Grade Twelve Learners' Understanding of the Concept of Derivative.

by

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PREFACE

The work described in this thesis was carried out in the School of Science, Mathematics and Technology Education, University of KwaZulu-Natal, from January 2006 to December 2008 under the supervision of Dr Sarah Bansilal.

This study represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

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ABSTRACT

This was a qualitative study carried out with learners from a grade twelve Standard Grade mathematics class from a South Durban school in the province of KwaZulu-Natal, South Africa. The main purpose of this study was to explore learners' understanding of the concept of the derivative. The participants comprised one class of twenty seven learners who were enrolled for Standard Grade mathematics at grade twelve level. Learners' responses to May and August examinations were examined. The examination questions that were highlighted were those based on the concept of the derivative. Additionally semi-structured interviews were carried out with a smaller sample of four of the twenty seven learners to gauge their perceptions of the derivative.

The learners' responses to the examination questions and semi-structured interviews were exhaustively analysed. Themes that ran across the data were identified and further categorised in a bid to provide answers to the main research question. It was found that most learners' difficulties with the test items were grounded in their difficulties with algebraic manipulation skills. A further finding was that learners overwhelmingly preferred working out items that involved applying the rules. Although the Higher and Standard grade system of assessing learners' mathematical abilities has been phased out, with the advent of the new curriculum, the findings of this study is still important for learners, teachers, curriculum developers and mathematics educators because calculus forms a large component of the new mathematics curriculum.

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Dedication

This thesis is dedicated to my parents, the late Mr and Mrs Naidoo and my dear brother, the late Colin Naidoo.

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CHAPTER ONE
INTRODUCTION

Calculus is a rich subject and has wide applications in natural and social sciences. The two fundamental concepts that underpin calculus are the derivative and the integral. According to Tall (1996) there is a growing desire to "research the learning process to understand how individuals conceptualise calculus concepts" (p, 291) and Parameswaran (2007) maintains that calculus is rich in abstraction and requires a high level of conceptual understanding which many students find difficult to cope with. Ferrini-Mundy & Graham (1991) argue that students' understanding of central calculus concepts is very primitive as they "demonstrate virtually no intuition about the concepts and processes of calculus … they diligently mimic examples and their attempts to adapt prior knowledge to a new situation" (p, 631) frequently results in very persistent and often inadequate conceptions.

White & Mitchelmore (1996) concern themselves with the memorisation that takes place by the large number of students who take calculus. Tall (1992) agrees with this as he mentions that students prefer procedural methods rather than conceptual methods especially when they encounter difficulties in calculus. According to Bowie (1998):

> "research studies on students' misconceptions have provided insights into difficulties students have in dealing with concepts in mathematics, … analysing misconceptions in calculus could provide information about the difficulties they face in learning calculus concepts" (p, 5).

Merely listing and detailing misconceptions is insufficient, however, more reflection on learner difficulties to calculus questions may be beneficial. Tall (1992), in his discussion on students' difficulties in calculus, points out that reflecting "on the difficulties encountered by students' of differing abilities and experience, to obtain empirical evidence to build and test theories of learning to enable more fruitful learning experiences for students in calculus" (p, 12) might be useful. As a way of exploring learners' understanding of the concept of the derivative, my study aimed to, not only provide a detailed description of learners' difficulties, but , probe the probable underlying reasons for their particular misconceptions and determine how previously acquired concepts impact on their understanding of the concept of the derivative. This has been achieved through an exhaustive analysis of learners' responses to examination questions and semi-structured interviews.

1.1 Rationale and purpose

Research studies concerning basic calculus concepts have been well documented (Orton, 1983a, Orton, 1983b, Williams, 2001). Many of these studies have been conducted mainly at universities with students who are engaged with a first year mathematics course (Palmiter, 1991, Bowie, 1998, Baker, Cooley & Trigueros 2000, Bezuidenhout, 2001, Parameswaran, 2007). In South African Schools, calculus is encountered for the first time by grade twelve learners and research at this level is very limited. According to White & Mitchelmore (1996) research into students' understanding of calculus reveals that there are various concepts that cause problems for students. Students' difficulties with abstract concepts of rate of change (Orton, 1984), limit (Tall & Vinner, 1981) and function (Even, 1993; Vinner, 1983) are well documented. The study by White & Mitchelmore (1996) shows that students have a very primitive understanding of the variable which impacts on their understanding of word problems involving rate of change. Their study revealed that students treated the variables as symbols to be manipulated rather than quantities to be related. Furthermore, many could not symbolise rate of change in items that required modelling a situation using algebraic variables. The concept of the derivative, although not abstract, also impacts on students' understanding of calculus.

According to Zandieh (1997, 2000) the concept of the derivative is a multi-faceted concept with a limit, ratio and function layer. She maintains that it is not appropriate to ask whether or not a student understands the concept of the derivative but one should ask for a description of a students' understanding of the concept of derivative. It is necessary to ask about the aspects of the derivative concept a student knows and the relationships a student sees between these aspects. Such an approach to the understanding of the derivative requires an in-depth study of the limit concept. In South African schools at grade twelve level, an in-depth study of the limit concept is not undertaken. Therefore the derivative concept is introduced dynamically and sometimes through the concept of average gradient. Thus the various aspects of the derivative concept, that Zandieh (1997, 2000) describes, may escape the grade twelve mathematics learners but nevertheless the study of calculus still remains underpinned by the concept of the derivative for them.

I have been teaching grade twelve calculus for more than twenty years to learners' with differing abilities and experiences and came to understand that while learners can solve routine problems involving the rules for differentiation, they find it difficult to solve problems requiring the understanding and application of the derivative concept. This was clearly evident as many learners experienced difficulties with problems requiring maximisation and minimisation and rate of change problems as this required the understanding of the derivative concept. It is precisely for this reason that I explored grade twelve learners' understanding of the concept of the derivative. My study aimed to achieve this through focusing, firstly, on some difficulties the grade twelve learners experienced when answering examination type questions related to the concept of the derivative and secondly, on their perceptions about the derivative. Furthermore, the issue about calculus teaching and learning has become of great concern to me and I wanted to know more about learners' conceptual understanding of calculus.

Through the Masters Degree, I was introduced to research in mathematics education and this exposure has persuaded and influenced me to look at these misconceptions differently and I found a need to look insightfully into the way learners were learning and understanding the calculus concepts. Furthermore, I was persuaded to look deeper into possible reasons as to why they were not answering calculus questions correctly, the way we, the mathematical community, deemed it necessary. Reading reports on research about mathematics teaching and learning has certainly made me look deeper and more insightfully into the way learning and understanding of calculus concepts was taking place and in particular, I wanted to research and explore grade twelve learners' understanding of the concept of derivative. Knowledge about learners' understanding and the strength of learners' understanding of the concept of the derivative will be beneficial when developing curriculum materials that aim to facilitate improved learning and understanding.

My study has been conducted with learners doing a Standard Grade mathematics course, which was a subject in the previous curriculum designed for learners who could not cope with the Higher Grade mathematics course. The Higher Grade mathematics course was a more advanced course and was a pre-requisite for learners that pursued a career in Engineering, Science or Mathematics. The focus of the Standard Grade mathematics course was mainly application of theory learnt.

Approximately 70 000 learners enrolled in the Standard Grade mathematics course in Kwa-Zulu Natal in 2007. In the new Revised National Curriculum Statement (RNCS) curriculum, it is compulsory for every learner in the Further Education and Training (FET) Phase to do either Mathematics or Mathematics Literacy as a subject and these are offered as two separate subjects. A bigger cohort of learners, approximately 80 000 in Kwa-Zulu Natal were enrolled for the subject Mathematics and calculus is still a major portion of the curriculum and hence a major examination component. In the November Examination 2008, the first examination of the RNCS curriculum, 33 marks of the 150 marks was allocated to calculus questions. Therefore, my study will be more useful in the new RNCS as the majority of the learners that would have been in the Standard Grade mathematics course are now in the Mathematics course. Furthermore, the concept of average gradient gains prominence and priority in the RNCS as it makes use of this concept of average gradient to introduce the derivative concept that underpins calculus. The understanding of calculus concepts, in particular the understanding of the derivative concept, is built via the average gradient concept. Thus, it is important for teachers to be knowledgeable about how learners understand the concept of the derivative and be aware of any difficulties that learners might encounter as this may help in the design of learning strategies and materials. Therefore my study is more relevant, pertinent and useful in the new RNCS as more learners are pursuing the Mathematics course and calculus is still a major aspect of mathematics curriculum at grade twelve level with the main focus being on the derivative.

1.2 Details of the study

A description of Vinner's (1997) framework for pseudo-conceptual thought processes in mathematics learning is provided in the theoretical framework as it is used to analyse the data. At the outset, the terms concept image, concept definition and Zandieh's (1997, 2000) multiple representation of the derivative concept is discussed. The process object duality in concepts with particular reference to Sfard's (1991) model for learning is also presented, as understanding learning is particularly relevant to my study. Furthermore, research about misconceptions can be used to inform us about learning and understanding and thereby develop curricula that aim to foster learning and improve understanding.

The survey of literature relevant to my study is presented in the literature review. Some calculus studies, Zandieh's (1997, 2000) description of the various layers of the derivative concept, some teaching experiments that provide insights into students' concept image of the derivative and students' algebraic misconceptions in basic calculus is discussed. The derivative is the basis for grade twelve Standard Grade calculus studies. Any further work with related aspects like sketching the graph of a cubic function and applying calculus concepts to problems involving maxima and minima is based on the understanding of the derivative. Therefore, understanding the derivative concept is of utmost importance in the grade twelve year.

The sample used in my study is a grade twelve class of learners pursuing the Standard Grade mathematics course. This is a convenience sample as it is their initial exposure to the study of calculus and they encounter the concept of the derivative for the first time. Learners' responses to examination questions in the calculus sections were collected at two different points, during May and August. Thereafter semi-structured interviews were conducted with four learners. The interpretive paradigm is most suited to my study as I wanted to see the participants' reality from the participants' point of view.

In the data analysis, I provide a detailed description of learners' responses to the examination questions. The various categories arise inductively as a result of the detailed descriptions and the categories set the stage for each of the critical questions to be addressed and answered. The analysis of learners' responses to the two examination questions addressed the various issues related and relevant to the first critical question. A detailed analysis of the interview is also provided as the interview analysis provided some answers to the second critical question.

The findings and conclusion provide some answers to the two research questions and the various categories in the data analysis chapters are used as a basis to answer these questions. Learners perceived the derivative as a rule and the stimuli within examination questions triggered certain association for learners as they formed a link with previously acquired concepts. Thus, learners resorted to pseudo-conceptual thought processes as a way of building a manageable set of techniques for dealing with the questions related to the concept of the derivative. Since learners understanding of the derivative was not underpinned and not grounded in conceptual understanding, they resorted to pseudo-conceptual thought processes whereby they tried to find associations with the representation from the given questions. Some of their responses to examination questions related to the concept of the derivative were obtained via these associations. Furthermore, learners' lack of procedural fluency in algebra contributed largely to their difficulties they experienced when they answered examination questions.

1.3 Outline of thesis

Chapter two describes the theoretical framework and Vinners' (1997) theoretical framework for conceptual behaviour and pseudo-conceptual behaviour is used. The theoretical framework comes before the research questions and the literature review. This chapter also focuses on how conceptual understanding is developed.

Chapter three describes the literature survey related to my research and also focuses on how my research is related to this literature. In particular an overview of Zandieh's (1997, 2000) theoretical framework for exploring learners' understanding of the derivative concept is provided as this is the focus of my study. In this chapter, I describe how my research will accommodate some gaps.

The chapter on methodology comes after the literature review and in this chapter I describe the method used for data collection, the interviews, the interviewees as well as some methodological issues.

In chapter five I present an analysis of learners' responses to May and August examination questions. This chapter describes the data in the various categories that emerged inductively. Thereafter, in chapter six, I analyse the transcribed interviews from each interviewee. Finally the discussion and conclusion chapter is presented.

CHAPTER TWO
THEORETICAL FRAMEWORK

At the outset, the terms concept image, concept definition and Zandieh's (1997, 2000) multiple representation of the derivative concept is discussed. Thereafter the process object duality in concepts with particular reference to Sfard's (1991) model for learning is presented. Merely presenting a list of learners' difficulties in understanding the concept of the derivative or categorising them was not adequate therefore it became necessary to introduce Vinner's (1997) theoretical framework for conceptual behaviour and pseudo-conceptual behaviour. The distinction between pseudo-conceptual behaviour and misconceptions is also discussed.

2.1 Concept image and concept definition

Tall & Vinner (1981) developed a learning theory which is particularly useful and relevant to learners' understanding of the concept of derivative. They introduce the notions of concept image and concept definition. Tall & Vinner (1981) use the term concept image to describe the total cognitive structure that is associated with a mathematical concept that includes all the mental pictures, properties, associations and processes related to a given concept. A concept image is continually constructed as learners mature and it changes with new stimuli and experiences of all kinds. An important aspect of a concept image is the concept definition.

The concept definition is a statement in words used to specify that concept. An individual may learn this concept definition in rote fashion or in a meaningful way or even personally reconstruct this definition. Tall & Vinner (1981) term this as a personal concept definition and it can be different from the formal concept definition. The formal concept definition is the definition accepted by the mathematical community and a learners' concept definition may or may not be consistent with the formal mathematical definition. Furthermore a learner's concept image may or may not include the formally correct mathematical definition. Vinner & Dreyfus (1989) suggest that concept images are not formed merely by definition but by experience. Thus concept images are personal and experiential. The derivative concept when encountered for the first time by grade twelve learners may be achieved not through the formal mathematical definition but through the concept of the gradient. Learners' experience with the concept of gradient, like walking uphill, may influence their understanding of the concept of derivative.

Barnard & Tall (1997) introduce the term "cognitive unit" to describe a part of the concept image that a learner focuses attention on at a given time and this could be any aspect of the concept image. This may be symbols, a specific or a general fact, a relationship, a step in an argument, theorems, representations, properties or any other aspects of the related concepts. For example, the derivative can be conceived as a rate of change. According to Barnard & Tall (1997), a student's ability to conceive and manipulate "cognitive units" is a vital facility for mathematical thinking and a rich concept image would include not only the formal mathematical definition, but many linkages within and between "cognitive units".

Vinner & Dreyfus (1989) show that learners have different associations for the concept of a function and learners do not always come up with the necessary associations that is most useful for solving a given task. When this occurs, the term compartmentalisation is used to refer to these errors. According to Zandieh (1997, 2000), a part of a learner's concept image is compartmentalised, or separated from other parts of the concept image, when the learner does not connect this idea in question to other aspects of the concept image. Sometimes a given problem may not stimulate the schema that is most relevant to solving the problem. Instead a less relevant schema is activated, which may not be appropriate in solving the problem. Probably students may have encountered this less relevant schema in prior interaction and for some underlying reasons, students opted to make use of them. My study aimed to explore learners' understanding of the concept of the derivative and analysing learners' schemas may provide insights to learners' conceptual understanding of the derivative.

The notion of a concept image can be used to examine a learners' understanding of the derivative but it is not sufficient because the concept of the derivative is a multi-faceted concept. Therefore, it is important to look at what aspects of the concept a learner knows and the relationship a learner sees between these aspects.

2.2 Multiple representation of the concept of the derivative

Researchers (Vinner & Dreyfus, 1989, Tall & Vinner, 1981) report that a learner's concept image very often includes a number of different representations of the concept. For example functions can be represented analytically or symbolically, graphically, numerically, verbally, and physically. As a way to develop learners' understanding of a particular concept, there should be an emphasis on the use of multiple representation of that concept. This idea is very relevant to the derivative concept as it is a multi-faceted concept.

According to Zandieh (1997, 2000) the concept of the derivative can be represented:

- graphically as the slope of a tangent line to the curve at a point,
- verbally as the instantaneous rate of change,
- physically as speed or velocity and
- symbolically as the limit of the difference quotient.

Structures underlying each of these representations can be described using the process-object duality.

2.3 Process object duality in concepts

Sfard's (1991) model for learning of abstract mathematical concepts can be conceived in two fundamentally different ways: operationally when a mathematical concept is seen as a process and structurally when a mathematical concept is seen as an object. For example, the algebraic expression $(x+3)$ can be seen as the process of adding 3 to the variable *x*. However, the algebraic expression $(x+3)$ can be conceived of as an object because it is possible to perform actions on it and these actions transform it, like in the expression $2(x+3)-1$. When the algebraic expression $2(x+3)-1$ is read as a series of operations then the computational process gives meaning to the symbols. However, Sfard & Linchevski (1994) point out that the same mathematical concept or the same representation may sometimes be conceived as operational and at other times as structural. This dual nature of mathematical constructs is inherent in most mathematical activities.

Sfard (1991) further emphasizes that operational and structural conceptions of the same mathematical concept are not mutually exclusive but they are in fact complementary. She argues that the ability of learners to see a mathematical concept both as a process and as an object "is indispensable for a deep understanding of mathematics, whatever the understanding of mathematics is" (p, 5). This dual nature of a mathematical construct is present in various kinds of symbolic representation and verbal descriptions of a mathematical concept. Sfard (1991) presents a strong argument for the hierarchical nature of mathematical constructs. She contends that in the process of a concept formation, the operational conception precedes the structural conception, as in computational mathematics, a vast majority of the ideas originate in the process rather than in the objects. For example, the process of counting eventually leads to the concept of a number and the process of adding 3+4 leads to the concept of sum. Sfard

(1991) identifies three stages in concept development which follows the route from process to object, which sometimes may be difficult.

The three stages are interorisation, condensation and reification. To illustrate these stages I will consider the following, finding the average gradient $\frac{f(b)-f(a)}{b-a}$ for a given function $f(x)$, between the points with co-ordinates $(a; f(a))$ and $(b; f(b))$ as an example. The concept of average gradient is studied at grade twelve level as a precursor to introducing the derivative. In the interiorisation stage, the learner continually substitutes numerical values in $\frac{f(b)-f(a)}{b-a}$ and then simplifies or evaluates this expression. This is the process where the learner performs the operations. By repeatedly doing this, the learner acquires the skill of substitution and simplification and this allows the learner to become familiar with the process and to carry it out through mental representations, thus enabling the process to become interiorised. Thus, the first stage of interiorisation occurs when learners familiarise themselves with the processes of substitution and simplification which eventually gives rise to the concept of average gradient, through understanding the elements of the process.

The second stage is that of condensation which is when a learner becomes more capable of thinking about a given process as a whole without necessarily considering its component steps. Being able to explain in words the meaning of $\frac{f(b)-f(a)}{b-a} > 0$ or the meaning of $\frac{f(b)-f(a)}{b-a} < 0$ is to think of the process as a whole. When a learner becomes more capable of working with $\frac{f(b)-f(a)}{b-a}$ as a whole, for a given function, without needing to do the actual substitution of values then the learner is regarded as being advanced in the process of condensation. Thus, this enables the learner to investigate average gradient by working out when the average gradient is positive, negative, zero or even undefined. A learner may also be capable of drawing a sketch of a linear graph to illustrate each of the above situations. Nevertheless, condensation still remains a process and lasts as long as the new entity remains tightly connected to a certain process.

When a learner is capable of conceiving $\frac{f(b)-f(a)}{b-a}$ as a fully fledged object then the concept is reified. This occurs when the learner is able to detach the processes that produce the new entity and see it as an object and draw it's meaning from the fact that this new object is a member of a certain category. Now $\frac{f(b)-f(a)}{b-a}$ becomes the object, average gradient. A learner can investigate general properties of average gradient or new mathematical objects can be constructed out of the present one. For example $\lim_{a\to b}\frac{f(b)-f(a)}{b-a}$ gives rise to the concept of the derivative of a function or gradient of a curve at a point.

2.4 Pseudo-structural approach

Sfard & Linchevski (1994) provide an explanation for what can happen if the desired operational-structural operation does not occur. According to Sfard's (1992) research, a high proportion of students exhibited a strong operational conception of a function as they described a function in terms of some computational formula. She points out that similar findings obtained by other researchers (Dreyfus & Vinner, 1989, Kieran, 1989) show that learners' tendency to associate functions with algebraic expressions is strong and common. This tendency may indicate an "operational conception (the student may perceive a formula as a short description of a computational algorithm)" as well as a structural conception "(the formula may be interpreted as a static relationship between ordered pairs)" (Sfard, 1992, p, 75) and sometimes it may be neither one of them. According to Sfard (1992), such a tendency may show a "semantically debased conception" and the term pseudo-structural conception is used to describe this phenomenon when it occurs.

Many learners are not able to see the mathematical object that they are required to see because it is not clear to them for a variety of reasons. However, these learners are now required to perform some complex operations on this virtually non-existing object. They develop a way of dealing with them by creating their own meaning and these meanings may not be appropriate at all. The mathematical object is now identified with its representation. A symbol, formula or graph becomes the object that is dealt with and this new "knowledge remains detached from its operational underpinnings and from the previously developed system of concepts. In these circumstances, the secondary processes must seem totally arbitrary" (Sfard & Linchevski, 1994, p, 117). Learners may still be able to perform these processes, but their understanding may remain instrumental (Skemp, 1976) as Sfard & Linchevski (1994) argue that meaningfulness comes when the learner is able to see the abstract ideas hidden behind the symbols. The result of learners adopting a pseudo-structural approach may lead to them developing a conception of mathematics that is not coherent and lacks rich relationships. Learners who adopt the pseudo-structural approach and confuse the abstract objects with their representations "do not realise that the symbols cannot perform the magic their referents are able to do: they cannot glue together lots of detailed pieces of knowledge into one powerful whole" (Sfard & Linchevski, 1994, p, 224). Inherent in this pseudo-structural outlook there may be methods that students use, which may not be in keeping with the desired instructional method, but may be as a result of some thinking process or some association process and warrants a conceptual framework which analyses and describes this pseudo-structural outlook.

2.5 Pseudo-conceptual mathematical thought processes in mathematics

Vinner (1997) suggests a framework where two of the main notions are conceptual behaviour and pseudo-conceptual behaviour. Central to the idea of Vinner's (1997) conceptual behaviour is the understanding of conceptual knowledge. Hiebert & Lefevre (1986) note that conceptual knowledge is learned meaningfully and Vinner (1997) maintains that talking about conceptual knowledge implies talking about conceptual understanding and conceptual thinking. Conceptual understanding is very similar to Skemp's (1976) relational understanding, which is understanding both what to do and the reason for doing so. Vinner (1997) coins the term conceptual behaviour because behaviour is what one sees and it is assumed that behaviour is as a result of some thought processes and he confines the behaviour to be either verbal behaviour or written. "Conceptual behaviour is based on meaningful learning and conceptual understanding. It is a result of thought processes in which concepts were considered, as well as relations between the concepts, ideas in which the concepts are involved, logical connections and so on" (Vinner, 1997, p, 100). According to Vinner (1997), it is more important to understand the mental processes that produce the behaviour than the behaviour itself. Even in trying to understand the former is mere speculation but nevertheless it may be a starting point. One of the goals of mathematics education in general is to achieve conceptual behaviour. Some operational measures may be required to inform whether this conceptual behaviour is achieved or not.

The most common form of operational measures are written examinations, class discussions, or any other means. My study used learners' responses to examination questions, as a way of observing learner behaviour, which may be helpful in determining whether the desired conceptual behaviour has been achieved or not. If the expected behaviour is produced by alternate (undesirable) thought processes, this may give rise to conceptual behaviour which may not be in keeping with the acceptable mathematical behaviour. The term pseudo-conceptual behaviour is used by Vinner (1997) to describe a behaviour that may seem like conceptual behaviour, but is actually produced by mental processes which do not characterise conceptual behaviour. According to Vinner (1997), "pseudo-conceptual behaviour is produced by pseudo-conceptual thought processes". The mental processes of "pseudo-conceptual thought processes" is based on the assumption that it is “simpler, easier, and shorter than the true conceptual processes” (Vinner, 1997, p, 101). Students tend to look for ways that will enable them to perform the task that is presented to them and may resort to pseudo-conceptual thought processes as it may seem easier for them. These thought processes often are formed spontaneously and they are sometimes natural cognitive reactions to certain cognitive stimuli. Students tend to use the pseudo-conceptual thought processes without reflecting on it.

Vinner (1997) makes use of specific cases to highlight some characteristics of pseudo-conceptual thinking. One of the characteristics of pseudo-conceptual thought processes is its uncontrolled associations. This happens when a stimulus is used to evoke certain associations in the student’s mind which then results in one way or another in their reactions (verbal or written). If students lack understanding of a concept or topic, they will not be in a position to examine these associations and know whether they are going to produce a correct answer or not. Furthermore, students are not in a position to determine that this uncontrolled reaction is a negative one especially if they do not practice critical thinking or lack reflective abilities. This uncontrolled association, triggered by some stimuli, fails to become a meaningful framework for further thought processes. For example, students may know how to solve for *x* in the quadratic equation $x^2 - 7x + 12 = 0$ but may not necessarily know the underlying reason in solving for *x*. If a quadratic algebraic expression occurs in another context like $y = x^2 - 7x + 12$ and students refer to it as a quadratic equation instead of a quadratic relationship, then students reacted to their first association without checking their thoughts which may be representative of typical pseudo-conceptual behaviour. This reveals that there may be confusion in the student’s mind between an equation and a relationship.

Confusions seem to be unavoidable, natural and common and Vinner (1997) presents a strong argument that they should not be considered as a negative phenomenon in learning. He maintains that the only negative phenomenon is that students do not have any control over the way associations are evoked but their reactions can be controlled. Associations can be examined by the students and they can determine whether they fit the context or not. The study of pseudo-conceptual modes of thinking is important since the aim is to replace them through the process of learning by true conceptual thought processes. Therefore, we need to know how they are formed and how to overcome the factors that dominate them. The pseudo-conceptual mode of thinking and misconceptions are closely related, however there are distinctions between the two.

Vinner (1997) presents a strong argument that misconceptions occur within a cognitive framework and the pseudo-conceptual mode occurs outside the cognitive framework. He maintains that a misconception involves cognition and is as a result of cognitive efforts which lead to a wrong idea while pseudo-conceptual behaviour does not reside in cognition. When pseudo-conceptual behaviour occurs, there is no cognitive involvement since the student looks for a satisfactory reaction to a certain stimulus.

2.6 Conclusion

The main focus of this chapter has been on concept understanding, concept definition, conceptual behaviour and pseudo-conceptual behaviour. Vinner's (1997) theoretical framework is the lens through which the data will be analysed as it provides an explanation for pseudo-conceptual behaviour, which explains what can happen when students' learning is not grounded in conceptual understanding.

CHAPTER THREE
LITERATURE REVIEW

In the literature review, I discuss some calculus studies. Zandieh's (1997, 2000) description of the various layers of the derivative concept, some teaching experiments that provide insights into students' concept image of the derivative and students' algebraic misconceptions in basic calculus is highlighted. In the last three decades research exploring student understanding of the derivative and the various aspects of the derivative has been well documented (Heid, 1988, Ubuz, 2001, Zandieh, 1997, 2000, Likwambe, 2006). The derivative is the basis for grade twelve Standard Grade calculus studies and any further work with related aspects like sketching the graph of a cubic function and applying calculus concepts to problems involving maxima and minima is based on the understanding of the derivative. Therefore, understanding of the derivative concept is of utmost importance in the grade twelve year.

3.1 Basic calculus concepts and the derivative

Heid (1988) provides useful information about student understanding of the derivative which was performed on two groups of college calculus students. One group used computer software extensively in the course while the other group was taught using more traditional methods. In the experimental group, the focus was on the concepts of the derivative and slope. Furthermore, the teaching of the concepts preceded the teaching of the skills and concepts were developed in greater depth and through the use of a variety of representations as well. Concepts were taught first and students used the relationship between the derivative and the slopes of secant lines to evaluate the derivatives on a number of occasions. An understanding of the derivative was the focal point of the experimental group and they acquired considerable experience in reasoning from the concept of the derivative. For example students, interpreted statements about derivatives in the context of applied situations, giving meaning to negative function values as well as negative slopes in applied situations. Students also described derivatives as approximations for slopes of tangent lines rather than as being equal to the slope itself. Heid (1988) reports that students from the experimental group demonstrated general understanding of most course concepts and misunderstandings surfaced in “limits” and “rates of change.”

The concept of the limit is the cornerstone of several related concepts such as understanding of the derivative, continuity and differentiability, etc. According to Parameswaran (2007) the precise formal definition of the concept of the limit is so complex and counter-intuitive that it fails to bring out readily the simple and intuitively obvious ideas that led to it in the first place. One of the significant conclusions of his research was that students rounded off to zero such small numbers while evaluating limits because they viewed limits as a process of approximation. Tall (1996) comments that in some countries calculus is studied in an intuitive form in school, where the limit concept is introduced dynamically in terms of a variable quantity ‘getting close to’ a fixed limiting value. Schools in South Africa adopt this approach of the limit concept to introduce the derivative concept in the study of calculus. Although understanding of the limit concept is crucial to understanding of the derivative, my study focused mainly on learners’ understanding of the derivative which is the main focal point at school level. Since an in-depth study of the limit concept is not done, the various layers of the derivative concept that Zandieh (1997, 2000) describes, escapes the grade twelve learners as the second layer is dependent on the limit concept.

3.2 The various layers of the derivative concept

The study by Zandieh (1997, 2000) focused on the understanding of the various layers of the derivative concept of nine high school students who were in an

Advanced Placement calculus class, of which six were national merit finalists. It is reported that these students preferred the graphic representation of the derivative with slope as the main focus and they also preferred to interpret the derivative as the rate of change. The representation focusing on slope was mentioned most often by six students and three students preferred the representation focusing on rate as a preferred interpretation.

According to Zandieh (1997, 2000) the derivative concept consists of a progression of three process-object layers which are the ratio, limit and function layer and these can be described in various representations. I find it necessary to provide a description of the various layers in symbolic, rate and graphic representation, as presented by Zandieh (1997, 2000), as these various layers of each of the above representation are present in questions which receive emphasis in the May and August examinations.

3.2.1 *Symbolic representation of the derivative*

The expression for the average gradient is also referred to as the symbolic difference quotient and written as $\frac{f(b)-f(a)}{b-a}$ or $\frac{f(a+h)-f(a)}{h}$ where *a* and *b* are values in the domain of the function and *h* is the distance between *a* and *b*. This quotient may be seen as a process or an object. As a process, it is substituting values for *a* and *b* in the function and thereafter evaluating it. To view it as an object means that in the second layer it is acted on by a limiting process:

$$f'(a)=\lim_{b\to a}\frac{f(b)-f(a)}{b-a} \text{ or } f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

These expressions give the value of the derivative function at *a* and is called the limit layer which may be seen both as a process and an object. The limit as a process is seen as taking the limit and as an object it is seen as the limit value. To progress to the third layer, the derivative function, the limiting process must be seen as consolidated and repeated for every value in the domain of f'. The formula:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is the symbolic representation of the derivative function which represents the gradient of a curve at every point $(x; f(x))$. Vinner & Dreyfus (1989) describe the process conception of a function as a correspondence between two non-empty sets and as a object they describe it as a set of ordered pairs. In the derivative's function layer, the ordered pairs are $(x; f'(x))$.

In both the May and August examination learners had to answer a question based on the symbolic representation of the derivative. They were required to find $f'(x)$ from first principles given a function $f(x)$. This question clearly illustrated and required learners to progress through each of these layers to arrive at the derivative function. Thus, it is apparent that both these examinations emphasize the need for learners to be at least aware of these layers as they ought to exhibit this in their response.

3.2.2 *The derivative as rate of change*

The difference quotient may also be used to measure the average rate of change of the dependent variable with respect to change in the independent variable. The calculation of this ratio of differences is a process and one may represent this ratio in Leibniz notation as $\frac{\Delta y}{\Delta x}$.

The average rate may be used as an object in the limiting process, the second process. The limiting process consists of a sequence of average rates of change as Δx approaches zero. In Leibniz notation, it may be written as $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Born out of the limiting process is the instantaneous rate of change which may be represented as $\frac{dy}{dx}$.

The instantaneous rate of change at each input value is now used as an object in the construction of the derivative function. Zandieh (1997, 2000) points out that the function process that receives emphasis is the “co variation” of the input values to the output values which is the instantaneous rate of change values. Furthermore, she also notes that the function as a process and object is not easily represented by the Leibniz notation.

Calculus questions involving rate of change problems occur frequently in many examinations and this rate problem appeared in both examinations. In the May examination a formula $B(t) = -3t^2 + 30t + 1500$ was given where $B(t)$ represents the number of bacteria present t hours later. Learners had to calculate the rate of change of bacteria at ten hours. This is the instantaneous rate of change represented by $B'(10)$.

Other types of problems concerning rate of change were tested as well which were combined with the sketch of the cubic function. Learners were required to calculate the co-ordinates of the turning point. For this, learners had to apply the rate of change at the turning point, which is zero.

3.2.3 *Graphic representation of the derivative*

The graphic interpretation of the derivative can also be described using the three layers. The first layer is the slope of the straight line that actually joins two points on the curve itself described by the graph of the function in question. The process is the calculation of the difference between the vertical distance divided by the difference between the horizontal distance and the object is the slope. This is actually the average gradient, which is also a ratio and can be represented in Leibniz notation as $\frac{\Delta y}{\Delta x}$.

In order to move to the second layer, the limit layer, one of the two points on the curve needs to be fixed. Connecting the fixed point to each of the other points in turn produces a number of secant lines which will approach the tangent line of the curve at the fixed point. This is the limit process and the object is the slope of the tangent line at the fixed point.

The third layer is the same. The limiting process may be seen as to have occurred for every point on the curve of the original function's graph. The idea of running through every point on the original curve and finding the instantaneous slope is the function process and the object is the derivative function, whose graph may also be seen as a curve itself.

In the May examination, the function $f(x) = -x^2 + 1$ was given and learners were required to find the average gradient of the curve between $x = -1$ and $x = 3$. Learners were not required to sketch the graph of the function but graphically this function represents the parabola and one needed to find the average gradient between the two given points. This actually represents the ratio layer of the graphic representation. Learners were also required to find the gradient of the curve of f at $x = 2$. This question emphasizes the limit layer of the graphic representation. It may be seen as one point $x = 2$, which is kept fixed and one can choose other points that are increasingly close to the first point, 2, so that a number of secant lines are drawn which approaches the tangent to the curve at the fixed point.

Although the various layers of the derivative of these representations were present in both examinations, seemingly the learners were not aware of these layers as their main focus was on the process and the output values. Sfard (1992) terms this as pseudo-structural conception which "merely denotes that the object a person is using does not refer to the underlying process of the true object" Zandieh (1997, 2000). Therefore, according to Zandieh (1997, 2000), it may be possible for learners to have a pseudo-structural conception of the derivative which refers exclusively to the value of the derivative that is the end result of the derivative without actually recognising the process that leads to the end result. The grade twelve learners, enrolled in Standard Grade mathematics in this study, fit the above category since the focus of the examination questions was mainly on the application of the derivative and less on the understanding of the various layers of the derivative.

Zandieh (1997, 2000) provided a theoretical framework to explore the understanding of the concept of the derivative focusing on the various layers of the derivative. Likwambe (2006) used this framework in her study which focused on the understanding of the various layers of the derivative of five students who were practising teachers engaged in the Advance Certificate in Education (ACE) course at a South African University. Likwambe's (2006) study revealed that the concept images of the derivative held by these students improved substantially from the first interview to the third interview with the main representation of the derivative being that of slope. She concluded that even upon completion of the ACE module on calculus, these practising teachers have concept images of the derivative which did not encompass all the layers. Furthermore they did not have more than two representations of the derivative. Likwambe (2006) maintained that:

> "with the function layer absent, it can be difficult to make sense of maximization and minimization tasks. With the limit layer absent or pseudo-structural, the concept itself and the essence of calculus escapes the teachers-and therefore also will be out of reach of our learners"(p, 86).

Likwambe (2006) argues strongly that many learners and educators have procedural knowledge (distinct from the process aspects of the concept) of the derivative as the study revealed that some subjects worked the derivative of the given function without necessarily showing concept images falling within the scope of Zandieh's (1997, 2000) framework. This aspect was captured by expanding Zandieh's (1997, 2000) theoretical framework to include instrumental understanding which is understanding the rules without reasons (Skemp, 1976).

Likwambe's (2006) study is very similar to Zandieh's (1997, 2000) study as they both explored student understanding of the concept of the derivative but Zandieh's (1997, 2000) results did not show any connections across representations as there were no research instruments to show this. Likwambe's (2006) study focused also on connections between representations within the layers of the derivative. However, the focus of grade twelve Standard Grade calculus syllabus is not on the understanding of the various layers of the derivative, but, on the understanding of the derivative mainly for the application of the derivative to solve problems on rate, maxima and minima, to find equations of tangents to curves at a point and to sketch the cubic function.

3.3 Some misconceptions in basic calculus concepts

Studies by Orton (1983a), Amit & Vinner (1990) and Ubuz (2001) focused on students' misconceptions of and errors involving the concept of the derivative. In the study by Amit & Vinner (1990), a questionnaire comprising of eleven questions which were test like items were used. This questionnaire was administered to 130 students at the end of their university calculus course and the analysis is a coherent interpretation of the students' answers to the questions. Amit & Vinner (1990) reported on the notational conflict that was present in one of student's answers. The student calculated the equation of the tangent and represented it as $y = \frac{2}{5}x + 1$ and later the student represented $f'(x)$ as $\frac{2}{5}x + 1$ and calculates $f'(5)$ as $\frac{2}{5}(5) + 1$. The student in this study calculated the equation of the tangent line and treated it as if it was the derivative function. Thus the derivative has become the equation of the tangent at a certain fixed point. This notational conflict can cause confusion and might impact on students especially when they are solving non-routine problems. Similar findings was reported by Ubuz (2001) who investigated whether and how computers in realistic classroom settings could influence first year engineering students' learning of calculus.

Ubuz's (2001) study consisted of 147 students enrolled in calculus courses in four different universities in England. The research results revealed that students had some misconceptions which were as follows:

- the derivative at a point gives the function at a point,
- the tangent equation is the derivative function,
- the derivative at a point is the tangent equation and
- the derivative at a point is the value of the tangent equation at that point.

According to (Ubuz, 2001), misconceptions of this nature seem to indicate a lack of understanding of derivative, tangent and the value of a function at a point.

The main goal of the study by Ubuz (2001) was to analyse in a systematic way, the scope and nature of first year engineering students' errors that occurred on specific tasks related to point of tangency, the derivative at a point and the approximate value of a function at a point. However, a detailed analysis of difficulties was not presented. Instead students' errors to questions in the pre-test and post-test were described for the purpose of categorisation. No further analyses of students' errors were provided which may have been useful to ascertain the cause of their errors.

Some researchers (Orton, 1983a, Bowie, 1998, 2000) focused on the analysis of students' errors to gain insight into students' understanding of basic calculus concepts. Bowie (1998) offers helpful insight into students' understanding of calculus concepts through the analysis of students' errors in answers to examination questions. She drew upon the notion of a pseudo-structural conception, a term used by Sfard (1992) and Sfard & Linchevski (1994), which provides an explanation for what can happen when students do not achieve the desired operational- structural cycle. According to Sfard & Linchevski (1994) the process of reification is difficult as it often takes considerable amount of time and effort. When students are not able to see the mathematical objects and structures because it is not clear to them, they develop a way of dealing with them by creating their own meaning. Students adopt a pseudo-structural approach where the mathematical object is identified with merely its representation. For example, students will focus on a symbol, formula or a graph and this becomes the object that students deal with and they separate it from the operations and process that lead to that result. The analysis of students' errors led Bowie (1998) to develop a model of learning which provided a possible explanation of how students who have a pseudo-structural approach to algebra develop an understanding of basic calculus concepts and some of the consequences of this.

Bowie's (1998) study also aimed to see whether learning theories, offered by Sfard (1992, 1991) and Dubinsky (1991), were useful in analysing the development of students' understanding of basic calculus concepts. The students' misconceptions were seen through the lens of the above learning theories. Bowie (1998) found it necessary to extend the work of Sfard (1992, 1991) and Dubinsky (1991) as the data in her study showed that the starting point for the students' construction of calculus concepts was based on pseudo-objects. Thus, this led Bowie (1998) to develop her own "model for understanding these misconceptions in the light of the mechanisms students used to construct their knowledge of calculus" (p, 122). The analysis of students' algebraic errors revealed that students showed strong evidence of this pseudo-structural approach to calculus studies and this was the underlying reason for the development of the model by Bowie (1998).

While Dubinsky's (1991) model of a schema gives a coherent description of modes of construction of mathematical knowledge, Bowie's (1998) model provides a description of the construction of mathematical knowledge based on a pseudo-structural conception of base knowledge. Furthermore, in this model Bowie (1998) presents a strong argument that actions on pseudo-objects become rehearsed into rules and these rules involve the “mechanistic manipulation” of pseudo-objects. In addition, the model suggests that students’ errors originate from their attempt to build a manageable set of techniques for dealing with calculus questions. Since these techniques are not rooted in deep conceptual understanding of calculus, it renders itself inadequate for dealing with calculus questions.

Bowie’s (1998) work provided a comprehensive audit of students’ errors with respect to various categories. However, possible reasons for students choosing a possible technique to solve a particular problem were not provided. Furthermore, there is no information suggesting at which stage of mathematical learning the student encountered the techniques or where it is emanating from and the possible reason for students using these particular techniques. In my study, I see myself as taking Bowie’s (1998) model further by exploring the possible reasons as to why learners choose a particular technique to solve a problem.

3.4 Calculus and algebra

Studies involving algebra and calculus (Orton, 1983a, Heid, 1988, Habre & Abboud, 2006) have shown significant improvement in students’ performance when taught in multiple representational environments.

Habre & Abboud (2006) conducted a study on the understanding of a function and its derivative as viewed by students at a University in Beirut. Concepts were introduced using multiple representations with the emphasis on visualisation and geometrical representation. The geometric aspect of the function concept was emphasized through the use of computers and the graphing calculator. Thereafter, the derivative was introduced via the idea of an average rate of change becoming the instantaneous rate of change. The researchers used a textbook in their study which introduced the derivative by first discussing the rate of change of a function at a given point as the limit of an average rate of change. Thereafter it proceeds to relate the result to the slope of a tangent line and finally arrive at the analytical definition of the derivative, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. Together with this reformed textbook, computers and graphing calculators were frequently used during teaching and assessment and students had access to computers in a computer laboratory. However, neither calculators nor computers were used during examinations. When interviewees, who comprised of above average students, were asked about the meaning of the derivative, six of the ten students spoke of "spontaneous rate of change at a point" and or "slope of the tangent at a point". According to the researchers these results showed an improvement in the students' conception of the derivative in favour of its geometric interpretation but it must be noted that these were above average students.

The findings in this research revealed that the general approach adopted in the course proved to be unpopular for the majority of the students, but rewarding for others. Furthermore, a study of students' performance on very specific exam questions revealed that for most students, the algebraic representation of a function still dominated their thinking of a function. However, these students showed an almost complete understanding of the derivative, particularly the idea of instantaneous rate of change of a curve at a given point. Thus, students' thinking remained algebraic although they were exposed to sections during the teaching and learning that were virtually geometrically taught. This study also showed that for some students the algebraic formulation of a function was a prerequisite for them to visualize it. Tall (1991a) and Vinner (1989) showed that students' understanding was typically algebraic and not visual, as their study revealed that visual information was more difficult for students to learn and was considered less mathematical. Thus, it seems that students' strong algebraic foundation dominated their thinking thereby impacting on their understanding of any future concepts taught.

Studies conducted by (Santos & Thomas, 2005, Judson & Nishimori, 2005) consisted of students that were in school doing a basic calculus course. Judson & Nishimori's (2005) study consisted of students from above average high schools in Japan and the United States doing a calculus course. The aim of the study was to determine any differences in students' conceptual understanding of calculus and their ability to use algebra to solve traditional calculus problems. The findings revealed that most Japanese and American high school calculus students had a solid grasp of the mechanics of calculus, and they understood the derivative as the rate of change and how it could be used to sketch the graphs of functions. There was little difference in the students' conceptual understanding of calculus between the two groups of students, but the Japanese students demonstrated much stronger algebra skills than the American students. The American students lacked fluency in manipulating algebraic expressions containing radicals and had difficulty with problems the Japanese students found to be straight forward.

3.5 Algebra and simplification

Habre & Abboud (2006) also administered a diagnostic examination to assess the mathematical background of the students. This study recommended that a large number of students were in need of an algebra course prior to a calculus course as these students performed poorly. For example in one question, students were asked to solve the inequality $3x < 9x + 4$ and sketch the solution set. 87% of all answers were wrong. No further description of students' answers were presented so one can not ascertain the exact nature of the difficulties that they might have experienced in answering this question. Thus, it is unclear whether students found the algebraic part which is the solving of the inequality, or the actual sketch of the solution set difficult. A detailed description of actual students' difficulties may have been more useful in gaining insight into student understanding and thought processes.

Kieran (1992) provided a detailed analysis of some students' thought processes during the simplification process of a complex algebraic expression. Through the interviews, Kieran (1992) was able to a certain extent, gain some insight into the thought processes students used in the simplification process. Students were required to simplify $\frac{x^2+3x-10}{x^2+2x-8}$ and it is reported that when students see x^2, it is a cue for them to factorise. This showed that students reduce this to a symbol manipulation task as the x^2 acts as a factorising cue for many of them. Probably because their previous encounters with expressions containing x^2 were mainly in the context of factorising and solving quadratic equations.

The study by Orton (1983a) investigated students' understanding of elementary calculus. The analysis of responses to tasks concerning differentiation and rate of change led to detailed data concerning the degree of understanding attained and common errors and misconceptions. It was reported that questions concerning understanding of differentiation and graphical approaches to rate of change were more difficult for students. The errors made by students were classified according to the scheme described by Donaldson (1963) as this proved to be more useful as it provided a broader categorisation than the one based on mathematical skills and concepts.

Donaldson (1963) describes three types of error, namely structural, arbitrary and executive. Structural errors are those errors which arise due to a student's failure to grasp some fundamental concepts essential to the solution or a failure to see the relationships involved in the problem. Arbitrary errors occur when students do not show any logic in their answers and show disregard for any information given to them. Executive errors occur when there is a failure to carry out manipulations, although the principles may have been understood. While (Orton, 1983b) used this scheme to classify errors on integration, Likwambe (2006) maintains that classifying errors using this scheme does not provide insights into the underlying structures of students' concept images as it stops at classification and does not lend itself to developing further insights into why students made those errors. My study aimed to provide a detailed description of learners' difficulties in understanding the concept of the derivative.

Orton (1983a) reported that some of the difficulties that students experienced were algebraic in nature. One of the tasks required the solution of the equation $3x^2 - 6x = 0$ and twenty four students were unable to solve this correctly. Twelve students incorrectly cancelled x or divided throughout by x and lost one solution, $x = 0$. Six students incorrectly factorised $3x^2 - 6x = 0$ into $3x(x - 6) = 0$. These types of difficulties that students experienced may be obscuring the very fundamental concept that calculus is built on in the South African School curriculum, namely that of the derivative, as learners' previous misconceptions are taken forward, hindering them from appreciating fundamental calculus concepts. Orton (1983a) suggested that the extent to which algebra is used to introduce calculus should be kept to a minimum. There are other aspects of algebra, for example, the concept of variables and algebraic symbolism that may impact on students' understanding of basic calculus concepts.

The study by White & Mitchelmore (1996), showed that students had a very primitive understanding of the variable. Their study involved first year university students who studied calculus in secondary schools. Students were presented with word problems involving rate of change that could be solved using algebra and calculus. In solving these word problems, they had to identify the appropriate concepts needed to solve the problem. In addition some algebraic relationships among the variables or the selection of some calculus concepts involving variables (such as the derivative) and its expression in symbolic form were needed. White & Mitchelmore (1996) referred to the process of selecting a calculus concept and expressing it in symbolic form as a symbolisation process.

Results of the analysis showed that very few students were able to correctly symbolise at any one time with the more complex rate of change problems. However, those who did were almost always correct. This study also revealed that there was a tendency by some students to focus on manipulation where they based their decisions about which procedure to apply on the given symbols. In doing so they ignored the meanings behind the symbols. This was further highlighted during the interview comments as students were actively "looking for symbols to which they could apply known manipulations" (White & Mitchelmore, 1996, p, 88).

Students' responses to these word problems were collected on four occasions during and after 24 hours of concept-based calculus instruction. Detailed analysis of the results revealed three main categories of errors, one being where the variables were treated as symbols to be manipulated rather than as quantities to be related. White & Mitchelmore (1996) identify three examples of such manipulation focus and they are:

> "failure to distinguish a general relationship from a specific value, searching for symbols to which known procedures are applied regardless of what the symbols refer to and remembering procedures solely in terms of the symbols used when they were first learned" (p, 91).

Their findings highlighted that students showing the manipulation focus had a concept of a variable that was limited to algebraic symbols. They had learned to operate with symbols without any regard to their contextual meaning. The term "abstract-apart" is used to describe this situation as it does not involve any true abstraction on the part of the student.

White & Mitchelmore (1996) did not provide any information as to why students resorted to this tendency of manipulation focus when presented with word problems. One possible reason as suggested by Bowie (1998) is perhaps that students' base knowledge may be pseudo-structural therefore they tend to focus on manipulation technique as a means of coping with providing a solution to the problem. It seems that our preoccupation with algebraic manipulation may hinder learners from developing a coherent understanding of the concepts in calculus.

3.6 Conclusion

There has been a concentration of research in basic calculus at university level (Bowie, 1998, Orton, 1983a, Ubuz, 2001, Likwambe, 2006, Habre & Abboud, 2006). Research in basic calculus conducted at school level in other countries has flourished (Zandieh, 2000, Hahkioniemi, 2005, Judson & Nishimori, 2005 and Santos & Thomas, 2005). It seems that research in basic calculus at South African Schools is very limited as calculus teaching was only introduced at schools in the 1980's. Therefore, I find it pertinent to undertake this research to explore learners' understanding of the concept of the derivative and how the difficulties they encounter in examination questions impacts on their conceptual understanding of the derivative.

CHAPTER FOUR
DESIGN OF STUDY AND RESEARCH METHODOLOGY

This chapter addresses the methodology. The research approach and the research instruments are discussed in detail. Thereafter some of the strengths and weaknesses of this study are presented. A limitation of this study is discussed as well.

4.1 Research approach

The aim of my study was to explore, in a grade twelve Standard Grade mathematics class, learners' understanding of the concept of derivative. The paradigm most suited to this, being a case study, is that of a naturalistic inquiry with particular reference to interpretive approach as the main goal of this study was to understand the learners' interpretations of reality (Cohen, Manion & Morrison, 2000). Qualitative research uses a naturalistic approach that aims to understand phenomena in context specific settings (Golafshani, 2003, Hoepfl, 1997). The setting was the grade twelve mathematics classroom where learners were studying the concept of the derivative for the first time.

I used the qualitative approach to interpret the data in my study. Cresswell (2003, p, 20) maintains that researchers using a qualitative approach seek to establish the meaning of a phenomenon from the views of participants. I gave meaning to the learners' responses to examination questions, which was the primary source of data, by finding out how they have solved the test items. I interpreted the data by analysing learners' responses to the examination questions and thereby explored what understanding of the derivative learners emerged with, what strategies they used in providing a response and how, if any, previously encountered mathematical concepts impacted on their answers and thereby on their conceptual understanding of the derivative. Therefore, in the process of interpreting the data, I provided detailed descriptions of learners' responses to the examination questions and semi-structured interviews. The analysis of data, from the semi structured interviews, was used to gain deeper insight into learners' understanding of the concept of the derivative. Thus my position as an interpretive researcher accords with Neuman's (2000) view that:

> "a qualitative researcher interprets data by giving them meaning, translating them, or making them understandable. However the meaning he or she gives begins with the point of view of the people being studied. He or she interprets the data by finding out how the people being studied see the world, how they define the situation, or what it means for them" (p, 148).

One of the distinguishing features of the interpretive approach by Hammersley & Atkinson (1983, cited in Cohen et al., 2000) is that the social world should be studied in its natural state without intervention of or manipulation by the researcher of the phenomenon of interest (Golafshani, 2003, p, 600). My study conforms to this aspiration since the field is a natural setting being my own learners and examinations are natural phenomena in any school situation. Lincoln & Guba (1985) maintain that the naturalistic inquirer "elects to carry out research in the natural settings" (p, 39). According to Neuman (2000), qualitative researchers' analyses emphasize contingencies in "messy" natural settings and they tend to use a case oriented approach which places the case at the centre stage and not the variables (Ragin, 1992a, p, 5, cited in Cohen et al., 2000).

My study is a case study as I explored a class of grade twelve learners' understanding of the concept of derivative. According to Cohen et al., (2000) a case study is a specific instance in action, my study being a class of grade twelve Standard Grade mathematics learners who were exposed to the concept of derivative for the first time. This provided a unique example of real learners in an examination situation which represents a real situation. "Yin (1984) defines a case study research method as an empirical inquiry that investigates a contemporary phenomenon within its real-life context …in which multiple sources of evidence is used” (cited in Soy, 1997, p, 1).

I also provided a rich detailed description of learners’ responses to the examination questions and Cohen et al., (2000) point out that Hitchcock & Hughes (1995) consider this as one of the hallmarks of a case study. Kohlbacher (2006) presents a strong argument in favour of both case study research as a research strategy and qualitative content analysis as a method of examination of data material. Furthermore, he seeks to encourage the integration of qualitative content analysis into the important step of data analysis in case study research (p, 5).

According to Neuman (2000), the “passage of time is an integral part of qualitative research. Qualitative researchers look at the sequence of events and pay attention to what happens first, second, third and so on” (p, 148). In a nutshell, qualitative researchers examine the same case over time and they sometimes see an issue evolve. In my study, the passage of time is shorter as the data for the examinations was obtained in May 2006 and August 2006 and the interviews took place during October 2006 and November 2006. Cresswell (2003) alludes to case studies being bounded by time and activity, with researchers collecting detailed information using a variety of data collection methods over sustained period of time (Stake, 1995, as cited by Cresswell, 2003). However, in using a case study, I was aware of some of its strengths and weaknesses and I made reference to Nisbet & Watt’s (1984) strengths and weaknesses of case study. One of the strengths of case study is that it speaks for itself therefore it is immediately intelligible. My study provided a detailed description of learners’ responses to examination test items during the analysis phase. I chose to undertake a case study in order to make a case understandable. How the grade twelve learners responded to calculus questions was significant, as it provided me, the researcher, with insight into the real dynamics of learners’ understanding of the concept of derivative. Case studies are strong on reality and provide insights into other similar situations and cases, thereby assisting interpretation of other similar cases. They can be undertaken by a single researcher without needing a full research team. The strengths of case study outweigh the weaknesses. One of the weaknesses of case study is that the results may not be generalisable except where researchers see their application and it is not my intention to generalise any of my results.

The design of the study comprised of three stages from which data was collected. Stage one is learners' responses to calculus questions in May examination which is more commonly known as Mid-year examination. Stage two is also learners' responses to calculus questions in August examination which is referred to as the Trial examination. Learners' marks to both these examinations form a major component of the continuous assessment marks. Stage three is the data collected from the semi-structured interviews which was conducted during October and November. The data from the above three stages of collection were used to answer the research questions of my study.

4.2 Research questions

This study aimed to explore learners' understanding of the concept of the derivative and this was done through the following two research questions.

1. What are some difficulties learners' experience when answering examination questions related to the concept of the derivative?
2. What are learners' perceptions about the concept of the derivative?

With research question one I was particularly interested in exploring the following issues:

- The impact of previously encountered concepts.
- The influence of the sequencing of topics taught.
- Learners' difficulties related to algebraic symbolism as well as the symbolism associated with calculus.

4.3 Context of the study

This study was conducted at a secondary school in Amanzimtoti (previously an all white area and the school is now classified as an ex-model C school) in the South of Durban in KwaZulu-Natal. This is a co-educational school with an enrolment in excess of a thousand learners. The learner population comprised of approximately 45% White, 38% African, 16% Indian and 1% Coloured. The learners reside in areas which are within a 20km radius of the school. They come mainly from Kwa-Makutha, Amanzimtoti, Umlazi and Isipingo. The socio-economic backgrounds range from middle to upper class. My association with the school for the past six years was mainly that of a parent. At the beginning of 2006, I was transferred to this school as a senior mathematics teacher.

This grade twelve mathematics class was allocated to me by the principal. The class comprised of fourteen African, two Indian, and thirteen White learners. The learners in grade twelve mathematics classes were graded according to their grade eleven mathematics final examination results. Learners in this class obtained marks between 30% and 40%, this being the lowest in the grade eleven sections. As the mathematics teacher of this class I have known the participants of this study since 24 January 2006. My relationship with the participants was mainly as their mathematics teacher aiming to help them achieve their goal, to obtain a good pass in mathematics at the Standard Grade level at the end of the year. I chose this school and the participants through convenience sampling.

I chose this sample because I had easy access as I was their mathematics teacher for 2006. Cohen et al.,(2000) terms this as convenience sampling as it does not represent any group apart from itself and it does not seek to generalise about the wider population. Furthermore, they also propose the selection of convenience sampling as a sampling strategy for case study research. These grade twelve learners chosen for this study is a sample that is extremely rich in information since they encounter calculus for the first time.

4.4 Data collection instruments

Examinations and semi-structured interviews were the main source of data collection. Being an insider at the school gave me the opportunity of gathering data from participants in a natural setting. In the section to follow, I explain in detail the different method for data collection, their rationale and some of the advantages and disadvantages.

4.4.1 *May examination and August examination*

The calculus section of the May and August examination was chosen as the main source of data as both these examinations covered a wide range of content and skills that fulfills the requirement of the calculus curriculum for grade twelve syllabus. The term that best describes these examinations is criterion referenced as the main objective for the learner is to fulfill a given set of criteria (Cohen et al., 2000). These criterion referenced examinations was the key source of data as it provided information about exactly how a learner has responded to calculus questions. These two examinations formed the main component of continuous assessment mark for the learner in mathematics as the final mark was submitted to the Department of Education and Culture at the end of grade twelve year. I chose to use the calculus section of the May and August examinations as the main source of data as it encompassed all work studied in calculus and it contributed largely to the passing or failing of mathematics.

I was appointed examiner for the May examination by the head of mathematics department of the school at the end of the first term and the due date for the first draft examination paper was at the end of April. In the design of the examination, no indication was given that it would be used as a research instrument and I obtained permission from the mathematics department only after the examinations were written. The examination paper was set for two hours and covered a wide range of topics, with some of the topics being done in the grade eleven year. The examination paper was set in accordance with the criteria set out by the Department of Education and Culture and the school's policy on examinations. Possible answers to the examination questions were then compiled by the examiner and this was termed as the marking memorandum where marks were allocated to the various steps. A 'method mark' was awarded to the correct method used and an 'accuracy mark' was for accuracy in the various steps leading to the correct answer. A 'carried accuracy' mark was awarded to the various steps that were consistent with any error that might have been incurred. Thereafter, the examination paper was moderated by another mathematics teacher from the mathematics department of the school. The remaining mathematics teachers involved in teaching the course moderated the paper to ascertain whether the questions reflected the competencies that they believed learners should have after completing the section on calculus. The head of the mathematics department did the final moderation of the paper and thereafter any changes to the paper had to be done in consultation with the head of the mathematics department.

The August examination is termed as the Trial examination and the criteria for this examination were very similar to that of the final examination as laid by the Department of Education and Culture. The duration of this paper was three hours and it covered a wider range of topics to that of the May examination. For this examination my role was now reversed as that of the moderator and not the examiner. A similar process as for the May examination was followed for moderation and checking.

The examinations were written under examination conditions as for the final examination. In the final examination learners were seated in the hall and they were placed one meter apart from each other. There was one invigilator per thirty learners. The examination papers were marked by the teachers teaching grade twelve learners and the subject head monitors the marking for consistency. The moderator of the paper checked ten percent of the marked scripts and reported the marking to be consistent, fair and accurate. Thereafter each learner checked his/her paper to ensure that all marking was done accurately. The papers were available for analysis only after this process was completed. I believe that the stringent factors encompassing the policy for examinations which was inclusive of setting, marking and moderating of examination papers and the administration of examinations addressed the issue of validity of the instrument in terms of identifying the sources of difficulties that impact on the learners' understanding of the derivative concept.

4.4.2 *Semi-structured interviews*

Atkinson and Delamont (2006) draw a clear distinction between qualitative research interviews and quantitative research interviews. They maintain that in qualitative interviewing there is great interest in the interviewee's point of view and the researcher wants rich, detailed answers whilst in quantitative interviewing the interview reflects the researchers concern and the interview is supposed to generate answers that can be coded and processed quickly (p, 313). Semi-structured interviews were used for this study as it explored learners' understanding of the concept of the derivative. Researchers beginning with an investigation with a fairly clear focus are more likely to use semi-structured interviews so that the more specific issues can be dealt with (Atkinson & Delamont, 2006, p, 315). This was clearly the case in my study. During the interview, learners were first probed about their understanding of the concept of slope or gradient, and later about their understanding of the derivative concept. Depending on learners' responses to questions asked, it sometimes became necessary to include questions that were not included initially but was essential at that point in time. This is in keeping with Atkinson & Delamont's (2006) notion of semi-structured interviews (p, 314) which describes some of its advantages and disadvantages.

The advantage of interviews is that they allow for probing, in-depth answers, flexibility and free response. In interviews people have a tendency to disclose thoughts and feelings more readily to a person than they would with other methods of data collection. One of the advantages of interviews mentioned by Atkinson & Delamont (2006), is that the response rate is quite high with the respondents answering all the questions because of their close involvement with the interviewer who can obtain more meaningful information because the interviewer can rephrase questions that are unclear to the respondent. However, there are some disadvantages. Firstly, if the sample size is small, this means that the findings may not be fully representative. Secondly, interviews are prone to subjectivity and bias on the part of the interviewer (Cohen et al., 2000, p, 277). The researcher can influence the data in terms of ‘leading on’ or influencing the respondent’s responses. If the researcher and the subject know each other, as is the case in my situation, there might be a tendency for the respondent to give information that he/she knows that the interviewer would want to hear. I tried not to influence the interviewees.

Before I could interview the learners, I outlined clearly the purpose of my interview and informed my subjects that they were being interviewed for research purposes. Furthermore, I told them in advance that I would be recording the interviews using a digital voice recorder because it would not be possible to write down everything that was said.

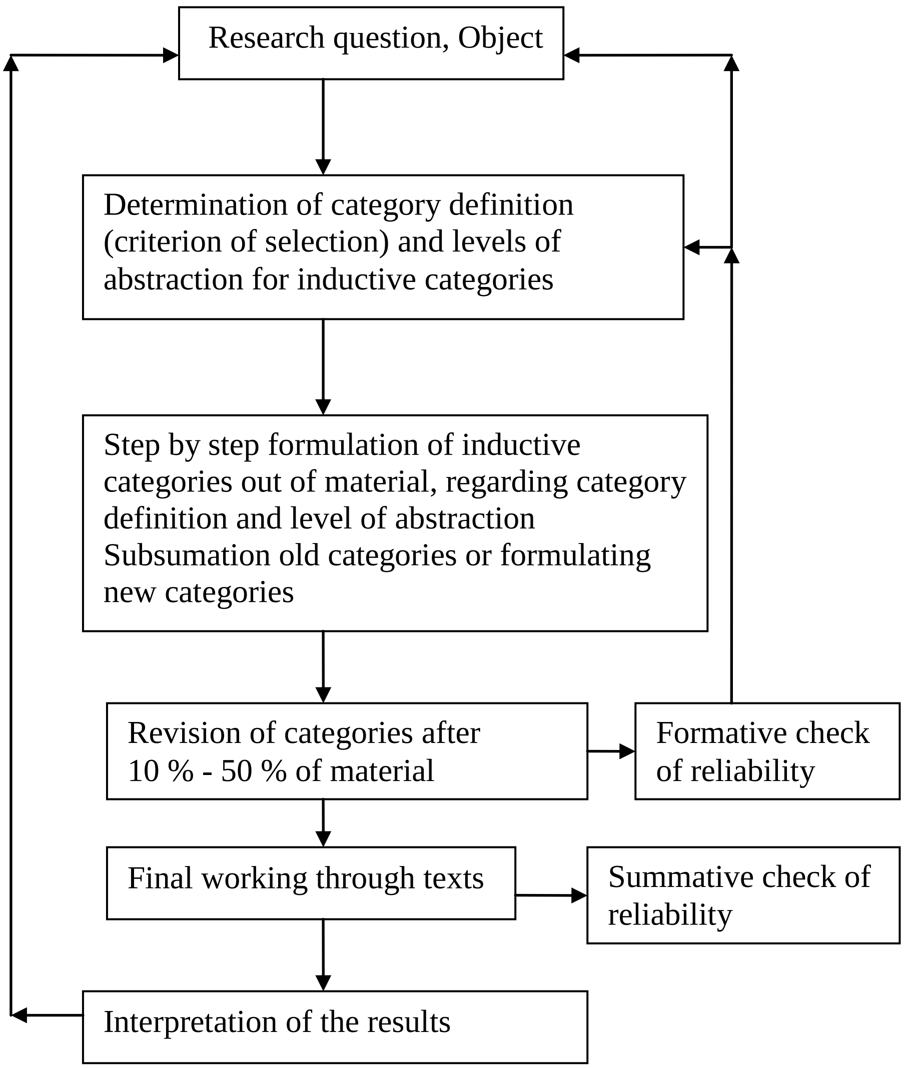
The interviews were recorded using the digital voice recorder in order to capture all the words spoken. The advantage of recording the interviews was that it allowed for greater accuracy than taking down notes. Instead of focusing on note taking, I listened carefully to what the subjects were saying so that I could ask more complex questions about their understanding of the derivative. At the same time I was also aware of the effect the presence of a digital voice recorder would have on the subject’s responses but the advantages of the recording clearly outweighed the disadvantages. I listened to the digital voice recordings via the computer and transcribed the data, beginning with a rough hand written transcription. I then typed the data and went back to the recordings to fill in missing or unclear passages. I tried to determine whether any patterns emerged for each of the questions.

4.5 Analysis of data

The first step in analysing the data was to categorise learners' responses according to the percentage marks allocated. Thereafter, learners' responses were further analysed and placed into various categories which was closely linked to the data. The development of the various categories is best described by Mayring's (2000) "step model of inductive category".

Kolbacher (2006) alludes to Mayring's (2000) argument that within the framework of qualitative approaches, it becomes essential to develop aspects of interpretation, the categories, as closely as possible to the material, and to formulate them in terms of the material. As a consequence, the procedures of inductive category development were compiled. The categories for the responses to the examination questions in my study developed inductively and Mayring's (2000) "step model of inductive category development" in Figure 4.1 aptly describes the development of the categories.

Figure 4.1 Model of inductive category development



The above model proposes that:

> "the main idea of the procedure is to formulate a criterion of definition, derived from the theoretical background and research question, which determines the aspects of the textual material taken into account. Following this criterion the material is worked through and categories are tentative and step by step deduced. Within a feedback loop those categories are revised, eventually reduced to main categories and checked in terms of their reliability. If the research question suggests quantitative aspects (e.g. frequencies of coded categories) can be analyzed" (Mayring, 2000, p, 6).

In keeping with the above model the initial step in the analysis was to note the frequency of the coded categories. Thereafter, I worked with the data from the examinations questions and deduced the categories step by step. The categories were continuously revised until it closely suited the data. A detailed description of the various categories is provided in chapter five.

4.6 Validity, reliability and trustworthiness

According to (Cohen et al., 2000, p, 104), there are many different types of validity and different types of reliability. They also maintain that reliability is a necessary precondition for validity which is aptly summed up by Lincoln & Guba (1985) when they state that there can be no validity without reliability and a demonstration of validity is sufficient to establish reliability. Golafshani (2003) contends that validity and reliability from the qualitative researchers' perspectives can be conceptualised as trustworthiness, rigor and quality in the qualitative paradigm (p, 604). (Cohen et al., 2000) and Lincoln & Guba (1985) maintain that there are several ways in which validity and reliability can be addressed. My study addressed the issue of validity and reliability through triangulation.

4.6.1 *Triangulation*

By observing something from different viewpoints or different angles one can sometimes gain different perspectives of it. This process is referred to as triangulation and there are several types of triangulation. According to Neuman (2000) the most common type of triangulation is triangulation of measures where something is measured in more than one way. My study addressed triangulation since data to address the research questions was collected at three different stages. Firstly, data from the examinations was collected at two different points, in May, the half- year examination and in August, the Trial examination. Secondly, data from the interviews was collected in October and November.

Cohen et al., (2000) defines triangulation "as the use of two or more methods of data collection in the study of some aspect of human behaviour" (p, 112). My study used examination questions and semi-structured interviews as methods of data collection to explore learners' conceptual understanding of the derivative. The term that best describes the triangulation that was used in my study is methodological triangulation as the study used the same method on different occasions and different methods on the same object of study (Cohen et al., 2000, p, 113). Triangulation is a strategy or a test for improving the validity and reliability of research or evaluations of findings (Golashani, 2003).

I have also selected criteria for judging the overall trustworthiness of a qualitative study (Hoepfl, 1997, Krefting, 1991). The table below adapted from (Krefting, 1991, p, 217) gives the summary of the strategies and the criteria my study used to address each of the strategy with which to establish trustworthiness.

Table 4.1 Strategies with which to establish trustworthiness in this study

STRATEGY	CRITERIA	APPLICATION
Credibility	Prolonged and varied field experience Triangulation Interview technique	Participants' written responses to May and August examinations. Digital voice recordings of interviews.
Transferability	Dense description	Extraction of participants' written responses to examination questions. Verbatim quotes from interviews.
Dependability	Dependability audit Triangulation	Interview transcripts. Participants' written responses to May and August examinations.
Confirmability	Confirmability audit Triangulation	Transcripts to be checked. Participants' written responses to be checked. Examination questions to be checked.

4.7 Ethical considerations

Ethical clearance was granted for my study. The ethical clearance approval number is HSS/06202A. Thereafter written permission for the conduction of my study at this school was obtained from the Principal. I informed him that the Department of Education and Culture approved of me conducting the study at this school provided it did not impact on school time. The principal expressed enthusiasm and was delighted that such a study was being undertaken by one of the staff members at his school.

Participants in this study were notified that their participation in this study was completely voluntary and they could withdraw at any stage if they wished to. All participants in this study were promised confidentiality and anonymity.

A meeting was held in my classroom during a lunch break before I began the research process. The nature, process and purpose of my study was outlined to all the participants. Learners were also invited to ask questions to seek clarity on any issue or uncertainty that they were experiencing at this stage. At this stage, letters of informed consent addressed to the learners and their parents were given. The letter requested permission from both the parents of the learner and the learner to participate in the study and to be interviewed at a time convenient to the learner.

4.8 Limitation of the study

My study used convenience sampling, where the participants were learners from my grade twelve mathematics class. Since the sample in my study is small it may not allow for the drawing of generalisations. Cohen et al., (2000) are also of the view that convenience sampling does not represent any group apart from itself, thus it does not seek to generalise about the wider population (p.103) but I, as the researcher provided sufficient information that could be used by the reader to determine whether the findings are applicable to similar situations. Merriam

(2000) states that "what we learn in a particular situation we can transfer to similar situations encountered" (p, 28). Therefore I do believe that my study will highlight grade twelve learners' experience when they encounter, for the first time, the concept of the derivative.

4.9 Conclusion

This chapter outlines the main aspects of the research methodology, the critical questions and the design used in my study. The techniques for data collection as well as how the data was analysed were discussed in detail. Triangulation and a limitation of my study were also discussed.

CHAPTER FIVE
PRESENTATION OF DATA FROM MAY AND AUGUST EXAMINATIONS

The main objective of this chapter is to discuss the data collected from the May and August examinations, and to give the researcher's interpretations supported mainly by evidence from learners' responses. Therefore, learners' responses to calculus questions in both examinations will be examined.

The data presentation reflects the steps I followed in 'getting into' the data. There are three layers of analysis that is presented. First an initial analysis was done to determine overall trends with respect to learners' performance. This is presented under the heading "Performance of learners" and tabulated using four categories of performance. Thereafter these categories are interrogated and an exhaustive classification of the different approaches and techniques used by the learners is presented. This second layer of analysis is presented under the heading "Interrogation of learners' responses". Finally the numerous categories are subsumed under four main headings which are suitable to describe the learners' difficulties across most of the questions in both examinations. Thus, the analysis of the data is firstly broadly categorised and followed by a more detailed, in-depth fine grain analysis.

5.1 Performance of learners

This section reveals how the learners performed in both examinations. Two learners were absent for either one of the examinations and they were excluded. The calculus section of the examination scripts of twenty seven learners who wrote the May and August examinations were scrutinised question by question. The marks obtained by each learner for each question was then recorded in a table form. Thereafter, depending on the mark that each learner obtained, the learner's responses were categorised.

The four categories are:

i) Correct response. Total marks.
ii) Response partially correct. Partial marks.
iii) Response provided. Zero marks.
iv) No Response. Zero marks.

The category, correct response total marks, represents learners who obtained total marks for their response. These learners provided solutions to questions that were totally correct and in keeping with the marking memorandum. A description of the marking memorandum was provided in chapter four. The calculus examination questions appear in appendix A and appendix B and some of the questions are also presented in the discussion of learners' responses. The second category represents learners who answered the questions but the solutions were not according to the marking memorandum for various reasons. Thus, learners obtained marks according to the marking memorandum for those aspects of the solution that were correct. Hence the category, response partially correct partial marks, was created.

Many learners provided responses to questions that were not according to the marking memorandum. These solutions were awarded no marks and thus categorised as attempted but obtained zero marks which was the third category. The fourth category represented learners who did not answer the question and no marks were given. Although category three and category four represents zero marks it is important to note that in category three, learners provided a response. This category also became the source of further analysis later on. Furthermore, for category four I did not probe any learner as to why they did not answer the question. There could be a variety of reasons for learners not answering the question.

The starting point of my analysis is recording learners' responses into these four categories. Table 5.1 shows the percentage of learners in each category for each question in the May examination. A similar process was used for the August examination and is represented in Table 5. 2.

Table 5.1 Classification of learners' responses to questions in May examination

May examination					
Question	Correct response. Total marks.	Response partially correct. Partial marks.	Response provided. Zero marks.	No response. Zero marks.	Total
1.1.	52	30	11	7	100
1.2.1.	44	22	22	12	100
1.2.2.	4	37	48	11	100
2.1.1.	89	0	11	0	100
2.1.2.	44	26	22	8	100
2.1.3.	15	11	52	22	100
2.1.4.	0	22	41	37	100
2.1.5.	15	26	22	37	100
2.2.1.	30	0	66	4	100
2.2.2.	7	70	19	4	100
2.2.3.	44	0	56	0	100
2.2.4.	26	0	70	4	100
2.2.5.	26	0	59	15	100

Table 5.2 Classification of learners' responses to questions in August examination

August examination					
Question	Correct response. Total marks.	Response partially correct. Partial marks.	Response provided. Zero marks.	No response. Zero marks.	Total
7.1.1.	30	30	33	7	100
7.1.2.	11	52	22	15	100
7.1.3.	8	33	33	26	100
7.2.1.	41	26	14	19	100
7.2.2.	7	37	26	30	100
7.2.3.	4	26	37	33	100
7.3.1.	22	4	63	11	100
7.3.2.	11	30	37	22	100
7.3.3.	15	33	26	26	100
7.4.	4	33	15	48	100

The above two tables reflect a trend in the learners' responses and shows that many learners have done well in questions which were straight forward and which required direct application of the rules for differentiation. For example forty four percent of learners provided a correct response to question 1.2.1 of the May examination and forty one percent of learners provided a correct response to question 7.2.1 of the August Examination. For these two questions, learners were first required to use the distributive law to multiply out the brackets and thereafter apply the rules of differentiation to find the derivative.

Less than ten percent of learners provided a correct response to questions 1.2.2, 7.2.2 and 7.2.3. Question 1.2.2 contained a fraction, part of question 7.2.2 contained a fraction and part of question 7.2.3 was in surd form. These three questions had to be written in simplified form first before finding the derivative.

No learner had question 2.1.4 from the May examination correct. The question itself had some ambiguity which may be a probable reason why no learner obtained the correct response.

Table 5.1 shows that more than fifty percent of learners' responses to questions 2.2.1, 2.2.3, 2.2.4 and 2.2.5 were incorrect. Question 2.2 was a problem on rate of change and the sub-questions were related to rate of change. Table 5.2 shows that a small percentage of learners obtained correct responses to question 7.4 of the August examination which required the application of the derivative to a problem on maxima and minima. The information in the above two tables revealed that many learners had some difficulties with questions that required application of the derivative. Further probing of learners' responses gave rise to the category "Interrogation of learners' responses" thereby hosting a more in-depth analysis.

5.2 Interrogation of learners' responses

A detailed analysis of categories partial response partial marks, and incorrect response zero marks, was then undertaken. Each learner's response in these two categories was scrutinised for any similarities or patterns that might exist. For the purpose of this analysis, five questions from the May examination are paired with a matching or similar question from the August examination. The detailed analysis using expanded categories for each of these pairs is presented in the five tables that follow. The questions in the tables are from the May and August examination and are similar to each other as the same concept is tested. Table 5.3 represents learners' responses to finding $f'(x)$ from first principles. I have used the symbol S_1 which represents student one, S_2 which represents student two and this format has been maintained throughout.

Table 5.3 Learners' responses to finding $f'(x)$ from first principles

	MAY EXAMINATION QUESTION 1.1.		AUGUST EXAMINATION QUESTION 7.1.1.	
	If $f(x)=3x^2+2$ find $f'(x)$ from first principles.		If $f(x)=-x^2+1$ find $f'(x)$ from first principles.	
	Student	Total	Student	Total
Perfect solution	S_{11} S_{13} S_{14} S_{15} S_{22} S_{26} S_{27}	7	S_9 S_{13} S_{14} S_{24} S_{26} S_{27} S_{29}	7
Solution correct. But wrote $f'(x)\lim_{h\to 0} = \ldots$	S_1 S_4 S_7 S_{10} S_{16} S_{20} S_{25}	7	S_4	1
Correct method, correct substitution but has computational errors either during expansion of brackets or simplification.	S_{12} S_{18}	2	S_{15} S_{25}	2
Used rules. $f'(x)=3x^2+2=6x$	S_3	1	S_3	1
Used rules. Incorrect answer.	S_9	1		
Correct formulae Incorrect substitution.	S_8 S_{24} S_{28} S_{29}	4	S_2 S_5 S_6 S_7 S_{10} S_{11} S_{12} S_{16} S_{18} S_{20} S_{22} S_{28}	12
No response.	S_2 S_{19} S_{23}	3	S_8 S_{23} S_{19}	3
Incorrect from step one.	S_6	1	S_1	1
Calculated $f(1)=3(1)^2+2=5$	S_5	1		
Total		27		27

The table shows that there are some trends. In May examination four learners chose the correct formula but substituted incorrectly into the formula and for the August examination this number increased to twelve learners. Seven learners in the May examination had some difficulty with the symbolism and wrote "$f'(x)\lim\limits_{h\to 0} = \ldots$" and this decreased to one learner in the August examination. Possible trends could also be detected from table 5.4 which shows learners responses to questions that required application of rules for differentiation.

Table 5.4 Learners' responses, requiring application of rules for differentiation, to questions containing products

	MAY EXAMINATION QUESTION 1.2.1		AUGUST EXAMINATION QUESTION 7.2.1	
	Find $\frac{dy}{dx}$ if $y=(x+1)(2-3x)$		Find $\frac{dy}{dx}$ if $y=-7x(x-2)$	
	Student	Total	Student	Total
Correct answer.	S_1 S_4 S_7 S_{10} S_{11} S_{13} S_{14} S_{15} S_{22} S_{25} S_{27} S_{29}	12	S_3 S_4 S_7 S_{10} S_{11} S_{13} S_{14} S_{22} S_{27} S_{29}	10
Multiplied brackets out correctly. No further details.	S_8 S_{16} S_{18} S_{26}	4	S_{12} S_{15} S_{16} S_{26}	4
Multiplied correctly. Differentiated incorrectly.	S_3	1	S_1 S_2 S_{25}	3
Multiplied incorrectly. Differentiated accordingly.			S_9 S_{19}	2
Multiplied incorrectly. Differentiated incorrectly.			S_5 S_6	2
Multiplied incorrectly. Factorised incorrectly. Solved for *x*.	S_{28}	1		
Solved for *x*.	S_2 S_9 S_{23}	3		
Multiplied correctly. Used $x=\frac{-b}{2a}=\frac{-(-1)}{2(3)}=\frac{1}{6}$	S_{20}	1		
Totally incorrect.	S_5 S_{12}	2		
No Response.	S_6 S_{19} S_{24}	3	S_8 S_{18} S_{20} S_{23} S_{24} S_{28}	6
Total		27		27

The above table shows that four learners multiplied the brackets out correctly with no further working details in the May and August Examination. Two learners, S_{16} and S_{26} repeated this in the August examination. In the May examination a further four learners solved for *x* and one learner used the formula to find the *x* co-ordinate of the turning point of the parabola. Aspects of quadratic theory were also evident in table 5.5 which represents learners' responses also requiring the application of rules for differentiation, but these questions contained a denominator.

Table 5.5 Learners' responses requiring application of rules for differentiation to questions containing a denominator

	MAY EXAMINATION QUESTION 1.2.2		AUGUST EXAMINATION QUESTION 7.2.2	
	Find $\frac{dy}{dx}$ if $y = \frac{x^2 + 6x - 4}{x}$		Find $f'(x)$ if $f(x) = \frac{3x}{x^3} - \sqrt{x}$	
	Student	Total	Student	Total
Totally correct.	S_4 S_{11}	2	S_{11} S_{14} S_{27}	3
Differentiated either only numerator or only denominator.	S_7 S_{22}	2	S_7	1
Differentiated the numerator. Differentiated the denominator.	S_1 S_{15} S_{27}	3	S_9	1
Correctly divided each term by *x*. No further working details.	S_{18}	1	S_{19}	1
Correct simplification. Incorrect differentiation.			S_{13} S_{25}	2
Correct method. Minor **slip.**	S_{13}	1	S_4	1
Incorrectly factorised the numerator. Solved for x .	S_3 S_9 S_{26}	3		
Incorrectly factorised the numerator. Simplified.	S_{12} S_{16}	2		
Incorrect simplification. Differentiated accordingly.	S_{10} S_{14} S_{25} S_{29}	4	S_{10} S_{22} S_{26} S_{29}	4
Incorrect simplification. Incorrect differentiation.			S_1 S_3 S_{15}	3
Attempted to factorise the numerator.	S_{23}	1		
Solved for x . Used the quadratic formula and solved for *x*.	S_2	1		
No response.	S_6 S_8 S_{19} S_{24} S_{28}	5	S_2 S_6 S_8 S_{12} S_{16} S_{18} S_{20} S_{23} S_{24} S_{28}	10
Totally illogical.	S_5 S_{20}	2	S_5	1
Total		27		27

Table 5.5 shows that in the May examination five learners incorrectly factorised the expression in the numerator, three of the five learners solved for *x* and S_2 used the quadratic formula to solve for *x*. More than thirty three percent of the learners did not provide a response in the August examination and a further six

learners simplified incorrectly. Some difficulties in simplification process is echoed in table 5.6, which represents learners' responses to finding the *x*- intercepts of the cubic function.

Table 5.6 Learners' responses to finding the *x*-intercepts of the cubic function

	MAY EXAMINATION QUESTION 2.1		AUGUST EXAMINATION QUESTION 7.3	
	The figure represents the graph of h where $h(x) = -x^3 + 3x + 2$ $= (x+1)^2(2-x)$		Given the function defined by $f(x) = -x^3 + 3x + 2$ $= -(x+1)^2(x-2)$	
	Write down the co-ordinates of A and D, the points of the X-intercepts.		Calculate $f(x) = 0$.	
Totally correct.	S_4 S_6 S_{10} S_{11} S_{13} S_{14} S_{16} S_{19} S_{23} S_{24} S_{25} S_{26} S_{27}	13	S_{14} S_{22} S_{24} S_{26} S_{27}	5
Totally incorrect.	S_{15} S_{18} S_{28}	3	S_{11} S_{18} S_{19} S_{28} S_{29}	5
No response.	S_8	1	S_1 S_6 S_8 S_{15} S_{16}	5
Substituted 0 in both forms of the equation. Has errors.			S_2 S_5 S_{20}	3
Used $b^2 - 4ac = 0$			S_{23}	1
Expanded $-(x+1)^2(x-2)$. Has computational errors.			S_3	1
Calculated $f(0)$.			S_9 S_{10} S_{12} S_{25}	4
Used either midpoint or distance formula	S_2 S_9 S_{20}	3		
Determined three roots of equation.			S_4 S_7 S_{13}	3
Merely wrote down incorrect ordered pairs.	S_3 S_5	2		
Correct method. Computational errors.	S_1 S_{22} S_{29}	3		
Correct method. Solved correctly for *x* only.	S_7 S_{12}	2		
Total		27		27

The above table shows that in the May examination thirteen learners had the *x*-intercepts of the cubic function correct and this decreased to five learners in the August examination. The table shows that many learners' responses to this question in the August examination were incorrect and a detailed analysis is provided in the various categories. Learners had to find the turning point of the cubic function and table 5.7 displays the categories that best describes their responses to this question.

Table 5.7 Learners' responses to finding the turning point of the cubic function

	MAY EXAMINATION		AUGUST EXAMINATION	
	Prove that the co-ordinates of C, the local maximum turning point of the graph of *h* are (1;4).		Determine $f'(x)$ and the local turning points of $f(x)$.	
Correct	S_4 S_6 S_{22} S_{27}	4	S_6 S_{13} S_{27}	3
Incorrect	S_9 S_{18} S_{25}	3	S_2 S_{19}	2
Differentiated correctly but did not equate $f'(x) = 0$	S_{11} S_{14}	2	S_3 S_4 S_{11} S_{14} S_{29}	5
Differentiated incorrectly and did not equate $f'(x) = 0$.			S_{10}	1
No Response	S_2 S_3 S_8 S_{16} S_{24} S_{28}	6	S_7 S_8 S_{15} S_{23} S_{25} S_{28}	6
Substituted 3 in $f(x)$.			S_{18}	1
Correct but did not equate $f'(x) = 0$. Answer correct.			S_{26}	1
Only one turning point found.			S_{22}	1
Only one turning point found. Incorrect.			S_1 S_{12}	2
Used $\frac{-b}{2a}$.	S_7 S_{10} S_{15} S_{20} S_{29}	5	S_9 S_{16} S_{20}	3
Used formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$			S_5 S_{24}	2
Used $x^2 + y^2 = r^2$	S_5	1		
Correct differentiation, Equated to 0 Solved for *x*	S_{26}	1		
Used midpoint formula	S_{12}	1		
Substituted (1;4) in original equation.	S_{19} S_{23}	2		
Correct differentiation. Equated to 0. Substituted (1;4)	S_{13}	1		
Used formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	S_1	1		
TOTAL		27		27

The above table shows that less than twenty percent of the learners were able to find the correct turning points. Of concern are the many learners who did not provide responses to questions and no probing of the category, No response. No mark, was undertaken. Some of the examination questions did not warrant the above tables as it may have appeared only in the May examination or only in the August examination. However a similar process was used to conduct a detailed analysis of these various questions and is presented in the various categories.

5.3 Common trends

Following exhaustive checks for similarities and trends in every examination question, the following four categories were identified as being suitable for analysis across the test items, enabling me to find common trends across the different test items. It was found that learners:

- Used inappropriate algorithms.
- Carried out incorrect algebraic simplification.
- Presented partial solution and
- Had problems with the symbolism associated with calculus.

Some responses to some of the questions were difficult to categorise as it did not fall into any of the above categories. This did not occur on a large scale and was rather isolated but warranted attention. Furthermore, some responses belonged to more than one category. For example a response could contain an inappropriate algorithm as well as incorrect algebraic simplification. Whenever this occurred, I placed it in the more dominating category. However, in the analysis of data from both examinations, it may sometimes be included in the discussion of both categories.

A brief description of each of the category is given. Thereafter, each category was further scrutinised for the main purpose of facilitating a fine grain analysis of learners' responses to the examination questions.

5.3.1 *Use of inappropriate algorithm*

In this category learners used an inappropriate algorithm to solve the given problem. The most commonly used algorithms was the use of quadratic theory, confusion between rules for differentiation and first principles and the use of formulae from co-ordinate geometry.

Use of quadratic theory

Very often learners drew on the notion of the theory relating to the quadratic relationship and quadratic equation. In grade eleven learners do an in-depth study of quadratic equations and the graph of the quadratic function. Learners often applied quadratic theory and quadractic formulae to inappropriate situations.

In the May examination learners were required to find $\frac{dy}{dx}$ if $y=(x+1)(2-3x)$. This question required learners to first expand the brackets before proceeding to find $\frac{dy}{dx}$ as the product rule for differentiation is not known at grade twelve Standard Grade level. Thus, learners had to use the rules for differentiation for this type of question. Learners apply the rule for finding $f'(x)$ given $f(x)=x^n$, the rule being $f'(x)=nx^{n-1}$. Some learners associated $y=(x+1)(2-3x)$ with the quadratic equation, $(x+1)(2-3x)=0$, which they know how to solve. For example S_2, S_9 and S_{23} solved for x and concluded that $\frac{dy}{dx}=-1$ and $\frac{dy}{dx}=\frac{2}{3}$. It seems as if the cue to solve for x was initiated by the factor form of the question. These two learners seemed to have associated the y with $\frac{dy}{dx}$ since they replaced the y with $\frac{dy}{dx}$. S_{28} expanded the brackets, $(x+1)(2-3x)=0$ incorrectly to

$3x^2 - 2x + 3 = 0$ and thereafter incorrectly factorised it to $(3x+1)(x-2) = 0$. This learner now solved for *x* and left the solution as $x = -\frac{1}{3}$ and $x = 2$.

S_{20} expanded the brackets, $(x+1)(2-3x)$, to $3x^2 - x + 2$ and thereafter proceeded to find $\Delta = \frac{-b}{2a} = \frac{-(-1)}{2(3)} = \frac{1}{6}$ seemingly not recognising the difference in the form of the question. This could be possibly linked to the discriminant which describes the roots which are *x* values. $\Delta = \frac{-b}{2a}$ is a formula used to find the axis of symmetry of a quadratic function and learners encounter this in mathematics in their grade eleven year. Another student, S_6, worked similarly in the August examination where learners were required to find $\frac{dy}{dx}$ if $y = -7x(x-2)$. S_6 expanded $-7x(x-2)$ to get $-7x^2 - 14x$ and proceeded to "find" $\Delta = \frac{-b}{2a} = \frac{-(14)}{2(-7)} = 1$.

The second type of question which required learners to apply the rules for differentiation was an algebraic expression containing a denominator. Since the quotient rule for differentiation is not known in grade twelve Standard Grade level, learners have to first simplify the given algebraic expression to free it from any denominator and then proceed to find the derivative of the resulting expression.

Learners were required to find $\frac{dy}{dx}$ if $y = \frac{x^2 + 6x - 4}{x}$. S_3 S_9 and S_{26} incorrectly factorised the numerator and 'solved' for x. S_3 'factorised' the numerator of $y = \frac{x^2 + 6x - 4}{x}$ to $\frac{dy}{dx} = \frac{(x+5)(x+1)}{x}$ and thereafter 'concluded' $x = -1$ or $x = -5$.

S_9 and S_{26} incorrectly factorised the numerator of $y = \frac{x^2 + 6x - 4}{x}$ to $\frac{x(x+6-4)}{x}$.

Both then 'cancelled' the x from the numerator with the x from the denominator and 'concluded' that $x = -2$.

S_2 used the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the roots of the algebraic expression in the numerator of $y = \frac{x^2 + 6x - 4}{x}$. This learner correctly identified the values for a, b, and c and substituted these values in the formula but made a computational error as $\frac{-6 \pm \sqrt{52}}{2}$ was simplified to $-6 \pm \sqrt{26}$ and 'concluded' that $x = -0{,}91$ or $x = 0{,}91$. A similar scenario is exhibited for the question on graphs.

At grade twelve level, the study on calculus usually includes a section where learners have to either draw or interpret the graph of a cubic function. In the May examination, the sketch of the cubic function was given and learners had to interpret the given sketch through a series of questions. In the August examination, learners had to draw the graph of the cubic function by first answering a series of questions. In both these examinations the sketch of the graph that resulted from the cubic function was exactly the same. But, the form of the algebraic representation of the function form varies slightly. The given function is shown in Table 5.8.

Table 5.8 Representation of the cubic function

May examination	August examination
$h(x) = -x^3 + 3x + 2 = (x+1)^2(2-x)$	$f(x) = -x^3 + 3x + 2 = \; -(x+1)^2(x-2)$

Learners were required to prove that one of the turning points of the cubic function is (1; 4) in the May examination and for the August examination learners had to find the co-ordinates of the turning point. In the May examination

S_{10} S_{20} S_{29} began their solution with the formula $x = \frac{-b}{2a}$. Thereafter S_{10} 'substituted' $b = -2$ and $a = 1$ into the formula $x = \frac{-b}{2a}$ which was simplified to 1. S_{10} then took this value as the *x* co-ordinate of C, the turning point of the cubic function. S_{20} 'substituted' $b = 3$ and $a = -1$ into the formula and simplified this to $\frac{3}{2}$. S_{29} substituted $b = 3$ and $a = 2$ into the formula, obtained $x = \frac{-3}{4}$ and 'simplified' further to $x = 4 - 3 = 1$. S_7 began the solution with an incorrect formula, $\frac{b^2}{2a}$, 'substituted' $b = 3$ and $a = -1$ and simplified this to $\frac{-9}{2}$. S_{15} merely wrote the formula $\frac{-b}{2a}$ with no further working details. The use of the formula, $x = \frac{-b}{2a}$ is also evident in the August examination but to a lesser degree.

It is important to note that only three learners (S_9 S_{16} S_{20}) used the above formula in the August examination as compared to five learners (S_7 S_{10} S_{15} S_{20} S_{21}) in the May examination. This decrease in the number of learners resorting to the use of the formula may be due to the phrasing of the question. In the August examination the question required learners to first determine $f'(x)$ and then find the local turning points of the cubic function whilst in the May examination learners had to "prove that the co-ordinates of C, the local maximum turning point of the graph of *h* are (1;4)". Learners were given precise direction in the August examination to first find $f'(x)$ and thereafter had to find the co-ordinates of the turning point of the function. In the May examination learners were left to use their own strategy to find the co-ordinates of the turning point. This formula $x = \frac{-b}{2a}$ is encountered in mathematics in grade eleven in the study of the graphical representation of the quadratic function and is used to find the *x* co-ordinate of the turning point of the parabola.

The fact that graphs of parabolas have just one turning point seemed to have influenced learners when trying to find the turning point of the cubic function. S_9 and S_{16} 'found' one turning point and 'sketched' the parabola and S_{20} also 'found' one turning point but did not provide a sketch. S_1 S_2 S_3 S_{12} and S_{22} 'calculated' only one turning point, which was incorrect, and 'drew' the graph of the parabola. S_5 also 'sketched' the graph of the parabola but did not use the above learners approach. Instead this learner 'used' the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the roots of the cubic function and substituted $a = 1, b = 3$ and $c = 2$. Two roots were obtained and the learner 'sketched' a parabola with these two roots.

In the May examination learners were given a question concerning a word problem on rate which required the application of derivative. The word problem was accompanied by a formula, $B(t) = -3t^2 + 30t + 1500$ where $B(t)$ represents the number of bacteria present, *t* hours later, was given. Learners had to provide a response to how many bacteria was present at the beginning of the observation. Instead of calculating $B(0)$, S_{12}, S_{19}, S_{25} and S_{28} equated $-3t^2 + 30t + 1500$ to zero and then incorrectly factorised and solved for *t*. Their responses were as follows:

S_{12}:

$$
\begin{aligned}
B(t) &= -3t^2 + 30t + 1500 \\
&= -3(t + 30)(t + 50) \\
&= -30 \quad or \quad -50
\end{aligned}
$$

S_{25}:

$$
\begin{aligned}
0 &= -3t^2 + 30t + 1500 \\
0 &= t(-3t + 30 + 1500) \\
0 &= t(-3t + 1530) \\
0 &= t(t - 510) \\
t &\neq 0 \quad or \quad t = 510
\end{aligned}
$$

S_{19}:

$$B(t) = -3t^2 + 30t + 1500$$
$$= (-3t + \quad)(t + \quad)$$

S_{28}:

$$-3t^2 + 30t + 1500 = 0$$
$$-3t - 30 - 1500 \quad = 0$$
$$(3t - 30)(t - 50) \quad = 0$$
$$3t - 30 \quad or \quad t - 50 = 0$$
$$t = 10 \quad or \qquad t = 510$$

For the above four learners it seems that "x^2" was a cue for them to factorise. S_{12}, S_{25}, and S_{28} factorised the quadratic aspect of the function and 'solved' for *t*. S_{19} also followed the same procedure but the factorisation of the algebraic expression is incomplete. Furthermore, S_{25} and S_{28}, equated the quadratic expression to zero instead of calculating B(0).

In summary, the influence of quadratic theory encountered in grade ten and eleven, was evident in many of the learners' responses to the examination questions and it was very strong. This was evident when learners were required to find the turning point of the cubic function and some learners used the formula, $x = \frac{-b}{2a}$ to calculate the *x* co-ordinate of the turning point. This formula is useful to find the *x* co-ordinate of the turning point or the axis of symmetry of the parabola. However, the number of learners that made use of this formula for the question in the August examination decreased. For this question, learners were first led to find the derivative before being asked for the turning point. Thus, it seems that when learners are guided by the question, they tend to cope with answering the question in a more meaningful way.

For questions that contained algebraic expressions of second degree, some learners factorised the expression. It seems that "x^2" acts as a cue for some learners to factorise and immediately after factorising they proceed to "solving" the equation. Also the form of the quadratic function, $y = (x+1)(2-3x)$ was taken as an instruction to solve the related quadratic equation instead of finding the derivative of *y* with respect to *x*. Solving of quadratic equations and sketching of parabolas are encountered in their grade ten and eleven year of study.

Some learners merely found only one turning point of the cubic function and then drew the sketch of the parabola. Aspects of the quadratic theory were a dominant feature in many of the learners' responses to the examination questions that had some resemblance to "quadratic" aspects. For example, the cubic function, $f(x) = -x^3 + 3x + 2$, which was given in both examinations had three terms and a quadratic function written in standard form, $f(x) = ax^2 + bx + c$, also has three terms.

Use of rules for differentiation instead of first principles and vice versa.

These responses were evident when learners were required to find $f'(x)$ from first principles given a function $f(x)$. In the May examination $f(x) = 3x^2 + 2$ and in the August examination $f(x) = -x^2 + 1$ was given. To answer this type of question the formula, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, had to be used, because of the stipulation that the derivative must be found from first principles. Some learners used the rules for differentiation to answer this question instead of finding $f'(x)$ from first principles. For example S_3's response to the question in the May examination was $f'(x) = 6x$ and to the August examination, $f'(x) = -2x$. S_9 also used the rules for differentiation in her/his response to this question but did not differentiate the function correctly.

On another occasion, learners were asked to find $\frac{dy}{dx}$ using the rules for differentiation. S_{12} began his solution to the question $y = (x+1)(2-x)$ with a formula $\frac{f(x+h)-f(x)}{h}$ while S_5 started his solution to $y=\frac{3x^2}{x}-\sqrt{x}$ with $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Learners had to find the co-ordinates of the turning points of the cubic function in the August examination. S_5 and S_{24} started their response to this question with the formula, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, continued to substitute incorrect values into the formula and obtained an answer of 0.

The conceptual understanding of the derivative lies within the formula, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, as working with this formula brings out and represents the various layers of the derivative that Zandieh (1997, 2000) describes. Some learners' responses did not show them working with the formula and it may be seen as if the various layers of the derivative was not present in their understanding of the derivative. But understanding the various layers of the derivative does not receive attention at the Standard Grade level as pointed out in chapter three.

Use of other inappropriate formulae

Some learners either used the midpoint formula or the distance formula inappropriately. Interestingly use of these formulae was only evident in questions which made reference to a point shown on the graph. For example, the May examination had a cubic function drawn and points A, C, and D was shown on the graph. Learners were required to find the co-ordinates of A and D and to prove the co-ordinates of C (1; 4).

To find the *x*-intercepts of the graph of the cubic function S_2 and S_{20} responded to this question by starting with $\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$. S_2 continued to substitute values for the variables and obtained incorrect co-ordinates for A and D while S_{20} did not proceed after the formula. S_9 began her/his response with $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, the distance formula, continued to substitute values for the variables and obtained incorrect co-ordinates for A and D.

To prove that the co-ordinates of the turning point was (1;4), S_{12} began her/his response with the midpoint formula, $\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$ and then continued to substitute some chosen values for the variables such that when simplified the correct ordered pair (1;4) was obtained.

Inappropriate formulae were used for the volume of a rectangular prism which is referred to as a rectangular box, in the given question. Learners were required to find the volume of this rectangular box in terms of x. S_7 and S_{18} began their solution to this question by writing $A=\frac{1}{2}b\times ht$ and S_{12} started with $A=L\times B$. The first formula is used to calculate the area of a triangle and learners are familiar with this as they encounter this from grade nine level. The *b* represents the base and *ht* represents the perpendicular distance between a side of the triangle and the corresponding opposite vertex. The second formula is used to calculate the area of a rectangle where *L* represents the length and *B* represents the breadth of the rectangle respectively. S_3's formula for the volume of a box was $V=(L+b)h$. Some learners used inappropriate formula when answering questions on rate.

The word problem on rate which is accompanied by a formula, $B(t)=-3t^2+30t+1500$ where $B'(t)$ represents the number of bacteria present, *t* hours later, was given. Learners had to calculate $B'(10)$, the rate of change at 10 hours. Some learners began their response with an inappropriate formula to the question. S_2 began the response with the formula, $S_n = \frac{n}{2}(a+l)$ while S_5 wrote $T_n = a(n-1)d$. It is pertinent to note that this question appeared in the May examination, directly after the learners completed a section on sequences and series which included the study and application of the formulae, $S_n = \frac{n}{2}(a+l)$ and $T_n = a(n-1)d$.

It is pertinent that formulae from analytical geometry and sequences and series were prevalent in some learners' responses and both these sections were undertaken by the learners just prior to the May examinations and they used these inappropriately.

5.3.2 *Incorrect algebraic simplification*

Three instances of incorrect algebraic simplification were identified. Many learners' responses revealed that they were not at ease working with the distributive law, when simplifying a fraction that contained algebraic expressions or when substituting values into formulae.

Distributive Law

Many learners' responses contained computational errors which were mainly related to expansion of brackets. For example S_{18}, S_{12} and S_{28} 'simplified' $-(3x^2+2)$ to $-3x^2+2$, S_{15} 'simplified' $-(-x^2+1)$ to $+1-x^2-1$, S_2 'simplified' $-(-x^2+1)$ to $-x^2-1$ and S_1 'simplified' $+(-x^2+1)$ to $+x^2+1$. S_{10} 'simplified' $-x(x+h)^2+1-x^2+1$ to $-x^2-xh+1+x^2+1$. S_9 and S_{19} 'expanded' $-7x(x-2)$ to $-7x^2+14$ while S_5 andS_6 'simplified' the expression to get $-7x^2-14x$.

Simplification of fractions

Learners had difficulty with simplifying fractions that contained algebraic expressions in the denominator. S_{25} simplified $\frac{-2hx-h^2}{h}$ by 'cancelling' the h^2 from the numerator with h from the denominator and obtained $-2hx-h$. S_{29} 'simplified' $\frac{4-h}{h}$ to 4 by 'cancelling' h from the numerator with h from the denominator. S_1 'simplified' $\frac{2hx+h^2+2}{h}$ to $2x+h+2$ by 'cancelling' h from the denominator with h from the first two terms of the numerator. S_{25} 'simplified' as follows:

$$\frac{x^2+6x-4}{x}$$
$$=\frac{x^2+6x}{x}-\frac{4}{x}.$$
$$=x+2.$$

S_{25} just 'cancelled' x from the numerator with x from the denominator of $\frac{x^2+6x}{x}$ and obtained $x+2$.

Substitution

Learners were given $f(x)=3x^2+2$ in the May examination and $f(x)=-x^2+1$ in the August examination and were required to find $f'(x)$ from first principles. Some learners had the correct formula to find $f'(x)$ but substituted incorrect values into the formula. For $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ S_2 and S_{22} 'substituted' $\frac{f(-x+h)^2+1-f(-x^2+1)}{h}$, S_5 'substituted' $\lim_{h\to 0}\frac{f(x^2+1)-f(x^2+1)}{h}$, S_6 'substituted' $\lim_{h\to 0}\frac{f(-x+1)--x^2}{h}$ and S_{10} 'substituted' $\lim_{h\to 0}\frac{-x(x+h)^2+1-x^2+1}{h}$

and S_{12} 'substituted' $\frac{-(x+h)^2 - -(x)^2}{h}$. Although only four learners exhibited this in the May examination, it increased to twelve in the August examination most probably because the algebraic expression given in the August examination contained a negative sign before x^2. Many learners experienced difficulties in the process of substitution especially with this particular type of algebraic expression that had a negative sign.

A diagram of a rectangular prism with the dimensions of the various sides in terms of *x* was presented to learners and they were required to find firstly the volume of the box in terms of *x*. Some learners wrote $V = L \times B \times H$, which is the formula for the volume of a rectangular prism, but they did not substitute the correct values into the formula. For example S_{16} begins the response as follows:

$$V = L \times B \times H$$
$$= 2x \times 2x \times 2x.$$

Learners' difficulties with algebraic simplification were clearly evident in their responses as shown in the above instances.

5.3.3 *Partial solution*

Many learners' responses were a partial solution to the examination question. Learners had either only one step written or had a partial solution with more than one step.

Only one step is written

This category is concerned with responses to questions that require multi-step working details, and only the first step is written with no further working details. For example some learners wrote down only the first step for the question that required $f'(x)$ from first principles. S_7 S_{19} and S_{28} wrote $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ with no further working details. For the question, find

$\frac{dy}{dx}$ if $y = -7x(x-2)$, five learners merely expanded the brackets of $y = -7x(x-2)$ and obtained $y = -7x^2 + 14x$ and showed no further working details. In the August examination learners were given a function $f(x) = -x^2 + 3x = 2$ and were required to determine $f'(x)$ and the local turning points of $f(x)$. S_3's response is as follows:

$$-x^3 + 3x + 2$$
$$f'(x) = -3x^2 + 3x.$$

This response contains only the first step of the solution as the learner has only differentiated correctly and did not find the co-ordinates of the turning points. In order to do this the learner had to equate $f'(x)$ to zero, solve for *x*, then substitute the zeros of the polynomial of $f'(x)$ into $f(x)$, simplify, and write the final answer as ordered pairs. S_2's response also follows a similar pattern, but contains computational errors.

Solution is incomplete but has more than one step

Learners were required to find $\frac{dy}{dx}$ and were presented with $y = (x+1)(2-3x)$. S_8, S_{16} and S_{26} simplified as follows:

$$\begin{aligned} y &= (x+1)(2-3x) \\ &= 2x - 3x^2 + 2 - 3x \\ &= -3x^2 - x + 2 \end{aligned}$$

The final step of their response was left as $y = -3x^2 - x + 2$ and did not contain any further working details as all three learners did not find the required derivative. S_{18} also followed a similar procedure but the response had computational errors as well.

Some learner's response shows more than one step for the question where they had to find $\frac{dy}{dx}$ if $y = \frac{x^2 + 6x - 4}{x}$. S_{18}'s response is as follows:

$$y = \frac{x^2}{x} + \frac{6x}{x} - \frac{4}{x}$$
$$y = x + 6 - 4x^{-1}.$$

This was the final answer with no further working details. S_{12} and S_{18}'s responses also contained more than one step but they simplified incorrectly. Their final answer was an algebraic expression which was free of a denominator and they did not differentiate with respect to *x*.

For the question on maximisation, learners had to find the dimensions of the rectangular box such that the volume is maximised. Firstly, the correct formula for the volume of a rectangular prism was needed. Thereafter, the volume had to be calculated in terms of *x* since the dimensions of the box is given in terms of *x*. Learners had to find $\frac{dV}{dx}$ before proceeding to find the dimensions of the box that would result in maximum volume. This question required multi-steps and was presented in the form of a word problem. S_4, S_{10}, S_{11}, S_{12} and S_{19} determined the volume of the rectangular box in terms of *x* and did not show any further working details. A similar trend was displayed in S_{18}'s response but the volume of the rectangular box was incorrect as the formula was incorrect.

5.3.4 *Problems with symbolism*

There were two main problems with symbolism namely the incorrect use of the limit notation and the incorrect use of the derivative notation.

Incorrect representation of limit notation

Although some learners' responses were correct there were problems with the limit notation as they either wrote, $f'(x)\lim_{h\to 0} = \ldots$ or $\lim_{h\to 0} = \ldots$ or $\overset{h\to 0}{\lim} = \ldots$ and this format was maintained throughout their solution. Altogether seven learners exhibited this in the May examination and only one of the seven learners repeated this in the August examination.

Incorrect use of derivative notation

Incorrect use of notation occurred especially in questions that required multi-step responses. Learners were required to find $\frac{dy}{dx}$ when $y=(x+1)(2-3x)$. S_3, S_8, S_{10}, S_4 and S_7 started their response as $\frac{dy}{dx}=(x+1)(2-3x)$ then simplified to $\frac{dy}{dx}=2x-3x^2+2-3x=-3x^2-x+2$. This notation was maintained throughout the various steps in their response and it ended with the correct notation as well as the correct answer as $\frac{dy}{dx}=-6x-1$. S_1 displayed a similar trend in the response but did not follow the same sequence of steps. This pattern was also displayed in responses to $y=-7x(x-2)$ as S_4, S_7, S_{12} and S_{26} began their response as follows:

$$\begin{aligned} y &= -7x(x-2) \\ \frac{dy}{dx} &= -7x^2+14 \\ &= -14x+14. \end{aligned}$$

S_{15} wrote $\frac{dy}{dx}=-7x^2+14x$ with no further working details.

For the question find $\frac{dy}{dx}$ if $y=\frac{x^2+6x-4}{x}$, S_4, S_{10} and S_{11} began their response with $\frac{dy}{dx}=\frac{x^2}{x}+\frac{6x}{x}-\frac{4}{x}$ and maintained this notation throughout the steps. S_4 and S_{11} differentiated correctly, S_{10} incorrectly simplified but correctly differentiated accordingly.

Some learners ignored notation and just continued answering the question. For example S_{29} response was as follows:

$y=(x+1)(2-3x)=2x-3x^2+2-3x=-3x^2-x+2=-6x-1$. According to this $y=(x+1)(2-3x)$ and $y=-6x-1$ but $\frac{dy}{dx}=-6x-1$. A similar trend was exhibited in the response to $y=-7x(x-2)$ as S_{10} S_{13} and S_{19} wrote $y=-7x^2+14x=-14x+14$. The answer represents the derivative of *y* with respect to *x* and not *y* itself. This showed that learners carried out certain rules in a series of disconnected steps with no understanding of the reasons for the steps. Learners could not distinguish between need for algebraic simplification as compared to differentiation of algebraic expressions.

For question 2.2, the formula $B(t)=-3t^2+30t+1500$ where $B(t)$ represents the number of bacteria present t hours later, was given and learners were required to calculate $B'(10)$, the rate of change at 10 hours. Fourteen of the twenty seven learners' responses were as follows:

$$\begin{aligned} B(10) &= -3(10)^2+30(10)+1500 \\ &= -300+300+1500 \\ &= 1500 \end{aligned}$$

S_{18} and S_{19} also calculated $B'(10)$, but their answer contains computational errors as well. Learners were familiar with calculation of function values when given a function as it was done in the grade ten and eleven year. $B(10)$ and $B'(10)$ are very similar in their representation. This association between $B(10)$ and $B'(10)$ is not based on a deep understanding between the function value at a point and the derivative or gradient of the function at a point, but rather on similarity in representation.

5.4 Conclusion

Learners' written responses to examination questions was the primary source of data collection. The analysis of the data within the various categories clearly portrays some difficulties learners experienced. Therefore, it was necessary to probe learners' thinking during their attempts in solving the problems and to probe their perceptions about the concept of the derivative. Semi-structured interviews were conducted to support the data from the written examinations and the analysis of the interviews are presented in chapter six.

CHAPTER SIX
PRESENTATION OF DATA FROM INTERVIEWS

The main objective of this chapter is to discuss the data from the semi-structured interviews conducted with four learners. Firstly an overview of the interview is presented. Thereafter, the researcher's interpretation of learners' understanding of the concept of the derivative is presented, supported mainly by evidence from learners' responses to the semi-structured interview questions.

6.1 Brief overview

Twenty seven learners wrote the two examinations. There were 10 learners in the upper half (that is learners whose combined results for both examinations was in the range 50%-90%) and 17 learners in the lower half (that is those learners whose combined results was in the range 0%-49%). Three learners from the upper half and three learners from the lower half were selected for the interviews. Unfortunately two learners did not arrive for their interviews due to some unforeseen problems. Therefore only four learners, coded as, Moti-M, Ners-N, Bran-B and Trav-R were interviewed. The learners in the upper half consisted of Moti-S_{13}-M, Ners-S_{22}-N, Bran-S_{27}-B and the learners in the lower half consisted of Trav-S_{25}-R. For Trav it became necessary to use R as T was used to represent the teacher.

The main purpose of the interview was to understand each learners' thinking during his or her attempts in solving problems concerning application of the derivative in the examinations and to thereby determine their understanding of the concept of the derivative that emerged. The interviews were seen as a means of producing data that could support the analysis of some difficulties experienced by these learners. The data arising out of the interviews was used in conjunction with learners' responses to the examination questions in order to provide a much deeper insight into learners' understanding of the concept of the derivative. Thus the main purpose of the interviews was to probe learners' perceptions and understanding of the concept of the derivative.

6.2 Conceptual understanding of the derivative

The main aim of my study was to explore the learners' understanding of the concept of the derivative and this was probed during the interviews. When Trav was asked about his understanding of calculus he replied:

> R: Mam, from what I know, from what I've been taught to do is, calculus is just finding x. Finding all forms of x and what x is unknown and just different forms and different like methods and finding different ways all these like, there's many, many things like things in calculus and you just sub.

He explained further:

> R: Well, just in this year we've learnt about derivative and finding the principle of $f'(x)$ that was the big new thing in grade twelve. We had to do in calculus other than that it's just simple finding x that has been taking up most of our calculus year.

When probed about his understanding of the derivative he replied:

> R: Honestly mam, I don't know what the derivative is. I don't know what it means but it's easy enough to work out, once you taught us how to do it. I don't know what it actually means.
>
> T: No ideas at all.
>
> R: No ideas at all what it really means.

During the interview Trav, on his own, mentioned derivative and finding $f'(x)$ but states that he cannot explain what it is as he has no idea what it means. For Trav, it seemed that the method and finding different ways of 'doing something' was more important to him as his focus of calculus was:

> "Finding all forms of x and what x is unknown and just different forms and different like methods and finding different ways all these like, there's many, many things like things in calculus and you just sub".

When Bran was asked about his understanding of the derivative he responded as follows:

> B: Derivative is basically just, like say we get given that number.
>
> T: Which number is it?
>
> B: Like ($-x^2$).
>
> T: Right.
>
> B: You can find out with two ways first with the first principle or just differentiation.
>
> T: Okay. When you say first principle, what do you mean by first principle?
>
> B: It's $f'(x)$, limits, $\dfrac{f(x+h)-x}{h}$, whatever.

It is evident from Bran's response that his focus was also on 'doing something' as his response to "*what is your understanding of the derivative?*", is that "*You can find out with two ways, first with the first principle or just differentiation*". He associated the derivative with its "mechanistic manipulation" of symbols which manifests itself when we use first principles to find $f'(x)$ or the rules for differentiation which learners are familiar with as it is in mathematics in the grade twelve syllabus.

Moti's response to the question,

> T: What is your understanding of the derivative?

is

> M: To derive where that point is on the graph.

Moti seemed to have coined this phrase 'to derive' as she made use of it several times during the interview which was revealed in the following extract:

> T: You say you derived it. How do you derive it? When you say derive what do you mean?
>
> M: I took a …, my calculus method.
>
> T: Right.
>
> M: You take the number that is on top and you and you multiply it by the number …, and you minus 1 from the exponent.

It seemed that for Moti 'to derive' was to use the rules of differentiation. This was also evident from the following extract when Moti was asked to talk about her experience with calculus.

> M: Not really![long silence]. Like the maximum and minimum part of it was a bit difficult.

She was probed further and she said:

M: That part I didn’t really understand … I think it was the maximum and minimum part: But I don’t know how to derive the …”

T: But you are using a very important word derive all the time. What does it mean, to you?

M: I don’t know how to explain it.

T: No, you can explain it in whichever way you want to explain it. You said quite a few times “derive”.

M: It’s like a derivative from a certain equation that you have and you derive another equation from it.

When probed further she replied similarly:

M: What did it mean to me? I don’t know. [long silence] It meant getting another equation but you asking in what way.

This extract revealed that Moti was able ‘to derive’ another equation from the original equation although she does not mention how she was going ‘to derive’ the second equation. She also stated that she does not know the meaning of the ‘derived’ equation. Furthermore, for Moti ‘deriving’ was making use of the rules for differentiation.

Ners response to the question,

T: So, what’s your understanding of a derivative now?

N: To find the average gradient’

T: To find the average gradient. The average gradient, you said is between two points.

N: Yes.

Ners associated the derivative with the average gradient but just prior to this she stated that average gradient was between two points.

Moti, Ners and Bran are learners who represented the upper half (50%-90%) and yet they find it difficult to explain their understanding of the derivative. Trav and Bran explanation of the derivative was in terms of ‘doing something’. Moti explained her understanding of the derivative in terms of the rules for differentiation and Ners stated that it is the average gradient. Obtaining good marks in an examination does not guarantee or does not indicate that a learner has developed conceptual understanding of the derivative. Trav was the learner from the lower half and he was quite upfront that he has no idea what the derivative means to him.

6.3 Application of the derivative in word problem

According to Table 5.1 question 2.2 requiring the application of the derivative seemed to have been answered poorly by many learners. The four learners were probed about this question in the interviews. Trav’s response was as follows:

R: Um, he is, okay, you are trying to find out, okay they’ve just put this into a word problem. They’ve put calculus into a word problem. And you have to find out how to work out different things in this equation through different signs, where as $t =$ time and $b =$ bacteria. And then the first question they are asking you is. How much bacteria which is in the beginning of your equation, is in the very beginning?

T: What do you think is in the very beginning? That means before they actually started this experiment.

R: Well, if you, they are talking about time you have to replace (t) with zero because you haven't started your experiment yet so it's at zero. I know that now because in the exams I was nervous, I was very muddled up and I did not know very much how to do it.

It seems that Trav's nervousness arose because of examinations and this has contributed to him being 'muddled up' when answering the question and this may have contributed to him not knowing what to do. Bran also maintained that this question on the microbiologist was a bit confusing to him. This was clearly evident as he said:

B: They just get confusing sometimes. Like you get confused when you first read it but you just try and make sense of it, but they're not too bad.

T: Why do you get confused when you read it for the first time? Is it the words or

B: Because there are lot of words and stuff you are trying to take in … you just gotta read it carefully and think about it.

For Bran it seemed that too many words in a problem does affect him as he said that he gets confused. For the question 2.2.2 Bran substituted 10 into the original function, thus finding $B(10)$ instead of $B'(10)$.

Moti also revealed that this type of problem caused confusion which was shown in the following extract.

M: Okay it was a bit confusing.

T: Yes.

> M: Because I am not exactly familiar with this kind of stuff. But eh [long silence], I tried to use the formula, that you showed us, … to work out the question.

It seemed that Moti's confusion arose because she was not familiar with this kind of problem.

Ners was also probed about her response to this question and her response was as follows:

> N: Well, this question was a bit confusing, but I haven't really worked with it before, but when I saw the $B'(10)$, I realised that it had to do with gradient and differentiation.

All four learners expressed some notion of uneasiness when working with problems which required the application of the derivative. They either stated that problems of this nature tends to be confusing for them (Moti, Ners and Bran) or as Trav stated that he gets 'muddled up' and it seemed that the examinations was also a contributing factor as he maintained that it made him nervous.

6.4 Preference for rules and formulae

The aim of my study was to determine learners' understanding of the concept of the derivative. During the interviews, the learners exhibited that they have a preference. The following extract demonstrated their preferences.

Trav feels that it was important to remember the formulae and rules for differentiation as revealed in the following extract:

> R: Calculus was definitely the most fun I had with maths, everything else not very much, but calculus because there's always a way to find out you can try with many different ways with one equation and finally you'll find your answer

because it's as hard as you'd think it is. You just have to remember your formulae which are not that hard either and remember your rules with derivative.

T: I'm glad you say derivative very often.

R: Ja!

T: Was there another name you had for derivative?

R: $\frac{dy}{dx}$

For Trav the fun part seemed to be working with the rules and formulae. He expresses delight as he maintained he has had the most fun with calculus.

Bran also maintained that working with the rules is much easier for him. It was also much quicker for him when working with the questions although at the same time it had very little meaning to him and it seemed that this was not much of an issue for him. This was clearly illustrated in the following extract:

T: Right, the next part is finding $\frac{dy}{dx}$. How did you manage with these problems?

B: Ah it's easy

T: What kind of methods you used and what strategy you used?

B: I just, eh, like say I just multiplied by, what you call it?

T: The exponent.

B: The exponent. Ja that's it, and then subtract 1 from the exponent.

T: So you basically using the rules, you would say.

B: Ja.

T: You applied the rules of the derivative or rules for differentiation.

B: Ja.

T: And this problem here. What did you do there?

B: I break it up like $\frac{9x^4}{3x} - \frac{6}{3x}$

T: That's very good. And you would say you were adequately prepared for question like this.

B: Yes, I find this stuff the easiest thing to do.

T: So would say it's easier for you to work from rules, with the derivative or with $f'(x)$ from first principles.

B: It's a lot easier with the rules, you could see that it was going to be $-2x$.

T: What meaning did the whole thing have to you?

B: Not much.[laughs]

He was then probed further:

T: What kind of sense it made to you if you learnt it as a set of rules?

B: It was a lot easier … a lot quicker and simpler. Like this whole thing. There is a lot more chance of you making a mistake as opposed to you like normal using the rules.

Bran has reasons for working with formulae and rules as it provided a lesser chance of him making a mistake.

For Moti working with rules was her comfort zone as her reply to:

> T: And this question here, using the rules to obtain the derivative, did you enjoy working with it?
>
> M: Ja! That was nice. The first principle and all that was so easy.

When asked why she preferred working with rules she said:

> M: I don't know, I just like to be … there is something to do and there you have to just do it that way. I don't like finding any round about way of doing it.

For Ners working with rules was easier and shorter as she said to find $B'(10)$:

> N: You could have used the rules its shorter.
>
> T: Alright and how did you react to this section, calculus?
>
> N: It was generally easier than the other sections.
>
> T: Why?
>
> N: Ah, I think because of the problems that they used, the formulae were easy to remember and to work with calculus compared to other sections.

These extracts revealed that these four learners had a preference for working with rules and formulae. Firstly, for them it seemed that working with rules and formulae was much easier. Secondly, it seemed that they are more comfortable working with the rules and formulae. If learners had a preference for this kind of activity then this will always take precedence over their conceptual understanding.

6.5 Conclusion

This chapter discussed the data from the semi-structured interviews. The data from these interviews was used to particularly answer research question two which is learners' perception about the concept of the derivative which is presented in chapter seven.

CHAPTER SEVEN
DISCUSSION OF FINDINGS AND CONCLUSION

The aim of this study was to determine learners' understanding of the concept of the derivative. This chapter aims to accomplish this by providing some answers. Firstly, a brief summary of the findings are presented. Thereafter some answers to both research questions are presented. The implications for learners, teachers, tertiary institution, curriculum developers and education departments are discussed and finally a limitation of this study is presented.

7.1 Findings

In unpacking the first question, it was found that most of the difficulties could be attributed to problems associated with previously encountered concepts or with algebraic symbolism.

Aspects of quadratic theory which learners encountered in the grade eleven year featured very strongly in learners' responses to the examination questions. Some concepts and formulae from analytical geometry and sequences and series, which were the most recent sections that learners encountered just prior to the May examination, also impacted on learners' responses to examination questions.

Some learners' responses demonstrated a lack of procedural fluency in carrying out algebraic procedures and this featured very often. For example, some learners had difficulty in applying the distributive law to expand the brackets that contained algebraic expressions. Learners also had difficulty simplifying algebraic fractions that contained algebraic expressions in the denominator.

Some learners experienced difficulties in providing a complete answer to examination type questions related to the derivative. Their responses contained a partial solution to questions that required multi-step working details. In many instances learners merely performed a single process like multiplying out a bracket only with no further working details. This rendered their response incomplete as it was only a partial solution to the examination question related to the concept of the derivative.

To find the derivative some questions required the application of the rules for differentiation while other questions required finding the derivative from first principles. Some learners experienced difficulties with these type of questions as they either used the rules for differentiation instead of using first principles and vice-versa.

A summary of the findings is presented below as answers to the research questions.

7.2 Research Question One

What are some difficulties learners experience when answering examination questions related to the concept of derivative?

7.2.1 *Learners applied aspects of quadratic theory inappropriately*

Aspects of the quadratic theory dominated learners' responses to examination type questions related to the concept of the derivative and this featured very often. One instance of the influence of the quadratic theory surfaced in the May examination where learners were required to find $\frac{dy}{dx}$ given $y = (x+1)(2-3x)$. Many learners viewed the quadratic function as a quadratic equation and 'solved the quadratic equation', $0 = (x+1)(2-3x)$ and 'concluded' $\frac{dy}{dx} = -1$ and $\frac{dy}{dx} = \frac{2}{3}$.

A second example was when learners were required to find $\frac{dy}{dx}$ when $y = \frac{x^2 + 6x - 4}{x}$ and some learners incorrectly factorised the numerator to obtain $\frac{dy}{dx} = \frac{(x+5)(x+1)}{x}$ and thereafter 'concluded' $x = -1$ and $x = -5$. It is clear that for these learners the factor form and standard form of the quadratic expression has evoked certain associations in their minds as these associations determined the way they reacted, and in this case they reacted by solving the quadratic aspect of the given question as a quadratic equation. The learners' response that is visible is the external reaction to the association.

A third instance of the influence of quadratic theory was demonstrated when the function, $h(x) = -x^3 + 3x + 2 = (x+1)^2(2-x)$ was given in the May examination and in the August examination another form of the function, $f(x) = -x^3 + 3x + 2 = -(x+1)^2(x-2)$, was given. One of the questions for this function, learners were asked to prove that the co-ordinates of the turning points of the function was (1;4). Many learners began their solution with the formula $x = \frac{-b}{2a}$ and thereafter 'substituted' values for a and b. It is pertinent to note that this particular cubic function had only three terms as with the case of the standard form of a quadratic expression, namely $ax^2 + bx + c$. It seemed that learners associated $f(x) = -x^3 + 3x + 2$ with that of the quadratic function on the basis of it having three terms without taking particular note of $f(x)$ being of third degree. There was eight instances of learners opting to use $x = \frac{-b}{2a}$. This common response of using the formula $x = \frac{-b}{2a}$ can be seen as a demonstration of their external reaction to their first association without checking their thought processes which is a typical pseudo-conceptual behaviour.

Vinner (1997) distinguishes between conceptual behaviour and pseudo-conceptual behaviour. He maintains that conceptual behaviour is based on meaningful learning and understanding where words are associated with ideas and ideas involve concepts as well. In pseudo-conceptual behaviour, words are associated with words, ideas are not involved therefore concepts are not involved as well. Furthermore, pseudo-conceptual thought processes is based on the assumption that it is simpler, easier and shorter than true conceptual processes. According to Vinner (1997), when learners are faced with a task, they start looking for ways that will enable them to perform the task.

Sometimes learners' natural cognitive reaction may be as a result of certain cognitive stimulus and in the above three examples, it is likely the stimulus was the similarity in representation of aspects of the given question to aspects of the quadratic theory. The above three instances show that the learners used the cognitive stimuli, namely similarity in representation with aspects of the quadratic expression or quadratic function, "without going through any reflective procedure, control procedure or analysis of any kind" (Vinner, 1997, p, 101). Thereafter, they applied it to a new situation where they had to find the co-ordinates of the turning point of the cubic function by usually, first finding the derivative of the cubic function. This demonstrates that these learners based their associations on the face value of the questions and reacted to their first association without checking their thoughts which can be seen as a typical pseudo-conceptual behaviour Vinner (1997).

7.2.2 *The sequencing of topics influence learners' responses to examination questions related to the concept of the derivative*

It was noted that the sequencing of concepts taught also appeared to influence learners' responses. My study showed that some learners used formulae that were inappropriate to calculus concepts, but appeared in the most recent concepts that were covered in class. For example, learners were required to find the co-ordinates of the x –intercepts of the cubic function, which was shown on the sketch graph. Some learners used the midpoint formula, $\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$, which is used to find the co-ordinates of the midpoint given two points on a straight line. This formula was encountered in analytical geometry towards the end of grade eleven year and was revised just prior to the May examination.

Some learners also used formulae pertaining to sequences and series inappropriately. To calculate $B'(10)$, some learners used the formula, $S_n = \frac{n}{2}(a+l)$ which is used to find the sum of an arithmetic series and some used the formula $T_n = a(n-1)d$ which is used to find the term of an arithmetic sequence. A likely explanation is the use of the phrase "$B(t)$ represents the number of bacteria" which may have being interpreted as a calculation leading to a certain number. It is pertinent to note that the section, sequences and series, was completed just prior to the May examination and learners would have encountered these formulae at this stage. The learners reacted to their first associations where they associated some aspect of the question, most likely the number of bacteria, which required the use of the above formulae without checking their thought processes which according to Vinner (1997), shows a typical pseudo-conceptual behaviour.

7.2.3 *Learners experience some difficulties with the symbolism and notation associated with the concept of the derivative*

My study showed that there were three areas where students struggled with symbolism and notation. These were with the symbols $B'(10)$, $\frac{dy}{dx}$ and with $f'(x)$. The underlying meaning and ideas of many mathematical concepts is conveyed through its symbolism and notation. There are a variety of ways of denoting the derivative of a given function. One way of denoting the derivative is as $f'(x)$ when $f(x)$ is given and this notation is used most frequently in mathematics in grade twelve in the Standard Grade curriculum.

The first instance where many learners experienced difficulties was with the calculation of $B'(10)$. When learners were presented with a formula, $B(t) = -3t^2 + 30t + 1500$, where $B(t)$ represents the number of bacteria present t hours later, and asked to calculate $B'(10)$, the rate of change at 10 hours they calculated $B(10)$ instead. The underlying conceptual meaning of substituting $t = 10$ into the original equation is finding the number of bacteria present at 10 hours. The learners would have encountered this in their grade ten and eleven year of study as they do an in-depth study of functions. It appears that the symbolism $B'(10)$ has evoked an association with $B(10)$ and the learners' reacted by substituting 10 into the original equation without reflecting on or without analysing what it means. The learners did not consider the ideas in which the concept of the derivative is involved, in this case the derivative as a rate of change. The learners went through the process of substituting 10 into the original function, carried out the computational aspects and produced an answer. White & Mitchelmore (1996) argue that students have a primitive understanding of the variable and they very often search for symbols to which known procedures are applied regardless of what the symbol refers to. In this particular instance learners seem to have remembered the procedures solely in terms of the symbol used when they first encountered $B(10)$. This demonstrates that learners have according to White & Mitchelmore (1996), an "abstract apart" concept which shows the learners manipulation focus where they "have a concept of variable that is limited to algebraic symbols; they have learned to operate with symbols without any regard to their possible contextual meaning" (White & Mitchelmore, 1996, p, 91). Thus it may be seen that learners were not able to distinguish between $B'(10)$ and $B(10)$.

As Vinner (1997) maintains, “pseudo-conceptual behaviour is produced by pseudo-conceptual thought processes” and “pseudo-conceptual thought processes are simpler, easier and shorter than true conceptual processes. Under these circumstances, it is only reasonable to assume that many learners will prefer the simpler, easier, and shorter processes to the more complicated conceptual processes” (p, 101). The learners when presented with the task of calculating $B'(10)$, looked for ways that would enable them to perform the task. The way these learners responded to this question was not necessarily the way the examiners expected when they designed the task. The examiners intended learners to use conceptual thought processes but they used pseudo-conceptual thought processes as reflected in their responses. The learners resorted to finding associations with previously encountered concepts in order to respond to examination questions related to the concept of the derivative. For the learner the most important issue was to provide a response to the question by whatever way is the simplest, easier and shorter route.

The second instance where learners struggled with the notation was with finding $\frac{dy}{dx}$. Learners were required to find the derivative of a function expressed as a product of two linear factors. Some learners determined $\frac{dy}{dx}$ by only multiplying out the brackets and concluded that it was $\frac{dy}{dx}$ without actually doing the process of differentiation. These learners associated the symbol $\frac{dy}{dx}$ with the process of only multiplying out the brackets as the question contained brackets. White & Mitchelmore (1996) contend that “students treat the visible symbols as candidates for well- known manipulation rules instead of considering the meaning of the symbols” (p, 91). Clearly in this situation, the learners identified the expression within the brackets as a candidate for the known procedure of multiplying out the brackets. The symbol $\frac{dy}{dx}$ was totally disregarded by these learners. These learners have learned to operate with symbols without any regard to their possible contextual meaning. Furthermore, they did not reflect on their answer and the representation of the given question was the stimulus that seemed to have influenced their responses.

7.2.4 *Learners lack procedural fluency in algebraic procedures*

For learners to be successful in mathematics learning they must be "mathematically proficient," a term used by Kilpatrick (2001). Mathematical proficiency consists of five interwoven and inter-dependent strands which are:

- Conceptual understanding – learners can comprehend mathematical concepts, operations and relations.
- Procedural fluency – learners acquire skill in carrying out procedures flexibly, accurately, efficiently and appropriately.
- Strategic competence – learners' ability to formulate, represent and solve mathematical problems.
- Adaptive reasoning – learners' capacity for logical thought, reflection, explanation and justification.
- Productive disposition – learners are inclined to view mathematics as sensible, useful and worthwhile, combined with a belief in diligence and one's own efficacy (Kilpatrick, 2001, p, 115).

The analysis in chapter five, revealed that many learners had difficulty in carrying out procedures flexibly, accurately, efficiently and appropriately which can be linked to Kilpatrick's (2001) procedural fluency. For example learners were not able to accurately carry out the simplification process that was necessary in some questions containing algebraic expressions as fractions. Learners were unable to accurately multiply out the brackets in questions with algebraic expressions containing brackets. Furthermore, learners inappropriately 'factorised' cubic functions, using the procedure for factorising quadratic functions because the cubic function contained three terms which is synonymous with the standard quadratic expression representation. Many learners also solved and determined the roots of algebraic expressions as if they were equations. Thus it seemed that these learners lacked procedural fluency in basic algebraic procedures as they did not possess the knowledge of when and how to use these known procedures appropriately.

7.3 Research Question Two

What are learners' perceptions about the concept of the derivative?

In providing some answers to this question data from chapter six has been used. The words and phrases that are in italics has reference as it appears in chapter six from pages 79-89.

The interview analysis revealed that learners placed more emphasis on the rules to find the derivative than on conceptual understanding of the derivative. This was exhibited when Trav maintained that, "*I don't know what the derivative is … but it is easy enough to work out … you just have to remember your formulae … and remember your rules with derivative*". Bran equated his understanding of the derivative with the rules for finding the derivative as he said "*derivative is basically … you can find out with two ways first with first principle or just with differentiation*". Once again this was referring to the rules to find the derivative. Moti has coined the phrase, "*to derive where the point is on the graph*" as her understanding of the derivative. With further probing, Moti revealed that "to derive" for her is to use the rules for differentiation as she says " *I took a …, my calculus method … You take the number that is on top and you and you mutiply it by the number …, and you minus one from the exponent*". This extract showed Moti's explanation of the rule for finding the derivative of x^n. For the above learners, it seemed that they found it easier working with the rules than trying to understand the concept of the derivative. Applying the rules to find the derivative produced the correct answer which was of more importance and value to the learners as it got them more marks. Therefore, for the above learners understanding the concept of the derivative was of secondary importance.

Learners also indicated a strong preference for working with rules and formulae as it was much easier. Trav maintained that, "*you just have to remember your formulae which are not that hard*" and Bran maintained that it was "*quicker and simpler*". The learners also expressed some joy working with the rules as it produced the desired answer. This was clearly pointed out by Trav when he said "*calculus was the most fun I had with maths, everything else not very much, but calculus because there's always a way to find out*". Bran enjoyed working with rules as it was a lot easier for him. Moti also expressed joy working with the rules as she said "*there is something to do and there you have to just do it that way. I don't like finding any round about way of doing it*". Ners felt that " *the formulae were easy to remember and to work with calculus compared to other sections*". All four learners in their attempt to convey their conceptual understanding of the derivative revealed that they preferred working with the rules as it was much easier. These four learners encapsulated Vinner's (1997) argument that "*pseudo-conceptual thought processes are simpler, easier and shorter then true conceptual processes ... it is reasonable to assume that many students will prefer the simpler easier and shorter processes to the more complicated conceptual processes*" (p, 101). Learners strong preference for working with rules and formulae has led them to perceive the derivative as a rule. This may be adequate to answer basic questions related to the concept of the derivative but when learners were asked to respond to the word problem on rate of change, many of them experienced many difficulties, as Table 5.1 revealed that learners' performance in this question was poor. The average percentage of correct responses for the five questions was 26,6 %. White & Mitchelmore (1996) provide an explanation as to why students become more comfortable with de-contextualised problems than with contextualised problems.

All the de-contextualised problems look very similar which students can solve and the appropriate procedures are thus easy to follow. When students are successful in this narrow context, this can even lead to a sense of satisfaction as displayed by Trav, Bran, Moti and Ners when they work with the rules for differentiation. But in contexts where there are no visible cues, like the word problem on rate of change, where learners have to use the conceptual knowledge to solve the various questions they find it intellectually more demanding. Learning the derivative as a rule may have been adequate to deal with routine symbolic procedures but the limitations of such procedural knowledge became apparent when learners were required to solve problems on rate of change. Learners' perception of the derivative as a rule and their sense of satisfaction of obtaining correct answers to de-contextualised problems were not adequate to deal with problems that required conceptual understanding of the derivative. Although these four learners portrayed a strong preference for rules, because it is "easier, simpler and shorter" (Vinner 1997), it has failed them badly when it came to applying the concept of the derivative.

7.4 Implications of this study

This study has implications for learners, teachers, tertiary institutions and education departments as curriculum developers. Some of the implications are discussed.

7.4.1 *Learners*

Learners must be aware that if they shift their focus to learning of rules to cover their lack of conceptual understanding they will have difficulties with problems that require conceptual understanding. Therefore it is in their best interests to develop a deeper understanding of the concepts and they should endeavour to foster conceptual understanding during their learning process as this has many benefits.

7.4.2 *Teachers*

Teachers must not assume that the trajectory that they lay out during the teaching and learning process goes according to plan. Although teachers want learners to form associations based on ideas that foster conceptual understanding, teachers must acknowledge that learners may also form inappropriate associations that are triggered of by some stimulus that may be present in learning material. Therefore, when teachers design learning and assessment activities, they must be aware that learners may form some association with the face value of the question.

7.4.3 *Higher Education*

The lecturers from Tertiary Institutions must not assume that students who do a course in mathematics at university possess a robust understanding of the concept of the derivative since they have done calculus at school level. While students may be able to cope with applying the rules to answer simple differentiation problems they may lack the conceptual understanding of the derivative. Furthermore, obtaining good mathematics results in grade twelve does not really mean that learners have acquired the necessary conceptual understanding of the concept of the derivative.

7.4.4 *Department of Education*

The type of assessment tasks given by the department in examinations has certain implications for learners. This was clearly revealed in the interviews where the learners showed their concern for the examinations. If the focus of learning the concept of the derivative is for passing examinations then this may foster and perpetuate rote learning. Therefore, it seems that the relevant authorities need to re-consider the kind of assessment tasks that is administered to learners to ascertain their progression to the next grade or next phase. It seems that the type of assessment tasks may be a way forward to developing conceptual understanding of the concept of the derivative since it is assessment which seems to be a key factor in determining why people learn and fail. Another way of developing and fostering conceptual understanding of the derivative, is through the curriculum.

7.4.5 *Curriculum Developers*

In the Revised National Curriculum Statement (Department of Education, 2003) the concept of average gradient receives greater emphasis. One of the assessment standards at grade eleven requires learners to investigate numerically the average gradient between two points and thereby develop an intuitive understanding of the concept of the gradient of a curve at a point. In the grade twelve year learners are required to investigate and use instantaneous rate of change. They are required to accomplish this by first developing an intuitive understanding of the limit concept in the context of approximating the gradient of a function. This sets the stage for establishing the derivatives of various functions from first principles. With the emphasis now being on the concept of average gradient, leading to the concept of the derivative it is hoped that learners will develop a better understanding of the concept of the derivative and thereby reduce some of their difficulties that they have when answering examination type questions related to the concept of the derivative. But curriculum changes alone may not produce the desired change which is to develop conceptual understanding of the derivative. Curriculum change should be accompanied together with a change in the type of assessment tasks that are given to learners. The focus should now be on the assessment tasks that allows and fosters conceptual understanding of the derivative. Further research should be undertaken at school level to determine to what extent the development of the conceptual understanding of the gradient concept leading to the development of the concept of the derivative impacts on learners' understanding of the concept of the derivative as this is the focus of the RNCS for the learning of calculus.

7.4.6 *My personal professional growth*

The learners in my study, lacked procedural fluency, as they did not possess the knowledge of when and how to use known procedures appropriately. With the absence of such procedural fluency it has become difficult for learners to attain conceptual understanding as evident in many learners' responses to examination questions. According to Kilpatrick (2001), procedural fluency is especially needed to support and sustain conceptual understanding. "Without sufficient procedural fluency, students have trouble deepening their understanding of mathematical ideas or solving mathematical problems" (Kilpatrick, 2001, p, 122). The lack of procedural fluency of participants in this study formed a barrier to conceptual understanding leading to the development and understanding of the concept of the derivative. Furthermore:

> "when skills are learned without understanding, they are learned as isolated bits of knowledge. Learning new topics then becomes harder since there is no network of previously learned concepts and skills to link a new topic to. This practice leads to a compartmentalization of procedures that can become quite extreme, so that students believe that even slightly different problems require different procedures" (Kilpatrick, 2001, p, 113).

It seems that this belief arose in some learners as they associated aspects of quadratic theory to some visual representation especially when three terms or two linear factors in questions containing algebraic expressions were given.

Engaging in research of this nature has influenced me as a mathematical practioner. My focus on teaching and learning has shifted to fostering conceptual understanding in mathematical concepts as this is far more beneficial. Furthermore, I feel strongly that there is ample data at school level which can be used to research and improve mathematics teaching and learning in our country. One way of accomplishing this is to engage current mathematics teachers in research of this nature which hopefully will lead to an improved understanding of how learners learn and thereby improve mathematics teaching and learning.

7.5 Limitation of this study

My study has a limitation. Since the sample in my study was small the results may not allow for generalisations, however, the intention of a qualitative research is not to generalise. The result in findings may be different for other studies since only two sets of examination were considered. It would have been more fruitful to see the responses to the final examination calculus questions, but that was not possible. Nevertheless, I as the researcher provided sufficient information that can be used by the reader to determine whether the findings are applicable to similar situations.

7.6 Conclusion

In summary it is evident that this sample of grade twelve Standard Grade learners had many difficulties with the conceptual understanding of the derivative and their conceptual understanding of the derivative was very limited. With the conceptual understanding of the derivative being limited and sometimes absent, learners find it difficult to make sense of problems involving rate of change, maxima and minima. Bowie (1998, 2000) provides a model that explains what happens when students construct mathematical knowledge based on pseudo-structural conception of base knowledge. Students' actions on pseudo-objects become rehearsed into rules and:

> "the model suggests that students' errors are not arbitrary, but originate from an attempt to build a manageable set of techniques for dealing with calculus questions. Because this set of techniques is not grounded in a conceptual understanding of calculus, it provides an inadequate range of problem solving tools. The students' creative use of gap closing strategies is the way in which they render a given calculus problem manageable via this set of techniques"
> (Bowie, 1998, p, 123).

Bowie's (1998, 2000) model and Vinner (1997) suggest that learners' errors are not arbitrary, but arises when they try to build a manageable set of techniques for dealing with the task at hand. In my study learners resorted to pseudo-conceptual thought processes as a way of building a manageable set of techniques for dealing with the questions related to the concept of the derivative. Since learners understanding of the derivative is not underpinned and not grounded in conceptual understanding, they resorted to pseudo-conceptual thought processes whereby they tried to find associations with the representation of the given question. Their response to questions related to the concept of the derivative was obtained via these associations, which were inadequate for the answer to the question. But understanding why learners resorted to making these associations is important and relevant, as it helps to better understand how learners learn. It also helps us, perhaps for the future, to design learning activities that are underpinned by conceptual understanding and is more meaningful to learners.

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APPENDICES

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APPENDIX A MAY EXAMINATION QUESTIONS

QUESTION ONE

1.1 If $f(x) = 3x^2 + 2$ find $f'(x)$ from first principles. (6)

1.2 Find $\frac{dy}{dx}$ if

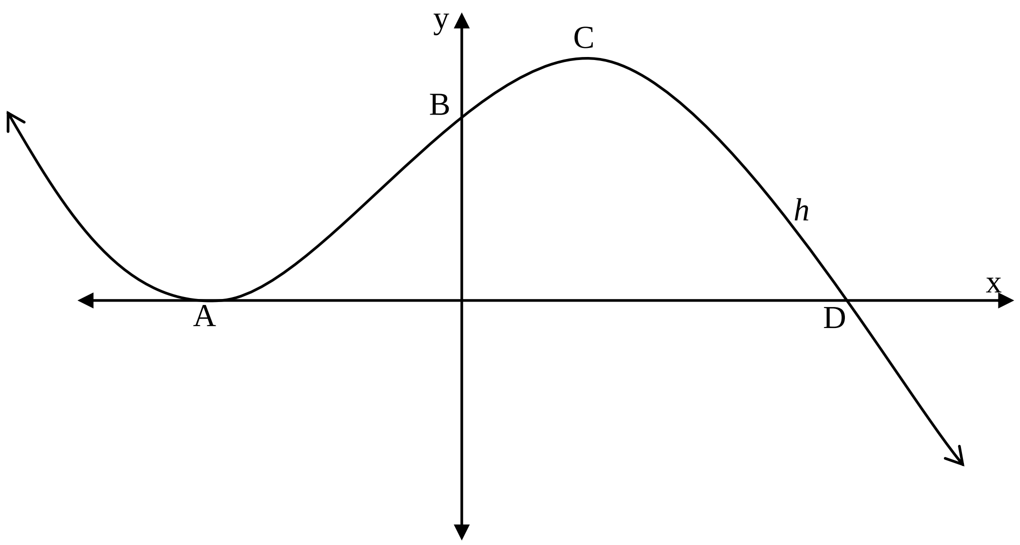
1.2.1. $y = (x+1)(2-3x)$ (3)

1.2.2. $y = \frac{x^2 + 6x - 4}{x}$ (4)

QUESTION TWO

2.1 The figure represents the graph of *h* where

$$h(x) = -x^3 + 3x + 2$$
$$= (x+1)^2(2-x)$$



2.1.1 Write down the co-ordinates of B where the graph of *h* cuts the *y* axis. (1)

2.1.2 Write down the co-ordinates of A and D, the points at which the graph cuts the *x*-axis. (3)

2.1.3 Prove that the co-ordinates of C, the local maximum turning point of the graph of *h*, are (1;4). (5)

2.1.4 The graph of *h* is rotated through an angle of 180^0 about the *x*-axis through the point (0;0) to produce the graph of a function *g*. Write down the equation that defines *g*. (2)

2.1.5 Write down the co-ordinates of the local turning point of the graph of *g*. (2)

2.2 A Micro-Biologist claims that a certain kind of anti-bacteria is added to a culture of bacteria, the number of bacteria is given by the formula $B(t) = -3t^2 + 30t + 1500$ where $B(t)$ represents the number of bacteria present *t* hours later.

2.2.1 How many bacteria was present at the beginning of the observation? (2)

2.2.2 Calculate $B'(10)$, the rate of change at 10 hours. (3)

2.2.3 Was the bacteria increasing or decreasing at the time $t = 10$ hours? (1)

2.2.4 At what instant was the maximum number of bacteria present? (2)

2.2.5 What was the maximum number of bacteria present? (3)

APPENDIX B AUGUST EXAMINATION QUESTIONS

QUESTION 7

7.1 If $f(x) = -x^2 + 1$

7.1.1 Determine $f'(x)$ from first principles. (5)

7.1.2 Find the gradient of f *at* $x = 2$. (2)

7.1.3 Find the average gradient of the curve between the points when $x = -1$ and $x = 3$ (3)

7.2 Determine the following:

7.2.1 $\dfrac{dy}{dx}$ if $y = -7x(x-2)$ (3)

7.2.2 $f'(x)$ if $f(x) = \dfrac{3x}{x^3} - \sqrt{x}$ (4)

7.2.3 $D_x[x.\sqrt[3]{8x^6}]$ (4)

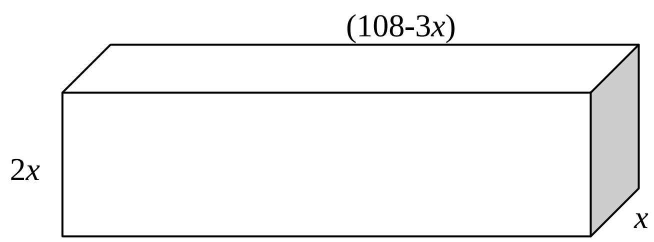
7.3 Given the function defined by $f(x) = -x^3 + 3x + 2 = -(x+1)^2(x-2)$

7.3.1 Calculate $f(x) = 0$ (2)

7.3.2 Determine $f'(x)$ and the local turning point of $f(x)$ (4)

7.3.3 Draw the sketch of the function $f(x) = -x^3 + 3x + 2$ showing clearly the points of intersection with the axes as well the co-ordinates of the turning points. (4)

7.4 If a rectangular box has a length, breadth and height of $2x$ mm, x mm, and $(108-3x)$ mm respectively, find an expression for the volume of the box. Then find x for the maximum volume of the box. (5)



APPENDIX C CONSENT FORM- PERMISSION

University of UKZN
Date: 01 Febuary 2006

The parent(s)/guardian

……………………….

……………………….

Dear………………….

REQUEST FOR PERMISSION FOR YOUR CHILD/WARD TO PARTICIPATE IN MY RESEARCH STUDY AT SCHOOL.

I am currently doing my Masters Degree in the field of Mathematics Education, through the University of KwaZulu-Natal. I am required to collect data relevant to my study. I have chosen your child/ward through the sampling process.

I therefore seek permission from you for your child/ward to participate in my research. Your child's/ward's May and August examination scripts will be used and your child/ward will participate in an interview. The interview will be conducted at school from 14H00 to 14H30.

All data collected will remain confidential and anonymity is assured. Furthermore participation is voluntary and your child/ward is free to withdraw at any time for any reason. All ethical issues will be adhered to.

My project supervisor is Dr S Bansilal. Contact details are
………………………….
My contact details are
………………………………………………………………

Thank you.
Yours faithfully
………………………….

……………………………………………………………………………………
……

Reply slip

I,
……………………………………………………………………………………..
(parent's/guardian's full name) hereby grant permission/do not grant permission for my child/ward to participate in the above research.

………………………………..   …………………...
Signature of parent/guardian   Date

APPENDIX D ETHICAL CLEARANCE