent on the way graphs are presented. Graphs by their very diagrammatic nature and symbolic forms are

impregnated with visual imagery, whether concrete or abstract. The relationship of mathematical concepts to something else or something similar is a common trend in mathematics. Therefore, the application of analogies, metaphors, metonyms and images to mathematical reasoning is linked to the way learners mentally construct mathematical ideas (Presmeg, 1997). These have a bearing on visualisation since analogies and metaphors are used to comprehend images and diagrams by providing meaning of the words and visual representations (Kadunz & Sträßer, 2004). The analogies, metaphors and images are referred to as "vehicles of thought" as they contribute to conceptual understanding (English, 1997).

Although imagery has been traditionally associated with the formation of pictures in the mind, an image is expressed in various ways. It is defined as a "mental construct depicting visual or spatial information" (Presmeg, 1997), a "mental construction which can be constructed, re-presented and transformed" (Wheatley, 1997) and a representation of many relations (Kadunz & Sträßer, 2004). Pylyshyn (2003) outlined how a visual image is formed through this simple mathematical problem: "John is taller than Mary, but shorter than Susan. Who is the tallest?" Imagery can be perceived according to the distinction of concrete-abstract, static-dynamic and holistic-sequential (Presmeg, 1997). In her 1985 research study on the role of visual processes in Grade 11 mathematics tasks in algebra, geometry and trigonometry, Presmeg focussed on Grade 11 learners who preferred visual thinking. These learners were referred to as visualisers. Her study revealed the following:

- Imagery is commonly used by learners in mathematical reasoning and problem solving.
- The learners displayed concrete, pattern, kinaesthetic, dynamic and memory imagery.

- The first forms of imagery that learners formed (prototypical imagery) such as in triangles, circle theorems in geometry and the parabola, were obstacles to their cognitive development. This imagery is referred to as uncontrollable imagery.
- Learners interpreted lines as parallel if they looked parallel.
- Learners who used dynamic imagery found them useful in solving mathematical problems.
- Learners often use visualisation in constructing mathematical meaning.

Mathematical learning is viewed as taking place through the continual complementary and metaphorical link between images and diagrams (Kadunz & Sträßer, 2004). In this situation, learners can formulate analogies with the representations they work with, or transform these representations metaphorically. They may also want to use algorithms in this process; the choice is really theirs.

Metaphors (normally used in literary circles as a figure of speech) are seen to be transporters of meaning in mathematics, creating similarities and new knowledge. Its validity is dependent on the creation of a mental conceptualization when comparing to another (English, 1997; Kadunz & Sträßer, 2004). The relation of metaphors to the cognitive domain is further confirmed by describing metaphors as “a matter of thought” (Lakoff & Núňez, 1997). An analogy is a relational correspondence of transferring information from one part to another by matching and mapping processes (English, 1997). Graphical representations on the Cartesian plane are diagrams that have been created according to recognised rules and procedures.

However, according to Wheatley (1997), “diagrams do not communicate, they evoke thoughts in an imagistic form”. As an example, when a student encounters $y = x^2 - 5x$, he undergoes image-based reasoning and formulates an image of the parabola. However, diagrams serve a recursive function, as they lead to formation of images, then diagrams, then images and so on (Wheatley, 1997).

This process of mathematical reasoning may be depicted by the following diagram:

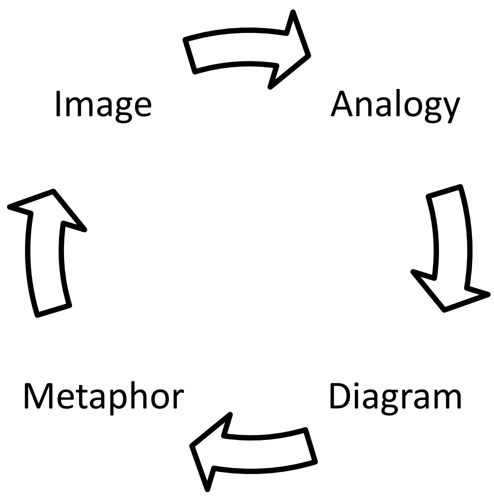


Figure 9: The mental process of mathematical reasoning.

Metaphors such as function, finite, infinite, reflect, rotate, continuous, increase, decrease, rise and fall impact on the interpretation of graphs. The Cartesian plane is regarded as a metaphorical structure consisting of axes, number line and symbols (Lakoff & Núňez, 1997). The understanding of a graph as a collection of points allows for the conceptualization of equations with x and y co-ordinates and infinity (∞) as a point metaphor allows for the conceptualization of infinity (English, 1997). The number line in the Cartesian plane is complex line metaphor belonging to the number system with points on a line and demands visual and symbolic understanding (English, 1997). Lakoff & Núňez (1997) relates the development of the Cartesian function as a metaphorical unit originating in the function concept as a set of ordered pairs, proceeding to the co-ordinate system as consisting of co-ordinate lines with points as intersection metaphors and eventually the graph as a collection of points.

2.8 Educators' mathematical knowledge of graphical functional relationships

Research on educators' mathematical knowledge and how their knowledge influences the teaching and learning of mathematical concepts has been scarce (Norman, 1992; Hill, Deborah & Schilling, 2008). In particular, Norman (1992), remarks on the insufficient investigations conducted on educators' knowledge and application of functions. Ascertaining the educators' conceptual understanding of functional relationships is related to understanding the way they teach and organise their classroom activities around the concept of functions (Norman, 1992). Hill, Deborah and Schilling (2008) present a model of mathematical knowledge for teaching that distinguishes between the pedagogical content knowledge (PCK) and the subject matter knowledge. Pedagogical content knowledge comprises of the knowledge of content and students (KCS), the knowledge of content and teaching (KCT) and the knowledge of the curriculum. However, these authors maintain that there has been little attempt to study the effect of PCK on student success in mathematics.

Cooney and Wilson (1993) maintain that the conceptual understanding held by educators, affects their classroom practise. Norman (1992) classifies the understanding of functional relationships by educators according to three categories. These are the exemplification and characterization of functions, application of functions and functional reasoning. In the first category understanding is related to definitions and characteristics of functions such as continuity, domain and range. The second category comprises the application to modelled contexts while the third entails analytical and interpretive skills. In the study with ten educators on their interpretation of the function concept, Norman (1992) concluded that while educators display diverse cognitive views, they display certain types of graphical actions. Educators were fixed on single techniques in graphical interpretation of the function concept, preferred graphical rather than algebraic forms and employed the traditional methods when commencing with functions.

Presmeg (2006) refers to the teaching visuality levels of educators in her study to classify the role of educators in using visualisation techniques in the classroom. She examined visual thinking strategies such as pictorial representations, use of educator’s own imagery, spatial inscriptions, pattern seeking methodology and delayed use of symbolism. It was concluded that educators’ teaching visuality score did not correspond to their level of knowledge of mathematical visuality.

2.9 Conclusion

The literature review provides a basis for the positioning of visual literacy in mathematics teaching and learning. The relationships between graphical interpretation and visualisation skills have been endorsed in some areas of mathematical research such as through computer technology and the calculus reform process. Furthermore, graphs by their very nature of its Cartesian presentation and syntactical significance have been well recognised in mathematics. The development of graphical functional relationships over the years has lead to a demand for new skills, such as visual literacy, to be included in graphical interpretation. The various types of graphical visual literacy skills intertwined with imagery, analogies, metaphors and diagrams impacts on visualisation techniques.

CHAPTER THREE

THEORETICAL FRAMEWORK

3.1 Introduction

A theoretical framework associated with the visual aspects of mathematics teaching and learning is necessary in providing a theory for functional graphical relationships in respect to graphical interpretation and conceptual understanding. The impact of visualisation in mathematics has been neglected in the past and various recent attempts at developing a theory based on the cognitive and affective domains have been investigated. According to Presmeg (2008) these attempts include visualisation in problem solving, three-dimensional geometry, levels of imagery and inscriptions, and Peircean semiotics. Zazkis et al (1996) proposed the Visualiser/Analyzer model that connects visual and analytical processes. However, the APOS Theory (actions, process, object and schema), an extension of Piaget's reflective abstraction, has been related to function relationships. Although the emphasis with this theory is largely based on cognitive processes, it is linked to the formation of graphical schemas. Berger (2006) classifies the APOS model which incorporates encapsulation and reification levels as neo-Piagetian because it focuses only on interiorised processes and neglects the role of language, signs and tools as social interactive processes. These have an impact on conceptual understanding. This standpoint entrenches the role of semiotics from a Peircean perspective, as well as Vygotsky's semiotic mediation as useful visual literacy theoretical frameworks.

The above visualisation-related theories are enshrined in the overarching theoretical perspective of constructivism since they entail the construction of mathematical knowledge. According to Harel and Dubinsky (1992) constructivism is defined as "a theory of learning in which knowledge is constructed by the individual in the mind, as opposed to being intrinsic from birth

or existing independently of human interaction". In constructivism learners are identified as active participants of constructing and reconstructing their mathematical meaning (Orton, 1992). The learners' existing knowledge is adapted through their conscious interaction with the educator and the study material. In this way they build new knowledge. Mathematics becomes social in nature and knowledge is therefore constructed through social interactions, referred to as social constructivism (Ernest, 1991). The theories on visual literacy can be represented as follows:

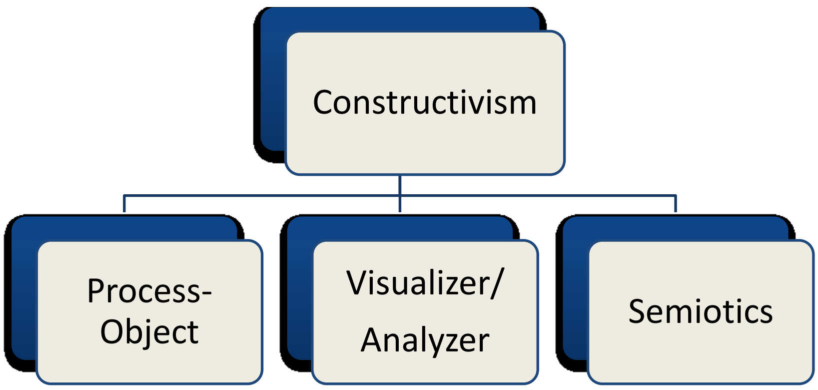


Figure 10: Theoretical perspectives of visual literacy

3.2 The APOS perspective

Piaget outlined the process of interiorisation as the utilisation of symbols, language, pictures and mental images to construct internal thought processes (Dubinsky, 1991). According to the APOS theory, in order to understand a mathematical concept and construct mathematical knowledge, the learner moves between different stages of the process-object theory (Berger, 2006). Bowie (2000) describes the following four conceptions in the APOS theory:

- The action conception is the transformation of objects by repeated physical or mental operations.

- Reflection upon an action with subsequent mental transformation and interiorisation is referred to as the process.
- A process becomes an object conception when actions and processes become a totality.
- The co-ordination of a collection of objects and processes result in the formation of a schema.

Reflective abstraction guides the construction of mathematical knowledge from a lower to a higher level as from action to representation (Cooley, Trigueros & Baker, 2007). Dubinsky (1991) outlines five notions of constructions that characterise advanced mathematical thinking. Firstly, interiorisation occurs through the conscious reflection on actions and inter-relationships. When two or more processes combine to form a new one, co-ordination takes place. Once the process has been mastered, the process becomes detached and the object is conceptualised through reification (Sfard, 1992) or encapsulated according to Dubinsky (1991). When schemas are transferred to different contexts generalisation occurs. Finally, reversal takes place when a new process is constructed by reversing an interiorised thought process.

The development of the graphical schema is demonstrated by Yerushalmy and Schwartz (1993) using the process-object standpoint. Initially, an action is performed on an object such as the symbolic expression. The object of the action transcends to the process stage which thereafter becomes encapsulated into a new object. According to these authors the process perspective is highlighted through the symbolic representation, while the graphical representation produces an entity appearance. The process-object view is further elaborated by Moschkovich, Schoenfeld and Arcavi (1993) where the process aspect is related to the x and y links in functions. The object perspective is only considered by executing actions on it through transforming functional relationships. Graphical actions such as rotation and translation portray the object aspect. The process actions are depicted by examples such as:

- Given $y = 3x + 2$, determine the value of y when $x = 5$.
- Why is 4 the y intercept in $y = 3x + 4$?
- Determine the x intercept in $y = 2x - 2$.

In accordance with Skemp's (1976) description of understanding, these examples might depict procedural or instrumental understanding. The following example which is used to highlight the object perspective, is similar to relational or conceptual understanding:

- Given $f(x) = x^3 - 3x^2 + 2x$, determine the y values when $x = 0; 1; 2; 3$ and 4.
- If $g(x) = x^3 - 3x^2 + 2x + 1$, determine the y values when $x = 0; 1; 2; 3$ and 4.

If one computes g(x) without referring to f(x), then one is functioning in the process level and if one ascertains that $g(x) = f(x) + 1$, then functioning occurs at the object level. In interpreting functional relationships, connection and flexibility between both the process and object perspectives allows for movement between graphs and equations (Moschkovich et al, 1993).

Sfard (1992) describes the process-object model in terms of the process occurring at the operational level and the object at the structural level. Concept development occurs through interiorisation, condensation and reification processes, of which reification portrays advanced, structural mental constructions. Tall et al (1998) describes Sfard's operational-structural dichotomy when "boys outnumber the girls by four" is regarded as operational in "add four to the number of girls" and as structural in "x = y + 4". Sfard (1992) regards the structural approach as being strengthened by visual imagery and considers the graph of a function as being structural and the algebraic expression as being both operational and structural. In the structural conception, the objects can be manipulated and further actions can be performed. The claim that students possess a limited object conception of the

function may be attributed to the traditional method of sketching graphs from algebraic expressions and less of transposing from graphs to formulas and tables (Sfard, 1992). This is confirmed by many authors such as Asiala et al (1997) who maintain that the continuous use and dependence of representing functions through algebraic formulas plays a role in a limited concept image of functions. The development of encapsulation of a process into an object is useful in describing the formation of a graphical schema where complex and abstract concepts are constructed (Cooley et al, 2007).

3.3 The Visualizer/Analyzer model.

The Visualizer/Analyzer model of Zazkis et al (1996) serves to draw attention to the inter-relationship between visual and analytical methods of thinking. Rather than distinguishing visualisation and analytical as separate cognitive strategies, they propose that the two are "mutually dependent" in solving mathematical problems. They therefore suggest the integration and synthesis of visual and analytical thinking in developing conceptual understanding. This pedagogical framework draws on Piaget's explanation of perception as involving the analysis of visual stimuli by focussing attention on components and features of objects and simultaneously interpreting and organising salient features such as patterns. The integration of logical thought with visual thinking is paramount and this position parallels Presmeg's (1997) emphasis on "logical rationality" as a function of visualisation.

However, the Visualizer/Analyzer model adopts the stance that visualisation is the connection between the mental and the external objects or events and that visualisation does not take place without external media (Zazkis et al, 1996). The use of concrete objects such as pictures, diagrams and computer images are therefore essential tools for visualisation. Analytical thinking entails the mental manipulation of objects by forming relationships among parts which eventually leads to a synthesised whole.

The Visualisation/Analysis (VA) model is depicted by the diagram shown below:

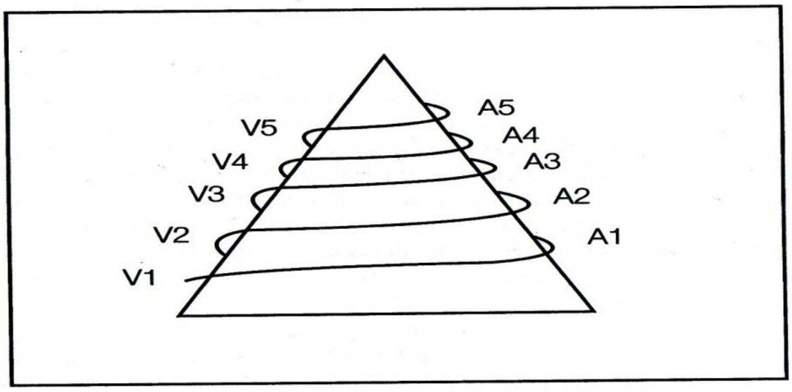


Figure 11: Visualisation/Analysis Model

The process of visualisation commences at V_1 with the formation of mental constructions by merely looking at external objects. It proceeds to the analysis level A_1, where actions are performed on the V_1 process and leads to further mental constructions. In the next stage of visualisation V_2, the learner interacts with the same visual stimuli as in V_1, but due to A_1, the image undergoes transformation. The progressive nature of further visualisation and analysis results in refined visualisations and advanced mental thinking. The narrowing to the apex in the figure demonstrates that as both types of thinking become closer interiorisation and integration takes place.

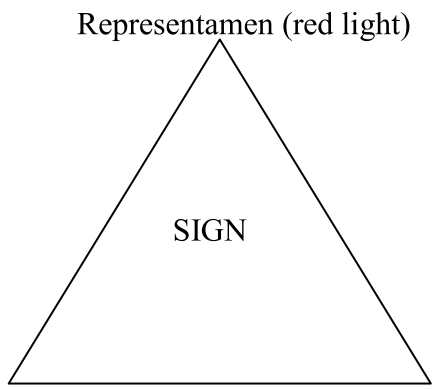
3.4. Semiotics

3.4.1. The inter-relationship between Peircean semiotics and vehicles of reasoning.

The study of graphical representations must be located within the scope of Ernest's (1997) description of mathematics as the "quintessential study of abstract sign systems". Graphical representations are textual forms, consisting of signs, symbols and geometrical formations. Zazkis et al (1996) consider visualisation to take place when the nature of symbols and their configurations are taken into the mental constructions. Semiotics presents a conceptual approach to interpretation of texts and signs and the creation of meaning (Ernest, 1997). This sign – meaning

connection can be described as semiotic activity (Van Oers, 2000). Radford (2001) regards the relationship between signs and cognition as one where signs are regarded as "helpers" of thought. There have been many contributions to the field of semiotics of which the signifier-signified relationship of Saussure and the Triadic model of Peirce are used extensively (Chandler, 2002). However I wish to explore and relate Peirce's semiotics to imagery and vehicles of reasoning which Presmeg (2008) proposes as a relevant theoretical direction in terms of visualisation.

Chandler (2002) defines semiotics as the study of signs or a reference to standing for something where the emphasis is on meaning-making. The Triadic model of Peirce describes the relationship between the sign, referred to as the representamen (that which represents something else or the form of the sign), the object (that which it stands for or to which the sign refers) and the interpretant (possible meaning or the sense made of the sign) (Chandler, 2002; Johansen & Larsen, 2002). Chandler (2002) provides the example of the traffic signal for 'stop' to depict this relationship. The red light denotes the representamen or sign and the vehicles halting refer to the object. The interpretant (meaning created), within the context of traffic signals, implies that the red light indicates that the vehicle must stop. This can be represented as follows:



object interpretant

(vehicles halting) (red light meaning to stop)

Figure 12: Peirce's Triadic model

Sfard (2000) describes the Triadic relationship in terms of graphical representations by making use of the parabolic function. The symbol $f(x) = x^2$ is the representamen which generates a reaction to draw a parabola. The parabola is the interpretant of the sign. Peirce developed trichotomies of sign relationships that exist within the signs (Smith, 1997; Chandler, 2002). These include the relationship between:

- the sign and itself (Firstness) - qualisign, sinsign and legisign.
- the sign and its object (Secondness) - icon, index and symbol.
- the sign and its interpretant (Thirdness) - rheme, dicisign and argument.

Sfard (2000) refers to the Secondness relationship between the sign and its object as object mediation, described as "a set of discursive competencies and underlying psychological processes". Through the link between the iconic, indexical and symbolic modes meaning could be obtained by object mediation. According to Chandler (2002) iconic signs represent its object mainly by similarity, for example, the picture of a mountain directs one to an idea of how the mountain appears. Therefore, the physical resemblance between the sign vehicle and the object is applicable (Presmeg, 2008). Indexical signs focus the attention to the objects by contiguity and a physical connection and not through intellectual operations. For example, the label on a medicine bottle or a sign bearing a street name is in 'direct connection' to the object (Johansen & Larsen, 2002). The arrows at the end of the x and y axes conjuring up a continuous axes is an indexical property. The symbolic forms are conventional signs which refer to the object by rule, association or law. This refers to all the mathematical conventions and truths that are established in the mathematics community. According to Van Oers (2000) the functions of symbols incorporates a reference to actions, replacement of actions and the provision of meaning.

A sign may be explained as "the interpreted relationship between the representamen, called the sign vehicle, and the object that it represents or stands for" (Presmeg, 2008). The representamen is a sign vehicle (Chandler, 2002) and on the basis of its relationship with the object, may lead to an interpretation emanating from signs being iconical, indexical or symbolic. Presmeg (2008) points out that in a mediated manner, sign vehicles can be understood and shows how the nature of signs could stimulate various responses and interpretation from different people. She noted that the calculation of the roots of the quadratic function $ax^2 + bx + c = 0$ using the formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ could induce signs that were iconic, indexical or symbolic. She found that students memorised the formula because of its shape and form, an iconic property. Substituting values for a, b and c is a directional act and therefore depicts the indexical mode. The conventional nature of the formula makes its interpretation symbolic. Relating this to the hyperbolic function $y = \frac{k}{x}$, would suggest that memorising this formula as a hyperbolic function is due to its iconic characteristic. Substituting values for x and y to determine the value for k is a computational skill, a directional act, and therefore yields the indexical function, while the symbolic form is predetermined by the rule appearing in texts and taught by the educator. However, Presmeg (2008) maintains that classification of signs is dependent on the contextual interpretation.

Johansen and Larsen (2002) differentiate between Peirce's three types of iconic signs - images, diagrams and metaphors. While images are related to our sensory perception, the researcher applies the types of visual imagery of Presmeg (2006) and classified as internal representations, to the learners conceptual understanding of graphs. The five types of imagery are: concrete imagery, pattern imagery (based on relationships), memory imagery, kinaesthetic imagery (physical movement) and dynamic imagery (transforming an image). While diagrams represent structural relationships with the objects, metaphors are signs "which represent the character of the representamen by representing a parallelism in something else" (Johansen & Larsen, 2002). According to Presmeg (2008) metaphors link the source (tenor) to the target (sign vehicle). The metaphorical usage of graphical terminology may be depicted as follows:

- increasing → rising
- decreasing → falling
- maximum → high
- minimum → low

The use of metonymy as sign vehicles in mathematics is depicted through the example 'Washington stands for the Government of the USA' (Presmeg, 2008). This might be correlated to f(x) stands for the y as in f(x) > 0, the y values greater than zero.

Presmeg (2008) makes reference to Marcou and Gagatsis' taxonomy of inscriptions that are connected to the iconic, indexical and symbolic sign vehicles. Three such sign vehicles are examined and these include the descriptive-depictive, polysemic-monosemic and the autonomous-auxiliary. Descriptive and depictive sign vehicles refer to symbolic and iconic modes respectively. Therefore, words and symbols may be regarded as descriptive and the visual forms that characterise structure as depictive. The polysemic-monosemic dichotomy is demonstrated through polysemic sign vehicles, where the same sign produces different but connected interpretations. Mathematical concepts that can stand on their own are autonomous in structure while diagrams are auxiliary sign vehicles.

3.4.2 Vygotsky's semiotic mediation

For Vygotsky, “the central fact about our psychology is the fact of mediation” (Vygotsky, 1999). The formation of mathematical meaning and construction of mathematical concepts is theorised by Vygotsky in terms of higher and lower mental functions, semiotic mediation and the use of psychological tools (Nicholl, 1998). Signs are regarded as the “prostheses of the mind” (Radford, 2001). Vile (1993) describes mathematical signs such as words, symbols, diagrams, graphs and schemas as psychological tools that link the lower mental function to higher, more complex thinking through mediated social interaction. These tools and signs perform the mediatory task that link the external reality to internal mental processes. This process is referred to as semiotic mediation. Conceptual development originates with the social and then undergoes internalisation through semiosis (Vile, 1993). Berger (2006) maintains that the usage of these mathematical signs is the first step in the conceptual process that occurs before deriving mathematical meaning. Language is perceived as a significant psychological tool that mediates action and thoughts (Vygotsky, 1999). The process is depicted as follows: symbolisation constitutes objects and language makes the appearance of an object possible. According to Vygotsky “the relationship to an object is mediated from the start by some other object” (Vygotsky, 1999).

Making meaning of mathematical objects in Vygotskian terms occurs through mediation where psychological instruments or signs act as stimuli that influence another stimulus (Van der Veer & Valsiner, 1993). The use of signs implies psychological structural changes such as the reformulation of cognitive structures. The implication here is that external signs lead to internal psychological change. This theory, therefore emphasises mathematical learning as a social activity brought about by language, signs and tools in contrast to learning viewed as interiorised and encapsulated (Berger, 2006). The central tenet of Vygotsky’s theory of concept formation is the focus on the individual and the social relations that the individual engages in. This social, interactive process is the underlying factor for the individual to perform higher mental functions. Since this study involves the role of visual literacy in graphical representations as presented in textual form and the pedagogical relationship with the educator, conceptual understanding depends on societal influences. Berger (2006) highlights the essential components of the functional use of signs:

- The use of words is essential for communication and conceptual development prior to concepts being fully developed.
- The meaning of concepts in the social world, as determined by experts in texts and mathematical dialogue influences conceptual understanding.
- Semiotic mediation is necessary for advanced mathematical thinking where mathematical knowledge is both cognitively and socially constructed.

The conceptual development stages of Vygotsky contain certain elements that have a bearing on conceptual understanding. In the preconceptual stage, development takes place in heaps, complexes and potential concepts (Berger, 2006). Complex thinking is non-logical and includes actions such as matching, associations, imitations and manipulations. Associations occur in ways such as the eccentric use of mathematical signs such as transferring properties of $f(x)$ to $f'(x)$. Complex thinking leads to the formation of potential concepts. The use of pseudoconcepts in Vygotskyian terms is a channel to conceptual understanding. Quasi or pseudoconcept is a potential concept that leads to higher complex psychological functions (Vygotsky, 1999). Pseudoconcepts are similar to the true concept. For example, a student uses the definition of the derivative to compute the derivative of a function before understanding the derivative concept (Berger, 2006). Nicholl (1998) maintains that during the course of semiotic mediation, the semiotic potential is accomplished through the process of decontextualisation. Decontextualisation, normally applied to the linguistic domain, refers to the abstraction of words or texts from the main source, thereby these becoming the objects of reflection.

Vile (1993), in a research study with students, provides a semiotic analysis of: What does x mean in $x - 3 = 5$? A student responded that x is the sign for multiply, while another saw x as a single number as in $8 - 3 = 5$. He further describes the following example as an interpretant on a process level, where the steps denote procedural knowledge through rote manipulation:

$$3y + 2 = y + 6$$
$$3y - y = 6 - 2$$
$$2y = 4$$
$$y = 2$$

Furthermore, a distinction is made between high level of semiotic demand and low level of semiotic understanding. This is demonstrated when a student solves for A in $7 + 2A = 4A - 11$. He arrives at the answer $A = 9$ by substituting various values for A and equating the left hand side of the equation to the right hand side. Vile (1993) regards the student as displaying low level of semiotic understanding, when the equation had high semiotic demand.

3.5 Conclusion

The exposition of the theoretical framework incorporating the APOS theory, Visualizer/Analyzer model and semiotics serves a useful foundation to make sense of learners' responses. The significant contribution of these theories emphasises their association with visual literacy. The APOS theory highlights the internal cognitive processes focusing on actions as processes or objects. The Visualizer/Analyzer model describes the relationship between the external images and the internal thought processes. A connection between the APOS and Visualizer/Analyzer theories might be perceived in terms of their similarity in stressing the attainment of advanced mathematical thinking. The link between the external and the internal processes is further perceived through the semiotic model, whereby semiotic activity is paramount due to the impact of signs, symbols, notation and textual forms on learners' understanding and interpretation.

CHAPTER FOUR

RESEARCH DESIGN AND METHODOLOGY

4.1. The research design

The research study is based on identifying what conceptual understandings Grade 11 learners have of the various characteristics of graphs. Graphs emanate from the section on functional relations as stipulated in the FET Mathematics Curriculum Statement (Department of Education, 2003). Through this study the researcher aims to discover learners' construction of mathematical meaning pertaining to graphical functional relationships. The examination of their understanding and interpretation is conducted from a visual literacy and visualisation perspective. Recent trends in mathematical inquiry have distinguished the social construction of mathematical knowledge in relation to textual and social interactions (Long, 2005). A qualitative research methodology suits the study well as it encompasses in depth inquiry, subjectivity and provides thick descriptions of phenomena (Cohen, Manion, & Morrison, 2007; Creswell, 1998; Henning, 2004). Qualitative research is descriptive and focuses on meaning and explanations (McEwan & McEwan, 2003). Through qualitative methodology, the researcher intends probing into learners' interpretations and cognitive constructions to gain a better pedagogical understanding of the role of visual literacy in mathematics classrooms.

This interpretive standpoint serves to guide the study in providing a theoretical framework based on the interpretivist paradigm. Henning (2004) provides an epistemological basis (how we come to know about this world) of interpretivist philosophy that indicates that knowledge is constructed by describing peoples actions, beliefs, values, understanding and construction of meaning. In this way the researcher endeavours to explore how, why and what meaning learners attribute when sketching and interpreting graphs and in particular how they create knowledge of mathematical concepts. Thick data is used mostly in qualitative research where the aim is to acquire as much meaning and interpretation of situations and contexts as possible (Henning, 2004). According to Maree (2007) the interpretive paradigm encompasses a descriptive analysis with the aim of providing a deeper understanding of the social and human relationships. This is relevant to the construction of mathematical knowledge.

4.2. Population and sample

Since the study pertains to the FET curriculum for Grades 10-12, the researcher decided to focus on the Grade 11 learners as they would have encountered graphs in Grade 10. In doing so, the intention is to somehow inform and locate in the entire FET phase, the role of visual literacy in graphical relationships and therefore Grade 11 learners would have suited this purpose adequately.

Qualitative sampling techniques such as non-probability sampling and purposive sampling were used in the selection process because the research enquiry did not aim to represent or make generalisations of the wider population (Cohen et al, 2007). According to Creswell (1998), in qualitative research purposeful sampling involves the selection of participants who are knowledgeable of what is to be investigated. Grade 11 learners were purposely selected as they had covered the sections on graphs sufficiently enough to be able to respond to a range of properties of graphs, thereby satisfactorily providing data on graphical interpretation. The sample size consisted of ten grade 11 learners with five learners each from two secondary schools. All grade 11 mathematics' learners in each school were ranked according to their performance level in the June examination. Learners were systematically sampled and selected by: frequency $= \frac{Number\ of\ learners}{5}$. This sample consisted of different ability levels of learners so that rich and varied data could be obtained. Two mathematics educators who were teaching the ten learners were also interviewed. These educators were experienced in teaching mathematics in the FET phase Grades 10-12. Educators

were used as a means to provide an overview of key issues around the learners' conceptual understanding and visual literacy. The two secondary schools were randomly selected from the eThekwini region in the province of KwaZulu-Natal. The study commenced after learners had completed the section on functional relationships at their respective schools.

4.3. Data collection methods

The data collection process was conducted at each of the two schools and is summarized as follows:

- The five learners and the educator each completed three worksheets based on graphs (Appendix B). This took approximately forty-five minutes per participant.
- Thereafter, the researcher interviewed each learner individually, followed by interviewing the educator. The answers written on the worksheets were used during the interview. For both groups of participants an interview schedule (Appendix C) was used in the interview process. This process took approximately one hour per participant. Each interview was audio taped.

Qualitative data collection strategies include observations, interviews and document analysis (Cohen et al, 2007). The data collection method is based on qualitative interviews and in particular semi-structured interviews. An interview schedule which comprised of open-ended questions based on graphical interpretation was formulated and used in the interviews. The interviews reveal the subjective reality of the learners through their thoughts, feelings and actions (Henning, 2004). Niewenhuis in Maree (2007) illustrates semi-structured interviews as consisting of a set of pre-determined questions, where the interviewer and the interviewee have a degree of freedom to probe and clarify questions and responses. This is similar to what Cohen et al (2007) refer to as the

less formal interview where the interviewer adapts the standardised questions to suit the dialogue process. Open-ended questions allow for flexibility through in depth responses, clarity of misunderstanding and obtaining true responses from interviewees (Cohen et al, 2007). The questions were based on graphical content and were structured in a manner to initiate dialogue that revealed graphical understanding. The standardized wording and asking the same questions to more interviewees increases the level of comparability and reduces interviewer effects and bias (Cohen et al, 2007) and increases intersubjectivity (Henning, 2004). Although questionnaires are similar to interviews (Cohen et al, 2007), the use of open-ended questions in questionnaires rarely provide qualitative data. Through the interviews the empirical data are related to the research aims. The researcher probes learners' responses to the written worksheets by asking them to explain and provide reasons or meanings for their understandings.

The worksheets were designed by the researcher intended to facilitate the learner's understanding of the different characteristics of graphs. This assisted the researcher in introspecting on the effect of visualisation strategies in conceptual understanding. Throughout the interview, reference was made to the three worksheets in respect to how they create knowledge of mathematical concepts. The worksheets were criterion referenced and content related. Criterion-referenced is normally associated with tests (Cohen et al, 2007). However the items on the worksheet bear similarity with tests, in that it aims to provide evidence on what the learner knows and can do with graphical concepts. The worksheets contained graphs and graphical instructions that were based on the way learners encountered graphical representations in mathematics textbooks and examination papers. The worksheets were therefore within the ability level of the learners.

4.4. Triangulation

To explore more fully the conceptual understanding of learners and to produce credible data, the researcher triangulated the findings of the learners through

educator interviews. Triangulation is the use of the multi- method strategy of two or more methods of collecting data (Cohen et al, 2007). However, the researcher employed methodological triangulation as stipulated in Cohen et al (2007), where the same method of interviewing was used for different participants. In this case the same questions as per interview schedule were asked of the two educators. Through triangulation "one comes from various points towards a measured position" (Henning, 2004). Therefore, the role of visual literacy is observed from the position of the learners and the educators, with the intention of providing a unified approach to conceptual understanding of graphical representations. The internal validity issues of credibility, dependability and authenticity are ensured through triangulation of methods and respondent validation. Detailed explanations are given in the research of the processes involved. Thick, in depth descriptions of the learners and teachers responses are accounted for in the data analysis.

4.5. Data analysis methods

The purpose of the research is to describe, interpret, discover, generate, explore and understand learners' conceptual understandings of graphs. Therefore, analytical inductive techniques are used in the analysis of data. These include categorising the data according to themes and providing detailed summaries of individual transcripts (Niewenhuis in Maree, 2007). The data is organised according to the research questions and around particular themes.

The presentation and analysis of data are aligned to the **key research questions**:

- How do the learners' understand the graphical representation of functions?
- Is the learners' visual understanding of the different characteristics of graphs procedural or conceptual?
- Is the educator's visual understanding of graphs different from those of learners?

When analysing the data of the learners and educators, the researcher focuses on the following for research question one:

- How do learners interpret the following signs and symbols: the Cartesian plane, the geometric curves of graphs and functional notation such as $f(x) > 0$?
- What is the learners' interpretation of the graphical terminology they use?
- What is the effect of graphical representations on learners' visual literacy skills?

For research question two the researcher identifies areas where learners employed procedural and conceptual understanding. For research question three the researcher examines the data of two educators that teach these learners in terms of their similar or different visual understanding to that of the learners. Maree (2007) sums up the analysis procedures that relate the gathered data with the primary and secondary research questions. Firstly, the researcher identifies themes, categories and subcategories. Thereafter, the researcher engages in a pattern seeking exercise. Lastly, the researcher compares the data with other theories. This is the mode in which the researcher conducts the data analysis.

CHAPTER FIVE

DATA ANALYSIS

5.1. Introduction

The focus area in the study is on the relationship between visual literacy and conceptual understanding and the interconnectedness of these two constructs. The process-object theories focus on interiorisation and encapsulation as triggered off by actions. Therefore, they emphasise the actions performed in cognition. The analysis aim firstly to determine the extent learners understand graphs at the process and object levels. This is related to the procedural-conceptual dichotomy. Secondly, the responses are examined in terms of the visualisation-analytical thinking dichotomy. Thirdly, the researcher claims that the visual literacy tools that learners encounter or experience when working with graphical representations cannot be separated from their semiotic character. Both the external, concrete visual material and the internal psychological domains are relevant to their conceptual understanding. Therefore, in the analysis, the researcher makes meaning of their understanding of graphical representations in terms of both external and internal objects as a means of creating an interpretation of the role of visual literacy in mathematics. Mathematical signs, symbols and tools have inherent potential to bring about mental constructions.

5.2. A statistical presentation

The following table 2 shows the correct responses, indicated by (1), and incorrect responses, indicated by (2), written by the ten learners on the worksheets (Appendix B). The responses are rated according to graphical skills determined by the researcher. The ten learners are indicated by the letters A to J.

LEARNER	A	B	C	D	E	F	G	H	I	J
GRAPHICAL SKILLS										
Worksheet One : The parabola										
1. sketch of parabola	1	1	1	1	1	1	1	1	1	1
2. reflection in x axis	1	2	2	2	1	1	1	1	2	1
3. determine x values for g(x) > 0	2	2	2	2	2	2	2	1	2	2
4. state the domain	2	2	2	2	1	1	1	1	2	1
5. state the range	2	2	2	2	2	1	1	1	2	2
Worksheet Two: Hyperbola & Exponential										
1. match graph with equation	1	1	1	1	1	1	1	1	1	1
2. determine equation from graph										
2.1. hyperbola	1	1	1	1	1	1	1	1	2	2
2.2. Exponential	1	2	2	2	1	1	1	1	2	2
3. reflection of exponential in y axis	2	2	2	2	1	2	1	2	1	1
Worksheet Three: Trigonometric graphs										
1. determine equation from graph										
1.1. sine function	2	1	2	1	1	1	2	1	2	2
1.2. cosine function-horizontal shift	2	2	2	2	1	2	2	1	2	2
2. new range after vertical shift	2	2	2	2	2	1	1	1	1	2
3. period of sine function	2	2	2	2	1	1	1	2	2	2

Table 2: Statistical analysis of learner responses

The analysis procedure bears a similar resemblance to that conducted by Asiala et al (1997). The above data, though not convincing enough to make any significant conclusions about learners' interpretation of graphical concepts and utilisation of graphical skills, may provide some commonalities that educators may be familiar with in terms of learners' difficulties. The following analysis might not sufficiently demonstrate learners' failing in conceptual understanding because

understanding of concepts is not reducible to correct and incorrect responses. However, it may be useful in identifying the following:

- Graphical skills where learners display poor graphical understanding.
- Graphical skills where learners display good graphical understanding.

These may be classified as follows:

- Poor understanding:
  - Determining values of x for which $g(x) > 0$.
  - Stating the domain and range of functions.
  - Reflection of graphs in the y axis.
  - Determining the equations of trigonometric graphs that depict horizontal transformation.
  - Stating the period of trigonometric graphs.

- Good understanding:
  - Sketching of parabola.
  - Reflection in the x-axis.

- Matching hyperbolic and exponential graphs with their equations.
- Determining the equation of the hyperbolic function from the given graph.

The table also indicates that as the graphical skills progress from the application of rules and procedures to the interpretation level, the responses reveal an increasing procedural understanding and a decline in conceptual understanding.

5.3 Learners' understanding of graphical representations of functions

5.3.1 The understanding of the Cartesian plane, notation and symbols

The purpose of this analysis is to determine whether learners understood basic notation, symbols and the use of the Cartesian plane. This understanding is determined in terms of the graphs of $f(x) = x^2$ and $g(x) = x^2 - 9$, which they sketched on worksheet 1 (Appendix B).

The following are learner responses when asked the question: Why did you use arrows or why did you not use arrows at the end of your graph/axis?

Learner A: Yes, we had to use arrows. It shows that the graph is going up. There will be no ending for the graph.
Researcher: Is there any need for the arrows at the end of your axis?
Learner A: Yes, the graph is not stopping.
Researcher: What does it mean to say that the graph is not stopping?
Learner A: I don't know, it just continues.

It is evident that learner A knew that arrows should be used, but had no idea why. His claim that it never ends is not sufficient to demonstrate an understanding of the symbol, although this might be perceived as partial understanding. This is further illustrated by learner B, who claimed that arrows are used where the graph ends and further confuses his response by then stating that the arrows can also mean that the graph continues. These are two contradictory statements. Graphs should either end at a point or continue forever. If the graph stops then it shows that the function has a finite domain and range. This is quite different from an infinite domain and range.

Learner B: It would be where the graph ends. Sometimes it can mean the graph continues.

Learner F: The graph goes to infinity.
However when asked why the arrows are pointing upwards:
Learner F: Because the gradient is positive, so it points upwards. If the gradient is negative it will point downwards.
Researcher: So you are relating the arrows with positive and negative gradients?
Learner F: Yes.

The above responses indicate the variety of interpretations learners possess of the arrows at the end of graphs and the axes. The many varied linguistic meanings offered indicate the role of language in mathematics as a means of thought. While generally most of the ten interviewed learners used words like "no ending, continues and infinity" as the purpose of the arrows, learner F related the arrows to the concept of positive and negative gradients. This may have actually been her relationship of the visual understanding of increasing and decreasing aspects of the graph to the gradient. Learners therefore displayed limited understanding of the arrows in terms of Cartesian functions and Cartesian planes and did not actually relate the arrows to the depiction of all possible x and corresponding y values that satisfy the Cartesian function. This issue will later be explored from the educators' perspective.

The understanding of the axes was extended to the real number concept as exemplified on the number line through the question: Are there any points between 2 and 3 on the axis?

Five of the ten learners initially responded that there are nine numbers and these are 2.1 ; 2.2 ; 2.3 ; 2.4 ; 2.5 ; 2.6 ; 2.7 ; 2.8 and 2.9. The researcher then asked:
Researcher: Are there any other points?
Learner C: No, only these numbers.

The researcher further probed with:
Researcher: What about fractions such as $2\frac{1}{2}$, $2\frac{3}{4}$ and other decimal numbers.

Learner C: Yes, when you put fractions, you can put fractions in between.

While there is some indication of understanding the real number concept by some learners, there appears to be the general trend of identifying nine numbers between the numbers 2 and 3. This implies that learners actually believed that there were a finite number of numbers between 2 and 3. This further contributes to the idea that learners may not understand why the graph of $f(x) = x^2$ is continuous. The apparent misconception of learners might be due to the concept image of the use of the ruler used in measurement with millimetre demarcations or perhaps the identification of decimal fractions on a number line taught in primary school. Tall and Vinner (1981) refer to concept image as those cognitive structures in the learners' mind, consisting of mental images and associated procedures built up over years. However, this understanding confirms their failure to regard the number line according to Dörfler's (2000) operational prototype, a conception that irrational and rational numbers can be extracted from the number line.

In relation to the curves the learners sketched, and in particular the graph of the parabola, the researcher asked the following question: What does it mean to join points on your graph?

Learner A: I just took the points from the table to the graph. My meaning of this is that it's a happy face….My equation was positive, there was no negative, it was x^2 only. I don't think I know why.

Learner C: So that I can get the shape of the graph. I don't know why. I just know it's the shape, so I join the dots.

Learner E: I am not sure. I just know how the parabola is supposed to look.

Learner H: To make a continuous graph. It is the shape of the parabola, so I just join the points. The x^2 tells us that it would be a parabola.

Learner G: It means average gradient.
The researcher did not at this stage enquire from the learner his understanding of average gradient. It would have been useful to have further probed this understanding of the learner.

This shows that their understanding of the topology of the real line as illustrated by Williams (1993) in terms of the relationship between points and the curve is limited. While learner G correlated the average gradient between two points as his understanding of curves on planes, the others related their understanding to how the parabola looks as their meaning of joining two points. This might be due to the learners' visualisation of the prototype of the parabolic graph and therefore all graphs could be drawn in a similar way. Very often, this is the result of simply mimicking what the teacher does. Often, learners use the same points that their teachers used and many have very little understanding of why they do it. It is significant within the context of the previous responses that they actually believed that there were a finite number of values between 2 and 3. This shows their misunderstanding of the number line and its values does not correlate with the actual continuous curve which has an infinite number of values between 2 and 3.

The concept of curves is related to the following investigation and was further extended through the question based on the learners' method for sketching the parabolas. Learners' A, B, C, D and H had substituted values for x (table method) and calculated the corresponding y values by using a calculator. These x and y values were then plotted on the Cartesian plane.

Researcher: Explain why you used only a few points and what your reasons are for joining the dots?
Learner A: Because of the time and of the size of the axis and my equation didn't have big numbers. I just chose the values that I use in class when I'm doing class work and when I practise my maths.
Researcher: That is why you just went up to four and minus four? What made you join all the points on your graph?

Learner A: Because of the type of graph. It was a parabola and when you have to draw graphs you need to attach the points.

Learner A had demonstrated his actions which he frequently uses in the class. The misconception that for graphs one needs to always join the points might conflict when he needs to draw graphs, with a domain of, for example, $x \in Z$. Learner H also accounted for his limited domain of x values due to "taking more time" if he took more x values.

Researcher: Why did you take only a few points to draw the graphs? Could you have taken more points?
Learner H: Yes, but it would take more time.

Learner D: My teacher taught me to draw the graph, a smiling face; therefore I joined the dots.

Learner B did not use the table method and had calculated the x and y intercepts for $g(x) = x^2 - 9$ and plotted these points on the axes to sketch the graph.

Researcher: Why did you join the x = -3 and x = +3 to the y = -9 ? Are there any other points on your curve?
Learner B: Because it's a happy face, that's why I used a parabola. There are no more points on the curve, just the three points I joined.

The strong process conception of both learner A and B indicate that a graph is merely a process of calculating and plotting a few points. The next action is to join the points because, according to the learners, they have to be joined.

Learners E and F also calculated the x and y intercepts, but learner E made reference to $f(x) = x^2$ as being the "parent graph".

Researcher: I see that you also did working for the graph of g where you determined the x intercepts and the y intercept. You did not take any points for x and plot these points.
Learner E: I felt that there is no need to take any points. The parent graph tells you that it will be a parabola.

Researcher: I see that you joined the three points -3, -9 and +3. Why did you join them ?
Learner F: The equation is a quadratic function, so it will have that shape.
Researcher: Are there any more points on this curve? If you take the entire curve of g, how can we get more points?
Learner F: We can use the table method. We can take -3, -2, -1, 0, 1, 2, 3 for x and substitute in the equation to find y.

Learners used procedural understanding in joining the points to form a curve with responses ranging from 'my teacher taught me', 'it's a quadratic function', 'for parabolas you don't really need to take many points' to 'the parent graph is the parabola'. This limited understanding of joining the dots contrasts to them understanding that the curve is continuous and consists of an infinite number of points. However, though a strong visual understanding of the x^2 allows them to form a parabola, there is also the tendency of their action to be based on memory or rote manipulation. Learner F exemplified the process level as stated by Moschkovich et al (1993) when she explained substitution as a method of identifying other points on the graph.

The procedural understanding was also ascertained when learners were asked how they would determine f(3); f($4\frac{1}{2}$) and f(20) and also to show on their graphs what this really means:

Researcher: Can you show me by means of your graph what you understand by f(3) and where you would read off f(3)?

Learner A: I would plot it on the x axis between 0 and 5.
Researcher: Why are you plotting it on the x axis?
Learner A: Because it is the x value.
Researcher: And what does this mean?
Learner A: It is the x.

While learners B and D showed no understanding of f(3), learners E, F and G when asked to show f(3) on the graph used analytical means and substituted x = 3 to obtain y = 9. When learners were asked to show f($4\frac{1}{2}$) they also pointed x = $4\frac{1}{2}$ on the x axis. However, when asked to show f(20) they responded that they might have " to extend the axis" or "we need to have a scale".

Most learners were able to deduce that the 3, $4\frac{1}{2}$ and 20 represented the value for x. However learners failed to understand that f(3) actually depicted the value for y. According to Sfard (1991), learners possess a process conception of a function more strongly than an object conception of the function. Although there might be some resemblance to this, mistaking f(3) for 3 shows that the learner does not understand the functional notation. While they could substitute 3 in $f(x) = x^2$ and get 9, they did not understand that this represented the co-ordinates (3; 9). Only two of the ten learners could correctly show what this aspect of functional notation meant with reference to the graph. Learners did not realise that there is no need for scale or extending the axes when sketching graphical functional representations.

Perhaps the use of a large figure f(20) prompted them to think in terms of scale. They could hence, not explain adequately what $f(4\frac{1}{2})$ meant and for the learners, f(20) meant that they would have to extend the axis.

The understanding of the relationship between functional notation through symbols and the corresponding correlation to graphs was also investigated through the learners' responses:

Researcher: Explain the meaning of the values of x for which $g(x) > 0$ in
$g(x) = x^2 - 9$. How did you arrive at the answer 2; 1; 0; -1; -2?
Learner A: I take the numbers from my table that are greater than 0.
Researcher: What are these numbers?
Learner A: It would be the y values greater than 0 and that would be 7.

Learner A had clearly not understood this question. However, he used the table method to sketch the graph of $g(x) = x^2 - 9$, by substituting x values and determining the y values. So when $x = 4$, he deduced that $y = 7$. Therefore, he replied that the answer is $y = 7$. However, there might be some indication that he knew that g(x) represented the y values. This was in direct contrast to learner C who listed positive whole numbers greater than zero.

Researcher: Explain the meaning of the values of x for which $g(x) > 0$?
Learner C: One, two three and so on.

Although learner E had the correct graphical understanding, the $x < -3$ was incorrectly stated as $x > -3$.
Learner E: It would be where the $x > 0$. It would be positive answers.

Learner R: Why positive?

Learner E: Because it must be greater than zero. On the positive side of the x axis we get positive answers and the graph is going up, so x > 3 and x > -3. For x > 3 we are going up and for x > -3 we are going up because we get positive y values.

However, learner G, who had also displayed a good graphical understanding for g(x) > 0, failed to realise the meaning of the functional notation and replied as:

Learner G: For the x values less than -3 the y is greater than 0, and for the x values greater than 3 the y is greater than 0.

Researcher: Why did you write your answer as 3 < x < -3?

Learner G: It's the same reason I told you sir.

The meaning of the functional notation was further probed in the question on the interpretation of the trigonometric graphs (Appendix B - Worksheet 3) where learners were asked to show on the given sketch of the graphs of f(x) = a sin b x and g(x) = cos(x + q), the following: f(x) = g(x), f(x) > 0, g(x) > 0 and f(x) > g(x). Many of the learners could indicate on the graph where the relationship f(x) = g(x) occurs. However, when probed on how they would arrive at a solution to this problem their responses varied:

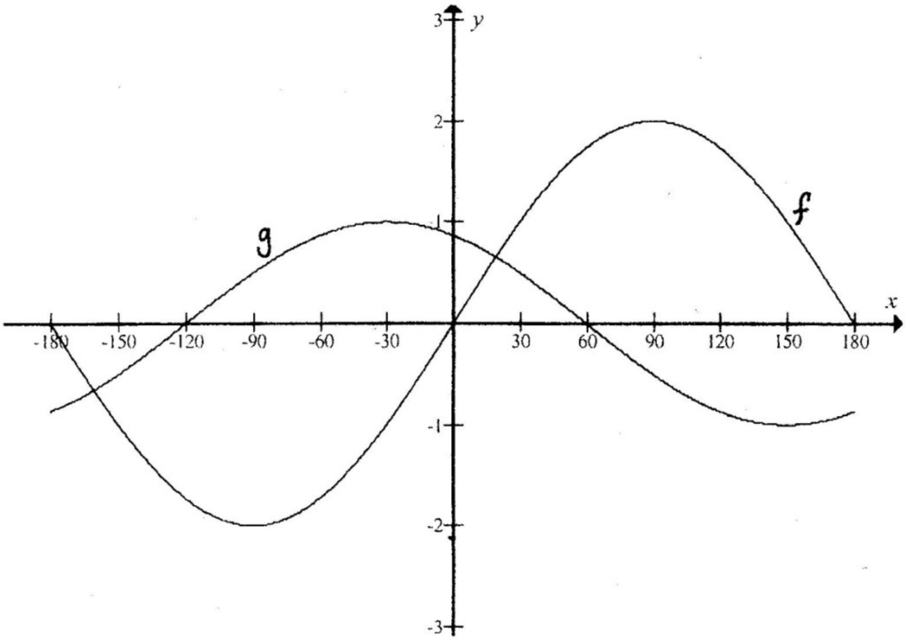


Figure 13: Trigonometric Graphs – worksheet 3

Learner B: I don't know how to calculate this.

Learner D did not understand this question at all.

Learner G responded that he would draw a table and "see where I get similar values".

When they were asked where they would read off $f(x) > 0$ and $g(x) < 0$, for learner B this meant $x > 0$ and $x < 0$:

Learner B: For f it would be on the positive X axis and for g it would be on the negative X axis.
Researcher: Why are you saying positive and negative?
Learner B: Because numbers greater than zero are positive and number less than zero are negative.

This conception was also confirmed by learner F when she responded:

Researcher: And $f(x) > 0$?

Learner F: It's the part of the graph greater than 0 on the x axis, the x values.

Researcher: And $g(x) < 0$?

Learner F: It's the negative side of the axis.

However learners' E and G did show a good level of understanding of $f(x)$ when they responded that $f(x) > 0$ is "above the x axis". Learner E was asked:

Researcher: How would you solve for $f(x) > g(x)$?

Learner E: It would be from the points of intersection where the f graph is above the g graph?

Researcher: Why are you saying above?

Learner E: I learnt that when something is greater than the other, I just go to where the first graph is greater than the other graph.

Researcher: What is your understanding of this?

Learner E: I do not know.

This shows that although learner E is correct in the analytical manipulation, she lags behind in her graphical understanding. Although the learners provided different conceptual views on $f(x) = g(x)$, $f(x) > 0$, $g(x) < 0$ and $f(x) > g(x)$; they related the symbolic notation with the graphical representations.

5.3.2 Learners interpretation of the graphical terminology

In terms of graphical terminology, the use of words to convey meaning was explored with reference to the graphical representations. The graphically related words used were:

- Increasing and decreasing.

- Maximum and minimum.
- Domain and range.
- Asymptotes.
- Gradient.
- Reflection
- Period

Learners where asked to describe the shape of their graphs. For $f(x) = x^2$; they used words such as “happy face, horse-shoe, smiling face, positive graph” to illustrate their interpretation of the parabola. All of the learners assigned these descriptions to the $+x^2$. When asked to describe $f(x) = -x^2$ they used the words “sad face and frown”. These depictions are similar to the metaphors that educators use in mathematics classrooms.

Increasing and decreasing functions over a particular interval are applied in algebraic and trigonometric graphs in Grade 10 and Grade 11. In Grade 12 this aspect of graphical interpretation is extended to calculus in the section on curve sketching. The understanding of increasing and decreasing functions over interval periods is usually articulated as being dependent on the x value increasing from the left of the number line. If the y values decrease, then the graph is decreasing between x value intervals. If the y values increase, then the graph increases between x value intervals. This concept is very much a visual skill.

Learners were asked: What does it mean when you say f(x) is increasing or decreasing? Show on the graphs of $f(x) = x^2$ and $g(x) = x^2 - 9$, where the graphs are increasing or decreasing.

Learner A related increasing to positive x values and decreasing to negative x values. The researcher categorises this as a misconception, due to the learner’s weak concept image of regarding increasing as positive numbers and decreasing as negative numbers.

Researcher: Show me where your graph of f is increasing?
Learner A: It is increasing on the right hand side of the x axis.
Researcher: And where is it decreasing?
Learner A: On the left hand side.
Researcher: Why is it increasing on the right hand side and decreasing on the left?
Learner A: Because of the positive numbers on the right and the negative numbers on the left.
Researcher: What do you mean by negative numbers on the left?
Learner A: The negative numbers decrease, therefore the graph decreases.

In this particular parabola the learner could not articulate the fact that the f(x) increased as the x values decreased to the left of the zero. They did not understand nor could they explain the fact that f(x) increased when x decreased to the left of zero or when x increased to the right of the zero. Clearly f(x) increased as the x values moved away from zero.

Learner H had related the concept of increasing to the graph pointing to positive infinity on the y axis and decreasing when the graph points towards negative infinity on the y axis.

Learner H: Yes, the f graph is increasing.
Researcher: Why do you say that the graph of f is increasing.
Learner H: It is going upwards, positive numbers.

Researcher: When would the graph be decreasing?
Learner H: When it is going downwards.

This misconception was also evident in the hyperbolic and exponential functions (Worksheet 2 – Appendix B):

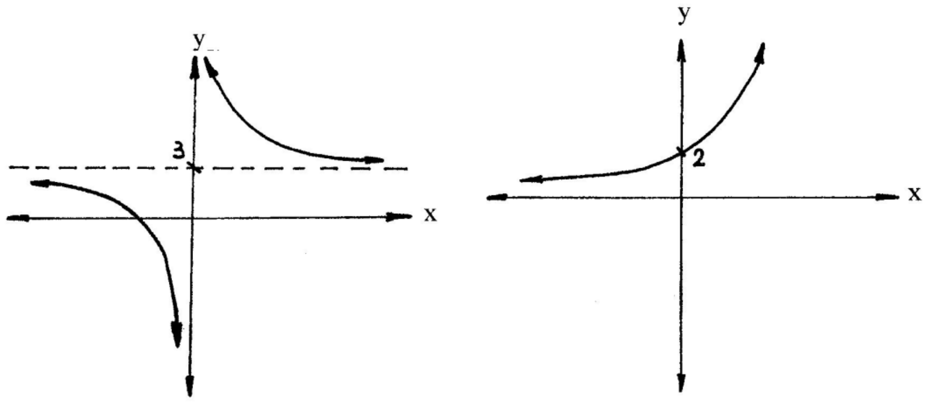
Learner A: For graph A, I don't know. For graph B it is increasing from the two upwards on the right hand side of the axis.
Researcher: What about the left of the y axis?
Learner A: That is negative, so the graph is decreasing.

Learners B, C, D and E conceptualised 'increasing as going up' and 'decreasing as going down'. This shows the linguistic correlation and depicts a literal configuration by interpreting the shape of the parabola as going up or coming down. Learner F was able to provide the following visual explanation for increasing and decreasing in $f(x) = x^2$:

Learner F: The graph of f first decreases and then it increases.
Researcher: What makes you say that?
Learner F: If you take the f graph on the left hand side of the axis, it is decreasing because the y values get smaller, like 10, 9, and 8 and so on. But on the right hand side it increases because the y values increase from 0,1,2,3 and so on.
Researcher: What tells you that the graph is increasing or decreasing?
Learner F: The values of the y.
Researcher: Only the y values? What do you see in these y values?
Learner F: Yes, only the y values. I look at the graph from the left and notice that as the graph comes down the y values decrease and as it goes up the y values increase.

However, when classifying the hyperbolic and exponential graph as increasing or decreasing, she provided a conflicting explanation.

Learner F: Graph A is increasing because it's going to the right. It's actually decreasing on the y axis and increasing on the x axis. Graph B is increasing.
Researcher: Why is graph B increasing?
Learner F: Because of the way it's sloping. From decreasing it goes to increasing, because the negative numbers are decreasing on the x axis and on the positive side it's increasing.



Graph A Graph B

Figure 14: The hyperbola and the exponential graphs-worksheet 2

Learner G conceptualised increasing with increasing x and increasing y values; and decreasing as decreasing x and decreasing y values. For the hyperbolic and exponential graphs:

Researcher: How would you describe graph A in terms of increasing and decreasing?
Learner G: Both, increasing and decreasing. The top curve is increasing because as the x increases the y is increasing and the other curve is decreasing because as the x decreases, the y is decreasing.
Researcher: And for the graph of B?
Learner G: One increasing and one decreasing. The y values are increasing and the x values are decreasing.

From the above, we can summarise that for some learners the understanding of increasing and decreasing functions are related to positive and negative numbers. For others interviewed, it may seem that that the figurative association of forming literal interpretations (Pimm, 1995) was utilised where the visual feature of the shape of the parabola was considered. The physical aspects of the graphs were used in their mental construction of the concepts of increasing and decreasing. No learner precisely stated that the graph increases as we move to the right and to the left of the x axis. In other words, no response indicated a real understanding of the function of $f(x) = x^2$. Learners should have actually recognised that the graph decreased as x increased from negative infinity to zero and increased from x equal to zero 0 to positive infinity.

Although the notion of increasing and decreasing functions is very much a visual judgement and linked to the physical features of graphs, the fixation of learners to physical features of graphs was also noticed in the examination of the effect of the words maximum and minimum on their graphical understanding. Learner E viewed maximum and minimum in terms of the way the graphs were presented and Learner F related the sign of the 'a' in $f(x) = ax^2 + q$ in terms of the shape of the graph.

Learner E: The graph is minimum because it is turning upwards. If it turns downwards, then we will get a maximum turning point.

Learner F: The graph of g is minimum at y = -9 and the graph of f at zero.
Researcher: When can we describe the graph as maximum?
Learner F: When the equation is negative and the graph points downwards.

However, learner G used the terms "starting low" to refer to minimum and starting "high and goes lower" to maximum.
Learner G: Yes, they are minimum graphs.
Researcher: Why do you say that?

Learner G: Because they are smiling face; they start at a lower point and goes higher, the y value starts low and gets bigger.
Researcher: When would you get maximum?
Learner G: Maximum is the other way around, the frown.
Researcher: Can you explain the meaning of maximum in terms of your graph?
Learner G: The y value starts high and goes lower.

Learners had to state the domain for the graph of $f(x) = x^2$ and the range for $g(x) = x^2 - 9$. From the statistical analysis conducted earlier learners showed limited graphical understanding of these two concepts. It would seem that learners are fixed on applying the domain as $x \in R$ to any graphical representation, due to the domain very frequently described as "all possible x values". Where some learners indicated that the domain for $f(x) = x^2$ as being $x \in R$, the same answer was given when stating the domain for the hyperbolic function $y = \frac{k}{x} + q$. Learners appeared to be fixed on the conception that the domain is always all the x values and that the range is always the y values irrespective of how the graphs were drawn on the Cartesian plane. The understanding of domain and range in terms of their graphs is exemplified in various conceptual forms.

Learner A extracts the domain and range from the table.
Learner A: Domain is the x values. I just take it from my table.
Researcher: What do you understand by range?
Learner A: It's the y value. I just take it from my table.

Learners E and F explained the domain as "all possible x values" for $f(x) = x^2$. Although this is true, they could not explain what it means in terms of their graphs, as values that satisfy the given function. Learner E explained that the range for $g(x) = x^2 - 9$ was given as $y \in Z$ because "the range will be all the y values".
Learner F: $x \in R$, these include counting numbers, integers and not surds. I don't know what it actually means in terms of the graph.

The learners understanding of asymptote was derived in terms of the graph A, the hyperbolic function and graph B, the exponential function- (Refer to worksheet 2):

Learners displayed good memory imagery in identifying that the formula y = $\frac{k}{x} + q$ represents the hyperbolic function and that the ‘q’ stands for the asymptote. The asymptote was identified as “the broken line”. When asked what their understanding of asymptote was, the learners gave these responses:

Learner B: That the graph must not touch the line.

Learner G: The graph would go close to it but never touches it.

Learner H: The graph will go close to the asymptote and never touches it. The asymptote is the q value in the equation.

This may be significant that learners associate the asymptote to the graph itself. In reality the asymptote is an imaginary line that is really not part of the graph. There is actually no need to construct this line. In terms of Vile’s (1993) categorization of low semiotic understanding, memorizing q as the asymptote and that the asymptote is a line that the graph will not touch, denotes low understanding. There was no evidence of any conceptual understanding for the asymptote. The misunderstanding of the asymptote was also evident when learner A classified the x and the y axis as being asymptotes of graph B. This might be due to the correlation with the asymptotes of y = $\frac{k}{x}$.

Researcher: For the equation of the graph of A, why did you write 3 for q?
Learner A: It is the asymptote. The equation says $\frac{k}{x} +$ ', the q stands for the asymptote.
Researcher: What do you understand by asymptote?
Learner A: If q is positive the broken line will be by positive 3.

Researcher: Can you show me any other asymptotes?
Learner A: Yes. The x and the y, the original, are the asymptotes for graph A. The x and y are also the asymptotes for graph B.
Researcher: Why are the curves not touching the x axis?

Learner A: I don't know why. The broken line means the graph shifted.

Learner C interpreted the asymptote as an indication that the graph is not continuous.
Learner C: It must be a dotted line. It cannot be a straight line. The graph won't carry on; it's where the graph has got to end.

There is a distinct difference in the way each of the learners A and C understood the idea of an asymptote. Whilst neither showed an understanding of what an asymptote represented, learner C thinks that an asymptote is the end of a graph. Learner C can correctly recognise or draw the graph, but her understanding of the asymptote is poor. There is a tendency of learners to learn mathematics by rote with little understanding of their actions. Perhaps their understanding of the asymptote arises from the description they encounter in text books. Goba and van der Lith (2005) in a Grade 10 mathematics text book describe the asymptote "as a line that the graph approaches but never meets, shown as a dotted line". This idea that the asymptote is always a dotted line bears consequences on the learners' understanding as depicted by learner H. Although graph A contained two asymptotes, $y = 3$ and $x = 0$ and graph B one asymptote, $y = 0$, learner H was fixed on the dotted line:

Researcher: How many asymptotes are there in graph A?
Learner H: Only one, the dotted line.
Researcher: Are there any asymptotes in graph B?
Learner H: No, there are no asymptotes.

Most of the interviewed learners indicated that the graph does not have a gradient and that gradient was only associated with straight line graphs. This probably

stems from the idea that gradient is the same as slope and slope is then associated with, for example, the incline of a hill. However, calculating the slope with the lower grades has always been associated with straight lines. Two of the learners did allude to the idea of an average gradient as an understanding of points on the graph, but this was not clear.

Learner E: Gradient is used for straight line, but you can get average gradient.
Researcher: What does average gradient mean?
Learner E: You can take any two points and find the average gradient.
Researcher: What does the gradient mean?
Learner E: The distance, maybe. I just know the formula we use to find gradient.
Researcher: You say that this graph does not have a gradient?
Learner E: No, it does have a gradient. It has an average gradient.

This procedural interpretation of gradient was confirmed by learner G:
Researcher: Can you explain gradient by means of your graph?
Learner G: You take any two points and substitute into the gradient formula.
Researcher: Where would the gradient be equal to zero?
Learner G: At the origin for the graph of f, because the y value is zero and for the graph of g at y = -9.
Researcher: Why at y = -9?
Learner G: I don't know.

Learner F had correlated the gradient with the arrows on the graph. The learner probably interpreted the arrows going upwards as an increasing graph and therefore the gradient is positive. The gradient of the linear function $y = mx + c$ was also related to the '*a*' in $y = ax^2$.

Learner F: Because the gradient is positive, so it points upwards. If the gradient is negative it will point downwards.
Researcher: What does gradient mean?

Learner F: The gradient means the slope. For the parabola graph it means the way the graph is *pointing* (the researcher's emphasis).
Researcher: What is the gradient of $y = x^2$?

Learner F: It is one… the number in front of x^2.

In a straight line graph the arrow would indicate the continuation of the graph in an identical way as it has done previously. It could be interpreted as the continuation of the gradient as well. Hence, by extrapolation, the learner assumed that all arrows on graphs indicate the continuation of the gradient. Learners responded in various ways to the concept of gradient. A procedural understanding was displayed when gradient meant substituting the co-ordinates into the gradient formula. For others, it referred to the distance on the curve, the arrows on the graph or the shape of the graph. All of these reflect that learners do possess a low conceptual understanding of gradient.

The metaphoric implication of 'reflection' in graphical representations was highlighted through the 'mirror-image' analogy, but emphasized by the learners through the positive and negative signs expressed in formulas.

Researcher: With regards to the reflection of $g(x) = x^2 - 9$ in the x axis, how did you arrive at $-x^2 + 9$?

Learner A: The reflection must be the opposite. It's a sad face.

Learner A draws the graph of the reflection correctly without a table.

Researcher: What helped you to get this answer?

Learner A: The signs. Because of the positive in x^2 the reflection would be a negative. The negative in the equation helped me to draw the sad face. I just used the equation to get the reflection.

Learner B: The shape. The reflection of the happy would be the sad, that's why I got $y = -x^2 + 9$.

Learner F: When the graph is reflecting, it is a mirror image. The signs are opposite $x^2 - 9$, that means that the graph will be sloping the other way and the y intercept will be +9. Instead of being a happy face, it will be a frown face.

Researcher: What helped you to get the answer?

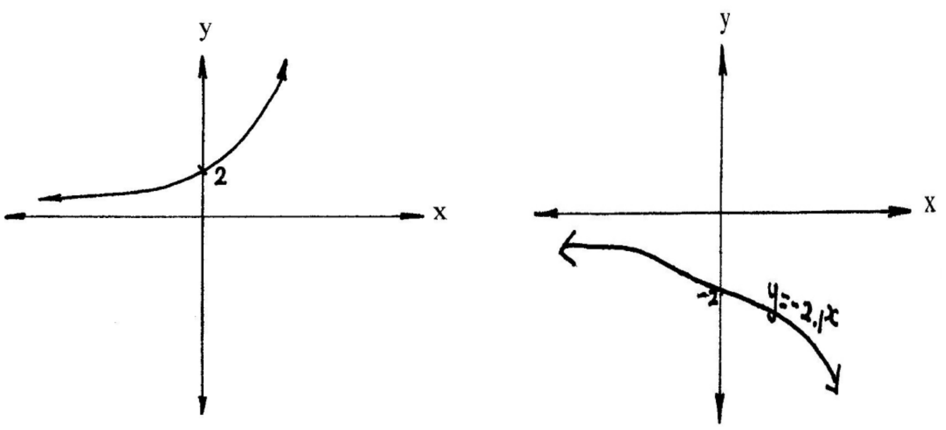
Learner F: I just changed the signs around.

Researcher: What do you understand by reflection?
Learner G: It is still the same image but the other side of it.
Researcher: How did you get your answer for the reflection of $g(x) = x^2 - 9$?
Learner G: When you reflect the g graph, it becomes sad and you just change the signs, negative becomes positive and positive becomes negative.
Researcher: How did you get the reflection for the graph of B?
Learner G: I looked at the graph and folded on the y axis.

Learner G drew the reflection of the exponential graph B using a visual process. The visual implication of reflection was further highlighted by Learner H who understood reflection to be "upside down". It would have been interesting to have probed the idea of having a reflection of $g(x) = x^2 – 9$ along line $y = -9$. This could have perhaps indicated the learners' actual visual understanding of reflections. However, for learners A, F and G the signs in the equation helped them to derive the equation of the reflection. For learner B it was the shape of the graph that assisted her. This indicates that both the formula and the graphical forms assist learners in their visual understanding.

The understanding of reflection as a method of changing the signs was used for the reflection of $y = a.b^x$ for graph B in worksheet 2 by both learners A and F. Both had drawn the reflection in the x axis when the question wanted the reflection in the y axis.

Learner A: I changed the positive two on the y axis into negative two on the y axis.
Learner F: I just did the same as previously. I just changed the signs.



Graph B   Answer of learner F

Figure 15: Reflection of Graphs

Although learners A and F were correctly able to reflect the parabola across the x axis when asked to reflect across the y axis, this was due to their application of rules and procedures learnt. Learner F had initially written the equation for graph B as $y = 2.1^x$. The answer given by learner F for the reflection of the graph of B in the y axis was hence given as $y = - 2.1^x$. She had followed the rule she used for the reflection of $g(x) = x^2 - 9$ across the x axis by changing the signs. In most cases, the learners associated 'reflection' with 'opposite', which in a sense, is exactly the way it appears. Thus, just having a visual conception of an idea is not sufficient. Together with 'seeing' must be a process of understanding. What appears to the eye to be opposite is really not the case. The process employed by some of the learners for determining the reflection of $g(x) = x^2 - 9$ worked well but there was little understanding involved. This was depicted by learners A and F who displayed procedural understanding which was applied correctly to one situation, but not to another. Therefore, learners could not work with more complex graphs.

The period is dependent on the b in y = a sin b x and calculated as period = $\frac{360^0}{b}$. However, the visual skill of interpreting the shape of the displayed graph assists in determining the period. Most learners indicated that the period of the graph of y = 2 sin x as $[-180°; 180°]$ where their reason cited was that it is where the graph starts and where it ends. Learner H regarded the period as the “length of the graph”. The following learners who indicated that the period was 360° related the shape of the sine graph to a “wave cycle”:

Learner E: The parent graph starts at 0^{o} and ends at 360^{o}. This graph starts at -180^{o} and ends at 180^{o}. So the period is 360^{o}.

The omission here is the fact that there is one complete cycle. Although this was not mentioned, learner E had correctly stated the period. However the researcher should have probed this understanding further with for example y = 2sinx, with the domain [-180°; 180°].

Learner F: There is one wave cycle from -180° to +180° and it covers 360°.

Learner G: The graph starts at -180°and ends at 180° and that gives 360°, the period for the sine graph is 360°.

It is perhaps a direct association to the domain in which they were expected to draw their graphs. The researcher cannot hypothesise what their responses would have been had the domain been [-180°; 270°].

5.3.3 The effect of graphical representations on learners’ visual literacy skills

The effects of $f(x) = x^2$ and $g(x) = x^2 - 9$ as symbols of mathematical instructions, evoked in many learners the use of the table method of substituting x values to obtain y values. Hence they plotted the co-ordinates on the Cartesian plane to sketch the graphs. This appears to be procedural understanding. However, learner E referred to $y = ax^2$ and $y = \frac{k}{x}$ as ‘parent graphs’ that she would always use or picture in her mind” to draw and analyse the characteristics of $y = ax^2 + q$ and $y = \frac{k}{x} + q$.

Researcher: Can you explain to me how you drew the graphs of f and g? I see that you did not show any working for the graph of f.

Learner E: First I pictured the parent graph; the parabola in my head...the x^2 indicated that it would be a parabola. I knew how the parent graph looks.

Researcher: Why is the graph of f turning at 0?

Learner E: It is the parent graph. My teacher told me that the parent graph will always cut at 0 and if you have a -9 like the second graph it is a vertical shift; that is why I started the second graph at -9.

Researcher: Why did you select graph A for 1.1?

Learner E: It's the hyperbola. I used the parent graph $\frac{1}{x}$ and this sign says that it is $\frac{k}{x}$. I knew that it was this graph. For all graphs, we get it from the parent graph. You can picture it in your mind and then you look at the shifts.

It is interesting to note that when learner A was asked to sketch the reflection of $g(x) = x^2 - 9$ in the x axis which he correctly indicated as $y = -x^2 + 9$, he did so *without a table* (the researcher's emphasis), but when questioned initially on the use of the tabular form of sketching $f(x) = x^2$ and $g(x) = x^2 - 9$:

Researcher: Could you have drawn the graphs without the table?

Learner A: Yes, but it would be difficult.

Researcher: Why do you say difficult?

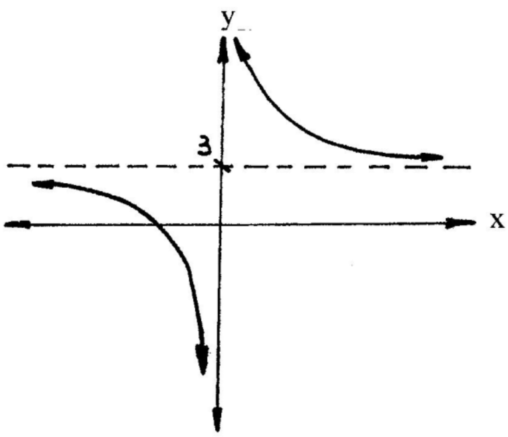
Learner A: The table shows clearly the points.

Researcher: Do you think the points are important?

Learner A: Yes, they are important.

The role of visual clues in co-ordinating visual sequences is an important feature of visual literacy. Arcavi (2003) refers to visually-moderated sequences (VMS) as a continuous process between visual clues (V) and procedures (P). The following data, though not depicting a continuous process may identify with VMS. For matching the hyperbolic and exponential graphs with their standard forms, all learners linked the structure of the graphs with the conventional form. For the hyperbolic function, the y = $\frac{k}{x}$ and the asymptote y = 3 shown on the graph were the visual clues. The exponential form of b^x and the shape of the graph directed all learners to the correct equation. Therefore, one may say that the visual clues led to learners performing the procedure of identifying the equation.

The following are the learners' responses for matching Graph A with the hyperbolic equation as shown on worksheet 2:



Equation: y = $\frac{k}{x} + q$

Figure 16: Matching the graph with the equation

Researcher: Why did you select graph A for 1.1.?
Learner A: It is a fraction…a fraction is a hyperbola. The graph has two curves.
Researcher: Why did you choose graph B for 1.2?
Learner A: It is the exponent. The equation has the exponent. The exponent graph has one curve.

Here the learner had memorized the shapes of the graphs. It may have been useful to have probed further by asking the learner to draw y = $\frac{x^2}{2}$. Also, by indicating that the exponential graph has one curve, it does not say much because if the hyperbola was drawn with a limited domain, only one half of the graph could have been drawn.

Learners C, D and F made use of the asymptotes:
Learner C: Because there's an asymptote.

Learner D: Because you get a broken line.

Researcher: What indicated to you that it is a hyperbolic function?
Learner F: The graph and the equation.
Researcher: What in the graph assisted you?
Learner F: Because of the way it is shaped, and it has two asymptotes.
Researcher: Which are the asymptotes?
Learner F: 3, the q value and the y axis, $x = 0$
Researcher: So, you indicated what helped you in the graph. What in the equation is related to the graph?
Learner F: I don't know.

The role of visual clues was further evident in the identification of the equations of trigonometric graphs $f(x) = a \sin bx$ and $g(x) = \cos(x + q)$ on worksheet 3 in Annexure B. When given the sketch of these graphs they elicited these various conceptual responses:

Researcher: How did you arrive at your answers for the equations of f and g? How did you get $a = 2$ and $b = 1$ for the graph of f?
Learner A: The graph turns at $y = 2$. The wave, the curve just turns at two. I got the information from the graph…from 90°. The $b = 1$ because the graph did not shift.

Learner B: Because of the amplitude.

Researcher: How did you get the answer $a = 2$ for graph f?

Learner E: I knew the 'a' in front of the sin…what effect it has on the graph. It is the amplitude. It means the maximum and minimum values of the graph.

The fact that learners see the 'a' value at the turning point is indicative of good visual literacy. But, it would be sheer guesswork if the researcher has to hypothesise what the learners would have said if the equation was $y = 2 \sin x + 1$. However, learner E demonstrates the 'visually-moderated sequence' as illustrated by the Visualizer/Analyzer model, where visual thinking progresses from the visual to the analytical levels in a cyclical process. Learner E on identifying the equation $g(x) = \cos(x + q)$ demonstrates the various levels of visual and analytical thinking which I classify as a visual literacy process.

Researcher: How did you get $q = +30^o$ for graph g?
Learner E: I knew that it will be the horizontal shift. When you add or minus, it will be the horizontal shift. I knew that the cos graph will be 90^o at 0. When you go to the left it will be plus and when you go to the right it will be minus.
Researcher: You just knew that as a rule?
Learner E: Yes, I knew that as a rule. So the 90^o went to 60^o… it went 30^o back, so it had to be plus 30^o.

Her thought processes may be classified according to the Visualiser/Analyzer model of Zaskis et al (1996) and depicted as follows:

V_1: $y = \cos x$	→	A_1: Passes through (90°; 0)
	↓	
V_2: $(x \pm \theta)$	→	A_2: horizontal shift-left or right
	↓	
V_3: $y = \cos(x + 30°)$	→	A_3: horizontal shift of 30° to the left

Figure 17: Example of the visualisation/analysis process

The above learner first experiences (V_1), the external image of y = cos x. This mental formation directs her thoughts to an internal analytical process A_1 by deducing that the x intercept is (90°; 0). In the subsequent phase (V_2), the addition sign in ($x \pm \theta$) is the focal point that indicates the transformation of the graph (A_2). In the following phase, the learner visualises the horizontal shift in relation to y = cos(x + 30°) advancing to 60° (V_3) and prompts her to construct +30° to the left (A_3).

The following procedure illustrated by learner F, though erroneously indicating that the equation was g(x) = cos (x + 60°), depicts visual and analytical processes similar to that of learner E.

Researcher: For the graph of g, how did you get q = +60°?

Learner F: First I took cos x, it would be one wave cycle passing through the origin. But now it shifted and it's on 60°. So, it shifted 60°. I took +60° because of the plus in the equation.

The transformation of graphs includes translating, reflecting and rotating graphs on the Cartesian plane. The researcher discussed reflection as a visualisation tool earlier. In the following discussion, the researcher provides insight into the learners' understanding of translating graphs in relation to the graphs $f(x) = x^2$ and $g(x) = x^2 - 9$:

Researcher: I see you drew separate tables for f and g. Is there a relationship between f and g?

Learner A: Yes the x^2. The both are parabolas.

Researcher: Can I get the graph of g from the graph of f?

Learner A: Yes, from f(x) you just have to minus nine.

Researcher: Let's say I gave you the f(x) graph and I told to use this to draw the g(x), what would you do?

Learner A: I'll still use the table; just take the domain I used for f.

Researcher: Can you see the relationship with the minus nine.

Learner A: Yes, the minus nine makes the graph go down by nine units.

Researcher: Is there any need still for your table?

Learner A: Yes, there is a need.

It is necessary to note that this learner learnt the drawing of graphs in a very procedural way (he had to use tables), although he might have done so for accuracy. This is despite the fact that he knew that “you have to minus 9”. Most of the other learners had good symbol-sense for firstly, the x^2 depicting the parabola, secondly that the y intercept is -9 derived from the equation $y = x^2 - 9$ and thirdly adding or subtracting a number at the end of the equation meant a vertical shift of the graph. However, as with learner A and many of the other learners, the sketching of graphs was done using the table of values. This might still be due to what Janvier (1987) described as translating from tabular to graphical, a process that is largely emphasised in mathematics classrooms.

Learner F: Yes, the f(x) does not have a y intercept. It is in the form $y = ax^2$ and the g(x) has a y intercept and is in the form $y = ax^2 + c$.
Researcher: Now, tell me, can we get the graph of g from the graph of f?
Learner F: Yes, you’re transforming it. You just draw it below that.
Researcher: But why did you not do this after you sketched f(x). You should have transformed it then.
Learner F: I did not think of that.

Although this is not clearly stated, the learner recognised that it was a transformation and it lay below. A closer look at Learner G, who had also used a table of integer x values from -5 to +5 for $f(x) = x^2$, shows us the various ways learners want to transform graphs.
Researcher: Can I draw the graph of g from the graph of f?
Learner G: Yes, for the f graph there’s no x intercepts and y intercepts. The y intercept of g is -9. You decrease all the points on the f graph by +9. The 25 comes down to 16, the 16 comes down to 7 and so forth.

Translation as a visual skill was further evident when learners had to state the new range of the graph of $f(x) = 2 \sin x$ after it underwent a vertical shift of 1 unit.
While learner A used the range of [2;-2] to assist in arriving at the new range of [3;-1], learner F used the graph:

Researcher: What assisted you in getting the new range for f in question two?
Learner F: I used the graph. When we get a vertical shift the graph moves up one unit, the -2 becomes -1 and the 2 becomes 3.
Researcher: Can you write down the new equation of f?
Learner F: It will be y = 2sinx +1.

On examination of the visualisation processes that learner F engaged in, the Visualizer/Analyzer model is suitably aligned to the cognitive processes. To be able to arrive at the equation after transformation, the following steps might be apparent:

$$\begin{array}{lcl} V_{1:}\ y = \sin x & \rightarrow & A_1\text{: amplitude and wave cycle} \\ & \downarrow & \\ V_2\text{: } y = 2\sin x & \rightarrow & A_2\text{: vertical shift of 1} \\ & \downarrow & \\ V_3\text{: } y = 2\sin x + 1 & & \end{array}$$

The learner's first external picture was the trigonometric equation. The equation y = sin x resulted in the analysis of amplitude and wave cycle. This made the learner advance to the image of a vertical shift of one unit which resulted in the manipulation of the graph to deduce the new range. The learner then visualised the image of the function of y = 2 sin x together with the vertical shift, to arrive at the equation of the new function.

5.3.4 A semiotic analysis

The signs and symbols that learners experience in graphical representations, referred to as the representamen according to Peirce's Triadic model, are sign vehicles that convey meaning to learners (Presmeg, 2008; Chandler, 2002). In the following analysis in table 3, a summary of some of the previous results is provided. This is according to the representamen, object and interpretant levels of Peirce. The interpretant column consists of the learners' interpretations and meanings of the signs they encounter in graphical representations.

Sign/Representamen *(that which represents something else)*	**Object** *(that which it stands for)*	**Interpretant** *(sense made by the learners of the signs)*
1. $f(x) = x^2$	quadratic function/ parabola	• parabola • happy-face, horse-shoe, positive graph, parent graph
2. Arrows at end of graphs	Infinity	• no ending • not stopping, graph continues
3. x axis	number line/real numbers	• Nine decimals between two whole numbers
4. shape of parabola	continuous curve	• it's the shape, smooth curve • happy-face • average gradient
5. f(3)	y value of function at x=3	• plot three on x axis
6. $g(x) > 0$	The function is greater than 0, is positive or y values are positive.	• $x > 0$
7. $y = \frac{k}{x} + q$	hyperbolic function	• q is asymptote • graph of hyperbola • graph does not touch axis • it is a fraction, therefore hyperbola • $\frac{k}{x}$ is parent graph
8. y = a sin b x	sine graph	• 'a' is amplitude • 'b' is wave cycle
9. $y = a.\, b^x$	exponential graph	• b^x is exponent • it has one curve
10. $f(x) = x^2 - 9$	Parabola	• y intercept is -9 • add/subtract number in the end means vertical shift

Table 3: Semiotic analysis of learner responses

The object perspective of the Triadic model resembles the concept definition of Tall and Vinner (1981) that is based on rules, laws and definitions that delineates mathematical truths. Vile (1993) refers to the process of semiosis as a continuous cycle where the interpretant develops into another sign and the semiotic cycle resumes. The response provided by learner E to sketch the graphs of $f(x) = x^2$ and $g(x) = x^2 – 9$ depicts this semiotic cycle:

Learner E: First I pictured the parent graph; the parabola in my head...the x^2 indicated that it would be a parabola. I knew how the parent graph looks.
Researcher: Why is the graph of f turning at zero?
Learner E: It is the parent graph. My teacher told me that the parent graph will always cut at zero and if you have a -9 like the second graph it is a vertical shift; that is why I started the second graph at -9.

The first sign she experiences is the $f(x) = x^2$, where the object is the quadratic function and this leads her to deduce that it will be a parabola. However, upon proceeding to $g(x) = x^2 - 9$, the previous interpretation becomes the sign whereby she relates this to the parent graph $f(x) = x^2$. The object of the graph of g is to translate f(x) which she interprets as shifting the graph nine units down. The influence of semiotic behaviour is further noticed when learners interchange between the equation and graph when making deductions. This was evident when learners matched the equations of the hyperbola, exponential and the trigonometric graphs to the semiotic character of the way these graphs were presented. For example, relating the 'broken line' with the asymptote, the $\frac{k}{x}$ with the 'fraction' and then the $\frac{k}{x} + q$ to the shape of the graph are semiotic cognitive structures that impact on visual literacy. Therefore, the sign vehicles are significant components of visual literacy.

The semiotic behaviour of the learners might be due to the effect of the iconic, indexical and symbolic nature of the signs and symbols as presented in graphical representations. The iconic effect occurs when the $f(x) = x^2$ was related to the parabola due to the shape and structure of the formula. The same sign invoked the table method when learners substituted values for x to draw the graph. This depicts the directional, indexical approach. The symbolic mode is depicted as of the learners' cognitive structure. The iconic mode forms a significant part of the learners' cognitive domain. The structure of $y = \frac{k}{x} + q,$ $y = a.b^x$, y = a sin b x and y = cos (x + q) influenced their visual thinking. In terms of the taxonomy of inscriptions of Marcou and Gagatsis as cited by Presmeg (2008), the descriptive-depictive dichotomy as representative of the symbolic and iconic modes, plays a key role in the learners semiotic actions.

Iconic signs are distinguished as they appear in images, diagrams and metaphors. The researcher classifies some of the learners' responses in terms of the imagery identified by Presmeg (2006). Concrete imagery is depicted in many of the learners understanding of graphs. Some of these include the effect of the physical structure of the Cartesian plane, the shape of graphs and the structure of the equations. A further example is the recognition of the parabola by the presence of x^2. Pattern imagery is evident when learners formed relationships between the graph of $f(x) = x^2$ and $g(x) = x^2 – 9$ and between $y = \frac{k}{x} + q$ and $y = \frac{k}{x}$. The use of memory imagery by learners also largely influenced their level of understanding. The rote application of the table method to sketch graphs, the identification of q as the asymptote, memorising the various equations and the changing of signs in the equation to determine the reflection of graphs denote aspects of memory imagery. Lastly kinaesthetic imagery, where learners moved images with the sense of sight assisted in their visual analysis. The determination of the new range of f(x) = 2 sin x when f(x) made a positive vertical shift of one unit, engaged learners in a physical manipulation of the graph with the corresponding amplitude change to arrive at [3;-1].

The iconic characteristic of metaphorical imagery was apparent in the learners' interpretation of the various metaphors. Metaphors, according to Presmeg (2008), link two domains by providing meaning to one domain through the similarity of the other domain. The use of the words increasing, decreasing, maximum, minimum and reflect provide conceptual understandings in learners through concrete, analogous structures. Metaphors such as the ‘mountain is maximum’ or the ‘valley is minimum’ can be related to the concept image the learners possess of maximum as ‘start high and goes lower’ or ‘turning downwards’. The metaphor that the ‘mirror-image is a reflection’ assists learners to visually transform graphs on the Cartesian plane.

5.3.5 Semiotic mediation

Semiotic mediation takes place within the context of the socio-cultural theory. The use of words, symbols, diagrams, graphs and schemas underline the significance of social activity as a key instrument of learning and conceptualisation as exemplified by Vygotsky (Berger, 2006). The use of words in mathematical communication, the meaning of mathematical truths as displayed in text materials and the signs and psychological tools constitute semiotic mediation. This therefore suggests that the learners’ interpretation of graphs also be scrutinized according to these lines. That language occupies a crucial component of their understanding of graphs is evident in their description of the various characteristics of graphs which have been alluded to in the previous analyses. The use of language mediated the sign with their internal cognitive structure. Their conceptual understanding of the structure of arrows on graphs was mediated through the arrows pointing upwards, therefore as learner A indicated “the graph is going up, there is no ending”. Learner B when asked how she would arrive at the reflection of $g(x) = x^2 - 9$, indicated that “the reflection of the happy would be a sad face”. Therefore, the interaction between language and signs as a means of mathematical learning is pertinent.

The interaction of the learners with the texts of graphs that they confront directs them into a social interaction between themselves and the texts. According to Berger (2006) the learner and the text can be characterized as a social relation, due to the fact that the text books have mathematical expert authority. This framework therefore mediates the individual learner's construction of meaning with the socially accepted text book and the educators that teach them. The impact of the educator is noticed in many instances such as in "my teacher told me":

Researcher: Why are you saying that $x \in R$ for the domain of f(x)? What do you understand by this?

Learner E: My teacher told me that for the parabola it will be $x \in R$. If you had an asymptote then the domain will change.

Learner D: My teacher taught me to draw the graph, a smiling face, a parabola. Therefore I joined the dots.

The effect of classroom instruction and practice was further demonstrated by learner A when he justified his use of the table method to sketch $f(x) = x^2$:

Learner A: I just chose the values that I use in class when I'm doing class work and when I practise my maths.

Berger (2006) stresses that "it is not *how* the student uses the signs but *that* she uses the signs" (italics-her emphasis). This is given prominence due to Vygotsky's emphasis on using words associated with a concept before the concept is internalised or understood. In terms of Vygotsky's (1999) preconceptual stages, the cognitive processes occur in heaps, complexes and potential concepts. However for Grade 11 learners to be dependent on these cognitive processes, then their forms of thinking would be 'primitive' in Vygotsky's terms. To the researcher this might be perceived in terms of Vile's (1993) low level of semiotic understanding while the interpretation pertains to high semiotic demand.

The interpretations such as the arrows at the end of graphs means 'no ending, not stopping' and that the curve on $f(x) = x^2$ is 'the shape' or 'the parabola' portrays thinking at the heap or low semiotic level. An example of complex thinking, classified as non-logical, is when learner E regarded the gradient of $y = x^2$ as one, because of the co-efficient of x^2, a concept transferred from the linear function of $y = mx + c$.

However, according to Vygotsky (1999) the transition from complex thinking to conceptual thinking is made possible through pseudoconcepts. These are similar to the real concepts although thinking is still largely in the complex level. The use of pseudoconcepts occurs just before full understanding of a concept. Examples provided by the learners include calculating the average gradient between two points before understanding what average gradient meant and stating that the domain for $y = x^2$ is $x \in R$ without understanding this concept. However, for Vygotsky, this procedure is vital for conceptual thinking to occur. It will be appropriate to mention here that some learners displayed high semiotic understanding in instances such as relating $f(x) = x^2$ to $g(x) = x^2 - 9$ and $\frac{k}{x}$ to $\frac{k}{x} + q$; while the example on finding q in g(x) = cos (x + q), although not answered well by many learners, had high semiotic demand because interpretation of the signs and notation was required.

5.4 Educators' understanding of graphical representation of functions

5.4.1 The understanding of the Cartesian plane, notation and symbols

The following analysis seeks to determine to what extent the educators' visual understanding of graphs are different from those of the learners.

In terms of their understanding of the use of arrows at the end of graphs:
Educator T1: It means that the graph will continue as you take bigger x and smaller x values.
Educator T2: The domain is real and the graph will go on indefinitely. As the x increases the y increases. The arrows are pointing upwards because it's going in that direction.

The two educators used words such as “continue” and “go on indefinitely” which are similar to the learners’ use of language as a means of describing this concept. However, the reference by educator T2 to “domain is real, as x increases the y increases and pointing upwards because it’s going in that direction” depict a diverse description of his understanding. It is also significant that educator T2 thought in a similar way to the learners when he stated that “as x increases the y increases”. This is incorrect because for $f(x) = x^2$ this is only true for $x > 0$. The opposite is true for values less than 0.

In terms of indicating the values that are between 2 and 3, educator T2 indicated that there will be an infinite number of points. However, though educator T1 understood that there will be many numbers, he seemed to parallel the learners’ interpretation that there are nine numbers between 2 and 3, although he could have been referring to particular decimal places, when he replied:

Educator T1: Yes, there is a whole lot of numbers between 2 and 3.
Researcher: About how many?
Educator T1: Depends how many decimal places you want to go. If you go one decimal place, there will be ten numbers. If you go two decimal places, there will be more numbers.

The understanding of the Cartesian plane was broadened to their understanding of the parabola as a curve with a set of infinite values. Educator T1 understood the parabola to be a set of corresponding x and y values, although no learner indicated this, and that a continuous curve would be formed. He also alluded to the points on the parabola not being in proportion.

Educator T1: Okay, if you take these points between 0 and 1, they won’t be direct proportion there. $(0.1)^2$ and $(0.2)^2$ they won’t be in proportion. That would be the reason for getting a curve and not necessarily a straight line.
Researcher: And what is your reason for joining the five points?

Educator T1: Between say 1 and 2 there are corresponding points. Instead of just leaving dots, you would have a continuous curve.
Researcher: What does it mean to join the two points to form a curve?
Educator T1: Those are all your corresponding x and y values.

Educator T2 related the concept of joining the two points to "you just know it has to be a parabola". This might be the reason why some learners asserted that their meaning of joining the two points is because they knew that it was a parabola.
Researcher: What made you join the x intercepts to the y intercept to form a curve?
Educator T2: Because you're teaching it for so long and you just know that it has to be a parabola and the x^2 indicated that it will be a parabola.

However the curve was related to his understanding of the gradient, which was similar to the two learners' explanation of gradient.
Researcher: What do you understand by the curve that joins the two points?
Educator T2: Gradient, I would say, that's why we talk of a curve, because the gradient is changing all the time.

The educators displayed different methods for sketching the graphs of $f(x) = x^2$ and $g(x) = x^2 - 9$. The educators used the table method and the x and y intercept methods, which are the same methods that the learners employed.
Educator T1: For the first graph f(x) I used the table method and for the second one I used the x and y intercept.
Researcher: For the first graph you said you used the table method; did you take points?
Educator T1: Yes, I took x values from -2 to +2.
Educator T2: I looked at the equation of f(x) and what the rules are about the parabola, about the impact of the 'a' and so on. Just to get a rough idea about where it lies in the Cartesian plane, I just took two points x = 1 and x = -1 and plotted them.
Researcher: And how did you sketch the graph of g?

Educator T2: I looked at the equation as well and I calculated the x and y intercept.

Therefore, the procedural understanding of sketching graphs was also evident with the educators. For the researcher, a conceptual understanding of sketching $g(x) = x^2 - 9$, is to translate $f(x) = x^2$ nine units down. While educator T1 understood the meaning of f(3) for $f(x) = x^2$ in terms of the graph, educator T2 had to be guided in his understanding:

Researcher: What does f(3) mean in terms of the graph?

Educator T1: That your corresponding y value for 3 is 9.

Researcher: And $f(4\frac{1}{2})$?

Educator T1: It would be $(4\frac{1}{2})^2$.

Researcher: And f(20), how would you show that on your graph?

Educator T1: Well, not on this one, it will be a large number, depends on your scale.

Educator T2: By substitution, when x = 3 then y = 9.

Researcher: In terms of your graph, what does this mean?

Educator did not immediately answer. Researcher pointed out x = 3 and projected on graph to find 9.

Educator T2: Yes, it is a point on the graph.

Researcher: How would you determine $f(4\frac{1}{2})$.

Educator T2: By substitution.

Researcher: How would you determine f(20) and explain this in terms of your graph?

Educator T2: By substitution, but we cannot show it on this particular graph, it is not accommodating. You can get 20 but you just have to adjust your scale.

Both educators understood that $g(x) > 0$ meant positive y values greater than zero and displayed good visual understanding of this concept, as well as the $f(x) > g(x)$

when comparing the trigonometric graphs $f(x) = 2 \sin x$ and $g(x) = \cos (x+30°)$. When comparing these graphs, they referred to one graph "above" the other graph that assists them in arriving at a solution.

Researcher: How would you get $f(x) > 0$?
Educator T1: You look above the x axis, for positive y values.
Researcher: And $f(x) > g(x)$?
Educator T1: You look at your graphs physically, where your f graph is above the g graph because that is where your values for f are bigger than the y values for g.

Researcher: How would you explain $f(x) > g(x)$?
Educator T2: Where the sine graph is lying above the cos graph and you can see it at these two points.
Researcher: Why are you saying above?
Educator T2: Because of the greater than.
This level of visual understanding was not evident in most of the learners.

5.4.2 The educators' interpretation of the graphical terminology.

While educators also used the terms happy face and sad face to describe the shape of the parabolas, they displayed better visual understanding for increasing, decreasing functions, domain and range, period, reflection and asymptote than the learners. The educators displayed good visual skills in determining increasing and decreasing functions: While educator T1 used "going down and goes up" that assists him, educator T2 related this to the gradient. These are their responses for $f(x) = x^2$ and $g(x) = x^2 – 9$.

Educator T1: We look at the graph from left to right. The positive gradient will give you increasing. Increasing will be from $x > 0$ and decreasing from $x < 0$ on the x axis. As you take x values smaller than 0, the graph is going down, therefore it is decreasing and as you take numbers bigger than 0, the graph goes up, increasing.

Educator T2: It's related to the gradient. The function is decreasing when its gradient is negative and increasing when its gradient is positive. Also we can say that as the x increases, the y decreases and it will be a decreasing function. As the x increases and the y increases, it will be an increasing function.

However, educator T1 indicated that the graph of $f(x) = x^2$ does not have a gradient but "it can have average gradient". Learner E had also mentioned this. Although, calculus is not taught in Grade 11, an explanation relating the derivative of a curve to the gradient of the curve at a point or relating infinite tangents to the curve with gradients, would have been useful although no question in this regard was asked.

The concepts of domain and range was understood quite well by both educators and explained with adequate visual understanding.

Educator T2: The domain is the x values and since I put my arrows in here the domain would be real. It means that every point on the x axis is taken into consideration.

The maximum and minimum was related to the highest and lowest points. The asymptote of $y = \frac{k}{x} + q$ was explained in terms of the graph shifting.

Educator T1: It means that for the graph there is no corresponding y value at that point, at y = 3 there is no x value; x is the denominator. The 3 means the graph has shifted 3 units up.

Educator T2: The graph is undefined at that point y = 3. The other asymptote is the y axis.

Researcher: Can you explain this concept of undefined?

Educator T2: I relate this to my primary school knowledge of division by 0 is undefined, other than that there is nothing.

The learners did not explain the asymptote as the educators did. Although educator T2 had explained the asymptote for graph A, for graph B he responded as follows:
Researcher: Are there any asymptotes in the second graph?
(After much hesitation)
Educator T2: It does not cut the x axis, so I would assume that the x axis is the asymptote.

For the concept of period a visual understanding was offered in terms of wavelength and wave motion. Some learners also mentioned this:

Educator T2: I looked at half the graph of sin x, which is 180°, so the wavelength is 360°.
Educator T1: If you look at what is given, it is one complete wave motion, starts at -180° and ends at +180°, so the total is 360°.

5.4.3 The effect of graphical representations on educators' visual literacy skills

Both educators did see the relationship between $f(x) = x^2$ and $g(x) = x^2 - 9$ in terms of the f(x) shifting down nine units, a similar action by the learners. However, they too, like the learners did not use this concept when sketching the graphs.

Educator T1: Yes, it's an identical graph, with adding -9; the graph has gone down nine places.
Educator T2: Yes, we can; we shift it down nine units. It might be a parallel shift to the curve of f(x) because of the average gradient. Here the 'a' is constant in both graphs, so I'm just assuming the graphs are parallel.
Researcher: If we give learners to sketch $h(x) = x^2 + 3$, would they see a relationship with $f(x) = x^2$?

Educator T2: I think they would. With the new FET syllabus I think they would see the shift. Previously we never taught like that. The transformations and the shifts are making learners understand graphs better.

Although educator T2 made this comment, it would seem that learners are still fixed with the table method of sketch graphs. This to the researcher is a very low semiotic application of graphs, and hence a low visual literacy skill for Grade 11 learners. But, the educators' visual understanding of 'reflection' depicted by the reflection of g(x) to be $-x^2+9$ serves to highlight the role of visual literacy in graphical interpretation. It must be noted however, that the educators did not mention changing of signs as that of the learners.

Educator T1: I looked at the original graph, and I reflected those points on the x axis. I looked at the new graph, and because it is a sad face, the 'a' is going to be negative. The -9 on the original, if you reflect -9 it will be +9 on the y axis.
Educator T2: I just flipped it on the x axis and I saw this point being -9 and it will just flip to 9. The reflection is a mirror-image. Every point is transformed about the x axis in this case.

For establishing the equations of the trigonometric graphs as well as the new range, the educators displayed the simultaneous use of visualisation and analytical skills as displayed in the Visualizer/Analyzer model as described previously. While learner E referred to the "parent graph", the educators used the term "normal graph". It would seem that these graphs play a significant role in visual literacy.

Researcher: Explain your answers f(x) = 2 sin 1x and g(x) = cos (x+30°)?
Educator T1: A normal sine graph goes up to 1 and -1, in terms of amplitude; and this graph goes up to 2 and -2, so the amplitude is the a = 2. The sine graph if it continues on the positive side it will end up at 360°, so the normal period of the sine graph is 360°, so I did not have to interfere with the 'b' at all; I just left it as it is.
Researcher: How did you get q=30°?

Educator T1: Normally the cos graph at 0 is +1 on the y axis, so when I look at -30°, it is giving me +1, so that graph has shifted 30° to the left, so the y=cos x helped me to get this answer.

Educator T2: It is the cos graph which has transformed horizontally. I had the picture of the normal cos in my mind. The normal cos would have cut the x axis at 90°, so I looked at this one cutting the x axis at 60° which told me it's being transformed to the left by 30°, so I inserted that into the equation.

5.5 Conclusion

The data analysis reflect the impact of signs, symbols, the Cartesian plane, the geometric curves of graphs, notation and graphical terminology on learners' and educators' understanding of graphs. The learners' interpretation of graphical representations and understanding of graphical terminology as they refer to graphs, depict much procedural understanding. While this is a necessary skill in graphical interpretation, the application of conceptual skills appears to be wanting in learners. However, the effect of graphical visual clues and sign vehicles on the learners' visual literacy skills is also adequately demonstrated by the learners.

CHAPTER SIX

CONCLUSION

6.1 Introduction

The research study on functional graphical representations reveals the following findings about the ten learners and the two educators used in the study. These findings do not in any way generalise to the wider mathematics community. However, it does contribute to the fast evolving role of visualisation in mathematics teaching and learning. In terms of the key research questions and other pertinent issues, the findings reveal to a certain extent that:

- Learners understand graphical functional representations mainly through rules and procedures. Therefore, the procedural understanding of graphical concepts is more prevalent than their conceptual understanding.
- Learners display misconceptions in graphical interpretation.
- Visual literacy does play a role in the visual understanding of graphs of learners and educators.
- The educators' utilise visual skills in the understanding of graphs, though sometimes different from that of the learners, and this contributes to the way learners interpret graphs.
- The procedural-conceptual dichotomy is a useful tool to describe learners' interpretation of functional notation and the Visualizer/Analyzer Model offers insight into the ways learners understand reflections, and shifting of graphs. Thus the semiotic theory provides a visualization-based theoretical framework that can be used to better understand learners' interpretation of graphs.

6.2 Main arguments

The misconceptions that learners display point towards a low or literal level of understanding of the Cartesian plane, the graphs they sketch and functional notation. These are depicted in their comprehension of the arrowheads, the number line and the joining of points to form graphs. This indicates that the relationship between real numbers and the Cartesian plane has not been fully established into their conceptual framework. The conceptualisation of the number line as an operative prototype, as a process to act upon, and the parabola as a relational prototype, a process to relate the parabola to aspects of graphical understanding, was not evident in most of the learners. As a result, for them there were nine decimals between the numbers 2 and 3 and that the parabola was merely a curve. The understanding of notation such as f (3), $g(x) > 0$ and $f(x) > g(x)$ suggests a strong procedural conception and a weak conceptual understanding of functional notation and graphical relevance. The fixation to the physical features of graphs such as in "the graph is not stopping; I know it's a happy face, it's increasing because it's going up" contribute to a basic understanding of graphs.

The procedural understanding is further demonstrated by a strong tendency to draw graphs using the table method and substituting integer values for x that they use in class. Most of the learners reacted doubtfully when asked to explain f (20) and f $(4\frac{1}{2})$. The assumption here is that the learners did not work with these numbers in the classroom environment. Other examples include the understanding of f (3) as substituting into the equation; and that of the gradient as substitution into the gradient formula. This might be due to a strong emphasis in algebraic computations in the classroom. The predominance of the concept image consisting of the learners' personal beliefs and cognitive structures influenced their understanding of graphical representations. Language played a significant part in understanding graphical concepts as demonstrated through their personal use of increasing, decreasing, maximum, minimum and reflection. The use of their words in mediating their thoughts plays a vital role in the formation of a proper understanding. The analysis indicates that learners to a great extent depend on procedural understanding of sketching of graphs, as well as in determining the equation of graphs when given visual information on the graph. However, learners also display a weak conceptual understanding for concepts such as domain, range, period, reflection, increasing, decreasing, asymptote and functional notation.

The level of visual understanding displayed by the learners indicates the significant role visual literacy play in graphical comprehension. These are exemplified in relating the name of the graph to its equation and then to its shape. The description of the asymptote in relation to the graph and the equation also indicates the application of visual literacy; though they did not demonstrate a conceptual understanding of asymptotes. The learners also exhibit a good visual understanding of reflecting graphs and translating graphs vertically on the axis. The concept of amplitude was visually established and correctly related to the trigonometric equation. The ability to perform advanced mathematical thinking using visual literacy skills was illustrated by a few learners in explaining the transformation of algebraic and trigonometric graphs.

From a theoretical perspective, learners displayed that they were able to construct knowledge in their interaction with graphs through signs and psychological tools. Graphical functional relationships represented in graphical, symbolic and tabular forms possess sign vehicles that assist in the construction of meaning. The numerous and varied meaning portrayed by the learners display signs as essential components of visual literacy and conceptual understanding. The forms of imagery such as iconic, indexical and symbolic influences their understanding; as well as the types of imagery such as concrete, pattern, memory and kinaesthetic. Thus, learners' understanding of graphical representations depends on the physical form of equations and graphs, pattern identification, retrieving from memory and transforming graphs. Therefore, semiotic mediation between the external concrete image of signs, words and the text, and the internal psychological image, constitute visual literacy processes. Graphs have a high semiotic demand that require a good semiotic understanding. Learners were able to appreciate this aspect of graphs to a certain extent.

The educators on the other hand, demonstrated good visual understanding of the algebraic and trigonometric graphical representations. They also understood the various graphical terminologies in relation to the graph. However, they did show signs of similar understanding to those of the learners. Therefore, it is evident that the way learners conceive graphical relationships is partly due to the way they are taught in the classroom. The literal understanding of the arrows as “the graph will continue and the graph is going in that direction” corresponds to learners interpretations of the arrows. The educator’s utterance of “you just know that it has to be a parabola, because you’re teaching it for so long” would certainly affect the learners’ conceptualisation of why points are joined to form a graph. Perhaps the reason behind these answers is that educators feel that it is not important to conceptualise these with the learners as they are not important for the examinations.

6.3 Limitations

From a research perspective certain shortcomings in the research process could affect the credibility and dependability of the data. According to Cohen, Manion and Morrison (2007) the sample size should be large enough to generate ample data. The study is also not a prolonged study. Prolonged engagement on the field will ensure trustworthiness of research process. The type of questions on the worksheet and the interview schedule might not be totally representative of visual literacy concepts. The researcher is a male, mathematics educator. As a result bias and subjectivity could have affected the interview process, such as in asking leading questions. The ability by the researcher to use relevant prompts and probes to allow the learners to explain, clarify and elaborate, produces rich, first hand information (Maree, 2007). This is one of the shortcomings experienced in the interview process. At certain instances the interview took on a question and answer format. The researcher should have pursued the responses to the questions further in order to acquire thick data.

From a mathematics perspective other visual literacy skills in graphical interpretation could have been omitted in the research process. Since graphical functional relationships is such a vast section, focussing on skills on one particular graph excludes other graphs. As an example, the skill of curve sketching was limited to parabolas. The conclusions made in respect to this graph might not be representative of other graphs, such as the exponential graph.

6.4 Recommendations

The following are some recommendations for further research and consideration by mathematics practitioners:

- The use of graphic calculators and computers in improving graphical skills has been widely researched in other countries. A research study of similar type should be considered in the South African context as a means of investigating visual literacy skills.

- A longitudinal study involving Grades 10-12 might provide a more detailed role of visual literacy on learners' conceptual understanding of graphs.

- In terms of curriculum planning and development, it should be recognised that visual literacy and visualisation skills affect the conceptual understanding of graphical relationships. This should be emphasised in study materials and in the teaching and learning situation. While procedural understanding is still necessary, the role of visual literacy in advanced mathematical thinking involving graphs is essential.

- Educators should emphasise visual literacy skills in mathematics classrooms when teaching functional graphical relationships. These include:

- The comprehension of graphs in respect to the Cartesian plane.

- Sketching of graphs without the table method.

- The meaning of graphical characteristics such as domain, range, asymptote, increasing and decreasing functions.

- Developing a conceptual relationship between the graph and the equation with emphasis on the effect of the parameters and transformation of graphs.

- Emphasising the visual relationships between the symbolic and graphical forms.

BIBLIOGRAPHY

Afamasaga-Fuata'i, K. (2004). *Concept Maps & Vee Diagrams as Tools for Learning New Mathematical Topics.* Paper presented at the First International Conference on Concept Mapping, Pamplona, Spain.

Anderson, G. M. (2007). The evolution of the "Cartesian Connection". *Mathematics Teacher, 102*(2), 107-111.

Arcavi, A. (2003). The Role of Visual Representations in the learning of Mathematics. *Educational Studies in Mathematics, 52*, 215-241.

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). The Development of Students' Graphical Understanding of the Derivative. *The Journal of Mathematical Behavior, 16*(4), 399-431.

Asli, O. (2001). The Graphing Skills of Students in Mathematics and Science Education.ERIC Digest [Electronic Version]. Retrieved 29 February, 2008 from http:www.ericdigests.org

Barwise, J., & Etchemendy, J. (1991). Visual Information and Valid Reasoning. In W. Zimmermann & S. Cunningham (Eds.), *Visualisation in Teaching and Learning Mathematics* (pp.9-24). Washington, DC: Mathematical Association of America.

Beigie, D. (2005). Exploring Graphs in a Target-Oriented Environment. *Micromath, 21,* 24-29.

Berger, M. (2006). Making mathematical meaning: from preconcepts to pseudoconcepts to concepts. *Pythagoras, 63*, 14-21.

Bergsten, C. (2000). From Sense to Symbol Sense [Electronic Version]. Retrieved 10 July 2008 from http://www.fmd.uni-osnabrueck.de/ebooks/erme/cerme1-proceedings/html

Blanche, M.T., & Durrheim, K. (2002). *Research in Practice – Applied methods for the social sciences*. Cape Town: University of Cape Town Press.

Boshoff, H. (2007, 2-6 July). *Generate graphs and generalise the effect of parameters on graphs by means of Autograph and Sketchpad.* Paper presented at the 13th Annual National Congress of the Association for Mathematics Education of South Africa, Mpumalanga, South Africa.

Bowie, L. (2000). A learning theory approach to students' errors in calculus. *Pythagoras, 52*, 2-14.

Bremigan, E. G. (2005). An Analysis of Diagram Modification and Construction in Students' Solutions to Applied Calculus Problems. *Journal for Research in Mathematics Education, 36*(3), 248-277.

Burns, M. (2006). A Thousand Words: Promoting Teachers' Visual Literacy Skills. Retrieved 1 July, 2008, from file: // : \ visual literacy 12.htm

Cates, B. (2002). CBL Activities and Students' Conceptual Knowledge of Functions. Retrieved 29 February, 2008, from http://archives.math.utk.edu

Chandler, D. (2002). *Semiotics: The Basics*. London: Routledge.

Cohen, L., Manion, L., & Morrison, K. (2007). *Research methods in education.* (6th ed.). London: Routledge.

Cooley, L., Trigueros, M., & Baker, B. (2007). Schema Thematization: A Framework and an Example. *Journal for Research in Mathematics Education, 38*(4), 370-392.

Cooney, T. J., & Wilson, M. R. (1993). Teachers' thinking about functions: historical and research perspectives. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp.133-158). New Jersey: Lawrence Erlbaum Associates.

Creswell, J. W. (1998). *Qualitative inquiry and Research design- Choosing among 5 Traditions.* London: Sage.

Department of Education. (2002). *National Curriculum Statement, Grades R-9: Mathematics*. Pretoria, South Africa: Department of Education.

Department of Education. (2003). *National Curriculum Statement, Grades 10-12 (General): Mathematics*. Pretoria, South Africa: Department of Education.

Department of Education. (2007). Grade 11 National Exemplar Paper 3. Pretoria, South Africa: Department of Education.

Dörfler, W. (2000). Means for Meaning. In P. Cobb, E. Yackel & K. Mclain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms- Perspectives on discourse, tools and instructional design* (pp.99-132). Mahwah,NJ: Lawrence Erlbaum Associates.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp.95-123). Dordrecht: Kluvwer Academic Publishers.

Eisenberg, T. (1992). On the development of a Sense for Functions. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function - Aspects of Epistemology and Pedagogy* (pp.153-174). Washington, DC: Mathematical Association of America.

Eisenberg, T., & Dreyfus, T. (1991). On the Reluctance to Visualise in Mathematics. In W. Zimmermann & S. Cunningham (Eds.), *Visualisation in Teaching and Learning Mathematics* (pp.25-38). Washington, DC: Mathematical Association of America.

Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education, 24*(1), 8-40.

Elkins, J. (2003). *Visual Studies - A Skeptical Introduction.* London: Routledge.

English, L. (Ed.). (1997). *Mathematical Reasoning-Analogies,Metaphors and Images*. Mahwah: Lawrence Erlbaum Associates.

Ernest, P. (1994). *Constructing mathematical knowledge: epistemology and mathematical education.* . London: The Falmer Press.

Ernest, P. (1997). Introduction: Semiotics, Mathematics and Mathematics Education. *Philosophy of Mathematics Education*, Journal 10. [Electronic Version]. Retrieved 4 November, 2008, from http://people.exeter.ac.uk/ PErnest/pome10/art1.htm

Fennema, E., & Romberg, T. A. (Eds.). (1999). *Mathematics Classrooms That Promote Understanding*. London: Erlbaum.

Fisher, R. (2005). *Teaching Children to Think.* New Jersey: Erlbaum.

Friel, S. N., Curcio, F. R., & George, W. B. (2001). Making Sense of Graphs:Critical Factors influencing Comprehension and Instructional Implications. *Journal for Research in Mathematics Education, 32*(2), 124-155.

Goba, B., & van der Lith, D. (2005). *Study & Master Mathematics-Grade 10 Learner's book.* Cape Town: Cambridge University Press.

Goldin, G., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. Cuoco (Ed.), *The Roles of representations in school mathematics* (pp.1-23). Reston Virginia: National Council of Teachers of Mathematics.

Harel, G., & Dubinsky, E. (Eds.). (1992). *The Concept of Function- Aspects of Epistemology and Pedagogy.* Washington, DC: Mathematical Association of America.

Henning, E. (2004). *Finding your way in qualitative research*. Pretoria: van Schaik.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Janvier, C. (Ed.). (1987). *Problems of Representations in the Teaching and Learning of Mathematics.* London: Erlbaum.

Johansen, J. D., & Larsen, S. E. (2002). *Signs in Use: An introduction to semiotics.* London: Routledge.

Julie, C., Cooper, P., Daniels, M., Fray, B., Fortune, R., Kasana, Z., et al. (1998). "Global Graphs": a window on the design of learning activities for outcomes-based education. *Pythagoras, 46/47*, 37-44.

Kadunz, G., & Sträßer, R. (2004). *Image-Metaphor-Diagram:Visualisation in Learning Mathematics.* Paper presented at the 28th Conference of the International Group for the Psychology of Mathematics Education, Universitaet Klagenfurt - Austria/ Lulea tekniska universitet - Sweden.

Kieran, C. (1993). Functions, graphing and technology. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp.190-237). New Jersey: Lawrence Erblaum Associates.

Kilpatric, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Kirshner, D., & Awtry, T. (2004). Visual Salience of Algebraic Transformations. *Journal for Research in Mathematics Education, 35*(4), 224-257.

Kleiner, I. (1993). Functions: Historical and pedagogical aspects. *Science & Education, 2*, 183-209.

Kwari, R. (2007). *An investigation into the development of the function concept through a problem-centred approach by Form 1 pupils in Zimbabwe, a thesis submitted for the degree of Master of Education,* UNISA, Durban.

KwaZulu-Natal Department of Education. (2008). Mathematics Paper 2 -Grade 11 Common Test - September 2008, Durban, South Africa: Department of Education.

Lakoff, G., & Núñez, R. E. (1997). Cognitive foundation for a mind-based mathematics. In L. D. English (Ed.), *Mathematical Reasoning-Analogies, metaphors and images* (pp.21-92). New Jersey: Lawrence Erlbaum Associates.

Long, C. (2005). Maths concepts in teaching: procedural and conceptual knowledge. *Pythagoras, 62*, 59-65.

Maree, K. (Ed.). (2007). *First steps in research*. Pretoria: van Schaik.

McEwan, E. K., & McEwan, P. J. (2003). *Making Sense of Research. What's Good, What's Not, and How to Tell the Difference.* Thousand Oaks: Sage.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp.69-100). New Jersey: Lawrence Erlbaum Associates.

Mudaly, V. (2008, 6-13 July). *Visual Literacy and Visualisation in Mathematics Education*. Paper presented at the Eleventh International Congress for Mathematics Education(ICME 11), Universidad Autónoma de Nuevo León,Monterrey, Mexico.

Nicholl, T. (1998). Vygotsky. Retrieved 5 March, 2008, from http://www.massey. ac.nz/alock/virtual/trishvyg.htm

Norman, F. A. (1992). Teachers' mathematical knowledge of the concept of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function-aspects of epistemology and pedagogy* (pp.215-232). Washington, DC: Mathematical Association of America.

Noss, R., Healy, L., & Hoyles, C. (1997). The Construction of Mathematical Meanings: Connecting the Visual with the Symbolic. *Educational Studies in Mathematics., 33*(2), 595-610.

O'Callaghan, B. R. (1998). Computer-Intensive Algebra and Students Conceptual Knowledge of Functions. *Journal for Research in Mathematics Education, 29*(1), 21-4.

Orton, A. (1992). *Learning Mathematics - Issues, Theory and Classroom Practice.* (2nd ed.). New York: Casell.

Pimm, D. (1995). *Symbols and Meanings in School Mathematics*. London: Routledge.

Presmeg, N. C. (1997). Generalization using imagery in mathematics. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors and images* (pp.299-312). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Presmeg, N. C. (2006). Research on visualisation in learning and teaching mathematics. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp.205-235). Rotterdam: Sense Publishers.

Presmeg, N. C. (2008, 6-13 July). *An overarching theory for research in visualisation in mathematics education.* Paper presented at the Eleventh International Congress for Mathematics Education(ICME 11), Universidad Autónoma de Nuevo León, Monterrey, Mexico.

Pylyshyn, Z. W. (2003). *Seeing and Visualising - It's Not What You Think.* Cambridge: MIT Press.

Radford, L. (2001, 12 July). On the relevance of semiotics in mathematics education. Paper presented at the 25th PME International Conference, University of Utrecht, Netherlands.

Rivera, F. (2007). Visualising as a Mathematical Way of Knowing: Understanding Figural Generalization. *Mathematics Teacher, 101*(1), 69-75.

Romberg, T. A., Fennema, E., & Carpenter, T. P. (Eds.). (1993). *Integrating research on the graphical representation of functions*. Hillsdale,New Jersey: Erlbaum.

Sfard, A. (1992). Operational origins of mathematical objects and the quandry of reification-the case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of Function: Aspects of Epistemology and Pedagogy* (pp.59-84). Washington, DC: Mathematical Association of America.

Sfard, A. (2000). Symbolizing mathematical reality into being-or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel & K. Mclain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools and instructional design* (pp.37-98). Mahwah,NJ: Lawrence Erlbaum Associates.

Sierpinska, A. (1992). On understanding the notion of the function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: aspects of epistemology and pedagogy.* (pp. 23-58). West La Fayette: Mathematical Association of America.

Sierpinska, A. (1994). *Understanding in Mathematics*. London: RoutledgeFalmer.

Silverman, L. (2002). The Power of Images: Visual-Spatial Learners. Retrieved 6 March, 2008, from http://www.visualspatial.org

Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education, 35*(5), 305-329.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, the bulletin for the Association of teachers of Mathematics, 77*, 1-20.

Skemp, R. (1986). *The Psychology of Learning Mathematics*. London: Penguin Books.

Smith, H. A. (1997). Peirce's Sign and Mathematics Education: An introduction. Philosophy of Mathematics Education, Journal 10 Retrieved 4 November, 2008, from http://people.exeter.ac.uk/PErnest/pome10/art3.htm

Stokes, S. (2002). Visual Literacy in Teaching and Learning: A literature Perspective. *Electronic Journal for the Integration of Technology in Education*, vol. 1, no. 1. Retrieved 3 June, 2007, from http://ejite.isu.edu/ Volume1 No1/pdfs/stokes.pdf

Tall, D. (1991). Intuition and Rigour: The role of visualisation in Calculus. In W. Zimmermann & S. Cunningham (Eds.), *Visualisation in Teaching and Learning Mathematics* (pp.105-120). Washington, DC: Mathematical Association of America.

Tall, D., Thomas, M., Davis, G., Gray, E., & Simpson, A. (1998). What is the object of the encapsulation of a process? Retrieved 4 November, 2008, from http://www.crme.soton.ac.uk/publications/gdpubs/object.html

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics, 12*, 151-169.

Thornton, S. (2002). A Picture is Worth a Thousand Words. Retrieved 20 March, 2007 from http://www.math.unipa.it/~grim/AThornton 251.

Van der Veer, R., & Valsiner, J. (1993). *Understanding Vygotsky: A quest for Synthesis*. Cambridge: Blackwell.

Van Doreen, W., De Bock, D., Janssens, D., & Verschaffel, L. (2008). The linear imperative: An inventory and conceptual analysis of student's overuse of linearity. *Journal for Research in Mathematics Education, 39*(3), 311-342.

Van Dyke, F. (2002). *A Visual Approach to Functions*. California: Curriculum Press.

Van Oers, B. (2000). Appropriation of Mathematical Symbols. In P. Cobb, E. Yackel & K. Mclain (Eds.), *Symbolizing and Communicating in Mathematics classrooms-Perspectives on discourse, tools and instructional design* (pp.133-176). Mahwah, NJ: Lawrence Erlbaum Associates.

Verhage, H., Adendorff, S., Cooper, P., Engel, M., Kasana, Z., Roux, P., et al. (2000). The interrelationship between graphs, formulae and tables: exploring the relationship between mathematics and outcomes based education. *Pythagoras, 51*, 37-42.

Vile, A. (1993). *Is this a sign of the times? A semiotic approach to meaning-making in mathematics education.* Paper presented at the Annual meeting of the American Educational Research Association, Chicago.

Vygotsky, L. S. (1999). The Function of Signs in the development of Higher Mental Processes. In R. W. Rieber (Ed.), *The collected works of L.S.Vygotsky-Volume 6 Scientific Legacy*. New York: Kluwer Academic.

Wheatley, G. H. (1997). Reasoning with images in mathematics activity. In L. D. English (Ed.), *Mathematical reasoning-Analogies, metaphors and images* (pp.281-298). New Jersey: Lawrence Erlbaum Associates.

White, P., & Mitchelmore, M. (1996). Introductory Calculus. *Journal for Research in Mathematics Education, 27*(4), 70-95.

Williams, L. V. (1983). *Teaching for the Two Sided Mind - A Guide to Right Brain/Left Brain Education.* New Jersey: Prentice Hall.

Williams, S. R. (1993). Functions, graphing and technology. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp.310-337). New Jersey: Lawrence Erblaum Associates.

Yerushalmy, M., & Schwartz, J. L. (1993). Seizing the opportunity to make algebra mathematically and pedagogically interesting. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representations of functions* (pp.42-68). New Jersey: Lawrence Erlbaum Associates.

Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Co-ordinating visual and analytical strategies: A study of of students' understanding of the Group D_4. *Journal for Research in Mathematics Education, 27*(4), 435-457.

Zimmermann, W., & Cunningham, S. (Eds.). (1991). *Visualisation in Teaching and Learning Mathematics.* Washington, DC: Mathematical Association of America.

APPENDIX A

Letter of Consent from Parent

Dear ______________________________________

I am presently an educator at a secondary school teaching Mathematics in the FET phase. I am also a student at the University of KwaZulu-Natal enrolled for the Masters in Education (M.Ed) degree with specialisation in Mathematics Education. I am interested in the role of visual literacy (visual skills) on Grade 11 mathematics learners' conceptual understanding of graphs. Graphs form an important part of the Grade 10-12 mathematics curriculum. As part of my research I would like to interview your son/daughter _______________________________ on his/her understanding of the graphs they experience in class.

If you agree, he/she will be interviewed by me after completing three mathematics worksheets based on graphs. The Grade 11 mathematics learners in your child's school will be ranked according to their ability level. I will be interviewing five selected learners from the school. It is hoped that your child's understanding of graphs, together with other learners understanding, will be identified and thereby enable educators and the departmental authorities to assist learners to improve by emphasising aspects in the teaching and learning of graphs.

The interview will take approximately 60 minutes and will be tape recorded. The interview will take place during the first week in August 2008. The data from the interview will only be used for my research purposes and will not be used for any other purpose without your consent. The recorded tapes will be lodged with the university authorities. Participation is voluntary and your child is not obliged to answer all the questions that I ask him/her and is free to withdraw from the interview at any time. Please note that no real names will be used in any material that I write up and every attempt will be made to keep the material confidential.

Thank you for your assistance. If you require any further information, please contact my course supervisor, Dr Vimolan Mudaly at 082 9770577 or contact me. My details are – Rajesh Rampersad at 073 953 1018.

Yours sincerely

____

R.Rampersad

<u>DECLARATION</u>

I, ______________________________________ (full name of parent/guardian) confirm that I understand the contents of this document and agree to allow my child__________________________________ to participate in the above mentioned research study. I understand that my child's real name will not be used in any write-up and that his/her responses will be treated confidentially. I also understand that he/she will not be under any threat to participate and is at liberty to withdraw from the study at any time.

Signature: ______________________________ Date: ______________

Thank you for your kind co-operation.

Letter of Consent from Mathematics Educator

Dear colleague

I am presently an educator at a secondary school teaching Mathematics in the FET phase. I am also a student at the University of KwaZulu-Natal enrolled for the Masters in Education (M.Ed) degree with specialisation in Mathematics Education. Currently I am engaged in conducting research in the role of visual literacy (visual skills) on Grade 11 mathematics learners' conceptual understanding of graphical functional relationships. Graphs form an important part of the Grade 10-12 mathematics curriculum. As part of my research, I would like to interview you on your understanding and interpretation of the Grade 11 graphs you teach in class. If you agree you will be interviewed by me after completing three mathematics worksheets based on graphs.

Five Grade 11 mathematics learners that you teach will also be interviewed. It is hoped that your understanding of graphs, together with the learners understanding will be identified and thereby enable educators and the departmental authorities to assist learners to improve by emphasising aspects in the teaching and learning of graphs.

The interview will take approximately 60 minutes and will be tape recorded. The interview will take place during the first week in August 2008. The data from the interview will only be used for my research purposes and will not be used for any other purpose without your consent. The recorded tapes will be lodged with the university authorities. Participation is voluntary and you are not obliged to answer all the questions that I ask you and am at liberty to withdraw from the interview at any time. Please note that no real names will be used in any material that I write up and every attempt will be made to keep the material confidential.

Thank you for your assistance. If you require any further information, please contact my course supervisor, Dr Vimolan Mudaly at 082 9770577 or contact me.

Yours sincerely

____

R.Rampersad

DECLARATION

I, _____________________________________ (full name of educator) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participating in the research project. I understand that my real name will not be used in any write-up and that my responses will be treated confidentially. I also understand that I will not be under any threat to participate and is at liberty to withdraw from the study at any time.

Signature: _____________________________ Date: ______________

Thank you for your kind co-operation.

APPENDIX B

Name: ____________________________ School: ______________________

Grade: ____________ Date: _______________

WORKSHEET ONE

1. Draw the following graphs on the same set of axes: $f(x) = x^2$ and $g(x) = x^2 - 9$.

y

x

2. Determine the equation of the reflection of g(x) in the x axis.

____

3. For which values of x is $g(x) > 0$?

____

4. State the domain of f. ____

5. State the range of g. ____

WORKSHEET TWO

Shown below are two graphs, A and B:

A B

1.Match these equations with the above graphs. Write down the correct letter A or or B.

1.1. y = $\frac{k}{x} + q$	
1.2. y = $a \cdot b^{x}$	

2. Write down one possible equation for each of the above graphs by substituting values for a, b, k and q. State the name of each graph.

Graph	Possible equation	Name
A		
B		

3. If graph B is reflected in the y axis, sketch this new graph below.

y

x

WORKSHEET THREE

Sketched below are two trigonometric functions for x ∈ [-180°; 180°]

f(x) = *a* sin *b* x

g(x) = cos (x + *q*)

1. Determine, by finding *a*, *b* and *q*, the equations of f and g.

f: ____

g: ____

2. Give the new range of the graph of f if f(x) undergoes a positive vertical shift of 1 unit?

____

3. What is the period of f(x)?

____

APPENDIX C

INTERVIEW SCHEDULE

Worksheet One

1. Describe why you selected this particular method in sketching the graphs?
2. You used only a few points to draw the graphs. Explain why you used only a few points, and what your reasons are for joining the dots?
3. What is the meaning of the curve between the two points?
4. Point using your pencil where on graph of $f(x) = x^2$, you would read off values for $f(3)$, $f(4\frac{1}{2})$ and $f(20)$?
5. Why did you/ did not use arrows at the end of your graph/axis? What do the arrows indicate on your graphs?
6. Why are the arrows pointing upwards?
7. Are there any points between 2 and 3 on the x axis? If yes, how many?
8. What words can you use to describe your graphs?
9. What does it mean when you talk of $f(x)$ increasing or decreasing? Show me on the graphs of $f(x)$ and $g(x)$ where the graphs are increasing.
10. Does the graph of $f(x)$ have a gradient? Where is the gradient equal to 0?
11. Describe the transformation from $f(x)$ to $g(x)$. How can we get $f(x)$ from $g(x)$?
12. Explain your answers obtained for question 2 to question 5 on your worksheet.

Worksheet Two

1. Explain with reasons your choices for questions 1 and 2? What visual information assisted you?
2. How did you draw the reflection? Explain.
3. How are the two graphs similar or different? Briefly describe what you know about these types of graphs.
4. What is the broken line in graph A called? What does it mean to us? Why is the graph of B not touching the x axis?
5. Which curves are increasing or decreasing? Show me with your pencil on both graphs where the graphs are increasing or decreasing.

Worksheet Three

1. Explain how you arrived at your answers for the equations of f and g?
2. What do you understand by range?
3. How did you arrive at your answer for question 2?
4. Explain your answer for question 3.
5. If the graph of g(x) goes 30º to the right, what would the equation of g(x) be?
6. For which values of x is f(x) increasing and g(x) decreasing?
7. Show me on your graphs where you would read off for $f(x) = g(x)$, $f(x) > 0$ and $f(x) > g(x)$.

APPENDIX D

TRANSCRIPTS OF INTERVIEWS

INTERVIEW BETWEEN RESEARCHER (R) AND LEARNER (A)

Worksheet One

R: Why did you select this particular method of sketching the graph?

A: Sir, the table makes it easier for me to write my graph because I just write down the f(x) points below the x points.

R: These x values that you took. Did you take any values for x?

A: I just chose the values that I used to use in class when I'm doing class work and when I practise my maths.

R: I see you used a few points for drawing the graph. Why did you use only a few points?

A; Sir, because of the time and of the size of the axis and my equation didn't have big numbers.

R: That is why you just went up to four and minus four? And why did you join all the points on your graph?

A: Because of the type of graph. It was a parabola and when you draw graphs you need to attach the points.

R: Can you tell me what the meaning of this curve between the two points is?

A: Sir, the meaning, I just took the points from the table to the graph. My meaning is that it's a happy face….My equation was positive, there was no negative, it was x^2 only. I don't think I know the meaning of this curve.

R: Show me on your graph where you would read off f (3). Which axis would you look at?

A: I would plot it on the x axis between 0 and 5.

R: Show me where you would plot f $(4\frac{1}{2})$ and f (20)

Learner shows on the x axis.

R: Now where is 3, 8 on the x axis.

A: It is here below 4.

R: I see you didn't use arrows at the end of your graph. Did we have to use arrows? What is the purpose of the arrows?

A: Yes we had to use arrows. It shows that the graph is going up. I made a mistake.

R: Is there going to be any ending for the graph?

A: There will be no ending for the graph.

R: If you take f (20) would it touch your graph?

A: Yes it would touch.

R: Is there any need for the arrows at the end of your axis?

A: Yes, the graph is not stopping.

R: Show me on your graph where 2 and 3 are. Tell me; are there any numbers between 2 and 3?

A: Yes sir, 2, 1 up to 2, 9. You get 2,1 ; 2,2 ; 2,3 ; 2,4 ;2,5 ; 2,6 ; 2,7 ; 2,8 ; 2,9 and then 3. It will be up to nine numbers altogether.

R: Let's take the shape of your graphs. What words can you use to describe the shape of your graph?

A: It's a happy face.

R: Why are you saying it's a happy face?

A: The equation of this graph is positive.

R: Are there any other words?

A: Yes the parabola is like a horse shoe.

R: Why are the graphs turning on the y axis? What in the equations relates to these turning points?

A: It's the number in the end. Minus nine of the g graph is the turning point and there is no number for the f graph after x^2, so it turns at zero.

R: Could you have drawn the graphs without the table?

A: Yes, but it would be difficult.

R: Why do you say difficult?

A: The table shows clearly the points.

R: Do you think the points are important?

A: Yes, they are important.

R: Have you heard of these two words, increasing and decreasing?

A: Yes.

R: Show me where your f graph is increasing?

A: It is increasing on the right hand side of the x axis.

R: And where is it decreasing?

A: On the left hand side.

R: Why is it increasing on the right hand side and decreasing on the left?

A: Because of the positive numbers on the right and the negative numbers on the left.

R: Okay, have you heard of gradient? What does it mean to you?

A: It is used with the straight line.

R: Can you get a gradient with these curves?

A: Yes. At any point, this point has x and y, you can get the gradient.

R: I see you drew separate tables for f and g. Is there a relationship between f and g?

A: Yes the x^2. The both are parabolas.

R: Can I get the g graph from the f graph?

A: Yes. From f(x) you just have to minus nine.

R: Let's say I gave you the f(x) graph and I told to use this to draw the g(x), what would you do?

A: I'll still use the table; just take the domain I used for f.

R: You can't see the relationship with the minus nine.

A: Yes, the minus nine makes the graph go down by nine units.

R: Is there any need still for your table?

A: Yes, there is a need.

R: With regards to your answer for number two, the reflection of g(x) in the x axis. How did you arrive at $-x^2 + 9$?

A: The reflection must be the opposite. It's a sad face.

Learner draws the graph of the reflection correctly without a table.

R: What helped you to get this answer?

A: The signs. Because of the positive in x^2 the reflection would be a negative. The negative in the equation helped me to draw the sad face. I just used the equation to get the reflection.

R: What do you understand by $g(x) > 0$?

A: I take the numbers from my table that are greater than 0.

R: What do you understand by domain and how did you get your answer of

$-4<x<4$?

A: Domain is the x values. I just take it from my table.

R: Even if you missed your arrows, the domain is still -4 to +4?

A: Yes, I take it from my table.

R: What do you understand by range?

A: It's the y value. I just take it from my table.

Worksheet Two

R: What are the reasons for your selection for 1.1 and 1.2?

A: Graph A is the hyperbola.

R: How did you match this with 1.1?

A: It is a fraction…a fraction is a hyperbola. The graph has two curves.

R: Why did you choose graph B for 1.2?

A: It is the exponent. The equation has the exponent. The exponent graph has one curve.

R: For the equation of the graph of A, why did you write 3 for q?

A: It is the asymptote. The equation says $\frac{k}{x} + \;'$, the q stands for the asymptote.

R: What do you understand by asymptote?

A: If q is positive the broken line will be by positive 3.

R: Can you show me any other asymptotes?

A: Yes. The x and the y, the original, are the asymptotes for graph A. The x and y are also the asymptotes for graph B.

R: For the equation of B, why did you choose 2 for the value of a?

A: It is the two from the graph.

R: How did you draw the reflection?

A: I changed the positive two on the y axis into negative two on the y axis.

R: How are the two graphs similar or different?

A: One has the fraction, the other is the exponent.

R: Can you draw for me the graph of y $= \frac{6}{x}$.

Learner draws on the page freehand.

R: Where are the asymptotes?

A: The x and y axis, the originals.

R: What are the equations of the axis?

A: The y axis is y=0 and the x axis is x=0.

R: Why are the curves not touching the x axis?

A: I don’t know why. The broken line means the graph shifted.

R: Show me on your graphs where they are increasing or decreasing.

A: For graph A, I don’t know. For graph B it is increasing from the two upwards on the right hand side of the axis.

R: When will the y intercept of Graph B be one?

A: When it is written as 1.2^x .

Worksheet Three

R: How did you arrive at your answers for f and g? How did you get a=2 and b=1 for the f graph?

A: The graph turns at y=2. The wave, the curve just turns at two. I got the information from the graph…from 90°. The b=1 because the graph did not shift.

R: Draw the graph of y= 10 sin x.

(Learner draws freehand and plots y=10 on the y axis in line with the turning point.)

How would you draw y= 10sin x +2.

A: The graph would start at 2 on the y axis.

R: How did you get q= -1 for the g graph?

A: The graph is in line with -1 on the y axis.

R: Draw the graph of y = cos x. Where does this graph start?

A: It starts at one.

R: Can you use this graph to draw y= cos x -2.

A: Yes, the graph will shift two units down because of the minus two.

R: Can you give me the range for the f graph?

A: It will be 2 and -2.

R: Now what will be the range if the graph shifts one unit upward?

A: It will be 3 and -1.

R: What helped you to get this answer?

A: The previous range, I’m getting the range from the graph.

R: How did you get your period for the graph of f to be -180°; 180°?

A: It is the starting and the ending.

INTERVIEW BETWEEN RESEARCHER (R) AND LEARNER (B)

Worksheet One

R: What method did you use to draw the two graphs?

B: I looked at the x^2 and saw it was positive, and then I knew it was a happy face. For the g graph I found the x intercepts x=3 and x=-3, and y intercept=-9 and I plotted these on the graph.

R: Why did you join the -3 to -9 and then to +3? Are there any points on your curve? Why did you not use a straight line to draw the graph?

B: Because it's a happy face, that's why I used a parabola. There are no more points on the curve, just the three points I joined.

R: If I put a negative in front of the x^2, what graph would we get? Can you draw that for me?

B: It would be a sad face.

Learner draws the graph correctly without any working.

R: Can you show me on your graph f (3)? Do you understand the meaning of f (3)?

B: I don't understand f (3).

Researcher shows learner how to get point on the graph.

R: Can we get any numbers between 2 and 3?

B: Yes we can get 2, 1; 2, 2; 2, 3; up to 2, 9 and then 3.

R: Show me where we can get f ($4\frac{1}{2}$) and f (20).

Learner shows $4\frac{1}{2}$ on X axis.

B: For f (20) we need to have a scale, like 5, 10, 15 20.

R: Why did you not use arrows on your graph?

B: I just stopped.

R: If I put arrows, what would be the meaning of the arrows?

B: It would be where the graph ends. Sometimes it can mean the graph continues.

R: What is the relationship between the f and the g graphs?

B: The x^2 and the x^2. They are parabolas. The turning point of the f is 0 and the g is -9.

R: Can you get these points from the equation?

B: Yes, you can.

R: If I give you the f graph, can you get the g graph from the f?

B: The g graph came wider and the turning point changed.

R: Did you hear of increasing and decreasing? Can we get it from your two graphs?

B: The f and the g is increasing because its going up and the $y= -x^2+9$ is decreasing because its going down.

R: Can we get gradient on these curves?

B: No. We get it only for straight lines.

R: What helped you to get the answer for the reflection?

B: The shape. The reflection of the happy would be the sad, that's why I got $y=-x^2-9$.

Worksheet Two

R: What helped you to get the answer A for 1.1.?

B: I knew the shape of the hyperbola. This is the equation of the hyperbola.

R: Why did you choose B for 1.2.?

B: Because the exponential graph, the equation is like that.

R: How did you get the +3 in $y = \frac{1}{x-} +$ for the equation of A?

B: This broken line, the asymptote.

R: If I changed this to -3 what will happen?

B: It will be down, the asymptote.

R: Do we have asymptotes on graph B?

B: No, only the dotted line in graph A is the asymptote.

R: What helped you to draw the reflection of graph B in the y axis?

B: The +2 on the y axis. I just took -2 on the y axis.

R: Why is graph B not touching the X axis?

B: I don't know.

R: What does the asymptote mean?

B: That the graph must not touch the line.

R: How would you draw $y=2^x$ and what would the y intercept be?

B: I would use a table. It would be like graph B. The y intercept would be 2, because of the 2 in $y=2^x$.

Worksheet Three

R: How did you get y = 2 sin x for the graph of f?

B: Because of the amplitude.

R: Show me on your graph where f(x) = g(x).

Learner rings the two points where the graphs intersect each other correctly, but could not show where to read off the values.

R: Where would you read off f(x) > 0 and g(x) < 0?

B: For f it would be on the positive X axis and for g it would be on the negative X axis.

INTERVIEW BETWEEN RESEARCHER (R) AND LEARNER (C)

Worksheet One

R: Can you explain to me how you did the sketch of f?

C: I used my calculator. Then I drew a table and used x values. I then found the y values and drew my graph. The equation told me that it would be a happy face.

R: What helped you to say that it would be a parabola?

C: The x^2.

R: How did you draw the graph of g?

C: I used the x^2 and the minus nine. When you minus nine the graph would go down by nine units and when there's a positive sign here it's going up.

R: I see you used dots. Let's take two dots. You joined them using a curve. What is the meaning of the curve?

C: So that I can get the shape of the graph. I don't know the meaning of the curve. I just know it's the shape, so I join the dots.

R: Can you show me how you would get f(3), f(4½) and f (20)?

C: I can plot the 3 and the 4½ on the X axis and go up. The 20, no the graph is small. I think if it was a straight line, yes it would touch.

R: What is the meaning of the arrows on the graph?

C: It means that the graph continues.

R: Are there any points between two and three on the X axis?

C: Yes, there are ten points, 2, 1; 2, 2; 2, 3; 2, 4 up to 2, 9 and then 3.

R: What words can you use to describe your graphs?

C: It's a happy face.

R: Have you heard of maximum and minimum? What would these graphs be?

C: Yes, I've heard of it. This graph is a maximum.

R: Why do you say that?

C: When something starts there and carries on. It's a maximum. There's no minimum.

R: Have you heard of increasing and decreasing?

C: Yes, I have. The graph is increasing because it's going up.

R: Is there a relationship between f and g?

C: Yes. From the normal graph, if you add or subtract a number, the graph goes up or comes down.

R: I see you did not write down the reflection. Can you draw a sad face? That will be the reflection.

Learner draws the reflection buts write down the equation as $f(x) = -x^2-9$.

R: What numbers are greater than 0 as in $g(x) > 0$?

C: One, two three and so on.

Worksheet Two

R: Why did you choose the graph of A for 1.1.?

C: Because there's an asymptote.

R: What does the asymptote mean?

C: It must be a dotted line. It cannot be a straight line. The graph won't carry on, It's where the graph has got to end.

R: You are getting the meaning from the graph. Can you get it from the equation?

C: Yes. The q number is the asymptote. You just add it in the end.

R: Are the graphs increasing or decreasing?

C: In A these graphs are increasing. In graph B ….I don't know.

Worksheet Three

R: I see that you did not write down the equations of f and g. It does not matter. But, when you see the graph of f, what name comes into your head?

C: I don't know.

R: Draw the graph of y=sinx from 0^o to $360^{o.}$

Learner draws the graph correctly.

R: Can you now draw $y = 2\sin x$.

Learner draws the graph and starts from 2.

C: Yes. The graph moves upwards…starts from 2.

R: How would the graph of $y= \sin x + 2$ look.

C: Oh! I made a mistake with the previous one. It would be for this equation $y= \sin x +2$…the graph moves up.

INTERVIEW BETWEEN RESEARCHER (R) AND LEARNER (D)

Worksheet One

R: What method did you use to sketch the two graphs?

D: I used the table method. I took -2, -1, 0, 1, 2 for x and substituted into the equation and plotted the points.

R: I see that you joined the five dots. Can you tell me why you joined the dots?

D: My teacher taught me to draw the graph-a smiling face-a parabola.

R: What in the equation led you to draw a parabola?

D: The x^2.

R: What is the meaning of the curve between the two dots?

D: Sir, I don't know.

R: Are there more points on this curve besides the five you plotted?

D: Yes there are. I started at -2. If I start at -3 there will be more dots.

R: Can you take any points between 1 and 2?

D: Yes I can take 1½.

R: Do you know what the meaning of f (3) is?

D: No, I don't know.

R: Why did you use arrows at the end of your graph?

D: To show that the graph can still increase if there are more numbers.

R: Why did you not take more numbers to draw your graph?

D: Sir, because my teacher taught me to take the five numbers.

R: Could you have taken -3 and -4?

D: Yes. I'll still get the same shape.

R: Are there any numbers between 1 and 2 on the X axis?

D: Yes, we can get nine numbers 1,1; 1,2; 1,3; 1,4; up to 1,9.

R: What words can you use to describe your graph?

D: A smiling face.

R: Show me when we can get a sad face.

D: When we get $y = -x^2$.

R: What do you think is the relationship between a sad and a happy face?

D: The negative makes it a sad face.

R: Are your graphs increasing or decreasing?

D: I think the one with the negative is decreasing..it's going down.

R: Can we get the graph of g from the graph of f? Is there any relationship between the f and the g graphs?

D: Yes-the x^2.

Worksheet Two

R: Why did you select graph A for 1.1.?

D: Because you get a broken line.

R: What is the relationship between the two curves?

D: The second curve is the exponential- it only has one curve. The hyperbola has two curves.

R: Do you know the name of the broken line?

D: It's the asymptote.

R: What does asymptote mean to you?

D: No, I don't know.

R: Does the B graph have an asymptote?

D: No, it doesn't have- asymptote is only a broken line.

R: I see for your graph of A you had the value 3 for q. Why did you use 3?

D: Because the asymptote was 3.

R: How would you describe your graphs in terms of increasing or decreasing?

D: The hyperbola, the second curve here is going down so it is decreasing.

Worksheet Three

R: For the graph of f you wrote 2 sin x. Why did you write the 2 in front?

D: From the y axis.

R: If the graph of f was in line with y=1, what number would you place in front of sinx ?

D: It will be one.

R: Can you draw the graph of y = 10 sin x for me on this page?

Learner draws the graph correctly and indicates 10 and -10 on the y axis.

R: Can you show me where on your graph f(x) = g(x)?

D: No.

INTERVIEW RESEARCHER (R) AND LEARNER (E)

Worksheet One

R: Can you explain to me how you drew the graphs of f and g? I see that you did not do any working for the graph of f.

E: First I pictured the parent graph; the parabola in my head...the x^2 indicated that it would be a parabola. I knew how the parent graph looks.

R: Why is the graph of f turning at 0?

E: It is the parent graph. My teacher told me that the parent graph will always cut at 0 and if you have a -9 like the second graph it is a vertical shift-that is why I started the second graph at -9.

R: I see that you also did working for the graph of g where you determined the x intercepts and the y intercept. You did not take any points for x and plot these points.

E: I felt that there is no need to take any points. The parent graph tells you that it will be a parabola.

R: What is the meaning of f(x)?

E: It is the y.

R: Can we get any points on your graph of f?

E: Yes, you can get. You get an infinite number of points. You can get positive numbers; you cannot get negative points because the graph is turning at zero.

R: What is meaning of the curves on the parabola?

E: I am not sure. I just know how the parabola is supposed to look.

R: Can you show me where on your graph you would get f (3)?

E: First I would put 3 for x in x^2. Then I would get 9. So where x=3, y=9.

R: Can you get $f\,(4\frac{1}{2})$ on your graph?

E: Yes you can, by going up.

R: But your vertical line from $4\frac{1}{2}$ will not touch your graph.

E: Yes it can touch. The arrows at the end mean that the graph goes on, so we can extend the line.

R: Can we get f (20) on the graph?

E: Yes, we can. We need to extend our axis and plot 20 on the x axis. We can also substitute 20 for x in the equation and get y = 400.

R: Can we put arrows on the axis and what would the arrows mean?

E: Yes. It means that the positive side goes to positive infinity and the other side goes to negative infinity.

R: Are there any numbers between 2 and 3?

E: Yes, there are lots of numbers, when you put fractions, you can put fractions in between.

R: How would you describe your graph in terms of maximum and minimum?

E: The graphs are minimum because it is turning upwards. If it turns downwards, then we will get a maximum turning point.

R: Would you say that your f graph is increasing or decreasing?

E: The f graph is increasing, because it carries on to positive infinity.

R: When would the graph be referred to as decreasing?

E: When it is facing downwards, it goes to negative infinity.

R: Can we get the gradient on the curve?

E: No. Gradient is used for straight line, but you can get average gradient.

R: What does average gradient mean?

E: You can take any two points and find the average gradient.

R: What does the gradient mean?

E: The distance, maybe. I just know the formula.

R: You say that this graph does not have a gradient?

E: No, it does have a gradient. It has a average gradient.

R: I see that you indicated earlier that we can get the graph of g from f. Can we get f from g?

E: Yes. I'll just drop the -9 and draw it, because I know it's the parent graph.

R: Can you explain how you got the answer $g(x) = -x^2+9$ for question 2, the reflection of g in the x axis? Your answer is correct.

E: I knew that it would be the opposite of $x^{2-}9$.

R: Explain to me the meaning of $g(x)>0$?

E: It would be where the $x >0$. It would be positive answers.

R: Why positive?

E: R: What does the domain mean?

E: All possible x values.

R: Why are you saying that x $\in$R? What do you understand by this?

E: My teacher told me that for the parabola it will be x$\in$R. If you had an asymptote then the domain will change.

R: How did you get the range?

E: The range will be all the y values. That is how I got y $\in$Z… all the y values. Because it must be greater than zero. On the positive side of the x axis we get positive answers and the graph is going up, so x>3 and x>-3. For x>3 we are going up and for x>-3 we are going up because we get positive y values.

Worksheet Two

R: Why did you select graph A for 1.1?

E: It's the hyperbola. I used the parent graph $\frac{1}{x}$ and this sign says that it is $\frac{k}{x}$. I knew that it was this graph. For all graphs, we get it from the parent graph. You can picture it in your mind and then you look at the shifts.

R: And the second graph?

E: It is the exponential. The x in y=ab^{x} is the exponent and the shape of the exponential graph goes like that.

R: Why did you write 3 for the q value in question 2?

E: I know that from the parent graph this graph went 3 up, so q has to be 3, the asymptote is 3.

R: What do you understand by asymptote?

E: The graph will come closer and closer to the asymptote but it will never touch.

R: Are there any other asymptotes in graph A?

E: Yes, it's the x and y, but for this graph it's the y, because it is not touching the y.

R: And for graph B, does it have asymptotes?

E: Yes, it's the x axis.

R: Do you know why it is not touching?

E: Not really.

R: Do you know of any other graph that has asymptotes?

E: Yes, the tan graph, it does not touch at $90^o$and 270^o.

R: Why?

E: Because it contains the shape of the tan graph.

R: How did you get 2.2^x for graph B?

E: I took the 2 from the graph. The ‘a’ has to be 2 because b^x is one1 for any number of b because it will be raised to the 0 and that will be one.

R: Would you say that graph A and graph B is increasing or decreasing?

E: Both graphs are increasing and decreasing, increasing where it goes up and decreasing where it goes down.

Worksheet Three

R: How did you get the answer a=2 for graph f?

E: I knew the ‘a’ in front of the sin…what effect it has on the graph. It is the amplitude. It means the maximum and minimum values of the graph.

R: Where did you get the 2 from?

E: From the y axis.

R: How did you get b=5?

E: Oh! I guessed that one.

R: How did you get q=+30^o for graph g?

E: I knew that it will be the horizontal shift. When you add or minus, it will be the horizontal shift. I knew that the cos graph will be 90^o at 0. When you go to the left it will be plus and when you go to the right it will be minus.

R: You just knew that as a rule?

E: Yes, I knew that as a rule. So the 90^o went to 60^o… it went 30^o back, so it had to be plus 30^o.

R: How did you get your range to be 3;-3 for question 2?

E: When you go up one unit from 3, it will be 3 and -1. So your maximum is 3 and your minimum is -1.

R: So, if you get 3 on top, you will get -1 at the bottom. If you get 4 at the top what will you get at the bottom?

E: Yes, you will get -4 at the bottom.

R: Explain how you got the period to be 360^o for the graph of f?

E: The parent graph starts at 0^o and ends at 360^o. This graph starts at -180^o and ends at 180^o. So the period is 360^o.

R: Where would you solve for f(x) =g(x)? Where would you read your answer?

E: At where the graphs intersect each other. I will read from the x axis.

R: How would you solve for f(x)>0?

E: It would be above the x axis… all the positive values.

R: How would you solve for f(x) > g(x)?

E: It would be from the points of intersection where the f graph is above the g graph?

R: So if the second point of intersection is 15^o, what would be your answer?

E: It would be $x > 15^o$.

INTERVIEW BETWEEN RESEARCHER (R) AND LEARNER (F)

Worksheet One

R: Describe the methods you used in sketching the graphs of f and g.

F: For my f graph I looked for the x intercept and since the equation is $y=x^{2,}$ I knew that it would start at the origin and since it is a positive it went upwards. For the g graph I also worked the x and the y intercept. I worked out the x intercepts to be +3 and -3and I plotted these on the axis. Then I worked out the y intercept to be -9. Since its positive it goes upwards.

R: I see that you joined the three points -3,-9 and +3. Why did you join them and make a curve?

F: The equation is a quadratic function, so it will have that shape.

R: Let me take two points that you joined on the g graph, -3 and -9. There is a curve from -3 to -9. Do you know the meaning of this curve?

F: It is increasing. I don't know the meaning.

R: Are there any more points on this curve?

F: We can get two more points from -3 to -9.

R: If you take the entire curve of g, how can we get more points?

F: We can use the table method. We can take -3,-2,-1, 0,1,2,3 for x and substitute in the equation to find y.

R: Now let us take f (3). What does this mean?

F: You substitute 3 for x in $y = x^2$and you get 9.

R: What does this mean?

F: That the y value is 9.

R: Can we get f ($4\frac{1}{2}$) and f (20)?

F: Yes. But for f (20) we have to draw a longer x axis and extend the graph.

R: I see that you inserted arrows at the end of your graphs. What do these arrows mean?

F: That it goes to infinity.

R: Why are the arrows pointing upwards?

F: Because the gradient is positive, so it points upwards. If the gradient is negative it will point downwards.

R: What does gradient mean?

F: The gradient means the slope. For the parabola graph it means the way the graph is pointing.

R: What is the gradient of $y = x^2$?

F: It is one… the number in front of x^2.

R: If it was $y = 3x^2$, what would the gradient be?

F: It would be three.

R: Let us take the number two and three on the x axis. Are there any points between these points?

F: Yes, about ten, I think you can get $2\frac{1}{2}$, $2\frac{1}{4}$ and so on.

R: What words can you use to describe your graphs?

F: Positive graphs.

R: How would you describe the graphs in terms of maximum and minimum?

F: The graph of g is minimum at y=-9 and the f graph at 0.

R: When can we describe the graph as maximum?

F: When the equation is negative and the graph points downwards.

R: What does increasing and decreasing mean in terms of your graphs?

F: The graph of f first decreases and then it increases.

R: Why makes you say that?

F: If you take the f graph on the left hand side of the axis, it is decreasing because the y values get smaller, like 10, 9, and 8 and so on. But on the right hand side it increases because the y values increase from 0,1,2,3 and so on.

R: What tells you that the graph is increasing or decreasing?

F: The values of the y.

R: Can you see any relationship between f and g?

F: Yes, the f(x) does not have a y intercept. It is in the form $y=ax^2$ and the g(x) has a y intercept and is in the form $y=ax^2 +c$.

R: Now, tell me, can we get the graph of g from the graph of f?

F: Yes, you're transforming it. You just draw it below that.

R: Let us now take your answers for questions two to question five. What do you understand by reflection?

F: When the graph is reflecting…mirror image.

R: Explain your answer $-x^2+9$.

F: The signs are opposite x^2+9, that means that the graph will be sloping the other way and the y intercept will be +9. Instead of being a happy face, it will be a frown face.

R: What helped you to get the answer?

F: I just changed the signs around.

R: Explain your answer of $x>3$ and $x<-3$ for finding values of x where $g(x)>0$.

F: For $x>3$ it will be values like 4, 5, 6 and -4,-5,-6 for $x<-3$.

R: Let us now take the next concept, the domain. What do you understand by domain?

F: It is all the x values.

R: How are these values related to the graph of f?

F: $x\varepsilon R$, these include counting numbers, integers and not surds. I don't know what it actually means in terms of the graph.

R: And what would you say the range of the graph of g is and why?

F. The range is the y values, $y\geq-9$; because the graph is going upwards from -9.

Worksheet Two

R: In the matching exercise why did you select Graph A for 1.1?

F: Because it's the hyperbolic function.

R: What indicated to you that it is a hyperbolic function?

F: The graph and the equation.

R: What in the graph assisted you?

F: Because of the way it is shaped, and it has two asymptotes.

R: Which are the asymptotes?

F: 3, the q value and the y axis, $x=0$

R: So, you indicated what helped you in the graph. What in the equation is related to the graph?

F: I don’t know.

R: You mentioned asymptotes; tell me what asymptotes mean to you?

F: The graph, when it goes near to the line, it does not touch.

R: Do you know why it would not touch?

F: No, I don’t know.

R: For 1.2 what told you that it would be the graph of B?

F: It’s an exponential function. The graph assisted me.

R: What in the equation assisted you?

F: It’s the x exponent.

R: Are these two graphs similar or different in any way?

F: They are similar and different. Graph A has two curves and graph B has one curve, and graph B is increasing.

R: Since you mentioned increasing, tell me whether the two graphs are increasing or decreasing?

F: Graph A is increasing because its going to the right. It’s actually decreasing on the y axis and increasing on the x axis. Graph B is increasing.

R: Why is graph B increasing?

F: Because of the way it’s sloping. From decreasing it goes to increasing, because the negative numbers are decreasing on the x axis and on the positive side it’s increasing.

R: Can we get the y intercept of graph B is equal to one?

F: You mean where the graph touches?

R: Yes.

F: I don’t know.

R: You explained the q=3 for 1.1. Can the k value be any number to get the same shape as graph A?

F: Yes it can be any number, but just that the intercept would be different.

R: Explain your reflection for graph B in the y axis?

F: I just did the same as previously. I just changed the signs.

Worksheet Three

R: What helped you to get a=2 and b=1 for the graph of f?

F: ‘a’ is the amplitude and the amplitude for the graph is 2 from the y axis.

R: And the b=1?

F: There, I saw how many wave cycles there are and there’s one wave cycle.

R: For the graph of g, how did you get q=+60°?

F: First I took cos x, it would be one wave cycle passing through the origin. But now it shifted and it’s on 60°. So, it shifted 60°. I took +60° because of the plus in the equation.

R: You spoke of a wave cycle in the graph of f. Can we get many cycles?

F: Yes, we can get many cycles.

R: What assisted you in getting the new range for f in question two?

F: I used the graph. When we get a vertical shift the graph moves up one unit, the -2 becomes -1 and the 2 becomes 3.

R: Can you write for me the new equation of f?

F: It will be y= 2sinx +1.

R: That is correct. Tell me how did you get the period = 360° for the graph of f?

F: There is one wave cycle from -180° to +180° and it covers 360°.

R: If the graph of g goes 30° to the right, what would be the new equation?

F: It would be cos(x+90°).

R: What is the meaning of the curve of f?

F: That x>0.

R: What I really mean, like 1°, 30, 5° and so on.

F: Yes, we can get that. It will be a really big graph.

R: Let us describe f in terms of increasing and decreasing?

F: It decreases then increases and then decreases.

R: Where on the graph is f(x) =g(x)?

F: At the points of intersection.

R: And g(x)>0?

F: It’s the part of the graph greater than 0 on the x axis, the x values.

R: And g(x) <0?

F: It’s the negative side of the axis.

INTERVIEW BETWEEN RESEARCHER (R) AND LEARNER (G)

Worksheet One

R: Can you describe to me the method that you used to sketch the graphs of f and g? What helped you to sketch these graphs?

G: For the f graph, seeing that there was no y value, I just took a few points for x and substituted into the equation. I used my calculator.

R: What values did you use for x?

G: I started with 0 and then went up to 5 and -5. I also did the same for the graph of g.

R: Can you tell me why you took only a few points to draw your graphs? Could you have taken more points?

G: For parabolas you don't really need to take many points. It just goes the same always and continues with the arrows.

R: I see that you joined the dots with curves. Why did you do use curves and not straight lines?

G: We learnt that parabolas are not really straight. You have a sad face or a smiley face.

R: What indicated to you that it would be a parabola?

G: Because of the x^2.

R: If you take any two points on the curve, what is the meaning of the curve between the two dots?

G: It means average gradient.

R: What do you understand by average gradient?

G: The x and the y values, as it increases it goes like that kind of way.

R: Does the curve mean anything to you?

G: No, it does not mean anything.

R: If I asked you to find f(3), how would you do that?

G: I'd just substitute x with 3 in the equation and I would get 9.

R: Can you show me the meaning of f(3) on the graph.

Learner points to 3 on the x axis and goes across to 9.

R: Now can you show me on the graph f $(4\frac{1}{2})$

G: Same way as we got f(3).

R: Show me how you would get f(20)?

G: You'd have to extend the x axis.

R: Would it touch your graph?

G: What do you mean by touch?

R: Like how we did f (3).

G: Yes it would.

R: I see that you did not use arrows at the end of your graph?

G: Oh! I was supposed to use arrows?

R: What would the arrows mean?

G: It would mean that the graph continues up to infinity.

R: Can the axes also have arrows?

G: Yes, it would mean that it can continue to infinity.

R: Let us take two points on the x axis, 2 and 3. Tell me; are there any points between these two points?

G: Yes, many , comma something, comma many zeros.

R: Can you use any words to describe your graphs?

G: Yes, they are minimum graphs.

R: Why do you say that?

G: Because they are smiling face; they start at a lower point and goes higher, the y value starts low and gets bigger.

R: When would you get maximum?

G: Maximum, is the other way around, the frown.

R: Can you explain the meaning of maximum in terms of your graph?

G: The y value starts high and goes lower.

R: Can you describe your graphs in terms of increasing and decreasing?

G: If you take the f graph, it increases and decreases, decreases down this way and increases up this way or it can decrease this way and increase the other way on the negative side of the x axis.

R: What tells you that the graph is increasing or decreasing?

G: The decreasing the x values increase and the increasing the x values increase.

R: Where would you get gradient on the your graph?

G: You take any two points and substitute into the gradient formula.

R: Where would the gradient be equal to 0?

G: At the origin for the graph of f, because the y value is 0 and for the graph of g at y=-9.

R: Why at y=-9?

G: I don't know.

R: Is there any relationship between the graphs of f and g?

G: Yes, they are the same graphs, just that the g graph has gone through a vertical shift.

R: Explain vertical shift to me?

G: Moving up or down and in this case moving down because of the -9 in the g graph.

R: Can I draw g from the f?

G: Yes, the f graph there's no x intercepts and y intercepts. The y intercept of g is -9. You decrease all the points on the f graph by -9. The 25 comes down to 16, the 16 comes down to 7 and so forth.

R: What do you understand by reflection?

G: It is still the same image but the other side of it.

R: How did you get your answer?

G: When you reflect the g graph, it becomes sad and you just change the signs, negative becomes positive and positive becomes negative.

R: How did you get the answer for the values of x when g(x)>0?

G: For the x values less than -3 the y is greater than 0, and for the x values greater than 3 the y is greater than 0.

R: Explain the meaning of domain?

G: All the x values in your graph. For the f graph xεR because for any value of x you would get a y value.

Worksheet Two

R: Can you explain why you selected A for 1.1?

G: Because of the hyperbola in the graph.

R: Anything in the equation helped you?

G: Yes, the q. The q would be the asymptote. The asymptote is at 3.

R: What do you understand by asymptote?

G: The graph would go close to it but never touches it.

R: Do you know of any other meaning?

G: No, it is just the q value.

R: How many asymptotes are there?

G: There are two, the other one is the y axis, the x-p =0, so p=0, therefore x-0=0, so x=0, which is the y axis.

R: For 1.2, why did you select graph B?

G: The equation has something to the power of x, so that means it's the exponential graph.

R: Describe the graph of A?

G: The standard form of the hyperbola is $y=\frac{k}{x}$ and the x and y axis are the asymptotes.

R: How did you get the equation for the graph of A?

G: The +3 is the asymptote, so the q value is +3.

R: How did you get the equation for the graph of B?

G: The answer is $y = 2.2^x$. The first 2 I got from the graph because when x=0, anything raised to the power 0 is equal to 1, so 2 times 1 is 2.

R: How did you get the reflection for the graph of B?

G: I looked at the graph and folded on the y axis.

R: How would you describe graph A in terms of increasing and decreasing?

G: Both, increasing and decreasing. The top curve is increasing because as the x increases the y is increasing, and the other curve is decreasing because as the x decreases, the y is decreasing.

R: And the graph B?

G: One increasing and one decreasing. The y values are increasing and the x values are decreasing.

Worksheet Three

R: For the graph of f, how did you get a=2?

G: Because of the amplitude, the highest point on the graph, which I would read from the y axis.

R: For the value of b, how did you do that?

G: I did not know.

R: Can we continue with the graph of f?

G: Yes.

R: Why is the graph stopping at 180°?

G: Because of the domain.

R: The given g graph, from where can we derive this graph?

G: The cos graph is stretched.

R: What would be the range of the f graph?

G: It would be from 2 to -2 on the y axis.

R: How did you get the answer for question 2?

G: The vertical shift of I unit, the -2 goes to -1 and the 2 goes to 3.

R: What would be the new equation of f for the vertical shift?

G: The a=2 would become a=3, it would affect the amplitude.

R: Explain the period of f(x)?

G: The graph starts at -180°and ends at 180° and that gives 360°, the period for the sine graph is 360°.

R: How would you solve for f(x) = g(x)?

G: Here I would see where they intersect.

R: Where would you read off the values?

G: I would draw a table and where I see similar values I get my answers.

R: And where is f(x)>0?

G: Above the x axis, from 0° to 180°.

INTERVIEW BETWEEN RESEARCHER (R) AND LEARNER H

Worksheet One

R: Describe to me the method that you used to draw the graphs of f and g.

H: I used my calculator. I used -4 to +4 as the domain and I found the y values and I plotted the points on the graph.

R: You did that for both the graphs?

H: Yes.

R: Why did you take only a few points to draw the graphs? Could you have taken more points?

H: Yes, but it would take more time.

R: Can you tell me the reason for joining two points on the graph?

H: To make a continuous graph. It is the shape of the parabola, so I just join the points. The x^2 tells us that it would be a parabola.

R: Do you know the meaning of the curve between the two points?

H: I don't know what the meaning is.

R: Okay, can I get any more points on this curve?

H: Yes, by putting in more x values, you can get more points.

R: I see that you put arrows at the end of your graph. What is the meaning of the arrows?

H: It shows that the graph is continuing, goes on.

R: What does f(3) mean?

H: I don't know. I think you would solve for f.

Researcher shows learner what it means in terms of the graph and substituting 3 into equation we get y = 9.

R: Can we get $f(4\frac{1}{2})$?

H: Yes, by substituting into equation.

R: Can we get f(20)?

Learner pauses.

H: We can get it somewhere in the end of the x axis.

R: Let us take two of your x values, 2 and 3. Are there any numbers between these two numbers?

H: There are many smaller numbers.

R: Why are the arrows pointing upwards?

H: It is pointing to positive.

R: Can you use any words to describe your graph?

H: A continuous graph, a smooth curve.

R: Can you use the words increasing and decreasing for your graphs?

H: Yes, the f graph is increasing.

R: Why do you say that the graph of f is increasing.

H: It is going upwards, positive numbers.

R: When would the graph be decreasing?

H: When it is going downwards.

R: Where are the graphs maximum or minimum?

H: These graphs are minimum at 0 and -9. If it is going down, then we would get a maximum turning point.

R: Can we find the gradient of the graph?

H: Yes, in between any two points.

R: What does the gradient mean in terms of the curve?

H: I don't know.

R: Can I get the graph of g from the graph of f?

H: Yes, -9 means the f graph comes 9 units down.

R: What do you understand by reflection?

H: Something that is upside down.

R: How did you arrive at the answer for g(x) > 0 as x < -3 and x > 3?

H: For x < -3 the y values are positive and for x > 3 the y values are also positive.

R: How can you relate the domain of the graph of f, $x \in R$, to your graph?

H: If you go on the positive x axis, we would get real numbers and also on the negative axis we would get real numbers.

Worksheet Two

R: Can you explain why you selected graph A for 1.1?

H: I saw the asymptotes on the graph and the equation is the standard form for the hyperbola.

R: What is the meaning of the asymptote?

H: The graph will go close to the asymptote and never touches it. The asymptote is the q value in the equation.

R: How many asymptotes are there in graph A?

H: Only one, the dotted line.

R: Are there any asymptotes in graph B?

H: No, there are no asymptotes.

R: How did you get the answer $y = 2.2^x$ for graph B?

H: I used the 2 from the y axis for the 2^x and then I used my calculator to get the rest of the answer.

R: Can we get the y intercept to be one? If so, what would the equation be?

H: Yes, the equation would be $y = 1^x$.

R: How did you draw the reflection of graph B?

H: I put a negative in front of $y = 2.2^x$ and then I used my calculator by taking x values and finding y values.

R: How would you describe your graphs A and B in terms of increasing and decreasing?

H: Graph A is decreasing because as the x increases, the y values decrease. Graph B is increasing because as the x increases, the y values also increase.

Worksheet Three

R: How did you arrive at the answer f(x) = 2 sin 1 x?

H: For the a =2, I got this from the amplitude on the graph and for b = 1, since the x values do not change, the b = 1.

R: How did you get q = 30° for the graph of g?

H: The graph is shifted horizontally to the left. The highest point of the cos graph is normally 1 on the y axis. This point shifted to the left 30°.

R: How did you arrive at the answer of y = 2 sin x + 1 for the vertical shift of one unit?

H: The graph moves up one unit and for vertical shift the number goes in the end. The new range is now 3 and -1.

R: What do you understand by period?

H: It is the length of the graph. Therefore the period for f(x) is [-180°; 180°]. It starts at 180° and ends at 180°.

R: How would you describe the graph of f in terms of increasing and decreasing?

H: It is increasing for positive x values and decreasing for negative x values.

R: How would you get f(x) > 0?

H: It would be above the x axis, the y values greater than 0, from 0° to 180°.

INTERVIEW BETWEEN RESEARCHER (R) AND EDUCATOR/TEACHER (T1)

Worksheet One

R: Can you describe to me what method you used to sketch these two graphs?

T1: For the first graph f(x) I used the table method and for the second one I used the x and y intercept.

R: For the first graph you said you used the table method; did you take points?

T1:Yes, I took x values from -2 to +2.

R: I notice that you joined the points for the graph of f. Why did you join these points?

T1: Okay, If you take these points between 0 and 1, they won't be direct proportion there. $(0,1)^2$ and $(0,2)^2$ they won't be in proportion. That would be the reason for getting a curve and not necessary a straight line.

R: And what is your reason for joining the five points?

T1: Between say 1 and 2 there are corresponding points. Instead of just leaving dots, you would have a continuous curve.

R: What do you mean by continuous curve?

T1: For all x values you would get corresponding y values.

R: What does it mean to join the two points to form a curve?

T1: Those are all your corresponding x and y values.

R: How would you determine f (3)?

T1: I would substitute 3 for x in the equation.

R: What does f (3) mean in terms of the graph?

T1: That your corresponding y value for 3 is 9.

R: And f $(4\frac{1}{2})$?

T1: It would be $(4\frac{1}{2})^2$.

R: And f (20), how would you show that on your graph?

T1: Well, not on this one, it will be a large number, depends on your scale.

R: I see that you used arrows at the end of your graph. What is the meaning of the arrows?

T1: It means that the graph will continue as you take bigger x and smaller x values.

R: Let us take the number 2 and 3 on the x axis. Are there any numbers between these two numbers?

T1: Yes, there is a whole lot of numbers between 2 and 3.

R: About how many?

T1: Depends how many decimal places you want to go. If you go one decimal place, there will be ten numbers. If you go two decimal places, there will be more numbers.

R: Can we get a set number of figures in between?

T1: No.

R: What words can you use to describe the shape of the graphs?

T1: Parabolic, happy face.

R: What is indicating to you that it will be a parabola?

T1: Yes, if you got x^2, then that will indicate that it will be a parabola.

R: In terms of your graph, what does increasing and decreasing mean?

T1: If we look at the graph from left to right, the positive gradient will give you increasing. Increasing will be from $x>0$ and decreasing from $x<0$ on the x axis. As you take x values smaller than 0, the graph is going down, therefore it is decreasing and as you take numbers bigger than 0, the graph goes up, increasing.

R: Does the graph have a gradient?

T1: No, but it can have average gradient.

R: Can we draw the graph of g from the graph of f?

T1: Yes, it's an identical graph, with adding -9, the graph has gone down nine places.

R: Can we get the graph of f from the graph of g?

T1: Yes, you could, you just ignore the constant.

R: Can you explain how you arrived at the reflection of g(x) to be $g(x) = -x^2+9$?

T1: I looked at the original graph, and I reflected those points on the x axis. I looked at the new graph, and because it is a sad face, the a is going to be negative. The -9 on the original, if you reflect -9 it will be +9 on the y axis.

R: Did you use the equation to help you?

T1: Yes.

R: What do you understand by $g(x)>0$?

T1: Where you are getting positive y values, When I look at my graph, from -3 to +3 the graph is below the x axis, you are getting negative y values, but as you take numbers >+3 and <-3, you are getting positive y values.

R: Explain your domain of f, $x \in Z$.

T1: For all values of x on the x axis there will be a y value.

R: What are integers?

T1: This should be real numbers.

R: How did you get the range of $y \geq -9$ for the graph of g?

T1: I looked at the lowest point -9 and I took the values greater than -9 on the y axis.

Worksheet Two

R: Why did you match graph A with 1.1?

T1: That's the standard form for the hyperbola and when you look at graph A, it is the hyperbola.

R: And for 1.2?

T1: It's the same. The equation is the standard form for the exponential and I matched it with the exponential graph B.

R: Is there any other information that helped you?

T1: No, I just related the equation to the graph.

R: If you take graph A, what is the name of the dotted line?

T1: Asymptote.

R: What do you understand by asymptote?

T1: It means that for the graph there is no corresponding y value at that point, at y=3 there is no x value; x is the denominator. The 3 means the graph has shifted 3 units up.

R: Which graph?

T1: The hyperbola, normally at y=0 you get the asymptote, so it means the asymptote has shifted 3 units up.

R: And what does asymptote generally mean?

T1: That the graph is undefined for that value y=3.

R: Explain your possible equations for Graphs A and B?

T1: I took the 3 as the asymptote for the q value and any value for k. For the graph of B I wrote $y = 2.2^x$; normally for the exponential graph where you get a^x, the y intercept is 1, but this y intercept is 2, so the a in $y=a.b^x$ is 2.

R: How would you describe graph A in terms of increasing and decreasing?

T1: It is decreasing. As you take bigger x values the y values decrease.

R: So, this is different from the $f(x) =x^2$.

T1: Yes.

R: Can we not use this same explanation for the parabola.

T1: Yes we can.

R: What helped you to draw the reflection of the graph of B in the y axis?

T1: I looked at the y line and if you take corresponding points, you will be able to get an idea how the reflection will be.

R: What would be the equation of this reflection?

T1: It would be $y = 2.\left(\frac{1}{2}\right)^x$.

R: It won't be 2.2^{-x}.

(after some hesitation)

T1:Yes.

Worksheet Three

R: Explain your answers f(x)= 2 sin 1x and g(x)= cos (x+30°)?

T1: A normal sine graph goes up to 1 and -1, in terms of amplitude; and this graph goes up to 2 and -2, so the amplitude is the a = 2. The sine graph if it continues on the positive side it will end up at 360°, so the normal period of the sine graph is 360°, so I did not have to interfere with the b at all, I just left it as it is.

R: How did you get q=30°?

T1: Normally the cos graph at 0 is +1 on the y axis, so when I look at -30°, it is giving me +1, so that graph has shifted 30° to the left, so the y=cos x helped me to get this answer.

R: How did you get the range for question 2?

T1: This graph will now shift 0ne unit up. The lowest point on the original is -2, so if it shifts up the lowest point is -1 and the highest point is 3.

R: How did you arrive at the period of 360° for the graph of f?

T1: If you look at what is given, it is one complete wave motion, starts at -180° and ends at +180°, so the total is 360°.

R: Why is f(x) ≠3?

T1: Because it is outside the range.

R: For which values of x is f(x) increasing?

T1: It will be from -90° to +90°?

R: What is helping you to get the answers so quickly?

T1: Just the x values and I look at the lines going up and coming down.

R: How would you get f(x)>0?

T1: You look above the x axis , for positive y values.

R: And f(x)>g(x)?

T1: You look at your graphs physically, where your f graph is above the g graph because that is where your values for f is bigger than the y values for g.

INTERVIEW BETWEEN RESEARCHER (R) AND EDUCATOR/TEACHER (T2)

Worksheet One

R: Can you describe to me the method you used to sketch the graphs of f and g?

T2: I looked at the equation of f(x) and what the rules are about the parabola, about the impact of the a and so on. Just to get a rough idea about where it lies in the Cartesian plane, I just took two points x=1 and x=-1 and plotted them.

R: And how did you sketch the graph of g?

T2: I looked at the equation as well and I calculated the x and y intercept.

R: What made you join the x intercepts to the y intercept to form a curve?

T2: Because you're teaching it for so long and you just know that it has to be a parabola and the x^2 indicated that it will be a parabola.

R: What do you understand by the curve that joins the two points?

T2: Gradient, I would say, that's why we talk of a curve, because the gradient is changing all the time.

R: Can we get more points on this curve?

T2: Yes we can.

R: About how many?

T2: Numerous.

R: So you say that your understanding of the curve is related to the gradient?

T2: I would say that because of the gradient and of the equation having $x^{2.}$

R: How would you determine f (3)?

T2: By substitution, when x=3 then y=9.

R: In terms of your graph, what does this mean?

Educator did not immediately answer. Researcher pointed out x=3 and projecting on graph to find 9.

T2: Yes, it is a point on the graph.

R: How would determine f $(4\frac{1}{2})$.

T2: By substitution.

R: How would you determine f (20) and explain in terms of your graph?

T2: By substitution, but we cannot show it on this particular graph, it is not accommodating. You can get 20 but you just have to adjust your scale.

R: What is your understanding of the arrows at the end of the graphs?

T2: The domain is real and the graph will go on indefinitely.

R: Can we say that for any value for x we will get a corresponding y.

T2: Yes, as the x increases the y increases. The arrows are pointing upwards because it's going in that direction.

R: Are there any points between 2 and 3?

T2: Yes, infinite number of points.

R: Let's take the shape of the graph-what words can you use to describe the shape?

T2: We normally use the smiling, happy and sad face.

R: Maximum and minimum? Did you use these words?

T2: Yes, looking at the shape of these graphs, you can see the lowest point, therefore it's a minimum, at the turning point...the graph is changing direction.

R: And maximum?

T2: The highest point...the highest y value.

R: How do you normally teach this section on graphs?

T2: I start with the table method. Then I move on to the five point method, the x and y intercept, turning point and axis of symmetry. Learners would use their calculators to get the values for the table…now the calculators can do the table as well, where learners just have to punch in the function and so on.

R: How would you describe the graph in terms of increasing and decreasing?

T2: It's decreasing to the left of 0 and increasing on the right.

R: Can you explain increasing and decreasing?

T2: It's related to the gradient. The function is decreasing when its gradient is negative and increasing when its gradient is positive. Also we can say that as the x increases, the y decreases and it will be a decreasing function. As the x increases and the y increases, it will be an increasing function.

R: Is there a relationship between the graphs of f and g?

T2: The f has shifted 9 units down vertically.

R: Can we get the graph of g from the graph of f?

T2: Yes, we can- we shift it down nine units. It might be a parallel shift to the curve of f(x) because of the average gradient. Here the a is constant in both graphs, so I'm just assuming the graphs are parallel.

R: If we give learners to sketch $h(x) = x^2+3$, would they see a relationship with $f(x) = x^2$?

T2: I think they would. With the new FET syllabus I think they would see the shift. Previously we never taught like that. The transformations and the shifts is making learners understand graphs better.

R: How did you get the reflection of g(x) in the x axis?

T2: I just flipped it on the x axis and I saw this point being -9 and it will just flip to 9. The reflection is a mirror-image. Every point is transformed about the x axis in this case.

R: Can we say that when we flip this over, we just change the signs?

T2: Yes, we can, because the y changes sign.

R: What does g(x)>0 means?

T2: It's the y value greater than 0. We look above the x axis, because y>0 above the x axis. So just looking at that I interpreted this question.

R: How did you get the domain?

T2: The domain is the x values and since I put my arrows in here the domain would be real. It means that every point on the x axis is taken into consideration.

R: How did you work out the range/

T2: I looked at the lowest point on the graph and I can see that this graph is going Up and I saw that it can only encounter numbers greater than or equal to -9.

Worksheet Two

R: Describe why you selected graph A for 1.1.?

T2: I looked at the asymptote line which is y=3. This stands for the q value in the equation. Therefore I selected graph A.

R: Explain why you selected Graph B for 1.2.?

T2: Graph B has the shape of an exponential function. It is an increasing curve. The equation is also in the exponent form. I therefore related both.

R: You mentioned asymptote; what do you understand by asymptote?

T2: The graph is undefined at that point y=3. The other asymptote is the y axis.

R: Can you explain this concept of undefined?

T2: I relate this to my primary school knowledge of division by 0 is undefined, other than that there is nothing.

R: Are there any asymptotes in the second graph?

(After much hesitation)

T2: It does not cut the x axis, so I would assume that the x axis is the asymptote.

R: When would graph B have no asymptotes?

T2: After it has transformed.

R: Explain your possible equation for graph A, y = $\frac{1}{x}$ + 3?

T2: This graph does not have a y intercept. It has an x intercept. That is why I wrote $\frac{1}{x}$ + 3. There is no horizontal shift. The 3 is the asymptote.

R: Explain your answer y = 2.2^x for graph B?

T2: The first 2 is due to the y intercept being 2, because 2^0 is 1 and 1x2 =2.

R: Can the b in y =a b^x be any value:

T2: No it cannot be anything. The graph is increasing, so b>1.

R: What would be the equation of the reflection of Graph B in the y axis?

T2: It would be y = 2.2^{-x}.

Worksheet 3

R: How did you get the equation of f(x)?

T2: The amplitude is 2, so a=2. For b=1, I looked at the period of the graph which is 360°.

R: Explain the answer for the graph of g?

T2: It is the cos graph which has transformed horizontally. I had the picture of the normal cos in my mind. The normal cos would have cut the x axis at 90°, so I looked at this one cutting the x axis at 60° which told me it's being transformed to the left by 30°, so I inserted that into the equation.

R: Explain your range for question 2?

T2: I looked at the amplitude again and we shifting this graph one unit up, so I transformed this graph one unit up.

R: What do you understand by range?

T2: The range would be all the y values that the graph encounters.

R: Explain the period of to be 360°?

T2: I looked at half the graph of sin x, which is 180°, so the wavelength is 360°.

R: Why is f(x) ≠3?

T2: Because the amplitude in this case is 2, you won't get a value of 3 unless the amplitude is 3.

R: How would you explain f(x) > g(x)?

T2: Where the sine graph is lying above the cos graph and you can see it at these two points.

R: Why are you saying above?

T2: Because of the greater than.