

# Parameters related to fractional domination in graphs

D.J.Erwin

Durban, November 1995

Submitted in partial fulfilment of the requirements for the degree of Master of  
Science in the Department of Mathematics and Applied Mathematics, University of  
Natal, Durban.

## Abstract

The use of characteristic functions to represent well-known sets in graph theory — such as dominating, irredundant, independent, covering and packing sets — leads naturally to fractional versions of these sets and corresponding fractional parameters.

Let  $S$  be a dominating set of a graph  $G$  and  $f : V(G) \mapsto \{0, 1\}$  the characteristic function of that set. By first translating the restrictions which define a dominating set from a set-based to a function-based form, and then allowing the function  $f$  to map the vertex set to the unit closed interval, we obtain the *fractional generalisation* of the dominating set  $S$ . In chapter 1, known domination-related parameters and their fractional generalisations are introduced, relations between them are investigated, and Gallai type results are derived. Particular attention is given to graphs with symmetry and to products of graphs.

If instead of replacing the function  $f : V(G) \mapsto \{0, 1\}$  with a function which maps the vertex set to the unit closed interval we introduce a function  $f'$  which maps the vertex set to  $\{0, 1, \dots, k\}$  (where  $k$  is some fixed, non-negative integer) and a corresponding change in the restrictions on the dominating set, we obtain a  $k$ -dominating function. In chapter 2 corresponding  $k$ -parameters are considered and are related to the classical and fractional parameters. The calculations of some well-known fractional parameters are expressed as optimization problems involving the  $k$ -parameters.

An  $e = 1$  function is a function  $f : V(G) \mapsto [0, 1]$  which obeys the restrictions that (i) every non-isolated vertex  $u$  is adjacent to some vertex  $v$  such that  $f(u) + f(v) = 1$ , and (ii) every isolated vertex  $w$  has  $f(w) = 1$ . In chapter 3 a theory of  $e = 1$  functions and parameters is developed. Relationships are traced between  $e = 1$  parameters and those previously introduced, some Gallai type results are derived for the  $e = 1$  parameters, and  $e = 1$  parameters are determined for several classes of graphs. The  $e = 1$  theory is applied to derive new results about classical and fractional domination parameters.

# Preface

The research on which this thesis is based was carried out in the Department of Mathematics and Applied Mathematics, University of Natal, Durban from December 1994 to November 1995, under the supervision of Professor Henda C. Swart and the co-supervision of Dr P. Dankelmann.

This thesis represents original work by the author and has not been submitted in any other form to another university. Where use was made of the work of others it has been duly acknowledged.

## Acknowledgements

I would like to thank the following organisations for financial support during my work: the Foundation for Research Development, the University of Natal, Standard Bank, and the Department of Mathematics and Applied Mathematics at the University of Natal, Durban.

My thanks go to Professor J.H. Swart for allowing me to register in his department for postgraduate study in Mathematics after finishing an honours degree in Physics.

I am eternally indebted to both of my supervisors. My thanks to Dr Peter Dankelmann for his keen interest in my work (even before his first cup of coffee) and for applying his well-developed critical powers to both mathematics and typesetting. I would also like to thank Henda Swart for her support since my first months at university: without her unflagging interest and superb talent (both as a teacher and a mathematician) I would never have contemplated doing what I have done.

Thanks to my girlfriend, Andrea, for being so understanding about the amount of time I've spent typing recently. Thanks to my dad, Geoff, for inspiration beyond words and many interesting conversations; thanks to my mum, Jan, for her selfless love and support.

*This thesis is dedicated to the memory of my dog, Scamper, who died on December 21st, 1994.*

David Erwin  
Durban  
November 1995

# Contents

<b>1</b>	<b>An introduction to fractional parameters</b>	<b>1</b>
1.1	Introductory remarks and some notation . . . . .	1
1.2	Introduction to some established classes of sets . . . . .	3
1.3	Introduction to fractional set functions . . . . .	6
1.4	Definitions of fractional set functions . . . . .	7
1.4.1	The fractional dominating function . . . . .	8
1.4.2	The fractional packing function . . . . .	8
1.4.3	The fractional irredundance function . . . . .	9
1.4.4	Other fractional set functions . . . . .	9
1.5	Some preliminary results . . . . .	12
1.6	Further results . . . . .	16
1.7	Fractionally well-dominated graphs . . . . .	20
1.8	Relation to the 2-packing number . . . . .	24
1.8.1	The independence of $P_2, \gamma_f$ and $\gamma$ . . . . .	26
1.8.2	$P_2$ in connected block graphs . . . . .	28
1.9	Gallai type results . . . . .	30
1.9.1	A Gallai type theorem for $\alpha_f$ and $\beta_f$ . . . . .	30
1.9.2	A Gallai type theorem for $\alpha_f^1$ and $\beta_f^1$ . . . . .	32
1.9.3	Other Gallai type results . . . . .	34
1.10	Fractional set functions in graphs with symmetry . . . . .	35
1.10.1	Complete multipartite graphs . . . . .	36
1.10.2	The automorphism class theorem . . . . .	39
1.11	Fractional parameters and graph products . . . . .	42
1.11.1	Definitions of graph products . . . . .	42
1.11.2	Vizing's conjecture for fractional domination . . . . .	43
1.11.3	Other forms of Vizing's conjecture . . . . .	47
<b>2</b>	<b><math>k</math>-parameters</b>	<b>53</b>
2.1	Definitions of $k$ -parameters . . . . .	53

2.1.1	The $k$ -dominating function . . . . .	53
2.1.2	The $k$ -packing function . . . . .	54
2.1.3	The $k$ -irredundance function . . . . .	54
2.2	Some relations between $k$ -set functions and well-known classes of sets	55
2.3	Inequalities between $\gamma_k(G), \Gamma_k(G), ir_k(G), IR_k(G)$ . . . . .	57
2.4	Relationships between $k$ -set and $k + 1$ -set parameters . . . . .	58
2.5	Relating $k$ -parameters to fractional parameters . . . . .	60
2.6	Equivalent $k$ -parameter theorems for some established fractional results	63
<b>3</b>	<b><math>e = 1</math> functions</b>	<b>67</b>
3.1	Introduction . . . . .	67
3.2	Observations . . . . .	68
3.3	Some preliminary results . . . . .	69
3.4	The relationship between $\gamma$ and $\underline{\gamma_{e=1}}$ . . . . .	72
3.5	Some Gallai type results . . . . .	74
3.6	More on the inequalities of (3.2.3) . . . . .	77
3.7	Inequalities for subgraphs . . . . .	78
3.8	$e = 1$ functions for different classes of graphs . . . . .	80
3.8.1	The complete graphs . . . . .	81
3.8.2	The complete bipartite graphs . . . . .	82
3.8.3	Paths . . . . .	83
3.9	A characterization of $\gamma_f$ and $\gamma$ . . . . .	86
3.10	Conclusion . . . . .	91
<b>A</b>	<b>Notation</b>	<b>93</b>

# Chapter 1

## An introduction to fractional parameters

### 1.1 Introductory remarks and some notation

This thesis deals with a family of parameters which are fractional versions of more familiar parameters. The original parameters are, in general, of the form: the minimum/maximum cardinality of a minimal/maximal set of vertices/edges such that for each vertex/edge the number of elements of the set in the neighbourhood of the vertex/edge is at most/least one. The parameters studied here are the extension where one places weights on the vertices/edges and the weight of a neighbourhood is the sum of the weights of the neighbours. The parameters studied include fractional domination, packing, irredundance, independence, and more. We begin by examining some of the well-known set-based parameters which we shall later fractionalise. The field of fractional parameters is a large one and we shall not attempt to survey all of it. We shall concentrate upon the theoretical aspects of some parameters closely linked to fractional domination. The algorithms and complexity issues related to these fractional parameters will not be considered.

The notion of domination in graphs is generally traced back to 1862 when C.F. de Jaenisch [dJ62] posed the problem of finding the smallest number of queens that could be placed on a chessboard to attack (i.e. dominate) each square on the board. Related problems, concerning other chess pieces, were sporadically considered, but the formal graph-theoretical formulation of the concept of domination was introduced much later, by Berge in 1958 [Ber58] and Ore in 1962 [Ore62] (Ore first referred to the ‘domination number’ of a graph, which Berge called its ‘coefficient of external stability’).

During the past two decades the study of various domination parameters and concepts related to them has developed into a major field of graph-theoretical research. The first survey of results on domination was published in 1977 by Cockayne and Hedetniemi [CH77]; since then a vast body of literature has been produced. In 1994 a provisional list of papers on domination in graphs, compiled by Teresa Haynes [Hay94], comprised 878 titles.

We shall not attempt to survey the field of domination but will indicate briefly the three main directions followed by researchers in order to generalise or specialise domination-related concepts.

Let  $G$  be a graph and  $S$  a set of vertices of  $G$  such that every vertex in  $G$  is in  $S$  or adjacent to at least one vertex in  $S$ . Then  $S$  is called a dominating set of  $G$  and the smallest cardinality of such a dominating set of  $G$  is known as the domination number of  $G$ ,  $\gamma(G)$ .

The earliest variations of these concepts were obtained by placing restrictions on the dominating sets  $S$  under consideration. We cite a few instances: in [CH77]  $S$  is required to be independent (and the associated minimum cardinality of an independent dominating set of  $G$  is the independent domination number of  $G$ ,  $i(G)$ ); in [CDH80] it was required that the subgraph  $\langle S \rangle$  induced by  $S$  should contain no isolated vertex (yielding the total domination number of  $G$ ,  $\gamma_t(G)$ ); the requirement that the dominating sets should induce connected subgraphs [HL84a] yields the connected domination number of  $G$ ,  $\gamma_c(G)$ . For a connected graph  $G$  that contains no induced path or cycle of order 5, dominating sets that induce complete subgraphs of  $G$  exist and the smallest cardinality of such a dominating clique is called the clique domination number of  $G$ ,  $\beta_k(G)$  [CK90]. If the induced graph  $\langle S \rangle$  is to contain a perfect matching,  $S$  is called a paired dominating set and the minimum cardinality of such a paired dominating set is the paired domination number of  $G$ ,  $\gamma_p(G)$  [Sla95].

A further class of parameters may be obtained by modifying the requirement that every vertex of a graph  $G$  should be adjacent to at least one vertex of a dominating set  $S$  or in that set. By requiring that each vertex of  $G$  not in  $S$  should be adjacent to at least  $n$  vertices of  $S$ , the  $n$ -domination number of  $G$  was introduced by Fink and Jacobson [FJ85]. If each vertex of  $G$  is contained in a subgraph of  $G$  which is isomorphic to a given graph  $F$ , the requirement that each vertex of  $G$  should be contained in such a subgraph that also contains at least one vertex of  $S$  leads to the definition of an  $F$ -domination number of  $G$ ,  $\gamma_F(G)$ . Distance-domination parameters are obtained if it is required that each vertex of  $G$  be within (prescribed) distance  $n - 1$  from a vertex of  $S$  [HOS91]. The requirement that each vertex in  $G$  should be adjacent to the vertices of a subset of  $S$  which is unique to that vertex leads to the



definition of locating-domination parameters [Sla88]. Many further variations on the above two themes are possible (see, for instance, [HL91]).

We are particularly interested in the class of parameters that arise as follows: a dominating set  $S$  of a graph  $G$  determines and is determined by its characteristic function  $f : V(G) \mapsto \{0, 1\}$ , defined, for  $v \in V(G)$ , by  $f(v) = 1$  if  $v \in S$ ,  $f(v) = 0$  if  $v \notin S$ . Changing the codomain  $\{0, 1\}$  to  $\{-1, 1\}$  or  $\{-1, 0, 1\}$  subject to appropriate restrictions on  $f$  has generated signed domination and minus domination parameters (see [DHHS], [DHHM]). We shall investigate a range of domination-related parameters that result when  $f : V(G) \mapsto \{0, 1\}$  is replaced by  $f : V(G) \mapsto [0, 1]$  and suitable constraints are imposed on  $f$  to yield generalisations of domination, independence, packing and irredundance numbers. References will be provided in the relevant sections.

For notation we follow Chartrand and Lesniak [CL86], and for convenience all the non-standard notation used is presented in appendix A. In particular, let  $G$  be a graph, then we denote by  $V(G)$  the vertex set of  $G$  and by  $E(G)$  the edge set of  $G$ . The *order* of  $G$  is  $p = |V(G)|$  and the *size* is  $q = |E(G)|$ . The degree of a vertex  $v$  is  $\deg v$ ; the smallest degree in  $G$  is  $\delta(G) = \min\{\deg v : v \in V(G)\}$  and the largest degree is  $\Delta(G) = \max\{\deg v : v \in V(G)\}$ . We denote by  $\text{iso}(G)$  the number of isolated components of a graph  $G$ , i.e. vertices of degree 0, and by  $\text{niso}(G)$  the number of non-trivial components of  $G$ . Thus

$$\text{iso}(G) + \text{niso}(G) = k(G).$$

If  $M$  and  $N$  are  $(m \times n)$ -vectors then we define the notation  $M \geq N$  to mean that, for  $1 \leq i \leq m$  and  $1 \leq j \leq n$ ,  $M_{ij} \geq N_{ij}$ . We denote by  $\vec{1}_n, \vec{0}_n$  the  $(n \times 1)$ -vectors consisting of  $n$  1's and  $n$  0's, respectively. The notation  $e \rightsquigarrow v$  means that the edge  $e$  is incident with the vertex  $v$ .

## 1.2 Introduction to some established classes of sets

Let  $G = (V, E)$  be a graph. Then a *dominating set*  $S \subseteq V(G)$  of vertices is a set of vertices chosen in such a way that every vertex not in the set is adjacent to at least one vertex in the set. We say that  $S$  is a dominating set of  $G$  if

$$V(G) \subseteq N[S].$$

Obviously,  $V(G)$  is a dominating set of any graph but we are most interested in those dominating sets which have small cardinalities. A dominating set  $S \subseteq V(G)$  is said to be *minimal* if the removal of *any* vertex from that set results in a non-dominating

set. The *domination number*  $\gamma(G)$  is the cardinality of a smallest of all the minimal dominating sets of  $G$ ,

$$\gamma(G) = \min\{|S| : S \text{ is a minimal dominating set of } G\},$$

and the *upper domination number* is the cardinality of a largest of the minimal dominating sets of  $G$ :

$$\Gamma(G) = \max\{|S| : S \text{ is a minimal dominating set of } G\}.$$

The formal theory of domination was introduced by Ore [Ore62] and Berge [Ber58].

An *irredundant set*  $S \subseteq V(G)$  is a set of vertices in which every member  $v \in S$  has within its closed neighbourhood a ‘private neighbour’: some vertex which is not in the closed neighbourhood of any other member of  $S$ :

$$(\forall v \in S) N[v] - N[S - \{v\}] \neq \emptyset.$$

Any vertex  $v$  is an irredundant set in  $G$ , but we are interested in those irredundant sets which are *maximal*. An irredundant set is maximal if the addition of any vertex yields a set which is no longer irredundant. The *irredundance number*  $ir(G)$  is then defined as the cardinality of a smallest of these maximal irredundant sets, and the *upper irredundance number* of  $G$  is the cardinality of a largest of these maximal irredundant sets (see [CFPT81]):

$$ir(G) = \min\{|S| : S \text{ is a maximal irredundant set}\},$$

$$IR(G) = \max\{|S| : S \text{ is a maximal irredundant set}\}.$$

In any minimal dominating set  $S \subseteq V(G)$ ,  $N[S] = V(G)$  and the closed neighbourhood of each  $v \in S$  contains at least one vertex which is not in the closed neighbourhood of any other member of  $S$  (if this is not true for some  $v \in S$  then  $S - \{v\}$  is a smaller dominating set, contradicting the minimality of  $S$ ). Thus every minimal dominating set is also a maximal irredundant set. Hence  $ir(G) \leq \gamma(G)$ . This result was first proved by Cockayne and Hedetniemi [CH77].

An *independent set*  $S \subseteq V(G)$  is a set of pairwise-nonadjacent vertices, i.e.

$$u, v \in S \implies uv \notin E(G)$$

and  $S$  is a *maximal independent set* of  $G$  if, furthermore, it is not properly contained in any independent set. The *independence number*  $\beta(G)$  is defined to be the cardinality of a biggest possible *maximal independent set* in  $G$ , where by maximal we mean that the addition of any vertex to that set ruins the independence;

$$\beta(G) = \max\{|S| : S \text{ is an independent set of } G\}. \quad (1.2.1)$$

Note that every maximal independent set of  $G$  is a minimal dominating set of  $G$  (let  $I \subseteq V(G)$  be a maximal independent set of  $G$ , and suppose that, for some  $v \in I$ ,  $I - \{v\}$  is a dominating set of  $G$ ; then  $N[v] \subseteq N[I - \{v\}]$  which implies that  $v \in N[I - \{v\}]$  and hence  $I$  is not independent). Hence we get the well-known result (see, for example, [CHM78]) that

$$\gamma(G) \leq \beta(G).$$

A *packing set*  $S \subseteq V(G)$  satisfies the restriction that no closed neighbourhood of any vertex in  $V(G)$  contains more than one element of that set  $S$ , i.e.

$$\forall v \in V(G) |N[v] \cap S| \leq 1,$$

and a *maximal packing set* is a packing set of  $G$  which is not contained as a proper subset in any packing set of  $G$ . The *upper packing number* and *lower packing number* are defined, respectively, as

$$\begin{aligned} P(G) &= \max\{|S| : S \text{ is a maximal packing set of } G\}, \\ p(G) &= \min\{|S| : S \text{ is a maximal packing set of } G\}. \end{aligned}$$

Packing sets have been studied by Meir and Moon [MM75].

A set  $S \subseteq V(G)$  is called a *vertex cover* of  $G$  or just a *cover* of  $G$  if

$$uv \in E(G) \implies u \in S \text{ or } v \in S,$$

and we define the *vertex-covering number* (or just the *covering number*) of  $G$  to be

$$\alpha(G) = \min\{|S| : S \text{ is a vertex cover of } G\}.$$

We can also consider those sets which are subsets of the edge set of some graph  $G$ . Let  $v \in V(G)$ , then we recall that the notation  $e \rightsquigarrow v$  means that the edge  $e$  is incident with the vertex  $v$ , i.e. there is a vertex  $u$  such that  $e = uv$ . Let  $S \subseteq E(G)$ , then we say that  $S$  is

- an *edge cover* of  $G$  if

$$\forall v \in V(G), S \cap (\cup_{e \rightsquigarrow v} e) \neq \emptyset,$$

and the *edge-covering number* of  $G$  is

$$\alpha^1(G) = \min\{|S| : S \text{ is an edge cover of } G\}.$$

- a *matching* of  $G$  if

$$v \in V(G) \implies |S \cap (\cup_{e \rightsquigarrow v} e)| \leq 1;$$

associated with this class of sets is the *matching number* of  $G$

$$\beta^1(G) = \max\{|S| : S \text{ is a matching of } G\}.$$

### 1.3 Introduction to fractional set functions

Definitions of classes of sets which make reference to set-based conditions — as in the dominating, independent and irredundant sets — can be replaced by equivalent definitions of classes of *functions* and corresponding *function-based* conditions. We set up functions mapping the vertex set  $V(G)$  to some set  $X \subseteq \mathbb{R}$  and then place restrictions on properties of those functions. The simplest way to do this, for any set  $S$  obeying whichever of the above restrictions we choose, is to use the *characteristic function* of  $S$  (here  $X = \{0, 1\}$ ):

$$g(v) = \begin{cases} 0 & v \notin S \\ 1 & v \in S. \end{cases} \quad (1.3.1)$$

We must now translate the restrictions which define our classes of sets into ‘function-based’ rather than ‘set-based’ terminology. The resulting definition varies with the type of set we are considering. To produce a *dominating function* (see [GS90]) we must require that, for any vertex  $v \in V(G)$ , the sum — over the closed neighbourhood of  $v$  — of the values of  $g$  in that neighbourhood must be at least one (since then any vertex  $v \in V(G)$  is in the closed neighbourhood of at least one vertex in  $S$ ):

$$(\forall v \in V(G)) \sum_{u \in N[v]} g(u) \geq 1. \quad (1.3.2)$$

An *irredundant function* must fulfill the requirement

$$(\forall v \in V(G)) g(v) > 0 \implies \exists u \in N[v] \text{ such that } \sum_{w \in N[u]} g(w) = 1, \quad (1.3.3)$$

and an *independent function* must meet the condition

$$uv \in E(G) \implies g(u) + g(v) \leq 1. \quad (1.3.4)$$

We could seek other sets  $X$  and corresponding functional requirements to reproduce functional versions of our ordinary dominating, irredundant and independent sets. However, a question which arises naturally at this point is : what properties of a graph can we examine if we now drop the implicit requirement that our functions — when suitably restricted — reproduce the well-known sets  $S$  which we had encountered earlier, and instead vary  $X$  (while retaining the requirements (1.3.2), (1.3.3) or (1.3.4) or some equivalent of them) and see what results? Perhaps the simplest problem in this class occurs when  $X = [0, 1]$ . Those functions  $g$  which map the vertex set to the unit closed interval,  $g : V(G) \mapsto [0, 1]$ , we shall call *fractional set functions*.

We shall also encounter functions  $h$  which map the *edge set* of  $G$  to the unit closed interval  $h : E(G) \mapsto [0, 1]$ : these we shall call *fractional edge set functions*.

An alternative motivation for studying fractional parameters based on familiar classes of sets is that of complexity: problems of the type ‘determine a subset  $S \subseteq V(G)$  which has the following property’ are often NP-complete: for example, finding dominating, covering, and independent sets are NP-complete problems [GJ79]. However, the related fractional problems can often be formulated as a linear programming problem and thus be solved in polynomial time, i.e. fractional domination is the linear relaxation of the integer programming problem for (normal) domination. The values of these related fractional parameters can sometimes be used to characterize or provide bounds for their integer counterparts.

## 1.4 Definitions of fractional set functions

It has been the case (see for example [DHL88, DHLF91]) that the terms *dominating function* and *fractional dominating function*, *irredundant function* and *fractional irredundant function*, and so on, have been used interchangeably. For the sake of clarity, throughout this document the two will be taken to be different. Thus by a *dominating function* we shall always mean (1.3.1) together with (1.3.2), while a *fractional dominating function* is a function  $f : V(G) \mapsto [0, 1]$  satisfying (1.3.2).

Let  $S$  be a subset of the vertex set  $V(G)$ , and let  $g : V(G) \mapsto [0, 1]$ . Then we define

$$g(S) = \sum_{v \in S} g(v)$$

and

$$|g| = g(V).$$

If  $f, g : A \mapsto B$  ( $B \subseteq \mathbb{R}$ ), we denote by  $f < g$  (or  $f > g$ ) that  $f(a) \leq g(a)$  (respectively  $f(a) \geq g(a)$ ) for all  $a \in A$  and  $f(x) < g(x)$  (respectively  $f(x) > g(x)$ ) for at least one  $x \in A$ . A function  $f : A \mapsto B$  which satisfies a condition  $C$  is said to be minimal (or maximal) with respect to that condition  $C$  if no function  $g : A \mapsto B$  exists which satisfies  $C$  such that  $g < f$  (respectively  $g > f$ ).

We shall also adopt the following convention: when a function (fractional or otherwise) of a certain class  $C$  has a cardinality equal to one of the bounds associated with that class  $C$  then we shall simply refer to it by the name of that bound. For example, if  $f : V(G) \mapsto [0, 1]$  is a fractional dominating function and  $|f| = \gamma_f(G)$  then we shall call  $f$  a  $\gamma_f(G)$ -function or, if no ambiguity is possible, a  $\gamma_f$ -function.

### 1.4.1 The fractional dominating function

Fractional domination appears to have been introduced by Farber [Far83] though it was first studied by Hedetniemi, Hedetniemi and Wimer [HHW83]. It has also been studied by Grinstead and Slater [GS90], Domke, Hedetniemi and Laskar [DHL88], and Domke, Hedetniemi, Laskar and Fricke [DHLF91]. Currie and Nowakowski [CN91] studied fractionally well-dominated graphs. We say that  $g : V(G) \mapsto [0, 1]$  is a *fractional dominating function of  $G$*  if

$$(\forall v \in V(G)) g(N[v]) \geq 1; \quad (1.4.1)$$

hence  $g$  is a *minimal fractional dominating function* if, furthermore, for any  $v \in V(G)$

$$g(v) > 0 \implies \exists u \in N[v] \text{ such that } g(N[u]) = 1. \quad (1.4.2)$$

(If there is a vertex  $v$  for which (1.4.2) is not true, i.e. every vertex  $u$  in the closed neighbourhood of  $v$  obeys  $g(N[u]) > 1$ , then we can decrease  $g(v)$  to obtain a smaller fractional dominating function and so  $g$  is not a minimal fractional dominating function. Conversely, (1.4.2) obviously implies minimality of a fractional dominating function  $g$ ).

The *fractional domination number of  $G$* ,  $\gamma_f(G)$ , and *upper fractional domination number of  $G$* ,  $\Gamma_f(G)$ , are defined by

$$\begin{aligned} \gamma_f(G) &= \min\{|g| : g \text{ is a minimal fractional dominating function of } G\}, \\ \Gamma_f(G) &= \max\{|g| : g \text{ is a minimal fractional dominating function of } G\}. \end{aligned}$$

### 1.4.2 The fractional packing function

We call  $g : V(G) \mapsto [0, 1]$  a *fractional packing function of  $G$*  if

$$(\forall v \in V(G)) g(N[v]) \leq 1.$$

A *maximal fractional packing function* is a fractional packing function for which, for any  $v \in V(G)$ ,

$$g(v) < 1 \implies \exists u \in N[v] \text{ such that } g(N[u]) = 1.$$

Notice that a vertex  $x$  with  $g(x) = 1$  induces a ‘dead-zone’: all those vertices  $y$  with  $0 < d(x, y) \leq 2$  must have  $g(y) = 0$ . This suggests a relationship with 2-packings which we shall examine later (Section 1.8).

The *lower fractional packing number* of  $G$ ,  $p_f(G)$ , and *upper fractional packing number* of  $G$ ,  $P_f(G)$ , are defined by

$$p_f(G) = \min\{|g| : g \text{ is a maximal fractional packing function of } G\},$$

$$P_f(G) = \max\{|g| : g \text{ is a maximal fractional packing function of } G\}.$$

The concept of a fractional packing function was introduced by Domke, Hedetniemi and Laskar [DHL88].

### 1.4.3 The fractional irredundance function

We call  $g : V(G) \mapsto [0, 1]$  a *fractional irredundance function* if for any  $v \in V(G)$

$$g(v) > 0 \implies \exists u \in N[v] \text{ such that } g(N[u]) = 1. \quad (1.4.3)$$

A *maximal fractional irredundance function* of  $G$  is a fractional irredundance function which is maximal with respect to (1.4.3). The *fractional irredundance number* of  $G$ ,  $ir_f(G)$ , and *upper fractional irredundance number* of  $G$ ,  $IR_f(G)$ , are defined as

$$ir_f(G) = \min\{|g| : g \text{ is a maximal fractional irredundance function of } G\},$$

$$IR_f(G) = \max\{|g| : g \text{ is a maximal fractional irredundance function of } G\}.$$

Comparing the conditions (1.4.2) and (1.4.3) we see that every minimal fractional dominating function is also a fractional irredundance function. This is analogous with the relationship between dominating and irredundant sets: every vertex  $v$  in a minimal dominating set  $S$  of a graph  $G$  has a private neighbour and hence that set  $S$  is also an irredundant set of  $G$ . Fractional irredundance was introduced by Domke, Hedetniemi and Laskar [DHL88].

Examples of fractional dominating, packing and irredundance functions are shown in Figure 1.1.

### 1.4.4 Other fractional set functions

Besides those described above we can define fractional set functions corresponding to other classes of sets which we have already discussed (Section 1.2). We say that  $g : V(G) \mapsto [0, 1]$  is a

- *fractional vertex covering function* if

$$uv \in E(G) \implies g(u) + g(v) \geq 1. \quad (1.4.4)$$

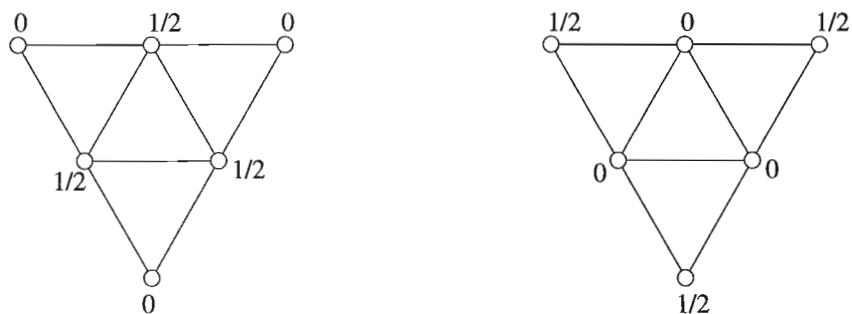


Figure 1.1: Examples of fractional dominating, packing and irredundance functions. The first graph shows a  $\gamma_f(G) = \frac{3}{2} = ir_f(G)$  function, and the second a fractional packing function with value  $P_f(G) = \frac{3}{2}$ .

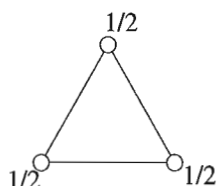


Figure 1.2: The values on the vertices are both a minimal fractional covering function and a maximal fractional independence function.

- *fractional vertex independence function* if

$$uv \in E(G) \implies g(u) + g(v) \leq 1. \tag{1.4.5}$$

Examples of fractional covering and fractional independence functions are shown in Figure 1.2.

In addition to those functions  $g$  which assign weights to the *vertices* of  $G$ , we can also consider fractional edge set functions. We say that  $g : E(G) \mapsto [0, 1]$  is a

- *fractional edge covering function* if

$$(\forall v \in V(G)) \sum_{e \sim v} g(e) \geq 1,$$

- *fractional matching function* if

$$(\forall v \in V(G)) \sum_{e \sim v} g(e) \leq 1. \tag{1.4.6}$$

Examples of fractional edge covering and fractional matching functions are shown in Figure 1.3.



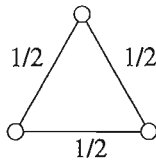


Figure 1.3: The values on the edges are both a minimal fractional edge covering function and a maximal fractional matching function.

Fractional matchings have been studied by Pulleyblank [Pul87]. We introduce the following related fractional parameters:

$$\alpha_f(G) = \min\{|g| : g \text{ is a fractional vertex covering function}\}, \quad (1.4.7)$$

$$\beta_f(G) = \max\{|g| : g \text{ is a fractional independence function}\}, \quad (1.4.8)$$

$$\alpha_f^1(G) = \min\{|g| : g \text{ is a fractional edge covering function}\}, \quad (1.4.9)$$

$$\beta_f^1(G) = \max\{|g| : g \text{ is a fractional matching}\}. \quad (1.4.10)$$

These four parameters (1.4.7), (1.4.8), (1.4.9) and (1.4.10) were independently introduced by Grinstead and Slater [GS90] and Domke, Hedetniemi and Laskar [DHL88]. A *minimal fractional vertex covering function*  $g : V(G) \mapsto [0, 1]$  is characterized by (1.4.4) together with

$$(\forall v \in V(G))(\exists u \in N(v)) \text{ such that } g(u) + g(v) = 1,$$

and thus the *upper fractional vertex covering number* is

$$\alpha_f^+(G) = \max\{|g| : g \text{ is a minimal fractional vertex covering function}\}. \quad (1.4.11)$$

A *maximal fractional vertex independence function*  $g : V(G) \mapsto [0, 1]$  is characterized by (1.4.5) together with

$$(\forall v \in V(G))(\exists u \in N(v)) \text{ such that } g(u) + g(v) = 1,$$

and the *lower fractional vertex independence number* is

$$\beta_f^-(G) = \min\{|g| : g \text{ is a maximal fractional vertex independence function}\}. \quad (1.4.12)$$

Analogously, we can define similar parameters,  $\alpha_f^{1+}(G)$  and  $\beta_f^{1-}(G)$ , for fractional edge set functions  $g : E(G) \mapsto [0, 1]$ . These two edge set function parameters together with (1.4.11) and (1.4.12) were introduced by Domke, Hedetniemi and Laskar [DHL88].

## 1.5 Some preliminary results

1. As noted in [DHL88], every minimal dominating set induces a minimal dominating function, and every minimal dominating function is itself a minimal fractional dominating function. This implies that

$$\gamma_f(G) \leq \gamma(G) \leq \Gamma(G) \leq \Gamma_f(G). \quad (1.5.1)$$

In [CFHJ90] an example of a graph  $G$  is provided for which  $\Gamma(G) < \Gamma_f(G)$  and it is shown that  $\Gamma_f(T) = \Gamma(T)$  for every tree,  $T$ . The latter result is extended in [CF93] in which classes of graphs  $G$  for which  $\beta(G) = \Gamma(G) = \Gamma_f(G) = IR(G)$  are obtained: a subset  $S$  of vertices of  $G$  is called a *stable transversal* of  $G$  if  $S$  contains exactly one vertex from each maximal clique in  $G$ ;  $G$  is said to be *strongly perfect* if  $G$  and each of its induced subgraphs has a stable transversal. It is shown in [CF93] that  $\beta(G) = \Gamma(G) = \Gamma_f(G) = IR(G)$  for all strongly perfect graphs  $G$ , and hence for even cycles, trees and all bipartite, permutation, comparability, chordal, co-chordal, peripheral, parity, perfectly orderable, Gallai and Meyniel graphs. For all simplicial graphs  $G$ ,  $\beta(G) = \Gamma(G) = \Gamma_f(G)$ , but simplicial graphs exist for which  $\Gamma(G) < IR(G)$  [CHHL88]. It is shown in [CF93] that if  $\xi, \eta \in \{\beta, \Gamma, \Gamma_f, IR\}$  and  $G_1, G_2$  are two graphs for which  $\xi(G_i) = \eta(G_i)$ ,  $i = 1, 2$ , then  $\xi(G_1 + G_2) = \eta(G_1 + G_2)$  (where  $+$  denotes the join operation). This result provides the means for constructing infinite classes of graphs for which the above inequalities hold.

2. The following results are derived in [DHL88] and [GS90]. We define the *neighbourhood matrix* of a graph  $G$ ,  $N(G) = A(G) + I(p)$  where  $A(G)$  is the adjacency matrix of  $G$  and  $I(p)$  the  $(p \times p)$  unit matrix, where  $p = |V(G)|$ . For any graph  $G$  with  $V(G) = \{v_1, \dots, v_p\}$ , the determination of  $\gamma_f(G)$  and  $P_f(G)$  is reducible to two linear programming problems. Let  $g$  be a fractional set function on  $G$ . Then we define the *characteristic vector* of  $g$ ,  $\vec{x}_g$ , to be a  $(p \times 1)$  vector the entries of which are  $\vec{x}_{g_i} = g(v_i)$ . Denote by  $\vec{x}$  an arbitrary  $(p \times 1)$  vector. The two problems are then

$$\gamma_f(G) \quad P_f(G) \quad (1.5.2)$$

$$\text{minimize } \vec{1}^T \cdot \vec{x} \quad \text{maximize } \vec{1}^T \cdot \vec{x} \quad (1.5.3)$$

$$\text{subject to } N\vec{x} \geq \vec{1} \quad \text{subject to } N\vec{x} \leq \vec{1} \quad (1.5.4)$$

$$\vec{0} \leq \vec{x} \leq \vec{1} \quad \vec{0} \leq \vec{x} \leq \vec{1} \quad (1.5.5)$$

where  $\vec{0}$  and  $\vec{1}$  are  $(p \times 1)$  vectors with zero and unit entries, respectively. A feasible solution to either of these two problems is the characteristic vector  $\vec{x} = \vec{x}_g$  of some fractional dominating or fractional packing function  $g$ . The optimal solutions of these problems are obviously  $\gamma_f(G)$  and  $P_f(G)$ . Furthermore, since  $N$  is symmetric, the two problems are duals of one another and so by the strong duality principle their solutions are equal. Thus, for any graph  $G$ ,

$$\gamma_f(G) = P_f(G). \quad (1.5.6)$$

This result has the following useful consequence: suppose that  $g : V \mapsto [0, 1]$  is a fractional dominating function and that  $h : V \mapsto [0, 1]$  is a fractional packing function. Then by definition  $|g| \geq \gamma_f(G) = P_f(G) \geq |h|$ . Thus, if we find two such functions  $g$  and  $h$  such that  $|g| = |h|$ , it must then be the case that  $\gamma_f(G) = |g| = |h| = P_f(G)$  and we have determined the fractional domination and upper fractional packing numbers of  $G$ . This is true even if there are vertices  $v_i \in V(G)$  where  $g(v_i) \neq h(v_i)$ .

3. The next three results were derived, independently, by Domke, Hedetniemi and Laskar [DHL88] and by Grinstead and Slater [GS90]. For any  $v \in V(G)$ , define a function  $g : V(G) \mapsto [0, 1]$  by

$$g(v) = \frac{1}{\Delta + 1},$$

where  $\Delta = \Delta(G)$ . Let  $v \in V(G)$ ; then

$$\begin{aligned} g(N[v]) &= \frac{\deg v + 1}{\Delta + 1} \\ &\leq \frac{\Delta + 1}{\Delta + 1} \\ &= 1. \end{aligned}$$

So  $g(N[v]) \leq 1$  for all  $v \in V(G)$  and  $g$  is a fractional packing function of  $G$ . Now  $|g| \leq P_f(G)$  and  $|g| = \frac{p}{\Delta + 1}$  imply that

$$P_f(G) \geq \frac{p}{\Delta + 1}. \quad (1.5.7)$$

4. For any  $v \in V(G)$ , define a function  $h : V(G) \mapsto [0, 1]$  by

$$h(v) = \frac{1}{\delta + 1},$$

where  $\delta = \delta(G)$ . Then

$$\begin{aligned} h(N[v]) &= \frac{\deg v + 1}{\delta + 1} \\ &\geq \frac{\delta + 1}{\delta + 1} \\ &= 1. \end{aligned}$$

So  $h(N[v]) \geq 1$  for all  $v \in V(G)$  and  $h$  is a fractional dominating function. Now  $|h| \geq \gamma_f(G)$  and  $|h| = \frac{p}{\delta+1}$  imply that

$$\gamma_f(G) \leq \frac{p}{\delta + 1}. \quad (1.5.8)$$

This result (1.5.8) and the previous result (1.5.7) produce the following chain of inequalities

$$\frac{p}{\Delta(G) + 1} \leq P_f(G) = \gamma_f(G) \leq \frac{p}{\delta(G) + 1}. \quad (1.5.9)$$

5. This last result (1.5.9) has the immediate consequence that if  $G$  is an  $r$ -regular graph then

$$P_f(G) = \gamma_f(G) = \frac{p}{r + 1}. \quad (1.5.10)$$

This implies that

- (a) for complete graphs ( $r = p - 1$ )

$$P_f(K_p) = \gamma_f(K_p) = 1,$$

- (b) for cycles ( $r = 2$ )

$$P_f(C_p) = \gamma_f(C_p) = \frac{p}{3}.$$

The linear programming formulation presented above prompts the definition of other parameters first introduced by Grinstead and Slater [GS90]. Let  $g : V(G) \mapsto [0, 1]$  be a fractional set function, and consider the sum over closed neighbourhoods

$$\sum_{v \in V(G)} g(N[v]). \quad (1.5.11)$$

We are summing over the weights of the closed neighbourhoods. In general, the weight of any vertex  $v$ ,  $g(v)$ , is counted more than once in this sum over the neighbourhoods of  $G$ . Each vertex  $v \in V(G)$  is counted in (1.5.11) once for the term

$g(N[v])$  corresponding to its own closed neighbourhood, and once again for each of the neighbourhoods of its  $\deg v$  neighbours. Thus we can rewrite (1.5.11) as

$$\sum_{v \in V(G)} g(N[v]) = \sum_{v \in V(G)} (1 + \deg v)g(v).$$

Define the *influence* of  $g$  to be

$$I(g) = \sum_{v \in V(G)} (1 + \deg v)g(v), \quad (1.5.12)$$

and the two related parameters

$$R_f(G) = \min_g I(g) \quad \text{subject to } N\vec{x}_g \geq \vec{1}, \quad (1.5.13)$$

$$F_f(G) = \max_g I(g) \quad \text{subject to } N\vec{x}_g \leq \vec{1}. \quad (1.5.14)$$

Notice that  $R_f(G)$  is a type of domination parameter and  $F_f(G)$  a type of packing parameter. Functions obeying the constraints in (1.5.13) and (1.5.14) are respectively fractional dominating functions and fractional packing functions but these new parameters related to them we shall call the *minfluence*,  $R_f(G)$ , and *maxfluence*,  $F_f(G)$ , of  $G$ . For any set  $S \subseteq V(G)$  we define

$$R(S) = \sum_{v \in S} (1 + \deg v),$$

and define the *redundance number* of  $G$  [GS] to be

$$R(G) = \min\{R(S) : S \text{ is a dominating set of } G\}.$$

Examples of  $F_f(G)$  and  $R_f(G)$  functions are shown in Figure 1.4, which is taken from [GS90].

The following theorem is due to Grinstead and Slater [GS90].

**Theorem 1.5.1.** *If there exists a fractional packing function,  $g : V(G) \mapsto [0, 1]$ , such that  $N\vec{x}_g = \vec{1}$ , then*

$$F_f(G) = p.$$

*If there exists a fractional dominating function,  $h : V(G) \mapsto [0, 1]$ , such that  $N\vec{x}_h = \vec{1}$ , then*

$$R_f(G) = p.$$

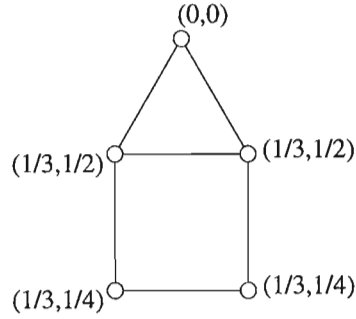


Figure 1.4: The figures in brackets are  $F_f(G)$  and  $R_f(G)$  functions, respectively. For this graph  $F_f(G) = \frac{14}{3} < R_f(G) = \frac{11}{2}$ .

*Proof.* Let  $g$  be a fractional packing function. Then

$$\begin{aligned} I(g) &= \sum_{v \in V(G)} g(N[v]) \\ &\leq p, \end{aligned}$$

and obviously if  $N\vec{x}_g = \vec{1}$  (which implies that  $g(N[v]) = 1, \forall v \in V(G)$ ) then  $F_f(G) = p$ . A similar proof holds for the fractional dominating case.  $\square$

## 1.6 Further results

The following theorem is due to Domke, Hedetniemi and Laskar [DHL88], but the proof has been modified.

**Theorem 1.6.1.** *For any graph  $G$*

$$\gamma_f(G) = 1 \iff \Delta(G) = p - 1.$$

*Proof.*

1. *First, we assume that  $\Delta(G) = p - 1$  and prove that  $\gamma_f(G) = 1$ .* If  $\Delta(G) = p - 1$  then  $G$  possesses a dominating vertex. Hence (referring to (1.5.1)) we find that  $1 \leq \gamma_f(G) \leq \gamma(G) = 1$ .
2. *Now, we assume that  $\gamma_f(G) = 1$  and prove that  $\Delta(G) = p - 1$ .* By reference to equation (1.5.9), this statement is obviously true. However, we shall present

a longer proof so as to obtain the Corollary 1.6.2. Let  $v$  be any vertex in  $G$ , and let  $g$  be a fractional dominating function of  $G$  such that  $|g| = \gamma_f(G) = 1$ . Let  $P = \{v \in V(G) : g(v) > 0\}$ . Then, since  $\gamma_f(G) = 1$ ,  $g(P) = 1$ . Since  $g(N[v]) \geq 1$  for all  $v \in V(G)$ , we must have that  $P \subseteq N[v]$ . Hence, if  $u \in P$ , then  $u$  is adjacent to every other vertex of  $G$ , so that  $\deg u = p - 1$ .

□

The following corollary, not due to the above authors, is a direct result of the proof of Theorem 1.6.1.

**Corollary 1.6.2.** *Let  $G$  be any graph with  $\Delta(G) = p - 1$ , and let  $g$  be any fractional dominating function with  $|g| = \gamma_f(G)$ . Then only those vertices  $v \in V(G)$  with  $\deg v = p - 1$  satisfy  $g(v) > 0$ .*

□

We remarked earlier that Cockayne and Hedetniemi [CH77] have shown that for any graph  $G$ ,  $ir(G) \leq \gamma(G)$ . Cockayne, Hedetniemi and Miller [CHM78] have proved an extended result: that for any graph  $G$ ,

$$ir(G) \leq \gamma(G) \leq \beta^-(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G). \quad (1.6.1)$$

We can prove a corresponding theorem, though not including as many parameters, for fractional parameters. We start with a result first proved by Domke, Hedetniemi and Laskar [DHL88].

**Theorem 1.6.3.** *Let  $G$  be any graph. Then*

$$ir_f(G) \leq \gamma_f(G) \leq \Gamma_f(G) \leq IR_f(G).$$

*Proof.* To prove the first and last inequalities, it suffices to show that any minimal fractional dominating function is a maximal fractional irredundant function. Let  $g$  be a minimal fractional dominating function, then,

$$(\forall v \in V(G))g(v) > 0 \implies \exists u \in N[v] \text{ such that } g(N[u]) = 1$$

and hence  $g$  is also a fractional irredundant function. We must show that  $g$  is maximal. For suppose, to the contrary, that there exists a fractional irredundant function  $h : V(G) \mapsto [0, 1]$  with

$$\begin{aligned} h(v) &> g(v), \\ h(u) &\geq g(u) \quad \forall u \in V(G) - \{v\}. \end{aligned}$$

Since  $g$  is a fractional dominating function,  $g(N[u]) \geq 1$  and thus  $h(N[u]) > g(N[u]) \geq 1$  for all  $u \in N[v]$ , so  $h$  cannot be an irredundant function. Hence  $g$  must be a maximal irredundant function.  $\square$

Every  $IR(G)$ -set induces a maximal fractional irredundance function and hence  $IR(G) \leq IR_f(G)$ . This remark, together with the result from Theorem 1.6.3 that  $\Gamma_f(G) \leq IR_f(G)$ , poses a question as to the relation between  $\Gamma_f(G)$  and  $IR(G)$ . The following theorem is due to Domke, Hedetniemi, Laskar and Fricke [DHLF91].

**Theorem 1.6.4.** *For any graph  $G$ ,*

$$\Gamma_f(G) \leq IR(G).$$

*Proof.* Let  $g : V(G) \mapsto [0, 1]$  be a  $\Gamma_f(G)$ -function, and denote by  $S = \{v_1, \dots, v_m\} \subseteq V(G)$  the set of vertices with  $g(N[v_i]) = 1$  for  $i = 1, \dots, m$ . Let  $P = \{u \in V(G) : g(u) > 0\}$ . Since  $g$  is minimal, every  $u \in P$  is either in  $S$  or adjacent to an element of  $S$ , i.e.  $S$  dominates  $P$ . Let  $D \subseteq S$  be a minimal set that dominates  $P$ . Then  $D$  is an irredundant set of  $\langle D \cup P \rangle$ , hence  $D$  is an irredundant set of  $G$ , and hence

$$IR(G) \geq |D|.$$

Now since  $D$  dominates  $P$ , every vertex  $u$  having  $g(u) > 0$  satisfies  $u \in N[v_i]$  for some  $1 \leq i \leq m$ . Let  $D = \{w_j : j = 1, \dots, n, n \leq m\}$ . Notice that  $g(N[w_j]) = 1$  for all  $1 \leq j \leq n$ . Hence

$$\begin{aligned} |D| &= \sum_{j=1}^n 1 \\ &= \sum_{j=1}^n g(N[w_j]) \\ &\geq \sum_{v \in V(G)} g(v) \\ &= |g| \\ &= \Gamma_f(G). \end{aligned}$$

Hence  $IR(G) \geq \Gamma_f(G)$  and the result follows.  $\square$

Every  $ir(G)$ -set induces a maximal fractional irredundant function, hence  $ir(G) \geq ir_f(G)$ . This remark in combination with Theorem 1.6.3 might seem to prompt a theorem similar to 1.6.4 to establish an analogous result for  $\gamma_f(G)$  and  $ir(G)$ .



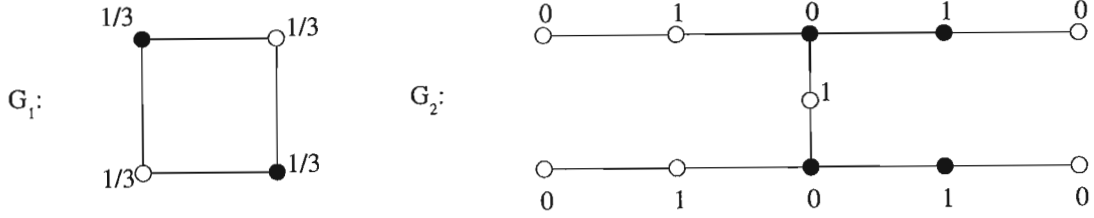


Figure 1.5: There is no relation between  $\gamma_f(G)$  and  $ir(G)$ . The graph  $G_1$  has  $\gamma_f(G_1) = \frac{4}{3} < 2 = ir(G_1)$ . The graph  $G_2$  has  $\gamma_f(G_2) = 5 > 4 = ir(G_2)$ . In the diagram, the shaded vertices are  $ir$ -sets of  $G_1$  and  $G_2$ , and the values on the vertices are  $\gamma_f$ -functions of  $G_1$  and  $G_2$ .

However, reference to Figure 1.5 shows that there is no relation between  $\gamma_f(G)$  and  $ir(G)$ .

We thus have the following chain of inequalities:

**Corollary 1.6.5.** *For any  $G$ ,*

$$ir_f(G) \leq \gamma_f(G) \leq \Gamma_f(G) \leq IR(G) \leq IR_f(G).$$

The following theorem is original.

**Theorem 1.6.6.** *For any graph  $G$ ,*

$$\gamma_f(G) \leq \beta_f^-(G) \leq \beta_f(G).$$

*Proof.* Let  $G$  be a graph. The second inequality is obvious. To prove the first, it suffices to show that every maximal fractional independence function of  $G$  is a fractional dominating function of  $G$ . Let  $f : V(G) \mapsto [0, 1]$  be a maximal fractional independence function of  $G$ , and let  $v \in V(G)$ . If  $\deg v = 0$  then  $f(v) = 1$  otherwise  $f$  is not maximal, so suppose that  $\deg v \geq 1$ . Then there must be a vertex  $u \in N(v)$  such that  $f(u) + f(v) = 1$ , otherwise for all  $w \in N(v)$ ,  $f(v) + f(w) < 1$  and we can increase  $f(v)$  and still have a fractional independence function of  $G$ , which contradicts the maximality of  $f$ . Thus for each  $v \in V(G)$ ,

$$f(N[v]) \geq 1,$$

and hence  $f$  is a fractional dominating function of  $G$ . □

We would expect from (1.6.1) that for any  $G$ ,  $\beta_f(G) \leq \Gamma_f(G)$ . However, this is not the case. Figure 1.2 shows a fractional set function for a graph  $G$  which is a maximal fractional independence function of  $G$ :  $f(u) + f(v) = 1$  for all  $uv \in E(G)$  and hence no  $f(u)$  can be increased. Hence  $\beta_f(G) \geq \frac{3}{2}$ . However, any minimal fractional dominating function of  $G$  for any  $v \in V(G)$  has  $f(N[v]) = |f| = 1$  and thus  $\Gamma_f(G) = 1$ ; so  $\Gamma_f(G) < \beta_f(G)$ . This is regarded as a serious deficiency in the theory of fractional domination and independence by Fricke [Fri95] and seems to indicate the need to extend the theory of fractional independence. One such approach is dealt with in Chapter 3 where  $e = 1$  functions are introduced, both as intrinsically interesting parameters and for their obvious relation to maximally independent functions.

## 1.7 Fractionally well-dominated graphs

Plummer [Plu70] calls a graph  $G$  *well-covered* if every maximal independent set of  $G$  has the same size, and *well-dominated* if every minimal dominating set of  $G$  has the same size. Currie and Nowakowski [CN91] call a graph  $G$  *fractionally well-covered* if every minimal fractional dominating function  $g : V(G) \mapsto [0, 1]$  (what they call a minimal fractional cover) has the same weight,  $|g|$ ; to preserve our terminology we shall call a graph with this property *fractionally well-dominated*. They prove the next theorem for the more general case where  $g : V(G) \mapsto [0, \infty)$ , but their proof is easily modifiable to cover the normal fractional dominating function which we are interested in. Before we prove this result, we shall need a lemma:

**Lemma 1.7.1.** *For  $i = 1, \dots, k$  (where  $k$  is some positive integer), let  $f_i : V(G) \mapsto [0, 1]$  be a fractional dominating function of  $G$ . Then the function defined as (for all  $v \in V(G)$ )*

$$f(v) = \frac{1}{k} \sum_{i=1}^k f_i(v)$$

*is also a fractional dominating function of  $G$ .*

*Proof.* Let  $v \in V(G)$ , then

$$\begin{aligned} f(N[v]) &= \frac{1}{k} \sum_{i=1}^k f_i(N[v]) \\ &\geq \frac{1}{k} \sum_{i=1}^k 1 \\ &= 1, \end{aligned}$$

and  $f$  is a fractional dominating function of  $G$ . □

Let  $G$  be a graph, then  $v \in V(G)$  is called a *simplicial vertex* of  $G$  if  $N[v] \cong K_{\deg v + 1}$ . The graph  $G$  is called a *simplicial graph* if it is possible to partition  $V(G)$  into the closed neighbourhoods of simplicial vertices; i.e. into vertex-disjoint, maximal, complete subgraphs of  $G$ , each subgraph containing at least one simplicial vertex.

**Theorem 1.7.2.** *Let  $G$  be a graph. Then  $G$  is fractionally well-dominated if and only if  $G$  is simplicial.*

*Proof.*

1. *Assume  $G$  is simplicial.* Let  $H_1, \dots, H_k$  be a decomposition of  $G$  into vertex disjoint, complete subgraphs, and for  $1 \leq i \leq k$  denote by  $x_i$  a simplicial vertex of  $H_i$  and  $V_i = V(H_i)$ . Let  $f$  be a minimal fractional dominating function of  $G$ . Suppose that there is some  $i$ , where  $1 \leq i \leq k$ , and some  $m > 0$  such that

$$f(V_i) = 1 + m.$$

Choose a vertex  $y \in V_i$  with  $f(y) > 0$ , and define a fractional set function  $f' : V(G) \mapsto [0, 1]$  as

$$f'(v) = \begin{cases} f(y) - \min\{f(y), m\} & \text{for } v = y, \\ f(v) & \text{for } v \in V(G) - \{y\}. \end{cases}$$

Let  $v \in V(G)$ , then

- if  $v \in V_j$ , where  $1 \leq j \leq k$  and  $j \neq i$ , then

$$\begin{aligned} f'(N[v]) &\geq f'(N[x_j]) \\ &= f(N[x_j]) \\ &\geq 1; \end{aligned}$$

- if  $v \in V_i$ , then

$$\begin{aligned} f'(N[v]) &\geq f'(N[x_i]) \\ &= f'(V_i) \\ &= 1 + m - \min\{f(y), m\} \\ &\geq 1. \end{aligned}$$

Hence our assumption that such an  $m$  exists is wrong and it must be the case that  $f(V_i) = 1$  for all  $1 \leq i \leq k$ . Thus all minimal fractional dominating functions of  $G$  have weight  $k$  and  $G$  is fractionally well-dominated.

2. *Assume  $G$  is fractionally well-dominated.* It is obvious that connected graphs of order one and two are fractionally well-dominated and simplicial. There are two connected graphs of order three:  $K_3$  (which is obviously simplicial and for which any fractional dominating function  $f$  has  $|f| = 1$ ) and  $P_3$ . There are two possible decompositions of  $P_3$  into vertex disjoint complete subgraphs (one of which has components isomorphic to  $K_1$  and the other components isomorphic to  $K_1$  and  $K_2$ ), neither of which yields a simplicial graph. It is obvious that  $P_3$  is not fractionally well-dominated (assign weights of  $\frac{1}{2}$  to all the vertices and a minimal fractional dominating function of weight  $\frac{3}{2}$  results; assign a weight of 1 to the vertex of degree 2 and weights of 0 to the end-vertices and a minimal fractional dominating function of weight 1 results). Thus the statement is true for all graphs  $G$  with  $p(G) = 1, 2, 3$ . The proof will be by reductio ad absurdum; assume that there exists a class of graphs which are fractionally well-dominated but not simplicial and let  $G$  be a graph from that class having minimum  $p$ .

(a) *Suppose that  $G$  contains no simplicial vertices.* Then any  $v \in V(G)$  is adjacent to (at least) two vertices which are mutually non-adjacent. For each  $v \in V(G)$  choose a maximal independent set  $J(v)$  of  $G$  which includes at least two vertices adjacent to  $v$ , and note that this maximal independent set is also a minimal dominating set. Let  $f_v : V(G) \mapsto [0, 1]$  be the characteristic function of the set  $J(v)$  and notice that, since all the maximal independent sets of  $G$  have the same cardinality,  $|f_v|$  is well-defined. Define a function ( $\forall v \in V(G)$ )

$$f(v) = \frac{1}{p} \sum_{u \in V(G)} f_u(v).$$

We know from Lemma 1.7.1 that  $f$  is a fractional dominating function of  $G$ . Suppose that this function  $f$  is a *minimal* fractional dominating function of  $G$ . Since  $G$  is fractionally well-dominated, for all  $v \in V(G)$ ,

$$|f| = |f_v|.$$

Let  $u, v \in V(G)$  and  $u \neq v$ ; because each  $f_v$  is a fractional dominating function of  $G$  it follows that

$$f_v(N[u]) \geq 1;$$

because of the manner of choice of each set  $J(v)$  and hence the associated  $f_v$  it is also the case that

$$f_v(N[v]) \geq 2.$$

Then for any  $v \in V(G)$ ,

$$\begin{aligned} f(N[v]) &= \frac{1}{p} \sum_{u \in V(G)} f_u(N[v]) \\ &\geq \frac{1}{p}((p-1) + 2) \\ &= 1 + \frac{1}{p}. \end{aligned}$$

Hence reducing any vertex by an amount  $\frac{1}{p}$  produces a smaller fractional dominating function which contradicts our assumption that  $f$  is minimal. Hence  $f$  is not minimal and there exists a minimal fractional dominating function  $g$  such that  $|g| < |f| = |f_v|$  ( $\forall v \in V(G)$ ), from which it follows that  $G$  is not fractionally well-dominated, contrary to our hypothesis. Hence  $G$  must contain at least one simplicial vertex.

- (b) *Suppose that in  $G$  there is a maximal, complete subgraph  $H_0$  with simplicial vertex  $x$ . If we take any minimal fractional dominating function  $f_1$  of  $G - H_0$  then  $f_1$  together with a function  $f_2$  which maps  $x$  to 1 and the other vertices of  $H_0$  to 0 is a minimal fractional dominating function of  $G$ . Then since  $G$  is fractionally well-dominated it must be the case that all the possible functions  $f_1$  must have the same weight. Therefore  $G - H_0$  is also fractionally well-dominated, and hence (by the minimality of  $p$ )  $G - H_0$  is simplicial. Let  $H_1, \dots, H_k$  be a partition of  $G - H_0$  into vertex disjoint, maximal complete subgraphs of  $G - H_0$ , and for each  $1 \leq i \leq k$  denote by  $x_i$  a simplicial vertex of  $H_i$ . We have thus partitioned  $G$  into complete subgraphs  $H_0, \dots, H_k$ . For  $i = 1, \dots, k$  let  $F_i$  be a maximal complete subgraph of  $G$  which contains  $H_i$ : obviously for each  $i$ ,  $F_i - H_i \subset H_0$ . It must be the case that  $H_0, F_1, F_2, \dots, F_k$  do not form a partition of  $G$  into vertex disjoint, maximal complete subgraphs each of which contains a simplicial vertex. Therefore there must be some  $1 \leq i \leq k$  for which  $F_i - H_i \neq \emptyset$  or for which  $F_i$  does not contain a simplicial vertex.*

- *Suppose that, for some  $1 \leq i \leq k$ ,  $F_i - H_i \neq \emptyset$ . Suppose then that  $z \in F_i \cap H_i$ . Let  $J$  be a maximal independent set of  $G - F_i \cap H_i$ . Now  $J \cup \{z\}$  is a dominating set of  $G$  and  $J \cup \{x, x_i\}$  is a maximal*

independent set of  $G$ ; hence  $G$  is not well-dominated and hence  $G$  is not fractionally well-dominated, which is a contradiction.

- Suppose that for some  $i$ , which we can without loss of generality suppose to be  $i = k$ ,  $F_k$  has no simplicial vertices in  $G$ . Thus every vertex in  $F_k$  must be adjacent to some vertex not in  $F_k$ . If  $y \in F_k$  and  $z$  is adjacent to  $y$  but  $z$  is not in  $F_k$  then  $z$  must be in one of the complete subgraphs  $H_1, \dots, H_{k-1}, H_0$ . For each  $y \in V(F_k)$  form a set by taking  $z \in V(H_i)$  for some  $0 \leq i \leq k-1$  which is adjacent to  $y$  but not in  $H_k$ , together with a vertex in  $H_k$  which is not adjacent to  $z$  and one vertex from each of  $H_0, \dots, H_{k-1}$  except for  $H_i$ . Every vertex in  $G$  is adjacent to at least one element of this set, which contains  $k+1$  vertices, hence it is a minimal dominating set of  $G$ . Let  $p_y : V(G) \mapsto [0, 1]$  be the characteristic function of this set (so  $p_y$  is a minimal fractional dominating function of  $G$ ), and define a new fractional set function as ( $\forall v \in V(G)$ ):

$$p(v) = \frac{1}{|V(H_k)|} \sum_{y \in H_k} p_y(v).$$

We claim that this set function  $p$  is not minimal: for each  $v \in H_k$ ,

$$p(N[v]) \geq \frac{|V(H_k)| + 1}{|V(H_k)|}$$

because for each fractional dominating function  $p_y$  we have  $p_y(N[y]) \geq 1$  but  $p_y(N[v]) \geq 2$ . Now choose a vertex  $w \in V(H_k)$  which satisfies  $p(w) > 0$ , and reduce  $p(w)$  by  $\frac{1}{|V(H_k)|}$  to form  $p' : V(G) \mapsto [0, 1]$ . Then, for each vertex  $v \in V(H_k)$ ,  $p'(N[v]) \geq 1$ . For  $v \notin V(H_k)$ ,  $p'(N[v]) \geq 1$ , because for each function  $p_y$  each  $v \in V(H_i)$ ,  $i < k$ , is either in the corresponding dominating set or else is adjacent to a vertex in  $H_i$ . This new function has smaller weight than the original functions; hence none of the original functions were minimal.

It must therefore be the case that the postulated graph  $G$  does not exist. Hence  $G$  is simplicial.

□

## 1.8 Relation to the 2-packing number

The *2-packing number*  $P_2(G)$  (also known as the upper packing number), originally defined by Meir and Moon [MM75], is the maximum cardinality of a set of vertices

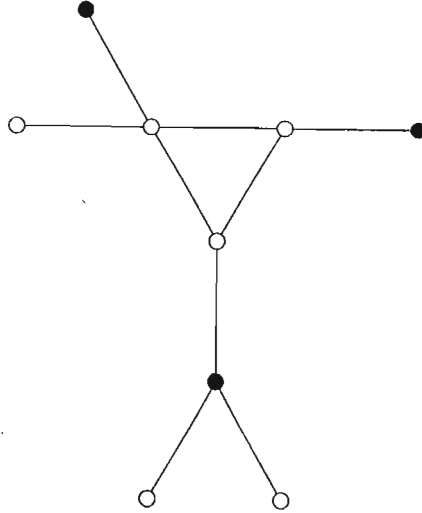


Figure 1.6: The shaded vertices are a  $P_2(G)$ -set.

$S \subseteq V(G)$  which obeys

$$u, v \in S \implies d(u, v) > 2.$$

A  $P_2(G)$ -set for a graph  $G$  is shown in Figure 1.6.

The following theorem is due to Domke, Hedetniemi and Laskar [DHL88].

**Theorem 1.8.1.** *Let  $G$  be any graph. Then*

$$p_f(G) \leq P_2(G) \leq P_f(G).$$

*Proof.* We shall prove both inequalities simultaneously by showing that any 2-packing set  $S$  with  $|S| = P_2(G)$  induces a maximal fractional packing function. Let  $S \subseteq V(G)$  be a 2-packing set for which  $|S| = P_2(G)$ . We define a function  $g$  (the characteristic function of  $S$ ):

$$g(v) = \begin{cases} 0 & v \notin S \\ 1 & v \in S. \end{cases}$$

Then  $|g| = \sum_{v \in S} 1 = P_2(G)$ . The next step will be to show that  $g$  is a maximal fractional packing function. Let  $v \in V(G)$ , then there are two possibilities:

1.  $v \in N[S]$ , say  $v \in N[u]$  where  $u \in S$  and  $v$  and  $u$  are not necessarily distinct. Since  $S$  is a 2-packing set, a nearest vertex to  $v$  in  $S$  besides  $u$  is at least distance 2 from  $v$ ; so  $N[v] \cap S = \{u\}$  and hence  $g(N[v]) = 1$ .
2.  $v \notin N[S]$ , then  $v$  must be distance 2 from some  $u \in S$ , otherwise (if  $v$  were farther away)  $S$  would not be maximal. Then there is a vertex  $w \in N[v] \cap N[u]$  and, as was discussed in 1,  $g(N[w]) = 1$ .

Hence, for any  $v \in V(G)$ , there is a  $u \in N[v]$  where  $g(N[u]) = 1$  and hence  $g$  is a maximal fractional packing function. Now since  $S$  induces a maximal fractional packing function  $g$  and  $|g| = P_2(G)$  it must be the case that

$$p_f(G) \leq P_2(G) \leq P_f(G). \quad \square$$

Combining this result 1.8.1 with previous results we obtain the following sequence of inequalities:

**Corollary 1.8.2.** *For any graph  $G$*

$$1 \leq p_f(G) \leq P_2(G) \leq P_f(G) = \gamma_f(G) \leq \gamma(G) \leq \Gamma(G) \leq \Gamma_f(G).$$

### 1.8.1 The independence of $P_2, \gamma_f$ and $\gamma$

The following was noted by Fisher [Fis94]. Four sets of relationships are allowed between  $P_2(G)$ ,  $\gamma_f(G)$  and  $\gamma(G)$  by the inequalities 1.8.2:

$$\begin{aligned} P_2(G) &= \gamma_f(G) = \gamma(G), \\ P_2(G) &< \gamma_f(G) = \gamma(G), \\ P_2(G) &= \gamma_f(G) < \gamma(G), \\ P_2(G) &< \gamma_f(G) < \gamma(G). \end{aligned}$$

The independence of the three parameters can be seen by considering the examples shown in Figure 1.7, for which the values of the parameters are:

$$\begin{aligned} P_2(G_1) &= 1 = \gamma_f(G_1) = \gamma(G_1), \\ P_2(G_2) &= 1 < \gamma_f(G_2) = 2 = \gamma(G_2), \\ P_2(G_3) &= 2 = \gamma_f(G_3) < \gamma(G_3) = 3, \\ P_2(G_4) &= 1 < \gamma_f(G_4) = \frac{4}{3} < \gamma(G) = 2. \end{aligned}$$



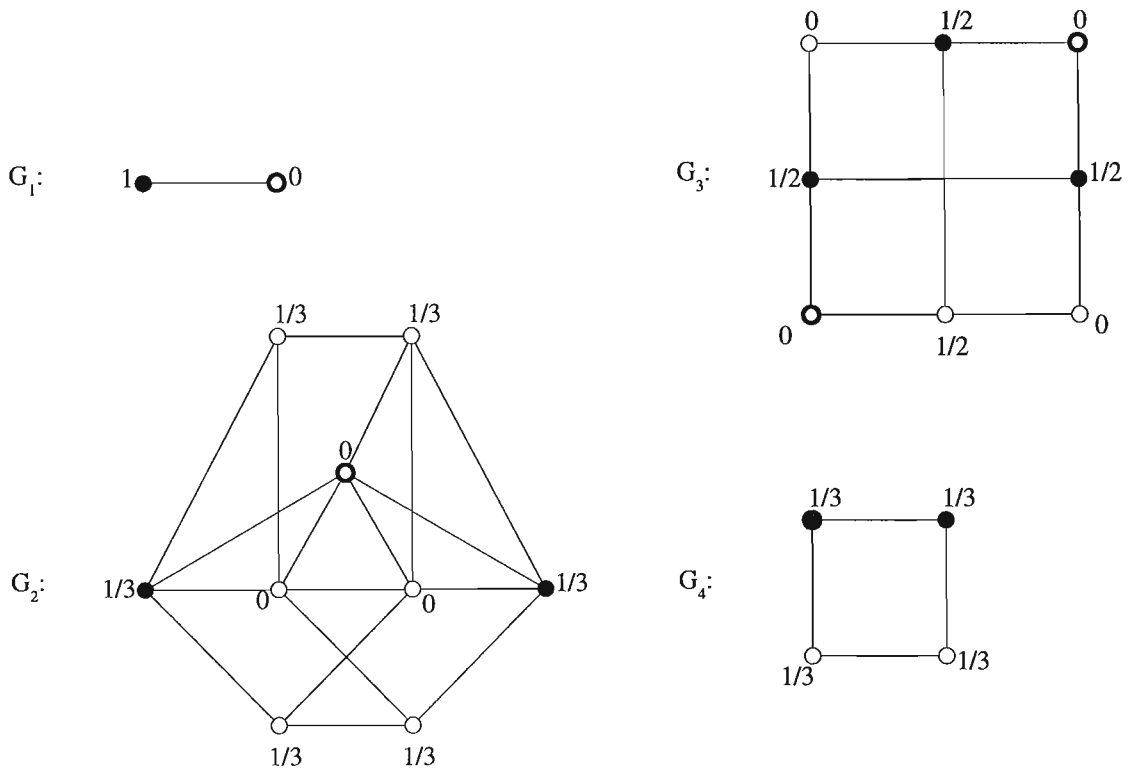


Figure 1.7: Examples showing equality and inequality of  $P_2(G)$ ,  $\gamma_f(G)$  and  $\gamma(G)$ . In each graph  $G_i$ , for  $i = 1, \dots, 4$ , the  $\gamma(G_i)$ -set is shown as the shaded vertices and the  $P_2(G_i)$ -set as the vertices with the heavy borders. The values on the vertices are a  $\gamma_f(G_i)$ -function. The quoted results are obvious for  $G_1$ , and follow from the result (1.5.10) for the graph  $G_4$ . In graph  $G_2$ , no two vertices are more than distance 2 apart; hence  $P_2(G_2) = 1$ . The weights are both a fractional packing and a fractional dominating function and so, by (1.5.6),  $\gamma_f(G_2) = 2$ ; that  $\gamma(G_2) = 2$  follows from Theorem 1.6.1 and the inequality (1.5.1). No two-vertex set dominates  $G_3$  but the indicated vertices are a dominating set. Thus  $\gamma(G_3) = 3$ , and the function values are once again both a fractional dominating and fractional packing function of  $G_3$  and so, by the discussion above,  $\gamma_f(G_3) = 2$ .

### 1.8.2 $P_2$ in connected block graphs

Meir and Moon [MM75] proved that if  $T$  is a tree then  $P_2(T) = \gamma(T)$ . Since every tree is also a connected block graph, this result is a consequence of a more general result due to Domke, Hedetniemi, Laskar and Allan [DHLA88], viz. that  $P_2(G) = \gamma(G)$  if  $G$  is a connected block graph. In order to prove this we shall first need to prove a related result. Recall that a *block graph* is a graph in which every block is complete.

**Lemma 1.8.3.** *Let  $G$  be any connected block graph. Let  $u, v \in V(G)$  such that  $d(u, v) > 1$ , and let  $P = u, x_1, \dots, x_n, v$  be a shortest  $u - v$  path in  $G$ . Then*

1. *all of the vertices  $x_1, \dots, x_n$  are cut vertices of  $G$ ,*
2.  *$P$  is the unique shortest  $u - v$  path in  $G$ .*

*Proof.* Since  $G$  is a block graph, every vertex in a block is at distance 1 from every other vertex in that block. Thus, if one of the vertices  $x_1, \dots, x_n$  is not a cut vertex of  $G$  then  $P$  is not a  $u - v$  path of shortest length. Furthermore, since each of the vertices  $x_1, \dots, x_n$  is a cut vertex of  $G$  and since no two cut vertices in distinct blocks can lie on the same cycle, this shortest  $u - v$  path  $P$  is unique.  $\square$

We are now ready to prove the previously mentioned theorem due to Domke, Hedetniemi, Laskar and Allan [DHLA88].

**Theorem 1.8.4.** *If  $G$  is a connected block graph then*

$$P_2(G) = \gamma(G).$$

*Proof.* Let  $G$  be a connected block graph of order  $p = |V(G)|$ . The proof will be by induction on  $p$ . Now if  $p \in \{1, 2, 3\}$  it follows immediately that  $P_2(G) = 1 = \gamma(G)$ . So let  $G$  be a graph of order  $p \geq 4$ , and assume that the relation  $P_2(G) = \gamma(G)$  holds for all connected block graphs of order less than  $p$ . Let  $m = \text{diam}(G)$ , if  $m = 1$  then  $G \cong K_p$  and  $P_2(G) = \gamma(G) = 1$ . Also, if  $m = 2$  then no vertex is more than distance 2 from any other and also, as  $G$  is a connected block graph,  $\text{rad}(G) = 1$ , thus  $P_2(G) = \gamma(G) = 1$ . Hence we assume that  $\text{diam}(G) = m \geq 3$ .

Let  $x_0, x_m \in V(G)$  be such that  $d(x_0, x_m) = \text{diam}(G) = m$ . Note that  $x_0$  and  $x_m$  cannot be cut vertices —  $x_0$  and  $x_m$  must be contained in end blocks of  $G$ , every block is complete and every block must contain at least two vertices. By Lemma 1.8.3

we know that there is a unique shortest path  $x_0, x_1, \dots, x_{m-1}, x_m$  between  $x_0$  and  $x_m$  and, furthermore, each of the vertices  $x_1, \dots, x_{m-1}$  is a cut vertex. Consider the components of  $\langle V(G) - \{x_1\} \rangle$  and denote by  $C(x_0)$  and  $C(x_m)$  the components containing  $x_0$  and  $x_2, \dots, x_m$ , respectively. Notice that any vertex  $v \notin V(C(x_m))$  must be adjacent to  $x_1$  (otherwise  $d(v, x_m) > m = \text{diam}(G)$  which is a contradiction). We define a set  $W \subseteq C(x_m)$  as

$$W = V(G) - N[x_1],$$

and note that  $x_m \in W$  because

$$\begin{aligned} d(x_m, x_1) &= m - 1 \\ &\geq 2. \end{aligned}$$

Note also that  $\langle W \rangle$  is a block graph. Let  $D$  denote a dominating set of  $\langle W \rangle$  chosen such that  $|D| = \gamma(\langle W \rangle)$ . Then  $D \cup \{x_1\}$  is a dominating set of  $G$  and thus  $\gamma(G) \leq \gamma(\langle W \rangle) + 1$ . Let  $P$  denote a 2-packing set of  $\langle W \rangle$  of maximum cardinality,  $|P| = P_2(\langle W \rangle)$ . Notice that, however we choose  $P$ , because every vertex in  $N[x_1]$  is excluded from  $W$ ,  $\forall y \in W, d(x_1, y) > 1$  and hence  $d(x_1, y) > 1$  for any  $y \in P$ . For any  $y \in P$

$$\begin{aligned} d(x_0, y) &= d(x_0, x_1) + d(x_1, y) \\ &> 2, \end{aligned}$$

and thus  $P \cup \{x_0\}$  is a 2-packing set of  $G$  and  $P_2(G) \geq P_2(\langle W \rangle) + 1$ .

Now  $|W| < |V(G)|$  so, by the inductive hypothesis,  $P_2(\langle W \rangle) = \gamma(\langle W \rangle)$ . Then

$$\begin{aligned} \gamma(G) &\geq P_2(G) \\ &\geq P_2(\langle W \rangle) + 1 \\ &= \gamma(\langle W \rangle) + 1 \\ &\geq \gamma(G) \end{aligned}$$

where the first step follows from the Corollary 1.8.2. Hence for any connected block graph  $G$ ,  $\gamma(G) = P_2(G)$ . □

The results 1.8.2 and 1.8.4 have as their immediate consequence

**Corollary 1.8.5.** *If  $G$  is a connected block graph then*

$$P_2(G) = P_f(G) = \gamma_f(G) = \gamma(G),$$

and also, since every tree is a connected block graph,

**Corollary 1.8.6.** *If  $T$  is a tree then*

$$P_2(T) = P_f(T) = \gamma_f(T) = \gamma(T).$$

## 1.9 Gallai type results

Recall the definitions of the covering, independence, edge-covering and matching numbers in Section 1.2. Gallai [Gal59] proved that:

**Theorem 1.9.1.** *Let  $G$  be any graph and  $H$  be any graph with  $\delta(H) \geq 1$ . Then*

$$\begin{aligned}\alpha(G) + \beta(G) &= p(G), \\ \alpha^1(H) + \beta^1(H) &= p(H).\end{aligned}$$

For a proof of this theorem, see (for example) [CL86]. If  $S$  is a  $\beta$ -set of  $G$  then no edge in  $G$  can have both its ends in  $S$ . Thus the complement of  $S$ ,  $V(G) - S$ , must have the property that every edge has at least one end vertex in  $V(G) - S$ , and hence  $V(G) - S$  is a covering set of  $G$ . It is this complementarity which produces Theorem 1.9, and the question immediately arises as to whether a similar result holds for the fractional parameters  $\beta_f(G)$  and  $\alpha_f(G)$ . For this fractional case the concept of complementarity will be slightly wider: the complement of a function  $f : V(G) \mapsto [0, 1]$  analogous to the complement of a set will be achieved by defining a function  $f' : V(G) \mapsto [0, 1]$  as  $f'(v) = 1 - f(v)$  for any  $v \in V(G)$ . Grinstead and Slater [GS90] announced without proof the next two theorems; the proofs are my own. The theorems are split across several smaller results to make the proofs of some of the results which follow more straightforward.

### 1.9.1 A Gallai type theorem for $\alpha_f$ and $\beta_f$

**Lemma 1.9.2.** *Let  $G$  be a graph,  $g : V(G) \mapsto [0, 1]$  a fractional set function on  $G$ , and define a function  $g' : V(G) \mapsto [0, 1]$  as*

$$g'(v) = 1 - g(v),$$

for all  $v \in V(G)$ . Then

$$\begin{aligned} &g \text{ is a fractional covering function of } G \\ \iff &g' \text{ is a fractional independence function of } G. \end{aligned}$$

*Proof.* Let  $g : V(G) \mapsto [0, 1]$  be a fractional vertex covering function of  $G$ , and define  $g' : V(G) \mapsto [0, 1]$  as above. Since  $g$  is a fractional vertex covering function of  $G$ ,

$$uv \in E(G) \implies g(u) + g(v) \geq 1,$$

and hence for any  $uv \in E(G)$

$$\begin{aligned} g'(u) + g'(v) &= (1 - g(u)) + (1 - g(v)) \\ &= 2 - (g(u) + g(v)) \\ &\leq 1; \end{aligned}$$

so  $g'$  is a fractional vertex independence function. It is obvious from this proof and the fact that  $(g')'(v) = g(v)$  that the converse implication is true; i.e., if  $g'$  is a fractional independence function of  $G$  then  $g$  is a fractional covering function of  $G$ .  $\square$

We may now state the fractional version of Theorem 1.9.

**Theorem 1.9.3.** *If  $G$  is any graph then*

$$\alpha_f(G) + \beta_f(G) = p.$$

*Proof.* If  $g$  is an  $\alpha_f(G)$ -function, then  $g'$  is a fractional independence function and so  $\beta_f(G) \geq p - \alpha_f(G)$ . If  $f$  is an  $\beta_f(G)$ -function, then  $f'$  is a fractional covering function and so  $\alpha_f(G) \leq p - \beta_f(G)$ . Hence the result follows.  $\square$

An immediate consequence of this result is the following.

**Corollary 1.9.4.** *Let  $G$  be a graph,  $g : V(G) \mapsto [0, 1]$  a fractional set function on  $G$ , and define a function  $g' : V(G) \mapsto [0, 1]$  as*

$$g'(v) = 1 - g(v),$$

for all  $v \in V(G)$ . Then

$$|g| = \alpha_f(G) \iff |g'| = \beta_f(G).$$

### 1.9.2 A Gallai type theorem for $\alpha_f^1$ and $\beta_f^1$

If  $f : E(G) \mapsto [0, 1]$  is a fractional edge set function of a graph  $G$  with  $\text{iso}(G) = 0$  then we define the notation (for any  $v \in V(G)$ )

$$f[v] = \sum_{e \rightsquigarrow v} f(e).$$

The following result was stated without proof by Grinstead and Slater; the proof is due to work by Dankelmann and Swart [DS95] and myself.

**Theorem 1.9.5.** *Let  $G$  be any graph with  $\delta(G) \geq 1$ . Then*

$$\alpha_f^1(G) + \beta_f^1(G) = p$$

*Proof.* Let  $G$  be any graph with  $\delta(G) \geq 1$ .

1. We prove that  $\alpha_f^1(G) \leq p - \beta_f^1(G)$ . Let  $m : E(G) \mapsto [0, 1]$  be a  $\beta_f^1(G)$ -function,  $V^< = \{v \in V(G) : m[v] < 1\}$ , and for each  $v \in V^<$  choose an edge incident with  $v$ ; denote by  $e_v$  the edge chosen for vertex  $v$ , and let  $E_{V^<} = \{e_v : v \in V^<\}$ . Notice that since  $m$  is a maximal fractional matching function, each  $v \in V^<$  is adjacent only to vertices in  $V(G) - V^<$ : if this were not true, say there existed a  $u \in V^<$  and  $uv \in E(G)$ , then  $m[v], m[u] < 1$  and hence  $m(uv) < 1$ , and we could increase  $m(uv)$  to obtain a larger fractional matching function, which contradicts the maximality of  $m$ . Since each  $v \in V^<$  is adjacent only to vertices in  $V(G) - V^<$ , each element of  $E_{V^<}$  can be chosen at most once and thus  $|V^<| = |E_{V^<}|$ . Define a function  $c$  as

$$c(e) = \begin{cases} m(e) + 1 - m[v] & \text{if } e = e_v \in E_{V^<} \\ m(e) & \text{if } e \notin E_{V^<}. \end{cases}$$

Notice that if  $e \in E_{V^<}$ , say  $e = e_v$ , then  $0 \leq m(e_v) \leq m[v] < 1$ , and if  $e \notin E_{V^<}$  then for any  $v$  satisfying  $e \rightsquigarrow v$ ,  $0 \leq m(e) \leq m[v] \leq 1$ . Then  $m(e) + 1 - m[v] \leq 1$ . Notice that for any  $e \in E(G)$ ,  $c(e) \geq m(e)$ . Hence  $c : E(G) \mapsto [0, 1]$ , and also

- if  $v \in V^<$ , then

$$\begin{aligned} c[v] &= \sum_{e \rightsquigarrow v} c(e) \\ &= \sum_{e \rightsquigarrow v} m(e) + 1 - m[v] \\ &= m[v] + 1 - m[v] \\ &= 1. \end{aligned}$$

- if  $v \notin V^<$ , then

$$\begin{aligned}
c[v] &= \sum_{e \rightsquigarrow v} c(e) \\
&\geq \sum_{e \rightsquigarrow v} m(e) \\
&= 1.
\end{aligned}$$

Hence  $c$  is a fractional edge covering function of  $G$ , and

$$\begin{aligned}
\alpha_f^1(G) &\leq c(E) \\
&= \sum_{e \in E_{V^<}} c(e) + \sum_{e \notin E_{V^<}} c(e) \\
&= \sum_{e \in E_{V^<}} m(e) + \sum_{v \in V^<} (1 - m[v]) + \sum_{e \notin E_{V^<}} c(e) \\
&= m(E) + \sum_{v \in V(G)} (1 - m[v]) \\
&= m(E) + p - 2m(E) \\
&= p - \beta_f^1(G).
\end{aligned}$$

2. We prove that  $\beta_f^1(G) \geq p - \alpha_f^1(G)$ . Let  $c : E(G) \mapsto [0, 1]$  be an  $\alpha_f^1(G)$ -function and let  $m : E(G) \mapsto [0, 1]$  be a fractional matching function of  $G$  which is a maximum among those fractional matching functions  $m'$  which have the property that  $m' \leq c$ . Define the following sets:

$$\begin{aligned}
V_{=} &= \{v \in V(G) : c[v] = 1\} \\
V_{+} &= \{v \in V(G) : c[v] > 1\} \\
E_{+} &= \{e \in E(G) : c(e) > m(e)\}.
\end{aligned}$$

Note the following:

- (a)  $c$  is a minimal fractional edge-covering function of  $G$ , and hence if there is any edge  $e = uv \in E(G)$  where  $u, v \in V_{+}$ , then it must be the case that  $c(e) = 0$  (otherwise we could decrease  $c(e)$  to obtain a smaller fractional edge-covering function) and thus  $0 \leq m(e) \leq c(e) = 0$ .
- (b) Furthermore, all of the edges in  $E_{+}$  must belong to the set  $[V_{=}, V_{+}]$ : because of the previous discussion no edges in  $E_{+}$  can join two vertices in  $V_{+}$ , and if there were an edge  $uv \in E(\langle V_{=} \rangle)$  then  $m(uv) < c(uv) \leq 1$ , and  $m[u] < c[u] = 1$  and also  $m[v] < c[v] = 1$ . Hence we can increase  $m(uv)$  to obtain a larger fractional matching  $m'$  which still obeys  $m' \leq c$ ; this contradicts our choice of  $m$  and hence every edge in  $E_{+}$  joins a vertex in  $V_{=}$  and a vertex in  $V_{+}$ .

- (c) Every  $v \in V_+$  satisfies  $m[v] = 1$ . Suppose that there exists a vertex  $u \in V_+$  for which  $m[u] < 1$ . Then there must exist an edge  $uw \in E_+$  and hence  $w \in V_+$ . Since  $w \in V_+$  it follows that  $m[w] < c[w] = 1$  and we can increase  $m(uw)$  to obtain a larger fractional matching. This is a contradiction, and hence  $m[v] = 1$ .

Using these three observations, we can now evaluate

$$\begin{aligned}
m(E) + c(E) &= m(E) + \alpha_f^1(G) \\
&= \sum_{e \in [V_+, V_+]} (m(e) + c(e)) + \sum_{e \in [V_+, V_=]} (m(e) + c(e)) + \sum_{e \in [V_=, V_=]} (m(e) + c(e)) \\
&= 0 + \sum_{e \in [V_+, V_=]} (m(e) + c(e)) + \sum_{e \in [V_=, V_=]} 2c(e) \\
&= \left( \sum_{e \in [V_+, V_=]} c(e) + \sum_{e \in [V_=, V_=]} 2c(e) \right) + \sum_{e \in [V_+, V_=]} m(e) \\
&= \sum_{v \in V_=} c[v] + \sum_{v \in V_+} m[v] \\
&= \sum_{v \in V_=} 1 + \sum_{v \in V_+} 1 \\
&= p.
\end{aligned}$$

Hence

$$\begin{aligned}
\beta_f^1(G) &\geq m(E) \\
&= p - \alpha_f^1(G).
\end{aligned}$$

Combining these two results, the theorem follows. □

### 1.9.3 Other Gallai type results

The next theorem is due to Domke, Hedetniemi and Laskar [DHL88]; the proof has been modified.

**Theorem 1.9.6.** *If  $G$  is any graph with no isolated vertices, then*

$$\alpha_f^+(G) + \beta_f^-(G) = p.$$



*Proof.* Let  $g$  be a minimal fractional covering function chosen so that  $|g| = \alpha_f^+(G)$ . Define a fractional set function  $g' : V(G) \mapsto [0, 1]$  as

$$g'(v) = 1 - g(v)$$

for every  $v \in V(G)$ . By Lemma 1.9.2, since  $g$  is a fractional vertex covering function,  $g'$  must be a fractional vertex independence function. All that remains is to show that  $|g'| = \beta_f^-(G)$ : once this is proven, we can (by referring to the proof of Theorem 1.9.3), infer that

$$\alpha_f^+(G) + \beta_f^-(G) = p.$$

First of all, since  $g$  is a *minimal* fractional vertex covering function we assert that  $g'$  is a *maximal* fractional vertex independence function. Suppose to the contrary that this is not the case — then there is a function  $h$  derived from  $g'$  by increasing one of the function values  $g(w)$  of  $g'$  by a positive number  $\epsilon > 0$  while not decreasing any other function values of  $g'$ :

$$\begin{aligned} h(w) &= g'(w) + \epsilon, \\ h(v) &\geq g'(v), \end{aligned}$$

where  $v \in V(G)$ . Consider the function  $h'$ , defined as

$$h'(v) = 1 - h(v).$$

By Lemma 1.9.2, since  $h$  is by assumption a fractional vertex independence function,  $h'$  is a fractional vertex covering function. Furthermore

$$\begin{aligned} |h'| &= p - |h| \\ &< p - |g'| \\ &= |g| \end{aligned}$$

which contradicts the assumption that  $g$  is minimal since we could modify  $g$  to achieve  $h$  by lowering  $g(w)$  by  $\epsilon$  and thus obtaining a smaller fractional covering function. Thus no such  $h$  exists and hence  $g'$  is a minimal fractional independence function. It follows from Corollary 1.9.4 that  $|g'|$  must then be the maximum cardinality of the minimal vertex covering functions, and hence the theorem follows.  $\square$

## 1.10 Fractional set functions in graphs with symmetry

We can use symmetry in a graph to characterize certain fractional set functions. We begin by examining the class of complete multipartite graphs which possess a high

degree of internal symmetry.

### 1.10.1 Complete multipartite graphs

The following result is due to Grinstead and Slater [GS90] and, although implied by it, serves to introduce the more general automorphism class theorem which follows.

**Theorem 1.10.1.** *Let  $G = K_{m_1, \dots, m_t}$  be a complete multipartite graph. Then there exists a  $\gamma_f$ -function of  $G$  which is constant on each of the  $t$  partite sets.*

*Proof.* Denote by  $B_i$  the  $i^{\text{th}}$  partite set in  $G$ ; i.e.,  $|B_i| = m_i$ . Let  $g : V(G) \mapsto [0, 1]$  be a  $\gamma_f$ -function on  $G$  and consider the following function  $g' : V(G) \mapsto [0, 1]$  defined by:

$$g'(v) = \begin{cases} g(v) & v \in V(G) - B_1 \\ \frac{1}{m_1} \sum_{u \in B_1} g(u) & v \in B_1. \end{cases}$$

There are now two possibilities:

1. If  $v \notin B_1$  then  $N(v)$  contains all  $m_1$  elements of  $B_1$  and hence  $g'(N[v]) = g(N[v]) \geq 1$ .
2. Let  $u \in B_1$  be chosen such that  $g(u) = \min\{g(y) : y \in B_1\}$ . Define  $C = B_2 \cup \dots \cup B_t$ . Let  $v$  be any vertex in  $B_1$ , then

$$\begin{aligned} g'(N[v]) &= \sum_{w \in N[v]} g'(w) \\ &= g'(v) + \sum_{w \in C} g'(w) \\ &= g'(v) + \sum_{w \in C} g(w) \\ &\geq g(u) + \sum_{w \in C} g(w) \\ &= \sum_{w \in N[u]} g(w) \\ &\geq 1. \end{aligned}$$

Hence  $g'$  is a fractional dominating function of  $G$ .

Notice that because we are averaging over a subset of the vertices,

$$|g'| = |g| = \gamma_f(G).$$

We can of course repeat the algorithm: pick any partite set from  $\cup_{i=2}^t B_i$  and define a function  $g'' : V(G) \mapsto [0, 1]$  by

$$g''(v) = \begin{cases} g'(v) & v \in V(G) - B_2 \\ \frac{1}{m_2} \sum_{u \in B_2} g'(u) & v \in B_2. \end{cases}$$

By the same reasoning employed above we deduce that  $g''$  is a  $\gamma_f(G)$ -function. We can continue this process until we produce a  $\gamma_f(G)$ -function  $g^{(t)} : V(G) \mapsto [0, 1]$  which is constant on each of the  $t$  partite sets of  $G$ .  $\square$

The following result is due to Grinstead and Slater [GS90]. Recall that in the linear programming formulation of fractional set functions (Section 1.5),  $N(G)$  is the neighbourhood matrix of a graph  $G$ , and that fractional domination is achieved with any  $(p \times 1)$ -vector  $\vec{x}$  such that  $N\vec{x} \geq 1$ . We show next that for a complete multipartite graph  $G = K_{m_1, \dots, m_t}$  there is a  $\gamma_f$ -function,  $g$ , of  $G$  obeying the following two constraints:

1.  $g$  is constant on each of the  $t$  maximally independent sets in  $G$ . Each vertex  $v \in B_i$  has a weight  $g(v) = a_i$ .
2.  $g$  is such that  $N\vec{x} = 1$ .

**Theorem 1.10.2.** *Let  $K_{m_1, \dots, m_t}$  have partite sets  $B_1, \dots, B_t$  with  $|B_i| = m_i$ ,  $i = 1, \dots, t$ . Then there exists a  $\gamma_f$ -function  $g$  of  $K_{m_1, \dots, m_t}$  and numbers  $a_1, \dots, a_t$  such that  $g(v) = a_i$  for all  $v \in B_i$ ,  $g(N[v]) = 1$  for all  $v \in V(G)$ , and*

$$\gamma_f(K_{m_1, \dots, m_t}) = \sum_{i=1}^t a_i m_i.$$

where

$$a_1 = \frac{1}{1 + (m_1 - 1) \left[ \frac{m_2}{m_2 - 1} + \frac{m_3}{m_3 - 1} + \dots + \frac{m_t}{m_t - 1} \right]}$$

and  $a_i = \frac{m_1 - 1}{m_i - 1} a_1$  for  $i = 2, \dots, t$ .

*Proof.* We determine what the weights  $a_i$  must be: because of the symmetries in  $G$  and  $g$  the linear program can be simplified so that we consider one equality for each of the  $t$  maximally independent sets rather than one for each of the  $m_1 + m_2 + \dots + m_t$

vertices:

$$\begin{pmatrix} 1 & m_2 & m_3 & \cdots & m_t \\ m_1 & 1 & m_3 & \cdots & m_t \\ m_1 & m_2 & 1 & \cdots & m_t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & m_3 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \quad (1.10.1)$$

Subtracting row 1 from row  $i$ :

$$\begin{aligned} m_1 a_1 + m_2 a_2 + \cdots + a_i + \cdots + m_t a_t - a_1 - m_2 a_2 - \cdots - m_i a_i - \cdots - m_t a_t \\ = (m_1 - 1)a_1 + (1 - m_i)a_i \\ = 0 \end{aligned}$$

and thus

$$a_i = \frac{m_1 - 1}{m_i - 1} a_1. \quad (1.10.2)$$

Substituting back into (1.10.1) we obtain

$$\begin{pmatrix} 1 & m_2 & m_3 & \cdots & m_t \\ m_1 & 1 & m_3 & \cdots & m_t \\ m_1 & m_2 & 1 & \cdots & m_t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & m_3 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{m_1-1}{m_2-1} \\ \frac{m_1-1}{m_3-1} \\ \vdots \\ \frac{m_1-1}{m_t-1} \end{pmatrix} a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Multiplying out the first row we find that

$$a_1 = \frac{1}{1 + (m_1 - 1) \left[ \frac{m_2}{m_2-1} + \frac{m_3}{m_3-1} + \cdots + \frac{m_t}{m_t-1} \right]}. \quad (1.10.3)$$

Thus, given the set  $\{m_1, \dots, m_t\}$  we can determine the weights  $a_i$  on the partite sets  $B_i$ . The result follows immediately.  $\square$

The complete multipartite graph  $K_{2,3,3}$  is shown in Figure 1.8 together with a fractional dominating function which is constant on each of the partite sets.

An immediate corollary of this result follows for the case where  $m_1 = m_2 = \cdots = m_t$ .

**Corollary 1.10.3.** *Denote by  $K_n^t$  the complete balanced  $t$ -partite graph on  $nt$  vertices. Then,*

$$\gamma_f(K_n^t) = \frac{nt}{nt - n + 1}.$$

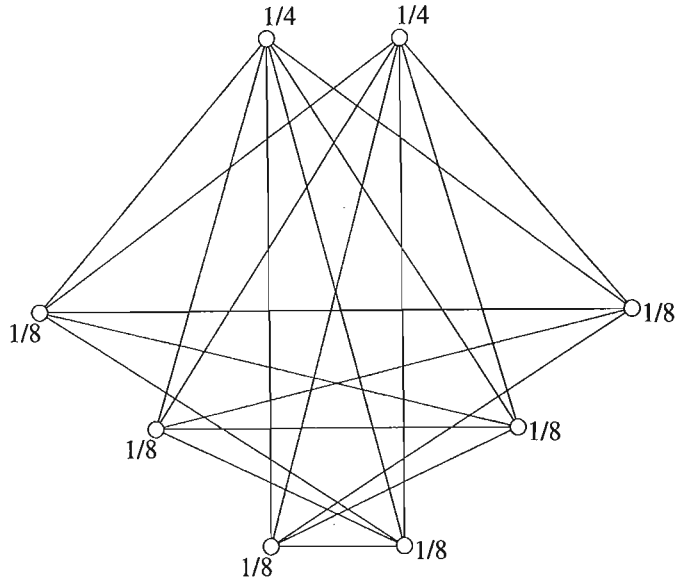


Figure 1.8: A fractional dominating function on  $K_{2,3,3}$  which is constant on each of the 3 partite sets.

*Proof.* From (1.10.3):

$$\begin{aligned} a_1 &= \frac{1}{1 + (n-1)\frac{n}{n-1}(t-1)} \\ &= \frac{1}{1 + n(t-1)}. \end{aligned}$$

And from (1.10.2)

$$a_i = \frac{n-1}{n-1} a_1 = a_1,$$

and so

$$\gamma_f(K_n^t) = \frac{nt}{1 + n(t-1)}.$$

□

### 1.10.2 The automorphism class theorem

The symmetry of the complete multipartite graphs has allowed us to produce the useful results 1.10.1, 1.10.2, 1.10.3; we can also characterize graphs not in this class provided that they possess some degree of symmetry. An *automorphism* of a graph  $G$  is an isomorphism of  $G$  with itself: a rearrangement of the vertex labels which

preserves adjacency. The set of all automorphisms of a graph  $G$  forms the *automorphism group* of  $G$ ,  $\mathbf{A}(G)$ . If  $\mathbf{A}(G) = \bigcup_i \phi_i$  and  $v \in V(G)$  then we call the set  $\{\phi_i(v) : \phi_i \in \mathbf{A}(G)\}$  the *automorphism class* of  $v$  and denote it by  $[v]$ . The following result is due to Grinstead and Slater [GS90].

**Theorem 1.10.4.** *Let  $G$  be any graph, and  $g : V(G) \mapsto [0, 1]$  a  $\gamma_f(G), P_f(G), R_f(G)$  or  $F_f(G)$  function. Then the fractional set function  $g' : V(G) \mapsto [0, 1]$ , defined as (for any  $v \in V(G)$ )*

$$g'(v) = \frac{1}{|[v]|} \sum_{w \in [v]} g(w),$$

*is also such a function.*

*Proof.* It is obvious that  $|g| = |g'|$  and (referring to (1.5.12)) that  $I(g) = I(g')$  since all the vertices in an automorphism class have the same degree. It is thus only necessary to demonstrate that the appropriate sums over vertex neighbourhoods satisfy the relevant inequalities for the fractional set-function  $g'$  under consideration. We must show the following to be true:

1.  $g(N[v]) \geq 1 \implies g'(N[v]) \geq 1$ , and,
2.  $g(N[v]) \leq 1 \implies g'(N[v]) \leq 1$ ,

where  $v$  is any vertex in  $V(G)$ . Let the automorphism classes of  $G$  be  $C_1, \dots, C_t$  with  $v = u_1 \in C_1$  and  $u_i \in C_i$  for  $2 \leq i \leq t$ . Let  $h_j = |N[v] \cap C_j|$  for  $1 \leq j \leq t$ , and let  $C_1 = \{v = u_1 = v_1, \dots, v_k\}$ . Note that, because each  $C_i$  is an automorphism class, for each fixed  $j$  we have  $h_j = |N[v_i] \cap C_j|$  for  $1 \leq i \leq k$  and if  $u_i$  and  $u'_i$  are distinct vertices in  $C_i$ , then  $|N(u_i) \cap C_1| = |N(u'_i) \cap C_1|$ . In particular, the number of edges connecting  $C_1$  and  $C_j$  for  $2 \leq j \leq t$  is  $kh_j = |N(u_j) \cap C_1| \cdot |C_j|$ . We have

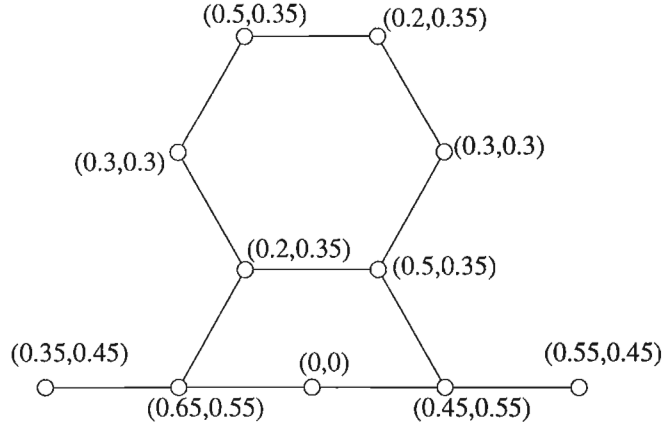


Figure 1.9: Application of the automorphism class theorem. The figures in brackets are (respectively) a  $\gamma_f$ -function of the graph in the figure, and another  $\gamma_f$ -function of that graph which has been derived from the first by averaging over the automorphism classes of the graph.

the following:

$$\begin{aligned}
k \sum_{w \in N[v_1]} g'(w) &= k \left( \sum_{w \in N[v_1] \cap C_1} g'(w) + \cdots + \sum_{w \in N[v_1] \cap C_t} g'(w) \right) \\
&= k(h_1 g'(u_1) + \cdots + h_t g'(u_t)) \\
&= \sum_{j=1}^t k h_j g'(u_j) \\
&= \sum_{j=1}^t |N[u_j] \cap C_1| |C_j| g'(u_j) \\
&= \sum_{j=1}^t |N[u_j] \cap C_1| |C_j| \sum_{w \in C_j} \frac{g(w)}{|C_j|} \\
&= \sum_{j=1}^t \sum_{w \in C_j} |N[u_j] \cap C_1| g(w) \\
&= \sum_{i=1}^k \sum_{w \in N[v_i]} g(w)
\end{aligned}$$

Now if  $\sum_{w \in N[v_i]} g(w) \geq 1$  it then follows that  $\sum_{w \in N[v_1]} g'(w) \geq 1$ , and if  $\sum_{w \in N[v_i]} g(w) \leq 1$  then  $\sum_{w \in N[v_1]} g'(w) \leq 1$ . Since our choice of  $v_1$  was arbitrary, this concludes the proof.  $\square$

An example of the application of this theorem is shown in Figure 1.9. Note that, as discussed before, Theorem 1.10.1 is a consequence of this result.

## 1.11 Fractional parameters and graph products

### 1.11.1 Definitions of graph products

Let  $G, H$  be graphs with vertex and edge sets  $V(G), V(H), E(G)$  and  $E(H)$  respectively; let  $P$  and  $Q$  be arbitrary matrices. We define the following products:

- The *cartesian product* of two graphs  $G$  and  $H$ , which we denote

$$G \times H,$$

has vertex set  $V(G \times H) = V(G) \times V(H)$  and edge set  $E(G \times H)$ . Two vertices  $(u, v)$  and  $(u', v')$  are adjacent in  $G \times H$  if and only if either

1.  $u = u'$  and  $vv' \in E(H)$ , or
2.  $v = v'$  and  $uu' \in E(G)$ .

- The *strong direct product* of two graphs  $G$  and  $H$ , which we denote

$$G \cdot H,$$

has vertex set  $V(G \cdot H) = V(G) \times V(H)$  and edge set  $E(G \cdot H)$ . Two vertices  $(u, v)$  and  $(u', v')$  are adjacent in  $G \cdot H$  if and only if either

1. Those vertices are adjacent in  $G \times H$ , or
2.  $uu' \in E(G)$  and  $vv' \in E(H)$ .

The cartesian and strong direct products are illustrated in Figure 1.10.

- The *tensor product* of two matrices  $P$  and  $Q$ , of dimension  $(m \times n)$  and  $(s \times t)$  respectively, which we denote

$$P \otimes Q,$$

is an  $(ms \times nt)$  matrix:

$$P \otimes Q = \begin{pmatrix} p_{11}Q & p_{12}Q & \cdots & p_{1n}Q \\ p_{21}Q & p_{22}Q & \cdots & p_{2n}Q \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}Q & p_{m2}Q & \cdots & p_{mn}Q \end{pmatrix}$$

where we denote by  $p_{ij}$  the elements of  $P$ .



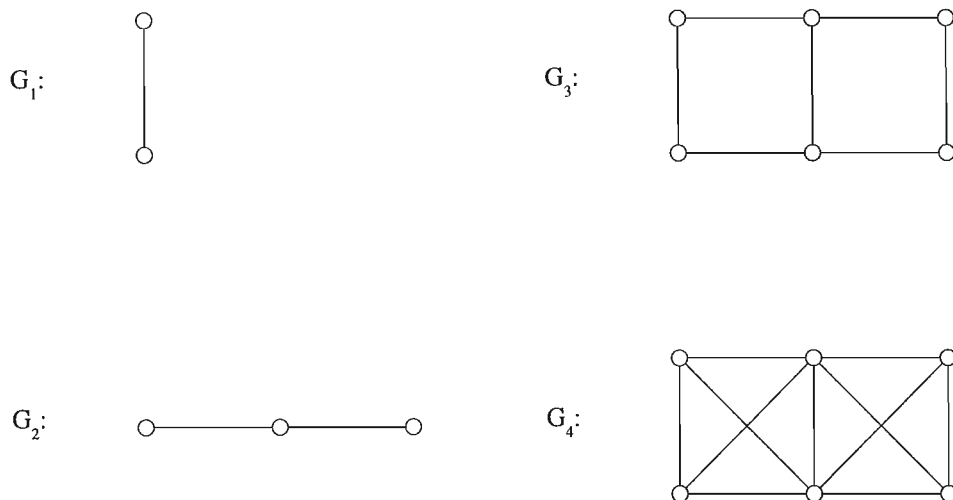


Figure 1.10: The graph  $G_3$  is the cartesian product  $G_3 = G_1 \times G_2$ , and the graph  $G_4$  is the strong direct product  $G_4 = G_1 \cdot G_2$ .

### 1.11.2 Vizing's conjecture for fractional domination

Vizing [Viz63] conjectured that, for all graphs  $G, H$ :

$$\gamma(G \times H) \geq \gamma(G)\gamma(H).$$

Vizing's conjecture is arguably the most well-known outstanding conjecture in domination theory. We shall prove a variation of Vizing's conjecture for fractional domination. Before we do, however, we need a preliminary result. This was stated without proof by Fisher, Ryan, Domke and Majumdar [FRDM94].

**Lemma 1.11.1.** *Let  $P$  be an  $(m \times n)$  matrix,  $Q$  be an  $(s \times t)$  matrix,  $\vec{x}, \vec{z}$  be  $(n \times 1)$ -vectors,  $\vec{y}, \vec{w}$  be  $(t \times 1)$ -vectors. Then*

1.  $(P \otimes Q)(\vec{x} \otimes \vec{y}) = (P\vec{x}) \otimes (Q\vec{y})$ .
2. If  $\vec{x} \geq \vec{z} \geq \vec{0}_n$  and  $\vec{y} \geq \vec{w} \geq \vec{0}_t$  then

$$\vec{x} \otimes \vec{y} \geq \vec{z} \otimes \vec{w}.$$

3. Let  $G$  and  $H$  be graphs. Then

$$N(G \cdot H) = N(G) \otimes N(H).$$

*Proof.*

1.

$$\begin{aligned}
(P \otimes Q)(\vec{x} \otimes \vec{y}) &= \begin{pmatrix} p_{11}Q & p_{12}Q & \cdots & p_{1n}Q \\ p_{21}Q & p_{22}Q & \cdots & p_{2n}Q \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}Q & p_{m2}Q & \cdots & p_{mn}Q \end{pmatrix} \begin{pmatrix} x_1\vec{y} \\ x_2\vec{y} \\ \vdots \\ x_n\vec{y} \end{pmatrix} \\
&= \begin{pmatrix} p_{11}x_1Q\vec{y} + p_{12}x_2Q\vec{y} + \cdots + p_{1n}x_nQ\vec{y} \\ p_{21}x_1Q\vec{y} + p_{22}x_2Q\vec{y} + \cdots + p_{2n}x_nQ\vec{y} \\ \cdots \\ p_{m1}x_1Q\vec{y} + p_{m2}x_2Q\vec{y} + \cdots + p_{mn}x_nQ\vec{y} \end{pmatrix} \\
&= \begin{pmatrix} p_{11}x_1 + p_{12}x_2 + \cdots + p_{1n}x_n \\ p_{21}x_1 + p_{22}x_2 + \cdots + p_{2n}x_n \\ \cdots \\ p_{m1}x_1 + p_{m2}x_2 + \cdots + p_{mn}x_n \end{pmatrix} \otimes (Q\vec{y}) \\
&= (P\vec{x}) \otimes (Q\vec{y}).
\end{aligned}$$

2.

$$\vec{x} \otimes \vec{y} = \begin{pmatrix} x_1y_1 \\ x_1y_2 \\ \vdots \\ x_ny_{t-1} \\ x_ny_t \end{pmatrix} \geq \begin{pmatrix} z_1w_1 \\ z_1w_2 \\ \vdots \\ z_nw_{t-1} \\ z_nw_t \end{pmatrix} = \vec{z} \otimes \vec{w}.$$

3. Suppose that  $|V(G)| = m$ ,  $|V(H)| = n$ . Then  $N(G \cdot H)$  is an  $(mn \times mn)$  matrix and so is  $N(G) \otimes N(H)$ . We need only verify that each element of  $N(G \cdot H)$  is the same as the corresponding element in  $N(G) \otimes N(H)$ . Let  $((u, v)(u', v'))$  denote an arbitrary element of  $N(G \cdot H)$ . There are two possibilities:

- $((u, v)(u', v')) = 0$ . Two sets of circumstances must hold:
  - (a) One or both of  $uu'$  and  $vv'$  is not in  $E(G)$  or  $E(H)$ , respectively. In this case one of  $N(G)_{uu'}$  or  $N(H)_{vv'}$  is zero and the corresponding element in  $N(G) \otimes N(H)$  is zero.
  - (b) If  $u = u'$  then  $vv' \notin E(H)$ . Hence  $N(H)_{vv'} = 0$  and the result follows. It follows similarly for the case when  $v = v'$ .

- $((u, v)(u', v')) = 1$ . Once again there are two sets of circumstances under which this is possible:
  - (a)  $uu' \in E(G)$  and  $vv' \in E(H)$ . Then both  $N(G)_{uu'}$  and  $N(H)_{vv'}$  are equal to one and the corresponding element in  $N(G) \otimes N(H)$  is one.
  - (b) If  $u = u'$  then  $vv' \in E(H)$ . Hence  $N(H)_{vv'} = 1$  and since all neighbourhood matrices have 1's down the main diagonal,  $N(G)_{uu'} = 1$ . The product of the two elements is thus one and the result follows. It follows similarly for the case when  $v = v'$ .

□

We are now ready to prove a result which has as its immediate consequence the fractional version of Vizing's conjecture. This is due to Fisher, Ryan, Domke and Majumdar [FRDM94].

**Theorem 1.11.2.** *Let  $G$  and  $H$  be any two graphs. Then*

$$\gamma_f(G \cdot H) = \gamma_f(G)\gamma_f(H).$$

*Proof.*

1. We first prove that  $\gamma_f(G \cdot H) \geq \gamma_f(G)\gamma_f(H)$ :

Let  $\begin{cases} \vec{x} \text{ be an optimal solution to the fractional packing problem (1.5.2) for } G, \\ \vec{y} \text{ be a optimal solution to the fractional packing problem (1.5.2) for } H. \end{cases}$

Then

$$\begin{aligned} N(G \cdot H)(\vec{x} \otimes \vec{y}) &= (N(G) \otimes N(H))(\vec{x} \otimes \vec{y}) \\ &= (N(G)\vec{x}) \otimes (N(H)\vec{y}) \\ &\leq \vec{1}_{p(G)} \otimes \vec{1}_{p(H)} \\ &= \vec{1}_{p(G)p(H)}. \end{aligned}$$

Hence  $\vec{x} \otimes \vec{y}$  is a feasible solution to the fractional packing problem for  $G \cdot H$ .  
Thus

$$\begin{aligned}
\gamma_f(G \cdot H) &= P_f(G \cdot H) \\
&\geq \bar{I}_{p(G)p(H)}^T(\vec{x} \otimes \vec{y}) \\
&= (\bar{I}_{p(G)}^T \otimes \bar{I}_{p(H)}^T)(\vec{x} \otimes \vec{y}) \\
&= (\bar{I}_{p(G)}^T \vec{x}) \otimes (\bar{I}_{p(H)}^T \vec{y}) \\
&= P_f(G)P_f(H) \\
&= \gamma_f(G)\gamma_f(H).
\end{aligned}$$

2. We now prove that  $\gamma_f(G \cdot H) \leq \gamma_f(G)\gamma_f(H)$ :

Let  $\begin{cases} \vec{z} \text{ be an optimal solution to the fractional domination problem (1.5.2) for } G, \\ \vec{w} \text{ be a optimal solution to the fractional domination problem (1.5.2) for } H. \end{cases}$

Then

$$\begin{aligned}
N(G \cdot H)(\vec{z} \otimes \vec{w}) &= (N(G) \otimes N(H))(\vec{z} \otimes \vec{w}) \\
&= (N(G)\vec{z}) \otimes (N(H)\vec{w}) \\
&\geq \bar{I}_{p(G)} \otimes \bar{I}_{p(H)} \\
&= \bar{I}_{p(G)p(H)}.
\end{aligned}$$

Hence  $\vec{z} \otimes \vec{w}$  is a feasible solution to the fractional domination problem for  $G \cdot H$ . Thus

$$\begin{aligned}
\gamma_f(G \cdot H) &\leq \bar{I}_{p(G)p(H)}^T(\vec{z} \otimes \vec{w}) \\
&= (\bar{I}_{p(G)}^T \otimes \bar{I}_{p(H)}^T)(\vec{z} \otimes \vec{w}) \\
&= (\bar{I}_{p(G)}^T \vec{z}) \otimes (\bar{I}_{p(H)}^T \vec{w}) \\
&= \gamma_f(G)\gamma_f(H).
\end{aligned}$$

Hence  $\gamma_f(G)\gamma_f(H) \leq \gamma_f(G \cdot H) \leq \gamma_f(G)\gamma_f(H)$  and the result follows.  $\square$

Since  $G \cdot H$  contains all of the edges of  $G \times H$  it is the case that any fractional dominating function of  $G \times H$  is also a fractional dominating function of  $G \cdot H$ . Hence  $\gamma_f(G \cdot H) \leq \gamma_f(G \times H)$ . The fractional version of Vizing's conjecture follows immediately.

**Corollary 1.11.3.** *Let  $G$  and  $H$  be any two graphs. Then*

$$\gamma_f(G \times H) \geq \gamma_f(G)\gamma_f(H).$$

The inequality in this Corollary 1.11.3 is sharp: for  $k \geq 2$  let  $G_k$  denote the set of graphs with vertex sets  $\{1, \dots, 2k\}$  where the subgraph induced by  $\{1, \dots, k\}$  has no isolated vertices, and for  $1 \leq i \leq k$  vertex  $i + k$  is connected only to vertex  $i$ . Then, for any  $G \in G_k$  and  $H \in G_l$  we have that  $\gamma_f(G) = k$ ,  $\gamma_f(H) = l$ , and  $\gamma_f(G \times H) = kl$ .

### 1.11.3 Other forms of Vizing's conjecture

We shall present forms of Vizing's conjecture involving the parameters  $P_2(G), \gamma_f(G)$  and  $\gamma(G)$ . In order to do this, however, we need to introduce a different formulation for  $\gamma(G \cdot H)$  and  $P_2(G \cdot H)$ . This formulation was presented without explanation by Fisher [Fis94]; the lengthy motivation below is my own. Using the linear programming formulation of Section 1.5, we can state the calculation of  $\gamma(G)$  and  $P_2(G)$  as integral programs:

$$\gamma(G) \quad \mathbf{P}_2(G) \quad (1.11.1)$$

$$\text{minimize } \vec{1}_p^t \cdot \vec{x} \quad \text{maximize } \vec{1}_p^t \cdot \vec{y} \quad (1.11.2)$$

$$\text{subject to } N\vec{x} \geq \vec{1}_p \quad \text{subject to } N\vec{y} \leq \vec{1}_p \quad (1.11.3)$$

$$\vec{0}_p \leq \vec{x} \leq \vec{1}_p \quad \vec{0}_p \leq \vec{y} \leq \vec{1}_p \quad (1.11.4)$$

$$\vec{x} \text{ integer} \quad \vec{y} \text{ integer} \quad (1.11.5)$$

Let  $\mathbb{Z}(m, n)$  denote the set of all  $(m \times n)$  matrices which have non-negative, integer elements. Let  $G, H$  be any two arbitrary graphs. Note that if  $g : V(G \times H) \mapsto \{0, 1\}$  is a  $\gamma(G)$ -function then we can record the weights assigned by  $g$  to the vertices of  $G \cdot H$  in a matrix  $Z \in \mathbb{Z}(p(G), p(H))$ , i.e. for any  $u_i \in V(G)$  and any  $v_j \in V(H)$ ,  $Z_{ij} = g((u_i, v_j))$ . It is then obvious that the weight of  $g$  is the sum of the elements in  $Z$ :

$$|g| = \vec{1}_{p(G)}^T Z \vec{1}_{p(H)}.$$

Let  $P = N(G)ZN(H)$ , noting that  $N(G), N(H), Z$  and  $P$  are  $(p(G) \times p(G)), (p(H) \times p(H)), (p(G) \times p(H))$  and  $(p(G) \times p(H))$  matrices, respectively, and denote their  $(i, j)$ <sup>th</sup> elements by  $g_{ij}, h_{ij}, z_{ij}$  and  $p_{ij}$ , respectively.

Then for  $m \in \{1, \dots, p(G)\}$  and  $n \in \{1, \dots, p(H)\}$ , recalling that  $N(G)$  and

$N(H)$  are symmetric matrices we have that

$$\begin{aligned} p_{mn} &= \sum_{i=1}^{p(G)} (g_{mi} \sum_{j=1}^{p(H)} z_{ij} h_{nj}) \\ &= \sum_{i=1}^{p(G)} \sum_{j=1}^{p(H)} z_{ij} g_{mi} h_{nj}. \end{aligned}$$

We note that, for  $i \in \{1, \dots, p(G)\}$ ,  $j \in \{1, \dots, p(H)\}$ , the product  $g_{mi} h_{nj}$  is non-zero (equal to one) if and only if  $g_{mi} = 1$  and  $h_{nj} = 1$ , which is the case if and only if one of the following four conditions is satisfied:

1.  $i \neq m, j \neq n$  and  $u_m u_i \in E(G), v_n v_j \in E(H)$ ;
2.  $i = m, j \neq n$  and  $v_n v_j \in E(H)$ ;
3.  $i \neq m, j = n$  and  $u_m u_i \in E(G)$ ;
4.  $i = m$  and  $j = n$ ;

i.e., if and only if  $(u_i, v_j) \in N_{G \cdot H}[(u_m, v_n)]$ . Hence

$$\begin{aligned} p_{mn} &= \sum_{(u_i, v_j) \in N_{G \cdot H}[(u_m, v_n)]} z_{ij} \\ &= \sum_{(u_i, v_j) \in N_{G \cdot H}[(u_m, v_n)]} g(u_i, v_j) \\ &= g(N_{G \cdot H}[(u_m, v_n)]). \end{aligned}$$

From this reworking it is immediately evident that

$$\gamma(G \cdot H) = \min_Z \vec{1}_m^T Z \vec{1}_n \tag{1.11.6}$$

subject to the restraint

$$N(G)ZN(H) \geq 1_{mn}, \quad Z \in \mathbb{Z}(m, n) \tag{1.11.7}$$

where  $1_{mn}$  denotes the  $(m \times n)$  matrix with unit entries, and

$$P_2(G \cdot H) = \max_Z \vec{1}_m^T Z \vec{1}_n \tag{1.11.8}$$

subject to the restraint

$$N(G)ZN(H) \leq 1_{mn}, \quad Z \in \mathbb{Z}(m, n). \tag{1.11.9}$$

This rephrasing of the question in terms of the matrices  $\mathbb{Z}(m, n)$  leads to the following theorem (see [Fis94]).

**Theorem 1.11.4.** *Let  $G$  and  $H$  be graphs, then*

$$\gamma(G)\gamma_f(H) \leq \gamma(G \cdot H) \leq \gamma(G)\gamma(H).$$

*Proof.*

- *We prove that  $\gamma(G)\gamma_f(H) \leq \gamma(G \cdot H)$ . Let  $Z \in \mathbb{Z}(m, n)$  be an optimal solution to the optimisation problem (1.11.6) and (1.11.7). Then*

$$N(G)ZN(H) \geq \mathbf{1}_{mn},$$

and thus the  $j^{\text{th}}$  column of the matrix satisfies

$$(N(G)ZN(H))_j \geq \vec{\mathbf{1}}_m.$$

Now

$$(N(G)ZN(H))_j = N(G)(ZN(H))_j$$

and thus  $(ZN(H))_j$  is the characteristic vector of a dominating set of  $G$ . Hence

$$\vec{\mathbf{1}}_m^T (ZN(H))_j \geq \gamma(G),$$

and so

$$\vec{\mathbf{1}}_m^T ZN(H) \geq \gamma(G)\vec{\mathbf{1}}_n.$$

Transposing we obtain

$$N(H)Z^T\vec{\mathbf{1}}_m \geq \gamma(G)\vec{\mathbf{1}}_n,$$

which after a little rearrangement becomes

$$N(H)\left(\frac{1}{\gamma(G)}Z^T\vec{\mathbf{1}}_m\right) \geq \vec{\mathbf{1}}_n.$$

Thus  $\frac{1}{\gamma(G)}Z^T\vec{\mathbf{1}}_m$  is the characteristic vector of a fractional dominating function of  $H$ . Then

$$\begin{aligned} \gamma_f(H) &\leq \vec{\mathbf{1}}_n^T \gamma^{-1}(G)Z^T\vec{\mathbf{1}}_m \\ &= \gamma^{-1}(G)\vec{\mathbf{1}}_n^T Z^T\vec{\mathbf{1}}_m \\ &= \gamma^{-1}(G)\gamma(G \cdot H) \end{aligned}$$

and hence the result follows.

- We prove that  $\gamma(G \cdot H) \leq \gamma(G)\gamma(H)$ . Let  $\vec{x}$  and  $\vec{y}$  be the  $(p(G) \times 1)$  and  $(p(H) \times 1)$  characteristic vectors of dominating sets of  $G$  and  $H$  with cardinality  $\gamma(G)$  and  $\gamma(H)$ , respectively. Then

$$\begin{aligned} N(G)\vec{x}\vec{y}^T N(H) &= (N(G)\vec{x})(N(H)\vec{y})^T \\ &\geq \vec{1}_m \vec{1}_n^T. \end{aligned}$$

Thus  $\vec{x}\vec{y}^T$  is a feasible solution of the optimisation problem (1.11.6) and (1.11.7):

$$\begin{aligned} \gamma(G \cdot H) &\leq \vec{1}_m^T \vec{x}\vec{y}^T \vec{1}_n \\ &= (\vec{1}_m^T \vec{x})(\vec{1}_n^T \vec{y}) \\ &= \gamma(G)\gamma(H), \end{aligned}$$

which is as required. □

There is an analogous result for the 2-packing number (see [Fis94]).

**Theorem 1.11.5.** *Let  $G$  and  $H$  be graphs, then*

$$P_2(G)P_2(H) \leq P_2(G \cdot H) \leq P_2(G)\gamma_f(H)$$

*Proof.*

- We prove that  $P_2(G)P_2(H) \leq P_2(G \cdot H)$ . Let  $\vec{x}$  and  $\vec{y}$  be the  $(p(G) \times 1)$  and  $(p(H) \times 1)$  characteristic vectors of 2-packing sets of  $G$  and  $H$  with cardinality  $P_2(G)$  and  $P_2(H)$ , respectively. Then

$$\begin{aligned} N(G)\vec{x}\vec{y}^T N(H) &= (N(G)\vec{x})(N(H)\vec{y})^T \\ &\leq \vec{1}_m \vec{1}_n^T. \end{aligned}$$

Thus  $\vec{x}\vec{y}^T$  is a feasible solution of the optimisation problem (1.11.8) and (1.11.9):

$$\begin{aligned} P_2(G \cdot H) &\geq \vec{1}_m^T \vec{x}\vec{y}^T \vec{1}_n \\ &= (\vec{1}_m^T \vec{x})(\vec{1}_n^T \vec{y}) \\ &= P_2(G)P_2(H), \end{aligned}$$

which is as required.



- We prove that  $P_2(G \cdot H) \leq P_2(G)\gamma_f(H)$ . Let  $Z \in \mathbb{Z}(m, n)$  be an optimal solution to the optimisation problem (1.11.8) and (1.11.9). Then

$$N(G)ZN(H) \leq 1_{mn},$$

and thus the  $j^{\text{th}}$  column of the matrix satisfies

$$(N(G)ZN(H))_j \leq \vec{1}_m.$$

Now

$$(N(G)ZN(H))_j = N(G)(ZN(H))_j$$

and thus  $(ZN(H))_j$  is the characteristic vector of a 2-packing set of  $G$ . Hence

$$\vec{1}_m^T(ZN(H))_j \leq P_2(G),$$

and so

$$\vec{1}_m^T ZN(H) \leq P_2(G)\vec{1}_n^T.$$

Transposing we obtain

$$N(H)Z^T\vec{1}_m \leq P_2(G)\vec{1}_n,$$

which after a little rearrangement becomes

$$N(H)\left(\frac{1}{P_2(G)}Z^T\vec{1}_m\right) \leq \vec{1}_n.$$

Hence  $P_2^{-1}(G)Z^T\vec{1}_m$  is the characteristic vector of a fractional packing function of  $H$  with value  $\vec{1}_n^T P_2^{-1}(G)Z^T\vec{1}_m$ . It follows from 1.5.6 that

$$\begin{aligned} \gamma_f(H) &= P_f(H) \\ &\geq \vec{1}_n^T P_2^{-1}(G)Z^T\vec{1}_m \\ &= P_2^{-1}(G)\vec{1}_n^T Z^T\vec{1}_m \\ &= P_2^{-1}(G)P_2(G \cdot H) \end{aligned}$$

and hence the result follows. □

The results we have obtained, when viewed together, give us a better picture of the relationships between the products of parameters of graphs and the parameters of products of graphs. All that we are missing is the relationship between the studied parameters and  $P_2(G \times H)$ ,  $\gamma_f(G \times H)$  and  $\gamma(G \times H)$ . Now since  $N(G \times H) \leq N(G \cdot H)$ , it follows that  $P_2(G \times H) \geq P_2(G \cdot H)$ ,  $\gamma_f(G \times H) \geq \gamma_f(G \cdot H)$  and  $\gamma(G \times H) \geq \gamma(G \cdot H)$ . The relationships are shown in Figure 1.11, which is taken from [Fis94].

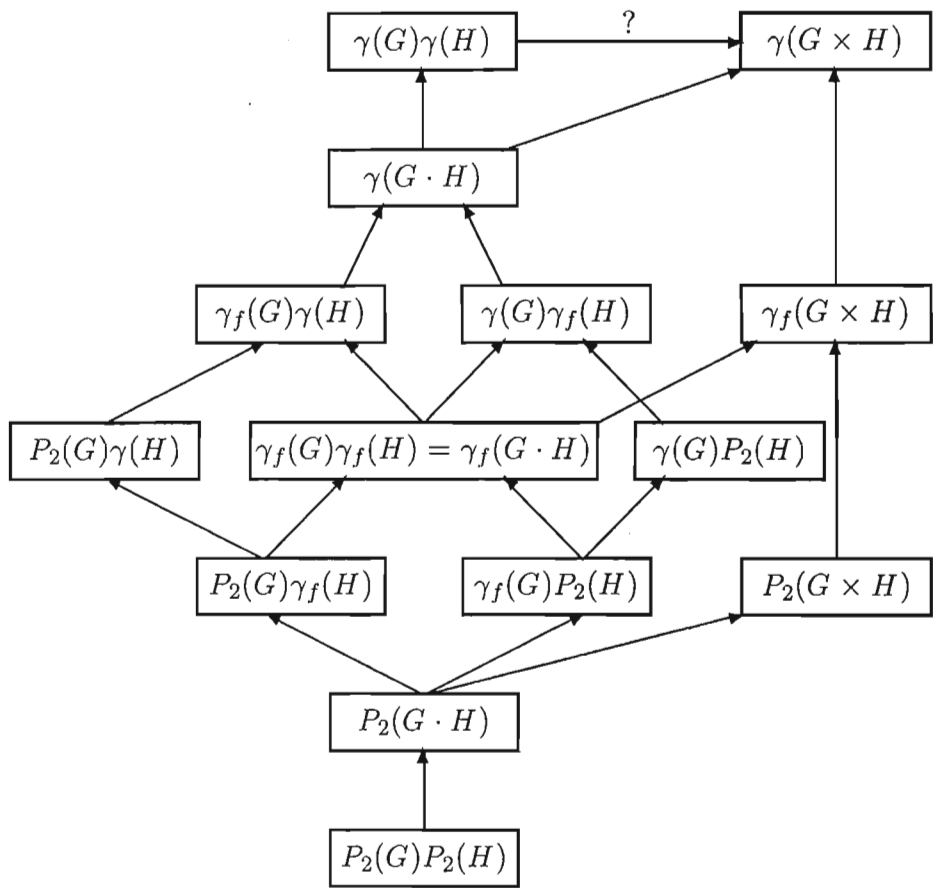


Figure 1.11: Relationships between the products of parameters and the parameters of products. Each arc indicates that the quantity at its head is greater than or equal to the quantity at its tail. The arc with the question mark is Vizing's conjecture. Where there is no *path* between two quantities, the two are independent: examples exist where each is larger than the other.

# Chapter 2

## $k$ -parameters

### 2.1 Definitions of $k$ -parameters

Let  $k$  be a fixed positive integer, then we call  $g : V(G) \mapsto \{0, 1, \dots, k\}$  a  $k$ -set function of  $G$ . We can define classes of  $k$ -set functions and related parameters analogous to those fractional set functions and parameters introduced in Chapter 1. The relationship between  $k$  parameters and fractional parameters has been studied by Domke, Hedetniemi, Laskar and Fricke [DHLF91].

#### 2.1.1 The $k$ -dominating function

The idea of  $k$ -domination was introduced by Hare [Har90] while trying to find values for the fractional domination number for  $P_m \times P_n$ , and has been studied by Domke, Hedetniemi, Laskar and Fricke [DHLF91]. We say that  $g : V(G) \mapsto \{0, \dots, k\}$  is a  *$k$ -dominating function of  $G$*  if

$$(\forall v \in V(G))g(N[v]) \geq k; \quad (2.1.1)$$

hence  $g$  is a *minimal  $k$ -dominating function* if, furthermore, for any  $v \in V(G)$

$$g(v) > 0 \implies \exists u \in N[v] \text{ such that } g(N[u]) = k. \quad (2.1.2)$$

Thus if  $S \subseteq V(G)$  is a dominating set of  $G$  then the characteristic function of  $S$  is a 1-dominating function of  $G$ .

The  *$k$ -domination number of  $G$* ,  $\gamma_k(G)$ , and *upper  $k$ -domination number of  $G$* ,  $\Gamma_k(G)$ , are defined by

$$\begin{aligned} \gamma_k(G) &= \min\{|g| : g \text{ is a minimal } k\text{-dominating function of } G\}, \\ \Gamma_k(G) &= \max\{|g| : g \text{ is a minimal } k\text{-dominating function of } G\}. \end{aligned}$$

It should be noted that  $k$ -domination is distinct from the concept of an  $n$ -dominating set (see [FJ85]) for which, if  $v \in V(G)$  and  $S$  is an  $n$ -dominating set of  $G$ ,  $v \notin S \implies |N(v) \cap S| \geq n$ .

### 2.1.2 The $k$ -packing function

We call  $g : V(G) \mapsto \{0, \dots, k\}$  a  $k$ -packing function of  $G$  if

$$(\forall v \in V(G))g(N[v]) \leq k.$$

A *maximal  $k$ -packing function* is a  $k$ -packing function for which, for any  $v \in V(G)$ ,

$$g(v) < k \implies \exists u \in N[v] \text{ such that } g(N[u]) = k.$$

The *lower  $k$ -packing number* of  $G$ ,  $\pi_k(G)$ , and *upper  $k$ -packing number* of  $G$ ,  $\Pi_k(G)$ , are defined by

$$\begin{aligned} \pi_k(G) &= \min\{|g| : g \text{ is a maximal } k\text{-packing function of } G\}, \\ \Pi_k(G) &= \max\{|g| : g \text{ is a maximal } k\text{-packing function of } G\}. \end{aligned}$$

### 2.1.3 The $k$ -irredundance function

We call  $g : V(G) \mapsto \{0, \dots, k\}$  a  $k$ -irredundant function if for any  $v \in V(G)$

$$g(v) > 0 \implies \exists u \in N[v] \text{ such that } g(N[u]) = k. \quad (2.1.3)$$

A *maximal  $k$ -irredundant function* of  $G$  is a  $k$ -irredundant function from which no larger  $k$ -irredundant function can be obtained by increasing the function values on some or all of the vertices. The  *$k$ -irredundance number* of  $G$ ,  $ir_k(G)$ , and *upper  $k$ -irredundance number* of  $G$ ,  $IR_k(G)$ , are defined as

$$\begin{aligned} ir_k(G) &= \min\{|g| : g \text{ is a maximal } k\text{-irredundant function of } G\}, \\ IR_k(G) &= \max\{|g| : g \text{ is a maximal } k\text{-irredundant function of } G\}. \end{aligned}$$

Comparing the conditions (2.1.2) and (2.1.3) we see that every minimal  $k$ -dominating function is also a  $k$ -irredundant function.

$k$ -packing and  $k$ -irredundant functions were introduced by Domke, Hedetniemi, Laskar and Fricke [DHLF91]. Examples of  $k$ -dominating,  $k$ -packing and  $k$ -irredundant functions are shown in Figure 2.1.

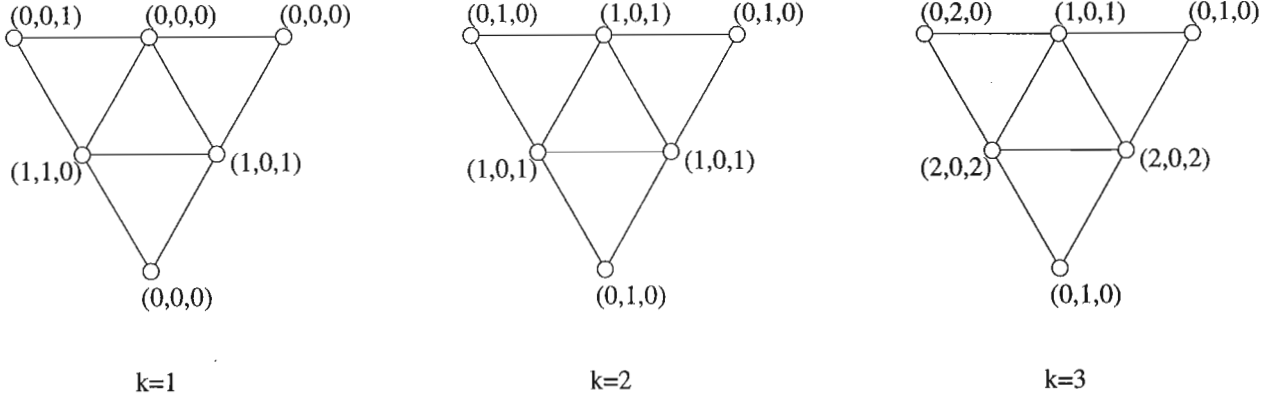


Figure 2.1: The figures in brackets show  $\gamma_k(G)$ ,  $\Pi_k(G)$  and  $ir_k(G)$  functions for the graph  $G$ , respectively; the functions in the left-most graph are for  $k = 1$ , the functions in the center graph are for  $k = 2$ , and those on the right are for  $k = 3$ . Hence:  $\gamma_1(G) = 2$ ;  $\gamma_2(G) = 3$ ;  $\gamma_3(G) = 5$ ;  $P_1(G) = 1$ ;  $P_2(G) = 3$ ;  $P_3(G) = 4$ ;  $ir_1(G) = 2$ ;  $ir_2(G) = 3$ ;  $ir_3(G) = 5$ .

## 2.2 Some relations between $k$ -set functions and well-known classes of sets

We now relate the  $k$ -set functions to some of the well-known parameters introduced in Chapter 1 and to each other. The results in this section are due to [DHLF91].

**Theorem 2.2.1.** *For every graph  $G$  and every positive integer  $k$ ,*

$$\gamma_k(G) \leq k \gamma(G) \leq k \Gamma(G) \leq \Gamma_k(G).$$

*Proof.* Let  $S$  be a minimal dominating set of vertices in  $G$ . Define a  $k$ -set function  $g : V(G) \mapsto \{0, \dots, k\}$  as  $(\forall v \in V(G))$ :

$$g(v) = \begin{cases} 0 & v \notin S \\ k & v \in S. \end{cases}$$

It is then obvious that  $g$  is a  $k$ -dominating function of  $G$ . We now show that this  $g$  must be a *minimal*  $k$ -dominating function of  $G$ . Assume to the contrary that there is a vertex  $v \in V(G)$  for which  $g(v) > 0$  and

$$\forall u \in N[v], g(N[u]) > k,$$

which implies — because  $g : V(G) \mapsto \{0, k\}$  — that for any  $u \in N[v]$ ,  $g(N[u]) \geq 2k$  and also  $g(v) = k$ . However, this implies that  $S' = S - \{v\}$  is a dominating set of  $G$ , and  $|S'| < |S|$  which contradicts the minimality of  $S$ . Hence  $g$  is a minimal  $k$ -dominating function and  $\gamma_k(G) \leq \min_g |g| = k\gamma(G)$ ; also  $\max_g |g| = k\Gamma(G) \leq \Gamma_k(G)$ , which establishes the required result.  $\square$

**Theorem 2.2.2.** *For every graph  $G$  and every positive integer  $k$ ,*

$$ir_k(G) \leq k\,ir(G) \leq k\,IR(G) \leq IR_k(G).$$

*Proof.* Let  $S$  be any maximal irredundant set of  $G$  and define a  $k$ -set function  $g : V(G) \mapsto \{0, \dots, k\}$  as  $\forall v \in V(G)$ :

$$g(v) = \begin{cases} 0 & v \notin S \\ k & v \in S. \end{cases}$$

Let  $v \in V(G)$ . If  $g(v) > 0$  then  $g(v) = k$  and  $v \in S$ . Now since  $S$  is an irredundant function there is a vertex  $u \in N[v]$  such that  $N[u] \cap S = \{v\}$ , and thus

$$\begin{aligned} g(N[u]) &= k|N[u] \cap S| + 0|N[u] - (N[u] \cap S)| \\ &= k, \end{aligned}$$

and so  $g$  is a  $k$ -irredundant function. We must now prove that  $g$  is a maximal  $k$ -irredundant function. Suppose that there is a  $k$ -irredundant function  $h$  obtained from  $g$  by increasing the function values of some or all of the vertices in  $G$ . Thus there is a  $v \in V(G)$  such that

$$\begin{aligned} g(v) &= 0, \\ h(v) &= m, \quad 0 < m \leq k. \end{aligned}$$

Now since  $h$  is a  $k$ -irredundant function there is a  $u \in N[v]$  satisfying  $h(N[u]) = k$ . So, since  $g(v) < h(v)$ , it follows that  $g(N[u]) < k$  and consequently  $g(N[u]) = 0$ . Hence  $N[u] \cap S = \emptyset$ . But, as  $S$  is a maximal irredundant set of  $G$ ,  $N[u] \subseteq N[S]$ , a contradiction. Then, since  $S$  was chosen to be an arbitrary maximal irredundant set of  $G$ , it follows that  $ir_k(G) \leq \min_g |g| = k\,ir(G)$ ; also  $\max_g |g| = k\,IR(G) \leq IR_k(G)$ , which establishes the required result.  $\square$

## 2.3 Inequalities between $\gamma_k(G)$ , $\Gamma_k(G)$ , $ir_k(G)$ , $IR_k(G)$

Obviously, for any graph  $G$ ,  $\gamma_k(G) \leq \Gamma_k(G)$  and  $ir_k(G) \leq IR_k(G)$ . We now address the relationship between the parameters  $\gamma_k(G)$ ,  $\Gamma_k(G)$  and  $ir_k(G)$ ,  $IR_k(G)$ . So as to order these parameters, we prove a little result first. The results in this section are due to [DHLF91].

**Lemma 2.3.1.** *Let  $G$  be any graph and  $k$  any positive integer; then every minimal  $k$ -dominating function of  $G$  is a maximal  $k$ -irredundant function of  $G$ .*

*Proof.* That every minimal  $k$ -dominating function of  $G$  is a  $k$ -irredundant function of  $G$  follows immediately from a comparison of (2.1.2) and (2.1.3). We must thus prove the maximality of the obtained  $k$ -irredundant function. Let  $g$  be a minimal  $k$ -dominating function of  $G$  and  $h$  a  $k$ -irredundant function of  $G$  such that for some  $u \in V(G)$ :

$$\begin{aligned} h(v) &\geq g(v), \quad \forall v \in V(G), \\ h(u) &= g(u) + m, \quad 0 < m \leq k. \end{aligned}$$

Since  $h$  is by assumption a  $k$ -irredundant function and  $h(u) > 0$ , there must be a vertex  $w \in N[u]$  such that  $h(N[w]) = k$ . However, for any  $w \in N[u]$  (since  $g$  is a  $k$ -dominating function),

$$\begin{aligned} h(N[w]) &\geq g(N[w]) + m \\ &\geq k + m \\ &> k \\ &= h(N[w]) \end{aligned}$$

which is a contradiction. Hence  $g$  is a maximal  $k$ -irredundant function. □

An immediate consequence of this Lemma 2.3.1 is the result

**Corollary 2.3.2.** *For any graph  $G$  and any positive integer  $k$ ,*

$$ir_k(G) \leq \gamma_k(G) \leq \Gamma_k(G) \leq IR_k(G).$$

## 2.4 Relationships between $k$ -set and $k + 1$ -set parameters

The results in this section are due to [DHLF91].

**Theorem 2.4.1.** *For any graph  $G$  and any positive integer  $k$ ,*

$$\gamma_k(G) < \gamma_{k+1}(G).$$

*Proof.* Let  $G$  be any graph and  $g : V(G) \mapsto \{0, \dots, k + 1\}$  a  $\gamma_{k+1}(G)$ -function. We consider two cases:

- *There exists  $w \in V(G)$  such that  $g(w) = k + 1$ . Define  $g' : V(G) \mapsto \{0, \dots, k\}$  as  $(\forall v \in V(G))$*

$$g'(v) = \begin{cases} k & \text{if } g(v) = k + 1 \\ g(v) & \text{if } g(v) \leq k. \end{cases}$$

Let  $v \in V(G)$ :

1. If there is a  $u \in N[v]$  such that  $g(u) = k + 1$ , then  $g'(u) = k$  and

$$\begin{aligned} g'(N[v]) &\geq g'(u) \\ &= k. \end{aligned}$$

2. If there is no  $u \in N[v]$  such that  $g(u) = k + 1$  then

$$\begin{aligned} g'(N[v]) &= g(N[v]) \\ &\geq k + 1 \\ &> k. \end{aligned}$$

Hence  $g'(N[v]) \geq k$  and  $g'$  is a  $k$ -dominating function of  $G$ ; also  $|g'| < |g|$  and so

$$\gamma_k(G) \leq |g'| < |g| = \gamma_{k+1}(G).$$

- *$g(v) \leq k$  for all  $v \in V(G)$ . Choose a vertex  $w \in V(G)$  such that  $g(w) > 0$  and define a function  $g'' : V(G) \mapsto \{0, \dots, k\}$  by*

$$g''(v) = \begin{cases} g(w) - 1 & \text{for } v = w \\ g(v) & \text{for } v \in V(G) - \{w\}. \end{cases}$$

Now  $g$  is a  $k + 1$ -dominating function, so for any  $v \in V(G)$ ,  $g(N[v]) \geq k + 1 > k$ .



1. If  $w \in N[v]$  then

$$\begin{aligned} g''(N[v]) &= g(N[v]) - 1 \\ &\geq k. \end{aligned}$$

2. If  $w \notin N[v]$  then

$$\begin{aligned} g''(N[v]) &= g(N[v]) \\ &\geq k + 1 \\ &> k. \end{aligned}$$

Thus for any  $v \in V(G)$ ,  $g''(N[v]) \geq k$  and  $g''$  is a  $k$ -dominating function of  $G$ ; also  $|g''| < |g|$ . Hence,

$$\gamma_k(G) \leq |g''| < |g| = \gamma_{k+1}(G).$$

□

**Theorem 2.4.2.** *Let  $G$  be any graph and  $k$  any positive integer. Then,*

$$\Pi_k(G) < \Pi_{k+1}(G).$$

*Proof.* Let  $g : V(G) \mapsto \{0, \dots, k\}$  be a  $\Pi_k(G)$ -function. Then it is obvious that  $g$  is also a  $k + 1$ -packing function of  $G$ , and

$$\begin{aligned} \Pi_{k+1}(G) &\geq |g| \\ &= \Pi_k(G). \end{aligned}$$

Let  $w \in V(G)$  and define a  $k + 1$ -set function  $g' : V(G) \mapsto \{0, \dots, k + 1\}$  as

$$g'(v) = \begin{cases} g(w) + 1 & \text{for } v = w \\ g(v) & \text{for } v \in V(G) - \{w\}. \end{cases}$$

Let  $v \in V(G)$ , then there are two cases to consider:

- If  $w \in N[v]$  then

$$\begin{aligned} g'(N[v]) &= g(N[v]) + 1 \\ &\leq k + 1. \end{aligned}$$

- If  $w \notin N[v]$  then

$$\begin{aligned} g'(N[v]) &= g(N[v]) \\ &\leq k \\ &< k + 1. \end{aligned}$$

Thus  $g'$  is a  $k + 1$ -packing function of  $G$ ; also  $|g'| > |g|$ . Thus

$$\Pi_{k+1}(G) \geq |g'| > |g| = \Pi_k(G).$$

□

## 2.5 Relating $k$ -parameters to fractional parameters

The results in this section are due to [DHLF91].

**Theorem 2.5.1.** *Let  $G$  be any graph and  $k$  any positive integer. Then*

$$\gamma_f(G) \leq \frac{\gamma_k(G)}{k} \leq \frac{\Gamma_k(G)}{k} \leq \Gamma_f(G).$$

*Proof.* To prove the first inequality, let  $g : V(G) \mapsto \{0, \dots, k\}$  be a  $\gamma_k(G)$ -function. Divide each weight of  $g$  by  $k$ : the result is a fractional dominating function of  $G$ . To prove the inequality for  $\Gamma_f(G)$  and  $\Gamma_k(G)$ , suppose that we had defined  $g$  to be a  $\Gamma_k(G)$ -function. The proof we have given is then as valid for this case as for the previously considered case with the single exception that we must prove that the function  $g'$  is a *minimal* fractional dominating function. Let  $v \in V(G)$  such that  $g'(v) > 0$ . Then  $g(v) > 0$  and there exists  $u \in N[v]$  such that

$$\begin{aligned} g'(N[u]) &= \frac{g(N[u])}{k} \\ &= \frac{k}{k} \\ &= 1, \end{aligned}$$

and hence  $g'$  is a minimal fractional dominating function of  $G$ ; thus

$$\frac{\Gamma_k(G)}{k} = |g'| \leq \Gamma_f(G).$$

□

We can prove a similar result for fractional packing and  $k$ -packing functions.

**Theorem 2.5.2.** *Let  $G$  be any graph and  $k$  any positive integer. Then*

$$p_f(G) \leq \frac{\pi_k(G)}{k} \leq \frac{\Pi_k(G)}{k} \leq P_f(G).$$

*Proof.* Similar to the proof of Theorem 2.5.1. □

From Theorems 2.5.1 and 2.5.2 and the result (1.5.6) we obtain the following useful result.

**Corollary 2.5.3.** *Let  $G$  be any graph and  $k$  any positive integer. Then,*

$$\Pi_k(G) \leq k P_f(G) = k \gamma_f(G) \leq \gamma_k(G).$$

**Theorem 2.5.4.** *Let  $G$  be any graph and  $k$  any positive integer. Then,*

$$IR_f(G) \geq \frac{IR_k(G)}{k}.$$

*Proof.* Let  $g : V(G) \mapsto \{0, \dots, k\}$  be a  $IR_k(G)$ -function. Define a fractional set function  $g' : V(G) \mapsto [0, 1]$  by (for all  $v \in V(G)$ )

$$g'(v) = \frac{g(v)}{k}.$$

Let  $v \in V(G)$ , and suppose that  $g'(v) > 0$ . Then  $g(v) > 0$  and there exists  $u \in N[v]$  such that

$$\begin{aligned} g'(N[u]) &= \frac{g(N[u])}{k} \\ &= \frac{k}{k} \\ &= 1, \end{aligned}$$

and hence  $g'$  is a fractional irredundance function of  $G$ ; thus

$$\frac{IR_k(G)}{k} = |g'| \leq IR_f(G).$$

□

We can obtain better results than the Theorems 2.5.1 and 2.5.2 relating the  $k$ -set functions to the fractional set functions.

**Theorem 2.5.5.** *Let  $G$  be any graph. Then*

$$\gamma_f(G) = \min\left\{\frac{\gamma_k(G)}{k} : k \text{ a positive integer}\right\}.$$

*Proof.* From Theorem 2.5.1 we know that, for any positive integer  $k$  and any graph  $G$ ,

$$\gamma_f(G) \leq \frac{\gamma_k(G)}{k}$$

and hence

$$\gamma_f(G) \leq \min\left\{\frac{\gamma_k(G)}{k} : k \text{ a positive integer}\right\}.$$

Recall from Section 1.5 that  $\gamma_f(G)$  can be determined using a linear program with rational coefficients and thus there exists an optimal solution to the linear program to calculate  $\gamma_f(G)$  which has rational function values. Let  $g : V(G) \mapsto [0, 1]$  be such an optimal solution; thus  $|g| = \gamma_f(G)$ . Let  $k'$  be the smallest integer for which  $k'g(v) \in \mathbb{Z}$  for all  $v \in V(G)$ , and define ( $\forall v \in V(G)$ )

$$h(v) = k'g(v).$$

Since, for any  $v \in V(G)$ ,  $0 \leq g(v) \leq 1$  it follows that  $0 \leq h(v) \leq k'$  and by construction  $h(v)$  is an integer and thus  $h : V(G) \mapsto \{0, \dots, k'\}$ . Since  $g$  is a fractional dominating function it is obvious that  $h$  is a  $k'$ -dominating function and

$$\begin{aligned} |g| &= \frac{|h|}{k'} \\ &\geq \frac{\gamma_{k'}(G)}{k'} \\ &\geq \min\left\{\frac{\gamma_k(G)}{k} : k \text{ a positive integer}\right\}. \\ &\geq \gamma_f(G) \\ &= |g|. \end{aligned}$$

Thus the result follows. □

**Theorem 2.5.6.** *Let  $G$  be any graph. Then*

$$P_f(G) = \max\left\{\frac{\Pi_k(G)}{k} : k \text{ a positive integer}\right\}.$$

*Proof.* Similar to the proof of 2.5.5. □

## 2.6 Equivalent $k$ -parameter theorems for some established fractional results

The results in this section are due to [DHLF91].

Recall the result 1.6.1 that

$$\gamma_f(G) = 1 \iff \Delta(G) = p - 1.$$

We can prove an equivalent result for  $k$ -parameters. This theorem was first proved by Domke, Hedetniemi, Laskar and Fricke [DHLF91], but the proof has been modified so as to produce the Corollary 2.6.2, which is not due to the above authors.

**Theorem 2.6.1.** *Let  $G$  be any graph and  $k$  any positive integer. Then*

$$\gamma_k(G) = k \iff \Delta(G) = p - 1.$$

*Proof.*

1. *First, we assume that  $\Delta(G) = p - 1$  and prove that  $\gamma_k(G) = k$ .* Let  $g$  be any  $k$ -dominating function of  $G$ . Since, for any vertex  $v \in V(G)$ ,  $g(N[v]) \geq k$ , it must be the case that  $|g| \geq k$  and hence, since  $g$  is arbitrary,  $\gamma_k(G) \geq k$ . Let  $u$  be a vertex in  $V(G)$  such that  $\deg u = \Delta(G) = p - 1$ , and define a  $k$ -set function  $h$  by

$$h(v) = \begin{cases} k & v = u \\ 0 & v \neq u. \end{cases}$$

Now  $N[u] = V(G)$  (so obviously  $h$  is a  $k$ -dominating function). Furthermore,  $|h| = k$ . Thus  $\gamma_k(G) \leq |h| = k \leq \gamma_k(G)$ , and hence  $\gamma_k(G) = k$ .

2. *Now, we assume that  $\gamma_k(G) = k$  and prove that  $\Delta(G) = p - 1$ .* Let  $v$  be any vertex in  $G$ , and let  $g$  be a  $k$ -dominating function of  $G$  such that  $|g| = \gamma_k(G) = k$ . Let  $P = \{v \in V(G) : g(v) > 0\}$ . Then, since  $\gamma_k(G) = k$ ,  $g(P) = k$ . Since  $g(N[v]) \geq k$  for all  $v \in V(G)$ , we must have that  $P \subseteq N[v]$ . Hence, if  $u \in P$ , then  $u$  is adjacent to every other vertex of  $G$ , so that  $\deg u = p - 1$ .

□

**Corollary 2.6.2.** *Let  $G$  be any graph with  $\Delta(G) = p - 1$ , and let  $g$  be any  $k$ -dominating function with  $|g| = \gamma_k(G)$ . Then only those vertices  $v \in V(G)$  with  $\deg v = p - 1$  satisfy  $g(v) > 0$ .  $\square$*

The following corollary is due to the authors of [DHLF91]. If  $g : V(G) \mapsto \{0, \dots, k\}$  is a maximal  $k$ -packing function then (for any  $v \in V(G)$ )  $g(v) < k$  implies that there is a  $u \in N[v]$  such that  $g(N[u]) = k$  and hence  $\Pi_k(G) \geq k$ . Similarly, if  $h$  is a maximal  $k$ -irredundant function then (for any  $v \in V(G)$ )  $h(v) > 0$  implies that there is a  $u \in N[v]$  such that  $h(N[u]) = k$  and hence  $ir_k(G) \geq k$ . Referring to the Corollaries 2.3.2 and 2.5.3 we see that

**Corollary 2.6.3.** *For any positive integers  $k, p$ ,*

$$ir_k(K_p) = \Pi_k(K_p) = \gamma_k(K_p) = k.$$

Domke, Hedetniemi and Laskar proved Theorem 1.8.1; i.e. that

$$p_f(G) \leq P_2(G) \leq P_f(G).$$

We can prove an analogous result for the  $k$ -parameters.

**Theorem 2.6.4.** *Let  $G$  be any graph and  $k$  any integer. Then*

$$\pi_k(G) \leq kP_2(G) \leq \Pi_k(G).$$

*Proof.* Let  $G$  be any graph and  $S \subseteq V(G)$  a  $P_2(G)$ -set. Define a  $k$ -set function  $g : V(G) \mapsto \{0, \dots, k\}$  as  $(\forall v \in V(G))$

$$g(v) = \begin{cases} 0 & v \notin S \\ k & v \in S. \end{cases}$$

Then

$$|g| = kP_2(G).$$

Now, any  $x, y \in S$  must satisfy the requirement that  $d(x, y) > 2$  and hence

$$N[x] \cap N[y] = \emptyset.$$

Thus, if  $v \in V(G)$ , then  $|N[v] \cap S| \leq 1$ , which implies that

$$\begin{aligned} g(N[v]) &= k|N[v] \cap S| \\ &\leq k, \end{aligned}$$

and hence  $g$  is a  $k$ -packing function of  $G$ . We claim that  $g$  is a maximal  $k$ -packing function of  $G$ ; there are three cases to consider:

- If  $v \in S$  then  $g(N[v]) = k$  and we cannot increase  $g(v)$ .
- If  $u \in S$  and  $v \in N(u)$  then  $g(u) = k$ ,  $g(v) = 0$ ,  $g(N[v]) = k$  and hence  $g(v)$  cannot be increased.
- If  $v \notin N[S]$  then  $d(u, v) \geq 2$  for all  $u \in S$ . Then it must be the case that there is a  $u \in S$  such that  $d(u, v) = 2$ , otherwise  $S' = S \cup \{v\}$  is a 2-packing set of  $G$  of larger cardinality than  $S$  which contradicts  $|S| = P_2(G)$ . Let  $w \in N(u) \cap N(v)$ , then by the discussion above  $g(N[w]) = k$  and hence we cannot increase  $g(v)$ .

Thus  $g$  is a maximal  $k$ -packing set of  $G$ , and hence

$$\pi_k(G) \leq |g| = kP_2(G) \leq \Pi_k(G).$$

□

From this result 2.6.4, the Corollary 2.5.3 and Theorem 2.2.1 we obtain the following:

**Corollary 2.6.5.**

$$k \leq \pi_k(G) \leq kP_2(G) \leq \Pi_k(G) \leq \gamma_k(G) \leq k\gamma(G).$$

From this result and Theorem 1.8.4 we get that

**Corollary 2.6.6.** *Let  $G$  be a connected block graph and  $k$  any positive integer. Then*

$$kP_2(G) = \Pi_k(G) = \gamma_k(G) = k\gamma(G).$$

By combining 1.6.5 with Theorems 2.2.1, 2.2.2, 2.5.1, 2.5.2, 2.5.4 and 2.6.4 we obtain the following chains of inequalities:

**Corollary 2.6.7.** *For any graph  $G$  and any positive integer  $k$ ,*

$$\begin{aligned} \frac{ir_k(G)}{k} \leq ir(G) \leq \gamma(G) \leq \beta^-(G) \leq \beta(G) \leq \Gamma(G) \leq \frac{\Gamma_k(G)}{k} \leq \Gamma_f(G) \leq IR(G) \\ \leq \frac{IR_k(G)}{k} \leq IR_f(G). \end{aligned}$$

**Corollary 2.6.8.** *For any graph  $G$  and any positive integer  $k$ ,*

$$\begin{aligned}
 1 \leq p_f(G) \leq \frac{\pi_k(G)}{k} \leq P_2(G) \leq \frac{\Pi_k(G)}{k} \leq P_f(G) = \gamma_f(G) \leq \frac{\gamma_k(G)}{k} \leq \gamma(G) \leq \beta^-(G) \\
 \leq \beta(G) \leq \Gamma(G) \leq \frac{\Gamma_k(G)}{k} \leq \Gamma_f(G) \leq IR(G) \leq \frac{IR_k(G)}{k} \leq IR_f(G).
 \end{aligned}$$



# Chapter 3

## $e = 1$ functions

### 3.1 Introduction

Except where noted otherwise, all of the work in this chapter is original. Let  $G = (V, E)$  be a graph. We say that a fractional set function  $f : V(G) \mapsto [0, 1]$  is an  $e = 1$  function of  $G$  (which we shall call just an  $e = 1$  function if no ambiguity is possible) if

$$\forall v \in V(G) \begin{cases} \deg v = 0 & \implies f(v) = 1, \\ \deg v > 0 & \implies \exists u \in N(v) \text{ such that } f(u) + f(v) = 1. \end{cases}$$

The rationale behind the terminology ‘ $e = 1$  function’ is that, for any  $e = 1$  function  $g : V(G) \mapsto [0, 1]$ , each non-isolated vertex  $v$  is incident with some edge  $e = uv$  such that the sum of  $g(u)$  and  $g(v)$  (which can be thought of as the sum ‘over the edge’,  $e$ ) is equal to 1. The concept of an  $e = 1$  function was introduced by Fricke and first considered by Fricke and Swart [FS95].

$f$  is a *maximal*  $e = 1$  function if no  $e = 1$  function  $g$  exists such that

$$\begin{aligned} g(v) &\geq f(v) \quad \forall v \in V(G), \\ g(u) &= f(u) + m \quad \text{for some } 0 < m \leq 1 \text{ and some } u \in V(G). \end{aligned}$$

$f$  is a *minimal*  $e = 1$  function if no  $e = 1$  function  $g$  exists such that

$$\begin{aligned} g(v) &\leq f(v) \quad \forall v \in V(G), \\ g(u) &= f(u) - m \quad \text{for some } 0 < m \leq 1 \text{ and some } u \in V(G). \end{aligned}$$

Figure 3.1 shows two  $e = 1$  functions for  $K_3$ : the first is maximal and the second minimal. We define

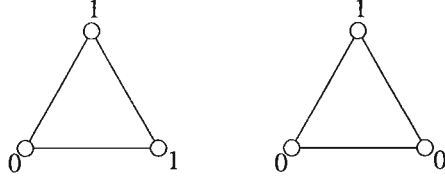


Figure 3.1: Maximal and minimal  $e = 1$  functions for  $K_3$

$$\gamma_{\overline{e=1}}(G) = \min\{|f| : f \text{ is a maximal } e = 1 \text{ function}\}, \quad (3.1.1)$$

$$\Gamma_{\overline{e=1}}(G) = \max\{|f| : f \text{ is a maximal } e = 1 \text{ function}\}, \quad (3.1.2)$$

$$\gamma_{\underline{e=1}}(G) = \min\{|f| : f \text{ is a minimal } e = 1 \text{ function}\}, \quad (3.1.3)$$

$$\Gamma_{\underline{e=1}}(G) = \max\{|f| : f \text{ is a minimal } e = 1 \text{ function}\}. \quad (3.1.4)$$

Notice that every minimal fractional vertex covering function of a graph  $G$  with  $\delta(G) \geq 1$  is an  $e = 1$  function (see Chapter 1).

## 3.2 Observations

1. If  $G$  is empty then, for any  $e = 1$  function  $f$  and  $\forall v \in V(G)$ ,  $f(v) = 1$ , and thus

$$\gamma_{\overline{e=1}}(G) = \gamma_{\underline{e=1}}(G) = \Gamma_{\overline{e=1}}(G) = \Gamma_{\underline{e=1}}(G) = p. \quad (3.2.1)$$

2. If  $G$  is non-empty then  $\exists uv \in E(G)$  such that  $f(u) + f(v) = 1$ . Since for any  $w \in V(G)$ ,  $0 \leq f(w) \leq 1$ , we have

$$\begin{aligned} |f| &\leq 1 + p - 2 \\ &= p - 1. \end{aligned}$$

Therefore,

$$\Gamma_{\overline{e=1}}(G) \leq p - 1. \quad (3.2.2)$$

Now let  $S$  be a  $\beta(G)$ -set where  $\beta(G)$  is the cardinality of a maximum independent set. Define a fractional set function  $f : V(G) \mapsto [0, 1]$  by

$$f(v) = \begin{cases} 0 & v \in S \text{ and } \deg v > 0 \\ 1 & v \notin S \text{ or } \deg v = 0. \end{cases}$$

- $p_i > 1 \implies \Gamma_{\underline{e=1}}(\langle V_i \rangle) \leq p_i - 1$ ,
- $p_i = 1 \implies \Gamma_{\underline{e=1}}(\langle V_i \rangle) = 1$ ,

and thus

$$\begin{aligned}
\Gamma_{\underline{e=1}}(G) &= \sum_{i=1}^{k(G)} \Gamma_{\underline{e=1}}(\langle V_i \rangle) \\
&\leq \sum_{i=1}^{\text{niso}(G)} p_i - \text{niso}(G) + \text{iso}(G) \\
&= p - \text{iso}(G) + \text{iso}(G) - \text{niso}(G) \\
&= p - \text{niso}(G).
\end{aligned}$$

It is obvious that since each component, isolated or otherwise, contributes at least one to the value of any  $e = 1$  function, that  $\gamma_{\underline{e=1}}(G) \geq k(G)$ . This completes the proof.  $\square$

Our chain of inequalities is now

$$k(G) \leq \gamma_{\underline{e=1}}(G) \leq \gamma_{\underline{e=1}}(G) \leq \frac{p(G) + \text{iso}(G)}{2} \leq \Gamma_{\underline{e=1}}(G) \leq \Gamma_{\underline{e=1}}(G) \leq p - \text{niso}(G).$$

**Theorem 3.3.2.** *For any graph  $G$ ,*

$$\gamma_{\underline{e=1}}(G) \leq \gamma(G) \leq \Gamma(G) \leq \Gamma_{\underline{e=1}}(G).$$

*Proof.* It is sufficient to prove the statement for the case when  $\text{iso}(G) = 0$  and to show that every minimal dominating set of  $G$  induces a minimal  $e = 1$  function. Let  $S$  be a minimal dominating set of  $G$ , and let  $f$  be the characteristic function of  $S$ :

$$f(v) = \begin{cases} 0 & v \notin S \\ 1 & v \in S. \end{cases}$$

Then, for  $v \in V(G)$ , either  $f(v) = 0$  or  $f(v) = 1$ :

1. If  $f(v) = 0$ , then  $v \notin S$ , but since  $S$  is a dominating set there is a vertex  $u \in S \cap N(v)$ . Since  $u \in S$ ,  $f(u) = 1$  and hence  $f(u) + f(v) = 1$ .
2. If  $f(v) = 1$ , then  $v \in S$  and because  $S$  is minimal and  $\deg v > 0$  there must be a vertex  $u \in (V(G) - S) \cap N(v)$ . Since  $u \notin S$ ,  $f(u) = 0$  and hence  $f(u) + f(v) = 1$ .

Thus,  $f$  is an  $e = 1$  function. We must now show that  $f$  is a minimal  $e = 1$  function. Suppose to the contrary that there exists a vertex  $u \in V(G)$ , a number  $0 < m \leq 1$ , and an  $e = 1$  function  $f' : V(G) \mapsto [0, 1]$ , such that,  $(\forall v \in V(G) - \{u\})$ ,

$$\begin{aligned} f'(u) &= f(u) - m, \\ f'(v) &\leq f(v). \end{aligned}$$

Since  $f$  assigns only the values 0 and 1 and  $0 \leq f'(u) < f(u)$ , it must be the case that  $f(u) = 1$ ; thus  $f'(u) = 1 - m$ , also  $u \in S$  and  $u$  is not isolated in  $G$  by assumption. Since  $S$  is a minimal dominating set there must be a private neighbour  $w \in V(G)$  such that  $S \cap N[w] = \{u\}$ . Let  $x$  be any vertex in  $N(w)$ , then:

$$\begin{aligned} f'(x) + f'(w) &\leq f'(N[w]) \\ &= f'(u) + \sum_{y \in N[w] - \{u\}} f'(y) \\ &\leq 1 - m + \sum_{y \in N[w] - \{u\}} f(y) \\ &= 1 - m + 0 \\ &< 1, \end{aligned}$$

and thus  $f'$  is not an  $e = 1$  function. Hence no such function  $f'$  exists and  $f$  is minimal.  $\square$

Notice that the next theorem follows directly from 3.4.1.

**Theorem 3.3.3.**

$$\gamma_f(G) \leq \gamma_{e=1}(G).$$

*Proof.* It is sufficient to show that every  $e = 1$  function is a fractional dominating function of  $G$ . Let  $f : V(G) \mapsto [0, 1]$  be an  $e = 1$  function. Now:

1.  $\text{deg}v = 0 \implies f(v) = 1 \implies f(N[v]) = 1$ ,
- 2.

$$\begin{aligned} \text{deg}v > 0 &\implies \exists u \in N[v] \text{ such that } f(u) + f(v) = 1 \\ &\implies f(N[v]) \geq 1. \end{aligned}$$

Thus, for any  $v \in V(G)$ ,  $f(N[v]) \geq 1$  and thus  $f$  is a fractional dominating function.  $\square$

It is not true that  $\Gamma_{\overline{e=1}}(G) \leq \Gamma_f(G)$ : in the graph  $K_3$ , any fractional dominating function must have total weight 1 and hence  $\Gamma_f(K_3) = 1$ ; however, by setting the function values of two of the three vertices equal to one and the other equal to zero we obtain an  $e = 1$  function of weight 2, and thus  $\Gamma_{\overline{e=1}}(K_3) \geq 2$ . The converse appears to be true:

**Conjecture 3.3.4.** *Let  $G$  be any graph. Then*

$$\Gamma_f(G) \leq \Gamma_{\overline{e=1}}(G).$$

We now have a second chain of inequalities:

$$\gamma_f(G) \leq \gamma_{\overline{e=1}}(G) \leq \gamma(G) \leq \Gamma(G) \leq \Gamma_{\overline{e=1}}(G). \quad (3.3.1)$$

### 3.4 The relationship between $\gamma$ and $\gamma_{\overline{e=1}}$

**Theorem 3.4.1.** *For any graph  $G$ ,*

$$\gamma_{\overline{e=1}}(G) = \gamma(G).$$

*Proof.* Since  $\gamma_{\overline{e=1}}(K_1) = 1 = \gamma(K_1)$ , it suffices to prove the statement for graphs  $G$  with  $\text{iso}(G) = 0$ . Let  $G$  be any graph with  $\text{iso}(G) = 0$ , and let  $f : V(G) \mapsto [0, 1]$  be a  $\gamma_{\overline{e=1}}(G)$ -function. Let  $E^=(G) = \{uv \in E(G) : f(u) + f(v) = 1\}$  and  $G' = \langle E^=(G) \rangle$ ; note that  $V(G') = V(G)$ . Denote by  $G'_i, i = 1, \dots, k(G')$  the components of  $G'$  and let  $V'_i = V(G'_i), p_i = |V'_i|, i = 1, \dots, k(G')$ . We observe that

$$|f| = \sum_{i=1}^{k(G')} f(V'_i),$$

and that  $f|_{V'_i}$ , the restriction of  $f$  to  $V'_i$ , is a  $\gamma_{\overline{e=1}}$  function of  $G'_i$  for  $i = 1, \dots, k(G')$ . Now let  $i \in \{1, \dots, k(G')\}, u \in V'_i$  and  $f(u) = a$ . Then  $f(v) = 1 - a$  if  $v \in N_{G'}(u)$ ,  $f(w) = a$  if  $w \in N_{G'}(v)$  and, in general, for  $v \in V'_i, f(v) = a$  (or  $f(v) = 1 - a$ ) if  $d_{G'}(u, v)$  is even (or  $d_{G'}(u, v)$  is odd, respectively).

We consider two cases:  $a = \frac{1}{2}$  and  $a \neq \frac{1}{2}$ .

- If  $a = \frac{1}{2}$ , then

$$\begin{aligned} \gamma_{\overline{e=1}}(G'_i) &= \sum_{v \in V'_i} f(v) \\ &= \frac{p_i}{2} \\ &\geq \gamma(G'_i). \end{aligned}$$

Hence we have shown that for any graph  $G$  there exists a  $\gamma_{e=1}(G)$ -function  $f : V(G) \mapsto [0, 1]$  such that  $f(v) \in \{0, 1\}$  for all  $v \in V(G)$ ; so  $f$  is the characteristic function of a set  $F \subseteq V(G)$ . We note that  $N[F] = V(G)$ ; i.e.,  $F$  is a dominating set of  $G$ , whence it follows that, by equation (3.3.1),

$$\begin{aligned}\gamma(G) &\leq |F| \\ &= \gamma_{e=1}(G) \\ &\leq \gamma(G)\end{aligned}$$

and the required result follows. □

### 3.5 Some Gallai type results

Let  $f : V(G) \mapsto [0, 1]$  be a fractional set function and define the function  $f_{not} : V(G) \mapsto [0, 1]$  as

- $\deg v = 0 \implies f_{not}(v) = 1$ ,
- $\deg v > 0 \implies f_{not}(v) = 1 - f(v)$ .

This leads us to the following result.

**Lemma 3.5.1.** *Let  $f : V(G) \mapsto [0, 1]$  be a fractional set function. Then*

$$f \text{ is an } e = 1 \text{ function} \iff f_{not} \text{ is an } e = 1 \text{ function.}$$

*Proof.* Since  $f_{not}(f_{not}(v)) = f(v)$ , it is sufficient to show that if  $f$  is an  $e = 1$  function then  $f_{not}$  is an  $e = 1$  function. Assume  $f$  is an  $e = 1$  function, then:

1. If  $\deg v = 0$  then  $f_{not}(v) = 1$ .
2. If  $\deg v > 0$  then there is a vertex  $u \in N(v)$  such that  $f(v) + f(u) = 1$ , and hence

$$\begin{aligned}f_{not}(v) + f_{not}(u) &= 1 - f(v) + 1 - f(u) \\ &= 1.\end{aligned}$$

Hence  $f_{not}$  is an  $e = 1$  function and the result follows. □

This can be rewritten, using Theorem 3.4.1, as

**Corollary 3.5.5.** *For any graph  $G$ ,*

$$\gamma(G) + \Gamma_{e=1}(G) = p(G) + iso(G).$$

**Theorem 3.5.6.** *Let  $f : V(G) \mapsto [0, 1]$  be an  $e = 1$  function of a graph  $G$ . Then*

$$f \text{ is a minimal } e = 1 \text{ function} \iff f_{not} \text{ is a maximal } e = 1 \text{ function.}$$

*Proof.* It suffices to prove the result for the case where  $iso(G) = 0$ . Let  $f$  be a minimal  $e = 1$  function. Suppose to the contrary that  $f_{not}$  is not a maximal  $e = 1$  function, which means that there exist a vertex  $u \in V(G)$ , an  $e = 1$  function  $f'$  and a number  $0 < m \leq 1$  such that, for any  $v \in V(G)$

$$\begin{aligned} f'(u) &= f_{not}(u) + m, \\ f'(v) &\geq f_{not}(v); \end{aligned}$$

so  $|f'| > |f_{not}|$ . Since  $f'$  is an  $e = 1$  function,  $f'_{not}$  is an  $e = 1$  function (Lemma 3.5.1). Now:

$$\begin{aligned} f'_{not}(u) &= 1 - f'(u) \\ &< 1 - f_{not}(u) \\ &= f(u), \end{aligned}$$

and also, for  $v \in V(G) - \{u\}$ ,

$$\begin{aligned} f'_{not}(v) &= 1 - f'(v) \\ &\leq 1 - f_{not}(v) \\ &= f(v). \end{aligned}$$

This contradicts our assumption of the minimality of  $f$ . Hence no such  $f'$  exists and hence  $f_{not}$  is maximal and the first part of the theorem follows. The proof of the second part of the theorem, the converse implication of the first, is structurally identical to this proof. Hence the theorem follows.  $\square$

**Theorem 3.5.7.** *Let  $f : V(G) \mapsto [0, 1]$  be an  $e = 1$  function of a graph  $G$ . Then*

$$|f| = \gamma_{\overline{e=1}}(G) \iff |f_{not}| = \Gamma_{\underline{e=1}}(G).$$

*Proof.* Let  $f$  be a  $\gamma_{\overline{e=1}}(G)$ -function, i.e.  $f$  has the smallest value of all of the maximal  $e = 1$  functions. Since  $f$  is a maximal  $e = 1$  function,  $f_{not}$  is a minimal  $e = 1$  function (Theorem 3.5.6). Suppose to the contrary that  $|f_{not}| < \Gamma_{\underline{e=1}}(G)$ . Let  $g$  be a  $\Gamma_{\underline{e=1}}(G)$ -function and (so as not to contradict the minimality of  $|f|$ )  $|g_{not}| \geq |f| = \gamma_{\overline{e=1}}(G)$ , remembering that  $g$  is a minimal  $e = 1$  function if and only if  $g_{not}$  is a maximal  $e = 1$  function. Then

$$\begin{aligned} |g| + |g_{not}| &> |f_{not}| + |f| \\ &= p + iso(G) \end{aligned}$$

which is impossible if  $g$  is an  $e = 1$  function. Hence no such  $g$  exists and the theorem follows.  $\square$

It follows immediately from 3.5.2 and 3.5.7 that

**Corollary 3.5.8.** *Let  $G$  be any graph. Then*

$$\gamma_{\overline{e=1}}(G) + \Gamma_{\underline{e=1}}(G) = p(G) + iso(G).$$

### 3.6 More on the inequalities of (3.2.3)

Using the results of Section 3.5 we can further elaborate the relationships between the parameters  $\gamma_{\underline{e=1}}(G), \gamma_{\overline{e=1}}(G), \Gamma_{\underline{e=1}}(G)$  and  $\Gamma_{\overline{e=1}}(G)$  as first stated in the chain of inequalities (3.2.3). First, we shall prove a small result.

**Lemma 3.6.1.** *For any graph  $G$ ,*

$$\gamma_{\underline{e=1}}(G) = \gamma_{\overline{e=1}}(G) \iff \Gamma_{\underline{e=1}}(G) = \Gamma_{\overline{e=1}}(G).$$

*Proof.* Let  $G$  be any graph. If  $\gamma_{\underline{e=1}}(G) = \gamma_{\overline{e=1}}(G)$ , then by the Corollaries 3.5.4 and 3.5.8

$$\begin{aligned} \gamma_{\underline{e=1}}(G) = \gamma_{\overline{e=1}}(G) &\iff (p(G) + iso(G) - \gamma_{\underline{e=1}}(G)) = (p(G) + iso(G) - \gamma_{\overline{e=1}}(G)) \\ &\iff \Gamma_{\overline{e=1}}(G) = \Gamma_{\underline{e=1}}(G). \end{aligned}$$

$\square$



Thus there are only four allowed sets of inequalities:

$$\begin{aligned}\gamma_{\underline{e=1}}(G) &< \gamma_{\overline{e=1}}(G) < \frac{p(G) + \text{iso}(G)}{2} < \Gamma_{\underline{e=1}}(G) < \Gamma_{\overline{e=1}}(G), \\ \gamma_{\underline{e=1}}(G) &< \gamma_{\overline{e=1}}(G) = \frac{p(G) + \text{iso}(G)}{2} = \Gamma_{\underline{e=1}}(G) < \Gamma_{\overline{e=1}}(G), \\ \gamma_{\underline{e=1}}(G) &= \gamma_{\overline{e=1}}(G) < \frac{p(G) + \text{iso}(G)}{2} < \Gamma_{\underline{e=1}}(G) = \Gamma_{\overline{e=1}}(G), \\ \gamma_{\underline{e=1}}(G) &= \gamma_{\overline{e=1}}(G) = \frac{p(G) + \text{iso}(G)}{2} = \Gamma_{\underline{e=1}}(G) = \Gamma_{\overline{e=1}}(G).\end{aligned}$$

Figure 3.2 shows four graphs which display these four types of inequalities; hence the parameters  $\gamma_{\underline{e=1}}(G)$  and  $\gamma_{\overline{e=1}}(G)$  are independent. The vertices in each are labelled with the function values corresponding to the  $\gamma_{\underline{e=1}}$  and  $\gamma_{\overline{e=1}}$  functions, respectively. The values of the parameters are:

$$\begin{aligned}\gamma_{\underline{e=1}}(G_1) &= 1 < \gamma_{\overline{e=1}}(G_1) = 2 < \frac{p(G_1)}{2} = 4 < \Gamma_{\underline{e=1}}(G_1) = 6 < \Gamma_{\overline{e=1}}(G_1) = 7, \\ \gamma_{\underline{e=1}}(G_2) &= 2 < \gamma_{\overline{e=1}}(G_2) = 3 = \frac{p(G_2)}{2} = \Gamma_{\underline{e=1}}(G_2) < \Gamma_{\overline{e=1}}(G_2) = 4, \\ \gamma_{\underline{e=1}}(G_3) &= 1 = \gamma_{\overline{e=1}}(G_3) < \frac{p(G_3)}{2} = 3 < \Gamma_{\underline{e=1}}(G_3) = 5 = \Gamma_{\overline{e=1}}(G_3), \\ \gamma_{\underline{e=1}}(G_4) &= 1 = \gamma_{\overline{e=1}}(G_4) = \frac{p(G_4)}{2} = \Gamma_{\underline{e=1}}(G_4) = \Gamma_{\overline{e=1}}(G_4),\end{aligned}$$

### 3.7 Inequalities for subgraphs

The results in this section are due to Fricke and Swart [FS95]. Let  $G$  be any graph and let  $H$  be a spanning subgraph of  $G$ , neither  $G$  nor  $H$  possessing isolated vertices. Let  $f : V(H) \mapsto [0, 1]$  be a maximal  $e = 1$  function of  $H$ . Then obviously  $f$  is an  $e = 1$  function of  $G$ , but not necessarily a maximal  $e = 1$  function of  $G$ . For instance, the graph  $G$  shown in Figure 3.3 contains all edges, dashed and non-dashed, and a spanning subgraph  $H$  of  $G$  consists of  $V(G)$  together with all of the non-dashed edges of  $G$ . The  $e = 1$  function shown is maximal on  $H$  but not on  $G$ .

There are now two sets of relations to consider:

- $\Gamma_{\overline{e=1}}(G) \geq \Gamma_{\overline{e=1}}(H)$ . It is obvious that this is true. We can obtain both equality and strict inequality of the parameters for different graphs and subgraphs. Figure 3.4 shows graphs  $G_1$  and  $G_2$  with the respective subgraphs  $H_1$ ,  $H_2$  under consideration consisting of everything in  $G_1$  or  $G_2$  with the exception of the dashed edges. The numbers shown are the values for the  $\Gamma_{\overline{e=1}}$  functions.

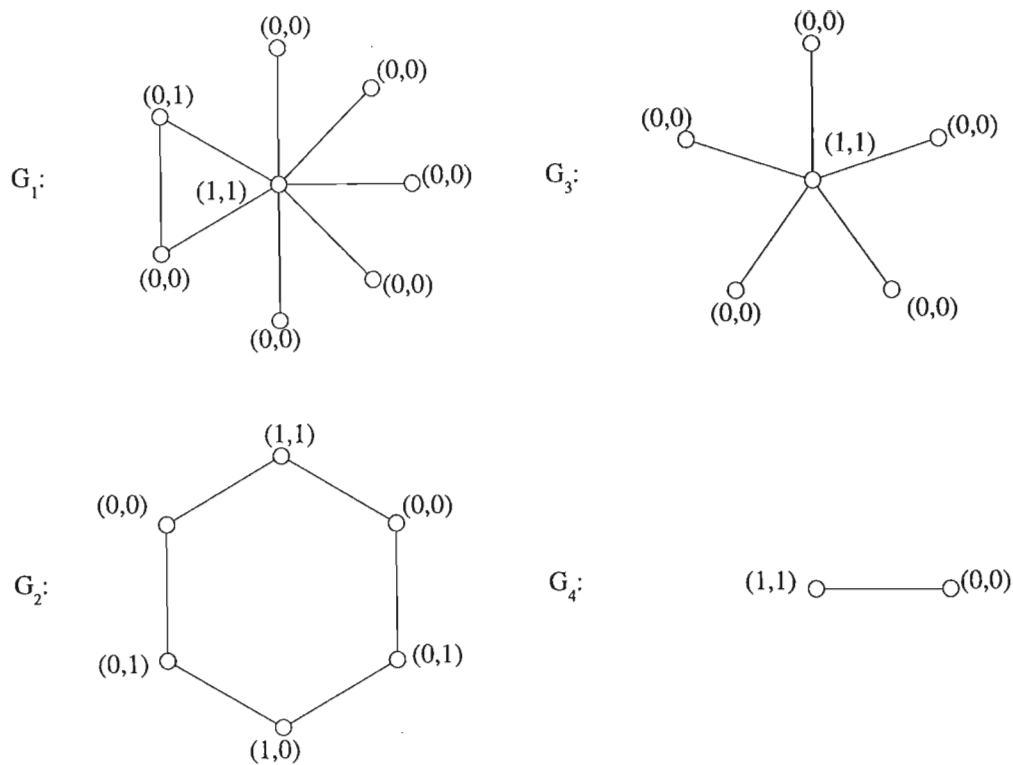


Figure 3.2: Equalities and inequalities for the parameters  $\gamma_{\underline{e=1}}$  and  $\gamma_{\overline{e=1}}$ .

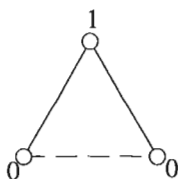


Figure 3.3: An  $e = 1$  function which is maximal on the subgraph consisting of all the vertices and the solid edges but not maximal on the graph consisting of all the vertices and all the edges.

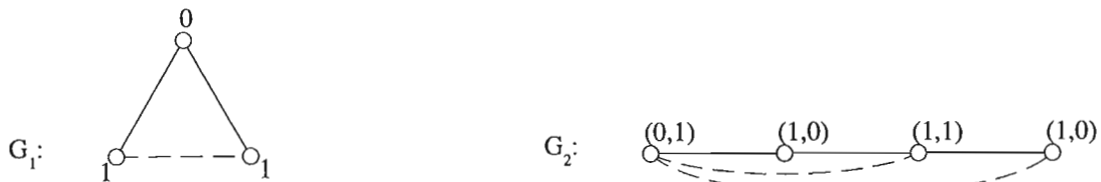


Figure 3.4: Equalities and inequalities for  $\Gamma_{e=1}(G)$  and  $\Gamma_{e=1}(H)$

In graph  $G_i$ ,  $i = 1, 2$ , the first number in brackets refers to the graph  $G_i$  and the second number to the subgraph  $H_i$ .

$$\begin{aligned}\Gamma_{e=1}(G_1) &= 2 = \Gamma_{e=1}(H_1), \\ \Gamma_{e=1}(G_2) &= 3 > 2 = \Gamma_{e=1}(H_2).\end{aligned}$$

- $\gamma_{e=1}(G)$  and  $\gamma_{e=1}(H)$ . The parameters  $\gamma_{e=1}(G)$  and  $\gamma_{e=1}(H)$  vary independently as  $G$  and  $H$  are varied. That the case  $\gamma_{e=1}(H) < \gamma_{e=1}(G)$  may occur can be seen by considering  $G = K_3$ ,  $H = P_3$ , for which  $\gamma_{e=1}(H) = 1 < \frac{3}{2} = \gamma_{e=1}(G)$ . To see that the case  $\gamma_{e=1}(H) > \gamma_{e=1}(G)$  may occur, we consider the following example: let  $G$  be a graph of order 12 consisting of an 8-cycle,  $u_1, u_2, \dots, u_8, u_1$ , a path  $u_1, v_1, v_2, v_3$  and a path  $u_6 w$ ; let  $H = G - \{u_3 u_4\}$ . Then a maximal  $e = 1$  function  $f$  may be defined on  $G$  by letting  $f(u_1) = f(u_2) = \frac{1}{2}$ ,  $f(u_3) = \frac{1}{4}$ ,  $f(u_4) = \frac{3}{4}$ ,  $f(u_5) = f(w) = \frac{3}{8}$ ,  $f(u_6) = \frac{5}{8}$ ,  $f(u_7) = \frac{3}{8} + \epsilon$ ,  $f(u_8) = \frac{5}{8} - \epsilon$ ,  $f(v_1) = f(v_3) = \frac{1}{2} + \epsilon$ ,  $f(v_2) = \frac{1}{2} - \epsilon$ ; hence  $\gamma_{e=1}(G) \leq |f| = \frac{47}{8} + \epsilon$ , where  $\epsilon$  is a small positive real number. It is easily shown that  $\gamma_{e=1}(H) = 6$ , attained by assigning a weight of  $\frac{1}{2}$  to each vertex in  $H$ , and so  $\gamma_{e=1}(H) > \gamma_{e=1}(G)$ .

### 3.8 $e = 1$ functions for different classes of graphs

We know from Sections 3.5 and 3.6 that of the four parameters originally considered, (3.1.1), (3.1.2), (3.1.3) and (3.1.4), only two of these may be considered as independent of one another. In determining the four parameters for a class of graphs we can thus consider only the maximal parameters (3.1.1) and (3.1.2), the others being determined by the results of Section 3.5. We now determine these maximal parameters for several classes of graphs. The results in this section are essentially due to Fricke and Swart [FS95].

### 3.8.1 The complete graphs

It is clear that the complete graphs  $K_1$  and  $K_2$  have equality for all  $e = 1$  parameters:  $\gamma_{\underline{e=1}}(K_i) = \gamma_{\overline{e=1}}(K_i) = \Gamma_{\underline{e=1}}(K_i) = \Gamma_{\overline{e=1}}(K_i) = 1$  if  $i = 1, 2$ . We can prove a more general result for an arbitrary complete graph  $K_p$ .

**Theorem 3.8.1.** *For any complete graph  $K_p$  with  $p \geq 2$ ,*

$$\begin{aligned}\gamma_{\overline{e=1}}(K_p) &= \frac{p}{2}, \\ \Gamma_{\overline{e=1}}(K_p) &= p - 1.\end{aligned}$$

*Proof.* The second result follows immediately from Corollary 3.5.5. Let  $V(K_p) = \{v_1, \dots, v_p\}$  and let  $f : V(K_p) \mapsto [0, 1]$  be a maximal  $e = 1$  function with  $f(v_1) \leq f(v_2) \leq \dots \leq f(v_p)$ . Note that the function  $f(v_1) = f(v_2) = \dots = f(v_p) = \frac{1}{2}$  yields a maximal  $e = 1$  function on  $K_p$  and hence  $\gamma_{\overline{e=1}}(K_p) \leq \frac{p}{2}$ . Furthermore, if for some  $v_i \in V(K_p) - \{v_1\}$  it is the case that  $f(v_i) + f(v_1) > 1$ , then  $f(v_i) + f(v_j) > 1$  for all  $v_j \in V(K_p)$  which contradicts our assumption that  $f$  is an  $e = 1$  function. Thus it must be the case that  $f(v_i) \leq 1 - f(v_1)$  for all  $v_i \in V(K_p) - \{v_1\}$ , and because  $f(v_1)$  is the lowest function value assigned by  $f$ ,  $0 \leq f(v_1) \leq \frac{1}{2}$ . Define a function  $g : V(K_p) \mapsto [0, 1]$  as

$$g(v_i) = \begin{cases} f(v_1) & i = 1 \\ 1 - f(v_1) & 2 \leq i \leq p. \end{cases}$$

Thus  $g(v_i) \geq f(v_i)$  for  $2 \leq i \leq p$ , and  $g$  is obviously an  $e = 1$  function of  $K_p$ . By our assumption that  $f$  is a maximal  $e = 1$  function, it must be the case that, for any  $1 \leq i \leq p$ ,  $f(v_i) = g(v_i)$ . Thus

$$\begin{aligned}f(v_1) &= g(v_1), \\ f(v_i) &= 1 - f(v_1) \text{ for } 2 \leq i \leq p.\end{aligned}$$

Then,

$$\begin{aligned}|f| &= f(v_1) + \sum_{i=2}^p (1 - f(v_1)) \\ &= p - 1 - (p - 2)f(v_1).\end{aligned}$$

It follows that  $|f|$  attains its minimum value for  $f(v_1) = \frac{1}{2}$ , in which case  $f(v_i) = \frac{1}{2}$  for any  $2 \leq i \leq p$  and  $\gamma_{\overline{e=1}}(K_p) = \frac{p}{2}$ .  $\square$

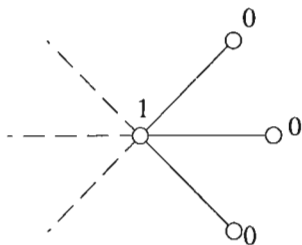


Figure 3.5: A  $\gamma_{e=1}$  function on the star  $K_{1,p-1}$ .

### 3.8.2 The complete bipartite graphs

If  $p_1 = 1$  then the graph is a star on  $p = p_1 + p_2$  vertices. Let  $f : V(K_{1,p-1}) \mapsto [0, 1]$  be defined such that  $f(v_1) = 1$  and  $f(v_i) = 0$  for all  $2 \leq i \leq p$  (see Figure 3.5). Note that  $f$  is maximal as any increase in  $f(v_i)$  for  $2 \leq i \leq p$  ruins the  $e = 1$  character of the function. Thus  $\gamma_{e=1}(K_{1,p-1}) = 1$  and the upper maximal bound  $\Gamma_{e=1}(K_{1,p-1})$ , is obviously equal to  $p - 1$ . We examine now the more interesting case where  $p_1, p_2 \geq 2$ .

**Theorem 3.8.2.** *For any two integers  $2 \leq p_1 \leq p_2$ ,*

$$\begin{aligned}\gamma_{e=1}(K_{p_1,p_2}) &= p_1, \\ \Gamma_{e=1}(K_{p_1,p_2}) &= p_1 + p_2 - 2.\end{aligned}$$

*Proof.* The second result follows immediately from Corollary 3.5.5 as  $\gamma(K_{p_1,p_2}) = 2$ . Let  $G \cong K_{p_1,p_2}$ , label the vertices of the two partite sets  $S_1, S_2$  (where  $V(G) = S_1 \cup S_2$ ) as  $V(S_1) = \{u_1, \dots, u_{p_1}\}$  and  $V(S_2) = \{v_1, \dots, v_{p_2}\}$ . Let  $f : V(G) \mapsto [0, 1]$  be a maximal  $e = 1$  function of  $G$  with  $f(u_1) \leq f(u_2) \leq \dots \leq f(u_{p_1})$  and  $f(v_1) \leq f(v_2) \leq \dots \leq f(v_{p_2})$ . Then  $f(u_1) + f(v_j) \leq 1$  and  $f(v_1) + f(u_i) \leq 1$  for  $1 \leq i \leq p_1$  and  $1 \leq j \leq p_2$ . Define a function  $g : V(G) \mapsto [0, 1]$  as

$$\begin{aligned}g(u_i) &= \begin{cases} f(u_1) & i = 1 \\ 1 - f(v_1) & 2 \leq i \leq p_1, \end{cases} \\ g(v_j) &= \begin{cases} f(v_1) & j = 1 \\ 1 - f(u_1) & 2 \leq j \leq p_2. \end{cases}\end{aligned}$$

It is obvious that  $g$  is an  $e = 1$  function, and  $g(u_i) \geq f(u_i)$  ( $1 \leq i \leq p_1$ ) and  $g(v_j) \geq f(v_j)$  ( $1 \leq j \leq p_2$ ). Thus  $|g| \geq |f|$ . By our assumption of the maximality of

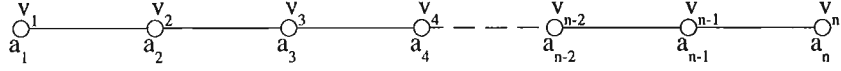


Figure 3.6: Vertex labelling  $v_i$  and function values  $a_i$  for the path  $P_n$ .

$f$ , this implies that  $f = g$  and hence

$$\begin{aligned}
 |f| &= (f(u_1) + f(v_2)) + (f(v_1) + f(u_2)) + \sum_{i=3}^{p_1} f(u_i) + \sum_{j=3}^{p_2} f(v_j) \\
 &= 1 + 1 + (p_1 - 2)(1 - f(v_1)) + (p_2 - 2)(1 - f(u_1)) \\
 &= 2 + p_1 - 2 - (p_1 - 2)f(v_1) + p_2 - 2 - (p_2 - 2)f(u_1) \\
 &= p_1 + p_2 - 2 - (p_1 - 2)f(v_1) - (p_2 - 2)f(u_1) \\
 &\geq p_2 - f(u_1)(p_2 - p_1).
 \end{aligned}$$

The minimum value is easily seen to be  $\gamma_{\overline{e=1}}(G) = p_1$  by setting  $f(u_i) = 1$  ( $1 \leq i \leq p_1$ ) and  $f(v_j) = 0$  ( $1 \leq j \leq p_2$ ).  $\square$

### 3.8.3 Paths

Throughout this section we use the convention that  $n = |V(P_n)|$  to avoid confusion. It is clear that  $\gamma_{\overline{e=1}}(P_i) = \gamma_{\overline{e=1}}(P_i) = \Gamma_{\overline{e=1}}(P_i) = \Gamma_{\overline{e=1}}(P_i) = 1$  for  $i = 1, 2$ . Let  $f$  be a maximal  $e = 1$  function on  $P_n$ , and label the vertices as  $V(P_n) = \{v_1, \dots, v_n\}$  and the function values of these vertices as  $f(v_i) = a_i$ , as shown in Figure 3.6. We can easily determine  $\gamma_{\overline{e=1}}(P_n)$  for the following cases:

- $n = 3$ , then

$$\begin{aligned}
 |f| &= a_1 + a_2 + a_3 \\
 &= (a_1 + a_2) + (a_2 + a_3) - a_2 \\
 &= 2 - a_2.
 \end{aligned}$$

Hence we obtain a minimal value by setting  $a_2 = 1$ , then

$$\begin{aligned}
 \gamma_{\overline{e=1}}(P_3) &= 1 \\
 &= \left\lfloor \frac{3}{2} \right\rfloor.
 \end{aligned}$$

Note that we can produce a maximal  $e = 1$  function with any value between 1 and 2 by varying  $a_2$ .

- $n = 4$ , then

$$\begin{aligned}
 |f| &= a_1 + a_2 + a_3 + a_4 \\
 &= (a_1 + a_2) + (a_3 + a_4) \\
 &= 2,
 \end{aligned}$$

and so

$$\begin{aligned}
 \gamma_{e=1}^-(P_4) &= 2 \\
 &= \left\lfloor \frac{4}{2} \right\rfloor.
 \end{aligned}$$

- $n = 5$ , then

$$\begin{aligned}
 |f| &= (a_1 + a_2) + a_3 + (a_4 + a_5) \\
 &= 2 + a_3.
 \end{aligned}$$

Note that we can produce a maximal  $e = 1$  function with any value between 2 and 3 by varying  $a_3$ . Thus,

$$\begin{aligned}
 \gamma_{e=1}^-(P_5) &= 2 \\
 &= \left\lfloor \frac{5}{2} \right\rfloor.
 \end{aligned}$$

Notice that, throughout the above discussion,  $n \geq 2 \implies \gamma_{e=1}^-(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$ . For the more general case  $n \geq 2$  we present the following theorem:

**Theorem 3.8.3.** *Let  $n \geq 2$  be an integer. Then*

1.

$$\begin{aligned}
 \gamma_{e=1}^-(P_n) &= \left\lfloor \frac{n}{2} \right\rfloor, \\
 \Gamma_{e=1}^-(P_n) &= \left\lfloor \frac{2n}{3} \right\rfloor.
 \end{aligned}$$

2. *Furthermore, for each  $x \in \left[ \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{2n}{3} \right\rfloor \right]$ , there exists a maximal  $e = 1$  function  $f$  on  $P_n$  for which  $|f| = x$ .*

*Proof.*

1. The result for  $\Gamma_{e=1}^{\overline{}}(P_n)$  follows immediately from Corollary 3.5.5, since  $\gamma(P_n) = \lfloor \frac{n}{3} \rfloor$ . Now observe that, for all  $n \geq 2$ , a maximal  $e = 1$  function  $f : V(P_n) \mapsto [0, 1]$  can be found by setting

$$f(v_i) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

Since  $|f| = \lfloor \frac{n}{2} \rfloor$ , then  $\gamma_{e=1}^{\overline{}}(P_n) \leq \lfloor \frac{n}{2} \rfloor$ . From the discussion above we know that for  $2 \leq n \leq 5$ ,  $\gamma_{e=1}^{\overline{}}(P_n) = \lfloor \frac{n}{2} \rfloor$ . Now assume that there exists a smallest integer  $n > 5$  such that  $\gamma_{e=1}^{\overline{}}(P_n) < \lfloor \frac{n}{2} \rfloor$ . Note that a maximal  $e = 1$  function  $f$  of  $P_n$  having cardinality  $|f| = \gamma_{e=1}^{\overline{}}(P_n)$  has  $f(v_{n-1}) + f(v_n) = 1$ ; hence  $\sum_{i=1}^{n-2} f(v_i) < \lfloor \frac{n-2}{2} \rfloor$  and so  $f^* = f|_{\{v_1, \dots, v_{n-2}\}}$  is not a maximal  $e = 1$  function of  $P_{n-2}$ . If  $f^*$  is an  $e = 1$  function, then there must exist an  $e = 1$  function  $g^* : V(P_n) \mapsto [0, 1]$  with  $g^* > f^*$  and (say)  $g^*(v_{n-2}) = b_{n-2}$ . Define a function  $g : V(P_n) \mapsto [0, 1]$  as

$$g(v_i) = \begin{cases} g^*(v_i) & \text{for } i = 1, \dots, n-2, \\ f(v_i) & \text{for } i = n-1, n. \end{cases}$$

Then we obtain an  $e = 1$  function  $g$  such that  $|g| > |f|$ , a contradiction. Hence it must be the case that  $f^*$  is not an  $e = 1$  function, and thus  $f(v_{n-3}) + f(v_{n-2}) \neq 1$ . There are now two cases to consider.

- If  $f(v_{n-3}) + f(v_{n-2}) < 1$  then we can increase  $f(v_{n-2})$  to obtain an  $e = 1$  function on  $V(P_{n-2})$ . This gives us an  $e = 1$  function on  $P_n$  with cardinality greater than  $|f|$  — a contradiction, as  $f$  is maximal.
- If  $f(v_{n-3}) + f(v_{n-2}) > 1$ , then  $f(v_{n-2}) > 0$  and  $f(v_{n-2}) + f(v_{n-1}) = 1$  so that  $f(v_{n-2}) = 1 - f(v_{n-1}) = f(v_n)$ ; also  $f(v_{n-3}) + f(v_{n-4}) = 1$ . We now construct a function  $f^{**} : V(P_n) \mapsto [0, 1]$  as (see Figure 3.7, where we use the notation  $f(v_{n-3}) = b$  and  $f(v_{n-1}) = a$ ):

$$\begin{aligned} f^{**}(v_i) &= f(v_i) \text{ for } 1 \leq i \leq n-3, \\ f^{**}(v_{n-2}) &= 1 - f(v_{n-3}), \\ f^{**}(v_{n-1}) &= f(v_{n-3}), \\ f^{**}(v_n) &= 1 - f(v_{n-3}). \end{aligned}$$

Notice that  $f^{**}$  is maximal and  $|f^{**}| < |f| = \gamma_{e=1}^{\overline{}}(P_n)$  which is a contradiction.



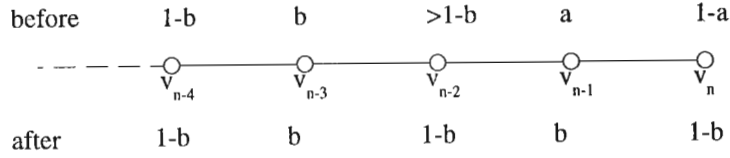


Figure 3.7: Replacing function values on the vertices.

It follows that  $\gamma_{e=1}(P_n) = \lfloor \frac{n}{2} \rfloor$  for all  $n \geq 2$ .

2. By referring to the discussion at the beginning of this section, it is clear that the result is true for  $n = 2, 3, 4, 5$ . Thus suppose that the result holds for  $2 \leq m < n$  and let  $x \in [\lfloor \frac{n}{2} \rfloor, \lfloor \frac{2n}{3} \rfloor]$ . There are two cases to consider:

- $x \in [\lfloor \frac{n-2}{2} \rfloor + 1, \lfloor \frac{2(n-2)}{3} \rfloor + 1]$ . Let  $f : V(P_{n-2}) \mapsto [0, 1]$  be a maximal  $e = 1$  function of  $P_{n-2}$  with  $f(V(P_{n-2})) = x - 1$  and let  $f(v_{n-1}) = 1$  and  $f(v_n) = 0$  to obtain a maximal  $e = 1$  function of  $P_n$  with  $f(V(P_n)) = x$ .
- $x > \lfloor \frac{2(n-2)}{3} \rfloor + 1 = \lfloor \frac{2n-1}{3} \rfloor$ . Then  $n \equiv 0 \pmod{3}$ , say  $n = 3k$  for some integer  $k$ , and  $x \in (2k - 1, 2k)$ , so let  $x = 2k - 1 + a$  where  $0 < a < 1$ . Now define a function  $g : V(P_n) \mapsto [0, 1]$  as

$$f(v_i) = \begin{cases} 1 & \text{if } i \equiv 1, 3 \pmod{3} \text{ and } i \leq n - 3 \\ 0 & \text{if } i \equiv 2 \pmod{3} \text{ and } i \leq n - 4 \\ a & \text{if } i = n - 2, n \\ 1 - a & \text{if } i = n - 1. \end{cases}$$

The cardinality of this function is  $|g| = 2k - 1 + a = x$  and it is a maximal  $e = 1$  function.

□

### 3.9 A characterization of $\gamma_f$ and $\gamma$

We can use the chain of inequalities (3.3.1) involving the  $e = 1$  function functions to provide a sufficient condition for a graph  $G$  to have the property that  $\gamma(G) = \gamma_f(G)$ . Before we can obtain the desired result it is necessary to prove a weaker result characterizing a certain class of  $e = 1$  functions.

**Lemma 3.9.1.** *Let  $G$  be any graph on which there can be defined a fractional set function  $f : V(G) \mapsto [0, 1]$  which is simultaneously an  $e = 1$  function and a fractional packing function. Then for any vertex  $v \in V(G)$ :*

- *If  $f(v) = 1$  then all vertices  $w \neq v$  with  $d(v, w) \leq 2$  satisfy  $f(w) = 0$ .*
- *If  $0 < f(v) < 1$  then there is exactly one vertex  $u \in N(v)$  with  $f(u) > 0$  and all other vertices  $w \in N(v) \cup N(u)$  satisfy  $f(w) = 0$ .*
- *If  $f(v) = 0$  then there is a  $u \in N(v)$  with  $f(u) = 1$ .*

*Proof.* Let  $f : V(G) \mapsto [0, 1]$  be such a function. Let  $v$  be any vertex in  $V(G)$ . Then:

- If  $\deg v = 0$  then  $v$  has no neighbours,  $f(v) = 1$  and the result follows.
- Suppose that  $\deg v > 0$ . Let  $u \in N(v)$  be any vertex such that  $f(v) + f(u) = 1$ . By the definition of  $f$ ,

$$\begin{aligned} f(v) + f(u) &= 1 \\ &\geq f(N[v]) \\ &\geq f(v) + f(u). \end{aligned}$$

It follows immediately that  $f(v) + f(u) = f(N[v])$  and by a similar argument that  $f(u) + f(v) = f(N[u])$ . Thus  $f(u)$  and  $f(v)$  have the only positive function values in  $N(u) \cup N(v)$ . If  $f(v) = 1$  then each  $u \in N(v)$  has  $f(u) = 0$  and no neighbour  $w$  of any  $u$  can have positive value  $f(w) = \epsilon > 0$  else

$$\begin{aligned} f(N[u]) &= f(v) + f(w) + f(N[u] - \{v, w\}) \\ &\geq 1 + \epsilon \\ &> 1 \end{aligned}$$

and  $f$  is not a fractional packing function. If  $f(v) = 0$  then because  $f$  is an  $e = 1$  function there must be a neighbour  $u \in N(v)$  with  $f(u) = 1$ . Hence the result follows. □

Notice that it is not possible to define such a function for all graphs (we return to this later; see Figure 3.9) and that there are graphs which possess both  $e = 1$  functions which are not packing functions and packing functions which are not  $e = 1$  functions (see Figure 3.8). Notice that any fractional set function  $f$  which is both an  $e = 1$

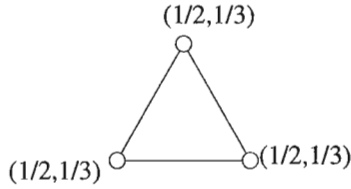


Figure 3.8:  $K_3$  possesses  $e = 1$  functions which are not packing functions and packing functions which are not  $e = 1$  functions. The figures in brackets are (respectively) an  $e = 1$  function  $f$  which is not a packing function since for any  $v \in V(K_3)$ ,  $f(N[v]) = \frac{3}{2} > 1$ , and a packing function  $g$  which is not an  $e = 1$  function since for any  $uv \in E(K_3)$ ,  $g(u) + g(v) = \frac{2}{3} < 1$ .

function and a fractional packing on  $G$  obeys the inequalities

$$\gamma_{\underline{e=1}}(G) \leq |f| \leq P_f(G).$$

**Theorem 3.9.2.** *Let  $G$  be a graph. If it is possible to define a fractional set function  $f : V(G) \mapsto [0, 1]$  on  $G$  which is both an  $e = 1$  function and a fractional packing function, then*

$$\gamma(G) = \gamma_f(G).$$

*Proof.* Let  $G$  be such a graph and let  $f : V(G) \mapsto [0, 1]$  be a fractional packing function which is also an  $e = 1$  function. Denote by  $V_1, \dots, V_k$  the vertex sets of the components of  $G$ , i.e.  $\bigcup_{i=1}^k V_i = V(G)$ .

- *First we show that if there is a  $v \in V_h$  (for some  $h$ ) with  $f(v) = 1$  then  $f : V_h \mapsto \{0, 1\}$ .*

1. If  $V_h = \{v\}$  is a trivial component,  $f(v) = 1$  and the result follows.
2. Suppose that  $|V_h| \geq 2$  and that  $f(v) = 1$ , and let the neighbours of  $v$  be  $v_1, \dots, v_{\deg v}$ . By Lemma 3.9.1, all the neighbours  $v_m$  of  $v$  must satisfy  $f(v_m) = 0$ . If  $e(v) = 1$  then the result follows. If this is not the case, then all of vertices in  $N(v_m) - \{v\}$  (denote them by  $v_{m1}, \dots, v_{m \deg v_m - 1}$ ) with  $\deg v_m > 1$  must satisfy  $f(v_{mn}) = 0$ , and now if  $e(v) = 2$  then the result follows. Supposing once again that this is not the case, each of these  $v_{mn}$  must be adjacent to exactly one vertex  $v_{mno}$  with positive function

value and that  $v_{mno}$  must satisfy  $f(v_{mno}) = 1$ . Hence each vertex  $x$  with  $f(x) = 1$  is adjacent only to vertices  $x_m$  with  $f(x_m) = 0$ , and each vertex  $y$  with  $f(y) = 0$  is adjacent to exactly one vertex  $y_n$  with  $f(y_n) = 1$ , all the other neighbours of  $y$  must have function value zero.

- *Now we show that no vertex  $v$  can exist with  $0 < f(v) < 1$ , unless  $v$  is in a component of order 2.* Suppose that  $v \in V_h$ ; if  $|V_h| = 1$  then it must be the case that  $f(v) = 1$ . We delay consideration of the case  $|V_h| = 2$  until later, assume that  $|V_h| > 2$ , and suppose that  $v$  has function value  $f(v) = \epsilon$  where  $0 < \epsilon < 1$ . By Lemma 3.9.1, there is a vertex  $u \in N(v)$  with  $f(u) = 1 - \epsilon$  and every vertex in  $N(v) \cup N(u) - \{u, v\}$  has function value equal to zero. Let  $w$  be any such vertex and suppose that it lies in  $N(v)$ , then by the aforementioned lemma, there must be a vertex  $x \in N(w)$  having function value  $f(x) = 1$  and every vertex  $y$  satisfying  $d(x, y) \leq 2$  must have function value equal to zero. However,  $d(x, v) \leq d(x, w) + d(w, v) = 2$  which contradicts the fact that  $f(v) = \epsilon > 0$ . Hence no such  $v$  can exist and  $f : V_h \mapsto \{0, 1\}$ .
- *Lastly, we deal with the case when  $|V_h| = 2$ , i.e.  $\langle V_h \rangle \cong K_2$ .* Then any valid  $f$  has weight  $f(K_2) = 1$  and hence out of all the infinite number of  $f$ 's possible we can choose one such that  $f : V_h \mapsto \{0, 1\}$ .

Thus if  $G$  possesses a fractional packing function  $f'$  which is also an  $e = 1$  function then it possesses a fractional packing function  $f$  which is also an  $e = 1$  function and which maps the vertices of  $G$  to the set  $\{0, 1\}$ , and which can differ from  $f'$  only in components of  $G$  of order 2. We can then treat  $f$  as the characteristic function of a set  $F$ , and  $|f| = |F|$ . Let  $v \in V(G) - F$ ; since  $f$  is an  $e = 1$  function it follows that there must be a  $u \in N(v)$  with  $f(u) = 1$  and hence for any  $v$ ,  $|N(v) \cap F| \geq 1$ . Thus  $F$  is a dominating set of  $G$  and  $|f| = |F| \geq \gamma(G)$ . Now, from the chain of inequalities (3.3.1), it follows that

$$\begin{aligned} \gamma_f(G) &\leq \gamma(G) \\ &\leq |f| \\ &\leq P_f(G) \\ &= \gamma_f(G) \end{aligned}$$

and the result follows. □

We can obtain the same characterization in terms of a more well-known class of graphs — those with efficient dominating sets — by the following observation.

**Lemma 3.9.3.** *A function  $f : V(G) \mapsto [0, 1]$  which is both a fractional packing and an  $e = 1$  function can be defined on  $G$  if and only if  $G$  has an efficient dominating set.*

*Proof.*

- *Suppose that there is a function  $f : V(G) \mapsto [0, 1]$  which is both a fractional packing and an  $e = 1$  function. Derive from  $f$  a function  $f' : V(G) \mapsto \{0, 1\}$  by replacing all the values of  $f$  in any component of  $G$  isomorphic to  $K_2$  with the value 0 on one vertex and 1 on the other. Now  $f'$  is the characteristic function of some set which we denote by  $F'$  and  $|f'| = |F'|$ . Let  $v$  be any vertex in  $V(G)$ , then by Lemma 3.9.1 if  $f'(v) = 1$  then all vertices  $w$  within distance two of  $v$  satisfy  $f'(w) = 0$  and hence are not in  $F'$ , and if  $f'(v) = 0$  then there is exactly one vertex  $w$  in  $N(v)$  with  $f(w) = 1$  and every vertex  $x$  within distance two of  $w$  has function value  $f'(x) = 0$ . Hence every vertex in  $V(G)$  is in the closed neighbourhood of exactly one member of  $F'$  and  $F'$  is an efficient dominating set.*
- *Suppose that  $G$  has an efficient dominating set,  $F$ . Denote by  $f$  the characteristic function of  $F$ . Since  $F$  is an efficient dominating set, every  $v$  satisfying  $f(v) = 0$  is adjacent to one vertex  $u$  with  $f(u) = 1$  and every vertex  $w$  with  $f(w) = 1$  is adjacent only to vertices  $x$  satisfying  $f(x) = 0$ . Hence  $f$  is an  $e = 1$  function and a fractional packing function.*

□

**Corollary 3.9.4.** *If  $G$  has an efficient dominating set then*

$$\gamma_f(G) = \gamma(G).$$

This result has been independently derived by Goddard and Henning [GH]. It is well known that not all graphs have efficient dominating sets. In Figure 3.9 a graph  $G$  is shown which consists of all vertices and all solid edges, and we define  $G'$  to be  $G$  with the addition of the dashed edge  $bg$ . The shaded vertices  $b$  and  $g$  constitute an efficient dominating set of  $G$ , but do not form an efficient dominating set of  $G'$ . Denote by  $F'$  a postulated efficient dominating set of  $G'$ . If  $a \in F'$  then we must choose  $d$  or  $f$  to be in  $F'$  but once we have chosen one it is impossible to dominate the other efficiently. If  $b \in F'$  then we cannot efficiently dominate  $d$  or  $f$ , and if  $c \in F'$  then we cannot efficiently dominate  $a$ . By the symmetries in the graph we can then deduce that  $G'$  does not have an efficient dominating set.

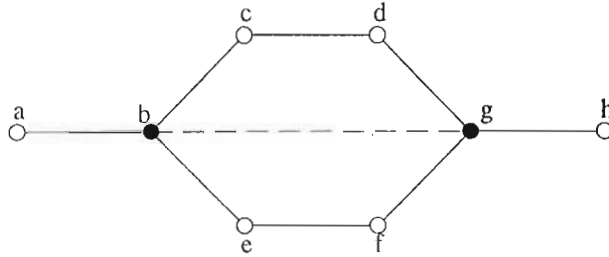


Figure 3.9: The graphs  $G$  (all vertices and solid edges) and  $G'$  (all vertices and all edges).  $G$  has an efficient dominating set,  $G'$  does not.

### 3.10 Conclusion

The study of fractional domination-type parameters is a relatively new and rapidly developing field. We suggest a few ideas which may bear investigation.

Fractional total dominating functions of a graph  $G$  may be introduced by replacing the requirements in (1.4.1) that  $f(N[v]) \geq 1$  by  $f(N(v)) \geq 1$  for any  $v \in V(G)$  and, correspondingly, the matrix in (1.5.2) by the adjacency matrix of  $G$ .

In attempting to establish a suitable fractional theory of independence one may consider  $e \leq 1$ -functions, i.e.  $f : V(G) \mapsto [0, 1]$  such that  $f(u) + f(v) \leq 1$  for every edge  $uv \in E(G)$ . We note that the characteristic functions of maximum independent sets and minimum independent dominating sets are indeed maximal  $e \leq 1$ -functions. Hence, if  $\Gamma_{e \leq 1}^-(G)$  and  $\gamma_{e \leq 1}^-(G)$  denote, respectively, the maximum and minimum values of  $f(V(G))$ , for all maximal  $e \leq 1$ -functions defined on  $V(G)$ , we have  $\gamma_{e \leq 1}^-(G) \leq \beta^-(G)$  and  $\Gamma_{e \leq 1}^-(G) \geq \beta(G)$ . It can be shown that, for connected graphs  $G$ ,  $\gamma(G) \leq \gamma_{e \leq 1}^-(G) \leq \beta_f^-(G)$ .

Goddard and Henning [GH] have considered *real domination* in graphs where the closed unit interval we have studied is replaced by an arbitrary subset  $R$  of the real numbers, and this technique could be used to generate real parameters corresponding to the fractional parameters discussed here. Goddard [God96] has also suggested that, for any graph  $G$ , any  $\gamma_{e=1}^-(G)$  function  $f : V(G) \mapsto [0, 1]$  can be replaced by a function  $f' : V(G) \mapsto \{0, \frac{1}{2}, 1\}$  such that  $|f'| = |f| = \gamma_{e=1}^-(G)$ , and also that if  $H$  is a bipartite graph then  $\gamma_{e=1}^-(H) = \alpha(H)$ . These two questions certainly deserve further investigation.

It is relatively easy to define new domination-related fractional functions, for instance, fractional connected dominating functions and fractional locating functions (extending concepts developed in [HL84b] and [Sla87]). It may be advisable to consider carefully the motivation for the introduction of each new "fractional" function

or parameter, and also to consider possible applications of the theory thus developed.

# Appendix A

## Notation

Throughout the text and throughout this section,  $G$  denotes a graph with vertex set  $V(G)$  and edge set  $E(G)$ . The cardinality of the vertex set we denote by  $p = |V(G)|$ , and the cardinality of the edge set by  $q = |E(G)|$ . Let  $v$  be any element of the vertex set; then the degree of  $v$  is denoted by  $\deg v$ . Throughout this section,  $H$  will denote a graph,  $u, v$  will denote vertices of  $G$ ,  $e$  will be an edge of  $G$ ,  $f : V(G) \mapsto [0, 1]$  a fractional set function of  $G$  and  $M, N$  will be  $(m \times n)$  matrices.

- $e \rightsquigarrow v$  means that the edge  $e$  is incident with the vertex  $v$ .
- $[v]$  denotes the automorphism class of  $v$ .
- $[V_1, V_2]$  denotes the set of all edges which join a vertex in  $V_1$  to a vertex in  $V_2$ .
- $G \cong H$  means that the graphs  $G$  and  $H$  are isomorphic to one another.
- $G \times H$  is the cartesian product of  $G$  and  $H$ .
- $G \cdot H$  is the strong direct product of  $G$  and  $H$ .
- $M \otimes N$  is the tensor product of  $M$  and  $N$ .
- $\langle S \rangle$  denotes the induced subgraph generated by a set  $S$ ;  $S$  can be a set of vertices or a set of edges.
- $\lfloor f \rfloor$  = the floor of  $f$ .
- $\lceil f \rceil$  = the ceiling of  $f$ .
- $\vec{1}_n$  is the  $(n \times 1)$  vector with unit entries.
- $\vec{0}_n$  is the  $(n \times 1)$  vector with zero entries.



- $\alpha(G)$  = the (vertex) covering number of  $G$ .
- $\alpha_f(G)$  = the fractional (vertex) covering number of  $G$ .
- $\alpha_f^+(G)$  = the upper fractional (vertex) covering number of  $G$ .
- $\alpha^1(G)$  = the edge covering number of  $G$ .
- $\alpha_f^1(G)$  = the fractional edge covering number of  $G$ .
- $\alpha_f^{1+}(G)$  = the upper fractional edge covering number of  $G$ .
- $\mathbf{A}(G)$  = the automorphism group of  $G$ .
- $\beta(G)$  = the independence number of  $G$ .
- $\beta^-(G)$  = the lower independence number of  $G$ .
- $\beta_f(G)$  = the fractional (vertex) independence number of  $G$ .
- $\beta_f^-(G)$  = the lower fractional (vertex) independence number of  $G$ .
- $\beta_k(G)$  = the  $k$ -independence number of  $G$ .
- $\beta^1(G)$  = the matching number of  $G$ .
- $\beta_f^1(G)$  = the fractional matching number of  $G$ .
- $\beta_f^{1-}(G)$  = the lower fractional matching number of  $G$ .
- $\beta^-(G)$  = the lower independence number of  $G$ .
- $\text{diam}(G)$  = the diameter of  $G$ .
- $d(u, v)$  = the distance between  $u$  and  $v$ .
- $\delta(G) = \min\{\text{deg}v : v \in V(G)\}$ .
- $\Delta(G) = \max\{\text{deg}v : v \in V(G)\}$ .
- $e(v)$  = the eccentricity of  $v$ .
- $F_f(G)$  = the maxfluence of  $G$ .
- $f_{\text{not}}(v) = 1$  ( $\text{deg}v = 0$ ) or  $1 - f(v)$  ( $\text{deg}v > 0$ ).
- $\gamma(G)$  = the domination number of  $G$ .

- $\Gamma(G)$  = the upper domination number of  $G$ .
- $\gamma_f(G)$  = the fractional domination number of  $G$ .
- $\Gamma_f(G)$  = the upper fractional domination number of  $G$ .
- $\gamma_{\underline{e=1}}(G)$  = the minimum weight of a minimal  $e = 1$  function of  $G$ .
- $\gamma_{\overline{e=1}}(G)$  = the minimum weight of a maximal  $e = 1$  function of  $G$ .
- $\Gamma_{\underline{e=1}}(G)$  = the maximum weight of a minimal  $e = 1$  function of  $G$ .
- $\Gamma_{\overline{e=1}}(G)$  = the maximum weight of a maximal  $e = 1$  function of  $G$ .
- $\gamma_k(G)$  = the  $k$ -domination number of  $G$ .
- $\Gamma_k(G)$  = the upper  $k$ -domination number of  $G$ .
- $I(f)$  = the influence of  $f$ .
- $ir(G)$  = the irredundance number of  $G$ .
- $IR(G)$  = the upper irredundance number of  $G$ .
- $ir_f(G)$  = the fractional irredundance number of  $G$ .
- $IR_f(G)$  = the upper fractional irredundance number of  $G$ .
- $ir_k(G)$  = the  $k$ -irredundance number of  $G$ .
- $IR_k(G)$  = the upper  $k$ -irredundance number of  $G$ .
- $iso(G)$  = the number of isolated vertices in  $G$ .
- $k(G)$  = the number of components of  $G$ .
- $niso(G)$  = the number of non-isolated components in  $G$ .
- $p(G)$  = the lower packing number of  $G$ .
- $P(G)$  = the upper packing number of  $G$ .
- $P_2(G)$  = the two-packing number of  $G$ .
- $p_f(G)$  = the lower fractional packing number of  $G$ .
- $P_f(G)$  = the upper fractional packing number of  $G$ .

- $\pi_k(G)$  = the lower  $k$ -packing number of  $G$ .
- $\Pi_k(G)$  = the upper  $k$ -packing number of  $G$ .
- $\mathbb{R}$  = the set of real numbers.
- $\text{rad}(G)$  = the radius of  $G$ .
- $R_f(G)$  = the minfluence of  $G$ .
- $\vec{x}_f$  = the characteristic vector of  $f$ .
- $\mathbb{Z}$  = the set of integers.
- $\mathbb{Z}(m, n)$  = the set of all  $(m \times n)$  matrices which have non-negative, integer elements.

# Bibliography

- [Ber58] Claude Berge, *Theory of graphs and its applications*, Dunod, Paris, 1958.
- [CDH80] E.J. Cockayne, R.M. Dawes, and S.T. Hedetniemi, *Total domination in graphs*, *Networks* (1980), no. 10, 211–219.
- [CF93] Grant A. Cheston and Gerd Fricke, *Classes of graphs for which upper fractional domination equals independence, upper domination, and upper irredundance*, *Discrete Appl. Math.* (1993), no. 55, 241–258.
- [CFHJ90] G.A. Cheston, G. Fricke, S.T. Hedetniemi, and D.P. Jacobs, *On the computational complexity of upper fractional domination*, *Discrete Appl. Math.* (1990), no. 27, 195–207.
- [CFPT81] E.J. Cockayne, O. Favaron, C. Payan, and A.G. Thomason, *Contributions to the theory of domination, independence and irredundance in graphs*, *Discrete Math.* (1981), no. 33, 249–258.
- [CH77] E.J. Cockayne and S.T. Hedetniemi, *Towards a theory of domination in graphs*, *Networks* (1977), no. 7, 247–261.
- [CHHL88] G.A. Cheston, E.O. Hare, S.T. Hedetniemi, and R.C. Laskar, *Simplicial graphs*, *Congr. Numer.* (1988), no. 67, 105–113.
- [CHM78] E.J. Cockayne, S.T. Hedetniemi, and D.J. Miller, *Properties of hereditary hypergraphs and middle graphs*, *Canad. Math. Bull.* **21** (1978), no. 4, 461–468.
- [CK90] Margaret B. Cozzens and Laura L. Kelleher, *Dominating cliques in graphs*, *Discrete Math.* (1990), no. 86, 101–116.
- [CL86] Gary Chartrand and Linda Lesniak, *Graphs and Digraphs*, second ed., Wadsworth and Brooks/Cole, Monterey, California, 1986.

- [CN91] James Currie and Richard Nowakowski, *A characterization of fractionally well-covered graphs*, *Ars Combin.* (1991), no. 31, 93–96.
- [DHHM] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, and A.A. McRae, *Minus domination in graphs*, preprint.
- [DHHS] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, and P.J. Slater, *Signed domination in graphs*, preprint.
- [DHL88] Gayla S. Domke, Stephen T. Hedetniemi, and Renu C. Laskar, *Fractional packings, coverings and irredundance in graphs*, *Congr. Numer.* (1988), no. 66, 227–238.
- [DHLS88] Gayla S. Domke, Stephen T. Hedetniemi, Renu Laskar, and Robert Allan, *Generalized packings and coverings of graphs*, *Congr. Numer.* (1988), no. 62, 259–270.
- [DHLF91] Gayla S. Domke, Stephen T. Hedetniemi, Renu C. Laskar, and Gerd Fricke, *Relationships between integer and fractional parameters of graphs*, *Graph Theory, Combinatorics and Applications* (Y.Alavi, G.Chartrand, O.R.Oellerman, and A.J.Schwenk, eds.), John Wiley and Sons, Inc., 1991, pp. 371–387.
- [dJ62] C.F. de Jaenisch, *Applications de l'analyse mathematique au jeu des echecs*, Petrograd, 1862.
- [DS95] P. Dankelmann and Henda C. Swart, private communication, 1995.
- [Far83] M. Farber, *Characteristics of strongly chordal graphs*, *Discrete Math.* (1983), no. 43, 173–189.
- [Fis94] David C. Fisher, *Domination, fractional domination, 2-packing, and graph products*, *SIAM J. Discrete Math.* **7** (1994), no. 3, 493–498.
- [FJ85] J.F. Fink and M.S. Jacobson, *n-domination in graphs*, *Graph Theory with Applications to Algorithms and Computer Science* (New York), Wiley-Intersci. Publ., New York, 1985, pp. 283–300.
- [FRDM94] David C. Fisher, Jennifer Ryan, Gayla S. Domke, and Aniket Majumdar, *Fractional domination of strong direct products*, *Discrete Appl. Math.* (1994), no. 50, 89–91.
- [Fri95] Gerd Fricke, private communication, 1995.

- [FS95] Gerd Fricke and Henda C. Swart, private communication, 1995.
- [Gal59] T. Gallai, *Über extreme Punkt- und Kantenmengen*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. (1959), no. 2, 133–138.
- [GH] W. Goddard and M.A. Henning, *Real and integer domination in graphs*, preprint.
- [GJ79] Michael R. Garey and David S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness*, W.H. Freeman and company, New York, 1979.
- [God96] W. Goddard, private communication, 1996.
- [GS] D.J. Grinstead and P.J. Slater, *A recurrence template for several domination related parameters in series-parallel graphs*, preprint.
- [GS90] D.L. Grinstead and P.J. Slater, *Fractional domination and fractional packing in graphs*, Congr. Numer. (1990), no. 71, 153–172.
- [Har90] E.O. Hare, *k-weight domination and fractional domination of  $P_m \times P_n$* , Congr. Numer. (1990), no. 78, 71–80.
- [Hay94] Teresa Haynes, private communication to H.C. Swart, 1994.
- [HHW83] S.M. Hedetniemi, S.T. Hedetniemi, and T.V. Wimer, *Linear time resource allocation algorithms for trees*, Tech. Report 428, Department of Mathematical Sciences, Clemson University, July 1983.
- [HL84a] Stephen T. Hedetniemi and Renu C. Laskar, *Connected domination in graphs*, Graph Theory and Combinatorics (London) (B. Bollobás, ed.), Academic Press, London, 1984, pp. 209–218.
- [HL84b] Stephen T. Hedetniemi and Renu C. Laskar, *Connected domination in graphs*, Graph Theory and Combinatorics (B. Bollobás, ed.), Academic Press, 1984, pp. 209–218.
- [HL91] S.T. Hedetniemi and R.C. Laskar (eds.), *Towards a theory of domination in graphs*, Elsevier, New York, 1991.
- [HOS91] M.A. Henning, O.R. Oellermann, and H.C. Swart, *Bounds on distance domination parameters*, J. Combin. Inform. System Sci. (1991), no. 16, 11–18.

- [MM75] A. Meir and J.W. Moon, *Relations between packing and covering numbers of a tree*, Pacific J. Math. **61** (1975), no. 1, 225–233.
- [Ore62] O. Ore, *Theory of graphs*, American Mathematical Society Publishers, Providence, 1962.
- [Plu70] M.D. Plummer, *Some covering concepts in graphs*, J. Combin. Theory (1970), no. 8, 91–98.
- [Pul87] W.R. Pulleyblank, *Fractional matchings and the Edmonds-Gallai theorem*, Discrete Appl. Math. (1987), no. 16, 51–58.
- [Sla87] P.J. Slater, *Domination and location in acyclic graphs*, Networks (1987), no. 17, 55–64.
- [Sla88] Peter Slater, *Dominating and reference sets in graphs*, J. Math. Phys. Sci. (1988), no. 22, 445–455.
- [Sla95] Peter Slater, private communication to H.C. Swart, 1995.
- [Viz63] V.G. Vizing, *The cartesian product of graphs*, Vychisl. Sistemy (1963), no. 9, 30–43.