ASPECTS OF SPHERICALLY SYMMETRIC COSMOLOGICAL MODELS

by

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Abstract

In this thesis we consider spherically symmetric cosmological models when the shear is nonzero and also cases when the shear is vanishing. We investigate the role of the Emden-Fowler equation which governs the behaviour of the gravitational field. The Einstein field equations are derived in comoving coordinates for a spherically symmetric line element and a perfect fluid source for charged and uncharged matter. It is possible to reduce the system of field equations under different assumptions to the solution of a particular Emden-Fowler equation. The situations in which the Emden-Fowler equation arises are identified and studied. We analyse the Emden-Fowler equation via the method of Lie point symmetries. The conditions under which this equation is reduced to quadratures are obtained. The Lie analysis is applied to the particular models of Herlt (1996), Govender (1996) and Maharaj *et al* (1996) and the role of the Emden-Fowler equation is highlighted. We establish the uniqueness of the solutions of Maharaj *et al* (1996). Some physical features of the Einstein-Maxwell system are noted which distinguishes charged solutions. A charged analogue of the Maharaj *et al* (1993) spherically symmetric solution is obtained. The Gutman-Bespal'ko (1967) solution is recovered as a special case within this class of solutions by fixing the parameters and setting the charge to zero. It is also demonstrated that, under the assumptions of vanishing acceleration and proper charge density, the Emden-Fowler equation arises as a governing equation in charged spherically symmetric models.

To all my brothers and sisters across South Africa,
May we never give up our struggle to liberate ourselves.

Preface and Declaration

The study described in this thesis was carried out in the Department of Mathematics and Applied Mathematics, University of Natal, Durban, during the period January 1997 to February 1998. This thesis was completed under the supervision of Professor S D Maharaj and Dr K S Govinder.

The research contained in this study represents original work by the author. It has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

Moodley

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Contents

1 Introduction

The behaviour of the gravitational field is at present best described by the theory of general relativity. Observations in cosmology and astrophysics are consistent with the theoretical predictions made by the theory of general relativity (Will 1981). This has been highlighted by recent observations using modern technology in spacecraft *eg* the Hubble space telescope. General relativity in the framework of a differentiable manifold is utilised to study various aspects of the gravitational field. Increasingly general relativistic models are being utilised for the analysis of strong gravitational fields; the conventional Newtonian models are not appropriate in this scenario *eg* neutron star models. The description of stellar structure and the problem of gravitational collapse are important areas of study in relativistic astrophysics. In cosmology we can answer many fundamental questions about the evolution of the universe by studying cosmological models which satisfy the Einstein field equations.

In this thesis we study spherically symmetric gravitational fields with a perfect fluid energy-momentum tensor for charged and uncharged matter. Spherically symmetric models are physically significant and are extensively utilised in a variety of applications. In astrophysics the collapse of a star can be accurately modelled by a spherically symmetric gravitational field (Shapiro and Teukolsky 1983). In cosmology spherically symmetric spacetimes have been used to model the behaviour and subsequent evolution of the early universe (Krasinski 1997). The spherically sym-

metric models provide an important generalisation of the Robertson-Walker models, the standard cosmological models which are homogeneous and isotropic. To describe many physical situations we need to incorporate anisotropies and inhomogeneities. Anisotropic cosmologies have been considered by a number of authors to study the effect of deviations from an isotropic universe (Ryan and Shepley 1975). The motivation for studying inhomogeneous models is to study the deviation from homogeneity based on observational evidence (Krasinski 1997).

The simplest inhomogeneous cosmological models are the spherically symmetric spacetimes which are invariant under the action of a three dimensional Lie algebra of rotational Killing vectors. If these models are also invariant under a three dimensional Lie algebra of translational Killing vectors, we regain the isotropic and homogeneous Robertson-Walker models. To analyse the behaviour of the gravitational field we need to solve the Einstein field equations which describe how curvature and matter are coupled. This system of highly nonlinear partial differential equations is not easily tractable in general and solutions are normally sought via simplifying assumptions on the nature of the matter content or the form for the gravitational potentials. Solutions to the Einstein field equations are listed by Kramer *et al* (1980) and Krasinski (1997). The study of exact solutions forms an important area of research in general relativity. They are important because many qualitative features of the gravitational field are obtained by analysing individual models. Without exact solutions to the Einstein field equations, it is not possible to consider many of the physical implications of the general field equations because of their nonlinearity. We are interested in obtaining exact solutions to the Einstein field equations in the case of spherically symmetric spacetimes, both for charged and uncharged matter in this thesis.

We review recent exact solutions of the Einstein field equations in spherically symmetric spacetimes which are applicable in cosmology and astrophysics. Note that there are few spherically symmetric solutions having nonzero shear because the system of partial differential equations that arises is more difficult to integrate; the shear-free case in contrast is easier to handle. We are principally concerned with the more general case of nonvanishing shear. An early known solution with shear is the Gutman and Bespal'ko (1967) solution which admits a stiff equation of state. Van den Bergh and Wils (1985) found exact solutions for nonstatic perfect fluid spheres with shear and a barotropic equation of state. All their models have nonzero expansion and some are also accelerating. Maharaj *et al* (1993) found a general class of solutions in terms of elementary functions with shear, expansion and acceleration which obey an equation of state which is a generalisation of the stiff equation of state. They incorporated the solutions of Gutman and Bespal'ko (1967), Hajj-Boutros (1985), Lake (1983), Shaver and Lake (1988) and Wesson (1978). Kitamura (1989, 1994, 1995a, 1995b), using the 'characteristic system' method devised by Takeno (1966), derived exact solutions for a perfect fluid with shear, expansion and acceleration which also involve the Gutman and Bespal'ko (1967), Sussman (1991), Van den Bergh and Wils (1985) and Wesson (1978) solutions as special cases. By imposing a conformal symmetry requirement on the manifold Herrera and Ponce de Leon (1985) produced conformally invariant exact analytical solutions of the field equations. Qadir and Zaid (1995) presented a complete classification of spherically symmetric spacetimes according to their isometries and line elements by solving the Killing equations. These references indicate that spherically symmetric gravitational fields with shear arise in a variety of applications and deserve clear scrutiny.

The difficulty in obtaining exact solutions lies primarily in the complicated system

of nonlinear partial differential equations that arises. It is therefore convenient when the solution of this system of partial differential equations reduces to the integration of a single differential equation. This is the case for shear-free fluids (Maharaj *et al* 1996) and we will demonstrate that a similar situation arises for particular classes of cosmological models where the shear is nonzero. In spherically symmetric spacetimes we discover that, in different physical situations, the system of field equations reduces to the study of the Emden-Fowler equation. The Emden-Fowler equation was first studied in an astrophysical context by Emden (1907). Fowler (1914) established its mathematical foundations and presented it in its present form. Kustaanheimo and Qvist (1948) first derived this equation as the principal equation in the study of shear-free spherically symmetric spacetimes. It has subsequently reappeared in the solution to the spherically symmetric field equations under different assumptions. It appears that the Emden-Fowler equation is of generic importance to spherically symmetric spacetimes.

The study of the mathematical properties of the Emden-Fowler equation and the techniques used to solve this equation are important in the context of general relativity. Since it was first derived, various methods have been used to solve this equation. Only recently has the more general systematic technique of the Lie analysis been applied to the Emden-Fowler equation (Leach 1981). The Lie analysis of differential equations was first formulated by Lie (1891) in his attempt to present a systematic and geometric approach to solve differential equations. The method employs the use of symmetry transformations of a differential equation to reduce the order of the differential equation. This allows us systematically to study the Emden-Fowler equation which, in different situations, appears as the principal equation in spherically symmetric spacetimes. Our intention in this thesis is to study the particular

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Emden-Fowler equation that arises in spherically symmetric systems with shear. We also identify particular classes of cosmological models where the behaviour of the gravitational field is governed by the Emden-Fowler equation.

In chapter 2 of this thesis we consider the basic kinematical and dynamical aspects of spherically symmetric spacetimes which establish the basis for later work. Those aspects of differential geometry relevant to this thesis are briefly discussed in §2.2. The general spherically symmetric line element for a perfect fluid source is specified in comoving coordinates and the kinematical quantities are obtained in §2.3. The nonvanishing components of the connection coefficients, the Ricci tensor, Ricci scalar and the Einstein tensor are explicitly calculated. In §2.4 the energy-momentum tensor is coupled to the Einstein tensor to generate the Einstein field equations. The conservation of energy-momentum is established and the relevance of an equation of state, in particular the barotropic equation of state, is discussed.

In chapter 3 we analyse the generalised Emden-Fowler equation using the method of Lie point symmetries. Special cases in this analysis are also considered. In §3.2 we define some important concepts intrinsic to symmetry groups of differential equations. The technique of the Lie analysis of differential equations is then discussed. In §3.3 we analyse the generalised Emden-Fowler equation via the method of Lie point symmetries and are able to reduce the equation to quadrature under certain conditions. Various special cases, which arise in this analysis, are considered in §3.4. A brief summary of the results of chapter 3 is included in §3.5.

In chapter 4 we investigate the importance of the Emden-Fowler equation in many cosmological models in general relativity. In §4.2 we obtain a specific case of the Emden-Fowler equation under the assumption of vanishing acceleration. Solutions for vanishing pressure are obtained by using the analysis of the Emden-Fowler equation

via the Lie method. In §4.3 we consider the case of vanishing shear in spherically symmetric spacetimes and discuss how the class of solutions obtained, is governed by a specific Emden-Fowler equation. In §4.4 we demonstrate how a particular Emden-Fowler equation arises under the assumption of a stiff fluid equation of state. Solutions for this accelerating, shearing and expanding class are obtained using the Lie analysis of the Emden-Fowler equation. In §4.5 we attempt to integrate the generalised Emden-Fowler equation via an *ad hoc* integration technique. We discuss the success of this technique and relate our results to previously published work.

In chapter 5 we examine the role of charge in spherically symmetric models by investigating the Einstein-Maxwell field equations. In §5.2 we derive the Einstein-Maxwell system of field equations for a spherically symmetric spacetime containing a charged perfect fluid. The conservation of energy-momentum follows from the field equations. In §5.3 we consider certain physical aspects of the structure of the Einstein-Maxwell system and generate qualitative results for the existence of charge. In §5.4 we derive a charged analogue of the spherically symmetric solution obtained by Maharaj *et al* (1993). On fixing the parameters in this class of solutions and setting the charge contribution to zero we regain the Gutman-Bespal'ko (1967) solution. In §5.5 we demonstrate that the Emden-Fowler equation arises as a governing equation in charged spherically symmetric spacetimes under the assumption of vanishing acceleration and proper charge density.

Chapter 6 outlines the conclusions arrived at in this thesis. The main results of the investigations are highlighted and possible extensions arising from these results are discussed.

2 Differential geometry, spherically symmetric space-times and the field equations

2.1 Introduction

The theory of general relativity provides the most successful description of the behaviour of the gravitational field. The differential equations that govern this behaviour arise from the idea that spacetime can be represented by a Riemannian manifold together with the description of the interaction between matter and curvature contained in the Einstein field equations. In this chapter we consider certain basic aspects of spherically symmetric spacetimes containing a perfect fluid source which are necessary for later work. The aspects of differential geometry relevant to general relativity and this thesis are briefly discussed in §2.2. In §2.3 we use the spherically symmetric line element given in comoving coordinates to obtain the kinematical quantities *viz* the acceleration, expansion, shear and vorticity. The non-vanishing components of the connection coefficients, the Ricci tensor, the Ricci scalar and the Einstein tensor are explicitly calculated for spherically symmetric spacetimes. The energy-momentum tensor for a perfect fluid matter distribution is introduced in §2.4. The coupling of the Einstein tensor and the energy-momentum tensor is used to generate the Einstein field equations. The conservation of energy-momentum follows from the field equations. The relevance of the equation of state, in particular the

barotropic equation of state, in cosmological models is also briefly discussed.

2.2 Differential Geometry

In this section we introduce the basic elements of differential geometry required to obtain the Einstein field equations. For a thorough discussion of these and other aspects of differential geometry, the reader is referred to do Carmo (1992). The application of differential geometry to general relativity is dealt with in greater detail by de Felice and Clark (1990), Hawking and Ellis (1973) and Misner *et al* (1973). We take spacetime to be a 4-dimensional differentiable manifold **M** with signature $(- + + +)$. On the manifold we define differentiable structures which can then be used to model the physics of our spacetime. The manifold is labelled by local coordinates $(x^a) = (x^0, x^1, x^2, x^3)$ where x^0 is timelike and x^1, x^2, x^3 are spacelike. A manifold is a topological space which locally has the structure of Euclidean space in that it may be covered with coordinate patches. The global structure of **M** may be very different from that of Euclidean space. The manifold **M** supports a differentiable structure by definition and the passage between the coordinate patches in overlapping coordinate neighbourhoods is smooth. For more information on manifolds refer to the texts given above.

To study physics in the manifold **M** it is necessary to measure the invariant separation of neighbouring points. This is done by introducing a symmetric, nondegenerate metric tensor field **g** onto the manifold. The fundamental line element, defining the invariant infinitesimal separation between neighbouring points on **M**, is given by

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b \tag{2.1}$$

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To characterise the curvature of the manifold we need to introduce additional structure on the manifold. The metric connection $\boldsymbol{\Gamma}$, also known as the Christoffel symbol of the second kind, is defined in terms of the metric tensor $\mathbf{g}$ in (2.1), and its derivatives by

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}) \tag{2.2}$$

where the comma denotes a partial derivative. The metric connection $\boldsymbol{\Gamma}$ is used to generalise the partial derivative in Minkowski spacetime to the covariant derivative in curved spacetime. For example we write the covariant derivative of a $(1,0)$ vector field $\mathbf{X}$ as

$$X^a{}_{;b} = X^a{}_{,b} + \Gamma^a{}_{bc}X^c$$

Similarly the covariant derivative of a $(2,0)$ tensor field $\mathbf{T}$ is defined by

$$T^{ab}{}_{;c} = T^{ab}{}_{,c} + \Gamma^a{}_{cd}T^{db} + \Gamma^b{}_{cd}T^{ad}$$

Clearly the covariant derivative reduces to the partial derivative in Minkowski spacetime.

The Riemann tensor is a $(1,3)$ tensor field which characterises the curvature of spacetime. Also known as the curvature tensor, the Riemann tensor vanishes in flat spacetime; there are always nonvanishing components in a curved spacetime. It is defined by the noncommutativity of the second covariant derivatives of a vector field $\mathbf{X}$ given by

$$X^a{}_{;bc} - X^a{}_{;cb} = R^a{}_{bcd}X^d \tag{2.3}$$

which is sometimes called the Ricci identity. On using the definition of the covariant derivative of the vector $\mathbf{X}$ in (2.3), we can write the Riemann tensor as

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} \tag{2.4}$$

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in terms of the metric connection (2.2). The components $R^a{}_{bcd}$ satisfy the following identities

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}$$

$$R_{abcd} + R_{acdb} + R_{adbc} = 0$$

$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0$$

On contracting the Riemann tensor (2.4) we obtain the Ricci tensor

$$R_{ab} = R^d{}_{adb}$$
$$= \Gamma^d{}_{ab,d} - \Gamma^d{}_{ad,b} + \Gamma^e{}_{ab}\Gamma^d{}_{ed} - \Gamma^e{}_{ad}\Gamma^d{}_{eb} \qquad (2.5)$$

which is a symmetric $(0,2)$ tensor. The Ricci scalar or curvature scalar is obtained by taking the trace of the Ricci tensor and is given by

$$R = g_{ab}R^{ab}$$
$$= R^a{}_a \qquad (2.6)$$

The Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} \qquad (2.7)$$

is defined in terms of the Ricci tensor and Ricci scalar. It is constructed such that it has zero divergence

$$G^{ab}{}_{;b} = 0 \qquad (2.8)$$

a result which follows from the definition (2.7) and the contracted Bianchi identities. The importance of **G** in gravity was first recognised by Einstein when developing

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the field equations for the theory of general relativity. The Bianchi identity (2.8) will be used later, together with the field equations, to derive the conservation of energy-momentum.

2.3 Spherically symmetric spacetimes

Spherical symmetry is a property inherent in many systems in nature. In general relativity it can be used to model both astrophysical and cosmological systems. A spacetime is said to be spherically symmetric if it admits a 3-dimensional Lie algebra of linearly independent rotational Killing vector fields; in a spherically symmetric spacetime the metric field $\mathbf{g}$ is invariant under rotations. The three rotational Killing vectors which the spacetime admit are

$$\boldsymbol{\xi}_1 = \partial_\phi$$

$$\boldsymbol{\xi}_2 = \cos\phi\partial_\theta - \sin\phi\cot\theta\partial_\phi$$

$$\boldsymbol{\xi}_3 = \sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi$$

in polar coordinates. The set of vectors $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$ spans a G_3 Lie algebra of motions and satisfy

$$\xi_{a;b} + \xi_{b;a} = 0$$

which is called Killing's equation.

With the condition of spherical symmetry and using comoving coordinates we can

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show that the line-element takes the form

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + Y^2(t,r)\left[d\theta^2 + \sin^2\theta d\phi^2\right] \qquad (2.9)$$

(Kramer *et al* 1980, Krasinski 1997). The functions $\nu(t,r)$, $\lambda(t,r)$ and $Y(t,r)$ represent the gravitational potentials.

A consequence of spherical symmetry is that the vorticity of the spacetime vanishes. From this result it follows that we can choose a comoving 4-velocity which is hypersurface orthogonal. The timelike, unit fluid 4-velocity **u** is defined by

$$u^a = \left(e^{-\nu}, 0, 0, 0\right)$$

which is orthogonal to the hypersurfaces $\{t = \text{constant}\}$. We measure the kinematical quantities, relative to the 4-velocity **u**, which are given by

$$\omega_{ab} = 0 \qquad (2.10a)$$

$$\dot{u}^a = (0, \nu', 0, 0) \qquad (2.10b)$$

$$\Theta = e^{-\nu}\left(\dot{\lambda} + \frac{2\dot{Y}}{Y}\right) \qquad (2.10c)$$

$$\sigma_1^1 = \sigma_2^2 = -\tfrac{1}{2}\sigma_3^3 = \tfrac{1}{3}e^{-\nu}\left(\frac{\dot{Y}}{Y} - \dot{\lambda}\right) \qquad (2.10d)$$

where ω_{ab} is the vorticity tensor, $\dot{u}^a$ is the acceleration vector, Θ is the expansion scalar and σ_{ab} is the shear tensor of the fluid. The technique of decomposing the covariant derivative of the velocity field $u_{a;b}$ relative to the 4-velocity to obtain the kinematical quantities, is discussed in more detail by Ellis (1971).

We can determine the curvature of the spacetime now that the line element (2.9) has been specified. The connection coefficients follow directly from (2.2) and the

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nonzero coefficients are listed below

$$\Gamma^0{}_{00} = \dot{\nu} \qquad\qquad\qquad \Gamma^0{}_{01} = \nu'$$

$$\Gamma^0{}_{11} = e^{2(\lambda-\nu)}\dot{\lambda} \qquad\qquad\qquad \Gamma^0{}_{22} = e^{-2\nu}Y\dot{Y}$$

$$\Gamma^0{}_{33} = \sin^2\theta\, e^{-2\nu}Y\dot{Y} \qquad\qquad\qquad \Gamma^1{}_{00} = e^{2(\nu-\lambda)}\nu'$$

$$\Gamma^1{}_{01} = \dot{\lambda} \qquad\qquad\qquad \Gamma^1{}_{11} = \lambda'$$

$$\Gamma^1{}_{22} = -e^{-2\lambda}YY' \qquad\qquad\qquad \Gamma^1{}_{33} = -\sin^2\theta\, e^{-2\lambda}YY'$$

$$\Gamma^2{}_{02} = \frac{\dot{Y}}{Y} \qquad\qquad\qquad \Gamma^2{}_{12} = \frac{Y'}{Y}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \qquad\qquad\qquad \Gamma^3{}_{03} = \frac{\dot{Y}}{Y}$$

$$\Gamma^3{}_{13} = \frac{Y'}{Y} \qquad\qquad\qquad \Gamma^3{}_{23} = \cot\theta$$

where dots denote differentiation with respect to t and primes denote differentiation with respect to r. We now generate the Ricci tensor (2.5), utilising the above connection coefficients, to yield the following nonzero components:

$$R_{00} = -\ddot{\lambda} - \dot{\lambda}^2 + \dot{\lambda}\dot{\nu} + 2\dot{\nu}\frac{\dot{Y}}{Y} - 2\frac{\ddot{Y}}{Y}$$
$$+ e^{2(\nu-\lambda)}\left(\nu'' + \nu'^2 - \nu'\lambda' + 2\nu'\frac{Y'}{Y}\right) \qquad (2.11a)$$

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$$R_{01} = 2\left(\dot{\lambda}\frac{Y'}{Y} + \nu'\frac{\dot{Y}}{Y} - \frac{\dot{Y'}}{Y}\right) \tag{2.11b}$$

$$R_{11} = -\nu'' - \nu'^2 + \lambda'\nu' + 2\lambda'\frac{Y'}{Y} - 2\frac{Y''}{Y}$$

$$+ e^{2(\lambda-\nu)}\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + 2\dot{\lambda}\frac{\dot{Y}}{Y}\right) \tag{2.11c}$$

$$R_{22} = e^{-2\nu}Y\dot{Y}\left(\dot{\lambda} - \dot{\nu} + \frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{\dot{Y}}\right)$$

$$+ e^{-2\lambda}YY'\left(\lambda' - \nu' - \frac{Y'}{Y} - \frac{Y''}{Y'}\right) + 1 \tag{2.11d}$$

$$R_{33} = \sin^2\theta\, R_{22} \tag{2.11e}$$

On using the Ricci tensor components (2.11) and the definition (2.6) we establish that the Ricci scalar has the form

$$R = 2e^{-2\nu}\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + 2\dot{\lambda}\frac{\dot{Y}}{Y} - 2\dot{\nu}\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} + 2\frac{\ddot{Y}}{Y}\right)$$

$$- 2e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' - 2\lambda'\frac{Y'}{Y} + 2\nu'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + 2\frac{Y''}{Y}\right) + \frac{2}{Y^2} \tag{2.12}$$

The corresponding nonzero components of the Einstein tensor (2.7) are generated from the Ricci tensor (2.11) and the Ricci scalar (2.12):

$$G_{00} = 2\dot{\lambda}\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} - e^{2(\nu-\lambda)}\left(-2\lambda'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + 2\frac{Y''}{Y}\right) + \frac{e^{2\nu}}{Y^2} \tag{2.13a}$$

$$G_{01} = 2\dot{\lambda}\frac{Y'}{Y} + 2\nu'\frac{\dot{Y}}{Y} - 2\frac{\dot{Y'}}{Y} \tag{2.13b}$$

$$G_{11} = 2\nu'\frac{Y'}{Y} + \frac{Y'^2}{Y^2} + e^{2(\lambda-\nu)}\left(2\dot{\nu}\frac{\dot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} - 2\frac{\ddot{Y}}{Y}\right) - \frac{e^{2\lambda}}{Y^2} \tag{2.13c}$$

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$$G_{22} = -e^{-2\nu} \left[\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} \right) Y^2 + \left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y} \right) Y \right]$$
$$+ e^{-2\lambda} \left[\left(\nu'' + \nu'^2 - \nu'\lambda' \right) Y^2 + \left(\nu'Y' - \lambda'Y' + Y'' \right) Y \right] \qquad (2.13\text{d})$$

$$G_{33} = \sin^2 \theta \, G_{22} \qquad (2.13\text{e})$$

These components specify the curvature and will be used to generate the field equations.

2.4   Einstein field equations

The Einstein field equations describe the coupling between the curvature of a space-time and the matter content. We assume the matter to be a perfect fluid of the form

$$T_{ab} = M_{ab}$$
$$= (\mu + p)u_a u_b + p g_{ab} \qquad (2.14)$$

where the energy density μ and the isotropic pressure p of the fluid are measured relative to the fluid 4-velocity **u**. We obtain the field equations by specifying the curvature through the Einstein tensor **G** and the matter content through the energy-momentum tensor **T**. The energy-momentum tensor is coupled to the Einstein tensor via the Einstein field equations

$$G_{ab} = T_{ab} \qquad (2.15)$$

where we have chosen units in which the value of the gravitational coupling constant is unity. These field equations were first formulated by Einstein to provide a description of gravitating systems.

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The perfect fluid form for the energy-momentum tensor (2.14) has the particular form

$$M_{ab} = \text{diag}\left(\mu e^{2\nu}, \ pe^{2\lambda}, \ pY^2, \ p\sin^2\theta \ Y^2\right) \qquad (2.16)$$

for the line element (2.9). From (2.16) and (2.13) we obtain the Einstein field equations

$$\mu = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - \lambda'Y' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}e^{-2\nu}\left(\dot{\lambda}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right) \qquad (2.17a)$$

$$p = -\frac{1}{Y^2} + \frac{2}{Y}e^{-2\lambda}\left(\nu'Y' + \frac{Y'^2}{2Y}\right) - \frac{2}{Y}e^{-2\nu}\left(\ddot{Y} - \dot{\nu}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right) \qquad (2.17b)$$

$$p = e^{-2\lambda}\left[\nu'' + \nu'^2 - \nu'\lambda' + \frac{1}{Y}\left(\nu'Y' - \lambda'Y' + Y''\right)\right]$$

$$- e^{-2\nu}\left[\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + \frac{1}{Y}\left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y}\right)\right] \qquad (2.17c)$$

$$\dot{Y}' - \dot{Y}\nu' - Y'\dot{\lambda} = 0 \qquad (2.17d)$$

for spherically symmetric gravitational fields.

The conservation of energy-momentum

$$T^{ab}{}_{;b} = 0$$

is a consequence of the field equations (2.15) and the Bianchi identity (2.8). The equation $T^{1b}{}_{;b} = 0$ is the conservation of momentum and the equation $T^{0b}{}_{;b} = 0$ is the conservation of energy. These conservation equations may be written respectively as

$$p' + (\mu + p)\nu' = 0 \qquad (2.18a)$$

$$\dot{\mu} + (\mu + p)\left(\dot{\lambda} + 2\frac{\dot{Y}}{Y}\right) = 0 \qquad (2.18b)$$

The equations (2.18) are first order differential equations which can also be derived directly from the field equations (2.17).

To describe fully the matter content we often specify an equation of state for cosmological and astrophysical systems. The equation of state is chosen on physical grounds. The barotropic equation of state $p = p(\mu)$, relating the energy density to the pressure, has many applications (Collins and Wainwright 1983). The simplest case is the linear form

$$p = (\gamma - 1)\mu$$

which allows for physically reasonable solutions in the range $1 \le \gamma \le 2$. The restriction on γ ensures that the energy conditions are satisfied and causality is not violated (Ellis 1971). Note that particular values of γ correspond to different types of matter. When $\gamma = 1$ the pressure vanishes and we have a dust solution. The case $\gamma = \frac{4}{3}$ corresponds to radiation. The case $\gamma = 2$ corresponds to the stiff equation of state for which the speed of sound equals the speed of light. Another example of an equation of state utilised in applications is the polytropic equation of state

$$p = k\mu^{1+\frac{1}{n}}$$

where k and n are constants (Shapiro and Teukolsky 1983).

3 A group theoretic approach to the Emden-Fowler equation

3.1 Introduction

Many of the numerous approaches and special techniques used to solve differential equations may be unified and extended via a general integration procedure. This procedure perfected by Lie (1891) is based on the invariance of a differential equation under a point transformation. The Lie theory of extended groups provides a powerful technique of solving many differential equations that arise in fields such as classical mechanics, continuum mechanics, hydrodynamics and relativity. In this chapter we analyse the generalised Emden-Fowler equation (Govinder and Leach 1996) for applications in general relativity in later chapters. Note that the study of the Emden-Fowler equation originated from earlier theories concerning gaseous dynamics in astrophysics. More recently the Emden-Fowler equation has been used to study chemically reacting systems, nuclear physics and relativistic mechanics (Wong 1975). One of the earliest attempts at solving the Emden-Fowler equation via the Lie method was performed by Leach (1981) when studying time dependent anharmonic oscillators. In section §3.2 we introduce the technique of the Lie analysis by first defining the important concepts intrinsic to symmetry groups of differential equations. The method of obtaining the symmetries and their use in the reduction of

differential equations to quadrature is then discussed with an illustrative example. In section §3.3 we analyse the generalised Emden-Fowler equation via the method of Lie point symmetries. The conditions under which this equation is reduced to quadratures are obtained. Various special cases which arise in this analysis are investigated in §3.4. For ease of reference the results obtained are briefly summarised in §3.5.

3.2   Lie symmetries and their application to differential equations

In this section we briefly introduce the technique of the Lie analysis. This method will be used to solve the ordinary differential equations that arise from relativistic models. The strength of the Lie analysis is that differential equations can be solved using their symmetries. By a symmetry we mean the generator of a transformation which leaves the form of the differential equation invariant. The symmetries are used to either reduce the order of the differential equation or transform the differential equation to a simpler form. Before describing the method in detail we define some important concepts intrinsic to Lie symmetries of ordinary differential equations.

One of the routes to the solution of differential equations is by transforming the dependent or independent variable, which results in a simpler equation to solve. When the transformation depends on the variables only, it is called a point transformation. We confine our attention to this type of transformation, although more general types of transformation can be used *eg* contact transformations (Mahomed and Leach 1991). We define a one-parameter group of point transformations

$$\tilde{x} = \tilde{x}(x, y; \epsilon)$$

$$\tilde{y} = \tilde{y}(x, y; \epsilon)$$

which depends on the arbitrary parameter ϵ and satisfies the group properties. We define an infinitesimal transformation

$$\tilde{x}(x, y; \epsilon) = x + \epsilon\xi(x, y) + \cdots = x + \epsilon Gx + \cdots$$

$$\tilde{y}(x, y; \epsilon) = y + \epsilon\eta(x, y) + \cdots = y + \epsilon Gy + \cdots$$

using the functions

$$\xi(x, y) = \frac{\partial \tilde{x}}{\partial \epsilon}\Big|_{\epsilon=0}$$

$$\eta(x, y) = \frac{\partial \tilde{y}}{\partial \epsilon}\Big|_{\epsilon=0}$$

and the operator

$$G = \xi(x, y)\partial_x + \eta(x, y)\partial_y$$

The quantities η and ξ are the components of the tangent vector G, which is called the generator of the infinitesimal transformation. There always exist coordinates such that the generator takes the normal form $G = \partial_s$, where s is the new independent variable.

Applying a point transformation to an ordinary differential equation

$$H(x, y, y', \cdots, y^{(n)}) = 0$$

requires knowledge of how the derivatives of y transform. This is calculated from

$$\tilde{y}' = \frac{\mathrm{d}\tilde{y}(x, y; \epsilon)}{\mathrm{d}\tilde{x}(x, y; \epsilon)} = \tilde{y}'(x, y, y'; \epsilon)$$

$$\tilde{y}'' = \frac{\mathrm{d}\tilde{y}'}{\mathrm{d}\tilde{x}} = \tilde{y}''(x, y, y', y''; \epsilon)$$

$$\vdots$$

The extension up to the n-th derivative of the generator G is given by (Mahomed and Leach 1990)

$$G^{[n]} = \xi(x,y)\partial_x + \eta(x,y)\partial_y + \left(\eta^{(n)} - \Sigma_{j=0}^{k-1}\binom{k}{j}y^{(j+1)}\xi^{(k-j)}\right)\partial_{y^{(n)}}$$

where the prime denotes the total derivative $\mathrm{d}/\mathrm{d}x$.

For our purposes a symmetry transformation of an ordinary differential equation is a point transformation that maps solutions into solutions. In mathematical notation we require that, under a symmetry transformation, the image $\tilde{y}(\tilde{x})$ of any solution $y(x)$ is again a solution. It is easily established that $G = \xi\partial_x + \eta\partial_y$ is a symmetry of $H(x,y,y',...,y^{(n)}) = 0$ if and only if

$$G^{[n]}H\,|_{H=0} = 0 \tag{3.1}$$

Condition (3.1) is an identity in the powers of y'. Equating coefficients of the different powers of y' to zero results in a system of linear partial differential equations in ξ and η. Solving this system explicitly for ξ and η yields the symmetry G.

Once the symmetries are known explicitly, they can be used to solve the differential equation. A single symmetry of a differential equation can be used to either reduce the order or to cast the differential equation into standard (or autonomous) form. We consider both these techniques below.

Consider the differential equation

$$y'' = y^2 \tag{3.2}$$

which is a special case of the generalised Emden-Fowler equation. This equation can be reduced to quadratures via the Lie approach if it possesses at least two symmetries. Using the package Program Lie (Head 1993), the symmetries are found to be

$$G_1 = \partial_x$$

$$G_2 = \tfrac{1}{2}x\partial_x - y\partial_y$$

where $[G_1, G_2] = \tfrac{1}{2}G_1$. We first reduce the order of (3.2) using the symmetry G_1. This choice is dictated by the fact that the nonzero commutator of G_1 and G_2 is proportional to G_1. Once reduced, the resulting equation will still possess the (transformed) symmetry G_2. If we reduce (3.2) using G_2 then the resulting equation will not possess any additional point symmetries (Olver 1993). The symmetry $G_1 = \partial_x$ has the associated Lagrange's system

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}y'}{0}$$

The two characteristics

$$\alpha = y, \qquad \beta = y'$$

can be used to determine the transformation

$$u = \gamma_1(\alpha) = \alpha = y$$

$$v = \gamma_2(\beta) = \beta = y'$$

where we have chosen the arbitrary functions γ_1 and γ_2 as indicated above and u is now the independent variable. Under this transformation (3.2) reduces to

$$vv'(u) = u^2 \tag{3.3}$$

which is a first order nonautonomous differential equation. (At this stage we could easily reduce (3.3) to quadrature. However, we choose to transform the equation into autonomous form to illustrate another use of symmetries.) In terms of the new variables the additional symmetry G_2 can be written as

$$G_2^{[1]} = \tfrac{1}{2}x\partial_x - y\partial_y - \tfrac{3}{2}y'\partial_{y'}$$

$$= -u\partial_u - \tfrac{3}{2}v\partial_v$$

An equation in autonomous form has the symmetry $\tilde{G} = \partial_X$ (Bluman and Kumei 1989) where we have chosen X as the independent variable and Y as the dependent variable. We seek to transform the symmetry

$$G_2^{[1]} = -(uX_u + \tfrac{3}{2}vX_v)\partial_X - (uY_u + \tfrac{3}{2}vY_v)\partial_Y$$

into $\tilde{G} = \partial_X$ where the subscript denotes the partial derivative with respect to that variable. On equating the coefficient of ∂_X to one and the coefficient of ∂_Y to zero we obtain the system

$$-uX_u - \tfrac{3}{2}vX_v = 1 \tag{3.4a}$$

$$-uY_u - \tfrac{3}{2}vY_v = 0 \tag{3.4b}$$

which is solved by the method of characteristics to obtain the required transformation. The associated Lagrange's system

$$\frac{\mathrm{d}v}{-\tfrac{3}{2}v} = \frac{\mathrm{d}u}{-u} = \frac{\mathrm{d}Y}{0}$$

for Y is obtained from (3.4b) and possesses the characteristic $\alpha = u^{-1}v^{\frac{2}{3}}$. On choosing the second characteristic as $Y = \beta = \beta(\alpha) = \alpha$ we obtain the transformation

$$Y = u^{-1}v^{\frac{2}{3}}$$

The associated Lagrange's system for X is obtained from (3.4a) as

$$\frac{\mathrm{d}v}{-\tfrac{3}{2}v} = \frac{\mathrm{d}u}{-u} = \frac{\mathrm{d}X}{1} \tag{3.5}$$

This yields the transformation

$$X = -\ln(u) + \gamma_1(\alpha)$$

$$= -\ln(u)$$

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on setting $\gamma(\alpha) = 0$ where $\alpha = u^{-1}v^{\frac{2}{3}}$ is also a characteristic of the system (3.5). The required transformation

$$X = -\ln(u)$$

$$Y = \gamma_2(\alpha) = \alpha = u^{-1}v^{\frac{2}{3}}$$

allows (3.3) to be written in the autonomous form

$$Y' - Y + \tfrac{2}{3}Y^{-2} = 0$$

which can again be reduced to an algebraic equation. More detailed discussions of these ideas and methods are available in texts by Bluman and Kumei (1989) and Olver (1993). We now apply these techniques to the Emden-Fowler equation.

3.3 The Lie analysis of the Emden-Fowler equation

We are concerned with the Emden-Fowler equation in the form

$$y'' + f(x)y^n = 0 \tag{3.6}$$

Our intention is to find those functions $f(x)$ such that (3.6) can be reduced to a first order equation or quadratures. The presence of the (initially) arbitrary function $f(x)$ prevents the direct use of Program Lie (Head 1993). Applying condition (3.1) with H given by (3.6), we obtain

$$\xi_{yy} = 0 \tag{3.7a}$$

$$\eta_{yy} - 2\xi_{xy} = 0 \tag{3.7b}$$

$$2\eta_{xy} - \xi_{xx} + 3f(x)y^n\xi_y = 0 \tag{3.7c}$$

$$\xi f_x y^n + n\eta f(x)y^{n-1} + \eta_{xx} - f(x)y^n(\eta_y - 2\xi_x) = 0 \tag{3.7d}$$

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which is an over-determined system of linear partial differential equations in ξ and η. Solving (3.7a) and (3.7b) yields

$$\xi(x,y) = a(x) + yb(x)$$

$$\eta(x,y) = y^2 b'(x) + yc(x) + d(x)$$

which upon substitution into (3.7c) results in the equation

$$2c'(x) - a''(x) + 3b''(x)y + 3f(x)d(x)y^n = 0 \tag{3.8}$$

This is an identity in the powers of y. From equation (3.8) we note that the cases $n = 0$ and $n = 1$ are special. Both these cases correspond to linear differential equations: the case $n = 0$ is easily solved by

$$y(x) = -\int^x \left(\int^u f(v)dv \right) du$$

and the case $n = 1$ corresponds to the linear harmonic oscillator equation. A more detailed discussion of linear differential equations is available in Ince (1956). For $n \neq 0, 1$ we have, on equating coefficients of the powers of y to zero in (3.8), the equations

$$b(x) = 0 \tag{3.9a}$$

$$b''(x) = 0 \tag{3.9b}$$

$$c'(x) = \tfrac{1}{2}a''(x) \tag{3.9c}$$

The first and second equations (3.9a, 3.9b) are satisfied by $b(x) = 0$. We integrate (3.9c) to obtain

$$c(x) = \tfrac{1}{2}a'(x) + C \tag{3.10}$$

where $C = $ constant. Substituting for $\xi = a(x)$ and $\eta = d(x) + (\tfrac{1}{2}a'(x) + C)y$ we rewrite (3.7d) as

$$\xi y^n [a(x)f_x + (n-1)f(x)(\tfrac{1}{2}a'(x) + C) + 2f(x)a'(x)]$$

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$$+y^{n-1}[nf(x)d(x)] + y[a'''(x)] + [d''(x)] = 0 \qquad (3.11)$$

where we have used (3.10). We equate coefficients of the powers of y to zero, to obtain

$$d''(x) = 0 \qquad (3.12a)$$

$$d(x) = 0 \qquad (3.12b)$$

$$a'''(x) = 0 \qquad (3.12c)$$

$$a(x)f_x + (n-1)(\tfrac{1}{2}a'(x) + C)f + 2f(x)a'(x) = 0 \qquad (3.12d)$$

(The special cases $(n = 2)$ and $(n = -3, C = 0)$ will be analysed later.) The equations (3.12a) and (3.12b) are satisfied by $d(x) = 0$. We integrate (3.12c) to obtain

$$a(x) = A_1 x^2 + A_2 x + A_3 \qquad (3.13)$$

which implies, via (3.10), that

$$c(x) = A_1 x + \tfrac{1}{2}A_2 + C$$

Integrating (3.12d) (excluding the possibilities of $f(x) = 0$ and $a(x) = 0$) yields

$$f(x) = Ka(x)^{-\frac{n+3}{2}} e^{-\int \frac{C(n-1)}{a(x)} dx} \qquad (3.14)$$

which is the form $f(x)$ must assume for the equation (3.6) to possess the symmetry

$$G = (A_1 x^2 + A_2 x + A_3)\partial_x + y(A_1 x + \tfrac{1}{2}A_2 + C)\partial_y \qquad (3.15)$$

To use this symmetry we restrict ourselves to considering the Emden-Fowler equation of the form

$$y'' + Ka(x)^{-\frac{n+3}{2}} e^{-\int \frac{C(n-1)}{a(x)} dx} y^n = 0 \qquad (3.16)$$

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Note that this equation possesses just the one symmetry G, *ie* (3.15), and not any more since the constants A_1, A_2, A_3 and C are not arbitrary but constrained by the equation (3.16). This symmetry is used to write (3.16) in autonomous form by performing a point transformation from variables (x, y) to (X, Y). We write the transformation as

$$X = E(x, y), \qquad Y = F(x, y)$$

We seek to transform the symmetry

$$G = a(x)\partial_x + y(\tfrac{1}{2}a'(x) + C)\partial_y$$

$$= (aE_x + (\tfrac{1}{2}a'(x) + C)yE_y)\partial_X + (aF_x + (\tfrac{1}{2}a'(x) + C)yF_y)\partial_Y$$

(where we have not substituted for $a(x)$) into $\bar{G} = \partial_X$. On equating the coefficients of ∂_X and ∂_Y to one and zero respectively we obtain the system

$$a(x)F_x + (\tfrac{1}{2}a'(x) + C)yF_y = 0 \qquad (3.17a)$$

$$a(x)E_x + (\tfrac{1}{2}a'(x) + C)yE_y = 1 \qquad (3.17b)$$

which is solved by the method of characteristics. The associated Lagrange's system

$$\frac{\mathrm{d}x}{a(x)} = \frac{\mathrm{d}y}{y(\tfrac{1}{2}a'(x) + C)} = \frac{\mathrm{d}F}{0}$$

for $F(x, y)$ is obtained from (3.17a). The first characteristic is

$$\alpha = y\mathrm{e}^{-\int \frac{\frac{1}{2}a'(x) + C}{a(x)}\mathrm{d}x}$$

and choosing the second characteristic as $F = \beta = \beta(\alpha) = \alpha$ results in

$$F(x, y) = y\mathrm{e}^{-\int \frac{\frac{1}{2}a'(x) + C}{a(x)}\mathrm{d}x}$$

The associated Lagrange's system for $E(x, y)$ follows from (3.17b) as

$$\frac{\mathrm{d}x}{a(x)} = \frac{\mathrm{d}y}{(\tfrac{1}{2}a'(x) + C)y} = \frac{\mathrm{d}E}{1}$$

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from which we obtain

$$E(x,y) = \int \frac{\mathrm{d}x}{a(x)} + \gamma(\alpha)$$

$$= \int \frac{\mathrm{d}x}{a(x)}$$

where we have set $\gamma(\alpha) = 0$. The required transformation is thus

$$X = E(x,y) = \int \frac{\mathrm{d}x}{a(x)}$$

$$Y = F(x,y) = y\mathrm{e}^{-\int \frac{\frac{1}{2}a'(x)+C}{a(x)}\mathrm{d}x} \tag{3.18}$$

Under this transformation the derivative $y'' = \mathrm{d}^2 y/\mathrm{d}x^2$ is written as

$$y'' = a(x)^{-2}\mathrm{e}^{\int \frac{\frac{1}{2}a'(x)+C}{a(x)}\mathrm{d}x}\left(Y'' + 2CY' + Y(C^2 + A_1A_3 - \tfrac{1}{4}A_2^2)\right)$$

where $Y' = \mathrm{d}Y/\mathrm{d}X$ and $Y'' = \mathrm{d}^2 Y/\mathrm{d}X^2$ while the remaining term in (3.16) trans-forms as

$$Ka(x)^{-\frac{n+3}{2}}\mathrm{e}^{-\int \frac{C(n-1)}{a(x)}\mathrm{d}x}y^n = KY^n a(x)^{-2}\mathrm{e}^{\int \frac{\frac{1}{2}a'(x)+C}{a(x)}\mathrm{d}x}$$

It follows that the equation (3.16) is transformed via (3.18) to

$$Y'' + 2CY' + DY + KY^n = 0 \tag{3.19}$$

where $D = C^2 + A_1A_3 - \tfrac{1}{4}A_2^2 = $ constant.

Equation (3.19) is easily reduced to quadrature if $C = 0$. For this case (3.19) simplifies to the form

$$Y'' + DY + KY^n = 0$$

which is reduced via

$$u = Y, \qquad v = Y' \tag{3.20}$$

to the first order equation

$$vv' + Du + Ku^n = 0$$

where u is the independent variable. Integrating this equation and transforming back to the original variables yields the quadrature

$$X - C_2 = \int \frac{dY}{\sqrt{C_1 - DY^2 - \frac{2K}{n+1}Y^{n+1}}}$$

for the case $n \neq -1$. In the case $n = -1$ we obtain

$$X - C_2 = \int \frac{dY}{\sqrt{C_1 - DY^2 - 2K\ln Y}}$$

On inverting (3.18) to transform the quadratures to the original variables x and y, we obtain the solution to the Emden-Fowler equation of the form

$$y'' + Ka(x)^{-\frac{n+3}{2}}y^n = 0$$

for the case $C = 0$, where $a(x)$ is given by (3.13).

For the general case $C \neq 0$, (3.20) is again used to reduce (3.19). In this case, we obtain

$$vv' + 2Cv + Du + Ku^n = 0$$

which is an Abel equation of the second kind. The solution of this first order equation, though it exists in principle, is not obvious. To proceed further we have to analyse (3.19) for additional symmetries.

The symmetry condition (3.1) for equation (3.19) yields the identity

$$Y'^3(-\xi_{YY}) + Y'^2(\eta_{YY} - 2\xi_{XY}) + Y'(2\eta_{XX} - \xi_{XX} + 3DY\xi_Y + 3KY^n\xi_Y)$$
$$+ \Big(D\eta + nK\eta Y^{n-1} + 2C\eta_X + \eta_{XX} + (2\xi_X - \eta_X)(DY + KY^n)\Big) = 0$$

On equating coefficients of the powers of Y' to zero we obtain the system

$$\xi_{YY} = 0 \qquad (3.21a)$$

$$\eta_{YY} - 2\xi_{XY} = 0 \qquad (3.21b)$$

$$2\eta_{XX} - \xi_{XX} + 3DY\xi_Y + 3KY^n\xi_Y = 0 \qquad (3.21c)$$

$$\eta D + nK\eta Y^{n-1} + 2C\eta_X + \eta_{XX} + (2\xi_X - \eta_X)(DY + KY^n) = 0 \qquad (3.21d)$$

Integrating equations (3.21a) and (3.21b) yields

$$\xi = a(X)Y + b(X)$$

$$\eta = (2Ca - a')Y^2 + c(X)Y + d(X)$$

which, upon substitution into (3.21c), gives

$$Y(10Ca' - 3Da - 5a'') + Y^n(3Ka) + (2c' + 2Cb' - b'') = 0 \qquad (3.22)$$

We equate coefficients of the powers of Y to zero in (3.22) to obtain

$$10Ca' - 3Da - 5a'' = 0 \qquad (3.23a)$$

$$a = 0 \qquad (3.23b)$$

$$2c' + 2Cb' - b'' = 0 \qquad (3.23c)$$

Equations (3.23a) and (3.23b) are satisfied by $a(X) = 0$ while (3.23c) relates $b(X)$ to $c(X)$. On substituting for ξ and η, equation (3.21d) simplifies to the identity

$$Y^n(2Kb' + c(n-1)K) + Y^{n-1}(Knd) +$$

$$Y(2Cc' + c'' + 2Db') + (d'' + 2Cd' + Dd) = 0$$

which yields the system

$$d'' + 2Cd' + Dd = 0 \qquad (3.24a)$$

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$$d = 0 \tag{3.24b}$$

$$2Cc' + c'' + 2Db' = 0 \tag{3.24c}$$

$$2b' + c(n-1) = 0 \tag{3.24d}$$

on equating the coefficients of the powers of Y to zero. The equations (3.24a) and (3.24b) are satisfied by $d(X) = 0$. We solve equations (3.23c) and (3.24d) to obtain

$$c(X) = M_1 e^{\frac{2C(n-1)}{(n+3)}X}$$

$$b(X) = -\frac{n+3}{4C} M_1 e^{\frac{2C(n-1)}{(n+3)}X} + M_2$$

(excluding the case $n = -3$, $C \neq 0$ which has no additional symmetries to (3.15)). Substituting $b(X)$ and $c(X)$ into equation (3.24c) yields the constraint

$$C^2 \left(\frac{n-1}{n+3}\right)^2 = \frac{A_2^2 - 4A_1 A_3}{4} \tag{3.25}$$

which forces $a(x)$ to have real roots. The symmetry of (3.19) is given by

$$\tilde{G} = \left(-\frac{n+3}{4C} M_1 e^{\frac{2C(n-1)}{(n+3)}X} + M_2\right) \partial_X + \left(Y M_1 e^{\frac{2C(n-1)}{(n+3)}X}\right) \partial_Y$$

In fact (3.19) has two symmetries because of the two arbitrary constants M_1 and M_2. The first symmetry $\tilde{G}_1 = \partial_X$ is obtained by setting $M_1 = 0$ and $M_2 = 1$. The additional symmetry

$$\tilde{G}_2 = \left(\frac{n+3}{4C} e^{\frac{2C(n-1)}{(n+3)}X}\right) \partial_X - \left(Y e^{\frac{2C(n-1)}{(n+3)}X}\right) \partial_Y$$

is obtained by setting $M_1 = 1$ and $M_2 = 0$ and arises only if the condition (3.25) holds. This symmetry is used to transform equation (3.19) into a simpler form. Under the transformation

$$w = E(X, Y) = \frac{2}{n-1} e^{-\frac{2C(n-1)}{(n+3)}X}$$

$$z = F(X, Y) = Y e^{-\frac{4C}{(n+3)}X} \tag{3.26}$$

equation (3.19) reduces to

$$z'' + Nz^n = 0 \qquad (3.27)$$

where $N = (\frac{n+3}{4C})^2 K = $ constant.

The conditions (3.14) and (3.25) combine to generate a stronger restriction on the form of $f(x)$ in (3.6). Condition (3.25) implies that the discriminant $\Delta\{a(x)\} = A_2^2 - 4A_1A_3 \geq 0$. This means that $a(x)$ can be factorised as $a(x) = A_1(x-\alpha)(x-\beta)$, where α and β are the two roots of $a(x)$. Substituting this form for $a(x)$ into (3.14) yields the form

$$f(x) = L(x-\alpha)^{-(n+3)}$$

where $L = KA_1^{\frac{n+3}{2}}$ and we have used the fact that $\alpha - \beta = \frac{2C(n-1)}{A_1(n+3)}$. This means that, with the exception of the special cases $n = 0, 1, 2, -3$, we have reduced the Emden-Fowler equation of the form

$$y'' + L(x-\alpha)^{-(n+3)}y^n = 0 \qquad (3.28)$$

to the simpler equation (3.27). Combining (3.18) and (3.26), we obtain the transformation

$$w = \frac{2}{n-1}\left(\frac{x-\beta}{x-\alpha}\right)$$

$$z = \frac{y}{\sqrt{A_1}}(x-\alpha)^{\frac{n+3}{n-1}} \qquad (3.29)$$

that allows this simplfication.

Equation (3.27) is easily reduced to quadrature. For the case $n = -1$ we obtain

$$w - C_2 = \int \frac{dz}{\sqrt{2C_1 - 2N\ln z}}$$

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while for the general case $(n \neq -3, -1, 0, 1, 2)$ we obtain

$$w - C_2 = \sqrt{\frac{n+1}{2N}} \int \frac{\mathrm{d}z}{\sqrt{C_1 - z^{n+1}}} \qquad (3.30)$$

where C_1 and C_2 are arbitrary constants. The solution of the Emden-Fowler equation (3.28) has been reduced to evaluating the above integral. Given a particular value of n, where $n \neq -3, -1, 0, 1, 2$, we obtain the solution of (3.28) by evaluating the integral

$$\int \frac{\mathrm{d}z}{\sqrt{C_1 - z^{n+1}}}$$

and using (3.29) to transform the solution back to the original variables.

Let us consider the case $n = 3$. Here the integral (3.30) that must be evaluated is an elliptic integral of the first kind. The solution

$$\int \frac{\mathrm{d}z}{\sqrt{C_1 - z^4}} = \frac{1}{\sqrt{C_1}} \mathrm{sn}^{-1} \left(\frac{z}{C_1} \right)$$

is written in terms of an elliptic function (Gradshteyn and Ryzhik 1994). On substituting this solution into (3.30) and transforming back to the original variables via (3.29), we obtain the solution to

$$y'' + L(x - \alpha)^{-6} y^3 = 0$$

as

$$y = C_1^{\frac{1}{4}} \sqrt{A_1} (x - \alpha)^{-3} \mathrm{sn} \left[C_3 \left(\frac{x - \beta}{x - \alpha} - C_2 \right) \right]$$

where $C_3 = \sqrt{\frac{NC_1}{2}}$, $N = \frac{9K}{4C^2}$ and $L = KA_1^3$.

3.4 Special cases

(a) $n = 2$

For the case $n = 2$ we are interested in solving

$$y'' + f(x)y^2 = 0 \tag{3.31}$$

The analysis is the same as for the general case up to equation (3.11). Equating coefficients of the powers of y to zero in (3.11) yields the system

$$d''(x) = 0$$

$$\tfrac{1}{2}a'''(x) + 2d(x)f(x) = 0$$

$$a(x)f_x + \tfrac{5}{2}a'(x) + Cf(x) = 0$$

This system is solved as

$$d = D_1 x + D_0$$

$$f(x) = Ka(x)^{-\frac{5}{2}}e^{-\int \frac{C}{a(x)}dx}$$

subject to $a(x)$ being a solution of

$$a''' = -4Kd(x)a(x)^{-\frac{5}{2}}e^{-\int \frac{C}{a(x)}dx} \tag{3.32}$$

When $d = 0$ ($D_1 = D_0 = 0$) the analysis is the same as for the general case since it follows from (3.32) that $a(x) = A_1 x^2 + A_2 x + A_3$. We will assume that $d \neq 0$. When $d = D_0$ ($D_1 = 0$), (3.32) can be differentiated to obtain

$$2aa^{(4)} + 5a'a''' + Ca''' = 0 \tag{3.33}$$

which is the differential equation that $a(x)$ must satisfy. In the general case $d = D_1 x + D_0$, Govinder and Leach (1996) have shown that the analysis reduces to

solving an equation of the same form as (3.33). They have in fact formally studied the symmetry properties of (3.33), the solution of which is not easy to obtain except in the case $C = 0$. Nevertheless without solving explicitly for $a(x)$ we use the symmetry $G = a(x)\partial_x + [d + y(\frac{1}{2}a' + C)]\partial_y$ to transform (3.31). Requiring that (3.31) be transformed to an autonomous equation, we obtain the transformation

$$X = \int \frac{\mathrm{d}x}{a(x)}$$

$$Y = ya(x)^{-\frac{1}{2}}e^{-\int \frac{C}{a(x)}\mathrm{d}x} - \int d(x)a(x)^{-\frac{3}{2}} \left(e^{-\int \frac{C}{a(x)}\mathrm{d}x}\right) \mathrm{d}x$$

where X is the independent variable and Y is the dependent variable. Under this transformation (3.31) becomes

$$Y'' + 2CY' + KY^2 + \left(\frac{A}{2} + C^2\right)Y + B = 0 \qquad (3.34)$$

where A and B are constants given by

$$A = aa'' - \tfrac{1}{2}a'^2 + 4KI$$

$$B = I(C^2 + \frac{A}{2} - 4KI) + a^{-1}(a' + 2C)I' + a^{-2}I''$$

with

$$I = \int d(x)a(x)^{-\frac{3}{2}} \left(e^{-\int \frac{C}{a(x)}\mathrm{d}x}\right) \mathrm{d}x$$

Multiplying (3.32) by $a(x)$ and integrating gives A as a constant of integration while multiplying (3.32) by $a(x)\int d(x)a^{-\frac{3}{2}}\mathrm{d}x$ and integrating gives B as a constant of integration. We remove the constant term B in (3.34) by the translation

$$Y = Z + W$$

where $W = $ constant and satisfies the equation

$$KW^2 + \left(\frac{A}{2} + C^2\right)W + B = 0$$

For a real W to exist we require that

$$\Delta = \left(\frac{A}{2} + C^2\right)^2 - 4BK \geq 0 \tag{3.35}$$

The differential equation that Z satisfies is given by

$$Z'' + 2CZ' + KZ^2 + \left(\frac{A}{2} + C^2 + 2KW\right)Z = 0 \tag{3.36}$$

This equation has an additional symmetry

$$\tilde{G} = e^{\frac{2}{5}CX}\partial_X - \left(\frac{4C}{5}Z + \frac{2C}{5K}(\frac{A}{2} + \frac{C^2}{25} + 2KW)\right)e^{\frac{2}{5}CX}\partial_Y$$

if the condition

$$C^2 = \frac{25}{24}\sqrt{\Delta} \tag{3.37}$$

holds, where Δ is given by (3.35). Requiring that the additional symmetry $\tilde{G}$ be transformed to the symmetry $\bar{G} = \partial_U$ yields the transformation

$$U = e^{-\frac{2}{5}CX}$$

$$V = (\frac{5K}{2C})^2 e^{\frac{4}{5}CX}Z$$

where U is the independent variable. Under this transformation (3.36) reduces to the simpler equation

$$V'' + V^2 = 0$$

This differential equation is easily reduced to the quadrature

$$\int \frac{dv}{\sqrt{U_1 - \frac{2}{3}V^3}} = U - U_0$$

where U_0 and U_1 are constants, which is recognisable as an elliptic integral. Equations (3.35) and (3.37) combine to yield

$$KB \geq \tfrac{1}{4}A^2$$

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This is the condition under which (3.31) is reduced to quadratures.

(b) $n = -3$, $C = 0$

For the case $n = -3$ and $C = 0$ the general analysis is valid up to equation (3.12d). With these values equation (3.12d) yields the condition $f_x = 0$ which implies that $f(x) = \bar{K} = $ constant. The Emden-Fowler equation then simplifies to

$$y'' + \bar{K}y^{-3} = 0 \tag{3.38}$$

which is a special case of the Ermakov-Pinney equation. A full discussion of this equation is available in Govinder (1993). We merely note that (3.38) reduces to the quadrature

$$x - x_0 = \int \frac{dy}{\sqrt{x_1 + \bar{K}y^{-2}}}$$

where x_0 and x_1 are constants of integration. This reduction to quadrature is possible because equation (3.38) has three symmetries which is more than the required two symmetries.

3.5 Summary

In order to solve the Emden-Fowler equation of the form

$$y'' + f(x)y^n = 0$$

we undertook a Lie symmetry analysis thereof. This equation possesses one symmetry (and thus can be reduced to a first order equation) provided $f(x)$ is given by

$$f(x) = Ka(x)^{-\frac{n+3}{2}}e^{-\int \frac{C(n-1)}{a(x)}dx} \tag{3.39}$$

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where $a(x) = A_1 x^2 + A_2 x + A_3$ and A_1, A_2, A_3, C and K are constants. The equation possesses an additional symmetry (now enabling reduction to quadratures) if the condition

$$C^2 \left(\frac{n-1}{n+3} \right)^2 = \frac{A_2^2 - 4A_1 A_3}{4} \tag{3.40}$$

holds. This implies that $a(x)$ must have real roots, α and β say. The two conditions (3.39) and (3.40) are combined to constrain $f(x)$ as

$$f(x) = L(x - \alpha)^{-(n+3)}$$

where L is a constant. It is therefore possible (via the use of the two symmetries) to reduce the Emden-Fowler equation of the form

$$y'' + L(x - \alpha)^{-(n+3)} y^n = 0 \tag{3.41}$$

to the quadrature

$$w - C_2 = \sqrt{\frac{n+1}{2N}} \int \frac{dz}{\sqrt{C_1 - z^{n+1}}} \tag{3.42}$$

(where C_1, C_2, N are constant), for the cases $n \neq -3, -1, 0, 1, 2$ via the transformation

$$w = \frac{2}{n-1} \left(\frac{x - \beta}{x - \alpha} \right)$$

$$z = \frac{y}{\sqrt{A_1}} (x - \alpha)^{\frac{n+3}{n-1}} \tag{3.43}$$

Once the quadrature has been evaluated the above transformation is inverted to obtain the solution in the original variables. The special mathematical cases ($n = 2$) and ($n = -3, C = 0$) were also studied. Similar results were obtained.

4 The Emden-Fowler equation in general relativity

4.1 Introduction

The Emden-Fowler equation is an equation of fundamental importance when analysing the gravitational behaviour in many cosmological models in general relativity. The equation of interest has the form

$$y'' = F(x)y^n \tag{4.1}$$

where n is constant and F is arbitrary. In cosmology a special case of (4.1) first arose in the analysis of Kustaanheimo and Qvist (1948) when studying shear-free spherically symmetric models. In this chapter we demonstrate that particular values of n are associated with different classes of solution to the Einstein field equations. In §4.2 we obtain an Emden-Fowler equation with $n = -\frac{1}{3}$ under the assumption of vanishing acceleration. The solution of the Emden-Fowler equation via the Lie analysis is used to obtain a class of solutions to the field equations for vanishing pressure. The case of vanishing shear in spherically symmetric spacetimes is discussed in §4.3 where an Emden-Fowler equation with $n = 2$ is obtained. This is the case that has attracted the greatest attention and interest of researchers (Maharaj *et al* 1996). In §4.4 an Emden-Fowler equation with $n = -2$ is derived under the assumption of a stiff fluid equation of state. Solutions are obtained for this accelerating, expanding

and shearing class by using the results of the Lie analysis. In §4.5 an attempt is made to integrate the Emden-Fowler equation for general n via an *ad hoc* integration technique utilised by Maharaj *et al* (1996) for the case $n = 2$. A conclusion is made about the uniqueness of the results of Maharaj *et al*.

4.2   The case $n = -\frac{1}{3}$

The Emden-Fowler equation (4.1) with $n = -\frac{1}{3}$ was obtained by Herlt (1996) in his attempt to find solutions of the Einstein field equations (2.17) for the case of vanishing acceleration. The expansion (2.10c) and shear (2.10d) are nonzero. On using (2.10b) we observe that vanishing acceleration is equivalent to

$$\nu' = 0 \tag{4.2}$$

Thus $\nu = \nu(t)$ and by rescaling the time coordinate in the line element (2.9) we can set $\nu = 0$ as long as $Y' \neq 0$ as pointed out by Kramer *et al* (1980). Then from the momentum conservation equation (2.18a) we deduce that $p' = 0$ so that $p = p(t)$. Under these assumptions the field equations become

$$\mu = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - \lambda'Y' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}\left(\dot{\lambda}\dot{Y} + \frac{\dot{Y}^2}{2Y}\right) \tag{4.3a}$$

$$p = -\frac{1}{Y^2} + e^{-2\lambda}\frac{Y'^2}{Y^2} - \left(2\frac{\ddot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2}\right) \tag{4.3b}$$

$$p = e^{-2\lambda}\left[\frac{1}{Y}\left(-\lambda'Y' + Y''\right)\right]$$
$$- \left[\ddot{\lambda} + \dot{\lambda}^2 + \frac{1}{Y}\left(\dot{\lambda}\dot{Y} + \ddot{Y}\right)\right] \tag{4.3c}$$

$$\dot{Y}' = \dot{\lambda}Y' \qquad \qquad (4.3d)$$

which are simpler than (2.17). Studies of nonaccelerating spherically symmetric gravitational fields, involving (4.2) and (4.3), have been pursued by a number of authors in the past including Govender (1996), Kitamura (1994), Maharaj *et al* (1993), McVittie and Wiltshire (1975) and Van den Bergh and Wils (1985). Their investigations involved *ad hoc* assumptions about the functional forms for the matter variables and gravitational potentials. Our treatment in contrast involves the systematic Lie analysis of differential equations and we show that the Emden-Fowler equation is the principal equation governing the behaviour of the gravitational field.

The field equation (4.3d) is integrated to yield

$$Y'^2 e^{-2\lambda} = 1 - \epsilon h^2(r) \qquad \qquad (4.4)$$

where $h(r)$ is an arbitrary function of integration and ϵ is an arbitrary constant which can be scaled to $0, 1$ or -1 (Herlt 1996). The field equation (4.3c) is satisfied if (4.3b) is satisfied. To demonstrate this we differentiate (4.3b)$\times Y^2$ with respect to r to obtain

$$2YY'p + p'Y^2 = 2Y'Y''e^{-2\lambda} - 2\lambda'e^{-2\lambda}Y'^2 - 2Y'\ddot{Y} - 2Y\ddot{Y}' - 2\dot{Y}\dot{Y}'$$

Clearly if we utilise $p' = 0$, $\dot{Y}' = \dot{\lambda}Y'$ and $\ddot{Y}' = \ddot{\lambda}Y' + \dot{\lambda}^2 Y'$ in the above equation we regain (4.3c). Thus we consider only the equations (4.3a), (4.3b) and (4.4) in solving the field equations. These are three equations in the four unknowns $p(t), \mu(r,t), \lambda(r,t)$ and $Y(r,t)$. In our approach we specify $p(t)$ and consequently obtain the other quantities. On substituting (4.4) into (4.3b) we obtain

$$p = \frac{1}{Y^2}\left(-\epsilon h^2(r) - 2\ddot{Y} - \dot{Y}^2\right) \qquad \qquad (4.5)$$

It is convenient to replace Y by

$$Y = z^{\frac{2}{3}}$$

so that (4.5) becomes

$$\ddot{z} + \frac{3}{4}p(t)z = -\frac{3}{4}\epsilon h^2(r)z^{-\frac{1}{3}} \tag{4.6}$$

in terms of the new dependent variable z. We remove the explicit r dependence by setting

$$z(r,t) = Z(r,t)E(r)$$

Under this transformation (4.6) becomes

$$\ddot{Z} + \frac{3}{4}p(t)Z = -\frac{3}{4}\epsilon E^{-\frac{4}{3}}(r)h^2(r)Z^{-\frac{1}{3}}$$

which simplifies to

$$\ddot{Z} + \frac{3}{4}p(t)Z = -\epsilon Z^{-\frac{1}{3}} \tag{4.7}$$

on choosing

$$E^{\frac{4}{3}}(r) = \frac{3}{4}h^2(r)$$

By specifying $p(t)$ in (4.7) we can solve this equation to obtain $Z(r,t)$ and then $Y(r,t)$ via the transformation

$$Y = \frac{\sqrt{3}}{2}h(r)Z^{\frac{2}{3}}$$

The gravitational potential $\lambda(r,t)$ follows from (4.4) and the remaining unknown function $\mu(r,t)$ is obtained from (4.3a). Therefore the equation (4.7) is the focal point of our investigation and we require a general solution. Note that we cannot

solve this equation by first specifying Z and then determining $p(t)$; this is because Z is a function of both the r and t coordinates. Even the simple form $Z = Z(t)$ does not generate useful solutions since the shear σ vanishes.

Some special cases of (4.7) were considered by Herlt (1996). For the cases

$$p = \text{constant},$$

$$p = 0$$

the differential equation (4.7) can be solved by means of quadratures. The case $p = $ constant was first solved by Ruban (1969). The case $p = 0$ corresponds to dust which was initially studied by Lemaitre (1933) and Tolman (1934). The value $\epsilon = 0$ in (4.7) corresponds to solving a simple second order linear differential equation. Bona and Stella (1987) and Leibowitz (1971) examined this case. Solutions for various functions $p(t)$, in (4.7) with $\epsilon = 0$, can be found by referring to handbooks of differential equations eg Kamke (1983). As these special cases have been studied comprehensively in the past by a variety of authors we do not pursue them further.

In our attempt to solve (4.7) in general we perform a point transformation

$$Z(r,t) = \jmath(t)y(x(t),r)$$

which results in (4.7) being transformed to

$$\dot{x}^2 \jmath y_{xx} + (2\jmath\dot{x} + \jmath\ddot{x})y_x + \left(\ddot{\jmath} + \frac{3}{4}p(t)\jmath\right) = -\epsilon\jmath^{-\frac{1}{3}}y^{-\frac{1}{3}} \tag{4.8}$$

The coefficients of y_x and y vanish if we specify the relation between x and t as

$$\jmath^2(x)\dot{x} = 1 \tag{4.9}$$

and we choose

$$\ddot{\jmath} + \frac{3}{4}p(t)\jmath(t) = 0 \tag{4.10}$$

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With the above assumptions (4.8) simplifies to

$$y_{xx} = F(x)y^{-\frac{1}{3}} \tag{4.11}$$

where

$$F(x) = -\epsilon \jmath^{\frac{8}{3}}(x)$$

This is an equation of the Emden-Fowler type. It is a partial differential equation because $y = y(x, r)$ but can be solved as an ordinary differential equation. If $F(x)$ is specified then the equation $F(x) = -\epsilon \jmath^{\frac{8}{3}}$ determines $\jmath(x)$. The form for $x(t)$ follows from integrating (4.9) which enables us to obtain $\jmath(t) = \jmath(x(t))$. The pressure $p(t)$ is then determined from (4.10). Note that if $\jmath(t)$ is a linear function of time then the pressure is zero. We now focus our attention on solving the Emden-Fowler equation (4.11).

For $n = -\frac{1}{3}$, the Emden-Fowler equation becomes

$$y'' + L(x - \alpha)^{-\frac{8}{3}} y^{-\frac{1}{3}} = 0 \tag{4.12}$$

The above equation is a special case of (3.41) and is solved in the manner described in §3.5. The quadrature (3.42) is first evaluated for $n = -\frac{1}{3}$. The solution to this quadrature is then inverted via the transformation (3.43) (with $n = -\frac{1}{3}$) to obtain the solution to (4.12). On inverting the solution of (4.12) via the transformations $Z(r, t) = \jmath(t)y(r, t)$ and $Y = \frac{\sqrt{3}}{2}h(r)Z^{\frac{2}{3}}$ we obtain the form for the metric potential $Y(r, t)$. It would appear that this procedure generates a new class of nonaccelerating solutions to the field equations. However this is not the case since the function

$$F(x) = -L(x - \alpha)^{-\frac{8}{3}} \tag{4.13}$$

implies that the pressure vanishes. Solutions with vanishing pressure have been obtained previously in the literature. We now demonstrate that the pressure is zero

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and then provide the solution of the spherically symmetric field equations for the case of vanishing pressure and acceleration by using the procedure described above.

Given the function (4.13) we obtain

$$\jmath(x) = \omega^{\frac{3}{8}}(x-\alpha)^{-1} \tag{4.14}$$

where $\omega = \frac{L}{\epsilon}$. This allows us to integrate (4.9) to obtain

$$(x-\alpha)^{-1} = -\omega^{-\frac{3}{4}}(t-t_0) \tag{4.15}$$

where t_0 is a constant of integration. Combining (4.14) and (4.15) we obtain

$$\jmath(t) = -\omega^{-\frac{3}{8}}(t-t_0)$$

from which we deduce that $\jmath = 0$. This is turn implies, using (4.10), that $p(t) = 0$ and we have vanishing pressure. The solution for the metric potential $Y(r,t)$, in the case of vanishing pressure and acceleration, is given by

$$N(t-t_0) = H D_1^{\frac{1}{2}}(r) \sin^{-1}\left(L(t-t_0)^{\frac{1}{3}}\tilde{Y}^{\frac{1}{2}}\right) + \frac{2}{3}D_2(r) - 1$$
$$- LHD_1^{\frac{1}{2}}(t-t_0)^{\frac{1}{3}}\tilde{Y}^{\frac{1}{2}}\sqrt{1 - L^2(t-t_0)^{\frac{2}{3}}\tilde{Y}} + \frac{2}{3}D_2(r) - 1$$

where

$$\tilde{Y}(r,t) = Y(r,t)D_1^{-1}(r)h^{-1}(r)$$

The arbitrary functions $D_1(r)$ and $D_2(r)$ result from integration and the constants H, N and L are given by

$$H = \frac{3\sqrt{3}}{4}C^3 K^{-\frac{3}{2}}$$

$$N = Ca_1^{-2}\left(-\frac{\epsilon}{K}\right)^{\frac{3}{4}}$$

$$L = 2K^{\frac{1}{8}}\left(-\frac{\epsilon}{9}\right)^{\frac{3}{8}}a_1^{-\frac{2}{3}}C^{-1}$$

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The remaining potential λ and the matter variables are deduced from the above form for Y.

4.3 The case $n = 2$

With the value $n = 2$, the Emden-Fowler equation (4.1) arises in the study of spherically symmetric cosmological models. However in this case the shear (2.10d) vanishes, the acceleration and expansion are nonzero, and the solution to the field equations (2.17) reduces to the integration of the single equation

$$y'' = F(x)y^2 \tag{4.16}$$

which is an Emden-Fowler equation. This equation has generated tremendous interest because of its important role in shear-free relativistic fluids. The first systematic analysis of (4.16) was performed by Kustaanheimo and Qvist (1948). Recent analyses of this equation include the treatments of Maharaj *et al* (1996), Stephani (1983), Stephani and Wolf (1996) and Srivastava (1987). Note that the related equation

$$y'' = F(x)y^2 + G(x)y^3$$

corresponding to shear-free fluids with an electromagnetic field arises in the analysis of the Einstein-Maxwell equations (Krasinski 1997, Srivastava 1992). We do not pursue this case in greater detail because the shear vanishes.

4.4 The case $n = -2$

The Emden-Fowler equation (4.1) with $n = -2$ arises in the analysis of Govender (1996) for shearing fluids in spherically symmetric models. This class of solutions is

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accelerating, expanding and shearing. Govender imposed the conditions

$$\lambda = \text{constant},$$

$$p = \mu$$

so that there is a stiff equation of state for his models. Under the above conditions the line element (2.9) can be written as

$$ds^2 = -\dot{Y}^2 dt^2 + dr^2 + Y^2[d\theta^2 + \sin^2\theta d\phi^2]$$

and the behaviour of the gravitational potentials depends on the function $Y(r,t)$. The explicit form for the physical and kinematical quantities is also obtained from $Y(r,t)$ (Govender 1996). The solution of the field equations (2.17) reduces to the integration of the differential equation

$$Y'' = f(r)Y^{-2} \tag{4.17}$$

where $f(r)$ is arbitrary. Again we observe that the Emden-Fowler equation arises in this class of shearing, spherically symmetric solutions.

The equation (4.17) is a special case of (3.41) with the independent variable r and the dependent variable Y corresponding to x and y respectively. The corresponding function $f(r)$ has the form

$$f(r) = -L(r - \alpha)^{-1}$$

The solution to this equation is obtained as discussed in §3.5 with the corresponding value $n = -2$. The quadrature (3.42) is first evaluated for $n = -2$. The solution to this quadrature is then inverted via the transformation (3.43) (with $n = -2$) to obtain the solution to (4.17). The gravitational potential $Y(r,t)$ is given by

$$PC_1^{\frac{3}{2}}(t)\left(\frac{2(r - \beta)}{3(r - \alpha)} + C_2(t)\right)$$

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$$= \log\left(\sqrt{1 + QC_1(t)(r-\alpha)^{-\frac{1}{3}}Y} - QC_1(r-\alpha)^{-\frac{1}{3}}Y\right)$$

$$- \log\left(\sqrt{1 - QC_1(r-\alpha)^{-\frac{1}{3}}Y}\right) - \log\left(\sqrt{QC_1(r-\alpha)^{-\frac{1}{3}}Y}\right)$$

$$- \frac{QC_1(r-\alpha)^{-\frac{1}{3}}Y}{\sqrt{1 + QC_1(r-\alpha)^{-\frac{1}{3}}Y}} \tag{4.18}$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions of integration and the quantities

$$P = \frac{8\sqrt{2}C^2}{9K}$$

$$Q = \frac{4C^2}{9K}a_1^{-\frac{1}{2}}$$

are constant. Again we have an example of the role played by the Lie analysis of chapter 3 in solving the field equations, via the Emden-Fowler equation. Note that the explicit solution (4.18) was not given by Govender (1996) in his treatment of shearing models.

4.5 Uniqueness of a method used to obtain a class of solutions

We now try to solve the Emden-Fowler equation (4.1) via an *ad hoc* technique of integration. This idea is motivated by the success of the approach as employed by Maharaj *et al* (1996) when integrating the Emden-Fowler equation (4.1) for the case $n = 2$. The first integral of (4.1) with $n = 2$ obtained by Maharaj *et al* has the form

$$\psi_0(t) = -y_x + F_I y^2 - 2F_{II}yy_x + 2F_{III}y_x^2 + 2\left[(FF_{II})_I - \tfrac{1}{3}K_0\right]y^3 \tag{4.19}$$

where $\psi_0(t)$ is an arbitrary function of integration where we use the notation

$$F_I = \int F(x)\,\mathrm{d}x$$

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for convenience. The existence of (4.19) is subject to the condition

$$2FF_{III} + 3(FF_{II})_I = K_0 \tag{4.20}$$

where K_0 is a constant. We can transform the integral equation (4.20) into an ordinary differential equation which is easier to work with. We introduce

$$\mathcal{F} = F_{III}$$

so that (4.20) becomes

$$2\mathcal{F}\mathcal{F}_{xxxx} + 5\mathcal{F}_x\mathcal{F}_{xxx} = 0$$

which can be integrated once to yield

$$\mathcal{F}_{xxx} = K_1\mathcal{F}^{-\frac{5}{2}} \tag{4.21}$$

where K_1 is a constant of integration. Integration of (4.21) yields the result

$$\mathcal{F}^{-1} = K_4 + K_3\left(\int \mathcal{F}^{-\frac{3}{2}}\mathrm{d}x\right) + K_2\left(\int \mathcal{F}^{-\frac{3}{2}}\mathrm{d}x\right)^2 - \frac{1}{6}K_1\left(\int \mathcal{F}^{-\frac{3}{2}}\mathrm{d}x\right)^3 \tag{4.22}$$

where K_2, K_3 and K_4 are constants of integration. We now let

$$u = \int \mathcal{F}^{-\frac{3}{2}}\mathrm{d}x, \qquad u_x = (\mathcal{F}^{-1})^{\frac{3}{2}}$$

which reduces (4.22) to the quadrature

$$x - x_0 = \int \frac{\mathrm{d}u}{\left(K_4 + K_3 u + K_2 u^2 - \frac{1}{6}K_1 u^3\right)^{\frac{3}{2}}} \tag{4.23}$$

where x_0 is constant. Thus the differential equation (4.21) has been reduced to the quadrature (4.23) which can be evaluated in terms of elliptic integrals in general (Gradshteyn and Ryzhik 1994). The first integral (4.19) with F given by (4.23) represents a new class of solutions of (4.16) found by Maharaj *et al* (1996).

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We now seek to solve the general Emden-Fowler equation (4.1), to determine for which other values of n the above technique yields a first integral. The method utilised is to integrate by parts until we are left with an integral which can be evaluated by placing a constraint on the form of $F(x)$. On integrating both sides of $y'' = F(x)y^n$ we obtain

$$y_x + \psi_0(t)$$

$$= F_I y^n - n \int F_I y^{n-1} y_x dx \tag{4.24}$$

where we utilised integration by parts. We continue with the procedure to integrate the term $\int F_I y^{n-1} y_x dx$ twice. This yields the result

$$y_x + \psi_0(t)$$

$$= F_I y^n - n F_{II} y^{n-1} y_x + n(n-1) \int F_{II} y^{n-2} y_x{}^2 dx$$

$$+ n(FF_{II})_I y^{2n-1} - n(2n-1) \int (FF_{II})_I y^{2n-2} y_x dx \tag{4.25}$$

A further integration by parts of (4.25) yields

$$y_x + \psi_0(t)$$

$$= F_I y^n - n F_{II} y^{n-1} y_x + n(n-1) F_{III} y^{n-1} y_x^2 + n(FF_{II})_I y^{2n-1}$$

$$- n(2n-1)(FF_{II})_I y^{2n-2} y_x - n(n-1)(n-2) \int F_{III} y^{n-3} y_x{}^3 dx$$

$$- \int \{2n(n-1)FF_{III} + n(2n-1)(FF_{II})_I\} y^{2n-2} y_x dx \tag{4.26}$$

If we proceed with integrating (4.26) by parts we generate the result

$$y_x + \psi_0(t)$$

$$= F_I y^n - n F_{II} y^{n-1} y_x + n(n-1) F_{III} y^{n-2} y_x^2 + n(FF_{II})_I y^{2n-1}$$

$$-n(2n-1)(FF_{II})_I y^{2n-2} y_x - n(n-1)(n-2) F_{IIII} y^{n-3} y_x^3$$

$$-\{2n(n-1)FF_{III} + n(2n-1)(FF_{II})_I\}_I y^{2n-2} y_x$$

$$+ (\{2n(n-1)FF_{III} + n(2n-1)(FF_{II})_I\}_I + 3n(n-1)(n-2)FF_{IIII})_I y^{2n-3} y_x^2$$

$$+ (\{2n(n-1)FF_{III} + n(2n-1)(FF_{II})_I\}F)_I y^{3n-2}$$

$$-(2n-3) \int (\{2n(n-1)FF_{III} + n(2n-1)(FF_{II})_I\}_I$$

$$+ 3n(n-1)(n-2)FF_{IIII})_I y^{2n-4} y_x^3 dx$$

$$+n(n-1)(n-2)(n-3) \int F_{IIII} y^{n-4} y_x{}^4 dx$$

$$-2 \int [(\{2n(n-1)FF_{III} + n(2n-1)(FF_{II})_I\}_I + 3n(n-1)(n-2)FF_{IIII})_I F$$

$$+(3n-2)(\{2n(n-1)FF_{III} + n(2n-1)(FF_{II})_I\}F)_I] y^{3n-3} y_x dx \quad (4.27)$$

It is possible to continue with the integration procedure but we stop at this point as (4.27) enables us to make our conclusions.

The objective in our analysis is to eliminate the integrals on the right hand sides of (4.24), (4.25), (4.26) and (4.27) and thereby generate a first integral for (4.1). With $n = 0$, (4.24) gives

$$y_x + \psi_0(t) = F_I \quad (4.28)$$

which is an obvious first integral of $y'' = F(x)$.

If $n = 1$ then (4.25) can be written as

$$y_x + \psi_0(t) = F_I - F_{II} y_x \quad (4.29)$$

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with the integrability condition

$$(FF_{II})_I = K_1$$

where K_1 is a constant. This integrability condition implies that $F(x) = 0$. The first integral of (4.29) is then

$$y_x + \psi_0(t) = 0 \qquad (4.30)$$

which is an obvious first integral of $y'' = 0$. For the case $n = 1$ this technique does not yield a first integral for nontrivial $F(x)$.

When $n = 2$, (4.26) yields the solution

$$\psi_0(t) = -y_x + F_I y^2 - 2F_{II} y y_x + 2F_{III} y_x^2 + 2\left[(FF_{II})_I - \tfrac{1}{3}K_0\right] y^3 \qquad (4.31)$$

which is the same as (4.19). The solution (4.31) is subject to the integrability condition

$$2FF_{III} + 3(FF_{II})_I = K_0$$

which is the same as (4.20). Therefore we have regenerated the first integral found by Maharaj et al (1996) in our general treatment. Note that although the first integrals (4.28) and (4.30) may be obtained using other techniques, the first integral (4.26) is not easily obtainable using other approaches because the differential equation $y'' = F(x)y^2$ is highly nonlinear. It is interesting to observe that the first integral (4.26) also arises as a special case in the Lie analysis of $y'' = F(x)y^2$ performed by Maharaj et al (1996). (Also refer to the general Lie analysis in chapter 3). With $F(x) = x^{-\frac{15}{7}}$, (4.31) takes the form

$$\psi_0(t) = -36y_x - \frac{9}{2}(7)x^{-\frac{8}{7}}y^2 - 9(7)^2 x^{-\frac{1}{7}}yy_x + \frac{3}{2}(7)^3 x^{\frac{6}{7}}y_x^2 - (7)^3 x^{-\frac{9}{7}}y^3 \qquad (4.32)$$

and with $F(x) = (ax + b)^{-\frac{15}{7}}$, (4.31) becomes

$$\psi_0(t) = -36y_x - \frac{9}{2}(\frac{7}{a})(ax + b)^{-\frac{8}{7}}y^2 - 9(\frac{7}{a})^2(ax + b)^{-\frac{1}{7}}yy_x$$

$$+\frac{3}{2}(\frac{7}{a})^3(ax + b)^{\frac{6}{7}}y_x^2 - (\frac{7}{a})^3(ax + b)^{-\frac{9}{7}}y^3 \tag{4.33}$$

The special cases (4.32) and (4.33) were obtained by Stephani (1983) and Srivastava (1987) respectively.

It would seem that our approach may be extended to generate other first integrals by subsequent iterations. However this is not possible as we can demonstrate by a careful analysis of (4.27). To obtain a first integral from (4.27) we need to eliminate the three integrals on the right hand side of this equation. The third integral is eliminated if we set

$$(\{2n(n - 1)FF_{III} + n(2n - 1)(FF_{II})_I\}_I + 3n(n - 1)(n - 2)FF_{IIII})_I F$$

$$+(3n - 2)(\{2n(n - 1)FF_{III} + n(2n - 1)(FF_{II})_I\}F)_I = K_2$$

where K_2 is a constant. With this integrability condition (which is the analogue of (4.20)) we find that (4.27) can be written as

$$y_x + \psi_0(t)$$

$$= F_I y^n - nF_{II}y^{n-1}y_x + n(n - 1)F_{III}y^{n-2}y_x^2 + n(FF_{II})_I y^{2n-1}$$

$$-n(2n - 1)(FF_{II})_I y^{2n-2}y_x - n(n - 1)(n - 2)F_{IIII}y^{n-3}y_x^3$$

$$-\{2n(n - 1)FF_{III} + n(2n - 1)(FF_{II})_I\}_I y^{2n-2}y_x$$

$$+(\{2n(n - 1)FF_{III} + n(2n - 1)(FF_{II})_I\}_I + 3n(n - 1)(n - 2)FF_{IIII})_I y^{2n-3}y_x^2$$

$$+(\{2n(n - 1)FF_{III} + n(2n - 1)(FF_{II})_I\}F)_I y^{3n-2}$$

$$-(2n - 3)\int(\{2n(n - 1)FF_{III} + n(2n - 1)(FF_{II})_I\}_I$$

$$+3n(n - 1)(n - 2)FF_{IIII})_I y^{2n-4}y_x^3 dx$$

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$$+n(n-1)(n-2)(n-3) \int F_{IIII} y^{n-4} y_x{}^4 \mathrm{d}x$$

$$-\frac{2}{3n-3+1} K_1 y^{3n-3+1} \tag{4.34}$$

It remains to eliminate the two integrals in (4.34). As they stand these integrals are impossible to evaluate because of the nonlinear powers of y_x. If we require that these integrals vanish then the following conditions must hold simultaneously:

$$2n - 3 = 0$$

$$n(n-1)(n-2)(n-3) = 0$$

The first condition gives the value $n = \frac{3}{2}$ and the second condition yields the values $n = 0, 1, 2, 3$ which is a contradiction. Hence we cannot eliminate both integrals on right hand side of (4.34) simultaneously. Attempting to further integrate (4.34) by parts will not allow us to obtain a new first integral. This is because each iteration produces integrals that contain higher powers of y_x. These integrals cannot be made to vanish simultaneously by choosing n: the quantity n would have to be both a fraction and a natural number which is not possible.

Therefore we have established that the integration procedure of Maharaj *et al* (1996) works only for the cases $n = 0, 1, 2$ and will not yield first integrals for other values of n. The integration by parts procedure yields the unique first integrals (4.28), (4.30) and (4.31). We believe that our result establishing the uniqueness of the procedure to obtain the first integrals of the Emden-Fowler equation (4.1), for $n = 0, 1, 2$ is new and has not been published previously. It is interesting to note that the values $n = 0, 1, 2$ also occur as special cases in the Lie analysis. The Lie analysis in chapter 3 is a more general technique however and it possible to obtain first integrals of the Emden-Fowler equation (4.1) for a wider range of values of n.

5 Charged, spherically symmetric solutions

5.1 Introduction

The role of charge in relativistic models has been a subject of investigation since the formulation of the general theory of relativity. The first charged solution was the Reissner-Nordstrom (1916,1918) spherically symmetric Einstein-Maxwell solution which generalised the Schwarzschild (1916) solution. The aim of finding solutions to the Einstein-Maxwell field equations is to obtain a physical understanding of the effect of an electromagnetic field in a curved spacetime. Despite the observed charge neutrality of the universe it is important to study the Einstein-Maxwell field equations for a number of reasons. For example the role of charge in gravitational collapse has been pointed out by a variety of authors (Joshi 1993). Also the electromagnetic field may play a role in preventing a big bang singularity (Vickers 1973). In §5.2 we derive the Einstein-Maxwell system of field equations for a spherically symmetric spacetime containing a charged perfect fluid. These equations are obtained for a particular choice of the electromagnetic gauge potential which is consistent with spherical symmetry. In §5.3 we discuss certain physical aspects of the structure of the Einstein-Maxwell system. A number of conditions are obtained from the general form of the electromagnetic gauge potential and the transition from the charged model to neutral matter is discussed. We derive a charged analogue of the spherically

symmetric solution obtained by Maharaj *et al* (1993) in §5.4. The shear, acceleration and expansion are nonvanishing in this class of solutions. The charged analogue of the Gutman-Bespal'ko (1967) solution, characterised by a stiff equation of state, is recovered as a particular case of this method. In §5.5 we demonstrate that the Emden-Fowler equation arises as a governing equation in charged spherically symmetric spacetimes under the assumption of vanishing acceleration and proper charge density. This emphasises the importance of studying the Emden-Fowler equation in general relativity.

5.2   Einstein-Maxwell field equations

The Einstein-Maxwell field equations describe the coupling between the curvature of spacetime and the matter content which now includes the electromagnetic field in contrast to §1.4. The Einstein equations are supplemented by the Maxwell equations which govern the behaviour of the electromagnetic field. Charge is introduced through the electromagnetic 4-potential **A** which defines the electromagnetic field tensor **F**. The electromagnetic field tensor is given by

$$F_{ab} = A_{b;a} - A_{a;b} \tag{5.1}$$

The electromagnetic contribution to the energy-momentum tensor is

$$E_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd} \tag{5.2}$$

which is defined in terms of **F**. The uncharged matter contribution to **T** is

$$M_{ab} = (\mu + p)u_a u_b + pg_{ab}$$

which is the perfect fluid (2.14). The total energy-momentum tensor is then given by

$$T_{ab} = M_{ab} + E_{ab} \qquad (5.3)$$

where the right hand side is the sum of (2.14) and (5.2). The Einstein-Maxwell field equations comprise the system

$$G_{ab} = T_{ab}$$
$$= M_{ab} + E_{ab} \qquad (5.4a)$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \qquad (5.4b)$$

$$F^{ab}{}_{;b} = J^a \qquad (5.4c)$$

where J^a is the 4-current density. We can write the 4-current as

$$J^a = \kappa u^a \qquad (5.5)$$

where u^a is the fluid 4-velocity and κ is the proper charge density.

We utilise the gauge freedom in choosing the 4-potential **A**. The form

$$A_a = (\phi(t,r), 0, 0, 0) \qquad (5.6)$$

for the 4-potential is consistent with spherical symmetry and has been extensively utilised in inhomogeneous cosmological models (Sussman 1987, Sussman 1988a, Sussman 1988b). The quantity $\phi(t,r)$ is called the electromagnetic gauge potential. Note that we have taken ϕ to be a function of both the radial and time coordinates analogous to the dependence of the metric potentials ν, λ and Y. We can calculate

the components of the electromagnetic field tensor from (5.1) and (5.6); the nonzero components are

$$F_{10} = -F_{01} = \phi'$$ (5.7)

On using these components we calculate the electromagnetic contribution to the energy-momentum tensor which is given by

$$E_{ab} = \text{diag}\left(\frac{1}{2}e^{-2\lambda}\phi'^2, \ -\frac{1}{2}e^{-2\nu}\phi'^2, \ \frac{1}{2}Y^2 e^{-2(\lambda+\nu)}\phi'^2, \right.$$
$$\left. \frac{1}{2}\sin^2\theta \ Y^2 e^{-2(\lambda+\nu)}\phi'^2\right)$$ (5.8)

The nonzero components follow from (2.16) and (5.8):

$$T_{ab} = \text{diag}\left(\mu e^{2\nu} + \frac{1}{2}e^{-2\lambda}\phi'^2, \ pe^{2\lambda} - \frac{1}{2}e^{-2\nu}\phi'^2, \ pY^2 + \frac{1}{2}Y^2 e^{-2(\lambda+\nu)}\phi'^2, \right.$$
$$\left. p\sin^2\theta \ Y^2 + \frac{1}{2}\sin^2\theta \ Y^2 e^{-2(\lambda+\nu)}\phi'^2\right)$$ (5.9)

which are for the total energy-momentum tensor **T** (5.3).

From (5.9) and (2.13) we obtain the Einstein field equations (5.4a) as

$$\mu = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - \lambda'Y' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}e^{-2\nu}\left(\dot\lambda\dot Y + \frac{\dot Y^2}{2Y}\right)$$
$$- \frac{1}{2}e^{-2(\lambda+\nu)}\phi'^2$$ (5.10a)

$$p = -\frac{1}{Y^2} + \frac{2}{Y}e^{-2\lambda}\left(\nu'Y' + \frac{Y'^2}{2Y}\right) - \frac{2}{Y}e^{-2\nu}\left(\ddot Y - \dot\nu\dot Y + \frac{\dot Y^2}{2Y}\right)$$
$$+ \frac{1}{2}e^{-2(\lambda+\nu)}\phi'^2$$ (5.10b)

$$p = e^{-2\lambda}\left[\nu'' + \nu'^2 - \nu'\lambda' + \frac{1}{Y}\left(\nu'Y' - \lambda'Y' + Y''\right)\right]$$

$$- \mathrm{e}^{-2\nu} \left[\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + \frac{1}{Y} \left(\dot{\lambda}\dot{Y} - \dot{\nu}\dot{Y} + \ddot{Y} \right) \right]$$

$$- \frac{1}{2} \mathrm{e}^{-2(\lambda+\nu)} \phi'^2$$

$$0 = \dot{Y}' - \dot{Y}\nu' - Y'\dot{\lambda} \tag{5.10c}$$

for a spherically symmetric model with charged matter. With $\phi = \phi(t)$ we regain the field equations (2.17) for uncharged matter.

Given the nonzero components of the electromagnetic field tensor (5.7), we obtain the Maxwell equations. The first Maxwell equation (5.4b) is identically satisfied for the particular electromagnetic field tensor (5.7). The second Maxwell equation (5.4c) is identically satisfied for $a = 2, 3$. We generate the conditions

$$\mathrm{e}^{-2\lambda-\nu} \phi' \left(\lambda' + \nu' - \frac{\phi''}{\phi'} - 2\frac{Y'}{Y} \right) = \kappa \tag{5.11a}$$

$$\phi' \left(\dot{\lambda} + \dot{\nu} - 2\frac{\dot{Y}}{Y} \right) = \dot{\phi}' \tag{5.11b}$$

from (5.4c) and (5.5) where we have set $a = 0$ and $a = 1$, respectively. We take (5.11a) as the definition of the proper charge density given in terms of the gravitational potentials and the electromagnetic gauge potential. The Maxwell equation (5.11b) does not arise in static spherically symmetric models as the potentials are functions only of the radial coordinate r (Humi and Mansour 1984, Pant and Sah 1979, Patel and Mehta 1995). The system of equations (5.10)-(5.11) comprise the Einstein-Maxwell equations for the spherically symmetric models, given by the line element (2.9), with our chosen form of **F** in (5.7).

For charged matter the conservation of energy-momentum is given by

$$T^{ab}{}_{;b} = (M^{ab} + E^{ab})_{;b} = 0$$

With $a = 1$ and $a = 0$ the conservation equations become

$$p' + (\mu + p)\nu' + e^{-2\lambda - 2\nu}\phi'^2 \left(\lambda' + \nu' - \frac{\phi''}{\phi'} - 2\frac{Y'}{Y} \right) = 0 \qquad (5.12a)$$

$$\dot{\mu} + (\mu + p) \left(\dot{\lambda} + 2\frac{\dot{Y}}{Y} \right) + e^{-2\lambda - 2\nu}\phi'^2 \left(-\dot{\lambda} - \dot{\nu} + \frac{\dot{\phi}'}{\phi'} + 2\frac{\dot{Y}}{Y} \right) = 0 \qquad (5.12b)$$

respectively. Analogous to the uncharged case we interpret (5.12a) as the momentum conservation equation and (5.12b) as the energy conservation equation. With $\phi = \phi(t)$ we regain the conservation equations (2.18) for uncharged matter.

5.3 Some physical aspects of the Einstein-Maxwell field equations

We now investigate some consequences that follow from the general structure of the Einstein-Maxwell system for a spherically symmetric spacetime. Certain important properties of this system are discussed which prove useful in describing solutions to the field equations from a physical perspective. The Einstein-Maxwell field equations (5.10)-(5.11) and the conservation equations (5.12) hold for a charged perfect fluid in a spherically symmetric spacetime for our chosen form of **F** given in (5.7).

The proper charge density κ is obtained in terms of ν, λ, Y and ϕ from the first Maxwell equation (5.11a). The condition (5.11b) arising from the second Maxwell equation is satisfied by

$$Y^2\phi' = A_1(r)e^{\nu + \lambda} \qquad (5.13)$$

Equation (5.13) gives the explicit relation between the electromagnetic gauge potential ϕ, the gravitational potentials and the arbitrary function $A_1(r)$. The relation

(5.13) is important because the electromagnetic gauge potential ϕ' is explicitly defined in terms of the potentials ν, λ and Y. It is important to note that the case

$$A_1(r) \neq 0$$

corresponds to charged spacetimes whereas on setting

$$A_1(r) = 0$$

we regain the uncharged models as $\phi = \phi(t)$ only. This condition is the criterion that enables us to recover the corresponding uncharged solution from a charged solution.

A consequence of the Maxwell equation (5.11b) is that the energy conservation equation (5.12b) is changed to

$$\dot{\mu} + (\mu + p) \left(\dot{\lambda} + 2\frac{\dot{Y}}{Y} \right) = 0 \tag{5.14}$$

which is exactly the form of the corresponding equation in the uncharged case in §2.4. It appears from (5.14) that the electromagnetic field $\mathbf{F}$ is not contributing directly to the energy density μ or the pressure p. This is an incorrect physical intepretation. From the Einstein field equations (5.10) and (5.13) we observe that the electromagnetic gauge potential ϕ is related to the gravitational potentials ν, λ and Y; thus there is an essential nonlinear link between ϕ and μ and p. From specific models where the functional form of ν, λ and Y are given it is clear that the potential ϕ is an integral component; the line element will be different from the uncharged case together with the energy density and pressure.

An interesting discussion arises from examining the condition $\kappa = 0$ corresponding to vanishing charge density. From (5.11a) we have

$$\phi' \left(\lambda' + \nu' - 2\frac{Y'}{Y} \right) - \phi'' = 0$$

which is satisfied by

$$Y^2 \phi' = A_2(t) e^{\lambda + \nu} \tag{5.15}$$

where $A_2(t)$ is an arbitrary function. From (5.13) and (5.15) we deduce that

$$A_1(r) = A_2(t) = \text{constant}$$

Another consequence of the condition $\kappa = 0$ is that the momentum conservation equation (5.12a) is simplified: the conservation equations (5.12) become

$$p' + (\mu + p)\nu' = 0$$

$$\dot{\mu} + (\mu + p)\left(\dot{\lambda} + 2\frac{\dot{Y}}{Y}\right) = 0$$

which are the same as in the uncharged model. Although ϕ' does not appear explicitly in the conservation equations, the electromagnetic field $\mathbf{F}$ does affect the matter variables (μ, p) and the gravitational potentials (ν, λ, Y). It is possible to give vanishing proper charge density κ a physical interpretation (Sussman 1988b). There exist uniform density solutions in which the matter is neutral but the electric field is nonzero. The interpretation here is that there are charges of oppposite sign located at opposite boundaries of spacetime which may be singular. For a comprehensive list of such solutions and a description of their physical properties refer to Krasinski (1997).

5.4 A charged generalisation of a spherically symmetric solution

In this case we generate the charged analogue of a class of solutions given by Maharaj *et al* (1993). This class is accelerating, shearing and expanding for spherically sym-

metric line elements (2.9). The Maharaj *et al* class of solutions has the advantage of containing the Gutman-Bespalko (1967) model with a stiff equation of state. We generalise the Gutman-Bespalko model to obtain its charged analogue.

The technique used to obtain the solution clearly follows the approach used by Maharaj *et al* (1993). We make the ansatz

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2T^2(t)(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.16}$$

Equation (5.13) can be written as

$$\phi' = A_1(r)e^{\nu+\lambda}r^{-2}T^{-2} \tag{5.17}$$

With the simplified form of the line element (5.16), the field equations (5.10) become

$$\mu = \frac{1}{r^2T^2} + e^{-2\lambda}\left(-\frac{1}{r^2} + \frac{2}{r}\lambda'\right) + e^{-2\nu}\frac{\dot{T}^2}{T^2} - \frac{A_1^2(r)}{2r^4T^4} \tag{5.18a}$$

$$p = -\frac{1}{r^2T^2} + e^{-2\lambda}\left(\frac{1}{r^2} + \frac{2}{r}\nu'\right) - \frac{1}{T^2}(2\ddot{T}T + \dot{T}^2)e^{-2\nu} + \frac{A_1^2(r)}{2r^4T^4} \tag{5.18b}$$

$$p = e^{-2\lambda}\left((\nu'' + \nu'^2 - \nu'\lambda') + \frac{1}{r}(\nu' - \lambda')\right) - e^{-2\nu}\frac{\ddot{T}}{T} - \frac{A_1^2(r)}{2r^4T^4} \tag{5.18c}$$

$$0 = 1 - r\nu' \tag{5.18d}$$

Equation (5.18d) is immediately integrated to yield

$$e^{2\nu} = a^2r^2 \tag{5.19}$$

where a is a constant. On equating (5.18b) to (5.18c), and using (5.19), we obtain

$$\frac{1}{T^2} + \frac{1}{a^2T^2}(\ddot{T}T + \dot{T}^2) - \frac{A_1^2(r)}{r^2T^4} = 2e^{-2\lambda}(1 + \lambda'r)$$

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which is often referred to as the pressure isotropy equation. If we make the choice

$$A_1(r) = a_1 r$$

where a_1 is a constant, then the variables r and t separate in the pressure isotropy equation. We obtain the result

$$\frac{1}{T^2} + \frac{1}{a^2 T^2}(\ddot{T}T + \dot{T}^2) - \frac{a_1^2}{T^4} = 2e^{-2\lambda}(1 + \lambda' r)$$

which is equivalent to

$$\frac{1}{T^2} + \frac{1}{a^2 T^2}(\ddot{T}T + \dot{T}^2) - \frac{a_1^2}{T^4} = 2k \tag{5.20a}$$

$$e^{-2\lambda}(1 + \lambda' r) = k \tag{5.20b}$$

where k is a constant. The differential equation (5.20b) for r is converted via the transformation

$$u = e^{2\lambda}$$

to the Riccati equation

$$u' + \frac{2}{r}u = \frac{2k}{r}u^2 \tag{5.21}$$

The Riccati equation (5.21) has the particular solution

$$u = \frac{1}{k}$$

To obtain a general solution we set

$$u = \frac{1}{k} + \frac{1}{v(r)}$$

which transforms (5.21) to the differential equation

$$v' + \frac{2}{r}v = -\frac{2k}{r}$$

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This is a linear equation in $v(r)$ and is easily integrated to yield

$$v = -\frac{k}{br}(k + br^2)$$

On transforming to the original variables the solution to (5.20b) is thus obtained as

$$e^{2\lambda} = \frac{1}{k + br^2}$$

Note that Maharaj *et al* (1993) obtained a similar form for $e^{2\lambda}$. However their analysis was valid only for uncharged matter. In addition their solution technique was *ad hoc* and the general solution of the Riccati equation was not demonstrated.

The time dependence in T is obtained from (5.20a). This equation is transformed via

$$T = Z^{\frac{1}{2}}$$

to the simpler differential equation

$$\ddot{Z} + 2a^2 - 4ka^2 Z - \frac{2a^2 a_1^2}{Z} = 0 \tag{5.22}$$

Equation (5.22) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{Z}^2 + 4a^2 Z - 4ka^2 Z^2 - 4a^2 a_1^2 \log Z\right] = 0$$

This equation is easily reduced to the quadrature

$$\int \frac{\mathrm{d}Z}{\sqrt{C_1 - 4a^2 Z + 4a^2 Z^2 + 4a^2 a_1^2 \log Z}} = t - t_0$$

where C_1 and t_0 are constant. On evaluating this quadrature we obtain the form of T via the transformation $T = Z^{\frac{1}{2}}$. Closed form solutions may be obtained for particular values of the constants in the integrand.

The matter variables μ and p are then obtained from (5.18a) and (5.18b) respectively as

$$\mu = -3b + \frac{k}{r^2} - \frac{1}{a^2 r^2} \frac{\ddot{T}}{T} - \frac{a_1^2}{2r^2 T^4} \tag{5.23a}$$

$$p = 3b + \frac{k}{r^2} + -\frac{1}{a^2 r^2} \frac{\ddot{T}}{T} + \frac{a_1^2}{2r^2 T^4} \tag{5.23b}$$

From (5.23) we note that p and μ are related by

$$p - \mu = 6b + \frac{a_1^2(r)}{r^2 T^4} \tag{5.24}$$

We regain the stiff equation of state if we set $a_1 = b = 0$. The electromagnetic gauge potential is given by

$$\phi' = \frac{a_1 a}{T^2} \frac{1}{\sqrt{k + br^2}}$$

and the proper charge density is

$$\kappa = -\frac{a_1 \sqrt{k + br^2}}{r^2 T^2}$$

Therefore we have generated an exact solution to the Einstein-Maxwell field equations (5.10)-(5.11) with the equation of state (5.24). This exact solution has nonvanishing acceleration, expansion and shear.

The choice $a_1 = 0$ implies that $\phi' = 0$ which corresponds to the uncharged case. With this assumption the differential equation (5.22) for Z becomes

$$\ddot{Z} + 2a^2 - 4ka^2 Z = 0$$

Depending on the sign of k this equation can be integrated to yield the following forms of the line element

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$k = 0:$

$$ds^2 = -a^2 r^2 dt^2 + \frac{1}{br^2} dr^2 + r^2(-a^2 t^2 + ct + d)d\Omega^2$$

$k = -n^2 < 0:$

$$ds^2 = -a^2 r^2 dt^2 + \frac{1}{-n^2 + br^2} dr^2 + r^2 \left(c\sin(2ant) + d\cos(2ant) - \frac{1}{2n^2} \right) d\Omega^2$$

$k = n^2 > 0:$

$$ds^2 = -a^2 r^2 dt^2 + \frac{1}{n^2 + br^2} dr^2 + r^2 \left(ce^{2ant} + de^{-2ant} + \frac{1}{2n^2} \right) d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and c and d are constants of integration. For the case $k \geq 0$ we require that $b > 0$ while for $k < 0$ we must have that $r \geq \sqrt{-k/b}$. These results were obtained by Maharaj $et\ al$ (1993). Thus we have recovered their solutions when the charge vanishes. The equation of state for this class of models is

$$p - \mu = 6b$$

This is a generalisation of the stiff equation of state $p = \mu$. If we set $k = 1, a = \frac{1}{2}$ and $b = 0$ we obtain the line element

$$ds^2 = -\frac{r^2}{4} dt^2 + dr^2 + r^2 \left(ce^t + de^{-t} + \frac{1}{2} \right) (d\theta^2 + \sin^2\theta d\phi^2)$$

which is the original Gutman-Bespal'ko (1967) solution. Note that the Maharaj $et\ al$ models contain this particular case.

5.5 The Emden-Fowler equation in charged spherically symmetric spacetimes

In this section we demonstrate that the Emden-Fowler equation also arises in charged spacetimes under specific conditions. This equation arises in analogy to chapter 3 and to the approach by Herlt (1996) in obtaining an Emden-Fowler equation for uncharged spherically symmetric spacetimes.

We analyse the case of vanishing acceleration. This assumption is equivalent to

$$\nu' = 0$$

Thus $\nu = \nu(t)$ and by rescaling the time coordinate in the line element (2.9) we can set $\nu = 0$. In addition we assume that

$$Y^2 \phi' = A e^\lambda$$

where A is a constant. This form for the electromagnetic gauge potential satisfies the condition (5.11b). Then equation (5.11a) yields the result

$$\kappa = 0$$

so that the proper charge density vanishes. Under these assumptions the momentum conservation reduces to

$$p' = 0$$

which implies that $p = p(t)$. The Einstein field equations (5.10) become

$$\mu = \frac{1}{Y^2} - \frac{2}{Y} e^{-2\lambda} \left(Y'' - \lambda' Y' + \frac{Y'^2}{2Y} \right) + \frac{2}{Y} \left(\dot{\lambda} \dot{Y} + \frac{\dot{Y}^2}{2Y} \right) - \frac{A^2}{2Y^4} \quad (5.25a)$$

$$p = -\frac{1}{Y^2} + e^{-2\lambda} \frac{Y'^2}{Y^2} - \left(2\frac{\dot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} \right) + \frac{A^2}{2Y^4} \quad (5.25b)$$

$$p = \mathrm{e}^{-2\lambda} \left[\frac{1}{Y} \left(-\lambda' Y' + Y'' \right) \right]$$

$$- \left[\ddot{\lambda} + \dot{\lambda}^2 + \frac{1}{Y} \left(\dot{\lambda}\dot{Y} + \ddot{Y} \right) \right] - \frac{A^2}{2Y^4} \tag{5.25c}$$

$$\dot{Y}' = \dot{\lambda} Y' \tag{5.25d}$$

The Maxwell equations (5.11) are satisfied because $Y^2 \phi' = A\mathrm{e}^\lambda$.

The field equation (5.25d) is integrated to yield

$$Y'^2 = \mathrm{e}^{2\lambda} f^2 \tag{5.26}$$

where $f = f(r)$ is an arbitrary function. We choose to set

$$f = 1$$

for simplicity. The field equation (5.25c) is satisfied if (5.25b) is satisfied. To demonstrate this we differentiate (5.25b)$\times Y^2$ with respect to r to obtain

$$2YY'p + p'Y^2 = 2Y'Y''\mathrm{e}^{-2\lambda} - 2\lambda'\mathrm{e}^{-2\lambda}Y'^2 - 2Y'\ddot{Y} - 2Y\ddot{Y}' - 2\dot{Y}\dot{Y}' - A^2\frac{Y'}{Y^3}$$

On utilising $p' = 0$, $\dot{Y}' = \dot{\lambda}Y'$ and $\ddot{Y}' = \dot{\lambda}^2 Y' + \ddot{\lambda}Y'$ in the above equation we regain (5.25c). Thus we consider only the equations (5.25a), (5.25b) and (5.26) in solving the field equations. These are three equations in the four unknowns $p(t), \mu(r,t), \lambda(r,t)$ and $Y(r,t)$. In our approach we specify $p(t)$ and consequently obtain the other quantities. On substituting (5.26) into (5.25b) we have

$$p = \frac{1}{Y^2} \left(-2\ddot{Y} - \dot{Y}^2 + \frac{A^2}{2Y^2} \right) \tag{5.27}$$

It is convenient to replace Y by

$$Y = z^{\frac{2}{3}}$$

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so that (5.27) becomes

$$\ddot{z} + \frac{3}{4}p(t)z = Pz^{-\frac{5}{3}} \tag{5.28}$$

in terms of the new dependent variable z, where $P = \frac{3}{8}A^2$ is constant.

To analyse (5.28) we perform the point transformation

$$z = \jmath(t)y(x(t), r)$$

(cf §4.2). Then (5.28) becomes

$$\dot{x}^2 \jmath y_{xx} + (2\jmath\dot{x} + \jmath\ddot{x})y_x + \left(\ddot{\jmath} + \frac{3}{4}p(t)\jmath\right) = P\jmath^{-\frac{5}{3}}y^{-\frac{5}{3}} \tag{5.29}$$

Under the assumptions

$$\jmath^2(x)\dot{x} = 1$$

$$\ddot{\jmath} + \frac{3}{4}p(t)\jmath(t) = 0$$

equation (5.29) simplifies to

$$y'' = F(x)y^{-\frac{5}{3}} \tag{5.31}$$

where

$$F(x) = P\jmath^{\frac{4}{3}}(x)$$

Equation (5.31) is an Emden-Fowler equation. We have seen other examples of the Emden-Fowler equation in chapter 4 for uncharged matter. Here we have demonstrated that an Emden-Fowler equation governs the behaviour of the Einstein-Maxwell system for charged matter. This latest appearance of the Emden-Fowler equation in charged, spherically symmetric spacetimes further demonstrates the importance of this equation in general relativity.

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6 Conclusion

The research conducted in this thesis investigated the aspects of spherically symmetric cosmological models, in particular the role of the Emden-Fowler equation. We demonstrated that the solution of the field equations reduced to the Emden-Fowler equation for different classes of shearing line elements with spherical symmetry. Our aim was to specify completely the behaviour of the gravitational field by solving the Emden-Fowler equation. Our analysis was initially applicable to a perfect fluid energy-momentum tensor for uncharged matter. Later we considered the Einstein-Maxwell equations for charged matter and showed that the Emden-Fowler equation plays a role in describing the behaviour of the gravitational field. We now highlight the main points and conclusions arrived at in this thesis.

After reviewing the differential geometry applicable to general relativity we obtained the Einstein field equations from the spherically symmetric line element for a perfect fluid source for uncharged matter in chapter 1. In chapter 2 we comprehensively analysed the Emden-Fowler equation via the geometric approach of the Lie analysis of differential equations. Solutions to the Emden-Fowler equation in particular cases were generated for application to the system of spherically symmetric field equations in later chapters. We demonstrated that the Emden-Fowler equation can be reduced via a symmetry transformation to quadratures under certain conditions. On evaluating this quadrature for the particular cases we generated a solution to the

Emden-Fowler equation by transforming back to the original variables.

The different physical situations within the spherically symmetric spacetimes, in which the Emden-Fowler equation arose as the principal equation, were then studied. The first situation concerned the assumption of vanishing acceleration. Solutions to the field equations, using the solution to the Emden-Fowler equation, were recovered under the additional condition of vanishing pressure. Our solution was related to the model of Herlt (1996). In the second situation we discussed the appearance of the Emden-Fowler equation in shear-free spherically symmetric spacetimes and highlighted the historical development of the equation for this case. The third physical situation involves the assumption of a constant gravitational potential and a stiff fluid equation of state. We demonstrated that the Emden-Fowler equation arises in this class of spherically symmetric models. An exact solution to the field equations was presented which extends the treatment of Govender (1996). The fourth situation involved the class of cosmological models presented by Maharaj *et al* (1996). We applied an *ad hoc* technique of integration to generate solutions to the Emden-Fowler equation and regained the Maharaj *et al* results. Our general approach enabled us to establish the uniqueness of this class of models.

We derived the Einstein-Maxwell system of field equations which governs the behaviour of a spherically symmetric model for a charged perfect fluid. The conservation equations follow from the field equations. We discussed the role of the proper charge density in the Einstein-Maxwell system and obtained a condition on the electromagnetic gauge potential. This condition enables us to characterise charged solutions which have an uncharged analogue when the gauge potential vanishes. We generalised the solution of Maharaj *et al* (1993) to obtain its charged analogue. By setting the charge contribution to zero we regained the Maharaj *et al* solution. We

demonstrated the uniqueness of this class of cosmological models by solving the non-linear Riccati eqaution. The famous Gutman-Bespal'ko (1967) solution was regained from our general class.

We now briefly discuss possible areas for future investigation that arise as a result of the findings in this thesis. The method of Lie point symmetries used to generate solutions of the Emden-Fowler equation in chapter 3 could be extended to include more general contact transformations (Mahomed and Leach 1991). We could apply the general Lie analysis to the Einstein-Maxwell system of field equations by using the Lie method for systems of partial differential equations to generate a similarity variable. This method, based on the approach used by Govender (1996) to solve the spherically symmetric field equations with neutral matter, reduces the system of partial differential equations to a simpler system of ordinary differential equations through the use of the similarity variable. We could also use a more general form for the similarity generator than the one chosen by Govender (1996), to obtain a wider class of solutions for both the uncharged and charged field equations. However this is a formidable problem and is a research initiative in its own right. We also notice that the Emden-Fowler equation appears to be generic in spherically symmetric spacetimes. We could investigate the precise relationship that exists between the Emden-Fowler equation and the geometric property of spherical symmetry in general. It is also necessary to consider the influence of the Emden-Fowler equation on gravitating systems which are not spherically symmetric. This could be done by analysing models which are not invariant under rotations *eg* the Bianchi models (Ryan and Shepley 1975). This investigation would provide additional information on the importance of the Emden-Fowler equation in general relativity. For the system of Einstein-Maxwell field equations we obtained a solution which reduced to the

solutions of Maharaj *et al* (1993) and Gutman-Bespal'ko (1967) in the uncharged limit. We could investigate whether it is possible to generate a wider class of solutions which have an appropriate uncharged limit and an equation of state other than the stiff fluid equation of state. This class of solutions would generalise all the exact solutions for spherically symmetric spacetimes in the uncharged limit. Also the role of the Emden-Fowler equation in the Einstein-Maxwell system has to be analysed, extending the simple example given in §5.4.

In conclusion it is hoped that this thesis has significantly highlighted the importance of studying spherically symmetric cosmological models, in particular the role of the Emden-Fowler equation in these models.

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