

Bounds on distance parameters of graphs

by

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To

My parents.

Thank you for all your love and support.

Preface and Declaration

The study described in this thesis was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban, during the period January 2003 to May 2007. This thesis was completed under the supervision of Professor H. C. Swart and Professor P. Dankelmann.

This study represents original work by the author and has not been submitted in any form to another university nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

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Abstract

This thesis contains an investigation of bounds on distance measures, in particular, radius and diameter in terms of other graph parameters.

In a graph G , the distance between two vertices is the length of a shortest path between them. The eccentricity of a vertex v is the maximum distance from v to any vertex in G . The radius of G is the minimum eccentricity of a vertex, and the diameter of G is the maximum eccentricity of a vertex.

Vizing established an upper bound on the size of a graph of given order and radius. In Chapter 2, we establish similar sharp bounds on the size of a bipartite graph of given order and radius.

The inverse degree $r(G)$ of a graph G is defined as $r(G) = \sum_{v \in V} \frac{1}{\deg v}$. In Chapter 3, we prove that, if G is connected and of order n , then the diameter of G is less than $(3r + 2 + o(1))r(G) \frac{\log n}{\log \log n}$. This improves a bound given by Erdős et al. by a factor of approximately 2.

A graph G is a minimal claw-free graph if it contains no $K_{1,3}$ as an induced subgraph and if, for each edge e of G , $G - e$ contains an induced claw. In Chapter 4, we establish an upper bound on the diameter of a minimal claw-free graph of given order.

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Chapter 1

Introduction

The purpose of this chapter is to define the most important terms that will be used in this thesis and to present motivation for our study as well as provide relevant background. Terms not defined in this chapter will be defined in subsequent chapters as the need arises.

1.1 Graph Theory Terminology

A *graph* $G = (V(G), E(G))$ consists of a finite non-empty set $V(G)$ of elements called *vertices* and a (possibly empty) set $E(G)$ of 2-element subsets of $V(G)$ called *edges*. The number of elements in $V(G)$ is called the *order* and the number of elements in $E(G)$ is called the *size* of G . If G has only one vertex then we say that G is *trivial*; otherwise G is *non-trivial*. If $e = \{u, v\} \in E(G)$, then we say that u and v are adjacent, while e is *incident* with u and v . We also say that e *joins* u and v . We often write $e = uv$ instead of $\{u, v\}$.

The *degree* $\deg_G(v)$ of a vertex v of G is the number of edges incident with v . A vertex of degree 1 is called an *end-vertex*. The *minimum degree* $\delta(G)$ and the *maximum degree* $\Delta(G)$ are defined as the minimum and maximum, respectively, of the degrees of vertices in G . The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v in G ; while the *closed neighbourhood* $N_G[v]$ is the union of $\{v\}$ and its neighbourhood. If there is no ambiguity, we may omit the subscript or argument G in the above notations. The *inverse degree*, $r(G)$, of G is defined as the sum of the inverses of the degrees of the vertices of G , that is $r(G) = \sum_{v \in V(G)} \frac{1}{\deg v}$.

A *walk* W in a graph G is an alternating sequence

$$W : v_0, e_1, v_1, \dots, v_{k-1}, e_k, v_k$$

of vertices and edges such that $e_i = v_{i-1}v_i$ for $i = 1, 2, \dots, k$. A walk that starts from v_0 and ends at v_i in a graph is referred to as a $v_0 - v_i$ *walk*. Since the vertices that appear in a walk determine the edges in the walk, we can omit the edges in the description of a walk, and denote the walk W by $v_0v_1 \dots v_k$. We call k the *length* of the walk, and write $l(W) = k$. A *path* is a walk in which no vertex is repeated. A path $v_0v_1 \dots v_k$ that begins at vertex v_0 and ends at vertex v_k is called a $v_0 - v_k$ *path*. A *cycle* of length k is a walk $v_0v_1 \dots v_k$ in which $k \geq 3$, $v_0 = v_k$ and the vertices $\{v_1, v_2, \dots, v_k\}$ are distinct. A cycle of length k is referred to as a k -*cycle*. A graph which contains no cycles and in which there is a walk from each vertex to every other vertex in the graph is called a *tree*.

For given sets S and T , $S \subseteq T$ means that S is a subset of T , and $S \subset T$ means that S is a *proper subset* of T , that is, $S \subseteq T$ and $S \neq T$. A graph $H = (V(H), E(H))$ is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is a *spanning subgraph* of G . If a spanning subgraph G' of a graph G is a tree, then G' is called a *spanning tree* of G . The *complement* \bar{G} of a graph G is the graph with $V(\bar{G}) = V(G)$, and such that uv is an edge of \bar{G} if and only if uv is not an edge of G . The cardinality of a set S is denoted by $|S|$. If $S \subseteq V(G)$ is non-empty, then the *subgraph induced by S* is the maximal subgraph of G with vertex set S , and is denoted by $\langle S \rangle_G$. For two subsets S and T of $V(G)$, $[S, T]$ denotes the set of all edges which join a vertex in S to a vertex in T .

A graph G is *connected*, if for any two vertices u and v , there is a $u - v$ path in G . A *component* of a graph G is a maximal connected subgraph of G . For a subset S of $V(G)$, $G - S$ is the graph obtained from G by deleting every vertex in S and all edges incident with it; if $S = \{v\}$, then we write $G - S = G - v$. A subset $S \subset V(G)$ is called a *cutset* if its deletion increases the number of components in G , and a vertex v whose deletion increases the number of components is called a *cut-vertex*, and a *non-cut vertex* or *ncv* otherwise. For a subset F of $E(G)$, $G - F$ is the graph obtained from G by deleting all edges of F ; if $F = \{e\}$, then we simply write $G - F = G - e$. A subset $F \subset E(G)$ whose deletion increases the number of components of a graph G is an *edge-cut*. The *edge-connectivity* $\lambda(G)$ of a connected non-trivial graph G is the minimum cardinality of an edge-cut of G . If G is disconnected

or trivial, then we define $\lambda(G) = 0$. We say that G is k -edge-connected if $k \leq \lambda(G)$.

A *block* B of a graph G is a maximal connected subgraph of G that has no cut-vertices. Hence, for any cut-vertex v of G , $B - v$ lies entirely in one component of $G - v$. A vertex x is said to be *separated* from a vertex y by a vertex v if v lies on every $x - y$ path (i.e., if x and y are in different components of $G - v$).

The *union* $G_1 \cup G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The union of k disjoint copies of G is denoted by kG . The *join* $G_1 + G_2$ of two vertex disjoint graphs G_1 and G_2 is the graph consisting of the union $G_1 \cup G_2$, together with all edges of the type xy , where $x \in V(G_1)$ and $y \in V(G_2)$. For $k \geq 3$ vertex disjoint graphs G_1, G_2, \dots, G_k , the *sequential join* $G_1 + G_2 + \dots + G_k$ is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup \dots \cup (G_{k-1} + G_k)$. The sequential join of k disjoint copies of G will be denoted by $[k]G$, while $k_1G_1 + [k_2]G_2 + k_3G_3$ will denote the sequential join $k_1G_1 + G_2 + \dots + G_2 + k_3G_3$.

A *complete graph* K_n of order n is the graph in which each vertex is adjacent to all the other $n - 1$ vertices of K_n . A graph G is *bipartite* if it can be partitioned into two (non-empty) subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 to a vertex of V_2 . If each vertex of V_1 is joined to every vertex of V_2 , then G is called a *complete bipartite graph*, and is denoted by $K_{n,m}$, where $n = |V_1|$ and $m = |V_2|$, or vice versa. A *path-complete graph* $PK_{n,m}$ of order n and size m is the graph obtained by joining one end-vertex of a (possibly trivial) path to at least one vertex of a complete graph. For convenience, we define $PK_{1,0} \cong K_1$. Swart [47] showed in 1996 that for any $n \in \mathbb{N}$ and $m \in \{n - 1, n, \dots, \binom{n}{2}\}$, $PK_{n,m}$ is unique up to isomorphism, where, for $r \in \mathbb{R}$ and $k \in \mathbb{N}$ $\binom{r}{k} = \frac{r(r-1)\dots(r-k+1)}{k!}$.

The graph $K_{1,n}$ is called a *star*. We refer to the star $K_{1,3}$ as a *claw* with the vertex of degree 3 as its *centre*. For any graphs G and H , G is said to be *H-free* if it does not contain H as an induced subgraph. Let G be a non-empty claw-free graph. If the removal of any edge of G produces a graph which is not claw-free, then G is a *minimal claw-free graph*, briefly denoted as an m.c.f. graph or m.c.f.g.

1.2 Distance Concepts

All graphs considered henceforth are connected and non-trivial, unless otherwise specified. Let G be a given graph of order n . The *distance* $d_G(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest $u - v$ path in G . The *eccentricity* $e_G(v)$ of a vertex $v \in V(G)$ is the distance from v to a vertex farthest from it in G . The *radius* of G , $\text{rad}(G)$, is the minimum eccentricity of a vertex in G , that is, $\text{rad}(G) = \min_{v \in V(G)} e(v)$. The *diameter* of G , $\text{diam}(G)$, is the maximum eccentricity of a vertex in G , that is $\text{diam}(G) = \max_{v \in V(G)} e(v)$. If $\{u, v\} \subseteq V(G)$ is a pair of vertices of G with $d_G(u, v) = \text{diam}(G)$, then $\{u, v\}$ is referred to as a *diametral pair of vertices*. Any shortest path joining two diametral vertices is called a *diametral path*.

A vertex c of G is called *central* if $e_G(c) = \text{rad}(G)$. The *centre* $C(G)$ is the set of all central vertices in G . An *eccentric vertex* of a vertex v is a vertex farthest away from v . If there is only one such vertex u , then u is called the *unique eccentric point* (or *uep*) of v . A *conjugate vertex* v^* of a vertex v is a central vertex which has v as its uep. (So a vertex might have more than one conjugate vertex, or none.) A *conjugate pair* is a pair of central vertices, each of which is the uep of the other.

The *distance of* v in G is defined as

$$\sigma(v, G) = \sum_{u \in V(G)} d_G(v, u),$$

and the *distance of* G as

$$\sigma(G) = \sum_{\{u, v\} \subseteq V(G)} d_G(v, u) = \frac{1}{2} \sum_{v \in V(G)} \sigma(v, G).$$

The *average distance* $\mu(G)$ of a graph G of order $n \geq 2$ is the average of the distances between all unordered pairs of vertices of G . In other words,

$$\mu(G) = \frac{2\sigma(G)}{n(n-1)}.$$

One sees that $\mu(G)$ is the arithmetic mean of all non-zero distances in G . This ensures that $1 \leq \mu(G)$ with equality if and only if G is a complete graph.

The *i -th distance layer* $N_i(v)$ of a vertex $v \in V(G)$ is the set of vertices at distance i from v . (So $N_1(v) = N(v)$). $N_{\leq i}(v)$ is the set of vertices at

distance at most i from v , that is

$$N_{\leq i}(v) = \{u \in V(G) \mid d_G(v, u) \leq i\}.$$

Similarly, $N_{\geq i}(v) = \{u \in V(G) \mid d_G(v, u) \geq i\}$.

For non-empty subsets $V_1, V_2 \subseteq V(G)$, the distance between V_1 and V_2 , $d(V_1, V_2)$, is defined as the minimum value of $d_G(a, b)$ taken over all vertices $a \in V_1, b \in V_2$. Hence $d(V_1, V_2) = 0$ means that $V_1 \cap V_2 \neq \emptyset$. If $V_1 = \{a\}$ has a single vertex, we write $d(a, V_2)$ instead of $d(V_1, V_2)$.

A subgraph H of G is said to be distance-preserving from a vertex v in G if $d_H(v, u) = d_G(v, u)$ for all $u \in V(H)$. A spanning tree T of G is said to be *radius-preserving* if $\text{rad}(G) = \text{rad}(T)$.

We define a non-trivial graph G to be *vertex-radius-decreasing* if $\text{rad}(G - v) < \text{rad}(G)$ for every ncv v of G .

We say that G is *radius-critical* if $\text{rad}(G - u) \neq \text{rad}(G)$ for every vertex $u \in V(G)$.

1.3 Literature Review

1.3.1 Motivation and Background

The purpose of this subsection is to give some motivation for our research and to provide background for relevant results. Proofs of some of the results will be given in the next subsection.

Since discrete structures are naturally modelled by graphs, this provides a motivation for studying distances in graphs, both theoretically and its applications. In fact, the distance between two vertices is one of the most thoroughly studied concepts in graph theory, and a book devoted to this subject was written by Buckley and Harary [3].

An important motivation to study distance concepts is the application of distance parameters in analyzing transportation networks. Consider a transport network consisting of locations (cities, computer processes or telephone receivers for instance) and transportation links (railway links, links for data transport or telephone lines for instance). A graph may conveniently model such a network, where vertices correspond to locations and edges correspond to the transportation links between the locations. Often the travel time between two locations, is proportional to the distance between the corresponding vertices in the graph. In such a network, decision problems involving the

optimal selection of one or more sites to locate emergency facilities arises. If one wishes to place, say, an emergency facility like a hospital or a fire station, then a primary concern in choosing such a location is that the travel time/distance from the emergency facility to a location farthest from it is as small as possible. If the best location for the emergency facility is chosen, then the radius of the graph is a measure that indicates the travel time from the emergency facility to a location farthest away. Thus the radius of a graph is an important measure of centrality. The travel time between two locations which are farthest apart in the network, that is, the maximum travel time between any two locations, is proportional to the diameter of the corresponding graph. If any two locations are chosen at random, then on average, the travel time between them is proportional to the average distance of the graph.

Such network applications are not solely limited to transportation systems, and in fact occur in many diverse areas such as metabolic and gene regulation networks in cells (see [51]), ecology and economic interactions.

The central vertices in a network are of particular interest because they may play the role of organization hubs. In addition to mathematicians, many biologists, sociologists, historians and geographers (see [45], [51] for references) have been interested in the concept of centrality.

Using Dijkstra's algorithm, which determines the distance from a vertex of the graph to every other vertex in the graph, we can for a given graph compute the radius, diameter and average distance in polynomial time. However, if the graph is not given but we know some of its properties, like size, order, or minimum degree, or we know, say, that the graph is bipartite or claw-free, then we may be interested in knowing bounds on the parameters radius, diameter and average distance in terms of some of the known properties of the graph. In Chapter 4, we determine an upper bound on the diameter of a m.c.f.g. given its order. In the remainder of this subsection, we give other examples of such bounds.

1.3.2 Radius and Diameter

It is well-known and easily proved that for a graph G , we have

$$\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G).$$

In 1973, Ostrand [41] showed that this is the only restriction on the diameter in terms of radius. He also showed that for any given $r, d \in N$,

with $r \leq d \leq 2r - 2$, there exists a graph with radius r and diameter d . The minimum order of such a graph is $r + d$ and moreover, there are exactly $\lfloor \frac{d-r}{2} \rfloor + 1$ non-isomorphic graphs of order $r + d$, radius r and diameter d . Each graph consists of two paths $u_0 u_1 \dots u_d$ and $u_s (= v_0) v_1 v_2 \dots v_{r-1} u_{r+s} (= v_r)$ with only the vertices u_s and u_{r+s} in common, where $0 \leq s \leq \lfloor \frac{d-r}{2} \rfloor$.

Erdős, Pach, Pollack and Tuza [18] proved that if G is a graph of order n and minimum degree δ , then

$$\text{diam}(G) \leq \left\lfloor \frac{3n}{\delta + 1} \right\rfloor - 1.$$

The bound on diameter was also proved by several authors ([1], [29], [32] and [36]).

Erdős, Pach, Pollack and Tuza [18] proved that if G is a connected graph of order n and minimum degree $\delta(G) \geq 2$, then

$$\text{rad}(G) \leq \frac{3(n-3)}{2(\delta+1)} + 5,$$

and also constructed graphs that, apart from the additive constant, attain the bound and, moreover, they gave improved bounds for K_3 -free and C_4 -free graphs. Using different methods, Dankelmann, Dlamini and Swart [9, 10, 15] obtained the slightly stronger bound

$$\text{rad}(G) \leq \frac{3}{2} \left(\frac{n}{\delta + 1} \right) + 1.$$

Dankelmann, Mukwembi and Swart [37] proved that if G is a 3-edge-connected graph of order n , then

$$\text{rad}(G) \leq \frac{1}{3}n + \frac{17}{3},$$

and also constructed graphs to show that the bound is sharp apart from the additive constant.

Many conjectures of the computer programme GRAFFITI [22, 23] led to the discovery of relations between parameters that seemed to have no obvious inter-dependence. GRAFFITI conjectured that the radius of a graph is not more than its independence number. This was successfully proved by various authors (e.g. [25]) and a slightly stronger result can be found in [20]. Let G be a connected graph. Recall that G is radius-critical if $\text{rad}(G - u) \neq \text{rad}(G)$ for

every vertex $u \in V(G)$. Gliviak [27], among others, gives a survey of results on critical graphs. Whilst the deletion of an edge from G never decreases the radius of G and the addition of an edge of the complement of G to G never increases the radius of G , the inequality $\text{rad}(G - u) \geq \text{rad}(G)$ for every vertex u of G is not always true. Segawa [44] proved that if G is a connected graph and F is a subset of $E(G)$ for which $G - F$ is connected, then

$$\text{rad}(G - F) \leq (|F| + 1)\text{rad}(G) - \frac{|F|}{2}.$$

Fajtlowicz [21] characterized r -critical graphs, which are an important special class of critical graphs. A graph G is r -critical if G has radius r and every proper induced connected subgraph of G has radius strictly smaller than r . Fajtlowicz [21] defines certain graphs of radius $r \geq 2$ called r -ciliates, as follows. Let $C_{p,q}$ be a graph obtained from p disjoint copies of the path P_{q+1} of order $q+1$ by linking together one end vertex of each P_{q+1} in a p -cycle C_p . An r -ciliate is the graph $C_{2q,r-q}$, and Fajtlowicz proved that a graph G of radius $r \geq 2$ is r -critical if and only if G is an r -ciliate.

A graph G is called *edge-radius-decreasing* or *erd* if $\text{rad}(G + e) < \text{rad}(G)$ for every $e \in E(\overline{G})$. For example, any cycle of even order is erd, while no path is (since its endpoints can be joined to form a cycle of the same radius.) Erd graphs have been studied by Nishanov [39, 40], Harary and Thomassen [31] and Glivak, Knor and Soltés [28], but no simple characterization is known. Vizing [49] considered a special class of erd graphs - viz., those graphs of given order and radius with the maximum possible number of edges (see Theorem 3). This proof was further refined in [47]. In Chapter 2, we establish a similar sharp upper bound on the size of a connected, bipartite graph of given radius and order (see Theorem 4).

1.3.3 Average Distance

The concept of the distance of a graph was first introduced in 1947, by Wiener [50], a chemist and is thus often referred to as the Wiener index. Since then there have been numerous papers in chemistry dealing with applications of the average distance (see, for example, [43].) In organic chemistry, the vertices of a graph might represent carbon atoms in a molecule and the edges represent the chemical bonds between them. Wiener himself observed that the melting point of certain hydro-carbons is directly proportional to the Wiener index of the corresponding graph.

One might expect the average distance of a graph to be dependent on the radius or the diameter of the graph, but this is not the case. Plesník [42] showed that apart from the obvious bound

$$1 \leq \mu(G) \leq \text{diam}(G),$$

the average distance of a graph is essentially independent of its radius and diameter. Specifically, Plesník showed that for any given $r, d \in \mathbb{N}$ and $t, \varepsilon \in \mathbb{R}$, where $r \leq d \leq 2r, 1 \leq t \leq d$, and $\varepsilon > 0$, there exists a graph G with $\text{rad}(G) = r, \text{diam}(G) = d$ and $|\mu(G) - t| < \varepsilon$.

Many conjectures of the computer program GRAFFITI [22, 23] involve average distance. A well known example is the inequality $\mu(G) \leq \alpha(G)$, where $\alpha(G)$ is the independence number of G . This was proved by Chung [7] and improved by Dankelmann [8]. A GRAFFITI conjecture involving two distance parameters, $\text{rad}(G) \leq \mu(G) + r(G)$, was disproved by Dankelmann, Oellermann and Swart [13]. The, less unexpected, GRAFFITI conjecture $\mu(G) \leq n/\delta(G)$, where $\delta(G)$ is the minimum degree of G , generated considerable interest. Asymptotically stronger inequalities were proved by Kouider and Winkler [33] and Dankelmann and Entringer [11]. The GRAFFITI conjecture was finally settled by Beezer et al. [2], which improved the result in [33].

GRAFFITI also made the attractive conjecture $\mu(G) \leq r(G)$ (see [22, 23]). This conjecture, however, turned out not to be true as Erdős, Pach and Spencer [19] disproved it by constructing an infinite class of graphs with average distance at least $(\frac{2}{3} \lfloor \frac{r(G)}{3} \rfloor + o(1)) \frac{\log n}{\log \log n}$ and diameter at least $(2 \lfloor \frac{r(G)}{3} \rfloor + o(1)) \frac{\log n}{\log \log n}$. Furthermore, they proved the upper bound, $\text{diam}(G) \leq (6r(G) + 2 + o(1)) \frac{\log n}{\log \log n}$. In Chapter 3, we improve upon the upper bound by Erdős, Pach and Spencer by a factor of two. We show that

$$\text{diam}(G) \leq (3r(G) + 2 + o(1)) \frac{\log n}{\log \log n},$$

and thus $\mu(G) \leq (3r(G) + 2 + o(1)) \frac{\log n}{\log \log n}$.

1.3.4 Survey of Important Results

In this subsection, we survey some important results that are related to distance concepts. Some of these results will be used as lemmas in subsequent

chapters, while others are included because they provide interesting bounds on distance parameters. We give proofs of those results whose proofs are neither long nor obvious. Some of these results have already been mentioned in the previous subsection. Most of these results can be found in most Graph Theory textbooks (see, for example [3] and [5]).

Proposition 1 *Every connected non-trivial graph contains at least two ncv's, and the only graphs containing exactly two ncv's are paths.*

Proposition 2 *For any connected graph G , the centre $C(G)$ is contained in one block of G .*

The next three results deal with trees.

Proposition 3 *Let T be a tree of radius r . Then either $\text{diam}(T) = 2r$ and $C(T)$ contains exactly one vertex, or $\text{diam}(T) = 2r - 1$ and $C(T)$ consists of two adjacent vertices.*

Proposition 4 *In a tree, no vertex can be equidistant from two adjacent vertices.*

Proposition 5 *Let G be any connected graph, and v any vertex in G . Then G contains a spanning tree which is distance-preserving from v .*

Such a tree can be found using the breadth-first-search algorithm with v as root (see, for example, [3]). We will usually denote it by T_v .

The next results deal with spanning trees.

Proposition 6 *If T is a radius-preserving spanning tree of a graph G then $C(T) \subseteq C(G)$.*

Proof Let c be any central vertex of T . Since removing edges cannot decrease the eccentricity of any vertex, $e_G(c) \leq e_T(c) = \text{rad}(T) = \text{rad}(G)$. It follows that $e_G(c) = \text{rad}(G)$; i.e., that $c \in C(G)$. ■

Note that if a spanning tree T of a graph G is *not* radius-preserving, then $C(T)$ is not necessarily contained on $C(G)$.

Proposition 7 *Let c be any central vertex of a connected graph G , and let T_c be a spanning tree of G which is distance-preserving from c . Then $c \in C(T_c)$, and $\text{rad}(T_c) = \text{rad}(G)$.*

Proof Since T_c is distance-preserving from c , $\text{rad}(T_c) \leq e_{T_c}(c) = e_G(c) = \text{rad}(G)$. Since removing edges cannot decrease the eccentricity of any vertex, it follows that $\text{rad}(T_c) = \text{rad}(G)$ and that $c \in C(T_c)$. ■

Not all radius-preserving spanning trees, however, are distance-preserving from some vertex. Proposition 7 has another useful consequence:

Proposition 8 *For any connected graph G of order n and radius r ,*

$$r \leq \left\lfloor \frac{1}{2}n \right\rfloor.$$

Proof Let c be any central vertex of G , and let T_c be a spanning tree of G which is distance-preserving from c . By Proposition 7, $\text{rad}(T_c) = r$, and hence $\text{diam}(T_c) = 2r$ or $2r - 1$. Now let P be any diametral path of T_c , and note that P has $\text{diam}(T_c) + 1 \geq 2r$ vertices. It follows that $n \geq 2r$, and hence that $r \leq \lfloor \frac{1}{2}n \rfloor$. ■

It is tedious but not difficult to show that equality holds if and only if

- (1) G is a path or cycle, or
- (2) n is odd and G consists of a path or cycle of order $2r$, a vertex w , and one, two or three edges joining w to vertices which are at most distance 2 apart in $G - w$.

Theorem 1 [20] *Let G be a connected graph of order n and minimum degree $\delta \geq 2$. Then*

$$(i) \text{ diam}(G) \leq \frac{3n}{\delta + 1} - 1;$$

$$(ii) \text{ rad}(G) \leq \frac{3n}{2(\delta + 1)} + 5.$$

Furthermore, (i) and (ii) are tight apart from the exact value of the additive constants, and for every $\delta > 5$ equality can hold in (i) for infinitely many values of n .

Proof (i) Denote $\text{diam}(G)$ by d and let v be a vertex of G such that $e_G(v) = d$. By the condition on minimum degree, $|N_{i-1}(v)| + |N_i(v)| +$

$|N_{i+1}(v)| \geq \delta + 1$ for all integers i with $0 \leq i \leq d$, where $N_{-1}(v) = \emptyset = N_{d+1}(v)$. Define the integer k by $d = 2k + r$, $r \in \{0, 1, 2\}$. Hence,

$$n \geq \sum_{i=1}^k (|N_{3i-1}(v)| + |N_{3i}(v)| + |N_{3i+1}(v)|) \geq \frac{k+1}{\delta+1}.$$

Rearranging and using $k = \frac{d-r}{3} \geq \frac{d-2}{3}$ yields (i).

(ii) Let z be a fixed central vertex of G and denote $\text{rad}(G)$ by r . Form a spanning tree T of G that is distance-preserving from z . Since $N_r(z) \neq \emptyset$, let z_r be a fixed vertex in $N_r(z)$. For $y \in V(G)$ denote a $z - y$ shortest path in T by $T(z, y)$. Then it can be shown, (see [20], for example) that there exists a vertex $y \in N_s(z)$, where $s \geq r - 5$, for which no two vertices $u \in (V(T(z, y)) \cap N_{\geq 5}(z))$ and $v \in (V(T(z, z_r)) \cap N_{\geq 5}(z))$ are such that $d_G(u, v) \leq 2$. For any i , let $N'_i = \{x \in N_i(z) \mid d_G(x, V(T(z, z_r)) \cap N_{\geq 5}(z)) \leq 1\}$ and $N''_i = \{x \in N_i(z) \mid d_G(z, V(T(z, y)) \cap N_{\geq 5}(z)) \leq 1\}$. It follows that

$$\left(\bigcup_{i=4}^r N'_i \right) \cap \left(\bigcup_{i=4}^r N''_i \right) = \emptyset$$

and by the condition on minimum degree, we have $|N'_{i-1}| + |N'_i| + |N'_{i+1}| \geq \delta + 1$ for all integers i with $5 \leq i \leq r$, and $|N''_{i-1}| + |N''_i| + |N''_{i+1}| \geq \delta + 1$ for all integers i with $5 \leq i \leq s$. Bounding n from below yields

$$n \geq |N_{\geq 3}(z)| + \sum_{i=4}^r |N'_i| + \sum_{i=4}^{s+1} |N''_i| \geq \frac{1}{3}(2r - 10)(\delta + 1) + 3,$$

and we arrive at (ii).

To show that (i) and (ii) are tight apart from the exact value of the additive constants, consider the following graph. Given integers n, k, δ with $k > 1, \delta > 5$ and $n = k(\delta + 1) + 2$, let $G_{n,\delta} = G_0 + G_1 + \cdots + G_{3k-1}$, where

$$G_i = \begin{cases} K_1 & \text{if } i \equiv 0 \pmod{3} \text{ or } i \equiv 2 \pmod{3}, \\ K_\delta & \text{if } i = 1, 3k - 2, \\ K_{\delta-1} & \text{otherwise.} \end{cases}$$

Clearly, $G_{n,\delta}$ has minimum degree δ , n vertices, $\text{diam}(G_{n,\delta}) = 3 \left(\frac{n-2}{\delta+1} \right) - 1$ and $\text{rad}(G_{n,\delta}) = \left\lceil \frac{3(n-2)}{2(\delta+1)} - \frac{1}{2} \right\rceil$. \blacksquare

The trivial sharp restriction $1 \leq \mu(G)$ on the average distance of an arbitrary graph G can be greatly improved for a graph of given order and size.

Theorem 2 [17] *Let G be a graph of order $n \geq 2$ and size m . Then*

$$\sigma(G) \geq n(n-1) - m,$$

with equality if and only if $\text{diam}(G) \leq 2$.

Proof There are $\binom{n}{2}$ ordered pairs of vertices of which m are at distance 1 apart and $\binom{n}{2} - m$ are at distance at least two apart. It follows that $\sigma(G) \geq m + 2(\binom{n}{2} - m) = n(n-1) - m$, with equality if and only if there are no pairs of vertices at distance 3 or more apart, that is, if $\text{diam}(G) \leq 2$. ■

Corollary 1 *Let G be a graph of order $n \geq 2$ and size m . Then*

$$\mu(G) \geq 2 - \frac{2m}{n(n-1)},$$

with equality if and only if $\text{diam}(G) \leq 2$.

Clearly, the lower bound provided in Corollary 1 is sharp for λ -edge-connected and k -vertex-connected graphs, as complete graphs attain the bound. The same corollary also implies that the average distance of a graph G is minimized if G has maximum size and diameter at most 2. This leads to the following corollary, which is found in [42].

Corollary 2 [42] *Let G be a graph of order $n \geq 2$. Then we have the following sharp lower bounds on $\mu(G)$:*

- (a) 1, for arbitrary G (see [16] and [17]);
- (b) $2 - \frac{2}{n}$ if G is a tree (see [17] and [35]);
- (c) $2 - \frac{6n-12}{n(n-1)}$, if G is planar and $n \geq 3$ (see [42]);
- (d) $2 - \frac{4n-6}{n(n-1)}$, if G is outerplanar (see [42]);
- (e) $\frac{3}{2} - \frac{1}{2(n-1)}$, if n is even and G is triangle-free or bipartite (see [42]);
- (f) $\frac{3}{2} - \frac{1}{2n}$, if n is odd and G is triangle-free or bipartite (see [42]);
- (g) $\mu(T_{n,k})$, where $T_{n,k}$ is the k -partite Turán graph (see [48]).

Proof By Corollary 1, (a), (c) and (d) follow from the fact that $m \leq \binom{n}{2}$ for all graphs, $m \leq 3n - 6$ for planar graphs and $m \leq 2n - 3$ for outplanar graphs while (b) follows from that the fact that $m = n - 1$ for every tree; (e), (f) and (g) are obtained from the well-known Turán Theorem (see [34]). ■

Cerf, Cowan, Mullin and Stanton [4] gave lower bounds on the distance of regular graphs whereas Plesík [42] gave lower bounds on distance in terms of the order and diameter of a graph.

The techniques for constructing spanning tree developed by Dankelmann and Entringer in [11] are generalized in [15].

Lemma 1 [15] *Let G be a graph of order n , and let $i \geq 1$ be a given integer. Suppose that $|N_{\leq i}(v)| \geq f \geq 1$ holds for every vertex $v \in V(G)$. Then G contains a spanning tree T with $\mu(T) \leq \frac{(2i+1)n}{3f} + \frac{14i+1}{3}$. Moreover,*

$$(i) \text{ diam}(G) \leq \frac{2i+1}{f}n - 1;$$

$$(ii) \text{ rad}(G) \leq \frac{2i+1}{2f}n + i.$$

Setting i to be 1 above gives $|N(v)| \geq \delta + 1$ which implies the following corollary:

Corollary 3 [15] *If G is a connected graph of order n and minimum degree δ , then*

$$(i) \text{ diam}(G) \leq \frac{3n}{\delta+1} - 1,$$

$$(ii) \text{ rad}(G) \leq \frac{3n}{2(\delta+1)} + 1.$$

The next theorem is due to Vizing [49], where he considered a special class of erd graphs - viz., those graphs of given order and radius with the maximum possible number of edges.

Theorem 3 [49] *Let n and r be any natural numbers such that $n \geq 2r \geq 2$. Define $f(n, r)$ to be the maximum possible number of edges in a graph of order n and radius r , and $C(n, r)$ to be the set of all graphs with order n , radius r and $f(n, r)$ edges.*

For any natural numbers n and r such that $n \geq 2r \geq 2$,

- a) $f(n, 1) = \frac{1}{2}n(n - 1)$
b) $f(n, 2) = \frac{1}{2}n(n - 1) - \lceil \frac{1}{2}n \rceil = \lfloor \frac{1}{2}n(n - 2) \rfloor$
c) $f(n, r) = \frac{1}{2}(n^2 - 4rn + 5n + 4r^2 - 6r)$ for $n \geq 2r \geq 6$.

Proof The graph with radius 1 and the maximum possible number of edges is the complete graph. $C(n, 2)$ consists of all graphs obtained from K_n by removing $\lceil \frac{1}{2}n \rceil$ edges covering $V(K_n)$. Graphs in $C(n, r)$, $n \geq 2r \geq 6$, consist of a complete graph K_{n-2r} and a cycle C_{2r} , where every vertex of the K_{n-2r} is joined to the same three consecutive vertices of C_{2r} .

We use double induction on n and r to show that $f(n, r) \leq \frac{1}{2}(n^2 - 4rn + 5n + 4r^2 - 6r)$ for $n \geq 2r \geq 6$. Let G be any graph in $C(n, r)$ and if G is not a vertex-radius-decreasing graph - i.e., G contains a vertex v such that $\text{rad}(G - v) \geq r$, then the result follows easily using the induction hypothesis. So, G must then be a vertex-radius-decreasing graph and Vizing, then, considers if G contains at least one cut vertex. If G does, then the result follows easily using the induction hypothesis. If G has no cut vertices, then by Menger's Theorem, each pair of vertices of G is contained on a cycle of length at least $2r$. Let M be a shortest cycle of length at least $2r$ in G and let M have length l . It can be shown that M is an induced cycle of G and that no vertex in $V(G) - V(M)$ can have more than three neighbours on M . Since $n \geq l \geq 2r$,

$$\begin{aligned} f(n, r) &= |E(G - M)| + |E(M)| + |[V(G - M), V(M)]| \\ &\leq \frac{(n-l)(n-l-1)}{2} + l + 3(n-l) \\ &\leq \frac{1}{2}(n^2 - 4rn + 5n + 4r^2 - 6r). \end{aligned}$$

■

Chapter 2

The Number of Edges in a Bipartite Graph of Given Order and Radius

2.1 Introduction

As remarked in Chapter 1, a graph G is called edge-radius-decreasing if $\text{rad}(G + e) < \text{rad}(G)$ for every $e \notin E(G)$. Vizing [49] considered a special class of erd graphs - viz., those graphs of given order and radius with the maximum possible number of edges (see Theorem 3).

Similarly, in this chapter, we establish a similar sharp upper bound on the size of a connected, bipartite graph of given radius and order (see Theorem 4).

We will make frequent use of the following definitions. A vertex v is called a cut-vertex if $\{v\}$ is a cutset, and a non-cut-vertex or ncv otherwise. An eccentric vertex of a vertex v is a vertex farthest away from v . If there is only one such vertex u , then u is called the unique eccentric point (or uep) of v . A conjugate vertex v^* of a vertex v is a central vertex which has v as its uep. (So a vertex might have more than one conjugate vertex, or none.) A conjugate pair is a pair of central vertices, each of which is the uep of the other. We define a non-trivial graph G to be vertex-radius-decreasing if $\text{rad}(G - v) < \text{rad}(G)$ for every ncv v of G .

2.2 Preliminary Results

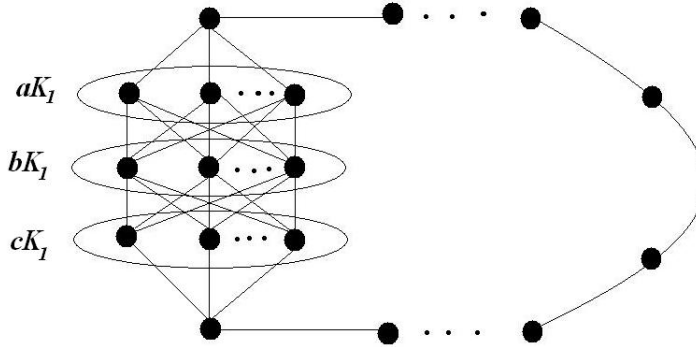


Figure 2.1: An example of a graph in \mathcal{B} .

Definition 1 The set $\mathcal{B}(n, r)$ consists of all graphs G obtained from C_{2r} with three consecutive vertices replaced by aK_1, bK_1, cK_1 , where $a + c = \lceil \frac{n-2r+3}{2} \rceil, b = \lfloor \frac{n-2r+3}{2} \rfloor$, or $a + c = \lfloor \frac{n-2r+3}{2} \rfloor, b = \lceil \frac{n-2r+3}{2} \rceil$. We shall use the notation $V'_1(G) = V(aK_1 \cup cK_1)$ and $V'_2(G) = V(bK_1)$. (See Figure 2.1).

Let $h(n, r) := \lfloor \frac{n^2}{4} \rfloor - nr + r^2 + 2n - 2r$ for $n \geq 2r \geq 8$.

In the proof of Lemma 2, we denote, for a vertex v of G , the star $\langle N_G[v] \rangle_G$ by $S_G(v)$.

Lemma 2 Let G be a connected bipartite graph of order n and radius at least $r \geq 4$. If $u, v \in V(G)$ with $d(u, v) \neq 2$, then $\deg u + \deg v \leq n - 2r + 4$. If $\deg u + \deg v = n - 2r + 4$ then $m(G) \leq h(n, r)$. If $\deg u + \deg v = n - 2r + 4$ and $m(G) = h(n, r)$, then G is one of the graphs in the family $\mathcal{B}(n, r)$.

Proof Let F be the union of the two stars $S_G(u)$ and $S_G(v)$. Since u and v have no common neighbours, F contains no cycle. Hence there exists a spanning tree T of G containing F . Let P be a diametral path of T . By $\text{rad}(T) \geq \text{rad}(G) \geq r$, we have $\text{diam}(T) \geq 2r - 1$; so P has at least $2r$

vertices. Since P contains at most two neighbours of u and v , respectively, we have

$$|V(T) - V(P)| \geq \deg_T(u) + \deg_T(v) - 4 = \deg_G(u) + \deg_G(v) - 4.$$

Hence

$$\deg_G(u) + \deg_G(v) \leq n - |V(P)| + 4 \leq n - 2r + 4,$$

as desired.

Now assume that $\deg u + \deg v = n - 2r + 4$. Then P has exactly $2r$ vertices, say, $P = w_0, w_1, \dots, w_{2r-1}$, $\text{rad}(T) = r$, and u and v are internal vertices of P , say $u = w_a$ and $v = w_b$; where (say) $a < b$. Moreover, T has the following properties:

- (a) each vertex not on a diametral path is an end-vertex of T and adjacent to u or to v ,
- (b) all vertices other than u or v have degree at most 2 in T .

To see that these two properties hold observe that, if one of them is violated, then a diametral path of T misses more than $\deg_G(u) + \deg_G(v) - 4$ vertices, and thus has fewer than $2r$ vertices, hence T has radius less than r , a contradiction.

It is clear that every spanning tree of G containing F has properties (a) and (b). We can choose T to also have the property of preserving the distance between u and v . This can be achieved by considering the union F' of F and a $u - v$ geodesic in G . Clearly F' is a (not necessarily spanning) subtree of G , so there exists a spanning tree T of G containing F' which has the desired property.

We now consider which edges G can contain, in addition to those of T . We show that, if $e \in E(G) - E(T)$, then either

- (i) $e = w_0w_{2r-1}$, or
- (ii) e joins a vertex in $N(u)$ to a vertex in $N(v)$, or
- (iii) $e = xw_{a+2}$ or $e = xw_{a-2}$ for some vertex $x \in N(w_a) - V(P)$, or
- (iv) $e = xw_{b+2}$ or $e = xw_{b-2}$ for some vertex $x \in N(w_b) - V(P)$.

Note that the indices are taken modulo $2r$, so if $a = 2r - 2$ then a vertex $x \in N(w_a)$ can be joined to w_0 .

First assume that e joins two vertices of P . Suppose that $e = w_iw_j$ with $w_iw_j \neq w_0w_{2r-1}$. Then at least one of the end points of e , say w_i , has degree at least 3 in $T + e$. Let w_i be such a vertex. Clearly, e is not incident with u or v since u and v have the same degree in G and in T , so $w_i \neq w_a, w_b$.

Consider the union of three stars $S_G(u)$, $S_G(v)$ and $S_{T+e}(w_i)$, which we denote by F_1 . First we show that F_1 contains a cycle. Suppose to the contrary that F_1 is a forest. Then there exists a spanning tree T_1 of G containing F_1 . In T_1 , vertices u and v have degree $\deg_G(u) + \deg_G(v)$, respectively, but w_i has degree at least 3, so T_1 does not have property (b), a contradiction. This shows that F_1 contains a cycle C_1 . Clearly, C_1 must contain w_i and either w_a and its two neighbours on P or w_b and its two neighbours on P . Without loss of generality we assume the former, so C_1 contains $w_a, w_{a+1}, w_i, w_{a-1}$. So $i = a + 2$ and $e = w_{a-1}w_{a+2}$ or $i = a - 2$ and $e = w_{a-2}w_{a+1}$. If $e = w_{a-1}w_{a+2}$ consider the tree $T' = T - w_{a+1}w_{a+2} + w_{a-1}w_{a+2}$. Clearly, u and v have full degree in T' , but w_{a-1} has degree 3, contradicting property (b). Similarly, if $e = w_{a-2}w_{a+1}$ the tree $T'' = T - w_{a-2}w_{a-1} + w_{a-2}w_{a+1}$ does not have property (b), a contradiction. Hence w_0w_{2r-1} is the only edge between two vertices of P present in G but not in T .

Now let $e \in E(G) - E(T)$ be an edge joining a vertex $x \in N(w_a) - V(P)$ to a vertex w_i on P . Suppose that e is not of type (iii), i.e., that $i \neq a - 2, a + 2$. Then either $i \geq a + 4$ or $i \leq a - 4$. (Note that in this part of the proof, subscripts are not taken modulo $2r$.)

CASE 1: w_i is not a neighbour of w_b on P .

So $i \neq b - 1, b + 1$. If $i \geq a + 3$ consider the graph $T + xw_i$, which has the unique cycle $w_a w_{a+1} w_{a+2} \dots, w_i x w_a$. Clearly, all edges in the set $E' := \{w_{a+1}w_{a+2}, w_{a+2}w_{a+3}, \dots, w_{i-2}w_{i-1}\}$ are on this cycle, so $T + xw_i - e' =: T(e')$ is a spanning tree of G for all $e' \in E'$. Since vertex w_i has degree 3 in $T(e')$, and vertex w_a has full degree, property (b) implies that in $T(e')$ vertex w_b does not have full degree. So each edge in E' is incident with vertex w_b . Since only two edges of E' can be incident with w_b , we have $E' = \{w_{a+1}w_{a+2}, w_{a+2}w_{a+3}\}$ and $w_b = w_{a+2}$. But then w_a and w_b are at distance 2, contradicting our hypothesis. If $i \leq a - 3$ then similar arguments lead to the same conclusion.

CASE 2: w_i is a neighbour of w_b on P .

So $i = b - 1$ or $i = b + 1$. Then $w_a x w_i w_b$ is a $(w_a - w_b)$ -path of length 3, so w_a and w_b are at distance 1 or 3 in T (and in G). First consider the case that w_a and w_b are at distance 1, so $b = a + 1$. Then $i = b + 1$ (since $i = b - 1 = a$ is not possible) and thus $i = a + 2$; so $e = xw_{a+2}$, as desired. Now consider the case that w_a and w_b are at distance 3; hence $b = a + 3$. But then $i \in \{b - 1, b + 1\} = \{a + 2, a + 4\}$. If $i = a + 2$ then $e = xw_{a+2}$, so e is of type (iii). That leaves the case $i = b + 1 = a + 4$. We show that $a + 4 = 2r - 1$, i.e., that w_{a+4} is an end-vertex of P . Suppose to the contrary

that $2r - 1 > a + 4$. In the tree $T - w_{a+1}w_{a+2} + xw_{a+4} =: T'$, vertices u and v have full degree and vertex w_{a+4} has three neighbours, contradicting property (b). Hence $a + 4 = 2r - 1$.

We now show that not all vertices in $N(w_b)$ are adjacent to a vertex in $N(w_a)$. Suppose to the contrary that each vertex $y \in N(w_b)$ has a neighbour $y' \in N(w_a)$. Then we can reduce the distance from w_a to the end-vertices in $N_T(w_b)$ as follows. Consider the tree

$$T'' = T - \{yw_b | y \in N_T(w_b), y \neq w_{b-1}\} + \{yy' | y \in N_T(w_b), y \neq w_{b-1}\}.$$

Since every end-vertex of T'' , except possibly w_0 , is within distance 3 of w_a , the distance from w_0 to any end-vertex of T'' is at most $d_{T''}(w_0, w_a) + 3 = 2r - 2$, while any two end-vertices of T'' , other than w_0 , are within distance at most 5. Hence the diameter of T'' is at most $2r - 1$, which implies $\text{rad}(T'') \leq r - 1$, a contradiction to $\text{rad}(G) \geq r$. This proves that there exist a vertex $y \in N_G(w_b)$ not adjacent to any vertex in $N_G(w_a)$. Hence, we can obtain, if necessary by renaming y and w_{2r-1} , that no vertex in $N_G(w_a)$ is adjacent to vertex w_{a+4} . Hence property (iv) holds.

We now show that in addition to properties (i)-(iv) the following holds:

- (v) if $x \in N(w_a)$, then at most one of the edges xw_{a-2}, xw_{a+2} is present in G ,
- (vi) if $y \in N(w_b)$, then at most one of the edges xw_{b-2}, xw_{b+2} is present in G ,
- (vii) if $xy \in E(G)$ for some $x \in N(w_a), y \in N(w_b)$, then $b = a + 1$ or $b = a + 3$.

To prove (v), suppose that a vertex $x \in N(w_a)$ is adjacent to w_{a-2} and to w_{a+2} . Then the tree $T' := T - \{w_{a-2}w_{a-1}, w_{a+1}w_{a+2}\} + \{xw_{a-2}, xw_{a+2}\}$ preserves the degrees of w_a and w_b , but has another vertex, namely x of degree 3. This contradicts property (b), and so (v) holds. Similarly, (vi) holds. Property (vii) follows directly from the fact that T preserves the distance between w_a and w_b in G .

Now the bound on the size of G follows easily. In addition to the edges of T , G can only have edges satisfying (i)-(vii). There is only one edge satisfying (i), namely the edge w_0w_{2r-1} . The graph G has at most $(\deg w_a - 2)(\deg w_b - 2) \leq \lfloor \frac{(n-2r)^2}{4} \rfloor$ edges of the form xy , where $x \in N(w_a) - V(P)$ and $y \in N(w_b) - V(P)$, that are not in T . Finally, each vertex not on P has at most one edge, not in T joining it to a vertex on P . Hence

$$\begin{aligned}
m(G) &\leq m(T) + 1 + (\deg w_a - 2)(\deg w_b - 2) + (n - |V(P)|) \\
&\leq n + \lfloor \frac{(n-2r)^2}{4} \rfloor + n - 2r \\
&= h(n, r),
\end{aligned}$$

as desired.

From the above proof it follows that, if $m(G) = h(n, r)$, then $\langle [N(w_a) - V(P) \text{ and } N(w_b) - V(P)] \rangle$ is a balanced, complete bipartite graph of order $n - 2r$, $w_0 w_{2r-1} \in E(G)$ and every vertex in $N(w_a) - V(P)$ (or in $N(w_b) - V(P)$) is adjacent to either w_{a+2} or w_{a-2} (or to either w_{b-2} or w_{b+2} , respectively.)

We show next that if $x \in N(w_a) - V(P)$ and $y \in N(w_b) - V(P)$, then it is impossible that both xw_{a-2} and yw_{b+2} are edges in G . Suppose to the contrary that $xw_{a-2}, yw_{b+2} \in E(G)$. Then $b = a + 3$ as otherwise $\text{rad}(G) < r$ and consider the spanning tree T''' of G , where

$$T''' =: T - \{w_{b+1}w_{b+2}, w_{a+1}w_{a+2}, w_{a-1}w_{a-2}\} + \{yw_{b+2}, xy, xw_{a-2}\}.$$

In T''' the vertices w_a and w_b have full degree, while x and y are both of degree 3, which contradicts (b). Consequently, it follows that $G \in \mathcal{B}(n, r)$. ■

We now present propositions that will be needed in the proof of our main result.

Proposition 9 [49] *For any connected graph G of order n , $\Delta(G) \leq n - 2\text{rad}(G) + 2$.*

Proof Let v be a vertex of maximum degree in G , and let T_v be a distance-preserving spanning tree of G with v as root, so $\deg_{T_v}(v) = \deg_G(v) = \Delta(G)$.

Let P be a diametral path of T_v ; then P has length $\text{diam}(T_v) \geq 2\text{rad}(T_v) - 1 \geq 2\text{rad}(G) - 1$. So P contains at least $2\text{rad}(G)$ vertices, at most two of which can be neighbours of v (since if P contained three neighbours of v , we would have a cycle in T). Hence, there must be at least $\Delta(G) - 2$ neighbours of v which are not on P . It follows that $n \geq 2\text{rad}(G) + \Delta(G) - 2$. ■

Definition 2 *Given integers n, d with $3 \leq d \leq n$, define the path-complete bipartite graph as follows:*

$$G(n, d) = [d - 1 - t]K_1 + \left\lfloor \frac{n - d + 1}{2} \right\rfloor K_1 + \left\lceil \frac{n - d + 1}{2} \right\rceil K_1 + [t]K_1,$$

where $1 \leq t \leq d - 2$.

Proposition 10 [15] *Let G be a bipartite graph of order n and diameter $d \geq 3$. Then*

$$m(G) \leq \left\lfloor \frac{n^2}{4} - \frac{nd}{2} + \frac{3n}{2} + \frac{d^2}{4} - \frac{d}{2} - \frac{7}{4} \right\rfloor,$$

and a path-complete bipartite graph $G(n, d)$ attains the bound.

Proposition 11 [21] *Let $\{v, v^*\}$ be any conjugate pair in a graph $G \not\cong K_2$. If $G - \{v, v^*\}$ is connected, then removing v and v^* from G cannot decrease the radius.*

Proof Let c be a central vertex of $G - \{v, v^*\}$, and let w be an eccentric vertex of c in G . Then $d_G(c, w) \geq \text{rad}(G)$. Since v and v^* are within distance $\text{rad}(G) - 1$ from all vertices in G except each other, w cannot be v or v^* . Since removing v and v^* cannot decrease the distance between c and w , it follows that $e_{G - \{v, v^*\}}(c) \geq d_G(c, w)$, and hence that $\text{rad}(G - \{v, v^*\}) \geq \text{rad}(G)$. ■

Proposition 12 [26, 21] *Let G be a graph containing an ncv v . Then $\text{rad}(G - v) < \text{rad}(G)$ if and only if v has a conjugate vertex, and in this case $\text{rad}(G - v) = \text{rad}(G) - 1$.*

Proof Let $\text{rad}(G - v) < \text{rad}(G)$, and let c be any central vertex of $G - v$. So $e_{G - v}(c) = \text{rad}(G - v) \leq \text{rad}(G) - 1 \leq e_G(c) - 1$. Since removing v cannot decrease the distance between any of the remaining vertices, it follows that v is the uep of c in G . Furthermore, since c is still at distance $e_G(c) - 1$ from the neighbours of v , $\text{rad}(G - v) = e_{G - v}(c) \geq e_G(c) - 1 \geq \text{rad}(G) - 1$. It follows that c is a central vertex of G and that $\text{rad}(G - v) = \text{rad}(G) - 1$.

Conversely, let v be the uep of some central vertex c in G . Then removing v cannot increase the distance between c and any other vertex w since v cannot lie on a shortest $c - w$ path. It follows that $e_{G - v}(c) < e_G(c)$, and hence that $\text{rad}(G - v) < \text{rad}(G)$. ■

Proposition 13 [21] *Let G be a vertex-radius-decreasing graph, and v a ncv of G . If v is not central, then all its conjugate vertices are cut-vertices. If v is central, then it has exactly one conjugate vertex v^* , and v^* is a ncv (so v and v^* form a conjugate pair).*

Proof By Proposition 12, v has a conjugate vertex v^* . If v^* is also a ncv of G , then, since G is vertex-radius-decreasing, v^* must have a conjugate

vertex v^{**} . Hence $d_G(v^*, v^{**}) = \text{rad}(G)$ - but the only vertex at distance $\text{rad}(G)$ from v^* is v . It follows that v must be central, and v^{**} must be v .

This proves firstly that if v is not central, then all its conjugate vertices are cut-vertices, and secondly that v cannot have two conjugate vertices which are ncvs. (Otherwise both would need to have v as a conjugate vertex; i.e., both would need to be the unique eccentric point of v .)

If v^* is a cut-vertex, let w be any vertex separated from v by v^* . Then $e_G(v) \geq d_G(v, w) = d_G(v, v^*) + d_G(v^*, w) \geq \text{rad}(G) + 1$; i.e., v is non-central. It follows that if v is central, then it has a unique conjugate vertex v^* and v^* is a ncv. ■

Proposition 14 [26, 21] *A graph G of order n is a vertex-radius-decreasing block if and only if G is self-centered, n is even, and $V(G)$ can be partitioned into conjugate pairs.*

Proof This follows as a direct consequence of Propositions 12 and 13. ■

Proposition 15 [21] *In any vertex-radius-decreasing graph containing at least one cut-vertex, every ncv has degree 1.*

Proof Let G be a vertex-radius-decreasing graph containing a ncv v of degree at least 2, and let x and y be any neighbours of v . We will prove that then G has no cut-vertices.

By Proposition 12, v has a conjugate vertex v^* such that $d_G(v^*, v) = \text{rad}(G)$ and $d_G(v^*, u) \leq \text{rad}(G) - 1$ for every $u \in V(G) - \{v\}$. Hence, $d_G(v^*, x) = \text{rad}(G) - 1$.

It follows that, if u is any vertex in $V(G) - \{v, x\}$, then no shortest $v^* - u$ path can contain x . In particular, $G - x$ contains a $v^* - y$ path and hence a $v^* - v$ path. So $G - x$ is connected.

Since $x \in N(v)$ was chosen arbitrarily, it follows that no neighbour of v is a cut-vertex. Since every neighbour of v has degree at least 2 (otherwise v would have been a cut-vertex), it follows in the same way that no vertex distance 2 apart from v is a cut-vertex, and so on. Hence, G contains no cut-vertices. ■

Proposition 16 *Let G be a bipartite graph and let v be a vertex in a partite set $V_i, i = 1, 2$. Then $\deg v \leq |V_{3-i}| - \text{rad}(G) + 2$.*

Proof Let T_v be a distance-preserving spanning tree of G with v as its root; so $\deg_{T_v}(v) = \deg_G(v)$. Let P be a diametral path of T_v . Then P has length $\text{diam}(T_v) \geq 2 \text{rad}(T_v) - 1 \geq 2 \text{rad}(G) - 1$. So P contains at least $2 \text{rad}(G)$ vertices, with at least $\text{rad}(G)$ of them in V_{3-i} . Moreover, at most two of them can be neighbours of v on P . So there are at least $\deg v - 2$ neighbours of v which are not on P . So

$$|V_{3-i}| \geq \text{rad}(G) + \deg v - 2,$$

and Proposition 16 follows. ■

2.3 The Main Result

In this section we shall obtain a bound on the size of a bipartite graph of order n and radius r .

The following lemma deals with the case $r = 4$ of our main theorem.

Lemma 3 *Let G be a bipartite graph of order n and radius 4. Then*

$$m(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor - 2n + 8 \text{ for } n \geq 8.$$

Moreover, if $m(G) = \left\lfloor \frac{n^2}{4} \right\rfloor - 2n + 8$, then $G \in \mathcal{B}(n, 4)$.

Proof Since $\text{rad}(G) = 4$, there exists a vertex $x \in V(G)$ such that $e(x) = 4$. Moreover, there is a vertex $x_4 \in V(G)$ such that $d(x, x_4) = 4$, having $xx_1x_2x_3x_4$ as a shortest $x - x_4$ path in G . For $1 \leq i \leq 4$, let N_i be the i th distance layer of x . So $x_i \in N_i$ for $1 \leq i \leq 4$. Since $e(x_1) \geq 4$, there is a vertex $\bar{x}_1 \in V(G)$ such that $d(x_1, \bar{x}_1) = 4$. Thus $\bar{x}_1 \in N_3$ and $x_2\bar{x}_1 \notin E(G)$. But \bar{x}_1 must have a neighbour in N_2 , say x'_2 , where $x'_2 \neq x_2$ and $x_1x'_2 \notin E(G)$. Moreover, x'_2 must have a neighbour in N_1 that is not x_1 , say x'_1 . Since $e(x_2) \geq 4$, there is a vertex $\bar{x}_2 \in V(G)$ such that $d(x_2, \bar{x}_2) = 4$, where $\bar{x}_2 \notin \{x, x_4\}$.

Suppose, without loss of generality, that $x \in V_1$. Then certainly $\{x, x_4\}$ and $\{x_2, \bar{x}_2\}$ are disjoint pairs of vertices in V_1 that are distance 4 apart. Since $e(x'_1) \geq 4$, there is a vertex $\bar{x}'_1 \in V_2$ such that $d(x'_1, \bar{x}'_1) = 4$, where $\bar{x}'_1 \notin \{x_1, \bar{x}_1\}$. Then certainly $\{x_1, \bar{x}_1\}$ and $\{x'_1, \bar{x}'_1\}$ are disjoint pairs of vertices in V_2 that are distance 4 apart.

So there exist four disjoint pairs of vertices, say u_i and v_i , such that $d(u_i, v_i) = 4$ for $1 \leq i \leq 4$, where $u_i, v_i \in V_1$ for $i = 1, 2$ and $u_i, v_i \in V_2$ for $i = 3, 4$. Denote by \overline{G} , the bipartite complement of G ; that is the graph with bipartition (V_1, V_2) such that for $u \in V_1, v \in V_2, uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. Let $V'_1 = V_1 - \{u_1, v_1, u_2, v_2\}$ and $V'_2 = V_2 - \{u_3, v_3, u_4, v_4\}$.

We show that $m(\overline{G}) \geq 2n - 8$.

For each vertex $w \in V_2$, there exist edges $e_1(w)$ and $e_2(w)$ joining w to a vertex in $\{u_1, v_1\}$ and $\{u_2, v_2\}$, respectively, in \overline{G} since otherwise $d_G(u_1, v_1) = 2$. Similarly, for each vertex $w \in V_1$, there exist edges $e_3(w)$ and $e_4(w)$ joining w to a vertex in $\{u_3, v_3\}$ and $\{u_4, v_4\}$. Clearly, the subsets

$$A = \{e_1(w)|w \in V_2\} \cup \{e_2(w)|w \in V_2\},$$

$$B = \{e_3(w)|w \in V'_1\} \cup \{e_4(w)|w \in V'_1\}$$

of $E(\overline{G})$ are disjoint. Hence,

$$m(\overline{G}) \geq |A| + |B| = 2|V_2| + 2(|V_1| - 4) = 2n - 8.$$

We have $m(G) + m(\overline{G}) \leq \lfloor \frac{n^2}{4} \rfloor$ since the maximum size of a complete bipartite graph is $\lfloor \frac{n^2}{4} \rfloor$. Hence

$$m(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor - m(\overline{G}) \leq \left\lfloor \frac{n^2}{4} \right\rfloor - 2n + 8,$$

as required.

We shall now show that if $m(G) = \lfloor \frac{n^2}{4} \rfloor - 2n + 8$, then $G \in \mathcal{B}(n, 4)$.

Suppose that $m(G) = \lfloor \frac{n^2}{4} \rfloor - 2n + 8$. Then, $m(\overline{G}) = 2n - 8$, and hence, $m(\overline{G}) = |A| + |B| = 2|V_2| + 2(|V_1| - 4)$. Hence, in G , every vertex in V_2 is adjacent to exactly one vertex in $\{u_1, v_1\}$ and exactly one vertex in $\{u_2, v_2\}$, and every vertex in V'_1 . Every vertex in V'_1 is adjacent to exactly one vertex in $\{u_3, v_3\}$ and exactly one vertex in $\{u_4, v_4\}$. Let x, y be an arbitrary adjacent pair of vertices in $V'_1 \cup V'_2$. Then $\deg_G(x) + \deg_G(y) = |V_1| - 2 + |V_2| - 2 = n - 4$. Hence, by Lemma 2, the result follows. \blacksquare

We now present our main theorem.

Theorem 4 *For natural numbers n and r such that $n \geq 2r \geq 2$, the maximum number of edges in a bipartite graph of order n and radius at least r is $b(n, r)$, where*

- a) $b(n, 1) = n - 1$,
- b) $b(n, 2) = \lfloor \frac{n^2}{4} \rfloor$,
- c) $b(n, 3) = \lfloor \frac{n^2}{4} \rfloor - \lfloor \frac{n}{2} \rfloor$,
- d) $b(n, r) = \lfloor \frac{n^2}{4} \rfloor - nr + r^2 + 2(n - r)$ for $n \geq 2r \geq 8$.

The bipartite graph with radius 1 and the maximum number of edges is the star $K_{1, n-1}$. The bipartite graph with radius 2 and the maximum number of edges is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$. The bipartite graph with radius 3 and maximum number of edges is obtained from the complete graph $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$, by the removal of a minimum edge cover. If G is a bipartite graph with radius 4 and the maximum number of edges, then $G \in \mathcal{B}(n, r)$.

Proof a) The only bipartite graph with radius 1 and order n is the star $K_{1, n-1}$, which has $n - 1$ edges.

b) The bipartite graph with radius 2 and the maximum number of edges is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$ which has $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n^2}{4} \rfloor$ edges.

c) Let G be a bipartite graph of order n , radius 3 and partite sets V_1 and V_2 . Since $\text{rad}(G) = 3$, every vertex in V_1 must be non-adjacent to at least one vertex in V_2 , and vice versa. Thus, $m(\overline{G}) \geq \lfloor \frac{n}{2} \rfloor$, and since the maximum size of a complete bipartite graph is $\lfloor \frac{n^2}{4} \rfloor$, we have $m(G) \leq \lfloor \frac{n^2}{4} \rfloor - m(\overline{G})$, and thus $m(G) \leq \lfloor \frac{n^2}{4} \rfloor - \lfloor \frac{n}{2} \rfloor$.

d) Let G be a bipartite graph of order n , radius at least r and maximum size with partite sets V_1 and V_2 .

By double induction, we prove that if G has order n and $\text{rad}(G) \geq r$, then $m(G) \leq b(n, r)$ for $n \geq 2r \geq 8$, and $m(G) = b(n, r)$ if and only if $G \in \mathcal{B}(n, r)$.

We first show the inequality for the case $n = 2r$, i.e., we show that $m(G) \leq b(2r, r)$ for $r \geq 4$.

Let G be a graph of radius r and order $2r$. By Proposition 9, $\Delta(G) \leq n - 2r + 2 = 2$. It follows that $m(G) \leq \frac{1}{2}n\Delta(G) \leq n = 2r = b(2r, r)$. Moreover, G must be a cycle of length $2r$ and thus $G \in \mathcal{B}(2r, r)$.

For the case $r = 4$, it has been shown in Lemma 3 that, for $n \geq 8$, $m(G) \leq b(n, 4)$ and if $m(G) = b(n, 4)$, $G \in \mathcal{B}(n, 4)$.

Now let n and r be natural numbers such that $r \geq 5$ and $n \geq 2r + 1$ and assume validity of the theorem for all bipartite graphs of order n' and radius

at least r' , where either $4 \leq r' \leq r - 1$ or else $r' = r$ and $2r \leq n' \leq n - 1$. Let G be any bipartite graph of order n and radius at least r .

Claim 1 *If $\{x, x^*\}$ is a conjugate pair of vertices in G , and the graph $G - \{x, x^*\}$ is disconnected, then $m(G) \leq b(n, r)$ and if $m(G) = b(n, r)$, then $G \in \mathcal{B}(n, r)$.*

Let $S = \{x, x^*\}$. Let G_1, G_2, \dots, G_k be the components of $G - S$. Let $G_x = \langle V(G_1) \cup S \rangle_G$ and $G_y = \langle V(G_2) \cup \dots \cup V(G_k) \cup S \rangle_G$. Note that G_y is connected for otherwise either x or x^* is not central. Suppose $n(G_x) = t$ and thus $n(G_y) = n - t + 2$. Moreover $\text{diam}(G_x), \text{diam}(G_y) \geq r$ and thus $r + 1 \leq t \leq n - r + 1$. Moreover by Proposition 10,

$$\begin{aligned} m(G_x) + m(G_y) &\leq \lfloor \frac{n^2}{4} - \frac{nt}{2} + \frac{5n}{2} + \frac{1}{2} + \frac{t^2}{2} - \frac{nr}{2} - 2r + \frac{r^2}{2} - t \rfloor \\ &= \lfloor \frac{n^2}{4} - nr + r^2 + 2n - 2r + \frac{1}{2}(t - r - 1)(t - n + r - 1) \rfloor \\ &\leq b(n, r) \end{aligned}$$

since $r + 1 \leq t \leq n - r + 1$ and therefore $\frac{1}{2}(t - r - 1)(t - n + r - 1) \leq 0$.

If $m(G) = b(n, r)$, then equality holds throughout the above inequalities, and it follows that G_x and G_y are both graphs of diameter r and maximum size, given their orders.

Moreover, $t = r + 1$ or $t = n - r + 1$. Without loss of generality, say $n(G_x) = n - r + 1$ and thus $n(G_y) = r + 1$. Since $\text{diam}(G_x) = r$ and by Proposition 10, $G_x \cong G(n - r + 1, r) = [r - 2]K_1 + \lfloor \frac{n - 2r - 2}{2} \rfloor K_1 + \lceil \frac{n - 2r - 2}{2} \rceil K_1 + K_1$. So G_x contains partite sets X and Y where $|X| = \lfloor \frac{n}{2} \rfloor - r + 1$, and $|Y| = \lfloor \frac{n}{2} \rfloor - r + 1$, where every vertex in X has degree $\lfloor \frac{n}{2} \rfloor - r + 1 + 1 = \lfloor \frac{n}{2} \rfloor - r + 2$, and every vertex in Y has degree $\lfloor \frac{n}{2} \rfloor - r + 1 + 1 = \lfloor \frac{n}{2} \rfloor - r + 2$. So G contains adjacent vertices, $x \in X$ and $y \in Y$, such that $\deg x + \deg y = n - 2r + 4$. It follows from Lemma 2 that $G \in \mathcal{B}(n, r)$.

Claim 2 *If G contains a conjugate pair of vertices then $m(G) \leq b(n, r)$. If $m(G) = b(n, r)$, then $G \in \mathcal{B}(n, r)$.*

Let $\{x, x^*\}$ be a conjugate pair of vertices in G . By Claim 1, we may assume that $G^* = G - \{x, x^*\}$ is connected. Then by Proposition 11, $\text{rad}(G^*) \geq r$. By Lemma 2, we need only consider the case where $\deg x + \deg x^* < n - 2r + 4$. Moreover, by the induction hypothesis, we know that

$m(G^*) \leq b(n-2, r)$. Hence,

$$\begin{aligned}
m(G) &\leq m(G^*) + \deg x + \deg x^* \\
&\leq b(n-2, r) + n - 2r + 3 \\
&= \left\lfloor \left(\frac{n-2}{2}\right)^2 \right\rfloor - (n-2)r + r^2 + 2(n-2-r) + n - 2r + 3 \\
&= \left\lfloor \frac{n^2}{4} \right\rfloor - nr + r^2 + 2n - 2r \\
&= b(n, r),
\end{aligned}$$

as required.

If $m(G) = b(n, r)$, then we have equality throughout i.e., $m(G^*) = b(n-2, r)$ and $\deg x + \deg x^* = n - 2r + 3$. Without loss of generality, say $\deg x \geq \deg x^*$. Then, $\deg x \geq \lfloor \frac{n}{2} \rfloor - r + 2$.

By the induction hypothesis, $G^* \in \mathcal{B}(n-2, r)$ and so in G^* , $|V'_1(G^*)| = \lceil \frac{n-2-2r+3}{2} \rceil = \lfloor \frac{n}{2} \rfloor - r + 1$ and $|V'_2(G^*)| = \lfloor \frac{n-2-2r+3}{2} \rfloor = \lceil \frac{n}{2} \rceil - r$ or $|V'_1(G^*)| = \lceil \frac{n}{2} \rceil - r$, $|V'_2(G^*)| = \lfloor \frac{n}{2} \rfloor - r + 1$.

Since $n(G^*) \geq 2r$ and $n(G^*) + 2 = n$, $n \geq 2r + 2$. Thus

$$\deg x \geq \left\lfloor \frac{n}{2} \right\rfloor - r + 2 \geq \left\lfloor \frac{2r+2}{2} \right\rfloor - r + 2 = 3.$$

Note that x can be adjacent to at most 2 vertices in $V(G^*) - (V'_1(G^*) \cup V'_2(G^*))$ as otherwise $\text{rad}(G) < r$. However, as $\text{rad}(G) \geq r$, it then follows that x cannot be adjacent to a vertex in $V'_1(G^*) \cup V'_2(G^*)$ and to two vertices in $V(G^*) - (V'_1(G^*) \cup V'_2(G^*))$. So x is adjacent to at most one vertex in $V(G^*) - (V'_1(G^*) \cup V'_2(G^*))$, and thus x is adjacent to at least $\lfloor \frac{n}{2} \rfloor - r + 2 - 1 = \lfloor \frac{n}{2} \rfloor - r + 1$ vertices in $V'_1(G^*) \cup V'_2(G^*)$, i.e., x is adjacent to every vertex in $V'_1(G^*)$ or x is adjacent to every vertex in $V'_2(G^*)$. Moreover, $\deg x = \lfloor \frac{n}{2} \rfloor - r + 2$, and thus $\deg x^* = \lceil \frac{n}{2} \rceil - r + 1$.

Since $\text{rad}(G) \geq 5$, $d_G(x, x^*) \geq 5$ and thus x^* cannot be adjacent to any vertex in $V'_1 \cup V'_2$ as otherwise $\text{rad}(G) < r$, and thus $\deg_G x^* = 2$. Hence, $n = 2r + 2$ since $\lceil \frac{n}{2} \rceil - r + 1 = 2$ and $n \geq 2r + 2$. Moreover, $n(G^*) = 2r$ and so $G^* \cong C_{2r}$. Hence, $\deg x = \lfloor \frac{2r+2}{2} \rfloor - r + 2 = 3$, and thus x must be adjacent to three vertices on $G^* \cong C_{2r}$, which is a contradiction as then $\text{rad}(G) < r$. Hence, equality cannot be attained in this case.

Claim 3 *If G is a vertex-radius-decreasing graph then $m(G) \leq b(n, r)$, and if $m(G) = b(n, r)$ then $G \in \mathcal{B}(n, r)$.*

By Claim 2, we need only consider the case where G has no conjugate pairs. Then, by Proposition 14, G must contain at least one cut-vertex and by Proposition 15, any ncv of G must have degree 1. Hence, G contains two end vertices x_1 and x_2 . Let $G' = G - \{x_1, x_2\}$, and note that if $\text{rad}(G') \leq r - 2$, then any central vertex c of G' is within distance $r - 2$ from every vertex in $V(G) - \{x_1, x_2\}$, including the neighbours of x_1 and x_2 . But then c is within distance $r - 1$ from x_1 and x_2 , contradicting $\text{rad}(G) = r$. Hence $\text{rad}(G') \geq r - 1$. So, by the induction hypothesis, $m(G') \leq b(n - 2, r - 1)$. Hence,

$$\begin{aligned} m(G) &= 2 + m(G') \\ &\leq 2 + b(n - 2, r - 1) \\ &= b(n, r), \end{aligned}$$

If $m(G) = b(n, r)$, we have equality throughout. So $m(G') = b(n - 2, r - 1)$ and thus by our induction hypothesis, $G' \in \mathcal{B}(n - 2, r - 1)$.

If $|V'_1(G)| \geq 3$ or $|V'_2(G)| \geq 2$, then G is not a vertex-radius-decreasing graph; thus $|V'_1(G)| = 2$ and $|V'_2(G)| = 1$. Hence, $n - 2r + 3 = 3$, and thus $n = 2r$ which is a contradiction as $n > 2r$. Hence, equality cannot be attained in this case.

Claim 4 *If v is a ncv of G with $\text{rad}(G - v) \geq r$ and $\deg v \leq \lfloor \frac{n}{2} \rfloor - r + 2$, then $m(G) \leq b(n, r)$. If $m(G) = b(n, r)$, then $G \in \mathcal{B}(n, r)$.*

By the induction hypothesis, $m(G - v) \leq b(n - 1, r)$, and hence,

$$\begin{aligned} m(G) &= m(G - v) + \deg v \\ &\leq b(n - 1, r) + \lfloor \frac{n}{2} \rfloor - r + 2 \\ &= \lfloor \frac{(n-1)^2}{4} \rfloor - (n - 1)r + r^2 + 2(n - 1 - r) + \lfloor \frac{n}{2} \rfloor - r + 2 \\ &= \lfloor \frac{n^2}{4} \rfloor - nr + r^2 + 2n - 2r \\ &= b(n, r), \end{aligned}$$

as required.

If $m(G) = b(n, r)$, we have equality throughout; so $m(G - v) = b(n - 1, r)$ and $\deg v = \lfloor \frac{n}{2} \rfloor - r + 2$. By the induction hypothesis, $G - v \in \mathcal{B}(n - 1, r)$.

If $n(G - v) = 2r$, then $G - v$ is a cycle of length $2r$, and moreover every vertex in $G - v$ has degree 2. Hence, any neighbour of v in G , say z , has degree 3 and thus G contains adjacent vertices v and z such that $\deg_G(z) + \deg_G(v) = 5 = n - 2r + 4$. Hence $G \in \mathcal{B}(n, r)$ by Lemma 2.

Since $n(G - v) \geq 2r + 1$ and $n = n(G - v) - 1$, $n \geq 2r + 2$. Hence $\deg_G(v) = \lfloor \frac{n}{2} \rfloor - r + 2 \geq \lfloor \frac{2r+2}{2} \rfloor - r + 2 \geq 3$. Note that v can be adjacent to at most one vertex in $G - \{v\} - (V'_1(G - v) \cup V'_2(G - v))$ as otherwise $\text{rad}(G) < r$. Thus v is adjacent to at least $\lfloor \frac{n}{2} \rfloor - r + 2 - 1 = \lfloor \frac{n}{2} \rfloor - r + 1$ vertices in $V'_1(G - v) \cup V'_2(G - v)$.

Let $w \in V'_i(G - v)$, $i = 1, 2$ such that $vw \in E(G)$, and let $y \in V'_{3-i}(G - v)$ such that $wy \in E(G - v)$. Then

$$\deg_{G-v}(w) + \deg_{G-v}(y) = |V'_1(G - v)| + |V'_2(G - v)| + 1,$$

and thus

$$\deg_G(w) + \deg_G(y) = |V'_1(G - v)| + |V'_2(G - v)| + 2 = (n - 1) - 2r + 3 + 2 = n - 2r + 4,$$

and hence $G \in \mathcal{B}(n, r)$ by Lemma 2.

Claim 5 *If w is a ncv of G with $2 \leq \deg w \leq \lfloor \frac{n}{2} \rfloor - r + 2$ and $\text{rad}(G - w) \leq r - 1$, then every neighbour of w is a ncv.*

By Proposition 12, w has a conjugate vertex w^* such that $d_G(w^*, w) = r$ and $d_G(w^*, u) \leq r - 1$ for every $u \in V(G) - \{w\}$. Let s and t be neighbours of w . It follows that if u is any vertex in $V(G) - \{w, s\}$, then no shortest $w^* - u$ path can contain s . In particular, $G - s$ contains a $w^* - t$ path and hence a $w^* - w$ path. So $G - s$ is connected. Since $s \in N(w)$ was chosen arbitrarily, it follows that no neighbour of w is a cut-vertex.

Claim 6 *If v is a ncv of G with $\text{rad}(G - v) \geq r$ and $\deg v > \lfloor \frac{n}{2} \rfloor - r + 2$, then $m(G) \leq b(n, r)$. If $m(G) = b(n, r)$, then $G \in \mathcal{B}(n, r)$.*

We shall first show that v has a neighbour that is a ncv.

Suppose to the contrary that every neighbour of v is a cut-vertex. Let T_v be a distance-preserving spanning tree of G with v as its root; so $\deg_{T_v}(v) = \deg_G(v)$. Let P be a diametral path of T_v . Then P has length

$$\text{diam}(T_v) \geq 2\text{rad}(T_v) - 1 \geq 2\text{rad}(G) - 1.$$

So P contains at least $2 \text{rad}(G)$ vertices. Moreover, the $(\deg v - 2)$ neighbours of v not on P cannot be end vertices because they are cut-vertices, and so they must be adjacent to a vertex that is non-adjacent to every other neighbour of v . Hence, since $\deg_{T_v}(v) \geq \lfloor \frac{n}{2} \rfloor - r + 3$,

$$\begin{aligned} n &\geq 2r + 2(\deg_{T_v}(v) - 2) \\ &\geq 2r + 2\left(\left\lfloor \frac{n}{2} \right\rfloor - r + 3\right) - 4 \\ &= 2\left\lfloor \frac{n}{2} \right\rfloor + 2, \end{aligned}$$

which is a contradiction.

Thus, v must have a neighbour, say x , which is a ncv. If $\deg x \geq \lfloor \frac{n}{2} \rfloor - r + 3$, then $\deg v + \deg x \geq 2\lfloor \frac{n}{2} \rfloor - 2r + 6$, which is a contradiction by Lemma 2. Hence, $\deg x \leq \lfloor \frac{n}{2} \rfloor - r + 2$. Moreover, since x is a ncv, $\text{rad}(G - x) \leq r - 1$ by Claim 4. By Proposition 12, x has a conjugate vertex, say \bar{x} . If $\text{rad}(G - \bar{x}) \leq r - 1$, then $\{x, \bar{x}\}$ would form a conjugate pair and the result follows by Claim 2. So $\text{rad}(G - \bar{x}) \geq r$ and since $d(\bar{x}, v) \neq 2$, $\deg \bar{x} \leq \lfloor \frac{n}{2} \rfloor - r + 2$ by Lemma 2. Hence, \bar{x} must be a cut-vertex by Claim 4, and so $G - \{\bar{x}\}$ has at least two components, say G_1 and G_2 .

Assume without loss of generality that $v, x \in V(G_1)$. Let x_1 be a neighbour of \bar{x} of degree at least 2 in $V(G_1)$.

Since $d(v, x_1) \neq 2$, $\deg_G(x_1) \leq \lfloor \frac{n}{2} \rfloor - r + 2$ by Lemma 2. Suppose x_1 is a ncv. Then, by Claim 4, $\text{rad}(G - x_1) \leq r - 1$. Applying Claim 5 to x_1 now yields that \bar{x} is not a cut-vertex, which is a contradiction. Hence, x_1 is a cut-vertex. Let H be the component of $G - x_1$ containing \bar{x} and denote by N'_i the i th distance layer of x_1 in H .

Since x_1 is a cut-vertex; it follows that every vertex in N'_1 is an end-vertex or a cut-vertex. By the same argument, if every vertex in N'_i , $i \geq 1$, is an end-vertex or a cut-vertex, then so is every vertex in N'_{i+1} (if any exists). Hence, by induction, each vertex in H is either an end-vertex or a cut-vertex.

Consider a distance preserving spanning tree T of $\langle V(H) \cup \{x\} \rangle$. Then either T is a path or T contains at least two end-vertices distinct from x_1 . In the former case, let x be the end-vertex of T , $x \neq x_1$, and y the neighbour of x , and in the latter case, let x and y be two end-vertices distinct from x . In both cases $G' =: G - x - y$ has $n - 2$ vertices, $\text{rad}(G') \geq r - 1$ and $m(G') = m(G) - 2$. Hence, by induction,

$$\begin{aligned}
m(G) &= m(G') + 2 \\
&\leq b(n-2, r-1) + 2 \\
&= b(n, r).
\end{aligned}$$

If $m(G) = b(n, r)$, we have equality throughout; so $m(G') = b(n-2, r-1)$. By the induction hypothesis, $G' \in \mathcal{B}(n-2, r-1)$. Hence G' contains vertices w, v such that $d_{G'}(w, v) \neq 2$ and $\deg_{G'}(w) + \deg_{G'}(v) = (n-2) - 2(r-1) + 4 = n - 2r + 4$. By Lemma 2, $G \in \mathcal{B}(n, r)$.

Claim 7 *If v is a ncv of G with $\text{rad}(G - v) \geq r$, then $m(G) \leq b(n, r)$. If $m(G) = b(n, r)$, then $G \in \mathcal{B}(n, r)$.*

This follows from Claims 4 and 6. ■

Chapter 3

Diameter and Inverse Degree

3.1 Introduction

This chapter is motivated by the GRAFFITI conjecture $\mu(G) \leq r(G)$ (see [22, 23]). Since $r(G) = \frac{n}{\delta^*}$, where δ^* is the harmonic mean of the degrees of the vertices of G , and since $\delta^* \geq \delta$, we have $r(G) \leq \frac{n}{\delta}$. Hence, this conjecture is a strengthening of the conjecture $\mu(G) \leq \frac{n}{\delta}$. Unfortunately, the conjecture turned out not to be true. Erdős, Pach and Spencer [19] disproved it by constructing an infinite class of graphs with average distance at least $(\frac{2}{3} \lfloor \frac{r(G)}{3} \rfloor + o(1)) \frac{\log n}{\log \log n}$ and diameter at least $(2 \lfloor \frac{r(G)}{3} \rfloor + o(1)) \frac{\log n}{\log \log n}$. Furthermore, they proved the upper bound, $\text{diam}(G) \leq (6r(G) + 2 + o(1)) \frac{\log n}{\log \log n}$, which is also an upper bound on the average distance since $\mu(G) \leq \text{diam}(G)$.

In this chapter, we improve upon the upper bound by Erdős, Pach and Spencer by a factor of two. We show that

$$\text{diam}(G) \leq (3r(G) + 2 + o(1)) \frac{\log n}{\log \log n},$$

and thus $\mu(G) \leq (3r(G) + 2 + o(1)) \frac{\log n}{\log \log n}$.

To enhance the readability of our inequalities, we will repeatedly use inequality chains like $a < b \geq c$, which are to be read as $a < b$ and $b \geq c$.

3.2 Results

Theorem 5 *Let G be a connected graph of order n with $r(G) \leq r$. Then, for constant r and large n ,*

$$\text{diam}(G) \leq \left(3r + 2 + o(1)\right) \frac{\log n}{\log \log n}.$$

Proof Let $x \in V(G)$ be a fixed vertex of eccentricity $d := \text{diam}(G)$. The i th distance layer $N_i(x)$ of x is the set of all vertices at distance i from x . Let $|N_i(x)| := n_i$ for $0 \leq i \leq d$, and $n_{-1} := n_{d+1} := 0$. Define $f(i) := \frac{n_i}{n_{i-1} + n_i + n_{i+1}}$ for $0 \leq i \leq d$. Since for any $y \in N_i$, where $0 \leq i \leq d$, the neighbourhood of y is contained in $N_{i-1} \cup N_i \cup N_{i+1}$, we have $\text{deg}y < n_{i-1} + n_i + n_{i+1}$, and thus $\frac{1}{\text{deg}y} > \frac{1}{n_{i-1} + n_i + n_{i+1}}$. Therefore,

$$r(G) = \sum_{y \in V(G)} \frac{1}{\text{deg}y} > \sum_{i=0}^d f(i).$$

If $n_d = n_{d-1} = \dots = n_1 = n_0$, then each distance layer has cardinality 1 and G is a path. Since, in this case, G has diameter $n - 1$ and inverse degree $r(G) = 1 + \frac{n}{2}$, the statement of the theorem holds. So we exclude this case from here onwards.

We now define two disjoint sets J and K . Let

$$J = \{i \mid 1 \leq i \leq d - 1 \text{ and } n_{i-1} < n_i \geq n_{i+1}\},$$

$$K = \{i \mid 1 \leq i \leq d - 1 \text{ and } n_{i-1} \geq n_i < n_{i+1}\}.$$

The elements of J and K can be considered peaks and troughs of the sequence n_0, n_1, \dots, n_d .

We now show that the elements of $J \cup K$ alternate, i.e., for every $s \in K$ ($s \in J$), there exists $t \in J \cup \{d\}$ ($t \in K \cup \{d\}$), with $t \geq s + 2$ and $s + 1, \dots, t - 1 \notin J \cup K$. Let $s \in K$ for $0 < s < d$ and thus by the definition of K , $n_s < n_{s+1}$. Let t be the first element following s such that $n_t \geq n_{t+1}$. Then $t \in J \cup \{d\}$. Note that such an element t exists since $n_{d+1} = 0$ and thus $n_d > 0 = n_{d+1}$. It is immediate from the definitions of J and K that $s + 1, \dots, t - 1 \notin J \cup K$. The proof for the case $s \in J$ is similar.

From the above proof it is clear that for any two consecutive elements s, t of $J \cup K$, the sequence n_s, n_{s+1}, \dots, n_t is monotonic. Moreover, if s is

the smallest and t is the largest element of $J \cup K$, then also n_0, n_1, \dots, n_s and n_t, n_{t+1}, \dots, n_d are monotonic. Hence we refer to sets $\{s, s+1, \dots, t\}$, with $s, t \in J \cup K \cup \{0, d\}$ and $s+1, s+2, \dots, t-1 \notin J \cup K$, as monotonic intervals. The length of such an interval is defined as $t - s$.

Clearly, there exists a monotonic interval of length at least $\frac{d}{|J|+|K|+1}$, the average length of the monotonic intervals. The main part of this proof is devoted to improving this bound and to expressing it in terms of d and r . For this aim we first partition J into two disjoint subsets A and B .

For $0 < i < d$, consider the values of i that belong to J . We now partition J further into two disjoint subsets, A and B , where

$$A = \{i \in J \mid n_{i-2} \leq n_{i-1} < n_i \geq n_{i+1} \geq n_{i+2}\} \quad \text{and} \quad B = J - A.$$

So an element $i \in J$ is in B if and only if $n_{i-1} < n_i \geq n_{i+1}$ and, in addition, $n_{i-2} > n_{i-1}$ or $n_{i+1} < n_{i+2}$.

We note that each $i \in B$ is an end point of a monotonic interval of length 1. Indeed, for $i \in B$ we have $n_{i-1} < n_i \geq n_{i+1}$ and, in addition, $n_{i-2} > n_{i-1}$, in which case $i-1 \in K$, or $n_{i+1} < n_{i+2}$, in which case $i+1 \in K$.

Since J and K alternate, no monotonic interval has both its end points in B . Hence there exists at least $|B|$ monotonic intervals of length 1, while the remaining $|J| + |K| - |B| + 1$ monotonic intervals have length at least 1.

If $i \in J$, then $2n_i > n_{i-1} + n_{i+1}$, and thus

$$f(i) = \frac{n_i}{n_{i-1} + n_i + n_{i+1}} > \frac{1}{3} \quad \text{for all } i \in J.$$

We now show that

$$f(j-1) + f(j) + f(j+1) > \frac{2}{3} \quad \text{for all } j \in A.$$

For $j \in A$, let $a = \frac{1}{f(j)} = \frac{n_{j-1} + n_j + n_{j+1}}{n_j}$ and thus, $a < 3$ since $\frac{1}{a} = f(j) > \frac{1}{3}$. By $n_{j-2} \leq n_{j-1}$ and $(a-1)n_j = n_{j-1} + n_{j+1}$,

$$\begin{aligned} f(j-1) &= \frac{n_{j-1}}{n_{j-2} + n_{j-1} + n_j} \\ &\geq \frac{n_{j-1}}{2n_{j-1} + n_j} \\ &= \frac{(a-1)n_{j-1}}{(2a-2)n_{j-1} + (a-1)n_j} \\ &= \frac{(a-1)n_{j-1}}{(2a-1)n_{j-1} + n_{j+1}}. \end{aligned}$$

Similarly, by $n_{j+2} \leq n_{j+1}$ and $(a-1)n_j = n_{j-1} + n_{j+1}$,

$$f(j+1) \geq \frac{(a-1)n_{j+1}}{(2a-1)n_{j+1} + n_{j-1}}.$$

Hence, in total,

$$\begin{aligned} f(j-1) + f(j) + f(j+1) &\geq \frac{(a-1)n_{j-1}}{(2a-1)n_{j-1} + n_{j+1}} + \frac{1}{a} + \frac{(a-1)n_{j+1}}{(2a-1)n_{j+1} + n_{j-1}} \\ &\geq \frac{(a-1)(n_{j-1} + n_{j+1})}{2a(n_{j-1} + n_{j+1})} + \frac{1}{a} \\ &= \frac{a+1}{2a}. \end{aligned}$$

Thus, by $a < 3$ we have $f(j-1) + f(j) + f(j+1) > \frac{2}{3}$ for all $j \in A$, as desired.

Now $r \geq r(G) > \sum_{i=0}^d f(i)$. Since the sets J and K alternate, J does not contain two consecutive integers. Hence $\{i-1, i, i+1\}$ and $\{j\}$ are disjoint for all $i \in A$ and $j \in B$. From the definition of A it is easy to see that $i+2 \notin A$ if $i \in A$. Hence also the sets $\{i-1, i, i+1\}$, $i \in A$, are disjoint. Therefore,

$$r > \sum_{i \in A} (f(i-1) + f(i) + f(i+1)) + \sum_{j \in B} f(j) > \frac{2}{3}|A| + \frac{1}{3}|B|,$$

or, equivalently,

$$2|A| + |B| < 3r.$$

We have $|J| + |K| + 1$ monotonic intervals of total length d . At least $|B|$ intervals have length 1, so the remaining $|J| + |K| + 1 - |B| = |A| + |K| + 1$ intervals have average length $\frac{d-|B|}{|A|+|K|+1}$. Hence there exists a monotonic interval of at least this length.

We now bound the average length in terms of d and r . Since J and K alternate, we have $|K| \leq |J| + 1$. Hence,

$$|A| + |K| + 1 \leq 2|J| + 2 - |B| = 2|A| + |B| + 2.$$

Hence there exists a monotonic interval of length at least

$$\frac{d-|B|}{|A|+|K|+1} \geq \frac{d-|B|}{2|A|+|B|+2} \geq \frac{d}{2|A|+|B|+2} - 1.$$

Let $\{a, a+1, a+2, \dots, b\}$ be such an interval. By $2|A|+|B| < 3r$, the interval has length

$$b - a \geq \frac{d}{3r + 2} - 1.$$

We assume that n_i is monotone increasing on $\{a, a+1, \dots, b\}$ (if n_i is decreasing the proof is analogous). Then $f(i) = \frac{n_i}{n_{i-1}+n_i+n_{i+1}} \geq \frac{n_i}{3n_{i+1}}$, and hence,

$$\sum_{i=a+1}^{b-1} \frac{n_i}{n_{i+1}} \leq 3 \sum_{i=a+1}^{b-1} f(i) < 3r.$$

Note that, for $S, x_i > 0$, the product $\prod_{i \in I} x_i$ is maximized subject to $\sum_{i \in I} x_i \leq S$ if $x_i = S/|I|$ for all x_i . So,

$$n_{a+1} = \left(\prod_{i=a+1}^{b-1} \frac{n_i}{n_{i+1}} \right) n_b < \left(\frac{3r}{b-a-1} \right)^{b-a-1} n_b.$$

Hence,

$$1 \leq n_{a+1} < \left(\frac{3r(3r+2)}{d-6r-4} \right)^{(d-6r-4)/(3r+2)} n.$$

Now let r be constant and let $x_1 = \frac{d-6r-4}{3r+2}$. If $x_1 \leq 3r/e$, i.e., if $d \leq \frac{3r(3r+2)}{e} + 6r + 4$, then the inequality of the theorem is satisfied for large n . So we can assume $x_1 > 3r/e$. Let $g(x) = \left(\frac{3r}{x}\right)^x = e^{x(\log 3r - \log x)}$. Then $g'(x) = \left(\frac{3r}{x}\right)^x (\log 3r - \log x - 1)$. If $x > \frac{3r}{e}$, then $g'(x) < 0$, and hence g is decreasing. It is shown in the previous page that $b-a-1 \geq x_1$ and that $n_{a+1} < ng(b-a-1)$. This implies $1 < ng(x_1)$. Define x_0 by $g(x_0) = 1/n$. Thus,

$$\left(\frac{x_0}{3r} \right)^{x_0} = n.$$

Observe that, in each case under consideration, we regard r as fixed, whereas n grows beyond all bounds if and only if x_0 does. Hence, $\log n = x_0 \log\left(\frac{x_0}{3r}\right)$, and so $\log \log n = \log x_0 + \log \log\left(\frac{x_0}{3r}\right)$. Thus,

$$\frac{\log n}{\log \log n} = \frac{x_0 \log \frac{x_0}{3r}}{\log x_0 + \log \log \frac{x_0}{3r}}.$$

Hence,

$$x_0 = \left(\frac{\log x_0 + \log \log \frac{x_0}{3r}}{\log x_0 - \log 3r} \right) \left(\frac{\log n}{\log \log n} \right),$$

and thus,

$$x_0 = \left(\frac{\log x_0 - \log 3r}{\log x_0 - \log 3r} + \frac{\log 3r + \log \log \frac{x_0}{3r}}{\log \frac{x_0}{3r}} \right) \left(\frac{\log n}{\log \log n} \right),$$

Hence $1 = \left(\frac{3r}{x_0}\right)^{x_0} n$, or equivalently $\left(\frac{x_0}{3r}\right)^{\frac{x_0}{3r}} = n^{\frac{1}{3r}}$, has the solution $x_0 = (1 + o(1)) \frac{\log n}{\log \log n}$. Since $x_0 = (1 + o(1)) \frac{\log n}{\log \log n} > 3r/e$ and $g(x_0) < fg(x_1)$, we have $x_1 < x_0$. Hence,

$$d - 6r - 4 = (3r + 2)x_1 < (3r + 2)(1 + o(1)) \frac{\log n}{\log \log n},$$

which yields the statement of the theorem. ■

Chapter 4

Diameter in Minimal Claw-free Graphs

4.1 Introduction

Graphs that do not contain a star on four vertices (claw) as an induced subgraph have received much attention, especially since the publication of the excellent survey paper [24] in 1997. This class of graphs includes, among others, line graphs, interval graphs, middle graphs, inflations of graphs and graphs with independence number equal to 2. Recently, Chudnovsky and Seymour found a structural characterization of claw-free graphs; that is, they defined certain classes of “basic” claw-free graphs and then showed that all claw-free graphs can be obtained by applying certain “expansion” operations. See [6].

Definition 3 *The graph $K_{1,n}$ is called a star. We refer to the star $K_{1,3}$ as a claw with the vertex of degree 3 as its centre.*

In [12], Dankelmann et. al considered graphs that are (edge-) minimal with respect to the property of being claw-free. This was motivated by questions about cycles in claw-free graphs, but has interest in its own right.

Definition 4 *Let G be a claw-free graph without isolated vertices. If the removal of any edge of G produces a graph that is not claw-free, then G is a minimal claw-free graph, briefly denoted as an m.c.f.g..*

That not every claw-free graph contains an m.c.f.g. as a subgraph may be seen, for example, by considering the line graph of K_4 : it is obviously claw-free but one can repeatedly remove its edges until an empty graph is obtained without creating a claw. On the other hand, the line graph of $K_{3,3}$ (equivalent to the cartesian product $K_3 \times K_3$) is an m.c.f.g.

In [12], Dankelmann et. al examined bounds on the minimum, average and maximum degrees of an m.c.f.g. and looked at the relationship between m.c.f.g.s and line graphs. For example, a 4-regular graph is m.c.f. if and only if it is the line graph of a $(K_4 - e)$ -free cubic graph.

We mention that a closely related concept, minimal line graphs, was considered by Sumner [46]. A graph is a *minimal line graph* if it is a line graph, but removal of any edge results in a graph that is not a line graph. Sumner proved that a graph G is a minimal line graph if and only if the following four conditions hold:

- (i) every edge of G lies in a triangle,
- (ii) every vertex of G has degree at least 3,
- (iii) if an edge e lies on a triangle whose vertices have an even degree sum, then e lies on another triangle,
- (iv) each 4-clique of G has at least two vertices adjacent to vertices outside the 4-clique.

Condition (i) clearly holds for m.c.f.g.s, and we will see that condition (ii) also holds for m.c.f.g.s.

An example of an m.c.f.g. is the 5-regular *icosahedron* on 12 vertices. Indeed, if we delete one, two or three vertices from the same triangle, then the result is still an m.c.f.g. The latter is depicted in Figure 4.1. (This is not a line graph.) An exhaustive computer search has shown that the smallest order of an m.c.f.g. is 9; apart from the above graph there are two others, namely the line graphs of the two cubic graphs of order 6.

In this chapter, we examine the diameter of m.c.f.g.s. We prove that the diameter $\text{diam}(G)$ of an m.c.f.g. G of order n satisfies

$$\text{diam}(G) \leq \frac{4}{9}(n - 20) + 7.$$

Moreover, we demonstrate that this bound is best possible.

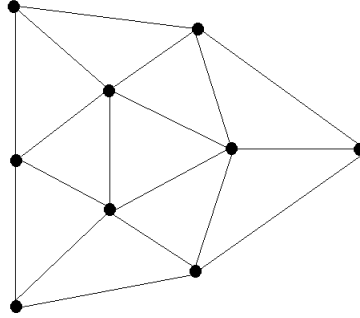


Figure 4.1: The m.c.f.g I_9 .

4.2 Preliminary Results

We will need the following concept.

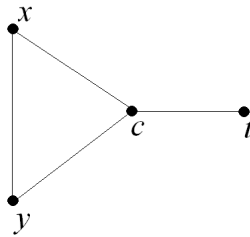


Figure 4.2: The near-claw $NC(xy, c, t)$.

Definition 5 A near-claw $NC(xy, c, t)$ is a graph obtained by removing from a complete graph K_4 , with vertex set $\{x, y, c, t\}$, the edges xt and yt . The vertices c and t are called the centre and tail of the near-claw $NC(xy, c, t)$, respectively.

It is immediately obvious that G is an m.c.f.g. if and only if every edge xy in G lies in a near-claw $NC(xy, c, t)$ as shown in Figure 4.2. Hence every edge of an m.c.f.g. is contained in a triangle. If $\langle \{a, b, c, d\} \rangle$ denotes a claw, it will be assumed that a is the centre of the claw.

Lemma 4 [12] *Let G be an m.c.f.g. Then G has maximum degree $\Delta(G) \leq n(G) - 3$.*

Proof Consider any vertex x . Let y be a neighbour of x . Then there exists a near-claw $NC(xy, c, t)$ with t non-adjacent to x but adjacent to c . Further, there exists a near-claw $NC(xc, c', t')$, where t' is non-adjacent to x and c . Hence there are at least two vertices non-adjacent to x . It follows that $\Delta(G) \leq n - 3$. ■

One can obtain an infinite family of m.c.f. graphs with $\Delta = n - 3$ by duplicating u as follows. The *duplication of a vertex u* in G , means the addition to G of a new vertex v , adjacent to u and all vertices in $N_G(u)$ (so that $N[u] = N[v]$). Clearly, G is claw-free if and only if G' is claw-free.

Lemma 5 [12] *Let G be a claw-free graph and suppose G' is formed by duplicating u to v . Then G' is an m.c.f.g. if and only if G is an m.c.f.g.*

Proof Assume G' is an m.c.f.g.

Let $e = ab \in E(G)$; then G' contains a near-claw $NC(ab, c, t)$ and $v \notin \{a, b\}$. If $NC(ab, c, t)$ is contained in G , then $G - e$ contains a claw.

Otherwise, suppose vertex v is in $NC(ab, c, t)$; hence $v \in \{c, t\}$. If $v = c$, then $u \notin \{a, b\}$, since otherwise, if $u = a$, then $vt \in E(G')$ and $ut \notin E(G')$, contradicting the assumption that $N[u] = N[v]$. Hence if $v = c$, then $NC(ab, u, t)$ is contained in G . On the other hand, if $v = t$, then, as $N[u] = N[v]$, $NC(ab, c, u)$ is contained in G . Hence G is an m.c.f.g.

Assume G is an m.c.f.g.

Clearly the removal of any edge of G' not incident with v produces a claw. So we need only to consider the edges incident with v . Let $v' \in V(G')$ such that $v' \in N(v)$ but $v' \neq u$, and let $e_1 = vv' \in E(G')$. So there exists the edge $uv' \in E(G)$ contained in, say, the near-claw $NC(uv', w, x)$ in G ; then vv' is contained in the near-claw $NC(vv', w, x)$ in G' , whence removal of vv' creates a claw.

Consider the edge $e_2 = uv \in E(G')$. By Lemma 4, there exists a vertex, say $w \in V(G') - N[u]$, that is adjacent to some vertex in $N(u)$, say v' .

Then $wv \notin E(G')$ and $v' \in N(v)$. Moreover, since $uv', vv', v'w \in E(G')$ and $uw, vw \notin E(G')$, $\{v', u, v, w\}$ induces a claw in $G' - e_2$. Hence G' is m.c.f. ■

Lemma 6 [12] *Let G be an m.c.f.g. Then G has minimum degree $\delta(G) \geq 3$.*

Proof Since every edge of G lies in a triangle, $\delta(G) \geq 2$. Now suppose that G contains a vertex v_0 of degree 2, adjacent to v_1 and v_2 , where $v_1v_2 \in E(G)$. Let $B = (N(v_1) \cap N(v_2)) - \{v_0\}$ and for $i = 1, 2$ $A_i = N(v_i) - (B \cup \{v_0, v_{3-i}\})$. Since v_1 and v_2 are not centres of claws, $A_1 \cup B$ and $A_2 \cup B$ induce complete subgraphs of G .

The edge v_0v_1 is contained in a near-claw with v_2 as centre, say $NC(v_0v_1, v_2, v_3)$, and v_0v_2 is contained in a near-claw $NC(v_0v_2, v_1, v_4)$; so $v_2v_3, v_1v_4 \in E(G)$ and $v_1v_3, v_2v_4 \notin E(G)$ and thus $v_3 \in A_2$ and $v_4 \in A_1$. The edge v_1v_2 is contained in a near-claw $NC(v_1v_2, v_5, v_6)$, where $v_1v_5, v_2v_5 \in E(G)$ and $v_1v_6, v_2v_6 \notin E(G)$; so $v_5 \in B$, $v_6 \notin N(v_1) \cup N(v_2)$.

Let $x \in A_2, y \in A_1$; then since $\langle \{v_5, v_1, x, v_6\} \rangle$ is not a claw and $v_1x, v_1v_6 \notin E(G)$, it follows that $xv_6 \in E(G)$; similarly as $\langle \{v_5, v_2, y, v_6\} \rangle$ is not a claw, it follows that $yv_6 \in E(G)$. Hence v_6 is adjacent to every vertex in $A_1 \cup A_2$. By the same argument it follows that

if $w \in N(B) - (N[v_1] \cup N[v_2])$ then w is adjacent to all of $A_1 \cup A_2$. (*)

The edge v_5v_6 is contained in a near-claw $NC(v_5v_6, c, t)$, say. If $c \in N(v_2)$, then as v_5 is adjacent to every vertex in $(N(v_1) \cup N(v_2)) - \{v_0\}$, it follows that $t \notin N(v_1) \cup N(v_2)$; hence as $\langle N(v_2) - \{v_0, v_1\} \rangle$ is complete, $v_2t \notin E(G)$. Also, $v_2v_6, v_6t \notin E(G)$ while c is adjacent to v_2, v_6 and t ; so $\langle \{c, v_2, v_6, t\} \rangle \cong K_{1,3}$, which is a contradiction. So $c \notin N(v_2)$. It follows similarly that $c \notin N(v_1)$.

So $c = v_7$ and $t = v_8$, where $v_7 \notin N[v_1] \cup N[v_2]$, and $v_5v_7, v_6v_7 \in E(G)$, while $v_5v_8, v_6v_8 \notin E(G)$. Note that, $v_1v_8 \notin E(G)$, since otherwise $\langle \{v_1, v_0, v_5, v_8\} \rangle \cong K_{1,3}$, and, similarly, $v_2v_8 \notin E(G)$. By (*), v_7 is adjacent to every vertex in $A_1 \cup A_2$.

That for $x \in A_2, xv_8 \notin E(G)$ follows from the observation that $\langle \{x, v_2, v_6, v_8\} \rangle$ is not a claw. So v_8 is non-adjacent to each vertex in A_2 and, similarly in A_1 . Furthermore, $xy \in E(G)$ for $x \in A_2, y \in A_1$, since $\langle \{v_7, x, y, v_8\} \rangle$ is not a claw. (See Figure 4.3.)

In conjunction with the fact that A_1 and A_2 induce complete graphs, we obtain that

$$\langle A_1 \cup A_2 \cup B \rangle \text{ is complete.} \quad (\dagger)$$

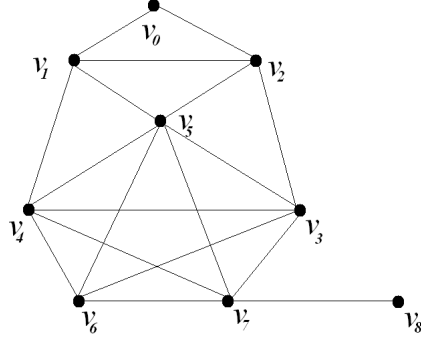


Figure 4.3: An induced subgraph.

The edge v_2v_3 is contained in a near claw, say $NC(v_2v_3, v_9, t)$. Clearly, $v_9 \notin B$, since otherwise v_3 and t would be adjacent by $(*)$. Hence $v_9 \in A_2$.

Consider t . Since t is not adjacent to v_3 but adjacent to v_9 , we have $t \notin N[v_1] \cup N[v_2] \cup \{v_6, v_7, v_8\}$, say, $t = v_{10}$ with $v_9v_{10} \in E(G)$, $v_3v_{10} \notin E(G)$.

By (\dagger) , $v_4v_9 \in E(G)$. Hence $v_4v_{10} \in E(G)$ since otherwise $\langle \{v_9, v_2, v_4, v_{10}\} \rangle \cong K_{1,3}$ is claw. But then $\langle \{v_4, v_1, v_3, v_{10}\} \rangle \cong K_{1,3}$, which is a contradiction. ■

By Lemma 6, we have the following corollary.

Corollary 4 [12] *Let G be an m.c.f.g. Then the vertices of degree 3 form an independent set.*

Proof Suppose that u and v are vertices of degree 3 in G such that $uv \in E(G)$.

If $N[u] = N[v]$, let $G' = G - \{v\}$. Then G' is m.c.f., but has a vertex of degree 2, which contradicts Lemma 6.

If the vertices u and v have different neighbourhoods, then since every edge lies in a triangle, $N(u) \cup N(v)$ induces the graph $K_1 + P_4$, where u and v are the interior vertices on P_4 . Let G' be the graph obtained by adding a vertex w adjacent only to u and v . Then G' is claw-free. The removal of the edge uw produces a claw centred at v , and the removal of the edge vw produces a claw centred at u and $G' - e$ contains an induced claw for each $e \in E(G)$. So G' is a minimal claw-free graph with a vertex of degree 2, which contradicts Lemma 6.

Hence $uv \notin E(G)$ and the result follows. ■

The following is an immediate consequence of Lemma 6 and Corollary 4.

Corollary 5 [12] *Let G be an m.c.f.g.. Then $\Delta(G) \geq 4$.*

We now look at the minimum number of edges in an m.c.f.g. We will need the following results.

Lemma 7 [12] *If an m.c.f.g. G contains a vertex v of degree 3, then v has a neighbour of degree at least 5.*

Proof Suppose to the contrary that no neighbour of v has degree exceeding 4. Then it follows from Lemma 6 and Corollary 4 that the neighbours of v , say v_1, v_2, v_3 , all have degree 4. The edge vv_1 is contained in a near-claw, say $NC(vv_1, v_2, t_1)$, so that $t_1 \in V(G) - N[v]$, $t_1v_2 \in E(G)$ and $t_1v_1 \notin E(G)$. A near-claw $NC(v_2t_1, c_2, t_2)$ exists in G ; here either $c_2 = v_3$ or c_2 is a new vertex.

If $c_2 = v_3$, then $v_2v_3, t_1v_3 \in E(G)$ and t_2 is a new vertex such that $v_3t_2 \in E(G)$, but $v_2t_2, t_1t_2 \notin E(G)$. Since $\deg v_3 = 4$, it follows that $v_3v_1 \notin E(G)$. A near-claw $NC(vv_3, c_3, t_3)$ exists in G , where $c_3 = v_2$. But $\deg v_2 = 4$, so $t_3 \in \{t_1, v_1\}$, a contradiction, as $vv_1, v_3t_1 \in E(G)$. Hence $c_2 \neq v_3$.

Thus c_2 is a new vertex. Then $v_2c_2, t_1c_2 \in E(G)$ and $N(v_2) = \{v, v_1, t_1, c_2\}$; hence the centre c_3 of a near-claw $NC(vv_3, c_3, t_3)$ must be v_1 and so $v_1v_3 \in E(G)$. A near-claw $NC(v_1v_2, c_4, t_4)$ exists in G , where $c_4 \in \{v, c_2\}$. If $c_4 = v$, then $t_4 = v_3$, a contradiction, as $v_1v_3 \in E(G)$. Hence $c_4 = c_2$ and $v_1c_2 \in E(G)$. A near-claw $NC(vv_2, c_5, t_5)$ exists in G , where $c_5 = v_1$ and $t_5 \in \{v_3, c_2\}$, which yields a contradiction as $vv_3, v_2c_2 \in E(G)$.

It follows that at least one neighbour of v is of degree exceeding 4. ■

Lemma 8 [12] *Let G be an m.c.f.g. If v is a vertex of degree 3 in G with only one neighbour of degree at least 5, say v_3 , then v_3 has no other neighbour of degree 3.*

Proof Suppose that the neighbours of v are vertices v_1, v_2 and v_3 , with $\deg v_1 = \deg v_2 = 4$. We show first that $v_1v_3, v_2v_3 \in E(G)$ and $v_1v_2 \notin E(G)$.

The edge vv_3 is contained in a near-claw, with say v_1 as centre, $NC(vv_3, v_1, t_1)$, where $t_1 \neq v_2$, and so $v_1v_3 \in E(G)$. Suppose $v_1v_2 \in E(G)$; then the edge v_1t_1 is contained in a near-claw which must have centre v_2 because $\deg v_1 = 4$, say $NC(v_1t_1, v_2, t_2)$ where $t_2 \neq \{v, v_3\}$, so that t_2 is a new vertex. However, $\langle \{v_2, v, t_1, t_2\} \rangle \cong K_{1,3}$, a contradiction, and so $v_1v_2 \notin E(G)$. Since vv_2 is contained in a near-claw which must have centre v_3 , it follows that $v_2v_3 \in E(G)$.

Further, consider this near-claw $NC(vv_2, v_3, x_1)$. Since v_3 is not the centre of a claw, it follows that $v_1x_1 \in E(G)$. Similarly, there is a vertex x_2 such that $v_2x_2, v_3x_2 \in E(G)$ but $v_1x_2 \notin E(G)$. Again because v_3 is not the centre of a claw, it follows that $x_1x_2 \in E(G)$. That is, the set $W = \{v, v_1, x_1, x_2, v_2\}$ induces a 5-cycle.

Now suppose v_3 has another neighbour z of degree 3. If $z \notin W$, then it has only two neighbours in W , and thus is part of a claw centered at v_3 . If $z \in W$, say $z = x_1$, then v_1 has no other neighbour, by the lack of claw centered at v_1 . But then v_1 has degree 3, a contradiction of Corollary 4. ■

Lemma 9 [12] *The size of an m.c.f.g. of order n is at least $2n$.*

Proof Let G be an m.c.f.g. of order n , T the set of vertices of degree 3 and let U denote the set of vertices of degree at least 5. Define H as the bipartite subgraph of G with vertex set $T \cup U$ whose edge set consists of all those edges with one end in T and one end in U .

By Lemma 7, in H every vertex of T has degree at least 1. Let A denote the vertices of T with degree 1 in H . By Lemma 8, the neighbours of A have degree 1 in H . Let $X = N(A)$. So every vertex in $T - A$ has degree at least 2 in H . On the other hand, since T is independent in G (by Corollary 4) and G is claw-free, every vertex in $U - X$ has degree at most 2 in H . Thus $|T - A| \leq |U - X|$ and so $|T| \leq |U|$.

Now, let d_i denote the number of vertices of degree i in G . Then

$$\sum_i id_i = 4n + \sum_i (i - 4)d_i \geq 4n + |U| - |T| \geq 4n,$$

as required. ■

4.3 The Main Result

Theorem 6 *Let G be an m.c.f.g. of order n and diameter $\text{diam}(G)$. Then*

$$\text{diam}(G) \leq \frac{4}{9}(n - 20) + 7.$$

Moreover, this bound is sharp.

Proof Let $P : x_0x_1 \dots x_d$ be a diametral path in G . For $i \in \{0, \dots, d-1\}$, x_ix_{i+1} is contained in a near-claw $NC(x_ix_{i+1}, c_i, t_i)$.

Claim 8 $c_i \notin V(P)$ for all i .

Suppose $c_i \in V(P)$. Then $c_i = x_j$. Clearly, $j \neq i, i+1$ and so $j \leq i-1$ or $j \geq i+2$. If $j \geq i+2$, then $x_i x_j$ is shorter than $x_i x_{i+1} \dots x_j$, which is a contradiction. If $j \leq i-1$, then $x_j x_{i+1}$ is shorter than $x_j \dots x_i x_{i+1}$, which is a contradiction.

Claim 9 $t_i \notin V(P)$ for all i .

Suppose $t_i \in V(P)$. Then $t_i = x_j$. Clearly, $j \leq i-2$ or $j \geq i+3$ since $x_i t_i, x_{i+1} t_i \notin E(G)$. If $j \geq i+3$, then $x_i c_i x_j$ is shorter than $x_i x_{i+1} x_{i+2} \dots x_j$, which is a contradiction. If $j \leq i-2$, then we have $x_j c_i x_{i+1}$ shorter than $x_j \dots x_{i-1} x_i x_{i+1}$, which is a contradiction.

Claim 10 If $k \geq i+2$ and $0 \leq i \leq d-1$ then $c_i \neq c_k$, and if $1 \leq i \leq d-2$, then $c_i \neq c_{i+1}$.

That $c_i \neq c_k$ if $k \geq i+2$ is obvious since otherwise $x_i c_i (= c_k) x_{k+1}$ is shorter than $x_i x_{i+1} \dots x_k x_{k+1}$.

Suppose $c_i = c_{i+1}$ and $1 \leq d-2$. Hence, $c_i x_i, c_i x_{i+1}, c_i x_{i+2} \in E(G)$. The near-claw $NC(x_i x_{i+1}, c_i, t_i)$ exists in G where, by Claim 9, $t_i \neq x_j$ for any j , and clearly $c_i \neq t_i$. So t_i is a new vertex in G such that $c_i t_i \in E(G)$ and $x_i t_i, x_{i+1} t_i \notin E(G)$. Since P is a diametral path, $x_i x_{i+2} \notin E(G)$ and thus $t_i x_{i+2} \in E(G)$ as otherwise $\langle \{c_i, t_i, x_i, x_{i+2}\} \rangle \cong K_{1,3}$. The near-claw $NC(x_{i+1} x_{i+2}, c_i, t_{i+1})$ exists in G where by Claim 9, $t_{i+1} \neq x_j$ for any j , and clearly $c_i \neq t_{i+1}$. Moreover, since $t_i x_{i+2} \in E(G)$, $t_i \neq t_{i+1}$ and hence t_{i+1} is a new vertex in G such that $c_i t_{i+1} \in E(G)$ and $x_{i+1} t_{i+1}, x_{i+2} t_{i+1} \notin E(G)$. Then $t_{i+1} x_i \in E(G)$ as otherwise $\langle \{c_i, x_i, x_{i+2}, t_{i+1}\} \rangle \cong K_{1,3}$ and $t_i t_{i+1} \in E(G)$ as otherwise $\langle \{c_i, t_i, t_{i+1}, x_{i+1}\} \rangle \cong K_{1,3}$. Since P is a diametral path, $x_{i+1} x_{i+3} \notin E(G)$ and thus $t_i x_{i+3} \in E(G)$ as otherwise $\langle \{x_{i+2}, t_i, x_{i+3}, x_{i+1}\} \rangle \cong K_{1,3}$. Since P is a diametral path, $x_{i-1} x_{i+1} \notin E(G)$, and thus $t_{i+1} x_{i-1} \in E(G)$ as otherwise $\langle \{x_i, t_{i+1}, x_{i-1}, x_{i+1}\} \rangle \cong K_{1,3}$. But then $x_{i-1} t_{i+1} t_i x_{i+3}$ is a shorter path than $x_{i-1} x_i x_{i+1} x_{i+2} x_{i+3}$, (see Figure 4.4) which is a contradiction.

Claim 11 If $t_i = c_j$, then $j = i-2$ or $j = i+2$ for $4 \leq i \leq d-5$.

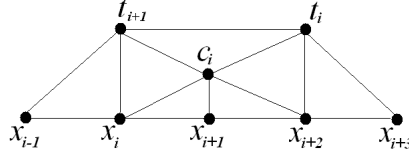


Figure 4.4: $x_{i-1}t_{i+1}t_ix_{i+3}$ is a shorter path than $x_{i-1}x_ix_{i+1}x_{i+2}x_{i+3}$.

If $t_i = c_j$, then clearly $j \neq i$. Moreover, since $c_{i-1}x_i, c_{i+1}x_{i+1} \in E(G)$ and $t_ix_i, t_ix_{i+1} \notin E(G)$, $j \neq i-1$ and $j \neq i+1$. So $j \leq i-2$ and or $j \geq i+2$. If $j \geq i+3$, then $x_ic_it_i(=c_j)x_{j+1}$ is shorter than $x_ix_{i+1}x_{i+2}x_{i+3} \dots x_{j+1}$, which is a contradiction. If $j \leq i-3$, then $x_jc_j(=t_i)c_ix_{i+1}$ is shorter than $x_j \dots x_{i-2}x_{i-1}x_ix_{i+1}$, which is a contradiction. Thus, $j = i-2$ or $j = i+2$.

Let $N_i = \{v \in V(G) : d(x_0, v) = i\}$ for $0 \leq i \leq d$. Then $x_i \in N_i$ for all i .

Claim 12 $c_i \in N_i \cup N_{i+1}$ for $0 \leq i \leq d-1$.

Since $x_i \in N_i$ and $c_ix_i \in E(G)$, $c_i \in N_j$ where $j \leq i+1$. Since $x_{i+1} \in N_{i+1}$ and $c_ix_{i+1} \in E(G)$, $c_i \in N_j$ where $j \geq i$, and thus the result follows.

Let $A_i = N_i \cup N_{i+1} \cup N_{i+2} \cup N_{i+3}$ for $0 \leq i \leq d-3$.

Claim 13 For $3 \leq i \leq d-6$, $|A_i| \geq 8$.

Since $x_i, x_{i+1}, x_{i+2}, x_{i+3} \in A_i$, and by Claim 12, $c_i \in N_i \cup N_{i+1}$, $c_{i+1} \in N_{i+1} \cup N_{i+2}$, and $c_{i+2} \in N_{i+2} \cup N_{i+3}$, $|A_i| \geq 7$. For $3 \leq i \leq d-6$, the near-claw $NC(x_{i+1}x_{i+2}, c_{i+1}, t_{i+1})$ exists in G where by Claim 11, $t_{i+1} \in \{c_{i-1}, c_{i+3}\}$ or t_{i+1} is a new vertex in A_i . If we have the latter case, then $|A_i| \geq 8$; so we need only consider the former case. If $t_{i+1} = c_{i-1}$, then $c_{i-1}c_{i+1} \in E(G)$ and, by Claim 12, $c_{i-1} \in N_i$ and $c_{i+1} \in N_{i+1}$ and thus $c_{i-1} \in A_i$, and hence $|A_i| \geq 8$. Similarly, if $t_{i+1} = c_{i+3}$, then $c_{i+3} \in N_{i+3}$ and hence $|A_i| \geq 8$.

Claim 14 *If $|A_i| = 8$, then $c_{i-1} \in N_{i-1}$ and $c_{i+3} \in N_{i+4}$ for $3 \leq i \leq d-6$.*

By Claim 13, $|A_i| \geq 8$. So if $|A_i| = 8$, then as shown in the proof of Claim 13, we have exactly one of the following three possibilities:

$$(i) \ t_{i+1} = c_{i-1} \text{ and } A_i^{(1)} = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}, c_{i-1}, c_i, c_{i+1}, c_{i+2}\},$$

$$(ii) \ t_{i+1} = c_{i+3} \text{ and } A_i^{(2)} = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}, c_i, c_{i+1}, c_{i+2}, c_{i+3}\},$$

$$(iii) \ t_{i+1} \notin A_i^{(1)} \cup A_i^{(2)} \text{ and } c_{i-1}, c_{i+3} \notin A_i.$$

We shall show that cases (i) and (ii) cannot occur:

Suppose $t_{i+1} = c_{i-1}$; then $c_{i+1} \in N_{i+1}$ and $A_i^{(1)} = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}, c_{i-1}, c_i, c_{i+1}, c_{i+2}\}$. Suppose, furthermore, that $t_{i-1} \notin A_i^{(1)}$; then $t_{i-1} \in N_{i-1}$ and thus $t_{i-1}c_{i+1} \notin E(G)$. But then $\langle \{c_{i-1}, x_{i-1}, t_{i-1}, c_{i+1}\} \rangle \cong K_{1,3}$, which is a contradiction. Hence, $t_{i-1} \in A_i^{(1)}$ and, by Claims 9 and 11, $t_{i-1} = c_{i+1}$. Thus $x_i c_{i+1} \notin E(G)$. The near-claw $NC(c_{i-1}c_{i+1}, c_a, t_a)$ exists where $c_a \in N_i \cup N_{i+1}$. Thus $c_a = c_i$, and so $c_{i-1}c_i, c_i c_{i+1} \in E(G)$. So $x_{i+1}t_a \in E(G)$ as otherwise $\langle \{c_i, t_a, c_{i-1}, x_{i+1}\} \rangle \cong K_{1,3}$ and $x_i t_a \in E(G)$ as otherwise $\langle \{c_i, t_a, x_i, c_{i+1}\} \rangle \cong K_{1,3}$. So $t_a \in N_i \cup N_{i+1}$ and thus $t_a \in A_i$. So $t_a = c_{i+2}$, but then by Claim 12, $t_a \in N_{i+2} \cup N_{i+3}$, which is a contradiction.

So $t_{i+1} \neq c_{i-1}$, and similarly $t_{i+1} \neq c_{i+3}$. Thus t_{i+1} is a new vertex in A_i . So $c_{i-1}, c_{i+3} \notin A_i$, and thus the result follows.

Claim 15 *If $|A_i| = 9$ and $c_{i-1} \in N_i$, then $c_{i+3} \in N_{i+4}$.*

Suppose to the contrary that $c_{i+3} \in N_{i+3}$; then $A_i = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}, c_{i-1}, c_i, c_{i+1}, c_{i+2}, c_{i+3}\}$. Suppose $t_{i+3} \notin A_i$ and let z be a neighbour of c_{i+3} in N_{i+2} . Then $\langle \{c_{i+3}, t_{i+3}, z, x_{i+4}\} \rangle \cong K_{1,3}$, a contradiction. So $t_{i+3} \in A_i$, and thus by Claims 9 and 11, $t_{i+3} = c_{i+1}$. Hence $c_{i+1}c_{i+3} \in E(G)$ and $c_{i+1}x_{i+3} \notin E(G)$. So, by Claim 12, $c_{i+1} \in N_{i+2}$. Since $c_{i-1} \in N_i$, $c_{i-1}c_{i+1} \notin E(G)$. The near-claw $NC(x_{i+1}x_{i+2}, c_{i+1}, t_{i+1})$ exists where $t_{i+1} = c_{i+3}$, and so $x_{i+2}c_{i+3} \notin E(G)$. The near-claw $NC(c_{i+1}c_{i+3}, c_a, t_a)$ exists where $c_a \in N_{i+2} \cup N_{i+3}$ and $c_a \neq x_{i+2}$. So $c_a = c_{i+2}$. Thus $c_{i+1}c_{i+2}, c_{i+2}c_{i+3} \in E(G)$. Moreover, $t_a \in A_i$, as otherwise $\langle \{c_{i+2}, x_{i+2}, c_{i+3}, t_a\} \rangle \cong K_{1,3}$ and $t_a = c_i$. So $c_i c_{i+2} \in E(G)$, $c_i c_{i+1} \notin E(G)$ and $c_i \in N_{i+1}$ by Claim 12. But then $\langle \{c_{i+2}, c_i, c_{i+1}, x_{i+3}\} \rangle \cong K_{1,3}$, a contradiction. So $c_{i+3} \in N_{i+4}$.

Claim 16 *If $d \geq 11$ and $|A_{4i}| \geq 9$ for every $i \in \{1, 2, \dots, \frac{t}{4}\}$, where $t = 4\lfloor \frac{d-7}{4} \rfloor$, then $|A_4| + |A_8| + \dots + |A_{t-4}| + |A_t| \geq \frac{9}{4}t$; otherwise $|A_4| + |A_8| + \dots + |A_{t-4}| + |A_t| \geq \frac{9}{4}t - 1$.*

If for every $4i$ such that $i \in \{1, 2, \dots, \frac{t}{4}\}$, $|A_{4i}| \geq 9$, then since we have $\frac{t}{4}$ terms $|A_{4i}|$, each of size at least 9, the result follows.

If for some $j \in \{4, 8, \dots, t-4, t\}$, $|A_j| = 8$ but for every $k \in \{4, \dots, j-4\} \cup \{j+4, \dots, t\}$, $|A_k| \geq 9$, then we have $(\frac{t}{4} - 1)$ terms $|A_{4i}|$ of size at least 9 together with one of size 8, and the result follows.

So we need consider the case where we have at least two values of i in $\{4, 8, \dots, \frac{t}{4}\}$ for which A_i is of size 8. Let j be the first value in $\{4, 8, \dots, t-4, t\}$ such that $|A_j| = 8$ and let $j+k$ be the first value in $\{j+4, \dots, t-4, t\}$ such that $|A_{j+k}| = 8$. If for any z in $\{j+4, \dots, j+k-4\}$, $|A_z| \geq 10$, then we are done, and so for every z in $\{j+4, \dots, j+k-4\}$, $|A_z| = 9$. By Claim 14, $c_{j+3} \in N_{j+4}$ since $|A_j| = 8$ and thus by Claim 15, $c_{j+7} \in N_{j+8}$. Thus, for any y in $\{j+4, \dots, j+k-4\}$, $|A_y| = 9$ and $c_{y-1} \in N_y$, and thus by Claim 15, $c_{y+3} \in N_{y+4}$. In particular, $c_{j+k-1} \in N_{j+k}$. But since $|A_{j+k}| = 8$, $c_{j+k-1} \in N_{j+k-1}$ by Claim 14, which is a contradiction, and so the result follows.

Claim 17 *If $c_3 \in N_4$, then $|A_0| \geq 10$.*

Suppose to the contrary that $|A_0| < 10$. We consider two cases:

CASE 1 $c_0 = c_1$

So $c_0 \in N_1$. The near-claw $NC(x_0x_1, c_0, t_0)$ exists where $t_0 \in N_2$ and $t_0x_2 \in E(G)$ as otherwise $\langle \{c_0, x_0, x_2, t_0\} \rangle \cong K_{1,3}$. The near-claw $NC(x_1x_2, c_0, t_1)$ exists where $t_1x_0 \in E(G)$ as otherwise $\langle \{c_0, x_0, x_2, t_1\} \rangle \cong K_{1,3}$. So $t_1 \in N_1$. Then $t_0t_1 \in E(G)$ as otherwise $\langle \{c_0, t_1, t_0, x_1\} \rangle \cong K_{1,3}$, and $t_0x_3 \in E(G)$ as otherwise $\langle \{x_2, x_1, x_3, t_0\} \rangle \cong K_{1,3}$. (See Figure 4.5.) The near-claw $NC(x_0c_0, c_a, t_a)$ exists where $c_a \in N_1$ and $t_a \in N_2$. Thus t_a is a new vertex in N_2 . Let $S = \{x_0, x_1, x_2, x_3, c_0, t_0, t_1, t_a\}$ and $S \subseteq A_0$; so $|A_0| \geq 8$ at this stage.

Now either $c_a \notin S$ or $c_a \in S$, in which case (since c_a is adjacent to c_0 and x_0) $c_a \in \{t_1, x_1\}$.

SUBCASE 1.1 Suppose $c_a = x_1$; then $x_1t_a \in E(G)$.

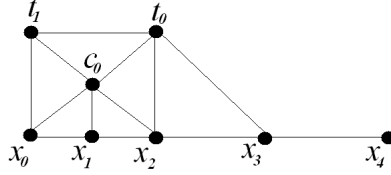


Figure 4.5: An induced subgraph.

Thus $x_2t_a \in E(G)$ as otherwise $\langle \{x_1, x_0, t_a, x_2\} \rangle \cong K_{1,3}$, and $x_3t_a \in E(G)$ as otherwise $\langle \{x_2, t_a, c_0, x_3\} \rangle \cong K_{1,3}$, and hence $t_0t_a \in E(G)$ as otherwise $\langle \{x_3, t_0, t_a, x_4\} \rangle \cong K_{1,3}$.

The near-claw $NC(t_0c_0, c_b, t_b)$ exists in G where $c_b \in N_1 \cup N_2$, and thus $t_b \in A_0$. Since every vertex in S is adjacent to c_0 or t_0 while $c_0t_b, t_0t_b \notin E(G)$, it follows that $t_b \notin S$ and is thus a new vertex in A_0 , $A_0 = S \cup \{t_b\}$, and $|A_0| = 9$. The near-claw $NC(c_0x_2, c_c, t_c)$ exists where $t_c \in A_0$, and thus $t_c = t_b$ as every vertex in S is adjacent to c_0 or x_2 and $|A_0| = 9$. So $t_bx_2 \notin E(G)$. Now, $c_b = t_1$ since no vertex in $A_0 - \{t_1\}$ is adjacent to c_0 , and to t_b . So $t_1t_b \in E(G)$, and thus $t_b \in N_1$ as otherwise $\langle \{t_1, t_b, t_0, x_0\} \rangle \cong K_{1,3}$. The near-claw $NC(x_1t_a, c_d, t_d)$ exists where $t_d \in A_0$ and so $t_d = t_b$ or t_1 . However, no vertex in A_0 is adjacent to x_1, t_a and t_b ; so $t_d \neq t_b$, and thus $t_d = t_1$ and $c_d = t_b$ as t_b is the only possible vertex in A_0 adjacent to x_1, t_a and t_1 . Thus $t_1t_a \notin E(G)$ and $x_1t_b, t_at_b \in E(G)$. The near-claw $NC(t_1t_b, c_e, t_e)$ exists where $c_e = x_0$, but t_e does not exist, a contradiction. So $x_1t_a \notin E(G)$.

SUBCASE 1.2 Suppose $c_a = t_1$; then $t_1t_a \in E(G)$.

Then $c_0t_a \notin E(G)$, $t_0t_a \in E(G)$ as otherwise $\langle \{t_1, t_0, t_a, x_0\} \rangle \cong K_{1,3}$, $t_ax_3 \in E(G)$ as otherwise $\langle \{t_0, x_3, c_0, t_a\} \rangle \cong K_{1,3}$ and $x_2t_a \in E(G)$ as otherwise $\langle \{x_3, x_2, t_a, x_4\} \rangle \cong K_{1,3}$. The near-claw $NC(c_0t_0, c_b, t_b)$ exists where $t_b \in A_0$, and thus, as every vertex in S is adjacent to c_0 and t_0 , t_b is a new vertex in A_0 such that $c_0t_b, t_0t_b \notin E(G)$. Hence $A_0 = S \cup \{t_b\}$, and thus $|A_0| = 9$. The near-claw $NC(t_1t_a, c_c, t_c)$ exists where $t_c \in A_0$, and so $t_c = \{t_b, x_1\}$ but $t_c \neq t_b$, as no vertex in A_0 is adjacent to t_1, t_a and t_b , so $t_c = x_1$ and $c_c = t_b$.

So $t_a t_b, t_1 t_b, x_1 t_b \in E(G)$. The near-claw $NC(x_1 t_b, c_d, t_d)$ exists where $t_d = t_0$ or x_3 , and thus $c_d = x_2$. So $t_b x_2 \in E(G)$. The near-claw $NC(c_0 x_2, c_e, t_e)$ exists where $t_e \in A_0$, but no vertex in $A_0 - \{c_0, x_0\}$ is non-adjacent to both c_0 and x_0 ; so t_e does not exist, which is a contradiction. So $t_1 t_a \notin E(G)$.

SUBCASE 1.3 Suppose $c_a \notin S$; so c_a is a new vertex in N_1 ,

Hence $|A_0| = 9$ since $A_0 = S \cup \{c_a\}$. Then $c_a c_0, c_a x_0, c_a t_a \in E(G)$. The near-claw $NC(c_0 t_0, c_b, t_b)$ exists where $c_b \in N_1 \cup N_2$, and thus $t_b \in A_0$. So $t_b = t_a$, and thus $t_0 t_a \notin E(G)$. Thus $c_a t_0 \notin E(G)$ as otherwise $\langle \{c_a, t_0, t_a, x_0\} \rangle \cong K_{1,3}$ and $t_a t_1 \notin E(G)$ as otherwise $\langle \{t_1, t_a, t_0, x_0\} \rangle \cong K_{1,3}$. So $c_b = x_2$ and thus $x_2 t_a \in E(G)$. So $t_a x_3 \in E(G)$ as otherwise $\langle \{x_2, x_3, c_0, t_a\} \rangle \cong K_{1,3}$, but then $\langle \{x_3, t_0, t_a, x_4\} \rangle \cong K_{1,3}$, which is a contradiction.

We therefore conclude that if $c_3 \in N_4$ and $c_0 = c_1$, then $|A_0| \geq 10$.

CASE 2: $c_0 \neq c_1$.

So $x_0, x_1, x_2, x_3 \in A_0$, and c_0 and c_1 are distinct vertices in A_0 . Since $c_3 \in N_4, c_3 \neq t_1$ by Claim 11, and thus t_1 is a new vertex (not in $\{x_0, x_1, x_2, x_3, c_0, c_1\}$) in A_0 . So $|A_0| \geq 7$. We proceed by first proving five propositions:

Proposition 17 $t_1 c_3 \notin E(G)$ and so $t_1 x_3, c_1 x_3 \notin E(G)$.

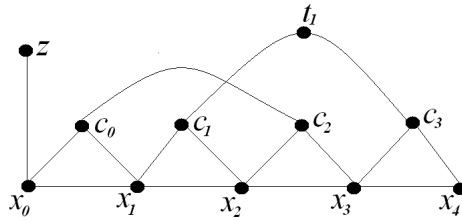


Figure 4.6: An induced subgraph.

Suppose to the contrary that $t_1 c_3 \in E(G)$. Then since $t_1 \in A_0$, it follows that $t_1 \in N_3$. Moreover, $c_1 \in N_2$, and thus $x_0 c_1 \notin E(G)$. Since c_2 is a

distinct vertex in $N_2 \cup N_3$, $|A_0| \geq 8$. By Lemma 6, $\deg x_0 \geq 3$, there is a new vertex in N_1 , say z and thus $|A_0| = 9$. So, $t_0 = c_2$, and thus $c_0c_2 \in E(G)$, $x_1c_2 \notin E(G)$ and $c_2 \in N_2$. See Figure 4.6.

Suppose $c_1x_3 \in E(G)$. Then $c_1c_2 \in E(G)$ as otherwise $\langle \{x_3, c_1, c_2, x_4\} \rangle \cong K_{1,3}$, $t_1c_2 \in E(G)$ as otherwise $\langle \{c_1, x_1, t_1, c_2\} \rangle \cong K_{1,3}$, $t_1x_3 \in E(G)$ as otherwise $\langle \{c_2, t_1, x_3, c_0\} \rangle \cong K_{1,3}$, and $c_0x_2 \in E(G)$ as otherwise $\langle \{c_2, c_0, x_2, t_1\} \rangle \cong K_{1,3}$. Thus $t_2 = z$, and so $x_2z \notin E(G)$ and $c_2z \in E(G)$. But then $\langle \{c_2, z, x_2, t_1\} \rangle \cong K_{1,3}$, which is a contradiction. So $c_1x_3 \notin E(G)$.

The near-claw $NC(x_0c_0, c_a, t_a)$ exists where $t_a \in N_2$. So $t_a = c_1$ or $t_a = x_2$. If $c_0x_2 \in E(G)$, then $t_a = c_1$, and thus $c_0c_1 \notin E(G)$. But then $\langle \{x_2, c_0, c_1, x_3\} \rangle \cong K_{1,3}$, which is a contradiction. So $c_0x_2 \notin E(G)$. Moreover, $c_0c_1 \notin E(G)$ as otherwise $\langle \{c_1, c_0, x_2, t_1\} \rangle \cong K_{1,3}$ and $c_1c_2 \notin E(G)$ as otherwise $\langle \{c_2, c_0, c_1, x_3\} \rangle \cong K_{1,3}$. The near-claw $NC(c_1t_1, c_b, t_b)$ exists where $c_b \in N_2 \cup N_3$, but no such vertex exists.

Thus $t_1c_3 \notin E(G)$. Then $t_1x_3 \notin E(G)$ as otherwise $\langle \{x_3, x_2, t_1, c_3\} \rangle \cong K_{1,3}$, and $c_1x_3 \notin E(G)$ as otherwise $\langle \{c_1, x_1, x_3, t_1\} \rangle \cong K_{1,3}$.

Proposition 18 $c_0x_2 \notin E(G)$

Suppose to the contrary that $c_0x_2 \in E(G)$. Then $c_0c_1 \in E(G)$ as otherwise $\langle \{x_2, c_0, c_1, x_3\} \rangle \cong K_{1,3}$, and $t_0 \neq t_1$, as otherwise $\langle \{c_0, x_0, x_2, t_1\} \rangle \cong K_{1,3}$; so t_0 is a new vertex in N_2 , and thus $|A_0| \geq 8$. Furthermore, $x_2t_0 \in E(G)$ as otherwise $\langle \{c_0, x_0, x_2, t_0\} \rangle \cong K_{1,3}$, and $x_3t_0 \in E(G)$, as otherwise $\langle \{x_2, x_1, x_3, t_0\} \rangle \cong K_{1,3}$.

Suppose $x_0c_1 \notin E(G)$. Then $c_1 \in N_2$, and $c_1t_0 \in E(G)$ as otherwise $\langle \{c_0, c_1, t_0, x_0\} \rangle \cong K_{1,3}$, $t_1t_0 \in E(G)$ as otherwise $\langle \{c_1, x_1, t_1, t_0\} \rangle \cong K_{1,3}$, $c_0t_1 \in E(G)$ as otherwise $\langle \{t_0, c_0, t_1, x_3\} \rangle \cong K_{1,3}$, and $t_1 \in N_1$ as otherwise $\langle \{c_0, x_2, t_1, x_0\} \rangle \cong K_{1,3}$. The near-claw $NC(x_0c_0, c_a, t_a)$ exists where $t_a \in N_2$. Thus t_a is a new vertex in N_2 , and thus $|A_0| = 9$. The near-claw $NC(c_0x_2, c_b, t_b)$ exists in G where $t_b = t_a$, and thus $x_2t_a \notin E(G)$. The near-claw $NC(c_0t_0, c_c, t_c)$ exists where $t_c = t_a$, and thus $t_0t_a, c_0t_a \notin E(G)$. Moreover, $t_ax_1 \notin E(G)$ as otherwise $\langle \{x_1, t_a, x_2, x_0\} \rangle \cong K_{1,3}$, and thus $t_at_1 \notin E(G)$ as otherwise $\langle \{t_1, t_0, t_a, x_0\} \rangle \cong K_{1,3}$. But this is a contradiction as t_a must have a neighbour in N_1 .

So $x_0c_1 \in E(G)$. So $c_1 \in N_1$ and $t_1 \in N_1$ as otherwise $\langle \{c_1, t_1, x_0, x_2\} \rangle \cong K_{1,3}$. The near-claw $NC(x_0c_0, c_d, t_d)$ exists where $t_d \in N_2$; thus t_d is a new vertex in N_2 with $t_dc_0 \notin E(G)$ and thus $|A_0| = 9$. The near-claw $NC(c_1x_2, c_e, t_e)$ exists where $t_e \in A_0$ and thus $t_e = t_d$ and so $t_dx_2, t_dc_1 \notin E(G)$.

$E(G)$. Thus $x_1t_d \notin E(G)$ as otherwise $\langle \{x_1, t_d, x_2, x_0\} \rangle \cong K_{1,3}$, and thus $t_dt_1 \in E(G)$ since t_d must have a neighbour in N_1 . The near-claw $NC(t_dt_1, c_f, t_f)$ exists where $c_f = t_0$, and thus $t_dt_0, t_1t_0 \in E(G)$. The near-claw $NC(c_0t_0, c_g, t_g)$ exists where $t_g \in A_0$ but t_g does not exist which is a contradiction.
So $c_0x_2 \notin E(G)$.

Proposition 19 $x_0c_1 \notin E(G)$ and thus $c_1 \in N_2$.

Suppose to the contrary that $x_0c_1 \in E(G)$. Then $t_1 \in N_1$ as otherwise $\langle \{c_1, t_1, x_0, x_2\} \rangle \cong K_{1,3}$, and so $t_0 \notin \{x_0, x_1, x_2, x_3, c_0, c_1, t_1\}$ and so $|A_0| \geq 8$.

Suppose $c_1t_0 \in E(G)$. Then $t_0x_2 \in E(G)$ as otherwise $\langle \{c_1, x_0, x_2, t_0\} \rangle \cong K_{1,3}$, $t_0x_3 \in E(G)$ as otherwise $\langle \{x_2, x_1, x_3, t_0\} \rangle \cong K_{1,3}$, $c_0c_1 \in E(G)$ as otherwise $\langle \{t_0, c_0, c_1, x_3\} \rangle \cong K_{1,3}$, and $c_0t_1 \in E(G)$ as otherwise $\langle \{c_1, c_0, t_1, x_2\} \rangle \cong K_{1,3}$. The near-claw $NC(c_1x_0, c_a, t_a)$ exists where $t_a \in N_2$ and $t_a \notin \{t_0, x_2\}$ and so t_a is a new vertex in N_2 . So $|A_0| = 9$, and $c_1t_a \notin E(G)$. The near-claw $NC(c_1x_2, c_b, t_b)$ exists where $t_b = t_a$, and thus $x_2t_a \notin E(G)$. Furthermore, $x_1t_a \notin E(G)$ as otherwise $\langle \{x_1, t_a, x_2, x_0\} \rangle \cong K_{1,3}$, and $t_ax_3 \notin E(G)$ as otherwise $\langle \{x_3, x_2, t_a, x_4\} \rangle \cong K_{1,3}$. But $\deg t_a \geq 3$, and so $c_0t_a, t_1t_a, t_0t_a \in E(G)$. The near-claw $NC(c_0t_0, c_c, t_c)$ exists; however, no vertex t_c exists which is non-adjacent to both c_0 and t_0 , which is a contradiction.

So $c_1t_0 \notin E(G)$.

Suppose $c_0c_1 \in E(G)$. Then $c_0t_1 \in E(G)$ as otherwise $\langle \{c_1, c_0, t_1, x_2\} \rangle \cong K_{1,3}$. The near-claw $NC(c_1x_2, c_d, t_d)$ exists. Suppose $c_d = x_1$, then t_d is a new vertex in A_0 , and thus $|A_0| = 9$. So $t_dx_1 \in E(G)$, $c_1t_d, x_2t_d \notin E(G)$ and furthermore, $t_d \in N_1$ as otherwise $\langle \{x_1, x_0, x_2, t_d\} \rangle \cong K_{1,3}$. Thus $c_2 = t_0$, and so $x_2t_0, x_3t_0 \in E(G)$. The near-claw $NC(c_0x_1, c_e, t_e)$ exists where $t_e = x_3$, but no c_e exists as $c_3 \in N_2$, which is a contradiction. Hence $c_d \neq x_1$, and thus c_d is a new vertex, $|A_0| = 9$ and thus $t_d = t_0$. So $x_2c_d, c_1c_d, c_dt_0 \in E(G)$ and $x_2t_0 \notin E(G)$. Thus $c_2 = c_d$, and so $t_0x_3 \in E(G)$ as otherwise $\langle \{c_d, x_3, t_0, c_1\} \rangle \cong K_{1,3}$. But then $\langle \{x_3, t_0, x_2, x_4\} \rangle \cong K_{1,3}$, which is a contradiction.

So $c_0c_1 \notin E(G)$.

Suppose $c_0t_1 \in E(G)$. Then $t_1t_0 \in E(G)$ as otherwise $\langle \{c_0, t_1, t_0, x_1\} \rangle \cong K_{1,3}$. The near-claw $NC(c_0x_1, c_f, t_f)$ exists. Suppose c_f is a new vertex in A_0 ; then $|A_0| = 9$ and $c_0c_f, x_1c_f \in E(G)$. Thus $t_f = x_3$ and hence, $c_f \in N_2$. Now, $c_fx_3 \in E(G)$ and $x_2c_f \in E(G)$ as otherwise $\langle \{x_3, c_f, x_2, x_4\} \rangle \cong K_{1,3}$; $t_1c_f \notin E(G)$ as otherwise $\langle \{c_f, t_1, x_1, x_3\} \rangle \cong K_{1,3}$ and $c_1c_f \notin E(G)$ as otherwise $\langle \{c_f, c_1, c_0, x_3\} \rangle \cong K_{1,3}$. The near-claw $NC(t_1c_1, c_g, t_g)$ exists where $c_g = x_0$ but no t_g exists, which is a contradiction. So c_f is not a new vertex and thus

$c_f = x_0$ with t_f a new vertex in N_1 . So $|A_0| = 9$ and $c_0t_f, x_1t_f \notin E(G)$. Then $c_2 = t_0$, and thus $x_2t_0, x_3t_0 \in E(G)$. Now, $x_2t_f \notin E(G)$ as otherwise $\langle \{x_2, t_f, x_1, x_3\} \rangle \cong K_{1,3}$ and $t_0t_f \notin E(G)$ as otherwise $\langle \{t_0, t_f, c_0, x_3\} \rangle \cong K_{1,3}$. By Lemma 6, $\deg t_f \geq 3$; so $t_1c_f, c_1t_f \in E(G)$. The near-claw $NC(c_1t_1, c_h, t_h)$ exists where $t_h = x_3$ but no c_h exists, which is a contradiction.

So $c_0t_1 \notin E(G)$.

Suppose $t_1t_0 \in E(G)$. Then $t_0x_3 \notin E(G)$ as otherwise $\langle \{t_0, t_1, c_0, x_3\} \rangle \cong K_{1,3}$, and $x_2t_0 \notin E(G)$ as otherwise $\langle \{x_2, x_1, t_0, x_3\} \rangle \cong K_{1,3}$. Thus c_2 is a new vertex in A_0 and so $|A_0| = 9$ and $c_2x_2, c_2x_3 \in E(G)$. The near-claw $NC(c_0t_0, c_i, t_i)$ exists where $c_i = c_2$ and so $c_0c_2, t_0c_2 \in E(G)$. The near-claw $NC(t_1t_0, c_j, t_j)$ exists where $c_j = c_2$, and so $t_1c_2 \in E(G)$. But then $\langle \{c_2, c_0, t_1, x_3\} \rangle \cong K_{1,3}$, which is a contradiction.

So $t_1t_0 \notin E(G)$. By Lemma 6, $\deg t_0 \geq 3$, so t_0 must be adjacent to at least one of x_2 and x_3 . If $t_0x_2 \in E(G)$, then $t_0x_3 \in E(G)$ as otherwise $\langle \{x_2, t_0, x_3, x_1\} \rangle \cong K_{1,3}$, and if $t_0x_3 \in E(G)$, then $t_0x_2 \in E(G)$ as otherwise $\langle \{x_3, x_2, t_0, x_4\} \rangle \cong K_{1,3}$. So $t_0x_2, t_0x_3 \in E(G)$. The near-claw $NC(c_0t_0, c_k, t_k)$ exists where c_k is a new vertex in A_0 , and so $|A_0| = 9$. Also, $c_0c_k, t_0c_k \in E(G)$.

Suppose $c_kc_1 \in E(G)$. The near-claw $NC(x_0c_1, c_l, t_l)$ exists where $t_l = t_0$. So $c_l = c_k$, and thus $c_k \in N_1$. The near-claw $NC(c_kc_1, c_m, t_m)$ exists where $t_m = x_3$ and thus $c_m = x_2$; so $c_kx_2 \in E(G)$. The near-claw $NC(c_kx_0, c_n, t_n)$ exists where $t_n \in N_2$, but no such vertex exists, which is a contradiction. So $c_kc_1 \notin E(G)$. So $t_k = t_1$, and thus $c_kt_1 \in E(G)$. The near-claw $NC(x_0c_1, c_p, t_p)$ exists where $t_p \in \{c_k, t_0\}$ and $c_p \in \{t_1, x_1\}$ and thus $t_p = c_k$ as $t_0t_1, t_0x_1 \notin E(G)$. So $c_k \in N_2$. The near-claw $NC(c_kt_1, c_q, t_q)$ exists but no c_q exists which is a contradiction.

So $x_0c_1 \notin E(G)$, and thus $c_1 \in N_2$.

Proposition 20 $c_0c_1 \notin E(G)$

Suppose to the contrary that $c_0c_1 \in E(G)$. Then $c_0t_1 \in E(G)$ as otherwise $\langle \{c_1, c_0, x_2, t_1\} \rangle \cong K_{1,3}$.

Suppose $x_0t_1 \notin E(G)$. Then $t_1 \in N_2$. Moreover c_2 must be a new vertex in $N_2 \cup N_3$. By Lemma 6, $\deg x_0 \geq 3$; hence there is a new vertex in N_1 , say z , and so $A_0 = \{x_0, x_1, x_2, x_3, c_0, c_1, c_2, t_1, z\}$ and $|A_0| = 9$. The near-claw $NC(zx_0, c_a, t_a)$ exists where $c_a = c_0$ or $c_a = x_1$ and thus z must be adjacent to at least one of these vertices. The near-claw $NC(c_0x_1, c_b, t_b)$ exists where $t_b = c_2$ or $t_b = x_3$. Suppose $t_b = x_3$, then $c_b = c_2, c_2 \in N_2$ and $c_0c_2, x_1c_2 \in E(G)$. The near-claw $NC(c_0c_2, c_c, t_c)$ exists where $t_c = z$,

and so $c_0z, c_2z \notin E(G)$, $c_a \neq c_0$, and thus $c_a = x_1$ and so $zx_1 \in E(G)$. Now, $zx_2 \in E(G)$ as otherwise $\langle\{x_1, z, c_0, x_2\}\rangle \cong K_{1,3}$ and $t_1c_2 \in E(G)$ as otherwise $\langle\{c_0, t_1, c_2, x_0\}\rangle \cong K_{1,3}$. The near-claw $NC(x_1c_2, c_d, t_d)$ exists, but no such vertex t_d exists which is a contradiction.

So $t_b = c_2$, and thus $c_0c_2, x_1c_2 \notin E(G)$.

Suppose $c_1c_2 \in E(G)$. Then $t_1c_2 \in E(G)$ as otherwise $\langle\{c_1, c_2, x_1, t_1\}\rangle \cong K_{1,3}$. The near-claw $NC(c_0c_1, c_e, t_e)$ exists where $t_e = \{z, x_3\}$. If $t_e = x_3$, then $c_e \in N_2$ but no c_e exists and so $t_e = z$. So $c_0z, c_1z \notin E(G)$ and $zx_1 \in E(G)$ since $c_a = x_1$. The near-claw $NC(zx_1, c_f, t_f)$ exists where $c_f = x_2$ and so $zx_2 \in E(G)$. But then $\langle\{x_2, c_1, z, x_3\}\rangle \cong K_{1,3}$, which is a contradiction.

So $c_1c_2 \notin E(G)$ and thus $c_b = z$. So $c_0z, x_1z, c_2z \in E(G)$ and so $c_2 \in N_2$. Now, $zc_1 \notin E(G)$ as otherwise $\langle\{z, c_1, c_2, x_0\}\rangle \cong K_{1,3}$. The near-claw $NC(c_0c_1, c_g, t_g)$ exists where $t_g = \{c_2, x_3\}$ and $c_g = \{t_1, x_1\}$. But x_1 is non-adjacent to c_2 and x_3 ; so $c_g = t_1, t_g = c_2$ and $t_1c_2 \in E(G)$. Hence, $t_g = c_2$ and thus $c_g = t_1$. Moreover, $t_1c_2 \in E(G)$ and $zt_1 \in E(G)$ as otherwise $\langle\{c_2, z, t_1, x_3\}\rangle \cong K_{1,3}$. The near-claw $NC(x_1z, c_h, t_h)$ exists where $t_h = x_3$ and $c_h = x_2$, and so $zx_2 \in E(G)$. But then $\langle\{x_2, x_3, z, c_1\}\rangle \cong K_{1,3}$, which is a contradiction.

So $x_0t_1 \in E(G)$, and thus $t_1 \in N_1$. So t_0 is a new vertex in A_0 with $c_0t_0 \in E(G), x_1t_0 \notin E(G)$ and $t_0 \in N_2$. Now, $c_1t_0 \in E(G)$ as otherwise $\langle\{c_0, c_1, t_0, x_0\}\rangle \cong K_{1,3}$, and $t_1t_0 \in E(G)$ as otherwise $\langle\{c_0, t_1, t_0, x_1\}\rangle \cong K_{1,3}$. At this stage, $\{x_0, x_1, x_2, x_3, c_0, c_1, t_0, t_1\} \subseteq A_0$. The near-claw $NC(x_0t_1, c_i, t_i)$ exists where $t_i \in N_2$; so t_i is a new vertex in A_0 , or $t_i = x_2$.

Suppose $t_i = x_2$. Then c_i is a new vertex in N_1 and so $|A_0| = 9$. Then $x_0c_i, t_1c_i, x_2c_i \in E(G)$. Now $c_1c_i \in E(G)$ as otherwise $\langle\{x_2, c_i, c_1, x_3\}\rangle \cong K_{1,3}$, and $c_ix_1 \in E(G)$ as otherwise $\langle\{x_2, x_1, c_i, x_3\}\rangle \cong K_{1,3}$. The near-claw $NC(c_0x_1, c_j, t_j)$ exists where $t_j = x_3$, but no vertex c_j exists which is adjacent to c_0, x_1 and x_3 , a contradiction.

So $t_i \neq x_2$, and so t_i is a new vertex in N_2 with $t_1t_i \notin E(G)$. Thus $|A_0| = 9$, and so $c_i = c_0$, and thus $c_0t_i \in E(G)$. Then $x_1t_i \in E(G)$ as otherwise $\langle\{c_0, x_1, t_i, t_1\}\rangle \cong K_{1,3}$, $t_it_0 \in E(G)$ as otherwise $\langle\{c_0, t_i, t_0, x_0\}\rangle \cong K_{1,3}$, $c_1t_i \in E(G)$ as otherwise $\langle\{c_0, t_i, c_1, x_0\}\rangle \cong K_{1,3}$, and $x_2t_i \in E(G)$ as otherwise $\langle\{x_1, x_0, x_2, t_i\}\rangle \cong K_{1,3}$. The near-claw $NC(c_0x_1, c_k, t_k)$ exists where $t_k = x_3$ and $c_k = t_i$. So $t_ix_3 \in E(G)$. The near-claw $NC(c_0t_i, c_l, t_l)$ exists but no t_l exists which is a contradiction.

So $c_0c_1 \notin E(G)$.

Proposition 21 $x_0t_1 \notin E(G)$.

Proof Suppose to the contrary that $x_0t_1 \in E(G)$. Then t_0 is a new vertex in N_2 and $\{x_0, x_1, x_2, x_3, c_0, c_1, t_0, t_1\} \subseteq A_0$. The near-claw $NC(c_0t_0, c_a, t_a)$ exists where c_a is a new vertex in A_0 , or $c_a = t_1$.

Suppose $c_a = t_1$. So $c_0t_1, t_1t_0 \in E(G)$ and thus $c_1t_0 \in E(G)$ as otherwise $\langle \{t_1, c_1, t_0, x_0\} \rangle \cong K_{1,3}$. Then t_a is a new vertex and so $|A_0| = 9$, with $t_at_1 \in E(G)$ and $c_0t_a, t_0t_a \notin E(G)$. The near-claw $NC(x_0t_1, c_b, t_b)$ exists where $t_b \in N_2$ and so $t_b = x_2$. Then $c_b = t_a$, and so $t_a \in N_1$ and $x_2t_a \in E(G)$. Now $x_1t_a \in E(G)$ as otherwise $\langle \{x_2, x_1, t_a, x_3\} \rangle \cong K_{1,3}$. The near-claw $NC(c_0x_1, c_c, t_c)$ exists where $t_c = x_3$ but no c_c exists, which is a contradiction.

So c_a is a new vertex and thus $|A_0| = 9$. So $c_0c_a, t_0c_a \in E(G)$. The near-claw $NC(c_0x_1, c_d, t_d)$ exists where $t_d = t_1$ or $t_d = x_3$.

Suppose $t_d = x_3$, and thus $c_d = c_a$. So $c_a \in N_2$, and $c_ax_1, c_ax_3 \in E(G)$. Then $c_ac_1 \notin E(G)$ as otherwise $\langle \{c_a, c_0, c_1, x_3\} \rangle \cong K_{1,3}$ and $x_2c_a \in E(G)$ as otherwise $\langle \{x_1, c_a, x_2, x_0\} \rangle \cong K_{1,3}$. The near-claw $NC(c_ax_1, c_e, t_e)$ exists where $t_e = t_1$ and $c_e = c_0$ and so $c_0t_1 \in E(G), c_at_1 \notin E(G)$. Now $t_1t_0 \in E(G)$ as otherwise $\langle \{c_0, t_1, t_0, x_1\} \rangle \cong K_{1,3}, c_1t_0 \in E(G)$ as otherwise $\langle \{t_1, c_1, t_0, x_0\} \rangle \cong K_{1,3}$ and $t_0x_3 \notin E(G)$ as otherwise $\langle \{t_0, c_0, c_1, x_3\} \rangle \cong K_{1,3}$. But then $\langle \{c_a, x_1, t_0, x_3\} \rangle \cong K_{1,3}$, which is a contradiction.

So $t_d = t_1$, and thus $c_0t_1 \notin E(G)$.

The near-claw $NC(x_0t_1, c_f, t_f)$ exists where $c_f = c_a$ and so $c_a \in N_1$ and $c_at_1 \in E(G)$. Thus $c_2 = t_0$, and so $x_2t_0, x_3t_0 \in E(G)$. Moreover, $t_1t_0 \notin E(G)$ as otherwise $\langle \{t_0, t_1, c_0, x_3\} \rangle \cong K_{1,3}$. The near-claw $NC(t_1c_1, c_g, t_g)$ exists where $c_g = c_a$; thus $c_ac_1 \in E(G)$. Moreover, $t_0c_1 \in E(G)$ as otherwise $\langle \{c_a, t_0, c_1, x_0\} \rangle \cong K_{1,3}$. But then $\langle \{c_1, t_0, t_1, x_1\} \rangle \cong K_{1,3}$, which is a contradiction.

So $x_0t_1 \notin E(G)$.

To continue with the proof of Claim 17 for the case in which $c_0 \neq c_1$, we note that we have shown thus far that A_0 contains the distinct vertices $x_0, x_1, x_2, x_3, c_0, c_1, t_1$, while $E(\langle A_0 \rangle)$ contains the edges in the induced path $x_0x_1x_2x_3$ as well as $x_0c_0, x_1c_0, x_1c_1, x_2c_1, t_1c_1$ while $t_1c_3, t_1x_3, c_1x_3, c_0x_2, x_0c_1, c_0c_1, x_0t_1, x_1t_1, x_2t_1, c_0x_3 \notin E(G)$.

By Lemma 6, $\deg x_0 \geq 3$, there is a new vertex in N_1 , say z , and c_2 is a new vertex in A_0 . Thus $|A_0| = 9$. See Figure 4.7.

The near-claw $NC(zx_0, c_a, t_a)$ exists where $c_a \in N_1$. So z is adjacent to at least one of the vertices c_0 and x_1 . The near-claw $NC(c_0x_1, c_b, t_b)$ exists where $c_b = \{c_2, z, x_0\}$. If $c_b = x_0$, then $t_b = z$ which is a contradiction as z is adjacent to at least one of the vertices in $\{c_0, x_1\}$.

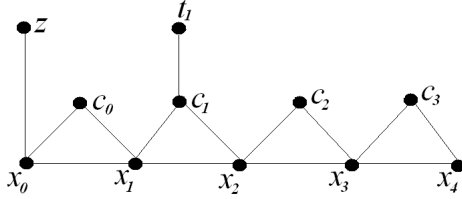


Figure 4.7: An induced subgraph.

Suppose $c_b = c_2$. Then $c_0c_2, x_1c_2 \in E(G)$ and $c_1c_2 \in E(G)$ as otherwise $\langle \{x_1, c_1, c_2, x_0\} \rangle \cong K_{1,3}$. But then $\langle \{c_2, c_0, c_1, x_3\} \rangle \cong K_{1,3}$, which is a contradiction.

So $c_b = z$, and thus $c_0z, x_1z \in E(G)$.

Suppose $c_0t_1 \in E(G)$, and thus $t_b = c_2$. So $zc_2 \in E(G), x_1c_2, c_0c_2 \notin E(G)$ and $c_2 \in N_2$. The near-claw $NC(c_0t_1, c_c, t_c)$ exists where $c_c = z$ and so $t_1z \in E(G)$. Thus $t_1c_2 \in E(G)$ as otherwise $\langle \{z, x_0, t_1, c_2\} \rangle \cong K_{1,3}$. So $t_c = x_2$ and so $zx_2 \in E(G)$. The near-claw $NC(zx_2, c_d, t_d)$ exists but no t_d exists which is a contradiction.

So $c_0t_1 \notin E(G)$.

Hence, $t_0 = c_2$; thus $c_0c_2 \in E(G), x_1c_2 \notin N_2$ and $c_2 \in E(G)$. Moreover, $c_2t_1 \notin E(G)$ as otherwise $\langle \{c_2, c_0, t_1, x_2\} \rangle \cong K_{1,3}$. But then $\deg t_1 < 3$, which is a contradiction by Lemma 8.

So $|A_0| \geq 10$.

Claim 18 *If $c_{d-4} \in N_{d-4}$, then $|A_{d-3}| \geq 10$.*

It follows similarly from Claim 17.

Claim 19 *If $c_3 \in N_3$, then $|A_0| \geq 11$.*

Suppose to the contrary that $|A_0| < 11$. We consider two cases:

CASE 1 $c_0 = c_1$

Since $x_0, x_1, x_2, x_3, c_0, c_3 \in A_0$, it follows that $|A_0| \geq 6$. We note that t_0 is a new vertex in A_0 with $c_0t_0 \in E(G)$ and $x_0t_0, x_1t_0 \notin E(G)$. Moreover,

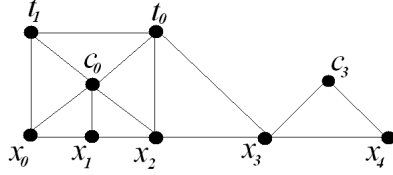


Figure 4.8: An induced subgraph.

$t_0x_2 \in E(G)$ as otherwise $\langle \{c_0, x_0, x_2, t_0\} \rangle \cong K_{1,3}$; so t_1 is a new vertex in A_0 , with $c_0t_1 \in E(G)$ and $x_1t_1, x_2t_1 \notin E(G)$. $t_1x_0 \in E(G)$ as otherwise $\langle \{c_0, t_1, x_0, x_2\} \rangle \cong K_{1,3}$, $t_1t_0 \in E(G)$ as otherwise $\langle \{c_0, t_0, t_1, x_1\} \rangle \cong K_{1,3}$ and $t_0x_3 \in E(G)$ as otherwise $\langle \{x_2, x_1, t_0, x_3\} \rangle \cong K_{1,3}$. So $|A_0| \geq 8$. See Figure 4.8.

If $t_3 \notin A_0$, then $\langle \{c_3, t_3, x_4, z\} \rangle \cong K_{1,3}$, where z is a neighbour of c_3 in N_2 , which is a contradiction. So $t_3 \in N_2 \cup N_3$, and is thus a new vertex in A_0 , and so $|A_0| \geq 9$.

If $t_3 \in N_3$, then $t_0t_3 \notin E(G)$ as otherwise $\langle \{t_0, t_1, x_3, t_3\} \rangle \cong K_{1,3}$, and $x_2t_3 \notin E(G)$ as otherwise $\langle \{x_2, x_1, x_3, t_3\} \rangle \cong K_{1,3}$. Since t_3 must have a neighbour in z in N_2 , z is a new vertex in N_2 such that $zt_3 \in E(G)$. Thus $A_0 = \{x_0, x_1, x_2, x_3, c_0, c_3, t_0, t_1, t_3, z\}$ and $|A_0| = 10$. The near-claw $NC(x_0c_0, c_a, t_a)$ exists where $t_a = z$ and so $c_0z \notin E(G)$; $zx_3 \notin E(G)$ as otherwise $\langle \{z, t_3, x_3, y\} \rangle \cong K_{1,3}$, where y is a neighbour of z in N_1 , and $x_2z \notin E(G)$ as otherwise $\langle \{x_2, c_0, z, x_3\} \rangle \cong K_{1,3}$, $t_0z \notin E(G)$ as otherwise $\langle \{t_0, c_0, x_3, z\} \rangle \cong K_{1,3}$, $t_1z \notin E(G)$ as otherwise $\langle \{t_1, t_0, z, x_0\} \rangle \cong K_{1,3}$ and $x_1z \notin E(G)$ as otherwise $\langle \{x_1, x_2, z, x_0\} \rangle \cong K_{1,3}$. But then c_a does not exist which is a contradiction.

So $t_3 \in N_2$.

Suppose $c_0t_3 \in E(G)$. Then $t_0t_3 \in E(G)$ as otherwise $\langle \{c_0, x_0, t_0, t_3\} \rangle \cong K_{1,3}$, $t_3x_2 \in E(G)$ as otherwise $\langle \{c_0, x_0, x_2, t_3\} \rangle \cong K_{1,3}$, $t_1t_3 \in E(G)$ as otherwise $\langle \{t_0, t_1, t_3, x_3\} \rangle \cong K_{1,3}$ and $x_1t_3 \in E(G)$ as otherwise $\langle \{x_2, x_1, t_3, x_3\} \rangle \cong K_{1,3}$. But then $\langle \{t_3, t_1, x_1, c_3\} \rangle \cong K_{1,3}$, which is a contradiction.

So $c_0t_3 \notin E(G)$, and so $x_2t_3 \notin E(G)$ as otherwise $\langle \{x_2, c_0, t_3, x_3\} \rangle \cong K_{1,3}$, $t_0t_3 \notin E(G)$ as otherwise $\langle \{t_0, c_0, x_3, t_3\} \rangle \cong K_{1,3}$, $t_1t_3 \notin E(G)$ as otherwise $\langle \{t_1, x_0, t_0, t_3\} \rangle \cong K_{1,3}$ and $x_1t_3 \notin E(G)$ as otherwise $\langle \{x_1, x_0, x_2, t_0\} \rangle \cong K_{1,3}$. By Lemma 6, $\deg t_3 \geq 3$, and so t_3 must be adjacent to at least two more vertices in $A_0 - \{x_0, x_1, x_2, x_3, c_0, t_1, t_0\}$ since $c_3t_3 \in E(G)$. Hence, $|A_0| \geq 7 + 4 = 11$.

CASE 2 $c_0 \neq c_1$.

Since $x_0, x_1, x_2, x_3, c_0, c_1, c_2, c_3$ are all distinct vertices in A_0 , it follows that $|A_0| \geq 8$. Note that if $t_3 \notin A_0$, then $\langle \{c_3, z, t_3, x_4\} \rangle \cong K_{1,3}$, where z is a neighbour of c_3 in N_2 , which is a contradiction. So $t_3 \in A_0$.

Proposition 22 $x_0c_1 \notin E(G)$, and thus $c_1 \in N_2$.

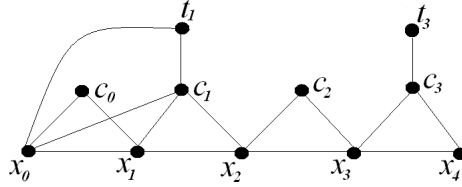


Figure 4.9: An induced subgraph.

Suppose to the contrary that $x_0c_1 \in E(G)$. Then $c_1 \in N_1$ and $x_0t_1 \in E(G)$ as otherwise $\langle \{c_1, x_0, x_2, t_1\} \rangle \cong K_{1,3}$, and thus $t_1 \in N_1$. So t_1 is a new vertex in N_1 , and so $|A_0| \geq 9$. Further $t_3 \in A_0$ and thus must be a new vertex in $N_2 \cup N_3$, and so $|A_0| = 10$. See Figure 4.9.

Suppose $t_3 \in N_3$. Then $x_2t_3 \notin E(G)$ as otherwise $\langle \{x_2, x_1, x_3, t_3\} \rangle \cong K_{1,3}$, and since t_3 needs a neighbour in N_2 , $t_3c_2 \in E(G)$, $c_2 \in N_2$ and $N_2 = \{x_2, c_2\}$. So $t_0 = c_2$, and so $c_0c_2 \in E(G)$. But then $\langle \{c_2, c_0, t_3, x_3\} \rangle \cong K_{1,3}$, which is a contradiction.

So $t_3 \in N_2$.

Suppose $c_1t_3 \in E(G)$. Then $x_2t_3 \in E(G)$ as otherwise $\langle \{c_1, x_2, t_3, x_0\} \rangle \cong K_{1,3}$, $x_1t_3 \in E(G)$ as otherwise $\langle \{x_2, x_1, x_3, t_3\} \rangle \cong K_{1,3}$ and $t_1t_3 \notin E(G)$ as otherwise $\langle \{t_3, t_1, x_1, c_3\} \rangle \cong K_{1,3}$. So $t_0 = c_2$, and thus $c_0c_2 \in E(G)$, $x_1c_2 \notin$

$E(G)$ and $c_2 \in N_2$. The near-claw $NC(c_3t_3, c_a, t_a)$ exists where $c_a \in N_2 \cup N_3$, and thus $c_a = x_2$ or $c_a = c_2$.

Suppose $c_a = x_2$. Then $c_3x_2 \in E(G)$. Now, $c_2c_3 \in E(G)$ as otherwise $\langle \{x_2, x_1, c_2, c_3\} \rangle \cong K_{1,3}$. So $t_a = c_0$, and so $c_0x_2 \in E(G)$ and $c_0t_3 \notin E(G)$ but then $\langle \{x_2, c_0, t_3, x_3\} \rangle \cong K_{1,3}$, which is a contradiction.

So $c_a = c_2$ and thus $t_3c_2, c_3c_2 \in E(G)$ and $c_0t_3 \in E(G)$ as otherwise $\langle \{c_2, c_0, x_3, t_3\} \rangle \cong K_{1,3}$. Thus $t_a = t_1$, and so $c_2t_1 \in E(G)$ but then $\langle \{c_2, t_1, x_3, t_3\} \rangle \cong K_{1,3}$, which is a contradiction.

So $c_1t_3 \notin E(G)$ and so $x_2t_3 \notin E(G)$ as otherwise $\langle \{x_2, c_1, t_3, x_3\} \rangle \cong K_{1,3}$ and $x_1t_3 \notin E(G)$ as otherwise $\langle \{x_1, x_0, x_2, t_3\} \rangle \cong K_{1,3}$. The near-claw $NC(t_3c_3, c_b, t_b)$ exists where $c_b \in N_2 \cup N_3$. So $c_b = c_2$ and thus $t_3c_2, c_3c_2 \in E(G)$, $x_1c_2 \notin E(G)$ as otherwise $\langle \{c_2, x_1, t_3, x_3\} \rangle \cong K_{1,3}$ and $c_2c_1 \notin E(G)$ as otherwise $\langle \{c_2, c_1, t_3, x_3\} \rangle \cong K_{1,3}$. The near-claw $NC(c_1x_2, c_c, t_c)$ exists where $c_c = c_0$ or x_1 .

Suppose $c_c = c_0$, and thus $c_0c_1, c_0x_2 \in E(G)$. Hence $t_c = t_3$ and so $c_0t_3 \in E(G)$ but then $\langle \{c_0, x_2, t_3, x_0\} \rangle \cong K_{1,3}$, which is a contradiction.

So $c_c = x_1$, and thus $t_c = c_0$ and so $c_0c_1, c_0x_2 \notin E(G)$. The near-claw $NC(t_1c_1, c_d, t_d)$ exists where $c_d = x_0$ and thus $t_d = c_0$. Hence, $c_0t_1 \notin E(G)$. Now, $x_2c_3 \notin E(G)$ as otherwise $\langle \{c_3, x_2, t_3, x_4\} \rangle \cong K_{1,3}$ and $c_0c_2 \notin E(G)$ as otherwise $\langle \{c_2, c_0, x_2, c_3\} \rangle \cong K_{1,3}$. Thus $t_0 = t_3$ and so $c_0t_3 \in E(G)$. The near-claw $NC(c_0t_3, c_e, t_e)$ exists, but no vertex c_e exists which is a contradiction.

Thus $x_0c_1 \notin E(G)$.

Proposition 23 $c_0x_2 \notin E(G)$.

Suppose to the contrary that $c_0x_2 \in E(G)$.

Recall by Claim 11 that $t_i = c_j$ implies $j = i - 2$ or $j = i + 2$. Suppose $t_0 = c_2$. Then $c_0c_2 \in E(G)$, $x_1c_2 \notin E(G)$ and $c_2 \in N_2$. Thus t_2 is a new vertex in A_0 and so $|A_0| \geq 9$, $c_2t_2 \in E(G)$ and $x_2t_2, x_3t_2 \notin E(G)$. Now, $c_0t_2 \in E(G)$ as otherwise $\langle \{c_2, c_0, t_2, x_3\} \rangle \cong K_{1,3}$, and so $t_2 \in N_1$ as otherwise $\langle \{c_0, x_0, x_2, t_2\} \rangle \cong K_{1,3}$.

Now suppose $c_0c_1 \notin E(G)$. Then, we have $c_1x_3 \in E(G)$ as otherwise $\langle \{x_2, c_0, c_1, x_3\} \rangle \cong K_{1,3}$ and so $t_3 \neq c_1$. Furthermore, $c_1c_2 \in E(G)$ as otherwise $\langle \{x_3, c_1, c_2, x_4\} \rangle \cong K_{1,3}$. Now $t_3 \in A_0$ and since $t_2 \in N_1$, $t_3 \neq t_2$, and so t_3 is a new vertex in $N_2 \cup N_3$, and thus $|A_0| = 10$, $t_3c_3 \in E(G)$ and $t_3x_3, t_3x_4 \notin E(G)$. Moreover, $t_1 \neq t_2$ as otherwise $\langle \{c_1, x_1, x_3, t_2\} \rangle \cong K_{1,3}$ and $t_1 \neq t_3$ as otherwise $\langle \{c_1, x_1, t_3, x_3\} \rangle \cong K_{1,3}$, and so $t_1 = c_3$. Hence $c_1c_3 \in E(G)$ and

$x_2c_3 \notin E(G)$ and thus $c_2c_3 \in E(G)$ as otherwise $\langle\{c_1, x_1, c_2, c_3\}\rangle \cong K_{1,3}$. But then $\langle\{c_2, t_2, x_2, c_3\}\rangle \cong K_{1,3}$, which is a contradiction.

So $c_0c_1 \in E(G)$ and thus $c_1c_2 \in E(G)$ as otherwise $\langle\{c_0, c_1, c_2, x_0\}\rangle \cong K_{1,3}$. The near-claw $NC(x_0c_0, c_a, t_a)$ exists where $t_a \in N_2$, and so t_a is a new vertex in N_2 such that $c_0t_a \notin E(G)$. Hence, $|A_0| = 10$.

Suppose $c_a = t_2$. Then $t_2t_a \in E(G)$ and $c_2t_a \in E(G)$ as otherwise $\langle\{t_2, t_a, c_2, x_0\}\rangle \cong K_{1,3}$, $x_3t_a \in E(G)$ as otherwise $\langle\{c_2, t_a, c_0, x_3\}\rangle \cong K_{1,3}$ and $x_2t_a \in E(G)$ as otherwise $\langle\{x_3, x_2, t_0, x_4\}\rangle \cong K_{1,3}$. The near-claw $NC(c_0x_2, c_b, t_b)$ exists where $t_b = c_3$ and so $x_2c_3 \notin E(G)$. The near-claw $NC(c_0c_2, c_c, t_c)$ exists where $t_c = c_3$ and so $c_2c_3 \notin E(G)$. Moreover, $c_c = c_1$ and so $c_1c_3 \in E(G)$. But then $\langle\{c_1, x_1, c_2, c_3\}\rangle \cong K_{1,3}$, which is a contradiction.

So $c_a = x_1$, and thus $x_1t_a \in E(G)$ and $t_ax_2 \in E(G)$ as otherwise $\langle\{x_1, x_2, t_a, x_0\}\rangle \cong K_{1,3}$, $t_ax_3 \in E(G)$ as otherwise $\langle\{x_2, c_0, t_a, x_3\}\rangle \cong K_{1,3}$ and thus $c_2t_a \in E(G)$ as otherwise $\langle\{x_3, c_2, t_a, x_4\}\rangle \cong K_{1,3}$. The near-claw $NC(c_0x_2, c_d, t_d)$ exists where $t_d = c_3$, and thus $x_2c_3 \notin E(G)$. The near-claw $NC(c_0c_2, c_e, t_e)$ exists where $t_e = c_3$ and so $c_2c_3 \notin E(G)$. Now $c_e = c_1$, and so $c_1c_3 \in E(G)$ but then $\langle\{c_1, x_1, c_2, c_3\}\rangle \cong K_{1,3}$, which is a contradiction.

So $t_0 \neq c_2$, and so t_0 is a new vertex in N_2 , and thus $|A_0| \geq 9$. By Lemma 6, $\deg x_0 \geq 3$; hence there is a new vertex z in N_1 , and so $|A_0| = 10$; $x_2t_0 \in E(G)$ as otherwise $\langle\{c_0, x_0, t_0, x_2\}\rangle \cong K_{1,3}$ and thus $t_0x_3 \in E(G)$ as otherwise $\langle\{x_2, x_1, t_0, x_3\}\rangle \cong K_{1,3}$. Since $t_3 \in A_0$, $t_3 = c_1$, and so $c_1c_3 \in E(G)$ and $c_1x_3 \notin E(G)$. Moreover, $c_0c_1 \in E(G)$ as otherwise $\langle\{x_2, c_0, c_1, x_3\}\rangle \cong K_{1,3}$, $t_0c_1 \in E(G)$ as otherwise $\langle\{c_0, t_0, c_1, x_0\}\rangle \cong K_{1,3}$ and $c_3t_0 \in E(G)$ as otherwise $\langle\{c_1, x_1, c_3, t_0\}\rangle \cong K_{1,3}$. The near-claw $NC(x_0c_0, c_f, t_f)$ exists where $t_f = c_2$. So $c_2 \in N_2$ and $c_0c_2 \notin E(G)$; $t_2 = z$ and so $c_2z \in E(G)$ and $zx_2 \notin E(G)$. Moreover, $t_1 \neq z$ as otherwise $\langle\{c_1, x_1, z, c_3\}\rangle \cong K_{1,3}$ and so $t_1 = c_3$. Thus $x_2c_3 \notin E(G)$, and so $c_1z \notin E(G)$ as otherwise $\langle\{c_1, z, x_2, c_3\}\rangle \cong K_{1,3}$, $t_0z \notin E(G)$ as otherwise $\langle\{t_0, z, x_2, c_3\}\rangle \cong K_{1,3}$. The near-claw $NC(zc_2, c_g, t_g)$ exists where $c_g = x_1$, and so $c_2x_1, zx_1 \in E(G)$. Then $c_1c_2 \in E(G)$ as otherwise $\langle\{x_1, c_1, c_2, x_0\}\rangle \cong K_{1,3}$ and $c_2c_3 \in E(G)$ as otherwise $\langle\{c_1, c_2, c_3, c_0\}\rangle \cong K_{1,3}$. So $t_g = c_0$ and thus $c_0z \notin E(G)$. But then $\deg z = 3$ and $\deg x_0 = 3$, which is a contradiction as the vertices of degree 3 form an independent set by Lemma 4

So $c_0x_2 \notin E(G)$.

Proposition 24 $c_0c_1 \notin E(G)$.

Suppose to the contrary that $c_0c_1 \in E(G)$.

Then $t_1 \neq c_3$ as otherwise $\langle \{c_1, c_0, x_2, c_3\} \rangle \cong K_{1,3}$, and thus t_1 is a new vertex in A_0 . So $c_0t_1 \in E(G)$ as otherwise $\langle \{c_1, c_0, x_2, t_1\} \rangle \cong K_{1,3}$.

Suppose $c_1x_3 \in E(G)$. Then $t_1x_3 \in E(G)$ as otherwise $\langle \{c_1, x_1, x_3, t_1\} \rangle \cong K_{1,3}$, and so $t_1 \in N_2$. But then $\langle \{x_3, t_1, x_2, x_4\} \rangle \cong K_{1,3}$, which is a contradiction.

So $c_1x_3 \notin E(G)$.

Now $t_1x_3 \notin E(G)$, otherwise $t_1 \in N_2$, and then $\langle \{x_3, x_2, t_1, x_4\} \rangle \cong K_{1,3}$, which is a contradiction.

Suppose $x_1c_2 \in E(G)$. Then $c_2 \in N_2$ and thus $c_1c_2 \in E(G)$ as otherwise $\langle \{x_1, x_0, c_1, c_2\} \rangle \cong K_{1,3}$ and $t_1c_2 \notin E(G)$ as otherwise $\langle \{c_2, x_1, x_3, t_1\} \rangle \cong K_{1,3}$.

Suppose t_2 is a new vertex in A_0 , and so $x_2t_2, x_3t_2 \notin E(G), c_2t_2 \in E(G)$ and $|A_0| = 10$. Then $x_1t_2 \in E(G)$ as otherwise $\langle \{c_2, x_1, x_3, t_2\} \rangle \cong K_{1,3}$ and $t_2 \in N_1$ as otherwise $\langle \{x_1, x_0, x_2, t_2\} \rangle \cong K_{1,3}$. So $t_0 = t_1$ and thus $t_1 \in N_2$. Moreover, $c_0c_2 \notin E(G)$ as otherwise $\langle \{c_0, x_0, c_2, t_1\} \rangle \cong K_{1,3}$. The near-claw $NC(c_0x_1, c_a, t_a)$ exists where $c_a \in \{c_1, x_0, t_2\}$ and thus $t_a = c_3$, and so $c_a = c_1$ since $c_a \in N_2$. $c_1c_3 \in E(G), x_2c_3 \in E(G)$ as otherwise $\langle \{c_1, c_0, x_2, c_3\} \rangle \cong K_{1,3}$ and $t_1c_3 \in E(G)$ as otherwise $\langle \{c_1, x_1, c_3, t_1\} \rangle \cong K_{1,3}$. But then $\langle \{c_3, x_2, t_1, x_4\} \rangle \cong K_{1,3}$.

So $t_2 = c_0$. Thus $c_0c_2 \in E(G)$ and $t_1 \in N_1$ as otherwise $\langle \{c_0, x_0, c_2, t_1\} \rangle \cong K_{1,3}$. So t_0 is a new vertex in N_2 and thus $|A_0| = 10$. Therefore, $c_2t_0 \in E(G)$ as otherwise $\langle \{c_0, c_2, t_0, x_0\} \rangle \cong K_{1,3}$, and $t_0x_3 \in E(G)$ as otherwise $\langle \{c_2, x_1, x_3, t_0\} \rangle \cong K_{1,3}$. Now $t_3 \neq c_1$ as otherwise $\langle \{c_1, t_1, x_1, c_3\} \rangle \cong K_{1,3}$ and since $t_3 \in A_0$, $t_3 = t_0$ which is a contradiction as $t_0x_3 \in E(G)$.

So $x_1c_2 \notin E(G)$.

Suppose $x_2c_3 \in E(G)$. Thus $c_2c_3 \in E(G)$ as otherwise $\langle \{x_2, c_2, c_3, x_1\} \rangle \cong K_{1,3}$, $t_1c_3 \notin E(G)$ as otherwise $\langle \{c_3, t_1, x_2, x_4\} \rangle \cong K_{1,3}$ and $c_1c_3 \notin E(G)$ as otherwise $\langle \{c_1, t_1, x_1, c_3\} \rangle \cong K_{1,3}$. Thus t_3 is a new vertex in $N_2 \cup N_3$, and so $|A_0| = 10$. By Lemma 6, $\deg x_0 \geq 3, t_1 \in N_1$ and so $x_2t_3 \in E(G)$ as otherwise $\langle \{c_3, x_2, x_4, t_3\} \rangle \cong K_{1,3}$ and $x_1t_3 \in E(G)$ as otherwise $\langle \{x_2, x_1, x_3, t_3\} \rangle \cong K_{1,3}$. So $t_0 = c_2$, and thus $c_0c_2 \in E(G)$ and $c_2 \in N_2$. Moreover, $c_2t_3 \in E(G)$ as otherwise $\langle \{c_3, c_2, t_3, x_4\} \rangle \cong K_{1,3}$. The near-claw $NC(c_0c_2, c_b, t_b)$ exists, but no vertex t_b exists which is a contradiction.

So $x_2c_3 \notin E(G)$. Thus $c_1c_3 \notin E(G)$ as otherwise $\langle \{c_1, c_0, x_2, c_3\} \rangle \cong K_{1,3}$.

Suppose $t_1c_3 \notin E(G)$. Then t_3 is a new vertex in $N_2 \cup N_3$, and so $|A_0| = 10$. By Lemma 6, $\deg x_0 \geq 3$, and hence $t_1 \in N_1$. The near-claw $NC(x_0t_1, c_c, t_c)$ exists where $c_c = c_0$. Now, $t_c \neq c_2$ as otherwise $c_0c_2 \in E(G), t_1c_2 \notin E(G)$ and $\langle \{c_0, t_1, c_2, x_1\} \rangle \cong K_{1,3}$. Thus $t_c = t_3$ and so $c_0t_3 \in E(G), t_1t_3 \notin E(G)$ and $t_3 \in N_2$. Then $c_1t_3 \in E(G)$ as otherwise $\langle \{c_0, c_1, t_3, x_0\} \rangle \cong K_{1,3}$ and $x_2t_3 \in$

$E(G)$ as otherwise $\langle\{c_1, t_1, t_3, x_2\}\rangle \cong K_{1,3}$. But then $\langle\{t_3, c_0, x_2, c_3\}\rangle \cong K_{1,3}$, which is a contradiction.

So $t_1c_3 \in E(G)$ and thus $t_1 \in N_2$. By Lemma 6, $\deg x_0 \geq 3$; so there is a new vertex z in N_1 , and so $|A_0| = 10$. The near-claw $NC(t_1c_3, c_d, t_d)$ exists where $c_d = c_2$. Thus $t_1c_2, c_3c_2 \in E(G)$. Now $c_0c_2 \notin E(G)$ as otherwise $\langle\{c_2, c_0, x_2, c_3\}\rangle \cong K_{1,3}$.

Suppose $c_1c_2 \in E(G)$. The near-claw $NC(c_1c_2, c_e, t_e)$ exists where $c_e = t_1, z$ or x_2 .

Suppose $c_e = x_2$, then $t_e = z$. So $zx_2 \in E(G)$ and $zc_1 \notin E(G)$, but then $\langle\{x_2, z, c_1, x_3\}\rangle \cong K_{1,3}$, which is a contradiction.

Suppose $c_e = t_1$, then $t_e = z$. So $t_1z \in E(G)$ and $zc_1 \notin E(G)$, but then $\langle\{t_1, z, c_1, c_3\}\rangle \cong K_{1,3}$, which is a contradiction.

So $c_e = z$, and so $c_1z, c_2z \in E(G)$. Then $zt_1 \in E(G)$ as otherwise $\langle\{c_2, t_1, z, x_3\}\rangle \cong K_{1,3}$ and $zx_2 \in E(G)$ as otherwise $\langle\{c_2, x_2, z, c_3\}\rangle \cong K_{1,3}$. The near-claw $NC(zx_0, c_f, t_f)$ exists where $t_f \in N_2$ but no such vertex exists.

So $c_1c_2 \notin E(G)$.

The near-claw $NC(c_0x_1, c_g, t_g)$ exists where $c_g = z$ or c_1 . So $c_g = z$, as otherwise if $c_g = c_1$, then no t_g exists. Moreover $t_g = c_2$, and thus $x_1z, c_0z, zc_2 \in E(G)$ and $c_2 \in N_2$. Now $zx_2 \in E(G)$ as otherwise $\langle\{c_2, z, x_2, c_3\}\rangle \cong K_{1,3}$ and $zt_1 \in E(G)$ as otherwise $\langle\{c_2, z, t_1, x_3\}\rangle \cong K_{1,3}$. The near-claw $NC(zx_0, c_h, t_h)$ exists where $t_h \in N_2$. Thus $t_h = c_1$ and so $zc_1 \notin E(G)$. But this is a contradiction since $\langle\{x_2, z, c_1, x_3\}\rangle \cong K_{1,3}$.

So $c_0c_1 \notin E(G)$.

Proposition 25 $x_1c_2 \notin E(G)$.

Suppose to the contrary that $x_1c_2 \in E(G)$, and so $c_2 \in N_2$.

Now $c_1c_2 \in E(G)$ as otherwise $\langle\{x_1, x_0, c_1, c_2\}\rangle \cong K_{1,3}$ and thus t_0 is a new vertex in N_2 . By Lemma 6, $\deg x_0 \geq 3$; so there is a new vertex z in N_1 and so $|A_0| = 10$. The near-claw $NC(c_0t_0, c_a, t_a)$ exists where $c_a = c_2$ or z .

Suppose $c_a = z$. Then $zc_0, zt_0 \in E(G)$ and $t_a \in N_2$. But then $\langle\{z, x_0, t_0, t_a\}\rangle \cong K_{1,3}$, which is a contradiction.

So $c_a = c_2$. Then $c_0c_2, t_0c_2 \in E(G)$. Now, $t_0x_3 \in E(G)$ as otherwise $\langle\{c_2, x_1, t_0, x_3\}\rangle \cong K_{1,3}$ and $c_1x_3 \in E(G)$ as otherwise $\langle\{c_2, c_0, c_1, x_3\}\rangle \cong K_{1,3}$. Now, $t_3 \in N_2 \cup N_3$, but no such vertex t_3 exists which is a contradiction.

So $x_1c_2 \notin E(G)$.

Proposition 26 $c_1x_3 \notin E(G)$.

Suppose to the contrary that $c_1x_3 \in E(G)$. Therefore, t_3 is a new vertex in $N_2 \cup N_3$ such that $c_3t_3 \in E(G)$ and $x_3t_3, x_4t_3 \notin E(G)$. By Lemma 6, $\deg x_0 \geq 3$; hence there is a new vertex z in N_1 and thus $|A_0| = 10$.

Suppose $c_1c_2 \notin E(G)$. Then $c_2x_4 \in E(G)$ as otherwise $\langle\{x_3, c_1, c_2, x_4\}\rangle \cong K_{1,3}$, and thus $c_2 \in N_3$. So $c_0c_2 \notin E(G)$ and thus $t_0 = t_3$ and so $c_0t_3 \in E(G)$ and $x_1t_3 \notin E(G)$. The near-claw $NC(c_0x_1, c_a, t_a)$ exists where $c_a = x_0$ or $c_a = z$. The near-claw $NC(x_0z, c_b, t_b)$ exists where $c_b = x_1$ or $c_b c_0$, and thus z is adjacent to at least one of the vertices in $\{x_1, c_0\}$. So, if $c_a = x_0$, then $t_a = z$, which is a contradiction. Thus $c_a = z$, and so $t_a \in N_2$ but then no t_a exists which is a contradiction.

So $c_1c_2 \in E(G)$. The near-claw $NC(c_0x_1, c_c, t_c)$ exists where $c_c = z$ or $c_c = t_3$.

Suppose $c_c = z$. Then $c_0z, x_1z \in E(G)$. The near-claw $NC(x_1c_1, c_d, t_d)$ exists where $t_d = t_3$ or $t_d = c_3$.

Suppose $t_d = c_3$, and so $c_1c_3 \notin E(G)$. Thus $t_1 = t_3$, but then $\langle\{c_1, x_1, x_3, t_3\}\rangle \cong K_{1,3}$, which is a contradiction.

So $t_d = t_3$, and so $c_1t_3, x_1t_3 \notin E(G)$. So $x_2t_3 \notin E(G)$ as otherwise $\langle\{x_2, t_3, x_1, x_3\}\rangle \cong K_{1,3}$, and thus $c_d = z$. Therefore $c_1z, t_3z \in E(G)$ and $t_3 \in N_2$, but then $\langle\{z, x_0, c_1, t_3\}\rangle \cong K_{1,3}$, which is a contradiction.

So $c_c = t_3$ and thus $c_0t_3, x_1t_3 \in E(G)$ and $t_3 \in N_2$. Then $t_0 = c_2$ and so $c_0c_2 \in E(G)$ and $c_2 \in N_2$. Then $t_3c_1 \in E(G)$ as otherwise $\langle\{x_1, x_0, c_1, t_3\}\rangle \cong K_{1,3}$, $t_3x_2 \in E(G)$ as otherwise $\langle\{x_1, x_0, x_2, t_3\}\rangle \cong K_{1,3}$, $c_2t_3 \in E(G)$ as otherwise $\langle\{c_0, x_0, c_2, t_3\}\rangle \cong K_{1,3}$ and $c_1c_3 \in E(G)$ as otherwise $\langle\{t_3, c_0, c_1, c_3\}\rangle \cong K_{1,3}$. The near-claw $NC(c_1x_1, c_e, t_e)$ exists where $t_e = z$ and $c_d = t_3$ or x_2 . If $c_d = t_3$, then $\langle\{t_3, x_1, z, c_3\}\rangle \cong K_{1,3}$ and if $c_d = x_2$, then $\langle\{x_2, x_1, z, x_3\}\rangle \cong K_{1,3}$, both of which are contradictions.

So $c_1x_3 \notin E(G)$.

Proposition 27 $x_2c_3 \notin E(G)$.

Suppose to the contrary that $x_2c_3 \in E(G)$. Then $t_1 \neq c_3$, and so $t_1 \notin \{x_i, c_i : i = 0, 1, 2, 3\}$; hence t_1 is a new vertex in A_0 and $c_2c_3 \in E(G)$ as otherwise $\langle\{x_2, x_1, c_2, c_3\}\rangle \cong K_{1,3}$. The near-claw $NC(c_0x_1, c_a, t_a)$ exists where $c_a = x_0$ or where c_a is a new vertex in A_0 .

Suppose $c_a = x_0$. Then $t_a \in N_1$ and thus $t_a = t_1$ or is a new vertex in N_1 . If t_a is a new vertex in N_1 , then the near-claw $NC(t_ax_0, c_b, t_b)$ exists where $c_b = t_1$ and so $t_1 \in N_1$, and if $t_a = t_1$, then the near-claw $NC(t_1x_0, c_c, t_c)$ exists where c_c is a new vertex in N_1 . In either case, $t_1 \in N_1$ and $|A_0| = 10$

and so $t_3 = c_1$. Thus $c_1c_3 \in E(G)$, but then $\langle\{c_1, x_1, t_1, c_3\}\rangle \cong K_{1,3}$, which is a contradiction.

So c_a is a new vertex in A_0 such that $c_0c_a, x_1c_a \in E(G)$, and so $|A_0| = 10$.

Suppose $c_1c_3 \in E(G)$. Then $t_1c_3 \in E(G)$ as otherwise $\langle\{c_1, x_1, t_1, c_3\}\rangle \cong K_{1,3}$ and $t_1x_4 \in E(G)$ as otherwise $\langle\{c_3, x_2, t_1, x_4\}\rangle \cong K_{1,3}$. Hence, $t_1 \in N_3$; thus $t_0 = c_2$ and so $c_0c_2 \in E(G)$. By Lemma 6, $\deg x_0 \geq 3$, $c_a \in N_1$ and so $t_a \in N_2$ but then t_a does not exist, which is a contradiction.

So $c_1c_3 \notin E(G)$ and thus $t_3 \neq t_1$ as otherwise $\langle\{c_3, t_1, x_2, x_4\}\rangle \cong K_{1,3}$. So $t_3 = c_a$ and thus $c_3c_a \in E(G)$, $c_a \in N_2$ and $x_3c_a \notin E(G)$. By Lemma 6, $\deg x_0 \geq 3$, so $t_1 \in N_1$. Thus $t_0 = c_2$, and so $c_0c_2 \in E(G)$ and $c_2 \in N_2$. The near-claw $NC(x_0t_1, c_a, t_d)$ exists where $c_d = c_0$ and thus $c_0t_1 \in E(G)$. Then $t_1c_2 \in E(G)$ as otherwise $\langle\{c_0, x_1, t_1, c_2\}\rangle \cong K_{1,3}$ and $c_1c_2 \in E(G)$ as otherwise $\langle\{t_1, c_1, c_2, x_0\}\rangle \cong K_{1,3}$. The near-claw $NC(c_0c_2, c_e, t_e)$ exists but no vertex t_e exists which is a contradiction.

So $x_2c_3 \notin E(G)$.

Proposition 28 $c_2x_4 \notin E(G)$.

Suppose to the contrary that $c_2x_4 \in E(G)$. Then $c_2 \in N_3$, and thus t_0 is a new vertex in N_2 . By Lemma 6, $\deg x_0 \geq 3$; so there exists a new vertex z in N_1 and so $|A_0| = 10$. The near-claw $NC(c_0x_1, c_a, t_a)$ exists where $c_a = z$ or $c_a = x_0$. The near-claw $NC(x_0z, c_b, t_b)$ exists where $c_b = x_1$ or $c_b = c_0$. Thus z is adjacent to at least one of the vertices in $\{x_1, c_0\}$. So, if $c_a = x_0$, then $t_a = z$, which is a contradiction. Hence, $c_a = z$. But then t_a does not exist, which is a contradiction.

So $c_2x_4 \notin E(G)$.

To continue with the proof of Claim 19 for the case in which $c_0 \neq c_1$ and $c_3 \in N_3$, we note that we have shown thus far that A_0 contains the distinct vertices $x_0, x_1, x_2, x_3, c_0, c_1, c_2, c_3$, while $E(\langle A_0 \rangle)$ contains the edges in the induced path $x_0x_1x_2x_3$ as well as $x_0c_0, x_1c_0, x_1c_1, x_2c_1, x_2c_2, x_3c_2, x_3c_3$ while $x_0c_1, c_0x_2, c_0c_1, x_1c_2, c_1x_3, x_2c_3, c_2x_4, c_0c_3 \notin E(G)$.

The near-claw $NC(c_0x_1, c_\alpha, t_\alpha)$ exists where $c_\alpha = x_0$ or c_α is a new vertex in A_0 .

Suppose $c_\alpha = x_0$. Then $t_\alpha \in N_1$, and so t_α is a new vertex in A_0 with $t_\alpha c_0, t_\alpha x_1 \notin E(G)$. The near-claw $NC(t_\alpha x_0, c_\beta, t_\beta)$ exists where $c_\beta \in N_1$, and is thus a new vertex in N_1 . Hence, $|A_0| = 10$. Since $t_3 \in N_2 \cup N_3$, $t_3 = c_1$. The near-claw $NC(c_1c_3, c_\gamma, t_\gamma)$ exists where $c_\gamma = c_2$; thus $c_1c_2, c_2c_3 \in E(G)$.

Then $c_1 t_\alpha \notin E(G)$ as otherwise $\langle \{c_1, x_1, t_\alpha, c_3\} \rangle \cong K_{1,3}$ and $x_2 t_\alpha \notin E(G)$ as otherwise $\langle \{x_2, x_1, t_\alpha, x_3\} \rangle \cong K_{1,3}$. Now, by Lemma 6, $\deg t_\alpha \geq 3$, and thus $c_2 t_\alpha \in E(G)$. But then $\langle \{c_2, c_1, t_\alpha, x_3\} \rangle \cong K_{1,3}$, which is a contradiction.

So $c_\alpha \neq x_0$ and so c_α is a new vertex in $N_1 \cup N_2$ such that $c_0 c_\alpha, x_1 c_\alpha \in E(G)$, and so $|A_0| \geq 9$.

Suppose $c_\alpha \in N_2$. By Lemma 6, $\deg x_0 \geq 3$, there exists a new vertex z in N_1 and so, $|A_0| = 10$. Now, $t_0 = c_2$ and thus $c_0 c_2 \in E(G)$, $c_2 \in N_2$, $c_2 c_\alpha \in E(G)$ as otherwise $\langle \{c_0, c_\alpha, c_2, x_0\} \rangle \cong K_{1,3}$ and $c_\alpha c_1 \in E(G)$ as otherwise $\langle \{x_1, c_\alpha, c_1, x_0\} \rangle \cong K_{1,3}$. Moreover, $c_\alpha x_3 \notin E(G)$ as otherwise $\langle \{c_\alpha, c_0, c_1, x_3\} \rangle \cong K_{1,3}$ and thus $t_\alpha = c_3$ and so $c_\alpha c_3 \in E(G)$. Hence, $c_1 c_3 \in E(G)$ as otherwise $\langle \{c_\alpha, c_0, c_1, c_3\} \rangle \cong K_{1,3}$, $c_2 c_3 \in E(G)$ as otherwise $\langle \{c_\alpha, c_2, c_3, x_1\} \rangle \cong K_{1,3}$ and $c_1 c_2 \in E(G)$ as otherwise $\langle \{c_3, c_2, c_1, x_4\} \rangle \cong K_{1,3}$. But then $\langle \{c_2, c_0, c_1, x_3\} \rangle \cong K_{1,3}$, which is a contradiction.

So $c_\alpha \in N_1$. Then $t_\alpha = c_2$ or t_α is a new vertex in A_0 .

Suppose $t_\alpha = c_2$. Then $c_\alpha c_2 \in E(G)$, $c_0 c_2 \notin E(G)$ and $c_2 \in N_2$. Then, $t_0 \neq c_2$ and t_0 is a new vertex in N_2 , and hence $|A_0| = 10$. The near-claw $NC(c_0 t_0, c_\delta, t_\delta)$ exists where $c_\delta = c_\alpha$ and so $c_\alpha t_0 \in E(G)$. But then, since $t_\delta \in N_2$, we have $\langle \{c_\alpha, x_0, t_0, t_\delta\} \rangle \cong K_{1,3}$ which is a contradiction.

So t_α is new vertex in A_0 with $c_\alpha t_\alpha \in E(G)$ and $c_0 t_\alpha, x_1 t_\alpha \notin E(G)$. Hence, $|A_0| = 10$. Then $t_0 = c_2$, and so $c_0 c_2 \in E(G)$ and $c_2 \in N_2$. The near-claw $NC(c_0 c_2, c_\epsilon, t_\epsilon)$ exists where $c_\epsilon = c_\alpha$, and so $c_\alpha c_2 \in E(G)$. Then $t_\alpha c_2 \in E(G)$ as otherwise $\langle \{c_\alpha, t_\alpha, x_1, c_2\} \rangle \cong K_{1,3}$. So $t_\epsilon = c_1$, and thus $c_\alpha c_1 \in E(G)$ and $c_1 c_2 \notin E(G)$. But then $\langle \{c_\alpha, x_0, c_1, c_2\} \rangle \cong K_{1,3}$ which is a contradiction.

So $|A_0| \geq 11$.

Claim 20 *If $c_{d-4} \in N_{d-3}$, then $|A_{d-4}| \geq 11$.*

This claim follows similarly.

Proof (of Theorem 6) Consider the vertex c_3 . By Claim 12, $c_3 \in N_3 \cup N_4$.

Suppose $c_3 \in N_3$. By Claim 19, $|A_0| \geq 11$. By Claim 12, $c_{d-4} \in N_{d-4} \cup N_{d-3}$, and thus by Claims 18 and 20, $|A_{d-4}| \geq 10$. By Claim 16, $|A_4| + |A_8| + \dots + |A_{t-4}| + |A_t| \geq \frac{9}{4}t - 1$. Hence

$$\begin{aligned} n &\geq |A_0| + |A_4| + |A_8| + \dots + |A_{t-4}| + |A_t| + |A_{d-4}| \\ &\geq 11 + 9 \lfloor \frac{d-7}{4} \rfloor + 10 \end{aligned}$$

$$> 20 + 9\lfloor \frac{d-7}{4} \rfloor.$$

So the result holds if $c_3 \in N_3$.

Now suppose $c_3 \in N_4$, and suppose to the contrary that the result does not hold. So

$$n < 20 + 9\lfloor \frac{d-7}{4} \rfloor,$$

and by Claims 16, 17 and 18,

$$n = 19 + 9\lfloor \frac{d-7}{4} \rfloor.$$

Thus $|A_0| = |A_{d-4}| = 10$ and $|A_4| + |A_8| + \dots + |A_{t-4}| + |A_t| = \frac{9}{4}t - 1$. So for some $j \in \{4, 8, \dots, t-4, t\}$, $|A_i| = 8$ but for every $k \in \{4, \dots, j-4\} \cup \{j+4, \dots, t\}$, $|A_k| = 9$. We shall now derive a contradiction.

Consider A_j . By Claim 14, $c_{j-1} \in N_{j-1}$. Since $|A_{j-4}| = 9$ and by Claim 15, $c_{j-5} \in N_{j-5}$. Similarly, since $|A_{j-8}| = 9$, $c_{j-9} \in N_{j-9}$. Continuing on we see that we must then have $c_3 \in N_3$, which is a contradiction.

So the result holds for $c_3 \in N_4$.

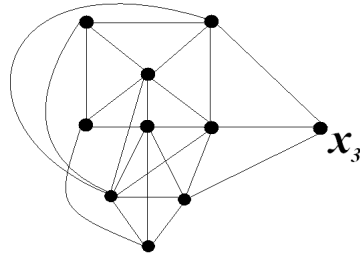


Figure 4.10: An induced subgraph.

To show sharpness, we consider the following family of graphs consisting of a copy of the graph in Figure 4.10, t copies of the graph in Figure 4.11 and a copy of the graph in Figure 4.12, where $1 \leq i \leq t$ and vertices to be identified are labelled on the figures.

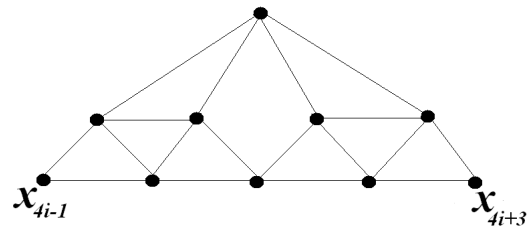


Figure 4.11: An induced subgraph.

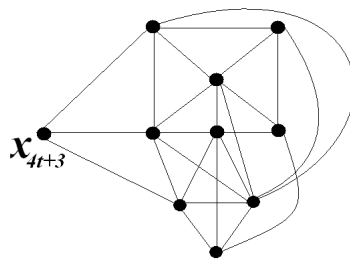


Figure 4.12: An induced subgraph.

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