Locally Finite Nearness Frames

by

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Declaration

I declare that the contents of this dissertation is the result of my own work apart from where references are made.

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Contents

Introduction

The concept of a *frame* was introduced in the mid-sixties by Dowker and Papert. Since then frames have been extensively studied by several authors, including Banaschewski, Pultr and Baboolal to mention a few. The idea of a *nearness* was first introduced by H. Herrlich in 1972 and that of a *nearness frame* by Banaschewski in the late eighties. T. Dube made a fairly detailed study of the latter concept.

The purpose of this thesis is to study the property of *local finiteness* and *metacompactness* in the setting of nearness frames. J. W. Carlson studied these ideas (including Lindëlof and Pervin nearness structures) in the realm of nearness spaces.

The first four chapters are a brief overview of frame theory culminating in results concerning regular, completely regular, normal and compact frames.

In chapter five we provide the definitions for various nearness frames : Pervin, Lindëlof , Locally Finite and Metacompact to mention a few. A particular locally finite nearness structure, denoted by μ_{LF}, is studied in detail. It is defined to be the nearness structure on a regular frame L generated by the family of all locally finite covers on the frame L. Also, a particular metacompact nearness structure, denoted by μ_{PF}, is studied in detail. It is defined to be the nearness structure on a regular frame L generated by the family of all point-finite covers of the frame L. Various theorems related the above nearness frames and these nearness structures are obtained.

Chapter 1

Preliminaries

1.1 Relations

For any set L, a *relation* on L denoted by "$\sim$", is any subset of $L \times L$. Given any pair $a, b \in L$, we write $(a, b) \in \sim$ to mean "a is *related* to b" *i.e.* a and b are *comparable* elements. The relation $\sim$ on the set L is called

(i) *reflexive*, if for each $a \in L$, $a \sim a$

(ii) *symmetric*, if for each $a, b \in L$, $a \sim b \Rightarrow b \sim a$

(iii) *antisymmetric*, if $a \sim b$ and $b \sim a$ in $L \Rightarrow a = b$

(iv) *transitive*, if $a \sim b$ and $b \sim c$ in $L \Rightarrow a \sim c$.

We call $\sim$ an *equivalence* relation on L if $\sim$ is reflexive, symmetric and transitive. A reflexive, antisymmetric, transitive relation on L is a *partial order* and is represented by the symbol "$\leq$". The set L equipped with a partial order is called a *partially ordered* set or a *poset* for short. We write $(L, \leq)$ for a poset.

Furthermore, a transitive relation satisfying

$$a \sim b \Rightarrow b \not\sim a \qquad \forall a, b \in L$$

is called a *strict order* and symbollically represented by "$<$". We say that the set L is *linearly ordered* or *totally ordered* by the partial order $\leq$ provided that exactly one of

$$a < b, \quad b < a \quad or \quad a = b$$

holds. If such is the case, then $(L, \leq)$ is a *linearly* or *totally ordered* set or is simply called a *chain*. Thus, a chain is a poset in which there are no *incomparable* elements.

Examples of Posets

1. The set of all subsets of the set X, *i.e.* the power set of X, $\mathcal{P}X$ partially ordered by set inclusion is a poset.

2. The set of all open sets in a topological space X, denoted by $\mathcal{D}X$ is a poset with the partial order as in (1).

3. Consider the set M and the set $\mathcal{R}M$ of all relations on M. Then $(\mathcal{R}M, \leq)$ with partial order $\leq$ defined as follows is a poset :

$$\Phi \leq \Psi \Leftrightarrow (x,y) \in \Phi \Rightarrow (x,y) \in \Psi \quad \forall\ \Phi, \Psi \in \mathcal{R}M, x, y \in M$$

For more examples and references see [11], [12], [13] or [14]. An ambiguous, although widely accepted practice, is to use L rather than $(L, \leq)$ to represent a poset, meaning that the partial ordering relation is understood. We will adopt this notion in what follows, unless otherwise stated.

Next we define the *join* and the *meet* of any arbitrary poset L. However, we first require the following essential concepts for an arbitrary poset L :

1. The *smallest* element of L, if it exists, is the element 0_L such that $0_L \leq a\ \forall\ a \in L$. We call 0_L the *zero* or the *bottom* of L if it exists.

2. The *largest* element of L, if it exists, is the element denoted by e_L or 1_L such that $a \leq 1_L\ \forall\ a \in L$. We call e_L or 1_L the *top* or the *unit* of L if it exists. We shall use the notation 1_L for the unit.

3. A *maximal* element of L, is an element b_1 such that for $b \in L$, $b_1 \leq b \Rightarrow b = b_1$.

4. A *minimal* element of L, is an element b_0 such that for $b \in L$, $b \leq b_0 \Rightarrow b = b_0$.

Minimal and maximal elements are unique by the antisymmetry property of $\leq$. Clearly, 0_L and 1_L if they exist, are minimal and maximal elements respectively and so are unique. Where no confusion arises we merely write 0 for the zero of L and 1 for the unit of L. The poset L is called *bounded* if both the zero and the unit exist in L. We are now ready to define the *join* and the *meet.*

Let $B \subseteq L$ be any subset of the poset L. Then

1. the *join* of B, if it exists, written $\bigvee B$, is the smallest element of the set $\{a \in L : b \leq a \ \forall \ b \in B\}$.

2. the *meet* of B, if it exists, written $\bigwedge B$, is defined dually as the largest element of the set $\{a \in L : a \leq b \ \forall \ b \in B\}$.

If B is a two-element set $\{a, b\}$ in L say, then we write $\bigvee\{a, b\}$ as $a \vee b$ and is read as "a join b", and $\bigwedge\{a, b\}$ as $a \wedge b$ and is read as "a meet b". If $B = \phi$, then $\bigvee \phi = 0_B$ and $\bigwedge \phi = 1_B$.

1.2 Lattices

The poset L is a *join-semilattice*, written as $\vee$-semilattice, if for each two-element set $\{a, b\}$ in L, $a \vee b$ exists. Dually, L is a *meet-semilattice*, written $\wedge$-semilattice provided that $a \wedge b$ exists for each two-element set $\{a, b\}$ in

L. L is called a *lattice* provided that both meet and join exist for every two-element set (hence, finite set) in L *i.e.* L is both a $\vee$-semilattice and a $\wedge$-semilattice.

In lattices, the $\vee$ and the $\wedge$ are finitary binary operations. So, for any pair $\{a, b\}$ in L, $a \vee b$ and $a \wedge b$ are elements of L. Furthemore, we have the following properties for a, b, c in any lattice L :

1. $a \wedge a = a$ *i.e. idempotency*
2. $a \wedge b = b \wedge a$ *i.e. commutativity*
3. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *i.e. associativity*

The $\vee$, being dual to $\wedge$, also satisfies the same properties. So, $\wedge$ and $\vee$ are idempotent, commutative and associative in any lattice L.

Types of Lattices

1. **Distributive Lattices** :

   The lattice L is *distributive* provided that it satisfies the ditributive law *i.e.* for each $a, b, c \in L$

   $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and

   $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

2. **Pseudocomplemented Lattices :**

   In a lattice L with 0, the *pseudocomplement* of an element $t \in L$ is the element t^* such that

   (i) $t \wedge t^* = 0$ and

   (ii) $a \wedge t = 0 \Rightarrow a \leq t^*$ for $a \in L$.

   Thus, t^* is the largest element of the set $\{a \in L : a \wedge t = 0\}$ *i.e.* $t^* = \bigvee\{a \in L : a \wedge t = 0\}$. So, t^* is unique. L is a *pseudocomplemented* lattice if every element in L has a pseudocomplement.

3. **Complemented Lattices :**

   In a bounded lattice L, the *complement* of an element $a \in L$ is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$. L is thus a *complemented* lattice if every element in L has a complement.

4. **Boolean Lattices :**

   A complemented, distributive lattice is a *Boolean* lattice.

5. **Complete Lattices :**

   L is a *complete* lattice if for any subset $S \subseteq L$, $\bigvee S$ and $\bigwedge S$ exist.

6. **Sublattices :**

   Any nonvoid subset S of a $\vee$-semilattice L in which $0 \in S$ and $a \vee b \in S$ whenever $a, b \in S$, where the $\vee$ on S being the restriction of the $\vee$ on L, is called a *sub-$\vee$-semilattice.* A *sub-$\wedge$-semilattice* can be dually defined. $\phi \neq S \subseteq L$ is a *sublattice* of the lattice L provided that S is both a sub-$\vee$-semilattice and a sub-$\wedge$-semilattice *i.e.* $a \wedge b$ and $a \vee b$ belong in S whenever $a, b \in S$.

Of greater interest in our exposition is that of complete lattices of which the foundations of our study is based on.

Lemma 1.1 :

A lattice L is complete $\Leftrightarrow \bigwedge S$ *exists* $\forall S \subseteq L$.

Proof :

($\Longleftarrow$) Let $T \subseteq L$.

Put $S = \{s \in L : t \leq s \text{ for each } t \in T\}$.

Then by the hypothesis $\bigwedge S$ exists. Let $s_0 = \bigwedge S$. Then $s_0 = \bigvee T$ and thus L is complete.

($\Longrightarrow$) Obvious. □

Remark 1.1 :

By this lemma, we need only that $\bigwedge$ *to exist for any subset* S *of* L *for* L *to be complete. For* $\bigvee S$ *to exist is redundant and, of course, this concept is naturally dual.*

1.3 Homomorphisms

A *meet-lattice homomorphism* (dually, *join-lattice homomorphism*) is a map $h : L \to M$ between lattices L and M which is $\wedge$-preserving (dually, $\vee$-preserving). If $h : L \to M$ preserves both $\wedge$ and $\vee$, then h is a *lattice homomorphism*. h is an *embedding* if it is one-to-one.

1.4 Ideals and Filters

A subset I of a $\vee$-semilattice L is an *ideal* if it satisfies the following properties

1. $0 \in I$
2. $a \vee b \in I$ whenever $a, b \in I$
3. I is a *down set* *i.e.* $a \in I$ and $b \leq a \Rightarrow b \in I$.

The properties 1 and 2 together imply that an ideal is a sub-$\vee$-semilattice.
The ideal I is a
(*i*) *proper* ideal if $I \neq L$ (*i.e.* $1 \notin I$)
(*ii*) *prime* ideal if I is proper and ($a, b \in L$ and $a \wedge b \in I \Rightarrow a \in I$ or $b \in I$).
The set $\downarrow a = \{b \in L : b \leq a\}$ is clearly an ideal of L and is called the *principal ideal generated by a*.

For any $\wedge$-semilattice L with unit, a subset $D \subseteq L$ is a down set provided that $\downarrow x \subseteq D$ whenever $x \in D$. Thus, I is an ideal provided that I is a sub-$\vee$-semilattice *and* a down set. We denote the set of all down sets of a lattice L by $\mathcal{D}L$. Using the notation in [12], the set of all ideals of the lattice L is denoted by $\mathcal{IDL}(L)$.

Remark 1.2 *:*

$\mathcal{IDL}(L) \subseteq \mathcal{D}L$.

It is an easy task to verify that an arbitrary intersection of down sets (respectively, ideals) is a down set (respectively, ideal), and an arbitrary union of down sets is a down set.

Then $\mathcal{D}L$ ordered by inclusion with $\bigwedge = \bigcap$ and $\bigvee = \bigcup$, is a complete lattice.

We call a collection of ideals $\{J_i\}_{i\in\Delta}$ *updirected*, written as $\hat{\bigvee} J_i$, provided that

$$i, j \in \Delta \Rightarrow \exists k \in \Delta \textit{ such that } J_i, J_j \subseteq J_k$$

Proposition 1.1 :

$\mathcal{IDL}(L)$ ordered by inclusion is a complete lattice for every distributive lattice L.

Proof :

Let $\{I, J\} \subseteq \mathcal{IDL}(L)$. Define $\wedge$ and $\vee$ as follows :
Let $I \wedge J = I \cap J$ and $I \vee J = \{a \vee b : a \in J \text{ and } b \in J\}$.
By *Remark* 1.2, $I \wedge J$ is an ideal of L. Thus we need to show that $I \vee J$ is an ideal of L. To this end we shall show
(i) $I \vee J$ is a sub-$\vee$-semilattice and
(ii) $I \vee J$ is a downset.

(i) : Since I and J are ideals $\Rightarrow$ $0 \in I, J$. Thus $0 = 0 \vee 0 \in I \vee J$. If $x, y \in I \vee J$, then $x = a \vee b$ and $y = \bar{a} \vee \bar{b}$ for some $a, \bar{a} \in I$ and $b, \bar{b} \in J$. However, I and J being ideals $\Rightarrow a \vee \bar{a} \in I$ and $b \vee \bar{b} \in J$. Thus,
$x \vee y = (a \vee b) \vee (\bar{a} \vee \bar{b}) = (a \vee \bar{a}) \vee (b \vee \bar{b}) \in I \vee J$.

(ii) : If $x \in I \vee J$, then $x = a \vee b$ for some $a \in I$ and $b \in J$. Since $\downarrow x = \downarrow (a \vee b) = (\downarrow a) \vee (\downarrow b)$, for any $y \in \downarrow x \Rightarrow y = t \vee s$ for some $t \in \downarrow a$ and $s \in \downarrow b$. Then $t \leq a \in I$ and I an ideal $\Rightarrow t \in I$. Similiarly, $s \in J$. Thus $y \in I \vee J$. So, $\downarrow x \subseteq I \vee J \Rightarrow I \vee J$ is a down set.

Hence, $I \vee J$ is an ideal of L.

It remains to show that $\mathcal{IDL}(L)$ is complete. By *Lemma* 1.1, it suffices to show for $\mathcal{S} = \{J_i\}_{i\in\Delta} \subseteq \mathcal{IDL}(L)$ that $\bigvee \mathcal{S}$ exists.

Let

$$\bigvee \mathcal{S} = \hat{\bigvee} J_i = \bigcup_{fin \Lambda \subseteq \Delta} \bigvee_{i \in \Lambda} J_i$$

Then, with the updirected property, $\bigvee \mathcal{S} \in \mathcal{IDL}(L)$. So, by *Lemma* 1.1, $\mathcal{IDL}(L)$ is complete. □

Quite easily, one can show that $I \wedge J = \{a \wedge b : a \in I \text{ and } b \in J\}$ is the ideal $I \cap J$ in the above Proposition.

By dualizing the concept of an ideal, we define *filters*, *prime* filters, *proper* filters and *principal* filters. For a comprehensive reference on Lattices, Ideals and Filters see [11], [12] or [13].

1.5 Some Categorical Concepts

A *category* $\mathcal{A}$ is a class of *objects* $ob\mathcal{A}$, and a class of *morphisms* or *arrows* $mor\mathcal{A}$ satisfying the following :

1. If $f \in mor\mathcal{A}$, then $\exists\ A = dom(f) \in ob\mathcal{A}$, called the *domain* of f and, $\exists\ B = cod(f) \in ob\mathcal{A}$, called the *codomain* of f and we write f as the arrow $A \xrightarrow{f} B$.

2. If $A \xrightarrow{f} B \xrightarrow{g} C$ are $\mathcal{A}$-morphisms, then $\exists$ an $\mathcal{A}$-morphism $A \xrightarrow{gf} C$ such that $h(gf) = (hg)f$ whenever h is defined with codomain A.

3. For each $\mathcal{A}$-object A, $\exists$ an $\mathcal{A}$-morphism $A \xrightarrow{1_A} A$ such that $f1_A = f$ and $1_A g = g$ whenever $f, g \in mor\mathcal{A}$ are defined with domain A and codomain A respectively.

Examples of Categories

1. The category **Set** with objects being sets and morphisms being all functions between sets.

2. The category **Grp** of groups and group homomorphisms.

3. The category **Top** of topological spaces and continuous maps.

4. The category **Slatt** of semilattices and semilattice homomorphisms.

5. The category **Latt** of lattices and lattice homomorphisms.

6. The category **BDLatt** of bounded distributive lattices and lattice homomorphism.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Then

(i) a *functor* $F : \mathcal{A} \to \mathcal{B}$ is a morphism of categories defining a function between the object class of $\mathcal{A}$ and the object class of $\mathcal{B}$

$F : ob\mathcal{A} \longrightarrow ob\mathcal{B}$

and a family of functions between the morphism class of $\mathcal{A}$ and the morphism class of $\mathcal{B}$

$F : mor\mathcal{A} \longrightarrow mor\mathcal{B}$

one for each ordered pair $(A, A') \in ob\mathcal{A}$ such that

1. $F1_A = 1_{FA} \quad \forall\ A \in ob\mathcal{A}$
2. $Fgh = FgFh \qquad$ for $g, h \in mor\mathcal{A}$

are satisfied.

The functor F is called *covariant* if property 2 is satisfied. If $Fgh = FhFg$ then the functor F is called *contravariant.*

(ii) for functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{A} \longrightarrow \mathcal{B}$, a *natural transformation* $\varphi : F \longrightarrow G$, is a class of morphisms $\varphi_A : FA \longrightarrow GA$, one for each $\mathcal{A}$-object A such that for all $\mathcal{A}$-morphisms $A \xrightarrow{f} A'$, $\varphi_{A'}Ff = Gf\varphi_A$ *i.e.* the following is a commutative diagram :

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\varphi_A"'] & FA' \arrow[d, "\varphi_{A'}"] \\
GA \arrow[r, "Gf"'] & GA'
\end{tikzcd}$$

(iii) the functor $F : \mathcal{A} \to \mathcal{B}$ is

1. an *isomorphism* if $\exists$ a functor $G : \mathcal{B} \to \mathcal{A}$ such that $FG = 1_{\mathcal{B}}$ and $GF = 1_{\mathcal{A}}$, where $1_{\mathcal{B}}, 1_{\mathcal{A}}$ are the *identity* functors which maps objects and morphisms, of the respective categories $\mathcal{B}$ and $\mathcal{A}$ identically to themselves.

2. an *equivalence* of categories if $\exists$ a functor $G : \mathcal{B} \to \mathcal{A}$ and natural isomorphisms $\varphi : FG \to 1$ and $\psi : 1 \to GF$.

3. *left adjoint* to the functor $G : \mathcal{B} \to \mathcal{A}$, written as $F \dashv G$, provided that $\exists$ natural transformations $\varphi : FG \to 1$ and $\psi : 1 \to GF$ such that $G\varphi\psi G = 1_G$ and $\varphi F F\psi = 1_F$ (1_G and 1_F being the identity maps of the functors G and F respectively) *i.e.* the following diagrams commute :

$$
\begin{tikzcd}
GFG \arrow[d, "G\varphi"'] & G \arrow[l, "\psi_G"'] \arrow[dl, "1_G"] \\
G &
\end{tikzcd}
\qquad
\begin{tikzcd}
FGF \arrow[d, "\varphi_F"'] & F \arrow[l, "F\psi"'] \arrow[dl, "1_F"] \\
F &
\end{tikzcd}
$$

The quadruple (F, G, φ, ψ) is called an *adjoint equivalence* of categories if φ and ψ are natural isomorphisms.

4. *full* if the functions $F : mor\mathcal{A} \to mor\mathcal{B}$ are onto.

5. *faithfull* if the functions $F : mor\mathcal{A} \to mor\mathcal{B}$ are one-to-one.

6. *representative* if $\forall$ $\mathcal{B}$-object B, $\exists$ an $\mathcal{A}$-object A such that $FA \simeq B$ *i.e.* FA is isomorphic to B.

(iv) Given the functor $F : \mathcal{A} \to \mathcal{B}$ and $B \in ob\mathcal{B}$, a *universal map* for B with respect to F is a pair (u, A) with $u : B \to FA$ a $\mathcal{B}$-morphism and $A \in ob\mathcal{A}$, such that for each $A' \in ob\mathcal{A}$, $\exists$ a unique $\mathcal{A}$-morphism $\bar{f} : A \to A'$ such that $F\bar{f}u = f$ *i.e.* the following diagram commutes :

$$\begin{tikzcd}
FA \arrow[d, "F\bar{f}"'] & B \arrow[l, "u"'] \arrow[dl, "f"] \\
FA' &
\end{tikzcd}$$

The category $\mathcal{B}$ is a *subcategory* of the category $\mathcal{A}$ if $ob\mathcal{B} \subseteq ob\mathcal{A}$ and $mor\mathcal{B} \subseteq mor\mathcal{A}$. $\mathcal{B}$ is a *full* subcategory of $\mathcal{A}$ if $mor(ob\mathcal{B}) = mor(ob\mathcal{A})$.

If $\mathcal{B}$ is a subcategory of $\mathcal{A}$ such that for each $\mathcal{A}$-object A, $\exists$ a universal map (f_A, B_A) for A with respect to the *inclusion* functor $inc : \mathcal{B} \to \mathcal{A}$, then the map f_A is called the *reflection* map. Diagrammatically, we have

$$\begin{tikzcd} B_A \arrow[d, "\exists \text{ unique } \bar{f}"'] & & A \arrow[ll, "f_A"'] \arrow[dll, "f"] \\ B & & \end{tikzcd}$$

Dually, we define the *coreflection* map by merely reversing the arrows above.

We require the following theorems in our study, the proofs of which can be found in [15].

Theorem 1.1 *:*

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between categories $\mathcal{A}$ and $\mathcal{B}$ such that for each $\mathcal{B}$-object B, $\exists$ a universal map (u_B, A_B), with $A_B \in ob\mathcal{A}$, for B with respect to F. Then

1. *there is a functor $G : \mathcal{B} \to \mathcal{A}$ satisfying*

   (a) $G(B) = A_B \quad \forall B \in ob\mathcal{B}$

   (b) $u = u_B : 1 \to FG$ *is a natural transformation*

2. *there is a unique natural transformation $v = v_A : GF \to 1$ such that*

(a) $F\mathrm{v}_A\mathrm{u}_{FA} = 1_{FA} \qquad \forall A \in ob\mathcal{A}$

(b) $\mathrm{v}_{GB}G\mathrm{u}_B = 1_{GB} \qquad \forall B \in ob\mathcal{B}.$

Remark 1.3 *The above theorem implies that $G \dashv F$ or F and G are* adjoint on the right.

Theorem 1.2 *:*

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Then the following statements are equivalent for a functor $F : \mathcal{A} \to \mathcal{B}$

1. *F is an equivalence of categories*
2. *F is part of an adjoint equivalence (F, G, φ, ψ)*
3. *F is full, faithfull and representative.*

For a more detailed analysis of the categorical concepts see [15].

Chapter 2

Frames

We define a *frame* to be a poset $(L, \leq)$ which satisfies the following two properties :

(i) L is a complete lattice

(ii) L satisfies the *infinite distributive law* :

$$x \wedge \bigvee_{i \in \Delta} x_i = \bigvee_{i \in \Delta} (x \wedge x_i)$$

$\forall x \in L$ and $\{x_i\}_{i \in \Delta} \subseteq L$.

As previously, we merely write L for the frame $(L, \leq)$. If $M \subseteq L$ is closed under finite $\wedge$ and arbitrary $\bigvee$ restricted from L, then M is a *subframe* of L.

2.1 Examples of Frames

1. All finite distributive lattices are frames.

2. All complete Boolean algebras are frames.

3. For any topological space X, the collection of open subsets of X, $\mathcal{D}X$, is a frame, where for any arbitrary subset $\{V_i\}_{i\in\Delta} \subseteq \mathcal{D}X$

$$\bigwedge_{i\in\Delta} V_i = int(\bigcap_{i\in\Delta} V_i)$$

with *int* being the topological *interior operator* and

$$\bigvee_{i\in\Delta} V_i = \bigcup_{i\in\Delta} V_i$$

We call $\mathcal{D}X$ the *frame of open sets of* X.

4. Every complete chain is a frame.

5. For any $\wedge$-semilattice L, $\mathcal{D}L$ is a frame, called the *frame of down sets of* X.

6. In a frame L with $a < b$, let $\langle a, b\rangle = \{x \in L : a \leq x \leq b\}$. Then $\langle a, b\rangle$ is a frame. So, $\langle 0, a\rangle = \downarrow a$ is a frame. Also $\uparrow b = \langle b, 1\rangle$ is a frame. However, $\downarrow a$ and $\uparrow b$ are not subframes of L.

Proposition 2.1 :

$\mathcal{IDL}(L)$ is a frame for any bounded distributive lattice L.

Proof :

We showed earlier, in *Proposition* 1.1, that $\mathcal{IDL}(L)$ is a complete lattice. So, we need only show that $\mathcal{IDL}(L)$ satisfies the infinite distributive frame law. Firstly, we shall show that $\mathcal{IDL}(L)$ is distributive.

Let $J, H, K \in \mathcal{IDL}(L)$. Since $J \cap H \subseteq J$ and $J \cap K \subseteq J$, we have $(J \cap H) \vee (J \cap K) \subseteq J$. Also $J \cap H \subseteq H$ and $J \cap K \subseteq K \Rightarrow (J \cap H) \vee (J \cap K) \subseteq H \vee K$. Thus, $(J \cap H) \vee (J \cap K) \subseteq J \cap (H \vee K)$.

Conversely, if $x \in J \cap (H \vee K)$, then $x \in J$ and $x \in H \vee K$. So, $x = h \vee k$ for some $h \in H$ and $k \in K$. Then
L distributive $\Rightarrow x = x \wedge x = x \wedge (h \vee k) = (x \wedge h) \vee (x \wedge k) \in (J \cap H) \vee (J \cap K)$.
Whence, $J \cap (H \vee K) = (J \cap H) \vee (J \cap K)$. So, $\mathcal{IDL}(L)$ is distributive.

Now, let $J \in \mathcal{IDL}(L)$ and $K = \{J_i\}_{i \in \Delta} \subseteq \mathcal{IDL}(L)$ be arbitrary. Then

$$\begin{aligned}
J \wedge \bigvee K &= J \cap (\hat{\bigvee}_{i \in \Delta} J_i) \\
&= J \cap (\bigcup_{fin \Lambda \subseteq \Delta} \bigvee_{i \in \Lambda} J_i) \\
&= \bigcup_{fin \Lambda \subseteq \Delta} \{J \cap (\bigvee_{i \in \Lambda} J_i)\} \\
&= \bigcup_{fin \Lambda \subseteq \Delta} \{\bigvee_{i \in \Lambda} (J \cap J_i)\} \\
&= \hat{\bigvee}_{i \in \Delta} (J \wedge J_i) \\
&= \bigvee (J \wedge K)
\end{aligned}$$

Thus, $\mathcal{IDL}(L)$ is frame. $\square$

We call $\mathcal{IDL}(L)$ the *frame of ideals of* L.

2.2 Frame Homomorphisms

For the frames L and M, a frame homomorphism is a map $h : L \to M$ such that

(*i*) $h(0_L) = 0_M$

(*ii*) $h(1_L) = 1_M$

(*iii*) $h(a \wedge b) = h(a) \wedge h(b) \qquad \forall a, b \in L$

(*iv*) For any $X \subseteq L, h(\bigvee X) = \bigvee h(X)$.

i.e. h preserves the unit, the zero, finite $\wedge$ and arbitrary $\bigvee$.

Examples of Frame Homomorphisms

1. For finite distributive lattices, any lattice homomorphism is a frame homomorphism.

2. For complete Boolean algebras, any Boolean homomorphism is a frame homomorphism.

3. In any frame L with $a \in L$, the maps

(i) $h : L \to \downarrow a$

$$x \rightsquigarrow x \wedge a \qquad \forall\, x \in L$$

(ii) $k : L \to \uparrow a$

$$x \rightsquigarrow x \vee a \qquad \forall\, x \in L$$

are frame homomorphisms.

Proposition 2.2 *:*

Let X and Y be topological spaces with $h : X \to Y$ any continuous map. Define

$\mathcal{D}h : \mathcal{D}Y \to \mathcal{D}X$ by

$$\mathcal{D}h(U) \rightsquigarrow h^{-1}(U) \qquad \forall\, U \in \mathcal{D}Y$$

Then $\mathcal{D}h$ is a frame homomorphism.

Proof :

Clearly, $\mathcal{D}h(\phi) = h^{-1}(\phi) = \phi$ *i.e.* $\mathcal{D}h(0_Y) = 0_X$. Also, $\mathcal{D}h(Y) = h^{-1}(Y) = X$ *i.e.* $\mathcal{D}h(1_Y) = 1_X$. If $U, V \in \mathcal{D}Y$, then

$$\begin{aligned} \mathcal{D}h(U \wedge V) &= h^{-1}(U \wedge V) \\ &= h^{-1}(U \cap V) \\ &= h^{-1}(U) \cap h^{-1}(V) \\ &= \mathcal{D}h(U) \wedge \mathcal{D}h(V) \end{aligned}$$

So, $\mathcal{D}h$ is closed under finite $\wedge$. Since h^{-1} preserves arbitrary unions and $\bigcup = \bigvee$, $\mathcal{D}h$ is closed under arbitrary joins. Indeed, $\mathcal{D}h$ is a frame homomorphism.

□

Proposition 2.3 *:*

For any frame L, the join-map

$$\begin{aligned} \bigvee : \mathcal{IDL}(L) &\to L \\ J &\rightsquigarrow \bigvee J \end{aligned}$$

is a frame homomorphism.

Proof :

Clearly, $\bigvee(0_{\mathcal{IDL}(L)}) = \bigvee\{0_L\} = 0$ and $\bigvee(1_{\mathcal{IDL}(L)}) = \bigvee L = 1$. If $J, H \in \mathcal{IDL}(L)$, then

$$\bigvee(J \wedge H) = \bigvee(J \cap H)$$

$$
\begin{aligned}
&= \bigvee_{a\in J}\bigvee_{b\in H}(a\wedge b)\\
&= (\bigvee_{a\in J} a)\wedge(\bigvee_{b\in H} b)\\
&= (\bigvee J)\wedge(\bigvee H)
\end{aligned}
$$

So, $\bigvee$ is closed under finite meet. We also have

$$
\begin{aligned}
\bigvee(J\vee H) &= \bigvee_{a\in J}\bigvee_{b\in H}(a\vee b)\\
&= \bigvee_{a\in J}(a\vee\bigvee_{b\in H} b)\\
&= (\bigvee_{a\in J} a)\vee(\bigvee_{b\in H} b)\\
&= (\bigvee J)\vee(\bigvee H)
\end{aligned}
$$

$\Rightarrow$ $\bigvee$ is closed under finite join. For arbitrary joins, let $\{J_i\}_{i\in I}\subseteq \mathcal{IDL}(L)$.

Then

$$
\begin{aligned}
\bigvee(\bigvee_{i\in I} J_i) &= \bigvee(\hat{\bigvee}_{i\in I} J_i)\\
&= \bigvee(\bigcup_{finF\subset I}\bigvee_{i\in F} J_i)\\
&= \bigcup_{finF\subseteq I}\bigvee(\bigvee_{i\in F} J_i)\\
&= \bigcup_{finF\subseteq I}\bigvee_{i\in F}(\bigvee J_i)\\
&= \hat{\bigvee}_{i\in I}(\bigvee J_i)
\end{aligned}
$$

Thus, $\bigvee$ is closed under arbitrary joins.

Hence, $\bigvee$ is a frame homomorphism. $\square$

It can also be shown that for any frame L, the join-map $\bigvee : \mathcal{D}L \to L$ is a frame homomorphism.

Compositions of frame homomorphisms are again frame homomorphisms, so we have the category **Frm** of frames and frame homomorphisms.

2.3 The category Frm and Top

Let X be any topological space. We know that $\mathcal{D}X$, the open sets of X, is a frame. So, we have the morphism

$\mathcal{D} : \mathbf{Top} \to \mathbf{Frm}$

$$X \rightsquigarrow \mathcal{D}X \qquad \forall\, X \in \mathbf{Top}$$

which takes every topological space X to the corresponding frame of open sets, $\mathcal{D}X$. For the continuous map $h : X \to Y$ between spaces X and Y we also showed in *Proposition* 2.2 that

$\mathcal{D}h : \mathcal{D}Y \to \mathcal{D}X$, where

$$\mathcal{D}h(U) \rightsquigarrow h^{-1}(U) \qquad \forall\, U \in \mathcal{D}Y$$

is a frame homomorphism. Thus, we have the following :

Proposition 2.4 *:*

$\mathcal{D}$ is a contravariant functor

Proof :

For any topological space X, $\mathcal{D}1_X(U) = U = 1_{\mathcal{D}X}(U)$ for each $U \in \mathcal{D}X$. So, $\mathcal{D}1_X = 1_{\mathcal{D}X}$. If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then

$gf : X \to Z$ is also continuous and thus $\mathcal{D}f : \mathcal{D}Y \to \mathcal{D}X$; $\mathcal{D}g : \mathcal{D}Z \to \mathcal{D}Y$; $\mathcal{D}f \circ \mathcal{D}g$ and $\mathcal{D}gf : \mathcal{D}Z \to \mathcal{D}X$ are frame homomorphisms. Then for any $W \in \mathcal{D}Z$ we have

$$\begin{aligned}
\mathcal{D}gf(W) &= (gf)^{-1}(W) \\
&= f^{-1}g^{-1}(W) \\
&= f^{-1}(\mathcal{D}g(W)) \\
&= \mathcal{D}f(\mathcal{D}g(W)) \\
&= \mathcal{D}f \circ \mathcal{D}g(W)
\end{aligned}$$

Thus $\mathcal{D}gf = \mathcal{D}f\mathcal{D}g$.

$\Rightarrow \mathcal{D}$ is a contarvariant functor. $\square$

We call $\mathcal{D}$ the *open set* functor.

For any frame L we the define the *spectrum* of L, ΣL, to be the set of all frame homomorphisms $\xi : L \to 2$, where $2 = \{0, 1\}$ the chain with just the unit and the zero.

Let $\Gamma_{\Sigma L} = \{\Sigma_a : a \in L\}$, where $\Sigma_a = \{\xi \in \Sigma L : \xi(a) = 1\}$.

Proposition 2.5 :

$\Gamma_{\Sigma L}$ is a topology on ΣL.

Proof :

Since $0 \in L \Rightarrow \Sigma_0 \in \Gamma_{\Sigma L}$. But, $\Sigma_0 = \{\xi \in \Sigma L : \xi(0) = 1\} = \phi$. Thus, $\phi \in \Gamma_{\Sigma L}$. Also, $\Sigma_1 = \{\xi \in \Sigma L : \xi(1) = 1\} = \Sigma L \in \Gamma_{\Sigma L}$. Now let Σ_a and $\Sigma_b \in \Gamma_{\Sigma L}$ for any $a, b \in L$. Since $\Sigma_a \cap \Sigma_b = \Sigma_{a \wedge b} \in \Gamma_{\Sigma L}$, $\Gamma_{\Sigma L}$ is closed under finite intersection. For any $S \subseteq L$, $\bigvee S \in L \Rightarrow \Sigma_{\bigvee S} \in \Gamma_{\Sigma L}$. But, $\bigcup_{a \in S} \Sigma_a = \Sigma_{\bigvee S} \in \Gamma_{\Sigma L}$. Thus, $\Gamma_{\Sigma L}$ is also closed under arbitrary unions. Hence, $\Gamma_{\Sigma L}$ is a topology on ΣL. $\square$

Now, for each frame homomorphism $h : L \to M$, let
$\Sigma h : \Sigma M \to \Sigma L$, where,

$$\Sigma h(\xi) \rightsquigarrow \xi h$$

Proposition 2.6 :

Σh is a continuous map.

Proof :

Let $\Sigma_a \in \Gamma_{\Sigma M}$ for any $a \in M$. Then,

$$\begin{aligned} \xi \in (\Sigma h)^{-1}(\Sigma_a) &\Leftrightarrow \Sigma h(\xi) \in \Sigma_a \\ &\Leftrightarrow \xi h \in \Sigma_a \\ &\Leftrightarrow \xi h(a) = 1 \\ &\Leftrightarrow \xi(h(a)) = 1 \\ &\Leftrightarrow \xi \in \Sigma_{h(a)} \end{aligned}$$

Thus $(\Sigma h)^{-1}(\Sigma_a) = \Sigma_{h(a)}$. But $\Sigma_{h(a)}$ is open in ΣL. So we have that for each Σ_a open in ΣM, $(\Sigma h)^{-1}(\Sigma_a)$ is open in ΣL.

$\Rightarrow \Sigma h$ is continuous. $\Box$

So, we have the following morphisms

$\Sigma : \mathbf{Frm} \to \mathbf{Top}$

$L \rightsquigarrow \Sigma L \qquad \forall F \in \mathbf{Frm}$

$h \rightsquigarrow \Sigma h \qquad$ for each $h \in mor\mathbf{Frm}$

which takes each frame L into its spectrum ΣL and each frame homomorphism h into Σh.

Proposition 2.7 :

Σ is a contravariant functor.

Proof :

For any frame L, $\Sigma 1_L(\xi) = \xi 1_L = \xi = 1_{\Sigma L}(\xi) \Rightarrow \Sigma 1_L = 1_{\Sigma L}$. If $h : L \to M$, $g : M \to N$ are frame homomorphisms, then $gh : L \to N$ is also a frame homomorphism. Then by *Proposition* 2.6, $\Sigma h : \Sigma M \to \Sigma L$; $\Sigma g : \Sigma N \to \Sigma M$; $\Sigma gh : \Sigma N \to \Sigma L$ are continuous maps. Also, $\Sigma h \circ \Sigma g$ is continuous. Then for any $\xi \in \Sigma N$

$$\begin{aligned} \Sigma gh(\xi) &= \xi(gh) \\ &= (\xi g)h \end{aligned}$$

$$
\begin{aligned}
&= \Sigma h(\xi g) \\
&= \Sigma h(\Sigma g(\xi)) \\
&= (\Sigma h \Sigma g)(\xi)
\end{aligned}
$$

Thus, $\Sigma gh = \Sigma h \Sigma g$.

$\Rightarrow \Sigma$ is a contravariant functor. $\square$

Proposition 2.8 :

$\mathcal{D}$ and Σ are adjoint in the right.

Proof :

We need to define natural transformations $\varphi : 1 \to \Sigma\mathcal{D}$ and $\psi : 1 \to \mathcal{D}\Sigma$ such that $\Sigma\psi\varphi_\Sigma = 1_\Sigma$ and $\mathcal{D}\varphi\psi_\mathcal{D} = 1_\mathcal{D}$. With regard to φ, for each topological space X we need to define $\varphi_X : X \to \Sigma\mathcal{D}X$. For each $x \in X$, let

$$\tilde{x} : \mathcal{D}X \to 2 \text{ by}$$

$$\tilde{x}(U) = \begin{cases} 0 & \text{if } x \notin U \\ 1 & \text{if } x \in U \end{cases}$$

Then $\tilde{x} \in \Sigma\mathcal{D}X$. So define

$\varphi_X : X \to \Sigma\mathcal{D}X$ by

$\varphi_X(x) \rightsquigarrow \tilde{x} \qquad \forall x \in X$

Then φ_X is a continous map and φ is a natural transformation. It is an easy task to verify the continuity of φ_X. We shall next show the naturality of φ.

Let X and Y be any topological spaces with $f : X \to Y$ continuous. The following diagram commutes :

$$\begin{tikzcd} X \arrow[r, "\varphi_X"] \arrow[d, "f"'] & \Sigma\mathcal{D}X \arrow[d, "\Sigma\mathcal{D}f"] \\ Y \arrow[r, "\varphi_Y"'] & \Sigma\mathcal{D}Y \end{tikzcd}$$

for if $x \in X$, we have $\Sigma\mathcal{D}f\varphi_X(x) = \Sigma\mathcal{D}f(\tilde{x}) = \tilde{x}\mathcal{D}f$ and $\varphi_Y f(x) = \widetilde{f(x)}$. However, for any V open in Y either $\widetilde{f(x)} = 1$ or $\widetilde{f(x)} = 0$. If $\widetilde{f(x)} = 1$, then $f(x) \in V$. So, $x \in f^{-1}(V) = \mathcal{D}f(V) \Rightarrow \tilde{x}\mathcal{D}f(V) = 1$. Similarly, if $\widetilde{f(x)} = 0$ then $\tilde{x}\mathcal{D}f(V) = 0$. Thus, in either case, $\widetilde{f(x)} = \tilde{x}\mathcal{D}f$. So, we have $\Sigma\mathcal{D}f\varphi_X = \varphi_Y f$.

Therefore φ is natural.

With regard to ψ, for any frame L we need to define $\psi_L : L \to \mathcal{D}\Sigma L$. Since $\mathcal{D}\Sigma L$ is the frame of open sets of ΣL, $\mathcal{D}\Sigma L = \{\Sigma_a : a \in L\}$. Then for $a \in L$, $\Sigma_a \in \mathcal{D}\Sigma L$. So, we define

$\psi_L : L \to \mathcal{D}\Sigma L$ by

$\quad \psi_L(a) \rightsquigarrow \Sigma_a$.

Then ψ_L is a frame homomorphism. We shall show that ψ is a natural transformation.

Let L and M be frames with the frame homomorphism $h : L \to M$. We need to show the commutativity of the following diagram :

$$\begin{tikzcd}
L \arrow[r, "\psi_L"] \arrow[d, "h"'] & \mathcal{D}\Sigma L \arrow[d, "\mathcal{D}\Sigma h"] \\
M \arrow[r, "\psi_M"'] & \mathcal{D}\Sigma M
\end{tikzcd}$$

To this end, for any $a \in L$,

$$\begin{aligned}
\mathcal{D}\Sigma h \psi_L(a) &= \mathcal{D}\Sigma h(\Sigma_a) \\
&= (\Sigma h)^{-1}(\Sigma_a) \\
&= \Sigma_{h(a)} \\
&= \psi_M h(a)
\end{aligned}$$

Thus, $\mathcal{D}\Sigma h \psi_L = \psi_M h$ and therefore ψ is natural.

Now, for any frame L and $\xi \in \Sigma L$

$\Sigma\psi_L\varphi_{\Sigma L}(\xi) = \Sigma\psi_L(\tilde{\xi}) = \tilde{\xi}\psi_L$. But, for any $a \in L$, $\tilde{\xi}\psi_L(a) = \tilde{\xi}\Sigma_a$. Thus,

$\tilde{\xi}\psi_L(a) = 0$ or $\tilde{\xi}\psi_L(a) = 1$. If $\tilde{\xi}\psi_L(a) = 0$, then $\xi \notin \Sigma_a \Rightarrow \xi(a) = 0$. If $\tilde{\xi}\psi_L(a) = 1$, then $\xi \in \Sigma_a \Rightarrow \xi(a) = 1$. So, in either case, $\tilde{\xi}\psi_L = \xi$. Thus, $\Sigma\psi_L\varphi_{\Sigma L}(\xi) = \tilde{\xi}\psi_L = \xi = 1_{\Sigma L}(\xi) \Rightarrow \Sigma\psi_L\varphi_{\Sigma L} = 1_{\Sigma L}$.

Also, for any topological space X and $U \in \mathcal{D}X$

$$\begin{aligned} \mathcal{D}\varphi_X\psi_{\mathcal{D}X}(U) &= \mathcal{D}\varphi_X(\Sigma_U) \\ &= \varphi_X^{-1}(\Sigma_U) \\ &= U \\ &= 1_{\mathcal{D}X} \end{aligned}$$

Thus $\mathcal{D}\varphi_X\psi_{\mathcal{D}X} = 1_{\mathcal{D}X}$.

Finally, we show that $(\psi_L, \Sigma L)$ is a universal map for L with respect to $\mathcal{D}$. Then it follows by *Theorem* 1.1 and *Remark* 1.2 that $\mathcal{D}$ and Σ are adjoint on the right.

Let X be any topological space with $h : L \to \mathcal{D}X$ any frame homomorphism. We need to show that there is a unique continuous map $\bar{h} : X \to \Sigma L$ such that the following diagram commutes :

$$\begin{tikzcd}
\mathcal{D}\Sigma L \arrow[d, "\mathcal{D}\bar{h}"'] & L \arrow[l, "\psi_L"'] \arrow[dl, "h"] \\
\mathcal{D}X &
\end{tikzcd}$$

i.e. $\mathcal{D}\bar{h}\psi_L = h$. Now, since Σ is a contravariant functor, we have that $\Sigma h : \Sigma\mathcal{D}X :\to \Sigma L$ and $\varphi_X : X \to \Sigma\mathcal{D}X$. Then we have

$$\begin{tikzcd}
\Sigma\mathcal{D}X \arrow[d, "\Sigma h"'] & X \arrow[l, "\varphi_X"'] \arrow[dl, "\bar{h}"] \\
\Sigma L &
\end{tikzcd}$$

So, define $\bar{h} : X \to \Sigma L$ by $\bar{h} = \Sigma h \varphi_X$.

Then

$$\begin{tikzcd}
\mathcal{D}\Sigma L \arrow[d, "\mathcal{D}\bar{h}"'] & & L \arrow[ll, "\psi_L"'] \arrow[dll, "h"] \\
\mathcal{D}X & &
\end{tikzcd}$$

for any $a \in L$

$$\begin{aligned}
\mathcal{D}\bar{h}\psi_L(a) &= \mathcal{D}\bar{h}(\Sigma_a) \\
&= (\bar{h})^{-1}(\Sigma_a) \\
&= (\Sigma h\varphi_X^{-1})(\Sigma_a) \\
&= \varphi_X^{-1}(\Sigma h)^{-1}(\Sigma_a) \\
&= \varphi_X^{-1}((\Sigma h)^{-1}(\Sigma_a)) \\
&= \varphi_X^{-1}(\Sigma_{h(a)}) \\
&= h(a)
\end{aligned}$$

$\Rightarrow \mathcal{D}\bar{h}\psi_L = h.$

For the unicity of $\bar{h}$, suppose that $g : X \to \Sigma L$ such that $\mathcal{D}g\psi_L = h$. Since

φ is a natural transformation, the following diagram commutes :

$$\begin{tikzcd}
X \arrow[r, "\varphi_X"] \arrow[d, "g"'] & \Sigma\mathcal{D}X \arrow[d, "\Sigma\mathcal{D}g"] \\
\Sigma L \arrow[r, "\varphi_{\Sigma L}"'] & \Sigma\mathcal{D}\Sigma L
\end{tikzcd}$$

i.e. $\Sigma\mathcal{D}g\varphi_X = \varphi_{\Sigma L}$. Also, $\Sigma\psi_L\varphi_{\Sigma L} = 1_{\Sigma L}$. Thus $\Sigma\psi_L\varphi_{\Sigma L}g = 1_{\Sigma L}g$. Then

$$\begin{aligned}
\Sigma\psi_L(\varphi_{\Sigma L}g) &= g \\
\Rightarrow \ \Sigma\psi_L(\Sigma\mathcal{D}g\varphi_X &= g \\
\Rightarrow (\Sigma\psi_L\Sigma\mathcal{D}g)\varphi_X &= g \\
\Rightarrow \quad (\Sigma_{\mathcal{D}g\psi_L})\varphi_X &= g \\
\Rightarrow \quad \Sigma h\varphi_X &= g \\
\Rightarrow \quad \bar{h} &= g
\end{aligned}$$

Hence $\bar{h}$ is unique.

Thus for each frame L $\exists$ a universal map $(\psi_L, \Sigma L)$. By *Theorem* 1.1 $\exists$ a unique functor G : **Frm** $\rightarrow$ **Top** such that

(i) $GL = \Sigma L$ and

(ii) $u : 1 \rightarrow \mathcal{D}G$ is natural.

However, Σ satisfies the same criteria. By the uniqeness of G, $\Sigma = G$. Hence, by *Theorem* 1.1 $\mathcal{D}$ and Σ are adjoint on the right. $\square$.

The above results are an exposition of those appearing in [1]. We also refer to [1] and [2] for some of the previous results as well as those that follow.

2.4 Spatial Frames and Sober Spaces

A frame L is called a *spatial* frame if ψ_L, as previously defined, is an isomorphism. The topological space X is a *sober* space provided that φ_X, as defined previously, is an homeomorphism. So, we have the category **SpFrm**$\subseteq$**Frm** of Spatial frames and frame homomorphisms. It is a full subcategory of **Frm**. We also have the category **Sob**$\subseteq$**Top** of Sober spaces and continuous maps which is a full subcateg ory of **Top**.

Theorem 2.1 *:*

The following statements are equivalent

1. *L is a spatial frame.*

2. *ψ_L is an isomorphism.*

3. *$\Sigma_a \subseteq \Sigma_b$ whenever $a \leq b$ in L.*

4. *$a < b \Rightarrow \exists\ \xi : L \to 2$ such that $\xi(a) = 0$ and $\xi(b) = 1$.*

Proof :

(1)$\Leftrightarrow$(2) : Obvious by definition.

(2)$\Leftrightarrow$(3) : Trivial.

(3)$\Leftrightarrow$(4):

Suppose that the result in (4) is true. Let $a < b$. Then by the hypothesis $\exists\ \xi : L \to 2$ such that $\xi(a) = 0$ and $\xi(b) = 1$. Then $\xi \in \Sigma_b$ and $\xi \notin \Sigma_a$. If $\eta \in \Sigma_a$, then $\eta(a) = 1$. Since $a < b$ and η is a frame homomorphism, we have $\eta(a) < \eta(b)$. Thus $1 < \eta(b)$
$\Rightarrow \eta(b) = 1 \Rightarrow \eta \in \Sigma_b$. Since $\xi \in \Sigma_b$ and $\xi \notin \Sigma_a \Rightarrow \Sigma_a \subset \Sigma_b$.

Conversely, if $a < b$, then $\Sigma_a \subset \Sigma_b$. Thus $\exists\ \xi \in \Sigma_b$ such that $\xi \notin \Sigma_a$. Thus, $\xi(b) = 1$ and $\xi(a) = 0$. $\square$

Corollary 2.2 :

For every topological space X, the frame of open sets $\mathcal{D}X$ is spatial.

Proof :

Suppose that $U, V \in \mathcal{D}X$ with $U \subset V$. Then $V - U \neq \phi$. Thus $\exists\ x \in V - U$. So, $x \in V$ but, $x \notin U$. Let

$$\tilde{x} : \mathcal{D}X \rightarrow 2 \text{ by}$$

$$\tilde{x}(W) = \begin{cases} 0 & \text{if } x \notin W \\ 1 & \text{if } x \in W \end{cases}$$

Thus, $\exists$ a frame homomorphism $\tilde{x} : \mathcal{D}X \rightarrow 2$ such that $\tilde{x}(U) = 0$ and $\tilde{x}(V) = 1$. So, by the previous theorem, $\mathcal{D}X$ is a spatial frame. □

Theorem 2.3 :

For each frame L, the space ΣL is sober.

Proof :

We need to show that $\varphi_{\Sigma L} : \Sigma L \rightarrow \Sigma\mathcal{D}\Sigma L$ is one-to-one and onto. Let $\varphi_{\Sigma L}(\eta) = \varphi_{\Sigma L}(\xi)$ for $\eta, \xi \in \Sigma L$. Then $\Sigma\psi_{\Sigma L}(\varphi_{\Sigma L}(\xi)) = \Sigma\psi_{\Sigma L}(\varphi_{\Sigma L}(\eta))$.
$\Rightarrow \Sigma\psi_L\varphi_{\Sigma L}(\xi) = (\Sigma\psi_L\varphi_{\Sigma L})(\eta)$. But, $\Sigma\psi_L\varphi_{\Sigma L} = 1_{\Sigma L}$. Thus $1_{\Sigma L}(\xi) = 1_{\Sigma L}(\eta) \Rightarrow \xi = \eta$. Thus $\varphi_{\Sigma L}$ is one-to-one.

Now, since $\Sigma\psi_L\varphi_{\Sigma L} = 1_{\Sigma L}$, we have $\Sigma\psi_L\varphi_{\Sigma L}\Sigma\psi_L = \Sigma\psi_L$. But, if $\xi, \eta \in \Sigma\mathcal{D}\Sigma L$ and $\xi \neq \eta$, then for some $\Sigma_a \in \mathcal{D}\Sigma L$ and $a \in L$, $\xi(\Sigma_a) \neq \eta(\Sigma_a)$. However, $\Sigma_a \in \mathcal{D}\Sigma L$ and ψ_L onto $\Rightarrow \exists\, b \in L$ such that $\psi_L(b) = \Sigma_a$. Then

$$\begin{aligned} \xi(\Sigma_a) &\neq \eta(\Sigma_a) \\ \Rightarrow \xi(\psi_L(a)) &\neq \eta(\psi_L(b)) \\ \Rightarrow \xi\psi_L &\neq \eta\psi_L \\ \Rightarrow \Sigma\psi_L(\xi) &\neq \Sigma\psi_L(\eta) \end{aligned}$$

Thus $\Sigma\psi_L$ is one-to-one. Then $\Sigma\psi_L(\varphi_{\Sigma L}\Sigma\psi_L) = \Sigma\psi_L(1_{\Sigma L}) \Rightarrow \varphi_{\Sigma L}\Sigma\psi_L = 1_{\Sigma L}$. But, $\varphi_{\Sigma L} : \Sigma L \to \Sigma\mathcal{D}\Sigma L$. Then for any $\delta : \mathcal{D}\Sigma L \to 2 \in \Sigma\mathcal{D}\Sigma L$, $\delta\psi_L \in \Sigma L$. Then

$$\begin{aligned} \varphi_{\Sigma L}(\delta\psi_L) &= \varphi_{\Sigma L}(\Sigma\psi_L(\delta) \\ &= (\varphi_{\Sigma L}\Sigma\psi_L)(\delta) \\ &= 1_{\Sigma L}(\delta) \\ &= \delta \end{aligned}$$

Thus $\varphi_{\Sigma L}$ is onto.

Hence, $\varphi_{\Sigma L}$ is one-to-one and onto.

$\Rightarrow \varphi_{\Sigma L}$ is an homeomorphism.

$\Rightarrow \Sigma L$ is sober. □

Proposition 2.9 *:*

Σ *and* $\mathcal{D}$ *induce a dual equivalence between* **SpFrm** *and* **Sob**.

Proof :

By *Theorem* 1.2 it suffices to show that Σ and $\mathcal{D}$ are full, faithfull and representative. We shall show the result for Σ. The proof for $\mathcal{D}$ is analagous.

(i) Σ is full :

Let L and M be spatial frames and $f : \Sigma M \rightarrow \Sigma L$ any frame homomorphism. We need to show that Σ is onto *i.e.* $\exists\ h : L \rightarrow M$ such that $\Sigma h = f$. Now, since $\mathcal{D}$ is contravariant, $\mathcal{D}f : \mathcal{D}\Sigma L \rightarrow \mathcal{D}\Sigma M$. But, $\psi_L : L \rightarrow \mathcal{D}\Sigma L$ and $\psi_M : L \rightarrow \mathcal{D}\Sigma M$. Also, with Σ and $\mathcal{D}$ being adjoint on the right, we have that ψ_M is an isomorphism. Thus $\psi_M \psi_M^{-1} = 1_M$. So, we have the following diagram :

$$\begin{tikzcd} L \arrow[r, "\psi_L"] & \mathcal{D}\Sigma L \arrow[d, "\mathcal{D}f"] \\ M & \mathcal{D}\Sigma M \arrow[l, "\psi_M^{-1}"] \end{tikzcd}$$

Define $h : L \rightarrow M$ by $h = \psi_M^{-1}\mathcal{D}f\psi_L$. Then

$$\begin{aligned} \Sigma h &= \Sigma(\psi_M^{-1}\mathcal{D}f\psi_L) \\ &= \Sigma\psi_L \Sigma\psi_M^{-1}\mathcal{D}f \end{aligned}$$

$$= \Sigma\psi_L \Sigma\mathcal{D}f \Sigma\psi_M^{-1}$$

But ΣL is sober $\Rightarrow \varphi_{\Sigma L}$ is an homeomorphism. Thus $\Sigma\psi_L \varphi_{\Sigma L} = 1_{\Sigma L} \Rightarrow \Sigma\psi_L = \varphi_{\Sigma L}^{-1}$. Also, $(\Sigma\psi_M)^{-1} = (\Sigma\psi_M^{-1})$. Then

$$\begin{aligned} \Sigma h &= \Sigma\psi_L \Sigma\mathcal{D}f \Sigma\psi_M^{-1} \\ &= (\varphi_{\Sigma L}^{-1})\Sigma\mathcal{D}f(\Sigma\psi_M^{-1}) \\ &= (\varphi_{\Sigma L}^{-1})\Sigma\mathcal{D}f((\varphi_{\Sigma L})^{-1})^{-1} \\ &= (\varphi_{\Sigma L}^{-1})\Sigma\mathcal{D}f\varphi_{\Sigma M} \end{aligned}$$

However, φ is a natural transformation. Thus the following diagram is commutative :

$$\begin{tikzcd} \Sigma M \arrow[r, "\varphi_{\Sigma M}"] \arrow[d, "f"'] & \Sigma\mathcal{D}\Sigma M \arrow[d, "\Sigma\mathcal{D}f"] \\ \Sigma L \arrow[r, "\varphi_{\Sigma L}"'] & \Sigma\mathcal{D}\Sigma L \end{tikzcd}$$

i.e. $\Sigma\mathcal{D}f\varphi_{\Sigma M} = \varphi_{\Sigma L}f$. Thus

$$\begin{aligned} \Sigma h &= (\varphi_{\Sigma L})^{-1}\Sigma\mathcal{D}f\varphi_{\Sigma L} \\ &= \varphi_{\Sigma L}^{-1}(\varphi_{\Sigma L}f) \\ &= 1_{\Sigma L}f \end{aligned}$$

$$= f$$

Hence, Σ is full.

(ii) Σ is faithfull :

Let L and M be spatial frames with $g, h : L \to M$ such that $\Sigma g = \Sigma h$. We require that $g = h$. Using the naturality of ψ and $g : L \to M$, we get

$$\begin{tikzcd} L \arrow[r, "\psi_L"] \arrow[d, "g"'] & \mathcal{D}\Sigma L \arrow[d, "\mathcal{D}\Sigma g"] \\ M \arrow[r, "\psi_M"'] & \mathcal{D}\Sigma M \end{tikzcd}$$

Thus $\mathcal{D}\Sigma g\psi_L = \psi_M g$.

Again, using the naturality of ψ and $h : L \to$ we get $\mathcal{D}\Sigma h\psi_L = \psi_M h$. Since $\Sigma h = \Sigma g \Rightarrow \mathcal{D}\Sigma h = \mathcal{D}\Sigma g$. Then

$$\begin{aligned} \mathcal{D}\Sigma h\psi_L &= \psi_M h \\ \Rightarrow \mathcal{D}\Sigma g\psi_L &= \psi_M h \\ \Rightarrow \psi_M g &= \psi_M h \end{aligned}$$

Since ψ_M is an isomorphism, we must have $h = g$. Hence, Σ is one-to-one and thus faithfull.

(iii) Σ is representative :

Let X be any sober space. Then $\mathcal{D}X$ is a spatial frame by *Corollary* 2.2. Then $\Sigma\mathcal{D}X$ is sober by *Theorem* 2.3. Thus $\varphi_X : X \to \Sigma\mathcal{D}X$ is a homomeomorphism. Thus $X \simeq \Sigma\mathcal{D}X$. Hence, Σ is representative.

Thus we have shown that Σ is full, faithfull and representative. Simimlarly, it can be shown that $\mathcal{D}X$ is full, faithfull and representative. So, by *Theorem* 1.2, Σ and $\mathcal{D}$ induce a dual equivalence between **SpFrm** and **Sob**. $\square$

Remark 2.1 *:*

With the above result $(\Sigma, \mathcal{D}, \varphi, \psi)$ *is a dual equivalence between* **SpFrm** *and* **Sob**. *Thus,* **Sob** $\subseteq$ **SpFrm*** $\subseteq$ **Frm*** *where* **SpFrm*** *and* **Frm*** *are the dual categories of* **SpFrm** *and* **Frm** *respectively. We call the* **Frm*** *the category* **Loc** *of* locales *and frame homomorphisms. Thus* **Loc** *is an extension of* **Sob**.

Remark 2.2 *:*

In **Frm**, $\psi_L : L \to \mathcal{D}\Sigma L$ *is the reflection map to* **SpFrm** *for any frame* L.
In **Top**, $\varphi_X : X \to \Sigma\mathcal{D}X$ *is the reflection to* **Sob**.

Proof :

In *Proposition* 2.8, we showed that $(\psi_L, \Sigma L)$ is a universal map for any frame L with respect to the functor $\mathcal{D} = 1_{\mathcal{D}}$. Thus, ψ_L is the reflection to **SpFrm**.

Similarly, $(\varphi_X, \mathcal{D}X)$ can be shown to be a universal map for any space X with respect to $\Sigma = 1_\Sigma$, so that φ_X is the reflection to **Sob**. $\square$

Chapter 3

Regular Frames

We aim to achieve an equivalent notion of the concept of regularity as in spaces for that in frames. We refer to [1], [2], [12] and [14].

Let L be any frame with $a, b \in L$. We say that a is *rather below* b, written as $a \prec b$, if there exists $y \in L$ such that $a \wedge y = 0$ and $y \vee b = 1$. We call the frame L a *regular* frame if

$$a = \bigvee \{b \in L : b \prec a\} \ \forall \ a \in L$$

Lemma 3.1 :

For any topological space X,

$$U \prec V \Leftrightarrow \overline{U} \subseteq V \quad \forall \ U, V \in \mathcal{D}X$$

where $\overline{U}$ is the closure of U in X.

Proof :

($\Longrightarrow$) Let $U, V \in \mathcal{D}X$ and suppose that $V \prec U$. Then $\exists\ W \in \mathcal{D}X$ such that $V \wedge W = 0_{\mathcal{D}X}$ and $W \vee U = 1_{\mathcal{D}X}$ *i.e.* $V \cap W = \phi$ and $W \cup U = X$. Then $V \subseteq X - W \subseteq U \Rightarrow \overline{V} \subseteq \overline{X - W}$. But, $X - W$ is closed in X. So, $\overline{X - W} = X - W$. Thus $\overline{V} \subseteq U$.

($\Longrightarrow$) If $\overline{V} \subseteq U$, then $X - \overline{V} \in \mathcal{D}X, U \in \mathcal{D}X$ and $V \cap (X - \overline{V}) = \phi$ and $(X - \overline{V}) \cup U = X$. Thus $V \prec U$. □

Theorem 3.1 :

A topological space X is a regular space $\Leftrightarrow$ $\mathcal{D}X$ is a regular frame.

Proof :

Suppose that X is a regular topological space. Let $U \in \mathcal{D}X$ be any open set of X. By the regularity of X, we have that $U = \bigvee\{V \text{open in } X : \overline{V} \subseteq U\}$. Then, by the previous lemma, $U = \bigvee\{V \in \mathcal{D}X : V \prec U\} \Rightarrow \mathcal{D}X$ is a regular frame.

Conversely, suppose that $\mathcal{D}X$ is regular. Let U be an open set of X and $x \in U$. Then$U \in \mathcal{D}X$ and $\mathcal{D}X$ regular $\Rightarrow U = \bigvee\{V \in \mathcal{D}X : V \prec U\}$. Since

$x \in U \Rightarrow x \in V$ for some $V \prec U$. Then $\overline{V} \subseteq U$. So, $x \in \overline{V} \subseteq U$. Thus X is a regular space. □

Lemma 3.2 :

Let L be any regular frame. Then for any $a, b, x, y, z \in L$

1. $x \prec a \Leftrightarrow x^* \vee a = 1$, *where x^* is the pseudocomplement of X.*
2. $z \leq x \prec a \Rightarrow z \prec a$.
3. $x \prec a \leq b \Rightarrow x \prec b$.
4. $x, y \prec a \Rightarrow x \vee y \prec a$.
5. $x \prec a, b \Rightarrow x \prec a \wedge b$.

Lemma 3.3 :

Any homomorphic image of a regular frame is a regular frame.

Proof :

Let L and M be frames with L regular and $h : L \to M$ any frame homomorphism. We require that $h(L)$ to be regular.

Let $y \in h(L)$ be arbitrary. Then $y = h(x)$ for some $x \in L$. But L being regular $\Rightarrow x = \bigvee\{b \in L : b \prec x\}$. Then $y = h(x) = h(\bigvee_{b \prec x} b) = \bigvee_{b \prec x} h(b)$.

But, $b \prec x \Rightarrow h(b) \prec h(x) = y$. Thus $y = \bigvee\{h(b) : h(b) \prec y\}$. Hence, $h(L)$ is regular. $\square$

Theorem 3.2 :

Let $\{M_i\}_{i\in I} \subseteq L$ be regular subframes of the frame L for each $i \in I$. Then the subframe M generated by $\{M_i\}_{i\in I}$ is again a regular subframe.

Proof :

Since M is the subframe generated by $\{M_i\}_{i\in I}$, each element in M must be of the form

$$\bigvee_{fin F \subseteq I} \bigwedge F$$

Thus M consists of joins of elements $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ for $a_k \in M_{i_k}$ for each $k = 1, 2, \ldots, n$. Now let $a \in M$. Then $a = \bigvee_{k=1}^{n}(\wedge a_k)$ for $a_k \in M_{i_k}$. But M_{i_k} is regular for each k and i. Thus $a_k = \bigvee\{x_k : x_k \prec a_k\}$ in M_{i_k}. Then

$$\begin{aligned}
a &= \bigvee_{k=1}^{n}(\wedge a_k) \\
&= \bigvee(a_1 \wedge a_2 \wedge \cdots \wedge a_n) \\
&= \bigvee\{(\bigvee_{x_1 \prec a_1}) \wedge (\bigvee_{x_2 \prec a_2}) \wedge \ldots (\bigvee_{x_n \prec a_n})\} \\
&= \bigvee(\bigvee\{x_1 \wedge x_2 \wedge \ldots \wedge x_n\})
\end{aligned}$$

But, $x_k \prec a_k$ in $M_{i_k} \Rightarrow \exists \bar{x_k} \in M_{i_k}$ such that $x_k \wedge \bar{x_k} = 0$ and $\bar{x_k} \vee a_k = 1$ for each $k = 1, 2, \ldots, n$. Then $\wedge \bar{x_k} \in M$ and $\wedge(x_k \wedge \bar{x_k}) = 0$. Also, $\wedge(x_k \vee \bar{x_k}) = 1$.

Thus $\wedge x_k \prec \wedge \bar{x_k}$. Then by *Lemma* 3.2, $\bigvee(\wedge x_k) \prec \wedge a_k$. Let $t = \bigvee(\wedge x_k)$. Then $t \prec \bigvee(\wedge a_k) \Rightarrow t \prec a$. Thus $a = \bigvee\{t : t \prec a\}$. Hence, M is regular. $\square$

M is the largest subframe generated by $\{M_i\}_{i \in I}$. We denote by $RegL$ the largest regular subframe generated by all the regular subframes of the frame L. So, we have the category **RegFrm** with objects being regular frames and frame hom omorphisms. It is a full subcategory of **Frm**.

Proposition 3.1 *:*

RegFrm *is coreflective in* **Frm** *with coreflection map* $i : RegL \longrightarrow L$ *, the inclusion* $RegL \subseteq L$ *for any frame* L*.*

Remark 3.1 *:*

In **Top***, regular spaces are a reflective subcategory.*

Chapter 4

Compact, Normal and Completely Regular Frames

We also get analagous results for the concepts of compactness, normality and complete regularity for frames, as exhibited by spaces. We merely give an excerpt of these concepts in our study. For more insight into them we suggest that the reader refer to [12].

4.1 Compact Frames

An element c in a frame L is called a *compact* element if $c \leq \bigvee X$ for $X \subseteq L \Rightarrow c \leq \bigvee F$ for finite $F \subseteq X$. Such elements are sometimes called *finite* elements. The frame L is called a *compact* frame provided that the unit of L, 1_L, is a compact element. Clearly, the frame $\mathcal{D}X$ is a compact frame $\Leftrightarrow$ the toplogical space X is compact.

4.2 Normal Frames

Normality is defined in a natural way. The frame L is a *normal* frame provided that if $a \vee b = 1$ in L,then $\exists\, x, y \in L$ such that $x \wedge y = 0$, $a \vee x = 1$ and $y \vee b = 1$. Easily, the frame $\mathcal{D}X$ is normal $\Leftrightarrow$ X is a normal topological space.

Theorem 4.1 :

Every compact regular frame is normal.

Proof :

Suppose that L is a compact regular frame with $a, b \in L$ such that $a \vee b = 1$. By regularity, $a = \bigvee\{x \in L : x \prec a\}$ and $b = \bigvee\{y \in L : y \prec b\}$. Then $1 = a \vee b = (\bigvee\{x \in L : x \prec a\}) \vee (\bigvee\{y \in L : y \prec b\}$. By compactness, $1 = (\bigvee_{x_i \prec a} x_i) \vee (\bigvee_{y_j \prec b} y_j)$ for $i \in I$ and $j \in J$ finite . Put $x = \bigvee_{x_i \prec a} x_i$ and $y = \bigvee_{y_j \prec b} y_j$. Then $x \vee y = 1$. But $x \prec a$ and $y \prec b$. Thus $\exists\, s, t \in L$ such that $x \wedge s = 0$ and $s \vee a = 1$, $y \wedge t = 0$ and $t \vee a = 1$. Then $s \wedge t = 0$ such that $a \vee s = 1$ and $b \vee t = 1$. Hence, L is normal. $\square$

4.3 Completely Regular Frames

In the frame L, a is said to be *completely below* b, written $a \prec\!\prec b$, if there exists a family $\{c_{i,k}\}$ where $i = 0, 1, 2, \ldots$ and $k = 0, 1, \ldots, 2^i$ such that for each i, k we have

(i) $c_{i,0} = a$

(ii) $c_{i,2^i} = b$

(iii) $c_{i,k} = c_{i+1,2k}$ and

(iv) $c_{i,k} \prec c_{i,k+1}$

The frame L is *completely regular* if for each $a \in L$

$$a = \bigvee \{b \in L : b \prec\!\prec a\}$$

We have thus the category **CRegFRm** of completely regular frames and frame homomorphisms. It is a full subcategory of **Frm**.

Chapter 5

Locally Finite and Metacompact Nearness Frames

5.1 Nearness Frames

The concept of *nearness* has been extensively studied for spaces in [16]. The frame counterparts have been developed in [2]. We give the essential concepts all of which can be found in [2].

Let L be any frame. We define for $A \subseteq L, B \subseteq L$ the *refinement* relation "$\leq$" as follows. We say that A *refines* B, written $A \leq B$ provided that for each $a \in A$ $\exists$ $b \in B$ such that $a \leq b$.

For $x \in L$, the "*A-star of x*", written $st(x, A)$ or Ax, is the set

$$Ax = \bigvee\{y \in A : y \wedge x \neq 0\}$$

and the star of B with respect to A is the set

$$AB = \{Ax : x \in B\}$$

We say that "*A star-refines B*", written $A^* \leq B$ provided that $AA \leq B$.

A *cover* on L is any subset whose join is the unit, *i.e.* $A \subseteq L$ is a cover of L provided that $\bigvee A = 1$. Any subset $B \subseteq A$ of the cover A is a *subcover* provided that $\bigvee B = 1$. We denote the set of all covers in a frame L by $cov(L)$. For a collection $\mu \subseteq cov(L)$, we say that for $x, y \in L$, x is *μ-strongly below y*, written $x \lhd_\mu y$, provided that $\exists A \in \mu$ such that $Ax \leq y$. The relation $\lhd_\mu$ is call ed the *strong incluson with respect to μ*. The collection μ is an *admissable* subcollection provided that for each $x \in L$,

$$x = \bigvee\{y \in L : y \lhd_\mu x\}$$

A *nearness* on the frame L is an admissable filter $\mu \subseteq cov(L)$. The frame L equipped with the nearness μ, written (L, μ) is then called a *nearness* frame. The members of μ are called *uniform* or *nearness* covers. If μ is a nearness on L such that for all $A \in \mu$ $\exists$ $B \in \mu$ such that $B^* \leq A$, then μ is called a *uniformity* on L and (L, μ) is called a *uniform* frame. The following lemma is essential for our study.

Lemma 5.1 *:*

A frame L has a nearness $\Leftrightarrow$ L is a regular frame.

Proof :

($\Longrightarrow$) Suppose that the frame L has a nearness μ say. Then each $x \in L$ has the representation

$$x = \bigvee\{y \in L : y \lhd_\mu x\}$$

by the admissability of μ. If $y \lhd_\mu x$, then $Ay \leq x$ for some $A \in \mu$. Then for $z = \bigvee\{t \in A : t \wedge y = 0\}$, we have $z \leq y^*$. So, $z \wedge y \leq y^* \wedge y = 0$. Thus $y \wedge z = 0$. Also, $z \vee Ay = (\bigvee\{t \in A : t \wedge y = 0) \vee (\bigvee\{t \in A : t \wedge y \neq 0\}) = \bigvee A$. Since A is a cover, $z \vee Ay = 1$. But, $Ay \leq x \Rightarrow 1 = z \vee Ay \leq z \vee x$. Thus $z \vee x = 1$. So, we have $y \wedge z = 0$ and $z \vee x = 1$. Thus $y \prec x$ and so, $x = \bigvee\{y \in L : y \prec x\}$. Hence L is regular.

($\Longleftarrow$) Suppose that L is regular. Let μ be the filter in L generated by all finite covers. Let $x \in L$ be any element. By regularity, $x = \bigvee\{y \in L : y \prec x\}$. If $y \prec x$, then $A = \{x, y^*\}$ is a μ-cover since it is clear that $y^* \vee x = 1$ Also, $Ay = \bigvee\{a \in A : a \wedge y \neq 0\} = x$. Thus $y \lhd_\mu x$, so $x = \bigvee\{y \in L : y \lhd_\mu x\}$. Hence, μ is admissable and thus is a nearness on L. $\square$

Remark 5.1 *For any regular frame L, any filter $\mu \subseteq cov(L)$ containing all finite covers is thus admissable and so is a nearness by the above lemma. Thus, $cov(L)$ itself is a nearness, which we call the* fine *nearness.*

A frame homomorphism $h : L \to M$ between nearness frames (L, μ_L) and (M, μ_M) is called a *uniform* or *nearness* homomorphism if $h(A) \in \mu_M$ for each $A \in \mu_L$. Thus we have the category **NFrm** of nearness fra mes and uniform homomorphisms. Also, **NFrm**$\subseteq$**RegFrm** is a subcategory of **RegFrm**.

We thus adopt the convention that all frames considered hereafter are assumed to be regular

5.2 Locally Finite Nearness Frames

We refer to [7], [9] and [17] and attempt to find generalisations to the concepts contained therein for nearness frames.

In a frame L, a subset A is *locally finite* provided that $\exists\ B \subseteq cov(L)$ such that each $b \in B$ meets only finitely many elements of A. The frame L is *paracompact* provided that each cover of L has a locally finite refinemen t, and L is *countably paracompact* provided that each countable cover of L has a locally finite refinement. A nearness $\mu \subseteq cov(L)$ is called *locally finite* provided that for each uniform cover $A \ \exists$ a locally finite cover $B \in \mu$ such that $B \leq A$. Then (L, μ) is called a *locally finite* nearness frame.

Let L be a regular frame. Put

1. $\mu_T = \{A \in cov(L) : \exists B \in cov(L) \text{ such that } B \leq A\}$

2. $\mu_P = \{A \in cov(L) : \exists$ finite $B \in cov(L)$ such that $B \leq A\}$

3. $\mu_L = \{A \in cov(L) : \exists$ countable $B \in cov(L)$ such that $B \leq A\}$

4. $\mu_{LF} = \{A \in cov(L) : \exists$ locally finite $B \in cov(L)$ such that $B \leq A\}$

Theorem 5.1 *:*

$\mu_T, \mu_P, \mu_L, \mu_{LF}$ are nearness structures on the frame L.

Proof :

μ_T is clearly the fine nearness, *i.e.* $\mu_T = cov(L)$. We shall show that

(i) μ_P

(ii) μ_L

(iii) μ_{LF}

are nearnesses.

(i) If $A, B \in \mu_P$, then $\exists$ finite covers A' and B' of L such that $A' \leq A$ and $B' \leq B$. Then $A' \wedge B'$ is a finite cover and $A' \wedge B' \leq A \wedge B$. So, $A \wedge B \in \mu_P$. Also, if $A \leq C$ then there exists A' finite such that $A' \leq A \leq C \Rightarrow C \in \mu_P$. Hence, μ_P is a filter on L. But, μ_P clearly contains all finite covers, so by *Remark* 5.1, μ_P is a nearness.

(ii) Let $A, B \in \mu_L$ be any. Then there exists countable covers A' and B' of L such that $A' \leq A$ and $B' \leq B$. Then $A' \wedge B'$ is a countable refinement of

$A \wedge B$, so $A \wedge B \in \mu_L$. Also if $A \leq C$, then $A' \leq A \leq C \Rightarrow A' \leq C$. Thus A' is a countable refinement of C and so $C \in \mu_L$. So, μ_L is a filter. Also, as every finite cover is countable, μ_L contains all finite covers and thus by *Remark* 5.1, μ_L is admissable. Hence μ_L is a nearness on L.

(*iii*) Let $A, B \in \mu_{LF}$ be arbitrary. Then $\exists$ locally finite covers A' and B' of L such that$A' \leq A$ and $B' \leq B$. Then $A' \wedge B'$ is a locally finite refinement of $A \wedge B$. Thus $A \wedge B \in \mu_{LF}$. Again, if $A \leq C$, then A' is a locally finite refinement of C. So, μ_{LF} is a filter. Again, as μ_{LF} contains all finite covers, by *Remark* 5.1, μ_{LF} is admissable and hence a nearness on L. $\square$

Remark 5.2 :

Using the notation in [7] and [9], we call μ_P the Pervin *nearness structure and μ_L the* Lindëlof *nearness structure on the regular frame L.*

Theorem 5.2 *:*

1. *μ_{LF} is a locally finite nearness on L.*
2. *If ν is any locally finite nearness on L, then $\nu \subseteq \mu_{LF}$.*

Proof :

(1) : Proved.

(2) : If ν is any locally finite nearness on L and $A \in \nu$, then $\exists$ locally finite $B \in \nu$ such that $B \leq A$. Then obviously, $B \leq B$ and B locally finite $\Rightarrow B \in \mu_{LF}$. Since μ_{LF} is a filter and $\mu_{LF} \ni B \leq A \Rightarrow A \in \mu_{LF}$. Thus $\nu \subseteq \mu_{LF}$. □

Remark 5.3 *:*

We call μ_{LF} the locally finite *nearness on L.*

The nearness frame (L, μ) is called

(i) *paracompact* provided that each uniform cover has a locally finite uniform refinement *i.e.* $\forall A \in \mu \ \exists B \in \mu$ such that B is locally finite and $B \leq A$.

(ii) *countably paracompact* provided that each countable uniform cover has a uniform locally finite refinement.

Remark 5.4 *It is quite vacuous that a regular frame L is paracompact provided that $(L, cov(L))$ is a paracompact nearness. Paracompact nearness* *spaces have been studied in [5].*

The following theorem compares the Locally Finite nearness, the Pervin nearness, the Lindëlof nearness and the Fine nearness.

Theorem 5.3 *:*

For any regular frame L

1. $\mu_P \subseteq \mu_{LF} \subseteq \mu_T$.

2. $\mu_{LF} = \mu_T \Leftrightarrow L$ *is a paracompact frame.*

3. $\mu_L \subseteq \mu_{LF} \Leftrightarrow L$ *is a countably paracompact frame.*

4. $\mu_{LF} \subseteq \mu_L \Leftrightarrow$ *every locally finite cover of L has a countable subcover.*

Proof :

(1) : If $A \in \mu_P$, then $\exists$ finite $B \in cov(L)$ such that $B \leq A$. As every finite cover is locally finite, $B \in \mu_{LF}$. Since $B \leq A$ and μ_{LF} is a filter $\Rightarrow A \in \mu_{LF}$. Thus $\mu_P \subseteq \mu_{LF}$. As $\mu_T = cov(L)$, clearly $\mu_{LF} \subseteq \mu_T$.

(2) : Suppose that $\mu_{LF} = \mu_T$. Let $A \in cov(L)$ be any. Then $cov(L) = \mu_T \Rightarrow A \in \mu_T = \mu_{LF}$. Then $\exists$ locally finite $B \in cov(L)$ such that $B \leq A$. Thus each cover of L has a locally finite refinement $\Rightarrow L$ is paracompact.

Conversely, if L is paracompact, then by (1) it suffices to show that $\mu_T \subseteq \mu_{LF}$. So, let $A \in \mu_T$ be arbitrary. Then $A \in cov(L)$ and $A \leq A$. Since L is paracompact and $A \in cov(L) \Rightarrow \exists\, B \in cov(L)$, B locally finite such that $B \leq A$. Then B is a locally finite refinement of A. Thus $A \in \mu_{LF}$. So, $\mu_T \subseteq \mu_{LF}$.

(3) : Suppose that $\mu_L \subseteq \mu_{LF}$. Let A be any countable cover of L. Then clearly, $A \in \mu_L$. Thus $A \in \mu_{LF}$. So, $\exists$ a locally finite $B \in cov(L)$ such that $B \leq A$. Thus A has a locally finite refinement and clear ly L is countably paracompact. For the converse, if $A \in \mu_L$ then there exists a countable cover B such that $B \leq A$. By countably paracompactness $\exists$ a locally finite refinement C of B and hence of A *i.e.* $A \in \mu_{LF}$. So, $\mu_L \subseteq \mu_{LF}$.

(4) : Suppose that $\mu_{LF} \subseteq \mu_L$. Let A be a locally finite cover of L. Then $A \in \mu_{LF} \Rightarrow A \in \mu_L$. Thus $\exists$ a countable $B \in cov(L)$ such that $B \leq A$. Then for each $b \in B\ \exists\ a_b \in A$ such that $b \leq a_b$. As B is countable cover, $A_B = \{a_b \in A : b \in B\} \subseteq A$ is a countable subcover of A.

Conversely, suppose that every locally finite cover of L has a countable subcover. Let $A \in \mu_{LF}$. Then $\exists$ a countable subcover A' of A. Since $A' \subseteq A$ we have $A \in \mu_L$, whence $\mu_{LF} \subseteq \mu_L$. $\square$

The nearness frame (L, ν) is called

(i) *totally bounded* provided that for each $A \in \nu\ \exists\ B \in cov(L)$ such that $B \leq A$ and B is finite *i.e.* ν is the nearness determined by all the finite covers of L.

(ii) *contigual* if for each ν-cover A, $\exists$ a ν-cover B such that B is a finite refinement of A.

Theorem 5.4 *:*

For a regular frame L, the following are equivalent

1. $\mu_P = \mu_{LF}$.

2. *Every locally finite cover of L has a finite subcover.*

3. *μ_{LF} is totally bounded.*

4. *μ_{LF} is contigual.*

Proof :

(1)$\Rightarrow$(2) :

Suppose that $\mu_P = \mu_{LF}$. Let A be any locally finite cover of L. Then $A \in \mu_{LF} \Rightarrow A \in \mu_P$. Thus $\exists$ a finite $B \in cov(L)$ such that $B \leq A$. Then for each $b \in B$, $\exists\ a_b \in A$ such that $b \leq a_b$. Then $\{a_b \in A : b \in B\}$ is a finite subcover of A.

(2)$\Rightarrow$(3) :

Suppose (2). For any $A \in \mu_{LF}$, $\exists$ a locally finite $B \in cov(L)$ such that

$B \leq A$. Then B has a finite subcover C. Then C is a finite refinement of A. Thus μ_{LF} is totally bounded.

(3)$\Rightarrow$(4) :

Suppose that μ_{LF} is totally bounded. Let A be a μ_{LF}-cover of L. Then by total boundedness $\exists$ a finite $B \in cov(L)$ such that $B \leq A$. Since every finite cover is locally finite, clearly B is a μ_{LF}-cover. Thus every μ_{LF}-cover has a finite μ_{LF}-refinement $\Rightarrow$ μ_{LF} is contigual.

(4)$\Rightarrow$(1) :

Suppose that μ_{LF} is contigual. By the previous theorem, it suffices to show that $\mu_{LF} \subseteq \mu_P$. Let $A \in \mu_{LF}$ be any. As μ_{LF} is contigual, A has a finite refinement $B \in \mu_{LF}$. So, clearly $A \in \mu_P$. Thus $\mu_{LF} \subseteq \mu_P$. $\square$

We call the nearness frame (L, ν) a *locally fine* nearness frame if whenever $A \in \nu$ and $\{B_a \in \nu : a \in A\} \subseteq \nu$ is any ν-subcollection, then $\{a \wedge b : a \in A$ and $b \in B_a\}$ is a ν-cover.

Theorem 5.5 :

(L, μ_{LF}) is a locally fine nearness frame.

Proof :

Let $A \in \mu_{LF}$ and $\{B_a : a \in A\}$ be a family of μ_{LF}-covers. We require that $\{a \wedge b : a \in A$ and $b \in B_a\} \in \mu_{LF}$.

Since $A \in \mu_{LF}$, $\exists$ a locally finite $S \in cov(L)$ such that $S \leq A$. Then for each $s \in S$, $\exists$ $a_s \in A$ such that $s \leq a_s$. Then $\mathcal{B} = \{B_{a_s} : s \in S\} \subseteq \{B_a : a \in A\}$. Thus $\mathcal{B} \subseteq \mu_{LF}$. Thus for each $s \in S$, $B_{a_s} \in \mu_{LF} \Rightarrow \exists$ T_s locally finite such that $T_s \leq B_{a_s}$. Since S and T_s are locally finite for each $s \in S$ we have $S \wedge T_s$ is locally finite and for each $x \in S \wedge T_s$, $x = u \wedge v$ for some $u \in S$ and $v \in T_s$. Then $\exists$ $a_u \in A$ such that $u \leq a_u$ as $S \leq A$. Also, $\exists$ $b \in B_{a_s}$ such that $v \leq b$ as $T_s \leq B_{a_s}$. Thus $x = u \wedge v \leq a_u \wedge b$. So, $S \wedge T_s \leq \{a \wedge b : a \in A$ and $b \in B_a\}$. Since μ_{LF} is a filter we have $\{a \wedge b : a \in A$ and $b \in B_a\} \in \mu_{LF}$. □

Corollary 5.6 :

Every locally finite nearness frame is locally fine.

Thus, we have the category **LFNFrm** of locally finite nearness frames and uniform homomorphisms. It is a full subcategory of **NFrm**.

Theorem 5.7 :

LFNFrm *is reflective in* **NFrm**.

Proof :

Let (L, μ) be any nearness frame. Define a map

$T : \mathbf{NFrm} \to \mathbf{LFNFrm}$ by

$$T((L,\mu)) \rightsquigarrow (L, T_\mu)$$

where $T_\mu = \{A \in \mu : \exists \text{ a locally finite } B \in \mu \text{ such that } B \leq A\}$. Then T is a uniform homomorphism. T is the reflection to **LFNFrm**.

Let $g : (L,\mu) \to (M,\nu)$ be any uniform homomorphism, where $(M,\nu) \in$ LFNFrm. We then have the following

$$\begin{tikzcd}
(L,T_\mu) \arrow[r, "h"] & (M,\nu) \\
(L,\mu) \arrow[u, "T"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Define $h : (L, T_\mu) \to (M, \nu)$ by

$$h(x) \rightsquigarrow g(x) \qquad \forall x \in L$$

Then h is a uniform homomorphism. Let $A \in T_\mu$. Then $\exists$ a locally finite

$B \in \mu$ such that $B \leq A$. Since g is a uniform homomorphism, $g(B) \in \nu$. Now $g(B) \in \nu$ and (M, ν) locally finite implies that there exists a locally finite $D \in \nu$ such that $D \leq g(B)$. Hence $D \leq g(B) \leq g(A) = h(A)$, *i.e.* $h(A) \in \nu$. Thus h is indeed uniform and is clearly unique. So, (T, T_μ) is a universal map. Hence T is the reflection map. $\square$

5.3 Metacompact Nearness Frames

We refer to [8] and try to generalise the concepts for nearness frames.

We define a cover A in a frame L to be *point-finite* $\Leftrightarrow$ whenever B is a subset of A and $\wedge B \neq 0$, then B must be a finite collection. The frame L is a *metacompact* frame provided that every cover of L has a point-finite refinement. Also, the frame L is *countably metacompact* provided that every countable cover of L has a point-finite refinement.

Proposition 5.1 :

If an open cover in a topological space X is topologically point-finite, then it is point-finite in the frame $\mathcal{D}X$.

Proof :

Let $\mathcal{U}$ be an open point-finite cover of the topological space X and $\mathcal{V} \subseteq \mathcal{U}$

with $\bigwedge \mathcal{V} \neq \phi$. Then $int(\bigcap_{V \in \mathcal{V}} V) \neq \phi$. Thus $\exists\ x \in X$ such that $x \in V$ for each $V \in \mathcal{V}$. Since $\mathcal{U}$ is point-finite, $\exists$ an open set O of X such that $x \in O$ and O meets finitely many members of $\mathcal{U}$. Then $\{U \in \mathcal{U} : O \cap U \neq \phi\}$ is a finite collection. Since $x \in V$ for each $V \in \mathcal{V}$ we have $O \cap V \neq \phi\ \forall\ V \in \mathcal{V}$. Thus $\mathcal{V} \subseteq \{U \in \mathcal{U} : O \cap U \neq \phi\}$ we have $\mathcal{V}$ is a finite collection. Thus $\mathcal{U}$ is point-finite in the frame $\mathcal{D}X$. $\square$

It would be desirable for our translation of point-finiteness to frames to be in accord with the topological equivalent when we consider $\mathcal{D}X$. However, this notion for frames seems to be weaker, but we nevertheless believe it is sufficiently interesting to merit further investigation.

Theorem 5.8 *:*

Every locally finite cover of a frame L is point-finite.

Proof :

Suppose that A is a locally finite cover of the frame L. Let $B \subseteq A$ such that $\bigwedge B \neq 0$. As A is locally finite there exists a cover T such that $T \leq A$ and each $t \in T$ meets only finitely many elements of A. Now since T is a cover, $(\bigwedge B) \wedge t \neq 0$ for some $t \in T$. Thus for each $b \in B$, $b \wedge t \neq 0$. Since $B \subseteq A$, we must have B finite. $\square$

A nearness μ on the frame L is *metacompact* if for each $A \in \mu\ \exists$ a point-finite

$B \in \mu$ which refines A. We then call (L, μ) a *metacompact* nearness frame. Naturally, a regular frame L is metacompact provided that $(L, cov(L))$ is a metacompact nearness frame.

Set $\mu_{PF} = \{A \in cov(L) : \exists$ point-finite $B \in cov(L)$ such that $B \leq A\}$.

Theorem 5.9 :

Let L be a regular frame. Then

1. *μ_{PF} is a metacompact nearness structure on L.*

2. *If ν is any metacompact nearness on L, then $\nu \subseteq \mu_{PF}$.*

Proof :

(1) : We show first that μ_{PF} is a filter on L. Take any $A, B \in \mu_{PF}$. Then $\exists$ point-finite covers A', B' such that $A' \leq A$ and $B' \leq B$. Then $A' \wedge B' \leq A \wedge B$. But, $A' \wedge B'$ is point-finite for if $T \subseteq A' \wedge B'$ with $\bigwedge T \neq 0$, then each $t \in T$ can be expressed as $t = a'_t \wedge b'_t$ for some $a'_t \in A'$ and $b'_t \in B'$. Since

$$\begin{aligned} \bigwedge T \neq 0 \quad &\Rightarrow \quad \bigwedge_{t \in T} t \neq 0 \\ &\Rightarrow \quad \bigwedge_{t \in t} (a'_t \wedge b'_t) \neq 0 \\ &\Rightarrow \quad (\wedge a'_t) \wedge (b'_t) \neq 0 \end{aligned}$$

Thus $(\wedge a'_t) \neq 0$ and $(\wedge b'_t) \neq 0$. However, as A' is point-finite and $\{a'_t : t \in T\} \subseteq A'$ and $(\wedge a'_t) \neq 0 \Rightarrow \{a'_t : t \in T\}$ is a finite collection. Similarly, since

B' is point-finite we g et that $\{b'_t : t \in T\}$ is also a finite collection. Thus $\{a'_t \wedge b'_t : t \in T\} = T$ is a finite collection. Whence, $A' \wedge B'$ is point-finite. Thus $A \wedge B \in \mu_{PF}$.

Also, for if $A \leq C$, then A' is a point-finite refinement of C. So, $C \in \mu_{PF}$. So, we do have that μ_{PF} is a filter. It is quite clear that every finite cover is point-finite. So, μ_{PF} contains all finite covers and so is admissab le. Hence, μ_{PF} is a nearness on L. For μ_{PF} metacompact is obvious by definition of metacompactness.

(2) : Suppose that ν is any metacompact nearness on L and $A \in \nu$. Then $\exists$ a point-finite $B \in cov(L)$ which refines A. So, clearly $A \in \mu_{PF}$ and thus $\nu \subseteq \mu_{PF}$. $\square$

Remark 5.5 *:*

We call μ_{PF} *the* Point Finite *nearness.*

A nearness μ on the frame L is *countably metacompact* provided that each countable μ-cover has a point-finite μ-refinement. Then (L, μ) is called a *countably metacompact* nearness frame.

The following theorem compares the Point Finite nearness, the Locally Finite nearness, the Pervin nearness, the Lindëlof nearness and the Fine nearness of any regular frame L.

Theorem 5.10 :

Let L be any regular frame. Then

1. $\mu_P \subseteq \mu_{PF} \subseteq \mu_T$.

2. $\mu_{LF} \subseteq \mu_{PF}$.

3. $\mu_{PF} = \mu_T \Leftrightarrow L$ *is a metacompact frame.*

4. $\mu_L \subseteq \mu_{PF} \Leftrightarrow L$ *is countably metacompact.*

5. $\mu_{PF} \subseteq \mu_L \Leftrightarrow$ *every point-finite cover of L has a countable subcover.*

Proof :

(1) : Clearly, as every finite cover is point-finite we must have that $\mu_P \subseteq \mu_{PF}$. Also, μ_T is just $cov(L)$ so, $\mu_{PF} \subseteq \mu_T$.

(2) : By *Theorem* 5.8, $\mu_{LF} \subseteq \mu_{PF}$.

(3) : Suppose that $\mu_{PF} = \mu_T$. Let $A \in cov(L)$ be arbitrary.Since $cov(L) = \mu_T = \mu_{PF}$, there exists a point-finite $B \in cov(L)$ such that $B \leq A$. Thus B is a point-finite refinement of A. So, L is metacompact.

Conversely, if L is a metacompact frame, then for any $A \in \mu_T = cov(L)$, there exists $B \in cov(L)$ which refines A. By metacompactness of L, for B, $\exists C \in cov(L)$ such that $C \leq B$ and C is point-finite. Then C is a point-finite refinement of A. So, $\mu_T \subseteq \mu_{PF}$. Together with (1), $\mu_{PF} = \mu_T$.

(4) : Suppose that $\mu_L \subseteq \mu_{PF}$. Let $A \in cov(L)$ be any countable cover. Then $A \in \mu_L = \mu_{PF} \Rightarrow \exists$ a point-finite $B \in cov(L)$ which refines A. So, each cover in L has a point-finite refinement. Thus L is countably metacompact.

Conversely, if L is countably metacompact and $A \in \mu_L$, then $\exists$ a countable cover $B \in cov(L)$ such that $B \leq A$. Since $B \in cov(L)$ and L countably metacompact $\Rightarrow \exists$ a point-finite $C \in cov(L)$ which refines B. Then C is a point-finite refinement of A. So, $C \in \mu_{PF}$. Thus $\mu_L \subseteq \mu_{PF}$.

(5) : Suppose that $\mu_{PF} \subseteq \mu_L$ and A is any point-finite cover of L. Then $A \in \mu_{PF} \subseteq \mu_L \Rightarrow \exists\ B \in cov(L)$ such that $B \leq A$. Then for each $b \in B$, $\exists$ $a_b \in A$ such that $b \leq a_b$.Then $\{a_b : b \in B\}$ is a countable subcover of A.

Conversely, if $A \in \mu_{PF}$, then there exists a point-finite cover B which refines A. Since B is point-finite, B has a countable subcover B'. Then B' is a countable refinement of A. So, $A \in \mu_L$. Thus $\mu_{PF} \subseteq \mu L$. □

We denote by **MNFrm** the category of metacompact nearness frames and uniform frame homomorphisms. It is full subcategory of **NFrm**.

Theorem 5.11 *:*

MNFrm *is reflective in* **NFrm**.

Proof :

Let (L, μ) be any nearness frame. Define a map

$T : \mathbf{NFrm} \rightarrow \mathbf{LFNFrm}$ by

$$T((L, \mu)) \rightsquigarrow (L, T_\mu)$$

where $T_\mu = \{A \in \mu : \exists \text{ a point-finite } B \in \mu \text{ such that } B \leq A\}$. Then T is a uniform homomorphism. T is the reflection to **MNFrm**.

Let $g : (L, \mu) \rightarrow (M, \nu)$ be any uniform homomorphism with (M, ν) metacompact. We then have the following

$$\begin{tikzcd}
(L, T_\mu) \arrow[r, "h"] & (M, \nu) \\
(L, \mu) \arrow[u, "T"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Define $h : (L, T_\mu) \rightarrow (M, \nu)$ by

$$h(x) \rightsquigarrow g(x) \quad \forall x \in L$$

Then h is a uniform homomorphism for if $A \in T_\mu$, $\exists$ a point-finite $B \in \mu$ such that $B \leq A$. Since g is a uniform homomorphism, $g(B) \in \nu$. Since (M, ν) is metacompact $\exists$ point finite $W \in \nu$ such that $W \leq g(B)$. Hence $W \leq g(B) \leq g(A) = h(A)$. As ν is a filter $h(A) \in \nu$. Clearly, h is unique. So, (T, T_μ) is a universal map with respect to the inclusion functor. Thus T is the reflection map. $\square$

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