THE DEVELOPMENT OF ADDITION PROBLEM SOLVING SKILLS IN GRADE ONE CHILDREN : A MICROGENETIC APPROACH

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ABSTRACT

This thesis replicates and explores some of the recent findings by Robert Siegler regarding the development of addition skills in grade one children. Siegler states that children employ a number of different strategies to solve single digit addition problems, these strategies coexist and compete, and cognitive variability is an essential aspect of cognitive development. He also advocates the use of the microgenetic approach in order to explore cognitive development. Many of Siegler's observations were replicated while the microgenetic approach produced valuable information. Consideration of Siegler's work resulted in two research questions being formulated, both concerning the actual selection of strategies.

First, a prediction analysis was employed to test the hypothesis that children attempt to match the most appropriate strategy to the problem presented according to a principle of least effort (defined as the attempt to maximise benefit and minimise cost). The predictions were stipulated prior to the analysis and were based on the arithmetic development literature. It was predicted that children would tend to retrieve the answers to small problems and tie-problems or calculate the answer by counting on from the larger addend by the amount indicated by the small addend (which involves reversing the order of the addends when the first addend is the smaller of the two). The strategy selections ($n=229$) made by a sample of 12 grade one learners on 21 single digit addition problems were categorised and compared to the predictions. The prediction analysis reduced the expected error by 63%, supporting the least effort model of strategy choice. The result is statistically significant ($Z=10.231$, $p<0.01$).

Second, a test of proportions was used to test the hypothesis that under memory demanding conditions children will resort to faster strategies, such as retrieval, in order to prevent memory decay, or execute their strategies in an overt manner, by using their fingers, in order to aid the memory limitations. The sample of selections used for the prediction analysis were compared to a sample collected under special conditions. These demanding conditions involved a simultaneous numerical memory task. The results suggest that retrieval is not used more frequently than normal. However, the strategies are more likely to be executed in an overt manner ($Z=5.123$, $p<0.01$).

The question of why and how the child develops an extensive addition strategy repertoire is considered. A particular discovery sequence, based on the data collected is proposed, while a constructivist model involving repetitive modifications is introduced as a possible account of strategy discovery. Siegler's theory, emphasising the variability of cognitive action, is contrasted to the Piagetian tradition, particularly the memory limitation neo-Piagetian models of development, emphasising how the structural development determines strategy use. It is proposed that future models should incorporate both aspects of development.

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DECLARATION

This whole thesis, unless specifically indicated to the contrary in the text, is my own original work.

Charles Stephen Young,
Pietermaritzburg,
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CHAPTER ONE

STATEMENT OF THE RESEARCH PROBLEM

There is still, and probably always will be, debate around the very nature of mathematics. There are those who, like Plato and his followers, believe that mathematics is an external reality awaiting discovery. On the other end of the spectrum there are those who contend that mathematics is no more than a game where symbols are manipulated according to precise rules. The mathematician Keith Devlin (2000b, p. 72) argues that mathematics is the "science of patterns". For him, mathematics is abstracted from the world around us, but those abstractions are shaped by the structure of our minds. Thus, according to this definition, mathematics is both discovered and invented. Arithmetic is a branch of mathematics that is concerned with the patterns of numbers. It is also the first branch of mathematics to which children are exposed. Devlin argues that higher mathematics becomes increasingly abstract but not necessarily more complex. Therefore, children need to master arithmetic before they are able to manage higher mathematics. If arithmetic is the foundation of higher mathematics, then addition is the foundation of arithmetic. The development of these addition skills, perhaps the foundation of all higher mathematics, is the focus of this work.

Yet, the development of this mathematical ability poses enormous obstacles to many learners, making the domain an important area of study. The difficulties that learners have with mathematics seems to be a particular problem in South Africa. A review of the annual matric results demonstrates the extent of the challenge that mathematics poses for South African learners. *The Third International Mathematics and Science Study* placed South Africa last out of the 41 participating countries[1]. South Africa was the only African country included in this study. A more recent study commissioned by the Department of Education revealed that South African grade four learners have the worst numeracy skills of the twelve African countries reviewed (Pretorius, 2000, July 16, p. 1). South Africa's substandard performance in mathematics education may have its origin in poor foundational teaching and learning. Thus, an appropriate place in which to focus any research is at the beginning, with simple addition. The research in the area may have significant implications for the way children are taught. Furthermore, the acquisition of knowledge in general, as a topic of cognitive science, is poorly understood. Gains made in the domain of arithmetic development may therefore translate into advances in the broader arena of cognitive science.

A further reason to study this particular domain is that there is a feeling that more of the cognitive literature should be based on familiar classroom activities (Siegler, 1987). One of Piaget's legacies is his collection of experimental tasks. Many of the neo-Piagetian researchers have adopted these tasks or close derivatives to them. Children are not familiar with most of these tasks. For example, the various class inclusion, conservation and transitivity tasks are very different from the usual early school activities. The tasks are unfamiliar for a good reason in that these experiments are designed to expose children's reasoning and not their factual knowledge. Siegler (1987) points out, however, that children may reason differently in familiar and novel domains. Therefore, it is as important to study familiar areas, such as basic arithmetic, an important aspect of the "other half of cognitive development" (Siegler, 1987, p. 731).

[1]This study was completed in 1995. A repeat of the study completed in 2000, and reported in *The Sunday Times*, revealed that South African students were outperformed by the students of 37 other countries, indicating that the situation has not improved (Pretorius, 2000, December 10).

A number of researchers have made important contributions to our understanding of how children develop their mathematical abilities. Two in particular have generated much work in the domain of numerical development. Piaget's domain general theory of cognitive development includes a description of how the child acquires the notion of number. For many years, until relatively recently, Piaget's general theory was the dominant one. The current neo-Piagetian theories have all emerged from this original work. Robert Siegler, arguably the leading current theorist of arithmetic development, entered the fray at a time when Piaget's dominance had already begun to decline. He has made a number of original contributions, most of which appear to substantially differ from Piaget's ideas. While Piaget's theory has been subjected to an enormous amount of empirical scrutiny, Siegler's has yet to face the same onslaught.

The discussion of children's numerical development is set against the backdrop of some of the broader cognitive development debates, particularly concerning the nature of early cognitive development. The *staircase metaphor,* best characterised by Piaget's theory and the neo-Piagetian theories, has and continues to dominate the way we regard cognitive development. (Even the modern information processing theories, although not stage theories, focus on the incremental relationship between age and thinking.) According to these staircase theories, children think in a particular way at each age related level and their thinking at each level has a certain stability. The relatively sudden shift from one level to the next, a step up the staircase, is poorly explained, while untidy evidence of variability in performance is swept under the carpet by means of labels such as *decalage*.

A recent alternative to this way of viewing development is Siegler's (1996) *overlapping waves metaphor*. According to this perspective, at any given age children will resort to a multitude of different ways of thinking. Here development is regarded as the gradual increase and decrease in the various ways of thinking, called *strategies*[2], as well as the discovery of new ones. Development involves changing patterns of strategy use. So, instead of a steplike progression, or a linear progression, Siegler (1996) proposes that development can best be characterised as a series of overlapping waves. The *staircase* theories have had a powerful influence on the teaching curriculum. Yet, if Siegler's alternative view offers a more realistic picture, then it may have useful corrective educational implications. (As a caution, it is noted that Sherman (1999) warns that cognitive theorists tend to leap to the educational implications of their research without empirical testing under realistic classroom conditions.) The debate between these two developmental models is the central theme of this thesis.

Robert Siegler (1996) proposes that people tend to employ multiple problem solving strategies, not only in mathematics, but also in other domains. The novice problem solver, in any particular domain, might initially employ a single basic strategy to solve the problems while the expert is able to decide which of many strategies is most appropriate to a particular problem (Siegler, 1996). Between these two poles, the intermediate problem solver resorts to a diverse arsenal of strategies. The moderate experience hypothesis (Siegler, 1996) suggests that the variability of strategies is greatest when the child has moderate experience with the task at hand. There are other models that describe a similar inverted-'U' relationship. McClelland and Atkinson's achievement motivation theory suggests that motivation is highest when the subjective probability of success is 0.5 (McClelland, 1985). In other words, when children expect to succeed on half of the trials presented their motivation is greatest. The reason for this is that easy tasks offer little

[2]For the time being, the strategy is defined as any non-obligatory goal directed procedure. The definition offered by Siegler and Jenkins (1989) will be discussed in Chapter Three.

sense of achievement while the difficult task provides an obvious excuse for failure. The Yerkes-Dodson law (Yerkes & Dodson, 1908, as cited in Lachenicht, 1987) describes a similar relationship between arousal and performance. Similarly, Karmiloff-Smith (as cited in Claxton, 1998) reports that in many domains children first acquire a basic level of competence and will then, if allowed, continue to play with different possibilities that may even reduce their competence for a while. In doing so, they may develop more advanced ways of solving problems or ways that allow them to cope in different situations. It is the development and use of this strategy arsenal that forms the focus of this work.

This study is concerned with Siegler's main theoretical ideas as well as his research method, the *microgenetic approach,* and attempts to evaluate them. A number of broader questions are also considered. First, why do children continue to develop a repertoire of addition strategies, even after they appear to be adequately served (adequately served in terms of a number of variables) by a few of their existing ones? Second, how do children acquire this assortment of strategies? Third, how do children select any particular strategy from their arsenal of existing ones? The first question refers to the *why* of cognitive development while the other two refer to the *how* of development. The *how* of development is a question that has been left, mostly, to the domain general theories, while the domain specific theories, such as the theory of number development, have focussed on the *what* of development. As a result, the field consists of many competing theories.

The third of the three questions mentioned above constitutes the main focus of this research. Two testable research questions were formulated in order to further examine the issue of strategy selection. The first research question proposes that children select their addition strategies according to the principle of *least effort*. In other words, they will attempt to match the most appropriate addition strategy with the problem presented. A large sample of strategy choices has been collected and subjected to a prediction analysis to test the hypothesis.

The principle of *least effort,* as it is defined in this thesis, refers to an attempt to maximise benefit and minimise cost. This has been used to describe a various aspects of animal and human behaviour and is equivalent to the Cost-Benefit Analysis in economics. The Cost-Benefit Analysis is a decision making tool, whereby one attempts to maximise benefit less the cost, often used for the allocation public funding for various projects (Brent, 1996). This decision making tool has been used in domains other than economics. For example, *foraging theory* implies that many animals resort to a Cost-Benefit Analysis when foraging for food. Accordingly, animals expend food-gathering energy in a way that maximises the energy returns (Stephens & Krebs, 1986). Researchers have used the principle to understand the behaviour involved searching for information on the internet as well as to describe how scholars accumulate articles for their literature surveys (as cited in Chalmers, 2000, November 11). Thus, the first research question suggests that children will attempt to select the strategy that provides the most accurate answer while attempting to minimise the amount of processing involved.

The second research question proposes that under conditions of cognitive stress children will resort to faster strategies, such as retrieval[3], in order to save limited working memory space. (Case (1985, as cited in Adams & Hitch, 1998) argues that speed reflects the amount of work space required.) Alternatively, children will attempt to execute the strategies in a covert manner, in this case by using their fingers, in order to overcome their memory limitations. A number of the neo-

[3]Retrieval involves a process were the answer to the problem is recalled without any apparent calculation.

Piagetians (Case, 1987b; Halford, 1987; Pascual-Leone, 1970) offer various memory limitation models of cognitive development. Case (1987b) argues that information storage and mental operations compete for limited working memory capacity[4]. Accordingly, if the task requirements exceed the available working memory capacity then a breakdown in information processing occurs. The experience of anxiety will also impact on this limited work space leaving less for mental processing. The data in this study was collected under two conditions. The first involved optimal conditions while the second involved an additional memory task making the execution of the strategy more difficult. Requiring children to store a number while simultaneously solving a simple addition problem reduces the work space available for the mental operations (Case, 1987b). The proportions of retrieval use and the proportions of overt and covert strategy use are compared for the two conditions in which the data was collected.

The literature on which this thesis is based comprises three parts. The first part, contained in chapter two, briefly covers Piaget's theory of cognitive development and, in particular, his account of how the child acquires a notion of number. Piaget's work is considered not only because he is one of the leading developmental theorists, but also that it provides a backdrop to some of the more recent work. Also, Piaget has had much to say about the development of logico-mathematical knowledge. Contrasting Piaget to Siegler allows us to not only compare staircase to waves, but to do this within the domain of numerical and arithmetical development. The memory limitation neo-Piagetian theories are briefly mentioned. The working memory limitation

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[4]Halford (1987) offers a similar theory to the one proposed by Case (1987b). However, Halford does not believe that processing competes with short term storage. This difference is discussed in greater detail in the following chapter.

hypothesis seems to play an important role in arithmetic processing.

The second part of the literature review, described in chapter three, reviews much of the work generated by Robert Siegler and his various colleagues. Siegler is probably the leading current theorist on the issue of early arithmetic development and offers an alternative to many of the existing theories of development. Most of Siegler's work involving simple addition is discussed. Siegler has shifted his focus to "extending the overlapping waves model to a broader range of tasks and age groups", and his most recent work is not relevant to the present study (personal communication, 2000, June 4).

The third and final component is detailed in chapter four. This chapter describes the overlap and the difference between Siegler's and Piaget's theories. This section explores the relationship between conceptual knowledge, the aspect of arithmetic development that Piaget has emphasised, and procedural knowledge, an aspect of arithmetic development that is particularly relevant to Siegler's theory.

The remainder of the thesis deals with the empirical research undertaken and, in particular, the two research questions described above. These chapters include the aims, methods, results and discussion of the results obtained.

CHAPTER TWO

THE STAIRCASE METAPHOR OF COGNITIVE DEVELOPMENT - JEAN PIAGET AND THE NEO-PIAGETIANS

Any review of the literature of cognitive development should begin with the work of Jean Piaget. In fact, no single theorist has made a greater contribution to the understanding of children's thinking. Piaget has been the most influential figure in the arena of cognitive development and probably in the field of developmental psychology as a whole. His theory, some of it nearly eighty years old, is the central point around which much of the current debate in cognitive development occurs. Despite many alternatives, Piaget's theory still occupies a prominent place in the understanding of the development of children's thinking. Also, Piaget has a fair amount to say about the nature and development of mathematics, however, for the purposes of this thesis the focus has been limited to Piaget's version of how the child acquires a basic notion of number (since this is essentially what arithmetic development is about). This description offers an important foundation on which to examine the development of addition skills. This chapter includes a brief summary of a very small part of Piaget's work, beginning with his general theory followed by a review of his work on the child's conception of number.

2.1 Piaget's General Theory

Piaget believed that the purpose of all behaviour and thought is to enable the organism to better adapt to the environment. Accordingly, adaptation and its complementary process of organisation are regarded as invariant functions essential for survival (Vuyk, 1981). Adaptation, according to

Piaget, is made possible through the two complimentary processes of assimilation and accommodation.

For adaptation to occur the organism must be able to assimilate aspects of the environment as well as accommodate elements of it. Assimilation is the process where environmental input is adjusted to fit the child's existing structure, whereas accommodation is the process where the child's existing structure is adjusted to fit the environmental inputs. According to Piaget (1983), assimilation ensures the continuity of the existing structure and the integration of new structures but does not cater for any variation of the structure. Accommodation is the modification of the existing structure by the elements it assimilates (Piaget, 1983). These two complimentary processes cannot be separated. Assimilation can not take place without some degree of accommodation. This process builds the cognitive structure, which in turn actively attempts to assimilate new elements of the child's interaction with the environment.

Equilibration is the dynamic interaction between the processes of assimilation and accommodation and refers to the progression of the cognitive structure towards a stable equilibrium when interacting with the environment. Equilibration is a process while equilibrium is a state, albeit a temporary one. Thus, equilibrium is not a final resting point, but rather becomes a new point of departure (Piaget, as cited in Battro, 1973). Equilibration can be regarded as a continuous sequence of the disequilibrium - equilibrium cycle. Accordingly, the organism has adapted to the environment when it has reached a state of equilibrium where the assimilatory needs are balanced by the accommodatory capacity. Equilibration involves the modification of the child's mental structures. Development is the progression of successive internal states of higher equilibrium. The infant starts the process of cognitive development by assimilation and succeeds because of accommodation (Russell, 1978).

While adaptation refers to the relationship between the individual and the environment, organisation, the internal aspect of adaptation, refers to the relationship between the organism's internal structures (Chapman, 1988; Piaget, as cited in Battro, 1973). Organisation is the action exerted by the whole onto the parts, which serves to prevent incoherence. Organisation strives for conservation. The child's existing structures are not abandoned for the sake of equilibrium (Vuyk, 1981). Accommodation takes place while the existing structures are maintained as far as possible. Organisation strives for a balance between differentiation and integration. (In terms of the process of organisation, Piaget's idea is compatible with some of the evolutionists such as Stephen Jay Gould (2000) who argue the importance of holistic structural principles. Siegler, as we shall see, is closer to the ideas of Richard Dawkins (1976) who emphasises only the adaptation through variation and selection.)

Piaget's theory proposes that all children progress through an invariant path of development. Accordingly, children progress through four stages of development that are both quantitatively and qualitatively different from each other. (At different times Piaget distinguished three, four or five stages, each being slightly different combinations of the various sub-stages. However, it is generally recognised that there are four separate stages (Vuyk, 1981).) Children do differ in terms of the tempo at which they proceed through the stages and some do not reach the highest stage. The sequence of the four stages is as follows: the sensorimotor period, the preoperational period, the concrete operational period and finally the formal operations period.

The sensorimotor period occurs between birth and approximately two years of age. During this stage development reflects children's increasingly sophisticated interactions with objects. For Piaget, sensorimotor adaptation is the foundation for conceptual adaptation (Russell, 1978). The preoperational stage takes place between the ages of two and seven years. This period is the preschool stage and is marked by the acquisition of representational skills, the most obvious of these being the acquisition of language. The concrete operations stage occurs approximately between the ages of seven and twelve years. Children at this level of cognitive development are no longer ruled by their perceptions. Children are able to solve concrete problems but are unable to tackle abstract concepts. Finally, many children reach the formal operations period at around twelve years of age. This is the highest level of development and is associated with the ability to reason about theoretical concepts in addition to the concrete ones. Therefore, development is from the dependence on perception and appearance to independence through the growth of an understanding of structure, or put differently, from figurative to operative knowledge.

The development through the stages is not always even. A concept may be acquired in one form, yet take considerable time before it has been extended across its full range (Gruber & Voneche, 1977). Piaget refers to this as *horizontal decalage,* which he incorporates into his theory as a force of development. The coexistence of more and less developed structures results in disequilibrium, which then drives the individual towards equilibrium by developing the less advanced structures. One of the controversial aspects of Piaget's theory is the issue the of structural unevenness, and whether this can be tolerated without compromising the stages. Gruber and Voneche suggest the stage concept could be weakened to refer to an orderly progression rather than a universal pathway.

Piaget's theory has endured considerable attack. The attempts to replicate his findings using his brilliant methodology, have, for the most part, been successful (Siegler, 1986). According to Halford (1989), most of the basic phenomena documented by Piaget do occur, however, it is Piaget's interpretations that have generated a great deal of controversy.

Not all theorists agree that the Piaget's developmental sequence is, for the most part, adequate. Some argue that Piaget has underestimated the abilities of young children and overestimated the abilities of adolescents and adults (people who fall into the formal operations period). For example, Dehaene (1997) reports that children are able to conserve at an age far younger than when Piaget's theory credits them with this ability. Others argue that most people do not ever operate at the level suggested by the formal operations level (Halford, 1989), and when they do, it is only in domains where they have a great deal of experience.

If some theorists believe that the invariant course of development has endured the criticism relatively intact, few would argue the same for Piaget's causes of development. Halford (1989), amongst others, argues that Piaget's mechanisms of development are largely unspecified even though he believes that course of development has endured the attentions of many researchers seeking to challenge the theory. The accommodation-assimilation interaction is somewhat vague and possibly an inadequate explanation of intellectual development. However, in Piaget's defence, he, towards the end of his life, attempted to further develop his mechanisms of development with limited success.

2.2 The Child's Conception of Number

Despite being a prolific writer, Piaget devoted relatively little of his work to the question of how children learn mathematics (Hughes, 1986). For Piaget, the focus has been on the development of children's number concept, which, he argues, is closely and inseparably intertwined with the development of logic (Piaget, 1953). Piaget (1953) believed that young children, to a large degree, acquire the notions of number and mathematical concepts spontaneously and independently through their interactions with the world. He argued that if we try to impose mathematical concepts prematurely, their learning will be merely verbal, without a true understanding of the concept. Also, it has been argued, that by trying to do so, we might only succeed in developing a lifelong aversion to mathematics.

An integral aspect in the development of the number concept is the concept of conservation (Piaget, 1952; 1953). In fact, according to Piaget (1952; 1953), conservation is a condition of all rational thought and mathematics is no exception. Conservation is a logical concept rather than a mathematical one, which must be acquired before any true understanding of number is possible (Piaget, 1953). For Piaget, conservation refers to the logical operations where children maintain magnitude and relations despite perceptual transformations (Gruber & Voneche, 1977). The development of conservation, according to Piaget (1952; 1953), is not innate, but rather acquired inseparably with the development of number. Children must first come to understand the principle of conservation of quantity before they can develop their number concept.

In *The Child's Conception of Number* (1952) Piaget describes the genesis of number and logical ability. Essentially, Piaget describes this early development through an assortment of experimental tasks involving the comparisons of two sets. Accordingly, children progress through three distinct stages (Piaget, 1952). At the first level, reached at the age of four-and-a-half to five, children rely on their perceptual evaluations of quantity. Thus, if a set of objects is spread out, children believe that the quantity has increased. At the second level, these children conserve in certain cases and are confronted by the conflict between one-to-one correspondence and the perceptual relations. At the third level, reached at the age of six and a half to seven, children do not have to reflect on the result of perceptual transformations but realise that the quantity is conserved. Thus, at the first level perceptions govern children's evaluation of quantity, at the second level the perceptions compete with conservation while at the third level the ability to conserve frees children from their perceptual limitations. The development of number is, for Piaget (1952; 1953), a gradual synthesis, through these three stages, of inclusion of categories and of serial order.

At the first stage of this development, children treat discontinuous quantities in the same way that they would treat continuous ones (Piaget, 1952). For example, when a beaker of water is poured into one of a different shape, children at this level believe that the quantity of water has changed. Similarly, when this water is replaced by a set of discrete objects, such as a collection of beads, these children arrive at the same conclusion. While the first example demonstrates that these children fail to conserve in the physical sense, the second demonstrates that they fail to conserve in the mathematical sense. At this level, children's judgements are based on the perceptual features of the set. Magnitude or quantity is believed to change when the objects within a set are displaced. One of the most famous of Piaget's experiments illustrating the conservation of number is the one-to-one correspondence experiment. If these children are asked to compare two rows of evenly spaced objects and of equal length, then most of them will agree that each row contains the same number of objects. When the length of one of the rows is stretched or compressed level one children will now claim that the longer row contains more objects (Piaget, 1952; 1953). According to Piaget, these children are non-conservers. These children make their judgements according to one perceptual criterion only; their judgement will be based on either the length of the row (usually) or the density. Thus, these children are not yet capable of recognising the relationship between the two features.

When level one children are asked to reproduce a figure made up by a number of objects, they will usually only succeed in making a general imprecise replication of the original figure (Piaget, 1952). Children's figure will usually differ from the model figure in terms of the number of objects that make up the figure, especially as the model figure becomes more complex. Furthermore, if the objects in the original figure are displaced, children at this level will assume that the quantity of objects has changed. In order to reproduce this new figure they will add or remove objects from their copy of the original design. Children at this age are unable to reverse the operations that transformed the sets, their thought being characterised by this irreversibility. This irreversibility being an important characteristic of the earlier stages. On the whole, the reasoning of children at this level is governed by his perception. They are not able to quantify, with quantification being limited to judgements of 'more', 'less' or 'the same'.

At the next level, children show a number of improvements in their reasoning ability. Their analysis of the figures that they are asked to reproduce is more precise. Children experience a significant improvement in their practical skill, a hallmark of the concrete operational stage. They may arrive at contradictory conclusions regarding comparisons of quantity corresponding with the different features that they use. For example, when comparing two rows of an equal number objects but differing in length they will notice that they arrive at contradicting conclusions when they consider the length of the rows and when they consider the density of the rows. These children no longer concentrate on outstanding features of the sets, but do compare sets according to the resemblance or the difference between the two. They begin to recognise the relationship between features of the sets but this is not yet well organised. For example, they are able to conserve at times, such as when they have observed the one-to-one correspondence, but revert to their perceptual judgements in other situations. When comparing continuous quantities children at this level are able to allow for the height and width of a container. However, this ability breaks down when one of these containers is further subdivided, because their evaluation is not based on conservation and the relationship between height and width has become too complicated.

At the third level of this development, children's reasoning is freed from their perceptions (Piaget, 1952). These children do not have to reflect on the result of the transformation, but rather know that any perceptual transformations or the displacement of some of the objects does not change the quantity of the set. They do not have to consider the result of the displacement, but know beforehand that this does not affect the quantity of the set. These children are able to conserve number. Piaget's notion of the conservation of number has had a large influence on the educational curriculum, being the benchmark from which children are able to develop their arithmetic skills (Hughes, 1986).

According to Piaget (1952), the development of cardinal and ordinal concepts follows three stages corresponding to the stages discussed above. At the first level of development, cardination and ordination do not yet exist. These children make cardinal evaluations based on their global judgement of quantity. At the second stage, an accurate cardinal evaluation is achieved by a one-to-one correspondence, which also requires some sort of ordination. However, this cardinal evaluation is not lasting, while ordination is not differentiated from qualitative seriation. At the third level, operations replace children's perceptual intuitions. Both cardination and ordination are achieved. Number is both a hierarchical class and a series.

According to Piaget (1952) the construction of positive integers is completed when children discover the additive and multiplicative operations. He emphasises that the development of number and class are complimentary processes. From Piaget's perspective, numbers have logical aspects while logic contains the notion of number. Children's mathematical concepts are generated spontaneously from his or her logical operations. The development of the additive composition of classes is demonstrated by one of Piaget's most famous experiments, the class inclusion test. This experiment tests children's ability to compare a set with a subset of itself and is another important yardstick in the child's developing abilities. The experiment involves giving children a number of wooden beads made up of two uneven subsets, for example, a larger subset of brown wooden beads and a smaller subset of white, but also wooden, beads. When they are asked if there are more wooden beads or brown beads, Children, until they are approximately seven or older, will say that there are more brown beads (Piaget, 1952). Piaget took this to indicate that younger children are unable to compare the part with the whole, but only part with part (brown with white, not brown with the total group of wooden beads). Similar results are obtained when these children are asked if there are more tulips (part) or flowers (whole). Again the acquisition of the ability to compare parts with whole proceeds through three distinct stages. At level one, children are unable to do this, at level two, they are successful under certain conditions only and at level three they are able to compare part to whole.

Piaget (1952) argues that the ability to recognise that two sets are equivalent through one-to-one correspondence is the beginning of the multiplicative operation. This becomes operational after children are able to recognise that three (or more) sets can be equivalent since one of these sets is multiplied by the other two (or more) when more than three sets are compared. Piaget (1952) argues that when children discovers that if x equals y and y equals z, then x equals z they have then grasped the two-to-one relationship and will be able to generalise this to larger numbers of sets.

Finally, for Piaget, measurement is a synthesis of division into parts and of substitution. The process of division allow children to conceive that the whole is composed of parts added together. Substitution allow children to apply one part on others which forms the basis of a system of units. Once children arrive at this level they have acquired the basic notion of number.

While Piaget has offered a detailed description of the *what* of development, the *how* of development is not as easily understood. His developmental pathway is very clearly specified, yet how children move from one level to the next is less clear. (Although this is a criticism that can be levelled at many other theorists including some of the modern ones.) For Piaget, sensorimotor actions become internalised and when reversibility is achieved, these internalised actions, or schemes, become operations (Rotman, 1977). These operations become the objects of formal thinking. The development of logico-mathematical knowledge, for Piaget, involves an operation on operations. New structures are constructed on the foundation of existing ones through the process of equilibration.

Piaget's (1952) description of children's number development incorporates a fair degree of variability in the way that children think. The second of Piaget's stages of number development is characterised by this variability. The children at this level employ many different and often contradictory ways of approaching the various tasks. Siegler's (1996) moderate experience hypothesis proposes that variability will be at its greatest when a child has had moderate exposure to the task, which is likely to coincide with the second stage, an inverted-'U' curve of experience. Number conservation is a logical task and children eventually reach a point where one logical approach dominates the other ways of comparing quantities. Level one children, according to Piaget's theory, are consistently fooled by their perceptions, while older children understand that perceptual transformations do not change the number of elements. For this reason level three thinkers usually resort to this single logical method of responding to the task making their thinking, according to Piaget, consistent. Or, as Siegler (1995, p. 251) has put it, "if children ever adopt a new strategy and use it consistently, they would seem likely to do it when the superiority of the new approach lies in its basic logic". The sequence then is from consistency to variability back to consistency. An obvious question is whether children's thinking is so consistent on numerical domains where there are many competing ways of approaching them as opposed to domains dominated by a single logical approach? Also, how well does this developmental pathway stand up to more modern research methodologies? The next chapter covers much of Siegler's work in numerical tasks in which children are more familiar.

2.3 Criticisms of Piaget's Theory of Numerical Development

As mentioned, Dehaene (1997) argues that children are able to conserve at an age far younger than Piaget would have believed. Dehaene claims that when children are asked to compare two sets they fail to understand the question rather than fail to conserve. These children witness the experimenter move the objects before asking the same question and therefore reason that the experimenter wants a different answer. Mehler and Bever (1967) replaced the objects in this type of experiment with a type of sweet. Children were allowed to pick up one of the rows and eat the sweets. Under these conditions, young children were usually able to choose the row with the greater number of elements regardless of the perceptual appearance. However, Dehaene concedes that Piaget's criteria for the concept of conservation are likely to be far more stringent. Being able to demonstrate conservation in one particular task does not necessarily mean that children have acquired the concept.

Siegler (1995) cites a number of other studies that expand Piaget's original findings on the classical number conservation task which support Dehaene's (1997) conclusion. These studies suggest that young children often perform better if the rows include fewer objects[1]; if the rows are transformed by adding or subtracting objects; if the wording of the questions is facilitative; if the transformation was 'accidental' rather than deliberate; and if the children are trained in various ways. This, however, as noted earlier, was only one of the many tasks to which Piaget's

[1]Chapter Six presents some evidence that suggests that the addition operation is performed more efficiently when small numbers are involved.

subjects were required to respond. Also, even if children are able to respond to the tasks in advanced ways under very specific conditions, one could not necessarily conclude that children acquire the number concept earlier than at the age proposed by Piaget or that they acquire this concept in a notably different way. Halford (1989) points out that despite the proliferation of studies designed to refute Piaget's findings, his account of the course of development remains relatively intact.

Piaget describes how young children, at the first level of numerical development, treat continuous and discontinuous quantities in the same way (and fail to conserve either). Piaget, however, does not demonstrate the role that language plays in this distinction between these two types of quantities. In *The Child's Conception of Number* (1952), the role that language plays in the acquisition of a number concept is not acknowledged. For example, the English language distinguishes between *mass* and *count* nouns. Mass nouns are superordinate terms such as butter, furniture and money (McShane, 1991). The discontinuous quantities in Piaget's (1952) experiment included beads (count nouns) while the continuous ones included water (mass noun). Markman (1985, 1989 as cited in McShane, 1991) argues the mass nouns assist children to develop their understanding of class inclusion relations. While on this point, there is other evidence, cited by Devlin (2000b), that indicates that Chinese and Japanese children outperform their English speaking counterparts in school mathematics. This disparity appears to be largely the result of language differences. A related finding, cited by Deheane (1997), indicates that children's digit spans differ according to their language, with Chinese children having an advantage over most of their western peers. Their advantage comes from speaking a language with short number words. Miller, Smith, Zhu, & Zhang (1995) report that Chinese children are better able to recite the counting sequence when compared to their American counterparts. The difference, once again, is to do with the short length of Chinese number words and the reality that

Chinese number grammar directly parallels the structure of the arabic system. This provides these children with a numerical head-start. Thus, language, in a number of different ways, plays an important role in the development of the number concept.

Russell (1978) offers a critique of the role that conservation plays in Piaget's theory of number development. In at least one conservation task involving the ability to judge the area bound by a loop of string, which is then elongated, adults believe that the changes in dimension are compensated while children are able to recognise the change in area (Russell, 1976 cited in Russel, 1978). Children appear to outperform adults on this particular task. Russell believes that children's success has to do with their greater familiarity with these sorts of tasks. But, as Russell (1978) suggests, if conservation indicates a structural equilibrium then it should really be insusceptible to the effects of familiarity.

Possibly the most damaging of the criticisms levelled against Piaget's theory has to do with his rejection of any innate numerical ability. A fundamental premise of Piaget's theory is that children enter the world a *tabula rasa,* whose contents are constructed with experience. The recent works by Dehaene (1997), Butterworth (1999), and Devlin[2] (2000b) all argue that children are born with an inherent *number module* or *number sense*. Their evidence comes from a collection of studies from a variety of domains. There is evidence to suggest that many animals are able to quantify small collections of objects, suggesting that this *number sense* is a feature of many different animal species. Also, Karen Wynn's (1992) famous experiment indicates that even infants as young as five months old will stare for longer at events that violate numerical concepts compared to those events that don't. This is a finding that strongly supports the 'innate' argument. Furthermore,

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[2]Devlin (2000b) uses the term *number gene*, but does so in a metaphorical sense. He seems to prefer the term used by Dehaene (1997), the *number sense.*

people who suffer specific brain injuries are rendered number blind, evidence supporting the modular approach to understanding the working of the mind.

This, according to the three authors, means that we are born (if not born then very soon afterwards) with the ability to see the world in numbers just as we perceive it in colour or shapes. The notion is similar to Noam Chomsky's concept of a *language acquisition device,* a hypothesised innate mechanism facilitating the learning of grammatical rules. Therefore, it appears that infants are not the blank slates that Piaget suggests, but rather that they enter the world with some core competencies. The concepts of the language and number modules are further supported by the proliferation of domain specific theories which have emerged as a consequence of the failure of the various domain general theories to adequately explain all aspects of cognitive development. If this view is correct, then it poses a serious challenge to Piaget's entire theory since he postulates that knowledge is constructed with higher concepts being built on the foundation of lower ones. However, some of the higher concepts appear to be present without the foundation of lower ones. Piaget vigorously dismissed any claims of *a priori* abilities. Nevertheless, there is growing evidence that in this respect he may have been wrong, although, perhaps not entirely wrong, since the innate numerical abilities may be very limited when compared with the final abilities[3].

2.4 The neo-Piagetians

The early 1970's and onwards saw the proliferation of the various neo-Piagetian theories. According to Case (1992), many of these theories were guided by the earlier work of Pascual-

[3] Annette Karmiloff-Smith (1992) offers a compromise between Piaget's constructivism and Jerry Fodor's nativism, arguing that both are important aspects of development.

Leone and have attempted to incorporate both domain general and domain specific aspects of cognitive development. The theories of Bickhard (1978), Bruner (1964), Case (1985), Fischer (1980), Halford (1982), McLaughlin, (1963) and Pascual-Leone (1970) are efforts to re-conceptualise Piaget's progression in light of some of the criticisms levelled at the theory and to incorporate more recent data (all cited in Halford, 1989). Furthermore, many of the theories have attempted to combine the information-processing accounts of development with Piaget's structural approach (Case, 1987a).

Most of these theorists, according to Halford (1989), have left Piaget's developmental sequence relatively intact. All of these theorists, and some others that do not fit into the category of the neo-Piagetian theories, propose that higher concepts are formed from the integration of lower ones, which is a fundamental Piagetian principle. However, Piaget's equilibration process is abandoned by all of the neo-Piagetian theorists in favour their own developmental mechanisms.

Miller (1956) argued, in a classical article, that there are memory imposed limits on our capacity to process information. He suggested that adults are able to process around seven units of information. Development involves 'chunking' increasing amounts of information into this fixed number of processing units. Therefore, it is not the amount of information that is limited, but rather the number of units into which the information is compressed. Some of the neo-Piagetian theorists appear to have incorporated a similar notion in their theories by emphasising children's developing processing capacities. They propose that development is linked to the child's short term storage space (STSS) and that the child's processing ability is constrained by their memory limitations.

Case (1987b), for example, is one of the theorists who argues that children's mental processing is constrained by the STSS. For Case (1987b), the total processing space comprises the operating space and the STSS. The operating space declines as the processing becomes more efficient and, since the total processing space remains constant, the STSS capacity increases. For Case (1987b) and others, the STSS is the workspace of higher cognitive processes and, therefore, the increase in storage results in the ability to process more complex information.

Halford (1987), like Case (1987b), argues that the information processing capacity increases with age, but unlike Case, he does not believe that the increases occur in a discontinuous fashion. For Halford (1987), development occurs as a result of increases in children's structure-mapping ability. Structure-mapping refers to the process where elements from an external structure are mapped to the internal representation of the structure, an essential aspect of reasoning. (He describes four levels of increasing structure-mapping ability.) Therefore, with age, children are able to map more complex relations between the elements of the structure. These more complex structure-mappings make greater information-processing demands on the child, but the trade-off is that children are increasingly able to manage more complex concepts. Thus, the structure-mapping level constrains children's conceptual understanding, which, in turn, constrains the strategies that they are able to generate. Halford, Maybery, O'Hare and Grant (1994) state that development involves the process of representing increasing complexity in parallel. Strategy development, therefore, may involve a gradual shift from serial to parallel processing.

Importantly, Halford (1987) does not believe that the STSS is the workspace of higher cognitive processes, and, accordingly, does not believe that Case's (1987b) trade-off occurs. Processing and STSS, for him, are at least partly distinct. In other words, increasing operational efficiency does not necessarily facilitate storage. Halford, Maybery, O'Hare and Grant (1994) report numerous studies that indicate that information can be held in passive short term memory without interfering with cognitive processes. They argue that the various memory studies support a multi-component view of working memory. Therefore, a more complete description of the various working memory components is required to better understand how memory and processing capacities are involved in development.

Two neo-Piagetian ideas make valuable contributions to our understanding of arithmetic development and are important for the purposes of this thesis. The first is the proposal that efficient problem solving methods free up working space (Case, 1987b). While the second is the suggestion that working space maturational development facilitate the use of more complex problem solving methods (Case, 1987b and Halford, 1987). Also important is the notion that this working space (whether this includes the STSS or not) is limited, which implies that the strategies children employ is, at least, partly dictated by their structural limits.

2.5 Summary

For Piaget, children's numerical development is a gradual process intertwined with their logical development. New mathematical structures are constructed from existing ones and it is believed that before the age of six or seven children are not ready for maths. Piagetian educators believe that children should be allowed to develop their concept of logic before they are taught mathematics (Dehaene, 1997). For this reason most pre-school activities involve playing with blocks of various colours and sizes. Also, Piaget describes a specific pathway in the acquisition of number concept where variability occurs only in the transition from the use of perceptually based reasoning to logical-mathematical reasoning. Piaget's sequence is invariant; children enter the world with no numerical ability and acquire a number concept in the first years of their lives. However, it appears that others factors affect the developmental pathway. Language plays a significant role in the acquisition of numerical ability (and perhaps other social factors play a similar role). Also, a great deal of new evidence suggests that the child is born with some limited numerical ability. The work of some of the neo-Piagetians emerged as attempts to re-conceptualise the pathway first observed by Piaget. Robert Siegler was initially regarded as a neo-Piagetian, however, as should become apparent in the next chapter, his current theory is too far removed from Piaget to be included in this category of theories.

CHAPTER THREE

THE OVERLAPPING WAVES METAPHOR OF COGNITIVE DEVELOPMENT - ROBERT SIEGLER AND OTHERS

Various researchers have, since the early 1980's, observed that children employ a number of different ways to solve addition problems (Siegler and Jenkins, 1989). These observations have resulted in the *strategy* becoming the most appropriate unit of analysis in the study of children's developing arithmetic skills. It does appear that the development of addition skills is not simply an issue of adding numbers faster than before, or more accurately than before or even both. Children acquire new ways of solving these problems, and, therefore, development is both a quantitative and qualitative process. Also, development is not a process of moving from one strategy to another. Rather, higher order strategies coexist and compete with lower order ones. The development of addition skills reflects the development of the child's overall strategy mix. To begin to understand children's arithmetic development one needs to explore the strategies that they use. Robert Siegler has conducted a large body of research examining the strategies that children use. In particular, he has focussed on arithmetic strategy development[1].

3.1 The Strategy

Siegler and Jenkins (1989) have defined the strategy as a nonobligatory *procedure*. Therefore, a procedure becomes a strategy when there are alternative procedures to choose from. Also, according to these authors, the strategy is not a *plan*. For them, plans are a conscious type of

[1]Although Siegler has also examined the domains of subtraction (Siegler, 1989b) and multiplication (Lemaire & Siegler, 1995), most of his effort has been directed at addition.

strategy or procedure. Strategies are either conscious or unconscious nonobligatory goal directed problem solving methods. However, since Siegler and Crowley (1991) advocate the use of retrospective verbal reports to determine the strategy used in his microgenetic studies, one can conclude that strategies, for the most part, must be conscious processes. The principles of strategy selection, on the other hand, could be unconscious.

Moreover, the strategy is not the same as a *heuristic*. A heuristic is a simple rule of thumb, which relies on a single piece of information rather than aggregating several. Heuristics reduce the number of possibilities, but sacrifice accuracy on the altar of economy. Nor are strategies the same as *algorithms*. These are, like plans, conscious problem solving methods that consist of exact steps leading to a precise answer. The strategy, for Siegler and Jenkins (1989), is a general and inclusive category of problem solving methods. Therefore, heuristics, plans and algorithms could all be strategies, but this does not necessarily apply the other way around. There has to be some overlap between these concepts. The term strategy fits the variation and selection model that Siegler (1996) appears to support. According to Siegler and Jenkins, there are at least eight different strategies that children use to solve single digit addition problems, one of these strategies being retrieval and the others being *backup* strategies (backup strategies are defined as all strategies other than retrieval).

The claim that children use multiple strategies to solve simple addition problems is a relatively new one. Groen and Parkman (1972) once proposed the min model (a model suggesting that children solve simple addition problems by exclusively using the min strategy, which involves counting on from the larger addend by the number indicated by the smaller addend) as a description of how first grade children solve these problems. Their findings revealed that the smaller addend was a good predictor of the child's solution time. They then concluded that children use the min strategy exclusively. Ashcraft (1982) extended this model after finding that the best predictor of solution times was the size of the smaller addend in first grade children, the size of the sum squared in fourth grade children and that the two variables were equally good at predicting solution times for children in the third grade. He then concluded that children in the first grade consistently use the min strategy, children in the fourth grade consistently use retrieval while the third grade saw the transition between the two strategies. (A sequence that sounds like one Piaget would have developed.)

However, according to Siegler and Jenkins (1989), the children's verbal reports on how they actually solved the problem was often not consistent with use of the min strategy. Siegler and Jenkins, using the microgenetic approach as opposed to a chronometric analysis, found that the smaller addend was an even better predictor of solution times then previously believed on problems where the children had reported using the min strategy. However, the smaller addend turned out to be a poor predictor of solution times on problems where children claimed to have used a different strategy. As a result, Siegler and Jenkins concluded that the verbal reports are accurate descriptions of the method used to solve the problem. Moreover, children use a variety of strategies, not only the min strategy or retrieval, when attempting to solve simple arithmetic problems.

Over the last two decades, Siegler and his colleagues presented much evidence supporting their model of strategy development. Siegler and Jenkins (1989) believe that children may use existing strategies to construct new ones. Also, they report that children are intrinsically motivated to develop new strategies even when their existing ones adequately serve them. However, the source of this motivation does not appear to be understood. New strategies may or may not provide greater accuracy and efficiency. Most young children use at least six strategies and are able to select adaptively amongst the available alternatives (Shrager & Siegler, 1998). (Selecting adaptively implies choosing the fastest and most accurate strategy given the problem, the child's procedural limitations and the task requirements.) While the discovery of new strategies occurs after correct answers as well as incorrect ones, and with difficult as well as not so difficult problems (Shrager & Siegler, 1998). These discoveries, according to Siegler, are generalised very slowly (Shrager & Siegler, 1998). However, generalisation can occur faster if the child is presented with *challenge problems,* which are problems that are best solved by the newly discovered strategy (Shrager & Siegler, 1998; Siegler & Jenkins, 1989). An important observation made by Siegler and his colleagues is that strategy discovery is not a trial and error process (Shrager & Siegler, 1998). Children do not appear to attempt flawed strategies when discovering new strategies. Thus, there appears to be some constraint on the discovery of new strategies which may facilitate learning (Siegler, 2000).

3.2 Variation and Selection

Siegler's cognitive developmental theory has emerged from his attempt to incorporate the many studies describing the variability of cognitive action. Traditionally, researchers have focussed on the way children think about particular topics at particular ages. This hunt for age related essences has resulted in theorists overlooking the extent of the variation that occurs in the way their subjects think (Siegler, 1996). Cognitive variation, in these theories, is limited to the period of transition between the stages. According to the stage theories, children progress through an invariant sequence of stages and function according to their particular developmental stage. (The concept of decalage has been proposed to explain findings that do not support this even progression.) The transition from one stage to another is a sudden phenomenon that does not appear to be well understood. Many recent studies have detected competencies earlier than some of the stage-theories predict and later incompetence at ages when people are expected to be operating at a level described by the most advanced stage. Siegler argues that these inconsistencies reflect the extent of the cognitive variation that occurs throughout our development. LeFerve, Sadesky and Bisanz (1996, cited in Siegler 1996) conclude that college students employ multiple strategies to solve simple addition problems, so this variation is not limited to novice or intermediate addition problem solvers. It is not yet known if this cognitive variation increases or decreases with competence, however, it seems likely that this pattern would be different for the various domains. Furthermore, Siegler argues that variation occurs between domains as well as within domains, which is a well-documented finding that has presented an ongoing obstacle to the stage theorists (although not to the modular model of the mind).

While this variation may, to an extent, explain some the findings that have not been adequately accommodated in the stage theories, it presents a new problem. If the child has a variety of methods to choose from then, somehow the child has to make a choice. Performance would be adversely affected if strategy selection were a random process. So, unless the selections are adaptive then this variation is a cognitive impediment. Siegler (1996) claims that children select adaptively. Yet explaining how the child chooses between the alternatives has proved to be a difficult task for the theorists. There has been a tendency to solve this problem by invoking higher order mechanisms. For example, Flavell's concept of *metacognition*, Sternberg's *metacomponents* and Case's *executive processes* have all, according to Siegler (1996), emerged, at least partly, to explain the problem of choice. Yet all that has really been achieved, according to Siegler, is the naming of things poorly understood. However, Siegler himself resorts to a metacognitive component when he attempts to describe a possible process of strategy discovery.

Siegler extends the observed processes of variation and selection (and inheritance) into a cognitive-developmental theory drawing from the principles of Darwin's evolutionary theory. Individual development, according to this argument, has much in common with the evolution of species. He believes that the evolutionary approach is likely to be the most appropriate and useful way of viewing cognitive development (Siegler, 1996). For the process of strategy development, two mechanisms are required. The first mechanism generates variability. The child would need to spontaneously generate new and different ways to solve the problems. It is insufficient to only record this broad choice of problem solutions. Any theory needs to explain how this variation occurs in the first place. Second, some sort of learning must occur where useful strategies are retained and less useful ones discarded.

Siegler is certainly not the first person to apply evolutionary principles to the field of cognitive development. Evolutionary psychology is a growing branch of the human sciences that attempts to use many of the principles and concepts that have been developed in the study of the evolution of species to understand the workings of the mind (Plotkin, 1998). Some of Siegler's ideas could be regarded as resembling the position put forward by evolutionists such as Dawkins (1976) who emphasise only the adaptation through selection and variation. For any Dawkins type model to succeed some equivalent of genes are needed. In Siegler's case, strategies are the gene equivalents. The question is, however, are such strategies granular enough to serve as the equivalents of genes?

3.3 The Overlapping Waves Metaphor of Cognitive Development

According to Siegler (1996), the overlapping waves model of cognitive development is a better reflection of the empirical data than previous staircase models have been. Different waves represent the changing frequency of individual strategy use. The direction of development is towards greater use of the more advanced strategies. New strategies are generated and added to the child's repertoire, becoming new waves in the model. This model represents the variation in thinking that previous models have failed to capture. The emphasis is placed on the variation that occurs throughout development, and not only between the stages. (The notion of cognitive variability is discussed in more detail in Chapter Four.)

The different waves in the overlapping waves metaphor of development refer to the different strategies that people employ, in the case of the present investigation, simple addition strategies. These addition strategies, based on those described by Siegler and Jenkins (1989), include the following:

1. *Retrieval* involves retrieving the answer directly from memory. This implies that arithmetic facts are stored in some kind of memory table.
2. The *sum strategy* involves counting each of the addends separately, then counting up to the first addend and continuing to count on by the number indicated by the second addend. This is one of the more basic strategies and often reflects the way children are initially taught. According to this definition, the completion time could be described for the addition problem $x + y$ by the formula $x + y + (x + y)$.
3. The *shortcut-sum* strategy involves counting from one up to the total of the two addends. Therefore, the completion time is described by $x + y$.
4. The *count from first* strategy involves counting on from the first addend by the number indicated by the second addend and the completion time is described by y.
5. The *min* strategy involves counting on from the larger of the two addends by the number indicated by the smaller of the addends. In other words, if the child is presented with the problem $x+y$ and y is greater than x, then the child starts at the

number y and counts on by the number x. If x is greater than y, then the child starts with x and counts on by y, in which case the strategy (at least for the present study) would be coded as the *count from first* strategy. The completion time would be indicated by the smaller of x and y.

6. *Decomposition* involves breaking the problem into more manageable parts. Decomposition could be described a class of strategies since there are many different ways in which problems can be decomposed and recombined. The completion times would probably be best indicated by a number less than the smaller of x and y.

7. *Guessing* is different to retrieval in that the child explains that he or she guessed. The child makes no attempt to retrieve an answer from memory, but simply provides any number that comes to mind, which implies that the number is randomly generated. There is evidence, however, that children spontaneously activate the sum of the two numbers, so it seems likely that guessing somehow involves consciously not attempting retrieve an answer.

8. *Finger recognition* involves putting up fingers to represent each of the addends and the child recognises the total. This is different to the case where the child uses her fingers to aid the execution of the particular backup strategy chosen.

The strategies described above are the strategies commonly used by children who are exposed to the base-ten arabic number system. Children who have learned the roman number system, for example, may develop different addition strategies. The list, however, is not exhaustive. Dixon, Smilek, Cudahy and Merikle (2000) describe the phenomenon of *coloured number synaesthesia*. They have studied a child who perceives numbers as colours. Each numeral has a specific and fixed hue. Thus, simple addition, for some, may involve the mixing of different colours. Consider another example from the addition domain, although not single digit addition. Devlin (2000a) describes a well known story concerning Karl Friedrich Gauss. When Gauss was a young child, one of his teachers instructed him to add all of the numbers from 1 to 100, no doubt to keep him occupied for some time. Gauss realised that the problem could be conceptualised differently. He decomposed the problem into a set of pairs as follows: (50 + 51) + (49 + 52) through to (1 + 100). He than took the number 101 (the sum of each of the pairs) and multiplied it by 50 to arrive at the correct answer in far less time than his teacher had hoped. This anecdote demonstrates that there are many different ways of decomposing addition problems. Therefore, there may be a number of other addition strategies not listed above. However, the strategies described by Siegler and Jenkins (1989) are likely to include all of the common ones.

Siegler's (1996) overlapping waves metaphor emphasises the variability of strategy use, which is in direct opposition to Halford's (1987) neo-Piagetian model (see Chapter Two), which suggests that there are structural constraints that determine the strategies that the child is able to use. Siegler's (1996) and Halford's (1987) views, however, are not necessarily mutually exclusive. It is possible that the cognitive variability that is described by Siegler (1996) occurs around structural constraints that are probably best accounted for from the Piagetian tradition. This point will be discussed in Chapter Seven.

3.4 The Strategy of Retrieval

The strategy of retrieval[2], as the name suggests, does not involve any actual calculation or counting. Retrieval is the most efficient of the strategies since it requires little effort to arrive at

[2]Some people would argue that retrieval is not actually a strategy since it can be used without any conceptual understanding of the domain. In syntax development, for example, retrieval often comes before understanding. Siegler and Jenkins (1989) deliberately define the strategy as any problem solving method in order to include retrieval.

an answer. (Although it could be argued that this strategy involves a number of memory demands, perhaps even more than some of the calculation strategies.) As children become more proficient with addition problems they tend to use retrieval more frequently. Siegler and his colleagues (1987; Siegler & Jenkins, 1989; Siegler, 1989a) attempt to explain the decision between retrieving an answer or computing one, with their so-called *distributions of associations* model. Before children select a strategy they attempt to retrieve an answer from memory. Siegler and Shrager (1984, cited in Siegler, 1989) report that even pre-schoolers decide between retrieval and a backup strategy. The probability of any particular answer being retrieved is proportional to the strength of that answer relative to the other possible answers associated with the problem (the peakedness of the distribution of associations). Thus, an answer associated with a peaked strength distribution is more likely to be retrieved than a possible answer associated with a flat strength distribution. With experience, the relative strength of a correct answer increases and the distribution becomes peaked. However, the child will only retrieve the answer if the strength of the answer exceeds the child's internal confidence criterion (a subjective threshold point that needs to be exceeded before the child will select the answer with any confidence – presumably a threshold point that varies according to different circumstances). If retrieval fails, the child resorts to what Siegler calls backup strategies. Siegler (1987) then offered the strategy choice model to explain how the child chose from the available backup strategies. The backup strategies are all strategies other than retrieval, the commonality being that the likelihood of using a backup strategy increases with more difficult problems, while retrieval is more likely to be used with less difficult problems (Siegler, 1996). It appears that the retrieval table grows as the child grows older.

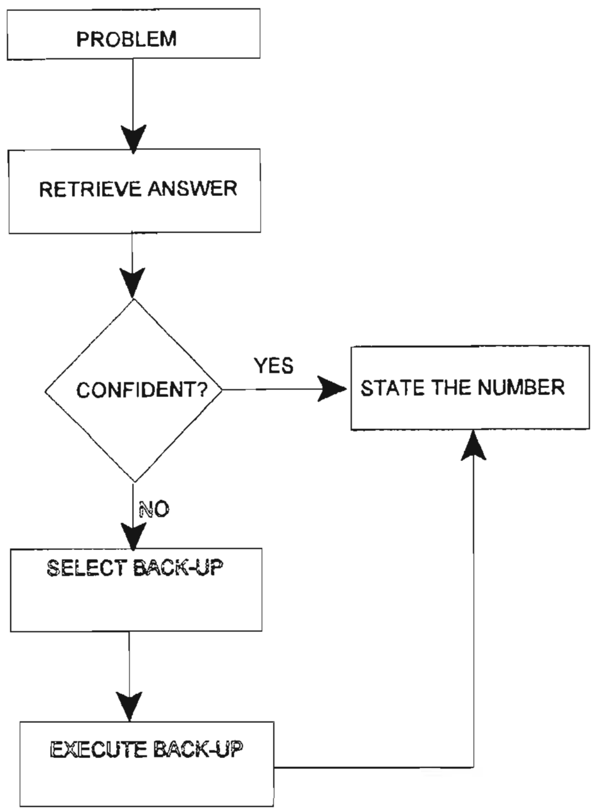

Figure 3-A. Flow diagram depicting Siegler's (1987; 1989a) addition process.

Anderson (1974, as cited in Anderson & Reder, 1999) argues that the organisation of human memory is associative. Accordingly, our memory includes a dense interconnected network where numbers are represented as nodes. The activation of any of the nodes results in a diminishing activation spread along the associative network. Therefore, closely related nodes are also activated. In many circumstances this is an advantage, but in the area of simple addition it may be the source of some of the difficulties that children have with retrieval. Dehaene (1997) demonstrates the advantage of this type of memory system by pointing out that when a person is faced with a dangerous animal, without having any previous experience with this type of animal, it is advantageous to be able to recall what one knows about other similar animals. However, when confronted with an addition problem the activation of all related facts only serves to make selecting the correct answer more difficult. The retrieval models assume that the operands and the problem as a whole trigger an activation of the memory network. A number of candidate answers are activated and the most strongly activated answer is the one that is retrieved (Niedeggen & Rosler, 1999). If one considers that $a+b$ equals $b+a$ then there are only forty five arithmetic facts that children have to be able to recall to cover all of the possible answers in single digit addition in base ten arithmetic. However, it takes considerable practice and time for these facts to be learned (Ashcraft, 1982). The reason for the difficulty is that the activation spread activates more than one possible answer resulting in arithmetic errors that are often related answers rather than unrelated ones. Also, the law that states that $a+b$ equals $b+a$ is probably a discovery too. In which case the retrieval problem becomes easier than the initial problem after the law is discovered. If one considers Piaget's notion of reversibility, a feature of the concrete operational stage, then the commutative law is likely to only be discovered sometime after children first begin to retreive.

LeFevre, Bisanz and Mrkonjic (1988) conducted an experiment that involved briefly presenting a pair of digits to their subjects, which they were required to memorise. Then a third digit was presented, after the other two had disappeared, and the subjects were asked if this digit was identical to one of the pair of digits already presented. The researchers noticed that on the trials where the third digit was equal to the sum of the digit pair, there was a significant increase in the response time compared to when the third digit was a neutral one. This suggests that we automatically activate the sum when presented with two single digits. The child takes longer to determine that the third digit was not a part of the digit pair, because this number has already been activated along with the original two numbers resulting in an interference effect. When the third digit is a neutral one, it is easier to differentiate it from the first set of numbers. Lemaire, Barrett, Fayol and Abdi (1994) obtained similar results with elementary school children. If we extend these findings to Siegler's (1987; 1989a) model, then we would conclude that the child does not choose between retrieval and a backup strategy, but rather that the child automatically activates a set of possible answers, and then chooses between stating the most active answer (if the child is sufficiently confident that the answer is correct) or calculating an answer using a backup strategy. Possibly this phenomenon is an aspect of our so-called number intuition. As we calculate an answer we may have already activated the answer to the problem. If the activated answer is consistent with the calculated one then perhaps we have what we describe as an intuitive feel that the answer is correct. This finding also suggests that children would have to partly ignore the two addends to generate an answer by guessing.

A further prediction, based on this model of retrieval, is that when children begin to learn multiplication, their retrieval time for addition problems would increase. This prediction is based on the assumption that children would start to associate a second answer with the single digit pair. For example, 2 and 3 would be strongly associated with the number 5 (2 plus 3 equals 5) and also strongly associated with the number 6 (2 multiplied by 3 equals 6). The child would experience some sort of interference effect between these two answers. Incorrect retrieval answers would be more likely to be related answers (the correct answer for the two operands but with a different operator). This prediction could be tested by comparing the response times and accuracy of the retrieval of addition facts of children immediately before they are introduced to multiplication and again when the school curriculum introduces multiplication. Miller and Paredes (1990) did exactly this and their results support the prediction. They concluded that there is a substantial interference effect when a new skill is integrated with existing knowledge. This is referred to as a *fan effect*, which is the phenomenon where retrieval times increase when additional facts are associated with concepts (Anderson, 1974 cited in Anderson & Reder, 1999). It is still not clear if this fan effect is the result of an interference effect, as claimed by Anderson and Reder or some type of inhibition mechanism, as argued by Radvansky (1999).

The distributions of associations model has been used to explain some of the observed individual differences between children (Siegler, 1989a). Siegler classified children into one of three groups, which were generated by a cluster analysis of a sample of children's cognitive performance. These groups include 'good students', 'not-so-good students' and 'perfectionists'. Siegler argues that the 'good students' and the 'perfectionists' solve the problems equally fast and accurately, both groups of children having peaked distributions. They differ, however, in that the 'perfectionists' set themselves higher confidence criteria, and thus use retrieval less often than the 'good students' do. 'Not-so-good' students, in contrast, have flatter distributions and lower confidence criteria. In other words, they use retrieval less accurately and more often than the other two groups. Siegler (1996) states that, although the 'perfectionist' group appears to represent the stereotypical girl while the 'not-so-good' students appear to represent typical boys, the two genders were evenly represented in all three groups.

Siegler (1990) reports that there is a significant correlation between the accuracy (defined as the percentage correct) of executing a backup strategy and the accuracy of using retrieval ($r=.43$ in simple addition). The observation that the 'perfectionists' are very accurate when using the backup strategies and set high confidence thresholds for retrieval is not a coincidence. Being able to solve the problems using a backup strategy accurately means that the child is able to develop peaked associative distributions. 'Not-so-good' students, on the other hand, are unable to set such a high confidence threshold, because their distibutions of associated answers is less peaked. Also, they do not need to set such a high confidence threshold because using retrieval at a lower threshold offers a fast strategy that is no less (or more) accurate than their backup methods. If accuracy is poor either way then the child may as well select the faster problem solving method. Siegler (1990) hypothesises that if one is able to teach the 'not-so-good' student to execute their backup strategies more accurately then the students would be able to build more peaked distributions of answer associations which would then allow more accurate retrieval. This was a hypothesis that Siegler (1990) said would soon be tested, however, to my knowledge this has not yet been done. Siegler's hypothesis offers a very optimistic view of individual differences. Perhaps the reason why the hypothesis has not been tested is that teaching a child to execute a backup strategy more effectively, as many teachers would remark, is not a very easy thing to do.

Carr and Jessup (1997) report that first grade girls are more likely to use the overt strategies, which involve finger counting, while boys are more likely to use more covert methods such as retrieval. There were, however, no significant gender differences in the overall performance of the children. This suggests that girls are more likely to be 'perfectionists', while boys are more likely to be 'good students' (or 'not-so-good students'). This finding suggests that Siegler's three categories of individual differences reflect, at least partly, gender differences. Siegler's (1989a) and Carr and Jessup's findings appear, for the time being, to be inconsistent.

Siegler's model is not the only model of the retrieval process. His model assumes that retrieval is always first attempted and if it fails to generate an answer in time or of sufficient strength then the child resorts to a backup strategy. A variant of this model assumes that retrieval and the computation occur simultaneously with the first to generate the answer being the strategy chosen, which is similar to the horserace model of word recognition (Logan, 1988 as cited in Schunn, Reder, Nhouyvanisvong, Richards, and Stroffolino, 1997). The commonality is that these models all postulate that if the answer is known it will retrieved otherwise one of the other computational strategies will be used. Schunn et al. argue that the *retrieve when the answer is known otherwise compute* models are unable to account for all of the empirical data.

Schunn et al. (1997) contend that the person first decides between retrieval and a backup strategy before attempting either method. Also, according to these authors, the decision to retrieve an answer is based on the familiarity of the problem and not the accessibility of the answer. They call this process a *feeling-of-knowing*, which is similar to Siegler's confidence criterion, the difference being that Siegler's is a measure of the answer strength while the *feeling-of-knowing* is a measure of problem familiarity. The *feeling-of-knowing* is possibly something like the *tip-of-tongue* phenomenon that occurs in the process of word retrieval. This model, referred to as the *source of activation confusion* (SAC) model, is not necessarily inconsistent with the finding by LeFevre et al. (1988; see Lemaire et al., 1994 for similar results) that suggests that we automatically activate the sum of a single digit pair. According to the SAC model the associated nodes are activated, but the *feeling-of-knowing* judgment is based on the activation level of the problem node and not the node representing the sum (Schunn et al.). This departure form Siegler suggests that children can decide to retrieve, even if the retrieval attempt is unsuccessful, or decide to compute the answer, presumably even when the answer is retrievable, without first attempting to retrieve the answer.

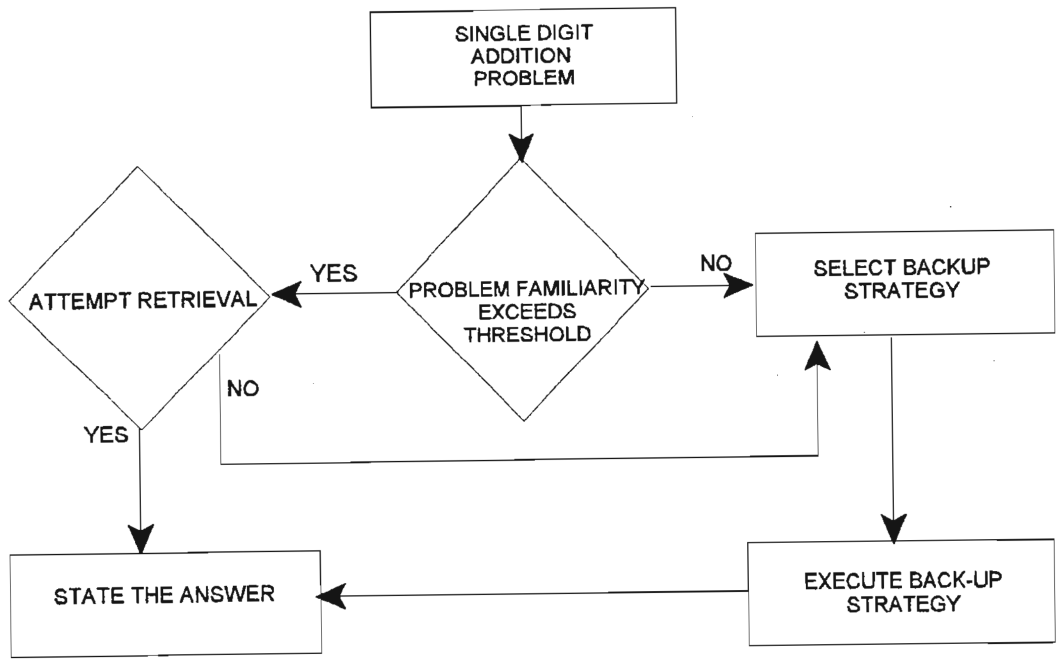

Figure 3-B. Flow diagram depicting Schunn et al.'s (1997) retrieval / compute decision process.

The various retrieval models do, to an extent, explain how a child might choose between retrieving an answer or calculating it. However, they do not describe how one selects from the various competing backup strategies after deciding not to retrieve. This is the focus of the next section.

3.5 Choosing between the Existing Backup Strategies

Siegler and Shipley (1995) claim that three generations of strategy choice models have emerged. The first generation of models refers to the various metacognitive models. Second generation models include the distribution of associations model. A third generation of models refers to the Adaptive Strategy Choice Model, although this is essentially a refinement of the distributions of associations model. Later models attempt to explain both the selection and the discovery process.

The first generation of models that were developed to account for strategy selection were the metacognitive models that proposed that the child's cognitive knowledge was used to govern their cognitive processes (Siegler, 1996). The term *metacognitive* has been used in different ways in the cognitive science literature. Siegler reserves the term for knowledge and processes that are explicit, rational, flexible, responsive to problem-solving goals and conscious (Crowley, Shrager, & Siegler, 1997; Siegler, 1996). The metacognitive mechanisms are potentially verbalisable and open to the process of reflection. They can be adapted to fit novel situations but the cost is that they are slow and require a great deal of the working memory resources. These early metacognitive models assert that the child is able to judge her intellectual capacity, the various strategies that she is able to choose from, and the demands of the task (Siegler, 1996). A rational selection is then made based on these three considerations. While some studies do indicate that explicit cognitive knowledge is related to performance (for example, Baroody & Gannon, 1994; Cowen & Renton, 1996; Canobi, Reeve, & Pattison, 1999), children appear to often make adaptive strategy choices without the explicit knowledge that these metacognitive models require (Siegler, 1996).

Associative mechanisms, on the other hand, refer to the part of human cognition that is " implicit, fast and responsive to nuances in the environment" (Crowley et al., 1997, p. 463). Strategy selection involves the learned correlations between the tasks, actions and their outcomes (Crowley et al.). Associative systems operate without the need of reflective awareness and thus do not place excessive demands on the working memory system. These processes become automised and are not conscious. The down side is that they require a great deal of problem solving experience to make the associative connections. Also, they are not easily adapted to novel situations. It is worth noting that these two types of mechanisms (associative and metacognitive) seem to have complimentary strengths and weaknesses.

The Siegler strategy choice model (1987; 1989a; Siegler and Jenkins, 1989) offers a description of how children choose between the competing backup strategies in the event that they did not retrieve an answer. Successful use of any particular backup strategy, in terms of speed and accuracy (or other possible variables), results in an increased strength of the strategy. The probability of choosing any particular strategy is proportional to its strength relative to the other competing strategies. New strategies have, according to Siegler, novelty points, which increases their likelihood of being used, but the points decrease over time. Novelty points ensure that new strategies are selected even though they have not yet established strength relative to the existing ones. Also, experience with different problems will result in a closer fit between the strategies and individual problems. Therefore strategy choice involves selecting the best strategy in terms of the problem presented.

The flow diagram illustrates some of the limitations of this model. The model suggests that one strategy will be used continuously on any given problem until it fails to generate an answer with sufficient speed and accuracy. However, this inflexibility is not a feature of children's addition (Siegler, 1996). Similarly, the model cannot account for the generalisation of strategies to new situations.

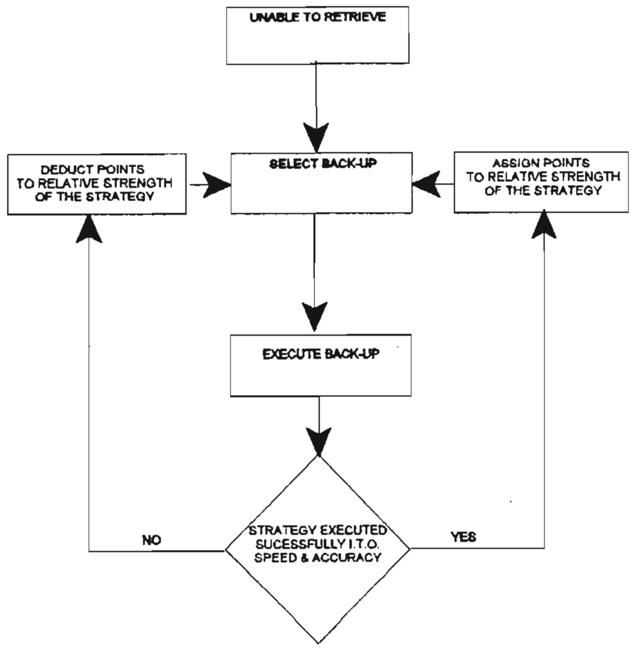

Figure 3-C. Flow diagram depicting Siegler's (1987; 1989a) strategy choice model

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Siegler and Shipley (1995) developed the Adaptive Strategy Choice Model (ASCM) as a modification of the earlier associative models. According to the ASCM, strategies operate on a problem to produce not only the answer, but also information regarding the effectiveness of the operation. This information is then retained in a data base under four headings: Global data refers to the history of the strategies effectiveness on all of the problems that it has been used, for example, all simple addition problems. Feature data refers to the effectiveness on specific classes of problems, such as simple addition problems with a large difference between the size of the addends. Problem-specific data refers to the strategy effectiveness on specific individual problems. Novelty data refers to the novelty points discussed earlier that allow new strategies to be competitive with old ones. As these new strategies are used their novelty points are gradually lost. This loss, however, is compensated by an increase in the strength associated with the different categories retained in the database. The database becomes a more accurate reflection of the effectiveness of the various strategies.

The Adaptive Strategy Choice Model differs from earlier associative models by explaining how successful use of a strategy in one situation is generalised to new situations. When a strategy is used on a problem that has not been solved using this particular strategy before, the selection would have been made according to the global and feature data associated with the strategy. If the strategy had never been used on the class of problem, then the decision would be made on the global data only. This model has been used to explain the selection of strategies in other domains (for example, Piaget's number conservation task in Siegler, 1995).

The Adaptive Strategy Choice Model assumes that strategy choice decisions are based on speed and accuracy only (although other variables could easily be incorporated into the model's decision making process). Other variables of strategy effectiveness could include the effort required to provide an answer (although effort is probably reflected by speed) or the ability to solve the problem under conditions of cognitive stress. Children may select their strategy on its aesthetic value. Also, ASCM offers a rational account of strategy selection and it is not clear how the model accounts for the variability that appears to characterise action.

While the various models encountered so far may, to a degree, explain how one chooses from existing strategies none of them explain how children develop these strategies in the first place. It is important to note the distinction between two types of strategy discovery. The first occurs when the difference between old and new methods has to do with the answers that they generate. This occurs when the child is motivated to discover new strategies because the existing ones are flawed. The advantages of generating new problem solving methods are obvious. The second type of discovery occurs when the difference between old and new is not in the answers that they generate but possibly in their efficiency or aesthetic value (Siegler & Jenkins, 1989). Siegler and Jenkins present evidence that indicates that children do not generate flawed simple addition strategies, so strategy discovery in this domain is of the second type. (According to Siegler (2000) if children construct flawed strategies, then this would indicate that they do not have an appropriate conceptual understanding of the domain.)

3.6 The Discovery of New Strategies

Crowley et al. (1997) return to the concept of a metacognitive system to account for their observations of strategy discovery. They argue that neither metacognitive mechanisms nor associative mechanisms alone can account for the observed patterns of discovery. Their alternative is an approach where the two mechanisms interact in a competitive negotiation. In returning to various metacognitive mechanisms Crowley et al. subject themselves to the same criticisms that Siegler has levelled at other metacognitive theorists.

Other theorists have noted that both explicit (metacognitive) and implicit (associative) mechanisms are involved in problem solving. Piaget (1976; 1978, cited by Crowley et al., 1997) and later Karmiloff-Smith (1992, cited by Crowley et al., 1997) both argued that problem solving starts as an implicit process becoming more explicit. In other words, children begin in a trial and error fashion eventually building a repertoire of useful strategies. Afterwards they are able to reflect on the existing strategies, adapting them or generalising them to new situations. Others reverse this sequence with the initial plans being explicit until later when the procedure is enacted without a metacognitive process governing it (Crowley et al., 1997). The important point being that both types of model involve both metacognitive and associative mechanisms.

Crowley et al. (1997) point to the discovery of the min strategy, a phenomenon that has been well documented (see Siegler and Jenkins, 1989), to argue their case for a competitive negotiation between both mechanisms. Based on their analysis of the data, the theorists reject both mechanisms alone. Essentially, their argument is based on the observation that children do not appear to generate flawed strategies (even if they sometimes fail to generate the correct answer). If an associative mechanism alone was responsible for the discovery of new strategies one would expect it to be a trial and error process. The errors leading to the construction of inherently flawed strategies. Addition strategies need to meet three goals, which are to represent the first addend, to represent the second addend and to represent the combined set of both addends (Crowley et al.,1997). Siegler and Crowley (1994, cited in Crowley et al., 1997) found that children are able to evaluate strategies according to these goals without being able to state why one particular strategy would be better than the other. Children seem to have some sort of metacognitive understanding of what the strategies must be able to do, although they have difficulty in verbalising this. The evidence suggests that some sort of metacognitive process is involved in the use of addition strategies.

However, the fact that new strategies are not initially generalised to situations where they would be most effective suggests that it takes time for the associative links to be of sufficient strength for the strategy to be able to compete with the existing ones, which is evidence for an associative mechanism. Furthermore, Siegler and Jenkins (1989) noticed that the presentation of addition problems best solved using the min strategy did not accelerate the discovery of the min strategy but rather accelerated the generalisation of the strategy for children who had already discovered it. They presented their subjects with a number of *challenge problems* during the eighth week of their eleven-week study. These challenge problems were addition problems consisting of one large and one small addend, and are best solved using the min strategy. Surprisingly, exposure to these challenge problems did not elicit the use of the min strategy in any of the children who had not yet discovered it. The problems did, however, result in children who had already discovered the strategy using it more frequently. This appears to be evidence that is best accommodated in an associative model. Therefore, both types of mechanism are somehow required to explain the data.

Two computer simulations have been developed that indicate that although either of the associative or the metacognitive mechanisms can account for the discovery of the min strategy alone, the simulated data does not reflect the empirical data that has been collected so far (Crowley et al., 1997). The associative and metacognitive models have complimentary strengths and weaknesses when compared to the data obtained in the Siegler and Jenkins (1989) study (Crowley et al.). Therefore, perhaps the solution lies in some sort of combination between the two mechanisms.

Crowley et al. (1997) propose that strategy discovery occurs through a competitive negotiation between metacognitive and associative mechanisms. They propose that a metacognitive system and an associative system act independently in the problem solving process. The first system to produce a satisfying decision usually wins the negotiation. Each system has a separate representation of the problem (although they must somehow constrain each other). In familiar domains, the faster associative system is likely to produce decisions before the slower yet more broadly applicable metacognitive system. For this reason, when we are familiar with a task we tend to rely on the associative system to generate implicit non-verbalisable decisions. The metacognitive system is freed from the burden of micro-managing the problem solving process which provides the opportunity for this system to notice interesting aspects of the strategy that might not be directly involved with the immediate problem solving goal. The metacognitive system can intervene by increasing its bid to manage the problem solving process. In unfamiliar domains, the associative system usually fails to provide a satisfying decision, giving the metacognitive system the opportunity to manage the process in a more explicit and verbalisable manner.

Since the two systems are independent and self contained, they do not require a meta-metacognitive system to manage the metacognitive-associative interaction. The model does not have a separate arbitration mechanism, rather it appears that the metacognitive system assumes control when it intervenes. A useful analogy is a horse and its rider. The horse represents the associative mechanism while the rider represents the metacognitive system, at times intervening to change the pace and direction.

There is other evidence for this type of dual system in the cognitive science literature. Guy Claxton (1998) in his recent book *Hare Brain Tortoise Mind* cites numerous research studies and other anecdotal evidence spanning the last few centuries that argues in favour of two different types of thinking. The first is the fast, deliberate and conscious *hare brain* which appears to resemble Crowley et al.'s (1997) definition of a metacognitive system, while the second is the slower, unconscious and more playful *tortoise mind* which, in many ways, resembles the associative system from Crowley et al.'s (1997) model. He argues that many, if not most, adults rely almost exclusively on the first while neglecting the second, to their detriment. Children fortunately employ this second type of thinking, which is perhaps at least part the reason why they spontaneously expand their strategy repertoire. The *tortoise mind* is able to detect elaborate patterns which the child can use to select and discover strategies, however, it may be some time before they are able to articulate this knowledge.

Shrager and Siegler (1998) tested the Crowley et al. (1997) model by developing a computer simulation based on their metacognitive-associative interaction. Their Strategy Choice And Discovery Simulation (SCADS) is an extension of the earlier Adaptive Strategy Choice Model.

SCADS differs from ASCM in that it represents strategies as a sequence of components rather than unitary mechanisms, records a more detailed trace of the execution of the strategies and includes three metacognitive mechanisms into the model. The metacognitive system is made up of an attentional spotlight, strategy change heuristics and goal sketch filters. Basically, the attentional spotlight allocates resources to the running of new strategies and later, when the simulation is more proficient in the execution of the strategy, allocates resources to the strategy change heuristics. These heuristics reassemble the different components of the strategies while the goal sketch filter weeds out flawed strategies. They report that the simulation selects and discovers strategies in a way that is consistent with children's behaviour.

SCADS involves trial and error mechanisms (associative mechanisms), a filter mechanism (goal sketch) and other mechanisms that are more responsive to the immediate problem solving goals, such as the change heuristics that seek to eliminate any redundant processing (metacognitive mechanisms). All that can safely be concluded is that some sort of interaction between mechanisms that are responsive to immediate problem solving goals and mechanisms that are not generates patterns of data that resembles children's development. In other words, there is a compromise between short-term and long-term goals. (This type of compromise is discussed further in Chapter Seven.) Since a computer has been used to model human behaviour, one would need to be very cautious about drawing any further conclusions.

SCADS indicates that at least some new strategies may be constructed from existing ones. This is a claim that was earlier made by Siegler and Jenkins (1989). Re-assembling new strategies with sub-routines from existing strategies while meeting the objectives specified by the goal sketch would lead to new functional strategies. Children could also look to other similar numerical domains for strategy parts. In this regard, the important point is that new strategies apparently do not violate addition principles.

Siegler and Jenkins (1989) first introduced the goal sketch that, they claim, governs the construction of new strategies. This goal sketch specifies the objectives that a functioning strategy must meet before it is included in the repertoire of existing strategies. Earlier it was stated that the goals any addition strategy must meet are to represent the first addend, to represent the second addend and to represent the combined set of both addends (Crowley et al, 1997). Also, the two sets cannot overlap. Possibilities that do not meet all of these requirements are discarded. Siegler and Jenkins originally argued that the goal sketch drives the discovery process, yet it seems to be no more than a standard against which new possibilities are judged. The goal sketch resembles Chomsky's point that children do not attempt certain wrong grammatical ideas. He suggests that there are constraints that prevent them from doing so. Like Chomsky, Siegler (2000) argues that these constraints make the learning mechanisms more powerful.

This goal sketch is one way of explaining how children do not invent flawed strategies, but in solving one question it raises a number of others. Siegler and Jenkins (1989) argue that goal sketches are generated before actual strategies are discovered through a general knowledge of the domain. Possibly it is some type of innate mechanism. Three recent publications argue that humans possess a faculty referred to as a *number sense* (Dehaene, 1997; Devlin, 2000b) or *number module* (Butterworth 1999). This module provides us with a numerical intuition. All three of the authors argue that this number module or sense compels us to see the world in quantities, as much as we are compelled to see the world in colours. This predisposes children to the learning of mathematics. Children, according to this perspective, are not the clean slates that Piaget's constructivism suggests, but are born with an inherent arithmetic ability. Perhaps children are able to identify the objectives of their goal sketch before they have the language to refine their natural arithmetic ability (although, this is not a line of reasoning that Siegler has followed). Thus, the goal sketch may be an innate ability.

The modular faculty approach referred to above, currently promoted in the domain of mathematics by Butterworth (1999) was first introduced by Franz Gall in the field of phrenology where it became discredited. It has been revived by and is currently associated with Jerry Fodor (1983) who details his argument in the book, *The Modularity of the Mind*. There do seem to be some domains in the brain that work as modules, even if we do not know how they work (for example, face-recognition). The approach, however, does not come without its critics. A counter argument suggests that the modular approach simply serves to rename the thing that needs to be explained without much idea how the module may work, a similar criticism to the one levelled at the various metacognitive structures. Also, the modular approach still needs to explain how the various modules combine to form a unified conscious experience.

Annettte Karmiloff-Smith (1992; 2000) draws a distinction between the notion of pre-specified modules and the process of modularisation. She argues that the mind may very well become a modular structure, but only as a consequence of development. According to this line of reasoning, certain brain circuits are selected for the various domain specific processes and, eventually, the different encapsulated modules may be formed. This approach is congruent with the findings that indicate that the young child's brain is characterised by its plasticity.

The goal sketch may reflect the child's sophisticated number concept. Piaget (1952) and others argue that the number concept is acquired. For Piaget the number concept is a special kind of concept and refers to a cognitive *operative* knowledge. This concept, as it is constructed, may provide children with a sufficient conceptual understanding of numbers so that their addition strategies do not violate addition principles. In other words, when children begin to add single digit pairs, they do so in a way that complies with their number concept. If the child has a sufficiently rich number concept then defective strategies should never be generated. (As mentioned in the Second Chapter, it is also possible that the constructivist and the modular views are both partially correct. Children may be born with a very limited understanding of number and this ability may be further developed during the early years of their lives.)

The debate between those who believe that we have innate abilities and those who conclude that our abilities are acquired is not a new one. In the classical work, *The Meno* Socrates attempts to demonstrate to Meno that knowledge is acquired before birth (Plato, trans.1975). He does this by questioning one of Meno's young slaves on a geometrical problem. Through the questions that Socrates asks, the boy is able to solve the problem. Socrates assumes that this demonstrates that knowledge is not taught but rather that the boy had the knowledge all along. This knowledge exists from birth, however, in this case, it required Socrates' questioning before it was recalled. The debate, therefore, is almost two and a half thousand years old, at least.

A somewhat different explanation for the goal sketch phenomenon is that in the domain of addition skills, children are initially taught the sum strategy, which specifies the objectives that all future strategies must meet. In this way the sum strategy becomes the goal sketch for later strategy development. Perhaps it is no coincidence that the initial strategy that is taught tends to make the three objectives most explicit. The sum strategy could be regarded as a prototypical goal sketch. While this goal sketch determines whether a new strategy works, only experience will determine whether it works well.

3.7 Other Variables that may be Associated with Strategy Discovery

There have been a number of factors that have been investigated in order to determine the role that they may play in the discovery process. These include the so-called transitional strategies, the quantity of problems, the type of problems, the role of the unconscious mind, the context in which maths is practised and anxiety.

With the various strategy selection models the strategy has been the unit of analysis. However, for the strategy discovery models the various components of the strategies become the unit of analysis. This is because many of the addition strategies are closely related and one of the leading explanations of strategy discovery is that new strategies are constructed by reassembling routines from existing ones. Crowley et al. (1997) claim that it is unlikely that each strategy maintains independent subroutines. Strategies access the various components when they are executed. These authors demonstrate this by describing the steps that may be involved in the discovery of the min strategy. This highlights the role of transitional strategies in the discovery process.

Crowley et al. (1997) present a task analysis of the sum to min transition. The sum strategy incorporates six procedural steps. If we take the addition problem $x + y$ then the sum strategy involves:

1. Assigning an addend to be represented first (x);
2. Assigning an addend to be represented second (y);
3. Count out the number x;
4. Count out the number y;
5. Count out part of the total represented by x; and,

6. Count out the remainder of the total represented by y by counting on from x by the number indicated by y (Crowley et al., 1997, p.471).

According to these researchers, the first discovery that the child makes is to notice that the Steps 3 and 4 are not necessary because this counting produces the original value of each addend. This is the cardinal word principle. The result is the shortcut sum strategy that involves Steps 1,2,5 and 6. Now two more discoveries are needed for the min strategy. One of these is that the order in which the addends are presented does not necessarily need to be the order in which they are processed. The child can count out the larger of x and y first. This offers and advantage when using the shortcut sum strategy because Step 6 involves counting the counts, which would be easier with smaller numbers. (Although, these discoveries may involve additional memory demands, it is also likely that the gains outweigh the disadvantages.) The other discovery involved in this sum to min transition is that the counting out of the larger addend in Step 5 returns the subtotal that is equal to the same addend. This step is then deleted and the result is the min strategy. In this sense the shortcut sum strategy is a transitional strategy linking the sum and min strategies.

Other researchers have suggested that the count from first strategy plays the same role as the shortcut sum strategy. If the sequence of the discoveries is that Step 3 and 4 are deleted then Step 5 is deleted before the child notices that he can reverse the order of the addends, the result is discovery sequence starting with the sum strategy then the count from first strategy before the min strategy. A sequence that seems to be more intuitive than the shortcut sum transition since the discoveries involved in deleting Steps 3 and 4 and then 5 are very similar and both are linked to the cardinal word principle. However, in the Siegler and Jenkins (1989) study, most of the children discovered the shortcut sum strategy before they discovered the min strategy. This data, however, was obtained from a very small sample and it is possible that sequence could follow one of two routes. The difference between the two positions is the point where the child begins to reverse the order of the two addends.

These descriptions of the sum to min transition appear to undermine Siegler's (1996) argument for an evolutionary approach to understanding cognitive development. Here he is suggesting that there is some sort of defined problem space through which children progress in a relatively confined and logical way, resembling a Piagetian progression. Three discoveries are made in this description and appear to be made in a fairly constrained sequence. Even if there is no fixed order in which these three discoveries are made, there does appear to be a sequence to the discovery of the strategies. Therefore, the description appears to be closer to Piaget's principles than to the random variation and selection processes that we associate with evolution.

Case's theory (as cited in Siegler, 1986), argues that strategy discovery will most likely occur after the processes are automised and mental resources become available. Therefore, the overall number of problems to which the child is exposed may be an important factor in the discovery process. Siegler and Jenkin's (1989) study of arithmetic strategy development saw seven of the eight children discover the min strategy at some point during the eleven weeks of practice. When children are presented with a sufficient number of addition problems, they will gain experience using a particular strategy and perhaps automization occurs. This would then free up mental resources for other possible processes. Fewer resources are required to use the strategy, and more attention directed to the encoding of the problem. Improved encoding facilitates the learning process. More relevant features of the problem are encoded and combined resulting in a better understanding of the task. This could then lead to different and more efficient ways of solving the problem.

Case's automization process (as cited in Siegler, 1986) resembles the associative-metacognitive competitive negotiation process introduced by Crowley et al. (1997). In this model the child becomes familiar with the domain and begins to rely on his associative system, which becomes more and more efficient. The metacognitive system, relieved of its management duties, is provided with the opportunity to notice aspects of the process that are not necessarily involved with the immediate problem-solving goal. It is the metacognitive system that makes the three discoveries that may be involved in the sum to min transition. In Case's model there is a shift from a type of associative mechanism to a type of metacognitive mechanism, while Crowley et al.'s model is made up of a continuous competitive interaction between the two mechanisms.

It does not appear that the type of problem has much involvement in the discovery process. It has already been noted that *challenge problems*, defined as problems that are best solved using the min strategy, did not result in children who had not yet discovered the min strategy discovering it (Siegler & Jenkins, 1989). All that was achieved by presenting these problems was the faster generalisation of the min strategy for those who had already discovered it. However, nothing is known about the role that other types of simple addition problems may have in the discovery process.

The is some debate around the role that unconsciousness processing plays in the discovery of new strategies. Siegler and Jenkins (1989) report that children tended to become less articulate when attempting to explain how they solved the problem on the trial when the min strategy was first discovered. Apparently, children were often the unaware that they had used the min strategy. Similarly, Siegler and Stern (1998) report that a new strategy may be used before the child is aware that she is using it. Siegler and Jenkins offer some explanations for these observations. Firstly, a new strategy is likely to be more taxing of the mental resources than an existing and relatively well practised one. Thus, there would be fewer resources available to monitor and report on the processes involved with solving the problem. Also, Siegler and Jenkins argue that the new strategy may have been used amongst the partial use of the other existing strategy, and, as a result, confusing the child as to how she solved the problem. This is consistent with the idea that new strategies are constructed from existing ones. It is also possible that the child has not yet established descriptive labels to describe the new process. In other words, this unawareness may be no more than a language problem.

The longer solution times observed by Siegler and Jenkins (1989), on the trial immediately before the discovery of the min strategy suggests that there may be some cognitive interference or conflict taking place. Perhaps Siegler would now state that the inarticulate behaviour is the result of the competitive interaction between the associative and metacognitive systems. The metacognitive system has to increase its bid on the problem solving process leaving fewer resources to monitor the process and preventing the child from giving a coherent report. Presumably, the new strategy would be adopted after a successful bid and the slow response time immediately before the discovery is the result of increased competition between the associative and metacognitive systems.

The context in which the child is taught and practices her maths skills influences not only performance (in terms of the selection and execution of strategies) but also the development of maths abilities. It appears to do this in two different ways. Claxton (1998) makes the distinction between the deliberate conscious way of thinking and the playful unconscious way of thinking. It is the unconscious playful way of thinking that may be involved in the recognition of patterns that are used to discover new strategies. This way of thinking is associated with a more relaxed and contemplative mode. However, under conditions of stress, attention is constricted. The person switches to the deliberate mode of thinking and the *tortoise mind* shuts down. Claxton cites evidence that indicates that noisy and hot environments have the same type of effect. It seems possible that noisy and crowded classrooms may hinder the type of thinking that may be an important aspect of strategy discovery. Thus, the context seems to partly determine the type of thinking that the child employs.

The context influences the child's performance and development in at least one other way. It appears that the conditions in which the child tackles her problems can impact on her cognitive performance by reducing the available working memory resources. Case (as cited in Adams & Hitch, 1998) argues that the concept of working memory plays a critical role in both cognitive performance and development. He suggests the temporary information storage and mental operations compete with each other for limited working memory space. Case (1985, as cited in Adams & Hitch) believes that the speed of the operation reflects the amount of work space that was required, with fast operations indicating that less work space was needed. Towse and Hitch (1995, as cited in Adams & Hitch) offer an alternative explanation for the speed - work space relationship. They suggest that the observation is the result of working memory decay. Faster operations result in less decay (hence the finding, cited by Dehaene (1997) and reported in Chapter Two, that shorter number names enhances memory). This issue is not yet resolved, however, we can conclude that for whatever reason, faster operations will be less affected by working memory constraints. Procedural solutions will require more working memory space than a non procedural method would, such as retrieval. Situations where the child is required to attend to more than one task at a time may overload the working memory system and cognitive processing is disrupted.

Anxiety is one example of a factor that demands limited working memory resources (Ashcraft, Kirk & Hopko, 1998). Anxiety, whether it is specific maths anxiety or generalised anxiety, draws resources away from the working memory system, by focussing the system on the object of anxiety, or the consequence of failure, that otherwise would be available for processing. Anxiety disrupts cognitive processing that relies on working memory and it appears that the greater the reliance on working memory (the slower the operation) the greater the disruption. The anxious child is required to attend to the task as well as their anxiety which means that their mental processing and temporary memory operations have less of the work space. Ashcraft, Kirk and Hopko speculate that the anxious child will not develop a full repertoire of maths strategies because the extra load prevents full mastery and learning. Thus, it appears that classroom situations that involve excessive anxiety will hinder both the performance and development of the child. Also, this demonstrates how an unsuitable home environment may impact on classroom performance. Given South Africa's social, political and economic history, it seems reasonable to assume that many children suffer from anxiety in one form or another. This may be just another example of a disadvantage that South African children are required to overcome. How children respond to these cognitively demanding situations is a question that is considered in the empirical section of the present study.

The second research question formulated in this study tests whether children who temporarily experience cognitively stressful situations will be more likely to resort to strategies such as retrieval, because they require less of the work space and are less susceptible to decay or, alternatively, use some sort of external counting aid, such as their fingers, because this extends their limited working memory capacity. This hypothesis, however, is not supported by the Siegler's (1990) observation that children who are told to emphasise speed over accuracy, or vice versa, do not choose different strategies, but rather execute them faster (when speed is emphasised) or more carefully (when accuracy is emphasised).

The advances made in this domain of cognitive science have not yet been translated into benefits for the way children are taught arithmetic. If we assume that the discovery of the full repertoire of addition strategies, or at least all of the sophisticated ones, is associated with a greater understanding of the number concept, then understanding the factors that facilitate the discovery process would benefit educators. So it seems that one way of ensuring that children develop the advanced strategies is to ensure that they gain experience solving a sufficient quantity of problems under suitable conditions. In other words, practice makes perfect.

3.8 Summary

The staircase metaphor has historically been useful in describing the essences at the different ages. Children's abilities at the various stages have been well documented and have, for the main part, withstood the multitude of studies that have tested these age-related essences (Halford, 1989). This simplicity has come at a cost - the cost of overlooking the extent of the variation that occurs and, furthermore, makes understanding the shift from one stage to the next impossible. Explaining how the child progresses from one level to the next has not been achieved because the gulf between the stages does not exist.

Siegler's work is based on the strategy as his unit of analysis. His major area of concern is to explain the extent of the variation that he and others have observed. Something that the stage theorists have failed to capture. For Siegler the development of addition ability involves changes in the child's strategy mix, with new strategies being added and old ones being erased, while at the same time the individual strategies are used more effectively and more or less frequently. Siegler believes that the selection of the strategy is purposeful. Accordingly, children attempt to vary their choices in response to differences in the nature of the problem as well as the situation.

Siegler has adopted an evolutionary approach to explain this variation and hopes that this will end the stalemate that exists with regard to developmental mechanisms. However, the variation does not appear to be as random as one would expect from an evolutionary model but, instead, may be closer to Piaget's structured logico-mathematical progression. Accounting for how the child benefits from the variation is only one aspect of the puzzle. How the child develops or discovers new strategies in the first place is not yet well understood.

CHAPTER FOUR

THE RELATIONSHIP BETWEEN CONCEPTUAL UNDERSTANDING AND PROCEDURAL KNOWLEDGE IN CHILDREN'S ADDITION

Piaget has contributed much to the understanding of the acquisition of the number concept while for Siegler, the focus has been on the development of mathematical procedures, particularly addition procedures. This raises the question of whether the differences between Piaget's *staircase* theory and Siegler's *overlapping waves* theory have more to do with the different focus that each theorist has taken rather than any fundamental theoretical contradictions. A more complete understanding of mathematical development will come when we better understand the relationship between Siegler's and Piaget's theories or, put differently, the relationship between procedural and conceptual knowledge. Procedural knowledge refers to the actual methods that children use to solve arithmetic problems. Conceptual knowledge, on the other hand, refers to a deeper understanding of our number system. It is possible that this understanding is used to construct economical problem solving procedures. The issue has some practical significance in that educators have fluctuated between emphasising one type of knowledge over the other. If we are ever able to describe this relationship more precisely then we may begin to determine the most appropriate classroom mix.

4.1 The Relationship between Conceptual and Procedural Knowledge

It is generally accepted that the two types of knowledge develop in tandem, however, the exact nature of the interaction is not well understood. Rittle-Johnson & Siegler (1998) argue that the nature of this relationship is limited to four possibilities:

1. Procedural knowledge develops before conceptual understanding;
2. Conceptual understanding comes before procedural knowledge;
3. The two develop concurrently; or,
4. The two develop iteratively. This fourth point indicates that small increments in one would lead to small increments in the other, which would then make possible further gains in the first one. This description sounds very much like Piaget's concept of *horizontal decalage* where structural unevenness results in disequilibrium, which leads to further growth in the less developed structures. One could draw an analogy with the development of optics. This led to the development of the telescope which, in turn, greatly expanded our knowledge of astronomy. Each new discovery made others possible.

Rittle-Johnson and Siegler (1998) point out that the research that has been conducted in this domain has been limited to addressing the first two hypotheses only. The remaining two possibilities have, so far, not been examined. This may be because we do not yet have the methodological tools to do so. Or, put differently, exploring the exact nature of the relationship is a difficult research goal.

4.1.1 Counting

The relationship between conceptual and procedural knowledge has been extensively explored in the domain of counting (although only in terms of the first two possible relationships listed above). Despite this relatively thorough research there still remain two prominent and competing theories (Wynn, 1990; Rittle-Johnson & Siegler, 1998). The first of these positions is the *principles before* theory of counting developed by Gelman and her various colleagues (for example, Gelman & Meck, 1983; Gelman, Meck & Merkin, 1986) argues that concepts come before procedures. Gelman and Gallistel (1978 as cited in Gelman & Meck, 1983 and Gelman, Meck & Merkin, 1986) proposed that five counting principles guide the development of the child's counting ability. These are:

1. The one-to-one principle - every item in the set receives a unique tag;
2. The stable order principle - the tags must be assigned in the same sequence;
2. The cardinal principle - the final tag is equivalent to the symbol representing the number of items in the set;
4. The abstraction principle - any group of objects can be tagged; and,
5. The order irrelevance principle - the objects in the set may be tagged in any sequence as long as the other four principles are not violated.

This theory of counting contends that counting principles exist in children before they begin to count as some sort of innate ability. Perhaps this is similar to Siegler and Jenkin's (1989) concept of the goal sketch, although the *principles before* theory is not the one that Siegler supports. Also, it could also be argued that the goal sketch develops from these same 'innate' counting principles. In other words, the reason that children appear to understand addition principles as soon as they begin to add is because they already understand basic counting principles and addition is really an extension of their counting ability. The recent support for innate numerical ability (e.g. Butterworth, 1999; Dehaene, 1997) provides additional backing for the *principles before* position.

The current alternative is the *principles after* theory of counting suggesting that procedures come before concepts. Briars and Siegler (1984 as cited in Rittle-Johnson & Siegler, 1998) are amongst some of the supporters of this view. Essentially, they argue that counting is first learned, possibly through imitation, as a routine activity. This activity is later generalised to other situations and gradually these principles emerge. Many routine activities are learned in a rhythmical (rather than linear) fashion, and counting is probably no exception.

Both positions acknowledge that young children have an advanced conceptual understanding of the early developing numerical domains, regardless of whether this understanding comes before or after procedural competence (Rittle-Johnson & Siegler, 1998). However, this advanced conceptual understanding is lost as children progress further into the mathematical curriculum. This is regarded as a paradox, stimulating much research. (It is possible that this decreasing conceptual understanding is associated with children's declining mathematical enthusiasm.) In grammar learning, children are first correct, then incorrect before being correct again. But the initial correctness is learning without understanding. In this numerical domain, however, it appears that this early correctness is with understanding.

The *privileged domains hypothesis* maintains that our early developing numerical abilities are evolutionary privileged. These skills have evolutionary importance and support the survival of the species. It is suggested that somehow we have an innate ability to acquire them, which is why young children appear to be so conceptually advanced.

The *frequency of exposure hypothesis* is described by Rittle-Johnson and Siegler (1998), as well as others, who contend that these early developing competencies develop because children have a great deal of opportunity to observe and apply them. Perhaps the concepts are acquired early for no other reasons than that they are relatively simple concepts and children are often exposed to them. This seems to be a more easily tested and explained hypothesis.

4.1.2 Single Digit Addition

Single digit addition, like counting, is regarded as an early developing competency and, as such, a privileged domain. However, the relationship between children's conceptual understanding and their procedural knowledge has been less extensively examined. Some research has inferred children's conceptual understanding indirectly from their procedural ability. This is problematic since Siegler and Jenkins (1989) demonstrate that children use a variety of strategies on any given problem. This means that they do not necessarily choose an advanced strategy even when they are capable of doing so, which could result in an under estimation of their conceptual ability (referred to as a *production difficulty* in the literature (Kluwe, 1990)). On the other hand, Baroody and Gannon (1984) argue that using the min strategy may reflect nothing more than the recognition of an economical addition strategy. This implies that children may use strategies without the conceptual understanding associated with them. Both of these are possible.

Both Baroody and Gannon (1984) and Cowen and Renton (1996) have explored the relationship between the concept of *commutativity* (the mathematical principle that states that $a + b = b + a$) and the use of the min strategy (an addition procedure where the order of the addends is reversed when necessary so that the child counts on from the larger addend by the number represented by the smaller addend). Not surprisingly, both studies indicate that there is a positive correlation between understanding commutativity and using the min strategy. Both studies report that some children display an understanding of the commutative principle before they acquired the use of the min strategy suggesting that the concepts come before the procedures.

Canobi, Reeve and Pattison (1998) detected a similar relationship between commutativity and use of the min strategy but also extended their study to include an examination of the relationship between various numerical concepts and the strategy of decomposition (an addition procedure where the problem is broken apart and recombined in a more manageable way). They explored whether the concepts of *additive composition* (the principle that states that numbers are composed by addition and can be decomposed in various ways) and *associativity* (the principle that states that a + (b + c) = (a + b) + c) are related to the strategy of decomposition. The researchers presented a series of addition problems to their subjects on a computer screen. With each new problem the previous problem and correct answer would remain on the screen. The problems were arranged in a sequence that provided conceptual clues to the subsequent problem. The children's conceptual understanding was assessed by their ability to use the conceptual clues from the previous problem. Also, the subjects were later asked to explain how to solve the problems to a puppet. Their results support the view that an understanding of the concepts of *additive composition* and *associativity* are related to the use of the decomposition strategy and, as already noted, understanding the concept of *commutativity* is related to the use of order indifferent strategies (such as the min strategy). Also, the results support the view that the order indifferent strategy comes before decomposition. Furthermore, they found that an understanding of these number concepts is related to the greater the use of retrieval as well as faster response times and greater accuracy.

One interpretation of the relationship between conceptual knowledge and the use of retrieval is that children who are able to understand the relations between the problems are better equipped to store and retrieve arithmetic facts (Canobi et al. 1998). Perhaps decomposition and the ability to reverse the order of the addends facilitates the use of retrieval. The authors do not mention that being able to reverse the order of the addends would almost halve the number of simple arithmetic facts that must be stored. A second interpretation is that the ability to store and retrieve arithmetic facts may somehow facilitate the understanding of conceptual knowledge (Canobi et al.). A third interpretation, one not mentioned by the authors, is that perhaps conceptual knowledge and the ability to store and retrieve arithmetic facts both develop independently with greater problem solving practice. The finding that suggests that children who have a greater conceptual understanding also tend to be more accurate, faster and flexible in terms of strategy choice could be explained in the same way - as the consequence of greater practice.

It is usually assumed that the *cardinal word principle*, which refers to the discovery that the last word in a counting sequence represents the total of the set, is associated with the various counting on strategies (the count from first strategy and the min strategy). Fuson, 1982, (as cited in McShane, 1991) makes a distinction between a count-to-cardinal connection and a cardinal-to-count connection. The first is the cardinal word principle that has been discussed and refers to an action already completed. The second predicts an action that can be carried out. It is the second connection that is involved in the construction of these more economical strategies.

If we extend Claxton's (1998) argument that there are two ways of thinking - the faster *hare brain* or the slower *tortoise mind* - or Crowley et al.'s (1997) metacognitive-associative negotiated interaction argument to the conceptual-procedural debate, then we would predict that children who are able to verbalise, and thus be conscious of, certain whole number concepts would also be more likely to use them to solve addition problems. This however, does not mean that children who are unable to describe or explain the concepts are unable to use them. Presumably, their *tortoise mind* has long recognised the patterns that correspond to the concepts and perhaps has used the patterns before the child is even conscious of this. It is only later when the *hare brain*, to use Claxton's description, or the metacognitive system, from Crowley et al.'s model, eventually detects these patterns (that the *tortoise mind* or associative system have already recognised) that the knowledge can become conscious.

Other evidence for the concepts before procedures view comes from Siegler and Jenkin's (1989) observation that children do not generate flawed addition strategies. As noted, children appear to understand the principles of addition and to only generate addition strategies that comply with these principles as soon as they are able to add. They must have some sort of early conceptual understanding of addition. If we think of addition as type of fast counting, then the early conceptual understanding of addition could be a consequence of being able to count.

Table 4-A lists the conceptual advances associated with the various calculation backup strategies described above. Also, the table includes the predicted completion times for each of the strategies listed according to their definitions discussed in Chapter Three. Therefore, given the addition problem $x + y$, the strategies can be described according to Table 4-A below.

Table 4-A. The conceptual complexity and completion times for the backup strategies.

Strategy	Conceptual Complexity	Completion Time
Sum	The child is unsure about number. Need to count addends in order to determine their size.	$x + y + (x + y)$.
Shortcut Sum	Able to hold both addends and count.	$x + y$.
Count from first	Able to hold both addends and count. Apply the count-to-cardinal connection.	y.
Min	Able to hold both addends and count. Apply count-to-cardinal connection. Apply commutative principle.	The smaller of x and y.
Decomposition	Able to hold both addends and count. Apply count-to-cardinal connection. Apply commutative principle. Apply associative principle. Apply principle of additive composition.	Less than the smaller of x and y. Decomposition refers to various strategies and, therefore, the completion times will differ according to the way the problem is decomposed.

Table 4-A provides a conceptual difficulty metric along with the completion times for each of the backup calculation strategies. As the completion times for each strategy decreases, the conceptual complexity increases. Halford's (1987) theory argues that advances in conceptual understanding is achieved by greater structure-mapping ability. Therefore, the increasing conceptual complexity described in the table above parallels structure-mapping advances.

The Piagetian approach implies that concepts come before procedures and that concepts dictate the procedures that are used. Halford's (1987) neo-Piagetian theory, discussed in Chapter Two, proposes that structure-mapping advances facilitate greater conceptual understanding. The conceptual understanding determines procedural knowledge. The greater information-processing demands made by more complex structure-mappings is compensated for by increased conceptual ability and the use of more sophisticated strategies.This does not necessarily mean that children do not ever use procedures without the associated concepts. However, in this case they have knowledge without understanding.

This particular sequence of concepts before procedures is not a universal mathematical sequence. Historically, in the development of mathematics, procedures often come before concepts. An example of this is calculus. Rittle-Johnson and Siegler (1998) assert that procedures come first in the domains of counting and fraction multiplication (although, as noted, the domain of counting is still open to debate) while concepts may come first in single digit addition, fraction addition and proportional reasoning and the order of acquisition in the domain of multi-digit addition and subtraction appears to be variable. All that can be generally concluded is that there appears to be a relation between procedural and conceptual knowledge and perhaps an order of acquisition. Since two of the possible relations (the possibility that the two may develop concurrently and the possibility that they develop iteratively) have not been explored, the exact nature of the relationship is still not well understood. It could also be argued that procedures without the associated concepts is knowledge without understanding.

4.2 The Staircase versus the Overlapping Waves

This leaves us with the question of whether Siegler and Piaget's descriptions are complimentary or competing theories of cognitive development. Siegler demonstrates that children at a given age think in multiple ways on classroom tasks, however, one can not necessarily assume that they also do this on classical Piagetian tasks, particularly his logical tasks. How can the path of change on Piagetian tasks be best characterised? Siegler (1995) has attempted to answer this by revisiting one of Piaget's classical experiments, and in doing so, sheds more light on the union between his and Piaget's theories. Siegler adopts a microgenetic, or micro-developmental, study of Piaget's number conservation task. Microgenetic studies involve a high density of observations over a relatively small period. While traditional stage theories have depicted change as being abrupt, the various microgenetic studies have suggested that it is more gradual (Siegler, 1995).

4.2.1 The One-to-One Correspondence Experiment

Siegler's (1995) conclusions dispute Piaget's (1952) depiction of the development of number concept through three stages. For Piaget (1952) level one children's perceptions govern their evaluation of quantity, while at the second level the perceptions compete with conservation and at the third level the ability to conserve frees children from their perceptual limitations. Siegler's (1995) results indicate that individual children and the group as a whole are best described as using a variety of reasons for their quantity evaluations throughout the Piagetian sequence, both before and after the level three type reasoning has been discovered. Siegler (1995) claims that Piaget and his colleagues over simplified the developmental sequence and did so because of the limitations of the traditional experimental methods. Siegler (1995) states:

> The monolithic character of these depictions seems unlikely to have sprung from any deep conviction that all children progress through the same path of change. Instead, it seems attributable to traditional methods not yielding sufficiently rich data to differentiate among individual children's change patterns, and to investigations therefore having little to say about the variability of the change process (p.233).

4.2.2 Variability

For many years Piaget's theory offered a paradigm for the study of cognitive development. The cognitive variability, that Siegler (1994) claims is a important aspect of cognitive functioning, was previously regarded as measurement error, something to be eliminated or classified as *decalage.*

It is precisely this variability that has now become crucial, in a Gestalt-like switch that, according to Kuhn (1970 as cited in Grannott, 1998), often characterises a change in the dominant paradigm.

This variability, according to Siegler (1996), is a vital part of understanding and explaining cognitive development. Siegler (1995) claims that the children who learned best on the number conservation task described above, continued to display cognitive variability within and between trials. Also, Siegler (1995) found that learning appears to be enhanced by having children explain the reasoning of others. Thus, on a practical level, we may be able to improve children's learning by encouraging them to explain the reasoning of others.

Variability is a characteristic of human action and has been described by a number of different researchers. Harry Collins (1992), for example, illustrates the importance of the role that variability plays in human behaviour in his critique of artificial intelligence. While it is not necessary to detail Collin's entire critique, it is worth noting that variability in behaviour is one aspect that separates humans from machines. It takes a great deal of training for a sports player, for example, to deliver a consistent stroke. One only needs to think of the amount of time that golf players spend practising their shots. Beginners will tend to vary from stroke to stroke, while experts are able to deliver relatively consistent shots. Computers, on the other hand, are able to execute a procedure in exactly the same way over and over again. This topic is discussed by learning theorists such as Skinner, where it serves as the source of a variety of operants. The point is that variability is an integral aspect of human behaviour.

For Siegler (1994) cognitive variability has several aspects: it occurs between children of different ages; between different children of the same age; within the same child when presented with similar problems; within the same child when presented with the same problem on more than one occasion (and this can not necessarily be explained by learning since the more advanced strategy is often used first); and within the same child on the same problem. Indeed, a new generation of research has documented the extent of this variability across many different domains. Siegler (1994) proposes that variability contributes directly to cognitive development and cites varied evidence to support this.

He points out that variability in simple addition is especially pronounced on the trial immediately before the discovery of a new strategy and on the trial when this new strategy is first attempted (Siegler, 1994; Siegler & Jenkins, 1989). (The latter point should be obvious since the use of a new strategy must increase the variability.) Thus, there is a proximal relationship between variability and strategy discovery. Also, children who display greater variability tend to be better learners (Siegler 1994; 1995). Furthermore, Siegler (1994) believes that many modern researchers in the area of cognitive development have found it necessary to incorporate some degree of cognitive variability into their models. Moreover, Siegler argues that if variability is related to learning, then thinking would be most variable during the period in our development when learning is more important than performance (Siegler, 1994). Apparently this is the case during infancy and childhood.

Perhaps then, according to Siegler (1994), this is why infants and children up to the age of seven have more synaptic connections than older children and adults. It is conceivable that this abundance of neural pathways is the source of this observed variability. Greenough, Black and Wallace (1987) describe a Darwinian process of selection where the connections are gradually pruned. Moreover, this mechanism is involved in both the processes of learning and development, suggesting that these two processes are more closely related than previously thought.

The notion of cognitive variability appears to be a major difference between the Piagetian tradition and Siegler's theory. The difference may have much to do with the fact that Piaget adopted a macro-perspective by focussing on variation in cognition across childhood, while Siegler has adopted a micro-perspective by concentrating on the variation within the stages described by Piaget.

4.2.3 Learning and Development

Siegler and Piaget differ in terms of how they view the processes of development and learning. Piaget frequently made the distinction between learning and development, regarding the two as fundamentally different processes. For Piaget, development involves the active construction of knowledge while learning referred to the passive formation of associations (as cited in Siegler, 2000). Siegler recognises the difference between these two processes, however, the distinction, according to him, has become increasingly blurred. Development, according to Siegler (2000, p.32), refers to change that is universal, species specific, occurs over a longer period and in response to a "broader variety of experiences", while learning appears to be the opposite. The rise of Piaget's theory saw the decline of learning as an area of research. Siegler (2000) believes that his theory is one of the few modern theories that have addressed the issue of learning directly. He argues that the two processes share the same underlying psychological and physiological processes. Siegler (2000) cites the research produced by Greenough et al. (1987) that suggests that both learning and development involve an initial increase in the number of synaptic connections and a later pruning in support of his argument.

4.3 Summary

Cognitive development involves both procedural and conceptual advances. There is evidence that indicates that the two types of knowledge are to be linked to each other, however, the exact nature of the relationship between the two is not yet well understood. In the domain of simple addition, many conceptual discoveries appear to precede procedural breakthroughs. This suggests that children's conceptual understanding largely determines their strategy repertoire. It seems possible that advanced strategies can be taught to children who are not conceptually ready for them, however, it is also likely that these strategies would often be used inappropriately and would not generalised to new situations that warrant their use.

One of the fundamental differences between Piaget's and Siegler's theories has to do with the notion of cognitive variability. Children's cognitive actions appear far more variable than Piaget originally thought and this variability appears to play an important role in subsequent learning. Thus, Piaget's monolithic progression may be overly simplified, although Piaget was probably more concerned with a macro-perspective of cognitive development, which did not reveal the detailed patterns demonstrated by Siegler's micro-analysis.

CHAPTER FIVE

AIMS AND METHOD

This chapter outlines the aims of the study before describing the research methodology. The microgenetic approach, detailed in the method section, differs from the usual longitudinal study in that the data collection is much denser and usually occurs over a shorter period of time. This type of research method is beginning to generate useful developmental data. A recently developed statistical measure, prediction analysis, is described and, hopefully, justified. Prediction analysis appears ideally suited to the microgenetic method. A reliability measure, Cohen's Kappa, is also reported in the method section. This statistic was calculated in order to assess the degree of observer agreement in the categorisation of the participant responses and the result provides some reliability for the method as a whole.

5.1 The Aims of the Study

The broad objective of this study is to evaluate some of the essential features of Siegler's theory of addition strategy development. Although essentially exploratory, two research questions have been specified along with a number of other research objectives. The term *exploratory research*, unfortunately, is often used to justify poor research methodology. In this particular domain, however, the current models and methodology are generally new and it could be argued that they are not yet well developed. Therefore, the most appropriate approach to take is an exploratory one.

5.1.1 Objectives

The broad objective of the present study is to attempt a replication of some of Siegler's research findings. These include:

1. Determine whether the microgenetic methodology, that Siegler and Crowley (1991) advocate, offers a feasible method of studying development;
2. Examine the degree of cognitive variability in children's addition strategy choices;
3. Consider how well Siegler's classifications of 'perfectionist', 'good students' and not-so-good students' describe the participants in this study;
4. Explore why children develop an extensive repertoire of addition strategies;
5. Evaluate Siegler's (2000) depiction of the addition strategy development pathway; and,
6. Consider the benefit associated with developing more advanced strategies

During the course of this research it became possible to formulate two testable research questions.

5.1.2 Research Questions

1. Does the principle of least effort describe the selection of children's addition strategies?
2. Do children under conditions of cognitive stress, execute their strategies in a covert manner in order to extend their working memory capacity, or resort to faster strategies, specifically retrieval, that require less working memory resources or are less susceptible to decay?

5.2 The Research Method

5.2.1 The Microgenetic Approach

The *microgenetic method*, according to Siegler and Crowley (1991), is an appropriate means of closely observing cognitive change. The method involves examining the change process while it happens, which then provides information on the factors associated with the change. The method has three important aspects:

1. Children are observed through the change period;
2. There is a high density of data relative to the rate of change; and,
3. A trial-by trial analysis is used to infer the strategy used on each trial and the factors associated with strategy development (Siegler & Crowley, 1991).

Conventional longitudinal studies are described by Siegler (1995) as before and after *snap shots* of the child's abilities over a long period, while the microgenetic study involves a high density of *snap shots* over a relatively short period of time. These observations should begin as close as possible to the point immediately before change occurs and continue until this change has stabilized. Change refers to the point at which the child is observed to have altered her pattern of strategy use. The approach yields both qualitative and quantitative data (Siegler & Crowley, 1991). This project differs from the usual microgenetic approaches in that there has been an attempt to manipulate some of the factors that may be associated with strategy choice. The strategy developmental process has been observed, but the conditions in which the child solves the problems have also been manipulated in order to determine the effect that this has on the strategy selection.

The research design is contingent upon being able to determine the actual strategy chosen on a trial by trial basis. The strategy that the child employed on each particular trial is inferred in a number of different ways. First, the child's overt behavior is carefully observed and recorded. This offers a direct way of determining how the child solved the problem. Second, the children are asked how they solved the particular problem. Requesting verbal reports from the children has been shown not to influence strategy choice (McGilly & Siegler, 1990, cited in Siegler & Crowley, 1991). Third, the solution times are classified according to the strategy used and subjected to a chronometric analysis.

This approach has a number of weaknesses that can be separated into either practical or statistical difficulties. The design involves a trial by trial analysis of individual children's performance and thus involves a great deal of time. Also, attempting to infer the strategy that was used requires skill and careful observation. The practical requirements of using small samples conflict with the usual statistical requirements of large ones. Finally, according to Siegler and Crowley (1991), the analysis of repeated measures data poses some statistical difficulties.

5.2.2 Analysis of Verbal Protocols

Siegler and Crowley (1991) report that the validity of verbal data as a means of inferring cognitive processes has been questioned. Ericsson and Simon (1980) have attempted to determine when and when not to use verbal reports. They argue that if the subject reports information as it enters her short-term memory it will be accurate but possibly incomplete. Information that needs to be retrieved from her long-term memory will not be reliable. Furthermore, the processes that are reported must not be too brief in duration since they will not be represented in the child's short-term memory. (LeFevre, Sadesky and Bisanz (1996) suggest that children are possibly better than adults at describing the mental processes associated with solving addition problems because the time per unit problem is far greater for children.) Ericsson and Simon further argue that if the person is required to report on the processes involved with solving a problem as the child solves the problem, and if the processes involved are of sufficient duration, then the reports will correspond with other methodologies of studying cognitive development.

5.2.3 Chronometric Analysis

As already noted, Groen and Parkman (1972) report that the size of the smaller addend in simple addition problems is the best predictor of solution times. This study wrongly assumed that children employ the same strategy on each trial. More recent work indicates that children tend to employ a variety of strategies when solving addition problems (Siegler, 1989b). Thus the size of the smaller addend is the best predictor of solution times to addition problems, as suggested by Groen and Parkman, when the subject uses the min strategy, but is not necessarily the best predictor when the child uses one of the other possible strategies. Therefore, Siegler (1989b) warns that in situations where children employ multiple methods to solve addition problems the use of a chronometric analysis to infer the strategy used is problematic. If, however, the subjects do use multiple strategies and the solution times are classified according to the strategy reported or observed, then this chronometric data can be separately analyzed.

5.2.4 The Pilot Study

A pilot study was conducted to investigate some of the practical considerations of the design. This included the development of the recording sheet and recording procedure. Furthermore, this phase was used to select the final sample of children. The data collected during the pilot study was not used in the empirical section of this thesis, but was used to determine whether or not the children would be able to comply with the requirements of the study and if their responses appeared to resemble those collected in Siegler and Jenkin's (1989) research.

The Principal of a Junior Primary school in Pietermaritzburg was contacted and permission obtained to collect the data. The school is located in a middle class suburb in Pietermaritzburg and offers pre-primary, grades one, two and three education. The first grade teachers were consulted and asked to identify learners who, they believed, were of average arithmetic ability and would be willing to participate in the research. In addition, the teachers described how they typically teach simple addition[1]. The criteria for suitability were that the candidates would not be too intimidated by an outsider asking them to solve addition problems and that they would be likely to cope with most of the simple addition problems presented. A list of simple addition problems was generated. Each of the addends was generated randomly on a computer spreadsheet. A recording sheet was designed (a copy of this recording sheet is included in the appendix) listing the problems in the order in which they would be presented. A section was included to record the actual strategy used, the verbal protocol as well as the time taken to provide an answer.

Some time was spent in getting to know the children before introducing them to the tasks. The introductory patter involved describing the research and ensuring that they were willing to participate. Twenty-one problems were presented over three sessions and twelve children, two from each grade one class, were retained for the main study.

[1]It does not appear that the teachers specifically train grade one pupils to solve simple addition problems in a multitude of ways. It seems that children are initially taught to use the sum strategy and invent new ways to solve the problems on their own. Therefore, as Siegler and Jenkins (1989) suggest, strategy discovery appears to largely be a spontaneous process.

The verbal instructions for each session were as follows:

'I am going to give you some maths sums and I want you to solve these as fast as you can, but you must also try very hard to give me the right answer. I am going to say the sums for you one at a time. Once I have told you the sum then you must tell me the answer. After you have told me the answer I will then ask you to tell me how you got your answer.'

'*Are you ready to begin?* [Answer.] *Do you understand what we are going to do?* [Respond to any queries that they may have which may involve repeating the instructions.] *The first sum is....*'

After the child has stated the answer, '*How did you get* [answer given by the participant].'

Then, '*here is the next one, are you ready?*'.

Finally, '*Thank you, you have been a great help*'.

5.2.5 The Participants

The participants in the study were a selection of first time grade one pupils at a Junior Primary school in Pietermaritzburg during 1999 who appeared to display intermediate addition problem solving skills during the pilot study. The participants all turned six between the second half of 1998 or the first half of 1999 and the average age of the participants was six-and-a-half years (6:6) at the start of the study. The *moderate experience hypothesis* (Siegler, 1996) suggests that children are likely to display the greatest variation in terms of strategy use following moderate experience with the task. Research conducted elsewhere (Siegler & Jenkins, 1989) indicates that children in their first year of schooling are likely to have had moderate experience with single digit addition. Participants were required to be able to solve at least half of the problems presented in the pre-test trials and be willing and able to report how they solved the problems. A further requirement was that the participants should not employ a retrieval strategy on more than half of the problems presented during the pre-test trials. This requirement was included in an attempt to select moderate addition problem solvers. Since the actual data sampling is an intensive one-on-one interaction that requires a great deal of time, a relatively small sample of twelve children was selected. Also, since the study involves a very fine and detailed analysis it is sufficient to select a small sample. The final sample consisted of five boys and seven girls. This sample of children actively and enthusiastically participated in the research project.

An analysis of any possible gender differences was not included in the scope of this study. However, Hyde, Fennema and Lamon (1990) in a meta-analysis of the arithmetic performance literature concluded that there are no gender differences in the overall ability. Carr and Jessup (1997) report that although there are no significant differences in the overall ability between boys and girls, boys are more likely to use retrieval while girls tend to use the overt strategies. Siegler (1989a), in a more extensive study, claims that boys and girls were spread equally over his three categories of 'perfectionists', 'good students' and 'not so good students' challenging any claims of gender differences in strategy choice.

5.2.6 The Main Study

Each child who participated in the study was exposed to the same sequence of single digit addition problems. The two addends were generated randomly on a computer spreadsheet and a list of problems was compiled. The children were individually presented with seven problems per session, one problem at a time. The problems were presented to the individual participants in one of the offices at the school. The testing patter was standardised and each child was asked to solve the addition problem, which was presented verbally, as quickly and accurately as possible. The patter used for the pilot study was used for the first phase of the main study.

The problems were presented to the children individually while at school between 08h00 and 10h00. Approximately three children were interviewed in an hour and the data collection took around twenty hours to complete. However, the data was collected so as to minimise the disruption to the participant's normal school routine and was therefore collected over a five week period. Each child was observed during the trials and then asked to explain how they arrived at their answer. All observations, answers and times were entered onto a record sheet for analysis.

The first phase of the study involved exposing the children to a variety of single digit addition problems differing in complexity. This phase consisted of three sessions where each of the twelve children was interviewed individually. Their answers and the time taken to provide them were recorded, while the strategy was noted and recorded. Data from this phase will be used to discuss how well children fit their strategies to the problems.

This was followed by the second phase where the children were required to solve the problems while engaging with a separate memory task. The children were required to store and attempt to recall a maximum of a three-digit number while simultaneously attempting the addition problem. The children were given the number before being given the problem and were then required to provide the answer to the problem and state the number. This three-digit sequence was generated randomly and reduced to either a two-digit or one-digit sequence if child had difficulty with complying with the task requirements. The problems presented were similar to the ones presented during the first phase, although only two sessions of data sampling took place. The data obtained during the second phase was compared to the first phase to determine the impact that the more memory demanding conditions had on the strategy choice and use.

It proved relatively easy for the child to report whether he or she retrieved the answer or calculated the answer. The responses were either that they knew the answer or that they counted. However, the dual task made it very difficult for the child to describe the exact backup strategy used whenever reporting that the answer was calculated. Therefore, it was decided to keep to the retrieval - backup strategy dimension. The research question is based on the hypothesis that the child would resort to retrieval more frequently in order to free up storage space for the dual task. Hence, a finer analysis of the backup strategy used was not necessary. The backup strategy was classified according to whether the child executed the strategy in a covert or overt manner (finger counting), which was not difficult to observe, since the research question suggests a shift from covert to overt strategy use.

The verbal instructions for this phase were:

'Now we are going to do things a little bit differently. I am going to give you some maths sums and I want you to solve these as fast as you can, but you must also try very hard to give me the right answer - just as we have been doing all along. But, before I give you the sum, I am also going to give you a number to remember. Then, I am going to say the sums for you one at a time. Once I have told you the sum then you must tell me the answer and then tell me the number that I asked you to remember. After you have told me the answer and the number, I will then ask you to tell me how you got your answer. Let's try one and see how it goes.'

'Are you ready to begin? The number that I want you to remember is [three, two or one digit number presented at one digit per second]. *The first sum is....* ' After the child has stated the answer, *'How did you get* [answer given by the participant]. '

Then, '*here is the next one, are you ready?*'.

Finally, *'Thank you, you have been a great help'*.

5.3 Statistical Analysis

The statistical analysis includes both descriptive and inferential statistical techniques. The relationship between benefit and strategy complexity is considered in the descriptive section. For this, each of the calculation backup strategies are assigned a conceptual complexity score according to the number of conceptual advances associated with them as described in Table 4-A. These scores are plotted against the average completion time for the strategies and both the linear and nonlinear regression equations are analysed. The first research question is tested via a prediction analysis. The criteria for the predictions were based on the developmental literature and specified prior to the analysis. The data collected during the first phase of the project is compared to the data collected during the second phase of the project to test the second research question.

5.3.1 Prediction Analysis

A prediction analysis has been used to test the first research question (that suggests that children will select their strategies to suit the problem encountered). If children do select their strategies according to a principle of least effort, then one should be able to predict the likely strategy or strategies that the child will use given the addition problem. Prediction analysis is a relatively new statistical method developed by Hildebrand, Laing and Rosenthal (1977) for the analysis of tabular data. Froman and Hubert (1980), in a review for the *Psychological Bulletin*, state that the prediction analysis method is an appropriate technique for addressing many of the hypotheses of interest to developmental psychologists. They argue that this method allows the analysis of data that was previously wasted or underutilised. Since the method does not require the assumption that the data is independent (a difficulty associated with the X^2 and many other common techniques) data collected over time can be analysed. The data is presented in the form of a two-way table, with rows equal to *R*, columns equal to *C*, and the researcher then specifies the set of cells that she or he believes will contain the observed data (hits), the remainder being misses. Hildebrand et al. present a measure of prediction success, the delta value, which is defined as:

$$\nabla = \quad 1 - \quad \frac{\sum_i \sum_j w_{ij} P_{ij}}{\sum_i \sum_j w_{ij} P_{i.} P_{.j}}$$

The population probability for any event is represented by $P_{ij,}$ while the unconditional probabilities are represented by $P_{i.}$ and $P_{.j}$ (marginal probabilities). An error measure, w_{ij}, is assigned to each cell. If the prediction identifies the cell as a success, or hit, then, the error measure is 0, while if the cell represents a miss, the error value has been set as 1. The numerator represents the observed errors while the denominator represents the expected errors. Therefore, $\nabla = 1$ - (observed errors / expected errors). The delta value represents the proportionate reduction in error. Hildebrand et al. (1977) further claim that the correlation coefficient R^2 is also an indication of the proportionate reduction of error, suggesting that the two measures can be directly compared.

A related measure, U, refers to the precision with which the predicted outcome can be located on the dependent variable (Hildebrand et al., 1977, p.26). A prediction that specifies a unique state is more precise than a prediction that specifies that the observation will lie in one a number of states. For example, the prediction that states that children will retrieve the answer to a given problem is more precise than the prediction that states that children will either retrieve the answer or reverse the order of the addends and count on. Therefore, when one compares two ∇ values, it is important to consider the precision of the predictions. One should also consider the scientific plausibility of the prediction as measured by whether a "story goes with it"(Hildebrand et al., p.205).

The delta measure, the precision and the statistical significance of delta were calculated using computer software recently produced by Alexander von Eye (personal communication, August 14, 2000), one of the leading developers of this statistical technique. Although the delta value can be calculated relatively easily on a computer spreadsheet, von Eye's programme offers a number of other measures that are not as easily calculated, particularly the statistical significance, saving a great deal of time.

According to Froman and Hubert (1980), prediction analysis possesses desirable statistical properties. The method has been developed to manage the cross-classification data that is often collected in developmental research. Moreover, this statistical technique is an appropriate method for analysing ordinal data. Froman and Hubert (p.137) predicted that this technique would become "a major addition to the inference techniques routinely applied in developmental data analysis". Despite this, the technique does not yet appear to be a well known method of analysis.

5.3.2 Test for Proportions based on the Normal Approximation.

The data in the present study has been collected under two different conditions; initially without any additional cognitive load and then with the additional requirement of storing and reproducing either a three, two or single digit number. The hypothesis is that the children will resort to greater use of retrieval otherwise they will solve the problems in a covert manner. The proportions of retrieval and then the proportions of overt backup strategy use both with and then without the additional memory task were compared to determine if the difference in proportions is significant. According to Clarke and Cooke (1983), if one has a sufficiently large sample of observations, the binomial variable will follow the normal distribution. Although this is an approximation, it is an adequate one, particularly when n is large. Using this method, one can calculate the confidence intervals of the probability that the proportions differ. A two-tailed test is used, although one could argue that a one-tailed test is sufficient, to determine if the proportions compared deviate in either direction.

5.3.3 Linear and Nonlinear Regression Analysis

A scatter plot of the relationship between conceptual complexity, scaled as one conceptual advance equals one unit (see Section 4.1.2 and particularly Figure 4-A), and average completion time (benefit) is used to search for a relationship between these variables. An attempt is made to fit a curve to the scatterplot using both linear and nonlinear bivariate regression techniques as implemented in GraphPad Prism software. The software automatically fits the best curve to the data entered. Other curves and lines can be fitted and compared to the reported results.

5.3.4 Coding

Siegler and Jenkins (1989) describe the strategies that they observed in their study. However, they do not explicitly make the distinction, for example, between use of the min strategy where the order of the addends is reversed and where it is not (i.e. when the first addend is the larger and the child counts on from the first addend by the number indicated by the second addend). In the latter case, one is not certain whether the strategy should be classified as the count-from-first strategy or the min strategy. The distinction between cases where the child reverses the order to counts on from the larger addend by the number indicated by the smaller addend and cases where she does not has been made in this thesis. For the purposes of this study the strategies were coded as follows:

1. Retrieval;
2. Min, order reversed;
3. Min, order not reversed (count from first);
4. Count from first, the order is not reversed and the first addend is smaller than the second addend;
5. Count from second, the order is reversed and the second addend is smaller than the first;
6. Sum;
7. Shortcut sum;
8. Finger recognition (this is not the same as finger counting, which is a strategy aid rather than a strategy);
9. Decomposition; and,
10. Guessing.

Each of the strategies was also coded according to whether it was executed in a covert (coded a) or overt (coded b) manner. Overt was defined as the observed use, no matter how subtly, of fingers to aid the counting process. For example, a strategy where the child reverses the order of the addends to count on from the larger addend and uses her fingers to aid her counting would be coded as an example of *strategy 2b.*

5.4 Observer Agreement Reliability

A selection of the verbal protocols (*n=50*) were presented for categorisation to an independent judge to determine the reliability of the ratings. A Cohen's Kappa value, κ, of 0.90 was obtained, which suggests that the strategies were reliably identified. Landis and Koch (1977 as cited in Everitt,1996) have provided some arbitrary benchmarks for the evaluation of κ. A Value of 0.90 falls in the *perfect range* (0.81-1.00) of observer agreement. Most of the difference that existed between the raters were easily resolved after the comparison was made. Some examples of the verbal protocols include the following: "Put six in my head then thinked two more" for the problem 6+2, indicating the use of the count from first strategy; "counted seven then four in my head" for the problem 7+4, indicating the use of the shortcut sum strategy; "just knew it" suggesting that the child had retrieved the answer; and "seven plus one equals eight then counted another three" for the problem 4+7 indicating that the child had used the min strategy.

5.5 Summary

The methodology is based on the microgenetic approach which, essentially involves an extensive trial-by-trial analysis of the strategies that a small sample of children employ to solve a limited sample of simple addition problems. This type of micro-analysis necessitates a small sample size, and the results can not necessarily be claimed to represent all children's thinking. Thus, the findings will be need to be treated with caution. The design does pose some statistical difficulties, but these are alleviated by the use a relatively new technique, prediction analysis, which is not yet a well known statistical measure, and a test of proportions based on the normal approximation, along with the various descriptive methods. The statistical analysis is the focus of the next chapter.

CHAPTER SIX

RESULTS

The results of the study should be considered as the starting point of the discussion that follows. Since the present thesis employs a exploratory design, the results are, at times, the source of further hypotheses, which will hopefully be considered in future research projects.

6.1 The Descriptive Statistics

6.1.1 Individual Participant Performance

Table 6-A summarises each child's performance. Each of the participants employed only three or four of the possible addition strategies, which suggests that there may be some constraint on the number of strategies that are chosen. The degree of variation, in terms of the number of different strategies used, does not appear to be closely related to performance ($r = 0.26$). The use of covert strategies, however, is negatively correlated ($r = - 0.46$) to performance. Thus, children who used retrieval and the overt backup strategies tended to perform better than those who resorted to the covert strategies. This fits Siegler's description of the 'perfectionist' student.

Table 6-A. Summary of each child's performance

Child	Number of Strategies Used	Average Time	Correct (n=21)	Retrieval	Covert Backup	Overt Backup
A	4	8.13s	85.7%	10.0%	25.0%	65.0%
B	3	2.96s	95.2%	19.0%	76.2%	4.8%
C	4	3.59s	81.0%	30.0%	70.0%	0.0%
D	3	3.02s	95.2%	28.6%	38.1%	33.3%
E	3	4.15s	47.6%	25.0%	75.0%	0.0%
F	4	4.39s	95.2%	42.1%	36.8%	21.1%
G	4	6.03s	100%	33.3%	28.6%	38.1%
H	3	5.63s	100%	23.8%	28.6%	47.6%
I	3	4.43s	81.0%	21.1%	42.1%	36.8%
J	4	4.59s	95.2%	4.8%	85.7%	9.5%
K	3	2.73s	71.4%	9.5%	71.5%	19.0%
L	3	5.69s	95.2%	75.0%	10.0%	15.0%

6.1.2 Descriptive Statistics for the Strategies Observed

Table 6-B summarises the use of the different strategies. Although the use of decomposition (strategy 9) always resulted in the correct answer, it was only used 4 times. Thus, the descriptive statistics for the decomposition strategy were extracted from a very small sample. The count on strategy includes strategy 3 (count on from the first addend when they should) which was utilized 70 times (30.4%) and strategy 4 (count on from the first addend even though reversing the order of the addends would be more economical) which was utilized 17 times (7.4%).

Table 6-B. Summary for each strategy

	Retrieval (1)	Reversal (2)	Count on (3 & 4)	Shortcut Sum (7)	Decomposition (9)
% Correct* (n)	96.8% *(n=63)*	93.1% *(n=58)*	86.2% *(n=87)*	81.3% *(n=16)*	100% *(n=4)*
Average Time	2.00s	4.63s	5.03s	8.80s	3.59s
Standard Deviation	1.47s	3.84s	3.47s	4.84s	1.61s
% Use (n=230)	27.4%	25.2%	37.8%	7.0%	1.7%

* This summary does not include one case where the child reports that she guessed (strategy 10) and one case where the child reversed the order of the addends to count on from the smaller by the larger (strategy 5).

If one arbitrarily defines the conceptual complexity of each of the strategies as the number of conceptual advances associated with each of the calculation strategies described by Table 4-A, then we can establish a complexity index for each of the strategies as follows:

Table 6-C. Conceptual complexity and average time

	Shortcut sum	Count from first	Min	Decomposition
Conceptual Complexity	1	2	3	5
Average Time	2.00s	4.63s	5.03s	8.80s

Therefore, according to Table 6-C, the shortcut sum strategy is assigned the value of 1, the count from first strategy the value of 2, the min strategy the value 3, and the decomposition strategies the value 5. The decomposition is assigned a value of 5 because two additional conceptual advances separate this strategy from the min strategy.

This complexity index can now be used as one of the scales in a scatterplot where strategy complexity is plotted against the average time for each of the strategies. This plot, Figure 6-A, gives an indication of the benefits associated with discovering the more advanced strategies, and thus reveals something about how the principle of least effort drives strategy acquisition. This graph is based on the assumption that each conceptual increment is equal, which may not necessarily be the case. However, the graph does provide a general indication of the relationship between complexity and completion time in the slope and the curve revealed in the graph. Thus, the exercise is a useful one, but the results must be treated with caution.

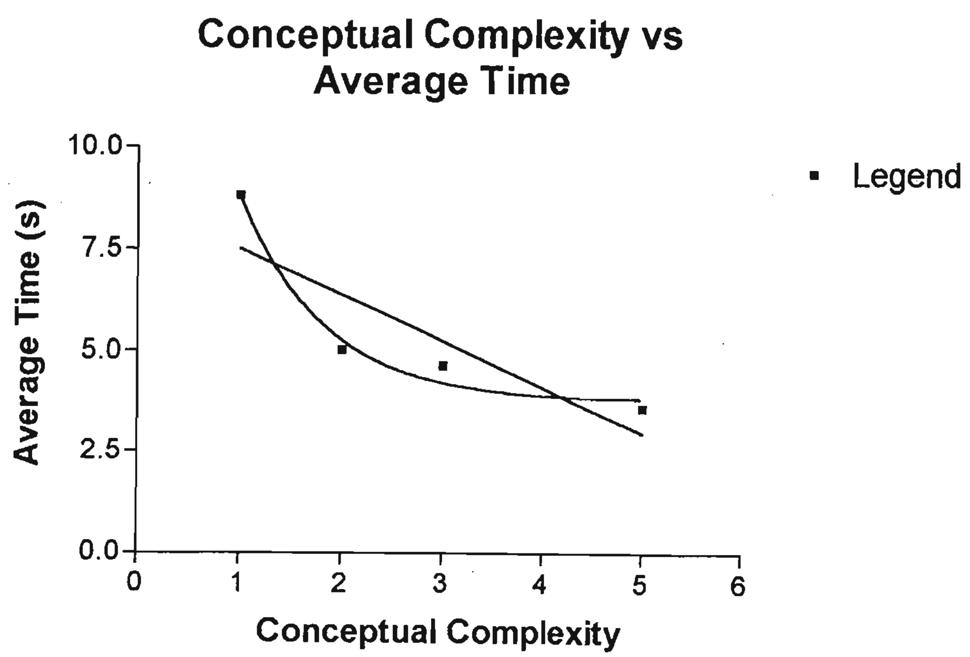

Figure 6-A. Conceptual complexity vs average time

Figure 6-A describes the relationship between conceptual complexity and the average time taken to execute each of the calculation backup strategies. The estimated linear regression line is:

$$y = -1.136x + 8.636,$$

while the goodness of fit fcr the linear equation is given by $r^2 = 0.7274$. Given that both Siegler (1996) and Halford (1987) indicate that the strategies are discovered a sequence of increasing complexity, the negative slope of the line indicates the overall benefit of developing more conceptually advanced strategies.

The estimated nonlinear regression is given by the equation:

$$y = 16.88e^{-1.218x} + 3.776,$$

which indicates that the benefit of changing strategy in the early stages is greater than the benefit of changing strategies in the later stages. The goodness of fit is given by $R^2 = 0.9823$. The graph is described an exponential decay equation suggesting that the returns (in the form of speed) associated with the conceptual advances diminish as complexity increases. This suggests that there is more benefit in terms of time saved for early advances than for later advances - a finding that makes sense if we consider that strategies like decomposition also involve additional manipulation and insight into the structure of the numbers. Later strategies like decomposition perhaps help pave the way for later arithmetic skills such as multiplication rather than simply conferring a great deal of extra speed in addition.

The correlation coefficient between conceptual complexity and average time is -0.85. Therefore, conceptual complexity, as it has been defined for the present study, accounts for 73% of the variance. It would be interesting to determine if conceptual complexity accounts for the same degree of variance in the other arithmetic domains.

6.1.3 Correlations between various Problem Indices and Latency

Table 6-D reports correlations between problem indices and latency. From the results presented in this table it seems that it is possible to correctly identify the different strategies. One would expect the min strategy to be correlated with the first addend (which would be the larger of the two) because the strategy involves counting out the first addend. Similarly, one would expect the count on strategy to be correlated with the second addend. The shortcut sum strategy should be correlated with the sum because the child begins counting at one and ends at the number representing the combined addends. Retrieval does not involve any counting and therefore should not necessarily be significantly correlated to either of the addends or the sum.

Squaring the sum improves the correlation coefficient for the strategies where time is expected to be best correlated to the size of the problem. This indicates that the sum squared is a better measure of problem size than the sum, suggesting that children solve small problems disproportionately faster than the larger ones.

Table 6-D. Correlations between various problem indices and latency

Index	Retrieval (1)	Reversal min (2)	Count on (3 & 4)	Shortcut sum (7)
Sum	0.25	0.49	0.26	0.60
Sum squared	**0.29**	0.53	0.28	**0.64**
1st addend	0.17	**0.70**	-0.23	0.06
2nd addend	0.08	0.15	**0.63**	0.59

6.2 The Inferential Statistics

6.2.1 Prediction Analysis

If children tend towards selecting their strategies to fit the problem encountered, then one should be able to predict the strategy that is likely to be used when given the actual problem. Hildebrand et al.'s (1977) method provides a means of evaluating the predictions. However, given the extent of the cognitive variation that is said to occur, one would not expect a perfect fit. The predictions are as follows:

6.2.1.1 First prediction

Retrieval when the sum of the two addends is less than or equal to 7. Siegler (1996) states that children of this age will sometimes retrieve the answers for small problems. However, he does not offer an exact definition of a small addition problem. Benford's law (described by Robert Mathews (1999) in the *New Scientist*) suggests that smaller numbers are more likely to be encountered than larger ones and, therefore, children will obtain more exposure to smaller numbers. This greater exposure means that children are more likely to retrieve the answers to problems comprised of small addends.

6.2.1.2 Second prediction

Order reversal when the first addend is less than the second addend. Groen and Parkman (1972) and Ashcraft (1982) claim that children at this age will always count on from the larger addend. Siegler and Jenkins (1989) reject this and demonstrate that the min strategy is used along with a number of other strategies. Therefore, the literature does indicate that children will employ this strategy, particularly for problems where there is a large difference between the two addends.

6.2.1.3 Third prediction

Count on from the first strategy when the first addend is greater than or equal to the second addend minus one. It is likely that reversing the order of the strategies makes some additional memory demands. If the first addend is one less than the second addend either of the min strategy (reversal) or the count from first strategy will be selected.

6.2.1.4 Fourth prediction

Retrieval when the two addends are equal (tie problems). This prediction is based on an exception to the *problem size effect*. The *problem size effect* is the most widely reported phenomenon in the mental arithmetic literature and refers to the observation that reaction times tend to be greater for larger arithmetic facts (Ashcraft, 1992). Tie problems, however, are not consistent with the size effect. Problems such as 7 + 7 are processed far more rapidly and accurately than their size would suggest (Ashcraft, 1992). More recently, researchers have suggested that the problem size effect is a misnomer (Ashcraft, 1992; Siegler, 1996). The reported relationship between problem size and response time exists only because size is related to problem difficulty. Tie problems do not fit the problem size prediction because they are easy problems to solve.

Table 6-E. Prediction analysis for the least effort hypothesis

Problem	Retrieval	Reverse	Count on	Count from one	Decomposition	Total
2+4	2	6	2	1	0	11
3+7	0	8	2	1	0	11
6+2	3	0	8	1	0	12
4+4	10	0	2	0	0	12
1+6	4	7	0	0	1	12
2+7	0	8	2	1	0	11
7+4	1	0	8	1	1	11
9+2	1	0	8	1	1	11
4+7	1	6	2	1	0	10
7+8	0	4	3	1	0	8
9+6	1	0	7	0	0	8
3+4	4	3	3	1	0	11
3+10	3	7	0	1	0	11
4+3	2	1	7	1	0	11
4+2	5	0	5	1	0	11
9+2	3	0	9	0	0	12
5+5	11	0	0	0	1	12
7+4	1	0	10	1	0	12
3+4	4	2	2	1	0	9
6+1	5	0	6	1	0	12
2+7	2	7	1	1	0	11
	63	59	87	16	4	229

The shaded cells in Table 6-E represent the *observed errors*. The *observed errors* total 45 while the *expected errors* total 122.23. The total number of observations is 229. The value of the prediction analysis, ∇, is 0.632 and the precision, U, is 0.534. Therefore, the model reduces the observed errors by slightly more than 63% when compared to an expected frequency distribution. The result is a significant one (Z = 10.231, the critical value at the 5% level is 1.645) and the hypothesis is strongly supported. Since this is a modelling technique, it would be useful to compare this to any other existing models. Groen and Parkman's (1972) min model, which states that the first grade child will always count on the larger addend by the number indicated by the smaller addend is a useful model for comparison and is reported in Table 6-F.

Table 6-F. Prediction analysis for the min model of strategy choice

Problem	Retrieval	Reverse	Count on	Count from one	Decomposition	Total
2+4	2	6	2	1	0	11
3+7	0	8	2	1	0	11
6+2	3	0	8	1	0	12
4+4	10	0	2	0	0	12
1+6	4	7	0	0	1	12
2+7	0	8	2	1	0	11
7+4	1	0	8	1	1	11
9+2	1	0	8	1	1	11
4+7	1	6	2	1	0	10
7+8	0	4	3	1	0	8
9+6	1	0	7	0	0	8
3+4	4	3	3	1	0	11
3+10	3	7	0	1	0	11
4+3	2	1	7	1	0	11
4+2	5	0	5	1	0	11
9+2	3	0	9	0	0	12
5+5	11	0	0	0	1	12
7+4	1	0	10	1	0	12
3+4	4	2	2	1	0	9
6+1	5	0	6	1	0	12
2+7	2	7	1	1	0	11
	63	59	87	16	4	229

The shaded cells in Table 6-F represent the *observed errors*. The total number of observations is 229. The value of the prediction analysis, ∇, is 0.326 and the precision, U, is 0.661.The result is also significant ($Z = 6.899$, the critical value at the 5% level is 1.645). Thus, Groen and Parkman's (1972) model, although slightly more precise than the least effort model, sees an error reduction of less than 35%. Therefore, the hypothesis that children select their strategies according to a principle of least effort results in fewer errors ($\nabla = 0.632$) than the min model of strategy choice ($\nabla = 0.326$). Also, the model that involves the adaptive selections is more 'scientifically plausible' than the min model, since much evidence has been collected to suggest that first grade children use multiple strategies and not only the min strategy. Therefore, since the ∇ value is an indication of how well the table fits the mode, one can conclude that the least effort hypothesis results in a far better fit of the data collected than the min model does.

Table 6-G. Evaluation of each of the predictions / partial hypotheses

Prediction	Partial Delta	Hits	Misses
Retrieval (1 & 4)	0.593	29 - First prediction 21 - Fourth prediction	13 - First prediction 3 - Fourth prediction
Reversal (2)	0.969	58	1
Count on (3)	0.713	76	11

Table 6-G presents the partial delta values for each of the three strategies in order to evaluate the individual predictions. For retrieval $\nabla = 0.593$, for the reversal strategy $\nabla = 0.969$ and for the count on strategy $\nabla = 0.713$. Thus, if we rank the predictions, they were most successful in identifying when the min strategy would be employed, a little less successful in identifying when the count from first strategy would be employed and least successful in identifying when retrieval would be employed. The retrieval strategy involved two predictions: The first being that children will retrieve the answers to small problems (if they do not employ a backup strategy instead); and that children will retrieve the answers to tie problems. The first prediction obtained 29 hits and 13

errors, while the tie problem prediction obtained 21 hits and only 3 errors. The prediction indicating that children will retrieve the answers to problems that are less than or equal to seven in problem size is the least effective of the predictions in reducing the error rate. The following chapter will consider where the model fits the data well and where it does not.

Finally, it is important to consider an alternative hypothesis, one that states that the previous strategy used is the best predictor of the subsequent strategy choice. It is already clear that this is not the only process involved, otherwise the success of the predictions reported above would not be expected. However, the most recently used strategy is likely to have some influence over subsequent strategy selections. This would be consistent with the various associative memory models discussed in Chapter Three and with much of the Cognitive Science literature in general. To investigate the hypothesis, the correlation between subsequent pairs of strategy choices is reported. This correlation is 0.37, which indicates that the previous strategy choice accounts for 13.9% of the variance. Therefore, since R^2 and ∇ can be compared, it is reasonable to conclude that the least effort model appears to be a better fit of the data collected when compared to the idea that children's selections are governed by their most recently used strategy. However, the least effort model is probably strengthened if this *recency effect* is incorporated. Moreover, this recency effect fits the least effort hypothesis to some extent, because using an already activated strategy saves the effort of finding another. Therefore, the recency effect is not a surprising result and does not necessarily undermine the least effort hypothesis.

6.2.2 Test for Proportions based on the Normal Approximation

The second research question is tested by comparing the trials that where collected under normal conditions with those collected under conditions of cognitive stress. The backup strategies, under both condition, were categorised according to how they were executed (covert versus overt). The hypothesis proposed that under the loaded conditions children will tend to use retrieval more frequently, or execute the backup strategy in a covert manner.

6.2.2.1 The proportions of retrieval and backup strategy use

Table 6-H. Test for retrieval and backup strategy proportions

Strategy	Normal	Loaded
Retrieval	63	38
Backup	173	117
	236	155

Table 6-H presents the frequency of retrieval and backup strategy use for both the normal and loaded conditions. The proportions of retrieval and backup strategy use is tested as follows:

NH: $\pi_A = \pi_B = \pi$(estimate); AH: π_A is not equal to π_B. A two tailed test resulted in $Z = 0.481$. The critical range for Z is -1.96 to 1.96 at the 5% significance level. Thus, there is no evidence to reject the null hypothesis. The loaded trials do not appear to influence the frequency of retrieval use.

6.2.2.2 *The proportions of overt and covert backup strategy use*

Table 6-I. Test for overt and covert backup strategy proportions

Backup Strategy	Normal	Loaded
Covert	114	35
Overt	59	82
	236	155

Table 6-H presents the frequency of covert and overt backup strategy use for both the normal and loaded conditions. The proportions of covert and overt backup strategy use is tested as follows: NH: $\pi_A = \pi_B = \pi$(estimate); AH: π_A is not equal to π_B. A two tailed test resulted in $Z = 5.123$. The critical range is -1.96 to 1.96 at the 5% significance level. The result is therefore significant and the null hypothesis is rejected. The loaded trials affect whether strategies are executed in a covert or overt manner. Children will be more likely to execute their strategies in an overt manner.

6.2.2.3 *Odds ratios*

Table 6-J. Odds ratios

Strategy	Normal	Loaded
Retrieval	0.27	0.25
Covert Backup	0.47	0.23
Overt Backup	0.25	0.53

Table 6-J presents the odds ratios to summarise the proportions of retrieval and overt backup strategy use for each of the two conditions. The child is 2.08 times more likely to use an overt

strategy in the trials where additional cognitive requirements are made, which is a statistically significant result. Children are 1.09 times more likely to use retrieval on the normal trials, which, as noted, is not a significant difference.

6.3 Summary

The results support the hypothesis that children select their strategies to suit the problem. They reverse the order of the addends when there is some advantage to doing so. Children count on from the larger addend (usually) by the number indicated by the smaller addend. They retrieve the answers to small problems. The results only partially support the hypothesis that under cognitively demanding situations children will resort to a fast strategy (retrieval) or a strategy aid (finger counting). While children do employ the finger aid counting when executing strategies in demanding conditions more frequently than usual, the same is not true for the strategy of retrieval.

CHAPTER SEVEN

DISCUSSION

This chapter reviews the results presented in chapter six in more detail. These results are addressed in a step-by-step fashion in line with the exploratory nature of the design. Each of the research questions is then addressed before turning to the general discussion. This includes some discussion on how and why children develop such an extensive repertoire of addition strategies before considering the issue of how well the present study supports Siegler's general theory. The implications of this study for educational practice as well as for future research is considered.

7.1 Review of the Descriptive Results

7.1.1 Individual Participant Performance

Table 6-A reveals that each of the children in the present study employed either three or four strategies. Considering the list of possibilities, this suggests that there may be some constraint on the number of different strategies that the child is able to use. The sample size of twelve children is very small and one cannot be very confident that this pattern is not simply a consequence of chance.

However, if this observation is accurate, then there are at least two different explanations. The first is that some strategies may simply be so superior to others that, after they are discovered, the others are rendered obsolete. One would then expect, once children have discovered these *superior* strategies, that they would all tend to use the same three or four strategies. This is partly supported by the observation that three of the strategies, all relatively sophisticated strategies, account for 83% of all strategy use.

It is possible, given the various memory limitation models of cognitive development, that there is a structurally imposed limit to the number of strategies that are simultaneously activated. In other words, the child may only be able to hold a few of the possible strategies available to immediate access. Children need to hold the strategy procedure and the details of the problem given in their limited working space. This explanation fits Halford's (1987) argument that processing capacity is a function of primary memory capacity, which is defined as memory for currently active information. Perhaps the primary memory capacity of these children is only sufficient to hold the problem details and three or four strategies available. Halford claims that primary memory increases with age and that this is what propels cognitive development.

The most significant predictor of performance appears to be the use of covert strategies. The degree of covert strategy use is negatively correlated to performance (measured as the percentage correct). Therefore, the use of retrieval and overt strategies is associated with better performance. This fits Siegler's classifications of 'perfectionist' and 'good-students'. 'Perfectionists' tend to mainly employ overt strategies, since these could be described as the safer options, while 'good-students' employ the strategies of retrieval along with the overt strategies. 'Not-so-good-students' typically resort to the covert strategies and retrieval at a lower confidence threshold, both of which are more susceptible to mistakes. Therefore, some support is offered to Siegler's categories, however, a more extensive evaluation would require a larger sample.

An interesting result concerning the notion of variability is that the number of different strategies used does not appear to be related to current performance. This, however, does not necessarily mean that the degree of variation is not related to future performance. We shall return to the issue of variation shortly.

7.1.2 Descriptive Statistics for the Strategies Observed

Table 6-B lists the descript've statistics for each category of strategy and indicates that the more advanced strategies (retrieval, min and decomposition) tend to produce answers with greater speed and accuracy than the less advanced strategies (shortcut sum and count from first). Retrieval was both the fastest and most accurate of all the strategies (if one excludes the decomposition strategy which was only used 4 times at 100% accuracy, but at a higher average latency). The ability to reverse the order of the addends to count on from the larger of the two addends is associated with better speed and accuracy than not reversing the order of the addends and counting on from the first addend. The most primitive of the strategies used (excluding the strategy of guessing) was the shortcut sum strategy which resulted in the slowest and least accurate records of all the calculation strategies used. Therefore, the use of the more conceptually sophisticated strategies appear to be associated with better performance in terms of both speed and accuracy.

There are at least two different explanations for this. First, if the discovery of new strategies is a function of experience, then the first graders who use the more advanced strategies are likely to be more experienced addition problem solvers and, therefore, more proficient. Similarly, a second explanation is that certain strategies may be more efficient than others and the use of these strategies results in better performance regardless of experience. For example, since the shortcut sum strategy involves more counting then the count from first strategy (which in turn involves more counting than the min strategy) it is more likely that the child will make a mistake using the shortcut sum strategy and take a longer time to arrive at the answer than if she used one of the more advanced strategies. Retrieval does not involve any counting and is the fastest and most accurate of the strategies used in this study. (Presumably, the accuracy of retrieval will depend on the child's internal confidence threshold - see the discussion in Chapter Three.) Therefore, having discovered the more economical strategies the child is able to solve simple addition problems more accurately and faster than before.

The latter explanation suggests that if one can teach the 'not-so-good student' to use more advanced strategies then their overall performance should improve, since performance may be related to the actual strategies that the child is able to use. However, many researchers suggest that strategies are a product of the child's conceptual knowledge (for example, Halford, 1987) and, therefore, using an advanced strategy without the conceptual foundation would be a process of learning without understanding. These strategies would probably be used inappropriately and are less likely to be generalised to other situations. Strategies may assist advances in understanding but perhaps only if the strategies are understood.

The ranking of the average latencies for the various backup strategies is consistent with Table 4-A, that lists the strategies in order of increasing complexity and decreasing completion times. Tables 4-A and 6-B both depict the order of decreasing completion times as follows: shortcut sum, count from first, min and decomposition. Therefore, this together with the sequence of strategy discovery discussed below, offers some support for Halford's (1987) structure-mapping theory.

Figure 6-A describes the relationship between the conceptual advances and average latency (Objective 5.1.1.6). These conceptual advances offer the benefit of decreased completion times. The linear regression line indicates the overall benefit of developing strategies of greater

conceptual complexity, while the nonlinear regression (curve) depicts the additional advantage of the early strategy advances. This suggests that strategies such as decomposition involve additional manipulation and insight into the structure of the numbers requiring greater time. Decomposition is the most advanced of the strategies developed and may be the most recently acquired strategy for the children who used it. Therefore, these children may not have had sufficient practice using the strategy and are not yet able to derive the full benefit from using it. Thus, at a more advanced stage of development, the relationship between complexity and average time may be better represented by a linear equation with a negative slope. Decomposition may also be a strategy that provides a foundation for later advances in arithmetic, such as multiplication, so the benefit does not only involve decreased solution times.

The graph described by Figure 6-A suggests that the discovery of new strategies fits the notion of least effort since the additional conceptual requirements result in increased benefit. The nonlinear regression, however, suggests that the benefit associated with additional strategies appears to tend towards a limit. This may be the point where new arithmetic operations are introduced to begin the process all over again.

As mentioned, of all the possibilities available only three strategies accounted for 83% of all strategy use. The strategies of retrieval, min (with reversal) and counting on from the first addend when the first addend is larger than the second were the three strategies favoured by the sample of first grade students. This is not really the pattern that one would expect given how Siegler has emphasised the notion of variability, or, alternatively, this suggests that cognitive variability occurs within limits.

7.1.3 Correlations between various Problem Indices and Latency

The results in Table 6-D generally support Siegler's microgenetic methodology in that it appears that the strategies were correctly identified (this may have been better if advanced recording and timing technology had been employed as Siegler suggests). For the strategies that involve counting, the latency (defined as the time taken to produce an answer) should be correlated with the amount of counting involved. According to Sternberg (1969, cited in Groen and Resnick, 1977), in the case where a mental process can be decomposed into a number of identical steps then the reaction time should be a linear function of the number of steps involved. For example, the shortcut sum strategy involves counting from one through to the total of the two addends. Therefore, the time taken to solve the problem should be best correlated the number of 'counts' which is indicated by the size of the sum. Similarly, for the min strategy (with reversal), latency should be best correlated with the size of the smaller addend, while for the count from first strategy, latency should be best correlated with the size of the first addend. If these patterns are not observed, then it seems unlikely that the strategies have been correctly identified.

The correlation scores calculated for the different strategies, for the most part, support the above expectations. However, an interesting deviation from this pattern is that the sum squared and latency resulted in a slightly higher correlation score than the correlation between sum and latency for the strategies where one would expect the size of the problem to be the best predictor of solution times. These results are consistent with other studies (for example, Siegler, 1996) that suggests that children are able to solve small addition problems disproportionately faster than larger ones. The relationship between size and latency appears to be curvilinear.

Obviously one would expect the child to solve smaller problems faster than larger ones since less processing is involved. Latency must be related to the amount of processing involved in solving any given addition problem. However, these small problems are solved even faster than one would expect given their size. This suggests that it is not simply the amount of processing that differentiates large problems from small ones, but also the nature of the processing. If we think of the processing involved in calculating the answer to a backup strategy as a step-by-step counting procedure, then it is difficult to understand why each step is slower when there are more steps involved assuming that the actual steps are not different.

Benford's law states that in natural contexts smaller numbers occur more frequently than larger ones (Mathews, 1999). The proportion of numbers beginning with the digit D is given by Log_{10} (1 + 1/D). Numbers beginning with the digit 1 occur in about 30% of cases, 2 in 18%, 3 in 12%, 4 in 9% and so on, with the proportion decreasing with larger numbers. One would intuitively expect that each number would be associated with a probability of 0.1, however, the distribution favours smaller numbers. Therefore, children are likely to gain more exposure with smaller numbers than they are with larger ones and, thus, be more proficient in solving them. This means that the association between problems and answers are stronger for smaller problems than they are for larger ones. According to Siegler's (1987, 1989a) distributions of associations model, the association between an addition problem and the candidate answers determines if the answer is retrieved or calculated[1]. This association may also impact on the time taken to decide between retrieval and calculation, increasing or decreasing the time taken for either process. This implies that since children are more familiar with smaller problems, they require less time to decide between retrieval or calculation.

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[1]Schunn, Reder, Nhouyvanisvong, Richards and Stroffolino, 1997 argue that the decision to retrieve an answer is based on the problem familiarity and not the accessibility of the answer. The same line of reasoning discussed in above can be applied to either model.

Dehaene (1997) argues that our mental number line represents numbers with decreasing accuracy. This number line becomes increasingly compressed. The larger the quantity, the less likely we are to express it precisely. Dehaene suggests that there is a universal decrease of number word frequency with number size. Apparently, we express small numbers more frequently than larger ones because our mental number line represents the larger ones with decreasing accuracy. Round numbers are the exception to this rule. The frequency of numbers such as 10, 12, 15, 20 and 100 are elevated compared to their neighbours. For example, the number 9, according to Dehaene, represents the exact quantity while the number 10 may include any quantity between 5 and 15. Therefore, it becomes more difficult to accurately navigate the line as the numbers increase. Thus, mental arithmetic involving small numbers is easier to process, but as the numbers get larger the processing is impeded by the compression of the mental number line.

A third factor that may be responsible for the curvilinear relationship between problem size and latency is offered by those who describe an innate number ability. Human infants and animals, according to the argument put forward by Dehaene (1997), Butterworth (1999) and Devlin (2000b), are able to perceive very small quantities without counting. This suggest that the answers to small problems are subsitized rather than retrieved. Subsitization may be a different process to reading the answer off an internal memory table and, therefore, occur disproportionately faster the process of retrieving the answer. In other words, the curvilinear relationship observed between latency and the size of the answer for retrieval may be explained as the consequence of assigning one label (retrieval) to two different strategies.

7.2 Review of the Inferential Results

7.2.1 Prediction Analysis

The results of the prediction analysis presented, which are used to consider the first research question, indicate that grade one children do attempt to select their strategies economically. In other words, grade one children employ a principle of least effort in the use of simple addition strategies. However, even though the model resulted in a significant reduction in the error, the selections are not always optimal. It is possible that children sacrifice a degree of their short-term optimality for long-term gains. Marian Stamp Dawkins (1986) discusses the link between short-term and long-term optimality, suggesting that behaviour may be optimal in the long-term sense if there is a compromise between short-term and long-term goals. Perhaps the child establishes some sort of compromise between optimal strategy selections and variability. Variability, in this sense, assists the longer-term goal of learning, but, to an extent, sacrifices the short-term goal of always producing the most accurate answer with the least amount of effort. Variability provides children with a broader spectrum of problem solving methods, which would assist them to adapt to new types of problems. Also, variability prevents stagnation. It is the variability of strategy use that enables children to approach optimal strategy selection in the first place. Therefore, one can conclude that children strive towards optimal strategy selections, but this is partially foiled by the variability of human action, which has long-term benefits.

The results also indicate that a recency effect is involved in the selection of strategies. If the child has narrowed the selection of strategies for a given problem to two options, then the most likely strategy will possibly be the one most recently used. Also, the recency effect fits the least effort hypothesis to some extent, since selecting the most recently used strategy saves the effort of finding an alternative one.

The selection of strategies, however, may be limited by children's procedural skills and their conceptual understanding, thereby restricting the choices that they are able to make. For example, if children are not sufficiently competent with the best backup strategy for the problem presented, they are likely to resort to the next best one. In other words, it seems likely that children would attempt to select the fastest strategy that they can execute successfully.

The support for the model as a whole suggests that the way grade one children approach simple addition tasks does not display the degree of variability that one would expect given the emphasis that has been placed on this particular observation. Siegler's overlapping wave model, however, would accommodate this pattern. According to this model, the variability would be determined by the number of closely competing waves at any point in time. At this stage in the individual children's development, their choices may be limited to the more advanced addition problem solving procedures, since these may be the only waves at their current position on the developmental continuum. The variation observed in the present study was often between the leading two and sometimes three strategies for any given problem. For example, the child may alternate between retrieval or the min strategy on a problem where both of these strategies are appropriate and, therefore, were predicted.

One could propose that the degree of variation decreases as the analysis becomes finer. If we examine all of the problems as a whole that were presented to the children then there is much variability in terms of how these problems were solved. However, if we examine each individual problem then the extent of the variation is much less. While this variation is certainly a feature of how children solve these types of problem, there is clearly a limit to the extent of this variation.

The patterns of strategy choice are not random, but appear to be restricted to the few best strategies for any given problem.

Before addressing each of the predictions it is important to note that there were 16 instances were a child employed the shortcut sum strategy, none of which were predicted. This child was the weakest of the participants and the use of the shortcut sum strategy suggests that she was somewhat behind the other participants in terms of her addition strategy development. Also, there where four cases were children employed various decomposition strategies, also not predicted. However, as noted, the predictions resulted in a proportionate reduction of the expected errors by 63% suggesting a relatively good fit.

7.2.1.1 *First prediction*

The prediction that children will tend to retrieve the answers to problems with a sum less than or equal to seven is only partially supported. The difficulty in predicting when the child retrieves an answer occurs because it is likely that there is a great deal of variation amongst different children in terms of which and how many of the possible answers they are able to retrieve. Retrieval is possibly related to the amount of exposure that the child has with addition problems and, therefore, reflect individual differences. Thus, children with greater exposure will tend to display the broader range of simple addition problems that they can solve with retrieval. Also, as discussed in Chapter Three, the ability to accurately retrieve answers is related to how accurately children execute the different backup strategies. The child that is able to execute backup strategies accurately is also able to build a distribution of peaked associations and, therefore, retrieve answers with greater confidence (Siegler, 1990). Also, the internal confidence threshold apparently varies from child to child which affects the range of problems that they are able to solve using retrieval.

To get an indication of where, and perhaps why, predictions are not well supported one needs to examine the error cells. This analysis will be limited to the cells with two or more errors. Three children reported that they retrieved the answer to the problem 10+3, which was not predicted. It is possible, however, that the strategy used was some type of fast notational strategy. Any number less than 10, say x, added to 10 equals a two digit number of the form $1x$. Perhaps these problems are similar to the tie problems that are easy to solve and, therefore, more likely to be retrieved. Two children reported that they retrieved the answer to the problem 2+7, while three reported retrieving the answer to the problem 6+2, neither of which were predicted. This indicates that some children will retrieve the answers to problems that are larger than a sum of seven. However, it is interesting that both of these cells are associated with problems that have answers of either 8 or 9 and are not much larger than 7. Possibly the retrieval prediction should be reformulated to state that children will tend to retrieve the answers to easy problems with the range increasing with greater experience. These problems are of sums of around 9, 8 or 7 and less, but may include larger problems that are, for whatever reason, easier to solve.

7.2.1.2 Second prediction

The prediction that if children do not retrieve an answer to a problem, then they will then calculate an answer by reversing the order of the addends so that they count on from the larger addend by the smaller addend is very well supported. This indicates that children will reverse the order of the addends when necessary. There were no cells with two or more errors in the second prediction column. Children are not likely to reverse the order of the addends unless there is an advantage to doing so. However, as will become apparent in the next section, children do sometimes fail to reverse the order of the addends when there is an advantage to doing so.

7.2.1.3 Third prediction

The prediction that children will count on from the first strategy when the first addend is greater than or equal to the second addend minus one, is well supported. This result suggests that reversing the order of the addends involves additional memory requirements and that children will do so only if there is a clear benefit.

There were, however, five cells containing two errors. Four of these error cells referred to instances where the child was expected to reverse the order of the addends but did not, choosing to rather use the count from first strategy. The other error cell containing two misses refers to an instance where the children were expected to retrieve the answer to a tie problem. This cell is associated with the only two misses that occur for the fourth prediction. The four error cells containing two misses indicates that reversing the order of the addends places additional requirements on the child and perhaps they need to be sufficiently competent at counting on from an arbitrary number before they are able to manage the process of reversing the order of the addends.

7.2.1.4 Fourth prediction

The data strongly supports the prediction that children will tend to retrieve the answers to tie-problems. The observation that children will tend to retrieve the answers to single-digit tie-problems has been well documented (for example, Ashcraft, 1992), but apparently not well understood. Tie-problems are the exception to the *problem size effect*. (The *problem size effect* refers to the observation that latency increases with the size of the problem.) Researchers have proposed that children retrieve the answers to these problems despite their size because they are somehow easier problems to solve. What is not clear is why children find these problems easier than any other addition problems of comparable size.

One could speculate that the answer has something to do with the notion of odd and even numbers. Even numbers, by definition, are composed of two equal integers. Children begin to develop a concept of odd and even numbers with their early number concept. The notion, however, is not readily generalised to multi-digit numbers (Frobisher, 1999). The concept is developed by partitioning objects into twos and noticing that sometimes there is a single object remaining (odd) and sometimes not (even). As children begin to understand the concept of even and odd numbers they may begin to associate even single digit numbers with the pair of equal sets that make up the total. Therefore, when presented with a problem comprised of a pair of equal integers, they are more likely to be able to retrieve the answer. Familiarity may have something to do with the a special patterns or characteristics associated with an addition problem and not only be about the amount of exposure that the child has had with the problem. Tie-problems are special since they are comprised of two equal whole parts and this feature means that they are stored and retrieved more easily.

Similarly, children begin to count in multiples greater than one. Children learn to count in twos, threes and so one. This would reinforce the tie problem familiarity and, therefore, increase the likelihood that their answers are retrieved. The issue of tie-problems appears to be unexplored.

On the whole, the four predictions resulted in a significant reduction in error of 63% supporting the least effort hypothesis of strategy selection. The measurement of the error reduction was made possible by prediction analysis, a technique that appears well suited to the type of data collected.

7.2.1.5 *The statistical technique*

Prediction analysis is not a technique that is frequently used in psychological research despite having been developed in order to analyse the cross-classification data that is often collected in developmental research (Froman and Hubert, 1980). Howell (1992) is commonly used as an advanced statistical text book in psychology and does not contain a single reference to the technique. The technique provided a valuable analysis of the data collected in this study, offering a type of measure that is not provided by the statistical measures that are typically used in psychological research. The use of the technique is facilitated by the software developed by, and obtainable from, Alexander Von Eye.

7.2.2 Test for Proportions based on the Normal Approximation

It is less clear how the context impacts on the selection of strategies. The hypothesis suggested by the second research question was only partially supported. Children do not appear to resort to greater use of retrieval when attempting addition problems under cognitively demanding situations. In fact, they may use retrieval less frequently than normal, although the difference was not significant. Halford et al. (1994) indicate that information can sometimes be held in short term memory without interfering with the cognitive processes. Given that there is a significant shift from the predominant use of covert strategies to the use of overt strategies for the more demanding conditions, then there must be some sort of interference taking place. However, it is beyond the scope of the present study to describe the exact nature of the interference in the multi-component working memory.

7.2.2.1 The proportions of retrieval and backup strategy use

The results of the test for proportions indicate that retrieval was not used more frequently under cognitively demanding conditions (and, therefore, the backup strategies were not used any less frequently). The nature of the task, however, may have interfered with the retrieval process reducing the frequency of its use. Halford (1987) warns that dual task experiments may involve an interference effect rather than competition for limited resources. The numbers that the children were required to recall would have been activated along with the numbers associated with the problem. This may have made choosing the correct answer more difficult than usual (LeFevre et al.,1988; Lemaire et al., 1994). Therefore, it is possible that children would resort to greater use of retrieval if the nature of the distraction task had been nonnumerical. On the other hand, the SAC model (Schunn et al., 1997) suggests that the decision to retrieve is not made on the accessibility of the answer, but rather on problem familiarity. Therefore, neither the demanding situation nor the additional activated candidate answers should affect the decision to retrieve or to compute (although, perhaps the additional anxiety associated with the dual task may affect the child's confidence threshold). This fits Siegler's (1990) observation that emphasising speed over accuracy, or vice versa, does not affect the actual strategies chosen. It may however, affect the way they are executed.

7.2.2.2 The proportions of overt and covert backup strategy use

The results do suggest that children execute their strategies in a manner that may aid their short term memory when required to solve the problems under cognitively demanding situations. They achieved this by executing the strategies in an overt rather than a covert manner. (This is a similar process where adults resort pen and paper for maths problems that may exceed their processing ability.) The use of fingers varied from the very obvious movement of the fingers to coincide with counting to the very subtle and slight finger movements that the child may not necessarily be aware of. These finger movements may be arithmetic's equivalent of speech hand gestures.

McNeil (1985) argues that in speaking, hand gestures and the speech output share the same computational stage. Gestures are a less transformed reflection of this inner process. Two hypotheses concerning the role that these gestures play have been proposed. McNeil believes that these hand gestures, or iconic gestures as he calls them, complement speech. For McNeil they serve to facilitate communication. Butterworth and Hadar (1989), on the other hand, propose that these hand gestures function to assist word retrieval. Devlin (2000b) argues that mathematical thought processes are not linguistic but visual. (There are other Mathematicians who may dispute this and the issue is not yet resolved.) It is possible then, that these finger movements serve a similar function to the speech gestures. They may be an external reflection of the internal visual process. More importantly, if we take this analogy seriously, the finger movements, no matter how subtle, may assist the child's working memory. This hypothesis is further supported by the finding that finger use and arithmetic share the same region of the brain and are, therefore, different manifestations of the same internal calculation process.

7.3 General Discussion

The study was designed to explore how children select from their repertoire of existing strategies and the above discussion is concerned with this issue. It would be useful to now return to the issues of why children have such an extensive repertoire of strategies (Objective 5.1.1.4) and how they develop this extensive repertoire in the first place. The first refers to the *why* of extensive repertoire development and the other to the *how* of development. Following this discussion we will then review the extent to which the present study supports Siegler's theory.

7.3.1 The Strategy Repertoire

The research conducted indicates that young children in their first year of school employ a vast array of different strategies and, according to the results just presented, attempt to select the best strategy for the problem presented. At an initial glance, this appears to be something of a cognitive luxury and is, therefore, not well understood. Groen and Resnick (1977), in a study of children's addition, propose that children are motivated to perform regularly encountered tasks as quickly and easily as possible. While, Klahr and Wallace (1976) argue that this is a general cognitive tendency and an important aspect of development. School going children do encounter many simple addition problems and are, therefore, likely to seek efficient problem solving methods.

A useful analogy would be the act of crossing a river. The person who is required to cross a river on a daily basis is more likely to construct a raft or a bridge. These structures correspond with the more advanced strategies. The person who is required to cross the river occasionally is likely to walk some distance to an easier known crossing point. This may not be as efficient as crossing a bridge, but does not involve as much construction. The person who is required to cross the river very infrequently and is not aware of a suitable crossing point may choose a strategy of walking upstream to find a point where the volume of water would be less. Perhaps not the most efficient strategy, but one that is bound to eventually assist the person in getting across.

Crowley et al. (1997) have recently developed a model involving both metacognitive and associative mechanisms locked in a competitive negotiation which can be used to further understand the development of more efficient addition strategies. This model is used in the present study to attempt an explanation of why and how children develop their full repertoire. Shrager and Siegler (1998) developed a computer simulation (SCADS) to test their metacognitive and associative theory. SCADS involves a mechanism allowing the mental processes to become automised. This, according to the model, then frees up 'change heuristics' that identify and eliminate redundant processing.

If we apply this process to the use of addition strategies a pattern of strategy development will emerge. The child begins to add using the first strategy taught and gradually become more proficient at using the strategy. With experience, the use of the strategy becomes automised and metacognitive resources that were previously involved in the execution of the strategy are freed and thus able to focus on the task of eliminating any redundant processing (the 'change heuristics' in SCADS). The initial strategy is modified accordingly and this new strategy is used in competition with the old one. The choice between the strategies is explained by the Adaptive Strategy Choice Model (ASCM) discussed in Chapter Three. Variability is likely to occur as the consequence of the competition between the closely matched existing strategies. In this case, the variability would probably not be entirely random but rather limited to the best strategies for the particular occasion. This idea is supported by the data collected in this study where three advanced strategies were used in 83% of all cases.

Gradually the execution of the new strategy becomes automised and the metacognitive system, once again, seeks to eliminate redundant processing. Accordingly, the first strategy used (the sum strategy) is modified in a step by step fashion with each step corresponding to a new strategy. All of the strategies coexist and compete. While the new strategies are being discovered and used, the child is able to establish stronger associations between the problems and the answer and, therefore, retrieve the answers to more and more problems.

A another useful analogy to this process of automatisation occurs in the act of learning to drive a motor car. Learner drivers have great difficulty driving because they need to consciously think of each act involved in driving the vehicle. They must consciously remember to disengage the clutch before changing gear then re-engage the clutch while, at the same time, steering the vehicle. As they become more proficient, these processes are automised and require less conscious control. Drivers become better able to review their driving and make the necessary improvements. With greater practice, the driving becomes better. Indeed, the automisation of driving is associated with the phenomenon of Driving Without Awareness, a cause of concern for traffic authorities.

Thus, the reason that children's repertoire is so extensive may be the consequence of their considerable exposure to, and practice with, small addition problems. The strategies that are generated comply with their conceptual understanding. The early conceptual understanding possibly defines the strategy parameters, while the development of new strategies facilitates deeper conceptual understanding. The processing modifications are related to the various conceptual advances discussed in Chapter Four.

This repetitive modifications process is not necessarily the only mechanism of strategy development. Children could be taught new strategies from their more capable peers and adults. (Although the use of an advanced strategy without the conceptual understanding associated with the strategy may be problematic.) It could also be argued that teaching assists the metacognitve mechanism of eliminating redundant processing by directly pointing out shortcuts and, in doing so, highlighting the unnecessary procedures.

Finally, it could also be argued that the strategy repertoire is not that extensive. If we decompose strategies into the different subroutines, then it is apparent that a few subroutines, recombined in different ways, account for the spectrum of strategies observed. For example, the cardinal-to-count connection (which involves the subroutine of counting from one of the addends), and the commutative principle (which involves the subroutine of swopping the order of the addends), and the principles of associativity and additive composition (which involve re-composing the problem into different parts), combined in various ways account for all of the backup strategies observed. With time, children are bound to discover the various strategy components and combine them in ways that make problem solving a more efficient process. This does not contradict the repetitive modification process described above, these subroutines correspond with each of the strategy improvements, the only difference being the unit of analysis. The actual sequence of strategy discovery is further described in the next section.

7.3.2 The Sequence of Strategy Discovery

This question is partly addressed by the previous one which offers a way of describing how new strategies are discovered. By applying the metacognitive and associative model to the development of new strategies it is possible to explain how the child's repertoire of strategies is extended. Moreover, if we begin with the sum strategy, it is possible to describe a number of different sequences of strategy discovery (Objective 5.1.1.5).

The child begins to add using the sum strategy. This may be because the young child learns to count with the aid of objects. When given an integer pair to add, she selects a set of the objects to represent the first addend by counting them out, does the same for the second addend, and then counts out the total made up by the two sets. Later, the child may use her fingers without the external objects. Either way, finger use is an important aspect of early addition. The part of the brain that is associated with the control of our fingers, the left parietal lobe, also happens to be the same region that is associated with arithmetic (Devlin, 2000b). (Perhaps this is why we refer to numbers as 'digits'.) With experience, the child becomes less reliant on her fingers, until this finger use is 'offline', indicating increasing automisation of the use of the strategy. This may also coincide with the discovery of more economical strategies. (However, it is possible that we do not ever totally disconnect our fingers from our thought process and these finger movements, as discussed above, may serve the same purpose as our speech gestures.) Eventually the child is able to use this addition strategy without any obvious external aids. Thus, it seems likely that the young child learns to count using the sum strategy, first in an overt manner, but becoming less overt with practice. The steps involved in the sum strategy are as follows (adapted from Crowley et al.,1997):

Step 1 - Count the first addend (x);

Step 2 - Count the second addend (y);

Step 3 - Count out the part of the total represented by x; and,

Step 4 - Count out the remainder of the total represented by y.

Once the child is proficient at using the sum strategy she may then discover that she does not need to count out each addend separately and delete Step 1 and Step 2. Thus, the child can count out the part of the total represented by the first addend (Step 3) before counting out the remainder of the total represented by the second addend (Step 4). This is the shortcut sum strategy. Siegler proposes that this strategy is a transitional strategy which links the development from the sum strategy to the min strategy. According to Siegler, the child at this point is able to reverse the order of the strategies and uses a combination between the shortcut sum and the min strategies. After this, according to Siegler's depiction, the child discovers the min strategy. Other theorists argue that the count from first strategy serves this transitional purpose. (For example, Groen and

Resnick (1977) suggest this possibility. Unfortunately, their methodology excluded the possibility of addressing the issue.) The data collected for this thesis suggests that both of these positions may be correct. The shortcut sum and the count from first strategies are possibly both transitional strategies.

The shortcut sum strategy may be the point where the developmental sequence splits, offering at least two different routes. If we consider the second view (where the count from first strategy comes before the min strategy), then after the child has discovered that she can delete Step 1 and Step 2, but follow Step 3 and Step 4 to obtain the total of the integer pair (shortcut sum strategy), she will soon be ready, after some practice at using the shortcut sum strategy, to observe that Step 3 is also redundant. This discovery is associated with the count-to-cardinal connection. The child does not need to count out the part of the total represented by the first addend (Step 3), but can rather begin at the number indicated by the first addend (x) and count on the remainder of the total indicated by the second addend (Step 4). This is the count from first strategy. Counting on from an arbitrary number greater than one is more difficult because it is harder to monitor the amount of counting already done (Fuson, Richards & Briars, 1982 as cited in Siegler & Jenkins, 1989), but the gains presumably outweigh the disadvantage. According to this sequence, both the shortcut sum strategy (without reversal) and then the count from first strategy complete the link between the primitive sum strategy and more advanced min strategy.

Having discovered these strategies and after becoming sufficiently proficient at using them, she could then begin to apply the addition principle of commutativity by discovering that the order of the addends is irrelevant to the answer. The result would be the ability to use the min strategy. The child can then reverse the order of the addends so that she begins to count from the larger addend by the number indicated by the smaller addend. In other words, they can modify Step 4 by ensuring that if x is less than y then the order of the addends is reversed. This is more economical than the count from first strategy because the child always counts from the larger addend by the number indicated by the smaller addend (and in doing so counts less). Being able to delete some of the original addition steps or apply new addition principles may involve further resource demands, however, it is likely that these are outweighed by the gains associated with less processing.

Finally, when the child has become proficient at using all of the strategies that she has discovered, she may begin to apply the related principles of associativity and additive composition to recombine units to facilitate the solution of the problem. Associativity means that the sum can be represented in different ways. Units can be shifted from one addend to the other without affecting the sum. Additive composition means that the addends can be broken down and recombined to create an additional addend. These principles, when applied, allow the child to discover the group of strategies referred as the decomposition strategies.

Thus, the sequence of discovery could be from the sum to shortcut sum to count from first to min to decomposition. This procedural sequence is associated with the following discoveries in the following order: Deletions of unnecessary sum strategy steps (shortcut sum), count-to-cardinal connection (count from first), order irrelevance of the addends (min) and the two related principles of associativity and additive composition (decomposition). This sequence is presented in Table 7-A.

Table 7-A. An alternative to Siegler's addition strategy developmental sequence

(1) Sum Strategy	(2) Shortcut Sum Strategy	(3) Count from first Strategy	(4) Min Strategy
	Drop unnecessary sum steps	Apply the cardinal to word principle	Apply the principle of commutativity
Count out the first then the second addend before counting out the sum of the two addends	Count from one to the sum of the two addends, beginning with the first addend	Counting on from the first addend by the second addend	Counting on from the larger addend by the smaller addend

The difference between the sequence described by Table 7-A, and the one that Crowley, Shrager, and Siegler (1997) have proposed, which is summarised in Table 7-B, essentially involves the point at which the child applies the commutativity principle (whether children apply the commutativity principle before or after the count-to-cardinal discovery). Reversing the order of the addends before applying the count-to-cardinal principle would certainly reduce processing when executing the shortcut sum strategy. For this strategy, the child needs to count up to the number indicated by the first addend then continue to count on by the number indicated by the second addend. In order to count on, the child needs to count her counts. This would certainly be easier if the second addend were the smaller of the two. Siegler suggests that children execute the shortcut sum strategy in this way. (They may initially count the first addend before the second and then later begin to reverse the order of the addends when necessary- thus employ two variations of the same strategy, these being shortcut without reversal and shortcut with reversal.)

However, if Siegler's sequence is correct, then we would expect children to reverse the order of the addends whenever the first addend was smaller than the second, regardless of whether they use the shortcut sum strategy or a count on strategy. They would progress from the shortcut sum strategy straight to the min strategy. However, the case where children count on from the first addend by the second addend when it would have been more economical to reverse the order (strategy 4) was observed 17 times (7.4% of all strategy use). Also, if Siegler's sequence is the only possible route then children who use the shortcut sum strategy should tend to reverse the order of the addends when necessary. However, the child that resorted to the shortcut sum strategy did not appear to reverse the order of the addends on any of the trials when it would have been appropriate to do so. The fact that we observe both the shortcut sum strategy, without reversing the order of the addends, and the count from first strategy, also without reversing the order of the addends, suggests that there is at least one other developmental pathway to the one Siegler has described. Another interpretation of this data is that children apply the principle of commutativity with the shortcut sum strategy, then stop doing so after applying the cardinal-to-count principle and then begin to apply the commutative principle again. There may be a range of pathways in the development of addition strategies.

Table 7-B. Siegler's addition strategy developmental sequence

(1) Sum Strategy	(2) Shortcut Sum Strategy (order irrelevant)	(4) Min Strategy
	1. Drop unnecessary sum steps 2. Apply the principle of commutativity	Apply the cardinal to word principle
Count out the first then the second addend before counting out the sum of the two addends	Count from one to the sum of the two addends, beginning with the smaller addend	Counting on from the larger addend by the smaller addend

The order of the other discoveries has not been disputed. None of the discoveries are useful unless the unnecessary sum strategy steps are discarded and this should therefore happen first. Also, the principles that allow for the decomposition of addition problems can be regarded as extensions of the order irrelevance principle. The order irrelevance principle involves shifting the difference between the two addends from the larger to the smaller addend (to do this, the child simply reverses the order of the two addends) while the decomposition strategies involve shifting any number of units from one addend to another. Thus, the reversal of the addends could be considered as a special case of the associative principle. Accordingly, being able to decompose problems implies that the child is also capable of swopping the addends (but not vice versa). Therefore, the order irrelevant strategies will be discovered before decomposition.

Furthermore, Halford's (1987) structure-mapping theory of cognitive development suggests that strategy discovery is constrained by the child's structure-mapping ability. This implies that strategies are discovered in order of increasing complexity and supports either sequence described above. Table 3-A lists the backup strategies in order of increasing complexity and decreasing completion time, a sequence possibly reflected by the order of acquisition. Therefore, the sequence of strategy discoveries, whether it is best described by Siegler's account or the alternative offered in the present study, seems to be congruent with a neo-Piagetian description of development.

The idea that strategy discovery involves repeated modifications of the existing strategies can account for either developmental sequence described. It is possible that some children will follow the sequence as described by Siegler while others may follow slight variations of this. Also, the sequence of discovery follows the child's increasing ability to map more complex relations.

7.3.3 The Implications for Siegler's Theory of Addition Strategy Development

Robert Siegler has made a number of new contributions to cognitive science in general and, in particular, to how we understand the development of children's early mathematical ability. On the whole, the data collected for this study seems to support many of Siegler's ideas.

7.3.3.1 The overlapping waves metaphor of cognitive development

One of Siegler's earlier contributions was the finding that children resort to a number of different strategies, even in a relatively simple domain such as single digit addition. These competing strategies correspond with the overlapping waves in his model of development. Development, according to this metaphor, is a process where children discover new strategies and discard old strategies while the existing strategies compete and the child becomes increasingly proficient at selecting the appropriate strategy for the problem given. Previous researchers believed that children would consistently use one strategy before moving on to the next (for example, Groen and Parkman, 1972; Groen and Resnick, 1977; Ashcraft, 1982). Siegler claims that young children will use around six different strategies and that the use of these strategies is variable. This study indicates that first grade children will use at least three or four different strategies and, therefore, lends support to Siegler's metaphor.

Siegler emphasises the variability of strategy use, which is an essential characteristic of cognitive development. The data collected for the present study supports the notion of cognitive variability (Objective 5.1.1.2). However, there appears to be some limit to the variability that does occur and, as the analysis becomes finer, the variability is reduced. Looking at all of the problems presented to the children one observes the use of many different strategies (at least eight different strategies were observed), but when one considers individual problems, the choices are often limited to two or three different strategies. This variability, for the most part, occurs between the most economical strategies for any given problem. However, cognitive variability does appear to be an important aspect of cognitive development and this agrees with Siegler's ideas.

Siegler's classifications of 'perfectionists', 'good students' and 'not-so-good students' is partially supported (Objective 5.1.1.3). The data collected suggests that the patterns of strategy use and the way the strategies are executed is related to performance. The use of overt strategies and retrieval is related to better performance. This pattern fits the 'perfectionists' and 'good students' classifications. To examine this issue more closely one would need to conduct a study with a larger sample of participants.

7.3.3.2 Strategy discovery

Crowley, Shrager and Siegler (1997) have proposed that strategy discovery involves both metacognitive and associative processes. The data collected in this study does not contribute any further understanding to the proposed combination between these two mechanisms. However, as discussed above, the findings support a particular sequence of simple addition strategy discovery.

Siegler proposes that the shortcut sum strategy serves as a transitional strategy instead of the count-from-first strategy. The data collected in the present study suggests that this may be the case, however, there is at least one other alternative to this sequence. It appears that the shortcut sum strategy (without reversal) and the count from first strategy both occur in the sequence presented.

A constructivist process of incremental modifications may prove to be a useful way of describing the discovery of new strategies. This could perhaps be regarded as a type of neo-Piagetian explanation. However, the description does not contradict the overlapping waves model of development. This constructivist process describes how and why new strategies are discovered, while the overlapping waves metaphor describes how the various competing strategies are used. This constructivist description is an extension of the 'change heuristic' mechanism employed in the SCADS software developed by Shrager and Siegler (1998). The constructivist models and the emphasis on variability need not be mutually exclusive. Perhaps the variability occurs around the underlying structural development. The fact that there is any debate about the exact developmental pathway suggests that a constructivist process of discovery may be a better model than the evolutionary approach to cognitive development associated with random variation and selection.

At one point, Siegler (1996) suggested that Darwinian evolutionary principles may be the best means for understanding cognitive development. He drew parallels between the history of our understanding of evolution with our increasing understanding of cognitive development, highlighting the similarities in the historical progression of our understanding and suggesting that the same conceptual breakthrough that occurred in evolution could be employed in the domain of cognitive science. Siegler may have been seduced by his observation that cognitive variation is an important feature of cognitive development and that it also holds an important role in Darwin's theory. This does not seem to be a position that he has subsequently pursued, perhaps wisely, since the application of evolutionary principles to psychology is not without controversy.

There are currently two different approaches to applying evolutionary principles to psychology. Evolutionists such as Steven Jay Gould (2000) believe that selection and variation are one of a number of different mechanisms of modification in the domains of both biology and psychology. Siegler (1996) could, perhaps, be regarded as being closer to the Dawkins (1976) approach, emphasising only the adaptation through selection and variation. These so-called ultra-Darwinians would apply a strict variation and selection approach to understanding the development of addition strategies. From this perspective, the strategy would be an equivalent to the gene, something at which Siegler (1996) hints. Through a process of variation, new strategies are generated, while through a process of selection, certain strategies are retained. Richard Dawkins

(1976) coined the term *meme* to refer to units of cultural transmission that includes things such as tunes, ideas, catch phrases and clothing fashions. It seems that the strategy may have been defined as a type of mathematical *meme*. Mary Midgely (2000), however, claims that thought cannot be granular (meaning that the unit cannot be further decomposed) and, therefore, there can be no identifiable gene equivalent. Midgely states that "if *memes* really correspond to Dawkinsian genes they must indeed be fixed units - hidden underlying causes of the changing items that appear round in the world". Strategies, like cultural *memes*, cannot be regarded as granular, since they have already been decomposed into smaller units by being described in terms of their subroutines.

A general critique of evolutionary psychology is that the approach seems to be better at describing the advantages of variation than it does at explaining the source. While biologists have had some success in explaining the source of the variation involved in the evolution of species, psychologists have still some way to go. The advantages of a degree of variation in thinking are clear. Variability appears to be related to learning. Numerous factors may play a role in motivating children to think in many different ways.

7.3.3.3 The microgentic method

Siegler has made contributions not only to developmental theory, but also to research methodology. The research methods described by Siegler and Crowley (1991) have been successfully employed in the present study yielding valuable data (Objective 5.1.1.1). The correlations between latency and the various problem indices reported in Table 6-D support the claim that children are able to accurately describe the strategies that they use to solve simple addition problems. Table 6-B, which reports the average time taken and the accuracy for each strategy further supports the validity of the classifications. Retrieval was the fastest and most accurate (excluding decomposition), the min strategy was the second fastest and second most accurate, the count from first was the third fastest and accurate strategy, while the shortcut sum was the least fast and accurate of the four strategies mentioned. This is the pattern that one would expect as well as being the pattern described in the literature (for example, Siegler, 1996). If the strategies were not correctly identified then it is unlikely that one would observe this performance hierarchy. Thus, the time and accuracy results suggest that the strategies were correctly identified.

7.3.4 Educational Implications

Since the present research involves a school task it is important to consider the educational implications, if any such implications exist. If we accept Devlin's (2000b) point that arithmetic is the foundation of higher mathematics, then obviously it is important to allow children to develop a full conceptual understanding of arithmetic. Halford (1987) suggests that the child's structure-mapping development constrains their conceptual development, which, in turn, constrains the strategies that the child is able to use. Therefore, one could argue that developing this repertoire is a necessary component of a complete conceptual understanding and that children should be encouraged to develop multiple problem solving techniques.

7.3.5 Future Research

It is assumed that higher mathematics is built on the foundation of arithmetic. A useful research question to consider would be the degree to which the early development of addition strategies predicts later mathematical success. It has been shown that children who display greater variety in the strategies chosen learn and perform better on that particular task. This question will consider whether this success is carried through to higher domains.

There are very definite advantages for children who are able to develop and understand the various strategies available. Thus, future research could focus on how to facilitate the development of this. It has been suggested in the literature that one way of doing this is to encourage children to explain how somebody who is older and more competent might solve a given problem. Perhaps greater emphasis could be placed on encouraging children not just to solve their problems but to do so in different ways. These ideas could be relatively easily tested.

The scientific study of arithmetic development would benefit from a longitudinal, but also very detailed, study tracking the sequence of strategy discovery in a sample of children as they progress from their pre-primary year to the second or third grade. The logistical difficulties of this type of study would be enormous. However, current knowledge is based on fragments collected at various developmental levels so this type of study would be a important contribution. Furthermore, the patterns of strategy use in the other branches of arithmetic (subtraction, multiplication and division) have not been well explored.

Children are increasingly required to use calculators at school. It would be interesting to explore whether this retards normal addition strategy development. Calculators may assist retrieval, but are unlikely to assist the various backup methods. Also, the addition strategies that children develop are linked to the number system. Therefore, it would be useful to explore the strategies that are used in other number systems.

The present research, and other research in the area, has been conducted on average children from middle class backgrounds (though not exclusively). It may be worth exploring the strategy patterns of children who come from less privileged backgrounds and who speak languages other than English. This is possibly the most common future research suggestion that appears in most theses, but is often left undone.

7.3.6 Critique

Perhaps the single most important critique of the study has to do with the sample size of the participants. Twelve grade one children participated in the study. The sample size was kept small in order to adopt a very fine analysis of their addition strategy selections. Other studies employing the same methodology have selected small samples of participants (for example, Siegler and Jenkins, 1989). The various studies are producing a relatively consistent picture and, taken together, provide further support for the general findings. However, a far more extensive study with a sample that reflects the greater population is overdue.

7.4 Summary

The results presented in Chapter Six support many of Siegler's main ideas. Children use a number of different problem solving methods, even for tasks such as simple addition. The microgenetic approach produced valuable data while prediction analysis offers a suitable means of analysing the this data.

The two research questions considered suggest that children attempt to match the most suitable strategy for the problem presented. However, optimal strategy choices are partly undermined by cognitive variability as well as limits to children's procedural and conceptual understanding. Furthermore, it appears that cognitively loaded situations affect the way that strategies are executed rather than which of the strategies is selected. The overt execution of strategies appears to offer an external memory aid that is used when the amount of processing required for a task threatens to exceed children's resources.

A strategy discovery process involving the repeated modifications of existing strategies is introduced. This is a process that is more compatible with the constructivist models of development than it is with the evolutionary models of development.

CHAPTER EIGHT

REPRISE

Throughout the present study, two approaches to cognitive development have been considered. The Piagetian tradition emphasises structural development and how this determines the way in which children approach simple addition and other types of problems. Halford (1987), in particular, regards cognitive development as a function of children's increasing structure-mapping abilities. Structure-mapping ability constrains children's conceptual understanding, which, in turn, dictates the strategies that they are able to develop. Siegler has positioned himself in direct opposition to the Piagetian tradition. His theory emphasises cognitive variability and the role it plays in development.

There is much evidence to suggest that children employ a host of different problem solving strategies, and not only the strategy associated with their current developmental level defined by their structural ability. In this respect, at least, Siegler appears to be correct. The selection of strategies from the available possibilities has been the empirical focus of this work. On the whole, the findings indicate that strategy choice is largely determined by the problem presented, as well as children's conceptual and procedural knowledge, but apparently not necessarily on the situation. The situation largely determines how the strategy will be executed.

The first research question, suggesting that a principle of least effort applies to the selection of strategies, was supported by the prediction analysis. The result indicates that children will attempt to match the problem presented with the most efficient strategy that they are able to execute successfully. Children are likely to retrieve the answer to small problems and easy problems such as tie-problems. With greater experience, children will be able to make more and more accurate associations between addition problems and their answers, thereby increasing the range of their retrieval strategy. The actual decision to retrieve or to resort to a backup strategy appears to be determined by problem familiarity. First grade children will tend to count on from the larger addend by the amount indicated by the smaller addend when the answer is not retrieved. If the smaller addend comes first they will tend to reverse the order of the addends, and are most likely to do this when the difference between the two addends is great or when the first addend is small and the second large.

However, children may never reach the level of perfect strategy selections, since selection-optimality is compromised by variability. Variability offers a short-term cost for a long-term benefit, the benefit being greater learning potential. Variability means that children are less likely to stagnate and also that they are better equipped to adapt to new situations.

The second research question considered whether, under conditions of cognitive stress, children execute their strategies in an overt manner in order to extend their working memory capacity, or resort to faster strategies, specifically retrieval, that minimise working memory decay or use less of the valuable workspace. The results do not support the prediction that children will use retrieval with greater frequency and, therefore, the context does not appear to influence the actual strategy selected. This conclusion is supported by Siegler's (1990) finding that emphasising speed over accuracy, or vice versa, does not influence the strategy selected but rather influences how the strategy is executed. The results do suggest that, under cognitively loaded conditions, children will execute their strategies in an overt manner. It is proposed that the overt execution of backup strategies aids the child's memory limitations. This finding supports the limited-processing-space hypothesis promoted by some of the neo-Piagetians. When children's cognitive processing resources are exceeded, they resort to external aids much in the same way as adults will resort to pen and paper or to a calculator. Given that the evidence supports the notion of structural constraints, then it seems likely that these structural constraints dictate the sequence of strategy discoveries.

It is further argued that the backup strategies are discovered in a sequence corresponding with increasing information-processing demands and decreasing completion times. This type of sequence, even if there is debate around the exact description of the sequence, fits the neo-Piagetian position on cognitive development. While the actual use of strategies is probably best described by Siegler's overlapping waves metaphor.

Therefore, the present study, as well as much of the other cognitive science literature, offers mixed support for the two competing positions of cognitive development. Perhaps the reason that the debate between the two traditions is ongoing is that both positions are partly correct. This suggests that it may be worthwhile to seek a compromise between the overlapping waves and the staircase metaphors of cognitive development. It is possible that the variability that Siegler has emphasised occurs around an orderly underlying structural progression compatible with the neo-Piagetian theories.

There are at least two ways of viewing cognitive variability. The first is discussed by Collins (1992) in his critique of artificial intelligence, where he claims that humans are simply unable to reproduce action in a perfectly consistent way. Something about our architecture prevents us from being able to do this. Perhaps children attempt to retrieve their best strategy from memory for any given problem, but since they are unable to follow the same retrieval path consistently, they sometimes retrieve close relatives instead of the strategy desired. Therefore, variability is the result of our cognitive architecture and occurs despite people's intentions.

The second explanation is offered by the evolutionary psychologists. Here variability in thinking is said to serve the same evolutionary purpose as genetic variation. Since the variability provides some adaptive advantage for individuals, the source of cognitive variability has been selected for over the course of our evolution. This may explain how our less than perfect architecture, described by Collins (1992), occurred in the first place.

Finally, if we accept that the children's conceptual levels determine the strategies that they discover, then the range of the strategy arsenal reflects the depth of their number concept. Devlin (2000b) points out that higher mathematics is increasingly abstract but not necessarily that much more complex. Therefore, children must master the complexities of arithmetic before they undertake higher mathematics. In other words, the basic arithmetic operations, of which addition may be the foundation, are likely to serve as the infrastructure for all higher mathematical concepts. Since cognitive variability appears to facilitate arithmetic development, children should be encouraged think in novel ways.

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APPENDIX

NAME				STRATEGY USED								
	PROBLEMS	ANSWER	CORRECT	retrieval	min	count 1st	decomp	s/cut	guess	finger rec	other	OBSERVATIONS	TIME
1													
2													
3													
4													
5													
6													
7													
8													
9													
10													
11													
12													
13													
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