YOUNG CHILDREN'S INTUITIVE SOLUTION STRATEGIES FOR MULTIPLICATION AND DIVISION WORD PROBLEMS IN A PROBLEM-CENTERED APPROACH

by

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DEDICATION

This thesis is dedicated to

My Late Parents : Mr and Mrs R. Naidu

in recognition of their love, devotion and encouragement through the years. This thesis would not have been possible without them.

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ABSTRACT

The intention of this research was to gather and document qualitative data regarding young children's intuitive solution strategies with regard to multiplication and division word problems.

In 1994, nineteen pupils from the Junior Primary Phase (i.e. Grade 1 and Grade 2), from a Durban school participated in this study, in which the instruction was generally compatible with the principles of the Problem-Centered mathematics approach proposed by Human et al (1993) and Murray et al (1992; 1993). Its basic premise is that learning is a social as well as an individual activity. The researcher's pragmatic framework has been greatly influenced by the views of Human et al (1993) and Murray et al (1992; 1993), on Socio-Constructivism and Problem-Centered mathematics.

Ten problem structures, five in multiplication and five in division which were adopted from research carried out by Mulligan (1992), were presented to the pupils to solve. The children were observed while solving the problems and probing questions were asked to obtain information about their solution strategies. From an indepth analysis of the children's solution strategies conclusions on the following issues were drawn:

1. the relationship between the semantic structure of the word problems and the children's intuitive strategies, and
2. the intuitive models used by the children to solve these problems.

The following major conclusions were drawn from the evidence:

1. Of the sample, 76% were able to solve the ten problem structures using a range of strategies without having received any formal instruction on these concepts and related algorithms.
2. There were few differences in the children's performance between the multiplication and division word problems, with the exception of the Factor problem type for the Grade 2 Higher Ability pupils.

3. The semantic structure of the problems had a greater impact on the children's choice of strategies than on their performance, with the exception of the Factor problems.
4. The children used a number of intuitive models. For multiplication, three models were identified, i.e. repeated addition, array, cartesian product with and without many-to-many correspondence. For division, four models were identified, i.e. sharing one-by-one, building-up (additive), building-down (subtractive), and a model for sub-dividing wholes.

CHAPTER ONE

INTRODUCTION

For many years researchers in mathematics education had relied mainly on the theories and methodologies of related disciplines such as cognitive psychology and educational psychology (Ginsburg, 1983). Recent developments have witnessed growing collaboration between mathematics educators, researchers and cognitive psychologists. Current approaches to research in mathematics education have been largely influenced by the work of the information processing theorists and the constructivists. While research in mathematics education benefits from both approaches, the constructivist paradigm has given much impetus to the growing body of research on children's numerical thinking (Mulligan, 1991: 1).

This study focuses on children's development in number concepts and processes from a psychology of mathematics education perspective. Word problems involving division and multiplication have traditionally been difficult for children to solve. The difficulty seems to reside mainly in children's predicament in choosing the "correct" operation. This situation is most probably the direct outcome of conventional school mathematics which formally first teaches all the "*prerequisite computational skills*" for solving multiplication and division word problems and then requires children to apply these skills to the solution of word problems (Olivier et al, 1992: 33).

On the other hand, researchers generally agree that young children enter school with a wide repertoire of informal mathematical problem-solving strategies that reflect and are based on their understanding of the problem situation and on their existing concepts (Olivier et al, 1990; Carpenter et al, 1982). Hughes (1986: 177) extends this belief by stressing the importance of building on children's own strategies and he warns of the possible negative effects of taught methods. "*Obviously we want children to move on eventually to new and more powerful strategies, but if these are forced upon children regardless of their own methods they will not only fail to understand the new ones but will feel ashamed and defensive about their own*".

There is a great deal of resistence against the direct teaching of formal arithmetic procedures because it is strongly believed by many researchers and educators that "*automatizing procedures in itself does not contribute to developing meaningful mathematical knowledge*" (Hiebert in Mulligan, 1991: 5). In other words children experience problems with automatic formal procedures because in most cases they have no understanding of these procedures and they don't know why they are using them. On the other hand, the children's own informal strategies are much more meaningful to them. It is also believed that these informal strategies can provide a meaningful basis for developing understanding of formal procedures (Hiebert, in Mulligan, 1991: 5).

The intention of this research is to gather and document qualitative data regarding young children's informal solution strategies with regard to multiplication and division word problems, and to make that available to the classroom teacher. This study will also supplement related research carried out in other parts of South Africa as well as internationally (Murray, Olivier and Human, 1989; 1992; 1993; Cobb et al, 1991; Fennema et al, 1991; Mulligan, 1991; 1992).

This study employs a socio-constructivist approach, of which the basic premise is that learning is a social as well as an individual activity. Socio-Constructivism per se covers a panoply of theoretical frameworks (Ernest in Cobb et al, 1991: 3) and it means a number of things to a number of people. The researcher's pragmatic / pedagogical framework is greatly influenced by the views of Human et al (1993: 1) and Murray et al (1992; 1993) on Socio-Constructivism and Problem-Centered mathematics; whose work is based on similar principles to that of Cobb et al (1991) and Fennema et al (1991). The stance of this thesis is that children possess a *"framework of knowledge*" for multiplication and division that has developed through everyday experiences (i.e. informal experiences) (Ausubel et al in Anghileri, 1989: 367).

Pupils from a Junior Primary School in Durban participated in this study in 1994. This sample school was one of a few pilot schools that had been attempting to implement the Problem-Centered approach for approximately five years, as proposed by the Research Unit in Mathematics Education at the University of Stellenbosch (RUMEUS) South Africa. The curriculum in a Problem-Centered classroom aims to build on children's informal knowledge and facilitate the development of their conceptual and procedural knowledge through the solution of real-life and other problems (Murray et al, 1989; Olivier et al, 1990). These classrooms are therefore ideal sites for research on the development of children's informal knowledge. The children involved in this study are representative of those in the "normal" or "mainstream" class.

RESEARCH QUESTIONS

Three main research questions will be addressed in this study:

1. What informal/intuitive solution strategies do children use in solving multiplication and division word problems?
2. What are the relationships between strategies used and the semantical structure of the problems?
3. What are the intuitive models for multiplication and division?

OVERVIEW

This thesis is organized into seven chapters. The first three chapters describe the research focus, theoretical framework and related research. Chapter Four deals with the research methodology employed in this study. The findings are discussed in Chapters Five and Six as general performance and strategy use; and solution strategies and problem type, respectively. Chapter Six also looks at the intuitive models and the role of social interaction. Chapter Seven summarizes the conclusions of the study; its strengths and limitations; the implications of the findings; and suggestions for further research.

CHAPTER TWO

REVIEW OF LITERATURE

INTRODUCTION

The review of literature will be dealt with in Chapters Two and Three under the following headings:

THEORIES OF LEARNING RELATED TO MATHEMATICS

The respective rationale underlying two major learning theories that have had a major impact on the learning and teaching of mathematics will be discussed in some detail. They are the traditional approach whose underlying learning theory is Behaviourism and the alternative or new approach whose learning theory is Socio-Constructivism.

ARITHMETIC WORD PROBLEMS IN GENERAL

This will involve a discussion of studies carried out on the solution of arithmetic word problems involving the four basic operations (e.g. Carpenter, Hiebert and Moser, 1981; Fuson, 1982; Gelman and Gallistel, 1978; Groen and Resnick, 1977; Hughes, 1986; Steffe et al, 1988; Murray et al, 1992).

MULTIPLICATION AND DIVISION WORD PROBLEMS

Five relatively recent studies that have had an impact on this thesis will be discussed (i.e. Fischbein et al 1985; Murray, Olivier, and Human, 1989, 1991, 1992, 1993; Kouba, 1989; Anghileri, 1985, 1989; and Mulligan, 1991, 1992).

THE PROBLEM-CENTERED APPROACH

This section concerning the research carried out by Murray, Olivier and Human in the Research Unit for Mathematics Education at the University of Stellenbosch (RUMEUS), will be dealt with in Chapter Three. The above research will be discussed in some detail because the subjects of this study were at a school that was attempting to implement this Problem-Centered approach.

THEORIES OF LEARNING RELATED TO MATHEMATICS

Over the past decades the views of and approaches to how children understand and learn mathematics have varied, in some instances, greatly. Various research perspectives on the learning and teaching of mathematics have been adopted. Learning theories such as Behaviourism, Piagetian theories, Cognitive Psychology and Constructivism have been dominant influences in education this century. Two of these theoretical approaches which have differing principles and approaches will be discussed. They are Behaviourism and Constructivism and they seem to have had the greatest impact on the teaching and learning of Mathematics, especially at the elementary level.

THE BEHAVIOURIST THEORY

This approach which is also referred to as the traditional approach or the connectionist theory of learning relates to an empiricist philosophy of science and it suggests that learning is the forming of habit which is based on reinforcement. The traditional empiricist motto is " *there is nothing in the mind that was not first in the senses"*. Its basic premise is therefore that all knowledge originates from experience (Olivier, 1989: 11). Behaviourists believe that the only scientific way to study learning is to base all conclusions on observations of how overt behaviour is influenced by forces in the environment (Biehler et al, 1990: 316). Behaviourism also assumes that pupils learn what they are taught because it is assumed that knowledge can be transferred intact from one person to another. This approach is based on the view that children have to be shown how to calculate by the conventional "right" methods and they have to replicate these exactly. In other words, learning in mathematics involves rote learning for writing mathematical symbols in some very specific ways, and this kind of knowledge can only be acquired by being told and by practicing it (Davis, 1990, 101). The basic view is that young children cannot fully understand computation or develop their own computational strategies and they have to therefore be taught by rote (Human et al, 1989).

CRITIQUE

Although the above view has been powerful and influential it does have significant short-comings. For many years this view has been challenged because it has long been known by researchers and educators that children make many mistakes in computation despite these very careful teaching methods. Human et al (1989) like many other researchers who oppose the traditional view, strongly believe that in the long run this direct teaching has drastic negative effects on children's ability to learn mathematics. The rationale / explanation offered is that if the teacher shows the child how to solve a problem, this knowledge is not properly constructed from within. Instead the child memorizes bits and pieces which s/he frequently assembles and applies in the wrong order for an unsuitable situation (Murray, 1992: 11). According to the behaviourist approach it is necessary to first present / teach the theory and then involve the children in practical problems. In other words, children are first taught the rules (skills) which they are expected to apply to solve problems. This teaching by imposition demands conformity and ignores children's methods; is therefore seen by children as being decontextualized, abstract, formal, prescriptive and meaningless. The major problem is that research clearly shows that skills learned in this fashion are not transferable and the children become mathematically illiterate (i.e. they are unable to translate from "mathematics" to the "real world" and vice versa (e.g. Fischbein et al., 1985; Carpenter, Hiebert and Moser, 1981; Murray et al, 1992). Pupils do well on straightforward calculations, but cannot choose the correct operations when the problem becomes a little more complex. For example, when asked to calculate the price of 0,53 litres of petrol at R2,60 a litre, the children are unsure whether to multiply or divide.

As the teacher is seen as the dispenser of knowledge the children are considered to be passive receivers / learners. The responsibility for learning according to this approach lies with the **teacher** and not the **child**. The children are therefore teacher dependent.

A theory that seems to be a powerful source for an alternative to direct instruction is that of constructivism. These theorists believe that the behaviourist theory has missed the *richness of human behaviour*. Constructivists believe that one has to examine the problem solving ability of learners, that is, the higher mental processes that they use, to deal with problems. They speculate on what people are thinking, on what strategies they are using and that these processes are not always observable. This is in direct contrast to the behavioural learning theory.

In support of the constructivist view Murray et al (1993: 73) state that contrary to the empiricist view of teaching as the transmission of knowledge and learning as the absorption of knowledge, research indicates that children construct their own mathematical knowledge irrespective of how they are taught. This approach will now be discussed in more detail.

THE CONSTRUCTIVIST THEORY

In the past few decades, i.e. particularly since the 1970's, constructivism rapidly gained recognition.The constructivist perspective on learning, (e.g. Piaget: 1970; Skemp: 1979) assumes that concepts are not taken directly from experience, but that a person's ability to learn from and what s/he learns from an experience depends on the ideas that s/he brings to that experience. In other words, knowledge does not simply arise from experience, it arises from the interaction between our experience and our current knowledge structures (Olivier, 1989: 18).

From a constructivist perspective the teacher cannot transmit knowledge ready-made to the child. Errors and misconceptions are viewed as the natural result of children's efforts to construct their own knowledge and these misconceptions are intelligent constructions based on correct or incomplete previous knowledge. Errors and misconceptions are thus regarded as an integral part of the learning process (Olivier,1989: 18). Central to this approach, is the idea of reflective abstraction. Merely completing a task is insufficient, reflection plays a vital role in the learning process (Wheatley, 1992: 529). Constructivism, as Ernest (1991) states, means many things to many people. However, according to Noddings (1990: 10) although there are conceptual differences, constructivists generally agree on the following:

- All knowledge is constructed. Mathematical knowledge is constructed, at least in part, through a process of reflective abstraction.
- Existing cognitive structures are activated in the process of construction. These structures account for the construction, that is, they explain the result of cognitive activity. Cognitive structures are under continual development. Purposive activity induces transformation of existing structures and the environment presses the individual to adapt.
- The acknowledgement of constructivism as a cognitive position leads to the adoption of methodological and pedagogical construction. Methodological construction in research develops methods of study that are consonant with the assumption of cognitive construction and pedagological construction suggests methods of teaching consonant with cognitive construction.

Broadly speaking there are two versions of constructivism, namely *"weak"* and *"radical"* constructivism (Lerman, 1989: 213). In support of the above discussion, but from a slightly different perspective Noddings (1990:14) suggests that constructivists should talk about weak and strong acts of construction instead of acts involving and not involving construction. In the mathematical environment strong acts of construction would be those that are recognized by mathematicians as mathematical. Weak constructions would be those recognized as limited in mathematical use.

Goldin (1990: 34) describes radical constructivism as a school of epistemology which emphasizes that one can never have access to a world of reality, only to the world that one constructs out of one's own experience. Thus the view that all knowledge is constructed and therefore all learning involves constructive processes and there is no such thing as pre-packaged knowledge. Radical constructivism also maintains that each person's world of experience is context-dependent. In other words, it is unique to that individual and inaccessible to others. Radical constructivism can therefore be viewed as a relativist epistemology.

The "*weak*" theory-in-practice version, which can also be termed "*empiricist-oriented*" constructivism assumes a different position. The differences between the forms of constructivism emerge when the source of knowledge and the process by which knowledge is constructed are considered. Unlike radical constructivists, empiricist-oriented constructivists locate knowledge in an external environment and see it as existing independently of a person's cognitive activity.

However, one can lead children to appreciate and understand mathematics as they acquire knowledge (Mulligan, 1991: 12). Like Mulligan, the researcher of this thesis assumes a constructivist viewpoint that maintains that children actively construct knowledge for themselves and assimilate it in their own way. In addition the researcher ascribes to a Socio-Constructivist viewpoint (like Olivier et al, 1992; Cobb et al, 1991) that learning is a social activity as well as an individual constructive activity. She also espouses the view postulated by Murray et al (1993: 73) on objective and subjective knowledge. According to them the traditional teaching approach leads to subjective knowledge that is largely re-constructed objective knowledge.

Whereas a Problem-Centered learning approach based on Socio-Constructivism leads to subjective knowledge that is a result of personal constructions. It must be emphasized that this does not mean that children are actually creating knowledge that does not exist as objective knowledge but that the children in this approach are actually constructing their knowledge as new. The researcher of this study, however, does not ascribe to a radical constructivist viewpoint and is more directed towards an empiricist-oriented approach.

Thus the research approach adopted by constructivist researchers and educators is mainly quasi-ethnographic which means that it includes clinical and task-based interviews, participant observation and descriptive case studies (Goldin, 1990; Julie, 1990). Researchers need to look beyond what children say, i.e. explicitly for knowledge that is implicit in what they do. In current research on children's understanding of mathematics, researchers look at the invented procedures children develop, as one kind of evidence of their implicit knowledge (Resnick, 1989: 162). The researcher of this study, has adopted a similar stance in this study. Research evidence indicates that prescriptive teaching of methods of computation and problem-solving (based on the principles of Behaviourism) necessarily induces a receptive, passive, dependent attitude towards learning. Whereas, the Socio-Constructivist approach induces an active, self-reliant, creative attitude towards learning (Human, 1990: 2).

ARITHMETIC WORD PROBLEMS IN GENERAL

Researchers today attribute much more mathematical knowledge and understanding to young children than they once did, giving them credit for their own informal methods which in the past would have been regarded as "inferior" or "not mathematical".

Due to children's poor performance in mathematics, much research has been conducted to determine the reasons for this. The observations and findings of much research (e.g. Carpenter, Hiebert and Moser, 1981; Carpenter and Moser, 1982; 1984; Fuson, 1982; Gelman and Gallistel, 1978; Groen and Resnick, 1977; Hughes, 1986; Steffe et al.; Murray et al, 1992) compels one to question the practices of the traditional approach. Is the traditional approach to learning and teaching mathematics really effective or is it to the detriment of the children? Their research has shown that young children bring an incredibly rich store of intuitive and informal knowledge to the learning situation. Ginsburg(1975: 1) refers to them as intuitive mathematicians or active meaning makers. He states that through their spontaneous interaction with the environment they develop various skills and techniques for coping with problems. When they enter school many are able to solve simple problems by direct modelling and/or informal counting procedures (Human et al., 1989).

According to Ginsburg (1975: 1) children's informal mathematics may be conceptualized according to two cognitive systems. *System One* involves patterns of perception and thought which are used to deal with quantitative problems but do not employ counting or other explicit forms of mathematics. *System Two* involves counting and related procedures by which children cope with quantitative problems in the absence of formal instruction. *Systems One* and *Two* are informal because they develop outside the formal school-setting. *System Three* on the other hand is formal because it involves concepts and techniques that are derived from a codified body of knowledge which the school attempts to inculcate in children. Ginsburg thus states that the onset of schooling raises several questions. A significant issue is the relationship between formal and informal knowledge. How does the child integrate what s/he learns at school with what s/he already knows? And what does the school do about this? In the traditional approach this informal knowledge is normally suppressed while the formal school methods are forced on the children.

Extensive research was therefore conducted either directly addressing this issue or other related issues. A common thread among many of these issues is that they were based on children's performance on word problems.

Studies carried out by Carpenter, Hiebert and Moser (1981) and Carpenter and Moser (1982) are popular examples of such research. Their research is based on an analysis of addition and subtraction word problems that distinguishes between different classes of problems based upon semantic characteristics of the problems (Fennema et al., 1991: 29). Their primary objective was to determine how successful children are at solving different types of addition and subtraction problems prior to formal instruction and to identify which types of problems are most difficult for them to solve. Their second major objective was to identify the intuitive strategies children use to solve these problems and to determine the factors that lead to their selection of different strategies.

The following is a summary of their results:

- Each type of problem provides a distinct interpretation of addition and subtraction. That is, children think about these problems not as addition or subtraction but as unique situations (Fennema et al, 1991: 29).
- Children are able to successfully solve basic addition and subtraction word problems **prior** to receiving formal instruction. Carpenter and Moser (1982) state that this suggests that verbal problems may give meaning to addition and subtraction. Thus verbal problems may represent a viable alternative for developing addition and subtraction concepts in school. Children tend to solve these problems by modelling and using several basic counting strategies.
- The tremendous variability between and within children in the solution processes used suggests that prior to receiving formal instruction, young children do not transform problems in a single way or apply a single strategy. As stated before, children have available a rich repertoire of informal strategies which they make use of to solve various types of problems (Carpenter et al.,1981: 38).

Research carried out by de Corte, Verschaffel and de Win (1985: 460) and de Corte and Verschaffel (1987: 379) support and supplement some of the above findings. They concluded firstly, that the semantic structure of verbal problems strongly influences the relative difficulty of such problems and the strategies used by young children to solve them. Secondly, that children apply a great variety of addition and subtraction strategies, some of which are never formally taught.

Research conducted by Olivier et al (1992: 33) supports most of the above findings. They found that the majority of children invent powerful non-standard algorithms alongside school-taught standard algorithms, and that they prefer to use these informal methods. The children's success rate was found to be significantly higher when they used their own algorithms as compared to when they used standard algorithms.

A summary of Bishop et al's (1991: 130) findings present an apt picture of children's informal knowledge. They state that children bring with them various kinds of mathematical knowledge e.g:

- knowledge about out-of-school situations where people use mathematics,
- knowledge about the social practices in which they engage and the "*emerged mathematical goals in those situations*" and
- knowledge about specific mathematical concepts such as the measurement of length and weight.

In view of the above, child generated algorithms should therefore be regarded as comprising a substantive part of the child's mathematical knowledge. Steffe (1983: 110) states that they should be nurtured and allowed to grow into increasingly powerful and sophisticated schemes.

Due to these types of results and conclusions there has been great support for a shift (in the past decade or two) in the emphasis in mathematics education, in certain parts of the world, from children mastering skills and understanding concepts in the traditional approach, to children making meaning of mathematics and becoming flexible mathematical thinkers with problem-solving as a central focus, in a Problem-Centered approach (Adler, 1992: 27).

STUDIES ON MULTIPLICATION AND DIVISION INVOLVING YOUNG CHILDREN

A considerable number of studies have been carried out on word problems involving addition and subtraction. It is only in recent years, i.e., approximately the past decade that more interest has been shown in young children's solution strategies to multiplication and division word problems. Five significant studies will be discussed.

STUDY BY FISCHBEIN, DERI, NELLO AND MERINO

Fischbein et al (1985) included in their sample children from Grades 5, 7 and 9. The students were asked to solve multiplication and division word problems. From their findings, Fischbein et al (1985: 4) concluded that "*each fundamental operation of arithmetic generally remains* *linked to an implicit, unconscious and primitive intuitive model. Identification of the operation needed to solve a problem with two items of numerical data takes place not directly, but as mediated by the model. The model imposes its own constraints on the search process".* According to them the implicit primitive models for multlplication and division are:

- the **repeated addition** model for multiplication: i.e. when adding the number in a group *n* times, where use of the term "and" is verbalized as a distinguishing feature, e.g. "three and three are six and three are nine";
- the **partitive** or **sharing** model for division: e.g. Eight apples are shared equally among two children, how many apples does each child get? In this case *n* number of items have to be shared equally among a specific given number of individuals. The child has to therefore break up/share the *n* items equally. For example, (Figure 2.1) a child could solve the above problem by first giving one apple to each of the two children, and another one to each, etc. until no apples are left. The child then adds up the apples given to each child:

Figure 2.1

A	B
1	1
1	1
1	1
$\underline{1}$	$\underline{1}$
$\underline{4}$	$\underline{4}$

- the **quotitive** (measurement) model for division which is only acquired with instruction: e.g. My mum has baked eight cakes. She gives two to each of my friends. How many friends do I have? Here the *n* items have to be shared out in specific given groups to an unknown number of individuals. The child has to therefore group the *n* items in the given groups. For example, (Figure 2.2) a child could solve the above problem by "taking away" groups of two until none are left. The child then counts the number of twos taken away:

Figure 2.2

$$8 - \underline{2} \rightarrow 6 - \underline{2} \rightarrow 4 - \underline{2} \rightarrow 2 - \underline{2} \rightarrow 0$$

However these intuitive models were based on traditional models of teaching multiplication and division and not on direct evidence of young children's intuitive modelling strategies when solving multiplication and division problems (Mulligan, 1992).

A number of researchers have exposed limitations of Fischbein et al's theory. These will be elaborated on during the discussion of work done by other researchers.

RESEARCH BY KOUBA ON MULTIPLICATION AND DIVISION

The study by Kouba (1986; 1989: 147-149) involved 128 pupils in Grades 1 to 3. The objectives of the study were:

- to determine a classification system for children's strategies based on common aspects of their solution strategies for equivalent set multiplication and division word problems.
- to identify differences in children's solution strategies that may be attributed to semantic differences in word problems,
- to examine the implications of the data for understanding children's intuitive models for multiplication and division, and
- to determine whether the underlying intuitive models for multiplication and division are those proposed by Fischbein et al in 1985.

Three types of common equivalent set problems were used:

- **multiplication** (product unknown), e.g. There are three boxes of sweets and in each box there are four sweets. How many sweets are there in all?
- **measurement division** (number of sets unknown), e.g. I have eight marbles. If I give each of my friends two, how many friends do I have ?
- **partitive division** (number of elements in each set unknown), e.g. If eight marbles are shared equally among two children how many does each child get?

While solving the problems children had access to concrete aids.

The following is a summary of the results and conclusions :

- It was found that 56 different strategies were used by the children. From an examination of these strategies two groups of common characteristics were derived.

One group lay in the abstractness involved in the solution strategy. These fell into five categories:

1. **direct representation**: the children used physical materials to model the problem and some form of one-by-one counting in calculating the answer,
2. **double counting**: the children kept a running count of the total number of objects in the groups while also counting out the objects to form groups,
3. **transitional counting**: the children calculated the answer to the problem by using a counting sequence based on multiples of a factor in the problem,
4. **additive and subtractive**: the child clearly identified the use of repeated addition or subtraction to calculate an answer, and
5. **recalled number fact**: the child obtained the answer by remembering the appropriate multiplication or division combination.

The other group lay in the manner in which concrete objects were used. These fell into three categories:

1. objects used as representations of the unique element in each set,
2. objects used as tallies or repeated references for the number spoken, and
3. no objects used.

Of the 333 appropriate strategies used by the children, the multiplication problems were solved mainly by assembling the given number of equivalent groups or by using recalled number facts. Measurement division problems were solved with recalled facts, double counting and equivalent groups. Partitive division problems were generally solved by dealing out objects and trial-and-error grouping (in Mulligan, 1991: 44).

Children's intuitive model for equivalent set multiplication is linked to their intuitive model for addition because both involve the actions of building sets and then putting these sets together. Multiplication, however, is much more complex than addition because for problems using whole numbers the children must recognize that one of the numbers given in the problem represents a set of equivalent sets (1989: 156).

- Children appear to employ two different intuitive models when solving division problems :
    1. **repeated taking away:** the children set out a number of objects to represent the dividend and then successively removed equivalent groups until the dividend was exhausted, and
    2. **repeated building up**: this was evidenced by double counting and counting by multiples.
- There are three different models for partitive division:

    1. sharing by dealing out: the children set out a number of objects to represent the dividend and then dealt those out one by one until the dividend was exhausted,
    2. sharing by repeated taking away: children set out a number of objects to represent the dividend, guessed at the number in each set and removed successive equivalent sets until the dividend was exhausted, and
    3. sharing by repeated building up: the children guessed at the number in a group and counted by multiples of that guess until the dividend was reached or exceeded.

Kouba's findings (1989), just as Olivier et al's (1992) discussed later on, thus refutes the intuitive models theory proposed by Fischbein et al (1985). Her study revealed that children appear to view multiplication as a different two-step process, i.e. they make several sets and then put them together. While this model is more consistent with Fischbein et al's (1985) initial definition of "*a number of the same size collections are put together*", it is not the same model implied by them when they discuss multiplication as defining a single referent set and operating on it by taking sets or partial sets (Kouba, 1989: 156). With division, there is not just one intuitive model as proposed by Fischbein et al, which can be seen from the discussion above.

JULIE ANGHILERI'S RESEARCH ON MULTIPLICATION

The studies by Anghileri (1985, 1989) focussed on the influence of problem structure, performance level and solution process. It looked at the development of understanding of multiplication from the early school years, before its formal introduction in school, through to the top primary age group. In 1985 Anghileri investigated different multiplicative structures, each associated with different situations and each involving particular language, i.e. (1989: 370-371):

1. **equal grouping / repeated addition**: e.g., the child was shown a *"pattern stick"* of structured cubes constructed from four different colours with two cubes of each colour. After some discussion the child was asked to make a "*pattern stick*" using five different colours with three cubes of each colour,
2. **allocation / rate 1**: e.g., the child was asked to take cotton reels from a bowl, one at a time and place them on the table. Each time one reel was placed on the table, the interviewer took three counters and concealed them in one hand. When five reels had been placed on the table (15 counters concealed) the child was asked to figure out how many counters were in the interviewer's hand.
3. **array**: e.g. after being shown a 6 x 3 array of coins fixed onto a card, and following some discussion the card was placed face down on the table so that the coins were hidden. The child was asked to figure out the total number of coins on the card,
4. **number line**: e.g. a demonstration was given by the interviewer of a small model character (man) jumping across 24 numbered stepping stones, illustrated on a card. The child was shown the man jumping "2 at a time" and "3 at a time" and was then asked which number stepping stone would be reached after 5 jumps, if the man was jumping "4 at a time",
5. **scale factor / rate 2**: e.g. illustrations of 2 lorries were shown to the child, one of which was three times as long as the other. Four square "boxes" were fitted (as a square) onto the smaller lorry and the child was asked to establish how many "boxes" would fit onto the larger lorry, and
6. **cartesian product**: e.g. the child was presented with a number of cardboard cut-out figures, sets of shorts in 3 different colours and sets of shirts in 4 different colours. The child was asked how many different outfits could be made from the given collection.

The individual interviews that she carried out showed that 4-8 year olds found problems that required a one-to-many matching easier than those based on the scale factor aspect of multiplication in which one object was referred to as "a number of times as big" as another. Both these aspects were found to be easier than the cartesian product problem which requires a many-to-many matching. Anghileri also found that the children's strategies varied according to their ages and abilities. The strategies that were used included the use of concrete materials, counting in groups (often using fingers), the reciting of number patterns and the direct application of a known multiplication fact. The results however did show children's lack of adequate understanding of multiplication.

Anghileri conducted further research in 1989. A similar procedure was followed as in her previous study. This study involved individual interviews of 234 pupils between 4 and 12 years of age. A summary of her findings are as follows:

1. The analysis of successful solution strategies revealed that there was a development from unitary counting, through rhythmic counting in groups, to the application of a single multiplication.
2. Anghileri distinguishes between counting procedures for addition and multiplication, where adding the same number in multiplication requires an "internal tally". According to her the children's ability to keep track by nodding or verbal cues would indicate the transfer from counting meaning to cardinal meaning.
3. She states that there is a link between the structure of a multiplication task and the solution strategy by the child, which may furnish further information about that child's developing understanding of multiplication.

Anghileri's work addresses the importance of investigating counting strategies, particularly in support of the notion that individual children use a variety of solution strategies, and specific schemes are formulated according to different aspects of multiplication as a result of different contexts and different modes of instruction. She also highlights the need for longitudinal studies which she states would provide the most effective method for monitoring the development of skills and understanding in multiplication.

RESEARCH BY MURRAY, OLIVIER & HUMAN ON DIVISION

The results and conclusions given by Murray et al (1991; 1992) and Olivier et al (1992) have been based on an ongoing research and development project since 1984, on the mathematics curriculum in the elementary phase (the RUMEUS Junior Primary Project).

The main objectives of this study on division were:

- a description and analysis of children's strategies,
- an analysis of the relationship between strategies used and the semantic structures of the problems,
- to identify the mechanisms of transition to more sophisticated strategies, and
- an analysis of the role of classroom social interaction in the construction and evolution of children's division schemes.

The following is a summary of their findings and conclusions:

- To a large extent the children's strategies corresponded with those identified by Kouba et al (1989) but the 22 children in their study used additional sophisticated strategies when working with larger numbers (1992: 36). They concluded that increased number sizes seemed to encourage students to develop more efficient strategies (1992: 158).
- Young children can solve both sharing and measurement problems at an intuitive level with no formal instruction (1992: 36). This finding refutes Fischbein et al's (1985: 14) theory that initially there is only one intuitive primitive model for division problems, i.e. the sharing (partitive) model; and only with instruction do children acquire a second intuitive model, i.e. the measurement (quotitive) model.
- Very few children naturally use subtraction (i.e. "taking away") when solving division problems. They prefer using building-up or addition strategies. Again this refutes Fischbein et al's notion that the implicit model for measurement division is repeated subtraction (1992: 36). The children's single general economical strategy develops from their measurement strategies. Their sharing strategies are progressively discarded (1992: 158).

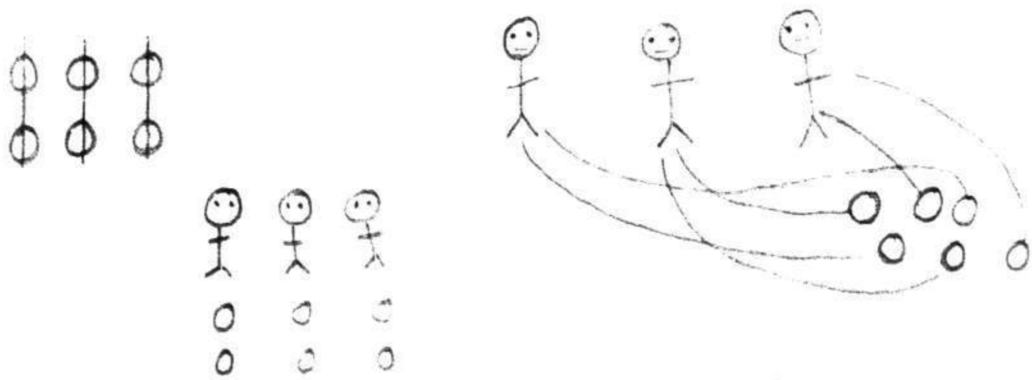
- Murray et al. (1992: 36) identified **four** factors that affect the children's evolution of strategies:
    1. Children's number concept development informs their strategies.
    2. Children's solution strategies are initially clearly determined by the semantic structure of the problem. Different strategies are used for sharing and measurement problems. This illustrates two different conceptions of division. However, they eventually construct an integrated concept of division and one general economical strategy for solving all division problems. This they referred to as " *distance from the problem"*.
    3. The children's solution strategies are parallelled by their level of awareness of the properties of operations or theorems-in-action.
    4. Children's strategies are not merely based on conceptual understanding, but simultaneously also inform their number concepts and awareness of theorems - in - action. This is an indication that conceptual and procedural development go hand in hand.
- Discussion leads firstly, to the improvement of strategies, because it enables children to reflect on their own strategies and the strategies of others, and secondly, to the prevention of misconceptions and the clarification of errors (Murray et al 1992: 158). At the end of the discussion children in the same class/group tend to arrive at strategies that are structurally the same. This is an illustration of the social element in the construction of knowledge.
- The following are examples of some of the strategies for division problems that the children in their study displayed:

    direct representation,
    numerical representation,
    subtraction,
    addition and multiplication, and transformations.

The following discussion is an elaboration of these strategies:

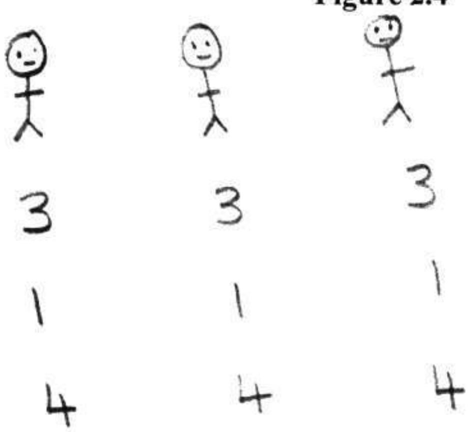
Direct representation: The problem context is drawn in greater or lesser detail and then solved by further drawing in the actions needed. For example, three children as shown in Figure 2.3 divide 6 sweets equally among 3 children :

Figure 2.3



Numerical Representation: The child models the structure of the problem using numerals, without employing arithmetical operations in their representation. For example, consider Figure 2.4 which shows a child dividing 12 marbles among 3 children using this strategy.

Figure 2.4



Subtraction: Subtraction as a strategy for division can represent three different conceptualizations:

- dealing out using estimation for sharing problems, e.g. a child divides 81 oranges among 3 boxes as follows:

  80 - 20 - 20 - 20 → 20 - 6 - 6 - 6 → 2 + 1 → 3 - 1 - 1 - 1 → 0

  $81 \div 3 = 27$

- subtracting the number of objects dealt out in each round to solve a sharing problem, e.g. a child divides 18 sweets among 3 children as follows:

  $18 - \underline{3} = 15 \rightarrow 15 - \underline{3} = 12 \rightarrow 12 - \underline{3} = 9 \rightarrow 9 - 3 = 6 \rightarrow 6 - \underline{3} = 3$

  $\rightarrow 3 - \underline{3} = 0$

- solving a measurement-interpreted problem by repeatedly subtracting the divisor, e.g. a child finds how many buses are needed to transport 350 children if there are 70 children per bus as follows:

  $350 - 70 \rightarrow 280 - 70 \rightarrow 210 - 70 \rightarrow 140 - 140 \rightarrow 0$

  $350 \div 70 = 5$

Addition and Multiplication: Addition and multiplication can be used for both sharing and measurement interpretations of division, e.g. a child divides 18 sweets among 3 children by repeated estimation:

$$4 + 4 + 4 = 12$$
$$6 + 6 + 6 = 18$$

Pupils progressively formalize such strategies, eventually expressing them as multiplication. An estimation dealing out strategy can also terminate in multiplication, e.g.

$$468 \div 12 = 39$$

initial version:

$$30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 +$$
$$30 + 30 = 360$$
$$7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 = 84$$
$$2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 24$$

final version:

$$12 \times \underline{30} = 360$$
$$12 \times \underline{7} = 84$$
$$12 \times \underline{2} = 24$$

therefore $468 \div 12 = 30 + 7 + 2 = 39$

Transformations: This method indicates the child's ability to reconceptualize a number as the sum of multiples of iterable units. It also involves intuitive use of the distributive law. Thus, for $51 \div 3$:

$$30 \div 3 = 10$$

$$12 \div 3 = 4$$

$$9 \div 3 = 3$$

$$10 + 4 + 3 = 17$$

Note that this strategy also includes estimation and the use of known number facts.

MULLIGAN'S LONGITUDINAL STUDY ON MULTIPLICATION AND DIVISION

Mulligan (1991, 1992) analysed children's solution strategies to a variety of multiplication and division problems. Intensive clinical interviewing was conducted at four stages over a two year period, and a classification scheme for problem structures and solution strategies was developed. Seventy children were followed from year two to year three, from the time where they had received no formal instruction in multiplication and division to the stage where they were formally taught basic multiplication facts. Ten problem structures, five for multiplication and five for division were classified according to differences in semantic structure. At each interview each child was asked to solve these ten different problems. She also examined the relationship between problem condition (i.e. small or large number combinations, use of counters or pictures) on performance and strategy use.

The broad objectives of this study were:

- an analysis of young children's solution processes prior to formal instruction,
- determine how children acquire informal strategies and whether these strategies develop and change over time,
- determine the relationship between informal and formal multiplication and division strategies.

The methodology was based on Carpenter and Moser's (1984) longitudinal study of children's solutions to addition and subtraction word problems.

In view of the research questions described above and the analysis of results, Mulligan structured the conclusions under three main sections:

Performance level: An overall analysis of the performance level, indicated that 75% of the sample were able to solve a range of multiplication and division word problems in the absence of formal instruction. The performance level generally increased for each interview stage over the two year period but it varied according to the difficulty of the problem structure and the size of the number combinations used. It was found that the children relied on a range of informal strategies that were based on their knowledge of counting and addition. There were few differences in the performance level between multiplication and division problem structures. A general decrease in the performance level for problems containing large number combinations was evident.

Strategy use: The evidence indicated that the semantic structure of the problem affected the children's choice of solution strategy more than it affected changes in their performance level. The solution strategies, in most cases, reflected the semantic structure of the problem where the action or relationship was modelled.

A wide range of solution strategies were identified. These were largely based on grouping, counting-all, skip counting and additive procedures for both multiplication and division problems. Few differences were found between the solution strategies for multiplication and division problems, except that sharing one-by-one, one-to-many correspondence and trial-and-error strategies were used exclusively for division. Some consistency in the primary strategies used, across interview stages, was found, but the use of known facts emerged strongly during the third and fourth interviews for most problem structures.

From an analysis of the variety of strategies used, three basic levels of strategy use were identified :

- strategies based on direct modelling and counting, using fingers or counters,
- strategies based on counting, addition and subtraction without direct modelling, and
- strategies based on known/derived addition and multiplication facts.

Mulligan concluded that there is a strong relationship between addition and the development of multiplication and division concepts.

Intuitive Models: The intuitive models that Mulligan (1991) identified were found to be more complex and varied than described by Fischbein et al (1985) and they were supported by the evidence in Kouba's (1989) study.

Based on an analysis of the children's solution strategies, Mulligan identified four underlying intuitive models for multiplication :

- **repeated addition**: the children formed or drew equivalent groups and counted them together;
- operating on a **referent** set: when solving a multiplying factor problem the children used a different model of operating on the referent set "to be timesed", but the repeated addition model was also used here;
- **an array**: This model was distinctly evident where children formed rectangular patterns but the repeated addition process was also used; and
- a **cartesian cross-product** model showing the cross product of two attributes, which was difficult to determine but there was evidence of children's thinking in this form

The two-step processing model of repeated addition, i.e. "*make equivalent sets and put them together*" was found to be predominant and consistent with the findings of Fischbein et al (1985) and Kouba (1989). The variety of models employed revealed that more than one model of repeated addition could be developed at an early age.

She also identified three underlying models for division with small and large number combinations: "*sharing one-by-one*", "*building-up*" (additive) and "*building-down*" (subtractive). A fourth underlying model for sub-dividing wholes was found where the notion of sub-dividing wholes was shown.

Mulligan designed and conducted a teaching experiment from a constructivist perspective in which teaching strategies that reflected the intuitive models for multiplication and division were successfully integrated.

This study has provided evidence that :

- young children can solve a variety of multiplication and division problems prior to formal instruction in these concepts,
- counting and additive procedures are very important for children to solve multiplication and division problems, prior to formal instruction, and
- teaching programmes could incorporate the development of informal strategies rather than focussing only on mastering number facts and computational skills that may not relate to the child's level of strategy development. Teachers could facilitate more meaningful learning by establishing links between children's intuitive strategies and formal teaching in the four basic operations.

CHAPTER THREE

THE PROBLEM-CENTERED APPROACH

Based on national and international research on young children's and adults' informal computational strategies, and small scale experiments from 1984 to 1987, a first version of a Socio-Constructivist programme for numeracy in the first three school years was developed in 1988 by a group of researchers at the HSRC Research Unit for Mathematics Education at the University of Stellenbosch (RUMEUS). This unit is attached to the university's Faculty of Education and initially it worked in conjunction with the Cape Education Department. This new/alternate approach is currently also being implemented in former NED school (i.e."white" schools) as well as voluntarily in several other schools from different departments around South Africa.

This research group (i.e. Hanlie Murray, Alwyn Olivier and Piet Human) is engaged in an ongoing research and development project on the mathematics curriculum at the elementary phase, attempting to build on children's informal knowledge in order to facilitate the development of conceptual and procedural knowledge. According to the researchers, their project is similar in content and approach to the CGI Project at the University of Wisconsin (e.g. Fennema, Carpenter and Peterson, 1991) and the Second Grade Project at Purdue University (e.g. Cobb, Yackel and Wood, 1988 in Olivier et al, 1992).

Their theoretical framework is based on Constructivism, the basic premise of which is that children actively build up their knowledge based on their own experience. Their approach is further inspired by Socio-Constructivism, i.e. that learning mathematics is a social as well as an individual constructive activity (Olivier et al, 1992: 30). The core of their approach is that pupils are actively discouraged from viewing computation as having to be done in prescribed ways, but that they must view it as something one does by using one's common sense.

Their baseline study indicated firstly, that children are immensely creative in inventing their own powerful, non-standard algorithms. Secondly, that the primary mathematics curriculum does not capitalize on the rich informal mathematics that young children bring to school. Thirdly, that children who rely on the standard algorithm experience a low rate of success. They therefore decided to implement their alternative approach to early arithmetic. Their main objective was to facilitate children's construction of the number concepts and computational strategies of the first three levels in their model, i.e. their semantic model, describing children's computational strategies for the basic arithmetical operations through four increasingly abstract levels, each level associated with its prerequisite understanding of number and numeration (Olivier et al, 1990 : 2), in contrast to teaching them Level 4, as the traditional curriculum does (Olivier, 1990: 2-3). A brief discussion of this model is necessary:

Level 1 : This is a pre-numerical phase. This level entails the ability to count or subitize a number of objects, and a knowledge of the number names and symbols of that particular interval, without assigning meaning to the individual digits of the numeral. The typical computational strategy at this level is "*counting all*" and the children are normally involved in direct modelling, using concrete objects or fingers, e.g. to solve 5 + 3 he will first count out five objects and then three, push them together, and count-all from the beginning "one, two, three, four, five, six, seven, eight," (Murray, 1988: 14; Human et al, 1989: 5; Murray et al, 1991: 5).

Level 2: The numerical phase now begins. At this level the child has acquired the "feel" of the numbers, i.e., their numerosities. This means that a child has a feel for the size and existence of a number. A number is no longer seen as just a part of a counting sequence (i.e., ordinal number) but as an "object" in its own right (i.e., cardinal number). S/he now has the ability to conceptualize a number in a particular interval as an abstract object independent of concrete referents or counting acts (Steffe et al in Olivier, 1990). The child no longer needs to count all or count from one. The typical computational strategies at this level are counting on, counting down, accelerated counting on and stepwise counting, e.g. to solve 5 + 3, s/he would say "five, six, seven, eight" (Olivier, 1990: 2; Human et al, 1989: 29; Murray, 1992; Murray, 1991: 5).

Level 3: This level involves the ability to replace a number in a particular number interval with two or more numbers which are more convenient to compute with. According to Cobb et al (in Olivier, 1990: 2) this provides the children with the computational basis to use thinking strategies, in other words to solve a computation by relating it to other known results.

Accelerated counting on is seen as a transition phase between levels two and three. The child who solves $51 \div 3$ at this level might "break up" 51 into $30 + 21$ or $30 + 12 + 9$, simply because these numbers are easier for him to divide by three (Olivier,1990; Human, 1989; Murray, 1992; Murray, 1991: 5).

Level 4: This level involves the ability to interpret a 2-digit number as consisting of some tens and some ones (e.g. 24 as two tens and four ones), without losing sight of the true meaning of the number. In other words to conceptualize tens as new "units", formed out of units/ones. They see Level 4 understanding as a prerequisite for the meaningful execution of the standard algorithm in its most sophisticated form (Human, 1989).

It appears that children "*pass through*" the levels of this model several times. For example, it seems that they experience the levels anew when they encounter larger ranges of numbers and for each new operation, albeit at an increasingly faster pace (Olivier, 1990: 7).

The **basic principles** of this problem-centered approach are:

1. Pupils are led to view all methods of computation as alternatives and not as prescriptions and they are allowed to exercise independent, individual choices of methods (Human, 1990: 1).
2. A didactical contract between the pupil and the teacher is established and maintained in which the teacher has a facilitative, consultative and managerial role, but refrains from direct teaching. S/he does not demonstrate a solution method or steer any activity in a direction that s/he had previously conceived as desirable (Murray et al, 1993: 74). The pupils are required to solve problems, construct methods of computation independently and explain their methods and share their solutions with their peers (Human, 1990). Their didactical approach also emphasizes the role of negotiation, interaction and communication between teacher and pupils and between pupils (Olivier, 1990: 4).
3. Emphasis is placed on how a novel problem is solved by the learner (Davis, 1990: 93).

4. Since it is not possible for conceptual knowledge to be transferred ready-made from one person to another, it must be actively re-built by every individual on the basis of his/her own experience (Olivier, 1990: 3).
5. The children are seen as active participants in the learning situation and their views are valued and respected (Olivier, 1990: 3). Their self-generated strategies are also viewed as important in their own right and not as transitional procedures towards the standard algorithm.
6. The standard vertical algorithm is not taught, instead children are encouraged to produce their own computational strategies. Their rationale for abandoning the standard algorithm is two-fold. Firstly, from their baseline data they concluded that number and number concepts develop slowly and that children need extensive experience of the first three levels of understanding as developmental underpinnings for true *Level 4* understanding. This approach therefore encourages instruction that is focussed on procedures which can be related to children's existing conceptual knowledge and at the same time, children's conceptual knowledge must be enriched to support the acquisition of more advanced procedures. They strongly believe that an instrumental understanding of the standard algorithm does not contribute to children's conceptual knowledge. Therefore its premature introduction at the syntactic level will inhibit understanding in mathematics. Pupils are able to develop sound *Level 3* conceptual knowledge on which to base their own thinking. Secondly, due to the advent of the calculator, the standard algorithm is seen as not having a legitimate place in the curriculum, definitely not in the first three school years (Olivier, 1990: 4). The calculator militates against the standard algorithm because its use requires understanding of exactly what needs to be entered. The use of a standard algorithm does not require much understanding, mere application. The basic purpose of the utilization of calculators is real understanding of methods of solution and underlying concepts (Human et al, 1993: xix).
7. Pupils solve problems while working in small groups, and they are required to explain their methods in writing as well as verbally, with the teacher providing the necessary support. They are also encouraged to discuss, compare and reflect on different strategies, trying to make sense of the other explanations offered, in this way learning from each other. Teachers are expected to spend much time listening to

the children and accepting their explanations and justifications in a non-evaluative manner in order to interpret and understand their cognitive development (Olivier, 1990: 4).

8. A key feature of this approach is the notion of differentiated progress according to individual ability. Continual assessment, of the children's knowledge and an attempt to provide appropriate learning experiences that will facilitate the child's development, by the teacher, is therefore essential (Olivier, 1990: 4).
9. Pupils' computational strategies are built on their number concept development; these are seen as inseparable. Provision is therefore made in the curriculum for facilitating the development and transition of number concepts through the three sequential levels of understanding of number (Olivier, 1990: 4).
10. All children do not always invent their own algorithms. Social interaction is therefore viewed as an alternative way of facilitating their conceptual development. The children's act of explaining and defending their computational strategies is seen as helping them to consolidate these strategies by making them overt so that they become objects of reflection (Olivier, 1990; 4).

The Problem-Centered approach to mathematics learning and teaching, as postulated by this research group, provides a radical alternative to both traditional transmission approaches which rely heavily on direct prescriptive exposition by the teacher and extensive drill and practice, and mediated learning (i.e. guided discovery) which relies heavily on skilful negotiation between an instructor and small groups of children (Human et al, 1993: 1).

CHAPTER FOUR

METHODOLOGY

INTRODUCTION

This study was designed to examine young children's intuitive solution strategies to multiplication and division word problems. The methodology involved instruction which was generally compatible with the principles of the Problem-Centered mathematical approach proposed by Olivier et al (1992), Human (1990) and Human et al (1993) (this approach has been discussed in detail in Chapter 3). The methodology also involved observations of the strategies used as well as the posing of probing questions in order to obtain information about the manner in which the children solved the multiplication and division problems.

The methodology was also partially based on Mulligan's (1991) research on An Analysis of Children's Solutions to Multiplication and Division Word Problems, who had in turn based her methodology on Carpenter and Moser's (1984) longitudinal research of Children's Solutions to Addition and Subtraction Word Problems. The researcher used Mulligan's (1991) classification scheme for multiplication and division word problems, in this study.

This research design was considered appropriate for this study because it enabled the researcher to directly examine the intuitive strategies children used in solving multiplication and division word problems with no formal instruction in these standard algorithms or associated concepts.

Details of procedure and type of instruction, selection of subjects, classification scheme for multiplication and division word problems, collection of data and analysis of data are discussed in this chapter.

PROCEDURE / TYPE OF INSTRUCTION

As the approach proposed by RUMEUS (which has been discussed in detail in Chapter Three) was followed at the school in which this study was conducted, a summary of some salient characteristics of the problem-centered approach (Murray et al, 1993: 74), discussed in the previous chapter is necessary:

1. Students are presented with problems that are meaningful and interesting to them, but which they cannot solve with ease using routinized procedures or drilled responses.
2. The teacher does not demonstrate a solution method, nor does she steer any activity (e.g. questions or discussion) in a direction that she had previously conceived as desirable, yet she expects every student to become involved with the problem and to attempt to solve it. Students' own invented methods are expected and encouraged.
3. It is expected of students to discuss, critique, explain, and if necessary justify their interpretations and solutions.

It must be stressed that the extent to which the above characteristics / principles were adhered to, at this school as a whole, is unclear. However the teachers of the few classes that the researcher did visit and observe seemed not to emphasize the third characteristic, especially the discuss and critique aspects.

For two block sessions of five weeks each, research was conducted with nineteen pupils. The ideal situation would have been to work with and observe the children in their classrooms, unfortunately this was not permitted by the school. The children involved in the study were therefore withdrawn from their original classes during the study. In the first five weeks for two periods (30 minutes each) per week, research involved a Grade 1 and a Grade 2 Mixed Ability group of six pupils each (i.e. two pupils each of Higher, Average and Lower Ability). During the second five week block session research involved two similar ability groups of four pupils each (i.e. Grade 1 - Average Ability and Grade 2 - Higher Ability). An Average Ability pupil was included in both block sessions (i.e. in the first five weeks in a Mixed ability group and in the second five weeks in the Average Ability group).

Initially a similar approach used by some of the teachers at this school was adopted. That is, the problem was read aloud to the group and the children worked on it in whichever ways they wished. They were then each given the opportunity to explain and justify (if necessary) their strategies to the rest of the group. The researcher's role was that of an observer in so far as she did not attempt to stimulate or steer the discussion in a particular direction.

In the fourth lesson (i.e., in the second week of the first five week block) the researcher changed her role as well as the classroom organization because she was concerned about the lack of communication. The groups were split into mixed ability dyads that were chosen by the researcher. They were instructed to work together. The researcher then adopted the role of a facilitator with the intention of attempting to establish the norms of social behaviour, moving around interacting with individual pairs. Her objectives were firstly, to establish the obligations and expectations for interaction within the groups which are important for meaningful discussion of mathematical solution strategies, and secondly, to establish and maintain the discourse within the dyads, hoping that the children would conduct the dialogue themselves. No demonstration of a method or strategy by the researcher was involved. Neither was there any attempt to steer the children's activity / thinking in a direction that was previously conceived as desirable. By asking certain questions or making certain comments she attempted to enable each child to realize that it was his/her responsibility to listen to and understand the other's explanation (Cobb et al, 1988; Wood and Yackel, 1990). Examples of these comments and questions were :

"Would \ Do you agree \ disagree with what Sally has said ?"
"Could you explain how you got that answer ?"
"Have you solved it differently ?"
"How is your's different from Calvin's ?"
"What do you think James is trying to say ?"
"Why don't you try it a different way?"

After the discussion in these sub-groups all pupils had to explain / describe and justify (if necessary) their solution strategies to the large group.

In the following lessons pupils were given the opportunity to work with whoever they chose. During one of these lessons pupils in each dyad were expected to work out the problems on their own, discuss it with their partners and then explain their partner's solution strategy to the rest of the group. The objective here was to ensure that each pupil listened to his/her partner's discussion carefully and attempted to understand it. The researcher's role as a facilitator remained unchanged.

During the lessons the researcher read out the problems to the children. They were then allowed to work out these problems in their groups. While the children attempted to solve the problems as individuals and in groups the researcher observed their strategies and recorded these. Each child was then expected to discuss his/her solution strategy to the rest of the group. At times it was necessary to ask the children to clarify their explanation. This method was employed by the researcher as she wanted to observe the children in an environment which resembled their classroom as closely as possible, i.e. the children working within groups. However, as each lesson period was thirty minutes long and there were four or six children in each group, it was not always possible to get all the children to clarify their explanations, as well as to observe every child in the group while s/he solved each problem. It is therefore necessary to make a distinction between the two types of explanations offered by the children, i.e. a child's explanation of what s/he is doing **while calculating** and a child's explanation of what s/he has done **after** calculating. It must be emphasized that there can be a difference.

Physical Aids

Previous studies of multiplication and division word problems (Bechtel and Weaver, 1976; Keranto, 1984; Hendrickson, 1979; Zweng, 1964) indicated that the use of physical objects made it easy to observe children's solution strategies. As one of the objectives of this research was to study the relationship between solution strategies and semantic structure, physical / concrete aids were made available.

Initially, i.e. in the first three lessons of the first five week session a set of one hundred coloured unifix cubes, abaci and the hundred squares were made available. The children were informed that they could use these aids to solve the problems but there was no compulsion to do so. Due to observations made during these first three lessons, it was decided that the above aids were not sufficient so specific aids were included from the fourth period onwards. The term specific aids was used to refer to aids that directly represented the objects and the subjects in the word problems, e.g. coins, scissors and cut-outs of children, oranges, pencils etc. At all times children were given paper and pencil to record their solution strategies. Again they were informed that they were not compelled to use these.

SELECTION OF PUPILS

From research carried out by Mulligan (1991) and Olivier et al (1992) it was deduced that children in Grade 2 would be the most suitable subjects for this study. As the intention of this study was to investigate children's intuitive strategies in multiplication and division it was seen appropriate to include Grade 1 pupils. The children were selected by their class teachers. Four criteria were used to select the subjects. They were ability grouping, mathematical background and prior experience, reading ability and gender differences. The latter three criteria are considered by Lester (1983) as three major factors that influence the problem solving process. These four criteria will be briefly discussed.

Ability Grouping : At the school where the research was carried out the children were grouped according to ability for mathematics. As it was the intention of the researcher to study the children in a setting that resembled their classroom as closely as possible, children from different ability groups from the same class were included in this study. As mentioned before, one Grade 1 pupil was included in both the mixed ability group and the homogeneous group. The reason for this was to investigate his performance in the different ability groups.

Mathematical Background and Prior Experience : As the intention of this research was to study children's intuitive problem solving strategies in multiplication and division word problems, only pupils who had not received any formal or informal instruction in multiplication and division in school were included in this study.

Reading Ability : Previous research had found that the ability to read and comprehend the language used in word problems was a factor that affected the solution process (Ballew et al; Cottrell et al in Mulligan, 1991: 99). As the researcher intended including children from Grades 1 and 2 and from all ability groups it was decided that the word problems would be read out to the children. So it was not necessary for the children to be able to read.

Gender : Research on gender differences has not yet established whether there are sex differences in the way young children solve word problems (Mulligan, 1991: 101). For this study an equal number of girls and boys were selected.

CLASSIFICATION SCHEME FOR MULTIPLICATION AND DIVISION WORD PROBLEMS

The classification scheme for multiplication and division word problems devised and trialled by Mulligan (1992: 28) was used. This included ten different problem structures, five having the semantic structure of multiplication and five having the semantic structure of division. According to Mulligan (1991: 103) these were adapted from the problem structures used by previous researchers (e.g. Anghileri, 1985; Bell et al, 1981; Brown, 1981; Kouba, 1986 and Vergnaud, 1983). As shown in Table 4.1 multiplication, repeated addition, rate, factor, cartesian product and array problems were included. For division, partition (sharing), quotition (measurement), rate, factor and sub-division (involving halves) problems were included. Table 4.1 represents an adapted version of Mulligan's (1992: 28) classification scheme of multiplication and division word problems.

During the experiment, which extended over a period of five weeks for each group some word problem structures were worded differently (i.e., the context of the problem was changed but not the semantic structure) when presented, and some were repeated when it was found that most of the children in the group experienced difficulty solving them.

In the first five week session, all ten problem types were presented to the two Mixed Ability groups. Examples of some problem types were presented more often than others depending on the children's performance and the time available. Table 4.2 indicates the number of times each problem type was dealt with.

TABLE 4.1

MULTIPLICATION AND DIVISION WORD PROBLEMS: CLASSIFICATION SCHEME

Multiplication	Division
1. Repeated Addition There are 2 boxes on the table. In each box there are 3 beads. How many beads are there altogether?	**1. Partition (sharing)** 12 apples are shared equally among 3 children. How many apples does each child get?
2. Factor My friend has 3 books and I have 2 times as many. How many books do I have?	**2. Factor** Pat has 6 marbles and this is 3 times as many as Sam has. How many marbles does Sam have?
3. Rate If you need 2 cents to buy one sticker, how much do you need to buy 3 stickers?	**3. Rate** My friend bought 4 pencils for 12 cents. If each pencil cost the same how much did one pencil cost?
4. Cartesian Product Marina has 3 skirts of different colours and 4 blouses of different colours that all match. In how many different ways can she dress?	**4. Quotition (measurement)** Mum has baked 8 buns. She puts them into plastic bags, 2 in each bag. How many plastic bags did she use?
5. Array There are 3 lines of children. In each line there are 4 children. How many children are there altogether?	**5. Sub-division** I have 1 apple to be shared evenly between 2 people. How much apple will each person get?

As can be observed from Table 4.2 the homogeneous groups (i.e., the Average Ability and the Higher Ability) who were involved in the second five week session of the research solved eleven and ten problems each. This was due to a number of disruptions at the school during this period. It was therefore not possible to see these children for the planned number of periods. As a result of the limited time, it was not always possible for the researcher to clarify certain issues with the children, related to their strategies.

TABLE 4.2

PROBLEM TYPES - FREQUENCY USED

MULTIPLICATION	G1-M.A.	G1-A.A.	G2-M.A	G2-H.A.
Repeated Addition	4	1	1	-
Factor	2	2	2	2
Rate	3	2	3	1
Cartesian Product	2	1	2	4
Array	3	1	2	-
DIVISION				
Partition	2	1	3	-
Factor	2	-	2	2
Rate	2	-	2	1
Quotition	2	1	2	-
Sub-Division	2	2	2	-
Total	*24*	*11*	*21*	*10*

Key: *G1 - Grade 1* *G2 - Grade 2*

M.A. - Mixed Ability *A.A - Average Ability* *H.A. - Higher Ability*

However the researcher did ensure that the Grade 1 pupils and the Grade 2 pupils as a whole were given the opportunity to solve all ten problem types at least twice. The above problem did not in any way affect the aims of this study as it was not the intention of the researcher to make a comparison within each grade, i.e. between the ability groups in each grade with regard to their solution strategies.

Number size

Mulligan (1991: 106) identified number size as a significant factor in determining problem difficulty, level of performance and strategy use. Taking this into consideration the following number triples were used: For Grade 1 multiplication (3,2,6; 2,3,6; 2,4,8; 2,5,10; 4,3,12; 3,4,12), for division (8,4,2; 8,2,4; 6,2,3; 9,3,3; 1,2,1/2; 3,6,1/2). For Grade 2 multiplication (6,4,24; 6,3,18; 3,4,12; 2,3,6; 3,5,5; 5,4,20; 5,6,30; 3,2,6; 5,5,25; 4,3,12), for division (12,3,4; 15,3,5; 18,3,6; 2,4,1/2; 9,3,3; 12,4,3; 20,4,5; 10,5,2; 16,4,4). The number triple: 2,2,4 was avoided in all cases except for the Cartesian Product problems, because the actual numbers involved were not evident in this problem type. The number size for Grade 1 was 2-12 and for Grade 2, 2-30. The number size was increased with performance.

COLLECTION OF DATA

The data in this study was gathered by qualitative research methodologies, including observation and interaction with small groups of children. Additional data sources include video tapes of some lessons, audio tapes of all lessons and copies of all the children's written work.

According to Edson (in Sherman et al, 1988: 3) qualitative inquiry is a form of "*moral discourse*", an attempt to "*understand ourselves in relation to the larger world*". The larger world includes both the past and the present and historical study is a way to reveal the relation. Experience is studied as a whole, not in isolation from the past and the present. According to Shimahara (in Sherman et al, 1988: 5) human behaviour / experience is shaped in context, i.e it is context-specific and that events cannot be adequately understood if isolated from their context. The contexts of inquiry are not to be contrived or constructed or modified, they are natural and must be taken as they are found. The aim of qualitative research is not verification of a predetermined idea, but discovery that leads to insights. Thus qualitative research focuses on natural settings not abstract or theoretical settings. Qualitative research presumes nothing but focuses on the perspective of those being studied. Qualitative implies a direct concern with experience as it is *"lived"* or *"felt"* or *"undergone"* (Sherman et al, 1988: 7).

Qualitative methodology makes it possible to get close to the data, thus allowing the data themselves to produce certain levels of explanation (Burgess, 1982; Marshall & Rossman, 1989). The implication of this is the opportunity for the researcher to interact with the respondents without imposing preconceived standards of behaviour on them.

Being concerned with the wholeness of experience however does not mean that qualitative research merely attempts to document all that can be known about an event or an individual in relation to the larger world. Such experience becomes relevant only when interpreted in terms of a frame of reference that can encompass them and give form and shape to a conception of the whole (Bellah et al, in Sherman et al, 1988: 46). Experience doesn't speak for itself, likewise there is no existing or determinate order that encompasses all experiences. Qualitative researchers must employ an interpretive frame of reference in order to bring meaning to experience. In this sense qualitative inquiry is not merely a search for knowledge for knowledge's sake, but a search for the significance of knowledge.

ANALYSIS OF DATA

The children's solution strategies will be analysed according to the categories identified by Mulligan (1992), Kouba (1989) and Murray et al (1992). After a detailed analysis of the strategies in order to identify the different intuitive strategies that children use when solving multiplication and division word problems, the levels of strategy use will be described by integrating the level of modelling and the level of abstractness of the solution strategies. This analysis will be synthesized as a general overview of common strategies. This information will be quantified and an indepth qualitative analysis of the solution strategies and problem types will be carried out to determine the relationship between semantical structure and strategies used. The role that social interaction played on the construction and evolution of children's problem solving strategies will be discussed briefly. The strategies will be analysed to identify intuitive models in multiplication and division word problems.

CHAPTER 5

RESULTS : GENERAL PERFORMANCE AND STRATEGY USE

This chapter focusses on a quantitative analysis of the children's solution strategies in multiplication and division word problems. An overview of the following three aspects of the analysis of results will be discussed:

- the children's individual profiles and their overall performance level,
- primary strategies used, and
- levels of strategy use.

A more detailed discussion on their intuitive strategies and performance will follow in Chapter Six

INDIVIDUAL PROFILES AND THE OVERALL PERFORMANCE LEVEL

As discussed in Chapter Four, no comparison between the ability groups with regard to performance in the different grades was intended. Therefore this chapter will focus on a discussion on the performance of each of the two grades. However, the results of the two ability groups in each grade will be presented separately but involve a common discussion. The reasons for this are firstly, that at times different word problems in each problem structure were solved by the different ability groups and secondly, the number of lessons held differed for each ability group. As discussed earlier, fewer problem structures were dealt with in the second week due to problems experienced at the school where the study was carried out.

As mentioned in Chapter 4 the classification scheme on multiplication and division word problems developed by Mulligan (1992: 28) which entails ten categories was used in this study. In some cases the exact same word problems were used, but in most cases they were varied according to number size and language, keeping in mind the ten categories. Table 4.1 represents some word problems used in each of the ten categories. A list of all the word problems used appears in the Appendix.

The results of the children's performance are tabulated and followed by a brief discussion. The symbols /, *x* and *a* in the tables stand for correct, incorrect and absent respectively.

GRADE 1 - MULTIPLICATION

Table 5.1 : Repeated Addition - Mixed Ability

Lesson Pupil	1	2	3	4	*Total Correct*
Kelvin	/	/	/	/	*4 (100%)*
Melloney	/	/	/	/	*4 (100%)*
Sara	x	x	/	/	*2 (50%)*
Byron	x	x	/	/	*2 (50%)*
Jolene	x	x	/	/	*2 (50%)*
Rowan	/	x	x	x	*1 (25%)*
Total Correct	*3 (50%)*	*2 (33%)*	*5 (83%)*	*5 (83%)*	*15 (63%)*

Table 5.2 : Repeated Addition - Average Ability

Lesson **Pupil**	**1**	***Total Correct***
Brandon	/	*1 (100%)*
Russel	/	*1 (100%)*
Roxanne	/	*1 (100%)*
Byron	/	*1 (100%)*
Total Correct	*4 (100%)*	*4 (100%)*

In the first and second lessons 50% and 33% respectively, of the children in the Mixed Ability group were able to solve the Repeated Addition problems (see Table 5.1). By the last lesson all the children except one (Rowan) were able to solve the problems. Rowan was only able to solve this problem structure on the first day. Kelvin and Melloney solved the problems on all four occasions. All the pupils in the Average Ability group solved this problem. There were 68% appropriate strategies for the Grade 1 as a whole (see Table 5.36).

Table 5.3 : Factor - Mixed Ability

Lesson Pupil	1	2	*Total Correct*
Kelvin	x	/	*1 (50%)*
Melloney	x	/	*1 (50%)*
Sara	x	x	*0 (0%)*
Byron	x	x	*0 (0%)*
Jolene	x	x	*0 (0%)*
Rowan	x	x	*0 (0%)*
Total Correct	*0 (0%)*	*2 (33%)*	*2 (17%)*

Table 5.4 : Factor - Average Ability

Lesson Pupil	1	2	*Total Correct*
Brandon	x	x	*0 (0%)*
Russel	x	x	*0 (0%)*
Roxanne	x	x	*0 (0%)*
Byron	x	x	*0 (0%)*
Total Correct	*0 (0%)*	*0 (0%)*	*0 (0%)*

In the first lesson on Factor problems none of the children in the Mixed Ability could solve the problem (see Table 5.3). By the second lesson 33% (i.e. Kelvin and Melloney - Higher Ability) were able to. Sara, Byron, Jolene and Rowan were very confused and they did not even attempt to solve the problems. None of the Average Ability pupils could solve this problem structure (see Table 5.4). They too expressed confusion and were not keen on persevering with the problem (see discussion on page 84). Only 10% of the strategies for Grade 1 were appropriate (see Table 5.36).

Table 5.5 : Rate - Mixed Ability

Lesson Pupil	1	2	3	*Total Correct*
Kelvin	x	/	/	*2 (33%)*
Melloney	x	/	/	*2 (33%)*
Sara	x	x	/	*1 (17%)*
Byron	x	x	/	*1 (17%)*
Jolene	x	x	/	*1 (17%)*
Rowan	x	x	a	*0 (0%)*
Total Correct	*0 (0%)*	*2 (33%)*	*5 (100%)*	*7 (41%)*

Table 5.6 : Rate - Average Ability

Lesson Pupil	1	*Total Correct*
Brandon	/	*1 (100%)*
Russel	x	*0 (0%)*
Roxanne	/	*1 (100%)*
Byron	/	*1 (100%)*
Total Correct	*3(75%)*	*3 (75%)*

In the first lesson on Rate problems none of the children in the Mixed Ability group solved the problem (see Table 5.5). By the second lesson 33% of the pupils were able to do so and by the third lesson all the children who were present solved the problem. Rowan was absent for the last lesson and he was unable to solve the problem in the previous two lessons. 75% of the Average Ability pupils solved this problem. Russel was unable to. On the whole Grade 1 had 48% appropriate strategies (see Table 5.36).

Table 5.7 : Cartesian Product - Mixed Ability

Lesson Pupil	1	2	*Total Correct*
Kelvin	/	/	*2 (100%)*
Melloney	x	/	*1 (50%)*
Sara	x	/	*1 (50%)*
Byron	x	/	*1 (50%)*
Jolene	/	/	*2 (100%)*
Rowan	/	x	*1 (50%)*
Total Correct	*3 (50%)*	*5 (83%)*	*8 (67%)*

Table 5.8 : Cartesian Product - Average Ability

Lesson Pupil	1	*Total Correct*
Brandon	/	*1 (100%)*
Russel	/	*1 (100%)*
Roxanne	/	*1 (100%)*
Byron	/	*1 (100%)*
Total Correct	*4 (100%)*	*4 (100%)*

Problem 1 of the Cartesian Product problems was solved by 50% of the children in the Mixed Ability group and problem 2 by 83% (see Table 5.7). Rowan solved the problem in the first lesson but not in the second lesson. Kelvin and Jolene solved the problems on both occasions. All pupils in the Average Ability group solved this problem structure (see Table 5.8). 75% of the Grade 1 strategies were appropriate (see Table 5.36).

Table 5.9 : Array - Mixed Ability

Lesson Pupil	1	2	3	*Total Correct*
Kelvin	/	/	/	*3 (100%)*
Melloney	/	/	/	*3 (100%)*
Sara	/	/	/	*3 (100%)*
Byron	/	/	/	*3 (100%)*
Jolene	/	/	/	*3 (100%)*
Rowan	x	a	x	*0 (0%)*
Total Correct	*5 (83%)*	*5 (100%)*	*5 (83%)*	*15 (88%)*

Table 5.10 : Array - Average Abilitry

Lesson Pupil	1	*Total Correct*
Brandon	/	*1 (100%)*
Russel	/	*1 (100%)*
Roxanne	/	*1 (100%)*
Byron	/	*1 (100%)*
Total Correct	*4 (100%)*	*4 (100%)*

All the children in both the abilities groups except Rowan in the Mixed Ability group solved all the Array problems (see Table 5.9 and Table 5.10). On the whole 90% of the responses were appropriate (see Table 5.36).

DIVISION

Table 5.11 : Partition - Mixed Ability

Lesson Pupil	1	2	*Total Correct*
Kelvin	/	/	*2 (100%)*
Melloney	x	/	*1 (50%)*
Sara	/	/	*2 (100%)*
Byron	x	/	*1 (50%)*
Jolene	/	/	*2 (100%)*
Rowan	x	/	*1 (100%)*
Total Correct	*3 (50%)*	*6 (100%)*	*9 (75%)*

Table 5.12 : Partition - Average Ability

Lesson Pupil	1	*Total Correct*
Brandon	/	*1 (100%)*
Russel	/	*1 (100%)*
Roxanne	/	*1 (100%)*
Byron	/	*1 (100%)*
Total Correct	*4 (100%)*	*4 (100%)*

Problem 1 on Partition problems was solved by 50% of the pupils in the Mixed Ability group, and problem 2 by all the children (see Table 5.11). Kelvin, Sara and Jolene solved this problem structure on both occasions. All the Average Ability pupils were able to solve the Partition problem (see Table 5.12). For Grade 1 as a whole 81% of the strategies were appropriate (see Table 5.36)

Table 5.13 : Factor - Mixed Ability

Lesson **Pupil**	**1**	**2**	***Total Correct***
Kelvin	x	x	*0 (0%)*
Melloney	x	x	*0 (0%)*
Sara	x	x	*0 (0%)*
Byron	x	x	*0 (0%)*
Jolene	x	x	*0 (0%)*
Rowan	x	x	*0 (0%)*
Total Correct	*0 (0%)*	*0 (0%)*	*0 (0%)*

In both lessons involving Factor problems none of the children in the Mixed Ability group were able to solve the problems (see Table 5.13). The children were confused and the problems were abandoned (see discussion on page 103-104). This problem structure was presented only to the Mixed Ability group.

The Rate problem, like the Factor problem was only presented to the Mixed Ability group. 33% of these pupils solved problem 1 and 83% problem 2 (see Table 5.14). Rowan was unable to solve the problems on both occasions. Kelvin and Jolene solved both the problems. The appropriate strategies made up 58% of the total responses.

In the Mixed Ability group 67% of the pupils solved the Quotition problem in the first lesson and 60% in the second lesson (see Table 5. 15). Kelvin and Melloney solved both problems. Rowan did not solve either. In the Average Ability group all pupils solved this problem structure (see Table 5.16). There were 73% appropriate strategies among the Grade 1 pupils (see Table 5.36).

Table 5.14 : Rate - Mixed Ability

Lesson **Pupil**	**1**	**2**	***Total Correct***
Kelvin	/	/	*2 (100%)*
Melloney	x	/	*1 (50%)*
Sara	x	/	*1 (50%)*
Byron	x	/	*1 (50%)*
Jolene	/	/	*2 (100%)*
Rowan	x	x	*0 (100%)*
Total Correct	*2 (33%)*	*5 (83%)*	*7 (58%)*

Table 5.15 : Quotition - Mixed Ability

Lesson Pupil	**1**	**2**	***Total Correct***
Kelvin	/	/	*2 (100%)*
Melloney	/	/	*2 (100%)*
Sara	x	/	*1 (50%)*
Byron	/	a	*1 (100%)*
Jolene	/	x	*1 (50%)*
Rowan	x	x	*0 (0%)*
Total Correct	*4 (67%)*	*3 (60%)*	*7 (64%)*

Table 5.16 : Quotition -Average Ability

Lesson Pupil	1	*Total Correct*
Brandon	/	*1 (100%*
Russel	/	*1 (100%)*
Roxanne	/	*1 (100%)*
Byron	/	*1 (100%)*
Total Correct	*4 (100%)*	*4 (100%)*

Table 5.17 : Sub-division - Mixed Ability

Lesson Pupil	1	2	*Total Correct*
Kelvin	/	/	*2 (100%)*
Melloney	x	/	*1 (50%)*
Sara	/	/	*2 (100%)*
Byron	x	/	*1 (50%)*
Jolene	x	/	*1 (50%)*
Rowan	x	a	*0 (0%)*
Total Correct	*2 (33%)*	*5 (100%)*	*7 (64%)*

Problem 1 of the Sub-division problems was solved by 33% of the children in the Mixed Ability group and problem 2 by all children present (see Table 5.17). Rowan was absent for lesson 2 and he was unable to solve problem 1. Kelvin and Sara solved the problems on both occasions. In the Average Ability group 50% of the children solved problem 1 and all were able to solve problem 2 (see Table 5.18). Of the total responses 68% were appropriate strategies (see Table 5.36).

Table 5.18 : Sub-division -Average Ability

Lesson Pupil	1	2	*Total Correct*
Brandon	/	/	*2 (100%)*
Russel	x	/	*1 (50%)*
Roxanne	x	/	*1 (50%)*
Byron	/	/	*2 (100%)*
Total Correct	*2 (50%)*	*4 (100%)*	*6 (75%)*

Discussion

An analysis of the individual profiles for Grade 1 indicates that the children were able to solve the problems 59% of the time, i.e. they solved 106 out of 180 problems using their own methods (see Table 5.36). All the pupils except Rowan were able to solve problems of all structures except the Factor problems at some stage or the other during the experimental period. Rowan could only solve 9% of the problems, i.e. 2 out of 23 problems. As mentioned earlier the Factor problems, both in multiplication and division seemed to pose the greatest problem for the children. Only 20% of the children, i.e. 2 out of 10 could solve the Multiplying Factor problem structure and none were able to solve the Division Factor problems. An obvious pattern with all the children except Rowan was an improvement in performance from the fourth lesson onwards. Rowan, a Lower Ability pupil, who displayed behavioural problems and was very easily distracted, only solved two problem structures, Cartesian Product and Partition, on one occasion each. Kelvin, a Higher Ability pupil was able to solve every problem structure except the Division Factor problem (i.e. 83% of the problems). Among the Average Ability pupils only Russel and Roxanne were unable to solve all the problems presented to them (i.e. 2 and 1 problems respectively). Byron, the Average Ability pupil (who according to his teacher was experiencing a problem in mathematics) only solved 11 out of 24 problems (i.e. 46%) while in the Mixed Ability group. However, while in the Average Ability group during the second block session he was able to solve all the problems presented to him except the Multiplication Factor problem.

GRADE 2

MULTIPLICATION

Table 5.19 : Repeated Addition - Mixed Ability

Lesson **Pupil**	**1**	**2**	***Total Correct***
Calvin	/	/	*2 (100%)*
Sally	/	/	*2 (100%)*
Kerry Lee	/	/	*2 (100%)*
James	/	/	*2 (100%)*
Michael M.	/	a	*1 (100%)*
Zoey	x	x	*0 (0%)*
Total Correct	*5 (83%)*	*4 (80%)*	*9 (82%)*

All pupils In the Mixed Ability group except Zoey were able to solve the Repeated Addition problems (see Table 5.19). Zoey was absent in the second lesson. This problem was only presented to the Mixed Ability group. Of the total responses 82% were appropriate (see Table 5.36).

None of the pupils in the Mixed Ability group could solve problem 1 on Factor problems (see Table 5.20). Only Calvin solved problem 2. All the pupils in the Higher Ability group were able to solve this problem structure (see Table 5.21). The appropriate strategies made up 47% of the total responses (see Table 5.36).

Table 5.20 : Factor - Mixed Ability

Lesson Pupil	1	2	*Total Correct*
Calvin	x	/	*1 (50%)*
Sally	x	x	*0 (0%)*
Kerry Lee	x	x	*0 (0%)*
James	x	x	*0 (0%)*
Michael M.	x	a	*0 (0%)*
Zoey	x	x	*0 (0%)*
Total Correct	*0 (0%)*	*1 (20%)*	*1 (9%)*

Table 5.21 : Factor - Higher Ability

Lesson Pupil	1	2	*Total Correct*
Michael S.	/	/	*2 (100%)*
Martin	/	/	*2 (100%)*
Samantha	/	/	*2 (100%)*
Angela	/	/	*2 (100%)*
Total Correct	*4 (100%)*	*4 (100%)*	*8 (100%)*

Table 5.22 : Rate - Mixed Ability

Lesson Pupil	**1**	**2**	**3**	***Total Correct***
Calvin	/	/	/	*3 (100%)*
Sally	/	/	/	*3 (100%)*
Kerry Lee	/	/	/	*3 (100%)*
James	/	/	/	*3 (100%)*
Michael M.	x	/	/	*2 (67%)*
Zoey	x	x	x	*0 (0%)*
Total Correct	*4 (67%)*	*5 (83%)*	*5 (83%)*	*14 (78%)*

Table 5.23 : Rate - Higher Ability

Lesson **Pupil**	**1**	***Total Correct***
Michael S.	/	*1 (100%)*
Martin	/	*1 (100%)*
Samantha	/	*1 (100%)*
Angela	/	*1 (100%)*
Total Correct	*4 (100%)*	*4 (100%)*

Problem 1 on Rate problems was solved by 67% of the pupils in the Mixed Ability group, problems 2 and 3 by all the pupils except Zoey (see Table 5.22). All the pupils in the Higher Ability group solved this problem structure (see Table 5.23). Of the total responses, 82% were appropriate strategies (see Table 5.36).

Table 5.24 : Cartesian Product - Mixed Ability

Lesson Pupil	1	2	*Total Correct*
Calvin	/	/	*2 (100%)*
Sally	/	/	*2 (100%)*
Kerry Lee	/	/	*2 (100%)*
James	x	/	*1 (50%)*
Michael M.	/	/	*2 (100%)*
Zoey	x	x	*0 (0%)*
Total Correct	*4 (67%)*	*5 (83%)*	*9 (75%)*

Table 5.25 : Cartesian Product - Higher Ability

Lesson Pupil	1	2	3	4	*Total Correct*
Michael S.	x	x	/	/	*2 (50%)*
Martin	x	/	/	/	*3 (75%)*
Samantha	x	/	/	/	*3 (75%)*
Angela	x	/	/	/	*3 (75%)*
Total Correct	*0 (0%)*	*3 (75%)*	*4 (100%)*	*4 (100%)*	*11 (69%)*

In the Mixed Ability group 67% of the pupils solved problem 1 on the Cartesian Produc problem and 83% problem 2 (see Table 5.24). Zoey did not solve a problem on either occasioı None of the children in the Higher Ability group could solve this problem structure in the fir lesson (see Table 5.25). However, in the subsequent lessons 92% of the strategies we appropriate, i.e. 11 out 12 responses. On the whole, Grade 2 had 71% appropriate responses (s Table 5.36).

Table 5.26 : Array - Mixed Ability

Lesson Pupil	1	2	*Total Correct*
Calvin	/	/	*2 (100%)*
Sally	/	/	*2 (100%)*
Kerry Lee	/	/	*2 (100%)*
James	/	/	*2 (100%)*
Michael M.	/	/	*2 (100%)*
Zoey	x	/	*1 (50%)*
Total Correct	*5 (83%)*	*6 (100%)*	*11 (92%)*

Only the Mixed Ability group was presented with the Array problems. Zoey was the only pupil who did not solve this problem structure on one occasion (i.e. in the first lesson) (see Table 5.26). The Array problem was one of two problems that she was able to solve during the experimental period. The appropriate responses made up 92% of the total responses (see Table 5.36).

DIVISION

Problem 1 on Partition problems was solved by 33% of the children in the Mixed Ability group, problem 2 by 67% and problem 3 by 83% (see Table 5. 27). Zoey did not solve the problem on all three occasions. Only the Mixed Ability group dealt with this problem structure. For Grade 2, there were 61% appropriate strategies (see Table 5.36).

None of the pupils in the Mixed Ability group could solve the Factor problem (see Table 5.28). Among the Higher ability pupils, three were able to (see Table 5.29). Only 15% of the strategies were appropriate (see Table 5.36). Refer to discussion on page 103-104.

Table 5.27 : Partition - Mixed Ability

Lesson **Pupil**	1	2	3	***Total Correct***
Calvin	/	/	/	*3 (100%)*
Sally	/	/	/	*3 (100%)*
Kerry Lee	x	/	/	*2 (67%)*
James	x	/	/	*2 (67%)*
Michael M.	x	x	/	*1 (33%)*
Zoey	x	x	x	*0 (0%)*
Total Correct	*2 (33%)*	*4 (67%)*	*5 (83%)*	*11 (61%)*

Table 5.28 : Factor - Mixed Ability

Lesson Pupil	**1**	**2**	***Total Correct***
Calvin	x	x	*0 (0%)*
Sally	x	x	*0 (0%)*
Kerry Lee	x	x	*0 (0%)*
James	x	x	*0 (0%)*
Michael M.	x	x	*0 (0%)*
Zoey	x	x	*0 (0%)*
Total Correct	*0 (0%)*	*0 (0%)*	*0 (0%)*

Table 5. 29 : Factor - Higher Ability

Lesson Pupil	**1**	**2**	***Total Correct***
Michael S.	x	x	*0 (0%)*
Martin	/	x	*1 (50%)*
Samantha	x	/	*1 (50%)*
Angela	x	/	*1 (50%)*
Total Correct	*1 (25%)*	*2 (50%)*	*3 (38%)*

Table 5.30 : Rate - Mixed Ability

Lesson **Pupil**	**1**	**2**	***Total Correct***
Calvin	x	/	*1 (50%)*
Sally	/	/	*2 (100%)*
Kerry Lee	/	/	*2 (100%)*
James	x	/	*1 (50%)*
Michael M.	x	/	*1 (0%)*
Zoey	x	x	*0 (0%)*
Total Correct	*2 (33%)*	*5 (83%)*	*7 (58%)*

Problem 1 of the Rate problem was solved by 33% of the pupils in the Mixed Ability group and problem 2 by 83% (see Table 5.30). Zoey could not solve the problems on both occasions. All pupils in the Higher Ability group solved this problem structure (see Table 5.31). The appropriate strategies made up 69% of the total responses on the Rate problem (see Table 5.36).

Table 5.31 : Rate - Higher Ability

Lesson Pupil	1	*Total Correct*
Michael S.	/	*1 (100%)*
Martin	/	*1 (100%)*
Samantha	/	*1 (100%)*
Angela	/	*1 (100%)*
Total Correct	*4 (100%)*	*4 (100%)*

Table 5.32 : Quotition - Mixed Ability

Lesson Pupil	1	2	*Total Correct*
Calvin	/	/	*2 (100%)*
Sally	/	/	*2 (100%)*
Kerry Lee	/	x	*1 (50%)*
James	x	/	*1 (50%)*
Michael M.	x	/	*1 (50%)*
Zoey	x	x	*0 (0%)*
Total Correct	*3 (50%)*	*4 (67%)*	*7 (58%)*

The Quotition problem structure was presented to only the Mixed Ability pupils, 50% of whom solved problem 1 and 67% problem 2 (see Table 5.32). Zoey did not solve either of the problems. 58% of the strategies were appropriate (see Table 5.36)

None of the pupils in the Mixed Ability group were able to solve problem 1 on Sub-division, but they were all able to solve problem 2 (see Table 5.33). This problem structure was only presented to the Mixed Ability group. 55% of the strategies were appropriate (see Table 5.36).

Table 5.33 : Sub-division - Mixed Ability

Lesson Pupil	1	2	*Total Correct*
Calvin	x	/	*1 (50%)*
Sally	x	/	*1 (50%)*
Kerry Lee	x	/	*1 (50%)*
James	x	/	*1 (50%)*
Michael M.	a	/	*1 (100%)*
Zoey	x	/	*1 (50%)*
Total Correct	*0(0%)*	*6(100%)*	*6 (55%)*

Discussion

During the experimental period all pupils except Zoey were able to solve all the problem structures except the Factor problems. Zoey, a Lower Ability pupil who experienced problems working with the other pupils in the group, only solved 2 out of 22 problems, i.e. 9%. Five problem structures were dealt with in the Higher Ability group. The problems that were selected were those that posed the greatest difficuty to the Grade 2 Mixed Ability group. All the pupils in the Higher Ability group were able to solve the Multiplication Factor problem on both occasions. This problem was only solved by the Higher Ability pupils in both ability groups, i.e. 9 out of 10 Higher Ability pupils (90%). Only three pupils in the Higher Ability managed to solve the Division Factor problems once. Appropriate strategies were used on 105 out of 169 multiplication and division word problems, i.e. 62% (see Table 5.36). As with the Grade 1 pupils, there was a general improvement in all the children's performance (except Zoey's) as the lessons progressed, and on all problem types except the Factor problems. Again as with the Grade 1 pupils the Factor problems (both multiplication and division) seemed to pose the greatest problem.

Summary

An obvious pattern, as mentioned earlier, with all the children in both grades except Rowan in Grade 1 and Zoey in Grade 2, was an improvement in performance from the fourth lesson onwards excluding the Factor problems. Some reasons (which have been discussed in detail elsewhere in this study) that could be attributed to this were:

- the children worked in interactive groups,
- they had the opportunity to use specific aids, and
- the teacher's role changed to that of facilitator.

No change was observed with Rowan and Zoey. As the lessons progressed fewer problems were dealt with because the children were spending more time on solving each problem. On the whole all groups performed poorly on the Factor problems. Albeit their performance was better on the Multiplication Factor problems than on the Division Factor problems.

The following discussion will focus on the primary strategies used by the children for the different problem structures.

STRATEGIES USED ACROSS PROBLEM STRUCTURES

Mulligan's (1991:127) idea of "primary strategies" was adopted. The main strategies used by the children, for each problem type is presented in three tables, i.e. two separate tables in which the primary strategies used by Grade 1 and 2 are reflected as shown in Table 5.34 and Table 5.35 respectively, and the appropriate strategies used by both the grades as shown in Table 5.36. The numbers in bold print refer to the percentages.

Table 5.36 reflects the number and percentage of appropriate strategies employed by the children across all the problem types. Refer to the key on page 63. The number of appropriate strategies is reflected above the total number of strategies for that specific grade and problem type, e.g. 19/28 for Repeated Addition means 19 out of 28.

Table 5. 34

Strategies - Grade 1

	M	PR	NR	KAF	KMF	MC	SR	G	S	CA	DC	SC	H	EA	*TO*
X															
RA	5 **12**	7 **17**	2 **5**	1 **2**	-	2 **5**	4 **10**	11 **26**	-	9 21	-	1 **2**	-	-	*42*
F	-	2 **40**	-	-	-	1 **20**	-	1 **20**	-	1 **20**	-	-	-	-	*5*
R	8 **30**	2 **7**	2 **7**	2 7	1 **4**	1 **4**	1 **4**	7 **26**	-	3 **11**	-	-	-	-	*27*
CP	7 **27**	3 **12**	1 **4**	-	-	-	-	9 35	-	6 **23**	-	-	-	-	*26*
A	10 **24**	3 **7**	-	3 **7**	-	-	1 **2**	12 **29**	-	12 **29**	-	-	-	-	*41*
÷															
P	5 **18**	5 **18**	-	-	-	2 **7**	-	5 **18**	3 **11**	4 **14**	4 **14**	-	-	-	*28*
F	-	-	-	-	-	-	-	-	-	-		-	-	-	*0*
R	4 **17**	1 **4**	2 **8**	1 **4**	-	1 **4**	-	4 **17**	2 **8**	4 **17**	4 **17**	-	-	1 **4**	*24*
Q	4 **14**	2 **7**	-	1 **4**	-	1 **4**	-	6 **21**	2 **7**	6 **21**	6 **21**	-	-	-	*28*
SD	4 **18**	2 **9**	-	-	-	-	-	-	4 **18**	6 **27**	-	-	6 **27**	-	*22*
TO	*47* ***19***	*27* ***11***	*7* ***3***	*8* ***3***	*1* ***0***	*8* ***3***	*6* ***2***	*55* ***23***	*11* ***5***	*51* ***21***	*14* ***6***	*1* ***0***	*6* ***2***	*1* ***0***	*243*

Key:

A - Array	*F - Factor*	*NR - Numerical Representation*	*S - Sharing*
CA - Counting All	*G - Grouping*	*OTMC - One-to-many correspondence*	*SD - Sub-division*
CO - Counting On	*H - Halving*	*P - Partition*	*SC - Skip Counting*
CP - Cartesian Product	*KAF - Known Addition Fact*	*PR - Pictorial Representation*	*SR - Symbolic Representation*
DC - Double Counting	*KMF - Known X Fact*	*Q - Quotition*	*TO- total*
E - Estimation	*M - Modelling*	*R - Rate*	*X - multiplication*
EA - Estimate-and-adjust	*MC - Mental Computation*	*RA - Repeated Addition*	*÷ - division*
	MTMC - Many-to-Many Correspondence		

Table 5. 35

Strategies - Grade 2

	M	P R	N R	K A F	K M F	M C	S R	G	S	C A	C O	SC	DC	R A	H	OT MC	M T M C	E A	E	*TO*
X																				
R **A**	5 **15**	4 **12**	1 **3**	-	-	1 **3**	1 **3**	10 **30**	-	1 **3**	-	9 **27**	-	1 **3**	-	-	-	-	-	*33*
F	8 **20**	-	-	-	8 **20**	-	8 **20**	8 **20**	-	7 **18**	-	1 **3**	-	-	-	-	-	-	-	*40*
R	11 **17**	9 **14**	-	-	2 **3**	-	4 **6**	18 **27**	-	15 **23**	4 **6**	-	1 **2**	2 **3**	-	-	-	-	-	*66*
C **P**	2 **4**	13 **23**	1 **2**	1 **2**	2 **4**	1 **2**	7 **13**	8 **14**	-	10 **18**	-	5 **9**	-	-	-	-	6 **1** **1**	-	-	*56*
A	7 **14**	8 **16**	-	-	-	2 **4**	-	15 **31**	-	15 **31**	-	-	-	-	-	-	-	2 **4**	-	*49*
÷																				
P	8 **19**	3 **7**	-	-	-	-	-	6 **14**	4 **10**	9 **21**	-	-	6 **14**	-	-	1 **2**	-	4 **1** **0**	1 **2**	*42*
F	3 **19**	-	-	-	1 **6**	-	1 **6**	3 **19**	-	3 **19**	-	-	3 **19**	-	-	2 **13**	-	-	-	*16*
R	8 **17**	1 **2**	-	2 **4**	-	-	3 **6**	7 **15**	6 **13**	6 **13**	-	2 **4**	6 **13**	-	-	2 **4**	-	3 **6**	2 **4**	*48*
Q	5 **10**	7 **13**	-	-	-	-	-	12 **21**	2 **4**	12 **21**	-	-	12 **21**	-	-	5 **9**	-	-	1 **2**	*56*
SD	3 **14**	2 **9**	-	-	-	-	-	-	4 **18**	3 **14**	-	-	-	-	5 **23**	5 23	-	-	-	*22*
T ***O***	*60* *14*	*47* *11*	*2* ***0***	*3* ***0***	*13* *3*	*4* *1*	*24* *6*	*87* *20*	*16* *4*	*81* *19*	*4* *1*	*17* *4*	*28* *7*	*3* ***0***	*5* *1*	*15* *4*	*6* *1*	*9* *2*	*4* *1*	*428*

Refer to key on previous page.

Table 5.36 - Appropriate Strategies

GRADES	1	2	*TO CORRECT*
X			
RA	19/28 **68%**	9/11 **82%**	*28/39* *72%*
F	2/20 **10%**	9/19 **47%**	*11/39* ***28%***
R	10/21 **48%**	18/22 **82%**	*28/43* ***65%***
CP	12/16 **75%**	20/28 **71%**	*32/44* *73%*
A	19/21 **90%**	11/12 **92%**	*30/33* ***91%***
TO X STRATEGIES CORRECT	*62/106* ***58%***	*67/92* *73%*	*129/198* ***65%***
÷			
P	13/16 **81%**	11/18 **61%**	*24/34* *71%*
F	0/12 **0%**	3/20 **15%**	*3/32* ***9%***
R	7/12 **58%**	11/16 **69%**	*18/28* ***64%***
Q	11/15 **73%**	7/12 **58%**	*18/27* ***67%***
SD	13/19 **68%**	6/11 **55%**	*19/30* ***63%***
TO ÷ STRATEGIES CORRECT	*44/74* ***59%***	*38/77* ***49%***	*82/151* ***54%***
TOTAL CORRECT	*106/180* ***59%***	*105/169* ***62%***	*149/243* ***61%***

On Table 5.36, it can be seen that the children's performance on multiplication was higher than for division. This is clearly evident for Grade 2. On the whole the children's performance on the Array problem type was far the best, followed by Cartesian Product, Partition and Repeated Addition. Grade 1 was able to solve a higher percentage of division problems than Grade 2 especially, for the Partition, Quotition and Sub-division problem types. On the other hand the Grade 2 pupils' performance in multiplication was much higher than the Grade 1 pupils'. They also solved a higher percentage of problems for all the multiplication problem types except Cartesian Product as well as for the Division Factor and Rate problems. Grade 2 also provided more appropriate strategies for both the Factor problem types. From the analysis of the children's strategies different levels of strategy use were identified. A discussion on this will follow.

LEVELS OF STRATEGY USE

The approach followed by Mulligan (1991: 131-134) with regard to the classification of levels of strategy use was closely followed. Six basic levels were identified from the children's intuitive strategies:

1. strategies based on direct modelling and counting, using concrete aids,
2. strategies based on counting with pictorial representation,
3. strategies based on counting with numerical representation (Olivier et al, 1992: 34),
4. strategies based on known or derived addition and multiplication facts,
5. symbolic representation, and
6. mental computation.

Table 5. 37 reflects the number of times each grade used the above six basic levels of strategies.

Table 5. 37 - Levels of Strategy Use

Levels	Grade 1	Grade 2	Total
1	47 **46%**	60 **39%**	107 **42%**
2	27 **26%**	47 **31%**	74 **29%**
3	7 **7%**	2 **1%**	9 **4%**
4 - Addition	8 **8%**	3 **1%**	11 **4%**
- Multiplication	0 **0%**	13 **8%**	13 **5%**
5	6 **6%**	24 **16%**	30 **12%**
6	8 **8%**	4 **3%**	12 **5%**
Total	*103*	*153*	*256*

Levels 1, 2 and 3 were further classified into one or more of the following **eight** categories:

1. sharing-one-by-one
2. one-to-many-correspondence
3. grouping with counting all
4. double counting
5. grouping with skip counting
6. estimate-and-adjust strategy
7. grouping with counting on, and
8. many-to-many correspondence.

Examples of the above categories are discussed below.

LEVEL I

Modelling with Counting

Modelling refers to the use of concrete material, e.g. unifix cubes, fingers, sticks or specific aids, to represent the action or relationship described in the problem.

1. Sharing-one-by-one: counting out the dividend and dealing out one by one to the specified number for the group, e.g. Sally in Grade 2 (page104) "*here's my twelve cents (coins) and I put 4 pencils and I gave them each a money and another one and another one... and then they equalled one, two, three cents. One pencil costs three cents"*.
2. One-to-many correspondence: a matching type strategy where both quantities or entities in the problem were modelled, e.g. Melloney in Grade 1 (page 84) "*I wanted to buy one pencil with three cents and I put the cents there and that was one pencil that cost three cents. And I wanted to buy another pencil so I took another pencil and it cost three cents so it was six cents"*. As she talked she placed the coins and the pencil cut-outs appropriately.
3. Grouping, counting all: formation of equivalent groups representing the quantities in the problem with one-by-one counting to calculate the total for multiplication, e.g. Sara in Grade 1 (page 72) after making 5 groups of two, the child counted "*one, two, three, four, five, six, seven, eight, nine, ten"*. For division, counting all may have occurred to check the dividend after grouping or before grouping, e.g. Martin in Grade 2 (page 102) " *one, two, three, four; five, six, seven, eight; nine, ten, eleven, twelve,* to verbalize " *now he's got three times more than his son, his son has four."*
4. Double counting: counting the number in the dividend and (simultaneously or afterwards) counting the number in each group (sharing division), e.g. Sally in Grade 2 (page 99) or the number of groups (measurement division), e.g. Melloney in Grade 1 (page 107) (Olivier et al, 1992: 34).
5. Grouping and skip counting: this is a more sophisticated counting strategy based on counting in multiples, e.g. Kelvin in Grade 1 (page 73) "*two, four, six, eight, ten*".
6. Estimate-and-adjust : formation of equivalent groups by estimation where the number in each group was unknown. Olivier et al (1992: 35) refer to this as an "*estimate-and-adjust*" strategy, e.g. James in Grade 2 (page 99) " *I put two here and I put ten, twelve and then I took three and then I put another three and then another three and then I had three left. So I gave them each one more and I got four*".

7. Grouping with counting on: formation of equivalent groups representing the quantities in the problem, emphasizing the number in the first group, then counting in ones, e.g. Calvin in Grade 2 (page 79) " *six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen*".

LEVEL 2

Pictorial Representation with Counting

For a pictorial representation the problem context is drawn in greater or lesser detail and then solved by further drawing in the actions needed (Olivier et al, 1992: 34).

1. Grouping with counting all: illustration of equivalent groups representing the quantities in the problem with one-by-one counting as used at Level 1.
2. One-to-many correspondence: illustration of both the entities in the problem and matching as used at Level 1.
3. Grouping with counting on: illustration of the problem and counting on as used at Level 1.
4. Grouping with skip counting: as used at Level 1 with illustration.
5. Many-to-many correspondence: this strategy was only identified for a specific Cartesian Product problem (i.e. the matching of two items of clothing). The children represented both objects in the problem, e.g. skirts and blouses. They then joined these by lines using the many-to-many correspondence strategy (see page 91-92 for examples).

LEVEL 3

Numerical Representation with Counting

Numerical representation refers to representation of the structure of the problem using numerals where no arithmetical operations are employed. (Olivier et al, 1992: 34).

Grouping, counting all: writing the number representing the equivalent groups in the problem with one-by-one counting as used at Level 1.

LEVEL 4

Known Facts

Known addition facts: indicated by retrieval of an addition fact for both multiplication and division, e.g. Melloney in Grade 1 (page 73) " *3 and 3 is 6*". This category also included some derived addition facts, e.g. Michael S.in Grade 2 (page 105-106) " *I knew 6 plus 6 was 12, so I added another 6 and I found that it was 18*".

Known multiplication fact: indicated by retrieval of a memorized multiplication fact, e.g. Angela in Grade 2 (page 92) "2 *times 3 is 6*".

LEVEL 5

Symbolic Representation

The solution strategy is represented as an appropriate number sentence, e.g. Michael M.in Grade 2 (page 87) for addition 6 + 6 + 6 + 6 = 24, and Martin in Grade 2 (page 87) for multiplication 6 x 4 = 24.

LEVEL 6

Mental Computation

When a child provided an answer, usually immediately after the presentation of the problem, with no obvious indication of the use of any of the above strategies.

Chapter Five has dealt with the children's general performance, by focussing on their overall performance level, the primary strategies used and the levels of strategy use. Chapter Six deals with an indepth analysis of the children's intuitive strategies, how the semantic structure of the problems affected the children's strategies, the intuitive models they used and briefly at the effect that social interaction had on the children's strategies.

CHAPTER SIX

RESULTS : SOLUTION STRATEGIES AND PROBLEM TYPES

Chapter Five focussed on the children's general performance and strategy use. This chapter will focus on the following:

1. An analysis of the children's intuitive strategies in relation to each problem type, i.e. at the different ways in which each problem structure was represented by the children as well as their explanations of their solution strategies. Examples of the strategies selected for discussion will reflect the range of appropriate / successful strategies used across problem types which are presented according to the abstractness (i.e., from least to most abstract) or in order of increasing sophistication. These strategies will be described in terms of their mathematical representation and their (implicit) underlying properties of operations (Olivier et al, 1992: 34). The common patterns among the strategies will be highlighted.
2. The effect of the semantic structure of each problem type on the children's strategy use.
3. The intuitive models used by the children.
4. The role of social interaction in the construction and evolution of these strategies.

Numbers 1 and 2 will be dealt with simultaneously followed by a discussion of numbers 3 and 4. An analysis of the primary strategies used revealed (as shown in the latter part of Chapter Five) great variations in the way the problem types were approached, i.e. with relation to the semantic structure of the problem and the use of concrete aids. Some children used the materials provided, others their fingers and still others a procedure in which no manipulation of concrete aids was evident. However what was clearly evident was the predominance of counting in some form or the other. This was also clearly evident in Anghileri's (1989: 376) study. Counting all was the most common counting strategy used.

ANALYSIS
MULTIPLICATION

As mentioned above the analysis of the children's intuitive strategies and the effect of semantic structure on their strategies will be discussed concurrently. Each problem structure will be looked at in detail. This will entail a discussion of the range of strategies used for each problem type and how the semantic structure of each of these problem types affected the children's choice of strategies. The analysis system used is based on those identified by Mulligan (1992), Kouba (1989) and Murray et al (1992). It must be **emphasized** firstly, that most of the strategies that are discussed for each of the problem types below, reflect the children's responses **after** they had calculated and at times discussed with the others in the group (refer to discussion on page 36). Secondly, that during the lessons there was discussion within the groups, which has not been included in this thesis. However, reference will be made to it when necessary.

REPEATED ADDITION (RA)

There was a range of solution strategies for the Repeated Addition problem type. As observed in Table 5.36, the majority of the children had appropriate solution strategies (i.e. 68% for Grade 1 and 82% for Grade 2). However, there were inconsistencies in the manner in which the children abstracted and explained their intuitive strategies.

1. Modelling (M)

The children modelled this problem type in two different ways. All three children whose strategies will be discussed grouped appropriately, Sara and Brandon counted all, whereas Kelvin skip counted.

Problem 1 : If I save 2 cents a day, how much will I save for 5 days?

Sara (G.1) *: I counted these* (placing the coins in twos) *and I counted them one, <u>two</u>, three, <u>four</u>, five, <u>six</u>, seven, <u>eight</u>, nine, <u>ten</u>.* Sara **grouped** one cent coins in twos and **counted all**

emphasizing the multiples of two. Anghileri (1989: 375) refers to this as tallying the groups (see page 120 for definition).

Kelvin (G.1) *: I counted in twos for five days, two, four, six, eight, ten and I wrote ten* (used coins). Kelvin placed five sets of two one cent coins (**grouping** in twos) and **skip counted**.

Problem 2 : There are 3 children at the table. Each child is given 4 crayons. How many crayons are there altogether?

Brandon (G.1) *: Twelve - I put four and four and I put another four and counted and I got twelve.* (He had placed three figures and on each figure he had placed four crayons). Brandon **grouped the specific aids** in fours, used **one-to-many correspondence** and **counted all** to calculate the total.

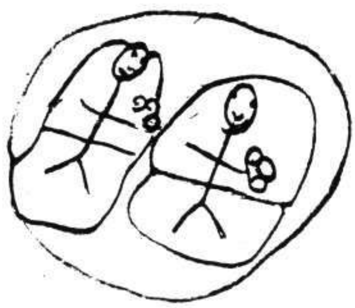
2. Pictorial Representation (PR)

Although the children drew the problem situation in some detail, there was a variety of ways in which this was done. All five children whose strategies will be discussed grouped appropriately. Four of them counted all and one skip counted.

Problem 3 : I have 2 friends. I give my friends 3 sweets each. How many sweets are there altogether?

Melloney (G.1) *: I put two men and I put three sweets in their hands and then I counted them altogether and then I said three and three is six* (Figure 6.1). Melloney represented the problem situation in exact detail, drawing two friends and placing three sweets in each one's hand. She **grouped** in threes, **counted all** and used a **known addition fact.**

Figure 6.1



Problem 4 : There are three boxes on the table. In each box there are two beads.

How many beads are there altogether?

***Calvin* (G.2)** : *I put one like big line and three big lines and then two dots for each one. Six.* (Figure 6.2). Calvin represented the object (boxes) as lines and drew two circles (**grouped** in twos) on each to represent beads, and he **counted all**.

Figure 6.2

Problem 5 : There are 3 tables in the classroom and 2 children at each table.

How many children are there altogether?

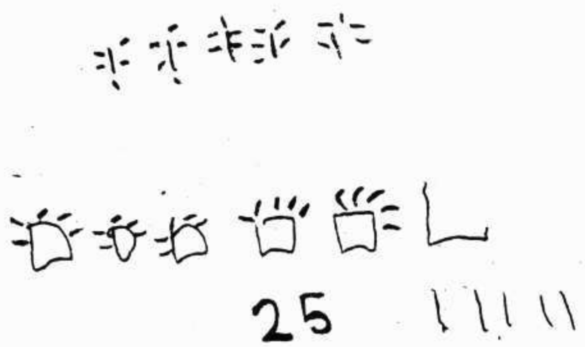
Kelvin (G.1) : *I put three tables and I put two over there and two over there and I said one, two, three, four, five, six* (pointing to the four children and then two imaginary figures - Figure 6.3). *I knew that there were two there* (pointing to the last table with no figures drawn). Kelvin's direct representation was incomplete. He combined visual and mental images to solve the problem. He **counted all** the figures that he had drawn (i.e. 2 groups of 2) as well as the two he visualized.

Figure 6.3

Problem 6 : There are 5 tables in the classroom and 5 children are seated at each table. How many children are there in the classroom?

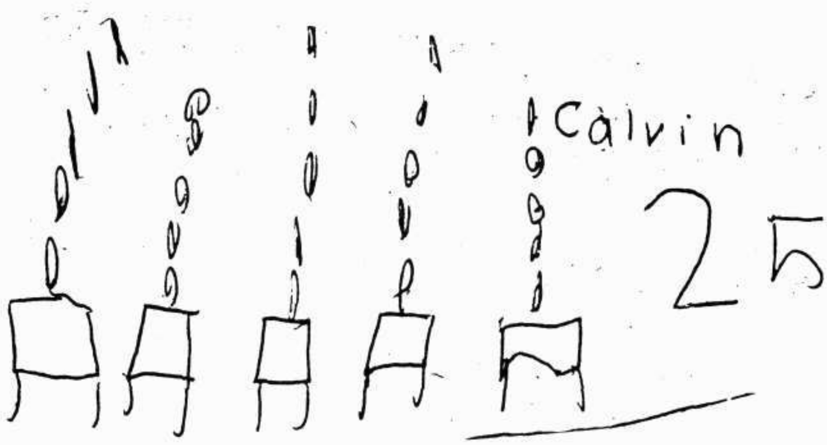
Sally (G. 2) : *I did five at each table and I got twenty five.* (Figure 6.4). Sally was observed **grouping in fives** and **counting all** to obtain the total. She represented the tables as box shaped objects and placed five strokes around each of them.

Figure 6.4



Calvin (G.2) : *I counted in fives. I put tables and I put five little dots at the back to make children and then I counted in fives, five, ten, fifteen, twenty, twenty-five. Twenty- five children.* (Figure 6.5). Calvin's representation was similar to Sally's but he drew five circles above each table as in an array. He also **skip counted** to obtain the total.

Figure 6.5



The Grade 1 children tended to represent the objects in the problem in more or less the same form in their drawings, i.e. tables and children. The Grade 2 children on the other hand, used more abstract figures, e.g. circles, lines and strokes. However one Grade 1 pupil (H.A.) seemed to use a strategy that was more sophisticated than those used by the Grade 2, i.e. Kelvin did not need to represent every child in his drawing in order to obtain the total. He seemed to visualize the rest.

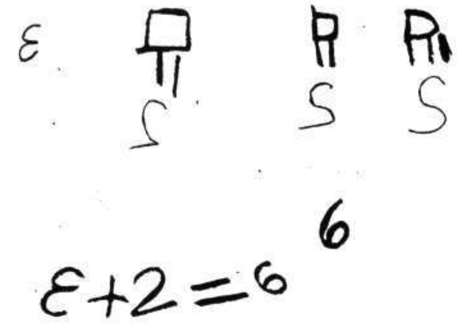
3. Numerical representation (NR)

The two strategies that are discussed belong to the same pupil, Melloney. In the second strategy she used an interesting vertical representation of the meaning of groups. This could possibly be related to her visualization of the addition algorithm in vertical form, i.e. repeated addition without the use of the addition symbol.

Refer to **Problem 5** (page 74)

Melloney : *I took 3 tables and I made one, two, three, four, five, six* (pointing to the legs on the tables drawn on her page - Figure 6.6). *I made two at each table and there was six so I put three plus two equals six.* Melloney **grouped in twos** to represent two children at each table. She **counted all** to obtain the total. Although she represented the children as a symbol (i.e. 2) she counted them as two single children. She also attempted to represent her solution as a number sentence, but used a + instead of an x sign (inappropriate **symbolic representation**). It is possible that she had merely made a slip or that she had no knowledge of the x sign at that time, however, this could only have been ascertained on questioning her further, which unfortunately did not occur due to lack of time.

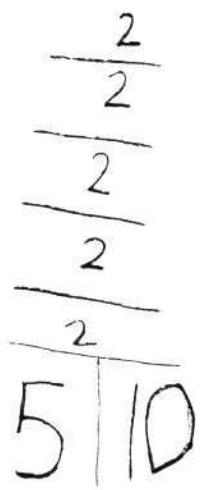
Figure 6.6



Refer to **Problem 1** (page 72)

Melloney (G.1) : *I put two on one day, two on another, two on another, two on another and two on another. I counted one, two, three, four, five, six, seven, eight, nine, ten and I put a five there for five days and I put a ten for there because I counted them altogether and they made ten. So I have five twos* (Figure 6.7). Melloney represented the groups numerically in a table form to indicate five consecutive days. She then **counted all** to obtain the total. Although in both instances under numerical representation, numerals were used, Melloney seemed to see each set as consisting of single or separate items, thus counting all to obtain the total.

Figure 6.7



4. Symbolic Representation (SR)

Refer to **Problem 6** (page 75)

James (G. 2) *: I put five plus five plus five plus five plus five and then I counted in fives and I got twenty five* (Figure 6.8). James, unlike Melloney used a more sophisticated strategy, he **skip counted** (a counting strategy which seems more appropriate for this type of representation). He also represented his solution **symbolically** as **repeated addition**.

Figure 6.8

$$5+5+5+5+5=25$$

5. Mental Computation (MC)

Refer to **Problem 4** (page 74)

Calvin (G.2) : (immediate response) *Six.* (when asked to elaborate) *I just knew.*

There were three instances of mental computation and all were Higher Ability pupils (see Tables 5.34 and 5.35). In all three instances, the children were unable or not eager to explain how they arrived at the answer. All the responses were similar to Calvin's when asked to elaborate, i.e. *I just knew.*

Discussion

For Repeated Addition the basic counting strategies involved grouping appropriately and then counting all to obtain the totals. The most common forms of representation were modelling with concrete material and pictorial representation. One Grade 1 and one Grade 2 pupil (both Higher Ability) computed mentally on two and one occasion/s respectively.There were only five cases of symbolic representation for the Repeated Addition problem structure. The solutions for this problem type were represented in five different ways:

1. Representing (modelling/pictorial representation), e.g. 5 figures and placing 2 crayons on each of these five figures. Where both the subject and the object in the problem are clearly shown (e.g. Brandon's work on page 73).
2. Representing (modelling/pictorial representation), e.g. 5 groups of twos. Where only the object in the problem is clearly shown (e.g. Sara's representation on page 72).
3. Representing (pictorial representation), e.g. 3 tables and placing 2 figures on each of the first two tables, but counting two on each of the three tables (imaginary figures on the third table).
4. Numerically representing 5 twos in the form of a table.
5. Representing as repeated addition and either counting all or skip counting.

Although the children represented their strategies in a number of different ways, the use of a common model was evident across all the strategies for Repeated Addition, i.e. the children grouped appropriately and either placed these groups with or without the corresponding objects/subjects in the word problem. In other words they expressed their grouping in repeated addition form. However, in most cases they counted all to obtain their total. There were only ten cases of skip counting (one in Grade 1, i.e. the only case of skip counting in Grade 1 and 9 in Grade 2 - see tables 5.34 and 5.35). Thus the intuitive strategies for Repeated Addition problems revealed that the children did relate their representations to the semantic structure of the problem. There obviously was an underlying model of repeated addition, however most of the children still depended on an addition (i.e. counting all) model.

FACTOR (F)

Only 10% of the Grade 1 solution strategies and 47% of the Grade 2 solution strategies were successful (see Table 5.36). Because of this there was not a variety of intuitive strategies. All the appropriate strategies belonged to Higher Ability pupils. The Factor problem was represented symbolically in **two** distinct ways: 3 x 6 = 18 (i.e. 3 times 6) and 6 x 3 = 18 (i.e. 6 times 3).

1. Modelling

Only the Grade 2 pupils modelled with concrete materials for this problem structure (see Tables 5.34 and 5.35). There were two versions.

> **Problem 7**: I have 6 cents and James has 3 times more than I have. How many cents does James have?

Calvin (G.2) *: I counted the money* (using coins) *six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen. Then I said three sixes and my answer was eighteen. Calvin* **counted on** from six stressing the multiples of six.

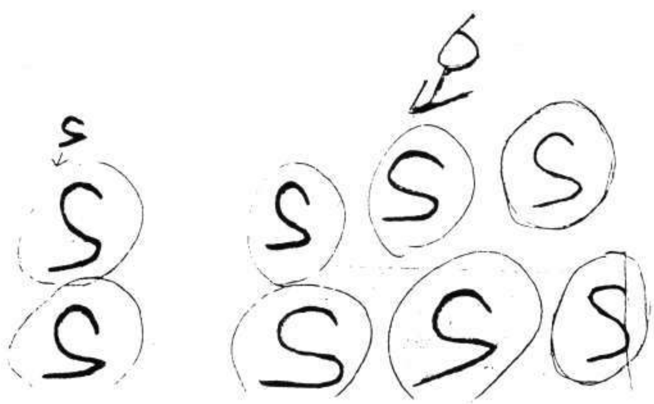
Samantha (G.2) *: O.K. six, twelve, eighteen. First of all I made six and two more sixes and I counted in six* (using cubes). *Six, twelve, eighteen.* Samantha **grouped** and **skip counted**.

2. Pictorial Representation

> **Problem 8** : Sam has 2 cents and I have 3 times as many. How many cents do I have?

Melloney (G.1) : *I took two cents and I took one and I took two and a one and a two and a one and a two. I put circles around them and then I added two more. Then I counted two, two, two* *and then I counted one, two,three, four, five, six. I put two coins and then six coins* (Figure 6.9). Melloney's representation was interesting in that she represented both the quantities (i.e., 2 and 6) involved in the problem.The two coins represented what one person (Sam) had and the six coins represented three times more than that. Clearly this is a correct relationship between the **number** of coins represented, which she had arrived at using **grouping** and **counting all.** However, the original problem did not deal with the number of coins, but with the **amount** (value) of money involved (e.g. Sam could have had one 2-cent piece and I could have had one 5-cent piece and one 1-cent piece). Having written a reversed two on each drawn coin, it appears as if she had simply ignored the value of the coins, just counting the number of coins. When asked to elaborate and clarify her answer further, she appeared confused and was unable to do so. It might be that she misinterpreted the problem situation thinking that Sam had two 2-cent coins, but this could only be ascertained by further questioning for which there was unfortunately no time available.

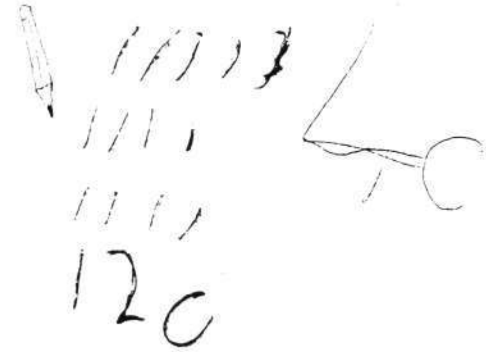
Figure 6.9



Problem 9 : Sam has 4 cents and I have 3 times as many. How much money do I have?

Calvin (G.2) : My friend had four cents and then I just drew three times four sticks and then I put twelve (Figure 6.10). Calvin's representation was different from Melloney's in that he only illustrated the product in his drawing, however he represented both the multiplier and the product numerically and in his verbalization included the multiplier. It is possible that he may not have used his representation to calculate the answer. From his explanation it seems like he had used a **known multiplication fact.**

Figure 6.10

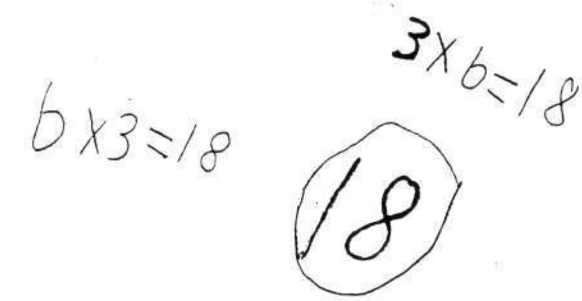


3. Symbolic Representation

Refer to **Problem 7** (page 79)

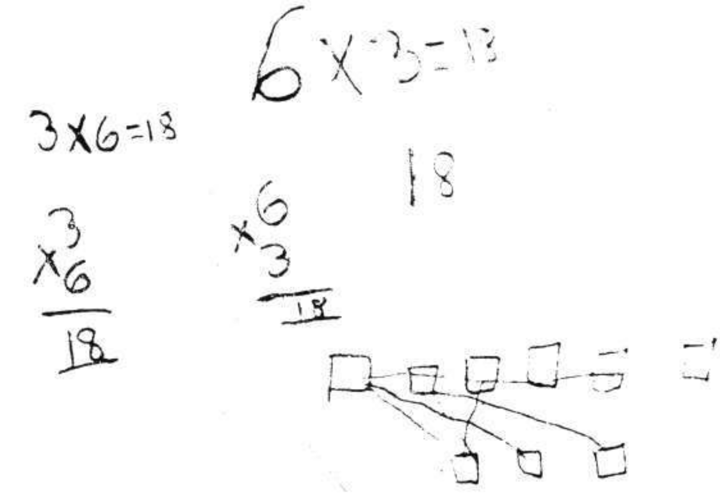
Michael S. (G.2) *: Six times three is eighteen. I've got it. Three sixes is* ***three****;* ***six****; seven, eight,* ***nine****; ten, eleven,* ***twelve****; thirteen, fourteen,* ***fifteen****; sixteen, seventeen,* ***eighteen****. I've got eighteen*. (Figure 6.11). Michael S. counted in multiples of three from three to six. He then counted in ones from seven to eighteen stressing the multiples of three. He also counted on from six up to eighteen, stressing the multiples of six. Although Michael S. verbalized *three sixes* he counted in threes. He presented his solutions **symbolically**. He has a concept of the commutative property of multiplication in this instance.

Figure 6.11



Samantha (G.2) : After modelling Problem 7 above she represented her solution **symbolically** both as vertical and horizontal number sentences. She attempted to do a **many-to-many** correspondence but abandoned the idea. (Figure 6.12).

Figure 6.12

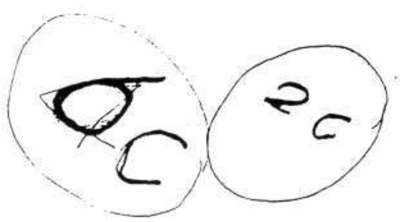


4. Mental Computation

Refer to **Problem 8** (page 79)

Kelvin *: I put two cents there and I put a big two and I put another circle around and then I put a six. Because you have three more than him so you had six* (Figure 6.13). Kelvin seemed to calculate this **mentally**. He could not offer any further explanation for this.

Figure 6.13



Discussion

Only two Grade 1 (H.A.) pupils were able to partially solve these problems. Both represented (pictorially) the coins possessed by both subjects in the problem. Their strategy reflected some sort of comparison. Melloney represented her solution in greater detail than Kelvin, showing the entire problem situation. This was as a result of her counting strategy, i.e. grouping and then counting all. Kelvin on the other hand merely wrote both the totals. This could have been as a result of his mental computation. Neither of the pupils made use of concrete material. Only five Grade 2 (H.A) pupils solved these problems. Even though all the pupils either counted all or skip counted they tended to either stress the appropriate multiples or verbalize the following *"times as many"* e.g. *three times four sticks.* When the children represented their solutions as multiplication number facts reflecting the commutative property, e.g. 3 x 6 = 18 and 6 x 3 = 18, they seemed to be functioning at the numerical level (which Murray et al, 1992, refer to as **Level 2** computational strategies in their semantic model). They all represented the multiplicand and then either drew or wrote *n* times as many. This is due to the semantic structure of the Multiplying Factor problem. The strategies above show that the Grade 2 pupils used more sophisticated strategies compared to the Grade 1 pupils. They counted on, skip counted and used known multiplication facts. However one Grade 1 (H.A) pupil computed mentally (this was the only case in both grades - see Tables 5.34 and 5.35).

As the performance on the Multiplying Factor was poor, some of the children's inappropriate stategies will be analysed in an attempt to explain this performance.

Refer to **Problem 8** (page 79)

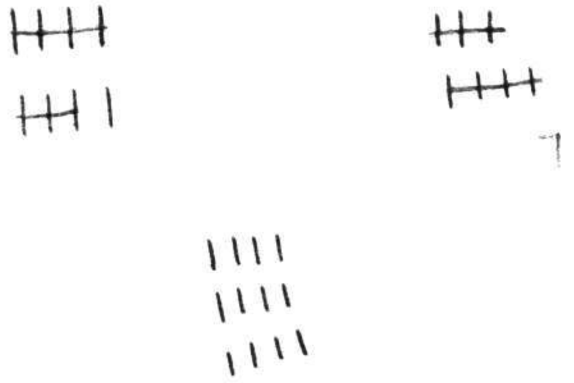
Roxanne (G.1) *: 5.* After a few minutes. *6, I put two* (placing two pencils) *then I put three plus another three, that equals six. No five.* Initially Roxanne had added two and three to get five. She then added three and three to get six, but was not sure of it so she decided that 5 was the correct answer. When she was questioned about it she just said *I don't know, it must be five.*

Refer to **Problem 9** (page 80)

James (G.2)*: It's four plus three, seven.* He wrote 4c + 3c = 7c.

Sally (G.2) *: What's times? I'm confused.* (She illustrated as shown in Figure 6.14 but did not want to elaborate).

Figure 6.14



In all three cases the children seemed to add, instead of multiply. Even though Roxanne and Sally did seem to obtain the correct answers, at some time or the other, they were not convinced about it. The basic problem seems to be one of language, i.e. it looks as if they had a problem with the meaning of "three times more". They seem to have interpreted it as three more. However, further investigation to corroborate this and/or to indentify other underlying causes is necessary.

RATE (R)

For Grade 1 there were 52% appropriate stategies and 82% for Grade 2 (see Table 5.36). Modelling was the predominant form of representation for the Multipliying Rate problem for both grades. The Grade 2 pupils also represented a large number of their strategies pictorially (i.e. 9 cases - see Tables 5.34 and 5.35). The predominant counting strategy was grouping and counting all.

1. Modelling

Problem 10 : If you need 3 cents to buy one pencil, how much would you need to buy 2 pencils?

Melloney (G.1) *: I wanted to buy one pencil with three cents and I put the cents there and that was one pencil that cost three cents. And I wanted to buy another pencil so I took another pencil and it cost another three cents so it was six cents. I wrote three for one sticker and three for the other sticker and I done three plus three equals six and I put a six there.* Working with specific aids (i.e. coins and pencil cut-outs), Melloney **grouped** in threes, used **many-to-one correspondence,** a **known addition fact** and **counted all** to obtain the total.

> **Problem 11** : If you need 6 cents to buy one pencil, how much do you need to buy 4 pencils?

Martin (G.2) *: I took one, two, three, four, five, six, over here and then one, two, three, four, five, six and I put that there and then one, two, three, four, five, six and left it there and one, two, three, four, five, six and then I counted one, two, three,....twenty four.* Using unifix cubes Martin **grouped in sixes** and **counted all** to obtain the total.

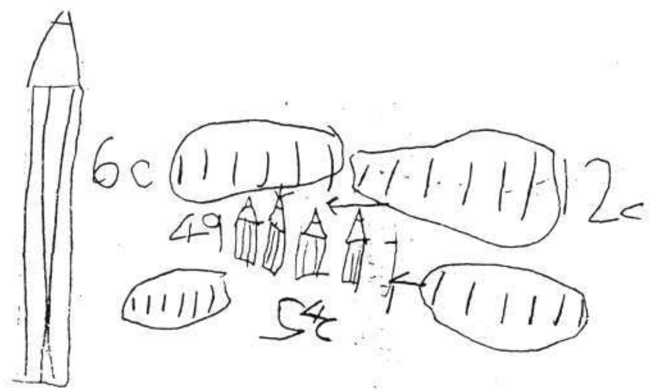
2. Pictorial Representation

There was only one type of pictorial representation where the objects were used as a representation of the unique elements of the set in the problem.

Refer to **Problem 11**

Calvin (G.2) *: I made twelve and I thought it wasn't twelve, so I put two circles around the sixes and I made another six and another six and I counted all and they were twenty four and I drew some pencils, 4. The p stands for pencils* (wrote 4q) *and this is the big pencil for six cents. They cost twenty four cents.* (Figure 6.15). Calvin initially placed two groups of six, he then realized his error and corrected himself. He **grouped,** used **one-to-many correspondence**, and **counted all.**

Figure 6.15



3. Numerical representation

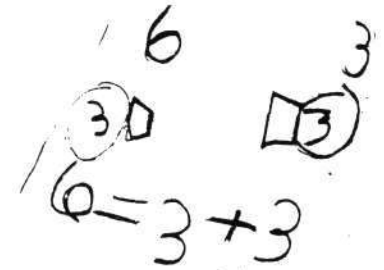
Melloney (G.1) : After modelling problem 10 she did the following numerical representation (Figure 6.16).

Figure 6.16

Problem 12 : If you need 3 cents to buy one sticker, how much money do you need to buy 2 stickers?

Kelvin (G.1) : *I put three cents then I put a sticker and two sticks and six cents. Two stickers would cost six cents* (Figure 6.17). Kelvin seemed to **calculate mentally**, he used a **known addition fact** and he **represented symbolically**.

Figure 6.17

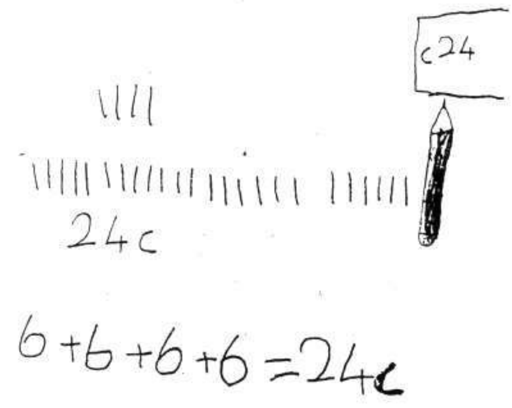


4. Symbolic Representation

Refer to **Problem 11** (page 85)

Michael M. (G.2) : *I did six plus six plus six plus six and that added up to twenty four cents* (Figure 6.18). Michael M. used **repeated addition** and he represented this **symbolically** (addition number sentence).

Figure 6.18



Martin (G.2) : After modelling Problem 11 he represented his solution **symbolically** (Figure 6.19) because he realized that multiplication is a shorter method . *It's quicker to do times. You don't have to go six plus six plus six plus six you just say four times six.* He stated this after looking at Michael M.'s repeated addition in Figure 6.18.

Figure 6.19

$$6 \times 4 = 24$$

Discussion

Again the basic counting strategy was grouping and counting all. Two pupils seemed to have calculated the totals mentally. One Higher Ability pupil wrote an appropriate addition number sentence. As discussed above a number of pupils used the objects as representations of the elements in each set. No particuliar preferred model could be identified for Multiplying Rate from the children's intuitive strategies. The underlying model was similar to that identified for Repeated Addition, i.e. the repeated addition model where the children relied on an addition (i.e. counting all) model. There was one instance where the "*timesed*" model was used. The only instance of double counting, as Mulligan (1991) described it, was identified for the Mutiplying Rate problem (i.e. counting all with a simultaneous count of the number of groups at the same time, e.g. *"one, two three (one);...four, five, six (two)... ").*

CARTESIAN PRODUCT (CP)

There were 75% appropriate strategies for Grade 1 and 71% for Grade 2 (see Table 5.36). The predominant form of representation for Cartesian Product was modelling among the Grade 1 children and Pictorial Representation among the Grade 2 children. The main counting strategy was again grouping and counting all for both groups. There were five instances of skip counting (in Grade 2 - see Tables 5.34 and 5.35). Initially the children experienced difficulties with the Cartesian Product problem, therefore specific aids were made available in subsequent lessons. This assisted in enabling most of the children to picture/figure out what the problems actually involved. When the children seemed to have an understanding of the problem situation they demonstrated that they had no use for these specific aids by making use of other concrete material and by representing pictorially.

1. Modelling

> **Problem 13** : The shop has black and white shirts in small and medium sizes. How many different choices can I make?

Melloney (G.1) : *I took one little white jacket and one little black jacket and the big white one and the big black one. Four.* Melloney used the appropriate specific aids (referring to them as jackets rather than as shirts as indicated in the problem) placing them as she spoke in **groups** of two according to their sizes. She then **counted all** to obtain the total.

Brandon (G.1) : *This is my big shirts and my little shirts here* (used single black and white cubes). *This is my big white and this is my big black and my little black. One, two, three, four. Four.* Brandon **grouped** the cubes in twos, paying special attention to the colour and not the size in his modelling. However his explanation was appropriate. He **counted all** to obtain the total.

Russel (G.1) *: I put two blacks and two whites. A big one. I put two together and a small one, one, two, three, four. Four* (used two cubes for big and one for small). Russel **grouped in twos** categorizing according to colour and size. He **counted all** to obtain the total.

Problem 14 : I can buy plain chips and salted chips in small, medium and large packets. How many different choices can I make?

Michael S. (G.2) : *Six. Look, two, four, six. You can buy two of these and two of these and two of these* (pointing to the different coloured and sized cubes). *I have big chips* (pointing to two sets of three cubes- one red (salted) and the other blue (plain)) *and I can buy two of these, two medium packets* (pointing to two sets of two cubes each - one red (salted) and one blue (plain)) *and two small packets* (pointing to two sets of single cubes - one red (salted) and one blue (plain)). *Look three salted, big, medium, small. Three plain, big, medium, small. One, two, three, four, five, six. These are big packets* (pointing to the two sets of three cubes), *the salted* (set of three red cubes) *and plain* (set of three blue cubes). *We've got two of them. Then you can buy them in small* (pointing to the two sets of single cubes) , *salted* (single red cube) *plain* (single blue cube). *You can buy two of these medium* (pointing to the two sets of two cubes), *salted* (set of two red cubes) *and plain* (set of two blue cubes). Michael S.'s representation was similar to Russel's but he modelled in greater detail. Although all four children modelled with concrete material, of some sort or the other, and grouped appropriately there were variations in the actual representations. For example, Russel and Michael S. arranged the cubes emphasizing the sizes whereas, Brandon did not seem to need to use different sized cubes to distinguish between the sizes.

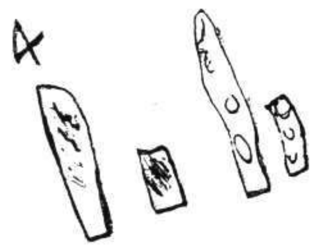
2. Pictorial representation

Problem 15 : I can buy plain chips and salted chips in small and large packets.

How many different choices can I make?

Kelvin (G.1) : *Four. Two big packets and two small packets* (Figure 6.20). He did not wish to elaborate. Kelvin **grouped** according to the flavours and sizes. In other words he systematically illustrated each flavour in the two different sizes. He seemed to calculate the total **mentally**.

Figure 6.20

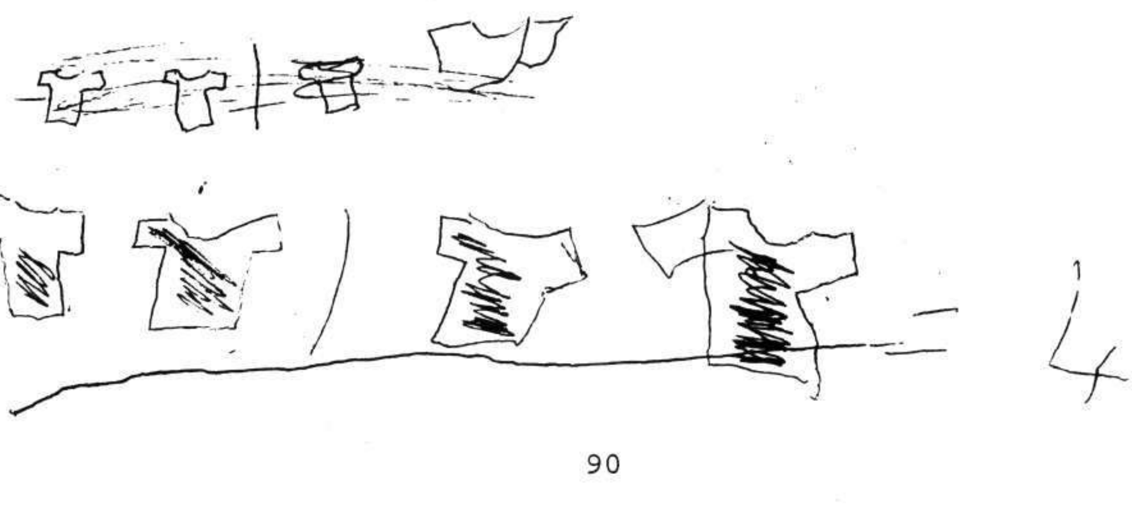


Problem 16 : The shop has black and white shirts in small and medium sizes.

How many different choices can I make?

Kerry Lee (G.2) : *I put two medium and two little and I counted them and it came to four* (Figure 6.21). When questioned about her explanation it was ascertained that although Kerry Lee had not represented the colours in her drawing and only focussed on the sizes, she had apparently reasoned that there were two medium (one black and one white) and two small (one black and one white). She then **counted all** to arrive at her answer.

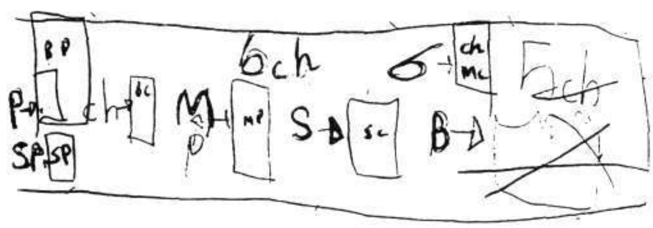
Figure 6.21



Problem 17 : I can buy plain chips or chicken chips in small, medium and large packets. How many different choices can I make?

Calvin (G.2) : *I put big chicken and this is small chicken and this is medium chicken and this is big plain, small plain and this is plain medium and I counted all and I got six* (pointing to each figure in the illustration respectively - Figure 6.22). Calvin attempted to represent the actual situation. However, his representation was inappropriate. His figures were not proportional but his explanation was appropriate.

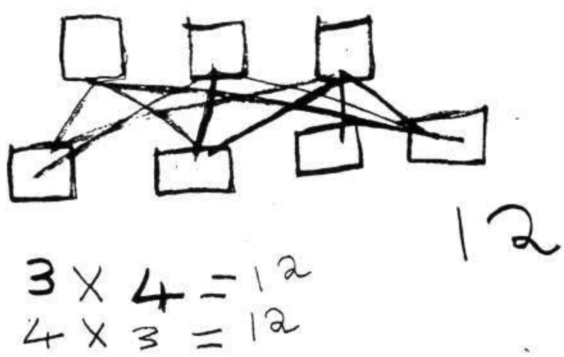
Figure 6.22



Problem 18 : Marina has 3 skirts and 4 blouses of different colours that match. How many different ways can she wear these?

Martin : *I put a line from there to there, and a line from there to there* (joined each top block to each bottom block). *I put one to each one and I counted it. It was very hard to count it because it was all muddled up and then I finally got it and it was twelve.* (Figure 6.23).

Figure 6.23

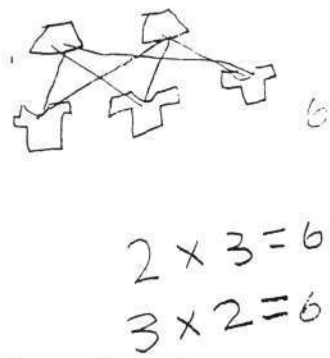


Problem 19 : Mum has 2 skirts and 3 blouses of different colours that match.

How many different ways can she wear these?

Angela (G.2) *: I made two skirts and three shirts and I joined them all up and I counted them and it was six. Here's the two and here's the three and we just have to join them up. For one skirt you can use it two times and this then with this skirt you can use it two times with the shirt and with this skirt you can use it two times too and then two plus two plus two equals six. It doesn't matter which way you put it, it's the same answer. Two times three is six and three times two is six.* (Figure 6.24).

Figure 6.24



After representing pictorially, both Martin and Angela wrote corresponding multiplication number sentences. Their representations were very interesting in that they used a model that involved many-to-many correspondence. Angela however, used a more sophisticated strategy than Martin. Where Martin matched on a one to one basis, Angela looked at it as "two times" each item. She initially interpreted this as repeated addition and then as multiplication. It is also noteworthy how both showed understanding of the commutative property representing different ways of pairing off the shirts and blouses.

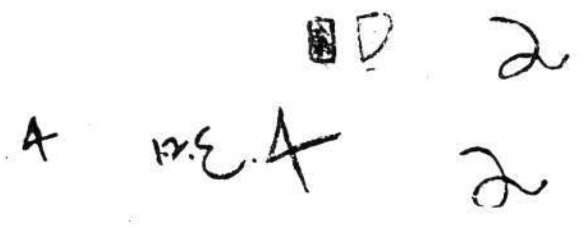
3. Numerical Representation

There were only two cases of Numerical Representation, one in each Grade. They differed slightly.

Refer to **Problem 13** (page 88)

Kelvin (G.1) *: I put a black and a white. I put one, two, three, four and that's all. Four choices, there are four jackets* (Figure 6.25). Kelvin first **grouped** in twos, then **counted all**. He seemed to have focussed on the colours, grouping them into different sizes. Like Melloney on page 89 Kelvin referred to the shirts as jackets. They had been working in the same group.

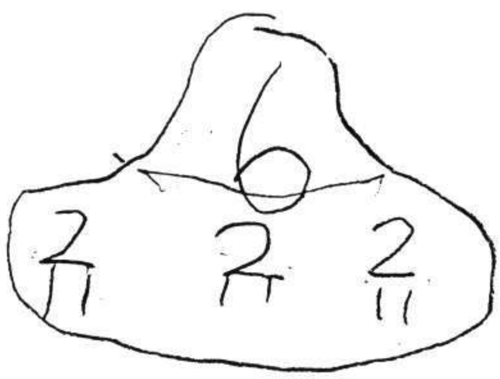
Figure 6.25



Refer to **Problem 17** (page 91)

Kerry Lee (G.2) *: I have two small chips, two medium chips and two big chips. Two, four, six. Six choices (*Figure 6.26). Kerry Lee **grouped** according to the sizes and **skip counted**.

Figure 6.26



4. Symbolic Representation

Only the Grade 2 pupils (7 cases - see Tables 5.34 and 5.35) represented symbolically for thi problem structure.

Martin and **Angela (G.2) :** Refer to Figure 6.23 and Figure 6.24. Initially Angela and Marti represented pictorially, followed by two multiplication number sentences, showing that the have an understanding of the commutative property in these instances.

5. Mental computation

Refer to **Problem 15** (page 90)

Michael M. (G.2) : *There was plain chips and salted chips and big and small so I counted four.*

He could not explain further.

Discussion

The children favoured modelling with concrete material. All the strategies seemed to involve grouping and counting all. The objects in the problems were represented (with concrete material or pictorially) in a range of ways. For example:

1. Categorizing according to size.
2. Categorizing according to colour or flavour.
3. Categorizing according to both colour/flavour and size.

Most of the pictorial representations were similar to those illustrated in the strategies involving modelling. However, the Grade 2 (H.A.) pupils used a different model which was peculiar to a specific type of Cartesian Product problem (i.e. matching of clothing). This model involved many-to-many correspondence. Because specific aids were used, initially it was not obvious whether the children were actually abstracting the two factors in the problem and cross multiplying. However, in their representations and explanations it could be seen that most of the children were able to integrate the two factors in the problem at once. This (i.e. the introduction of specific aids) showed that with minimal intervention (in this case the mere introduction of a more appropriate aid) young children can model, visualize and represent situations for Cartesian Product problems. On the other hand the model used by the Grade 2, Higher Ability pupils indicates that they had a clear understanding of the Cartesian Product situation. They were able to integrate both factors very efficiently and illustrate this very clearly pictorially as well as relate it to the appropriate multiplication number fact.

ARRAY (A)

Majority of the pupils in both grades could solve these problems (i.e. 90% appropriate strategies for Grade 1 and 92% for Grade 2 - see Table 5.36)). In Grade 1 the predominant form of representation was Modelling and in Grade 2, Modelling and Pictorial Representation. As with all the other problem types for multiplication, grouping and counting all was the main counting strategy.

1. Modelling

When modelling for the Array problems the children in both grades tended to prefer using the specific aids rather than any of the other concrete material. Some children used the abacus.

> **Problem 20** : In the classroom there are 2 rows of chairs with 3 chairs in each row. How many chairs are there altogether?

Jolene (G.1)*: I put chairs like this* (placing three chair cut-outs in a row) *three rows of chairs and I put another three* (placing another three under the first three) *and then I counted and I went one, two, three, four, five, six. Altogether there are six.* Jolene used the specific aids and packed out a row of three chairs first. Her verbalization of three **rows** of chairs (instead of a row of three chairs) was accidental (this was ascertained from further questioning). She then packed out another row of three chairs thus **grouping in threes** and **counted all** to obtain the total.

Melloney (G.1) *: I put two chairs down* (placing two cut-outs in a row) *and then I put a one and two* (placing another two under the first two) *and I knew that there had to be one more in each row. That's one, one* (placing one more in each row) *and then I counted them one, two, three, four, five, six and there were six.* Melloney placed the cut-outs one at a time ending up with two **groups** of three. She then **counted all** to obtain the total. The representations of Jolene and Melloney were similar but they grouped differently. Jolene placed a set of three chairs at a time whereas Melloney placed a set of two chairs at a time.

> **Problem 21** : In the classroom there are 5 rows of chairs, with 6 chairs in each row. How many chairs are there altogether?

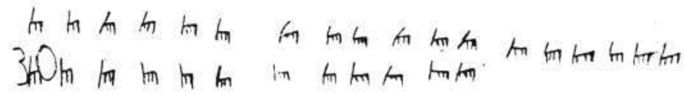
Zoey (G.2) *: I put six, six, six, six, six* (pointing to the five sets of chairs cut-outs) *and then I counted them and they make thirty.* During the five weeks of the study, Zoey only solved two problems. For both these, she made use of specific aids and she solved the problems only after observing other children solving the problems in this way. Zoey arranged the chair cut-outs in five rows of six chairs and she **counted all** to obtain the total.

2. Pictorial Representation

Refer to **Problem 21** (page 95)

James *: I done five rows of chairs, six in each row. I counted them and I got thirty.* (Figure 6.27). Although James' representation was similar to Zoey's, he however illustrated pictorially and **counted all** after **grouping** the chairs in sixes.

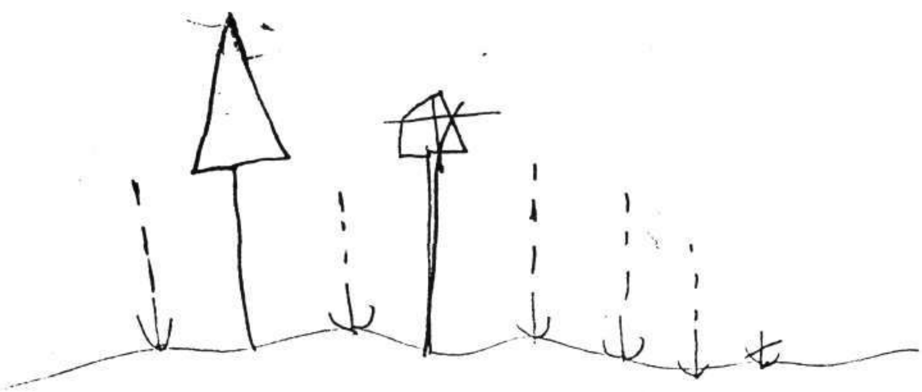
Figure 6.27



Problem 22 : A vegetable patch has 5 rows of onion plants, with 4 plants in each row. How many onion plants are there altogether?

Calvin (G. 2) *: I took one big vegetable patch and then I put five there and I put three there and I counted them all and I got twenty.* (Figure 6.28). Calvin first drew in the five onion plants, he then drew three strokes above each of the five onion plants, thus **grouping in fours**. He finally **counted all** the onion plants and the strokes to obtain the total, twenty.

Figure 6.28

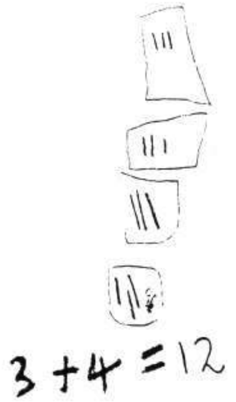


3. Symbolic Representation

Problem 23 : A vegetable patch has 4 rows of onion plants, with 3 plants in each row. How many onion plants are there altogether?

Kelvin (G.1) *: It's three plus three plus three plus three* (pointing to the illustration - Figure 6.29). Kelvin **grouped in threes** and he seemed to count in threes. He also attempted to represent the solution symbolically. Instead of using the multiplication sign he used the addition sign. It is possible that he had just made a writing error or he had no knowledge of the meaning of the x-sign and just used the +-sign which was the only one at his disposal at that time. This could only have been ascertained by further questioning, however this did not take place due to lack of time.

Figure 6.29



4. Mental Computation

Refer to **Problem 21** (page 95)

Kelvin (G.1) *: I drew two chairs and then I drew a six. I didn't have enough chairs because I don't draw chairs and because I know its three plus three* (Figure 6.30). It is not clear whether Kelvin calculated mentally or used the pictorial representation to arrive at his solution. Unfortunately he was not questioned further on this, due to lack of time. There are two possible explanations for the above representation, i.e. Kelvin could have calculated mentally and just drew two chairs to represent the objects in the problem, thus not really using the illustration to arrive at the solution; or he could have used his illustration as a reference set (see discussion on page 120-121). He also used a **known addition fact**.

Figure 6.30

Discussion

Although the Array problems had the highest success rate, hardly any of the strategies used were abstract. Instead they used concrete and semi-concrete representations. There were only three instances of the use of known addition facts among the Grade 2 pupils and one instance of symbolic representation by a Grade 1 pupil (H.A.). The children's strategies were represented in the following **three** ways:

1. *n* chairs in *m* rows placed exactly in corresponding position of an array,
2. groups of unifix cubes, placed in a random cluster rather than in lines or rows, and
3. one horizontal line of *n* chairs, where each chair represents a column of *m* chairs.

The model that was identified for the Array problems involved an arrangement of the objects in the form of an Array. The only other time this Array model was used, was by Calvin for the Repeated Addition problem as shown on page 75 in Figure 6.5.

DIVISION

According to Fischbein et al (1985: 14) "*...initially there is only one intuitive primitive model for division problems - the sharing model. With instruction, pupils acquire a second intuitive model - the measurement model*". The data below refutes the above conjecture. Like Mulligan (1991) and Olivier et al (1992), the research reported below has found that children can solve different types of division problems at an intuitive level prior to formal instruction.

PARTITION (P)

There were 81% appropriate strategies for Grade 1 and 61% for Grade 2. The main forms of representation for Grade 1 were Modelling and Pictorial Representation, and Modelling for Grade 2. The predominant forms of calculation were grouping, sharing, counting all and double counting.

1. Modelling

The Partition problems were modelled differently by different pupils.

> **Problem 24** : If 12 apples are shared equally among 3 children, how many apples would each child get?

Sally (G.2) *: I did one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve* (using counters). *I had three people and I gave one to him and one to him* (Sally **shared one at a time** until nothing was left). *We count and it is one, two, three, four; one, two, three, four; one, two three, four. Each one gets four.* Sally **double counted.**

Problem 25 : 4 boys have 8 marbles to share. How many marbles does each boy get?

Byron *: There's four boys and eight marbles and they make two.* (First he placed the four counters representing children and then he shared the counters one at a time). *I gave him one first and then I gave him one* (he went on **sharing one at a time**) *and then I counted. Each boy got two.* After sharing one-by-one Byron counted the number in each group , i.e. he **double counted**. Byron's representation differed from Sally's in that he used the objects as a representation of the elements of each set, i.e. Byron represented both the boys (i.e. the subject) and the marbles (i.e. the object) with the cubes, whereas Sally only represented the apples (i.e. the object) with the cubes.

Refer to **Problem 24** (page 98)

James (G.2) : *I put two here and I put ten, twelve and then I took three and then I put another three and then another three and then I had three left. So I gave them each one more and I got four.* Initially James shared **three at a time,** he then shared the remaining three, thus using the **estimate-and-adjust** strategy.

All the strategies for the Partition problem that were illustrated through modelling involved sharing or grouping in one form or another as well as the use of the building-up model (additive).

2. Pictorial Representation

There was only one way in which the Partition problem was represented pictorially.

Problem 26 : I want to share 20 marbles equally among 4 of my friends. How many will each friend get?

Calvin (G.2) : *I got four. I made four little boys and I did like four circles and I put four circles and each got four marbles.* When asked to elaborate. *I just guessed 4.* (Figure 6.31). Calvin

grouped in fours using **estimation**. His representation is interesting in that it is different from the others. He used a building-down model (subtractive) which was evident when he first drew sixteen sweets and then grouped into fours.

Figure 6.31

3. Mental Computation

Problem 27 : I want to share 6 sweets equally among 2 of my friends. How many will each friend get?

Kelvin (G.1) : *There were two friends and six sweets and we give each one three sweets* (he wrote the number three and would not elaborate).

As with the Array problems almost all the strategies for the Partition problems were at a concrete/semi-concrete level. There were only two instances of mental computation (one Higher Ability Grade 1 pupil). None of the other more sophisticated forms of representation were used. For the Partition problem structure a specific model was identified from the children's strategies, i.e. the sharing model. In most cases the children also double counted, i.e. they first counted the number in the dividend and then the number in each group. The building-up model was thus evident. However there was one instance of the use of the building-down model, e.g. Calvin's representation in figure 6.31 above.

FACTOR

Only three Grade 2, Higher Ability pupils were able to solve these problems. Only 9% of the strategies were appropriate. Because of this, as with the Multiplication Factor solution strategies, there was not a range of strategies. The only form of representation was modelling.

The children calculated by grouping and counting all (double counting). One child used a known multiplication fact and represented symbolically after modelling with concrete material.

1. Modelling

Problem 28 : I have 15 pencils and this is 3 times as many as Sam has. How many pencils does Sam have?

Samantha (G.2) *: So you have three times as many as Sam has. It is less because you said three times as many as Sam has. It is three times less than fifteen. I've got one, two, three; one, two, three; one, two, three; one, two, three; one, two, three; and then I counted one, two, three, four, five* (built the cubes up to fifteen).... *So it equals five.* Although Samantha said "*three times*" she operated on groups of three, obtaining five groups of three instead of three groups of five. She also attempted to write a multiplication number sentence but abandoned the idea. Samantha's strategy was inappropriate but she obtained the correct (numerical) answer. It appears as if she she was confusing five groups with five elements. When questioned about this, she was very confused and couldn't elaborate. However, it seems like she may have grouped in threes because of the term "three times" and built up to fifteen because this number arreared in the problem.

Angela (G.2) *: I have fifteen like this and I counted five* (referring to her set of fifteen cubes). *One, two, three, four, five and broke it and one, two, three, four, five and broke it and I have three fives* (Angela was unable to elaborate).

Angela's and Samantha's strategies were similar but they grouped differently and they used different models. Samantha grouped in threes using a building-up model whereas Angela grouped in fives using a building-down model.

Problem 29 : My father had 12 shirts and this is 3 times as many as I have. How many shirts do I have?

Martin *: Right here's the father's shirts, they are stacked upto here, it's twelve. And now you can see the son's here. The son's got only a little bit of that and the father's got three times more.*

Now he can show his son one, two, three, four; five, six, seven, eight; nine, ten, eleven, twelve. Now he's got three times more than his son. His son has four (Martin broke up the row of cubes as he explained and he wrote down the corresponding multiplication number fact, as shown in Figure 6.32).

Figure 6.32

$$4 \times 3 = 12$$

Discussion

One pupil used the *building-up model* and two the building-down model. The children experienced the greatest difficulty with this problem structure. On the first occasion, when this problem structure was presented to the Grade 2, Higher Ability pupils only one child (Martin) was able to solve it after much perseverence. The others resigned with great frustration. They were unable to understand Martin's explanation even though he tried to explain it to them a number of times. After much trial-and-error some of the pupils made the following remarks *"I don't know maths", "I don't like maths", I'm confused" and "What is times?"* . This was as a result of the confusion they were experiencing when presented with the Division Factor problem. On the second occasion Martin could not solve this problem structure, no matter how much he tried. Two other pupils were able to. Even they were not too confident about their strategies. Samantha could not write an appropriate number sentence and she was not eager to explain her strategy to the others and Angela could not elaborate on her explanation. The children's extremely poor performance on the Division Factor problem type deems it necessary to attempt to explain some reasons for the difficulty they experienced. In order to do this, an analysis of some of their inappropriate strategies will be done:

Refer to **Problem 29** (page 101)

Samantha (G.2) : *Is it 48?* (I asked: "Why?") *Because I did it like this. I put twelve and twelve and another twelve. It's more than twelve.* (Samantha insisted that the answer had to be more than twelve).

Refer to **Problem 28** (101)

Michael (G.2) : (Immediate response) - *It's 60. What's 15 times 3, its 60.* (Samantha said: " No you mustn't do that, you must have less"). After spending some time on the problem he stated: *6, it's six. 1, 2, 3,.....15. Then I cross out 3, 3, 3, then I count 1, 2, 3, 4, 5, 6.* (He refused to listen to the others and went on insisting that it was six, which he illustrated as shown in Figure 5.33).

Figure 6.33

Martin (G.2) : *1, 2, 3,......15* (counting 15 cubes). *This is funny, there's 15 and my friend got 3 times more, so I add on 3, <u>18.</u>*

The most common error pattern for the Division Factor problems was that the children were either multiplying or adding the numbers instead of dividing or subtracting. In other words instead of "making less" as Samantha said they were "making more". This could be due to the lack of understanding of the phrase used in these problems, i.e. " times as many". The children seem to see "times" as multiplication or making more. As with the Multiplying Factor problem type the basic problem seems to be one of language, but further investigation is necessary to ascertain this.

RATE (R)

Grade 1 had 58% and Grade 2 had 69% successful strategies (see Table 5.36). The main form of representation was modelling for both groups. The predominant calculation strategies were double counting (i.e. grouping and counting all), and sharing for both grades. There were two instances of skip counting, three instances of the use of known addition fact and one of mental computation (see Tables 5.34 and 5.35).

1. Modelling

Problem 30 : My friend bought 4 pencils for 12 cents. If each pencil costs the same, how much did one pencil cost?

Sally (G.2) : *Here's my twelve cents and I put four pencils and I gave them a money and another one and another one and another one* (Sally went on placing the coins one at a time on each pencil cut-out) *and then they equalled one, two, three cents. One pencil costs three cents.* Sally **shared one by one, counting all**.

Problem 31 : My friend bought 3 pencils for 9 cents. If each pencil costs the same, how much did one pencil cost?

Jolene (G.1) *: First I was holding these like this* (placing three pencil cut-outs alongside each other) *and then I put two* (placing two coins at each pencil) *and then I put one, two, three* (sharing the remaining three coins equally among the three pencils) *and now there's three* (referring to the pencils) *and now three cents*. Initially Jolene used **one-to-many correspondence** sharing out two at a time. She then realized that she had some remaining, so she shared these, making use of the **estimate-and-adjust strategy**.

Problem 32 : A man must walk 18 km in 3 hours. How many kilometres must he walk per hour to achieve this?

Angela (G.2) *: I took eighteen and then I just broke up six and then I counted another six and another six. I just broke them up into sixes.* While talking she broke the tower of eighteen cubes into three groups of six.When questioned about the multiplication number fact in Figure 6.34 she was unable to elaborate. Angela seems to have **estimated** and she used the building-down model.

Figure 6.34

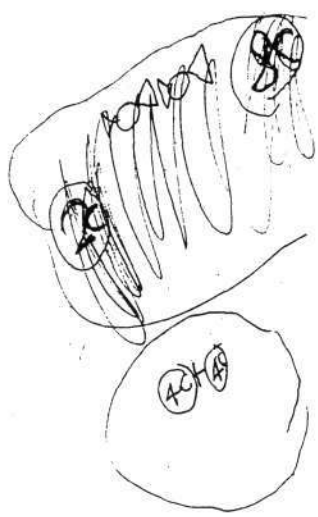
$6 \times 3 = 18$ (6)

2. Symbolic Representation

Problem 33 : My mother bought me 2 sweets for 8 cents. If each sweet costs the same amount, how much did one sweet cost?

Kelvin (G.1) *: I put two sweets there and then I put a two cent and a eight cent but then I scratched it and I put a four cent plus four cent. Each sweet is four cents* (Figure 6.35). He did not wish to elaborate. Kelvin seemed to compute this mentally. He represented his solution symbolically (incomplete).

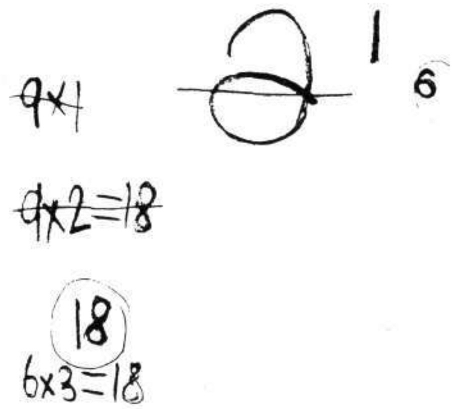
Figure 6.35



Refer to **Problem 32** (page 104)

Michael S. (G.2) *: I first went nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen and then I thought ah! I got the answer six. I started with nine times one but then I said no it's nine and then I went nine times two but the it was wrong also because now it is only two hours and then I went six, seven, eight, nine,ten, eleven, twelve. No! Six, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen. I knew six plus six was twelve so I added* *another six and I found that it was eighteen.* (Figure 6.36). Michael S.'s strategy was similar to Angela's strategy (on page 103), where they used the building-down model. While Angela modelled her solution, Michael S. visualized and worked symbolically. Both pupils also represented their solutions as appropriate number facts. The strategy Michael used was unusual in that he was actually "breaking up" or "building down" symbolically. He used the **estimate-and-adjust** strategy, **derived fact**, **known multiplication fact** and represented **symbolically**.

Figure 6.36



3. Mental Computation

Refer to **Problem 31** (page 104)

Kelvin (G.1) : (almost immediate response) *I took nine cents. Because he bought with all nine cents and they each costed three cents.* Again Kelvin seemed to compute mentally.

Discussion

The building up model was evident in majority of the children's strategies. However, there were a few instances of the use of the building-down model.

QUOTITION (Q)

For Grade 1, there were 73% successful strategies and 58% for Grade 2 (see Table 5.36). Th main form of representation for Grade 1 was modelling and for Grade 2 modelling and pictoria representation. Again double counting was the main counting strategy (see Tables 5.34 an 5.35).

1. Modelling

Problem 34 : My dad has 18 pens. He shares these equally among his children. If each child receives 3 pens, how many children does my dad have?

Kerry Lee (G.2) *: I put some children and I put three pencils near every children and then I done, I counted all the children and I got six. One, two, three, four, five, six.* Kerry-Lee worked with cut-outs of pencils and people. She first counted eighteen pencil cut-outs and placed ten people cut-outs. She then used **one-to-many correspondence** placing three pencils on each person and those people cut-outs that were not used were discarded. Her use of the building-down model was clearly evident.

Problem 35 : I have a strip of gum that is 6cm long. I gave each of my friends 2cm of the gum. How many friends do I have?

Brandon (G.1) *: I had six centimetres of gum and I gave two centimetres to my friend and two centimetres to my other friend and two centimetres to my other friend and I've got nought. Three friends.* (joined six cubes to represent the gum, and then took away two cubes at a time). Brandon **grouped in twos** and then shared these equally thus demonstrating the use of the building-down model.

Problem 36 : 8 toys are shared equally among children at the table. If each child has 4 toys, how many children are there?

Melloney (G.1) *: I had eight toys and there were two people and there's four for each person, one, two, three, <u>four</u>; one, two, three, <u>four</u>* (first placing unifix cubes in two groups of four and then counting the groups) *and I got four plus four is equal to eight. That's one, two. Two children.* When modelling with the cubes she counted them individually to make up two **groups of four,** thus using the building-up model.

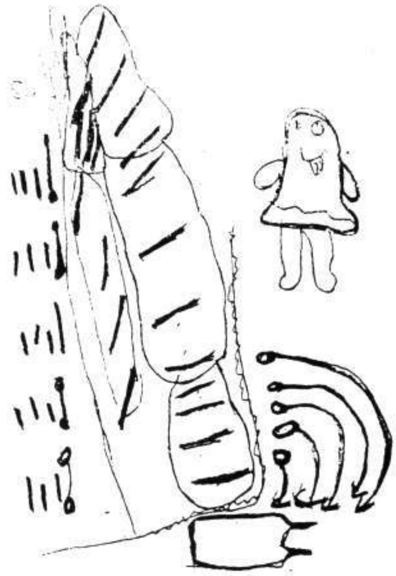
2. Pictorial Representation

There was a range of pictorial representations.

Problem 37 : Mum has baked 20 buns. She puts them into plastic bags, 4 in each bag. How many plastic bags did she use?

Calvin (G. 2) *: I made twenty sticks and I put circles around them and I got five bags.* (Figure 6.37). Like many of the other strategies for the Quotition problem Calvin also demonstrated the use of the building-down model. He **grouped in fours**.

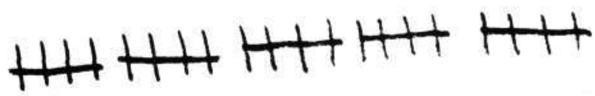
Figure 6.37



Refer to **Problem 38** (page 107)

Sally (G.2) *: I did twenty and then I crossed out four and then there were five packets.* (Figure 6.38). Like Calvin, Sally **grouped in fours** and used the building-down model.

Figure 6.38



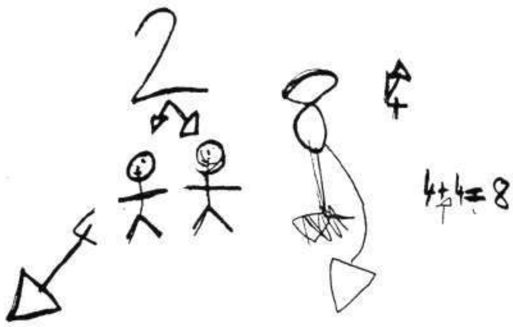
Problem 38 : 12 toys are shared equally among children at the table. If each child receives 3 toys, how many children are there?

Sally (G. 2) *: I put four people, then I tried, I gave them each three and I counted the three One, two, three, four, five, six seven, eight, nine, ten, eleven, twelve, and if it wasn't twelve could do another four. But it's four people.* (Figure 6.39). Sally's solution strategy was differen from many of the other's for the Quotition problem. She **estimated** and **grouped in three** ensuring that she **built up** to twelve.

Figure 6.39

Melloney (G.1) : After modelling Problem 36, Melloney did a pictorial representation as shown in Figure 6.40). As discussed on page 107, Melloney used the building-down model. Her use of this model was again evident in the addition number fact (i.e., 4 + 4 = 8) that she wrote.

Figure 6.40



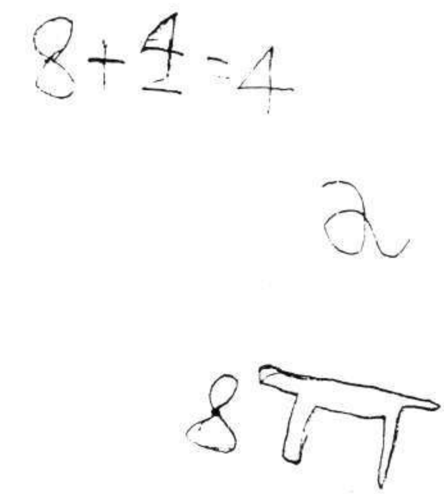
3. Mental Computation

There was only one case of mental computation.

Refer to **Problem 36** (page 107)

Kelvin (G.1) : *Because there were eight toys on the table and there were two children, that equals four toys each. That equals two children* (Figure 6.41). He did not want to elaborate. Again Kelvin seemed to compute mentally. He also attempted a **symbolic representation** which was inappropriate. When asked about this he could not clarify. Perhaps he did not know the appropriate sign for division which was not yet taught to them.

Figure 6.41



Discussion

Like the Array and Partition problems there were only concrete and semi-concrete forms of representation. There was evidence of both the building-up and the building-down models being used. However, for the Quotition problem type, the building-down model rather than the building-up model, was more prevalent. This was always accompanied by the use of the counting-all strategy. The building-down model was obvious in the following representations:

1. When modelling, the children very easily selected the total number required (i.e. the dividend) and then started splitting or physically breaking up the appropriate groups.
2. A similar approach was adopted when the children represented pictorially. They merely drew the total number (i.e., the dividend) and then either circled the groups or struck them off one group at a time. Then in both cases they totalled the number of groups.

There were a number of cases where the children used the objects as representations of the elements in each set. Another strategy that was quite prevalent for this problem structure was the one-to-many correspondence.

SUB-DIVISION (SD)

There were 68% successful strategies for Grade 1 and 55% for Grade 2. The only forms of representation were modelling and pictorial representation. Halving, sharing and counting all were the main types of calculation. (see Tables 5.34 and 5.35).

1. Modelling

The children only modelled with specific aids, i.e. cut-outs of oranges and apples; and scissors. On the first occasion that this problem structure was presented the majority of the children could not solve this problem structure (there were only 40% successful strategies for Grade 1 and 0% for Grade 2 - see Tables 5.17; 5.18 and 5.33). However on the second occasion when specific aids were available all the pupils had appropriate strategies.

> **Problem 39** : I have 3 apples to be shared equally among 6 people. How much apple will each person get?

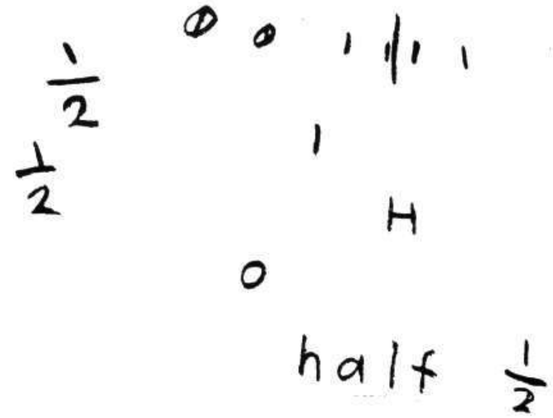
Byron (G.1) : *One, two, three, four, five, six* (placing six children cut-outs). (Picks up three orange cut-outs which he has cut up into two parts each and places them at each person cut-out). *Half on that one, half on that one and half on that one and half on that one and half on that man and half on that man. So each one gets half and there is nothing left.* Byron physically halved the orange cut-outs and matched six children to six halves. This could also be described as sharing them **one piece at a time** or **one-to-one correspondence**.

2. Pictorial Representation

> **Problem 40 :** I have 2 oranges to be shared equally among 4 people. How much orange will each person get?

Sally (G.2) : ***I took four people and I put them here and I took two oranges and I cut them and I got four people and I gave each one and they each get half.*** Sally then represented this as shown in Figure 6.42.

Figure 6.42



Discussion

For the Sub-division problems all the strategies were similar. The children halved the objects, either physically or pictorially and then matched these to the corresponding number of subjects in the problem. Their use of the one-to-one strategy was obvious.

From the indepth analysis of the chidren's intuitive solution strategies a number of intuitive models have been identified. A discussion on these intuitive models now follow.

INTUITIVE MODELS

From the analysis of the children's intuitive strategies **four** models were identified for multiplication and **three** models for division:

Intuitive Models for Multiplication

1. Equivalent Groups Model

Fischbein et al (1985) referred to this as the repeated addition model, which refers to *making equivalent sets and adding/putting them together*. The majority of the children's stategies for multiplication in this study reflected this model. It was also found that there were two levels of abstractness when this model was used. At the first level children made groups, counted the items in each group, put these together and then counted all the items. At the second more abstract level, the children counted the separate sets, e.g. one, <u>two</u>, three, <u>four</u>, five, <u>six</u>, seven, <u>eight</u>, nine, <u>ten</u> (see Sara's strategy for Repeated Addition on page 72). In some instances they skip counted (see Kelvin's strategy for Repeated Addition on page 73).

2. An Array Model (Mulligan, 1991: 162)

In this case the information in the problem was presented in the form of a rectangular array. This model occured only for Array and Repeated Addition problems. In all cases the children modelled or represented pictorial rows or lines (see examples on page 75 and 96).

3. Cartesian Product Model

This model was identified exclusively for the Cartesian Product problems. It was represented in two ways. At the lower, more concrete level the children clearly showed (with concrete material or pictorial representation) that they were operating on two measures, e.g. sizes and colours/flavours (see pages 90-91). At the higher more abstract level, the children used many-to-many correspondence which clearly reflects the use of the Cartesian Product model. However, this model was exclusively identified for the Cartesian Product problems that involved matching of two items of clothing. The children represented both objects of the problem, e.g. skirts and blouses. They then joined these by lines using the many-to-many correspondence strategy (see page 91-92 for examples). One pupil attempted to represent her Multiplying Factor solution strategy using this model but she was unable to do so (see Samantha's illustration on page 82). One reason for this could be that the Multiplying Factor problem structure does not lend itself to this specific type of representation. This only confirms the notion that the many-to-many correspondence model is more suitable to the Cartesian Product problem.

Intuitive Models for Division

1. The Sharing Model

This model was evident for all the division problems except the Factor problems, where the children shared one or more at a time or half at a time (see pages 98, 104 & 112 for examples).

2. The Building-up Model

This is also referred to as the "additive model" (Mulligan, 1991: 164-165). Contrary to the idea put forward by Fischbein et al (1985 : 14), that the *implicit model for measurement division is repeated subtraction,* this research confirms the findings of Olivier et al (1992: 36) and Mulligan (1991: 165) that many children prefer using a *building-up model* when solving division problems (see pages 99 and 104). This model incorporates the following strategies with and without direct modelling:

- grouping, counting all,
- one-to-many correspondence,
- trial and error grouping,
- grouping and skip counting,
- grouping and double counting, and
- known addition and multiplication facts.

3. The Building - down Model

Mulligan (1991: 165) also refers to this model as the "subtractive model". Fischbein et al (1985) refer to it as the "quotitive model". This model was evident when the children counted out or verbalized the dividend first, and then formed equivalent groups from the dividend - this action reflects "repeated taking away" from the dividend (see pages 108 and 110).

4. Halving

This model was evident when the children sub-divided wholes, then shared these portions evenly.

During the experimental period the researcher realized that the particular type of interaction and organization within the classroom environment had had a great impact on the children's problem solving strategies. Therefore, a brief discussion on the role that social interaction and the teacher as a facilitator played on the construction and evolution of the children's problem solving strategies will now be dealt with.

SOCIAL INTERACTION

In Chapter 4 on Methodology (pages 35-36) the idea of the teacher as a facilitator and social interaction within groups was introduced. As mentioned on page 35 the initial approach adopted was found to be ineffective, therefore a different approach was used after the third lesson. The following observations, made during the first three lessons, impacted both on the researcher's decision to change her approach as well as on the type of approach she adopted :

1. The majority of the pupils in both groups solved the problems individually. In most cases, the Higher Ability pupils completed their work as fast as they could, covered their work and began to talk, disturbing the others. In response the others then tended to rush without seeming to give much thought to what they were doing, in order to finish. Their main objective seemed to be to obtain an <u>answer</u> as <u>fast</u> as they could <u>individually</u>.
2. Three pupils in these groups, two of Lower Ability and one of Average Ability (Byron) tended to spend only a few seconds on-task playing with the different counters on the table. The Lower Ability pupils also tended to spend time peeking at their neighbour's work and very often copying it. They

appeared to be very uncertain of their ability to work successfully alone. When asked to explain to the rest of the group, the Higher Ability and sometimes the Average Ability pupils competed for the chance to do so first. Afterwards they very seldom paid attention to the explanation of the other pupils in the group. Apparently they saw no value in listening to each other. The Lower Ability pupils were usually the last to offer an explanation, which most often merely consisted of a one word answer or an inappropriate strategy. Their work usually just involved a few illustrations on the page and sometimes simple manipulations of the counters and the abacus.

3. Comments or questions on **any** explanations by other pupils were non-existent. It seemed apparant that although the children were arranged in groups they failed to work as co-operative groups. A similar finding to King's (1991: 50) in a traditional learning environment, was made, namely, that many children in this study as young as six already reflected a competitive vs a collaborative norm. The pupils did not seem to be interested in what the others had to offer.
4. In both the Mixed Ability groups the children tended to shun some of the other pupils. For example, in the Grade 1 group there was a Lower Ability pupil who would not spend more than a few seconds on a task and who did not get anything correct. For most of the time he would just sit back with his arms folded, talk to others, or play with the manipulatives, irritating some of the others. When it was his turn to explain, the rest of the group just wouldn't listen to him, even though his explanation lasted only a few seconds. They would sometimes say "Oh! He doesn't know", with exasperation. In the Grade 2 group a very shy, reserved, hesitant Low Ability pupil was treated in a similar manner. After the first lesson she began to respond with a "I don't know/understand", "I can't do it" or "I don't want to". Juvonen (1992) had similar findings. He stated that anger towards members in the groups seems to increase rejection and lack of social support, whereas sympathy seems to promote prosocial behaviours such as helping. This was observed within one of the groups. One Higher Ability girl who tended to sympathize with

Lower/Average achievers, normally chose one of them as her partner and explained how the problems were solved.

5. The Lower Ability pupils tended to talk less than their partners did. The Higher Ability pupils tended to tell and to show their partners what to do. As a result, when each child was expected to discuss his/her solution strategy to the whole group, the Lower Ability pupils, who were shown what to do, merely gave an answer and when asked to elaborate, responded with a "I don't know" or "she/he told/showed me". Mulryan (1992) also found that high achievers tended to either dominate in the group or chose to work alone. Low achievers were relatively passive, engaged in significantly more low-level or superficial attending behaviour and manifested more off-task behaviours than the high achievers did. The behaviours of some low achievers in small groups may be explained at least in part as "social-loafing", where individuals could "hide in the crowd", in order to evade work, or feel "lost in the crowd" because they are overwhelmed (Latane et al, 1979: 830).
6. The view expressed by Murray et al (1993: 75) that students can and ought to learn from each other by **listening** to and trying to make sense of other procedures and concepts being explained, is strongly supported. Therefore, the researcher's immediate reaction to the first lesson was :"How is this approach going to work if these children don't **listen** to each other?". This lack of attention, displayed by the children in this study did not change in the next two lessons. The researcher therefore decided to change her role from an observer to a facilitator (see pages 35-36 for discussion).

During and after the fourth lesson some of the following observations were made :

1. When the children worked in pairs that were chosen by the researcher, a difference was noted. This was attributed to her role as a facilitator. As Desforges (1987) states, the adult plays a vital role in sustaining the discourse in the collaborative small groups. The children clearly need the

teacher's assistance in order to establish a more productive collaborative relationship for learning mathematics (Cobb et al, 1989: 110).

2. There was no specific pattern when the children were asked to choose their partners. On different days they tended to choose different partners, irrespective of their ability. This finding is not consistent with that of Murray et al (1993), who stated that pupils seem to choose peers relatively equal in cognitive ability. The finding in this study was more consistent with Stable's (1992), who stated that pupils prefer to work in friendship groups, because it seems to offer them some sort of security. This finding also contradicts Vygotsky's (1978) statement about learning through collaboration with more capable peers.
3. A valuable strategy for making the children listen to each other was to make each one of them explain his/her strategy to the other pupil in the dyad, after which each pupil in the dyad had to explain his/her partner's method to the larger group. A marked improvement occurred. This could be attributed to the fact that the children now felt compelled to listen to their partners in order to explain. This could be one way of ensuring that they assume responsibility for the maintainance of the discourse.
4. Another positive observation was that the Lower Ability pupils stopped peeking into the work of the other pupils'. They most probably now felt more secure with the knowledge that they were working together and that the emphasis was now on the discussion of the solution strategy and not on the answer.
5. The maximum number of problems completed per thirty minute period during the first three lessons, were four as compared to the two during the rest of the lessons. The fact that the group was now working on fewer problems was not cause for concern, because it was obvious that they were now spending more time on the problem, working it out and discussing it. As Wheatley (1992: 532) stated persistence is a necessary factor in mathematics problem solving.

6. Two of the Lower Ability pupils in this study (i.e. the children who had initially been shunned by some of the others) made little, if no observable progress during this period. Their initial behaviour (i.e. lack of participation and confidence) did not change much.
7. The Average Ability pupil (Byron) who was included in both the five week sessions seemed to take on a different role in the Average Ability group as compared to his role in the Mixed Ability group. In the Mixed Ability group, he tended to be very reserved and offered only one word answers or very simple explanations. In some instances he showed no influence of the strategies used by the other members of the group. This finding is similar to that of Maher et al's (1992:72). When Byron worked in the paired groups he tended to be more actively involved in solving the problems and he offered more detailed and appropriate discussions of his solution strategies. A more striking change was observed when Byron worked in the single ability group (i.e. in the Average Ability group), he assumed a very confident role in that he was very willing to work on the problem, to discuss with his partner and with the rest of the group. He also tended to question and comment on what the others were doing and now, there was evidence of the influence of the previous group. There could be three explanations for this type of behaviour. Firstly, as similar problems were dealt with during both the sessions he may have felt more confident to handle these. Secondly, as Maher et al found (1992) he may have now been displaying the influences of the previous group. Thirdly, he may have felt more confident with children of a similar ability. This is consistent with the observation made by Murray et al (1992), that children of similar ability prefer to work together because they find it easier to establish a consensual domain. Lesh (in Bauersfeld, 1992:167) shares a similar point of view. He stated that the qualitatively different strategy systems of thought of higher achievers may be inaccessible to average ability children. In the researcher's opinion, Byron gained from being in both the mixed and the similar ability group. He may not have behaved in this manner if he had not been in that Mixed Ability group. It is

therefore important to note that a number of issues need to be considered when making decisions about grouping children. The suggestion by Palincsar et al (1989) that flexible groups should be maintained, is strongly supported.

The obsevations made during the study suggests that teachers need to become aware of individual patterns of responses among their pupils and take steps to promote more active involvement by all students, especially low-achievers (Mulryan, 1992). It also suggests that social interaction within the groups (i.e. among the pupils as well as between the pupils and the teacher, in the role of facilitator) is important, for the success of this approach. As all children do not always invent their own algorithms, social interaction is viewed as an alternative way of facilitating their conceptual development (Olivier et al, 1990: 7). It is the teacher who has to help the children to establish the norms for social interaction or collaborative dialogue.These are firstly, explaining their solutions to other pupils and answering questions. Secondly, listening to alternative solutions offered by their partners and weighing them up and thirdly, attempting a consensus (Wood et al, 1990). It must be stressed that there are times when children are not ready to agree on a solution so the teacher should allow for student disequilibration and rather than reach closure, revisit the problem (Maher et al, 1992: 73). These social norms need to be continually reconstructed during the course of instruction and it is advisable to establish social norms early in group work if work in these groups can be beneficial to all (Cobb et al, 1991).

Unfortunately, the time spent on this study was not sufficient to assess the long-term effects of the teacher's role as a facilitator and social interaction within small groups.

Summary

Chapter Six has discussed in detail the children's intuitive strategies and the effects of the semantic structure of each problem type on their strategy use. It was found that there was a variety of strategies (except for both the Factor problems and for the Sub-division problems). The majority of which were represented by both Grades either through modelling with concrete material or through pictorial representation (i.e., strategies at Level 1 and 2 - see Table 5.37). Among the Grade 2 pupils the use of known multiplication facts and symbolic representation was also quite prevalent. Two reasons that could be put forward for the lack in range of strategies for the Factor and Sub-division problems are: Firstly, as the majority of the children experienced difficulties with both the Factor problems (especially the Division Factor problems - see Table 5.36 for performance), the number of strategies for these specific problem types as compared to the number for the other problem types was very low. Secondly, initially the children experienced problems with Sub-division (see Tables 5.17 and 5.33), but when they were beginning to show that they had a better understanding of the problem structure there was unfortunately not enough time to do more problems of this type. The children therefore did not have the opportunity to demonstrate a range of strategies.

For both multiplication and division, only the children in Grade 2 used strategies at Level 2, i.e. the numerical phase. A large number of the children's responses were Level 1 computational strategies, i.e. grouping and counting all, which Murray et al (1991) refer to as the pre-numerical phase. However, this was done in two distinct ways:

1. Counting items in each group until the total is obtained (see Melloney's strategy on page 77).
2. Counting items in each group, pausing at the end of each set or emphasizing the last number in each set until the total is obtained (see Sara's strategy on page 72). Anghileri (1989: 375) refers to this as *tallying the groups*. The pause or emphasis between each group of words, in the counting sequence, e.g. : *1, <u>2</u>, 3, <u>4</u>, 5, <u>6</u>, 7, <u>8</u>, 9, <u>10</u>,* marks the attainment of each subtotal and indicates a change in the counting procedure from *storing* the counting word representing a subtotal, e.g. <u>4</u> to extending the counting sequence, i.e. 5, <u>6</u>. Schaeffer et al (1974) refers to this as the *cardinality principle* , which Fuson (1982) explains as the child having to transfer from the counting meaning to the cardinal meaning.

It was also found that the children may have used objects in two of the three ways that Kouba (1989: 152) had identified :

1. Objects are used as a **representation** of the unique elements in each set of the problem. Four versions of this were identified in this research, e.g. :

- Brandon on page 73 set out three cut-outs of children to represent children and on each he placed four crayon cut-outs to represent crayons (he used specific aids).
- On page 99, Byron modelled the problem by using unifix cubes. He first set out four cubes which represented the children, then shared out more cubes to represent marbles .
- On page 75 Sally represented the problem pictorially, drawing box shaped objects to represent tables and little marks/strokes to represent children.
- Melloney on page 76 represented pictorially. She drew tables to represent tables and wrote the number 2 under each table to represent a set of two children at each table.

2. Objects are used as a **reference**, e.g. Kelvin on page 97 when setting out two chairs (pictorially) could have done this to represent two sets of chairs and used this as two reference sets to count two groups of three to obtain six chairs. (However,as discussed on page 97, it is not certain what strategy Kelvin actually used). Mulligan (1991: 162) refers to this strategy as "operate on the set".

For the multiplication problems, addition and multiplication number facts (i.e. number sentences) were prevalent but for division there were only a few instances of their use. The division and subtraction number facts did not feature in any of the children's strategies. In a number of cases when a multiplication number fact was used, e.g. 3 x 6 =18, the children were not entirely certain in their interpretation of it. In other words they were not sure whether it meant 3 times 6 or 6 times 3.

From the analysis of the children's intuitive strategies it was clearly evident that the semantic structure of the problems influenced the children's choice of strategies, i.e. the models that they used. For multiplication, three intuitive models were identified, i.e. the "equivalent groups" model which was consistent with the findings of Fischbein et al (1985), Kouba (1989) and Mulligan (1991); the "array" problem was consistent with Mulligan's (1991) findings. However, the finding on the Cartesian Product model differed from that of Mulligan's (1991), in that in this research the use of the model was clearly evident especially the use of the many-to-many correspondence. For Division, four models were identified, i.e. 'sharing one-by-one", "building-up","building-down" and a model for sub-dividing wholes. All three models for division were also identified in Mulligan's (1991) study.

Although the *equivalent sets model,* as Fischbein et al (1985), Kouba(1989) and Mulligan (1991) had found, was used predominantly across all multiplication problem types, the above analysis of the multiplication intuitive models shows that children do use a variety of intuitive models. This is clearly seen in the children's solution strategies for the Array and the Cartesian Product problems. These models were very different from the *equivalent sets model.* This research thus shows that a change in the semantic structure of the problem does affect the intuitive model used.

There was a high occurrence of double counting, i.e. fourteen in Grade 1 and twenty-eight cases in Grade 2. This confirms Olivier et al's (1992 : 34) finding that the double counting strategy underlies all the strategies in division. The only problem structure in division in which double counting did not feature was Sub-division. This could be the result of the semantic structure of the Sub-division problem, which required halving.

The children also tended to prefer particular strategies for certain problem types, e.g. the Quotition problem type for which there were more grouping strategies than sharing strategies as compared to their use in the other division problem types. It also seemed that the children found it easier to use one-to-many correspondence for division problems than many-to-many correspondence (which was used only for the Cartesian Product problem where two items of clothing had to be matched). This finding confirms Julie Anghileri's , which states that children tend to find the one-to-many correspondence easier than the many-to-many correspondence.

This study has also found that the role played by the teacher and the particular organization of the children in the class has a great impact on the children's solution strategies. From the findings it can be concluded that social interaction within the groups with the teacher in the role of a facilitator is extremely important for the success of the Problem-Centered Approach, an environment which will encourage the construction and evolution of the children's solution strategies.

Another observation that was made showed that at times the children's explantions of their strategies changed during the course of the lesson. There were instances when the researcher observed the children's initial solution strategy and listened to their explanations while they were working and then listened to the explanation they offered to the rest of the group at the completion of the problem. Very often these two strategies differed to a certain degree. The final explanations were usually shorter and more refined; and there were also indications of the inclusion of other children's ideas in these final explanations.

CHAPTER SEVEN

CONCLUSIONS

INTRODUCTION

The main research problem investigated in this study was young children's intuitive strategies (and the intuitive models they used) and their relationship with the semantic structure of the different multiplication and division problem types. One significant observation made during the investigation (which was briefly discussed in Chapter Six of this thesis) was the impact of social interaction and the teacher in the role of facilitator on the children's strategy use.

The investigation was carried out over a ten week period, and it involved Grade 1 and 2 pupils in different ability groups. During these ten weeks, the researcher made certain changes, within the group as well as in her role, that were considered to be necessary for the children to work more productively.

This concluding chapter summarizes the main findings, strengths and limitations of the study implications of these findings and suggestions for future research.

SUMMARY OF MAIN FINDINGS

With reference to the research questions described in Chapter One and the analysis of results, the summary of findings will be discussed under **three** main sections: performance, strategy use and intuitive models.

Performance

An analysis of the overall performance level (i.e for both grades) indicated that 61% of the strategies were appropriate and 76% of the sample were able to solve the ten different problem structures for multiplication and division by the final lesson of the experimental period. This, was in spite of the children not having had any formal/informal instruction on these concepts. On the whole Grade 2 performed better than Grade 1 (see Table 5.36). For Grade 1 the appropriate strategies made up 59% and for Grade 2, 62% of the total strategies.

Children in both grades used a range of informal strategies which were obviously based on their knowledge of counting and addition. This was also the case for the division problems, i.e. the children tended to rely more on addition and counting except in the case of Quotition. As Mulligan (1991) had found, this research also revealed that the semantic structure of the different problems did not have as great an impact on the children's performance level as it had on their choice of strategies, with the exception of both the Factor problems. The Grade 2 (H.A.) pupils, however did not experience problems with the Multiplying Factor problem type. An investigation of their particular solution strategies reveals a range of strategies. With the Grade 1 pupils there was not much difference in their performance between the multiplication and division problem structure, but there was a sharp decrease in their performance for both the multiplication and division Factor problems. In fact none of the Grade 1 pupils could solve the Division Factor problems. As far as the Grade 2 was concerned their performance on the multiplication problems was higher than for the division problems. This was due to the sharp decrease in performance for the Division Factor problems. The analysis also revealed that on the whole 20% of the children could only solve 20% of these problems.

Strategy use

An indepth analysis of the children's intuitive strategies revealed that although they used a range of strategies (refer to Tables 5.34 and 5.35) **four** strategies featured predominantly across all problem structures, i.e. grouping, counting-all, modelling and pictorial representation. As mentioned under Performance, the semantic structure had a greater influence on the children's choice of strategy than on their performance, with the exception of the Factor problems. This is clearly shown on Tables 5.1 - 5.33. As mentioned earlier, 76% of the sample was able to solve all the problem structures by the final lesson. The children's solution strategies usually reflected the semantic structure of the problem. In other words the action or relationship depicted in the problem was either modelled or represented pictorially. This finding was consistent with that of Mulligan (1991). There was no great difference in strategy use between multiplication and division but it was found that certain strategies were used exclusively for certain problems, e.g. sharing, one-to-many correspondence, halving and estimate-and-adjust were used only for division. Whereas, for multiplication the many-to-many correspondence strategy was used exclusively specificially for the Cartesian Product problem that involved the matching of two items of clothing.

Another finding related to performance and semantic structure was that, when the children experienced difficulities with a problem they tended to use more concrete, less abstract forms of representation. In these instances they also showed a preference for the use of the specific aids. Problems such as the Cartesian Product, Sub-division and the Factor problems (especially the Division Factor) involved mainly modelling and pictorial representation (see Tables 5.34 and 5.35)

The analysis also revealed the children's preference for the use of counting and additive strategies across all problem types. There were only a few instances of the use of subtractive strategies (i.e. the "building- down" model) which was represented through modelling and pictorial representation. No other form of subtraction strategies (e.g. known subtraction fact or subtraction number fact) or division strategy featured in the children's solutions. This finding disputes Fischbein et al's (1985) finding that the only model for division (prior to instruction) is the partitive/sharing model. Although this model did feature in the children's work in this study, it was not the only model used. As Olivier et al (1992) stated, children tend to prefer using additive strategies for both multiplication and division word problems. This is also consistent with the findings of Mulligan (1991). She stated that the children's informal strategies reflect a strong relationship between addition and the development of multiplication and division concepts.

Intuitive Models

A number of intuitive models were identified from the analysis of the children's intuitive strategies. The majority of these were consistent with those identified by researchers such as Kouba (1989), Mulligan (1991) and Olivier et al (1992). As also reported by the above researchers, the evidence in this research showed that the intuitive models used by young children are much more varied and complex than Fischbein et al (1985) had indicated.

For multiplication, the underlying or overarching model was the "equivalent sets model" which was consistent with the findings of Fischbein et al (1985), Kouba (1989) and Mulligan

(1991). The children also used other models in relation to specific problem types, e.g. the Array model for Array and Repeated Addition problems; the Cartesian Product model for Cartesian Product problems with and without the use of many-to-many correspondence. The many-to-many correspondence was used only for a particular type of Cartesian Product problem, i.e. one that involved matching of two items of clothing problem. This also disputes Fischbein et al (1985), who stated that Repeated Addition is the only implicit model for multiplication.

For division, the underlying model was double counting. This is consistent with the finding of Olivier et al (1992). Children in this study as in Mulligan (1991) and Olivier et al (1992) displayed the use of the building-up model as well as the sharing model, the building-down model and a model involving halving. As mentioned earlier, they showed a preference for the building-up model, except for the Quotition problem structure for which the building-down model was more prevalent..

A brief discussion on the strengths and limitations of this study follows.

STRENGTHS AND LIMITATIONS

The findings of the present investigation endorse the constructivist views held by several researchers (e.g. Mulligan (1991), Kouba (1989) and Murray et al (1992)). It also disputes some of Fischbein et al's (1985) findings. This investigation indicated that young children have the ability to solve a range of multiplication and division word problems using a range of strategies prior to instruction. An advantage of carrying out this study with children who were accustomed to the Problem-Centered Approach was that it was not necessary to explain in detail what was required of them.

There were four main limitations of this study:

1. It was not possible to observe the children in their natural environment, i.e. their own classrooms, as it was not allowed by the school. Due to this the children were withdrawn from their classes during normal mainstream

lessons. This therefore restricted the researcher in that she was only able to observe them in small groups of 4 and 6 which were much smaller than the groups in their actual classes;

2. Due to the problem discussed above the children could only be with the researcher for a limited period of time. The consequence of this was that there were times when not all the children had the opportunity to fully clarify their explanations or for the researcher to probe their reasoning and strategies deeper; and all the groups were not able to work on all the problem types. In the second 5-week block session the school experienced a number of disruptions (due to elections countrywide). This compounded the problem of time;
3. Another consequence of the problem discussed above, was that it was not always possible to observe and question every child while s/he was working on each problem. Therefore much of the data presented (i.e. the children's strategies) reflects responses that were provided after they worked on the problems as well as after they may have discussed with or listened to others in their groups. Due to this, the final explanations offered by the children may have been different from their initial strategies and understanding of the problems; and
4. Some of the pupils seemed to experience difficulty with verbalizing their solutions. This was clearly evident when they were asked to clarify or elaborate. It would therefore have been more appropriate to observe every child individually while s/he was working on the problem, in order to have had a better idea of what s/he actually did. This however was not possible as the researcher wished to observe the children in as close a social environment as possible to their classrooms.

IMPLICATIONS OF FINDINGS

The evidence clearly indicates that young children are able to solve a range of multiplication and division problems intuitively, in a condusive environment, such as a Problem-Centered classroom. This therefore suggests that children in the Junior Primary Phase should not be restricted to certain word problems in multiplication and division; they should be given the opportunity to solve a range of word problems intuitively. Of course this has to take place in an environment that encourages this behaviour, i.e. a Problem-Centered rather than a Traditional classroom.

The evidence in this study also indicates that the children should be given the opportunity to work with a range of concrete material. This research found that for certain problem structures the children initially needed to use the specific aids until they were able to comprehend the problem situation. After which they opted to use other forms of representation.

SUGGESTIONS FOR FUTURE RESEARCH

1. Although this study analysed the children's strategies, it did not address individual children's range of intuitive strategies. Future research could investigate whether individual children have more than one model for multiplication and division and whether they consistently use the same model across different problem types.
2. Future research could also address the impact that social interaction, as well as the teacher in the role of facilitator, have on the construction and evolution of children's strategies.
3. Another type of study that could be carried out could focus on spatial development rather than arithmetical development, as there are not many studies in this area.
4. Longitudinal studies of children over, say, three years, focussing on how their strategies evolve and change need to be carried out.
5. Investigation on Factor problems to ascertain whether an explanation of the meaning of "*n* times more/as many or *n* times less" contributes to higher success rates.

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APPENDIX A

LIST OF PROBLEMS USED

Multiplication

Repeated Addition

1. There are 3/5 tables in the classroom and 2/5 children are seated at each table. How many children are there in the classroom?
2. There are 3 boxes on the table. In each box there are 2 beads. How many beads are there altogether?
3. I have 2 friends. I give my friends 3 sweets each. How many sweets are there altogether?
4. There are 3 children at the table. Each child is given 4 crayons. How many crayons are there altogether?

Factor

1. My friend has 3 books and I have 2/4 times as many. How many books do I have?
2. Sam has 2c/4c and I have 3 times as many. How much money do I have?
3. I have 6 marbles and Jane has 3 times more than I have. How many marbles does Jane have?
4. I have 2 pencils and my friend Jane has 3 times more than I have. How many pencils does Jane have?

Rate

1. I bought one sweet for 2c/3c. How much money do I need to buy 3/4 sweets?
2. If you need 3c/5c to buy one sticker, how much money do you need to buy 2/4 stickers?
3. If you need 3c/6c to buy one pencil, how much would you need to buy 2/4 pencils?
4. I bought one sweet for 3c. How much money do I need to buy 5 sweets?

Cartesian Product

1. Marina has 3 skirts and 4 blouses of different colours that all match. In how many different ways can she dress?
2. I can buy plain chips and salted chips in small, medium and large packets. How many different choices can I make?
3. Mum has 2 skirts and 3 blouses. How many different ways can she wear these?
4. I can buy plain chips or salted chips in small and large packets. How many different choices can I make?
5. The shop has black and white shirts in small and medium sizes. How many choices can you make?

Array

1. There are 2/3 lines of children. In each line there are 4 children. How many children are there altogether?
2. In the classroom there are 2/5 rows of chairs with 3/6 chairs in each row. How many chairs are there altogether?
3. A vegetable patch has 4/5 rows of onion plants, with 3/4 plants in each row. How many onion plants are there altogether?

Division

Partition

1. I want to share 6/10 sweets equally among 3/5 of my friends. How many will each friend get?
2. 4/12 apples are shared equally among 2/3 children. How many apples does each child get?
3. 2/4/4 boys have got 8/8/16 marbles to share. How many marbles does each boy get?

Factor

1. My father has 12 shirts and this is 3 times as many as I have. How many shirts do I have?
2. I have 6/9 books and this is 3 times as many as my friend has. How many books does my friend have?
3. Pat has 6 marbles and this is 3 times as many as Sam has. How many marbles does Sam have?
4. I have 15 pencils and this is 3 times as many as Sam has. How many marbles does Sam have?

Rate

1. My mother bought me 2/3 sweets for 8/9 cents. If each sweet cost the same amount, how much did one sweet cost?
2. My friend bought 3/4 pencils for 9/12 cents. If each pencil costs the same, what is the price of one pencil?
3. A man must walk 18km in 3 hours. How many kilometres must he walk per hour to achieve this?

Quotition

1. Mum has baked 8/20 buns. She puts them into plastic bags, 2/4 in each bag. How many plastic bags did she use?
2. My dad has 9/18 pens. He shares these equally among his children. If each child receives 3 pens, how many children does my dad have?
3. 8/12 toys are shared equally among children at the table. If each child receives 4/3 toys, how many children are there?
4. I have a strip of gum that is 6cm long. I give each of my friends 2cm of gum. How many friends are there?

Sub-division

1. I have 1/2 oranges/apples to be shared equally between/among 2/4 people. How much orange/apple will each person get?
2. I have 3 fruit to be shared equally among 6 of my friends. How much fruit will each friend get?

APPENDIX B

OPERATIONAL DEFINITIONS

The operational definitions used in this study were adopted from the following studies: Mulligan (1991), Kouba (1989), Anghileri (1989) and Olivier et al (1992).

Intuitive strategies: Those strategies intuitively developed by the children to solve the multiplication and division word problems. These may have developed **informally** prior to or at the time of the study. Examples of intuitive strategies are listed below.

Modelling: refers to the use of concrete material e.g. unifix cubes, fingers, sticks or specific aids to represent the action or relationship described in the problem.

Pictorial Representation: the problem context is drawn in greater or lesser detail and then solved by further drawing in the actions needed. Numerical representation refers to representation of the structure of the problem using numerals where no arithmetical operations are employed.

Sharing-one-by-one: counting out the dividend and dealing out one by one to the specified number for the group, e.g. Sally in Grade 2 (page 104) "*here's my twelve cents (coins) and I put 4 pencils and I gave them each a money and another one and another one... and then they equalled one, two, three cents. One pencil costs three cents*".

One-to-many correspondence: a matching type strategy where both quantities or entities in the problem were modelled, e.g. Melloney in Grade 1 (page 84).

Grouping with counting all: formation of equivalent groups representing the quantities in the problem with one-by-one counting to calculate the total for multiplication, e.g. Sara in Grade 1 (page 72) after making 5 groups of two, the child counted "*one, two, three, four, five, six, seven, eight, nine, ten*". For division, counting all may have occurred to check the dividend after grouping or before grouping, e.g. Martin in Grade 2 (page 102) " *one, two, three, four; five,* *six, seven, eight; nine, ten, eleven, twelve,* to verbalize " *now he's got three times more than his son, his son has four.*"

Double counting: counting the number in the dividend and (simultaneously or afterwards) counting the number in each group (sharing division), e.g. Sally in Grade 2 (page 99) or the number of groups (measurement division), e.g. Angela in Grade 2 (page 107).

Grouping and skip counting: this is a more sophisticated counting strategy based on counting in multiples, e.g. Kelvin in Grade 1 (page 73) "*two, four, six, eight, ten*".

Estimate-and-adjust : formation of equivalent groups by estimation where the number in each group was unknown. Olivier et al (1992: 35) refer to this as an "*estimate-and-adjust*" strategy, e.g. James in Grade 2 (page 99).

Grouping with counting on: formation of equivalent groups representing the quantities in the problem. emphasizing the number in the first group, then counting in ones, e.g. Calvin in Grade 2 (page 79) " *six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen*".

Many-to-many correspondence: this strategy was only identified for a specific Cartesian Product problem (i.e. the matching of two items of clothing). The children represented both objects in the problem, e.g. skirts and blouses. They then joined these by lines using the many-to-many correspondence strategy (see page 92 for examples).

Known addition facts: indicated by retrieval of an addition fact for both multiplication and division, e.g. Melloney in Grade 1 (page 73) " *3 and 3 is six*". This category also included some derived addition facts, e.g. Michael S. in Grade 2 (page 105) " *I knew 6 plus 6 was 12, so I added another 6 and I found that it was 18*".

Known multiplication fact: indicated by retrieval of a memorized multiplication fact, e.g. Angela in Grade 2 (page 92) "2 *times 3 is 6*".

Symbolic representation: The solution strategy is represented as an appropriate number sentence, e.g. Michael M. in Grade 2 (Page 87) for addition $6 + 6 + 6 + 6 = 24$.

Mental Computation: When a child provided an answer, usually immediately after the presentation of the problem, with no obvious indication of the use of any of the above strategies.

Skip counting: counting in a particular pattern or sequence, e.g. "*two, four, six*". This occurred with modelling and grouping and the number of groups may have been counted physically. Skip counting also occurred in situations where the child was visualizing.

Repeated addition: Adding the number in a group *n* times, where use of the terms "*and*" and "*plus*" were verbalized as distinguishing features, e.g. "*six plus six is twelve and another six is eighteen*".

Derived fact: Using a known fact to find another fact, e.g. "*two sixes are twelve and another six is eighteen*".