Spherically Symmetric Solutions in Relativistic Astrophysics

by

Anslyn J. John

Submitted in partial fulfilment of
the requirements for the degree of
Master of Science in the Faculty of Science,
University of Natal
Durban

March 5, 2002

Abstract

In this thesis we study classes of static spherically symmetric spacetimes admitting a perfect fluid source, electromagnetic fields and anisotropic pressures. Our intention is to generate exact solutions that model the interior of dense, relativistic stars. We find a sufficient condition for the existence of series solutions to the condition of pressure isotropy for neutral isolated spheres. The existence of a series solution is demonstrated by the method of Frobenius. With the help of MATHEMATICA (Wolfram 1991) we recovered the Tolman VII model for a quadratic gravitational potential, but failed to obtain other known classes of solution. This establishes the weakness, in certain instances, of symbolic manipulation software to extract series solutions from differential equations. For a cubic potential, we obtained a new series solution to the Einstein field equations describing neutral stars. The gravitational and thermodynamic variables are non-singular and continuous. This model also satisfies the important barotropic equation of state $p = p(\rho)$. Two new exact solutions to the Einstein-Maxwell system, that generalise previous results for uncharged stars, were also found. The first of these generalises the solution of Maharaj and Mkhwanazi (1996), and has well-behaved matter and curvature variables. The second solution reduces to the Durgapal and Bannerji (1983) model in the uncharged limit; this new result may only serve as a toy model for quark stars because of negative energy densities. In both examples we observe that the solutions may be expressed in terms of hypergeometric and elementary functions; this indicates the possibility of unifying isolated solutions under the hypergeometric equation. We also briefly study compact stars with spheroidal geometry, that may be charged or admit anisotropic pressure distributions. The adapted forms of the pressure isotropy condition can be written as a harmonic oscillator equation. Two simple examples are presented.

Preface

The study described in this thesis was carried out in the School of Mathematical and Statistical Sciences, University of Natal, Durban, during the period January 2001 to January 2002. This thesis was completed under the supervision of Prof. S. D. Maharaj.

The research contained in this thesis represents original work by the author. It has not been submitted in any form to another university, nor has it been published previously. Where use was made of the work of others, it has been duly acknowledged in the text.

Anslyn James John

January 2002

Acknowledgments

The author would like to thank sincerely and acknowledge gratefully the following for their generous cooperation in contributing to the success of this thesis:

- Prof. S. D. Maharaj for his excellent guidance and supervision throughout the preparation and compilation of this thesis. His confidence in my abilities and timely interventions of financial support were tremendously appreciated. I am also greatly indebted to him for kindling my interest in general relativity and cosmology, and actively facilitating my participation in these fields.

- Dr. K. S. Govinder for his expertise in the area of differential equations, which proved to be invaluable to this thesis. His boundless optimism and encouragement played no small role in my decision to pursue a life in mathematical physics, and for that, I am grateful.

- Dr. S. Moopanar, Dr. D. B. Lortan, Dr. S. M. Wagh and Dr.M. Govender for many useful discussions pertaining to this thesis.

- Mrs. Dale Haslop and Mrs. Faye Etheridge of the School of Mathematical and Statistical Sciences for their administrative assistance.

- The National Research Foundation for financial assistance through the award of a Grant-Holders Bursary in the Open Research Programme.

- The University of Natal for financial assistance in the form of a Graduate Assistantship.

- Many outstanding and supportive friends – especially Lynelle John and Seshini Pillay. I am also indebted to Aneta Pillay, whose patience and proofreading abilities had a marked influence on this thesis.

- The academics of the then-Departments of Physics and Mathematics of the University of Natal, Durban for many rewarding years of instruction.

Dedication

To Sir Karl Raimund Popper (1902-1994)
and the ideal of the Open Society

Contents

1 Introduction **1**

2 Static Relativistic Stars: an introduction **6**

2.1 Introduction 6

2.2 Differential Geometry 7

2.3 Fluids and electromagnetic fields 11

2.4 Static spherically symmetric spacetimes 13

2.5 The field equations 16

2.5.1 Neutral fluids 17

2.5.2 Charged fluids 19

2.6 Spheroidal Geometry 20

2.7 Criteria for physically viable stellar models 22

3 Computational aids and Series Solutions **26**

3.1 Introduction 26

3.2 An existence theorem 27

3.3 The Tolman VII solution 31

3.4 A new series solution via the method of Frobenius 33

4 A generalisation of Maharaj and Mkhwanazi **41**

4.1 Introduction 41

4.2 Specifying Z and E 42

4.3 The hypergeometric equation 44

4.4 Previous cases regained 47

4.5 New charged solutions 49

5 Spheroidal Geometry **54**

5.1 Introduction 54

5.2 Charged, isotropic Tikekar stars 55

5.3 Anisotropic Tikekar stars 58

6 Conclusion **61**

7 References **66**

Chapter 1

Introduction

General relativity is the modern theory of gravity as proposed by Einstein in 1915. What distinguishes general relativity from other physical theories is the idea that spacetime is no longer a passive stage upon which nature performs. Indeed the curvature of spacetime is coupled to its matter and energy content via the Einstein field equations in a highly non-linear manner. The theory makes predictions that differ from the classical Newtonian paradigm. Four important predictions, viz. perihelion advance, the bending of light, time delay of radar signals and the geodesic effect, have been shown to be consistent with observations (Foster and Nightingale 1998, Narlikar 1979). These tests were important in the acceptance of general relativity and have since been expanded to include other observations and experiments (Davies 1989, Will 1981). The Newtonian theory remains successful when $\frac{GM}{c^2R} \ll 1$, and is, indeed, preferrable in this regime due to its greater simplicity. In the limit of strong gravitational fields however, we must employ the Einstein field equations for an accurate description of the gravitational field.

Soon after its inception, the general theory was applied to problems in cosmology. Indeed the standard models of cosmology are relativistic theories. In contrast, astrophysics remained largely Newtonian until the 1960s. At this stage the discovery of quasars prompted the development of far-ranging hypotheses. The more conventional of these recognised the importance of general relativity (Zel'dovich and Novikov 1971). Furthermore, even stars with low densities corresponding to $M \sim 100$ solar masses, will eventually exhaust their nuclear fuel and become compact. At this stage of stellar evolution, relativistic effects become significant.

One of the themes of this thesis is to work towards models of compact stellar objects. Since $\frac{GM}{c^2R} \sim 1$, we do not utilise the parametrised post-Newtonian approximation (Narlikar 1979) or other other approximation techniques. Approximate solutions tend to lead to very simplified models and important relativistic features are often absent or masked. Also, since our long-term goal is to understand the physics of such bodies, we do not attempt to employ numerical techniques. The approach adopted is thus to explicitly solve the Einstein field equations. We are faced with the formidable task of obtaining exact solutions to the Einstein field equations – a system of coupled, non-linear partial differential equations. Fortunately the physics of our problem enables us to simplify our model substantially. As a simplifying approximation we assume that spacetime is static and spherically symmetric. Such an assumption is consistent with isolated stars that are compact and not radiating. We only consider objects void of shear or heat flow. Accordingly we model the relativistic star as a perfect fluid. However we do allow for the possibility of charged stars. For charged stars the Einstein field equations have to be adapted to include electromagnetic fields; the resultant system is the Einstein-Maxwell system of non-linear equations.

A number of important exact solutions have been discovered with applications in relativistic astrophysics:

(a) The Schwarzschild exterior solution. This describes the gravitational field outside a static spherically symmetric body. Historically this is the first exact solution of the field equations (Schwarzschild 1916a).

(b) The Schwarzschild interior solution. This describes the gravitational field inside a static spherically symmetric body. This solution (Schwarzschild 1916b) matches smoothly to the exterior Schwarzschild line element, and is a good model of small stars, where the pressures are not too large.

(c) The Reissner-Nordstrom solution. This models spacetime outside a charged, static spherically symmetric body. In the limit of vanishing electric fields, we recover the Schwarzschild exterior solution.

It is important to note that other exterior solutions, describing important astrophysical phenomena, exist. The most important of these are the Vaidya solution, which describes radiating bodies, and the Kerr solution, which describes rotating bodies. Any new interior solution describing static spherically symmetric bodies should match the appropriate exterior solution.

We attempt, through this thesis, to obtain new exact solutions that model the interior of charged and neutral stars. We thus have to solve the Einstein-Maxwell and Einstein field equations for these respective cases. Under high pressures stars may possess a non-zero charge during the early stages of their evolution (Stephani 1990). Stars may also acquire a net charge through accretion (Shvartsman 1971). The presence of an electric field can also counter the onset of gravitational collapse, as a net charge distribution produces a repulsive Coulomb force. This affects the formation of singularities (Treves and Turolla 1999). The occurence of charge does have consequences for the cosmic censorship hypothesis (Joshi and Dwivedi 1992a, 1992b, 1992c, Joshi 1993). This conjecture states that any singularity formed by gravitational collapse will always remain hidden behind an event horizon. The presence of charge may be crucial to the presence of naked singularities in cosmic censorship, and exact solutions are helpful in investigations of this hypothesis. An extensive literature of exact solutions to the Einstein field equations has been generated over the years; recent reviews are given by Delgaty and Lake (1998), Kramer *et al* and Krasinski (1997). Many solutions of the Einstein-Maxwell system have also been found but these have yet to be categorised systematically. (We suggest in chapter 4 the possibility of utilising the hypergeometric function to bring together some classes of charged solutions.) Particular exact solutions of the Einstein-Maxwell system, with physically realistic features have been found by Hansraj (1996), Herrera and Ponce de Leon (1985), Maartens and Maharaj (1990) and Tikekar (1984).

The Einstein and Einstein-Maxwell systems describing stellar objects are under-determined. There are different approaches to integrating this system. One approach is to assume that the spacetime admits a particular symmetry (Castejon-Amanedo and Coley 1992, Maharaj *et al* 1991). Another approach, which this thesis adopts, is directly to integrate the system by specifying forms for one or more of the variables on physical grounds.

We now briefly review the work conducted here.

In chapter 2 we introduce those aspects of differential geometry and general relativity relevant to this thesis. We then introduce the field equations for charged and neutral perfect fluids in static spherically symmetric spacetimes. We transform these equations to a more tractable form. The notion of spheroidal geometry is introduced, and the relevant systems of field equations are produced. We conclude with a review of conditions for physical admissability of interior solutions.

In chapter 3 we establish a sufficient condition for the existence of neutral static solutions. Bearing this condition in mind we specified forms for one of the gravitational potentials, Z, and attempted to find new solutions. We demonstrated the merits of software packages, and recovered the Tolman VII model for a quadratic form of Z. Using the method of Frobenius we obtained a series solution, for a cubic form of Z, which we believe has not been previously documented.

In chapter 4 we attempted to solve a form of the Einstein-Maxwell system by specifying the potential Z and the electric field intensity E. We obtained an electrostatic generalisation of a class of neutral solutions due to Maharaj and Mkhwanazi (1996). These solutions are expressed in terms of hypergeometric functions. For special cases we regained some documented uncharged solutions. We then demonstrated, what we believe to be, two new charged solutions, which we were able to express in terms of elementary functions.

In chapter 5 we briefly examined some problems with spheroidal geometry. We utilised a general transformation to rewrite the condition of pressure isotropy in a simpler form for charged stars. We also derived an analogous expression for anisotropic pressure distributions. Particular exact solutions are presented for specified forms of the electric field intensity E and pressure anisotropy factor $\triangle$.

There were four broad aims of this thesis. Firstly to obtain new exact solutions to the Einstein and Einstein–Maxwell systems. Secondly to identify classes of solutions that could lead to physically viable models of stars. Thirdly to demonstrate the possibility of unifying many seemingly disparate exact solutions of the Einstein-Maxwell system. And fourthly, to review the methodology of the field.

The results obtained in this thesis, are summarised in the concluding chapter, wherein suggestions for future work are outlined.

Chapter 2

Static Relativistic Stars: an introduction

2.1 Introduction

One of the fundamental aims in this thesis is to work towards a realistic description of compact stellar objects. For this purpose it is necessary to use the complete non-linear form of the Einstein field equations of general relativity. To this end we develop those aspects of differential geometry and general relativity crucial to our arguments. In §2.2 we define the concepts of the line element, the metric tensor field, the metric connection and the covariant derivative of an arbitrary tensor field. These definitions are not unique to general relativity, but form part of the wider subject of differential geometry. What is peculiar to general relativity is the idea that the gravitational field is a manifestation of the curvature of spacetime. The concept of curvature is clearly defined, and our conventions are specified with the introduction of the Riemann tensor, Ricci tensor, Ricci scalar and Einstein tensor. In §2.3 we describe a covariant formulation of Maxwell's laws, and introduce the idea of a relativistic fluid. We also introduce the notion of perfect fluids. The electromagnetic field and matter are then coupled to the gravitational field by means of the Einstein-Maxwell field equations. We restrict spacetime to be static and spherically symmetric. The geometry of this specific spacetime is determined by explicitly evaluating the Einstein tensor. We also considered the effect of this geometry on the energy-momentum tensor, and evaluate the relevant expressions for neutral and charged perfect fluids. In the latter case we assume a simple form for the electromagnetic field, consistent with static charged bodies. In §2.5 we derive the Einstein field equations for charged and neutral perfect fluids in static spherically symmetric spacetimes. We utilise a change of coordinates, due to Durgapal and Bannerji (1983), that simplifies the task of finding new exact solutions. We also transform the field equations to a form that allows comparison with the predictions of non-relativistic astrophysics. The Newtonian equation of hydrostatic equilibrium is derived, as a special case, from the field equations, and demonstrates the successful incorporation of classical results into the non-linear theory. We introduce the concept of spheroidal geometries in §2.6. Many physically reasonable models of dense stars have been obtained for spheroidal stars (Maharaj and Leach 1996, Tikekar 1990 and Vaidya and Tikekar 1982). We derive the relevant field equations for neutral and perfect fluids, and also explicitly state the conditions of pressure isotropy for both cases. Finally, in §2.7, we list two exterior spacetimes, that are of crucial importance in relativistic astrophysics viz. the exterior Schwarzschild line element (Schwarzschild 1916a) and the Reissner-Nordstrom line element (Reissner 1916, Nordstrom 1918). In addition we state the relevant features that realistic stellar models should exhibit. Any new exact solution purporting to describe relativistic stars should be assessed against these criteria.

2.2 Differential Geometry

We assume that spacetime is a four-dimensional differentiable manifold endowed with a metric tensor field g. Locally the manifold has the structure of Euclidean space in that it may be covered by coordinate patches. In the case of general relativity the metric tensor field is indefinite and the manifold is referred to as pseudo-Riemannian. The tensor field $\mathbf{g}$ is symmetric and non-singular with signature $(- + + +)$. We label points in spacetime with real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$, where $x^0 = ct$ (c being the speed of light in vacuum) is the temporal coordinate and x^1, x^2, x^3 are spatial coordinates. Throughout this thesis we have used the convention that the speed of light assumes the value 1. It is not our intention to give a detailed exposition of differential geometry, and the interested reader is referred to the standard texts on the subject, such as de Felice and Clark (1990), Hawking and Ellis (1973) and Misner *et al* (1973).

The invariant distance between neighbouring points on a manifold is defined by the line element

$$ds^2 = g_{ab}dx^a dx^b \tag{2.1}$$

where g_{ab} are the covariant components of $\mathbf{g}$. The Minkowski line element of special relativity, in Cartesian coordinates, has the form

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

which is a special case of (2.1). The inner product of the two contravariant vector fields $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ is defined by

$$\langle \boldsymbol{\lambda}, \boldsymbol{\mu} \rangle = g_{ab}\lambda^a \mu^b.$$

We require that the inner product remains constant along a curve in the manifold; this implies the condition

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}) \tag{2.2}$$

where Γ is the metric connection. This is the fundamental theorem of Riemannian geometry which implies the existence of a unique, symmetric connection Γ which preserves inner products under parallel transport. The importance of tensor calculus lies in the invariance of quantities under coordinate transformations. Such objects are referred to as tensorial. The directional derivative of a tensor field is not a tensorial quantity in general. It is, however, possible to introduce tensorial derivatives on the manifold. Let $V^{a_1a_2\cdots a_r}{}_{b_1b_2\cdots b_s}$ be an arbitrary (r,s) tensor field. The covariant derivative of $\mathbf{V}$ is defined by

$$V^{a_1a_2\cdots a_r}{}_{b_1b_2\cdots b_s;c} = V^{a_1a_2\cdots a_r}{}_{b_1b_2\cdots b_s,c}$$

$$+\Gamma^{a_1}{}_{dc}V^{da_2\cdots a_r}{}_{b_1b_2\cdots b_s} + \cdots + \Gamma^{a_r}{}_{dc}V^{a_1a_2\cdots d}{}_{b_1b_2\cdots b_s}$$

$$-\Gamma^{d}{}_{b_1c}V^{a_1a_2\cdots a_r}{}_{db_2\cdots b_s} - \cdots - \Gamma^{d}{}_{b_sc}V^{a_1a_2\cdots a_r}{}_{b_1b_2\cdots d} \tag{2.3}$$

where semi-colons and commas denote covariant and partial differentiation respectively. From (2.3) we observe that the covariant derivative is a generalisation of the partial derivative and, when operating on an (r,s) tensor field, produces an $(r,s+1)$ tensor field. In particular for a $(0,1)$ vector field V_a, (2.3) reduces to

$$V_{a;b} = V_{a,b} - \Gamma^c{}_{ab}V_c.$$

The curvature of spacetime is quantified by the Riemann (or curvature or Riemann-Christoffel) tensor $\mathbf{R}$. By considering the second covariant derivative of the $(0,1)$ tensor field $\mathbf{V}$, we generate the result

$$V_{a;bc} - V_{a;cb} = (\Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec})V_d$$

$$= R^d{}_{abc}V_d$$

where we have defined

$$R^a{}_{bcd} \equiv \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}. \tag{2.4}$$

Here $R^a{}_{bcd}$ are the components of the $(1,3)$ tensor field $\mathbf{R}$. The Riemann tensor provides a measure of curvature of a manifold, i.e. the deviation from the flatness characteristic of

Minkowski spacetime. The components of $\mathbf{R}$ have the following useful properties

$$R_{abcd} = -R_{bacd}$$

$$R_{abcd} = -R_{abdc}$$

$$R_{abcd} = R_{cdab}$$

$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0$$

which can be easily verified from the definition (2.4). Of particular importance is the symmetric Ricci tensor

$$R_{ab} = R^{c}{}_{acb}$$

$$= \Gamma^{c}{}_{ab,c} - \Gamma^{c}{}_{ac,b} + \Gamma^{c}{}_{dc}\Gamma^{d}{}_{ab} - \Gamma^{c}{}_{db}\Gamma^{d}{}_{ac} \tag{2.5}$$

obtained from a contraction of (2.4). Contraction of the Ricci tensor (2.5) yields

$$R = R^{a}{}_{a}$$

$$= g^{ab}R_{ab} \tag{2.6}$$

which is the Ricci, or curvature, scalar. We are now in a position to define, utilising (2.5) and (2.6), the Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} \tag{2.7}$$

which is necessarily symmetric. The Einstein tensor has vanishing divergence, so that

$$G^{ab}{}_{;b} = 0 \tag{2.8}$$

which is the contracted Bianchi identity.

2.3 Fluids and electromagnetic fields

The energy-momentum tensor $\mathbf{M}$ for a neutral fluid is

$$M^{ab} = (p+\rho)u^a u^b + pg^{ab} + q^a u^b + q^b u^a + \pi^{ab} \tag{2.9}$$

where p is the isotropic (kinetic) pressure, ρ is the energy density, q^a is the heat flux vector ($q^a u_a = 0$) and π^{ab} is the anisotropic pressure, or stress, tensor ($\pi^{ab}u_a = 0 = \pi^a_a$). All these quantities are measured relative to a comoving fluid four-velocity $\mathbf{u}$ which is unit and timelike ($u^a u_a = -1$). For a perfect fluid ($\pi^{ab} = 0 = q^a$), (2.9) reduces to

$$M^{ab} = (p+\rho)u^a u^b + pg^{ab}. \tag{2.10}$$

It is often required that the matter distribution satisfies a barotropic equation of state $p = p(\rho)$. Particular equations of state with the perfect fluid energy-momentum tensor (2.10) are often considered in applications in cosmology and astrophysics. The linear relation

$$p = \gamma\rho, \quad 0 \leq \gamma \leq 1$$

is known as the γ-law equation of state. The restriction $0 \leq \gamma \leq 1$ is required to preserve causality. Also widely used is the polytropic equation of state

$$p = k\rho^{\frac{n+1}{n}}$$

where k and n are constants.

In terms of the four-potential $\mathbf{A}$, we define the Faraday tensor $\mathbf{F}$ as

$$F_{ab} = A_{b;a} - A_{a;b}$$

which is skew-symmetric. The Faraday tensor can be interpreted in terms of the electric field, $\mathbf{E} = (E^1, E^2, E^3)$, and the magnetic field, $\mathbf{B} = (B^1, B^2, B^3)$, by means of the matrix representation

$$F^{ab} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}.$$

The electromagnetic stress tensor $\mathbf{E}$ is given by

$$E_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}. \tag{2.11}$$

In order to consider the effect of $\mathbf{E}$ on the gravitational field we require a covariant formulation of Maxwell's laws. The governing equations are given by

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{2.12a}$$

$$F^{ab}{}_{;b} = J^a. \tag{2.12b}$$

In the above $\mathbf{J}$ is the four-current density defined by

$$J^a = \sigma u^a$$

where σ is the proper charge density. For further information on Maxwell's field equations (2.12), see Misner *et al* (1973) and Narlikar (1979).

The total energy-momentum tensor $\mathbf{T}$ is the sum of $\mathbf{M}$ and $\mathbf{E}$:

$$T^{ab} = M^{ab} + E^{ab}$$

explicitly given by (2.9) and (2.11) respectively. We are now in a position to introduce the effect of gravitational interactions on matter and electromagnetic fields. This is given by the Einstein-Maxwell system of equations

$$G_{ab} = T_{ab}$$

$$= M_{ab} + E_{ab} \tag{2.13a}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{2.13b}$$

$$F^{ab}{}_{;b} = J^a. \tag{2.13c}$$

This is a highly non-linear system of coupled, partial differential equations governing the behaviour of a gravitating system in the presence of an electromagnetic field. Note that we utilise units in which the coupling constant in the Einstein equations (2.13a) is unity. From (2.8) and (2.13a) we obtain

$$T^{ab}{}_{;b} = 0 \tag{2.14}$$

which are the conservation equations.

2.4 Static spherically symmetric spacetimes

We shall deal with a form of (2.13) applicable to problems in relativistic astrophysics. Since our intention is to study stable, stellar objects it seems reasonable, on physical grounds, to assume that spacetime is static and spherically symmetric. This is clearly consistent with models utilised to study physical processes in compact objects as undertaken by Shapiro and Teukolsky (1983) amongst others.

The generic line element for static, spherically symmetric spacetimes is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.15}$$

where we have used spherical polar coordinates $(x^a) = (t, r, \theta, \phi)$. The quantities $\nu(r)$ and $\lambda(r)$ are related to the gravitational potentials. The connection coefficients, associated with the metric (2.15), are easily determined from (2.2); the non-vanishing components are given by

$$
\begin{array}{lllllll}
\Gamma^0{}_{01} & = & \nu' & \qquad & \Gamma^2{}_{12} & = & \frac{1}{r} \\
\Gamma^1{}_{00} & = & \nu' e^{2(\nu-\lambda)} & & \Gamma^2{}_{33} & = & -\sin\theta\cos\theta \\
\Gamma^1{}_{11} & = & \lambda' & & \Gamma^3{}_{13} & = & \frac{1}{r} \\
\Gamma^1{}_{22} & = & -r\ e^{-2\lambda} & & \Gamma^3{}_{23} & = & \cot\theta \\
\Gamma^1{}_{33} & = & -r\ e^{-2\lambda}\ \sin^2\theta & & & &
\end{array}
$$

where primes denote differentiation with respect to r. The connection coefficients $\Gamma^a{}_{bc}$ may also be obtained via a variational approach if one specifies the scalar $L = \frac{1}{2}g_{ab}(x^c)\dot{x}^a\dot{x}^b$ as a Lagrangian.

From (2.5) we write down the Ricci tensor components

$$R_{00} = e^{2(\nu-\lambda)}\left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{2\nu'}{r}\right) \tag{2.16a}$$

$$R_{11} = -\left(\nu'' + \nu'^2 - \nu'\lambda' - \frac{2\lambda'}{r}\right) \tag{2.16b}$$

$$R_{22} = 1 - e^{-2\lambda}(1 + r(\nu' - \lambda')) \tag{2.16c}$$

$$R_{33} = \sin^2\theta\ R_{22} \tag{2.16d}$$

$$R_{ab} = 0, \quad a \neq b.$$

We can now evaluate the Ricci scalar

$$R = 2\left(\frac{1}{r^2} - \left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{2\nu'}{r} - \frac{2\lambda'}{r} + \frac{1}{r^2}\right)e^{-2\lambda}\right) \tag{2.17}$$

from (2.6). The components of the Einstein tensor

$$G^{00} = e^{-2\nu}\frac{1}{r^2}[r(1-e^{-2\lambda})]' \tag{2.18a}$$

$$G^{11} = e^{-2\lambda}\left[-\frac{1}{r^2}(1-e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda}\right] \tag{2.18b}$$

$$G^{22} = \frac{1}{r^2}e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) \tag{2.18c}$$

$$G^{33} = \frac{1}{\sin^2\theta}G^{22} \tag{2.18d}$$

$$G^{ab} = 0, \quad a \neq b$$

follow from (2.7), (2.16) and (2.17).

We investigate various forms of the energy-momentum tensors M^{ab} and E^{ab} (note that $T^{ab} = M^{ab} + E^{ab}$) with the comoving four-velocity $u^a = e^{-\nu}\delta^a_0$ for the line element (2.15). For neutral ($E^{ab} = 0$) perfect fluids (2.10) we obtain the non-vanishing components

$$T^{00} = e^{-2\nu}\rho \tag{2.19a}$$

$$T^{11} = e^{-2\lambda}p \tag{2.19b}$$

$$T^{22} = \frac{1}{r^2}p \tag{2.19c}$$

$$T^{33} = \frac{1}{r^2\sin^2\theta}p. \tag{2.19d}$$

For charged perfect fluids we make the choice

$$A_a = (\phi(r), 0, 0, 0) \tag{2.20}$$

which is a simple form consistent with spherical symmetry. The only non-zero components of the Faraday tensor F^{ab} are

$$F_{01} = -\phi'(r)$$

and its skew-symmetric partner F_{10}. The associated contravariant component has the form

$$F^{01} = e^{-(\nu+\lambda)}E(r)$$

where we have defined

$$E(r) = e^{-(\nu+\lambda)}\phi'(r).$$

The quantity E may be interpreted as the electric field intensity in the manner prescribed by Herrera and Ponce de Leon (1985). The proper charge density takes the form

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)'. \tag{2.21}$$

Then the components of $\mathbf{E}$ may be found using (2.11), and the total energy-momentum tensor has the non-zero components

$$T^{00} = e^{-2\nu}\left(\rho + \frac{1}{2}E^2\right) \tag{2.22a}$$

$$T^{11} = e^{-2\lambda}\left(p - \frac{1}{2}E^2\right) \tag{2.22b}$$

$$T^{22} = \frac{1}{r^2}\left(p + \frac{1}{2}E^2\right) \tag{2.22c}$$

$$T^{33} = \frac{1}{r^2\sin^2\theta}\left(p + \frac{1}{2}E^2\right) \tag{2.22d}$$

for spherically symmetric spacetimes (2.15).

2.5 The field equations

We now generate the field equations, in various coordinate systems, for the case of perfect fluids in static spherically symmetric spacetimes. We consider both charged and neutral perfect fluids in the Einstein-Maxwell system (2.13).

2.5.1 Neutral fluids

On equating (2.18) and (2.19) we obtain

$$\frac{1}{r^2}[r(1-e^{-2\lambda})]' = \rho \tag{2.23a}$$

$$-\frac{1}{r^2}(1-e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} = p \tag{2.23b}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p. \tag{2.23c}$$

As a consequence of (2.14) we have the conservation equation

$$\frac{dp}{dr} = -(\rho + p)\frac{d\nu}{dr} \tag{2.24}$$

which may be used in place of any of the field equations. The system of equations (2.23) governs the gravitational behaviour of a neutral perfect fluid.

An equivalent form of the field equations is obtained if we use the transformation

$$x = Cr^2 \tag{2.25a}$$

$$Z(x) = e^{-2\lambda(r)} \tag{2.25b}$$

$$A^2y^2(x) = e^{2\nu(r)} \tag{2.25c}$$

due to Durgapal and Bannerji (1983), where C and A are arbitrary constants. Under the transformation (2.25), the system (2.23) becomes

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} \tag{2.26a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} \tag{2.26b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1)y = 0 \tag{2.26c}$$

where the overdot denotes differentiation with respect to the variable x. Note that (2.26) is a system of three equations in the four unknowns, ρ, p, y and Z. The advantage of this system lies in the fact that a solution can, upon a suitable specification of $Z(x)$, be readily obtained by integrating (2.26c) which is second order and linear in y.

We now consider a different form of the field equations which eases comparison with the Newtonian equations. Equation (2.23a) implies that

$$r(1 - e^{-2\lambda}) = \int_0^r \rho(\eta)\eta^2 d\eta + k$$

where k is a constant. This suggests that if we define a "mass" function $m(r)$ by

$$m(r) \equiv \frac{1}{2}\int_0^r \rho(\eta)\eta^2 d\eta$$

we then obtain

$$m(r) = \frac{1}{2}r(1 - e^{-2\lambda})$$

where we have set $k = 0$ to ensure the metric function $e^{-2\lambda}$ remains finite at the origin. The "mass" function $m(r)$ is proportional to the mass enclosed by a sphere of coordinate radius r. The true radius R is given by

$$R(r) = \int_0^r e^{\lambda(\eta)} d\eta.$$

The field equations (2.23) now assume the form

$$\frac{dm}{dr} = \frac{1}{2}r^2\rho \tag{2.27a}$$

$$\frac{d\nu}{dr} = \frac{\frac{1}{2}pr^3 + m}{r(r-2m)} \tag{2.27b}$$

$$\nu'' + \nu'^2 + \frac{r-(rm)'}{r(r-2m)}\nu' - \frac{rm'-m}{r^2(r-2m)} = \frac{pr}{2-m}. \tag{2.27c}$$

On substituting (2.27b) into (2.24) we have

$$\frac{dp}{dr} = -\frac{(\rho+p)(m+\frac{1}{2}pr^3)}{r(r-2m)} \tag{2.28}$$

which is often termed the Oppenheimer-Volkoff equation. In the limit

$$p \ll \rho, \quad m \ll r$$

the previous expression approaches the limiting equation

$$\frac{dp}{dr} = -\frac{\rho m}{r^2} \tag{2.29}$$

which is the equation of hydrostatic equilibrium for Newtonian stars. By comparing the relativistic result (2.28) with equation (2.29) we see that the effect of relativistic corrections is to steepen the pressure gradient predicted by Newtonian astrophysics.

2.5.2 Charged fluids

The results of this section depend on the choice for the four-potential, $A_a = (\phi(r), 0, 0, 0)$, made earlier. On equating (2.18) and (2.22), and taking into consideration (2.21), we obtain

$$\frac{1}{r^2}[r(1-e^{-2\lambda})]' = \rho + \frac{1}{2}E^2 \tag{2.30a}$$

$$-\frac{1}{r^2}\left(1-e^{-2\lambda}\right) + \frac{2\nu'}{r}\, e^{-2\lambda} = p - \frac{1}{2}E^2 \tag{2.30b}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p + \frac{1}{2}E^2 \tag{2.30c}$$

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)'. \tag{2.30d}$$

The system of equations (2.30) governs the behaviour of the gravitational field for a charged perfect fluid. Observe that the system reduces to (2.23) when $E = 0$. Note that as a consequence of (2.14), we have

$$p' + (\rho + p)\nu' = \frac{E}{r^2}(r^2 E)' \tag{2.31}$$

which is sometimes used as a starting point to solve the system (2.30).

An equivalent form of (2.30) is generated if we utilise the transformation (2.25), due to Durgapal and Bannerji (1983). With this transformation, the Einstein-Maxwell system becomes

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{E^2}{2C} \tag{2.32a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} - \frac{E^2}{2C} \tag{2.32b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + \left(\dot{Z}x - Z + 1 - \frac{E^2 x}{C}\right)y = 0 \tag{2.32c}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}(x\dot{E} + E)^2 \tag{2.32d}$$

where, as before, overdots indicate differentiation with respect to the variable x.

2.6 Spheroidal Geometry

To obtain an exact solution of (2.23) we need to specify one of the unknowns ρ, p, ν or λ. Here we adopt the procedure of choosing the gravitational potential $\lambda(r)$. The advantage of this approach is that we can then characterise the geometry of the hypersurfaces $\{t = \text{constant}\}$. We make the choice

$$e^{-2\lambda(r)} = \frac{1 - Kr^2/R^2}{1 - r^2/R^2} \tag{2.33}$$

where K is a constant. The form (2.33) was first utilised by Vaidya and Tikekar (1982) in their analysis of compact objects. This has the interesting geometrical interpretation that the hypersurfaces described by $\{t = \text{constant}\}$ are 3-spheroids. The quantity K is the spheroidal parameter. Substituting (2.33) into (2.23) yields

$$\rho = \frac{1-K}{R^2}\frac{3-Kr^2/R^2}{(1-Kr^2/R^2)^2} \tag{2.34a}$$

$$p = \left(\frac{2\nu'}{r}+\frac{1}{r^2}\right)\frac{1-r^2/R^2}{1-Kr^2/R^2}-\frac{1}{r^2} \tag{2.34b}$$

$$p = \left(\nu''+\nu'^2+\frac{\nu'}{r}\right)\frac{1-r^2/R^2}{1-Kr^2/R^2}-\left(\nu'+\frac{1}{r}\right)\frac{(1-K)(r/R^2)}{(1-Kr^2/R^2)^2}. \tag{2.34c}$$

On eliminating p from (2.34b) and (2.34c) we obtain

$$\begin{aligned} 0 = (1-Kr^2/R^2)(1-r^2/R^2)\left(\nu''+\nu'^2-\frac{\nu'}{r}\right) - \\ (1-K)\frac{r}{R^2}\left(\nu'+\frac{1}{r}\right)+\frac{1-K}{R^2}(1-Kr^2/R^2) \end{aligned} \tag{2.35}$$

which is the condition of pressure isotropy. For $K = 0$ we obtain the interior Schwarzschild solution (Schwarzschild 1916b), and for $K = 1$ the hypersurfaces $\{t = \text{constant}\}$ are flat. The general solution of (2.34), for integral values of K, was discovered by Maharaj and Leach (1996).

The choice of the metric function (2.33) can also be used to study the behaviour of charged stars. In this case we obtain the equations

$$\rho+\frac{1}{2}E^2 = \frac{1-K}{R^2}\frac{3-Kr^2/R^2}{(1-Kr^2/R^2)^2} \tag{2.36a}$$

$$p-\frac{1}{2}E^2 = \left(\frac{2\nu'}{r}+\frac{1}{r^2}\right)\frac{1-r^2/R^2}{1-Kr^2/R^2}-\frac{1}{r^2} \tag{2.36b}$$

$$p + \frac{1}{2}E^2 = \left(\nu'' + \nu'^2 + \frac{\nu'}{r}\right)\frac{1 - r^2/R^2}{1 - Kr^2/R^2} - \left(\nu' + \frac{1}{r}\right)\frac{(1-K)(r/R^2)}{(1 - Kr^2/R^2)^2} \tag{2.36c}$$

$$\sigma^2 = \frac{1}{r^4}\left(\frac{1 - Kr^2/R^2}{1 - r^2/R^2}\right)(r^2E')^2 \tag{2.36d}$$

from the Einstein-Maxwell system (2.30). On eliminating p from (2.36b) and (2.36c) we obtain

$$(1 - Kr^2/R^2)^2E^2 = (1 - Kr^2/R^2)(1 - r^2/R^2)\left(\nu'' + \nu'^2 - \frac{\nu'}{r}\right)$$

$$-(1-K)\frac{r}{R^2}\left(\nu' + \frac{1}{r}\right) + \frac{1-K}{R^2}(1 - Kr^2/R^2) \tag{2.37}$$

which is the condition of pressure isotropy extended to the electromagnetic field. In this thesis we seek solutions for various classes of the systems (2.34) and (2.36).

2.7 Criteria for physically viable stellar models

A number of solutions to the Einstein and Einstein-Maxwell equations exist that may be utilised to model the exterior spacetime of compact objects in relativistic astrophysics. Any new solution applicable to the interior of the body should be matched smoothly to the appropriate exterior solution subject to physical criteria. We state two exterior solutions used in this thesis.

The spacetime surrounding a static, spherically symmetric body of mass M is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.38}$$

which is the exterior Schwarzschild line element (Schwarzschild 1916a). The gravitational field outside a static, charged spherically symmetric body has the form

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.39}$$

for a body of mass M. Here Q is a constant related to the total charge of the sphere. The line element (2.39) is the exterior Reissner-Nordstrom solution. When $Q = 0$, (2.39) reduces to the exterior Schwarzschild line element (2.38).

We can obtain solutions to the Einstein and Einstein-Maxwell systems which may not be physically reasonable. We need to isolate those solutions which are physically reasonable, and consequently may be utilised to model astrophysical phenomena. To this end we stipulate a number of criteria for physical acceptability that have been used in previous investigations. We briefly outline a number of conditions which realistic stellar models should satisfy :

(a) The pressure and energy density should be positive and finite throughout the interior of the star:

$$0 \leq p \leq \infty$$

$$0 \leq \rho \leq \infty$$

(b) The pressure and energy density should be monotonically decreasing functions of r. The pressure must vanish at the stellar boundary $r = R$:

$$\frac{dp}{dr} \leq 0$$

$$\frac{d\rho}{dr} \leq 0$$

$$p(R) = 0$$

(c) The metric functions should be continuous at the boundary $r = R$. The interior line elements should smoothly match to the exterior Schwarzschild and Reissner-Nordstrom solutions for the case of neutral and charged spherically symmetric solutions respectively:

$$\text{neutral stars:} - e^{2\nu(R)} = e^{-2\lambda(R)}$$

$$= 1 - \frac{2M}{R}$$

$$\text{charged stars:} - e^{2\nu(R)} = e^{-2\lambda(R)}$$

$$= 1 - \frac{2M}{R} + \frac{Q^2}{R^2}$$

(d) In the case of charged solutions, the electric field intensity $E(r)$ must be continuous at $r = R$.

(e) The metric functions, $e^{2\lambda}$ and $e^{2\nu}$, and the electric field intensity E should be positive and non-singular throughout the star's interior.

(f) The speed of sound must remain subluminal throughout the star's interior. In our units this means

$$0 \leq \frac{dp}{d\rho} \leq 1.$$

This condition is necessary to preserve causality.

(g) The solution should be stable with respect to radial perturbations.

Not all models satisfy the above conditions. However these are useful as they provide qualitative features which represent many physical stars. Most solutions do not satisfy all the conditions (a)–(g). For example it is not clear from observational evidence whether

$$\frac{dp}{dr} \leq 0 \quad \text{and} \quad \frac{d\rho}{dr} \leq 0$$

throughout the star's interior; many researchers believe that this condition is too stringent and is not true for many compact objects. However it is important to compare the physical characteristics of individual stellar models with the conditions listed above. A comprehensive analysis of perfect fluid solutions to the Einstein field equations for static spherically symmetric models was compiled by Delgaty and Lake (1998). A proper and complete analysis of the physical features, including stability with respect to radial perturbations, is not trivial and can only be done for exact solutions with simple analytic representations. Such analyses in comoving and non-commoving coordinates, for classes of spherically symmetric perfect fluid distributions, have been conducted by Knutsen (1984, 1992, 1995, 2000).

Chapter 3

Computational aids and Series Solutions

3.1 Introduction

In this chapter we establish a sufficient condition for the existence of solutions to the field equation for pressure isotropy (2.26c). Given a general analytic form for the gravitational potential $Z(x)$, (2.26c) will possess a series solution, and the Einstein field equations (2.26) are thus integrable. The existence of regular and regular singular points is established in §3.2. We choose a quadratic form for Z in §3.3 that has the advantage of being continuous and non-singular in the stellar interior. To find an exact solution for $y(x)$ we exploited the symbolic manipulation capabilities of MATHEMATICA (Wolfram 1991). For various parameter values our attempts were unsuccessful at extracting the solutions about regular singular points of the equations. MATHEMATICA did however obtain the solution at the regular point. This exercise highlights the analytic limitations of computer software, and suggests that the algorithm employed has greater difficulty recovering series solutions near regular singular points than at regular points of differential equations. (A more thorough investigation of this point would be of invaluable service to applied mathematicians and theoretical physicists.) The quadratic solution found reduces to the Tolman VII stellar model. In §3.4 we specify a physically well-behaved cubic form for Z, which we believe had not been previously examined. We found no documented solutions to the equation in standard references (Kamke 1983, Zwillinger 1989), and MATHEMATICA returned a result in terms of hypergeometric functions with complex arguments. Since that form was not suitable to a physical analysis we attempted to find the series solution directly. This yields a third order recurrence relation, which we manage to solve from first principles. It is then possible to exhibit a new exact solution to the Einstein field equations. The curvature and thermodynamical variables appear to be well-behaved. We also demonstrate the possibility of an explicit barotropic equation of state $p = p(\rho)$. Most exact solutions fail to achieve this, and some exponents, e.g. Krasinski (1997), believe that all realistic models in general relativity should satisfy such an equation of state. We believe that a detailed physical analysis of our solution is likely to lead to a more realistic model for compact objects.

3.2 An existence theorem

The Einstein field equations, in the form (2.26), are under-determined. The standard approach is to specify the gravitational potential $Z(x)$ and attempt to solve (2.26c) for the potential y. In principle we can always choose $Z(x)$ such that the system is integrable. We establish below, the result that (2.26c) possesses a series solution for every analytic $Z(x)$.

Consider the general second-order, linear homogenous differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{3.1}$$

where a_0, a_1 and a_2 are functions of x. If the limits

$$\lim_{x\to 0}\frac{a_1}{a_0} \tag{3.2a}$$

$$\lim_{x\to 0}\frac{a_2}{a_0} \tag{3.2b}$$

are finite, then (3.1) is said to possess a regular point at $x = 0$. By the method of Frobenius, equation (3.1) thus possesses two linearly independent solutions of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

about $x = 0$. The differential equation (3.1) may possess regular points at other values of x. The existence of these of these points is easily established by appropriately translating the independent variable and evaluating the above limits. The equation (3.1) may still admit series solutions if the limits (3.2) do not exist. This scenario arises when (3.1) has a regular singular point at the origin. A necessary and sufficient condition for this to occur is the existence of the finite limits

$$\lim_{x \to 0} \frac{(x-0)a_1}{a_0} \tag{3.3a}$$

$$\lim_{x \to 0} \frac{(x-0)^2 a_2}{a_0}. \tag{3.3b}$$

In this case the form of the series solutions is more complicated (Powers 1987); the crucial point being that we are assured of the existence of such solutions when (3.1) possesses regular singular points. In practice one determines the coefficients c_n by substituting the solution into (3.1), and obtaining a recurrence relation for c_n - the solution of which *per se* is a non-trivial matter, and often presents the greatest obstacle in this approach.

Let $Z(x)$ be analytic about some neighbourhood of $x = 0$. This is equivalent to Z having a power series representation

$$\begin{aligned} Z(x) &= \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots \\ &= \sum_{i=0}^{\infty} \alpha_i x^i \end{aligned} \tag{3.4}$$

in the interval $|x - 0| < \delta$, where α_i and δ are real constants. It is easy to show that

$$\dot{Z} = \sum_{i=1}^{\infty} i \alpha_i x^{i-1}.$$

The quantities a_0 and a_1 follow on comparing (2.26c) and (3.1) since Z and $\dot{Z}$ are known. We now evaluate the limits (3.2) in the hope of establishing a regular point at the origin of our coordinate system

$$\begin{aligned}
\lim_{x\to 0}\frac{a_1}{a_0} &= \lim_{x\to 0}\frac{2\dot{Z}x^2}{4Zx^2} \\
&= \lim_{x\to 0}\frac{2x^2(\alpha_1+2\alpha_2 x+3\alpha_3 x^2+\cdots)}{4x^2(\alpha_0+\alpha_1 x+\alpha_2 x^2+\cdots)} \\
&= \lim_{x\to 0}\frac{(\alpha_1+2\alpha_2 x+3\alpha_3 x^2+\cdots)}{2(\alpha_0+\alpha_1 x+\alpha_2 x^2+\cdots)} \\
&= \frac{\alpha_1}{2\alpha_0}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0}\frac{a_2}{a_0} &= \lim_{x\to 0}\frac{\dot{Z}x-Z+1}{4Zx^2} \\
&= \lim_{x\to 0}\frac{\alpha_2 x^2+2\alpha_3 x^3+3\alpha_4 x^4+\cdots+1-\alpha_0}{4x^2(\alpha_0+\alpha_1 x+\alpha_2 x^2+\cdots)} \\
&= \lim_{x\to 0}\frac{x^2(\alpha_2+2\alpha_3 x+3\alpha_4 x^2+\cdots)+1-\alpha_0}{4x^2(\alpha_0+\alpha_1 x+\alpha_2 x^2+\cdots)}.
\end{aligned}$$

If we let $\alpha_0 = 1$ we find

$$\begin{aligned}
\lim_{x\to 0}\frac{\dot{Z}x-Z+1}{4Zx^2} &= \lim_{x\to 0}\frac{1}{4}\frac{\alpha_2+2\alpha_3 x+3\alpha_4 x^2+\cdots}{1+\alpha_1 x+\alpha_2 x^2+\cdots} \\
&= \frac{\alpha_2}{4}.
\end{aligned}$$

If $\alpha_0 \neq 1$, then

$$\lim_{x\to 0}\frac{\dot{Z}x-Z+1}{4Zx^2} \to \pm\infty.$$

Thus the limits (3.2) are finite if $\alpha_0 = 1$.

We now determine whether $x = 0$ is a regular singular point of (2.26c), when $\alpha_0 \neq 1$, by evaluating the limits (3.3):

$$\lim_{x\to 0} \frac{(x-0)a_1}{a_0} = \lim_{x\to 0} \frac{2\dot{Z}x^3}{4Zx^2}$$

$$= \lim_{x\to 0} \frac{2x^3(\alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + \cdots)}{4x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots)}$$

$$= \lim_{x\to 0} \frac{(\alpha_1 x + 2\alpha_2 x^2 + 3\alpha_3 x^3 + \cdots)}{2(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots)}$$

$$= 0$$

$$\lim_{x\to 0} \frac{(x-0)^2 a_2}{a_0} = \lim_{x\to 0} \frac{x^2(\dot{Z}x - Z + 1)}{4Zx^2}$$

$$= \lim_{x\to 0} \frac{x^2(\alpha_2 x^2 + 2\alpha_3 x^3 + 3\alpha_4 x^4 + \cdots + 1 - \alpha_0)}{4x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots)}$$

$$= \lim_{x\to 0} \frac{\alpha_2 x^2 + 2\alpha_3 x^3 + 3\alpha_4 x^4 + \cdots + 1 - \alpha_0}{4(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots)}$$

$$= \frac{1-\alpha_0}{4\alpha_0}$$

The limits (3.3) are thus finite if $\alpha_0 \neq 0$.

We can now state our result in terms of the following theorem:

Theorem: *The differential equation*

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1)y = 0$$

admits two linearly independent series solutions for the choice

$$Z = \sum_{i=0}^{\infty} \alpha_i x^i \quad (\alpha_0 \neq 0).$$

The point $x = 0$ is a regular point of the equation when $\alpha_0 = 1$. The point $x = 0$ is a regular singular point when $\alpha_0 \neq 1$.

3.3 The Tolman VII solution

We choose $Z(x)$ to be the general quadratic function

$$Z(x) = c + ax + bx^2 \tag{3.5}$$

where a, b and c are constants. Substituting (3.5) into (2.26c) yields

$$4x^2(c + ax + bx^2)\ddot{y} + 2x^2(a + 2bx)\dot{y} + (bx^2 + 1 - c)y = 0. \tag{3.6}$$

Clearly (3.5) satisfies the criteria of the above theorem, provided $c \neq 0$. Therefore (3.6) possesses a series solution. Having established the existence of a solution, we attempted to solve (3.6) analytically, using the computer algebra capabilities of MATHEMATICA (Wolfram 1991). The relevant input material is

$$\text{DSolve}[\left(1 - c + b\,x^2\right)\, y[x] + 2\,x^2\,(a + 2\,b\,x)\, y'[x] + 4\,x^2\,\left(c + a\,x + b\,x^2\right)\, y''[x] == 0, y[x], x].$$

As expected no solution was obtained for $c = 0$. Our attempt was unsuccessful for other specified values of the parameter c, viz. $-1, \pm 2$. MATHEMATICA failed to solve equation (3.6) for these cases – which clearly contradicts the result established in §3.2. The existence of solutions for these cases is guaranteed by our theorem. We hope that this exercise serves as a warning of the limits of the analytic equation solving capabilities of symbolic manipulation software.

We observe that (3.6) simplifies substantially if we assign the particular value to the parameter c:

$$c = 1$$

In this special case, (3.6) reduces to

$$4(bx^2 + ax + 1)\ddot{y} + (4bx + 2a)\dot{y} + by = 0 \tag{3.7}$$

which, again, in term of our earlier theorem, must possess two linearly independent solutions of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We again used MATHEMATICA (Wolfram 1991) in an attempt to find this solution. In this instance our efforts were successful, and we obtained the result which we rewrite as

$$y(x) = B_1 \cos(\log \xi) + B_2 \sin(\log \xi)$$

$$\xi = \sqrt{\frac{2(a+2bx)}{\sqrt{b}} + 4\sqrt{1+ax+bx^2}}$$

where B_1 and B_2 are integration constants. For suitable choices of B_1, B_2, a and b we recover the Tolman VII interior solution, which has the line element

$$ds^2 = -B^2 \sin^2 \ln \sqrt{\frac{\sqrt{1 - \frac{r^2}{R^2} + 4\frac{r^4}{A^4}} + 2\frac{r^2}{A^2} - \frac{1}{4}\frac{A^2}{R^2}}{C}}\, dt^2 + \left(1 - \frac{r^2}{R^2} + 4\frac{r^4}{A^4}\right)^{-1} dr^2 +$$

$$r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3.8}$$

as listed by Kramer *et al* (1980).

We have demonstrated the advantage of symbolic manipulation software, in this instance MATHEMATICA which quickly generated a solution to (3.7) when $c = 1$. Our solution (3.8) was shown to coincide with a well-known stellar solution, viz. Tolman VII, and the use of MATHEMATICA was certainly instrumental in preventing the rediscovery of a published result. Due to the proliferation of exact solutions discovered, it becomes increasingly difficult for exponents of the field to have knowledge of the vast literature produced. What has proven to be quite invaluable is the review by Kramer *et al* (1980) – a new edition of which is anticipated, at the time of writing. In addition, a number of databases of exact solutions to the Einstein field equations have been compiled in recent years, and these should prove similarly invaluable to researchers in the field. An excellent example of such a database is the "On-Line Invariant Classification Database" by Jim Skea (1997). These resources, in conjunction with computer algebra systems are also useful in determining the physical properties of these solutions. One important review on the physical acceptability of isolated, static, spherically symmetric, perfect fluid solutions that exemplifies this approach was compiled by Delgaty and Lake (1998).

3.4 A new series solution via the method of Frobenius

A large number of exact solutions are known for the system of equations (2.26) that model a relativistic star with no charge. Many of these are listed by Kramer *et al* (1980) and Krasinski (1997). A comprehensive list of static solutions, that satisfy stringent conditions for spherically symmetric perfect fluids, was compiled by Delgaty and Lake (1998). In an attempt to obtain a new solution to the system (2.26) we make the choice

$$Z = ax^3 + 1 \tag{3.9}$$

where a is a constant. As far as we are aware the solution of the Einstein equations (2.26), with the gravitational potential Z given in (3.9), has not been previously published. The cubic form of (3.9) is consistent with our analysis for series solutions in §3.2. We suspect that the cubic form (3.9) has not been considered before because the resulting differential equation in the dependent variable y is difficult to solve; quadratic forms for Z are listed by Delgaty and Lake (1998). The quadratic form of Z is simpler to handle and contains the familiar Tolman models. The choice (3.9) ensures that the potential Z is continuous and well-behaved in the interior of the star; Z has a finite value at the centre.

Substituting (3.9) into (2.26c) yields

$$2(ax^3 + 1)\ddot{y} + 3ax^2\dot{y} + axy = 0. \tag{3.10}$$

The linear second order differential equation (3.10) is difficult to solve when $a \neq 0$. We have not found a solution for $a \neq 0$ in standard references such as Kamke (1983). Software packages are sometimes helpful in solving linear equations. We utilised MATHEMATICA (Wolfram 1991) in an attempt to integrate (3.10). The relevant input material is

$$\text{DSolve}[2(ax^3+1)y''[x]+3ax^2y'[x]+axy[x]==0, y[x], x].$$

This produced the output

$$\{\{y(x) \rightarrow \text{C}(1)\,\text{Hypergeometric2F1}(\frac{1}{12}-\frac{i}{12}\sqrt{7}, \frac{1}{12}+\frac{i}{12}\sqrt{7}, \frac{2}{3}, -\left(a\,x^3\right))$$

$$+x\,\text{C}(2)\,\text{Hypergeometric2F1}(\frac{5}{12}-\frac{i}{12}\sqrt{7}, \frac{5}{12}+\frac{i}{12}\sqrt{7}, \frac{4}{3}, -\left(a\,x^3\right))\}\}.$$

This form of solution is not particularly useful as it is given in terms of a hypergeometric function with complex arguments. We require a real solution to describe a realistic star with a barotropic matter distribution. Consequently it is necessary to utilise other methods of solution for (3.10).

We attempt to find a series solution to (3.10) using the method of Frobenius. As the point $x=0$ is a regular point of (3.10), there exists two linearly independent solutions of the form of a power series with center $x=0$. We therefore can write

$$y(x)=\sum_{n=0}^{\infty} c_n x^n \tag{3.11}$$

where the c_n are the coefficients of the series. For a legitimate solution we need to determine the coefficients c_n explicitly. Substituting (3.11) into (3.10) yields

$$2(ax^3+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}+3ax^2\sum_{n=1}^{\infty} nc_n x^{n-1}+ax\sum_{n=0}^{\infty} c_n x^n=0.$$

This equation is equivalent to

$$2(2c_2+6c_3x+12c_4x^2)+3ac_1x^2+a(c_0x+c_1x^2)+$$

$$a\sum_{n=2}^{\infty}\{2n^2+n+1\}c_n x^{n+1}+2\sum_{n=5}^{\infty} n(n-1)c_n x^{n-2}=0.$$

The second series in the above equation can be written as

$$2\sum_{n=2}^{\infty}(n+3)(n+2)c_{n+3}x^{n+1}$$

on replacing the dummy variable. Finally we obtain the simplified equation

$$4c_2 + (12c_3 + ac_0)x + 4(6c_4 + ac_1)x^2 +$$

$$\sum_{n=2}^{\infty}\{a[2n^2 + n + 1]c_n + 2(n+3)(n+2)c_{n+3}\}x^{n+1} = 0.$$

For this equation to hold true for all x we require

$$4c_2 = 0 \tag{3.12a}$$

$$12c_3 + ac_0 = 0 \tag{3.12b}$$

$$6c_4 + ac_1 = 0 \tag{3.12c}$$

$$a[2n^2 + n + 1]c_n + 2(n+3)(n+2)c_{n+3} = 0,\ n \geq 2. \tag{3.12d}$$

It remains to obtain the coefficients c_n from the system (3.12).

Equation (3.12d) is a difference equation, or recurrence relation, which has to be solved. This particular equation is a linear recurrence relation with variable, rational coefficients of order three. General techniques of solution for difference equations are limited to the simplest cases (Durrell and Robson 1958). The equation (3.12d) does not fall into the known classes. We were, however, able to obtain a solution to (3.12d) from first principles.

Rewriting (3.12d) as

$$c_{n+3} = -\frac{a}{2}\frac{2n^2 + n + 1}{(n+3)(n+2)}c_n,\ n \geq 2 \tag{3.13}$$

we obtain the following results. Equations (3.12a) and (3.13) imply

$$c_2 = c_5 = c_8 = \cdots = 0. \tag{3.14}$$

From (3.12b) and (3.13) we generate the expressions

$$c_3 = -\frac{a}{12}c_0$$

$$= -\frac{a}{2}\frac{1}{3.2}c_0$$

$$c_6 = -\frac{a}{2}\frac{2.3^2+3+1}{6.5}c_3$$

$$= \frac{a^2}{2^2}\frac{2.3^2+3+1}{6.5}\frac{1}{3.2}c_0$$

$$= \frac{a^2}{2^2}\frac{2.3^2+3+1}{6.3}\frac{1}{5.2}c_0$$

$$c_9 = -\frac{a}{2}\frac{2.6^2+6+1}{9.8}c_6$$

$$= -\frac{a^3}{2^3}\frac{2.6^2+6+1}{9.8}\frac{2.3^2+3+1}{6.5}\frac{1}{3.2}c_0$$

$$= -\frac{a^3}{2^3}\frac{2.6^2+6+1}{9.6.3}\frac{2.3^2+3+1}{8.5.2}c_0.$$

It is clear that the coefficients $c_3, c_6, c_9, \ldots$ can all be written in terms of the coefficient c_0. These coefficients generate the following pattern

$$c_{3n+3} = (-1)^{n+1}\left(\frac{a}{2}\right)^{n+1} \times$$

$$\frac{[2(3n)^2+3n+1]\cdots[2(3.1)^2+3.1+1][2(3.0)^2+3.0+1]}{\{(3n+3)\cdots(3.1+3)(3.0+1)\}\{(3n+2)\cdots(3.1+2)(3.0+2)\}}c_0. \quad (3.15)$$

We can rewrite (3.15) in the form

$$c_{3n+3} = (-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\prod_{k=0}^{n}\frac{2(3k)^2+3k+1}{(3k+3)(3k+2)}c_0 \quad (3.16)$$

where we have utilised the conventional symbol $\prod$ to denote multiplication.

We can obtain a similar formula for the coefficients $c_4, c_7, c_{10}, \ldots$ From (3.12c) and (3.13) we have

$$c_4 = -\frac{a}{6}c_1$$

$$= -\frac{a}{2}\frac{2.1^2+1+1}{4.3}c_1$$

$$c_7 = -\frac{a}{2}\frac{2.4^2+4+1}{7.6}c_4$$

$$= \frac{a^2}{2^2}\frac{2.4^2+4+1}{7.6}\frac{2.1^2+1+1}{4.3}c_1$$

$$= \frac{a^2}{2^2}\frac{2.4^2+4+1}{7.4}\frac{2.1^2+1+1}{6.3}c_1$$

$$c_{10} = -\frac{a}{2}\frac{2.7^2+7+1}{10.9}c_7$$

$$= -\frac{a^3}{2^3}\frac{2.7^2+7+1}{10.9}\frac{2.4^2+4+1}{7.6}\frac{2.1^2+1+1}{4.3}c_1$$

$$= -\frac{a^3}{2^3}\frac{2.7^2+7+1}{10.7.4}\frac{2.4^2+4+1}{9.6.3}\frac{2.1^2+1+1}{1}c_1.$$

The coefficients $c_4, c_7, c_{10}, \ldots$ can all be written in terms of the coefficient c_1. These coefficients generate a pattern which is clearly of the form

$$c_{3n+4} = (-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\frac{[2(3.0+1)^2+(3.0+1)+1][2(3.1+1)^2+(3.1+1)+1]}{\{(3n+4)\cdots(3.1+4)(3.0+4)\}}\times$$

$$\frac{[2(3.2)^2+(3.2+2)+1]\cdots[2(3n+1)^2+(3n+1)+1]}{\{(3n+3)\cdots(3.1+3)(3.0+3)\}}c_1 \tag{3.17}$$

As for (3.15), we note that (3.17) can be expressed in the form

$$c_{3n+4} = (-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\prod_{k=0}^{n}\frac{2(3k+1)^2+(3k+1)+1}{(3k+4)(3k+3)}c_1 \tag{3.18}$$

where, again $\prod$ denotes multiplication.

From (3.14) we observe that the coefficients $c_2, c_5, c_8, \ldots$ all vanish. The coefficients $c_3, c_6, c_9, \ldots$ are generated from (3.16). The coefficients $c_4, c_7, c_{10}, \ldots$ are generated from (3.18). Hence the difference equation (3.12d) has been solved and all non-zero coefficients are expressible in terms of the leading coefficients c_0 and c_1. From (3.11), (3.14), (3.16) and (3.18) we establish

$$y(x) = c_0 + c_1x^1 + c_3x^3 + c_4x^4 + c_6x^6 + c_7x^7 + c_9x^9 + c_{10}x^{10} + \cdots$$

$$= c_0\left(1 - \frac{a}{2}\frac{1}{3.2}x^3 + \frac{a^2}{2^2}\frac{2.3^2+3+1}{6.3}\frac{1}{5.2}x^6 - \right.$$

$$\left.\frac{a^3}{2^3}\frac{2.6^2+6+1}{9.6.3}\frac{2.3^2+3+1}{8.5.2}x^9 + \cdots\right) +$$

$$c_1\left(x - \frac{a}{2}\frac{2.1^2+1+1}{4.3}x^4 + \frac{a^2}{2^2}\frac{2.4^2+4+1}{7.4}\frac{2.1^2+1+1}{6.3}x^7 - \right.$$

$$\left.\frac{a^3}{2^3}\frac{2.7^2+7+1}{10.9}\frac{2.4^2+4+1}{7.6}\frac{2.1^2+1+1}{4.3}x^{10} + \cdots\right)$$

$$= c_0\left(1 + \sum_{n=0}^{\infty} c_{3n+3}x^{3n+3}\right) + c_1\left(x + \sum_{n=0}^{\infty} c_{3n+4}x^{3n+4}\right) \tag{3.19}$$

where c_0 and c_1 are arbitrary constants. Equation (3.19) can be rewritten as

$$y(x) = c_0\left(1 + \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\prod_{k=0}^{n}\frac{2(3k)^2+3k+1}{(3k+3)(3k+2)}x^{3n+3}\right) +$$

$$c_1\left(x + \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\prod_{k=0}^{n}\frac{2(3k+1)^2+(3k+1)+1}{(3k+4)(3k+3)}x^{3n+4}\right) \tag{3.20}$$

where we have utilised (3.16) and (3.18). Clearly (3.20) is of the form

$$y(x) = c_0 y_1(x) + c_1 y_2(x) \tag{3.21}$$

where

$$y_1(x) = \left(1 + \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\prod_{k=0}^{n}\frac{2(3k)^2+3k+1}{(3k+3)(3k+2)}x^{3n+3}\right) \tag{3.22a}$$

$$y_2(x) = \left(x + \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\prod_{k=0}^{n}\frac{2(3k+1)^2+(3k+1)+1}{(3k+4)(3k+3)}x^{3n+4}\right) \tag{3.22b}$$

are linearly independent solutions of (3.10). Therefore we have found the general solution to the differential equation (3.10) for the particular gravitational potential Z given in (3.9). The advantage of the solutions in (3.20) – (3.22) is that they are expressed in terms of a series with real arguments unlike the complex arguments given by MATHEMATICA (Wolfram 1991).

From (3.21) and the Einstein field equations (2.26) we generate the solution

$$e^{2\lambda} = \frac{1}{ax^3+1} \tag{3.23a}$$

$$e^{2\nu} = A^2 y^2 \tag{3.23b}$$

$$\frac{\rho}{C} = -7ax^2 \tag{3.23c}$$

$$\frac{p}{C} = 4(ax^3+1)\frac{\dot{y}}{y} + ax^2 \tag{3.23d}$$

where A and C are constants. The quantity y is given by (3.22) and a is a constant. Note that in order to obtain non-negative energy densities ρ we require $a < 0$. The matter variables ρ and p are both bounded and continuous. The gravitational potentials ν and λ are well-behaved and have simple forms. Our solution has the interesting feature of admitting an explicit barotropic equation of state. We observe from (3.23c) that

$$x = \sqrt{\frac{\rho}{-7aC}}, \quad a < 0$$

and the variable x can be written in terms of ρ only. The function y in (3.22) can be expressed in terms of ρ and the variable x is eliminated. Consequently the pressure p in (3.23d) is expressible in terms of ρ only, and we can write

$$p = p(\rho).$$

Thus the solution in (3.23) obeys a barotropic equation of state. This highly desirable feature is unusual for most exact solutions as pointed out by Kramer *et al* (1980). This feature of the solution leads us to believe that it will lead to a realistic model of neutral stars. We point out that graphical plots of ν, λ, ρ and p are easy to generate using computer software packages even though the solution is given in terms of an infinite series. The approach used in this chapter can also be extended to the problem of charged stars, for relevant choices of the electric field intensity E, and such an area of investigation should be pursued in future.

Chapter 4

A generalisation of Maharaj and Mkhwanazi

4.1 Introduction

In this chapter we attempt to obtain new exact astrophysical solutions to the Einstein-Maxwell system (2.32). We adopt the approach of specifying the gravitational potential Z and the electric field intensity E, in order to integrate (2.32c). In §4.2 we choose a form of Z used by Maharaj and Mkhwanazi (1996). This choice includes the case of Durgapal and Bannerji (1983) that produces physically reasonable models for neutral matter which are consistent with neutron star observations. Upon transforming our equation, and making a physically reasonable choice for E, we obtain a hypergeometric equation. Our prescribed form of E appears to approximate an inverse square law, when viewed by an observer from infinity. In §4.3 we list some crucial features of the hypergeometric equation and its eponymous special function solution. These solutions can, in certain cases, reduce to more familiar elementary functions which greatly simplifies the analysis of their physical properties. Software packages, e.g. MATHEMATICA (Wolfram 1991), are particularly adept at demonstrating the equivalence, when they exist, between elementary functions and special cases of the hypergeometric function. In §4.4 we present two examples of known neutral exact solutions, which were originally given in terms of elementary functions; we demonstrate that they can also be written in terms of hypergeometric functions. In §4.5 we obtain two members of a new class of solutions to (2.32c), which we express in terms of both hypergeometric functions and elementary functions. We demonstrate that, in the limit of vanishing electric fields, the first example incorporates the solution of Maharaj and Mkhwanazi (1996). The matter and curvature variables are well behaved for the electrostatic generalisation; we believe that our solution forms a realistic model for charged stars. As a second example we find a charged generalisation of the Durgapal-Bannerji solution (1983). However the energy density ρ is negative, and violates the strong and weak energy conditions (Hawking and Ellis 1973) for barotropic matter. The approach outlined in this chapter may be used to obtain charged analogues of other static solutions. The hypergeometric function provides a vehicle to classify and categorise exact solutions obtained under different assumptions. Whilst we were able to unify a large class of neutral and charged static solutions under the hypergeometric equation, we should point out that were unable to recover some well-known models which are physically reasonable; a prime example being the solution of Finch and Skea (1989). The change of variables used in our treatment prohibits the limit required to obtain this particular solution, and thus excludes it from our transformed solution space.

4.2 Specifying Z and E

We examine a particular form of the Einstein-Maxwell field equations by making explicit choices for the gravitational potential $Z(x)$ and the electric field intensity $E(x)$. The system (2.32) comprises four equations in six unknowns, Z, y, ρ, p, E and σ. By specifying the gravitational potential $Z(x)$ and the electric field intensity $E(x)$, we are in a position to integrate (2.32c). The solution of the system then follows. We make the particular choice

$$Z(x) = \frac{1+kx}{1+x} \tag{4.1}$$

where k is a real constant. In (4.1) we take $k \neq 1$. If $k = 1$ then the metric function $e^{2\lambda} = 1$ and the energy density is

$$\rho = -\frac{E^2}{2}.$$

To avoid negative energy densities, which are unphysical for barotropic stars, we consequently take $k \neq 1$. The choice (4.1) was also made by Maharaj and Mkhwanazi (1996) and Mkhwanazi (1994) in their analyses of uncharged relativistic stars. Our objective is to confirm that this type of potential is also consistent with non-vanishing electromagnetic fields. Note that the choice (4.1) contains, as a special case, the Durgapal and Bannerji (1983) solution – which is widely utilised as a realistic model for neutron stars. Other physically reasonable choices of the gravitational potential Z are possible; we have chosen the form (4.1) as it produces a charged solution that necessarily reduces to a well-known model in the appropriate uncharged limit.

Upon substituting (4.1) into (2.32c) we obtain

$$4(1+kx)(1+x)\ddot{y} + 2(k-1)\dot{y} + \left(1 - k - \frac{E^2(1+x)^2}{Cx}\right) y = 0 \tag{4.2}$$

When $E = 0$, (4.2) reduces to

$$4(1+kx)(1+x)\ddot{y} + 2(k-1)\dot{y} + (1-k)\, y = 0$$

for uncharged stars. It is convenient at this point to introduce a new independent variable $\mathcal{X}$. This helps to simplify the second order equation (4.2). The relevant transformation is given by

$$1 + x = K\mathcal{X} \tag{4.3a}$$

$$K = \frac{k-1}{k} \tag{4.3b}$$

$$Y(\mathcal{X}) = y(x). \tag{4.3c}$$

With the help of the transformation (4.3), equation (4.2) can be written as

$$\mathcal{X}(1-\mathcal{X})\frac{d^2Y}{d\mathcal{X}^2} - \frac{1}{2}\frac{dY}{d\mathcal{X}} + \left(\frac{K}{4} + \frac{K^2(1-K)E^2\mathcal{X}^2}{C(K\mathcal{X}-1)}\right)Y = 0 \tag{4.4}$$

in terms of the new dependent and independent variables, Y and $\mathcal{X}$ respectively. Clearly (4.4) is integrable once E is specified. A variety of choices for E is possible: only a few are physically reasonable and generate solutions in closed form.

We observe that (4.4) is simplified if we make the choice

$$E^2 = \frac{\alpha}{4}\frac{C}{K^2(1-K)}\frac{K\mathcal{X}-1}{\mathcal{X}^2} \tag{4.5}$$

where α is a constant. The electric field intensity E in (4.5) vanishes at the centre of the star, and remains continuous and bounded in the interior of the star for a wide range of values of the parameter K. Thus this choice for E is physically reasonable and is a useful form to study the gravitational behaviour of charged stars. Equation (4.4) now assumes the simpler form

$$\mathcal{X}(1-\mathcal{X})\frac{d^2Y}{d\mathcal{X}^2} - \frac{1}{2}\frac{dY}{d\mathcal{X}} + \left(\frac{K}{4} + \frac{\alpha}{4}\right)Y = 0 \tag{4.6}$$

for the choice (4.5). When $\alpha = 0$, (4.6) becomes

$$\mathcal{X}(1-\mathcal{X})\frac{d^2Y}{d\mathcal{X}^2} - \frac{1}{2}\frac{dY}{d\mathcal{X}} + \frac{K}{4}Y = 0 \tag{4.7}$$

and there is no charge. Equation (4.7) was investigated in detail by Maharaj and Mkhwanazi (1996) and Mkhwanazi (1994). Their analyses of (4.7) produced physically viable models of uncharged relativistic stars; we expect that our investigation of (4.7) will produce useful models of charged relativistic stars. Note that equations (4.6) and (4.7) are special cases of the hypergeometric differential equation.

4.3 The hypergeometric equation

The general hypergeometric differential equation is given by

$$\mathcal{X}(1-\mathcal{X})\frac{d^2Y}{d\mathcal{X}^2} - [(a+b+1)\mathcal{X} - c]\frac{dY}{d\mathcal{X}} - abY = 0 \tag{4.8}$$

where a, b and c are real constants (Abramowitz and Stegun 1972). The solutions to (4.8) are given in terms of the hypergeometric function $F(a, b; c; \mathcal{X})$ and are categorised by the three regular singular points

$$\mathcal{X} = 0, 1, \infty$$

of the equation (4.8). The general theory of differential equations distinguishes between the following six cases

(i) None of the numbers $c, c - a - b, a - b$ is an integer.

(ii) One of the numbers $a, b, c - a, c - b$ is an integer.

(iii) $c - a - b$ is an integer, but c is not an integer.

(iv) $c = 1$.

(v) $c = m + 1$, where m is a natural number.

(vi) $c = 1 - m$, where m is a natural number.

The general properties of the solutions for each of the six cases given above are discussed by Abramowitz and Stegun (1972). We can relate (4.6) and (4.8) by setting

$$a = -(b + 1)$$

$$b = -\frac{1}{2} \pm \sqrt{1 + K + \alpha}$$

$$c = -\frac{1}{2}$$

We note that the solution will be real provided $-1 \leq K + \alpha$, which is equivalent to

$$(2 + \alpha)k \leq 1, \quad \text{or}$$

$$(2 + \alpha)k \geq 1.$$

Real solutions of (4.8) exist for a wide range of values for α and K.

The first solution of (4.8) is given as a hypergeometric series

$$Y_1 = 1 + \frac{ab}{c}\mathcal{X} +$$

$$\frac{a(a+1)b(b+1)}{c(c+1)}\frac{\mathcal{X}^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{\mathcal{X}^3}{3!} + \cdots$$

$$= F(a,b;c;\mathcal{X}) \tag{4.9}$$

The second solution is given by

$$Y_2 = \mathcal{X}^{1-c}\left[1 + \frac{(a-c+1)(b-c+1)}{(-c+2)}\mathcal{X} + \right.$$

$$\left. + \frac{(a-c+1)(a-c+2)(b-c+1)(b-c+2)}{(-c+2)(-c+3)}\frac{\mathcal{X}^2}{2!} + \cdots\right]$$

$$= \mathcal{X}^{1-c}F(a-c+1, b-c+1; 2-c; \mathcal{X}). \tag{4.10}$$

The hypergeometric series solutions (4.9) and (4.10) have been obtained from the treatment of Kreyzig (1972). The general solution to (4.8) is given by the sum

$$Y = c_1 F(a,b;c;\mathcal{X}) + c_2\mathcal{X}^{1-c}F(a-c+1, b-c+1; 2-c; \mathcal{X}) \tag{4.11}$$

where c_1 and c_2 are constants.

As (4.6) is a special case of (4.8), its solutions will necessarily be of the form (4.11) in general. Thus we have determined that gravitational potentials Z of the form (4.1) produce Einstein-Maxwell stars whose gravitational behaviour is governed by hypergeometric functions.

4.4 Previous cases regained

A variety of new solutions, in terms of elementary and special functions, are obtainable from (4.11) for particular values of a and b (or equivalently α and K) with $c = -\frac{1}{2}$. Some values of K may reduce (4.6) to solutions that have already been documented. Here we consider two such cases of known solutions obtainable from our general class. These correspond to neutral stars with no electromagnetic field ($\alpha = 0$). As a first example we take $\alpha = 0$ and $K = -1$ ($\Leftrightarrow k = \frac{1}{2}$). Then (4.6) becomes

$$\mathcal{X}(1-\mathcal{X})\frac{d^2Y}{d\mathcal{X}^2} - \frac{1}{2}\frac{dY}{d\mathcal{X}} - \frac{1}{4}Y = 0.$$

This differential equation admits the two, linearly independent solutions

$$Y_1 = F\left(-\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2}; \mathcal{X}\right)$$

$$Y_2 = \mathcal{X}^{3/2} F\left(1, 1; \frac{5}{2}; \mathcal{X}\right)$$

which are hypergeometric functions. It is possible to express these solutions in terms of elementary functions

$$y_1(x) = (2+x)^{1/2}$$

$$y_2(x) = (2+x)^{1/2} \ln[(1+x)^{1/2} + (2+x)^{1/2}]^2 - 2(1+x)^{1/2}$$

in terms of the variables y and x, used earlier. This solution was also found previously by Maharaj and Mkhwanazi (1996).

As a second example we take $\alpha = 0$ and $K = 3$ ($\Leftrightarrow k = -\frac{1}{2}$) so

$$\mathcal{X}(1-\mathcal{X})\frac{d^2Y}{d\mathcal{X}^2} - \frac{1}{2}\frac{dY}{d\mathcal{X}} + \frac{3}{4}Y = 0.$$

This equation has the two linearly independent solutions

$$Y_1 = F\left(\frac{1}{2}, -\frac{3}{2}; -\frac{1}{2}; \mathcal{X}\right)$$

$$Y_2 = \mathcal{X}^{3/2} F\left(2, 0; \frac{5}{2}; \mathcal{X}\right).$$

These hypergeometric functions are equivalent to the elementary functions

$$y_1(x) = (2 - x)^{1/2}(2x + 5)$$

$$y_2(x) = (1 + x)^{3/2}$$

in terms of the variables y and x. This solution was also found by Maharaj and Mkhwanazi (1996) and contains the neutron star model of Durgapal and Bannerji (1983). For completeness we list the solution of the Einstein system (2.26) in this case:

$$\frac{\rho}{C} = \frac{3(3 + Cr^2)}{2(1 + Cr^2)^2} \tag{4.12a}$$

$$\frac{p}{C} = \frac{9c_1(1 + Cr^2)^{1/2}(1 - Cr^2) - c_2(2 - Cr^2)^{1/2}(10Cr^2 + 13)}{2c_1(1 + Cr^2)^{5/2} + 2c_2(2 - Cr^2)^{1/2}(1 + Cr^2)(2Cr^2 + 5)} \tag{4.12b}$$

$$e^{2\lambda} = \frac{2(1 + Cr^2)}{2 - Cr^2} \tag{4.12c}$$

$$e^{2\nu} = A^2[c_1(1 + Cr^2)^{3/2} + c_2(2 - Cr^2)^{1/2}(2Cr^2 + 5)]^2 \tag{4.12d}$$

where c_1 and c_2 are constants. The solution (4.12) is widely used to describe dense stars (Delgaty and Lake 1998).

Note that in the above examples we have verified the equivalence between the hypergeometric functions and the elementary functions with MATHEMATICA (Wolfram 1991). Clearly it is possible to generate other examples where the hypergeometric function can be written in terms of elementary functions. We have restricted ourselves to examples that are simple and have physical significance. Our analysis in this section may be extended to other known solutions of the Einstein system (2.26). This approach provides a mechanism to collate and categorise different solutions in terms of the hypergeometric function in a simple fashion. This indicates that it is possible to bring together seemingly disparate solutions in a unified treatment.

4.5 New charged solutions

The particular solutions presented in §4.4 correspond to the system of equations (2.26) for a neutral relativistic star. It is possible, upon integrating the hypergeometric equation (4.6), to generate new solutions to the system (2.32) corresponding to a charged star. In this section we present simple solutions where the hypergeometric functions can be written in terms of elementary functions. We have chosen our parameter values so that we obtain the same hypergeometric equations as in §4.4. However these correspond to the Einstein-Maxwell system (2.32) and will generate different exact solutions as we will demonstrate.

We choose the particular parameter values

$$K = -2 \quad \left(\Leftrightarrow k = \frac{1}{3}\right)$$

$$\alpha = 1.$$

Then (4.6) becomes

$$\mathcal{X}(1-\mathcal{X})\frac{d^2Y}{d\mathcal{X}^2} - \frac{1}{2}\frac{dY}{d\mathcal{X}} - \frac{1}{4}Y = 0. \tag{4.13}$$

The general solution to equation (4.13) is given by

$$Y = c_1 F\left(-\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2}; \mathcal{X}\right) + c_2 \mathcal{X}^{-3/2} F\left(1, 1; \frac{5}{2}; \mathcal{X}\right) \tag{4.14}$$

where $F(-\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2}; \mathcal{X})$ and $\mathcal{X}^{-3/2}F(1, 1; \frac{5}{2}; \mathcal{X})$ are linearly independent hypergeometric functions, and c_1 and c_2 are constants. In terms of the variables x and y, we can rewrite the solution as

$$y(x) = (2+x)^{1/2}[c_1 + 2c_2 \ln [(1+x)^{1/2} + (2+x)^{1/2}]] - 2c_2(1+x)^{1/2}. \tag{4.15}$$

The equivalence between the hypergeometric functions in (4.14) and the elementary functions in (4.15) has been verified with MATHEMATICA (Wolfram 1991).

It is now possible to generate an exact solution to the Einstein-Maxwell system (2.32). This is given by

$$e^{2\lambda} = \frac{1+x}{1+\frac{1}{3}x} \tag{4.16a}$$

$$e^{2\nu} = A^2\{(2+x)^{1/2}(c_1 + c_2 \ln[(1+x)^{1/2} + (2+x)^{1/2}]) - 2c_2(1+x)^{1/2}\}^2 \tag{4.16b}$$

$$\frac{\rho}{C} = \frac{28+15x}{24(1+x)^2} \tag{4.16c}$$

$$\frac{p}{C} = \frac{-(15x+16)}{24(1+x)^2} + \frac{1}{2\sqrt{2+x}} \times$$

$$\frac{[c_1 + 2c_2 \ln(\sqrt{1+x} + \sqrt{2+x})]}{[-2c_2\sqrt{1+x} + c_1\sqrt{2+x} + 2c_2\sqrt{2+x}\ln(\sqrt{1+x}+\sqrt{2+x})]} \tag{4.16d}$$

$$\frac{E^2}{C} = \frac{1}{12}\frac{x}{(1+x)^2} \tag{4.16e}$$

$$\frac{\sigma^2}{C} = \frac{C(3+x)^3}{36(1+x)^5}. \tag{4.16f}$$

The form (4.16) is a new exact solution to the Einstein-Maxwell system of equations (2.32). The gravitational potentials ν and λ are well-behaved and continuous in the interior of the star. This is also true for the energy density ρ, the pressure p, the electric field intensity E and the charge density σ. These quantities remain finite and non-singular. The simple form of this solution makes a detailed analysis of the physical features of the model feasible.

As a second example we choose the parameter values

$$K = \frac{1}{2} \quad (\Leftrightarrow k = 2)$$

$$\alpha = \frac{5}{2}.$$

Then (4.6) assumes the form

$$\mathcal{X}(1-\mathcal{X})\frac{d^2Y}{d\mathcal{X}^2} - \frac{1}{2}\frac{dY}{d\mathcal{X}} + \frac{3}{4}Y = 0. \tag{4.17}$$

This equation possesses a general solution in terms of the two linearly independent hypergeometric functions

$$Y = c_1 F\left(\frac{1}{2}, -\frac{3}{2}; -\frac{1}{2}; \mathcal{X}\right) + c_2\mathcal{X}^{3/2}F\left(2, 0; \frac{5}{2}; \mathcal{X}\right). \tag{4.18}$$

As with the previous example, these special functions can be written in terms of elementary functions. Using MATHEMATICA (Wolfram 1991) we were able to express the solution (4.18) as

$$y(x) = c_1(2-x)^{1/2}(2x+5) + c_2(1+x)^{3/2} \tag{4.19}$$

in terms of the variables y and x.

The solution to the Einstein-Maxwell system (2.32) is given by

$$e^{2\lambda} = \frac{1+x}{1+2x} \tag{4.20a}$$

$$e^{2\nu} = A^2[c_1^2(2-x)(2x+5)^2 + c_2^2(1+x)^3 + 2c_1c_2(2-x)^{1/2}(1+x)^{3/2}(2x+5)] \tag{4.20b}$$

$$\frac{\rho}{C} = -\frac{3+\frac{13}{8}x}{(1+x)^2} \tag{4.20c}$$

$$\frac{p}{C} = \frac{5x}{8(1+x)^2} + \frac{1}{1+x} +$$

$$\frac{12(1+2x)(c_1 - 2c_1x + c_2\sqrt{2+x-x^2})}{(1+x)[-2c_1(-2+x)(5+2x) + 2c_2(1+x)\sqrt{2+x-x^2}]} \tag{4.20d}$$

$$\frac{E^2}{C} = \frac{5}{4}\frac{x}{(1+x)^2} \tag{4.20e}$$

$$\frac{\sigma^2}{C} = \frac{5C(3+x)^2(1+2x)}{4(1+x)^5}. \tag{4.20f}$$

The form (4.20) is a new exact solution to the Einstein-Maxwell system (2.32). The quantities λ, ν, p, E, and σ are well-behaved and non-singular in the interior of the star. In particular the quantities associated with the electromagnetic field viz. E and σ are bounded and have a simple, attractive form suggestive of classical results. However the energy density ρ is negative. Consequently this solution is not very useful for matter that has to satisfy the weak and strong energy conditions (Hawking and Ellis 1973). Even though the model admits this undesirable feature, the solution (4.20) may be utilised as a toy model to obtain qualitative features that may be useful to describe more realistic matter. In addition the solution may be used to describe quark stars and models that admit unconventional matter (Sharma *et al* 2001, Witten 1984). Note that negative values for ρ arise because of the particular choices of Z in (4.1) and E in (4.5). We need to consider other forms of Z and E such that ρ is positive. This example illustrates the difficulty of finding Einstein-Maxwell solutions that satisfy all the conditions for physical acceptability listed in §2.7.

The exact solutions (4.16) and (4.20) are new solutions to the Einstein-Maxwell system (2.32). As in §4.4 these solutions arise as special cases of the hypergeometric function which reduce to simple functions. The advantage of these solutions is that they are given in terms of elementary functions which greatly simplifies the analysis of the physical properties of the model. We do not pursue the physical analysis as it is outside the scope of this thesis. We expect that the charged solution (4.16) is well-behaved and satisfies the criteria for physical acceptability given in §2.7. The charged solution (4.20) has to be adapted to admit positive energy densities. These are areas for future investigation. We expect that other published charged solutions of the Einstein-Maxwell system are special cases of the hypergeometric equation. It may be a worthwhile exercise to identify these cases and unify such solutions under the hypergeometric function. This will help in preventing the duplication of solutions, and will assist in indicating under which conditions new solutions may exist.

Chapter 5

Spheroidal Geometry

5.1 Introduction

We now consider stellar problems exhibiting spheroidal geometry. The choice of the gravitational function (2.33), due to Vaidya and Tikekar (1982) and Tikekar (1990), implies that the hypersurfaces characterised by $\{t = \text{constant}\}$ are 3-spheroids. The results of Tikekar were incorporated in more general models of superdense stars subsequently proposed by Maharaj and Leach (1996) and Mukherjee *et al* (1997). Spheroidal geometries exhibit the important physical feature of being stable with respect to radial pulsations (Knutsen 1988) which suggests that spheroidal models describe realistic stars. Indeed the model of Tikekar was found to be physically viable and applicable to the latter stages of stellar evolution. Charged, static objects in the context of spheroidal geometry have been studied extensively (Patel and Koppar 1987, Patel *et al* 1997, Tikekar and Singh 1998 and Sharma *et al* 2001). In §5.2 we investigate charged, spheroidal bodies by utilising a transformation that linearises (2.37), the governing differential equation. We specify a form for the electric field intensity E, dependent on an arbitrary function f. For certain choices of f, (2.37) possesses solutions in terms of Legendre and gamma functions. It is possible to transform (2.37) to a form resembling the equation of simple harmonic motion. The advantage of this form of the equation is that its solutions are well-documented for a wide range of f. We demonstrate a simple case of such a solution. In §5.3 we briefly consider the phenomenon of anisotropy in stellar models. Deviations from pressure isotropy are believed to be an important feature in stellar models at very high densities. A detailed microscopic formulation of the origin of these anisotropies has yet to be discovered (Dev and Gleiser 2000). The presence of anisotropy does influence the critical mass for stability and affects values for the surface redshift. In §5.3 we utilise the linearising transformation of §5.2 on a modified form of the Einstein field equations. The resulting equation depends on an arbitrary function f – proportional to the anisotropy factor. We make a simple choice for f and obtain an exact solution. We conclude this chapter with an observation that, due to the mathematical similarities between the governing field equation, the techniques used in §5.2 are equally applicable in §5.3.

5.2 Charged, isotropic Tikekar stars

We utilise a form for the metric function λ which was first used by Vaidya and Tikekar (1982) and Tikekar (1990). This form of λ ensures that the hypersurfaces generated by $\{t = \text{constant}\}$ are spheroidal. Upon specifying the four-potential (2.20) and the metric function (2.33), the Einstein-Maxwell field equations for a charged, perfect fluid in a static, spherically symmetric spacetime reduce to (2.36). In addition, note that the condition of pressure isotropy (2.37) must hold for isotropic relativistic stars. We invoke the change of variables

$$x^2 = 1 - r^2/R^2 \tag{5.1a}$$

$$\psi(x) = e^{\nu(r)}. \tag{5.1b}$$

The transformation (5.1) enables us to write (2.37) as

$$(1 - K + Kx^2)\frac{d^2\psi}{dx^2} - Kx\frac{d\psi}{dx} + \left(\frac{(1 - K + Kx^2)^2 R^2 E^2}{x^2 - 1} + K(K-1)\right)\psi = 0 \tag{5.2}$$

which is linear in ψ. It is convenient at this stage to introduce the change of independent variable

$$u^2 = \frac{K}{K-1}x^2 \tag{5.3}$$

in the hope of simplifying (5.2). The variable (5.3) was also utilised by Patel (1999) in a comprehensive analysis of relativistic stars in a variety of higher dimensional theories of gravity. Equation (5.2) becomes

$$(u^2-1)\frac{d^2\psi}{du^2} - u\frac{d\psi}{du} + \left(\frac{(K-1)^2R^2(u^2-1)^2E^2}{(K-1)u^2-K} + K - 1\right)\psi = 0 \tag{5.4}$$

in terms of the new variable (5.3). We write the electric field intensity as

$$E^2 = \frac{(K-1)u^2-K}{(K-1)^2(u^2-1)^2R^2}f(u)$$

where $f(u)$ is an arbitrary function. Equation (5.4) can now be written as

$$(u^2-1)\frac{d^2\psi}{du^2} - u\frac{d\psi}{du} + \{f(u)+K-1\}\psi = 0 \tag{5.5}$$

which is clearly simpler than (5.2).

For specific choices of the function $f(u)$, equation (5.5) admits solutions in terms of special functions. These special functions are products of Legendre and gamma functions and are listed by Abramowitz and Stegun (1972, p 781). It is also possible to further transform (5.5) in the hope of obtaining elementary function solutions. This has the advantage of simplifying a detailed physical analysis. We observe that (5.5) can be transformed into a form reminiscent of the harmonic oscillator equation, $\ddot{q} + \tilde{f}(t)q = 0$ by the transformation

$$w(\vartheta) = \psi(u) \tag{5.6a}$$

$$\vartheta = \int_{u_0}^{u} \exp\left(\int_{\tilde{r}_0}^{\tilde{r}} \frac{u}{u^2-1}du\right)d\tilde{r}. \tag{5.6b}$$

We explicitly determine the relation between the old independent variable u and the new one ϑ by evaluating (5.6b):

$$\vartheta = \int_{u_0}^{u} \exp\left(\frac{1}{2}\int_{\tilde{r}_0}^{\tilde{r}} \frac{2u}{u^2-1}du\right)d\tilde{r}$$

$$= \frac{u}{2}\sqrt{u^2-1} - \frac{1}{2}\ln(u+\sqrt{u^2-1}) + C \tag{5.7}$$

where C is a constant. Utilising (5.6) we transform (5.5) into

$$\frac{d^2w}{d\vartheta^2} - [f(u(\vartheta)) + K - 1]w = 0 \tag{5.8}$$

where f is now a function of the new independent variable ϑ. The new variable ϑ is related to u by equation (5.7). A cursory glance at any of the standard reference texts on differential equations (Zwillinger 1989, Abramowitz and Stegun 1972) reveals that (5.8) admits a plethora of solutions for various choices of f. We demonstrate one such simple choice. If we let

$$f(u(\vartheta)) = -n^2 + 1 - K$$

then (5.8) reduces to the simple harmonic oscillator equation

$$\frac{d^2w}{d\vartheta^2} + n^2 w = 0$$

which possesses the general solution

$$w(\vartheta) = A\cos(n\vartheta) + B\sin(n\vartheta)$$

where A and B are constants. The equation (5.7) relates ϑ and u. Note that we have made our problem more tractable, at the expense of working in a less familiar coordinate system. It is often difficult, or impossible, to invert such transformations, and the advantage of increased integrability of the Einstein system has to be measured against this prospect. Also the Einstein field equations are non-linear, and each invoked transformation restricts its solution space in a non-trivial manner. This too is an important point for consideration when seeking exact solutions to the equations of general relativity. We expect that other choices of $f(u(\vartheta))$ may lead to interesting classes of solutions. We do not pursue this investigation here; our objective was to indicate that the condition of pressure isotropy (5.2) can be transformed to the simple harmonic form (5.8) which may lead to solutions of a simple form. This is clearly an area for further investigation.

5.3 Anisotropic Tikekar stars

We now briefly comment on extending the results of §5.2 to anisotropic stars which are uncharged, and possess the spheroidal geometry. As the pressure is anisotropic it is necessary to adapt the system (2.34). The Einstein field equations (2.13a) with anisotropic pressures become

$$\rho = \frac{1-K}{R^2}\frac{3-Kr^2/R^2}{(1-Kr^2/R^2)^2} \tag{5.9a}$$

$$p_r = \left(\frac{2\nu'}{r}+\frac{1}{r^2}\right)\frac{1-r^2/R^2}{1-Kr^2/R^2}-\frac{1}{r^2} \tag{5.9b}$$

$$p_\perp = \left(\nu''+\nu'^2+\frac{\nu'}{r}\right)\frac{1-r^2/R^2}{1-Kr^2/R^2}-\left(\nu'+\frac{1}{r}\right)\frac{(1-K)(r/R^2)}{(1-Kr^2/R^2)^2} \tag{5.9c}$$

where p_r is the radial component of the pressure, and $p_\perp$ is the tangential component. Stars with anisotropic pressure distributions (Herrera and Ponce de Leon 1985) are characterised by a non-zero anisotropy factor Δ, which implies

$$\Delta \equiv p_\perp - p_r$$

$$\neq 0.$$

Spherical symmetry forces all the pressure components to be strictly functions of the radial coordinate r.

The investigations of Ruderman (1972) and Canuto (1974) indicate that stellar models may be anisotropic over certain density ranges. Anisotropy in the pressure could be introduced by the existence of a solid core, presence of superfluidity or other physical processes (Glendenning 1997, Heiselberg and Jensen 2000) in the core regions which are not as yet completely understood. Bowers and Liang (1974) and Maharaj and Maartens (1989) have found particular anisotropic models that exhibit higher surface redshifts, that are consistent with observation, than those predicted by isotropic models. Gokhroo and Mehra (1994) point out the following interesting physical consequence of anisotropy. Pressure anisotropy leads to an extra term, proportional to $(p_\perp - p_r)/r$ in the conservation equation (2.14). This term represents a force due to the anisotropic nature of the fluid; and is directed outwards when $p_r < p_\perp$ and inwards when $p_r > p_\perp$. The former case implies the possibility of more compact stellar objects when considering anisotropic fluids as the extra force acts in opposition to gravitational collapse.

Taking the difference between (5.9b) and (5.9c) yields

$$(1 - Kr^2/R^2)^2(p_r - p_\perp) = (1 - Kr^2/R^2)(1 - r^2/R^2)\left(\nu'' + \nu'^2 - \frac{\nu'}{r}\right) -$$

$$(1-K)\frac{r}{R^2}\left(\nu' + \frac{1}{r}\right) + \frac{1-K}{R^2}(1 - Kr^2/R^2). \qquad (5.10)$$

Note the similarity between equations (2.37) and (5.10). Under the change of coordinates (5.1), equation (5.10) becomes

$$\frac{(1 - K + Kx^2)^2}{1 - x^2}R^2(p_r - p_\perp)\psi = (1 - K + Kx^2)\frac{d^2}{dx^2}\psi - Kx\frac{d}{dx}\psi + K(K-1)\psi. \qquad (5.11)$$

Without loss of generality we define the difference in radial and tangential pressures to be

$$p_r - p_\perp = \frac{(1-x)^2}{1 - K + Kx^2}\frac{1}{R^2}f(x) \qquad (5.12)$$

where we introduced the function $f(x)$. Equation (5.11) can now be written as

$$(1 - K + Kx^2)\frac{d^2\psi}{dx^2} - Kx\frac{d\psi}{dx} + [K(K-1) - f(x)]\psi = 0 \qquad (5.13)$$

which simplifies for various choices of $f(x)$. Observe that (5.13) can be transformed to a form similar to (5.5) using the transformation (5.3). Consequently the results of §5.2 can be easily adapted to this section. As the results obtained are very similar we do not list them. In particular it is possible to regain the equivalent of the simple harmonic equation (5.8).

We exhibit a simple solution to (5.13) in terms of elementary functions. For the choice

$$f(x) = K(K-1)$$

the field equation (5.13) simplifies to

$$(1-K+Kx^2)\frac{d^2\psi}{dx^2}-Kx\frac{d\psi}{dx}=0 \tag{5.14}$$

Equation (5.14) is separable and can be integrated in closed form. It has the general solution

$$\psi=\frac{1}{2C}\sqrt{x^2+\frac{1-K}{K}}+\frac{1-K}{2CK}\ln\left(x+\sqrt{x^2+\frac{1-K}{K}}\right)+C' \tag{5.15}$$

where C and C' are constants of integration. The gravitational potential ν now follows from (5.1b) and (5.15). The radial pressure p_r and the tangential pressure $p_\perp$ are derivable from (5.9b) and (5.9c) respectively. The Einstein system (5.9) has been solved as the remaining thermodynamic and gravitational functions are expressible in terms of ψ.

Other choices of $f(x)$ in (5.13) are possible. We do not pursue this possibility because it is outside the scope of our investigations. Our objective was to indicate that it is possible to generate solutions to the Einstein field equations for anisotropic models. We believe that we have achieved this with our simple solution (5.15). We have also shown that the methodology of §5.2 may be applied to this section. New exact solutions may be obtained for other choices of the function $f(x)$. A class of solutions that generated physically realistic models for neutron stars has been discovered using anisotropic fluids (Mak and Harko 2001). We believe that this is an area of research worthy of future pursuit. We hope that we have demonstrated the possibility of obtaining astrophysical models for spheroidal stars with anisotropic pressures.

Chapter 6

Conclusion

Our aim in this thesis was to attempt to find new exact solutions to the Einstein and Einstein-Maxwell systems. We sought solutions that could be used to form a description of realistic stars. Consequently we assumed that spacetime was static and spherically symmetric. We also attempted to demonstrate the possibility of unifying seemingly unrelated solutions of the Einstein-Maxwell system under special functions. We hope that some of the techniques and methods utilised in this thesis will be helpful to other practitioners. We generated a number of new solutions to the Einstein and Einstein-Maxwell systems which may be physically reasonable. We have found no references to these solutions, which can be expressed in terms of elementary functions, in the literature. We believe that a physical analysis of these solutions could lead to realistic models for dense, static stars.

We summarise our results below:

- We derived the Einstein and Einstein–Maxwell field equations for perfect fluids in static spherically symmetric spacetimes. We utilised a transformation due to Durgapal and Bannerji (1983) to cast these equations in a more tractable form. As a special case, we derived the field equations for charged and neutral perfect fluids in spacetimes exhibiting spheroidal geometry. The above forms of the field equations have led to physically acceptable models of dense stars which motivated our choice. A number of criteria for acceptable relativistic stellar models were stated. We also listed the two exterior solutions, viz. the Schwarzschild and Reissner-Nordstrom line elements, that any interior solution obtained in this thesis should be matched to.

- We established an existence theorem in §3.2 for neutral stars. Given the analytic form for the gravitational potential $Z(x)$:

$$Z(x) = \sum_{i=0}^{\infty} \alpha_i x^i \quad (\alpha_0 \neq 0)$$

equation (2.26c), the condition of pressure isotropy, will always possess a series solution about $x = 0$; the point $x = 0$ being a regular point or regular singular point of this equation. With this result, we attempted to solve the pressure isotropy equation by specifying quadratic and cubic forms for Z. The quadratic form was shown, with the aid of MATHEMATICA, to reduce to the familiar Tolman VII solution. For the cubic form

$$Z = 1 + ax^3$$

where a is a constant, MATHEMATICA returned a solution in terms of hypergeometric functions with complex arguments. Since this was clearly unphysical, we attempted to extract the series solution directly. This method depended on the solution of a difference equation, which we managed to solve from first principles. Our solution was thus expressed in terms of a power series, whose coefficients were explicitly determined. This solution possessed the interesting feature of admitting an explicit barotropic equation of state. We suspect that a detailed physical analysis will confirm the physical validity of our model. We believe that the exact solution corresponding to cubic Z is new. The limits of computational software were demonstrated by the inability of MATHEMATICA to recover solutions whose existence had been assured by the theorem given in §3.2.

- We attempted to find a charged analogue of the class of solutions discovered by Maharaj and Mkhwanazi (1996). To this end we made the choices

$$Z = \frac{1+kx}{1+x}$$

where k is a constant, and

$$E^2 = \frac{\alpha}{4}\frac{C}{K^2(1-K)}\frac{K\mathcal{X}-1}{\mathcal{X}^2}$$

where α, C and K are constants, and $\mathcal{X}$ is the new independent variable. We obtained a hypergeometric equation, whose properties were analysed. The solution of this equation depended on the parameters K and α. We illustrated two charged solutions, that we believe to be previously unpublished, in terms of elementary functions. We also regained the static solutions of Maharaj and Mkhwanazi (1996) and Durgapal and Bannerji (1983) in the limit of vanishing charge. All these solutions were special cases of the hypergeometric function. The special function solutions were shown to reduce to elementary functions for the examples considered. We believe that we have successfully demonstrated the feasibility of unifying various exact solutions under a single class of equations, and as such have addressed one of the aims of this thesis. One of the new solutions, obeying the barotropic equation of state $p = p(\rho)$, appears to satisfy the criteria for physical admissability, mentioned in §2.7. The other admits negative energy densities and may serve as a toy model for more exotic stellar material. A physical analysis of members of the class of charged solutions should be possible when they can be expressed in terms of elementary functions.

- We examined various forms of the field equations consistent with the spheroidal model of Vaidya and Tikekar (1982). We studied cases exhibiting charge, as well as those featuring pressure anisotropy. We demonstrated that the condition of pressure isotropy (adapted to include the electric field intensity E for charged stars) can be written as the harmonic oscillator equation

$$\frac{d^2w}{d\vartheta^2} - \{f(u(\vartheta)) + K - 1\}w = 0$$

where f is an arbitrary function. A similar equation is obtainable for anisotropic pressure distributions. We obtained solutions for particular choices of $f(u(\vartheta))$. Other choices are possible. Note that it is not always possible to invert the coordinate transformation to regain the solution in terms of the original variables; this is a drawback to the method used in chapter 5.

A number of possibilities for the extension of this work exist. Firstly we can investigate other forms of the gravitational potential Z that satisfy the existence theorem in §3.2. The relevant equations may be solved using symbolic software, transforming to a standard form, or attempting to obtain the series solution directly. It may also be possible to obtain a similar result for the charged equation (2.32c) for particular choices of the electric field intensity E. The use of symbolic software, whilst shown to be of limited reliability, is certainly advantageous in those situations where it is applicable. Our investigations hinted at a difficulty in extracting series solutions centred about regular singular points. This is an area that needs to be investigated. Our investigations into spheroidal geometry indicated that it is possible to generate solutions, via the harmonic oscillator transformations, for charged and anisotropic distributions. It is important, in future work, to determine the various classes of solution that are permitted under the relevant transformations. We successfully managed to unify various charged and neutral static solutions under the hypergeometric equation. We need to identify and classify seemingly disparate solutions, which have already been found, in terms of the hypergeometric equation. It may be possible to find a coordinate transformation that unifies other classes of exact solutions under some type of special function. Note, however that, we were unable to recover the Finch and Skea solution (1989), due to the change of coordinates needed to affect the unification. This hints at some of the difficulties in this important task. Spacetimes exhibiting spheroidal geometry have been shown to lead to physically reasonable models of compact stars. Other choices of the anisotropy factor than that pursued here, could be used to obtain new solutions.

All solutions should be subjected to a stringent physical analysis. This would greatly assist in reducing the class of exact solutions to those of astrophysical significance.

Chapter 7

References

1. Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions*(New York: Dover)
2. Bowers R L and Liang E P T 1974 *Astrophys. J.* **188** 657
3. Canuto V 1974 *Ann. Rev. Astron. Astrophys.* **12** 167
4. Castejon-Amenedo J and Coley A A 1992 *Class. Quantum Grav.* **9** 2203
5. Davies P 1989 *The New Physics* (Cambridge: Cambridge University Press)
6. Delgaty M S R and Lake K 1998 *Comput. Phys Commun.* **115** 395
7. Dev K and Gleiser M 2000 astro-ph/0012265
8. Durgapal M C and Bannerji R 1983 *Phys. Rev. D* **27** 328
9. Durrell C V and Robson A 1958 *Advanced Algebra: Vol.II* (London: G. Bell and Sons)
10. de Felice F and Clark C J S 1990 *Relativity on Manifolds* (Cambridge: Cambridge University Press)
11. Finch M R and Skea J E F 1989 *Class. Quantum Grav.* **6** 467

12. Foster J and Nightingale J D 1998 *A short course in General Relativity* (New York: Springer)

13. Glendenning N K 1997 *Compact Stars: Nuclear Physics, Particle Physics and General Relativity* (Berlin: Springer–Verlag)

14. Gokhroo M K and Mehra A L 1994 *Gen. Rel. Grav.* **26** 75

15. Hansraj S 1996 *Charged stars in General Relativity* (M.Sc. Thesis: University of Natal, Durban)

16. Hawking S W and Ellis G F R 1973 *The Large Scale structure of Spacetime* (Cambridge: Cambridge University Press)

17. Heiselberg H and Jensen M H 2000 *Phys. Rep.* **328** 237

18. Herrera L and Ponce de Leon J 1985 *J. Math. Phys.* **26** 2302

19. Joshi P S 1993 *Global aspects in Gravitation and Cosmology* (Oxford: Clarendon Press)

20. Joshi P S and Dwivedi I H 1992a *Gen. Rel. Grav.* **24** 129

21. Joshi P S and Dwivedi I H 1992b *Phys. Rev. D* **45** 2147

22. Joshi P S and Dwivedi I H 1992c *Commun. Math. Phys.* **146** 333

23. Kamke E 1983 *Differentialgleichungen: Lossungsmethoden und Losungen* (Stuttgart: B G Teubner)

24. Knutsen H 1984 *Gen. Rel. Grav.* **16** 777

25. Knutsen H 1988 *Mon. Not. R. Astron. Soc.* **232** 163

26. Knutsen H 1992 *Gen. Rel. Grav.* **24** 1297

27. Knutsen H 1995 *Class. Quantum Grav.* **12** 2817

28. Knutsen H 2000 *Gravitation and Cosmology* **4** 303

29. Kramer D, Stephani H, MacCallum M A H and Herlt E 1980 *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press)

30. Krasinski A 1997 *Inhomogenous Cosmological Models* (Cambridge: Cambridge University Press)

31. Kreyzig E 1972 *Advanced Engineering Mathematics* (New York: Wiley)

32. Maartens R and Maharaj M S 1990 *J. Math. Phys.* **31** 151

33. Maharaj S D and Leach P G L 1996 *J. Math. Phys.* **37** 430

34. Maharaj S D, Leach P G L and Maartens R 1991 *Gen. Rel. Grav.* **23** 261

35. Maharaj S D and Maartens R 1989 *Gen. Rel. Grav.* **21** 899

36. Maharaj S D and Mkhwanazi W T 1996 *Quaestiones Mathematicae* **19**, 211

37. Mak M K and Harko T 2001 gr-qc/0110103

38. Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman)

39. Mkhwanazi W T 1994 *Relativistic Spherical Stars* (M.Sc. Thesis: University of Natal, Durban)

40. Mukherjee S, Paul B C and Dadhich N K 1997 *Class. Quantum Grav.* **14** 3475

41. Narlikar J V 1979 *General Relativity and Cosmology* (Delhi: MacMillan)

42. Nordstrom G 1918 *Proc. K. Ned. Akad. Wet.* **20** 1238

43. Patel H M 1999 *A study of some non-empty spacetimes in general relativity* (Ph.D. Thesis, Gujerat University)

44. Patel L K and Koppar S K 1987 *Aust. J. Phys.* **40** 441

45. Patel L K, Tikekar R and Sabu M C 1997 *Gen. Rel. Grav.* **29** 489

46. Powers D L 1987 *Elementary differential equations* (Boston: Prindle, Weber and Schmidt)

47. Reissner H 1916 *Ann. Phys. Lpz.* **50** 106

48. Ruderman R 1972 *Ann. Rev. Astron. Astrophys.* **10** 427

49. Schwarzschild K 1916a *Sitzungsber Preuss. Akad. Wiss. Phys. Math.* **K1** 42

50. Schwarzschild K 1916b *Sitzungsber Preuss. Akad. Wiss. Phys. Math.* **K1** 42

51. Shapiro S L and Teukolsky S A 1983 *Black Holes, White Dwarfs and Neutron Stars* (New York: Wiley)

52. Sharma R, Mukherjee S and Maharaj S D 2001 *Gen. Rel. Grav.* **33** 999

53. Shvartsman V F 1971 *Soviet Physics-JETP* **33** 475

54. Skea J 1997 *On-line Invariant Classification Database* http://edradour.symbcomp.uerj.br/

55. Stefani H 1990 *General Relativity: An Introduction to the Theory of the Gravitational Field* (Cambridge: Cambridge University Press)

56. Tikekar R 1984 *J. Math. Phys.* **25** 1481

57. Tikekar R 1990 *J. Math. Phys.* **31** 2454

58. Tikekar R and Singh G P 1998 *Gravitation and Cosmology* **4** 294

59. Treves A and Turolla R 1999 *Astrophys. J.* **517** 396

60. Vaidya P C and Tikekar R 1982 *J. Astrophys. Astron.* **3** 325

61. Will C M 1981 *Theory and experimentation in gravitational physics* (Cambridge: Cambridge University Press)

62. Witten E 1984 *Phys. Rev. D* **30** 272

63. Wolfram S 1991 *MATHEMATICA Version 2.0* (Redwood City: Addison-Wesley)

64. Zel'dovich Y B and Novikov I D 1971 *Relativistic Astrophysics Vol I: Stars and Relativity* (Chicago: University of Chicago Press)

65. Zwillinger D 1989 *Handbook of differential equations* (London: Academic Press)