Complete Symmetry Groups: A Connection Between Some Ordinary Differential Equations and Partial Differential Equations

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by

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Thesis submitted in fulfilment of the academic requirements for the Degree of Doctor of Philosophy in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban, South Africa.

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Abstract

The concept of complete symmetry groups has been known for some time in applications to ordinary differential equations. In this Thesis we apply this concept to partial differential equations. For any $1+1$ linear evolution equation of Lie's type (Lie S (1881) Über die Integration durch bestimmte Integrale von einer Klasse linear partieller Differentialgleichung *Archiv für Mathematik og Naturvidenskab* **6** 328-368 (translation into English by Ibragimov NH in *CRC Handbook of Lie Group Analysis of Differential Equations* **2** 473-508) containing three and five exceptional point symmetries and a nonlinear equation admitting a finite number of Lie point symmetries, the representation of the complete symmetry group has been found to be a six-dimensional algebra isomorphic to $sl(2,R) \oplus_s A_{3,1}$, where the second subalgebra is commonly known as the Heisenberg-Weyl algebra. More generally the number of symmetries required to specify any partial differential equations has been found to equal the number of independent variables of a general function on which symmetries are to be acted.

In the absence of a sufficient number of point symmetries which are not solution symmetries one must look to generalized or nonlocal symmetries to remove the deficiency. This is true whether the evolution equation be linear or not. We report *Ansätze* which provide a route to the determination of the required nonlocal symmetry or symmetries necessary to supplement the point symmetries for the complete specification of the equations.

Furthermore we examine the connection of ordinary differential equations to partial differential equations through a common realisation of complete symmetry group. Lastly we revisit the notion of complete symmetry groups and further extend it so that it refers to those groups that uniquely specify classes of equations or systems. This is based on some recent developments pertaining to the properties and the behaviour of such groups in differential equations under the current definition, particularly their *representations* and *realisations* for Lie remarkable equations. The results seem to be quite astonishing.

Keywords

Lie Symmetries, Complete Symmetry Group, Implicit Complete Symmetry Group, Lie remarkable equations, Representations and Realisations.

Declaration

I, Senzosenkosi Mandlakayise Myeni, affirm that the material contained in Sections §3.3 and §3.4, Chapters 4, 5, 6 and 7 of this Thesis is original and has not (to my knowledge) been published elsewhere except where due reference has been made in the text. The rest of the Thesis is a synthesis of known results. I further avow that this Thesis is not being and has not been used for the award of any other degree or diploma in any University or institution.

S M Myeni
April, 2008

Summary

This work is divided into three parts, each with its own objective. Moreover each of these parts has its own individual preface to give the reader a good indication of what to expect in the subsequent Chapters comprising the respective parts.

Part I

This part is mainly concerned with the introduction of the necessary concepts which are of use in the subsequent parts of the Thesis. We begin with some historical foundations of symmetry in nature and the evolution of this beautiful notion of symmetry to infinitesimal transformation. We further introduce the concept of complete symmetry groups to differential equations. Lastly we review some of the recent results and applications of this notion to partial differential equations.

Chapter 1 gives a gentle introduction to the presentation of the Thesis. Here the author presents his views on Mathematics and its writing.

In **Chapter 2** we discuss the concept of transformation groups particularly in relation to symmetry. We confine ourself to point symmetries of differential equations and provide some already known results concerning transformation groups.

Chapter 3 introduces the concept of Complete Symmetry Group as was introduced by Krause in 1994 and give a simple illustration of how to find a Complete Symmetry Group for a second-order ordinary differential equation and extend the considerations to partial differential equations. We further present some of the recent developments in the evolution of the concept of complete symmetry groups of differential equations.

Part II

This part of the Thesis is mainly concerned with establishing the connection between ordinary differential equations and partial differential equations through a common realisation of their complete symmetry group. Various structures of equations are investigated and assorted partial differential equations are constructed from symmetries of different forms of ordinary differential equations.

In **Chapter 4** we construct a steady-state partial differential equation using the idea of complete symmetry groups from a second-order ordinary differential equation for free particle, $y'' = 0$. The nonlinear equation immediately realised is equivalent (in symmetry structure) to a system of a second-order ordinary differential equation and a first-order ordinary differential equation.

In **Chapter 5** we construct steady-state partial differential equations using the idea of complete symmetry groups from higher-order ordinary differential equations $y^{(n)} = 0, \ n \geq 3$. The nonlinear equations immediately realised are hyperbolic partial differential equations of Bateman type.

In **Chapter 6** we construct a steady-state partial differential equation using the idea of complete symmetry groups from a system of ordinary differential equations

$$\begin{aligned} y_i'' &= 0, \qquad i = 1, n-2, \\ u' &= 0. \end{aligned}$$

The equations immediately realised are the nonlinear zero determinant matrix partial differential equations of Bateman type.

Part III

This part serves as a Conclusion to the Thesis. Here we review and extend the definition complete symmetry group to mean exactly that which was originally intended.

In **Chapter 7** we revisit this notion and further extend it so that it refers to those groups that uniquely specify classes of equations or systems, based on some recent developments pertaining to the properties and the behaviour of such groups in differential equations, particularly their representations and realisations.

Preface

The inspiration for the work carried out in this Thesis, similarly in [46], came from an honours project of one of Peter Leach's students, Kostis Andrioupoulos. Having looked at the complete symmetry groups of ordinary differential equations, he concluded his project by raising an interesting but intrinsic question, '*What about the Complete Symmetry Groups for partial differential equations?*'

Having taken up the challenge partially to elucidate the answer to this question during the study for a Masters degree, the audacity of my thinking has resulted in the work presented in this Thesis. The enthusiasm to carry out this work has been largely influenced by an imagination of remarkable vivacity. The man commonly referred to as the Father of Lie groups, a Norwegian Mathematician, Marius Sophus Lie in the last third of the nineteenth century is quoted as having written '*It was the audacity of my thinking*'. Indeed it was his ability to think outside the conventional way of thinking that has uncovered the beauty and the far-reaching usefulness of Lie groups in symmetry analysis and differential equations, particularly in observing the connection between all the so-called sophisticated techniques that were used to solve differential equations prior to Lie group analysis.

I have always maintained the belief that there was something special but yet uncovered about complete symmetry groups. As I was thinking (out-of-the-box) deeply about complete symmetry groups, their properties started to exhibit some interesting characteristics. In particular the connection of ordinary differential equations to partial differential equations through the common realisation of their complete symmetry groups would be very difficult or even impossible in the absence of complete symmetry groups. The series of these surprising facts are contained in Chapters 3, 4, 5, 6 and 7 of this Thesis. Major parts of this Thesis have been accepted for publication in the cited journals and some are in preprint form. I still maintain that we have yet to experience the far-reaching beauty and ability of complete symmetry groups in explaining differential equations.

Contents

List of Figures

Acknowledgements

Firstly and foremostly I would like to express my sincere gratitude to the Almighty God (The Father, The Son Jesus Christ and The Holy Spirit) for the wisdom, perfect guidance and the ability to undertake this intrinsic task.

Professor PGL Leach, who in many ways is my academic father and in this sense I have many siblings. His kind of mathematics and his unwavering character have inspired me. Professor JG O'Hara for providing the much needed moral support thus making sure that the product of this journey and final version of this Thesis is so much better than the first draft.

National Research Foundation and University of KwaZulu-Natal for their continued financial support.

Lastly my lovely wife, Slie Myeni, my family, friends and my local church: The KCC family, for keeping me sane through these trying times. May God Bless you all!!

Dedication

This work is dedicated to the following important people in my life,

- my parents Muntuwokuhamba J Myeni and Lephina T Myeni, my lovely wife, Slindile Prisca Myeni,
- the KCC family and the Body of Christ and
- the African Lion! This one is for you!

Part I

Laying The Firm Foundation

Preface

This part is mainly concerned with the introduction of the necessary concepts which are of use in the subsequent parts of the Thesis. We begin with some historical foundations of symmetry in nature and evolution of this beautiful notion of symmetry to infinitesimal transformation. We discuss the concept of complete symmetry groups to differential equations and review some recent results and further applications of this notion in partial differential equations.

Chapter 1

Introduction

The beauty of writing a Thesis is that one is presented with a distinctively infrequent opportunity to articulate his opinions on certain issues. In this Chapter the author wishes to present his views on his understanding of Mathematics as a fundamental field of study and the way Mathematics is written.

> *Mathematics is a discipline – a domain of knowledge, an intellectual heritage with ancient roots, with language and methods for analysis and understanding of aspects of the worlds that we inhabit and experience. Mathematics is not a profession or community. The pursuance of Mathematics is a profession and those who practise the profession constitute a community dedicated to the generation, application, conservation and transmission of knowledge and interacting with other domains and institutions of learning and with society at large.*

Indeed this is what Mathematics and Mathematicians are all about, whether the area of interest is symmetry analysis or dynamical systems or even number theory. It is all a *discipline* and a *profession*.

Furthermore I still maintain that we are yet to discover the beauty of symmetry analysis particularly in the study of partial differential equations and more specifically those equations coming from the Mathematics of Finance. These equations for example, and surprisingly, are derived from stochastic differential equations using some fundamental theorems, such as

Feynman-Kac and Fokker-Planck [17], borrowed from Statistical Physics! I further propose that we, the young generation, should do as Peter Leach, whom I so honour, has done to me and many others by introducing us to his astounding world of wonders called *symmetry*, carry forward the tradition, improve where there is a need and make this conundrum called Mathematics accessible and known to all....

Chapter 2

Symmetry

2.1 Symmetry: Exploring the Foundations

Symmetry has long been exploited for its beauty and of course its science. For example the ancient Shang in China made use of symmetry in their bronze vessels, Spanish mosques exhibit much symmetry and, moreover, patterns used in Islamic culture fully reflect the complete symmetry of reflections and rotations on a two-dimensional plane! There are seventeen such *wall-paper* symmetry groups[1] [37].

By unravelling the historical exploration of symmetry from the Greeks to the Modern age, Yang [25], a Nobel prize winner in Physics (1957), showed how the concept of symmetry has contributed to the concept of invariance.

> *A symmetry can be defined as the invariance in the pattern that is observed when some transformation is applied to it.*

Generally we associate symmetry with geometrical forms, like the vase for example, and rightfully so. Though the concept of symmetry had its origin in geometry, it is possible to extend the concept of invariance with respect to transformations of other kinds an example of which is that the electromagnetic force remains unchanged if the positive and negative forces are interchanged. It is surprising to note that Albert Einstein formulated his Special Theory of Relativity from the symmetry of space and time!

[1]Wall-paper groups are classes of discrete groups of isometries with two independent translations.

Furthermore the laws of electromagnetism upon which the special theory of relativity rests are governed by Lorentz symmetry[2].

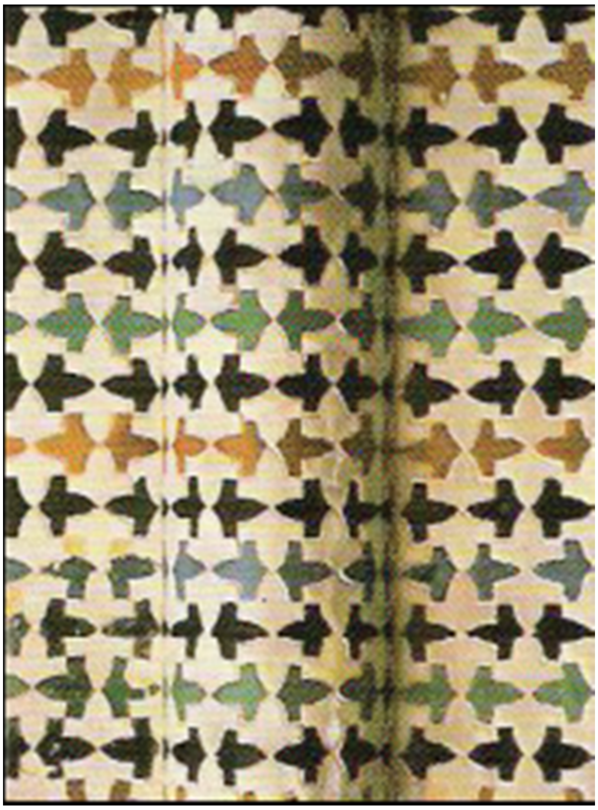

Figure 2.1: Paintings in Alhambra

The secret of nature is symmetry, but most observations in nature do not exhibit symmetry[3]. A profound way to hide symmetry is the phenomenon of spontaneous symmetry-breaking. There are two types of symmetries: Finite and infinitesimal. Finite symmetries can be discrete or continuous. Parity and time reversal are discrete symmetries[4] of Nature while space on the other hand is a continuous transformation. Mathematicians have been forever fascinated by patterns. Classifications of planar patterns and spatial patterns began seriously in the nineteenth century. Since then several classification methods have been developed and in particular one of the ways

[2]In fact Einstein resisted the idea of Lorentz symmetry, but subsequently embraced it and even generalised it into the Theory of General Relativity.

[3]Thus making symmetry a God-given phenomenon.

[4]A specific pattern is repeated at finite intervals in different directions; the pattern is the same and hence invariant under finite transformations.

to study patterns and lattices is to analyse the transformations that leave these patterns invariant through the concept of transformation groups (see the subsequent Sections). More precisely for one dimension there are seven such groups and for two dimensions all planar tessellations can be classified by the seventeen different plane symmetry groups [37].

As a matter of interest in discoveries made by mathematicians we take a brief tour down memory lane and look at the summary of interesting results there:

1. Mathematicians discovered the complete classification of the three-dimensional crystallographic or repeated patterns before they published the results for easier two-dimensional problem. The complete list of 230 three-dimensional crystallographic patterns was completed in 1890 by Fedorov *et al* [33]. It was only a year later that they worked out the details for the two-dimensional groups.

2. In 1944 Müller [52] studied patterns in the Islamic art. It happened that all seventeen types of two-dimensional repeated patterns were used creatively in Islamic culture. In fact she could only identify eleven of the seventeen wall-paper classes[5]. It was not until 1987 that mathematicians were able to document all seventeen types of in the incredibly beautiful artistry achieved by the builders in the Alhambra Palace in Spain [37] (see below).

3. When Hilbert [22] addressed the International Congress of Mathematicians in Paris in 1900, he proposed a number of outstanding problems upon which mathematicians could focus and possibly resolve. One of these was related to an understanding of higher-dimensional crystallographic groups. In 1910 Bieberbach [12] proved that there is only a finite number of such groups in any dimension.

[5]She also investigated other kinds of groups as well, in particular the eighty groups of the two-sided Euclidean plane. These are the topic of a short note by Müller [53], which includes also illustrations of mosaics from the Alhambra representing nine different wall-paper groups. Despite her mathematical beginnings Müller became a well-known astronomer and was for several years the General Secretary of the International Mathematical Union. In a letter of 1984 she mentioned that there is continuing interest in her Thesis and that there is a plan to republish it. Regrettably this seems not to have happened.

4. In 1948 Zassenhaus [63] provided an algorithm to determine the complete set of representatives for space groups in arbitrary dimensions. Today there are 4783 known symmetry groups in four dimensions. However, the issue of enumerating the exact number of symmetry groups in all dimensions remains an open and a challenging problem.

As an example taken from the Alhambra ornaments there is much of what could be called symmetry in the tiling in Figure 2.1. Given the shape of the tiles, they are arranged in the only possible way; it entails periodicity. Upon this the designer imposed coloration rules: Half the tiles are white; of the other half, half are black and the remainder are equally divided between green, blue and brown tiles. This is a way of looking that would have been understood by the Moorish artisans and may well have been their intention. One could say that we find there an example of colour symmetries (some horizontal mirrors preserve the white, black, and green tiles, while interchanging blue and brown ones, while other mirrors and glide axes lead to other permutations of colours), but this would have been totally extraneous to the thinking community of people over 500 years ago. Hence it is entirely irrelevant. There are many additional examples of similar assignments of colours as well as in the ornamentation of other cultures. For example Figure 2.2 shows an example in which one half of the tiles are white, one quarter black and the last quarter evenly divided between tan and green. A mathematical investigation of the possibilities would appear to be both interesting and achievable and possibly even of interest to anthropologists. On the other hand in many cases there is no such orderliness in the colors of the tiles; one has the feeling that the artists destroyed the symmetries to make the tiling less monotonous.

2.2 Differential Operators as Symmetries

When I was introduced to ordinary differential equations, they were presented with a wide variety of techniques designed to solve certain particular types of equations. These equations seemed unrelated in every respect, particularly types such as separable, homogeneous or exact equations. Of course this was state of the art around the middle of the nineteenth century until **Marius Sophus Lie**, son of Johann Herman Lie, a

Figure 2.2: Alhambra Ornaments

Lutheran minister, youngest of six brothers, having attended primary school in the city of Moss, a port in the Southeast of the Norway, in the eastern side of the Oslo Fjord spoiled the party when he made a profound and far-reaching discovery that these special methods were actually special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of transformations[6]. This elegant observation[7] immediately unified and extended significantly the available techniques of integration. These groups, now universally known as Lie groups, have had a profound impact on all areas of Mathematics, Physics, Engineering, Finance and other mathematically based sciences.

I was surprised to learn, some years later, that the original inspiration of these modern manifestations of Lie groups was in fact the field of differential equations! I also learnt that the full range of applicability of Lie

[6]A continuous transformation can be considered as a succession of very small transformations from the identity.

[7]Actually this was more than just an observation as Lie further showed how these methods were connected [31].

Figure 2.3: Marius Sophus Lie, son of Johann Herman Lie

group methods to differential equations is yet to be determined. Further it is impossible to overestimate the importance of Lie's contribution to modern Science and Mathematics. This Chapter is devoted to general concepts of Lie groups, in particular the results related to the original intention of Lie groups.

The treatment of differential equations usually requires one simultaneously to analyse a particular class of equations. For instance the class of second-order ordinary differential equations would be represented by

$$\frac{d^2y}{dx^2} = w(x, y, y'), \tag{2.2.1}$$

where $y' = \dfrac{dy}{dx}$, and a nonlinear diffusion equation by

$$u_t = [D(u)u_x]_x. \tag{2.2.2}$$

In both these cases w and D are arbitrary functions of their arguments, at least in some domain of interest. Hence the class would be characterised completely by allowing the arbitrary functions to take all possible functional forms.

In this Section we present the concept of *invertible point transformations* which act on the coordinate space of the independent and dependent variables, particularly the transformations named *symmetry* .

> *A* symmetry *of a differential equation is a transformation which maps every solution of the differential equation to a solution of the same equation.*

The concept of symmetry was informally understood centuries ago by considering an object; it would remain unceremoniously unaltered under some action of a particular transformation, *e.g.,* translation, dilation or reflection. Also the concept of a group, specifically the *symmetry group* of an object, is defined to be the set of all invertible transformations that leave the object invariant.

> *A* point symmetry *of a differential equation is an invertible point transformation which maps every solution of the differential equation to a solution of the same differential equation.*

Here the point transformation is the usual change of variables in differential equations which leaves the order unchanged[8]. In the nineteenth century the Norwegian mathematician **Marius Sophus Lie**, who had to give up his military career due to bad eyesight, discovered transformations, probably after having close contact with Darboux, Charles and Jordan in Paris. He further made a crucial discovery that a number of techniques of integration of ordinary differential equations, which had been perceived to be *ad hoc* and generally not related to each other, could be amalgamated by means of group theory. Hence Lie started looking at partial differential equations, hoping that a theory could be developed which was analogous to the Galois Theory of Equations. Galois Theory, the final product of many efforts by mathematicians to represent solutions of algebraic equations explicitly by radicals, deals with transformations of a finite group. Lie examined his contact transformations and considered how they affected a process due to Jacobi of generating further solutions of differential equations from an already known solution. This exercise led to combining the transformations

[8]The order is also preserved under contact transformation, but contact transformations are not germane to the present work.

in a way that is now called a Lie algebra. During the winter of 1873 Lie began to develop his theory of continuous transformation groups called Lie groups[9]. He showed that the symmetries of a differential equation form a group (the *admitted group* of the equation). The knowledge of this group was shown by Lie to have a tremendous amount of assistance in understanding and constructing solutions of the differential equation. The applications of symmetry groups to differential equations embrace, amongst others,

- mapping known solutions to (other) solutions,
- extensions to methods of integration of ordinary differential equations,
- construction of invariant solutions, i.e., solutions which are unaltered under the action of a subgroup of the admitted group and
- detection of a linearising transformation.

To perform any of these tasks it is imperative to find a reliable method for finding symmetries of differential equations. In principle one has to insert the arbitrary change of variables into the differential equation and force the new variables to satisfy the *same* differential equation. This usually produces a large number[10] of (usually nonlinear) differential equations, called the defining equations, to be satisfied by the transformation. However, this is too cumbersome to be of much use as it produces a large system of equations for which solution is out of the question on the grounds of tedium if not difficulty. For example the three scalar second-order ordinary differential equations of the Kepler problem have thirty-six independent defining equations which is the same number one obtains with infinitesimal transformations. A crucial observation of Lie was to consider the infinitesimal action of a group. Consider the rotations of the circle in figure 2.4 in the (x, y) plane

$$\begin{aligned}\bar{x} &= x\cos\theta - y\sin\theta \\ \bar{y} &= x\sin\theta + y\cos\theta\end{aligned} \tag{2.2.3}$$

[9]One should recall that Lie always wrote in terms of groups whereas these days one distinguishes between a group and an algebra.

[10]If one injects no constraints into the nature of the symmetry, the number of defining equations equals the number of equations of system under analysis. The specification of a particular type of symmetry leads to a greater number of determining equations.

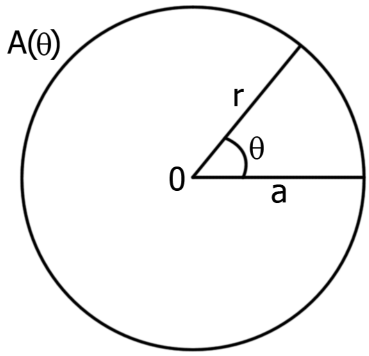

Figure 2.4: Rotations in the (x, y)-plane

which form a group the transformations of which are parameterized by the angle θ of rotation. When $\theta = 0$, we have the identity transformation. Since θ can vary continuously, these rotations are continuous transformations and the appearance of the circle is unchanged no matter what value of the angle through which it is rotated. The expansion in the neighborhood of identity $\theta = 0$ produces

$$\begin{aligned} \bar{x} &= x + \epsilon y + 0(\epsilon^2) \\ \bar{y} &= y - \epsilon x + 0(\epsilon^2). \end{aligned} \tag{2.2.4}$$

Lie regards (2.2.4) as an 'infinitesimally small' rotation of the plane. Lie's fundamental theorem is that the action of a group can be essentially *completely* recovered from the group's infinitesimal action and only involves the solution of an initial value problem for a finite system of ordinary differential equations. This moderates the task of solving a nonlinear system of differential equations to a system of linear differential equations, *i.e.,* the determining equations, for the infinitesimal group action. Once the infinitesimals are found, it suffices to solve the initial value problem of ordinary differential equations to recover the symmetry group. It cannot be overemphasised how imperative 'infinitesimalising' of symmetry calculation is: as Olver [57] page 47 states "almost the entire range of application of Lie groups to differential equations ultimately rests on this one construction". By way of contrast to the analysis of the nonlinear equations for the finite transforms[11] the calculation of the Lie point symmetries for the equations

[11]For a ghastly example of which see Krause & Aguirre [3, 4, 5]

which we consider in this Thesis is easily performed by one of the classic codes developed for this purpose.

2.3 Transformation Groups

In this Section we discuss the basic definitions and results on transformation groups, in particular those related to differential equations. These results are standard and we give no proofs or motivation. In the works of Bluman and Anco [13] and Olver [57] these results have been developed in great detail.

2.3.1 Transformations and Lie Groups

By *transformation* we mean a transformation of a space, $x = (x_1, x_2, \ldots x_n)$, which is a smooth (C^∞) mapping $\hat{x} = \tau(x)$ such that $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ is one-one and onto. The *inverse* transformation, τ^{-1}, of τ, if it exists, can always be defined.

Our main focus is on groups that are continuously parameterized by γ-real parameters, namely $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_\gamma)$.

Definition 2.1 (Lie Group)**.** Let γ real parameters, $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_\gamma)$, lie in a space P. The space P is a *γ-parameter Lie group* if there exists a binary operation, $*$, on P such that

- There is a unique *identity element,* $e \in P$, such that $\epsilon * e = e * \epsilon = \epsilon \quad \text{for all} \quad \epsilon \in P.$
- The operation $*$ is *associative*, i.e., $\epsilon * (\Delta * \gamma) = (\epsilon * \Delta) * \gamma$ for all ϵ, Δ and $\gamma \in P$.
- For every $\epsilon \in P$ there exists an *inverse* element, $\epsilon^{-1} \in P$, such that $\epsilon * \epsilon^{-1} = \epsilon^{-1} * \epsilon = e$. The inverse is unique for every $\epsilon \in P$.
- Both the binary operation $*$ and the map $\epsilon \mapsto \epsilon^{-1}$ are *analytic*.

In some cases the identity element, e, can be regarded as the origin, but this is not necessary. Also the binary operation, $*$, on a transformation group is always a composition.

We define a *subgroup* of P as a subset of P with the same law of operation, $*$.

Definition 2.2. A *Lie transformation group* on a space, $x = (x_1, x_2, \ldots, x_n)$, is a collection, G, of smooth transformations, τ of x, obtained as the homomorphic image of a Lie group of parameters. There exists a map, $\tau : P \to G$, such that

- $\tau(e)$ is the identity map of $x : \tau(e)(x) = x$ for all x.
- $\tau(\epsilon) \circ \tau(\Delta) = \tau(\epsilon * \Delta)$ for all $\epsilon, \Delta \in P$.
- $\tau(\epsilon^{-1}) = \tau(\epsilon)^{-1}$.
- The mapping, $x' = F(x, \epsilon) = \tau(\epsilon)(x)$, is smooth (C^∞) in x and ϵ.

We take, as a special case of a transformation group, a single real parameter ϵ to belong to additive group of real numbers and define:

Definition 2.3. A one-parameter (ϵ) Lie group acting on a space x is a transformation group on x with the properties that

- $\tau(0)$ is the identity transformation on x and
- $\tau(\epsilon) \circ \tau(\Delta) = \tau(\epsilon + \Delta)$.

Definition 2.4. A *linear group* A, the group consisting of all nonsingular complex $n \times n$ matrices, is called the complex general linear group, $PL(n, \mathcal{C})$, and the general linear group, $PL(n, \mathcal{R})$, comprises all nonsingular real $n \times n$ matrices. The general linear group, $PL(n, \mathcal{R})$, is a subgroup of $PL(n, \mathcal{C})$. The complex special linear group, $SL(n, \mathcal{C})$, consists of matrices with determinant one. The real special linear group[12], $SL(n, \mathcal{R})$, is the intersection of these two subgroups, *i.e.*,

$$SL(n, \mathcal{R}) = SL(n, \mathcal{C}) \cap PL(n, \mathcal{R}) \ .$$

Definition 2.5 (Rotation group)**.** The rotation group, $SO(n, \mathcal{R})$, is the special proper real orthogonal group given by the intersection of the group of orthogonal matrices, $O(n, \mathcal{R})$, and the complex special linear group, *i.e.*,

$$SO(n, \mathcal{R}) = O(n, \mathcal{R}) \cap SL(n, \mathcal{C}).$$

[12]Note that for the rest of the Thesis we use the notation $SL(N, \mathcal{R})$ and $sl(n, R)$ interchangeably to refer to same group.

2.3.2 Infinitesimal Operators

As was mentioned above, the key to successful construction of Lie transformation groups is infinitesimalisation, which replaces the nonlinear conditions for a group with linear ones. Consider a one parameter (ϵ) group of transformations acting on a space, $x = (x_1, x_2, \ldots, x_n)$. In a neighborhood of the identity $\epsilon = 0$ the transformation,

$$\bar{x} = F(x, \epsilon), \tag{2.3.5}$$

can be expanded as

$$\bar{x} = x + \epsilon\xi(x) + O(\epsilon^2), \tag{2.3.6}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ is given by[13]

$$\xi_i(x) = \frac{\partial F_i}{\partial \epsilon}(x, \epsilon)\bigg|_{\epsilon=0}. \tag{2.3.7}$$

The quantities ξ_i are called *infinitesimals* of the one-parameter group. Expansion (2.3.6) represents an 'infinitesimal transformation' from the group to (2.3.5).

Definition 2.6 (Group operator)**.** The group operator, $\mathbf{X}$, of a one-parameter group with infinitesimals, ξ_i, $i = 1, n$, is the first-order differential operator

$$\mathbf{X} = \xi_i(x)\partial x_i. \tag{2.3.8}$$

Note that an operator, $\mathbf{X}$, it is a coordinate-free object. It encodes the information on the rate of change of a function, f, with respect to the group parameter, ϵ, as a point, x, is dragged along by the one-parameter group associated with $\mathbf{X}$:

$$\frac{df}{d\epsilon}\left(x'(\epsilon)\right)\bigg|_{\epsilon=0} = \mathbf{X}f(x)\ .$$

[13] Here we use Einstein summation convention and also the notation ∂x_i and ∂_{x_i} are used interchangeably for the same purpose in the rest of the Thesis.

2.3.3 Extensions of group operators

Our primary goal is to develop results related or applicable to differential equations in arbitrary numbers of independent and dependent variables. So we adopt the following conventions. Let $x = (x_1, x_2, \ldots, x_n)$ be the n independent variables and $u = (u_1, u_2, \ldots, u_m)$ be the m dependent variables. We are interested in a transformation which acts on all such variables. The space (x, u) of independent and dependent variables is called the *base space* of the differential equation. Henceforth we further restrict ourselves to point transformations.

Theorem 2.1. *Let*

$$\mathbf{X} = \xi_i(x,u)\partial x_i + \eta_j(x,u)\partial u_j$$

be an operator for a transformation group, G, acting on basic space, (x,u). The kth extension of an operator, $\mathbf{X}$, is the operator

$$\mathbf{X}^{[k]} = \xi_i(x,u)\partial x_i + \eta_j(x,u)\partial u_j + \eta_j^{(i)}(x,u,u_{(1)})\partial u_j^{(i)}$$

$$+ \cdots + \eta_j^{(l)}(x,u,u_{(1)},\ldots,u_{(k)})\partial u_j^{(l)}, \qquad 0 \le l \le k,$$

where $\eta_j^{(l)}$, $1 \le l \le k$, are obtained from the recurrence relation

$$\eta_j^{(l)} = Dx_i \eta_j^{(l)} - u_j^{lq}(Dx_i \xi_q)$$

and Dx_i is the operator of total differentiation with respect to the variable x_i. Moreover $u_{(k)} = (u_x, u_y, u_z)$ for $u = u(x,y,z)$, $k = 1$ and is defined similarly for arbitrary u and k.

In subsequent Chapters, when we consider the connection of complete symmetry groups of ordinary differential equations to the complete symmetry groups of partial differential equations, we apply a special case of the above theorem to the situation in which there is one dependent variable u and three independent variables, say, x_1, x_2 and x_3, with an operator, $\mathbf{X}$, given by

$$\mathbf{X} = \xi_1(x_1,x_2,x_3)\partial x_1 + \xi_2(x_1,x_2,x_3)\partial x_2 + \xi_3(x_1,x_2,x_3)\partial x_3 + \eta(x_1,x_2,x_3,u)\partial u.$$

The relevant [13] $\eta_j^{(l)}$[14] are given by:

$$
\begin{aligned}
\eta_1^{(1)} &= \frac{\partial \eta}{\partial x_1} + \left[\frac{\partial \eta}{\partial u} - \frac{\partial \xi_1}{\partial x_1}\right] u_1 - \frac{\partial \xi_2}{\partial x_1} u_2 - \frac{\partial \xi_3}{\partial x_1} u_3 \\
\eta_2^{(1)} &= \frac{\partial \eta}{\partial x_2} + \left[\frac{\partial \eta}{\partial u} - \frac{\partial \xi_2}{\partial x_2}\right] u_2 - \frac{\partial \xi_1}{\partial x_2} u_1 - \frac{\partial \xi_3}{\partial x_2} u_3 \\
\eta_3^{(1)} &= \frac{\partial \eta}{\partial x_3} + \left[\frac{\partial \eta}{\partial u} - \frac{\partial \xi_3}{\partial x_3}\right] u_3 - \frac{\partial \xi_1}{\partial x_3} u_1 - \frac{\partial \xi_2}{\partial x_3} u_2 \\
\eta_{11}^{(2)} &= \frac{\partial^2 \eta}{\partial x_1^2} + \left[2\frac{\partial^2 \eta}{\partial x_1 \partial u} - \frac{\partial^2 \xi_1}{\partial x_1^2}\right] u_1 - \frac{\partial^2 \xi_2}{\partial x_1^2} u_2 - 2\frac{\partial \xi_3}{\partial x_1} u_{13} \\
&+ \left[\frac{\partial \eta}{\partial u} - 2\frac{\partial \xi_1}{\partial x_1}\right] u_{11} - \frac{\partial^2 \xi_3}{\partial x_1^2} u_3 - 2\frac{\partial \xi_2}{\partial x_1} u_{12}. \\
\eta_{22}^{(2)} &= \frac{\partial^2 \eta}{\partial x_2^2} + \left[2\frac{\partial^2 \eta}{\partial x_2 \partial u} - \frac{\partial^2 \xi_2}{\partial x_2^2}\right] u_2 - \frac{\partial^2 \xi_1}{\partial x_2^2} u_1 - 2\frac{\partial \xi_3}{\partial x_2} u_{23} \\
&+ \left[\frac{\partial \eta}{\partial u} - 2\frac{\partial \xi_2}{\partial x_2}\right] u_{22} - \frac{\partial^2 \xi_3}{\partial x_2^2} u_3 - 2\frac{\partial \xi_1}{\partial x_2} u_{12}. \\
\eta_{33}^{(2)} &= \frac{\partial^2 \eta}{\partial x_3^2} + \left[2\frac{\partial^2 \eta}{\partial x_3 \partial u} - \frac{\partial^2 \xi_3}{\partial x_3^2}\right] u_3 - \frac{\partial^2 \xi_1}{\partial x_3^2} u_1 - 2\frac{\partial \xi_2}{\partial x_3} u_{23} \\
&+ \left[\frac{\partial \eta}{\partial u} - 2\frac{\partial \xi_3}{\partial x_3}\right] u_{33} - \frac{\partial^2 \xi_2}{\partial x_3^2} u_2 - 2\frac{\partial \xi_1}{\partial x_3} u_{13}. \\
\eta_{13}^{(2)} &= \frac{\partial^2 \eta}{\partial x_1 \partial x_3} + \left[\frac{\partial^2 \eta}{\partial x_3 \partial u} - \frac{\partial^2 \xi_2}{\partial x_1 \partial x_3}\right] u_1 \\
&+ \left[\frac{\partial^2 \eta}{\partial x_1 \partial u} - \frac{\partial^2 \xi_3}{\partial x_1 \partial x_3}\right] u_3 + \left[\frac{\partial \eta}{\partial u} - \frac{\partial \xi_1}{\partial x_1} - \frac{\partial \xi_3}{\partial x_3}\right] u_{13} \\
&- \frac{\partial \xi_1}{\partial x_3} u_{11} - \frac{\partial \xi_3}{\partial x_1} u_{33} - \frac{\partial \xi_2}{\partial x_3} u_{12} - \frac{\partial \xi_2}{\partial x_1} u_{23} - \frac{\partial^2 \xi_2}{\partial x_1 \partial x_3} u_2. \\
\eta_{12}^{(2)} &= \frac{\partial^2 \eta}{\partial x_1 \partial x_2} + \left[\frac{\partial^2 \eta}{\partial x_2 \partial u} - \frac{\partial^2 \xi_2}{\partial x_1 \partial x_2}\right] u_1 + \left[\frac{\partial^2 \eta}{\partial x_1 \partial u} - \frac{\partial^2 \xi_1}{\partial x_1 \partial x_2}\right] u_2 \\
&+ \left[\frac{\partial \eta}{\partial u} - \frac{\partial \xi_1}{\partial x_1} - \frac{\partial \xi_2}{\partial x_2}\right] u_{12} - \frac{\partial \xi_2}{\partial x_1} u_{22} - \frac{\partial \xi_1}{\partial x_2} u_{11} - \frac{\partial \xi_3}{\partial x_2} u_{13} \\
&- \frac{\partial \xi_3}{\partial x_1} u_{23} - \frac{\partial^2 \xi_3}{\partial x_1 \partial x_2} u_3.
\end{aligned}
$$

[14]Upon taking into consideration that ξ_i, $i = 1, 3$, are not dependent upon u and the fact that η is linear in u for the operators considered in subsequent Chapters.

$$\begin{aligned}\eta_{23}^{(2)} &= \frac{\partial^2\eta}{\partial x_2\partial x_3}+\left[\frac{\partial^2\eta}{\partial x_3\partial u}-\frac{\partial^2\xi_2}{\partial x_2\partial x_3}\right]u_2+\left[\frac{\partial^2\eta}{\partial x_2\partial u}-\frac{\partial^2\xi_3}{\partial x_2\partial x_3}\right]u_3\\ &+ \left[\frac{\partial\eta}{\partial u}-\frac{\partial\xi_2}{\partial x_2}-\frac{\partial\xi_3}{\partial x_3}\right]u_{23}-\frac{\partial\xi_2}{\partial x_3}u_{22}-\frac{\partial\xi_3}{\partial x_2}u_{33}-\frac{\partial\xi_1}{\partial x_3}u_{12}\\ &- \frac{\partial\xi_1}{\partial x_2}u_{13}-\frac{\partial^2\xi_1}{\partial x_2\partial x_3}u_1.\end{aligned}$$

2.3.4 Lie algebra of operators

A γ-parameter Lie transformation group has associated γ group operators, $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_\gamma$, which are linearly independent and form a γ-dimensional vector space over $\mathbb{R}$ which has the additional structure of closure under operation of taking a Lie Bracket. Let $\mathbf{X} = \xi_i\,\partial x_i$ and $\mathbf{Y} = r_i \partial x_i$ be two group operators. Their Lie Bracket $[\mathbf{X},\mathbf{Y}]_{LB}$ is the first-order operator

$$[\mathbf{X},\mathbf{Y}] = \mathbf{XY}-\mathbf{YX} = \left(\xi_i\,\frac{\partial r_i}{\partial x_i}-r_j\,\frac{\partial \xi_j}{\partial x_j}\right)\frac{\partial}{\partial x_i}.$$

The Lie Bracket has the following properties

- Bilinearity. $[\mathbf{X}, a\mathbf{Y}+b\mathbf{Z}] = a[\mathbf{X},\mathbf{Y}]+b[\mathbf{X},\mathbf{Z}]$, where a and b are constants.
- Skew-symmetry. $[\mathbf{X},\mathbf{Y}] = -[\mathbf{Y},\mathbf{X}]$.
- Jacobi identity.

$$[\mathbf{X},[\mathbf{Y},\mathbf{Z}]]+[\mathbf{Y},[\mathbf{Z},\mathbf{X}]]+[\mathbf{Z},[\mathbf{X},\mathbf{Y}]] = 0.$$

Any vector space of operators satisfying the above three properties is called a Lie algebra of operators. A Lie algebra of operators contains all the information necessary to reconstruct a Lie group.

Theorem 2.2. *To every γ-parameter Lie transformation group, $G,$ there exists a corresponding γ-dimensional Lie algebra of operators, $L.$ A γ-dimensional vector space, $L,$ of operators derives from a Lie transformation group if and only if L is closed under the Lie Bracket, i.e.,*

$$[\mathbf{X},Y] \in L \;\; \textit{for all} \;\; \mathbf{X},\mathbf{Y} \in L.$$

A finite-dimensional Lie algebra can be resolved with respect to a basis, $\mathbf{X}_i$, in which case the closure condition becomes

$$[\mathbf{X}_i, \mathbf{X}_j] = C^k_{ij}\mathbf{X}_k$$

for some structure constants, C^k_{ij}, of L.

2.3.5 Direct and semidirect sum

A Lie algebra, L, is the *semidirect sum* of two subalgebras, I and J, and is denoted by $L = I \oplus_s J$ if L is a vector space which is the direct sum of I and J, and I is an ideal in $L : [I, J] \subseteq I$.

$[\ ,\ ]$	X_i	Y_j
X_i	$\{X_i\}$	$\{X_i\}$
Y_j	$\{X_i\}$	$\{Y_j\}$

A semidirect sum becomes a direct sum of ideals when the commutator table takes the form

$[\ ,\ ]$	X_i	Y_j
X_i	$\{X_i\}$	
Y_j		$\{Y_j\}$

i.e., the two ideals I and J, do not interact at all!
Note that a subalgebra, I, of L is an *ideal* if $[\mathbf{X}, \mathbf{Y}] \in I$ for all $\mathbf{X} \in I$ and all $\mathbf{Y} \in L$.

2.4 Differential Equations and Symmetry

The primary concern in this part of the Thesis is to realise the significance of symmetry in differential equations. In this Section we state some important and main results concerning symmetries of differential equations to enable this work to be as self-contained as possible. The results are in generalised form with essential special cases mentioned that are of use in subsequent chapters.

Point symmetry

> *A transformation acting on the base space (x, u) of a system E of differential equations is a* point symmetry *of the system E of differential equations if it maps every solution u of system E to a(nother) solution u' of system E.*

We write a system E of S kth-order differential equations in n independent variables and m dependent variables defined by a function $F = (f_1, f_2, \ldots, f_s)$ on the kth-extended space $(x, u, u_{(1)}, \ldots, u_{(k)})$ as

$$F(x, u, u_{(1)}, \ldots, u_{(k)}) = 0. \tag{2.4.9}$$

This is a system of algebraic equations with particular coordinates interpreted as coordinates of derivative spaces. The equation, $F = 0$, specifies a hypersurface embedded in the space $(x, u, u_{(1)}, \ldots, u_{(k)})$. In particular, when there is one dependent variable and three independent variables, say

$$u = u(x, y, z), \tag{2.4.10}$$

the spaces[15] involved become

$$\begin{aligned} x &= (x, y, z) \\ u &= (u) \\ u_{(1)} &= (u_x, u_y, u_z) \\ u_{(2)} &= (u_{xx}, u_{xy}, u_{yy}, u_{zz}, u_{xz}, u_{yz}). \end{aligned}$$

As a result the base space is (x, y, z, u) and the twice-extended space is

$$(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{xz}, u_{yz}).$$

By a solution of a differential equation we mean a function, $u = u(x)$, such that, when the function is substituted into (2.4.9), it satisfies the relation (2.4.9) for all x in some neighborhood of (x_0). The term mapping means mapping the graph of u. The symmetry according to the definition acts on functions which represent solutions of the differential equation. To know whether a point transformation is actually a symmetry, we must know all solutions of a differential equation. Of course this is clearly practically

[15] Here $(x, u, u_{(1)})$-space is of dimension $2n + 1$, $(x, u, u_{(1)}, u_{(2)})$-space is of dimension $\frac{1}{2}(n^2 + 5n + 2)$.

unviable. So the function criterion of the definition must be replaced with a 'point-by-point' criterion. The following theorem provides the basis for the practical calculation of symmetries. The point-by-point criterion allows us to treat the differential equations as algebraic equations in the extended space.

Theorem 2.3. *Let τ be a transformation of the base space (x, u) the extension of which, $\tau_1\tau_1^{[1]}, \ldots, \tau^{[k]}$, leaves the surface E invariant. Then τ is a symmetry of E.*

Proof:

> *Every solution, $u = u(x)$, of E has a graph lying in the surface E. The extension, $\tau^{[k]}$, of τ maps the extended graphs to extended graphs. So there is a function, $u' = u'(x')$, such that $\tau^{[k]}$ maps $\Gamma(u^{[k]})$ to the graph $\Gamma({u'}^{[k]})$ which lies in E. Hence u' is a solution of the differential equation and therefore τ is a symmetry of E.*

Jacobian condition

> *A system E of S differential equations, $F = 0$, satisfies the Jacobian condition if the Jacobian of F with respect to the variables $(u, u_{(1)}, u_{(2)}, \ldots, u_{(k)})$ is of full rank, S, at all points on E.*

Note that we do not allow the independent variable x to be bound by an algebraic relation since this leads to the system E having no solutions and thus constitutes an inconsistency. Further the Jacobian condition guarantees that a system of differential equations can be written in *solved form* in terms of one or S of the derivatives $(u, u_{(1)}, \ldots, u_{(k)})$.

Theorem 2.4. *Let E be a system of differential equations, $F = 0$, satisfying the Jacobian condition. Suppose G is a Lie transformation group such that*

$$X^{[k]}F = 0 \;\; \textit{whenever} \;\; F(x, u, u_{(1)}, \ldots, u_{(k)}) = 0 \qquad \textbf{(2.4.11)}$$

for every group operator X of G. Then G consists of symmetries of E.

The theorem above gives a constructive method for finding point symmetries of a differential equation. However, to ensure that all point symmetries are found, the system of differential equations, E, must satisfy some additional hypotheses, such as local solvability. Since there are two surfaces of interest in the space $(x, u, u_{(1)}, \dots, u_{(k)})$, *videlicet* the surface E specifying the differential equations and the surface generated by a union of all graphs of solutions of the differential equations, only for locally solvable systems can these two surfaces be defined. If this be not the case, there are portions of E through which there are no solutions. By *local solvability* we mean that for every point $(x, u, u_{(1)}, \dots, u_{(k)})$ there passes a graph of solution $u = u(x)$.

Theorem 2.5. *A locally solvable system of E of differential equations, $F = 0$, admits a symmetry τ if and only if $(\tau, \tau^{[1]}, \dots, \tau^{[k]})$ leaves invariant the surface E.*

The following theorem shows that the set of all operators X satisfying (2.4.11) generates the full[16] symmetry group of E. The problem arises when one tries to assert the local solvability criterion for which one would need to know all solutions of the differential equation. Hence we need a more convenient condition.

Regular system

A system of differential equations,

$$F(x, u, u_{(1)}, \dots, u_{(k)}) = 0,$$

is regular *if*

- *the function F is analytic in all its arguments,*
- *F satisfies the Jacobian condition, and*
- *no further relation of order k or less can be derived from E by differentiation or taking compatibility conditions.*

Theorem 2.6. *A regular system of differential equations is locally solvable.*

The results above lead to an algorithmic construction of the symmetry group, G, of a regular system of differential equations. They are also applicable to the case of one differential equation with two independent variables and one dependent variable which we consider in the subsequent parts of this Thesis.

[16]We use the term 'full' to avoid using the term 'complete' as this term is given another meaning in the subsequent Chapter.

Chapter 3

Complete Symmetry Groups: An innovative Approach to Specify Differential Equations

3.1 Introduction

During the late nineteen seventies, when more and more mathematicians started to realise that symmetry groups were actually intended for differential equations, the term 'Complete Symmetry Group' was used to describe a set of point symmetries of a differential equation (or system) attained by the Lie method rather than the use of the term 'Symmetry Group' which is known today and refers to a similar situation.

The complete symmetry group of a differential equation, equally a system of differential equations, was introduced by Krause [23, 24] to describe the group which in its algebraic representation completely specified the differential equation under consideration. The differential equation in this instance was that of the Kepler Problem for which Krause found it necessary to introduce nonlocal symmetries since the Lie point symmetries of the system of ordinary differential equations describing the Kepler Problem are insufficient to specify the system completely. Indeed the point symmetries corresponding to the angular momentum are not even included in the representation of the complete symmetry group. The reason for this is found in the concept of minimality of representation as an integral part of the definition of a complete symmetry group. Although nonlocal symmetries feature in a number of studies of the complete symmetry groups of certain problems [26, 27, 28, 55], this has been more an accident of the development of the study of complete symmetry groups and the theoretical treatment and applications in terms of point symmetries have been well established [6, 7, 8].

Krause's concept of a complete symmetry group of a differential equation (equally a system) was that the group associated with the set of symmetries, be they point, contact, generalised or nonlocal, required to specify the equation completely when acted upon the arbitrary function describing the general structure of the equation. Specifically he required, as was prescribed in his paper, that a realisation of a complete symmetry group must be endowed with the following properties

- *the group acts freely and transitively on the manifold of all permitted motions of the system (i.e., the manifold of solutions is a homogeneous space of the group) and*

- *the given equations of motion are the only ordinary differential equations that remain invariant under the specified action of the group, i.e., the group be specific to the differential equation or system and no other differential equation or system must admit it.*

This means that every system of differential equations can be entirely characterised by the symmetry laws it obeys, *i.e.*, different systems cannot have exactly the same representation of symmetry properties. If the expression be the same, the equation is the same. In this logic the group of symmetries of a given system is said to be *complete*.

3.2 Applications to Ordinary Differential Equations

The usual stratagem, in which one seeks a complete symmetry group of a given n^{th}-order ordinary differential equation, is that of starting at the general n^{th}-order equation and applying the pertinent extensions of the symmetries possessed by the original equation until one recovers the

n^{th}-order differential equation. It is, however, rather apparent that all of the above is only successful if the given equation possesses a sufficient number of symmetries[1] that it be completely specified by them. This implies that one can separate all problems into basic clusters. Firstly the *over-symmetric* cluster of problems have enough point symmetries to form a complete symmetry group. Secondly there is a situation where one is almost triumphant, *i.e.,* an *under-symmetric* cluster of problems for which there is an insufficient number of point symmetries to specify the equation completely and one has to search for nonlocal symmetries to remove the deficit[2] and, thirdly, a situation where absolutely *nothing* can be done due to the lack of explicit expressions for the symmetries.

To epitomise the procedure in the context of Lie point symmetries and in the simplest possible manner so that the idea is not full of twists and turns we consider the ordinary differential equation which is the emblematic representation of all linear and linearisable second-order equations:

$$y'' = 0 \tag{3.2.1}$$

which possesses the following Lie point symmetries

$$\begin{array}{rclrcl} G_1 &=& \partial_y & G_5 &=& x\partial_x \\ G_2 &=& x\partial_y & G_6 &=& x^2\partial_x + xy\partial_y \\ G_3 &=& y\partial_y & G_7 &=& y\partial_x \\ G_4 &=& \partial_x & G_8 &=& xy\partial_x + y^2\partial_y. \end{array}$$

The intent is to demonstrate how many symmetries listed above are required to specify (3.2.1) completely. Firstly we consider the general second-order ordinary differential equation

$$y'' = f(x, y, y'), \tag{3.2.2}$$

where f is as an yet unknown function to be specified.

[1]As all (systems of) ordinary differential equations possess an infinite number of symmetries, the notion of sufficiency in the number refers to the explicit representation of the symmetries. One commences with point symmetries as these are more convenient to compute

[2]Also treated in [43] for partial differential equations.

We apply the $SL(2,R)$ symmetries, *videlicet* G_4, G_5 and $G_6,$ to (3.2.2) to see if we can recover (3.2.1). The second extension of a general symmetry of a second-order ordinary equation is

$$G^{[2]} = \xi\partial_x + \eta\partial_y + (\eta' - y'\xi')\partial_{y'} + (\eta'' - 2y''\xi' - y'\xi'')\partial_{y''}$$

for a symmetry G given by

$$G = \xi\partial_x + \eta\partial_y.$$

The second extension of G_4 is

$$G_4^{[2]} = \partial_x + 0\partial_y + 0\partial_{y'} + 0\partial_{y''}.$$

The application of this to (3.2.2) yields

$$0 = \frac{\partial f}{\partial x}$$

from which we get

$$f = g(y, y') \tag{3.2.3}$$

so that

$$y'' = g(y, y').$$

The second extension of G_5 is given by

$$G_5^{[2]} = x\partial_x + \tfrac{1}{2}y\partial_y - \tfrac{1}{2}y'\partial_{y'} - \tfrac{3}{2}y''\partial_{y''}\ .$$

Its application to (3.2.3) gives

$$-\tfrac{3}{2}y'' = \tfrac{1}{2}y\frac{\partial g}{\partial y} - \tfrac{1}{2}y'\frac{\partial g}{\partial y'}.$$

The corresponding associated Lagrange's system is given by

$$\frac{dy}{y} = \frac{dy'}{-y'} = \frac{dg}{-3g}$$

from which we obtain the two characteristics

$$u = yy' \quad \text{and} \quad v = gy^3$$

so that

$$g = y^{-3}h(yy'), \tag{3.2.4}$$

i.e.,

$$y'' = y^{-3}h(yy').$$

Finally the second extension of G_6 is

$$G_6^{[2]} = x^2\partial_x + xy\,\partial_y + (y - xy')\partial_{y'} - 3xy''\partial_{y''}. \tag{3.2.5}$$

The action of $G_6^{[2]}$ on (3.2.4) yields

$$\begin{aligned} -3xy'' &= -3y^{-4}xy\,h + y^{-3}\frac{\partial h}{\partial u}[xyy' + y^2 - xyy'] \\ -3xy^{-3}h &= -3y^{-3}xh + y^{-1}\frac{\partial h}{\partial u} \\ 0 &= \frac{1}{y}\frac{\partial h}{\partial u}, \end{aligned} \tag{3.2.6}$$

but $y^{-1} \neq 0$ so that $\dfrac{\partial h}{\partial u} = 0$, *i.e.*, $h = k$, where k is a constant of integration, and so we have

$$y'' = \frac{k}{y^3} \tag{3.2.7}$$

which is an equation of Ermakov-Pinney class. To recover $y'' = 0$ we need another symmetry, say G_3, with its second extension given by

$$G_3^{[2]} = 0\partial_x + y\,\partial_y + y'\partial_{y'} + y''\,\partial_{y''}.$$

The action on (3.2.7), *videlicet*

$$y'' = -3ky^{-3}y,$$

forces $k = 0$. We have seen that G_3, G_4, G_5 and G_6 specify $y'' = 0$ completely. The four elements form the algebra

$$A_1 \oplus sl(2;R).$$

It was further discovered [6] that the sets of point symmetries G_3, G_7 and G_8, equally G_1, G_2 and G_3, were also sufficient to specify (3.2.1) completely. Further these groups had a common property that their point symmetries formed an $A_{3,3}$ algebra[3]. It was then concluded that not only is it possible to have multiple representations of the same group that uniquely specifies the

[3]In the description of the algebras we have used the Mubarakzyanov classification scheme [35, 38, 39, 40]. The algebra is the direct sum of $sl(2,R)$ and an abelian subalgebra.

ordinary differential equation but also that one can have different groups. It was then proposed that, when one talks about complete symmetry group of an ordinary differential equation, one should mean the group of *minimal dimension*, that is, the complete symmetry group of an ordinary differential equation is the group of the minimal number of symmetries required to specify the equation[4].

The definition of complete symmetry group was further extended [6] to first integrals of ordinary differential equations as the minimal number of symmetries required to specify the first integral up to an arbitrary function of itself. For example the action of one of the symmetries of

$$I_1 = xy' - y \tag{3.2.8}$$

on $f(x, y, y')$, say G_1, produces

$$x\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y'} = 0 \tag{3.2.9}$$

so that

$$f = f(y, xy')$$

upon taking a suitable choice of characteristics. The application of G_2 produces

$$f = f(xy' - y). \tag{3.2.10}$$

The action of any other symmetries renders $f \equiv 0$. This is to be expected that one can hope to achieve for complete symmetry groups of first integrals of ordinary differential equations since any integral is invariant under a subset of the Lie point symmetries of the ordinary differential equation. For the first integrals of ordinary differential equations it was concluded in the same paper [6], in relation to the complete symmetry symmetry group of first integrals, that:

[4]Note that the group, $A_1 \oplus sl(2; R)$, we have chosen is not of minimal dimension as it contains more symmetries than the other groups presented. Hence it cannot be a representation of complete symmetry group of (3.2.1) according to Krause's definition [23, 24] and the extended definition of complete symmetry group [6]. However, it suffices to illustrate the technique in general.

1. All n^{th}-order linear equations of maximal symmetry have a set of n linearly independent integrals linear in the dependent variable and its derivatives.

2. Two of the integrals admit $(n + 1)$ Lie point symmetries and others possess n Lie point symmetries. In all cases $(n-1)$ of the symmetries are solution symmetries and the other come(s) from the $sl(2, R)$ subalgebra.

3. The complete symmetry groups of the $(n - 2)$ integrals include all point symmetries of the differential equation.

4. For the two exceptional integrals, *i.e.* the integrals with $(n+1)$ Lie point symmetries, it is possible to specify uniquely the integral completely with just two symmetries, one of the solution symmetries and one of the $sl(2, R)$. However, the algebra closes only on the addition of the remaining solution symmetries.

5. In the case of a linear equation with less than maximal symmetry each of the set of n fundamental integrals of an equation with $(n+2)$ Lie point symmetries possesses n Lie point symmetries and is completely specified by those symmetries.

6. For a linear equation with only $(n + 1)$ Lie point symmetries each of fundamental integrals admits only $(n - 1)$ Lie point symmetries and the integral can only be specified completely by the determination of another symmetry, be it generalised or nonlocal.

The common feature and an accepted fact one encounters in the consideration of a complete symmetry group, from which seemingly one cannot escape, is the obfuscation of nonlocal symmetries. This type of symmetry has to be summoned for cases in which one has fewer than the required number of point symmetries admitted by an equation or a system to form a complete symmetry group. In particular Krause encountered this situation with the Kepler Problem, *videlicet*

$$\ddot{\mathbf{r}} + \mu\frac{\mathbf{r}}{r^3} = 0, \tag{3.4.16}$$

which has the five-dimensional algebra of Lie point symmetries

$$\begin{aligned} X_1 &= \partial_t & X_4 &= x_3\partial x_1 - x_1\partial x_3 \\ X_2 &= t\partial_t + \tfrac{2}{3}r\partial_r & X_5 &= x_1\partial x_2 - x_2\partial x_1 \\ X_3 &= x_2\partial x_3 - x_3\partial x_2, \end{aligned}$$

where x_1, x_2 and x_3 are the usual Cartesian components of the position vector, $\mathbf{r}$, of magnitude r. Unfortunately these five point symmetries are inadequate to specify (3.4.16) completely. Krause had resort to the introduction of the nonlocal symmetries,

$$\mathbf{Y} = 2\left(\int \mathbf{r}\, dt\right)\partial_t + \mathbf{r}r\,\partial_r,$$

for the Kepler Problem. The addition of these three symmetries was sufficient to specify the equations completely. Krause implied that the Complete Symmetry Group of the Kepler Problem comprised eight Lie symmetries, five point and three nonlocal. Subsequently Leach and Nucci [28] showed that the angular momentum operators were not needed so that the complete symmetry group of the two-dimensional Kepler problem was found to be five-dimensional. Leach proved that indeed the minimal number of symmetries required to specify the Kepler Problem completely is seven, by reducing the problem to a two-dimensional linear isotropic oscillator and a conservation law in terms of new variables related to the Ermanno-Bernoulli constants and the components of the angular motion vector [28].

Taking a glance at what type of behaviour of complete symmetry groups is observed for systems of ordinary differential equations, we note that [7] there are some interesting outcomes. Not only do we have different representations of the complete symmetry group of the higher-dimensional isotropic simple harmonic oscillator described by

$$\ddot{\mathbf{r}} + \mathbf{r} = 0 \tag{3.2.11}$$

but we also lose the minimality when we have to consider that the group must have closure. After the treatment of the two-dimensional oscillator in plane polar coordinates [7], it was observed that the set containing the minimal number of symmetries

$$\begin{aligned} \Phi_j &= x_j\partial x_j \\ \Phi_{2\pm} &= e^{\pm it}\sum_{j=1}^{n}\partial x_j \end{aligned} \tag{3.2.12}$$

completely specifies the set of equations

$$\ddot{x}_j + x_j = 0 \quad j = 1, n. \tag{3.2.13}$$

It was also observed that the operators in the above set do not form a closed algebra! To close the algebra one requires to add more symmetries. It was then decided that the complete symmetry group of (3.2.13) be given by

$$\begin{aligned} \Gamma_1 &= \sum_{j=1}^{n} x_j \partial x_j \\ \Gamma_{j\pm} &= e^{\pm it} \partial x_j \end{aligned} \tag{3.2.14}$$

and the symmetries concerned are the $2n$ solution symmetries and the diagonal homogeneity symmetry[5]. It was observed in general that $\frac{1}{2}(n^2 - n + 6)$ set of symmetries

$$\begin{aligned} \Psi_1 &= \partial t \\ \Psi_{j\pm} &= e^{\pm 2it} \left(\partial t \pm j \sum_{j=1}^{n} x_j \partial x_j \right) \\ \Psi_{kj} &= x_k \partial x_j - x_j \partial x_k \end{aligned} \tag{3.2.15}$$

with an algebra $sl(2, R) \oplus so(n)$ is sufficient to specify completely the n-dimensional isotropic oscillator.

Specifically for $n = 2, 3$ the complete symmetry group is given by (3.2.15). For $n \geq 4$ the complete symmetry group is given by (3.2.12). The most interesting case, as was observed in [7], is that both (3.2.14) and (3.2.15) have the same number of elements, *i.e.,* we observe that both groups have equal validity for the complete symmetry group!

All that being well and good for ordinary differential equations, a question still remained. What about the complete symmetry group of partial differential equations? Can we expect to find the similar properties to those observed for ordinary differential equations? This was left an open question until recently in a study of the complete symmetry group of the $1 + 1$ heat equation and some related equations which arise in Financial Mathematics

[5] The i in the exponential in equation (3.2.14) and following equations is the usual imaginary axis representation.

in which the authors [41] showed that the number of Lie point symmetries required to specify the $1+1$ heat equation is six. The classical heat equation, as a linear partial differential equation, possesses an infinite number of Lie point symmetries. The considerations have since been extended beyond just $1+1$ evolution equations [42, 43, 44, 46] and they now cover the complete symmetry group of steady-state partial differential equations. We present some of these astounding results in the next Section.

3.3 Applications to Partial Differential Equations

In this Section we turn our attention to the complete specification of a class of partial differential equations in terms of the Lie point symmetries possessed by the equations [41]. The equations we consider are $1+1$ evolution equations. The archetypal representative is the classical heat equation, but there are many others. To mention but a few examples there are the Burgess equation [15, 36] which describes the growth of a tumour in the brain, the Black-Scholes [10, 11] equation giving the value of an option or derivative under certain assumptions as to the nature of the market, many equations of Hamilton-Jacobi-Bellman type [54] which arise in the Mathematics of Finance and the Schrödinger equation which in a certain sense has motivated this work. In a mathematical sense the Schrödinger equation and the heat equation are the same in the field of complex variables. The nongeneric Lie point symmetries of the Schrödinger equation are closely related to the Noether symmetries of the corresponding Action Integral [30] and so there is a connection between the symmetries of evolution equations and those of ordinary differential equations derivable from a variational principle. This correspondence is not to be found with partial differential equations such as the wave equation or Laplace's equation.

A linear $1+1$ evolution equation can possess $\infty+1+n$ Lie point symmetries. The infinite set of symmetries is composed of the solutions of the linear equation. Then there is the homogeneity symmetry which in this case reflects the linearity of the equation. Both solution symmetries and homogeneity symmetry are generic to a linear evolution equation. The remaining n Lie point symmetries are nongeneric and their number depends upon the internal structure of the $1+1$ evolution equation[6]. The values which the number of nongeneric symmetries may take are confined to zero, one, three and five in the case of an $1+1$ evolution equation of standard form. These correspond to very specific algebras. Obviously the single symmetry can only be the one-dimensional abelian algebra, A_1. The three-dimensional algebra is $A_{3,8}$, commonly known as $sl(2,R)$ or $so(2,1)$. In the case that there are five nongeneric symmetries the algebra is $A_{3,8} \oplus_s A_{3,1}$, in which the homogeneity symmetry, although generic, is required to be added to the five to close the algebra. Although the solution of an equation plus its initial/boundary conditions is often not possible in terms of the Lie point symmetries of the differential equation due to incompatibility with the conditions, there is a class of equations to be found in the Mathematics of Finance for which this is not only possible but is also a very satisfying and elegant approach to the solution. For example the Black-Scholes equation possesses the maximal number of Lie point symmetries and its solution via Lie group theoretic methods has been presented by Gazisov and Ibragimov [21]. Evolution equations of the same genre but allowing for more complicated source terms, thereby reducing the number of nongeneric symmetries from five to three, are also solvable under the usual terminal condition associated with such models [29].

In this Section we consider the classical heat equation[7]. We should not be surprised that this equation is completely specified by its Lie point symmetries. The classical heat equation is the parabolic differential equation which corresponds to the ordinary differential equation of the free particle

[6]Naturally the discussion may be extended to linear evolution equations in more than one 'space' dimension and we make some points in the Conclusion. However, to maintain a clarity of presentation we confine our attention to $1+1$ evolution equations in the body of the text.

[7]The concept of complete symmetry group is introduced in the simplest setting by considering mostly linear or linearizable equations. We discuss these equations to introduce easily the idea of complete symmetry group to partial differential equations without the obfuscation of nonlinear equations. However, the same concept is applicable for those whose interest lies with nonlinear partial differential equations.

problem. The classical heat equation, *videlicet*

$$u_t = u_{xx}, \tag{3.3.16}$$

where

$$u_t = \frac{\partial u}{\partial t} \quad \text{and} \quad u_{xx} = \frac{\partial^2 u}{\partial x^2},$$

admits the Lie point symmetries

$$\begin{array}{lcl} \Gamma_1 & = & \partial_x \\ \Gamma_2 & = & t\partial_x - \frac{1}{2}xu\partial_u \\ \Gamma_3 & = & \partial_t \\ \Gamma_4 & = & t\partial_t + \frac{1}{2}x\,\partial_x - \frac{1}{4}u\,\partial_u \end{array} \qquad \begin{array}{lcl} \Gamma_5 & = & t^2\partial_t + tx\,\partial_x - \frac{1}{4}(x^2+2t)u\,\partial_u \\ \Gamma_6 & = & u\,\partial_u \\ \Gamma_7 & = & h(x,t)\partial_u, \end{array}$$

where $h(t,x)$ is a solution of the heat equation, (3.3.16). The corresponding Lie brackets are

$[,]$	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	$-\frac{1}{2}\Gamma_6$	0	Γ_1	Γ_2	0
Γ_2	$-\frac{1}{2}\Gamma_6$	0	$-\Gamma_1$	Γ_2	$-\frac{1}{2}\Gamma_2$	0
Γ_3	0	$-\Gamma_1$	0	Γ_3	$2\Gamma_4$	0
Γ_4	Γ_1	$-\frac{1}{2}\Gamma_2$	Γ_3	0	Γ_5	0
Γ_5	Γ_2	0	$2\Gamma_4$	Γ_5	0	0
Γ_6	0	0	0	0	0	0

The nongeneric Lie point symmetries $\Gamma_1 - \Gamma_5$ comprise two groups. The symmetries, Γ_3, Γ_4 and Γ_5, constitute the Lie algebra $sl(2,R)$ which is characteristic of ordinary differential equations of maximal symmetry and of Ermakov-Pinney systems. The two remaining symmetries, Γ_1 and Γ_2, correspond to the solution symmetries of the one-dimensional free particle. In general a scalar second-order ordinary differential equation derivable from a variational principle possesses at most five Noether point symmetries [32]. They are the counterparts of the five nongeneric Lie point symmetries of the heat equation. The connection is more easily seen through the corresponding time-dependent linear Schrödinger equation [30] to which the heat equation is related by a simple point transformation. The Lie algebra which characterises (3.3.16) comprises the five symmetries of (3.3.16) plus the homogeneity symmetry, $\Gamma_6 = u\partial_u$. The six Lie point symmetries split into two three-dimensional subalgebras. One is the algebra $sl(2,R)$ mentioned above.

The other is the three-dimensional Heisenberg-Weyl of the three symmetries Γ_1, Γ_2 and Γ_6. The six-dimensional algebra has the structure $sl(2,R)\oplus_s W$. In the more systematic notation of the Mubarakzyanov classification scheme this is written as $A_{3,8}\oplus_s A_{3,1}$. It so happens that these six symmetries are also a representation of the complete symmetry group of (3.3.16). We should emphasise that the number of Lie point symmetries of a given differential equation and the number of symmetries required to specify it completely have no particular relation (as the Ermakov-Pinney equation [27] illustrates clearly) in terms of the complete specification of a given equation.

Further note that the coefficient of ∂_u in Γ_6, which represents the homogeneity of the heat equation, is also a solution of the heat equation in that it is the dependent variable itself.

The aim is to use the symmetries above (or some of them as one sees below) to specify completely the heat equation. We proceed as follows.

Set

$$u_{xx} = f(x,t,u,u_x,u_t), \tag{3.3.17}$$

where f is an arbitrary function of its five independent variables. We apply the symmetries above to (3.3.17) in turn until we completely specify the arbitrary function.

The application of Γ_1 and Γ_3 in turn reduces (3.3.17) to

$$u_{xx} = f(u,u_x,u_t). \tag{3.3.18}$$

We note that Γ_2, Γ_4 and Γ_6 can be obtained from Γ_5 by taking the derivative with respect to a suitable variable (x, t and xx, respectively). Further Γ_2 and Γ_4 are just the products of the Lie Brackets $[\Gamma_1,\Gamma_5]_{LB}$ and $[\Gamma_3,\Gamma_5]_{LB}$, respectively, and it is known that the application of symmetries forming the Lie Bracket also includes in itself the application of the product of a Lie Bracket. So the application of Γ_2,Γ_4 and Γ_6 explicitly give no additional information. This suggests that we use Γ_5 to see how much information it yields. The relevant second extension of Γ_5 is

$$\begin{aligned}\Gamma_5^{[2]} &= xt\partial_x + t^2\partial_t - \tfrac{1}{4}(x^2+2t)u\,\partial_u - \left(\tfrac{1}{2}xu + \tfrac{1}{4}(x^2+2t)u_x + tu_x\right)\partial_{u_x}\\ &- \left(\tfrac{1}{2}u + \tfrac{1}{4}(x^2+2t)u_t + 2tu_t + xu_x\right)\partial_{u_t}\\ &- \left(\tfrac{1}{2}u + xu_x + \tfrac{1}{4}(x^2+2t)u_{xx} + 2tu_{xx}\right)\partial_{u_{xx}}\end{aligned}$$

and $\Gamma_5^{[2]}(u_{xx}-f)=0$ implies that

$$\begin{aligned}-\tfrac{1}{2}u - xu_x - \left[\tfrac{1}{4}(x^2+2t)+2t\right]u_{xx} = -\tfrac{1}{4}(x^2+2t)u\,\frac{\partial f}{\partial u}\\ +\left[-\frac{1}{2}xu - \left[\tfrac{1}{4}(x^2+2t)+t\right]u_x\right]\frac{\partial f}{\partial u_x}\\ +\left[-\frac{1}{2}u - \left[\tfrac{1}{4}(x^2+2t)+2t\right]u_t - xu_x\right]\frac{\partial f}{\partial u_t}.\end{aligned}$$

Since x and t are not in f, we can extract coefficients of x, t and not x or t separately. Taking coefficients as indicated on the left below we obtain

$$x^2 \ :\quad -\tfrac{1}{4}u_{xx} = -\tfrac{1}{4}u\frac{\partial f}{\partial u} - \tfrac{1}{4}u_x\frac{\partial f}{\partial u_x} - \tfrac{1}{4}u_t\frac{\partial f}{\partial u_t},$$

$$\textit{ie,}\quad f = u\frac{\partial f}{\partial u} + u_x\frac{\partial f}{\partial u_x} + u_t\frac{\partial f}{\partial u_t} \tag{3.3.19}$$

$$x \ :\quad -u_x = -\frac{1}{2}u\frac{\partial f}{\partial u_x} - u_x\frac{\partial f}{\partial u_t},$$

$$\textit{ie,}\quad u_x = \frac{1}{2}u\frac{\partial f}{\partial u_x} + u_x\frac{\partial f}{\partial u_t} \tag{3.3.20}$$

$$t \ :\quad -\frac{5}{2}u_{xx} = -\frac{1}{2}u\frac{\partial f}{\partial u} - \frac{3}{2}u_x\frac{\partial f}{\partial u_x} - \frac{5}{2}u_t\frac{\partial f}{\partial u_t},$$

$$\textit{ie,}\quad 5f = u\frac{\partial f}{\partial u} + 3u_x\frac{\partial f}{\partial u_x} + 5u_t\frac{\partial f}{\partial u_t} \tag{3.3.21}$$

$$- \ :\quad -\frac{1}{2}u = -\frac{1}{2}u\frac{\partial f}{\partial u_t},$$

$$\textit{ie,}\quad 1 = \frac{\partial f}{\partial u_t}. \tag{3.3.22}$$

The solution of the above system of equations is rather trivial and produces

$$f = u_t \tag{3.3.23}$$

whence

$$u_{xx} = u_t. \tag{3.3.24}$$

We have been able to specify (3.3.17) completely using just Γ_5. We have not needed to use Γ_2, Γ_4 and Γ_6 to specify (3.3.16) as suggested. The three 'unused' symmetries are still necessary for inclusion in the representation of the complete symmetry group of (3.3.16) due to the necessity to close the algebra under the operation of the Lie Bracket and thus compromising the minimum number by adding more symmetries to close the algebra[8]. The complete symmetry group of (3.3.16) is then[9] $\{\Gamma_i,\ i = 1, \ldots, 6\}$.

3.4 Under-symmetric Differential Equations and Integrals

The heat equation (3.3.16) is one of the *over-symmetric* equations for which there is no trouble to find a complete symmetry group. It transpires [43] that sometimes one can have an over-symmetric equation or system, but one could find that some of those point symmetries do not feature in the specification of the equation[10]. We illustrate this by considering the complete characterisation of a class of partial differential equations by its symmetry group involving nonlocal symmetries of the nonlinear heat equation of the form

$$u_t = F(u_x)u_{xx}. \tag{3.4.25}$$

In particular we consider the specific case mentioned in [2] for which F takes the form

$$F \;=\; u_x^n, \qquad\qquad n \neq 0, -2. \tag{3.4.26}$$

It is well understood from [2] that equation (3.4.25) has, *inter alia*, a realisation of the algebra $3A_1$, where

$$3A_1 = \langle \partial t, \partial u, \partial x \rangle, \tag{3.4.27}$$

[8]This is the same feature observed for systems of ordinary differential equations that one loses minimality when closure is to be taken into account[7]! Could this behaviour suggest a 'relationship' between systems of ordinary differential equations and partial differential equations?

[9]Note that this set is exclusive of the infinite number of solution symmetries as the application of these symmetries produces no useful information.

[10]In particular in [43] it has been shown that Λ_4, Λ_5 and Λ_7 are not needed in the specification of the equation. In fact their inclusion engenders another equation altogether!

for an arbitrary function, F. Also the maximal invariance algebra admitted by (3.4.25) according to Theorem 3.4 of [2] is the four-dimensional Lie algebra

$$\langle 2t\partial t + x\partial x + u\partial u\rangle \oplus_s 3A_1. \tag{3.4.28}$$

The class of equations is further reported to admit the following Lie point symmetry for different smooth functions F

$$F = u_x^n \quad : \quad \Gamma = nt\partial t - u\partial u \qquad n \neq 0, -2. \tag{3.4.29}$$

Furthermore it is accepted [42] that for the realisation of a complete symmetry group of a $1+1$ evolution equation one requires six symmetries, be they point, generalized or nonlocal. This therefore implies that one would require a nonlocal symmetry, possibly more than one, to specify (3.4.25) completely.

For the case $F = u_x^n$, $n = 0$, we obtain the classical heat equation, *videlicet*

$$u_t = u_{xx}. \tag{3.4.30}$$

The complete symmetry group of (3.4.30 has been treated in the previous Section and in greater detail in [42].

For $F = u_x^n$, $n = -2$, in (3.4.29) one immediately realises a linearisable equation[11],

$$u_t = u_x^{-2}u_{xx}, \tag{3.4.31}$$

which admits the following Lie point symmetries

$$\begin{aligned}
\Lambda_1 &= \partial t \\
\Lambda_2 &= \partial u \\
\Lambda_3 &= 2t\partial t + u\partial u \\
\Lambda_4 &= xu\partial x - 2t\partial u \\
\Lambda_5 &= t^2\partial t - \frac{1}{4}\left(2t + u^2\right)x\partial x + ut\partial u \\
\Lambda_6 &= x\partial x \\
\Lambda_7 &= F_5(u,t)\partial x,
\end{aligned} \tag{3.4.32}$$

[11] Linearisation is achieved by the hodograph transformation.

where F_5 a solution of the linear heat equation $x_t = x_{uu}$. We have a few important observations of equation (3.4.31) to note. Firstly this equation does not seem to admit point symmetries constituting the $3A_1$ algebra but rather the $A_{2,1}$ algebra given by Λ_1 and Λ_2. Secondly the point symmetries look very similar to those admitted by the heat equation (3.4.30) but with x in place of u. This reflects a particular behaviour in the geometry of the problem.

For $F = u_x^n$, $n \neq 0, -2$, in (3.4.29) one realises the not-so-simple equation,

$$u_t = u_x^n u_{xx}, \qquad n \neq 0, -2, \tag{3.4.33}$$

which is invariant under the action of the following Lie point symmetries

$$\begin{aligned} \Gamma_1 &= \partial t \\ \Gamma_2 &= \partial u \\ \Gamma_3 &= \partial x \\ \Gamma_4 &= (n+2)t\partial t + x\partial x \\ \Gamma_5 &= nt\partial t - u\partial u. \end{aligned} \tag{3.4.34}$$

We note here that equation (3.4.33) admits a finite number of Lie point symmetries, but this does not reflect any strange behaviour associated with the geometry as in the previous case. The homogeneity symmetry is lacking.

3.4.1 Complete Symmetry Group of $u_t = u_x^n u_{xx}$.

We now turn our attention to the construction of a complete symmetry group of the class of equations

$$u_t = u_x^n u_{xx}. \tag{3.4.35}$$

We construct the complete symmetry group of (3.4.35) for $n = -2$ and $n \neq -2$.

Specification of $u_t = u_x^n u_{xx}$ for $n = -2$

Consider the former case and start with a general second-order evolution equation, *videlicet*

$$u_{xx} = f(x, t, u, u_x, u_t), \tag{3.4.36}$$

where f is a function of its arguments to be specified. For equation (3.4.36) to be invariant under Λ_1 and Λ_2 it must be of the form

$$u_{xx} = f(x, u_x, u_t). \tag{3.4.37}$$

The second extension of Λ_6 is given by

$$\Lambda_6^{[2]} = x\partial x - u_x \partial u_x - 2u_{xx}\partial u_{xx}. \tag{3.4.38}$$

The application of (3.4.38) to (3.4.37) produces the following associated Lagrange's system upon taking (3.4.37) into account

$$\frac{df}{-2f} = \frac{dx}{x} = \frac{du_x}{-u_x} = \frac{du_t}{0}. \tag{3.4.39}$$

One may carefully select the characteristics so that (3.4.37) takes the form

$$u_{xx} = u_x^2 h(u_t, xu_x). \tag{3.4.40}$$

The application of the second extension of Λ_3, given by

$$\Lambda_3^{[3]} = u\partial u + 2t\partial t + u_x\partial u_x - u_t\partial u_t + u_{xx}\partial u_{xx}, \tag{3.4.41}$$

to (3.4.40) produces the associated Lagrange's system

$$\frac{dh}{-h} = \frac{dv}{v} = \frac{du_t}{-u_t} \tag{3.4.42}$$

upon letting $v = xu_x$. One's choice of characteristics may reduce (3.4.40) to

$$u_{xx} = u_t u_x^2 g(w), \tag{3.4.43}$$

where $w = xu_t u_x$. The application of the second extension of Λ_4, given by

$$\Lambda_4^{[2]} = xu\partial x - 2t\partial u - uu_x\partial u_x - 2u_t\partial u_t - 2uu_{xx}\partial u_{xx}, \tag{3.4.44}$$

to (3.4.43) produces, after some simplification,

$$g + wg' = 0, \tag{3.4.45}$$

upon the integration of which (3.4.43) becomes

$$u_{xx} = \frac{u_x}{x}\kappa, \tag{3.4.46}$$

where κ is a constant of integration. It clear that we are by no means obtaining the intended equation as we note that the term u_t has disappeared completely in equation (3.4.46) and there is no way to recover it from the application of Λ_5. This surely is not an desired outcome since we started with a general second-order *time-evolution* equation, applied point symmetries admitted by an *time-evolution* equation hoping to achieve an *time-evolution* equation. In fact what we obtain is politely expressed as absurd! Why?

The answer lies in the consideration of the geometry of the equation. The symmetries (3.4.32) of (3.4.31) reflect *rotation* and *reflection* in space. The reflection behaviour is amenable to continuous transformations only but not to continuous point transformations. This in turn translates into the fact that this equation is not connected to the heat equation via a continuous transformation but rather a discrete transformation[12], *videlicet*

$$t = -T, \quad x = U, \quad u = X, \qquad T \neq 0, U \neq 0, X \neq 0. \tag{3.4.47}$$

The transformation (3.4.47) clearly is a continuous but not a point transformation. One immediately observes a reflection symmetry and consequently (3.4.31) cannot be specified using point symmetries only and so we require nonlocal symmetries to remove the deficit and we do this in the next section[13].

Specification of $u_t = u_x^n u_{xx}$ for $n \neq -2$

As before we commence with a general second-order time-evolution equation

$$u_{xx} = f(x, t, u, u_x, u_t), \tag{3.4.48}$$

where f is a function to be specified. For the intended equation to be invariant under Γ_1, Γ_2 and Γ_3 of (3.4.34) it must be of the form

$$u_{xx} = f(u_x, u_t). \tag{3.4.49}$$

The second extension of Γ_5 is given by

$$\Gamma_5^{[2]} = nt\partial t - u\partial u - u_x\partial u_x - (n+1)u_t\partial u_t - u_{xx}\partial u_{xx} \tag{3.4.50}$$

[12]One need take heed when constructing similarity solutions of (3.4.25) with $F = u_x^n$, $n = -2$, using point symmetries to avoid disaster!

[13]This feature is in fact important to illustrate the fact that the maximal number of point symmetries possessed by a given equation is not adequate to imply its complete specification with point symmetries only as claimed in [41].

and its application to (3.4.49) produces

$$u_{xx} = u_x \frac{\partial f}{\partial u_x} + (n+1) u_t \frac{\partial f}{\partial u_t}. \tag{3.4.51}$$

With the associated Lagrange's system given by

$$\frac{df}{f} = \frac{du_x}{u_x} = \frac{du_t}{(n+1)u_t}$$

the characteristics may be chosen such that equation (3.4.49) takes the form

$$u_{xx} = u_x h\left(\frac{u_x^{n+1}}{u_t}\right). \tag{3.4.52}$$

The second extension of the remaining point symmetry is

$$(n+2)t\partial t + x\partial x - u_x \partial u_x - (n+2)u_t \partial u_t - 2u_{xx}. \tag{3.4.53}$$

Its application to (3.4.52) yields after some simplification

$$h = -wh', \tag{3.4.54}$$

where we have set $w = u_x^{n+1}/u_t$. The solution of this equation is

$$h = \frac{\iota}{w},$$

and consequently (3.4.52) takes the form

$$u_t = \iota u_{xx} u_x^n, \quad n \neq -2, \tag{3.4.55}$$

where ι is a constant to be specified. To specify the constant one requires a nonlocal symmetry as we have exhausted the available point symmetries of (3.4.34). Note that unlike the previous case equation (3.4.55) is in line with the equation we are trying to specify. The not-so-pleasant journey to search for nonlocal symmetries is inevitable in both these circumstances and for that reason we devote the entire 3.4.2 to this exercise[14].

[14] The constant can also be easily removed by rescaling, however the intention here is to completely specify the equation using symmetries, which warrants the search of a nonlocal symmetry as there are no point symmetries left.

3.4.2 Algorithmic Approach for Finding Nonlocal Symmetries

We now return to the question of finding nonlocal symmetries of (3.4.25). We begin with the case for which

$$F = u_x^n, \quad n = -2, \tag{3.4.56}$$

where we have the $\infty + 1 + 5$ Lie point symmetries (3.4.32), and later the case where $n \neq -2$. From the previous section it has become apparent that the two point symmetries Λ_4 and Λ_5 are not specifying equation (3.4.25). Hence one requires two more symmetries, which must be nonlocal, to remove the deficit so as to satisfy the requirements of the conjecture in [46]. To accomplish this we commence with the general second-order evolution equation

$$u_{xx} = f(x, t, u, u_x, u_t), \tag{3.4.57}$$

where f is a function yet to be determined. We proceed to apply the symmetries of the algebra $2A_1$, where

$$2A_1 \quad = \quad \langle \partial t, \partial u \rangle, \tag{3.4.58}$$

to equation (3.4.57) and consider both cases for n.

Nonlocal symmetry for $F = u_x^n,\ n = -2$

In this case we use the realisation of (3.4.58) to remove t and u from f to obtain

$$u_{xx} = f(x, u_x, u_t). \tag{3.4.59}$$

At this juncture we must specify the desired general structure of the remaining function and then work backwards to find the nonlocal symmetry which would specify that structure[15]. In this case we desire that the function f take the form

$$u_{xx} = u_t h(x, u_x) \tag{3.4.60}$$

[15]This is what makes the nonlocal symmetry immediately realised *generic* to a particular class of the equations having a similar underlying structure.

and we impose this structure and conduct a search for this miraculous nonlocal symmetry. This structure, (3.4.60), would have been obtained by setting $f = u_t h(x, u_x)$ which can only be specified by the choice of characteristics obtained from the associated Lagrange's system

$$\frac{df}{f} = \frac{du_t}{u_t} = \frac{dx}{0} = \frac{du_x}{0}. \tag{3.4.61}$$

The associated Lagrange's system above results from the application of the second extension of our as yet *unknown* nonlocal symmetry, say Λ_8, given by

$$\Lambda_8^{[2]} = \tau(\cdot)\partial t + \eta(\cdot)\partial u + \xi(\cdot)\partial x + \eta_t \partial u_t + (0)\partial u_x + u_{xx}\partial u_{xx}, \tag{3.4.62}$$

where τ and η are functions of unspecified arguments, to equation (3.4.59). From the associated Lagrange's system above we immediately deduce that $\xi(\cdot) = 0$. The extended infinitesimals of (3.4.62) suggest the system of partial differential determining equations [57],

$$\begin{aligned}
\eta_x : \quad & \frac{\partial \eta}{\partial x} - \frac{\partial \tau}{\partial x} u_t = 0, \\
\eta_t : \quad & \frac{\partial \eta}{\partial t} - \frac{\partial \tau}{\partial t} u_t = u_t, \\
\eta_{xx} : \quad & \frac{\partial^2 \eta}{\partial x^2} - 2\frac{\partial \tau}{\partial x} u_{tx} - \frac{\partial^2 \tau}{\partial x^2} u_t = u_{xx},
\end{aligned} \tag{3.4.63}$$

in which differentiation with respect to t or x is total, with the demand that η is free of u[16]. The first and third equations give

$$\frac{\partial \tau}{\partial x} = -\frac{u_{xx}}{u_{tx}}. \tag{3.4.64}$$

Differentiating the first equation with respect to t and the second equation with respect to x, taking (3.4.64) into account and after some manipulation we obtain

$$\frac{\partial \tau}{\partial t} = -1 - \frac{u_{xx} u_{tt}}{u_{tx}}. \tag{3.4.65}$$

With (3.4.65) the second equation of (3.4.63) produces

$$\frac{\partial \eta}{\partial t} = \frac{u_{xx} u_{tt} u_t}{u_{tx}}. \tag{3.4.66}$$

[16]This is merely to ensure the simplicity of computation of a nonlocal symmetry. One may choose to have a different demand and still arrive at a required nonlocal symmetry.

Also equation (3.4.64) with the first equation of (3.4.63) yields

$$\frac{\partial \eta}{\partial x} = -\frac{u_{xx}u_t}{u_{tx}}. \tag{3.4.67}$$

With (3.4.64), (3.4.65), (3.4.66) and (3.4.67) one immediately gathers the required coefficients

$$\begin{aligned} \tau(\cdot) &= -\left(\int \frac{u_{xx}}{u_{tx}}dx + \int \frac{u_{xx}u_{tt}}{u_{tx}}dt + t\right) \\ \eta(\cdot) &= \int \frac{u_{xx}u_{tt}u_t}{u_{tx}}dt - \int \frac{u_{xx}u_t}{u_{tx}}dx \end{aligned} \tag{3.4.68}$$

and the required nonlocal symmetry is given by

$$\Lambda_8 = \tau(\cdot)\partial t + \eta(\cdot)\partial u \tag{3.4.69}$$

with τ and η defined by (3.4.68). The search for a nonlocal symmetry[17] of equation (3.4.25) has been made possible by what we term the *Choice*[18] algorithm. Furthermore the nonlocal symmetry is said to be generic to a class of equations having a structure similar to (3.4.60). Accordingly, upon application of a nonlocal symmetry (3.4.69), equation (3.4.59) takes the form (3.4.60).

Nonlocal symmetry for $F = u_x^n, \quad n \neq -2$

In this case we apply the third realisation of (3.4.58) to obtain

$$u_{xx} = f(u, u_x, u_t) \tag{3.4.70}$$

and require that (3.4.70) take the form

$$u_{xx} = u_t h(u, u_x). \tag{3.4.71}$$

This can be obtained by a setting $f = u_t h(u, u_x)$ from the characteristics $\dfrac{f}{u_t}, u$ and u_x. These characteristics are obtained from the associated Lagrange's system

$$\frac{df}{f} = \frac{du_t}{u_t} = \frac{du}{0} = \frac{du_x}{0}.$$

[17]One can also use the methods treated in [14, 58, 59, 60, 62] to find nonlocal symmetries. However, for our purposes this is a more direct method to find nonlocal symmetries

[18]Choice since the nonlocal symmetry realised after the application of the algorithm depends on the choices made by the user. It must be noted that the *choice* algorithm produces a nonunique nonlocal symmetry due to different possible choices of assumptions.

The above system is the result of the application of an as yet unknown nonlocal symmetry, Γ_6 say. The system tells us that the nonlocal symmetry can only be of the form

$$\Gamma_6 = \zeta(\cdot)\partial t + \xi(\cdot)\partial x + \eta(\cdot)\partial u, \tag{3.4.72}$$

where ζ and ξ are to be specified and $\eta = 0$. The extended infinitesimals produce the following system of symmetry determining equations

$$\begin{aligned}
\eta_x : &\quad \frac{\partial \xi}{\partial x}u_x - \frac{\partial \tau}{\partial x}u_t = 0,\\
\eta_t : &\quad \frac{\partial \tau}{\partial t}u_t + \frac{\partial \xi}{\partial t}u_t = -u_t,\\
\eta_{xx} : &\quad \frac{\partial^2 \xi}{\partial x^2}u_x + \frac{\partial^2 \tau}{\partial x^2} + 2\frac{\partial^2 \xi}{\partial x}u_{xx} + 2\frac{\partial^2 \tau}{\partial x^2}u_{xt} = -u_{xx},
\end{aligned} \tag{3.4.73}$$

which is similar to the previous case and the required coefficients of ∂t and ∂x are

$$\begin{aligned}
\zeta &= -\left[\int \frac{u_{xx}}{u_{tx}}dx + \int \frac{u_{xx}u_{tt}}{u_{tx}}dt + t\right]\\
\xi &= \int \frac{u_{xx}u_{tt}u_t}{u_{tx}}dt - \int \frac{u_{xx}u_t}{u_{tx}}dx.
\end{aligned} \tag{3.4.74}$$

Consequently we have (3.4.72), which has been obtained by the choice algorithm. All that is left now is to specify the equations completely.

3.4.3 Complete Specification of the Equations

Complete specification of $u_t = u_x^n u_{xx}$ for $n = -2$

To specify completely the equation one applies the remaining useful symmetries to (3.4.60). The second extension of Λ_6 is given by

$$\Lambda_6^{[2]} = x\partial x - u_x\partial u_x - 2u_{xx}\partial u_{xx} \tag{3.4.75}$$

and its application to (3.4.60) produces the associated Lagrange's system

$$\frac{dh}{-2h} = \frac{dx}{x} = \frac{du_x}{-u_x} \tag{3.4.76}$$

so that after a suitable choice of characteristics the function h takes the form

$$h = u_x^2 g(xu_x) \tag{3.4.77}$$

and consequently equation (3.4.60) takes the form

$$u_{xx} = u_t u_x^2 g(x u_x). \tag{3.4.78}$$

A similar application of $\Lambda_3^{[2]}$to (3.4.78) reduces (3.4.78) to

$$u_{xx} = \gamma u_t u_x^2, \tag{3.4.79}$$

where γ is a constant of integration. At this point it seems we have run out of useful point symmetries as the application of any of the remaining symmetries does not produce the equation. We therefore have to make use of our algorithm to find a nonlocal symmetry to specify the arbitrary constant. We require that $\gamma = 1$. There are several ways to achieve this, one of which would be to demand that the coefficients of ∂u, ∂u_x, ∂u_t and ∂u_{xx} be 0, u_x, 0 and $-2u_{xx}$ respectively. Using the algorithm described above and after a certain amount of effort we obtain

$$\begin{aligned}\tau(\cdot) &= 2\left[\int\left(\frac{u_x u_{xx}}{u_{tx} - u_t u_{xx}}\right)dx - \int\left(\frac{u_x u_{xx}}{u_{tx} - u_t u_{xx}}\right)dt\right] + t \\ \xi(\cdot) &= 2\int\left(\frac{u_x u_{xx}}{u_{tx} - u_t u_{xx}}\right)dx + \int\left(\frac{u_t}{u_x}\left(1 + 2\frac{u_x u_{xx}}{u_{tx} - u_t u_{xx}}\right)\right)dt + t.\end{aligned} \tag{3.4.80}$$

The second nonlocal symmetry is given by

$$\Lambda_9 = \tau\partial t + \xi\partial x \tag{3.4.81}$$

with τ and ξ given by (3.4.80) and consequently equation (3.4.57) has been completely specified.

Complete specification of $u_t = u_x^n u_{xx}$ for $n \neq -2$

The complete specification (3.4.57) for the case $n \neq -2$ can be achieved by two routes. One may apply the nonlocal symmetry (3.4.81) to equation (3.4.55) to achieve the desired equation (3.4.33) since the nonlocal symmetry serves to specify a constant to equal one. Alternatively one may apply the remaining symmetries[19] of (3.4.34) to (3.4.71) as follows: The application of Γ_2 to equation (3.4.71) produces

$$u_{xx} = u_t g(u_x). \tag{3.4.82}$$

[19]Point symmetries remaining after finding a nonlocal symmetry.

The application of the second extension of Γ_5 produces

$$ng = -u_x g' \tag{3.4.83}$$

so that (3.4.82) becomes

$$u_{xx} = \iota u_t u_x^{-n}. \tag{3.4.84}$$

It is interesting to note that the remaining point symmetry gives no useful information when applied to specify the constant! This is one example to show that a complete symmetry group of a given partial differential equation is not *unique* once nonlocal symmetries have been summoned to specify a given equation. Hence one turns back to a nonlocal symmetry (3.4.81) to specify the constant and consequently equation (3.4.33) follows immediately.

The other unused symmetries of (3.4.32) and (3.4.34) are still necessary for inclusion in the representation of complete symmetry group of (3.4.57) due to meeting the requirement of the closure property. Consequently we represent them as

$$\langle\ \Lambda_i, i = 1, 9\rangle_{csg} \tag{3.4.85}$$

for the case $n = -2$ and

$$\langle\Gamma_j, j = 1, 6\rangle_{csg} \tag{3.4.86}$$

for the case $n \neq -2$. It must be mentioned that the Choice algorithm had been applied in [42, 46] before its formal introduction to the symmetry community in [43]. Further it was established that, when nonlocal symmetries found by the Choice algorithm have been summoned to specify the equation, the entire process of specifying the equation is then called *Implicit Complete Symmetry Group* if one nonlocal symmetry is used and *Quasi-implicit Complete Symmetry Group* if more that one such nonlocal symmetry was used. These processes were seen clearly in action in [42, 43].

The implicit and quasi-implicit complete symmetry group approach not only provides us with the number of symmetries sufficient to form a complete symmetry group but also provides a more direct way to find nonlocal symmetries. The nonlocal symmetries found are known to have specific roles in the development of a partial differential equation. Also some nonlocal symmetries found by the Choice algorithm are *extracting* nonlocal symmetries since they remove a variable from the arbitrary function while other nonlocal symmetries are *combining* nonlocal symmetries as they combine the variables inside the arbitrary function. It further happens that, when an arbitrary function we are trying to specify contains more than three arguments, for example containing either the space or the time variable in addition to the u and its derivatives, the extracting nonlocal symmetry simply becomes a Lie point symmetry. An interesting point to note is that the nonlocal symmetry, for example,

$$\Delta = \xi(x,t)\partial_x + x\,\partial_t,$$

for

$$\xi(x,t) = \int_R K(x-y;t)\xi_0(y)dy + \int_0^t \int_R K(x-y,t-s)g(y,s)dy\,ds, \quad (3.4.87)$$

where $K(x,t)$ is the diffusion kernel given by

$$K(x,t) = \left(\frac{1}{4\pi Dt}\right)^{\frac{1}{2}} \exp\left(\frac{-x^2}{4Dt}\right)$$

and ξ_0 and $g(x,t)$ are continuous bounded functions, is a nonlocal symmetry for all evolution equations which can be written in the form,

$$w_{xx} + w_t = h(w, w_x), \quad (3.4.88)$$

admit Δ as a nonlocal symmetry. These types of nonlocal symmetries are said to be *generic* to equations of the structures they specify and therefore one can proceed in a similar way to find other *generic* nonlocal symmetries for other structures of equations using the Choice algorithm[20].

As a matter of interest the Choice algorithm, though engineered to craft nonlocal symmetries for partial differential equations, is perfectly applicable

[20]The Choice algorithm is the remedy for the situation where there are no explicit expressions for symmetries for a given equation. One can construct a number of such nonlocal symmetries to observe at least a certain structure of a given equation. The beauty of it is that the role of a particular nonlocal symmetry is known exactly in the specification of the equation.

to ordinary differential equations and first integrals. For example consider the Emden-Fowler equation

$$y'' + 2y^3 = 0 \tag{3.4.89}$$

with the first integral

$$I = y'^2 + y^4. \tag{3.4.90}$$

This integral (3.4.90) admits only the symmetry ∂x. The general function for the first integral is $f(x, y, y')$. The application of the available symmetry ∂x, gives

$$f = f(y, y'). \tag{3.4.91}$$

With the knowledge that the complete symmetry group of a first integral of a second-order ordinary differential equation comprises two elements (not necessarily unique), instead of the approach followed by Andriopoulos *et al* in [6] we invoke the services of the Choice algorithm as follows: Firstly recall that the structure of the symmetry is given by

$$G = \xi\partial_x + \eta\partial_y.$$

We require a symmetry that sets[21]

$$f = g(y'^2 + y^4).$$

This would be as a result of the characteristics,

$$g, y'^2 + y^4,$$

which would be obtained from the following associated Lagrange's system

$$\frac{dy}{-2y'} = \frac{dy'}{4y^3} = \frac{dg}{0}.$$

The above associated Lagrange's system and the requirement that the operator be a symmetry of the underlying differential equation simply imply that

$$\begin{aligned} \eta &= -2y' \\ \eta' - y'\xi' &= 4y^3 \\ \eta'' - 2y''\xi' - y'\xi'' + 6\eta y^2 &= 0. \end{aligned} \tag{3.4.92}$$

[21]Note that the required structure is firstly specified and then the investigation of a nonlocal symmetry which would allow this structure is performed afterwards.

Finding the solution of the above system is rather a trivial matter and it produces

$$\xi = 0, \tag{3.4.93}$$

and consequently the required nonlocal, equally called generalised, symmetry is[22]

$$G_2 = -2y'\partial_y. \tag{3.4.94}$$

Since G_2 was obtained from its own application, though it was unknown, its application to (3.4.91) yields

$$f = g(y'^2 + y^4)$$

and thus the integral has been specified up to an arbitrary function of itself.

3.5 Conclusion

We have seen that although the equation under consideration may posses a sufficient number of point symmetries in terms of [46], some of those point symmetries may not be useful in the specification of the equation as observed in §3.4.1. Furthermore in §3.4.2 we provided a Choice algorithm which allows the determination of required nonlocal symmetries necessary to supplement the point symmetries. The nonlocal symmetry realised is *generic* to the structure imposed upon the equation. As a result all equations having the structure (3.4.60) and (3.4.71) admit the nonlocal symmetries, Λ_8 and Γ_6, respectively. Furthermore these nonlocal symmetries need not be unique.

We have also seen that, once nonlocal symmetries come into the picture, one loses the apparent uniqueness property of complete symmetry group for a given equation which was a nice feature for partial differential equations. What has transpired in this Section is that, with the help of the Choice algorithm, one can, if one so choose, completely specify the given equation using nonlocal symmetries only.

[22]The very first application of the Choice algorithm to craft nonlocal symmetries for ordinary differential equations and their first integrals was in [43].

The complete symmetry groups of ordinary differential equations have been intensively studied and it is now well understood [6] that the equation may possess many representations of the same group which specify uniquely the equation. Hence the introduction of the concept of minimal dimension under the operation of the Lie bracket was indispensable. Before we leave this Chapter we state some important observations on the properties of complete symmetry groups for ordinary and partial differential equations.

Corollary

The complete symmetry group of a nonhomogeneous linear n^{th}-order ordinary differential equation consists of $n+1$ symmetries.

$$\begin{aligned} G_i &= S_i\partial_y \;\;;\;\; i = 1, n, \\ G_{n+1} &= (y - \alpha(x))\partial_y \;, \end{aligned}$$

where the S_i are solutions of the corresponding homogeneous equation and $\alpha(x)$ is a particular solution of the nonhomogeneous equation.

Corollary

All linear equations have a complete symmetry group represented by an algebra consisting of Lie point symmetries only.

Further results were observed.

Corollary

All equations linearisable by a point transformation have a complete symmetry group represented by an algebra $A_1 \bigoplus_s nA_1$ of point symmetries.

In general [7] a system of n second-order ordinary differential equations requires $2n+1$ symmetries to specify it completely. The Newtonian equations for the Kepler Problem possess just the five Lie point symmetries of the algebra $A_2 \oplus A_{3,9}$ representing invariance under time translation and rescaling on the one hand and the rotational invariance of $so(3)$ on the other. Krause resorted to the use of nonlocal symmetries to remedy the deficit[23] and devised an ingenious scheme for their determination. Unfortunately nonlocal symmetries of differential equations in general have a property in common with symmetries of first-order differential equations, that is, although they are more numerous than the grains of sand by the sea, there is no finite algorithm for their general determination. We have seen in [44] that for the two-dimensional heat conduction equation the complete specification was achieved using even a lesser number of point symmetries than those admitted by the equation. However, we also observed that, when the number of point symmetries is insufficient, one seems to get into trouble very quickly. This is because between the application of point symmetries one has to apply nonlocal symmetries. With lack of experience, if nothing else, one would not know where to include a nonlocal symmetry. Just to provide a guideline, if the construction of an equation gets really unsightly, then it may be an indication that a nonlocal symmetry is required at that step. Considering the observations in [41, 42, 43, 46] we have established that

> *All second-order evolution partial differential equations of maximal nongeneric Lie point symmetries plus the homogeneity symmetry possesses a complete symmetry group comprising Lie point symmetries only,*

which is in parallel to the second corollary for the complete symmetry groups for ordinary equations above, but the story is very different when it comes to the number of symmetries required to specify the partial differential equation. It had been concluded that [43]

> *The number of symmetries required to form a complete symmetry group of minimal dimension, be they point, nonlocal or other, for any $1 + n$ evolution partial differential equation is equal to the number of arguments in the general second-order evolution partial*

[23]We must note that the use of nonlocal symmetries in the first application of this concept of a complete symmetry group should not be taken to imply that nonlocal symmetries are a necessary concomitant. That nonlocal symmetries have played an important role in the determination of the complete symmetry group in a number of instances [26, 27, 28, 55] should not obscure the reality that point symmetries have played an important role in the theoretical development as well as certain applications [6, 7, 8] of complete symmetry groups.

differential equation, videlicet

$$F(x, u, \partial_u, \partial_u^2, \ldots, \partial_u^k) = 0, \tag{3.5.95}$$

where $x = (x, x_2, \ldots, x_n)$ denotes the set of coordinates corresponding to its n independent variables, u denotes the dependent variable and $\partial_u^j,\ j = 1, k$, corresponds to all partial derivatives of u with respect to x. In terms of the coordinates, $x, u, \partial_u, \partial_u^2, \ldots, \partial_u^k$, the partial differential equation, (3.5.95), is an algebraic equation that defines an hypersurface in $(x, u, \partial_u, \partial_u^2, \ldots, \partial_u^k)$-space.

Part II

Whence Comes The Oaks

Preface

There is a very interesting story about the beginning of the work in subsequent Chapters that must be told to the reader for better enlightenment regarding the subsequent Chapters.

When I was at the beginning of the research for a doctoral programme, wondering about what I was about to do, I came into the office of one of my supervisors, Professor PGL Leach, for some guidance at Howard College, Room 308a, Desmond Clarence Building. Not to my surprise the supervisor did not make the time for the appointment – not for the first time – but luckily I had access to his office, thanks to his kindness and trust, I made myself very comfortable in one of the very few places one could have a seat. In fact a place to seat in most academics offices, including my own, could be easily described as an endangered scarce resource.

Inside this very full to capacity office there was a white board on the wall with about n things written on it. Of the n things written in that board, $n \sim \infty$, and in fact one should consider using the same symbol, there was a set of point symmetries written under the title *Whence Comes The Oaks*[24]. There was no reference to the function or an equation admitting these point symmetries, whether ordinary, partial or something else. Having recently worked on complete symmetry groups for partial differential equations [42, 46] I made an effort to find out without asking the supervisor who had not arrived after some time now.

I started by finding out whether these symmetries could have came from a partial differential equation and after a few pages of calculation and simplification, I found a nonlinear partial differential equation of great importance in the literature admitting very similar Lie point symmetries but slightly different as these symmetries were multiplied by arbitrary functions (see §4.2). I was later intrigued and in fact very

[24]Pity I had no camera to capture the event which has resulted in a revolution of complete symmetry groups.

surprised when Peter informed me that the symmetries had in fact came from a free particle problem which is an ordinary differential equation! It was clear that a lot was hidden behind this coincidence so I made it my duty partially to illuminate the lot. The interested booklover will read all about it in the subsequent Chapters comprising this part of the Thesis. Once again I cannot overemphasize the fact that we are yet to experience the far-reaching beauty and ability of complete symmetry groups in explaining differential equations.

Chapter 4

The connection of ordinary differential equations and partial differential equations

4.1 Introduction

Until recently the determination of complete symmetry groups has been confined to systems of ordinary differential equations. In a study of the complete symmetry group of the $1 + 1$ heat equation and some related equations which arise in Financial Mathematics we [41] showed that the number of Lie point symmetries required to specify the $1 + 1$ heat equation is six. The classical heat equation, as a linear partial differential equation, possesses an infinite number of Lie point symmetries. The considerations have since been extended beyond just $1 + 1$ evolution equations [42, 43, 44] and they now cover the complete symmetry group of two-dimensional second-order partial differential equations. If the evolution partial differential equation has the maximal number of point symmetries, then the complete symmetry group comprises all the point symmetries that the equation admits. If, however, the equation does not admit a maximal number of point symmetries, then the complete symmetry group contains nonlocal symmetries required to remove the deficit between the maximal number of point symmetries for the class to which the partial differential equation belongs and the admitted point symmetries [42]. This feature easily clarifies the second requirement above for complete symmetry group.

It is also a well known fact that the $1+1$ heat equation,

$$u_{xx} = u_t, \tag{4.1.1}$$

is reducible to free particle problem via reduction using the Lie point symmetries admitted by the heat equation. The heat equation admits amongst other symmetries the Lie point symmetry ∂t [57] which produces the associated Lagrange's system

$$\frac{dt}{1} = \frac{du}{0} = \frac{dx}{0}$$

with the similarity variables $v = x$ and $w = u$. Setting $u = y(x)$, we obtain $u_t = 0$ and $u_{xx} = y''$ so that the heat equation takes the form (4.1.2). In this Chapter we construct partial differential equations using Lie point symmetries admitted by an ordinary differential equation, *videlicet*

$$y'' = 0, \tag{4.1.2}$$

where $y = y(x)$ and the prime denotes differentiation with respect to x. It is therefore natural to expect (if successful) to obtain the heat equation or at least an equation in the same equivalence class as the heat equation. However, this be not the case [47].

4.2 Second-order Ordinary Differential Equations and Partial Differential Equations

The free particle problem

The free particle problem

$$y'' = 0 \tag{4.2.3}$$

is a fundamental equation in theory and applications. Equation (4.2.3) admits a wonderful supply of Lie point symmetries[1]

$$\begin{array}{rclrcl}
\Lambda_1 &=& \partial y & \Lambda_5 &=& 2x\partial x + y\partial y, \\
\Lambda_2 &=& x\partial y & \Lambda_6 &=& x^2\partial x + xy\partial y, \\
\Lambda_3 &=& y\partial y & \Lambda_7 &=& y\partial x, \\
\Lambda_4 &=& \partial x & \Lambda_8 &=& y^2\partial y + xy\partial x.
\end{array}$$

[1]Though these symmetries have already appeared, we list them again for the ease of reference.

The Lie Brackets of the above operators are given by

$[,]$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8
Λ_1	0	0	Λ_1	0	Λ_1	$-\Lambda_2$	$-\Lambda_4$	Λ_5
Λ_2	0	0	Λ_2	$-\Lambda_1$	$-\Lambda_2$	0	Λ_5	Λ_6
Λ_3	Λ_1	Λ_2	0	0	0	0	Λ_7	Λ_8
Λ_4	0	$-\Lambda_1$	0	0	$2\Lambda_5$	Λ_5	0	Λ_7
Λ_5	Λ_1	Λ_2	0	$2\Lambda_5$	0	$2\Lambda_6$	Λ_7	Λ_5
Λ_6	$-\lambda_2$	0	0	Λ_5	$2\Lambda_6$	0	$-\Lambda_8$	0
Λ_7	$-\Lambda_4$	Λ_5	Λ_7	0	Λ_7	$-\Lambda_8$	0	0
Λ_8	Λ_5	Λ_6	Λ_8	Λ_7	Λ_5	0	0	0

The idea is to use the point symmetries above to specify the general second-order partial differential equation with two independent variables and one dependent variable, *videlicet*

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{4.2.4}$$

From our knowledge of complete symmetry groups for partial differential equations this equation is indeed specifiable using eight symmetries be they point or nonlocal [44]. We use the above point symmetries to specify (4.2.4) written in solved form in terms of one of its arguments as

$$u_{xx} = f(x, y, u, u_x, u_y, u_{xy}, u_{yy}), \tag{4.2.5}$$

where f is a function to be determined. For equation (4.2.5) to be invariant under the action of Λ_1 and Λ_4 it must be of the form

$$u_{xx} = f(u, u_x, u_y, u_{xy}, u_{yy}). \tag{4.2.6}$$

The second extension of Λ_3 is given by [13, 57]

$$\Lambda_3^{[2]} = y\partial y - u_y\partial u_y - u_{xy}\partial u_{xy} - 2u_{yy}\partial u_{yy} \tag{4.2.7}$$

and its application to (4.2.6) produces the associated Lagrange's system

$$\frac{df}{0} = \frac{du}{0} = \frac{du_x}{0} = \frac{du_y}{u_y} = \frac{du_{xy}}{u_{xy}} = \frac{du_{yy}}{2u_{yy}}$$

for which the characteristics immediately realised are

$$u,\ u_x,\ \frac{u_{xy}}{u_y},\ \frac{u_{xy}^2}{u_{yy}}.$$

Consequently equation (4.2.6) takes the form

$$u_{xx} = f\left(u, u_x, \frac{u_{xy}}{u_y}, \frac{u_{xy}^2}{u_{yy}}\right). \tag{4.2.8}$$

For simplicity of notation we let $v = \dfrac{u_{xy}}{u_y}$ and $w = \dfrac{u_{xy}^2}{u_{yy}}$. It is efficacious to observe the Lie Brackets and particulary to note that the application of Λ_6 is efficient at this point[2] with its second extension given by

$$\begin{aligned}\Lambda_6^{[2]} &= x^2\partial x + xy\partial y - [2xu_x + yu_y]\,\partial u_x - xu_y\partial u_y - [2u_x + 4xu_{xx} + 2yu_{xy}]\,\partial u_{xx}\\ &- [u_y + yu_{yy} + 3xu_{xy}]\,\partial u_{xy} - 2xu_{yy}\partial u_{yy}.\end{aligned}$$

Its application to (4.2.8) produces

$$\begin{aligned}2u_x + 4xu_{xx} + 2yu_{xy} &= (2xu_x + yu_y)\frac{\partial f}{\partial u_x} + 2xv\frac{\partial f}{\partial v} + 4wx\frac{\partial f}{\partial w}\\ &+ (u_y + yu_{yy})\left(\frac{1}{u_y}\frac{\partial f}{\partial v} + 2\frac{u_{xy}}{u_{yy}}\frac{\partial f}{\partial w}\right).\end{aligned}$$

Since x and y are no longer part of the arguments of f, one can extract their coefficients to obtain the system of equations:

$$\begin{aligned}x &: 2f = u_x\frac{\partial f}{\partial u_x} + v\frac{\partial f}{\partial v} + 2w\frac{\partial f}{\partial w},\\ y &: 2u_{xy} = u_y\frac{\partial f}{\partial u_x} + u_{yy}\left(\frac{1}{u_y}\frac{\partial f}{\partial v} + 2\frac{u_{xy}}{u_{yy}}\frac{\partial f}{\partial w}\right),\\ - &: 2u_x = u_y\left(\frac{1}{u_y}\frac{\partial f}{\partial v} + 2\frac{u_{xy}}{u_{yy}}\frac{\partial f}{\partial w}\right).\end{aligned}$$

We can write this system in a more amenable manner by multiplying the second and third equations by $\dfrac{u_{xy}}{u_{yy}}$ and v, respectively, to obtain

$$\begin{aligned}x &: 2f = u_x\frac{\partial f}{\partial u_x} + v\frac{\partial f}{\partial v} + 2w\frac{\partial f}{\partial w},\\ y &: 2w = \frac{u_y u_{xy}}{u_{yy}}\frac{\partial f}{\partial u_x} + v\frac{\partial f}{\partial v} + 2w\frac{\partial f}{\partial w}\\ - &: 2vu_x = v\frac{\partial f}{\partial v} + 2w\frac{\partial f}{\partial w}.\end{aligned} \tag{4.2.9}$$

[2]This is due to our understanding of the Lie Bracket operations, *i.e.,* the action of the Lie Bracket carries the information which would be obtained from the individual symmetries making up the bracket.

The solution of the above system is a simple routine work. We obtain from the third equation

$$f = 2vu_x + h\left(u, u_x, \frac{v^2}{w}\right), \tag{4.2.10}$$

where h is an as yet arbitrary function. From the first equation we find that

$$h = u_x^2 g\left(u, \frac{v^2}{w}\right) \tag{4.2.11}$$

upon which (4.2.10) becomes

$$f = 2vu_x + u_x^2 g\left(u, \frac{v^2}{w}\right). \tag{4.2.12}$$

With (4.2.12) the second equation yields

$$g = -\frac{u_{yy}}{u_y^2} \tag{4.2.13}$$

and consequently we have found a partial differential equation specified by the set[3] of Lie point symmetries of an ordinary differential equation, (4.2.3). It is given by

$$u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = 0, \tag{4.2.14}$$

which is a two-dimensional Bateman equation[4]. Its general implicit solution is given by

$$x\phi_1(u) + y\phi_2(u) = C, \tag{4.2.15}$$

where C is a constant and ϕ_1 and ϕ_2 are arbitrary smooth functions[5]. The Bateman equation above is very useful in the Painlevè analysis of partial differential equations and in fact in the singularity analysis equation (4.2.14)

[3]Note that even though the set $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_6\}$ has specified (4.2.14) is minimal, it does not have closure. To achieve closure we must include Λ_5. This is the same phenomenon observed [50] in the complete symmetry group of both partial differential equations and systems of ordinary differential equations.

[4]The application of Λ_8 produces yet another multiple of equation (4.2.14) and by the mechanism of the Lie Bracket, the action of Λ_7 has already been carried out by the application of $\Lambda 8$, and consequently we have used all Lie point symmetries of (4.2.3) to specify (4.2.14).

[5]Not quite what we expected! As the free particle problem is obtainable from the heat equation via symmetries, we expected to obtain a heat equation or an equivalent at the very least.

was first obtained by Weiss [61] when he was investigating the integrability of the double sine-Gordon equation. Equation (4.2.14) is linearisable by a Legendre transformation and it is invariant under the Möbius group. Equation (4.2.14) is a parabolic equation and, as one would expect or not expect, admits the following infinite-dimensional set of Lie point symmetries.

$$\begin{array}{rclrcl} G_1 &=& H_1(u)\partial y & G_6 &=& H_6(u)x\partial x \\ G_2 &=& H_2(u)\partial x & G_7 &=& H_7(u)\left(x^2\partial x + xy\partial y\right) \\ G_3 &=& H_3(u)y\partial y & G_8 &=& H_8(u)\left(y^2\partial y + xy\partial x\right) \\ G_4 &=& H_4(u)y\partial x & G_9 &=& H_9(u)\partial u, \\ G_5 &=& H_5(u)x\partial y & & & \end{array}$$

where $H_i,\ i = 1, 9,$ are arbitrary functions of u. Taking suitable functions $H_i,\ i = 1, 9,$ we immediately achieve the $A_1 \bigoplus_s sl(3, R)$ symmetry group which is that of (4.2.3) with of course the added point symmetry G_9.

There are four classes of symmetry in terms of provenance. The two-dimensional Abelian algebra comprising two symmetries, *videlicet* $\partial x_i,\ \ i = 1, 2,$ the $[(3-1)^2 = 4]$-dimensional algebra, $x_i\partial x_j,\ \ i, j = 3, 6,$ and the two-dimensional Abelian algebra $x_ix_j\partial x_j,\ \ i, j = 7, 8,$ and G_9. These symmetries can be further classified into known algebras containing $A_1 \bigoplus_s 2A_1$, $sl(2, R)$, the so-called noncartan symmetries and G_9. An important but curious observation is the fact that for a parabolic equation one would expect a class containing the infinite number of Lie point symmetries comprising solutions of the equation. The difference here is due to the nonlinearity of the homogenous in u (4.2.14) since nonlinearity destroys the property of linear superposition.

4.3 Connection with a System of Ordinary Differential Equations

Another curious observation is that the group of Lie point symmetries of equation (4.2.14) is in fact equivalent to that of a system of ordinary differential equations[6], *videlicet*

$$\begin{aligned} y'' &= 0 \\ u' &= 0, \end{aligned} \tag{4.3.16}$$

and surprisingly exactly the same set

$$\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_6\}$$

of point symmetries that specified (4.2.14) can specify (4.3.16) as we show below. As before we start with the general system of ordinary differential equations

$$\begin{aligned} y'' &= f(x, y, u, y') \\ u' &= g(x, y, u, y'), \end{aligned} \tag{4.3.17}$$

where f and g are functions to be specified. For the system (4.3.17) to be invariant under Λ_1 and Λ_4 it must be of the form

$$\begin{aligned} y'' &= f(u, y') \\ u' &= g(u, y'). \end{aligned} \tag{4.3.18}$$

The second extension of Λ_3 is given by

$$\Lambda_3^{[2]} = y\partial y + y'\partial y' + y''\partial y''. \tag{4.3.19}$$

Its application to the upper equation of (4.3.19) leads to

$$f = y' \frac{\partial f}{\partial y'} \tag{4.3.20}$$

in which case the upper equation takes the form

$$y'' = F(u) y' \tag{4.3.21}$$

and, when the same operation is performed to the second equation of (4.3.19), it takes the form

$$u' = G(u). \tag{4.3.22}$$

[6]The term 'equivalent' here and henceforth unless stated otherwise, of differential equations (equally systems) is used in the context of referring to equations (equally systems) admitting the same set of Lie point symmetries and are specified by the same set of Lie point symmetry.

The second extension of Λ_6 is given by

$$\Lambda_6^{[2]} = x^2\partial x + xy\partial y + (y - xy')\partial y' - (xy'' + 2xy'')\partial y''. \tag{4.3.23}$$

Upon application of this to (4.3.21), the equation becomes

$$y'' = 0. \tag{4.3.24}$$

Similarly, when (4.3.23) is applied to (4.3.22), this equation becomes

$$u' = 0. \tag{4.3.25}$$

Consequently we have specified[7] (4.3.16). So we have two distinct models completely specified by the same set of Lie point symmetries!

This of course raises some interesting questions: Since differential equations are supposed to have physical interpretations, can one use either (4.3.16) or (4.2.14) in modelling? Do they carry the same information? Will the two solutions be the same or have the same effectiveness? In fact what we asking is, are the two models equivalent in the sense that the two equations describe the same reality[8]? What is apparent is that, though we are used to having one equation represented by different structures of symmetry, here we have one symmetry structure representing different equations, actually different kinds of equations and this remarkable fact compels us to revisit the statements by Krause regarding complete symmetry groups and we do this in the Conclusion.

[7]Note that though the set of point symmetries $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_6\}$ that has specified (4.3.16), contains the minimal number of point symmetries used to specify (4.3.16), it does not have closure. To achieve closure we must include Λ_5. Nevertheless it suffices to display the intended picture.

[8]One is reminded that a given system of stochastic differential equations satisfying appropriate conditions is equivalent to a second order evolution partial differential equation [17, 20]. Furthermore, both models describe the same reality and share the expectation (which is intuitively a constant) as a common solution and is unique to the system of stochastic differential equations, much the same as we observed with (4.3.16) and (4.2.14). However the questions here require much deeper analysis which is the subject of our future endevours, for now it suffices to note that some connection or equivalence (in the correct context of this Thesis) exists.

4.4 The Connection of Solutions

Returning to the question of whether the Bateman equation (4.2.14) and the system of ordinary differential equations (4.3.16) have the same solution or similar behaviour we examine the two solutions. It is well known that the general solution of the system of ordinary differential equations, (4.3.16), is given by

$$\begin{aligned} y &= Ax + B \\ u &= D, \end{aligned} \tag{4.4.26}$$

where A, B and D are arbitrary constants. The upper equation can be rewritten as

$$y\phi_2 + x\phi_1 = C, \tag{4.4.27}$$

where C, ϕ_1 and ϕ_2 are rescaled constants. The general solution of the Bateman equation is given by

$$x\phi_1(u) + y\phi_2(u) = C, \tag{4.4.28}$$

where C is a constant and $\phi_1(u)$ and $\phi_2(u)$ are arbitrary smooth functions.

Consider the Bateman solution and take $u = D$, D constant. Since $\phi_1(u)$ and $\phi_2(u)$ are smooth functions of u, if $u = D, D$ constant, then $\phi_1(u) = \phi_1$, ϕ_1 constant. Similarly $\phi_2(u) = \phi_2$, ϕ_2 constant. The Bateman solution immediately has the form

$$x\phi_1 + y\phi_2 = C, \tag{4.4.29}$$

where C, ϕ_1 and ϕ_2 are constants. It turns out that we have a particular solution of the Bateman equation when $u = D, D$ constant. This particular solution of the Bateman equation is equivalent, if not the same since constants can always be rescaled, to the solution of the system of ordinary differential equations (4.3.16). In fact one can ascertain that the general solution of the system of ordinary differential equations is a special case of a more general solution of the Bateman equation. One can deduce that the Bateman equation and the system of ordinary differential equations are exactly the same at some point in space, maybe for a surface $u = D, D$ constant in (u, x, y)-space[9].

[9]The relationship between a system of ordinary differential equations and a partial differential equation of the second-order we have established here would have been very

4.5 Conclusion

In this Chapter we commenced with a one-dimensional heat equation in which we used one of the Lie point symmetries admitted by the heat equation to obtain a free particle problem (4.1.2) through reduction of order in §4.1. We then went to apply the Lie point symmetries of the free-particle problem to the general second-order partial differential equation with the hope of obtaining something with the expectation that the something would be of a kind similar to the heat equation from which the free-particle problem was derived. We have obtained, quite unexpectedly, a Bateman equation (4.2.14) in §4.2. The Bateman equation happens to admit exactly the same structure of Lie point symmetries as the system of equations (4.3.16) and in fact the exact same Lie point symmetries that were used in the construction of the Bateman equation were used to construct the system of ordinary differential equations (4.2.14), *i.e.,* the two models are specified by the same group of Lie point symmetries.

We have observed, in many ways rather than one, that the application of complete symmetry groups, as was introduced by Krause in 1994 [23, 24] to ordinary differential equations to describe the group which in its algebraic representation completely specified the differential equation under consideration, has been extended beyond the scope of ordinary differential equations to partial differential equations. *See also* [41]. We have further observed that the second property in Krause's definition of complete symmetry group in §4.1 which requires that the group be specific to the differential equations or the system has been violated by equation (4.2.14) and the system of ordinary differential equations (4.3.16), that is, we have have found two distinct systems which are specified by the same group of Lie point Symmetries[10]! This violation may, however, not be seen

difficult or even impossible in the absence of Complete Symmetry Groups. One recalls that first-order partial differential equation and systems of first-order ordinary differential equations were regarded as the same in the theoretical works of the second lag of the nineteenth Century. This is just one of the numerous uses of Complete Symmetry Groups in differential equations.

[10]In fact, when one speaks of the complete symmetry group, one should specify whether the complete symmetry group is for an ordinary differential equation, a system or a partial differential equation.

in a very serious light as it happens to be quite the contrary, *i.e.,* it proves the equivalence of the systems seen in §4.3. That is the demonstration of the remarkable power of complete symmetry groups. It remains to be seen how higher-order ordinary differential equations behave when the same analysis is applied and also the behaviour of higher-dimensional Bateman equations as opposed to systems of ordinary differential equations and that is our next endeavour.

Chapter 5

The Connection of Higher-order Ordinary Differential Equations and Partial Differential Equations

5.1 Introduction

The previous Chapter on the connection of ordinary differential equations to partial differential equations through the realisation of complete symmetry group [47] has shown that the Bateman equation,

$$u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = 0, \tag{5.1.1}$$

is equivalent to the system of ordinary differential equations

$$\begin{aligned} y'' &= 0 \\ u' &= 0 \end{aligned} \tag{5.1.2}$$

and surprisingly the two are specified by the exact same set of Lie point symmetries. In fact we had used the Lie point symmetries of the free particle problem to construct (5.1.1) through complete symmetry group analysis.

In this Chapter we investigate how higher-order ordinary differential equations behave when a similar analysis to that used in the previous chapter is applied, *i.e.* that of constructing a second-order partial differential equation from the Lie point symmetries of a higher-order ordinary differential equation. Specifically we attempt to construct partial differential equations using Lie point symmetries admitted by third- and fourth-order ordinary differential equations, *videlicet*

$$y^{(3)} = 0, \tag{5.1.3}$$

and

$$y^{(4)} = 0, \tag{5.1.4}$$

where $y = y(x)$ and the prime denotes differentiation with respect to x. It is therefore natural to expect (if successful) to obtain an equation of Bateman-type and this turns out to be the case. We move to consider higher-order equations of the form

$$y^{(n)} = 0 \tag{5.1.5}$$

and construct a corresponding general partial differential equation.

5.2 Third-order Ordinary Differential Equation and Partial Differential Equations

We have seen that from the symmetries of a free particle problem we obtained a Bateman equation (5.1.1) [47]. In this Section we consider taking Lie point symmetries of a third-order ordinary differential equation, say

$$y''' = 0, \tag{5.2.6}$$

given by

$$\begin{aligned}
\Delta_1 &= \partial y & \Delta_5 &= x\partial x \\
\Delta_2 &= \partial x & \Delta_6 &= x^2\partial y \\
\Delta_3 &= y\partial y & \Delta_7 &= x^2\partial x + 2xy\partial y, \\
\Delta_4 &= x\partial y & &
\end{aligned}$$

for which the Lie Brackets of (5.2.6) are

$[,]$	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7
Δ_1	0	0	Δ_1	0	0	0	$-2\Delta_4$
Δ_2	0	0	0	Δ_1	Δ_2	$-2\Delta_4$	$-\Delta_3$
Δ_3	Δ_1	0	0	$-\Delta_4$	0	Δ_6	0
Δ_4	0	Δ_1	$-\Delta_4$	0	$-\Delta_4$	0	$-\Delta_6$
Δ_5	0	Δ_2	0	$-\Delta_4$	0	$-2\Delta_6$	$-\Delta_7$
Δ_6	0	$-2\Delta_4$	Δ_6	0	$-2\Delta_6$	0	0
Δ_7	$-2\Delta_4$	$-\Delta_3$	0	$-\Delta_6$	$-\Delta_7$	0	0

and constructing a second-order partial differential equation. As usual [41] we commence with a general second-order partial differential equation

$$u_{xx} = f(x, y, u, u_x, u_y, u_{xy}, u_{yy}), \tag{5.2.7}$$

where f is a function to be determined. For equation (5.2.7) to be invariant under the action of Δ_1 andΔ_2 it must be of the form

$$u_{xx} = f(u, u_x, u_y, u_{xy}, u_{yy}). \tag{5.2.8}$$

The second extension of Δ_3 is given by [13, 57]

$$\Delta_3^{[2]} = y\partial y - u_y\partial u_y - u_{xy}\partial u_{xy} - 2u_{yy}\partial u_{yy} \tag{5.2.9}$$

and its application to (5.2.7) produces the associated Lagrange's system

$$\frac{df}{0} = \frac{du}{0} = \frac{du_x}{0} = \frac{du_y}{u_y} = \frac{du_{xy}}{u_{xy}} = \frac{du_{yy}}{2u_{yy}},$$

for which the characteristics are immediately realised as

$$u, u_x, \frac{u_{xy}}{u_y}, \frac{u_{yy}}{u_y^2}.$$

Consequently equation (5.2.7) takes the form

$$u_{xx} = f\left(u, u_x, \frac{u_{xy}}{u_y}, \frac{u_{yy}}{u_y^2}\right). \tag{5.2.10}$$

For simplicity we let $p = \dfrac{u_{xy}}{u_y}$ and $v = \dfrac{u_{yy}}{u_y^2}$. The second extension of Δ_5 is given by

$$\Delta_5^{[2]} = x\partial x - u_y\partial u_y - u_{xy}\partial u_{xy} - 2u_{xx}\partial u_{xx} \tag{5.2.11}$$

and its application to (5.2.10) produces

$$u_{xx} = u_x^2 h\left(u, \frac{u_{xy}}{u_x u_y}, \frac{u_{yy}}{u_y^2}\right). \tag{5.2.12}$$

For ease of calculation we write $w = \dfrac{u_{xy}}{u_x u_y}$. It is efficacious to observe the commutative relations and particularly to note that the application of Δ_6 is efficient at this point with its second extension given by

$$\begin{aligned} \Delta_6^{[2]} &= \tfrac{1}{2}x^2\partial x + xy\partial y - \left[xu_x + yu_y\partial u_y\right]\partial u_x \\ &- xu_y\partial u_y - \left[u_x + 2xu_{xx} + 2yu_{xy}\right]\partial u_{xx} \\ &- \left[u_y + yu_{yy} + 2xu_{xy}\right]\partial u_{xy} - 2xu_{yy}\partial u_{yy}. \end{aligned} \tag{5.2.13}$$

The application of (5.2.13) to equation (5.2.10) gives

$$\begin{aligned} u_x + 2xu_{xx} + 2yu_{xy} &= \left(xu_x + yu_y\right)\frac{\partial f}{\partial u_x} + xu_y\frac{\partial f}{\partial u_y} \\ &+ \left(u_y + yu_{yy} + 2xu_{xy}\right)\frac{\partial f}{\partial u_{xy}} + 2xu_{yy}\frac{\partial f}{\partial u_{yy}} \end{aligned}$$

from which we obtain the following system of equations

$$\begin{aligned} x &: 2f &&= u_x\tfrac{\partial f}{\partial u_x} + u_y\tfrac{\partial f}{\partial u_y} + 2u_{xy}\tfrac{\partial f}{\partial u_{xy}} + 2\tfrac{\partial f}{\partial u_{yy}}, \\ y &: 2u_{xy} &&= u_y\frac{\partial f}{\partial u_x} + u_{yy}\frac{\partial f}{\partial u_{xy}}, \\ - &: u_x &&= u_y\frac{\partial f}{\partial u_{xy}}. \end{aligned}$$

Taking into account equation (5.2.12) we obtain from the third equation

$$\frac{\partial h}{\partial w} = 1. \tag{5.2.14}$$

Using this we obtain from the second equation

$$h = \frac{3}{2}\frac{u_{xy}}{u_x u_y} - \frac{1}{2}\frac{u_{yy}}{u_y^2}. \tag{5.2.15}$$

The first equation gives the immediate satisfaction of the relationship. Consequently the equation realised is[1]

$$u_y^2 u_{xx} - \frac{3}{2}u_x u_y u_{xy} + \frac{1}{2}u_x^2 u_{yy} = 0. \tag{5.2.16}$$

[1] Not too different from the two-dimensional Bateman equation (5.1.1) found by a similar analysis in Chapter 4 [47].

5.3 Fourth-order Ordinary Differential Equation and Partial Differential Equation

Following what transpired in §5.2 one becomes very curious as to what would happen if a similar analysis was applied to the fourth-order equation (5.3.17). We turn our attention to the task of finding what happens. In a similar way we can construct a second-order partial differential equation from a fourth-order ordinary differential equation, *videlicet*

$$y'''' = 0 \tag{5.3.17}$$

which admits the Lie point symmetries

$$\begin{array}{rlrl} \Phi_1 = & \partial x & \Phi_5 = & x\partial y \\ \Phi_2 = & \partial y & \Phi_6 = & x\partial x \\ \Phi_3 = & x^3\partial y & \Phi_7 = & y\partial y \\ \Phi_4 = & x^2\partial y & \Phi_8 = & x^2\partial x + 3xy\partial y. \end{array}$$

We observe that Φ_1, Φ_2, Φ_5 and Φ_6 and $\Delta_1, \Delta_2, \Delta_3$ and Δ_5 respectively are the same. Their consecutive application to

$$u_{xx} = f(x, y, u, u_x, u_y, u_{xy}, u_{yy}) \tag{5.3.18}$$

certainly, with suitable choice of characteristics when manouvering the process, produces

$$u_{xx} = u_x^2 h\left(u, \frac{u_{xy}}{u_x u_y}, \frac{u_{yy}}{u_y^2}\right). \tag{5.3.19}$$

All we need do is to apply the second extension of Φ_8 to equation (5.3.19). For ease of calculation we write $w = \dfrac{u_{xy}}{u_x u_y}$. The second extension of Φ_8 is given by

$$\begin{aligned} \Phi_8^{[2]} &= x^2\partial x + 3xy\partial y - \left[xu_x + \frac{3}{2}yu_y\partial u_y\right]\partial u_x \\ &- \frac{3}{2}xu_y\partial u_y - \left[u_x + 2xu_{xx} + 3yu_{xy}\right]\partial u_{xx} \\ &- \left[\frac{3}{2}u_y + \frac{3}{2}yu_{yy} + \frac{5}{2}xu_{xy}\right]\partial u_{xy} - 3xu_{yy}\partial u_{yy} \end{aligned} \tag{5.3.20}$$

and its application to (5.3.18) yields

$$\begin{aligned} u_x + 2xu_{xx} + 3yu_{xy} &= \left(xu_x + \frac{3}{2}yu_y\right)\frac{\partial f}{\partial u_x} + \frac{3}{2}xu_y\frac{\partial f}{\partial u_y} \\ &+ \left(\frac{3}{2}u_y + \frac{3}{2}yu_{yy} + \frac{5}{2}xu_{xy}\right)\frac{\partial f}{\partial u_{xy}} + 3xu_{yy}\frac{\partial f}{\partial u_{yy}} \end{aligned}$$

from which we obtain the following system of equations

$$\begin{array}{rcl} x & : & 2f & = & u_x\dfrac{\partial f}{\partial u_x}+\dfrac{3}{2}u_y\dfrac{\partial f}{\partial u_y}+\dfrac{5}{2}u_{xy}\dfrac{\partial f}{\partial u_{xy}}+3\dfrac{\partial f}{\partial u_{yy}}, \\ y & : & 3u_{xy} & = & \dfrac{3}{2}u_y\dfrac{\partial f}{\partial u_x}+\dfrac{3}{2}u_{yy}\dfrac{\partial f}{\partial u_{xy}}, \\ - & : & u_x & = & \dfrac{3}{2}u_y\dfrac{\partial f}{\partial u_{xy}}. \end{array}$$

Taking into account equation (5.3.19) we obtain from the third equation

$$\frac{\partial h}{\partial w}=\frac{2}{3}. \tag{5.3.21}$$

Using this we obtain from the second equation

$$h=\frac{4}{3}\frac{u_{xy}}{u_xu_y}-\frac{1}{3}\frac{u_{yy}}{u_y^2}. \tag{5.3.22}$$

The first equation gives the immediate satisfaction of the relationship. Consequently equation (5.3.19) takes the form

$$u_y^2u_{xx}-\frac{4}{3}u_xu_yu_{xy}+\frac{1}{3}u_x^2u_{yy}=0 \tag{5.3.23}$$

which seems to follow the pattern of the two-dimensional Bateman equation (5.1.1) found by a similar analysis in §5.2.

5.4 The General n^{th}-order equation and Partial Differential Equations

Seemingly one can proceed in a similar way to the previous sections to find a two-dimensional second-order partial differential equation from an n^{th}-order ordinary differential equation, $n \geq 3$,

$$y^{(n)}=0 \tag{5.4.24}$$

which admits in general these symmetries

$$\begin{array}{rclrcl} \Upsilon_1 & = & \partial x & \Upsilon_{n+1} & = & y\partial y \\ & \vdots & & \Upsilon_{n+2} & = & \partial y \\ \Upsilon_n & = & x^{n-1}\partial y & \Upsilon_{n+3} & = & x\partial x+\frac{1}{2}(n-1)y\partial y \\ & & & \Upsilon_{n+4} & = & x^2\partial x+(n-1)xy\partial y \end{array}$$

for $n \geq 3$. When we observe that Φ_1, Φ_2, Φ_5 and Φ_6 and $\Upsilon_1, \Upsilon_2, \Upsilon_3$ and Υ_4 respectively are the same, their consecutive application to

$$u_{xx} = f(x, y, u, u_x, u_y, u_{xy}, u_{yy}) \tag{5.4.25}$$

certainly, with suitable choice of characteristics when manoeuvering the process, produces

$$u_{xx} = u_x^2 h\left(u, \frac{u_{xy}}{u_x u_y}, \frac{u_{yy}}{u_y^2}\right). \tag{5.4.26}$$

All we need do is to apply the second extension of Υ_5 to equation (5.4.26). The consequent action of Υ_6 on (5.4.26) produces a much to be desired equation of the form

$$u_y^2 u_{xx} - \frac{n}{n-1} u_x u_y u_{xy} + \frac{1}{n-1} u_x^2 u_{yy} = 0. \tag{5.4.27}$$

Consequently we can conclude that, if an ordinary differential equation can be reduced to any form of $y^{(n)} = 0$, then it can be mapped to a second-order partial differential equation and it has the form (5.4.27). As a matter of interest we note that (5.4.27) can be written as

$$\begin{vmatrix} 0 & u_x & u_y \\ u_x & (n-1)u_{xx} & \frac{n}{2}u_{xy} \\ u_y & \frac{n}{2}u_{xy} & u_{yy} \end{vmatrix} = 0.$$

Furthermore the Lie point symmetries used in the construction of such equations come from the $sl(2, R)$ subalgebra plus the solution symmetry. Equations (5.3.23), (5.2.16) and (5.4.27) admit the following general Lie point symmetries

$$\begin{aligned} \Psi_1 &= \partial x & \Psi_4 &= x\partial x \\ \Psi_2 &= \partial y & \Psi_5 &= y\partial y \\ \Psi_3 &= \partial u & \Psi_6 &= u\partial u, \end{aligned}$$

from which we note that contrary to the two-dimensional Bateman equation which admits nine Lie point symmetries, *i.e.,* its point symmetries are preserved when moving from an ordinary differential equation (5.4.24) to partial differential equation (5.4.27), only four symmetries are preserved in higher-order equations. The nice thing to note is that the preserved symmetries, when applied to the general second-order partial differential equation, produce what we may call the basic structure of such partial differential equations, *videlicet*

$$u_{xx} = u_x^2 h\left(u, \frac{u_{xy}}{u_x u_y}, \frac{u_{yy}}{u_y^2}\right). \tag{5.4.28}$$

Moreover the discriminant (5.4.27) is given by

$$\Delta = (n-2)^2(u_x u_y)^2 \geq 0 \tag{5.4.29}$$

which immediately informs us that equation (5.4.27) is parabolic only if $n = 2$. Thereafter it becomes hyperbolic and is never be elliptic as $\Delta \geq 0$. Consequently one is not surprised by the number of point symmetries admitted by (5.4.27) for $n \geq 3$. In fact these Lie point symmetries are sufficient to give the basic structure (5.4.28) of (5.4.27) and one can use the *Choice* algorithm presented in §3.3 [43] to find the specific desired equation by finding and utilising the appropriate nonlocal symmetries.

5.5 Conclusion

We have seen two valuable outcomes in this Chapter. One is that the concept of complete symmetry group introduced by Krause in 1994, and the subsequent introduction of the condition of minimality of the complete symmetry group for a given equation or system by Andriopoulos *et al* have not taken into account that the same group can in fact specify different types of equation, in this case partial differential equations, (5.2.16) and (5.3.23), have been specified by Lie point symmetries of third- and fourth-order ordinary differential equations (5.2.6) and (5.3.17) in §5.2 and §5.3 [48]. We have also extended the notion of the application of Lie point symmetries admitted by ordinary differential equations to the construction of a second-order partial differential equation (5.4.27) from the point symmetries of any n^{th}-order ordinary differential equation (5.4.24) of maximal symmetry in §5.4. The equation immediately realised happened to be of Bateman-type but with coefficients involving the *order* of the ordinary differential equation from which the Lie point symmetries came. Specifically the order of an ordinary differential equation appears in the form of coefficients of the mixed derivative u_{xy} as $\dfrac{n}{n-1}$ and coefficient of one of the double derivatives u_{yy} as

$\left(\dfrac{1}{n-1}\right)$. Further to that we noted that, contrary to the Bateman equation (5.1.1) which is parabolic and admits a Lie algebra of dimension ninefold infinity [47], equation (5.4.27) is hyperbolic with six Lie point symmetries [48]. We have also noted that the Lie point symmetries of equation (5.4.27) are sufficient to construct a *basic* partial differential equation (5.4.28) for all equations of type (5.4.27) and the complete specification can be easily achieved by using the *Choice* algorithm described in §3.3 [43].

A further interesting observation to take from this Chapter is that the parabolic Bateman equation (5.1.1) is in fact a special case of a more general partial differential equation (5.4.27) for $n = 2$. As we saw in §5.4, equation (5.4.27) is hyperbolic with six Lie point symmetries for $n > 2$, but becomes parabolic with $9 \times \infty$ Lie point symmetries for $n = 2$. It would be interesting to see the if the solution to the Bateman equation (or its variation) can be used to determine the solution of the general equation from the n^{th}-order differential equation of type (5.4.27).

Chapter 6

The connection of systems of ordinary differential equations and partial differential equations

6.1 Introduction

We have seen in previous Chapters that there exists a connection between ordinary differential equations and partial differential equations through the common realisation of their complete symmetry groups. In particular we in §4.2 have shown that the Bateman equation,

$$u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = 0, \tag{6.1.1}$$

is equivalent to a system of ordinary differential equations

$$\begin{aligned} y'' &= 0 \\ u' &= 0 \end{aligned} \tag{6.1.2}$$

and surprisingly the two are specified by the exact same set of Lie point symmetries [47]. In fact there we used the Lie point symmetries of the free particle problem to construct (6.1.1) through complete symmetry group analysis[1]. We have further constructed in §5.2 and §5.3 a second-order partial

[1]One recalls that there is a connection between nonlinear partial differential equations that can be solved by the inverse scattering transform and nonlinear ordinary differential equations without movable critical points [1].

differential equations using Lie point symmetries admitted by third- and fourth-order ordinary differential equations, *videlicet*

$$y^{(3)} = 0 \tag{6.1.3}$$

and

$$y^{(4)} = 0, \tag{6.1.4}$$

which are

$$u_y^2 u_{xx} - \frac{3}{2} u_x u_y u_{xy} + \frac{1}{2} u_x^2 u_{yy} = 0 \tag{6.1.5}$$

and

$$u_y^2 u_{xx} - \frac{4}{3} u_x u_y u_{xy} + \frac{1}{3} u_x^2 u_{yy} = 0, \tag{6.1.6}$$

respectively [48]. We went further to construct a generalised nonlinear hyperbolic partial differential equation, *videlicet*

$$u_y^2 u_{xx} - \frac{n}{n-1} u_x u_y u_{xy} + \frac{1}{n-1} u_x^2 u_{yy} = 0 \tag{6.1.7}$$

from Lie point symmetries of a general higher-order equation of the form

$$y^{(n)} = 0. \tag{6.1.8}$$

In this Chapter we turn our attention to equation (6.1.1) and in particular we look at larger systems of ordinary differential equations

$$y_i'' = 0 \quad i = 1, n-2, \tag{6.1.9}$$

$$u' = 0, \tag{6.1.10}$$

to construct from their Lie point symmetries corresponding general partial differential equations. We later discuss the algebraic properties of the resulting equations in the context of a larger class of equations to which they belong. Some of the results observed in this Chapter have been presented in [49].

6.2 System of Second-order Ordinary Differential Equations and Partial Differential Equations

Following what transpired in Chapter 4 with the equivalence of a system of equations (6.1.2) to a nonlinear partial differential equation, commonly known as the Bateman equation (6.1.1), one becomes curious about the connection of other systems of ordinary differential equations and partial differential equations [47]. Consider the following system of second-order ordinary differential equations, *videlicet*

$$\begin{aligned} y'' &= 0 \\ z'' &= 0. \end{aligned} \tag{6.2.11}$$

The system above admits the following Lie point symmetries

$$\begin{aligned}
\Theta_1 &= \partial z & \Theta_9 &= x\partial y \\
\Theta_2 &= \partial y & \Theta_{10} &= y\partial y \\
\Theta_3 &= \partial x & \Theta_{11} &= z\partial x \\
\Theta_4 &= y\partial z & \Theta_{12} &= y\partial x \\
\Theta_5 &= x\partial z & \Theta_{13} &= x\left(x\partial x + z\partial z + y\partial y\right) \\
\Theta_6 &= z\partial z & \Theta_{14} &= y\left(y\partial y + z\partial z + x\partial x\right) \\
\Theta_7 &= z\partial y & \Theta_{15} &= z\left(z\partial z + y\partial y + x\partial x\right). \\
\Theta_8 &= x\partial x & &
\end{aligned}$$

One would be quick to note that in fact the structure of the Lie algebra of point symmetries is exactly the same as the Lie algebra of point symmetries admitted by higher-dimensional Bateman equation [50] given by[2]

$$\begin{aligned}
0 &= u_{x_1}^2\left(u_{x_2x_2}u_{x_3x_3} - u_{x_3x_2}^2\right) - 2u_{x_1}u_{x_2}\left(u_{x_1x_2}u_{x_3x_3} - u_{x_3x_1}u_{x_2x_3}\right) \\
&+ 2u_{x_1}u_{x_3}\left(u_{x_2x_1}u_{x_3x_2} - u_{x_3x_1}u_{x_2x_2}\right) - 2u_{x_2}u_{x_3}\left(u_{x_1x_1}u_{x_2x_3} - u_{x_3x_1}u_{x_1x_2}\right) \\
&- u_{x_2}^2\left(u_{x_1x_1}u_{x_3x_3} - u_{x_3x_1}^2\right) + u_{x_3}^2\left(u_{x_1x_1}u_{x_2x_2} - u_{x_1x_2}^2\right),
\end{aligned} \tag{6.2.12}$$

where $x_1 = x$, $x_2 = y$ and $x_3 = z$.

[2]The calculation of point symmetries and the Lie Bracket has been made readily possible by one of the classic codes which was developed in [16].

They are

$$\begin{array}{rclrcl}
\Gamma_1 & = & F_1(u)\partial z & \Gamma_9 & = & F_9(u)x\partial y \\
\Gamma_2 & = & F_2(u)\partial y & \Gamma_{10} & = & F_{10}(u)y\partial y \\
\Gamma_3 & = & F_3(u)\partial x & \Gamma_{11} & = & F_{11}(u)z\partial x \\
\Gamma_4 & = & F_4(u)\partial u & \Gamma_{12} & = & F_{12}(u)y\partial x \\
\Gamma_5 & = & F_5(u)y\partial z & \Gamma_{13} & = & F_{13}(u)x\partial x \\
\Gamma_6 & = & F_6(u)x\partial z & \Gamma_{14} & = & F_{14}(u)y\,(y\partial y + z\partial z + x\partial x) \\
\Gamma_7 & = & F_7(u)z\partial z & \Gamma_{15} & = & F_{15}(u)x\,(x\partial x + z\partial z + y\partial y) \\
\Gamma_8 & = & F_8(u)z\partial y & \Gamma_{16} & = & F_{16}(u)z\,(z\partial z + y\partial y + x\partial x)
\end{array}$$

which is a $(4)^2$-dimensional, $\{A_1 \bigoplus_s sl(4,R)\}$ algebra. As was observed in Chapter 4, the algebra of equation (6.2.12) is in fact equivalent (in structure) to that of a system of ordinary differential equations, *videlicet*

$$\begin{aligned} y'' &= 0 \\ z'' &= 0 \\ u' &= 0, \end{aligned} \qquad (6.2.13)$$

and the same questions that arose in the previous section can be asked for this sort of behaviour. Curiously the system (6.2.13) is specifiable using the point symmetries, $\Theta_1, \Theta_2, \Theta_3, \Theta_{10}, \Theta_7$ and Θ_{13}, or by using the corresponding Γs. This is achieved by applying Θ_1, Θ_2 and Θ_3 to the general system of ordinary differential equations, *videlicet*

$$\begin{aligned} y'' &= f(x,y,z,u,y',z') \\ z'' &= g(x,y,z,u,y',z') \\ u' &= h(x,y,z,u,y',z'), \end{aligned} \qquad (6.2.14)$$

to obtain

$$\begin{aligned} y'' &= f(u,y',z') \\ z'' &= g(u,y',z') \\ u' &= h(u,y',z'). \end{aligned} \qquad (6.2.15)$$

The application of $\Theta_{10} = y\partial y$ to (6.2.15) produces

$$\begin{aligned} y'' &= y'F(u,z') \\ z'' &= G(u,z') \\ u' &= H(u,z'). \end{aligned} \qquad (6.2.16)$$

Similarly the application of $\Theta_6 = z\partial z$ to (6.2.16) yields

$$\begin{aligned} y'' &= y'F_1(u) \\ z'' &= z'G_1(u) \\ u' &= H_1(u). \end{aligned} \tag{6.2.17}$$

The second extension of Θ_{13} is given by

$$\begin{aligned} \Theta_{13}^{[2]} &= x^2\partial x + xy\partial y + xz\partial z + (y - xy')\,\partial y' \\ &\quad + (z - xz')\,\partial z' - 3xy''\partial y'' - 3xz''\partial z''. \end{aligned}$$

Its application to (6.2.22) produces for the top equation,

$$-3xy'' = (y - xy')F_1(u), \tag{6.2.18}$$

for the middle equation

$$-3xz'' = (z - xz')H_1(u) \tag{6.2.19}$$

and for the last equation,

$$0 = G_1(u). \tag{6.2.20}$$

Separating coefficients of x, y and z in (6.2.18), (6.2.19) and (6.2.20) we obtain for x,

$$\begin{aligned} -3y'' &= y'F_1(u) \\ -3z'' &= z'G_1(u) \\ 0 &= H_1(u), \end{aligned} \tag{6.2.21}$$

and for y and z

$$\begin{aligned} 0 &= yF_1(u) \\ 0 &= zG_1(u) \\ 0 &= H_1(u), \end{aligned} \tag{6.2.22}$$

and consequently we obtain (6.2.13) after solving the resulting systems of equations. It is useful to observe the Lie Brackets in Appendix A in order to write the complete symmetry group of (6.2.13) which has closure. One immediately observes that in order for the set of point symmetries, $\Theta_1, \Theta_2, \Theta_3, \Theta_{10}, \Theta_6$ and Θ_{13}, to close in terms of the Lie Bracket one need add the Lie point symmetries, Θ_5, Θ_8 and Θ_9. It is not surprising to observe that the same set of Lie point symmetries,

$$\{\Theta_1, \Theta_2, \Theta_3, \Theta_5, \Theta_6, \Theta_8, \Theta_9, \Theta_{10}, \Theta_{13}\},$$

that specified the system of ordinary differential equations (6.2.13) indeed specifies the partial differential equation (6.2.12) with $x_1 = x, x_2 = y$ and $x_3 = z$. A similar phenomenon was observed in Chapter 4 with the Bateman equation, (6.1.1), and a system of ordinary differential equations, (6.2.13).

6.3 Family of Zero Determinant Equations

Following what transpired in Chapter 4 regarding the equivalence of the Bateman equation (6.1.1) to a system of ordinary differential equations (6.2.13), one would not be completing the picture if equation (6.1.1) is not examined and discussed in a broader context of the family to which it belongs and the equivalent systems therein. This enables one to make educated inferences and possibly to provide a way forward to the future encounter of such equations. Equation (6.1.1) is in fact the zero determinant of an 3×3 Bateman matrix given by

$$B_3 = \begin{bmatrix} 0 & u_{x_1} & u_{x_2} \\ u_{x_1} & u_{x_1x_1} & u_{x_1x_2} \\ u_{x_2} & u_{x_2x_1} & u_{x_2x_2} \end{bmatrix},$$

that is[3],

$$|B_3| = 0 = u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}, \tag{6.3.23}$$

and this equation is equivalent[4] to the system of ordinary differential equations,

$$\begin{aligned} y'' &= 0 \\ u' &= 0,. \end{aligned} \tag{6.3.24}$$

[3]Where we have used the notation $x_{(i)}$, $i = 1, \ldots, n$ instead of $x, y, z, \ldots$ in order to enable generalisation later in this section.

[4]Admit the same set of Lie point symmetries upon specifying that all the arbitrary functions to be one.

Surprisingly the two are specified by the exact same set of Lie point symmetries. Similarly equation (6.2.12) is the zero determinant of a 4×4 Bateman matrix given by

$$B_4 = \begin{bmatrix} 0 & u_{x_1} & u_{x_2} & u_{x_3} \\ u_{x_1} & u_{x_1x_1} & u_{x_1x_2} & u_{x_1x_3} \\ u_{x_2} & u_{x_2x_1} & u_{x_2x_2} & u_{x_2x_3} \\ u_{x_3} & u_{x_3x_1} & u_{x_3x_2} & u_{x_3x_3} \end{bmatrix},$$

that is,

$$\begin{aligned} |B_4| = 0 \quad = \quad & u_{x_1}^2 \left(u_{x_2x_2}u_{x_3x_3} - u_{x_3x_2}^2\right) - 2u_{x_1}u_{x_2}\left(u_{x_1x_2}u_{x_3x_3} - u_{x_3x_1}u_{x_2x_3}\right) \\ + \quad & 2u_{x_1}u_{x_3}\left(u_{x_2x_1}u_{x_3x_2} - u_{x_3x_1}u_{x_2x_2}\right) - 2u_{x_2}u_{x_3}\left(u_{x_1x_1}u_{x_2x_3} - u_{x_3x_1}u_{x_1x_2}\right) \\ - \quad & u_{x_2}^2\left(u_{x_1x_1}u_{x_3x_3} - u_{x_3x_1}^2\right) + u_{x_3}^2\left(u_{x_1x_1}u_{x_2x_2} - u_{x_1x_2}^2\right). \end{aligned} \tag{6.3.25}$$

Unexpectedly (6.3.25) is algebraically equivalent to a system of ordinary differential equations, *videlicet*

$$\begin{aligned} y'' &= 0 \\ z'' &= 0 \\ u' &= 0. \end{aligned} \tag{6.3.26}$$

Proceeding in a similar fashion one immediately finds the Bateman equation obtained from a zero determinant of an $(n+1) \times (n+1)$ Bateman matrix that admits the same Lie point symmetries as a system of equations given by

$$y_i'' = 0, \qquad i = 1, n-2, \tag{6.3.27}$$

$$u' = 0, \tag{6.3.28}$$

where there are $n-2$ second-order ordinary differential equations (6.3.27) coupled with one first-order equation (6.3.28) corresponding to the more general

$$|B_{(n+1)}| = 0, \tag{6.3.29}$$

where the $(n+1)\times(n+1)$ Bateman matrix $B_{(n+1)}$ is given by

$$B_{(n+1)} = \begin{bmatrix} 0 & u_{x_1} & u_{x_2} & u_{x_3} & \cdots & u_{x_n} \\ u_{x_1} & u_{x_1x_1} & u_{x_1x_2} & u_{x_1x_3} & \cdots & u_{x_1x_n} \\ u_{x_2} & u_{x_2x_1} & u_{x_2x_2} & u_{x_2x_3} & \cdots & u_{x_2x_n} \\ u_{x_3} & u_{x_3x_1} & u_{x_3x_2} & u_{x_3x_3} & \cdots & u_{x_3x_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{x_n} & u_{x_nx_1} & u_{x_nx_2} & u_{x_nx_3} & \cdots & u_{x_nx_n} \end{bmatrix}.$$

The Bateman matrix, $B_{(n+1)}$, was proposed by Fairlie [19]. The general solution of a $(1+n)$-dimensional Bateman equation is given by

$$\sum_{i=1}^{n} x_i\phi_i(u) = C, \tag{6.3.30}$$

where C is an arbitrary constant and $\phi_i(u)$, $i = 1, n$, are n arbitrary smooth functions. For $n \geq 3$ the singularity manifold condition for the n-dimensional generalisation of the sine-Gordon, Liouville and Mikhailov equations is given by the n-dimensional Bateman equation [18]. Note that the number of point symmetries of the Bateman equation in (6.1.1) is 9 which is 3^2 and such equations are called *minor Bateman* equations [47]. The Bateman equation, (6.1.1), is only a special solution of the polynomial field theory equations which were studied in two dimensions only .

As a matter of curiosity we look at what happens if we evaluate the Lie point symmetries of

$$|B_{(n+1)}| = u_t. \tag{6.3.31}$$

For $n = 2$ the equation admits the nine Lie point symmetries

$$\begin{array}{lcllcl} G_1 & = & \partial y & G_6 & = & u\partial u + x\partial x \\ G_2 & = & \partial x & G_7 & = & 2t\partial t - u\partial u \\ G_3 & = & u\partial u + y\partial y & G_8 & = & \partial t \\ G_4 & = & y\partial x & G_9 & = & \partial u. \\ G_5 & = & x\partial y & & & \end{array}$$

We quickly note here that we have lost three symmetries, but we have gained another three. What is even more evident is that the arbitrary functions of u have disappeared completely. The point symmetries that are modified are those involving diagonal terms. In fact they get $u\partial u$ added to them. For $n = 3$ the picture changes slightly as we obtain

$$\begin{array}{lcllcl}
\Gamma_1 & = & \partial z & \Gamma_9 & = & x\partial y \\
\Gamma_2 & = & \partial y & \Gamma_{10} & = & z\partial x \\
\Gamma_3 & = & \partial x & \Gamma_{11} & = & y\partial x \\
\Gamma_4 & = & \partial u & \Gamma_{12} & = & 3t\partial t - u\partial u \\
\Gamma_5 & = & y\partial z & \Gamma_{13} & = & 2u\partial u + 3z\partial z \\
\Gamma_6 & = & x\partial z & \Gamma_{14} & = & 2u\partial u + 3y\partial y \\
\Gamma_7 & = & \partial t & \Gamma_{15} & = & 2u\partial u + 3x\partial x, \\
\Gamma_8 & = & z\partial y & & &
\end{array}$$

i.e., the number of point symmetries is reduced by one compared to point symmetries of (6.2.12) as we loose all diagonal point symmetries to the new point symmetries involving $u\partial u$. Curiously the off-diagonal point symmetries remain unchanged!

6.4 Systems of Higher-order Ordinary Differential Equations and Partial Differential Equations

One becomes naturally curious about the connection between higher-order systems of ordinary and partial differential equations following what happened in §5.4 regarding the construction of a generalised hyperbolic partial differential equation, *videlicet*

$$u_y^2 u_{xx} - \frac{n}{n-1} u_x u_y u_{xy} + \frac{1}{n-1} u_x^2 u_{yy} = 0, \qquad (6.4.32)$$

from the Lie point symmetries of a general higher-order equation of the form

$$y^{(n)} = 0. \qquad (6.4.33)$$

The curiosity comes in light of what has transpired in the previous Sections of this Chapter with the equivalence of a system second-order ordinary differential equations to a second-order partial differential equation. Indeed we have seen the equivalence of a linear ordinary differential equation of maximal symmetry of any order to the second-order partial differential equation. We turn our attention to construct a three-dimensional second-order partial differential equation from the Lie point symmetries of a system of higher-order ordinary differential equations. Consider the system of two third-order equations,

$$\begin{aligned} y''' &= 0 \\ z''' &= 0, \end{aligned} \tag{6.4.34}$$

which admit the following Lie point symmetries

$$\begin{aligned} \Sigma_1 &= \partial z & \Sigma_9 &= x\partial y \\ \Sigma_2 &= \partial y & \Sigma_{10} &= y\partial y \\ \Sigma_3 &= \partial x & \Sigma_{11} &= x^2\partial z \\ \Sigma_4 &= y\partial z & \Sigma_{12} &= x^2\partial y \\ \Sigma_5 &= x\partial z & \Sigma_{13} &= x\left(x\partial x + 2z\partial z + 2y\partial y\right). \\ \Sigma_6 &= z\partial z \\ \Sigma_7 &= z\partial y \\ \Sigma_8 &= x\partial x \end{aligned}$$

The successive application of the set of Lie point symmetries,

$$\{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_5, \Sigma_6, \Sigma_8, \Sigma_9, \Sigma_{10}, \Sigma_{13}\},$$

to the general three-dimensional second-order partial differential equation, *videlicet*

$$F(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{xy}, u_{xz}, u_{yz}, u_{yy}, u_{zz}) = 0, \tag{6.4.35}$$

renders it as

$$\begin{aligned} 0 &= u_{x_1}^2\left(2u_{x_2x_2}u_{x_3x_3} - \frac{9}{4}u_{x_3x_2}^2\right) - 3u_{x_1}u_{x_2}\left(u_{x_1x_2}u_{x_3x_3} - \frac{3}{2}u_{x_3x_1}u_{x_2x_3}\right) \\ &+ 3u_{x_1}u_{x_3}\left(u_{x_2x_1}u_{x_3x_2} - \frac{3}{2}u_{x_3x_1}u_{x_2x_2}\right) - 3u_{x_2}u_{x_3}\left(u_{x_1x_1}u_{x_2x_3} - \frac{3}{2}u_{x_3x_1}u_{x_1x_2}\right) \\ &- u_{x_2}^2\left(2u_{x_1x_1}u_{x_3x_3} - \frac{9}{4}u_{x_3x_1}^2\right) + u_{x_3}^2\left(2u_{x_1x_1}u_{x_2x_2} - \frac{9}{4}u_{x_1x_2}^2\right), \end{aligned} \tag{6.4.36}$$

in which the fractions have purposely been left unsimplified. Similarly the system of two fourth-order ordinary differential equations, *videlicet*

$$\begin{aligned} y'''' &= 0 \\ z'''' &= 0, \end{aligned} \tag{6.4.37}$$

admits the Lie point symmetries

$$\begin{array}{rclrcl}
\Omega_1 &=& \partial z & \Omega_9 &=& x\partial y\\
\Omega_2 &=& \partial y & \Omega_{10} &=& y\partial y\\
\Omega_3 &=& \partial x & \Omega_{11} &=& x^2\partial z\\
\Omega_4 &=& y\partial z & \Omega_{12} &=& x^2\partial y\\
\Omega_5 &=& x\partial z & \Omega_{13} &=& x^3\partial z\\
\Omega_6 &=& z\partial z & \Omega_{14} &=& x^3\partial y\\
\Omega_7 &=& z\partial y & \Omega_{15} &=& x\left(x\partial x + 3z\partial z + 3y\partial y\right).\\
\Omega_8 &=& x\partial x & & &
\end{array}$$

The successive application of the set of Lie point symmetries,

$$\{\Omega_1, \Omega_2, \Omega_3, \Omega_5, \Omega_6, \Omega_8, \Omega_9, \Omega_{10}, \Omega_{15}\},$$

to the general second-order partial differential equation (6.4.35) produces

$$\begin{aligned}
0 &= u_{x_1}^2\left(3u_{x_2x_2}u_{x_3x_3} - \frac{16}{4}u_{x_3x_2}^2\right) - 4u_{x_1}u_{x_2}\left(u_{x_1x_2}u_{x_3x_3} - \frac{4}{2}u_{x_3x_1}u_{x_2x_3}\right)\\
&+ 4u_{x_1}u_{x_3}\left(u_{x_2x_1}u_{x_3x_2} - \frac{4}{2}u_{x_3x_1}u_{x_2x_2}\right) - 4u_{x_2}u_{x_3}\left(u_{x_1x_1}u_{x_2x_3} - \frac{4}{2}u_{x_3x_1}u_{x_1x_2}\right)\\
&- u_{x_2}^2\left(3u_{x_1x_1}u_{x_3x_3} - \frac{16}{4}u_{x_3x_1}^2\right) + u_{x_3}^2\left(3u_{x_1x_1}u_{x_2x_2} - \frac{16}{4}u_{x_1x_2}^2\right). \qquad (6.4.38)
\end{aligned}$$

It should by now be very clear to the reader that by proceeding in a similar fashion one would immediately obtain for any system of two n^{th}-order ordinary differential equations,

$$\begin{aligned}
y^{(n)} &= 0\\
z^{(n)} &= 0, \qquad (6.4.39)
\end{aligned}$$

the Lie point symmetries

$$\begin{array}{rclrcl}
\Xi_1 &=& \partial z & \Xi_8 &=& y\partial y\\
\Xi_2 &=& \partial y & \Xi_9 &=& x\partial x\\
\Xi_3 &=& \partial x & \Xi_{10} &=& x\partial y\\
\Xi_4 &=& y\partial z & \Xi_{11} &=& (x^2 + x^3 + \cdots + x^{n-1})\partial z\\
\Xi_5 &=& z\partial y & \Xi_{12} &=& (x^2 + x^3 + \cdots + x^{n-1})\partial y\\
\Xi_6 &=& z\partial z & \Xi_{13} &=& x\left(x\partial x + (n-1)z\partial z + (n-1)y\partial y\right),\\
\Xi_7 &=& x\partial z & & &
\end{array}$$

where Ξ_{11} and Ξ_{12} are $n-2$ solution symmetries. Following an analysis similar to the above one immediately obtains from the application of the set of Lie point symmetries

$$\{\Xi_1, \Xi_2, \Xi_3, \Xi_6, \Xi_7, \Xi_8, \Xi_9, \Xi_{10}, \Xi_{13}\}$$

to the general second-order partial differential equation, (6.4.35), a general three-dimensional second-order partial differential equation, *videlicet*

$$\begin{aligned} 0 &= u_{x_1}^2\left((n-1)u_{x_2x_2}u_{x_3x_3} - \frac{n^2}{4}u_{x_3x_2}^2\right) - nu_{x_1}u_{x_2}\left(u_{x_1x_2}u_{x_3x_3} - \frac{n}{2}u_{x_3x_1}u_{x_2x_3}\right) \\ &+ nu_{x_1}u_{x_3}\left(u_{x_2x_1}u_{x_3x_2} - \frac{n}{2}u_{x_3x_1}u_{x_2x_2}\right) - nu_{x_2}u_{x_3}\left(u_{x_1x_1}u_{x_2x_3} - \frac{n}{2}u_{x_3x_1}u_{x_1x_2}\right) \\ &- u_{x_2}^2\left((n-1)u_{x_1x_1}u_{x_3x_3} - \frac{n^2}{4}u_{x_3x_1}^2\right) + u_{x_3}^2\left((n-1)u_{x_1x_1}u_{x_2x_2} - \frac{n^2}{4}u_{x_1x_2}^2\right). \end{aligned} \tag{6.4.40}$$

Equation (6.4.40) can be written in a more readable, 4×4, zero determinant format, as

$$0 = \begin{vmatrix} 0 & u_{x_1} & u_{x_2} & u_{x_3} \\ u_{x_1} & (n-1)u_{x_1x_1} & -\frac{n}{2}u_{x_1x_2} & -\frac{n}{2}u_{x_1x_3} \\ u_{x_2} & \frac{n}{2}u_{x_2x_1} & (n-1)u_{x_2x_2} & \frac{n}{2}u_{x_2x_3} \\ u_{x_3} & -\frac{n}{2}u_{x_3x_1} & -\frac{n}{2}u_{x_3x_2} & u_{x_3x_3} \end{vmatrix}.$$

We note that in all the above considerations the off-diagonal Lie point symmetries involving the independent variables of the system of ordinary differential equations do not come to the party of specifying the partial differential equation[5]. More interestingly these symmetries are not even necessary to close the algebra as all the above sets form closed algebras. Furthermore the solution symmetries involving higher powers of x are also not necessary either to close the algebra or to specify the partial differential equation.

As was observed in §5.2 regarding the number of point symmetries admitted by equation (5.4.27) for any $n\geq 3$, equation (6.4.40) admits a fewer number of

[5]This has a parallel in the complete specification of the Kepler Problem. The three symmetries comprising of $so(3)$ - so relevant to the physical explanation of angular momentum - play no role in the complete symmetry group.

point symmetries for any $n \geq 3$ as opposed to the system of partial differential equation from which it was obtained and these are given by

$$\begin{array}{rclcrcl} \Phi_1 & = & \partial z & \quad & \Phi_5 & = & z\partial z \\ \Phi_2 & = & \partial y & & \Phi_6 & = & y\partial y \\ \Phi_3 & = & \partial x & & \Phi_7 & = & x\partial x \\ \Phi_4 & = & \partial u & & \Phi_8 & = & u\partial u. \end{array}$$

The above point symmetries comprise the algebra $4A_1 \bigoplus_s 4A_1$. We further note that equation (6.4.36) is a special case of (6.4.40) for $n = 2$. It goes without saying that to specify such equations requires the services of the nonlocal symmetries and possibly the $Choice$ algorithm would be very effective in the search of the much-required nonlocal symmetries to specify the equation.

6.5 Conclusion

In this Chapter we have observed that the number of Lie point symmetries admitted by equations of the Bateman family is n^2, $n \geq 3$, comprising four classes of Lie point symmetries, *i.e.,* n abelian symmetries of the type $\partial x_i,\ i = 1, n-1$, the $(n-1)^2$ solution symmetries of type $x_i\partial x_j,\ i,j = n, n^2 - n$, the $(n-1)$ abelian symmetries of type $x_i x_j \partial x_j\ i,j = n^2-n+1, n^2$, and, as one would expect, ∂u, when arbitrary functions are specified and symmetries arranged in the order given here. Furthermore it transpires that each equation resulting from the zero determinant of a particular $n \times n$ Bateman matrix admits the same Lie point symmetries as a corresponding system of ordinary differential equations given by

$$y_i'' = 0, \qquad i = 1, n-2, \tag{6.5.41}$$

$$u' = 0, \tag{6.5.42}$$

where there are $n-2$ second-order ordinary differential equations (6.5.41) coupled with one first-order equation (6.5.42) and in some cases the same set of point symmetries specifying the zero-determinant Bateman matrix partial differential equation specifies the corresponding system of ordinary differential equations. This can be verified by evaluating the symmetries of $|\mathrm{B_{n+1}}| = 0$ for $n \geq 3$ then using the symmetries to specify the required equations.

We have also seen the extension of the results regarding the construction a generalised hyperbolic partial differential equation (5.4.27) from an n^{th}-order ordinary differential equation (6.4.33) to take into account the construction of a general hyperbolic partial differential equation (6.4.40) from the systems of two ordinary differential equations, (6.4.39), of any order.

Another interesting conclusion one draws from the above considerations is that one needs to extend the definition of complete symmetry group to take into account the type of equation for which complete symmetry group is analysed as the same set of point symmetries may specify a partial differential equation, a system of ordinary differential equations or something else! We devote the next Chapter to some observations of the behaviour of complete symmetry group in order that we may be able to extend the definition of complete symmetry groups to mean exactly that to which we refer.

Part III

The 'The Evolution and Revolution'

Preface

The following Chapter is the concluding part of this work in which we revisit the notion of complete symmetry groups and further extend it so that it refers to those groups that uniquely specify classes of equations or systems, based on some recent developments observed in the preceding Chapters pertaining to the properties and the behaviour of such groups in differential equations, particularly their representations and realisations. The observations presented in the following have been largely influenced by a lunch meeting with Peter Leach in the School of Mathematical Science's Common Room the intentions of which had nothing to do with evolutionising and revolutionising the concept of complete symmetry groups.

Chapter 7

Complete Symmetry Groups: The Difference Between Realisation and Representation

7.1 Introduction

For the benefit of the self-containment of this Chapter and the better understanding of the matters arising in its Conclusion it would be useful for us to revisit the history behind the introduction of complete symmetry groups to differential equations.

The complete symmetry group of a differential equation, equally a system of differential equations, was introduced by Krause [23, 24] to describe the group which in its algebraic representation completely specified the differential equation under consideration. The differential equation in this instance was that of the Kepler Problem for which Krause found it necessary to introduce nonlocal symmetries since the Lie point symmetries of the system of ordinary differential equations describing the Kepler Problem are insufficient to specify the system completely. Indeed the symmetries corresponding to the angular momentum are not even included in the representation of the complete symmetry group. The reason for this is found in the concept of minimality of representation as an integral part of the definition of a complete symmetry group. Although nonlocal symmetries feature in a number of studies of the complete symmetry groups of certain problems [26, 27, 28, 55], this has been more an accident of the development of the study of complete symmetry groups and the theoretical treatment and applications in terms of point symmetries has been well established [6, 7, 8].

Krause's concept of a complete symmetry group of a differential equation (equally a system) was that the group associated with the set of symmetries, be they point, contact, generalised or nonlocal, required to specify the equation completely when applied to the arbitrary function describing the general structure of the equation. Specifically he required, as was prescribed in his paper, that a realisation of a complete symmetry group must be endowed with the following properties

a. the group acts freely and transitively on the manifold of all permitted motions of the system (i.e., the manifold of solutions is a homogeneous space of the group); and

b. the given equations of motion are the only ordinary differential equations that remain invariant under the specified action of the group, i.e., the group be specific to the differential equation or system and no other differential equation or system must admit it.

This means that every system of differential equations can be entirely characterised by the symmetry laws it obeys, *i.e.*, different systems cannot have exactly the same representation of symmetry properties. If the expression be same, the equation is the same. In this logic the group of symmetries of a given system is said to be *complete*. This notion of complete symmetry group has been extended to functions, first integrals [6] and partial differential equations [41], also in §3.3.

In this Chapter we turn our attention to examine in detail the applicability and the suitability of the above definition to differential equations. Following what transpired in preceding Chapters and also in [51] with regards to ordinary differential equations and partial differential equations, we look closely at the sufficiency of the above definition of complete symmetry groups towards yielding a group which uniquely specifies a given equation in real applications using Lie point symmetries with the view to refine the definition where necessary. We omit the discussions involving nonlocal symmetries as these are as many as grains of the sea sand, but the mechanism of their determination may require more than a certain amount of ingenuity particularly for partial differential equations [42, 43, 46].

7.2 Representations

7.2.1 Representations: Ordinary Differential Equations

The usual subterfuge, in which one seeks a complete symmetry group of a given n^{th}-order ordinary differential equation, is that of starting at the general n^{th}-order equation and applying the pertinent extensions of the symmetries possessed by the original equation until one recovers the n^{th}-order differential equation. For ordinary differential equations some interesting results have been observed [6, 50] in terms of *representation* of the complete symmetry group of a particular ordinary differential equation. To illustrate this in the context of Lie point symmetries and in the simplest possible manner so that the idea is not full of twists and turns we consider the ordinary differential equation which is the emblematic representation of all linear and linearisable second-order equations:

$$y'' = 0 \tag{7.2.1}$$

which possesses the following Lie point symmetries

$$\begin{array}{rclcrcl} G_1 &=& \partial_y && G_5 &=& x\partial_x \\ G_2 &=& x\partial_y && G_6 &=& x^2\partial_x + xy\partial_y \\ G_3 &=& y\partial_y && G_7 &=& y\partial_x \\ G_4 &=& \partial_x && G_8 &=& xy\partial_x + y^2\partial_y. \end{array}$$

The intent is to demonstrate how many of the symmetries listed above are required to specify (7.2.1) completely. Firstly we consider the general second-order ordinary differential equation

$$y'' = f(x, y, y'), \tag{7.2.2}$$

where f is as an yet unknown function to be specified.

We have seen in §3.2 that G_3, G_4, G_5 and G_6 specify $y'' = 0$ completely. The four elements form the algebra

$$A_1 \oplus sl(2; R).$$

It was further discovered [6] that the sets of point symmetries G_3, G_7 and G_8, equally G_1, G_2 and G_3, were also sufficient to specify (7.2.1) completely.

By way of example take G_1, G_2 and G_3 and apply them to (7.2.2). The application of G_1 to (7.2.2) yields

$$y'' = g(x, y'). \tag{7.2.3}$$

The application of G_3 to (7.2.3) produces

$$y'' = y'h(x). \tag{7.2.4}$$

Lastly the application of G_2 to (7.2.4) yields the desired equation (7.2.1).

We have seen that the two different sets of symmetries do specify (7.2.1). Furthermore these sets have a common property in that their point symmetries form an $A_{3,3}$ algebra. It was then concluded that not only is it possible to have multiple **representations** of the same group that uniquely specifies the ordinary differential equation but also that one can have different groups specifying the same ordinary differential equation as we have seen above. It was then proposed [6] that, when one talks about complete symmetry group of an ordinary differential equation, one should mean the group of

c. *minimal dimension,*

that is, the complete symmetry group of an ordinary differential equation is the group of the minimal number of symmetries required to specify the equation.

All that being well and good for complete symmetry groups of ordinary differential equations, a question still remains. What about the *representation* of the complete symmetry group of a partial differential equation? Can we expect to find properties similar to the ones observed for ordinary differential equations? In a study of the complete symmetry group of the $1+1$ heat equation in §3.3 we have shown that the number of Lie point symmetries required to specify the $1+1$ heat equation is six. The classical heat equation, as a linear partial differential equation, possesses an infinite number of Lie point symmetries and is completely characterised by them[1].

7.2.2 Representations: Partial Differential Equations

In this Section we turn our attention to the representations of complete symmetry groups of partial differential equations. As we mentioned above, we have observed different representations of the complete symmetry group of ordinary differential equations. The natural question would be: Is this the case for partial differential equations? We investigate below.

Consider the following nine-dimensional Lie group of symmetries

$$\begin{array}{rclrcl}\Lambda_1 &=& \partial x & \Lambda_5 &=& y\partial y,\\ \Lambda_2 &=& \partial y & \Lambda_6 &=& u\partial u,\\ \Lambda_3 &=& \partial u & \Lambda_7 &=& u\partial y,\\ \Lambda_4 &=& x\partial y & \Lambda_8 &=& x\partial x.\\ \Lambda_9 &=& y\partial u & & & \end{array}$$

This group of Lie point symmetries admits five distinct eight-dimensional subgroups. The application of the above Lie point symmetries with the omission of, say, Λ_3, to the general second-order partial differential equation with two independent variables and one dependent variable, *videlicet*

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}) = 0, \tag{7.2.5}$$

in the usual way [41, 50] produces the following partial differential equation

$$u_{xx}u_{yy} - u_{xy}^2 = 0, \tag{7.2.6}$$

which is of Monge-Ampère type. Equation (7.2.6), and in fact the whole family of Monge-Ampère equations, was examined in [56] and was shown to be Lie remarkable. The equation admits the following $sl(4,R)$ Lie point symmetries

[1]Such equations are termed Lie remarkable, a concept introduced by Oliveri *et al* [56], which refers to the property that the number of Lie point symmetries of the given equation equals the number required to specify it.

$$
\begin{array}{rclrcl}
\Theta_1 & = & \partial u & \Theta_9 & = & x\partial y \\
\Theta_2 & = & \partial y & \Theta_{10} & = & y\partial y \\
\Theta_3 & = & \partial x & \Theta_{11} & = & u\partial x \\
\Theta_4 & = & y\partial u & \Theta_{12} & = & y\partial x \\
\Theta_5 & = & x\partial u & \Theta_{13} & = & x\left(x\partial x + u\partial u + y\partial y\right) \\
\Theta_6 & = & u\partial u & \Theta_{14} & = & y\left(y\partial y + u\partial u + x\partial x\right) \\
\Theta_7 & = & u\partial y & \Theta_{15} & = & u\left(u\partial u + y\partial y + x\partial x\right) \\
\Theta_8 & = & x\partial x, & & &
\end{array}
$$

which is similar to the group above but with additional Lie point symmetries. Thus the group of point symmetries Λ_i is just a subgroup of the symmetries Θ_i and consequently equation (7.2.6) is said to be (strongly) Lie remarkable[2].

It is interesting to observe that the addition of the omitted point symmetry, Λ_3, produces no additional information. Similarly should we choose instead to omit say Λ_4 and follow the same analysis we obtain (7.2.6). In fact we can further choose to omit Λ_5 or Λ_8 and we realise the same equation (7.2.6). Similarly with these subgroups the inclusion of each of the omitted point symmetry produces no additional information and this is true for most equations of Monge-Ampère type.

7.2.3 Comment

We have observed that, parallel to the different representations of complete symmetry groups for ordinary differential equations, one can also have different representations of the complete symmetry group for the same partial differential equation. In this case all the sets we have chosen equally consist of eight Lie point symmetries which of course is the minimal required number of symmetries to specify a second-order partial differential equation of the type of (7.2.5) [42, 43, 44, 46]. Thus a question of different *representations* of complete symmetry groups for the equation, or equally a system, is a reality in both types of differential equation considered in this Section. However, we have yet to experience the unexpected with realisations.

[2]The property that the equation is completely characterised by a subset of the point symmetry it admits [34].

7.3 Realisations

We have seen that there can be multiple *representations* of the complete symmetry group of a given differential equation. In this Section we investigate, using observed results, the question whether the same can be said about *realisations*, that is having the same group of Lie point symmetries specifying multiple distinct equations.

7.3.1 Different-Types Same-Order Differential Equations

In Chapter 4 we observed the equivalence of a system of second-order ordinary differential equations,[3] *videlicet*

$$\begin{aligned} y'' &= 0 \\ u' &= 0, \end{aligned} \tag{7.3.7}$$

to a nonlinear second-order Bateman partial differential equation. The following set

$$\begin{aligned} \Upsilon_1 &= \partial y \\ \Upsilon_2 &= x\partial y \\ \Upsilon_3 &= y\partial y \\ \Upsilon_4 &= \partial x \\ \Upsilon_6 &= x^2\partial x + xy\partial y \end{aligned} \tag{7.3.8}$$

of point symmetries specifies (7.3.7). Furthermore the same set of Lie point symmetries above specifies another equation, a partial differential equation this time, *videlicet*

$$u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = 0, \tag{7.3.9}$$

which is a two-dimensional Bateman equation specified by the set of Lie point symmetries given above. Its general implicit solution is given by

$$x\phi_1(u) + y\phi_2(u) = C, \tag{7.3.10}$$

[3]Admits the same Lie point symmetries.

where C is a constant and ϕ_1 and ϕ_2 are arbitrary smooth functions. Equation (7.3.9) is a parabolic equation and, as one would or would not expect, admits the following infinite-dimensional set of Lie point symmetries.

$$\begin{array}{rclrcl} G_1 &=& H_1(u)\partial y & G_6 &=& H_6(u)x\partial x \\ G_2 &=& H_2(u)\partial x & G_7 &=& H_7(u)\left(x^2\partial x + xy\partial y\right) \\ G_3 &=& H_3(u)y\partial y & G_8 &=& H_8(u)\left(y^2\partial y + xy\partial x\right) \\ G_4 &=& H_4(u)y\partial x & G_9 &=& H_9(u)\partial u, \\ G_5 &=& H_5(u)x\partial y & & & \end{array}$$

where $H_i, \quad i = 1, 9,$ are arbitrary functions of u with the algebra $\{A_1 \bigoplus_s sl(3,R)\}_\infty$.

7.3.2 Same-Type Same-Order Differential Equations

Consider the same set of Lie point symmetries, Λ_i, considered in the previous Section. If for example we omit Λ_9 and replace it with Θ_{11} from the previous Section and successively apply this new group to general second-order partial differential equation with two independent variables and one dependent variable, *videlicet*

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}) = 0, \qquad (7.3.11)$$

in the usual way [41, 50] we obtain the equation [9]

$$u_{xx}u_{yy} - u_{xy}^2 = 0. \qquad (7.3.12)$$

More interestingly, applying the same group to equation (7.3.11) written in solved form in terms of one of its arguments as

$$u_{xx} = f(x,y,u,u_x,u_y,u_{xy},u_{yy}), \qquad (7.3.13)$$

one obtains the following equation

$$u_y^2u_{xx} - 2u_xu_yu_{xy} + u_x^2u_{yy} = 0, \qquad (7.3.14)$$

which is a two-dimensional Bateman equation [9].

Thus we have two distinct equations of the same type, *i.e.,* partial differential equations, specified by the same set of Lie point symmetries. This indeed was not expected! Note that this new set of point symmetries is another *representation* of the complete symmetry group for the Bateman equation. Further we note both equation (7.3.12) and (7.3.14) are (strongly) Lie remarkable.

7.3.3 Different-Types Different-Order Differential Equations

In §5.4 we observed a different phenomenon when there was a specification of a second-order partial differential equation

$$u_y^2 u_{xx} - \frac{n}{n-1} u_x u_y u_{xy} + \frac{1}{n-1} u_x^2 u_{yy} = 0. \tag{7.3.15}$$

written in a zero-determinant form as

$$\begin{vmatrix} 0 & u_x & u_y \\ & & \\ u_x & (n-1)u_{xx} & \frac{n}{2} u_{xy} \\ & & \\ u_y & \frac{n}{2} u_{xy} & u_{yy} \end{vmatrix} = 0$$

from the Lie point symmetries

$$\begin{array}{rclrcl} \Upsilon_1 & = & \partial x & \Upsilon_{n+1} & = & y\partial y \\ & \vdots & & \Upsilon_{n+2} & = & \partial y \\ \Upsilon_n & = & x^{n-1}\partial y & \Upsilon_{n+3} & = & x\partial x + \frac{1}{2}(n-1)y\partial y \\ & & & \Upsilon_{n+4} & = & x^2\partial x + (n-1)xy\partial y \end{array}$$

of any higher-order ordinary differential equation

$$y^{(n)} = 0, \; n \geq 3. \tag{7.3.16}$$

It was also observed in §6.4 that the system of second-order ordinary differential equations, *videlicet*

$$\begin{aligned} y'' &= 0 \\ z'' &= 0, \end{aligned} \tag{7.3.17}$$

possesses the following Lie point symmetries

$$\begin{array}{rclrcl} \Theta_1 & = & \partial z & \Theta_9 & = & x\partial y \\ \Theta_2 & = & \partial y & \Theta_{10} & = & y\partial y \\ \Theta_3 & = & \partial x & \Theta_{11} & = & z\partial x \\ \Theta_4 & = & y\partial z & \Theta_{12} & = & y\partial x \\ \Theta_5 & = & x\partial z & \Theta_{13} & = & x\left(x\partial x + z\partial z + y\partial y\right) \\ \Theta_6 & = & z\partial z & \Theta_{14} & = & y\left(y\partial y + z\partial z + x\partial x\right) \\ \Theta_7 & = & z\partial y & \Theta_{15} & = & z\left(z\partial z + y\partial y + x\partial x\right) \\ \Theta_8 & = & x\partial x & & & \end{array}$$

and is specified by the following set of Lie point symmetries

$$\{\Theta_1, \Theta_2, \Theta_3, \Theta_{10}, \Theta_7, \Theta_{13}\}.$$

However, the same set of point symmetries specifies the three-dimensional second-order partial differential equation

$$\begin{aligned} 0 &= u_{x_1}^2\left(u_{x_2x_2}u_{x_3x_3} - u_{x_3x_2}^2\right) - 2u_{x_1}u_{x_2}\left(u_{x_1x_2}u_{x_3x_3} - u_{x_3x_1}u_{x_2x_3}\right) \\ &+ 2u_{x_1}u_{x_3}\left(u_{x_2x_1}u_{x_3x_2} - u_{x_3x_1}u_{x_2x_2}\right) - 2u_{x_2}u_{x_3}\left(u_{x_1x_1}u_{x_2x_3} - u_{x_3x_1}u_{x_1x_2}\right) \\ &- u_{x_2}^2\left(u_{x_1x_1}u_{x_3x_3} - u_{x_3x_1}^2\right) + u_{x_3}^2\left(u_{x_1x_1}u_{x_2x_2} - u_{x_1x_2}^2\right) \end{aligned} \tag{7.3.18}$$

for $x_1 = x, x_2 = y$ and $x_3 = z$. Equation (7.3.18) admits the following Lie point symmetries

$$\begin{aligned} \Gamma_1 &= F_1(u)\partial z & \Gamma_9 &= F_9(u)x\partial y \\ \Gamma_2 &= F_2(u)\partial y & \Gamma_{10} &= F_{10}(u)y\partial y \\ \Gamma_3 &= F_3(u)\partial x & \Gamma_{11} &= F_{11}(u)z\partial x \\ \Gamma_4 &= F_4(u)\partial u & \Gamma_{12} &= F_{12}(u)y\partial x \\ \Gamma_5 &= F_5(u)y\partial z & \Gamma_{13} &= F_{13}(u)x\partial x \\ \Gamma_6 &= F_6(u)x\partial z & \Gamma_{14} &= F_{14}(u)y\left(y\partial y + z\partial z + x\partial x\right) \\ \Gamma_7 &= F_7(u)z\partial z & \Gamma_{15} &= F_{15}(u)x\left(x\partial x + z\partial z + y\partial y\right) \\ \Gamma_8 &= F_8(u)z\partial y & \Gamma_{16} &= F_{16}(u)z\left(z\partial z + y\partial y + x\partial x\right). \end{aligned}$$

7.3.4 Same-Order Different-Types Different-Dimensions Differential Equations

One does need to go very far in understanding studies involving rocket science to observe that the Lie point symmetries of equation (7.3.18) with $F_i,\ i = 1, 16$, being equal to one, are exactly the same as Lie point symmetries of (7.3.17) and are also the same as the Lie point symmetries of (7.2.6) with z replaced by u in latter case. The implications of this scenario is that one can use the exact same set of point symmetries, say

$$\{\Theta_1, \Theta_2, \Theta_3, \Theta_{10}, \Theta_7, \Theta_{13}\},$$

to specify either (7.3.18) or (7.3.17) or (7.2.6). This obviously depends on the structure of the general equation to be specified. If one commences with a two-dimensional partial differential equation, one obtains (7.2.6). Commencing with a system of ordinary differential equations one gets (7.3.17). Similarly, if one commences with a three-dimensional partial differential equation, one obtains (7.3.18). Most probably commencing with another general structure totally different to the above cases produces something totally different! This justifies the fact that the current definition of complete symmetry group needs to be revisited and possibly extended in order to prevent the occurrence of such eventualities!

7.4 Conclusion

The complete symmetry groups of ordinary differential equations have been intensively studied and it is now well understood [6] that an equation may possess many representations of the same group which specify uniquely the equation. Hence the introduction of the concept of minimal dimension under the operation of taking the Lie Bracket was imperative. For partial differential equations we note that the likelihood of multiple representation of the complete symmetry group of a given equation is much higher for equations which are **strongly** Lie remarkable than for those who are **weakly** Lie remarkable. In fact weakly Lie remarkable partial differential equations possess only one representation of complete symmetry group in terms of Lie point symmetries as the equation is specified by all the point symmetries it possesses.

We have also seen that, over and above the phenomenon of multi-representation of complete symmetry groups, there is also a notion of multirealisation in the sense that the same set of Lie point symmetries can specify distinct equations. This was an unexpected outcome and in fact very surprising that the equation realised may not be the intended one! This compels one to introduce further conditions necessary to ensure that the group obtained uniquely specifies the intended equation *before* seeking the complete symmetry group. The conditions are

d. *the **type** of intended equation be specified, that is, whether the intended outcome be an ordinary differential equation, a partial differential equation or system,*

e. *the **order** of an intended equation be it ordinary, partial or system must be specified, e.g. first-, second- or higher-order, and*

f. *the* ***number*** *of independent variables be known, that is, the dimensions of an intended equation be it ordinary, partial or system must be specified.*

This way one ensures that the complete symmetry group uniquely specifies the correct equation or at least the correct type and of the correct order with the correct dimensions. However, the problem of multirealisation of distinct equations of the same type, order and dimensions is still possible. This is because the notion of *minimality* of complete symmetry group is not very helpful to prevent multirealisation when it comes to partial differential equations particularly the ones which are strongly Lie remarkable. The problem of avoiding multirealisation for strongly Lie remarkable partial differential equations remains an open and a challenging one. Perhaps one should always keep in mind that more often than not, partial differential equations are accompanied by inital/boundary conditions. Further research should possibly consider characterising both the partial differential equation and its boundary conditions. This may ensure that the resulting complete symmetry group uniquely specifies the whole boundary value problem[4].

For now we propose an approach which has worked very well in the past [6, 43, 44, 46, 47, 48], that is, instead of writing the general equation on which point symmetries are to be acted as

$$F(x, u, \partial_u, \partial_u^2, \ldots, \partial_u^k) = 0, \tag{7.4.19}$$

where $x = (x, x_2, \ldots, x_n)$ denotes the set of coordinates corresponding to its n independent variables, u denotes the dependent variable and $\partial_u^j,\ j = 1, \ldots, k$, corresponds to all partial derivatives of u with respect to x and where in terms of the coordinates, $x, u, \partial_u, \partial_u^2, \ldots, \partial_u^k$, the partial differential equation, (7.4.19), is an algebraic equation that defines an hypersurface in $(x, u, \partial_u, \partial_u^2, \ldots, \partial_u^k)$-space, one may write the general equation in solved form in terms of one of its argument, say,

$$F\left(x, u, \partial_u, \partial_u^2, \ldots, \partial_u^{k-1}\right) = \partial^j u, \quad j = 1, \ldots, k. \tag{7.4.20}$$

[4]Unfortunately it is well known that boundary/inital conditions are usually not invariant under the symmetries of the equation itself. A notable exception is to be found in a number of evolution equations which arise in the Mathematics of Finance.

Final Experiential Comment

It is often said that mathematicians make horrible chefs. Of course I make an exception to the person reading this Thesis. However, writing this Thesis has been a rather peculiar experience. As the *The Mask of Zeus Dobie* thinks to himself in page **1089** volume **30** of the Notices of the American Mathematical Society:

> ‘It’s funny ... (but also sad), how many people imagine that Mathematics consists of interminably applying fixed formulae to clearly defined problems and so ‘*working them out*’. Because it’s not like that at all. Half the time you don’t even know what you’re looking for until you’ve found it. A great deal more than half the time you spend looking at a blank sheet of paper and chewing the end of a pencil – the blunt end, hopefully – while you’re trying to see what the bloody problem is. You know it’s just there all right, but no, you can’t grasp it, you can’t quite perceive how to formulate it ... mathematician’s block...’

I am a true bystander to this rather unpleasant fact. It happened to me countless times throughout this journey. Many times I never knew what I was looking for until....

Appendix A

The following table depicts the Lie Brackets of the Lie point symmetries of equation (6.2.12) in §6.2.

$[,]$	Γ_1	Γ_2	Γ_3	Γ_4
Γ_1	0	0	0	Γ_1
Γ_2	0	0	0	Γ_2
Γ_3	0	0	0	Γ_3
Γ_4	$-\Gamma_1$	$-\Gamma_2$	$-\Gamma_3$	0
Γ_5	0	$-\Gamma_1$	0	$-\Gamma_5$
Γ_6	0	0	$-\Gamma_1$	$-\Gamma_6$
Γ_7	Γ_1	0	0	$-\Gamma_7$
Γ_8	$-\Gamma_2$	0	0	$-\Gamma_8$
Γ_9	0	0	$-\Gamma_2$	$-\Gamma_9$
Γ_{10}	0	$-\Gamma_2$	0	$-\Gamma_{10}$
Γ_{11}	$-\Gamma_3$	0	0	$-\Gamma_{11}$
Γ_{12}	0	$-\Gamma_3$	0	$-\Gamma_{12}$
Γ_{13}	0	0	$-\Gamma_3$	$-\Gamma_{13}$
Γ_{14}	$-\Gamma_5$	$-\Gamma_7 - \Gamma_{10} - \Gamma_{13}$	$-\Gamma_{12}$	$-\Gamma_{14}$
Γ_{15}	$-\Gamma_6$	$-\Gamma_9$	$-\Gamma_7 - \Gamma_{10} - \Gamma_{13}$	$-\Gamma_{15}$
Γ_{16}	$-\Gamma_7 - \Gamma_{10} - \Gamma_{13}$	$-\Gamma_8$	$-\Gamma_{11}$	$-\Gamma_{16}$

Lie Brackets of the Lie point symmetries of equation (6.2.12) (continued)

$[,]$	Γ_5	Γ_6	Γ_7	Γ_8
Γ_1	0	0	Γ_1	Γ_2
Γ_2	Γ_1	0	0	0
Γ_3	0	Γ_1	0	0
Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
Γ_5	0	Γ_5	$\Gamma_7+\Gamma_{10}$	Γ_6
Γ_6	0	0	Γ_6	Γ_9
Γ_7	$-\Gamma_5$	$-\Gamma_6$	0	Γ_8
Γ_8	$-\Gamma_7-\Gamma_{10}$	$-\Gamma_9$	$-\Gamma_8$	0
Γ_9	$-\Gamma_6$	0	0	0
Γ_{10}	$-\Gamma_5$	0	0	$-\Gamma_8$
Γ_{11}	$-\Gamma_{12}$	$-\Gamma_7-\Gamma_{13}$	$-\Gamma_{11}$	0
Γ_{12}	0	$-\Gamma_5$	0	$-\Gamma_{11}$
Γ_{13}	0	$-\Gamma_6$	0	0
Γ_{14}	0	0	0	$-\Gamma_{16}$
Γ_{15}	0	0	0	0
Γ_{16}	$-\Gamma_{14}$	$-\Gamma_{15}$	$-\Gamma_{16}$	0

Lie Brackets of the Lie point symmetries of equation (6.2.12) (continued)

$[,]$	Γ_9	Γ_{10}	Γ_{11}	Γ_{12}
Γ_1	0	0	Γ_3	0
Γ_2	0	Γ_2	0	Γ_3
Γ_3	Γ_2	0	0	0
Γ_4	Γ_9	Γ_{10}	Γ_{11}	Γ_{12}
Γ_5	Γ_6	Γ_5	Γ_{12}	0
Γ_6	0	0	$\Gamma_7 + \Gamma_{13}$	Γ_5
Γ_7	0	0	Γ_{11}	0
Γ_8	0	Γ_8	0	Γ_{11}
Γ_9	0	Γ_9	Γ_8	$\Gamma_{10} + \Gamma_{13}$
Γ_{10}	$-\Gamma_9$	0	0	Γ_{12}
Γ_{11}	$-\Gamma_8$	0	0	0
Γ_{12}	$-\Gamma_{10} - \Gamma_{13}$	$-\Gamma_{12}$	0	0
Γ_{13}	$-\Gamma_9$	0	$-\Gamma_{11}$	$-\Gamma_{12}$
Γ_{14}	$-\Gamma_{15}$	$-\Gamma_{14}$	0	0
Γ_{15}	0	0	$-\Gamma_{16}$	$-\Gamma_{14}$
Γ_{16}	0	0	0	0

Lie Brackets of the Lie point symmetries of equation (6.2.12) (continued)

$[,]$	Γ_{13}	Γ_{14}	Γ_{15}	Γ_{16}
Γ_1	0	Γ_5	Γ_6	$\Gamma_7+\Gamma_{10}+\Gamma_{13}$
Γ_2	0	$\Gamma_7+\Gamma_{10}+\Gamma_{13}$	Γ_9	Γ_8
Γ_3	Γ_3	Γ_{12}	$\Gamma_7+\Gamma_{10}+\Gamma_{13}$	Γ_{11}
Γ_4	Γ_{13}	Γ_{14}	Γ_{15}	Γ_{16}
Γ_5	0	0	0	Γ_{14}
Γ_6	Γ_6	0	0	Γ_{15}
Γ_7	0	0	0	Γ_{16}
Γ_8	0	Γ_{16}	0	0
Γ_9	Γ_9	Γ_{15}	0	0
Γ_{10}	0	Γ_{14}	0	0
Γ_{11}	Γ_{11}	0	Γ_{16}	0
Γ_{12}	Γ_{12}	0	Γ_{14}	0
Γ_{13}	0	0	Γ_{15}	0
Γ_{14}	0	0	0	0
Γ_{15}	$-\Gamma_{15}$	0	0	0
Γ_{16}	0	0	0	0

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